Discrete Laplacian for Bounded Biharmonic Weights

Supplementary Material for HW3

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The goal of this cheetsheet is to derive the discrete version of laplacian operator for tetrahedral meshes, so that we can compute the bounded biharmonics weights for a mesh based on:

$$\min_{\omega} \sum_{j=1}^{m} \frac{1}{2} \int_{\Omega} \|\Delta \omega_j\|^2 dV$$

where $\omega_j \in \mathcal{R}^n$. As the energy for each ω_j is independent, we use ω instead of ω_j in the following text.

1 Weak Formulation

First we change the problem setting to that we want to solve the following equation:

$$f = \Delta \omega$$

We want to compute f for each point in the space to be the laplacian of our weight function. Because we are going to solve it in a discrete domain, we have to take some weak formulation for that.

Instead of solving $f = \Delta\omega$, we try to find a function f, so that for any function v,

$$\int_{\Omega} vf = \int_{\Omega} v\Delta\omega \tag{1}$$

By Green's Identity:

$$\int_{\Omega} v\Delta\omega = \int_{\partial\Omega} v\nabla\omega - \int_{\Omega} \nabla\omega\nabla v$$

We assume the function v has zero value on the boundary of the domain, so that

$$\int_{\Omega} v\Delta\omega = -\int_{\Omega} \nabla\omega\nabla v$$

By discretizing the space with first order finite element, we choose the linear hat function as basis function. The value of a point inside each triangle can be approximate by a linear combination of the values on three vertices. For example, if we know the value of $f(x)$ on each vertex, $f(v_i) = a_i$, we can approximate the function value inside the triangle as $f(x) = \sum_i a_i \phi_i(x)$, where $\phi_i(x)$ is the coefficient. We can do such finite element descritization for our weight function ω, function v and our goal function f.

For the left hand of equation (1), we discretize it as

$$\int_{\Omega} vf = \int_{\Omega} \sum_i v_i\phi_i \sum_j f_j\phi_j = \int_{\Omega} \sum_i \sum_j v_i f_j \phi_i \phi_j = v^{\top} M f$$

where $M_{ij} = \int_{\Omega} \phi_i \phi_j dV$.

For the right hand of equation (2), we discretize it as

$$\int_{\Omega} v\Delta\omega = -\int_{\Omega} \nabla\omega\nabla v = -\int_{\Omega} (\sum_i \omega_i \nabla\phi_i)(\sum_j v_j \nabla\phi_j) = v \top L\omega$$

where $L_{ij} = -\int_{\Omega} \nabla\phi_i \nabla\phi_j dV$

Because this hold for any function v, we have

$$Mf = L\omega \Rightarrow f \approx M^{-1}L\omega$$

(Note: this is an approximating way to derive the laplacian operator, strictly derivation needs a much more complicated technique called mixed finite element).

So given the discretization basis functions ϕ and the weights ω, we can compute the laplacian as

$$\Delta\omega = M^{-1}L\omega$$

Now, let's look at how to compute the matrices M and L.

2 Triangle Surface Mesh

For triangle surface mesh, the derivation for laplacian operator can be found in [laplacian operator]. Please read it before deriving for tetrahedral mesh.

3 Tetrahedral Mesh

3.1 Mass Matrix M

As $M_{ij} = \int_\Omega \phi_i \phi_j dV$, it involved the product of h_i and h_j, which would be quadratic. We usually approximate the mass matrix by a diagonal lumped mass matrix instead of compute the exact matrix.

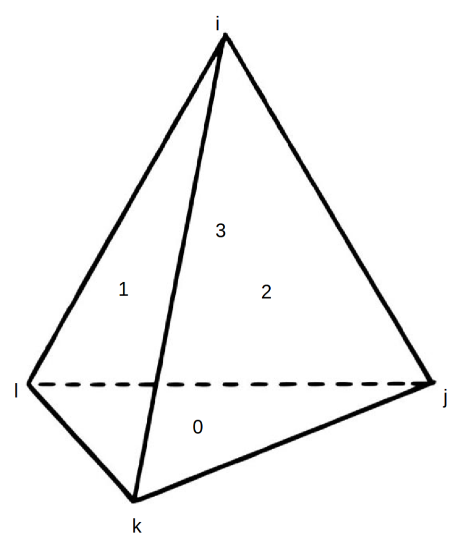
The most simple approach is that the value of the ith diagonal element is the surrounding volume of ith vertex.

$$M_{ii} = \sum_{T \in \mathcal{N}(v_i)} \frac{1}{4} V_T$$

where $\mathcal{N}(v_i)$ is the tetrahedron set in the ith vertex's neighbourhood and V_T is the volume of that tetrahedron.

3.2 Cotangent Laplacian Matrix L

The matrix $L_{ij} = -\int_\Omega \nabla\phi_i \nabla\phi_j dV$ is well defined. Since ϕ_i are piecewise linear functions in a tetrahedron, $\nabla\phi_i$ is a constant vector inside a tetrahedron, and thus $\int_\Omega \nabla\phi_i \nabla\phi_j$ yields one scaler per element.



Let's define some notation first. We label the four faces (F) of the tet from 0 to 4. The face 0 is opposite to vertex i and the face 3 is opposite to vertex j. S_k represents the area of the face k and θ_{ij} is the dihedral angle between face i and face j. l_{ij} is the length of the edge between face i and face j.

By following the derivation for the triangle mesh, we first evaluate the gradient $\nabla\phi_i$ in this tetrahedron. we know $\phi_i(i) = 1$ and $\phi_i(j) = \phi_i(j) = \phi_i(k) = \phi_i(l) = 0$. Because $\nabla\phi_i$ is constant inside the tet, we have

$$\nabla\phi_i \cdot (v_i - v_j) = 1$$
$$\nabla\phi_i \cdot (v_i - v_k) = 1$$
$$\nabla\phi_i \cdot (v_i - v_l) = 1$$

This yields:

$$\nabla\phi_i \cdot (v_j - v_k) = 0$$
$$\nabla\phi_i \cdot (v_l - v_k) = 0$$

which means $\nabla\phi_i \perp F_0$, which is the direction of the gradient.

Then we compute the magnitude:

$$\begin{aligned} 1 &= \nabla\phi_i \cdot (v_i - v_k) \\ &= \|\nabla\phi_i\| \cdot (\text{height on } F_0) \\ &= \|\nabla\phi_i\| \cdot \frac{3V}{S_0} \end{aligned} \tag{2}$$

So we get

$$\|\nabla\phi_i\| = \frac{S_0}{3V} \tag{3}$$

Now let's compute the element of matrix L. There are two cases, the first case is $i \neq j$ and i and j is connected by an edge. And the second case is $i = j$. We consider the contribution from one tet to L_{ij}.

For the first case, $i \neq j$.

$$\begin{aligned} \int_{tet} \nabla\phi_i \nabla\phi_j dV &= V \nabla\phi_i \cdot \nabla\phi_j \\ &= -V\|\nabla\phi_i\|\|\nabla\phi_j\| \cos\theta_{03} \\ &= -V(\frac{S_0}{3V})(\frac{S_3}{3V}) \cos\theta_{03} \\ &= -\frac{1}{9V} S_0 S_3 \cos\theta_{03} \\ &= -\frac{1}{6} l_{03} \cot\theta_{03} \end{aligned} \tag{4}$$

The last step is derived by applying the volume formula from Wolfram.

$$V = \frac{2}{3l_{03}} S_0 S_3 \sin\theta_{03}$$

For the second case, $i = j$.

$$\begin{aligned} \int_{tet} \nabla\phi_i \nabla\phi_j dV &= V\|\nabla\phi_i\|^2 \\ &= V(\frac{S_0}{3V})^2 \\ &= \frac{S_0^2}{9V} \\ &= \frac{1}{9V} S_0 (S_1 \cos\theta_{01} + S_2 \cos\theta_{02} + S_3 \cos\theta_{03}) \\ &= \frac{1}{6}(l_{01} \cot\theta_{01} + l_{02} \cot\theta_{02} + l_{03} \cot\theta_{03}) \end{aligned} \tag{5}$$

By combining equation (4) and (5), and considering all tets, we can get:

$$L_{ij} = \begin{cases} \sum_{e \in \mathcal{O}(ij)} \frac{1}{6} l_e \cot\theta_e & \text{if edge } ij \text{ exists} \\ -\sum_{i \neq j} L_{ij} & i = j \\ 0 & otherwise \end{cases} \tag{6}$$

$e \in \mathcal{O}(ij)$ means the edge e is opposite to edge ij in a tetrahedron.

4 Bounded Biharmonic Weights Optimization

Once we have the matrices ready, we can discretize our energy formula as

$$\begin{aligned}
\min_{\omega} \sum_{j=1}^{m} \frac{1}{2} \int_{\Omega} \|\Delta \omega_j\|^2 dV &= \min_{\omega} \sum_{j=1}^{m} \frac{1}{2} \int_{\Omega} (M^{-1} L \omega_j)^\top M (M^{-1} L \omega_j) \\
&= \min_{\omega} \frac{1}{2} \sum_{j=1}^{m} \omega_j^\top (L M^{-1} L) \omega_j
\end{aligned} \tag{7}$$