LECTURE 4, OPPENHEIM CONJECTURE II

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CONTENTS

1. OVERVIEW

In this lecture we prove Theorem 3.5 from lecture 3.

Theorem 1.1 (Theorem 3.5, Lec.3)**.** *Let $\Lambda \in \mathrm{X}_3$ be such that $\mathrm{SO}_{Q_0}(\mathbb{R}).\Lambda$ is bounded, then either $\mathrm{SO}_{Q_0}(\mathbb{R}).\Lambda$ is closed and hence compact, or the closure of $\mathrm{SO}_{Q_0}(\mathbb{R}).\Lambda$ contains a $\{\boldsymbol{v}_s\}_{s\geq 0}$-orbit or a $\{\boldsymbol{v}_s\}_{s\leq 0}$-orbit.*

By comparison, the ultimate knowledge regarding this is:

Theorem 1.2. *Every $\mathrm{SO}_{Q_0}(\mathbb{R})$-orbit in X_3 is either closed or dense.*

Notations and assumptions.

- $Q_0(x_1,x_2,x_3) = 2x_1x_3 - x_2^2$;
- $\mathrm{H} := \mathrm{SO}_{Q_0}(\mathbb{R}) \leq \mathrm{G} = \mathrm{SL}_3(\mathbb{R})$, $\mathrm{X}_3 := \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$;
- $$\boldsymbol{a}_t := \begin{bmatrix} e^t & & \\ & 1 & \\ & & e^{-t} \end{bmatrix},\quad \boldsymbol{u}_s := \begin{bmatrix} 1 & s & s^2 \\ & 1 & s \\ & & 1 \end{bmatrix}\quad \boldsymbol{v}_s := \begin{bmatrix} 1 & 0 & s \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$
- $$u_0 := \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix},\quad v_0 := \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$
- $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{h}^{\perp}$ denotes its orthogonal complement ;

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- Fix some $x_0 \in \mathrm{X}_3$ with $\mathrm{H}\cdot x_0$ being bounded and non-closed. Write Y_0 for the closure of $\mathrm{H}\cdot x_0$.

See last lecture for the precise definition of $\mathfrak{h}^{\perp}$.

Outline. Recall that from Lec.2, we wish to obtain something nontrivial that preserve Y_0 (or a minimal subset of Y_0) in the direction of the normalizer. The normalizer of H is basically H itself. We restrict our attention to a unipotent flow $\boldsymbol{u}_s$ of H, then we can start to apply the same argument as in Lec.2. Take a $\boldsymbol{u}_s$-minimal subset Y_1 of Y_0. The possibility of having a closed $\boldsymbol{u}_s$ orbit can be easily excluded. Then by Lec.2 we would have some additional element preserving Y_1 in the normalizer of $\boldsymbol{u}_s$. Under the current situation, two possibilities for this additional invariant exist. One is $\boldsymbol{a}_t$, the other is $\boldsymbol{v}_s$. So for our purpose we may assume that the $\boldsymbol{u}_s$-minimal set is $\boldsymbol{a}_t$-stable. And to treat this case, instead of considering those preserving Y_1, we consider the set of g mapping Y_0 to Y_1. The lack of group structure here would cause us some difficulty, taken care of by $\boldsymbol{a}_t$.

2. THE PROOF

Consider the following

$$\mathscr{O} := \left\{ y \in \mathrm{Y}_0 \,\middle|\, \mathrm{H}\cdot y \text{ is open in } \mathrm{Y}_0 \right\}$$

Thus $\mathscr{O}$ is an H-invariant open (possibly empty) subset of Y_0, in other words, $\mathrm{Y}_0 \setminus \mathscr{O}$ is an H-invariant compact set.

Lemma 2.1. *$\mathscr{O} \neq \mathrm{Y}_0$ unless $\mathrm{Y}_0 = \mathrm{H}\cdot x_0$.*

Proof. Otherwise each H-orbit is open, and hence closed in Y_0. In particular $\mathrm{H}.x_0$ is closed. But Y_0 is not closed, so here is a contradiction. □

Now we take Y_1 to be a nonempty $\{\boldsymbol{u}_s\}_{s\in\mathbb{R}}$-minimal set in $\mathrm{Y}\setminus\mathscr{O}$. There are three cases to consider

1. Y_1 is a closed $\{\boldsymbol{u}_s\}_{s\in\mathbb{R}}$-orbit;
2. Y_1 does not fall in case 1 and Y_1 is $\{\boldsymbol{a}_t\}_{t\in\mathbb{R}}$-invariant ;
3. Y_1 does not fall in case 1 or 2.

Actually, case 1 is not an option since Y_1 is bounded and hence has a lower bound on injectivity radius. So we are left with case 2 and 3.

2.1. **Case 2.** By definition, for every $x \in \mathrm{Y}_1$, there exists $y_n \to x$ with $y_n \in \mathrm{Y}_0$ and

$$y_n = \exp(h_n)\exp(w_n)x$$

where $h_n \in \mathfrak{h}$, $w_n \in \mathfrak{h}^{\perp}$ both converging to 0 and $w_n \neq 0$. Replacing y_n by $\exp(-h_n)y_n$ we assume $h_n = 0$. The case when w_n belongs to $\mathrm{Lie}(\{\boldsymbol{v}_s\})$ for infinitely many n's is easier and we assume this is not the case.

Lemma 2.2. *Assume w_n does not belong to $\mathrm{Lie}(\{\boldsymbol{v}_s\})$ for n large enough. For $\delta > 0$ small enough and n large enough, there exists $t_{n,\delta}$ such that*

1. $$\left\|\mathrm{Ad}(\boldsymbol{u}_{t_{n,\delta}})\cdot w_n\right\| \in [\frac{\delta}{10^{10}}, 10^{10}\delta];$$
2. *every limit point of $\left(\mathrm{Ad}(\boldsymbol{u}_{t_{n,\delta}})\cdot w_n\right)$ lies in $\mathrm{Lie}(\{\boldsymbol{v}_s\})$.*

See sec.3.3 for the proof. Now assume the lemma and choose $t_{n,\delta}$ as above. Define

$$x_{n,\delta} := \boldsymbol{u}_{t_{n,\delta}}.x_0, \quad y_{n,\delta} := \boldsymbol{u}_{t_{n,\delta}}.y_n,$$

then

$$y_{n,\delta} = \exp\left(\mathrm{Ad}(\boldsymbol{u}_{t_{n,\delta}}) \cdot w_n\right).x_{n,\delta}.$$

By passing to a subsequence depending on δ, we assume $\lim x_{n,\delta} = x_{\infty,\delta}$, $\lim y_{n,\delta} = y_{\infty,\delta}$ in Y_1 and $\lim \mathrm{Ad}(\boldsymbol{u}_{t_{n,\delta}}) \cdot w_n = s_\delta v_0$ in Y_0. Hence

$$y_{\infty,\delta} = \exp(s_\delta v_0)\, x_{\infty,\delta} = \boldsymbol{v}_{s_\delta}.x_{\infty,\delta}$$

with $s_\delta \to 0$. Thus

$$\begin{aligned} &\mathrm{Y}_0 \supset \{\boldsymbol{u}_t \boldsymbol{v}_{s_\delta}.x_{\infty,\delta}\}_{t\in\mathbb{R}} = \{\boldsymbol{v}_{s_\delta}\boldsymbol{u}_t.x_{\infty,\delta}\}_{t\in\mathbb{R}} \\ \implies &\mathrm{Y}_0 \supset \overline{\{\boldsymbol{v}_{s_\delta}\boldsymbol{u}_t.x_{\infty,\delta}\}_t} = \boldsymbol{v}_{s_\delta}\mathrm{Y}_1. \end{aligned}$$

The closed set

$$\left\{g \in \mathrm{H} \,\middle|\, g\mathrm{Y}_1 \subset \mathrm{Y}_0\right\}$$

is not necessarily a group. Hence we can not conclude the existence of a $\boldsymbol{v}_{\mathbb{R}}$ (or half of it) orbit inside Y_0 immediately. This is where the assumption that Y_1 is $\boldsymbol{a}_t$-stable steps in. Indeed,

$$\boldsymbol{v}_{e^{2t}s_\delta}\mathrm{Y}_1 = \boldsymbol{a}_t\boldsymbol{v}_{s_\delta}\boldsymbol{a}_t^{-1}\mathrm{Y}_1 = \boldsymbol{a}_t\boldsymbol{v}_{s_\delta}\mathrm{Y}_1 \subset \boldsymbol{a}_t\mathrm{Y}_0 = \mathrm{Y}_0, \quad \forall t \in \mathbb{R},$$

so depending on the sign of s_δ, Y_1 contains some $\boldsymbol{v}_{s\geq 0}$ or $\boldsymbol{v}_{s\leq 0}$-orbit. We are done.

2.2. **Case 3.** Take $x \in \mathrm{Y}_1$. Since $\{\boldsymbol{u}_s x\}$ is not closed, we can find $y_n = \exp(h_n)\exp(w_n)x$ with $h_n \in \mathfrak{h}$, $w_n \in \mathfrak{h}^\perp$, $h_n, w_n \to 0$ and $h_n + w_n \notin \mathrm{Lie}(\boldsymbol{u}_s)$. We can no longer assume $h_n = 0$.

Lemma 2.3. *For $\delta > 0$ small enough and n large enough, there exists $t_{n,\delta}$ and $s_{\delta,n}$ such that*

$$\boldsymbol{u}_{s_{\delta,n}} \cdot \boldsymbol{u}_{t_{n,\delta}} \exp(h_n)\exp(w_n)\boldsymbol{u}_{t_{n,\delta}}^{-1} = \exp(h_{n,\delta})\exp(w_{n,\delta}),$$

for some $h_{n,\delta} \in \mathfrak{h}$, $w_{n,\delta} \in \mathfrak{h}^\perp$ with

$$\max\left\{\left\|h_{n,\delta}\right\|, \left\|w_{n,\delta}\right\|\right\} \in \left[\frac{\delta}{10^{100}}, 10^{100}\delta\right]$$

and every limit point of $\left(h_{n,\delta} \oplus w_{n,\delta}\right)$ lies in $\mathrm{Lie}(\{\boldsymbol{a}_t\}) \oplus \mathrm{Lie}(\{\boldsymbol{v}_s\})$.

See sec.3.7 for the proof. Let $h_{\infty,\delta} \oplus w_{\infty,\delta}$ be a limit of $\left(h_{n,\delta} \oplus w_{n,\delta}\right)$. Write $g_\delta := \exp(h_{\infty,\delta})\exp(w_{\infty,\delta})$. Note that g_δ normalizes $\{\boldsymbol{u}_s\}_{s\in\mathbb{R}}$.

As in Lec.2, we arrive at

$$y_{\infty,\delta} = g_\delta.x_{\infty,\delta} \in \mathrm{Y}_1, \quad x_{\infty,\delta} \in \mathrm{Y}_1.$$

Hence

$$g_\delta\mathrm{Y}_1 = \overline{\{g_\delta\boldsymbol{u}_s.x_{\infty,\delta}\}_s} = \overline{\{\boldsymbol{u}_s.y_{\infty,\delta}\}_s} = \mathrm{Y}_1$$

As

$$\left\{g \in \mathrm{G} \,\middle|\, g\mathrm{Y}_1 = \mathrm{Y}_1\right\}$$

is a closed subgroup, if we write $g_\delta = \exp v_\delta$ with $v_\delta \to 0$ in $\mathrm{Lie}(\{\boldsymbol{a}_t\boldsymbol{v}_s\})$, then there exists some $v_{\neq 0} \in \mathrm{Lie}(\{\boldsymbol{a}_t\boldsymbol{v}_s\})$ such that

$$\exp(sv)\mathrm{Y}_1 = \mathrm{Y}_1, \quad \forall s \in \mathbb{R}.$$

If v has non-trivial $\mathrm{Lie}(\{\boldsymbol{v}_s\})$-component then we are done. Otherwise we go back to case 2. Hence the proof completes.

3. Proof of the two lemmas

The reader is encouraged to prove Lem.2.2 and 2.3 on his/her own since the proof presented here has simple ideas but messy details.

Both $\mathfrak{h}$ and $\mathfrak{h}^{\perp}$ are stable under the adjoint action of H. Hence we consider them separately. In matrix terms,

$$\mathrm{Ad}(g).M = gMg^{-1}, \quad \mathrm{ad}(X).M = XM - MX, \quad \exp(\mathrm{ad}(X)) = \mathrm{Ad}(\exp(X)).$$

3.1. **Computation, conjugacy by unipotents.** Take $w = (w_{ij}) \in \mathfrak{h}^{\perp}$, note that

$$\mathrm{Ad}(\boldsymbol{u}_s)\cdot w = \exp(s\,\mathrm{ad}(u_0))\cdot w = u_0 + s\cdot\mathrm{ad}(u_0)\,w + \frac{s^2}{2}\,\mathrm{ad}(u_0)^2\,w + \frac{s^3}{3!}\,\mathrm{ad}(u_0)^3\,w + \frac{s^4}{4!}\,\mathrm{ad}(u_0)^4\,w$$

where the higher order terms vanish.

Write

$$\begin{aligned} w &= \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & -2w_{11} & -w_{12} \\ w_{31} & -w_{21} & w_{11} \end{bmatrix} \\ &= w_{31}\cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + w_{21} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} + w_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\quad + \frac{-w_{12}}{3} \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} + \frac{w_{13}}{6} \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Also note that

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\mathrm{ad}\, u_0} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{\mathrm{ad}\, u_0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\mathrm{ad}\, u_0} \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\mathrm{ad}\, u_0} \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using this, one can compute that

$\mathrm{Ad}(\boldsymbol{u}_t)w =$

$$\begin{bmatrix} \frac{t^2}{2}w_{31} + tw_{21} + w_{11} & \frac{t^3}{3!}w_{31} + \frac{t^2}{2}w_{21} + tw_{11} + \frac{-w_{12}}{3} & \frac{t^4}{4!}w_{31} + \frac{t^3}{3!}w_{21} + \frac{t^2}{2}w_{11} + t\frac{-w_{12}}{3} + \frac{w_{13}}{6} \\ tw_{31} + w_{21} & * & * \\ w_{31} & * & * \end{bmatrix} \tag{1}$$

where the terms marked as $*$ are determined by the others, since the matrix is an element in $\mathfrak{h}^{\perp}$.

3.2. **Linear independence of characters.** Intuitively, one sees that the upper right corner of Equa.1 should dominate the rest. To turn this intuition into solid statement is not so direct due to the possible cancellations between terms. By modifying the value of t, though, we can avoid this. For simplicity let δ be such that

$$\delta := \max\left\{ \left|\frac{t^4}{4!}w_{31}\right|, \left|\frac{t^3}{3!}w_{21}\right|, \left|\frac{t^2}{2}w_{11}\right|, \left|t\frac{-w_{12}}{3}\right|, \left|\frac{w_{13}}{6}\right| \right\}. \tag{2}$$

Thus ($\|\cdot\|_{\sup}$ denotes the maximal value of the absolute values of entries of a matrix)

$$\|\mathrm{Ad}(\boldsymbol{u}_t)w\|_{\sup} \leq 5\delta. \tag{3}$$

For simplicity write

$$p_w(t) := \frac{t^4}{4!}w_{31} + \frac{t^3}{3!}w_{21} + \frac{t^2}{2}w_{11} + t\frac{-w_{12}}{3} + \frac{w_{13}}{6}.$$

Lemma 3.1. *With δ, t as above, we have that*

$$\max\left\{|p_w(t)|, |p_w(2t)|, |p_w(3t)|, |p_w(4t)|, |p_w(5t)|\right\} \geq \frac{\delta}{10^{10}}.$$

To prove this lemma, consider the matrix

$$M_0 := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2^4 & 2^3 & 2^2 & 2 & 1 \\ 3^4 & 3^3 & 3^2 & 3 & 1 \\ 4^4 & 4^3 & 4^2 & 4 & 1 \\ 5^4 & 5^3 & 5^2 & 5 & 1 \end{bmatrix}.$$

Lemma 3.2. $\det(M_0) = 4!3!2! \neq 0$*, and coefficients of* M_0^{-1} *satisfy*

$$|(M_0^{-1})_{ij}| \leq \frac{4!5^4 4^3 3^2 2}{4!3!2!} \leq 10^9$$

for every i, j.

Proof. M_0 is a Vandermonde matrix. Details left as exercise. □

Proof of Lemma 3.1.

$$\begin{bmatrix} p_w(t) \\ p_w(2t) \\ p_w(3t) \\ p_w(4t) \\ p_w(5t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2^4 & 2^3 & 2^2 & 2 & 1 \\ 3^4 & 3^3 & 3^2 & 3 & 1 \\ 4^4 & 4^3 & 4^2 & 4 & 1 \\ 5^4 & 5^3 & 5^2 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{t^4}{4!}w_{31} \\ \frac{t^3}{3!}w_{21} \\ \frac{t^2}{2}w_{11} \\ \frac{-w_{12}}{3} \\ \frac{w_{13}}{6} \end{bmatrix}$$

And

$$\delta = \left\| \begin{bmatrix} \frac{t^4}{4!}w_{31} \\ \frac{t^3}{3!}w_{21} \\ \frac{t^2}{2}w_{11} \\ \frac{-w_{12}}{3} \\ \frac{w_{13}}{6} \end{bmatrix} \right\|_{\sup} = \left\| M_0^{-1} \cdot \begin{bmatrix} p_w(t) \\ p_w(2t) \\ p_w(3t) \\ p_w(4t) \\ p_w(5t) \end{bmatrix} \right\|_{\sup} \leq 5\left\|M_0^{-1}\right\|_{\sup} \cdot \left\| \begin{bmatrix} p_w(t) \\ p_w(2t) \\ p_w(3t) \\ p_w(4t) \\ p_w(5t) \end{bmatrix} \right\|_{\sup}$$

Hence

$$\left\| \begin{bmatrix} p_w(t) \\ p_w(2t) \\ p_w(3t) \\ p_w(4t) \\ p_w(5t) \end{bmatrix} \right\|_{\sup} \geq \frac{\delta}{5\left\|M_0^{-1}\right\|_{\sup}} \geq \frac{\delta}{10^{10}}.$$

□

3.3. **Proof of Lemma 2.2.** Let $w_{ij}(n)$ be the matrix coefficients of w_n. Let $\delta>0$, for n large, we can find $t\in\mathbb{R}$ such that

$$\delta := \max\left\{\left|\frac{t^4}{4!}w_{31}(n)\right|, \left|\frac{t^3}{3!}w_{21}(n)\right|, \left|\frac{t^2}{2}w_{11}(n)\right|, \left|t\frac{-w_{12}(n)}{3}\right|, \left|\frac{w_{13}(n)}{6}\right|\right\}, \tag{4}$$

namely, Equa.2 holds. Let $t_{n,\delta}$ be one of $t,2t,...,5t$ such that the maximum in Lem.3.1 is attained. By Lem.3.1,

$$\left\|\mathrm{Ad}\,\boldsymbol{u}_{t_{n,\delta}}.w_n\right\|_{\sup} \geq \frac{\delta}{10^{10}}.$$

Also note that as $n\to\infty$, $t_{n,\delta}$ necessarily goes to $+\infty$. Equa.3 says that

$$\left\|\mathrm{Ad}\,\boldsymbol{u}_{t_{n,\delta}}.w_n\right\|_{\sup} \leq 5\delta.$$

From Equa.1, one sees that for $(i,j)\neq(1,3)$,

$$\left|\left(\mathrm{Ad}\,\boldsymbol{u}_{t_{n,\delta}}.w_n\right)_{i,j}\right| \leq \frac{4!\delta}{t_{n,\delta}},$$

which shows that as n goes to the infinity, only $\left(\mathrm{Ad}\,\boldsymbol{u}_{t_{n,\delta}}.w_n\right)_{1,3}$ survives. Now the proof is complete.

3.4. **From SL2 to SO(Q).** In this subsection we give an explicit morphism from $\mathrm{SL}_2(\mathbb{R})$ to $\mathrm{SO}_{Q_0}(\mathbb{R})$.

3.4.1. *$sl_2(\mathbb{R})$ as a quadratic space.* Note that $\mathrm{SL}_2(\mathbb{R})$ acts on

$$\mathfrak{sl}_2(\mathbb{R}) = \{2\times 2 \text{ trace zero matrices}\}$$

via the adjoint representation. And this action preserves the symmetric bilinear form

$$-\mathrm{Tr}: (X,Y)\mapsto \mathrm{trace}(X\cdot Y).$$

To identify $\mathfrak{sl}_2(\mathbb{R})$ with $\mathbb{R}^3$, consider the basis

$$E_1=\begin{bmatrix}0&1\\0&0\end{bmatrix},\quad E_2=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad E_3=\begin{bmatrix}0&0\\-1&0\end{bmatrix}.$$

And we fix an isomorphism $\mathbb{R}^3\cong\mathfrak{sl}_2(\mathbb{R})$ by sending e_i to E_i where (e_1,e_2,e_3) is the standard basis of $\mathbb{R}^3$.

Then one can check that

$$\left(-\mathrm{Tr}(E_i\cdot E_j)\right)_{i,j} = \begin{bmatrix}0&0&1\\0&-1&0\\1&0&0\end{bmatrix},$$

which means that $-\mathrm{Tr}$ is identified with Q_0 under the fixed isomorphism.

3.4.2. *adjoint action of sl2 in basis.* Denote by $\rho:\mathrm{SL}_2(\mathbb{R})\to\mathrm{SO}_{Q_0}(\mathbb{R})$ the morphism obtained by the above identification of $\mathfrak{sl}_2(R)\cong\mathbb{R}^3$. Let us compute $\mathrm{d}\rho:\mathfrak{sl}_2(R)\to\mathfrak{so}_{Q_0}(\mathbb{R})$.

Let

$$X=\begin{bmatrix}a&b\\c&-a\end{bmatrix}\in\mathfrak{sl}_2(\mathbb{R}).$$

Then

$$\begin{aligned}\mathrm{ad}(X)E_1 &= \begin{bmatrix}a&b\\c&-a\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}-\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}a&b\\c&-a\end{bmatrix}\\ &=\begin{bmatrix}0&a\\0&c\end{bmatrix}-\begin{bmatrix}c&-a\\0&0\end{bmatrix}=\begin{bmatrix}-c&2a\\0&c\end{bmatrix}=2aE_1+(-\sqrt{2}c)E_2+0E_3,\end{aligned}$$

$$\mathrm{ad}(X)E_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} a & -b \\ c & a \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} a & b \\ -c & a \end{bmatrix} = \begin{bmatrix} 0 & -\sqrt{2}b \\ \sqrt{2}c & 0 \end{bmatrix} = (-\sqrt{2}b)E_1 + 0E_2 + (-\sqrt{2}c)E_3$$

and

$$\mathrm{ad}(X)E_3 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$
$$= \begin{bmatrix} -b & 0 \\ a & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -a & -b \end{bmatrix} = \begin{bmatrix} -b & 0 \\ 2a & b \end{bmatrix} = 0E_1 + (-\sqrt{2}b)E_2 + (-2a)E_3$$

Hence we have that

$$\mathrm{d}\rho : \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mapsto \begin{bmatrix} 2a & -\sqrt{2}b & 0 \\ -\sqrt{2}c & 0 & -\sqrt{2}b \\ 0 & -\sqrt{2}c & -2a \end{bmatrix}.$$

(sanity check: RHS is indeed a matrix in $\mathfrak{so}_{Q_0}(\mathbb{R})$)

3.5. **Image of a unipotent flow.** Let

$$\boldsymbol{u}'_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}.$$

Then

$$\rho(\boldsymbol{u}'_s) = \exp\left(\mathrm{d}\rho\begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix}\right) = \exp\left(\begin{bmatrix} 0 & -\sqrt{2}s & 0 \\ 0 & 0 & -\sqrt{2}s \\ 0 & 0 & 0 \end{bmatrix}\right) = \boldsymbol{u}_{-\sqrt{2}s}$$

3.6. **Exponential of a lower triangular matrix.** Say we have

$$\exp\begin{bmatrix} x & 0 \\ y & -x \end{bmatrix} = \begin{bmatrix} (1+a) & 0 \\ b & (1+a)^{-1} \end{bmatrix},$$

we would like to express x, y in terms of a, b.

Indeed, by definition of exp,

$$\mathrm{LHS} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} x & 0 \\ y & -x \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & 0 \\ y & -x \end{bmatrix}^2 + \frac{1}{3!}\begin{bmatrix} x & 0 \\ y & -x \end{bmatrix}^3 + \dots$$

So we should compute the powers of this matrix first.

$$\begin{bmatrix} x & 0 \\ y & -x \end{bmatrix}^2 = \begin{bmatrix} x^2 & 0 \\ 0 & (-x)^2 \end{bmatrix} \implies \begin{bmatrix} x & 0 \\ y & -x \end{bmatrix}^{2n} = \begin{bmatrix} x^{2n} & 0 \\ 0 & (-x)^{2n} \end{bmatrix}.$$

And odd powers are

$$\begin{bmatrix} x & 0 \\ y & -x \end{bmatrix}^{2n+1} = \begin{bmatrix} x^{2n+1} & 0 \\ yx^{2n} & (-x)^{2n+1} \end{bmatrix}.$$

Thus

$$\exp\begin{bmatrix} x & 0 \\ y & -x \end{bmatrix} = \begin{bmatrix} e^x & 0 \\ y\left(\frac{e^x - e^{-x}}{2x}\right) & e^{-x} \end{bmatrix} = \begin{bmatrix} 1+a & 0 \\ b & (1+a)^{-1} \end{bmatrix}.$$

And thus

$$x = \ln(1+a),\ y = b\left(\frac{2\ln(1+a)}{(1+a)-(1+a)^{-1}}\right). \tag{5}$$

The equality for y is not needed. Also note that for $|a| < 1$

$$|\ln(1+a) - a| \le 2|a|^2 \tag{6}$$

and that if (a_n) is a sequence contained in some fixed compact sub-interval of $(-1,+\infty)$ and (b_n) is a sequence converging to 0 then the corresponding (y_n) should converge to 0.

3.7. **Proof of Lemma 2.3.** Define h'_n by $h_n = \mathrm{d}\rho(h'_n)$. Write

$$\exp(h'_n) = \begin{bmatrix} 1+a_n & b_n \\ c_n & 1+d_n \end{bmatrix}.$$

For $s,t\in\mathbb{R}$, write $s' := s/(-\sqrt{2})$, $t' = t/(-\sqrt{2})$. Hence $\rho(\boldsymbol{u}'_{s'}) = \boldsymbol{u}_s$ and $\rho(\boldsymbol{u}'_{t'}) = \boldsymbol{u}_t$.

Choose $s_{n,\delta}$ depending on $t_{n,\delta}$(to be determined later) such that

$$\boldsymbol{u}_{s'_{n,\delta}}\boldsymbol{u}_{t'_{n,\delta}}\exp(h'_n)\boldsymbol{u}^{-1}_{t'_{n,\delta}} = \begin{bmatrix} (1+d_n-t'_{n,\delta}c_n)^{-1} & 0 \\ c_n & 1+d_n-t'_{n,\delta}c_n \end{bmatrix}.$$

See Lec.2 for details. Define $h'_{n,\delta}$ by

$$\exp\left(h'_{n,\delta}\right)\begin{bmatrix} (1+d_n-t'_{n,\delta}c_n)^{-1} & 0 \\ c_n & 1+d_n-t'_{n,\delta}c_n \end{bmatrix}.$$

Write $w_n = (w_{ij}(n))$. Choose t such that

$$\delta = \max\left\{\left|d_n - t'c_n\right|, \left|\frac{t^4}{4!}w_{31}(n)\right|, \left|\frac{t^3}{3!}w_{21}(n)\right|, \left|\frac{t^2}{2}w_{11}(n)\right|, \left|t\frac{-w_{12}(n)}{3}\right|, \left|\frac{w_{13}(n)}{6}\right|\right\}.$$

Also let

$$\delta' := \max\left\{\left|\frac{t^4}{4!}w_{31}(n)\right|, \left|\frac{t^3}{3!}w_{21}(n)\right|, \left|\frac{t^2}{2}w_{11}(n)\right|, \left|t\frac{-w_{12}(n)}{3}\right|, \left|\frac{w_{13}(n)}{6}\right|\right\}.$$

We choose $t_{n,\delta}$ from $t, 2t, ..., 5t$ such that the maximum in Lem.3.1 is attained (with δ replaced by δ').

Define $h_{n,\delta} := \mathrm{d}\rho(h'_{n,\delta})$ and

$$w_{n,\delta} := \mathrm{Ad}(\boldsymbol{u}_{t_{n,\delta}}).w_n.$$

Now everything is defined and it remains to check the conclusion of Lem.2.3.

For n sufficiently large,

$$\left|d_n - t'_{n,\delta}c_n\right| \le 10\delta.$$

Now if $\delta = \delta'$, then the conclusion follows from the proof of Lem.2.2 and the sentence below Equa.6.

If $\delta > \delta'$, then $\delta = |d_n - tc_n|$. Now by Equa.6,

$$\left|\left(h'_{n,\delta}\right)_{2,2}\right| \ge \left|d_n - t'_{n,\delta}c_n\right| - 2\left|d_n - t'_{n,\delta}c_n\right|^2 \ge \frac{1}{2}\delta - (2\delta)^2$$

which is at least 0.1δ for δ sufficiently small. Again, the conclusion holds. Now we are done.