LECTURE 6, NONDIVERGENCE OF UNIPOTENT FLOW ON X3 AND OPPENHEIM CONJECTURE III

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CONTENTS

Notations:

- $\mathrm{H} := \mathrm{SO}_{Q_0}(\mathbb{R})$ with $Q_0(x_1,x_2,x_3) := 2x_1x_3 - x_2^2$;
- $\mathrm{X}_3 := \{$ unimodular lattices in $\mathbb{R}^3\}$;
- $\boldsymbol{u}_s := \begin{bmatrix} 1 & s & \frac{s^2}{2} \\ & 1 & s \\ & & 1 \end{bmatrix} = \exp\left(s \cdot \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}\right)$, $U := \{\boldsymbol{u}_s :\, s \in \mathbb{R}\}$;
- $\boldsymbol{v}_s := \begin{bmatrix} 1 & 0 & s \\ & 1 & 0 \\ & & 1 \end{bmatrix} = \exp\left(s \cdot \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}\right)$, $V := \{\boldsymbol{v}_s :\, s \in \mathbb{R}\}$;
- $\boldsymbol{a}_t := \begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} = \exp\left(t \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right)$, $A := \{\boldsymbol{a}_t :\, t \in \mathbb{R}\}$;
- $B^+ := \{\boldsymbol{a}_t \boldsymbol{u}_s :\, s, t \in \mathbb{R}\}$.

1. SUMMARY

Finally, in this section we prove the strong form of Oppenheim conjecture. The general case can be reduced to the case of three variables, which we now state

Theorem 1.1. *Let Q be a non-degenerate indefinite ternary real quadratic form that is not proportional to a rational quadratic form. Then $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$. Actually $Q(\mathrm{Prim}(\mathbb{Z}^3))$ is dense in $\mathbb{R}$.*

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Theorem 1.2. *For every non-closed orbit of* H *on* X_3*, its closure contains a* $\{\boldsymbol{v}_s\}_{s\geq 0}$ *or* $\{\boldsymbol{v}_s\}_{s\leq 0}$*-orbit.*

By similar arguments presented in Lec3, page 7, "proof of Thm.3.4 assuming", Thm.1.1 would follow from Thm.1.2 and the following (Exercise, you might wish to minic the proof before, but now should also make use of Thm.1.4 below.)

Theorem 1.3. *If an* H*-orbit is closed, then the stabilizer in* H *is discrete and of finite covolume in* H*. Also the corresponding quadratic form is a scalar multiple of some rational quadratic form.*

The key to promote the weak version to this one is the following non-divergence theorem

Theorem 1.4. *For every* $\varepsilon > 0$*, there exists a compact subset* $\mathscr{C}$ *of* X_3 *such that for every* $\Lambda \in \mathrm{X}_3$*, at least one of the followings is true*

1. *The portion of time for* $\boldsymbol{u}_s\Lambda$ *to spend outside* $\mathscr{C}$ *is smaller than* ε*, i.e.,*
$$\limsup_{T\to+\infty} \frac{1}{T}\,|\{s \in [0,T] \mid \boldsymbol{u}_s \cdot \Lambda \notin \mathscr{C}\}| \leq \varepsilon;$$
2. $\Lambda \cap \{(x,0,0),\ x \in \mathbb{R}\}$ *contains a non-zero vector of length smaller than* ε*;*
3. $\Lambda \cap \{(x,y,0),\ x,y \in \mathbb{R}\}$ *contains a lattice (of* $\mathbb{R}e_1 \oplus \mathbb{R}e_2$*) of covolume smaller than* ε*.*

(I should modify the statement to make it more aligned with the one from last lecture, proved by the same argument presented below)

Corollary 1.5. *Let* $\varepsilon \in (0,1)$ *and pick* $\mathscr{C}$ *as in the above theorem. Then every orbit of* B^+ *intersects non-trivially with* $\mathscr{C}$*.*

Finally let us make a convenient definition. Let $e_1 := (1,0,0)$ and $e_2 = (0,1,0)$.

Definition 1.6. *We say* $\mathbb{R}.e_1$ *is* Λ*-rational iff* $\Lambda \cap \mathbb{R}e_1$ *is a lattice in* $\mathbb{R}e_1$*, and* $\mathbb{R}e_1 \oplus \mathbb{R}e_2$ *is* Λ*-rational iff* $\Lambda \cap \mathbb{R}e_1 \oplus \mathbb{R}e_2$ *is a lattice in* $\mathbb{R}e_1 \oplus \mathbb{R}e_2$*. In either of these two cases, we say that the orbit* $U.\Lambda$ ***degenerates****.*

This notion is justified by the fact that in these cases the orbit is essentially contained in certain (embedded) $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 / \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, which is interpreted as the space of lattices of lower rank with a fixed volume together with a marked point in the quotient torus.

2. Proof of Theorem 1.2

Now let us prove Theorem 1.2. Start with Λ_0 with $\mathrm{H}.\Lambda_0$ non-closed. Let $\mathrm{Y}_0 := \overline{\mathrm{H}.\Lambda_0}$. Define $\mathscr{O}$ as in Lec.4, the union of all H-orbits in Y_0 that is open in Y_0. Note that $\mathscr{O} \neq \mathrm{Y}_0$.

The old argument takes care of the case when $\mathrm{Y}_0 \setminus \mathscr{O}$ contains no degenerate U-orbit.

Indeed under this assumption every U-orbit in $\mathrm{Y}_0 \setminus \mathscr{O}$, by Thm.1.4, intersects with some fixed compact set non-trivially. Hence we can find a nonempty U-minimal set Y_1 in $\mathrm{Y}_0 \setminus \mathscr{O}$. As in Lec.4, there are two cases:

1. Y_1 is A-stable, we consider $\mathrm{Map}(\mathrm{Y}_1, \mathrm{Y}_0) := \{g\mathrm{Y}_1 \subset \mathrm{Y}_0\}$;
2. Y_1 is not A-stable, we consider $\mathrm{Aut}(\mathrm{Y}_1) := \{g\mathrm{Y}_1 = \mathrm{Y}_1\}$.

The arguments in Lec.4 should go quite smoothly here. In case 1, you may need to do a further perturbation to guarantee the sequence you get has a convergent subsequence.

2.1. **New story, general assumption.** However, it is unavoidable that $\mathrm{Y}_0 \setminus \mathcal{O}$ may contain some degenerate U-orbit. Let us take a nonempty $B := B^+$ minimal set $\mathrm{Y}_1 \subset \mathrm{Y}_0 \setminus \mathcal{O}$ whose existence is guaranteed by the nondivergence corollary Coro.1.5. There is only one case (the most difficult one!) that we really need to use Y_1. Take some $\Lambda_1 \in \mathrm{Y}_1$ such that $U.\Lambda_1$ degenerates. We will assume $\mathbb{R}e_1 \oplus \mathbb{R}e_2$ is Λ_1-rational and leave the other case when $\mathbb{R}e_1$ is Λ_1-rational to the reader.

2.2. **Case 1, no closed U-orbits.** Assume Y_1 contains no closed U-orbit.

As we assumed, $U.\Lambda$ is stuck in the following closed set (for simplicity write $\mathbb{R}e_{1,2} := \mathbb{R}e_1 \oplus \mathbb{R}e_2$)

$$\mathrm{X}_3(\mathbb{R}e_{1,2}, c_1) := \left\{\Lambda \in \mathrm{X}_3 \,\middle|\, \mathbb{R}e_{1,2} \text{ is } \Lambda\text{-rational, } \left\|\Lambda \cap \mathbb{R}e_{1,2}\right\| = c_1\right\}$$

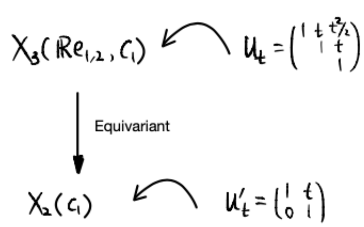
where $c_1 := \left\|\Lambda_1 \cap \mathbb{R}e_{1,2}\right\|$. Also let

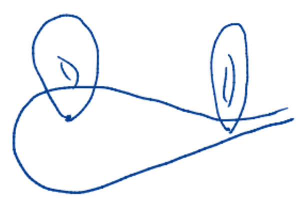
$$\mathrm{X}_2(c_1) := \left\{ \text{lattices in } \mathbb{R}^2 \text{ of covolume } c_1 \right\}.$$

Then we have a natural continuous surjection $\pi : \mathrm{X}_3(\mathbb{R}e_{1,2}, c_1) \to \mathrm{X}_2(c_1)$ with compact fibres that is equivariant with respect to

$$\begin{aligned}\rho_\pi : \{g \in \mathrm{SL}_3(\mathbb{R}),\, g \text{ preserves } \mathbb{R}e_{1,2},\, \det(g|_{\mathbb{R}e_{1,2}}) = 1\} &\to \mathrm{SL}_2(\mathbb{R}) \\ g &\mapsto g|_{\mathbb{R}e_{1,2}}.\end{aligned}$$

In particular we have





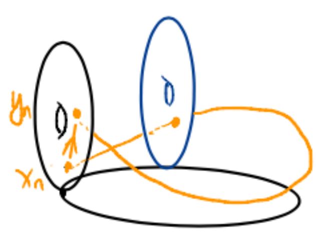
Now we wish to find a U-minimal set in Y_1.

2.3. **Case 1.1, some $\pi(U.\Lambda_2)$ is compact.** Assume for some $\Lambda_2 \in \mathrm{Y}_1$, $\pi(U.\Lambda_2)$ is closed and hence compact.

Then $\overline{U.\Lambda_2}$ is compact and let $\overline{U.\Lambda_3}$ be a nonempty minimal U-set in $\mathrm{Y}_2 := \overline{U.\Lambda_2}$.

Then we can find pairs (x_n, y_n) in Y_2 such that $y_n = \exp(w_n)x_2$ with

- $w_n = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$, $w_n \neq 0$, $w_n \to 0$.



First assume $w_n \in \mathrm{Lie}(V)$ for infinitely many n, then $\exp(w_n) \in \mathrm{Aut}(\mathrm{Y}_2)$ and since the later is a closed group, we have the full $V \subset \mathrm{Aut}(\mathrm{Y}_2)$.

Otherwise w_n is not fixed by $\mathrm{Ad}(U)$ and for any $\delta > 0$ and for n large enough we can find $t_{n,\delta}$ such that

- $\left\|\mathrm{Ad}(\boldsymbol{u}_{t_{n,\delta}}).w_n\right\| \approx \delta$;
- every limit of $\left(\mathrm{Ad}(\boldsymbol{u}_{t_{n,\delta}}).w_n\right)$ is in $\mathrm{Lie}(V)$.

And by taking a limit we find

- $x_{\infty,\delta}, y_{\infty,\delta} \in \mathrm{Y}_2$ and $w_{\infty,\delta} \in \mathrm{Lie}(V)$
- $w_{\infty,\delta} \neq 0$, $w_{\infty,\delta} \to 0$ as $\delta \to 0$.

Arguing as above, we have $V \subset \mathrm{Aut}(\mathrm{Y}_2)$.

2.4. **Case 1.2, $\pi(U.\Lambda)$ is never compact.** Assume for every $\Lambda \in \mathrm{Y}_1$, $\pi(U.\Lambda)$ is not compact.

Then there is some compact set such that every $U.\Lambda$ intersects non-trivially for every $\Lambda \in \mathrm{Y}_1$. Therefore there is a nonempty U-minimal set in Y_1 and the rest of the proof is not so different from Sec.2.3.

2.5. **Case 2, exists a closed U-orbit.** Assume Y_1 contains a closed U-orbit $U.\Lambda_2$.

2.6. **Case 2.1, recurrence in non-centralizer direction.** Assume there exists a sequence $(y_n) \subset \mathrm{Y}_0$ converging to Λ_2 such that

- $y_n = \exp(w_n)\Lambda_2$ with $w_n \in \mathfrak{h}^{\perp}$, $w_n \notin \mathrm{Lie}(V)$.

[Recall from Lec.4, $\mathfrak{h}^{\perp}$ is a complement of $\mathfrak{h} = \mathfrak{so}_{Q_0}(\mathbb{R})$ in $\mathfrak{sl}_N(\mathbb{R})$ that is stable under the adjoint action of $\mathrm{SO}_{Q_0}(\mathbb{R})$]

By assumption for every $\delta > 0$ and for n large enough we can find $t_{n,\delta}$ such that

$$\left\|\mathrm{Ad}(\boldsymbol{u}_{t_{n,\delta}}).w_n\right\| = \delta.$$

[The trick of doing perturbation as done in Lec.4 does not quite help here. We really want an equality here.]

By passing to a subsequence, for every $\delta > 0$, we get some $w_{\infty,\delta} \in \mathfrak{h}^{\perp}$ with norm exactly δ and $y_{\infty,\delta} = \exp(w_{\infty,\delta}).\Lambda_2 \in \mathrm{Y}_0$.

2.6.1. *Lucky case.* If there is some $\delta_0 > 0$ such that for all $\delta < \delta_0$, $w_{\infty,\delta} \in \mathrm{Lie}(V)$, then we get that

$$\boldsymbol{v}_{[0,\delta_0]}.\Lambda_2 \subset \mathrm{Y}_0, \text{ or } \boldsymbol{v}_{[-\delta_0,0]}.\Lambda_2 \subset \mathrm{Y}_0.$$

W.L.O.G, assume $\boldsymbol{v}_{[0,\delta_0]}.\Lambda_2 \subset \mathrm{Y}_0$. Hence for every t, s,

$$\boldsymbol{v}_{[0,e^{2t}\delta_0]}\boldsymbol{a}_t\boldsymbol{u}_s.\Lambda_2 = \boldsymbol{a}_t\boldsymbol{u}_s\boldsymbol{v}_{[0,\delta_0]}.\Lambda_2 \subset \boldsymbol{a}_t\boldsymbol{u}_s\mathrm{Y}_0 = \mathrm{Y}_0. \tag{1}$$

By Thm.1.4, there exists a compact set such that for every $t > 0$, there exists $s_t > 0$ such that $\boldsymbol{a}_t\boldsymbol{u}_{s_t}.\Lambda_2$ lives in this compact set. In particular we may select $t_n \to +\infty$ and $s_n \in \mathbb{R}$ such that $\lim a_{t_n}s_n.\Lambda_2$ exists and call it Λ_∞. Then by Equa.1 and a continuity argument, we have

$$\boldsymbol{v}_{[0,+\infty)}.\Lambda_\infty \subset \mathrm{Y}_0.$$

So we are done.

2.6.2. *Unlucky, try again!* If the assumption in Sec.2.6.1 does not holds. Then we can repeat what is done above Sec.2.6.1. So we get some $y^{(2)}_{\infty} = \exp(w^{(2)}_{\infty,\delta}).\Lambda_2$ with $\left\| w^{(2)}_{\infty,\delta} \right\| = \delta$. If lucky, then we go back to Sec.2.6.1. If not, then we can repeat this process again to get $w^{(3)}_{\infty,\delta}$. It suffices to note that this process should stop.

Indeed recall the computation we made in [Lec4, Sec 3.1]:

$\mathrm{Ad}(\boldsymbol{u}_t)w =$

$$\begin{bmatrix} \frac{t^2}{2}w_{31} + tw_{21} + w_{11} & \frac{t^3}{3!}w_{31} + \frac{t^2}{2}w_{21} + tw_{11} + \frac{-w_{12}}{3} & \frac{t^4}{4!}w_{31} + \frac{t^3}{3!}w_{21} + \frac{t^2}{2}w_{11} + t\frac{-w_{12}}{3} + \frac{w_{13}}{6} \\ tw_{31} + w_{21} & * & * \\ w_{31} & * & * \end{bmatrix}. \tag{2}$$

From this computation we sees right away that (if this process continues)

$$\begin{aligned}
(w_{\infty,\delta})_{3,1} =& 0;\\
(w^{(2)}_{\infty,\delta})_{3,1} =& (w^{(2)}_{\infty,\delta})_{2,1} = 0;\\
(w^{(3)}_{\infty,\delta})_{3,1} =& w^{(3)}_{\infty,\delta})_{2,1} = w^{(3)}_{\infty,\delta})_{1,1} = 0;\\
(w^{(4)}_{\infty,\delta})_{3,1} =& w^{(4)}_{\infty,\delta})_{2,1} = w^{(4)}_{\infty,\delta})_{1,1} = (w^{(4)}_{\infty,\delta})_{1,2} = 0 \implies w^{(4)}_{\infty,\delta} \in \mathrm{Lie}(V).
\end{aligned}$$

Thus we are always lucky at some point.

2.7. **Case 2.2, recurrence only in centralizer direction.** Assume the assumption made in Sec.2.6 is wrong. This can be rephrased as saying there exists some $\delta_0 > 0$ (assumed to be much smaller than $\mathrm{InjRad}(\Lambda_2)$) such that

$$\mathrm{Obt}^{-1}\left(\mathrm{Map}(\Lambda_2, \mathrm{Y}_0) \cap \mathcal{N}_{\mathrm{id}}(\delta_0)\right) \subset \mathfrak{h} \oplus \mathrm{Lie}(V)$$

where $\mathrm{Obt}: \mathfrak{h} \oplus \mathfrak{h}^{\perp} \to \mathrm{X}_3$ is a local diffeomorphism (about $(0,0)$) defined by

$$\mathrm{Obt}(h, w) := \exp(h)\exp(w).\Lambda_2.$$

This is the last and the most annoying case. And only in this case we use Y_1, defined as a B-minimal set. We are going to derive a contradiction and show that this case is not allowed. The argument below is more-or-less a reproduction of [Bekka-Mayer, Page 182].

2.7.1. *Step 1.* Y_1 is not a closed set.

Indeed, otherwise, one sees that Y_1 is even compact by Thm.1.4. But this is impossible by considering $a_t.\Lambda_2$ as $t \to -\infty$.

2.7.2. *Step 2.* Step 1 together with minimality implies that there exists $b_n = a_n u_n \in B$ with $a_n \to \infty$ such that $b_n.\Lambda_2 \to \Lambda_2$. Note that if $a_n = \boldsymbol{a}_{t_n}$ then $t_n \to +\infty$.

2.7.3. *Step 3.* Since $\mathrm{Y}_1 \subset \mathrm{Y}_0 \setminus \mathcal{O}$ and by our assumption made in this subsection, we find $(v_n) \subset \mathrm{Lie}(V)$ such that $v_n \neq 0$, $v_n \to 0$ and $\exp(v_n).\Lambda_2 \in \mathrm{Y}_0$ for all i.

2.7.4. *Step 4.* This is the key step.

Since $b_n.\Lambda_2 \to \Lambda_2$, we can find for every large n, a unique λ_n close to id such that $b_n.\Lambda_2 = \lambda_n.\Lambda_2$. By assumption one can write $\lambda_n = h_n \exp(v(\lambda_n))$ for some $h_n \in H$ and $v(\lambda_n) \in \mathrm{Lie}(V)$. We want to argue that $h_n \in \pm B$, the normalizer of U in H (since h_n is close to identity, we actually have $h_n \in B$).

Now fix some large n and take l large enough depending on n. We have

$$b_n.(\exp(v_l).\Lambda_2) = \exp(v'_l).b_n.\Lambda_2 = \exp(v'_l).\lambda_n.\Lambda_2$$

where $v'_l = \mathrm{Ad}(b_n).v_l$. When l is large, v'_l is small.

By assumption for n large and l larger,

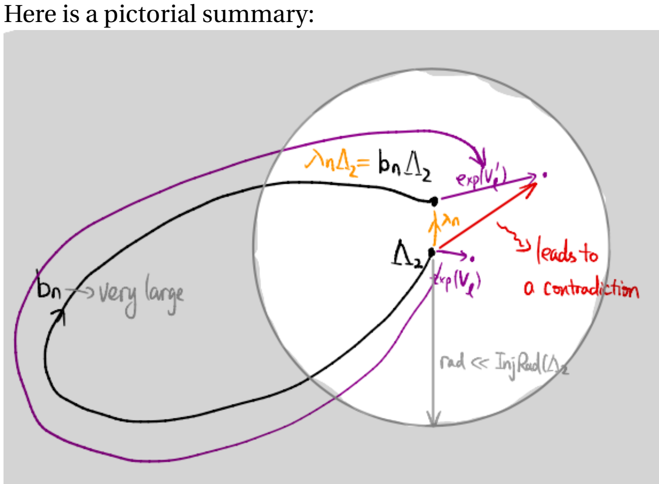
$$\exp(v_l') \cdot \lambda_n = \exp(v_l') h_n \exp(v(\lambda_n)) \in \mathcal{N}_{\mathrm{id}}(\delta_0) \cap H \cdot V.$$

Although the computation of $\log(\exp(X)\exp(Y))$ is usually hard, we still have (again, for l large)

$$\exp(v_l') h_n \exp(v(\lambda_n)) \in \mathcal{N}_{\mathrm{id}}(\delta_0) \cap H \cdot V \implies \exp(\mathrm{Ad}(h_n).v_l') \in \mathcal{N}_{\mathrm{id}}(\delta_0) \cap H \cdot V$$
$$\implies \exp(\mathrm{Ad}(h_n).v_l') \in \mathcal{N}_{\mathrm{id}}(\delta_0) \cap V \implies h_n \in B.$$

The last step follows from an explicit calculation similar to Equa.2 replacing $\boldsymbol{u}_t$ by the one opposite to it.

Here is a pictorial summary:



2.7.5. *Step 5.* Step 4 says that

$$b_n.\Lambda_2 = h_n \exp(v(\lambda_n)).\Lambda_2$$

for some $h_n \in B$ close to identity. This is impossible! Why? note that Λ_2 is a periodic U-orbit and everything here normalizes U. Hence both sides are U-periodic. However, the centralizer of U would preserve the period but $\boldsymbol{a}_{t_n}$ (recall $b_n = \boldsymbol{a}_{t_n} u_n$ with $t_n \to +\infty$) will make the period much larger. This is a contradiction.

3. Proof of Theorem 1.4

From now on we discuss how Thm.1.4 is proved. From the start assume item 2 and 3 do not happen and want to prove item 1 holds.

A direct computation shows that a vector $v \in \mathbb{R}^3$ is fixed by U iff $v \in \mathbb{R}e_1 \oplus \mathbb{R}e_1 \wedge e_2$. For a primitive subgroup Δ of $\Lambda_0 \in \mathrm{X}_3$, we still denote by Δ the vector (well-defined up to ± 1) representing Δ. For instance if $\Delta = \mathbb{Z}v \oplus \mathbb{Z}w$, then Δ is also viewed as a vector $\pm v \wedge w \in \wedge^2\mathbb{R}^3$. Now assume $U.\Lambda_0$ does not degenerate, then every nonzero subgroup Δ is not fixed by U and by the feature of polynomials,

$$\lim_{t\to-\infty} \|\boldsymbol{u}_t.\Delta\| = \lim_{t\to+\infty} \|\boldsymbol{u}_t.\Delta\| = +\infty.$$

We can ensure at least the trajectory under U of each subgroup can not be small for a long time:

Lemma 3.1. *There exist $C_2 > 0$ and $\alpha_2 > 0$ such that for every interval $[a,b]$ in $\mathbb{R}$, every $\boldsymbol{x} \in \mathbb{R}^3 \oplus \wedge^2\mathbb{R}^3$ and every $\rho \in (0,1)$, if $M_0 := \sup_{s\in[a,b]} \|\boldsymbol{u}_s\boldsymbol{x}\|$, then*

$$\frac{1}{b-a}\operatorname{Leb}\left\{s \in [a,b] \,\middle|\, \|\boldsymbol{u}_s\boldsymbol{x}\| < \rho M_0\right\} \le C_2\rho^{\alpha_2};$$

The proof is left as exercise.

The key observation we made last time does not hold anymore. The following notion is aimed to save the situation, providing a sufficient condition for being contained in a compact set.

Definition 3.2. *For $\delta,\rho \in (0,1)$, $\Lambda \in \mathrm{X}_3$ is said to be (δ,ρ)-protected (by the flag $\{\{0\} \ne \mathbb{Z}\boldsymbol{v} \subset \Delta \subset \Lambda\}$) iff there exists $0 \ne \mathbb{Z}\boldsymbol{v} \subset \Delta \subset \Lambda$ where $\mathbb{Z}\boldsymbol{v}$ and Δ are primitive subgroups of rank 1 and 2 such that*

$$\|\boldsymbol{v}\|, \|\Delta\| \in (\rho\delta, \delta).$$

Lemma 3.3. *Assume $\rho,\delta \in (0,1)$. If $\Lambda \in \mathrm{X}_3$ is (δ,ρ)-protected then $\Lambda \in \mathscr{C}_\rho$.*

Proof. It suffices to prove that every non-zero vector w in Λ has norm at least ρ. So we may assume that $\|w\| < 1$.

Pick $\boldsymbol{v}$ and Δ as in the definition. Because Λ is of covolume one, w has to be contained in Δ since $\|w\| < 1$. Moreover

$$\rho\delta \le \|\Delta\| \le \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\| \le \delta\,\|\boldsymbol{w}\| \implies \|\boldsymbol{w}\| \ge \rho.$$

□

Key observation. Here we have already employed the special feature of X_3 (not valid for $\mathrm{X}_{\ge 4}$): once we find $\mathbb{Z}.v$ and Δ two primitive subgroups such that $\|\mathbb{Z}.v\|, \|\Delta\| < 1$, then it is automatic that $\mathbb{Z}.v$ is contained in Δ. Therefore, in searching for a flag that (δ,ρ)-protects Λ we may look for $\mathbb{Z}.v$ and Δ in an independent way (the condition of being a flag automatically holds).

Thus Thm.1.4 follows from Lem.3.3, the observation and the following:

Lemma 3.4. *For every $\varepsilon > 0$, there exist $\varepsilon',\rho,\delta \in (0,1)$ such that for every Λ nondegenerate, there exists T_0 such that for all $T \ge T_0$,*

$$\frac{1}{T}\operatorname{Leb}\left\{t \in [0,T] \,\middle|\, \nexists \boldsymbol{x} \in \operatorname{Prim}^1(\boldsymbol{u}_t\Lambda),\ \|\boldsymbol{x}\| \in (\rho\delta,\delta),\ \boldsymbol{u}_t.\Lambda \notin \mathscr{C}_{\varepsilon'}\right\} \le \varepsilon,$$

and

$$\frac{1}{T}\operatorname{Leb}\left\{t \in [0,T] \,\middle|\, \nexists \Delta \in \operatorname{Prim}^2(\boldsymbol{u}_t\Lambda),\ \|\Delta\| \in (\rho\delta,\delta),\ \boldsymbol{u}_t.\Lambda \notin \mathscr{C}_{\varepsilon'}\right\} \le \varepsilon.$$

If we fix a compact set in X_3 from the beginning and allow ε',ρ,δ to depend on this compact set, then conclusion holds for $T_0 = 0$ and all Λ contained in this compact set.

Proof of Lemma 3.4. Fix some $\delta \in (0,1)$. Take $\varepsilon' := \delta/2$. Choose $\rho \in (0,1)$ small enough such that $C_2(2\rho)^{\alpha_2} < 0.5\varepsilon$. Assume that Λ contains no degenerate vectors. We are going to prove the first inequality and leave the second one as exercise.

By taking T_0 large enough, we assume that for every $\mathbb{Z}.v \in \operatorname{Prim}^1(\Lambda)$, for some $t \in (0,T)$, $\|\boldsymbol{u}_t.v\| \ge \delta$ (and we can forget about the non-degeneracy condition from now on).

Indeed, take $t = 1$, there are only finitely many $\mathbb{Z}.v \in \operatorname{Prim}^1(\Lambda)$ such that $\|\boldsymbol{u}_t.v\| < \delta$. List them as $\{\mathbb{Z}.v_1,\dots,\mathbb{Z}.v_l\}$. Since $\boldsymbol{u}_t$ does not fix v_i for every i by non-degeneracy condition, we have that $\|\boldsymbol{u}_t.v_i\| \to +\infty$ as $t \to +\infty$. So we can pick T_0 such that $\left\|\boldsymbol{u}_{T_0}.v_i\right\| > \delta$ for every i and this would do the job.

Consider the set $\{t \in (0,T), \boldsymbol{u}_t.\Lambda \notin \mathscr{C}_{\delta/2}\}$, which is open and hence can be written as a disjoint union of open intervals. Take one of them, say (a,b). Note that we have not excluded the possibility of $(a,b) = (0,T)$ yet.

For every $t \in (a,b)$, by definition, there is some $\mathbb{Z}.v \in \mathrm{Prim}^1(\Lambda)$ such that $\|\boldsymbol{u}_t.v\| < \delta/2$. For every such $\mathbb{Z}.v$ and t, define $\mathscr{O}(\mathbb{Z}.v,t)$ to be the maximal open interval in $\mathbb{R}$ containing t such that

$$s \in \mathscr{O}(\mathbb{Z}.v,t) \implies \|\boldsymbol{u}_t.v\| < \delta.$$

From the definition, it is possible that $\mathscr{O}(\mathbb{Z}.v,t)$ is not contained in (a,b), or even $(0,T)$. However, it is also impossible for $(0,T)$ to be contained in $\mathscr{O}(\mathbb{Z}.v,t)$ by the choice of T_0. Thus,

$$\sup_{(0,T)\cap\mathscr{O}(\mathbb{Z}.v,t)} \|\boldsymbol{u}_t.v\| \geq \delta.$$

If (a,b) contains some end point of $\mathscr{O}(\mathbb{Z}.v,t)$ then this also holds replacing $(0,T)$ by (a,b). Otherwise, we must have for $t=a$ or $t=b$, $\mathrm{sys}(\boldsymbol{u}_t.\Lambda) = \delta/2$. In either case, the following is always true

$$\sup_{(a,b)\cap\mathscr{O}(\mathbb{Z}.v,t)} \|\boldsymbol{u}_t.v\| \geq \frac{\delta}{2}. \tag{3}$$

As $\mathbb{Z}.v, t$ varies, $\{\mathscr{O}(\mathbb{Z}.v,t)\cap(a,b)\}$ covers (a,b). Now we claim that it is possible to select a subcovering with multiplicity at most 2 (the number 2 is not important, but it should be an absolute constant). The *multiplicity* of a covering refers to the maximal number of possible overlaps (not sure if this is standard terminology).

Here is one possible way of proving the claim, you may wish to find your own.

Since each of a,b belongs to some $\mathscr{O}(v,t)$, we can find a finite collection of $\{\mathscr{O}(\mathbb{Z}.v,t)\}$ that covers (a,b). By passing to a further sub-covering if necessary, we assume it is minimal and is given by $\{\mathscr{O}(\mathbb{Z}.v_i,t_i) = (a_i,b_i)\}$ with $a_i < a_{i+1}$. Then we must have

$$a_1 < a < a_2 < b_1 < a_3 < b_2 < a_4 < .... < a_l < b_{l-1} < b < b_l,$$

and the claim holds.

Let $I_i := (a_i,b_i)\cap(a,b)$. By Equa.3, $\sup_{s\in I_i}\|\boldsymbol{u}_s.v_i\| \geq \frac{\delta}{2}$. Then by Lem.3.1,

$$\frac{1}{|I_i|}\left|\left\{s\in I_i \,\middle|\, \|\boldsymbol{u}_s.v_i\| \leq (2\rho)\cdot\frac{\delta}{2}\right\}\right| \leq C_2(2\rho)^{\alpha_2} \leq 0.5\varepsilon.$$

Adding them together completes the proof. $\square$