EXERCISE SHEET 3

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截止日期：最迟在 5.20 提交作业。

评分标准：取 sup-norm ——只要做对一小道题，就能得到满分。当然，你也可以尝试说明题目出错了。

提示：你可以自由使用序号靠前习题的结果来解答序号靠后的习题。

如对习题（陈述，定义等）有任何的疑问，请联系我。

Contents

1. Mixing and equidistribution

- $G = \mathrm{SL}_2(\mathbb{R})$, $U = \left\{ \mathbf{u}_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, s \in \mathbb{R} \right\}$, $A = \left\{ \mathbf{a}_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, t \in \mathbb{R} \right\}$;
- $V = \left\{ \mathbf{v}_r = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, r \in \mathbb{R} \right\}$;
- Γ is a lattice in G, let $X := G/\Gamma$ and $\widehat{\mathrm{m}}_X$ be the unique G-invariant probability measure on X;
- Fix a right invariant Riemannian metric on G. Use this metric to induce a distance function $d(\cdot,\cdot)$ on G, let $d_X([g_1]_\Gamma, [g_2]_\Gamma) := \inf_{\gamma_1,\gamma_2 \in \Gamma} d(g_1\gamma_1, g_2\gamma_2)$;
- for every $\delta, s_0 > 0$, let
$$\mathrm{Box}(\delta, s_0) := (-\delta,\delta) \times (-\delta,\delta) \times (0, s_0);$$
- let $\mathrm{Leb}_{\delta,s_0}$ be the restriction of standard Lebesgue measure restricted to $\mathrm{Box}(\delta, s_0)$;
- by abuse of notation we also denote by $\mathrm{Leb}_{\delta,s_0}$ for its push-forward under the map $(r,t,s) \mapsto \mathbf{v}_r \cdot \mathbf{a}_t \cdot \mathbf{u}_s$;
- for $x \in X$, let Obt_x denote the map $G \to X$ defined by $g \mapsto g.x$.

Exercise 1.1. *Fix $x \in X$, $\delta, s_0 > 0$. Show that there exists a non-negative function $f \in L^\infty(X, \mathrm{m}_X)$ such that $(\mathrm{Obt}_x)_* \mathrm{Leb}_{\delta,s_0} = f \cdot \widehat{\mathrm{m}}_X$.*

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Exercise 1.2. *Show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $s_0 > 0$, $t > 0$, $(r, u, s) \in \mathrm{Box}(\delta, s_0)$ and $x \in X$ we have*

$$d_X\left(\mathbf{a}_t \cdot (\mathbf{v}_r \mathbf{a}_u) \cdot \mathbf{u}_s.x, \mathbf{a}_t \mathbf{u}_s.x\right) < \varepsilon.$$

Recall that mixing implies that for $\phi, \psi \in L^2(X, \widehat{\mathrm{m}}_X)$,

$$\lim_{t \to \pm\infty} \int \phi(\mathbf{a}_t.x)\psi(x)\widehat{\mathrm{m}}_X(x) = \int \phi(x)\widehat{\mathrm{m}}_X(x) \cdot \int \psi(x)\widehat{\mathrm{m}}_X(x).$$

Exercise 1.3. *For every $s_0 > 0$, $x_0 \in X$ and $f \in C_c(X)$, we have*

$$\lim_{t \to +\infty} \frac{1}{s_0} \int_0^{s_0} f(\mathbf{a}_t \mathbf{u}_s.x_0)\,\mathrm{ds} = \int f(x)\widehat{\mathrm{m}}_X(x).$$

Exercise 1.4. *Show that if $(U.x_n)$ is a sequence of compact U-orbits of periods $S_n \to +\infty$, then for every compactly supported continuous function f,*

$$\lim_{n \to +\infty} \frac{1}{S_n} \int_0^{S_n} f(\mathbf{u}_s.x_n)\,\mathrm{ds} = \int f(x)\widehat{\mathrm{m}}_X(x).$$

Exercise 1.5. *Show that the above convergence (in Exer. 1.3) is "uniform" in the following sense. For every $f \in C_c(X)$, $\varepsilon, s_0 > 0$ and $x_0 \in X$, there exists $\delta > 0$ such that for every $y \in X$ with $d_X(x_0, y) < \delta$, we have for all $t > 0$,*

$$\left| \frac{1}{s_0} \int_0^{s_0} f(\mathbf{a}_t \mathbf{u}_s.x_0)\,\mathrm{ds} - \frac{1}{s_0} \int_0^{s_0} f(\mathbf{a}_t \mathbf{u}_s.y)\,\mathrm{ds} \right| < \varepsilon.$$

Exercise 1.6. *Use the above exercise to give another proof of the equidistribution of horocycle flows. Show that if $U.x_0$ is not compact in X, then for every $f \in C_c(X)$,*

$$\lim_{S \to +\infty} \frac{1}{S} \int_0^{S} f(\mathbf{u}_s.x_0)\,\mathrm{ds} = \int f(x)\widehat{\mathrm{m}}_X(x).$$

2. NON-COMMENSURABLE LATTICES IN $\mathrm{SL}_2(\mathbb{R})$, II

This is a continuation of Exercise 2.1–2.6 from Exercise Sheet 2. Notations are inherited and here are a few more:

- Let $X := \mathrm{G}/\Gamma$ and $\widehat{\mathrm{m}}_X$ the unique G-invariant probability measure on X;
- Let Ω be a nonempty open bounded subset of UV^+ (or UV^-);
- Let $\widetilde{\mu}_0$ be the restriction of the Haar measure on UV to Ω. Fix $x_0 \in X$, let μ_0 be the push-forward of $\widetilde{\mu}_0$ under the map $g \mapsto g.x_0$. By multiplying by a scalar, we normalize μ_0 to be a probability measure $\widehat{\mu}_0$.

Exercise 2.1. *Show that $\widehat{\mathrm{m}}_X$ is A-mixing.*

Exercise 2.2. *Using mixing to show that $\lim_{t \to +\infty} (\mathbf{a}_t)_* \widehat{\mu}_0 = \widehat{\mathrm{m}}_X$.*

Exercise 2.3. *Let Y_0 be as in Exer 2.3 from Exer. Sheet 2. Show that $\mathrm{Y}_0 = X$.*

Thus we have shown that H-orbits on X are either closed or dense.

Now let Γ_1, Γ_2 be two discrete subgroups in $\mathrm{SL}_2(\mathbb{R})$ (later we will assume them to be cocompact).

Exercise 2.4. *The following two are equivalent*

1. $\Gamma_1 \cdot \Gamma_2$ *is closed in* $\mathrm{SL}_2(\mathbb{R})$;
2. $\mathrm{H} \cdot (\Gamma_1 \times \Gamma_2)$ *is closed in* G.

Exercise 2.5. *The following two are equivalent*

1. $\Gamma_1 \cdot \Gamma_2$ *is dense in* $\mathrm{SL}_2(\mathbb{R})$;

2. $\mathrm{H}\cdot(\Gamma_1\times\Gamma_2)$ *is dense in* G.

From now on we assume Γ_1, Γ_2 are both cocompact in $\mathrm{SL}_2(\mathbb{R})$.

Exercise 2.6. *The following two are equivalent*

1. $\Gamma_1\cdot\Gamma_2$ *is closed in* $\mathrm{SL}_2(\mathbb{R})$*;*
2. Γ_1 *is commensurable with* Γ_2 *(namely,* $\Gamma_1\cap\Gamma_2$ *is of finite-index in both* Γ_1 *and* Γ_2*).*

[It seems unclear to me how to prove this only assuming Γ_i's are lattices. There is an approach using random walk by Eskin–Margulis.]

Exercise 2.7. *The followings are equivalent*

1. Γ_1 *is commensurable with* Γ_2*;*
2. $\Gamma_1\cdot[\mathrm{id}]_{\Gamma_2}$ *is a finite subset of* $\mathrm{SL}_2(\mathbb{R})/\Gamma_2$*;*
3. $\Gamma_1\cdot\Gamma_2$ *is not dense in* $\mathrm{SL}_2(\mathbb{R})$.

3. Totally geodesic hyperbolic planes in H3, II

Notations and assumptions are inherited from Sec.3 from Exercise Sheet 2.

Exercise 3.1. *Show that* H*-orbits on* G/Γ *are either closed or dense.*

4. Mixing fails for non-semisimple groups

Notations

- $B=A\cdot U$ where $A:=\left\{\mathbf{a}_t=\begin{bmatrix} e^t & 0\\ 0 & e^{-t}\end{bmatrix}, t\in\mathbb{R}\right\}$ and $U=\left\{\mathbf{u}_s=\begin{bmatrix} 1 & s\\ 0 & 1\end{bmatrix}, s\in\mathbb{R}\right\}$;
- $\mathscr{H}$ is a separable Hilbert space and $\Phi: B\to\mathscr{U}(\mathscr{H})$ is a unitary representation of B.

Exercise 4.1. *Show that if* $\mathscr{H}$ *has no non-zero* $\Phi(U)$*-fixed vector ("*U*-ergodic"), then for every* $\phi,\psi\in\mathscr{H}$ *and* $t_n\to+\infty$, $\lim_n\langle\Phi(\mathbf{a}_{t_n}).\phi,\psi\rangle=0$ *("*A^+*-mixing").*

Exercise 4.2. *Same notations and assumptions as in last exercise. Show that for every* $\phi,\psi\in\mathscr{H}$ *and* $t'_n\to-\infty$, $\lim_n\langle\Phi(\mathbf{a}_{t'_n}).\phi,\psi\rangle=0$ *("*A^-*-mixing").*

Below is an example showing that "U-mixing" may not be true under the hypothesis made in last two exercises.

Let $\mathscr{H}_0:=L^2(\mathbb{R}_{>0},\mathrm{Leb})$. Define, for $t,s\in\mathbb{R}$ and $\phi\in\mathscr{H}_0$,

$$(\mathbf{a}_t.\phi)(x):=e^t\phi(e^{2t}x),\quad (\mathbf{u}_s.\phi)(x):=e^{2\pi i s x}\cdot\phi(x).$$

Exercise 4.3. *Show that the above defined action of* A *and* U *extends to a group homomorphism* $\Phi_0: B\to\mathrm{Hom}(\mathscr{H}_0,\mathscr{H}_0)$.

Here $\mathrm{Hom}(\mathscr{H}_0,\mathscr{H}_0)$ stands for linear maps from $\mathscr{H}_0$ to $\mathscr{H}_0$.

Exercise 4.4. *Show that image of* Φ_0 *consists of unitary operators.*

Exercise 4.5. *Show that* Φ_0 *defines a unitary representation of* B *(namely, one should check continuity w.r.t. strong operator topology).*

Exercise 4.6. *Show directly that* Φ_0 *is* A*-mixing. Namely, for a divergent sequence* $(a_n)\subset A$ *and* $\phi,\psi\in\mathscr{H}_0$, $\lim_n\langle\Phi_0(a_n).\phi,\psi\rangle=0$.

Exercise 4.7. *Show that there is no non-zero* $\Phi_0(U)$*-fixed vector. Yet* Φ_0 *is not* U*-mixing.*

5. Another example of Mautner phenomenon

Notations

- $N := \left\{ \begin{bmatrix} 1 & s & r \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \middle| s,t,r \in \mathbb{R} \right\}$, $Z := \left\{ \mathbf{z}_r := \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$;
- $W := \left\{ \mathbf{w}_t := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$, $U := \left\{ \mathbf{u}_s := \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$;
- $\mathscr{H}$ is a separable Hilbert space and $\Phi : N \to \mathscr{U}(\mathscr{H})$ is a unitary representation of N.

Exercise 5.1. *Verify the following*

$$\mathbf{w}_t \mathbf{u}_s \mathbf{w}_{-t} = \mathbf{u}_s \mathbf{z}_{-st}, \ \forall s,t \in \mathbb{R}.$$

Exercise 5.2. *Show that a $\Phi(W)$-fixed vector is $\Phi(Z)$-fixed.*

[Since $W \cdot Z$ is a normal subgroup of N with quotient group $\mathbb{R}$, there exists a unitary representation $(\Phi, \mathscr{H})$ of N and $v \in \mathscr{H}$ such that its stabilizer in N is exactly $W \cdot Z$.]

Now let Γ be a lattice in N.

Exercise 5.3. *Show that Γ is not commutative, and hence, not virtually commutative (namely, every finite-index subgroup of Γ is not commutative).*

Exercise 5.4. *Show that $\Gamma \cap Z$ is a lattice in Z.*

Let $p : N \to N/Z$ (Z is normal in N) be the natural quotient map.

Exercise 5.5. *Show that $p(\Gamma)$ is a lattice of N/Z.*

Let $\widehat{\mathrm{m}}_X$ be the N-invariant probability measure on N/Γ and let $\widehat{\mathrm{m}}_{\overline{X}}$ be the N/Z-invariant probability measure on $(N/Z)/p(\Gamma)$.

Exercise 5.6. *Show that $\widehat{\mathrm{m}}_X$ is W-ergodic iff $\widehat{\mathrm{m}}_{\overline{X}}$ is W-ergodic.*

Exercise 5.7. *Fix Γ, show that there exists some one-parameter unipotent subgroup $\{\mathbf{v}_s\}$ of N that acts ergodically on $\widehat{\mathrm{m}}_X$.*

One more example.

Let $G := \left\{ \begin{bmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{bmatrix} \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}),\ x,y \in \mathbb{R} \right\}$.

$\Gamma := \left\{ \begin{bmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{bmatrix} \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}),\ x,y \in \mathbb{Z} \right\}$.

Exercise 5.8. *Use mixing and non-divergence of unipotent flow to show that $\mathrm{SL}_2(\mathbb{Z})$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$.*

Exercise 5.9. *Show that Γ is a lattice in G.*

Let $\widehat{\mathrm{m}}_{G/\Gamma}$ be the unique G-invariant probability measure on G/Γ.

Exercise 5.10. *Show that $\widehat{\mathrm{m}}_{G/\Gamma}$ is $\mathrm{SL}_2(\mathbb{R})$-ergodic.*

Here we embed $\mathrm{SL}_2(\mathbb{R})$ in the left upper corner of G. By what has been proved in the class, this implies that $\widehat{\mathrm{m}}_{G/\Gamma}$ is $\mathrm{SL}_2(\mathbb{R})$-mixing.

6. Lattices and closedness of orbits

- G is a connected Lie group and Γ is a discrete subgroup of G;
- $H \leq G$ is a closed subgroup.

Exercise 6.1. *Assume $H \cap \Gamma$ is a lattice in H. Show that for a divergent sequence (x_n) in $H/H \cap \Gamma$,* $\mathrm{InjRad}(x_n) \to 0$.

Exercise 6.2. *Assume Γ satisfies the conclusion of the last exercise. Show that $H\Gamma/\Gamma$ is closed in G/Γ.*

- $U = \left\{ \mathbf{u}_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, s \in \mathbb{R} \right\}$, Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Exercise 6.3. *Assume $U \cap \Gamma$ is cocompact in U, by duality we know that $\Gamma U/U$ is closed in $\mathrm{SL}_2(\mathbb{R})/U$. The latter is homeomorphic to $\mathbb{R}^2 - (0,0)$ under $g \mapsto g.e_1$. Thus $\Gamma.e_1$ is closed in $\mathbb{R}^2 - (0,0)$. Show that, in fact, $\Gamma.e_1$ is closed in $\mathbb{R}^2$.*

Exercise 6.4. *Show that the conclusion might fail if we replace "$U \cap \Gamma$ is cocompact in U" by "$U\Gamma$ is closed in $\mathrm{SL}_2(\mathbb{R})$".*

Exercise 6.5. *Show that $B = A \cdot U$ with $A := \left\{ \mathbf{a}_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, t \in \mathbb{R} \right\}$ has no lattice.*