习题三

June 7, 2024

选取五道题解答。截止日期：6 月 21 日课前。中英文皆可。你们可以互相讨论（当然，我希望你们互相讨论！），或者查阅资料。但是写在纸上/latex 这一过程请务必独立完成。

For $\lambda \in [0,1]$, define

$$\mathbf{BAD}(\lambda, 1-\lambda) := \left\{(\alpha,\beta) \in \mathbb{R}^2 \,\middle|\, \exists c>0,\, \forall q \in \mathbb{Z}^+,\, \max\left\{q\langle q\alpha\rangle^{\frac{1}{\lambda}}, q\langle q\beta\rangle^{\frac{1}{1-\lambda}}\right\} > c\right\}$$

where by convention we set $\langle\cdot\rangle^{\frac{1}{0}} := 0$. Thus $\mathbf{BAD}(1,0)$ is just $\mathbf{BAD} \times [0,1)$.

Exercise A. *Show that if (α,β) is a counter-example to Littlewood conjecture, then there exists $\lambda \in [0,1]$ such that $(\alpha,\beta) \in \mathbf{BAD}(\lambda, 1-\lambda)$.*

Remark 0.1. *It seems unknown whether $\bigcap_{\lambda\in[0,1]} \mathbf{BAD}(\lambda, 1-\lambda)$ is empty or not. It would have full dimension if the intersection were taken over a countable subset.*

Recall:

$$\Lambda_{\alpha,\beta} := \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} .\mathbb{Z}^3 \in \mathrm{X}_3, \quad \mathbf{u}^+_{\alpha,\beta} := \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise B. *Given $\lambda \in [0,1]$ and $(\alpha,\beta) \in \mathbb{R}^2$, show that the following two are equivalent:*

1. *$(\alpha,\beta) \in \mathbf{BAD}(\lambda, 1-\lambda)$;*
2. $$\left\{\begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{(1-\lambda)t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} .\Lambda_{\alpha,\beta} \,\middle|\, t \geq 0\right\} \text{ is bounded in } \mathrm{X}_3.$$

Given $\lambda \in [0,1]$, we say that a linear functional $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is λ-**bad** iff

$$\exists\, c>0,\ \forall\, (l,m) \in \mathbb{Z}^2 \setminus \{(0,0)\},\ \langle\varphi(l,m)\rangle \max\{l^{\frac{1}{\lambda}}, m^{\frac{1}{1-\lambda}}\} > c.$$

By convention : $l^{\frac{1}{\lambda}} := +\infty$ if $l \neq 0, \lambda = 0$ and $l^{\frac{1}{\lambda}} := 0$ if $l = 0, \lambda = 0$. The convention for $m^{\frac{1}{1-\lambda}}$ is similar. Define

$$\Lambda'_{\alpha,\beta} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 1 \end{bmatrix} .\mathbb{Z}^3 \in \mathrm{X}_3$$

Exercise C. *Given $\lambda \in [0,1]$ and $(\alpha,\beta) \in \mathbb{R}^2$, show that the following two are equivalent:*

1. *The linear functional $\varphi_{\alpha,\beta}(x,y) := \alpha x + \beta y$ is λ-bad;*
2. $$\left\{\begin{bmatrix} e^{-\lambda t} & 0 & 0 \\ 0 & e^{-(1-\lambda)t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} .\Lambda'_{\alpha,\beta} \,\middle|\, t \geq 0\right\} \text{ is bounded in } \mathrm{X}_3.$$

Exercise D. *Given $\lambda \in [0,1]$ and $(\alpha,\beta) \in \mathbb{R}^2$, show that the following two are equivalent:*

1. *$(\alpha,\beta) \in \mathbf{BAD}(\lambda, 1-\lambda)$;*

2. *The linear functional $\varphi_{\alpha,\beta}(x,y) := \alpha x + \beta y$ is λ-bad.*

Hint: show that the map $g\mathbb{Z}^3 \mapsto g^{-\operatorname{tr}}\mathbb{Z}^3$ on X_3 is well-defined and continuous.

Exercise E. *Show that if $(\alpha,\beta) \in \mathbb{R}^2$ are linearly dependent over $\mathbb{Q}$, then $(\alpha,\beta) \notin$ $\mathbf{BAD}(\lambda, 1-\lambda)$ for every $\lambda \in (0,1)$.*

Exercise F. *Assume that α,β are two numbers contained in the same totally real number field. Moreover assume α,β are linearly independent over $\mathbb{Q}$. Then (α,β) is contained in* $\mathbf{BAD}(1/2,1/2)$.

Hint: In the proof, you may assume the following facts, which can be proved quickly if you know a little Galois theory. There exist real numbers t_i's, u_{ij}'s and $M_0 \in \mathbf{SL}_3(\mathbb{R})$ such that $\prod_{i=1}^3 t_i = 1$, the A-orbit $\mathrm{A}.(M_0.\mathbb{Z}^3)$ is compact and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha & -\beta & 1 \end{bmatrix} = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \cdot M_0$$

Indeed, if $\{\sigma_1 = \mathrm{id}, \sigma_2, \sigma_3\}$ denote the three distinct embeddings of the totally real field into $\mathbb{R}$, then M_0 can be chosen to be a scalar multiple of

$$\begin{bmatrix} -\sigma_3(\alpha) & -\sigma_3(\beta) & 1 \\ -\sigma_2(\alpha) & -\sigma_2(\beta) & 1 \\ -\alpha & -\beta & 1 \end{bmatrix}.$$

Recall: $d^{\mathbf{SL}_2(\mathbb{R})}(\cdot,\cdot)$ (or just write d) denotes a right $\mathbf{SL}_2(\mathbb{R})$-invariant metric on $\mathbf{SL}_2(\mathbb{R})$ that is compatible with the topology on $\mathbf{SL}_2(\mathbb{R})$ and d^{X_2} denotes the quotient metric on $\mathrm{X}_2 \cong \mathbf{SL}_2(\mathbb{R})/\mathbf{SL}_2(\mathbb{Z})$. Using this we defined

$$\operatorname{InjRad}(x) := \frac{1}{10}\inf\left\{d^{\mathrm{X}_2}(g,h) \;\middle|\; g \neq h \in \mathbf{SL}_2(\mathbb{R}),\ g.\mathbb{Z}^2 = h.\mathbb{Z}^2 = x\right\}.$$

Exercise G. *Show that a sequence $(x_n) \subset \mathrm{X}_2$ diverges iff* $\operatorname{InjRad}(x_n) \to 0$.

Hint: Find nilpotent integer matrices (N_n) such that $(g_n N_n g_n^{-1})$ converges to the zero matrix.

Exercise H. *Show that for $x \in \mathrm{X}_2$, if $\left\{\begin{bmatrix} 1 & r \\ 0 & 1\end{bmatrix}.x \;\middle|\; r \in \mathbb{R}\right\}$ is compact, then $\left(\begin{bmatrix} e^{t_n} & 0 \\ 0 & e^{-t_n}\end{bmatrix}.x\right)$ diverges in X_2 whenever $t_n \to -\infty$.*

Hint: use the exercise above.

From Exercise I to P, let $X := \{0,1\}^{\mathbb{Z}_{\geq 0}}$. Equipped with the product topology, X is compact. In concrete terms, a sequence $(x^k_\star)$ converges to $x_\star$ iff for every i, (x^k_i) converges to x_i as k tends to infinity. We usually denote an element of X by $x_\star$ or $(x_0x_1x_2....)$. Define a metric on X by

$$d(x_\star, y_\star) := 2^{-\inf\left\{n \in \mathbb{Z}_{\geq 0} \;|\; x_n \neq y_n\right\}}.$$

For instance $d(0101..., 1000....) = 2^{-0} = 1$. This metric is compatible with the product topology. Given $\mathbf{a} := (a_0, a_1, ..., a_{l-1}) \in \{0,1\}^l$ (we will refer to such things as **words of length** l), let

$$C_{\mathbf{a}} := \{x_\star \in X \mid x_i = a_i,\ \text{for } i = 0, ..., l-1.\}$$

For instance, $C_{(0,1)} = \{x_\star \in X \mid x_0 = 0, x_1 = 1\}$.

Finally, define a continuous map $\sigma : X \to X$ by $\sigma(x)_i := x_{i+1}$.

Exercise I. *Let X be equipped with the metric defined above,* $\dim_H(X) = 1$.

Exercise J. *Let X be equipped with the metric defined above,* $\dim_{\square}(X) = 1$.

For $\lambda \in [0,1]$, there is a Borel probability measure m_λ on X such that

$$\mathrm{m}_\lambda(C_{\mathbf{a}}) := \lambda^{l_0}(1-\lambda)^{l-l_0}.$$

for every word $\mathbf{a}$ of length l. Here $l_0 := \#\{i \in \{0,1,...,l-1\} \mid \mathbf{a}_i = 0\}$. Accept the fact that m_λ is σ-invariant for every $\lambda \in [0,1]$. Let $\mathcal{P}_0$ be the partition $\{C_{(0)}, C_{(1)}\}$.

Exercise K. *For every σ-invariant Borel probability measure ν on X, show $h_\nu(\sigma) = h_\nu(\sigma, \mathcal{P}_0)$.*

Exercise L. *Show that $h_{\mathrm{m}_\lambda}(\sigma) \leq \log 2$. And equality holds iff $\lambda = 1/2$.*

For $l \geq 0$, let $\mathcal{P}_0^{l-1} := \mathcal{P}_0 \vee \sigma^{-1}\mathcal{P}_0 \vee ... \vee \sigma^{-(l-1)}\mathcal{P}_0$. For $x_\star \in X$, let $\mathcal{P}_0^{l-1}(x_\star)$ be the unique element of $\mathcal{P}_0^{l-1}$ containing $x_\star$.

Exercise M. *Fix $\lambda \in [0,1]$. Show that for m_λ-almost every $x_\star \in X$, we have*

$$\lim_{l\to\infty} \frac{-\log \mu(\mathcal{P}_0^{l-1}(x_\star))}{n} = h_{\mathrm{m}_\lambda}(\sigma).$$

Exercise N. *Take $E \subset X$ such that $\dim_\square(E) > 0$. Let E' be the closure of the union of $\sigma^n(E)$ as n varies in $\mathbb{Z}_{\geq 0}$. Show that there exists a σ-invariant Borel probability measure ν supported on E' such that $h_\nu(\sigma) > 0$.*

In the reverse direction, we have,

Exercise O. *Let ν be a σ-invariant probability measure on X with $h_\nu(\sigma) > 0$, show that $\dim_{\mathrm{H}}(\mathrm{supp}(\nu)) > 0$.*

Recall that $x \in \mathrm{supp}(\nu)$ iff every open neighborhood of x has positive ν-measure.

For $x_\star \in X$ and $n \in \mathbb{Z}^+$, define

$$\mathbf{w}_n(x_\star) := \#\{f : \{0,1,...,n-1\} \to \{0,1\} \mid f(i) = x_{n+i},\ \exists\, n \in \mathbb{Z}_{\geq 0}\}$$

and its "word entropy" as

$$\mathrm{WH}(x_\star) := \limsup_{n\to\infty} \frac{\log \mathbf{w}_n(x_\star)}{n}.$$

Exercise P. *Let $x_\star \in X$ and E' be the closure of $\{\sigma^n(x_\star),\ n \in \mathbb{Z}_{\geq 0}\}$. Prove that if $\mathrm{WH}(x_\star) > 0$, then there exists a σ-invariant Borel probability measure ν supported on E' such that $h_\nu(\sigma) > 0$.*