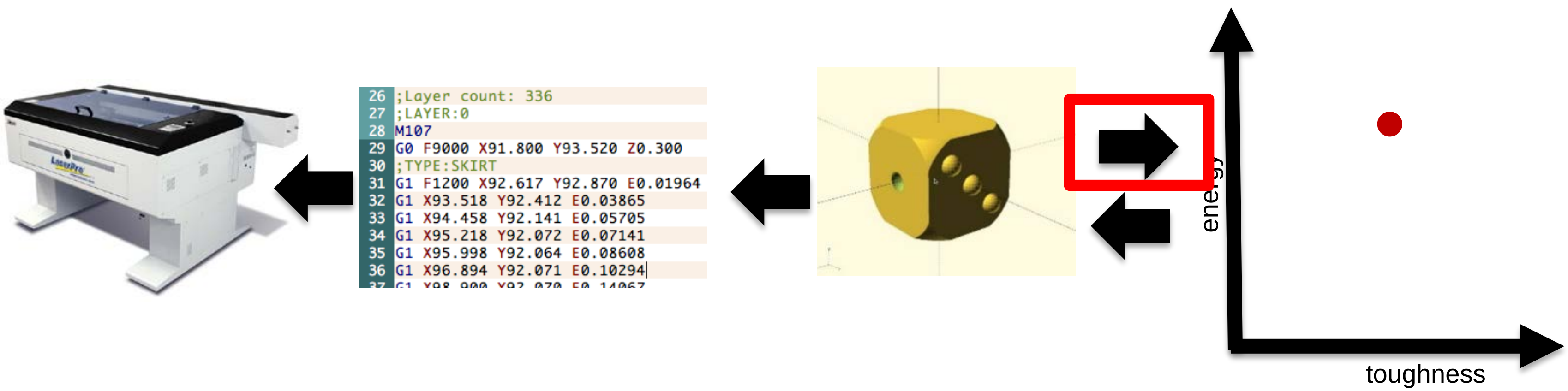
Statics on Deformable Bodies

Wojciech Matusik

Computational Design Stack



Hardware

Machine Code

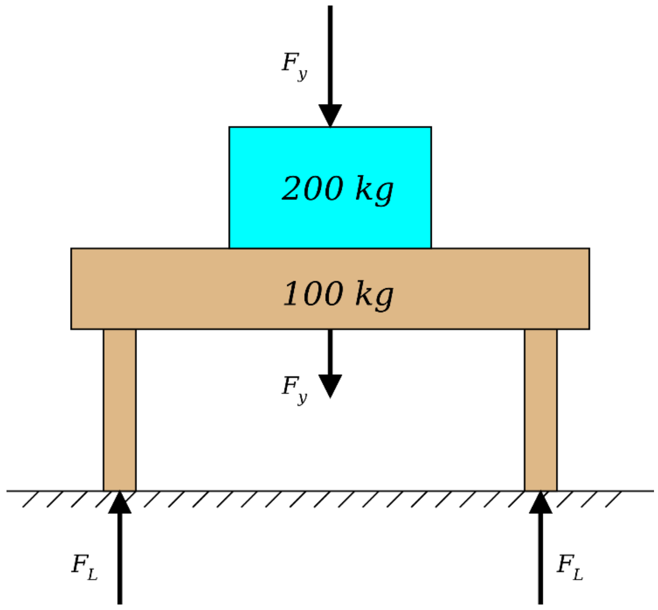
Direct Design

Functional Design

Statics vs. Dynamics

We call this time varying motion Dynamics

Statics is concerned with the case when net forces are balanced, acceleration is zero.



Statics on Deformable Bodies

- What can this help us measure?

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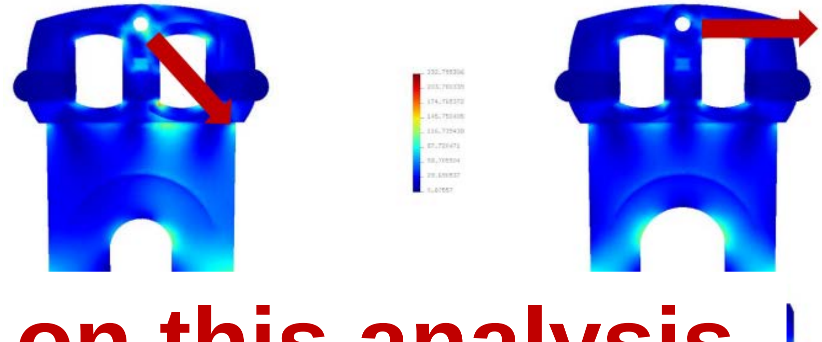
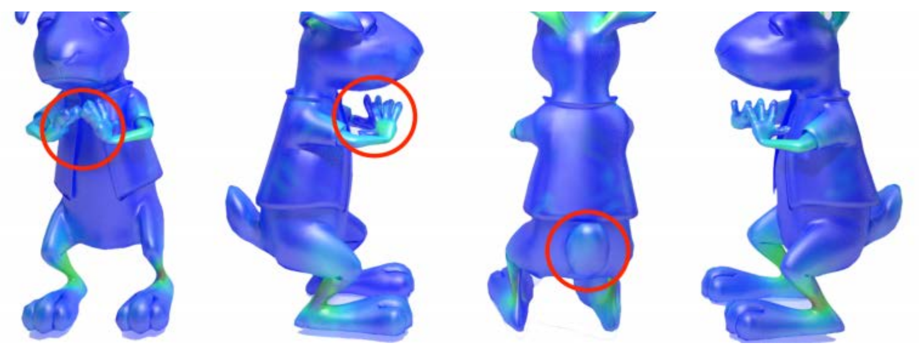


How objects

We will focus on this analysis in the next few lectures!



Where objects are likely to break

Statics on Deformable Bodies (Plan)

- Continuum Mechanics Intro
  - Spring Systems
  - Continuum Mechanics in 1D
- Continuum Mechanics in 3D
  - Strain
  - Stress
  - Material model (linear case)
- Discretization (3D)
  - Finite Elements
  - Solving for Static Equilibrium
- More material models

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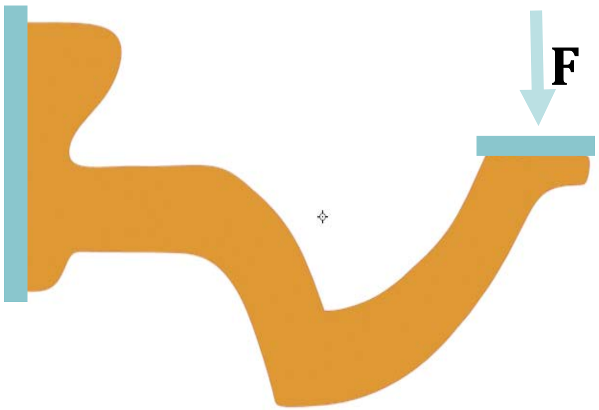
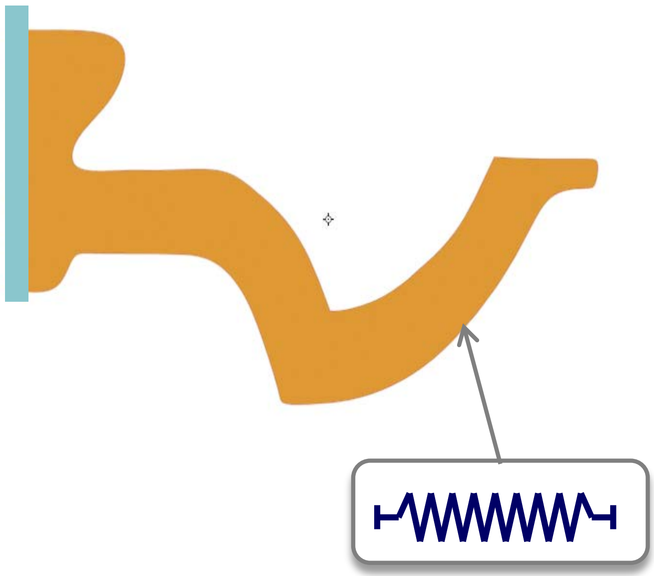
Model Problem – Coat Hanger



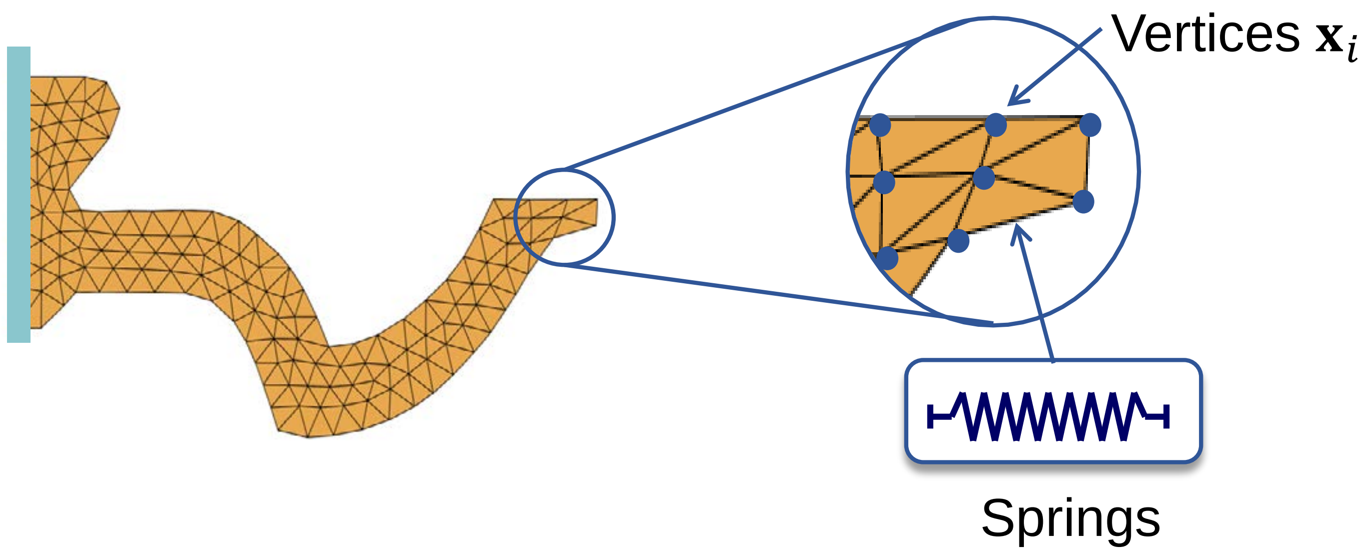
Modeling Elasticity



How to model elastic materials?

- Atomic or molecular mechanics
- Continuum mechanics
- Spring network abstraction

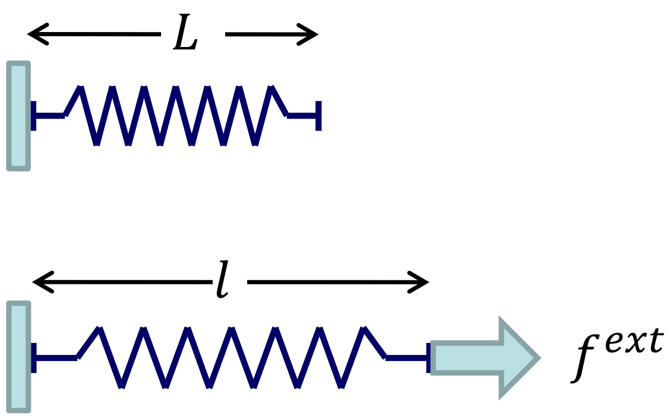
Spring Networks



Representation: 2D triangle mesh

- Vertices $\mathbf{x}_i \in \mathbf{R}^2$
- Edges E_{ij} connecting vertices $\mathbf{x}_i$ and $\mathbf{x}_j$

Hookean Springs



Elasticity: *Ability of a spring to return to its initial length when the deforming force is removed.*

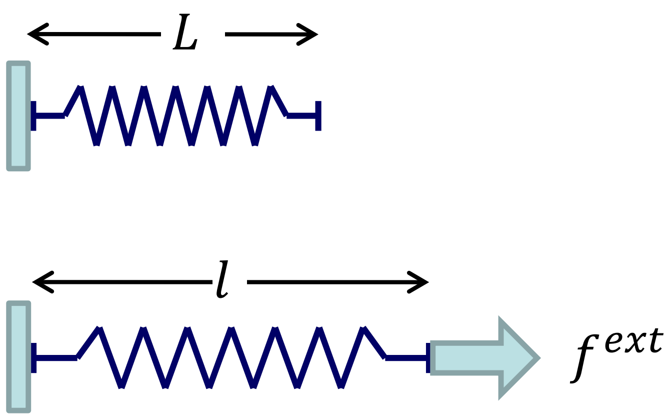
Spring Force:

$\rightarrow$ *Force is linear w.r.t. extension!*

$$\boxed{f^{ext} = k(l - L) \qquad \text{Hooke's Law}}$$

Length of undeformed spring L
Length of deformed spring l
Spring stiffness k

Hookean Springs



For elastic springs, forces are **conservative**, i.e., no energy is lost during deformation.

Work done by forces

$$W = \int_L^l f^{ext}(x)dx = \int_L^l k(x-L)dx$$

Stored energy of the spring is

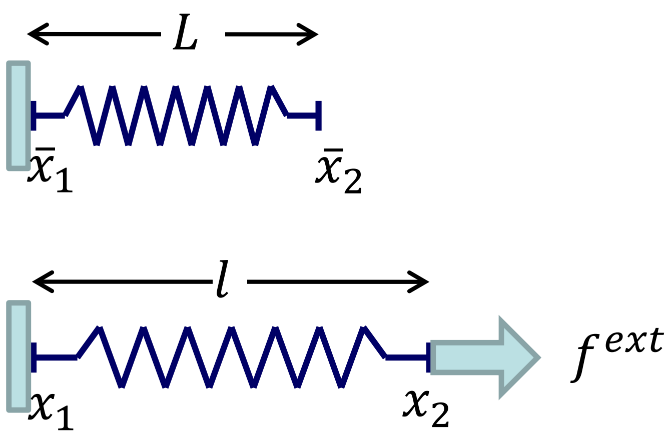
$$E = W = \frac{1}{2}k(l-L)^2$$

Force f^{int} exerted by spring follows as negative gradient of E,

$$f^{int} = -\frac{dE}{dx} = -k(l-L)$$

Length of undeformed spring	L
Length of deformed spring	l
Spring stiffness	k

Hookean Springs in $\mathbf{R}^n$



The configuration of a spring is determined by the position of its two endpoints.

We distinguish between

- Deformed positions $\boldsymbol{x}_1, \boldsymbol{x}_2 \in R^n$
- Undeformed positions $\overline{\boldsymbol{x}}_1, \overline{\boldsymbol{x}}_2 \in R^n$

Lengths are functions of positions, i.e.,

$l = |\boldsymbol{x}_2 - \boldsymbol{x}_1|_2$ and $L = |\overline{\boldsymbol{x}}_2 - \overline{\boldsymbol{x}}_1|_2$

$$l = |\boldsymbol{e}|_2 = \left(\boldsymbol{e}^T \boldsymbol{e}\right)^{\frac{1}{2}} \qquad \text{with} \quad \boldsymbol{e} = \boldsymbol{x}_2 - \boldsymbol{x}_1.$$

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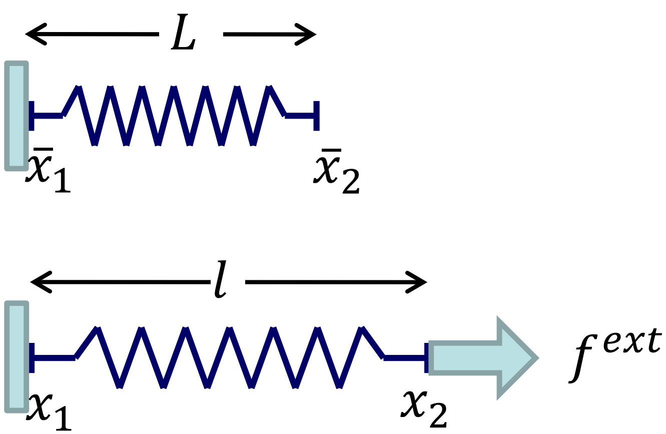

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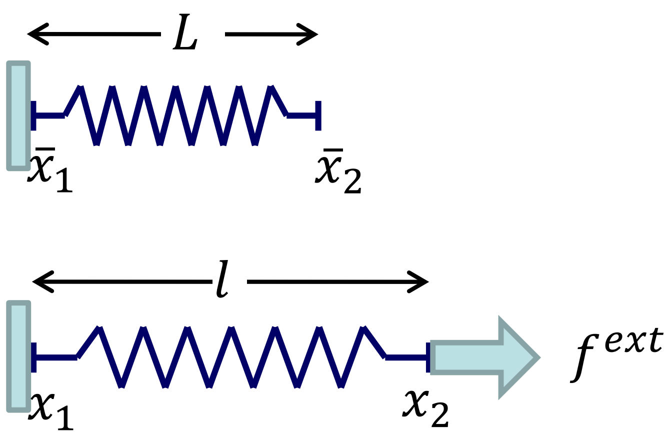
Hookean Springs in $\mathbb{R}^n$



Recall $E = \frac{1}{2}k(l-L)^2$ and
Force on $\boldsymbol{x}_1$ is:

$$\boldsymbol{f}_1 = -\frac{\partial E(\boldsymbol{x}_1, \boldsymbol{x}_2)}{\partial \boldsymbol{x}_1} = -\frac{\partial E(\boldsymbol{x}_1, \boldsymbol{x}_2)}{\partial l}\frac{\partial l}{\partial \boldsymbol{x}_1}$$

$$\frac{\partial E}{\partial l} = k(l-L)$$

$$\frac{\partial l}{\partial \boldsymbol{x}_1} = \frac{1}{2}\left(\boldsymbol{e}^T\boldsymbol{e}\right)^{-\frac{1}{2}}\frac{\partial\left(\boldsymbol{e}^T\boldsymbol{e}\right)}{\partial \boldsymbol{x}_1} = -\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{|\boldsymbol{x}_2-\boldsymbol{x}_1|}$$

$$\boxed{\boldsymbol{f}_1 = k(l-L)\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{|\boldsymbol{x}_2-\boldsymbol{x}_1|} \qquad \boldsymbol{f}_2 = -\boldsymbol{f}_1}$$

Hookean Springs – Generalization

- **Inconvenience**: springs with same material but different lengths will have different stiffness coefficients k:

  Rest length $L_1 = L$ *subject to* f *deforms to* $l_1 = l$.
  Rest length $L_2 = 2L$ *subject to* f *deforms to* $l_2 = 2l$.
  so $k_2 = \frac{1}{2} k_1$.

- **Idea**: use relative deformation $\varepsilon = \frac{l-L}{L}$ and stiffness $\tilde{k} = kL$. Then

$$f^{int} = -k(l - L) = -\tilde{k}\varepsilon \qquad \text{and} \qquad E = \frac{1}{2}\tilde{k}\varepsilon^2 L$$

- **Advantage**: $\tilde{k}$ is a material constant valid for all spring lengths L.

Spring Networks - Summation

Energy of spring network

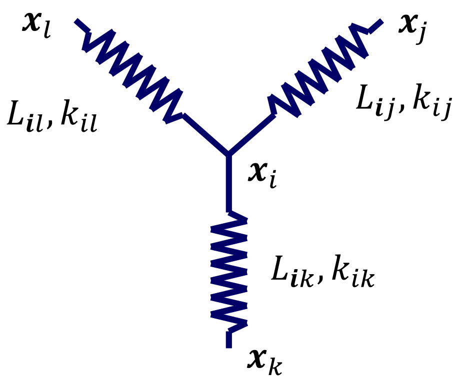
$$E = \sum_k E_k$$

Total spring force at given node

$$\boldsymbol{f}_i^{int} = -\frac{\partial E}{\partial \boldsymbol{x}_i} = -\sum_k \frac{\partial E_k}{\partial \boldsymbol{x}_i}$$

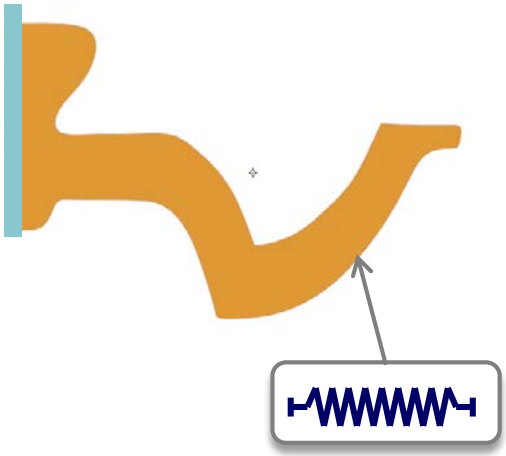
Total force at given node

$$\boldsymbol{f}_i = \boldsymbol{f}_i^{int} + \boldsymbol{f}_i^{ext}$$



Equilibrium Conditions - Forces

We can compute the total forces $\boldsymbol{f}(\boldsymbol{x})$ for a given configuration $\boldsymbol{x}$.



Given applied forces $\boldsymbol{f}^{ext}$, how to compute resulting configuration $\boldsymbol{x}$?

For static equilibrium, the acceleration has to be zero for all nodes,

$$\boldsymbol{a}_i(\boldsymbol{x}) = \boldsymbol{0} \ \forall i$$

From Newton's second law, we know that

$$\boldsymbol{f}_i(\boldsymbol{x}) = m_i \boldsymbol{a}_i(\boldsymbol{x}) = \boldsymbol{0}$$

Static Equilibrium Conditions

$$\boldsymbol{f}_i^{int}(\boldsymbol{x}) + \boldsymbol{f}_i^{ext} = \boldsymbol{0} \ \forall i$$

Equilibrium Conditions - Energy

Internal forces are negative gradient of internal energy E^{int}.

Assume that external forces derive from potential E^{ext}.

$$\boldsymbol{f}_i^{int} = -\frac{\partial E^{int}}{\partial \boldsymbol{x}_i} \qquad \boldsymbol{f}_i^{ext} = -\frac{\partial E^{ext}}{\partial \boldsymbol{x}_i}$$

Then, static equilibrium conditions

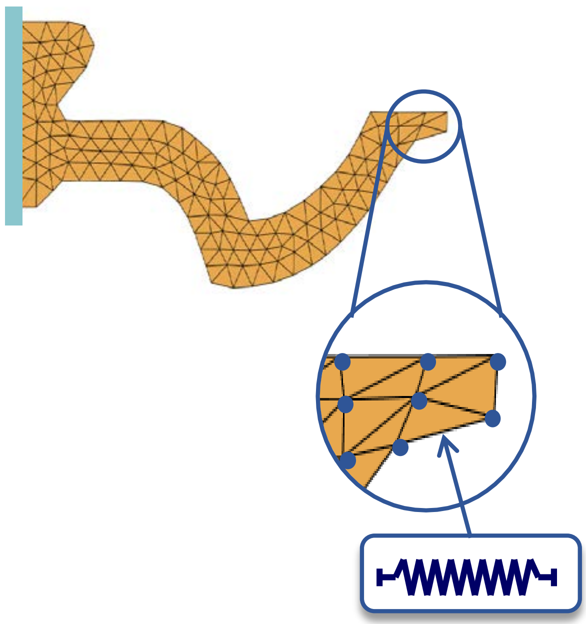
$$\boldsymbol{f}_i^{int}(\boldsymbol{x}) + \boldsymbol{f}_i^{ext} = \boldsymbol{0} \ \forall i$$

are equivalent to $\boldsymbol{x}$ being a **stationary point** for the total energy

$$E(\boldsymbol{x}) = E^{int}(\boldsymbol{x}) + E^{ext}(\boldsymbol{x}), \text{ i.e., } \frac{\partial E(\boldsymbol{x})}{\partial \boldsymbol{x}} = \boldsymbol{0}$$

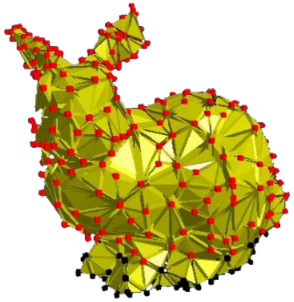
Mass Spring Systems

- Mass spring model
  - Mass points & spring forces
  - Easy to understand and implement
- Limited accuracy
  - Behavior depends on mesh
  - Finding spring stiffness coefficients to best approximate a given real material is difficult
  - No volume and area preservation



Continuum Mechanics and FEM

- Start from continuous model
  - Continuum mechanics
  - Equilibrium conditions
- Discretize with Finite Elements
  - Decompose model into elements (e.g., tetrahedra)
  - Formulate energy and derivatives per element
  - Minimize sum of per-element energies
- Advantages
  - Accurate material behavior
  - Largely independent of mesh structure



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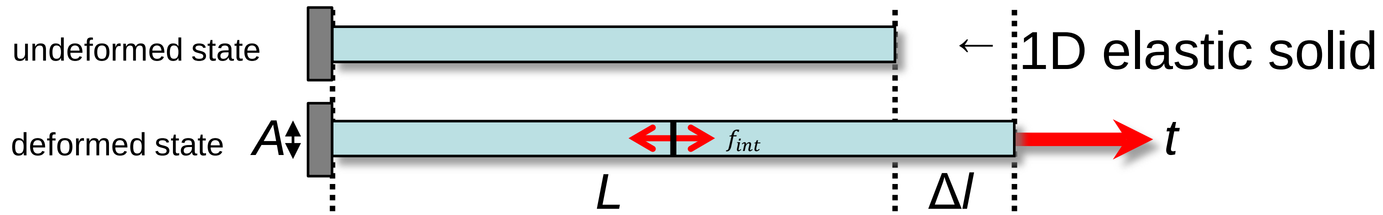
1D Continuous Elasticity

Given t, how to determine deformed configuration?

Principle of minimum potential energy

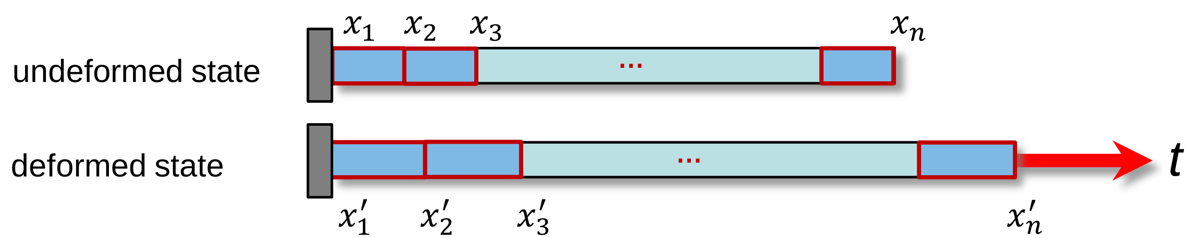
A mechanical system in static equilibrium will assume a state of minimum potential energy.

1D Continuous Elasticity



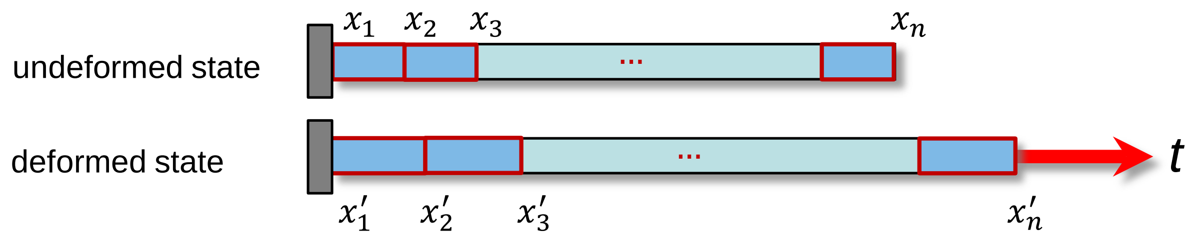
- Strain: $\varepsilon = \frac{\Delta l}{L}$ (*relative stretch*)
- Stress: $\sigma = \frac{f_{int}}{A}$ (*internal force density*)
- Hooke's law: $\sigma = k\varepsilon$ (k *material constant*)
- Strain energy density: $\Psi = \frac{1}{2}k\varepsilon^2$ (*postulate via* $\sigma = \frac{\partial\Psi}{\partial\varepsilon}$)

1D Continuous Elasticity



- Discretize domain into elements
- Element strain: $\varepsilon_i = \frac{x'_{i+1} - x'_i - L_i}{L_i}$ with $L_i = x_{i+1} - x_i$
- Element strain energy: $W_i = \Psi_i \cdot L_i = \frac{1}{2} k \varepsilon_i^2 \cdot L_i$
- Total strain energy: $W = \sum W_i$

1D Continuous Elasticity

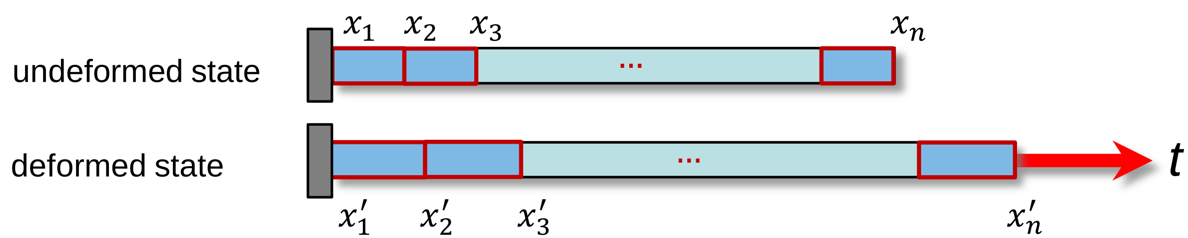


Minimum energy principle: at equilibrium

- system assumes a state of minimum total energy
- total forces vanish for all nodes

- $W_i = \frac{1}{2} k \varepsilon_i^2 \cdot L_i$ and $\varepsilon_i = \frac{x_{i+1}' - x_i' - L_i}{L_i} \rightarrow \frac{\partial W_i}{\partial x_i'} = \frac{\partial W_i}{\partial \varepsilon_i} \frac{\partial \varepsilon_i}{\partial x_i'} = -k\varepsilon_i$
- $f_i = -\frac{\partial W}{\partial x_i'} = -\frac{\partial W_{i-1}}{\partial x_i'} - \frac{\partial W_i}{\partial x_i'} = -k(\varepsilon_{i-1} - \varepsilon_i)$ for $i = 2 \ldots n-1$
- $f_1 = k\varepsilon_1$ and $f_n = -k\varepsilon_{n-1}$

1D Continuous Elasticity



Equilibrium conditions $f_i = \begin{cases} 0 & \forall i \in 2 \ldots n-1 \\ t & i = 1 \\ -t & i = n \end{cases}$

→ *n-2* linear equations for *n-2* unknowns x'_i

→ solve linear system of equations to obtain deformed configuration.

In this case (constant material, no body forces), deformation is constant.

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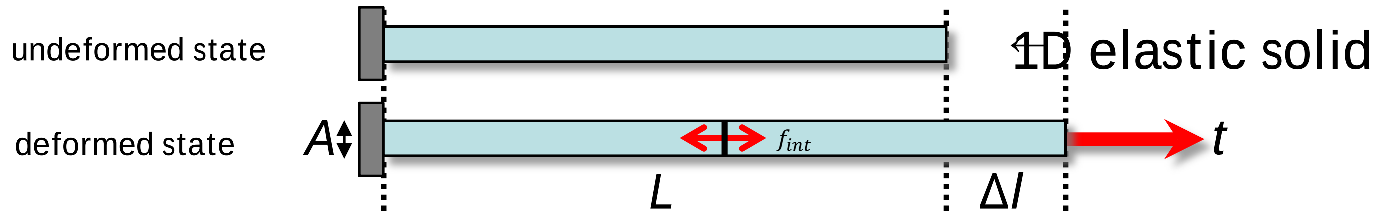
Recap 1D Continuous Elasticity



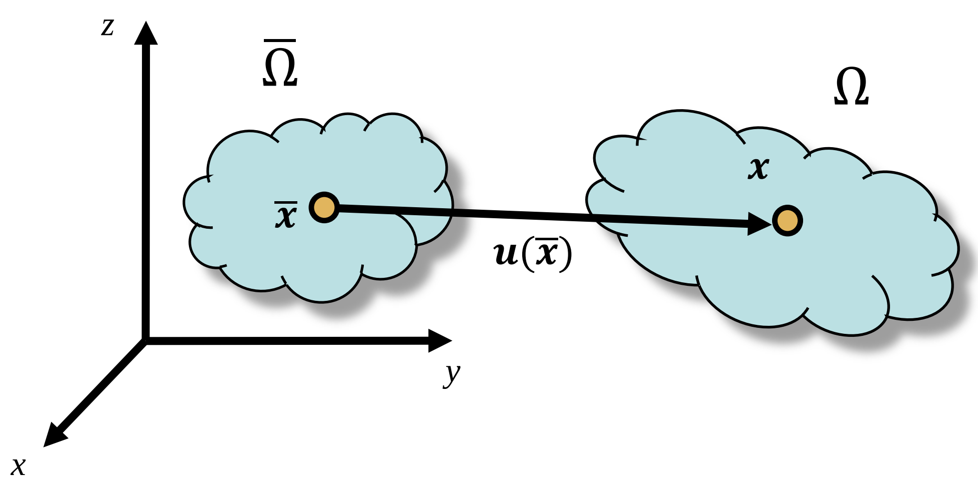
- Strain: $\varepsilon = \frac{\Delta l}{L}$ (*relative stretch*) (no unit)
- Stress: $\sigma = \frac{f_{int}}{A}$ (*internal force density*) (force per area)
- Hooke's law: $\sigma = k\varepsilon$ (k *material constant – how stiff is it!*)
- Strain energy density: $\Psi = \frac{1}{2}k\varepsilon^2$ (*postulate via* $\sigma = \frac{\partial\Psi}{\partial\varepsilon}$)

3D Deformations

- For a deformable body, identify the
  - undeformed state $\overline{\Omega} \subset \mathbf{R}^3$ described by positions $\overline{\boldsymbol{x}}$
  - deformed state $\Omega \subset \mathbf{R}^3$ described by positions $\boldsymbol{x}$

- Displacement field $\boldsymbol{u}$ describes $\overline{\Omega}$ in terms of Ω

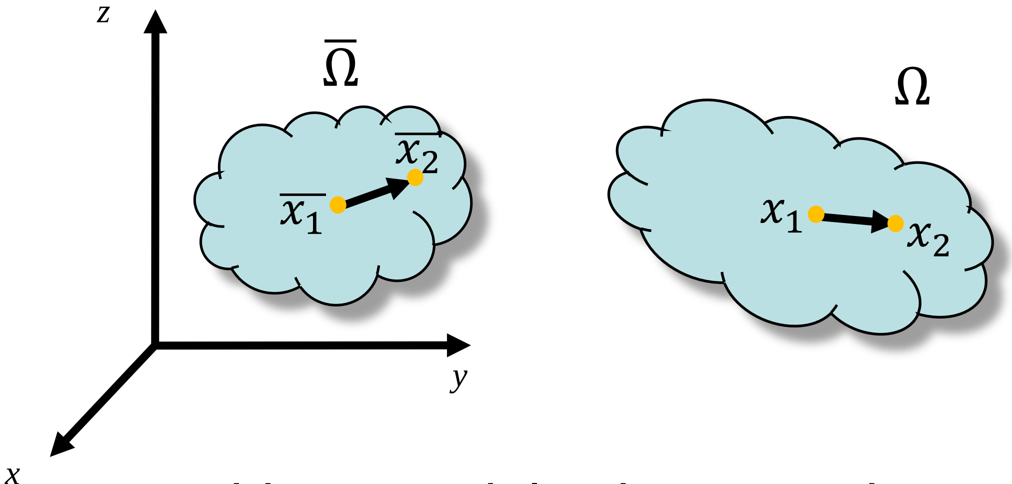
$$\boldsymbol{u}(\overline{\boldsymbol{x}}): \overline{\Omega} \to \Omega \quad , \quad \boldsymbol{x}(\overline{\boldsymbol{x}}) = \overline{\boldsymbol{x}} + \boldsymbol{u}(\overline{\boldsymbol{x}})$$



$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{pmatrix}$$

u is displacement in x direction
v is displacement in y direction
w is displacement in z direction

3D Deformations

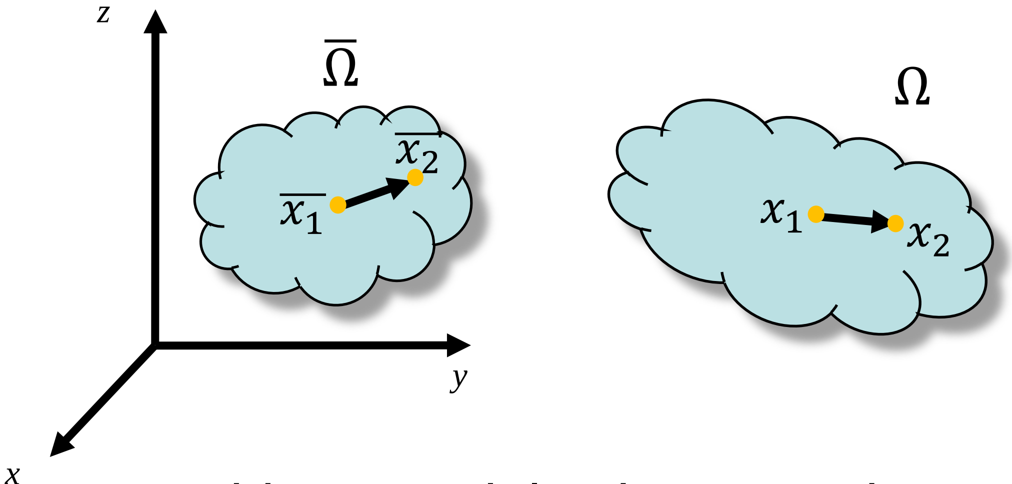


- Consider material points $\bar{\boldsymbol{x}}_1$ and $\bar{\boldsymbol{x}}_2$and $\bar{\boldsymbol{d}} = \bar{\boldsymbol{x}}_2 - \bar{\boldsymbol{x}}_1$such that $|\bar{d}|$ is infinitesimal

- Now consider deformed vector d

$$\boldsymbol{d} = \boldsymbol{x}_2 - \boldsymbol{x}_1 =$$

3D Deformations



- Consider material points $\bar{\boldsymbol{x}}_1$ and $\bar{\boldsymbol{x}}_2$and $\bar{\boldsymbol{d}} = \bar{\boldsymbol{x}}_2 - \bar{\boldsymbol{x}}_1$such that $|\bar{d}|$ is infinitesimal

- Now consider deformed vector d

$$\boldsymbol{d} = \boldsymbol{x}_2 - \boldsymbol{x}_1 = \bar{\boldsymbol{x}}_2 + \boldsymbol{u}(\bar{\boldsymbol{x}}_2) - \bar{\boldsymbol{x}}_1 - \boldsymbol{u}(\bar{\boldsymbol{x}}_1)$$
$$= \bar{\boldsymbol{d}} + \boldsymbol{u}(\bar{\boldsymbol{x}}_1 + \bar{\boldsymbol{d}}) - \boldsymbol{u}(\bar{\boldsymbol{x}}_1)$$
$$\approx \bar{\boldsymbol{d}} + \boldsymbol{u}(\bar{\boldsymbol{x}}_1) + \boldsymbol{\nabla u}\,\bar{\boldsymbol{d}} - \boldsymbol{u}(\bar{\boldsymbol{x}}_1) = \overbrace{(\mathbf{I} + \boldsymbol{\nabla u})}^{\text{Deformation gradient } \mathbf{F}}\,\bar{\boldsymbol{d}}$$

$$\nabla \mathbf{u} = \begin{pmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x w & \partial_y w & \partial_z w \end{pmatrix}$$

3D Nonlinear Strain

- Deformation gradient $\mathbf{F} = (\mathbf{I} + \nabla \mathbf{u})$ maps undeformed vectors to deformed vectors, $\boldsymbol{d} = \mathbf{F}\,\overline{\boldsymbol{d}}$.

How can we quantify deformation at a given point?

- Measure change in length (*squared*) in all directions

$$|\boldsymbol{d}|^2 - |\overline{\boldsymbol{d}}|^2 = \boldsymbol{d}^T \boldsymbol{d} - \overline{\boldsymbol{d}}^T \overline{\boldsymbol{d}}$$

$\Rightarrow$ **Green strain** $\quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$

3D Linear Strain

- Green strain is quadratic in displacements

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla\mathbf{u}^T\nabla\mathbf{u})$$

- Neglecting quadratic terms leads to the linear

Cauchy strain

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t) = \frac{1}{2}(\mathbf{F} + \mathbf{F}^t) - \mathbf{I}$$

- Written out:

$$\varepsilon = \frac{1}{2}\begin{pmatrix} 2\partial_x u & \partial_y u + \partial_x v & \partial_z u + \partial_x w \\ \partial_x v + \partial_y u & 2\partial_y v & \partial_z v + \partial_y w \\ \partial_x w + \partial_z u & \partial_y w + \partial_z v & 2\partial_z w \end{pmatrix}$$

Notation

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$$

3D Linear Strain

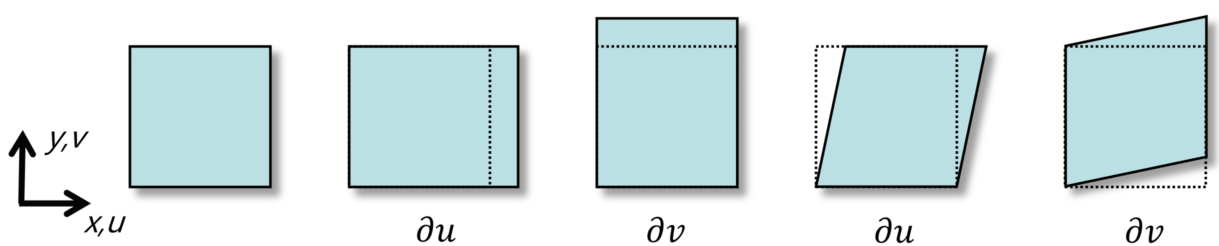
- Linear Cauchy strain

$$\varepsilon = \frac{1}{2}\begin{pmatrix} 2\partial_x u & \partial_y u + \partial_x v & \partial_z u + \partial_x w \\ \partial_x v + \partial_y u & 2\partial_y v & \partial_z v + \partial_y w \\ \partial_x w + \partial_z u & \partial_y w + \partial_z v & 2\partial_z w \end{pmatrix} =: \begin{pmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z \end{pmatrix}$$

ε_i : normal strains

γ_i : shear strains

- Geometric interpretation



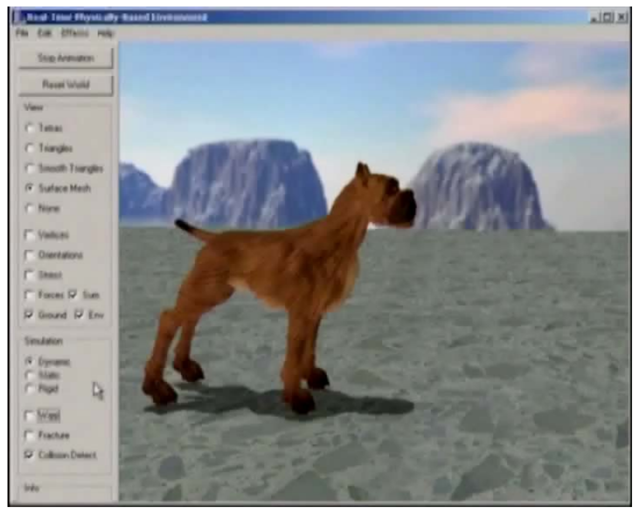
$\frac{\partial u}{\partial x}$ $\frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y}$ $\frac{\partial v}{\partial x}$

Cauchy vs. Green strain

- Nonlinear Green strain is rotation-invariant
  - Apply incremental rotation $\mathbf{R}$ to given deformation $\mathbf{F}$ to obtain $\mathbf{F}' = \mathbf{RF}$
  - Then $$\mathbf{E}' = \frac{1}{2}(\mathbf{F}'^T \mathbf{F}' - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T \mathbf{R}^T \mathbf{R} \mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}) = \mathbf{E}$$
- Linear Cauchy strain is not rotation-invariant

$$\varepsilon' = \frac{1}{2}\left(\mathbf{F}' + \mathbf{F}'^t\right) \neq \frac{1}{2}(\mathbf{F} + \mathbf{F}^t) = \varepsilon \quad \rightarrow \quad \text{artifacts for larger rotations}$$

Stiffness Warping



Mueller and Gross, Interactive Virtual Materials, Graphics Interface ‘04
http://matthias-mueller-fischer.ch/publications/GI2004.pdf

Cauchy vs. Green strain: Summary

- Green strain is quadratic in displacements

$$E = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T + \nabla\boldsymbol{u}^T\nabla\boldsymbol{u}\right)$$

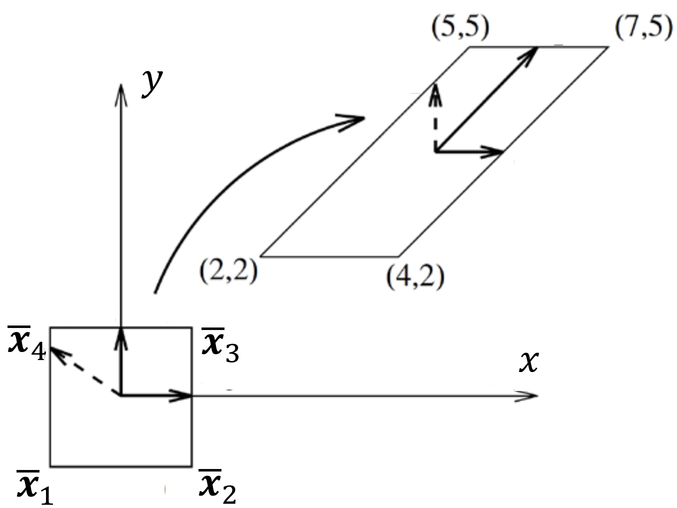
- Neglecting quadratic terms leads to the linear Cauchy strain

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{t}}) = \frac{1}{2}(\mathbf{F} + \mathbf{F}^{t}) - \mathbf{I}$$

Note:

- both Cauchy and Green strain are invariant under translation
- Green strain is invariant under rotation, but Cauchy strain is not

Example



- $\bar{\boldsymbol{x}}_1 = (-1, -1)$
- $\bar{\boldsymbol{x}}_2 = (1, -1)$
- $\bar{\boldsymbol{x}}_3 = (1,1)$
- $\bar{\boldsymbol{x}}_4 = (-1,1)$

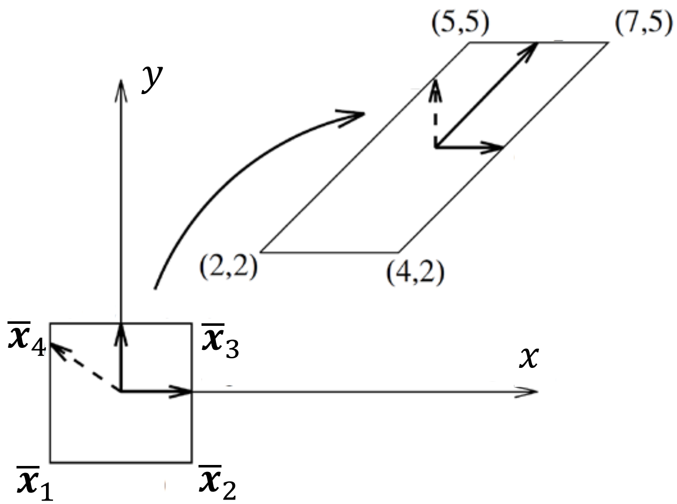
- Undeformed configuration $\bar{\boldsymbol{x}} = (\bar{x}, \bar{y})^T$
- Deformed configuration $\boldsymbol{x}(\bar{\boldsymbol{x}}) = (x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))^T$
- Displacement field $\boldsymbol{u}(\bar{\boldsymbol{x}}) = (u(\bar{x}, \bar{y}), v(\bar{x}, \bar{y}))^T$

$$\boldsymbol{u}_1 = \boldsymbol{u}(\bar{\boldsymbol{x}}_1) = \boldsymbol{x}(\bar{\boldsymbol{x}}_1) - \bar{\boldsymbol{x}}(\bar{\boldsymbol{x}}_1) = (3,3)^T$$
$$\boldsymbol{u}_2 = (3,3)^T$$
$$\boldsymbol{u}_3 = (6,4)^T$$
$$\boldsymbol{u}_4 = (6,4)^T$$

- Compute displacement as $\frac{\partial u}{\partial \bar{x}} = \frac{\boldsymbol{u}_{2x} - \boldsymbol{u}_{1x}}{\bar{\boldsymbol{x}}_{2x} - \bar{\boldsymbol{x}}_{1x}}$ etc.

$$\boldsymbol{F} = \nabla \mathbf{u} + \mathbf{I} = \begin{bmatrix} \frac{\partial u}{\partial \bar{x}} & \frac{\partial u}{\partial \bar{y}} \\ \frac{\partial v}{\partial \bar{x}} & \frac{\partial v}{\partial \bar{y}} \end{bmatrix} + \boldsymbol{I} = \frac{1}{2}\left[\quad \right] + \boldsymbol{I} =$$

Example



- $\bar{\boldsymbol{x}}_1 = (-1,-1)$
- $\bar{\boldsymbol{x}}_2 = (1,-1)$
- $\bar{\boldsymbol{x}}_3 = (1,1)$
- $\bar{\boldsymbol{x}}_4 = (-1,1)$

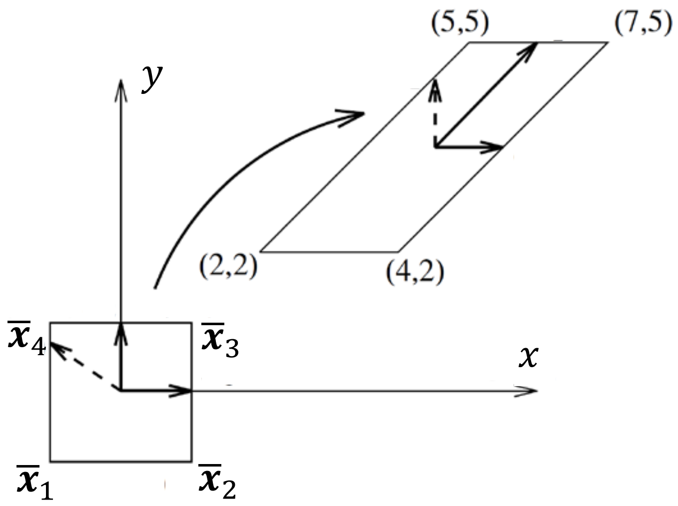
- Undeformed configuration $\bar{\boldsymbol{x}} = (\bar{x}, \bar{y})^T$
- Deformed configuration $\boldsymbol{x}(\bar{\boldsymbol{x}}) = (x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))^T$
- Displacement field $\boldsymbol{u}(\bar{\boldsymbol{x}}) = (u(\bar{x}, \bar{y}), v(\bar{x}, \bar{y}))^T$

$$\boldsymbol{u}_1 = \boldsymbol{u}(\bar{\boldsymbol{x}}_1) = \boldsymbol{x}(\bar{\boldsymbol{x}}_1) - \bar{\boldsymbol{x}}(\bar{\boldsymbol{x}}_1) = (3,3)^T$$
$$\boldsymbol{u}_2 = (3,3)^T$$
$$\boldsymbol{u}_3 = (6,4)^T$$
$$\boldsymbol{u}_4 = (6,4)^T$$

- Compute displacement as $\frac{\partial u}{\partial \bar{x}} = \frac{\boldsymbol{u}_{2x} - \boldsymbol{u}_{1x}}{\bar{\boldsymbol{x}}_{2x} - \bar{\boldsymbol{x}}_{1x}}$ etc.

$$\boldsymbol{F} = \nabla \mathbf{u} + \mathbf{I} = \begin{bmatrix} \frac{\partial u}{\partial \bar{x}} & \frac{\partial u}{\partial \bar{y}} \\ \frac{\partial v}{\partial \bar{x}} & \frac{\partial v}{\partial \bar{y}} \end{bmatrix} + \boldsymbol{I} = \frac{1}{2}\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} + \boldsymbol{I} = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}$$

Example



$$\boldsymbol{F} = \frac{1}{2}\begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}$$

$$\boldsymbol{F} \cdot (1,0)^T = (1,0)^T$$
$$\boldsymbol{F} \cdot (0,1)^T = (1.5,1.5)^T$$
$$\boldsymbol{F} \cdot (1,1)^T = (2.5,1.5)^T$$

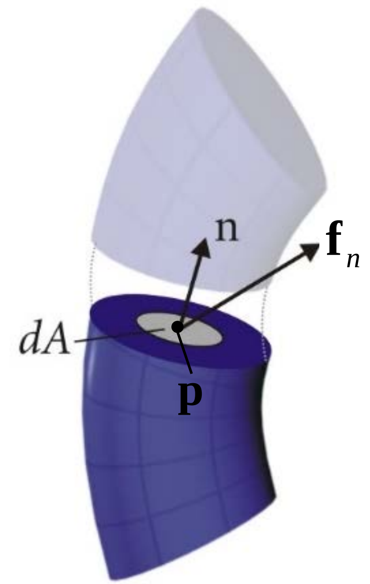
$$\boldsymbol{E} = ?$$

- $\bar{\boldsymbol{x}}_1 = (-1,-1)$
- $\bar{\boldsymbol{x}}_2 = (1,-1)$
- $\bar{\boldsymbol{x}}_3 = (1,1)$
- $\bar{\boldsymbol{x}}_4 = (-1,1)$

3D Stress

- Virtual experiment on deformed solid
  - Insert cut plane with normal $\mathbf{n}$ through $\mathbf{p}$
  - Observe traction force $\mathbf{f}_n(\mathbf{n},\mathbf{p})$ on area dA
  - Traction force density $\mathbf{t}_n(\mathbf{n},\mathbf{p}) = \frac{d\mathbf{f}_n}{dA}$ as $dA \to 0$

How does $\mathbf{t}_n$ *change with* $\mathbf{n}$ *?*



- Cauchy's stress theorem: $\mathbf{t}_n$ depends linearly on $\mathbf{n}$

$$\mathbf{t}(\mathbf{x},\mathbf{n}) = \sigma(\mathbf{x}) \cdot \mathbf{n}$$

Cauchy stress tensor

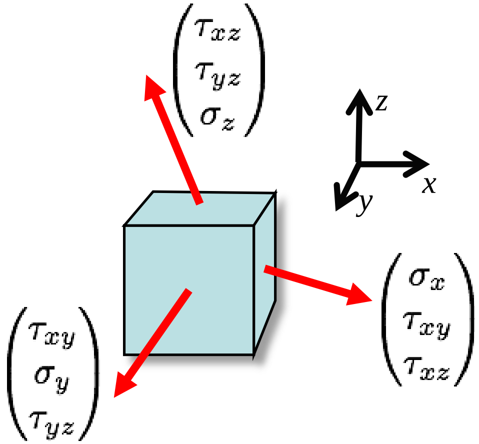
Cauchy Stress

- Cauchy stress tensor written out

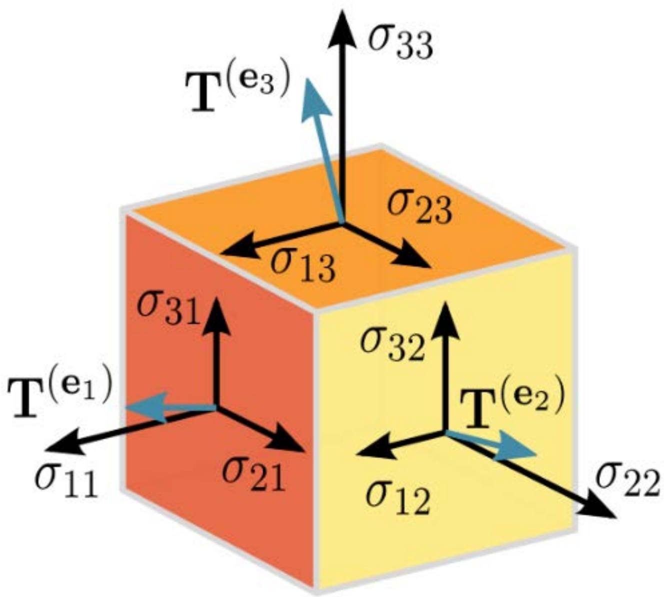
$$\mathbf{t}(\mathbf{x},\mathbf{n}) = \sigma(\mathbf{x}) \cdot \mathbf{n} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} \cdot \mathbf{n}$$

  - Normal stress components σ_i
  - Shear stress components σ_{ij}

- Entries of σ are force components on unit cube



What's a tensor?



The second-order Cauchy stress tensor in the basis (e1, e2, e3):

$$\mathbf{T} = \left[\, \mathbf{T}^{(\mathbf{e}_1)}\, \mathbf{T}^{(\mathbf{e}_2)}\, \mathbf{T}^{(\mathbf{e}_3)} \,\right],$$

$$\mathbf{T} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Still confused: watch this cute video:
https://www.youtube.com/watch?v=f5liqUk0ZTw

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  - Stress
  - **Material model (linear case)**
- Discretization (3D)
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Constitutive Laws

- Material model links strain to energy (and stress)

Hookean materials $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}.$

How big is $\mathbf{E}$?

Stress and strain are symmetric tensors

Linear Isotropic Materials

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix},$$

- The scalar E is Young’s modulus describing the elastic stiffness and
- the scalar $\nu \in \left[0 \ldots \frac{1}{2}\right)$ Poisson’s ratio, a material parameter that describes to which amount volume is conserved within the material

Material Model

- Linear isotropic material (*generalized Hooke's law*)
  - Energy density $\Psi = \frac{1}{2}\lambda \mathrm{tr}(\boldsymbol{\varepsilon})^2 + \mu \mathrm{tr}(\boldsymbol{\varepsilon}^2)$
  - Cauchy stress $\sigma = \frac{\partial \Psi}{\partial \varepsilon} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}$
  - Lame parameters λ and μ are material constants

$\boxed{\mathrm{tr}(\boldsymbol{\varepsilon}) = \sum \varepsilon_{ii}}$

Material Parameters

v · t · e	Elastic moduli for homogeneous isotropic materials	[hide]
	Bulk modulus (K) · Young's modulus (E) · Lamé's first parameter (λ) · Shear modulus (G, μ) · Poisson's ratio (ν) · P-wave modulus (M)	

Conversion formulas [hide]

Homogeneous isotropic linear elastic materials have their elastic properties uniquely determined by any two moduli among these; thus, given any two, any other of the elastic moduli can be calculated according to these formulas.

	$K=$	$E=$	$\lambda=$	$G=$	$\nu=$	$M=$	Notes
$(K,\, E)$	K	E	$\frac{3K(3K-E)}{9K-E}$	$\frac{3KE}{9K-E}$	$\frac{3K-E}{6K}$	$\frac{3K(3K+E)}{9K-E}$	
$(K,\, \lambda)$	K	$\frac{9K(K-\lambda)}{3K-\lambda}$	λ	$\frac{3(K-\lambda)}{2}$	$\frac{\lambda}{3K-\lambda}$	$3K-2\lambda$	
$(K,\, G)$	K	$\frac{9KG}{3K+G}$	$K-\frac{2G}{3}$	G	$\frac{3K-2G}{2(3K+G)}$	$K+\frac{4G}{3}$	
$(K,\, \nu)$	K	$3K(1-2\nu)$	$\frac{3K\nu}{1+\nu}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	ν	$\frac{3K(1-\nu)}{1+\nu}$	
$(K,\, M)$	K	$\frac{9K(M-K)}{3K+M}$	$\frac{3K-M}{2}$	$\frac{3(M-K)}{4}$	$\frac{3K-M}{3K+M}$	M	
$(E,\, \lambda)$	$\frac{E+3\lambda+R}{6}$	E	λ	$\frac{E-3\lambda+R}{4}$	$\frac{2\lambda}{E+\lambda+R}$	$\frac{E-\lambda+R}{2}$	$R=\sqrt{E^2+9\lambda^2+2E\lambda}$
$(E,\, G)$	$\frac{EG}{3(3G-E)}$	E	$\frac{G(E-2G)}{3G-E}$	G	$\frac{E}{2G}-1$	$\frac{G(4G-E)}{3G-E}$	
$(E,\, \nu)$	$\frac{E}{3(1-2\nu)}$	E	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	ν	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	
$(E,\, M)$	$\frac{3M-E+S}{6}$	E	$\frac{M-E+S}{4}$	$\frac{3M+E-S}{8}$	$\frac{E-M+S}{4M}$	M	$S=\pm\sqrt{E^2+9M^2-10EM}$ There are two valid solutions. The plus sign leads to $\nu \geq 0$. The minus sign leads to $\nu \leq 0$.
$(\lambda,\, G)$	$\lambda+\frac{2G}{3}$	$\frac{G(3\lambda+2G)}{\lambda+G}$	λ	G	$\frac{\lambda}{2(\lambda+G)}$	$\lambda+2G$	
$(\lambda,\, \nu)$	$\frac{\lambda(1+\nu)}{3\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	λ	$\frac{\lambda(1-2\nu)}{2\nu}$	ν	$\frac{\lambda(1-\nu)}{\nu}$	Cannot be used when $\nu=0 \Leftrightarrow \lambda=0$
$(\lambda,\, M)$	$\frac{M+2\lambda}{3}$	$\frac{(M-\lambda)(M+2\lambda)}{M+\lambda}$	λ	$\frac{M-\lambda}{2}$	$\frac{\lambda}{M+\lambda}$	M	
$(G,\, \nu)$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$2G(1+\nu)$	$\frac{2G\nu}{1-2\nu}$	G	ν	$\frac{2G(1-\nu)}{1-2\nu}$	
$(G,\, M)$	$M-\frac{4G}{3}$	$\frac{G(3M-4G)}{M-G}$	$M-2G$	G	$\frac{M-2G}{2M-2G}$	M	
$(\nu,\, M)$	$\frac{M(1+\nu)}{3(1-\nu)}$	$\frac{M(1+\nu)(1-2\nu)}{1-\nu}$	$\frac{M\nu}{1-\nu}$	$\frac{M(1-2\nu)}{2(1-\nu)}$	ν	M	

https://en.wikipedia.org/wiki/Lam%C3%A9_parameters

Material Model

- Material model links strain to energy (and stress)
- Linear isotropic material (*generalized Hooke's law*)
  - Energy density $\Psi = \frac{1}{2}\lambda \mathrm{tr}(\boldsymbol{\varepsilon})^2 + \mu \mathrm{tr}(\boldsymbol{\varepsilon}^2)$

$\boxed{\mathrm{tr}(\boldsymbol{\varepsilon}) = \sum \varepsilon_{ii}}$

  - Cauchy stress $\sigma = \frac{\partial \Psi}{\partial \varepsilon} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}$
  - Lame parameters λ and μ are material constants
- Interpretation
  - $\mathrm{tr}\left(\boldsymbol{\varepsilon}^2\right) = \mathrm{tr}\left(\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}\right) = \|\boldsymbol{\varepsilon}\|_F^2$ penalizes all strain components equally
  - $\lambda \mathrm{tr}(\boldsymbol{\varepsilon})^2$ penalizes dilatations, i.e., volume changes

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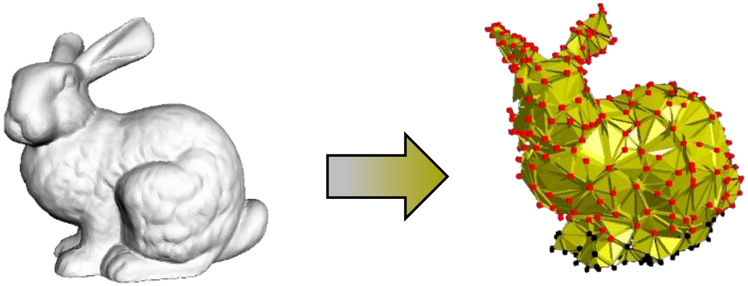
Finite Element Discretization

- Divide input model into elements (e.g., triangles in 2D, tetrahedra in 3D)



- For each element, evaluate its energy, the energy gradient, and the energy Hessian
- All quantities depend (only) on the deformation gradient $\mathbf{F}$

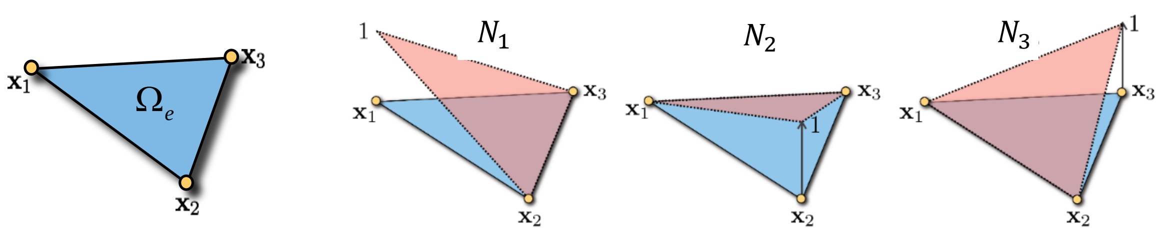
Finite Elements

What is a finite element?

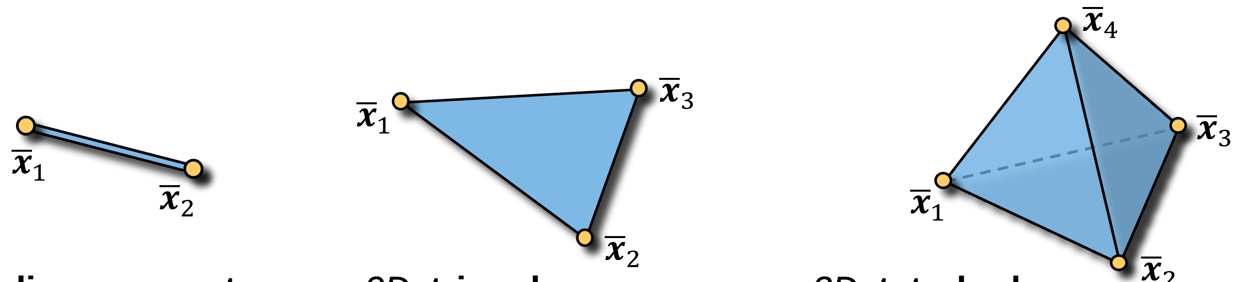
A finite element consists of

- a closed subset $\Omega_e \subset \boldsymbol{R}^d$ (*in d dimensions*)
- n nodal basis functions, $N_i: \Omega_e \to \boldsymbol{R}$
- n vectors of nodal variables $\overline{\boldsymbol{x}}_i \in \boldsymbol{R}^d$ describing the reference geometry
- n vectors of degrees of freedom (e.g., deformed positions $\boldsymbol{x}_i$)

Example: 3-node triangle with linear basis functions



Linear Simplicial Elements



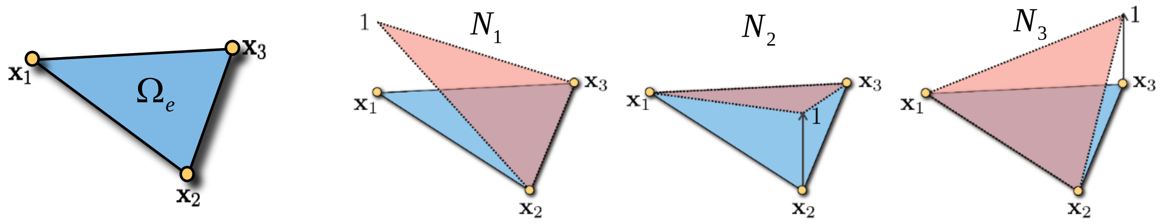
1D: **line segment** *2D*: **triangle** *3D:* **tetrahedron**

- Simplicial elements admit linear basis functions
- Basis functions are uniquely defined through
- reference geometry $\overline{\boldsymbol{x}}_i$ and
- interpolation requirement $N_i(\overline{\boldsymbol{x}}_j) = \delta_{ij}$

$\overline{\boldsymbol{x}}_i = \bar{x}_i$	in 1D
$\overline{\boldsymbol{x}}_i = (\bar{x}_i, \bar{y}_i)$	in 2D
$\overline{\boldsymbol{x}}_i = (\bar{x}_i, \bar{y}_i, \bar{z}_i)$	in 3D

Computing Basis Functions – 2D

Example: 3-node elements with linear basis functions

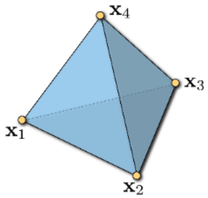


- Basis functions are linear: $N_i(x,y) = a_i x + b_i y + c$
- Due to $N_i(\mathbf{x}_j) = \delta_{ij}$, we have

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{bmatrix} \Rightarrow \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{bmatrix}$$

Computing Basis Functions - 3D

4-node tetrahedron with 4 linear basis functions

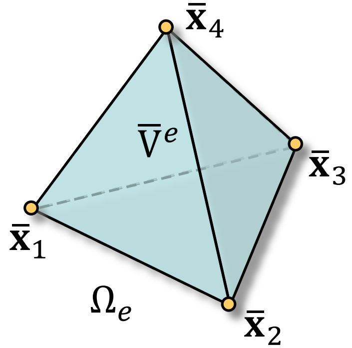


... ...

- Basis functions are linear, $N_i(\bar{x}, \bar{y}, \bar{z}) = a_i\bar{x} + b_i\bar{y} + c_i\bar{z} + d_i$
- From $N_i(\bar{\boldsymbol{x}}_j) = \delta_{ij}$ we obtain

$$N_i(\bar{x}, \bar{y}, \bar{z}) = a_i\bar{x} + b_i\bar{y} + c_i\bar{z} + d_i$$

$$\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} = \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ \delta_{4i} \end{pmatrix}$$



Deformation Gradient

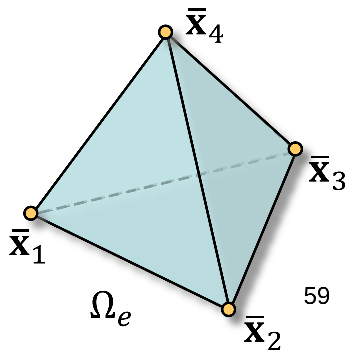
- Use basis functions to define continuous geometry of element as

$$\bar{\mathbf{x}}(\bar{x},\bar{y},\bar{z}) = \sum N_i(\bar{x},\bar{y},\bar{z})\bar{\mathbf{x}}_i \qquad \text{and} \qquad \mathbf{x}(\bar{x},\bar{y},\bar{z}) = \sum N_i(\bar{x},\bar{y},\bar{z})\mathbf{x}_i$$

- Deformation gradient

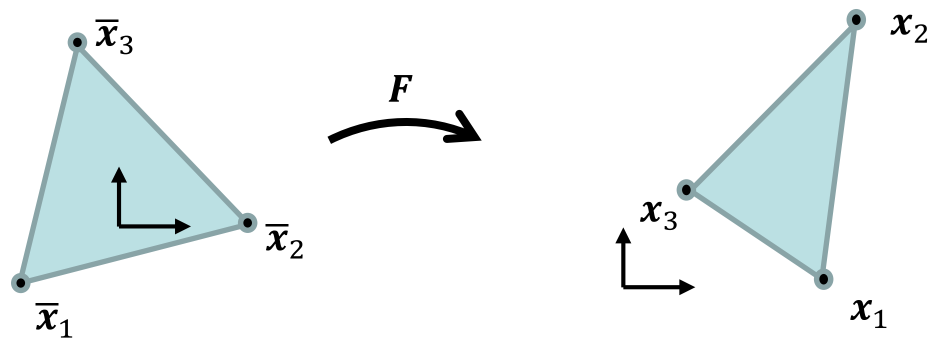
$$\mathbf{F} = \frac{\partial \mathbf{x}(\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}} = \sum_i \mathbf{x}_i \left(\frac{\partial N_i}{\partial \bar{\mathbf{x}}}\right)^T$$

- Note:
  – $\mathbf{F} \in \mathbf{R}^{3\times 3}$ and $\mathbf{F}$ is linear in $\mathbf{x}_i$
  – N_i are linear on element, $\mathbf{F}$ is constant
  – Hence, $W^e = \int_{\Omega_e} \Psi = \Psi(\mathbf{F}) \cdot \bar{V}^e$



Example

- $\bar{\boldsymbol{x}}_1 = (-2,-1)$
- $\bar{\boldsymbol{x}}_2 = (2,0)$
- $\bar{\boldsymbol{x}}_3 = (-1,3)$



- $\boldsymbol{x}_1 = (3,0)$
- $\boldsymbol{x}_2 = (4,5)$
- $\boldsymbol{x}_3 = (1,2)$

- Compute basis functions N_i
- Compute basis function derivatives $\frac{\partial N_i}{\partial \bar{x}} = \nabla_{\bar{\boldsymbol{x}}} N_i$
  - Compute $\mathbf{F}$ via $F_{kl} = \sum_i \mathrm{x}_{i,k}\, \nabla_{\bar{x}_l} N_i$
  - Compute $\mathbf{F}$ via $\mathbf{F} = \sum_i \mathbf{x}_i \left(\nabla_{\bar{\boldsymbol{x}}} N_i\right)^T$

Hint (the inverse)

$$\frac{1}{15}\begin{bmatrix} -3 & 4 & -1 \\ -3 & -1 & 4 \\ 6 & 7 & 2 \end{bmatrix}$$

Notation

Continuous case:

- Undeformed configuration $\bar{\boldsymbol{x}}(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})^T$
- Deformed configuration $\boldsymbol{x}(\bar{x}, \bar{y}) = (x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))^T$

Discretized:

- Undeformed configuration $\bar{\boldsymbol{x}}(\bar{x}, \bar{y}) = \sum_i N_i (\bar{x}, \bar{y})\bar{\boldsymbol{x}}_i$
- Deformed configuration $\boldsymbol{x}(\bar{x}, \bar{y}) = \sum_i N_i (\bar{x}, \bar{y})\boldsymbol{x}_i$

Interpreting F

- Polar decomposition $\boldsymbol{F} = \boldsymbol{RU}$, with $\boldsymbol{R}$ orthonormal (i.e. a rotation) and $\boldsymbol{U}$ positive definite
- If $\boldsymbol{F}$ is non-singular, i.e., $\det \boldsymbol{F} \neq 0$, then its PD exists and is unique.

Green strain: $E = \frac{1}{2}\left(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}\right)$

Cauchy strain: $\frac{1}{2}(\mathbf{F} + \mathbf{F}^t) - \mathbf{I}$

Green strain: $E = \frac{1}{2}(\boldsymbol{U}^{\boldsymbol{t}} \boldsymbol{U} - \boldsymbol{I})$

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Solving The Static Equilibrium Problem

- Necessary condition for static equilibrium

$$\mathbf{f}_i(\mathbf{x}) = \mathbf{f}_i^{ext} + \mathbf{f}_i^{el}(\mathbf{x}) = 0 \ \forall i$$

- Given $\mathbf{x}$ with $\mathbf{f}(\mathbf{x}) \neq 0$, find $\Delta\mathbf{x}$ such that $\mathbf{f}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{0}$
- First order approximation → $\mathbf{f}(\mathbf{x} + \Delta\mathbf{x}) \approx \mathbf{f}(\mathbf{x}) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \Delta\mathbf{x}$
- Therefore: we should solve for $-\mathbf{f}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \Delta\mathbf{x}$

Stiffness matrix

$$\mathbf{K} = \frac{\partial \mathbf{f}^{el}}{\partial \mathbf{x}}$$

$$\mathbf{f}_i^{el} = -\frac{\partial W}{\partial \boldsymbol{x}_i}$$

Newton's method

While not converged
  Compute $\mathbf{f}(\mathbf{x}), \mathbf{K}(\mathbf{x})$
  Solve $\mathbf{K}(\mathbf{x})\Delta\mathbf{x} = -\mathbf{f}(\mathbf{x})$
  line search $\alpha = \text{linesearch}(\mathbf{x}, \Delta\mathbf{x})$
  Update $\mathbf{x} \mathrel{+}= \alpha\Delta\mathbf{x}$
end

Solving The Static Equilibrium Problem

- Necessary condition for static equilibrium

$$\mathbf{f}_i(\mathbf{x}) = \mathbf{f}_i^{ext} + \mathbf{f}_i^{el}(\mathbf{x}) = 0 \;\forall i$$

- Given $\mathbf{x}$ with $\mathbf{f}(\mathbf{x}) \neq 0$, find $\Delta\mathbf{x}$ such that $\mathbf{f}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{0}$
- First order approximation → $\mathbf{K}(\mathbf{x})\Delta\mathbf{x} = -\mathbf{f}(\mathbf{x})$

Stiffness matrix

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While not converged
- Compute $\mathbf{f}(\mathbf{x}), \mathbf{K}(\mathbf{x})$
- Solve $\mathbf{K}(\mathbf{x})\Delta\mathbf{x} = -\mathbf{f}(\mathbf{x})$
- line search $\alpha = \mathrm{linesearch}(\mathbf{x}, \Delta\mathbf{x})$
- Update $\mathbf{x} \mathrel{+}= \alpha\Delta\mathbf{x}$

end

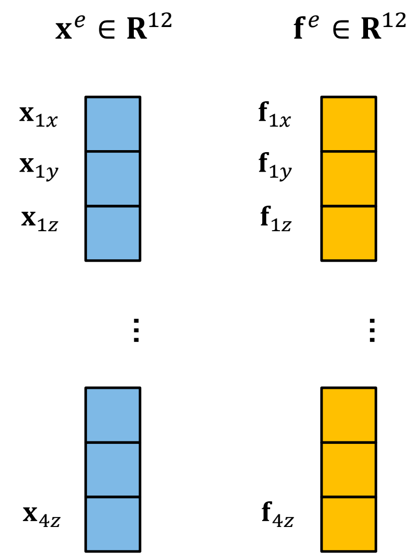
Linear Elasticity - Derivatives

- Computing the derivatives (*per element*)

- $\mathbf{f}^e_{mx} = -\frac{\partial W^e}{\partial \mathbf{x}_{mx}} = \sum_{ij} \frac{\partial W^e}{\partial \boldsymbol{\varepsilon}^e_{ij}} \frac{\partial \boldsymbol{\varepsilon}^e_{ij}}{\partial \mathbf{x}_{mx}}$

- $$\frac{\partial \mathbf{f}^e_{mx}}{\partial \mathbf{x}_{ny}} = -\frac{\partial^2 W^e}{\partial \mathbf{x}_{mx} \partial \mathbf{x}_{ny}} = -\sum_{ijkl} \frac{\partial^2 W^e}{\partial \boldsymbol{\varepsilon}^e_{ij} \boldsymbol{\varepsilon}^e_{kl}} \frac{\partial \boldsymbol{\varepsilon}^e_{ij}}{\partial \mathbf{x}_{mx}} \frac{\partial \boldsymbol{\varepsilon}^e_{kl}}{\partial \mathbf{x}_{ny}}$$

- $\frac{\partial \boldsymbol{\varepsilon}^e_{ij}}{\partial \mathbf{x}_{mx}} = const.$, $\frac{\partial W^e}{\partial \boldsymbol{\varepsilon}_{ij}} = \sigma_{ij}$, $\frac{\partial^2 W^e}{\partial \boldsymbol{\varepsilon}_{ij} \partial \boldsymbol{\varepsilon}_{kl}} = const.$



$m, n = 1 \dots 4$

$i, j, k, l = 1 \dots 3$

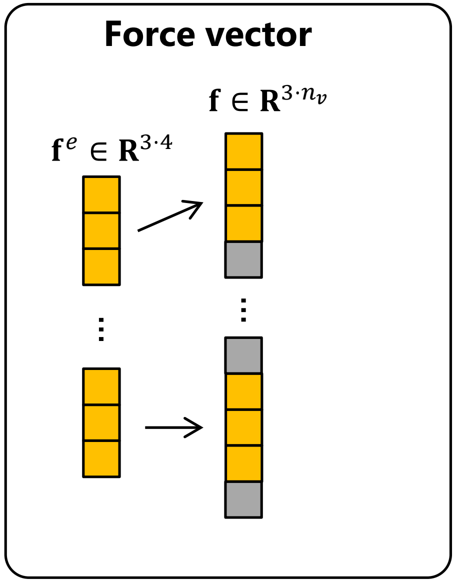
Stiffness matrix

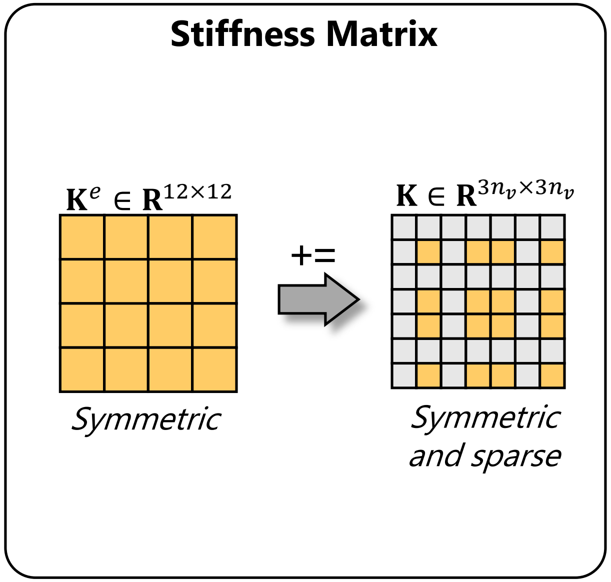
$$\mathbf{K} = \frac{\partial \mathbf{f}^{el}}{\partial \mathbf{x}}$$

$$\mathbf{f}^{el}_i = -\frac{\partial W}{\partial \boldsymbol{x}_i}$$

Linear Elasticity - Assembly

Assemble element contributions into global vector and matrix





Statics on Deformable Bodies (Plan)

- Continuum Mechanics Intro
  - Spring Systems
  - Continuum Mechanics in 1D
- Continuum Mechanics in 3D
  - Strain
  - Stress
  - Material model (linear case)
- Discretization (3D)
  - Finite Elements
  - Solving for Static Equilibrium
- **More material models**

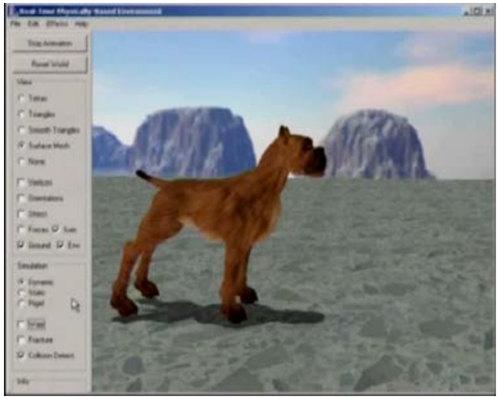
Linear Elasticity - Material Model

- Material model links strain to energy (and stress)
- Linear isotropic material (*generalized Hooke's law*)
  - Energy density $\Psi = \frac{1}{2}\lambda \mathrm{tr}(\boldsymbol{\varepsilon})^2 + \mu \mathrm{tr}(\boldsymbol{\varepsilon}^2)$
  - Cauchy stress $\sigma = \frac{\partial \Psi}{\partial \varepsilon} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}$
  - Lame parameters λ and μ are material constants

$\boxed{\mathrm{tr}(\boldsymbol{\varepsilon}) = \sum \varepsilon_{ii}}$

Linear Elasticity - Behavior

- For linear elements, $\boldsymbol{F}$ is constant and $W = \int_{\bar{\Omega}_e} \Psi(\boldsymbol{F}) = \Psi(\boldsymbol{F}) \cdot \bar{V}$
- For linear elasticity, W is quadratic in $\mathbf{x}$, $\mathbf{f}$ is linear in $\mathbf{x}$, and $\frac{\partial^2 W}{\partial \boldsymbol{x}^2}$ is constant → only solve one linear system for static equilibrium
- Problem: Cauchy strain is not invariant under rotations → inaccuracies for large rotations deformations
- Solution: use nonlinear deformation measure → nonlinear continuum mechanics



Nonlinear Elasticity

- Idea: replace Cauchy strain with Green strain

Nonlinear Elasticity

- Idea: replace Cauchy strain with Green strain

  $\rightarrow$ *St. Venant-Kirchhoff material* (StVK)

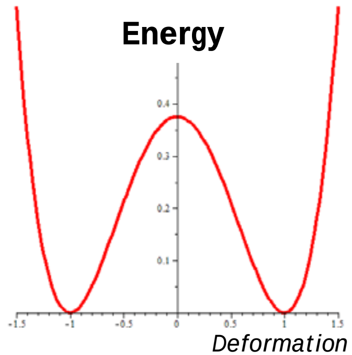
- Energy $\Psi_{StVK} = \frac{1}{2}\lambda \mathrm{tr}(\mathbf{E})^2 + \mu \mathrm{tr}(\mathbf{E}^2)$

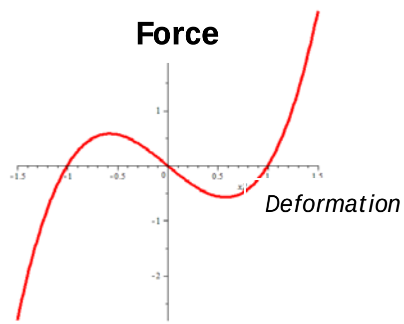
- Component l of force on node k is $\mathbf{f}_{kl}^e = -\frac{\partial W^e}{\partial \mathbf{x}_k} = -\sum_{ij} \frac{\partial W^e}{\partial \mathbf{F}_{ij}^e} \frac{\partial \mathbf{F}_{ij}^e}{\partial \mathbf{x}_{kl}}$

- Note:
  - Energy is quartic in $\mathbf{x}$, forces are cubic
  - Solve system of nonlinear equations

StVK Limitations

- *Problem*: StVK softens under compression





- *Reason*: Green strain $\mathbf{E} = \frac{1}{2}(\mathbf{F}^t\mathbf{F} - \mathbf{I}) \rightarrow -\frac{1}{2}\mathbf{I}$ *for* $\mathbf{F} \rightarrow \mathbf{0}$
- *Work around*: add volume term

$$\Psi_{StVK} = \frac{\lambda}{2}\mathrm{tr}(\mathbf{E})^2 + \mu\mathrm{tr}\left(\mathbf{E}^2\right) \qquad \rightarrow \qquad \Psi_{Mod} = \eta(\det(\mathbf{F}) - 1)^2 + \mu\mathrm{tr}\left(\mathbf{E}^2\right)$$

Isotropic Hyperelasticity

- **Hyperelasticity**: the stress-strain relationship derives from a strain energy density function Ψ; Ψ is a potential, i.e., only depends on state of deformation, not on the path travelled, and not on the rate of deformation.
- **Isotropy:** the material behavior is the same in any material direction, i.e., $\Psi(\boldsymbol{F}) = \Psi(\boldsymbol{QFQ}^T)$ for all orthogonal matrices $\boldsymbol{Q}$.
- For example, a uniaxial strain of given magnitude will lead to same energy, regardless of the axis.
- Rubbers and many biological materials are isotropic and (*nearly*) hyperelastic.

Isotropic Hyperelasticity

- If the material is isotropic, then the relationship between Ψ and $\boldsymbol{C} = \boldsymbol{FTF}$ must be independent of the choice of material axes.
- Consequently, Ψ can only depend on the invariants of $\boldsymbol{C}$, i.e.,

$$\Psi(\boldsymbol{C}) = \Psi(I_1(\boldsymbol{C}), I_2(\boldsymbol{C}), I_3(\boldsymbol{C}))$$

where the three invariants of $\boldsymbol{C}$ are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \qquad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \qquad I_3 = J^2 = \det(\boldsymbol{C})$$

Incompressibility

- Many materials such as biological tissue and rubbers strongly resist volumetric deformation
- To model (*nearly*) incompressible materials, decompose deformation $\boldsymbol{C}$ into volumetric and deviatoric (volume-preserving) parts,

volumetric: $J = \det(\mathbf{F})$ deviatoric: $\bar{\mathbf{C}} = \det(\mathbf{F})^{-2/3}\,\mathbf{C}$

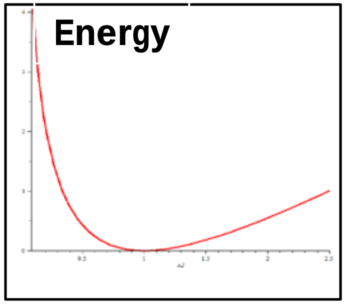
- Introduce deviatoric invariants, i.e., invariants of $\bar{\mathbf{C}}$ as

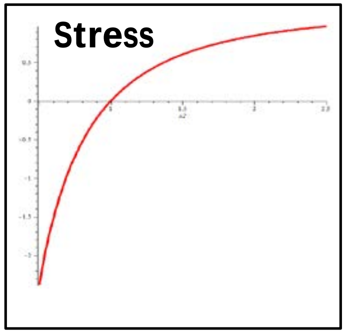
$$\bar{I}_1 = J^{-2/3} I_1 \quad \text{and} \quad \bar{I}_2 = J^{-4/3} I_2$$

Neo-Hookean Material

- The strain energy density for a *compressible* Neo-Hookean material is defined as

$$\Psi_{NH} = \frac{\mu}{2}(\mathrm{tr}(\mathbf{C}) - 3) - \mu \ln J + \frac{\lambda}{2}\ln(J)^2$$

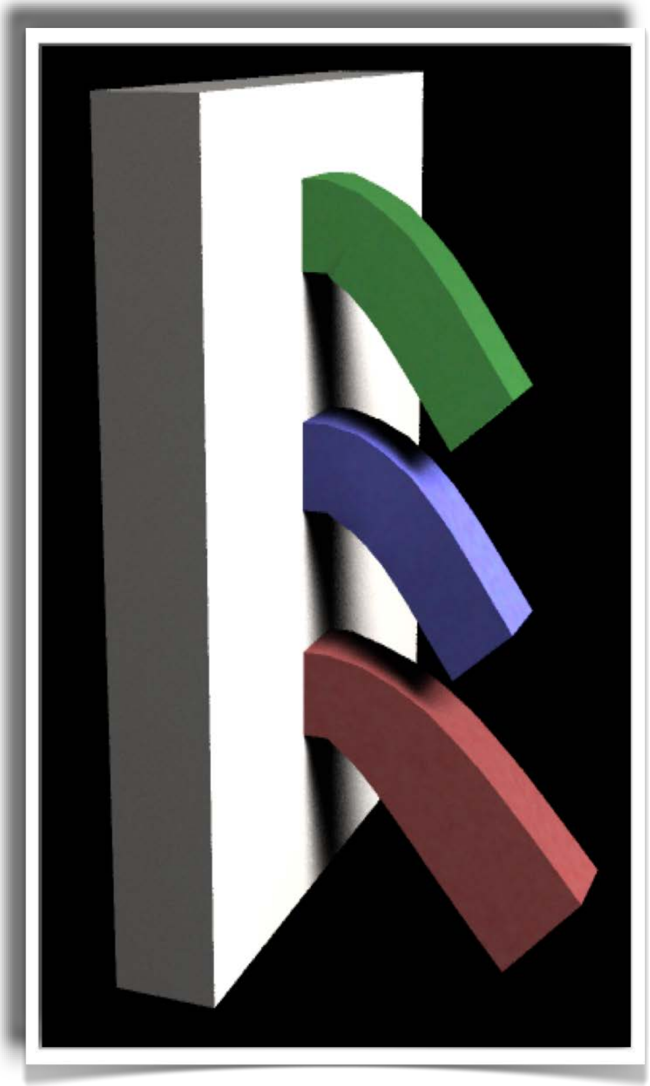




Observations:

- the first term penalizes all deformations equally (since $\mathrm{tr}(\boldsymbol{C}) = |\boldsymbol{F}|_F^2$)
- the third term goes to infinity for increasing compression (*faster than the second*)
- the stress-strain behavior is initially linear, but goes into plateau for larger deformations
- Rule of thumb: NH is good for deformations of up to 20%

Model Comparison

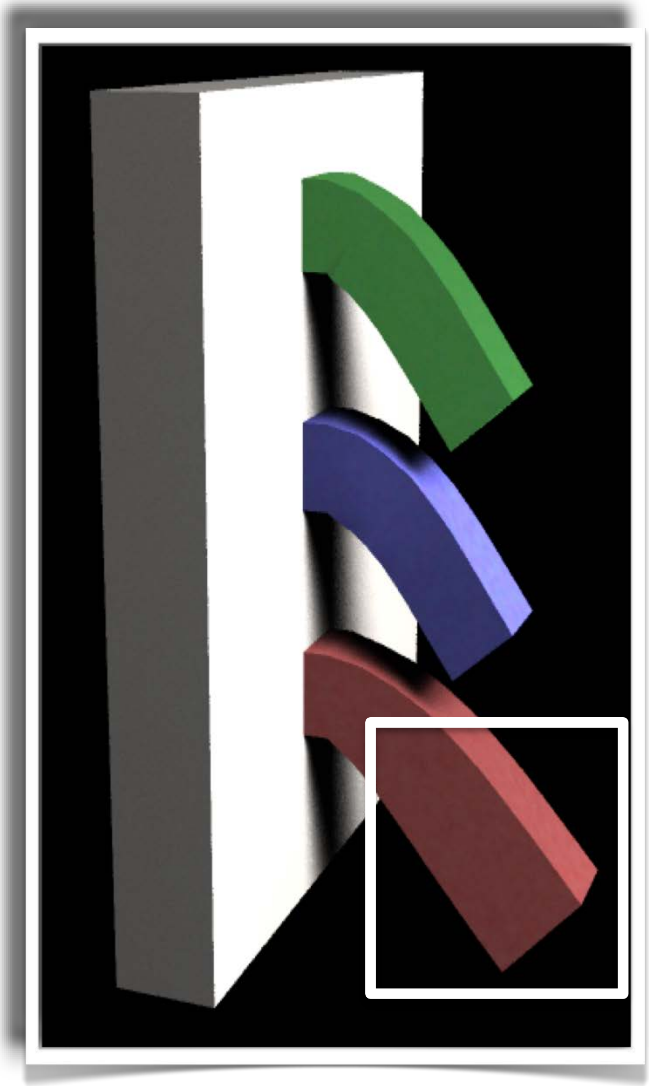


St. Venant-Kirchhoff

Neo Hookean

Linear

Model Comparison



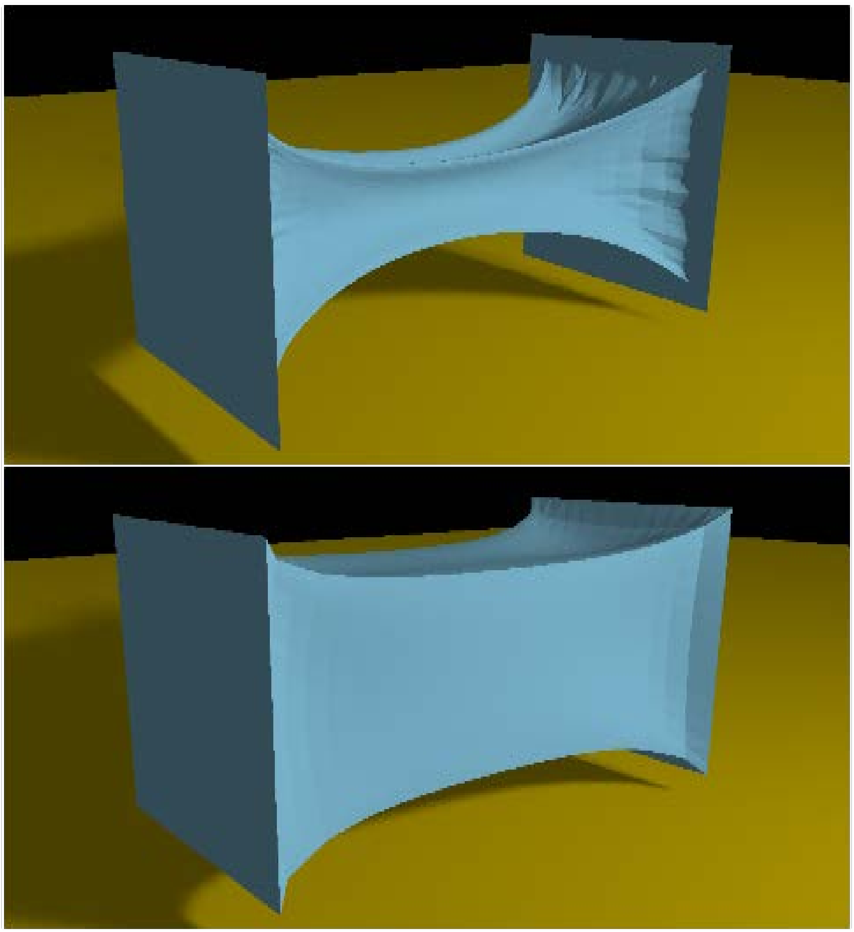
St. Venant-Kirchhoff

Neo Hookean

Linear

Choosing the wrong material model leads to artifacts!!!!

Hyperelastic Models: Differences



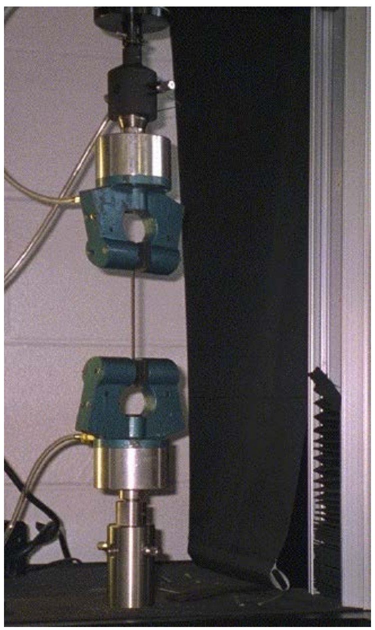
St. Venant-Kirchhoff

Neo Hookean

Material Measurement & Fitting

- Fit material coefficients to experimental stress-strain data
- Design experiments that lead to simple homogeneous states of deformation
- Multiple experiments are needed to extract all material coefficients
  - Uniaxial extension
  - Planar extension
  - Equibiaxial extension
  - ...
- Collect data (stress-strain curves), fit material coefficients
  - by solving analytical equations for stress-strain behavior (numerically)
  - by minimizing the difference between simulated and measured strains/stresses

Materials – Measurements



Uniaxial extension

Planar extension

- Simple tension
  - $\lambda_1 = \lambda, \quad \lambda_2 = \lambda_3 = 1/\sqrt{\lambda}$
  - $\sigma_{11} = \sigma, \quad \sigma_{22} = \sigma_{33} = 0$
- Pure shear
  - $\lambda_1 = \lambda, \quad \lambda_2 = \frac{1}{\lambda}, \quad \lambda_3 = 1$
  - $I_1 = I_2$

http://www.axelproducts.com/downloads/TestingForHyperelastic.pdf

The Limits of Hyperelasticity

- Real-world materials are not perfectly hyperelastic
  - Viscosity (*stress relaxation, creep*)
  - Plasticity (*irreversible deformation*)
  - Mullins effect (*stiffness depends on strain history*)
  - Fatigue, damage, ...

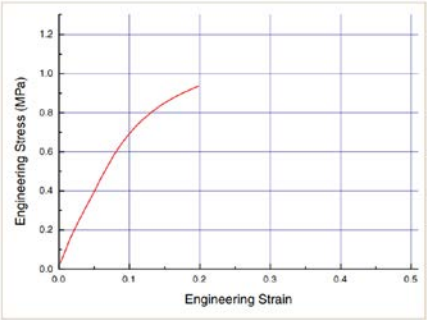


Figure 11. 1st Loading of a Thermoplastic Elastomer

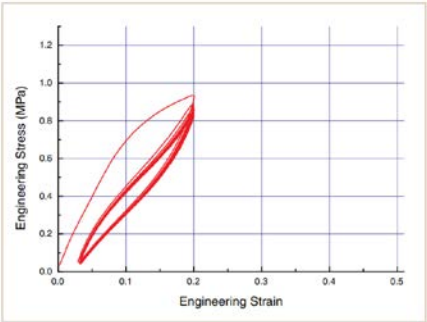


Figure 12. Multiple Strain Cycles of a Thermoplastic Elastomer

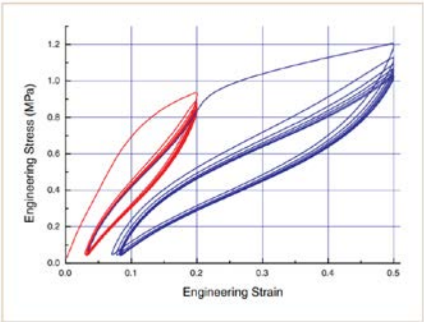


Figure 13. Multiple Strain Cycles of a Thermoplastic Elastomer at 2 Maximum Strain Levels

Further Reading

Textbook

- Bonet and Wood, Nonlinear Continuum Mechanics