LECTURE 5, UNIPOTENT FLOWS ON X_2 AND NONDIVERGENCE

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CONTENTS

Notation:

$$\boldsymbol{u}_s := \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{a}_t := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \ U := \{\boldsymbol{u}_s, s \in \mathbb{R}\}.$$

$$\mathrm{X}_2 = \left\{ \text{unimodular lattices in } \mathbb{R}^2 \right\}$$

1. SUMMARY

The main reference for Lecture 5,6,7 would be Kleinbock's Clay notes [Kle10]. My exposition differs slightly and is less efficient compared to the reference.

Theorem. *Assume Γ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. Let $\mathrm{X} := \mathrm{SL}_2(\mathbb{R})/\Gamma$ and $x_0 \in \mathrm{X}$. Then*

1. *$U.x_0$ is either compact or dense;*
2. *a sequence of compact orbits $(U.x_n)$ with period increasing to ∞ becomes dense in X;*
3. *a sequence of compact orbits $(U.x_n)$ with period decreasing to 0 diverges in X.*

We will prove the theorem in the case when $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and leave the general case as an exercise to the reader. But note that in the proof we won't use the fact that $\mathrm{SL}_2(\mathbb{Z})$ is a lattice.

Theorem 1.1. *Recall $\mathrm{X}_2 \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. Let $\Lambda_0 \in \mathrm{X}_2$. Then*

1. *$U.\Lambda_0$ is either compact or dense;*
2. *a sequence of compact orbits $(U.\Lambda_n)$ with period increasing to ∞ become dense in X_2;*
3. *a sequence of compact orbits $(U.\Lambda_n)$ with period decreasing to 0 diverge in X_2.*
4. *$U.\Lambda_0$ is compact iff $\boldsymbol{a}_t.\Lambda_0$ diverges as $t \to -\infty$ iff Λ_0 contains a horizontal vector (i.e., a vector of the form $(*,0)$ with $* \neq 0$).*

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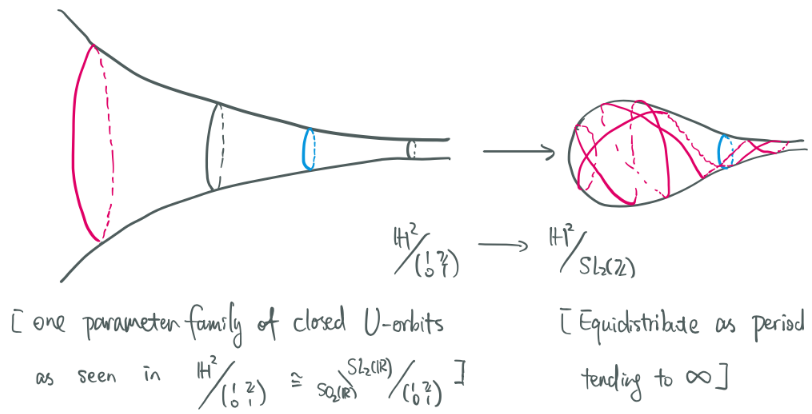
When $\{\boldsymbol{u}_s.\Lambda_0\}_{s\in\mathbb{R}}$ is dense, the discrete version $\{\boldsymbol{u}_s.\Lambda_0\}_{s\in\mathbb{Z}_{\geq 0}}$ is also dense (Exercise). In this lecture we will only prove item 1 from the theorem. Item 2 is left as an exercise (using similar proof to the one presented here). Item 3 follows from the fact that injectivity radius is bounded from below on compact sets. The nontrivial part of item 4 is proved in Lem.2.3. Also note that in the present case all compact U-orbits form a one-parameter family indexed by $\boldsymbol{a}_t$ as t varies.



2. Proof of main theorem

Definition 2.1. *For $\varepsilon > 0$, define*

$$\mathscr{C}_\varepsilon := \left\{\Lambda \in \mathrm{X}_2 \,\middle|\, \mathrm{sys}(\Lambda) \geq \varepsilon\right\}.$$

Somehow I decide to use the more suggestive notation

$$\mathrm{sys}(\Lambda) := \inf_{v\in\Lambda, v\neq 0} \|v\|.$$

By Lem.2.9 from lec.3 (Mahler's criterion), $\mathscr{C}_\varepsilon$ is a compact set and every compact set in X_2 is contained in $\mathscr{C}_\varepsilon$ for some $\varepsilon > 0$.

Lemma 2.2. *[Uniform non-divergence of unipotent flows for X_2] For every compact set $K \subset \mathrm{X}_2$ and $\varepsilon \in (0,1)$, there exists $\delta = \delta(K,\varepsilon) > 0$ such that the following holds. For every interval (a,b) with $a < b$ in $\mathbb{R}$ and $\Lambda_0 \in \mathrm{X}_2$ satisfying $\boldsymbol{u}_{s_0}.\Lambda \in K$ for some $s_0 \in (a,b)$, we have that*

$$\frac{1}{b-a}\mathrm{Leb}\{s \in (a,b) \,|\, \boldsymbol{u}_s.\Lambda_0 \notin \mathscr{C}_\delta\} \leq \varepsilon.$$

Actually the choice of δ is also independent of the unipotent flow we use.

Lemma 2.3. *If $\varepsilon \leq 1$ and $\Lambda \in \mathrm{X}_2$ are such that $\boldsymbol{u}_s.\Lambda \notin \mathscr{C}_\varepsilon$ for every s in some interval of infinite length (i.e., something like $(a,+\infty), (-\infty,b), (-\infty,+\infty)$), then Λ contains a horizontal vector of length less than ε. That is to say, $(v_1,0) \in \Lambda$ for some $0 < |v_1| < \varepsilon$.*

The reader might have noticed that the converse also holds since U-action fixes the horizontal direction. Also note that such U-orbits are closed and compact. In this case, one may think of U-action on Λ as "Dehn-twist" along the closed geodesic represented by $(v_1,0) \in \Lambda = \pi_1(\mathbb{R}^2/\Lambda)$.

Proof of Theorem 1.1 assuming Lem.2.2 and 2.3. These two lemmas basically allow us to repeat the argument from Lec.2.

Take some $x_0 \in \mathrm{X}_2$ such that $U.x_0$ is not compact. Let Y_0 be its closure. Consider

$$\left\{ \overline{U.y} \,\middle|\, y \in \mathrm{Y}_0,\ U.y \text{ is not compact} \right\}$$

Let Y_1 be a (nonempty) minimal element whose existence is guaranteed by Lem.2.2 and Zorn's lemma. Thus for every $y \in \mathrm{Y}_1$, $U.y$ is either compact or dense in Y_1.

There are two cases to discuss.

Case 1. Y_1 contains no compact U-orbit;

Case 2. Y_1 contains some compact U-orbit.

Let us start with Case 1. (of course, in the end we know that case 1 does not happen)

Take $x_1 \in \mathrm{Y}_1$. By Lem 2.2 and 2.3, there are $s_n \to \infty$ such that $\boldsymbol{u}_{s_n}.x_1 \in \mathscr{C}_1$. We may and do assume that $|s_n - s_m| > 1$ if $n \neq m$. As they are distinct from each other, we can find $x_n \neq y_n$ from this set such that $d(x_n, y_n) \to 0$. Thus we can find $A_n \in \mathrm{SL}_2(\mathbb{R})$ with $A_n \to \mathrm{id}$ such that

$$y_n = A_n.x_n.$$

For n large enough, $A_n \notin U$. Actually we are going to assume $c_n \neq 0$ and leave the other cases to the reader. Write

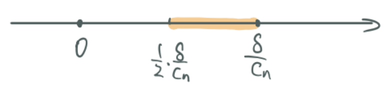
$$A_n = \mathrm{id} + \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \text{ with } a_n, b_n, c_n, d_n \to 0,\ c_n \neq 0.$$

(ok, I slightly deviate from the notation in lec.2, a_n there is replaced by $1 + a_n$ and d_n replaced by $1 + d_n$)

Just as in Lec.2, for some s to be determined, take $t = t(s)$ such that (see equa.(4) from Lec.2)

$$\boldsymbol{u}_t(\boldsymbol{u}_s A_n \boldsymbol{u}_s^{-1}) = \begin{bmatrix} (1 + d_n - s c_n)^{-1} & 0 \\ c_n & 1 + d_n - s c_n \end{bmatrix}.$$

Fix a small parameter $\delta > 0$. If we set $s_{n,\delta}$ as in lec.2, then there is no guarantee that $u_{s_{n,\delta}}.x_n$ would have a convergent subsequence. Thus we need to apply Lem.2.2 again, to $K = \mathscr{C}_1$ and $\varepsilon = 0.6$. So we get some $\delta_1 = \delta(\mathscr{C}_1, 0.6) > 0$ such that the conclusion there holds. Now we search for $s_{n,\delta}$ within $\frac{1}{c_n}\delta\lambda$ as λ varies from $(0.5, 1)$.



By our choice of δ_1, with Lem.2.2 applied to $(a, b) = (0, \frac{\delta}{c_n})$, we have that for some $\lambda_{n,\delta} \in (0.5, 1)$, if we set

$$s_{n,\delta} := \frac{1}{c_n}\delta\lambda_{n,\delta},$$

then $u_{s_{n,\delta}}.x_n \in \mathscr{C}_{\delta_1}$. As before, with $s = u_{s_{n,\delta}}$ and $t = t(s)$, let

$$x'_n := u_s.x_n, \quad y'_n := u_{t+s}.y_n.$$

By taking the limit along a subsequence, we get a pair $x_{\infty,\delta}, y_{\infty,\delta} \in \mathrm{Y}_0$ such that

$$y_{\infty,\delta} = \exp\left(\begin{bmatrix} (1 + \lambda_{n,\delta}\delta)^{-1} & 0 \\ 0 & 1 + \lambda_{n,\delta}\delta \end{bmatrix} \right).x_{\infty,\delta}.$$

Let $B_{n,\delta}$ be this diagonal matrix. Then by minimality of Y_1 and by assumption of case 1, $B_{n,\delta}.\mathrm{Y}_1 = \mathrm{Y}_1$. By letting $\delta \to 0$, we have that Y_1 is invariant under the group consisting of positive diagonal matrices. The rest of the proof is similar to Lec.2. (well, the proof of lem.2.8 is slightly different for non-cocompact lattices, but still ok; actually in the present case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, it should be even easier by regarding $\mathrm{SL}_2(\mathbb{R})/B$ as the space of lines in $\mathbb{R}^2$ and it suffices to observe that rational lines are dense among all lines.)

So now turn to case 2.

We more-or-less repeat the above proof with x_n being on a fixed closed U-orbit $U.x_2$ contained in Y_1. However, this time there is no need to, and we do not, modify the definition of $s_{n,\delta}$ from Lec.2. The end result would be

$$y_{\infty,\delta} = \exp\left(\begin{bmatrix} (1+\delta)^{-1} & 0 \\ 0 & 1+\delta \end{bmatrix}\right).x_{\infty,\delta}$$

where $x_{\infty,\delta} \in U.x_2$. Modifying by certain $\boldsymbol{u}_s$, we may and do assume that $x_{\infty,\delta} = x_2$. Thus, as δ varies, we get $B^+.x_2$ is contained in Y_1. The rest of the proof is the same as in case 1.

□

3. Proof of nondivergence

Lemma 3.1. *There exist $C_1 > 0$ and $\alpha_1 > 0$ such that for every interval (a,b) in $\mathbb{R}$, every $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$ and every $\rho \in (0,1)$, we have*

$$\frac{1}{b-a}\mathrm{Leb}\left\{s \in (a,b) \,\middle|\, \|\boldsymbol{u}_s.v\| < \rho M_0\right\} \le C_1 \rho^{\alpha_1}.$$

where $M_0 := \sup_{s\in(a,b)} \|\boldsymbol{u}_s.v\|$.

Proof. Take $C_1 = 2\sqrt{2}$ and $\alpha_1 = 1$.

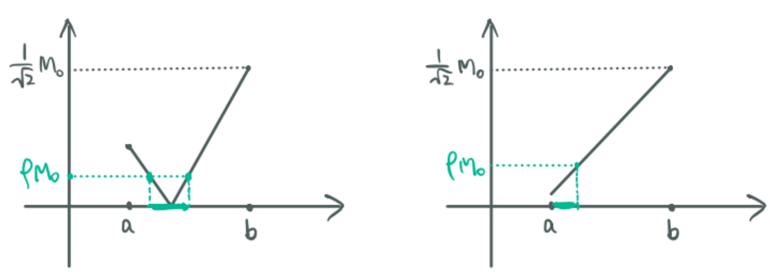
Note $\boldsymbol{u}_s.(v_1, v_2) = (v_1 + s v_2, v_2)$.

If $|v_2| \ge \frac{1}{\sqrt{2}} M_0$ then for every $s \in (a,b)$, $\|\boldsymbol{u}_s.v\| \ge |v_2| \ge \frac{1}{\sqrt{2}} M_0$. So if $\rho \le \frac{1}{\sqrt{2}}$, then we are already done. Otherwise, $C_1 \rho^{\alpha_1} \ge 1$. Also ok.

So now we are left with the case when $|v_1 + s_0 v_2| \ge \frac{1}{\sqrt{2}} M_0$ for some $s_0 \in (a,b)$. Refer to the picture below, we see that

$$\frac{1}{b-a}\mathrm{Leb}\left\{s \in (a,b) \,\middle|\, |v_1 + s v_2| < \rho M_0\right\} \le 2\frac{\rho}{1/\sqrt{2}} = C_1 \rho.$$

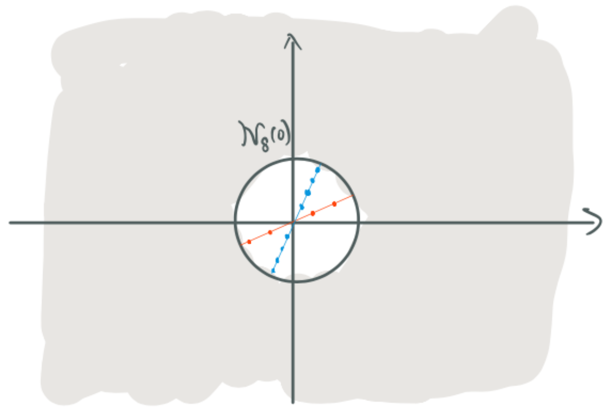
It remains only to note that $|v_1 + s v_2| < \|\boldsymbol{u}_s.(v_1, v_2)\|$. □



Key observation. A rank 2 unimodular lattice $\Lambda \in \mathrm{X}_2$ is not allowed to contain two linearly independent vector of length strictly smaller than 1. For otherwise, if v, w is such a pair,

$$\|\Lambda\| \le \|\mathbb{Z}v \oplus \mathbb{Z}w\| \le \|v\|\,\|w\| < 1,$$

contradicting against the assumption that Λ is unimodular.



No matter what the unimodular lattice is, you at most see a single line in a small nbhd about $\vec{0}$.

Let
$$\mathrm{Prim}(\Lambda) := \left\{v_{\neq 0} \in \Lambda \,\middle|\, \mathbb{R}.v \cap \Lambda = \mathbb{Z}.v\right\}$$
be the set of primitive vectors.

Proof of Lem.2.2. Find $\delta_0 \in (0,1)$ such that $K \subset \mathscr{C}_{\delta_0}$. We shall determine δ later, depending on δ_0 and ε.

Take $\Lambda_0 \in K \subset \mathscr{C}_{\delta_0}$. Let
$$I(\Lambda_0, \varepsilon_0) := \left\{s \in (a,b) \,\middle|\, \mathrm{sys}(\boldsymbol{u}_s.\Lambda_0) < \delta_0\right\}$$
which decomposed as a disjoint union of open intervals
$$I(\Lambda_0, \varepsilon_0) = \bigsqcup_{\alpha \in \mathscr{A}} I_\alpha$$
with certain index set $\mathscr{A}$.

Take one $I_\alpha = (x_\alpha, y_\alpha)$. By the remark right before the proof, for $s \in I_\alpha$, there exists a unique v_s (up to ± 1) in $\mathrm{Prim}(\Lambda_0)$ with
$$\|\boldsymbol{u}_s.v_s\| < \varepsilon_0.$$
By connectedness, this v_s has to be independent of $s \in I_\alpha$. For this reason denote it by v_α. By Lem.3.1,
$$\frac{1}{|I_\alpha|}\mathrm{Leb}\left\{s \in I_\alpha \,\middle|\, \|\boldsymbol{u}_s.v_\alpha\| < \rho\delta_0\right\} < C_1\rho^{\alpha_1}.$$
We take $\rho = \rho(\varepsilon)$ such that $C_1\rho^{\alpha_1} < \varepsilon$. Let $\delta := \rho\delta_0$.
$$\{s \in (a,b) \mid \|\boldsymbol{u}_s.v_\alpha\| < \delta\} = \bigsqcup_{\alpha \in \mathscr{A}} \left\{s \in I_\alpha \,\middle|\, \|\boldsymbol{u}_s.v_\alpha\| < \rho\delta_0\right\}$$

implying

$$\mathrm{Leb}\{s \in (a,b) \mid \|\boldsymbol{u}_s.v_\alpha\| < \delta\} = \sum_{\alpha \in \mathscr{A}} \mathrm{Leb}\left\{s \in I_\alpha \,\middle|\, \|\boldsymbol{u}_s.v_\alpha\| < \rho\delta_0\right\} < \sum_{\alpha \in \mathscr{A}} |I_\alpha| \cdot \varepsilon \leq (b-a)\varepsilon.$$

□

Proof of Lem.2.3 . Let I be this infinite interval. Since for each $s \in I$ there exists a unique (up to ± 1) v_s in $\mathrm{Prim}(\Lambda)$ with $\|\boldsymbol{u}_s.v_s\| < 1$. By connectedness argument, this $v = v_s$ is independent of $s \in I$. Thus $\|\boldsymbol{u}_s.v\| < 1$ for all $s \in I$. This happens only if U fixes v and we are done. □

REFERENCES

[Kle10] Dmitry Kleinbock, *Quantitative nondivergence and its Diophantine applications*, Homogeneous flows, moduli spaces and arithmetic, Clay Math. Proc., vol. 10, Amer. Math. Soc., Providence, RI, 2010, pp. 131–153. MR 2648694