EXERCISE SHEET 2

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截止日期：最迟在 4.19 提交作业。

评分标准：取 sup-norm ——只要做对一小道题，就能得到满分。当然，你也可以尝试说明题目出错了。

提示：你可以自由使用序号靠前习题的结果来解答序号靠后的习题。

如对习题（陈述，定义等）有任何的疑问，请联系我。

CONTENTS

1. (C,α)-GOOD FUNCTIONS

Let $C,\alpha>0$ and J be an interval in $\mathbb{R}$, recall a function $f: J\to\mathbb{R}$ is said to be (C,α)-good on J iff for every interval $I\subset J$ of finite length and every $\rho\in(0,1)$,

$$\frac{1}{|I|}\mathrm{Leb}\left\{t\in I \,\middle|\, \left|f(t)\right|\leq \rho M_I\right\}\leq C\rho^{\alpha}. \tag{1}$$

where $M_I := \sup_{t\in I}\left|f(t)\right|$.

In this set of exercises we show that there are constants (C,α) such that every polynomial of degree at most three is (C,α)-good on $\mathbb{R}$. The general case would follow from the same proof with some constant depending only on the degree.

Given four distinct points $\boldsymbol{v}=(v_0,v_1,v_2,v_3)$ in $\mathbb{R}$, for $k=0,1,2,3$, define

$$L^k_{\boldsymbol{v}}(x) := \prod_{i\neq k}\frac{x-v_i}{v_k-v_i}.$$

Exercise 1.1. *Fix such a $\boldsymbol{v}$ as above. Prove that for any choice of four real numbers (w_0,w_1,w_2,w_3), there exists at most one polynomial p of degree at most 3 such that $p(v_i)=w_i$.*

Exercise 1.2. *Same assumption as in last exercise. Show that $p(x) := \sum_{k=0}^{3} w_k\cdot L^k_{\boldsymbol{v}}(x)$ satisfies $p(v_i)=w_i$ for every $i=0,1,2,3$.*

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Exercise 1.3. *Same assumption as in last exercise. Let $\varepsilon, \delta > 0$ be two positive real numbers. Assume further that $|v_i - v_j| \geq \delta$ for every pair (i, j) with $i \neq j$. Also assume $|w_i| \leq \varepsilon$ for all i. Show that for every $x \in [0,1]$, $\left|p(x)\right| \leq 4\varepsilon\delta^{-3}$ where p is as in the last exercise.*

Exercise 1.4. *Let $I \subset [0,1]$ be a measurable subset with $\mathrm{Leb}(I) = 9\delta > 0$. Show that there exists four points (v_0, v_1, v_2, v_3) in I such that $|v_i - v_j| \geq \delta$ for every pair (i, j) with $i \neq j$.*

Exercise 1.5. *Find $C, \alpha > 0$ such that for every polynomial of degree at most three and $\rho \in (0,1)$, Equa.1 holds when $I = [0,1]$.*

Exercise 1.6. *Show that every polynomial of degree at most three is (C, α)-good on $\mathbb{R}$ with C, α same as in the last exercise.*

Let J be an interval of finite length. Let

$$\mathscr{A} := \{f = ae^x + be^{-x},\ a, b \in \mathbb{R}\}.$$

Exercise 1.7. *Show that there exist $C, \alpha > 0$ (depending on J and $\mathscr{A}$) such that for every function $f \in \mathscr{A}$ is (C, α)-good on J.*

Exercise 1.8. *If f_1, f_2 are (C, α)-good on J, then $x \mapsto \max\left\{\left|f_1(x)\right|, \left|f_2(x)\right|\right\}$ is also (C, α)-good on J.*

2. Non-commensurable lattices in $\mathrm{SL}_2(\mathbb{R})$, I

We apply ideas in Lec.4 to a different example. Our ultimate goal is to show that two cocompact lattices in $\mathrm{SL}_2(\mathbb{R})$ is either commensurable or their product is dense in $\mathrm{SL}_2(\mathbb{R})$, which will (hopefully) be achieved in the next set of exercises.

Notations:

- $\mathrm{G} := \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, $\mathrm{H} := \Delta(\mathrm{SL}_2(\mathbb{R}))$ and Γ is a cocompact lattice in G;
- $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ and $\mathfrak{h} := \mathrm{Lie}(\mathrm{H})$;
- $A := \left\{\left(\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\right),\ t \in \mathbb{R}\right\} = \{\Delta \boldsymbol{a}_t,\ t \in \mathbb{R}\}$;
- $U := \left\{\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right),\ t \in \mathbb{R}\right\} = \{\Delta \boldsymbol{u}_t,\ t \in \mathbb{R}\}$;
- $V := \left\{\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}\right),\ t \in \mathbb{R}\right\} = \{\boldsymbol{v}_t,\ t \in \mathbb{R}\}$;
- $V^+ := \{\boldsymbol{v}_t,\ t \geq 0\}$, $V^- := \{\boldsymbol{v}_t,\ t \leq 0\}$;
- $W := AUV$, $W^+ := AUV^+$, $W^- := AUV^-$.

Exercise 2.1. *Show that W is a group and W^+, W^- are semigroups.*

Exercise 2.2. *Let*

$$\mathfrak{h}^{\perp} := \{(X, -X) \mid X \in \mathfrak{sl}_2(\mathbb{R})\} \subset \mathfrak{g}.$$

Show that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ and this decomposition is preserved by $\mathrm{Ad}(\mathrm{H})$.

Now take $\Lambda_0 \in G/\Gamma$ such that $\mathrm{H}.\Lambda_0$ is not closed. Define $\mathrm{Y}_0 := \overline{\mathrm{H}.\Lambda_0}$ and

$$\mathscr{O} := \left\{y \in \mathrm{Y}_0 \,\middle|\, \mathrm{H}.y \text{ is open in } \mathrm{Y}_0\right\}.$$

Exercise 2.3. *Show that $\mathscr{O} \neq \mathrm{Y}_0$.*

Let Y_1 be a nonempty U-minimal set in $\mathrm{Y}_0 \setminus \mathscr{O}$.

Exercise 2.4. *Show that Y_1 is not a closed U-orbit.*

Exercise 2.5. *Assume Y_1 is not preserved by A. Show that Y_0 contains a W-orbit.*

(Hint: consider $\mathrm{Aut}(\mathrm{Y}_1)$.)

Exercise 2.6. *Assume* Y_1 *is preserved by* A. *Show that* Y_0 *contains a* W^+*-orbit or a* W^-*-orbit.*

(Hint: consider $\mathrm{Map}(\mathrm{Y}_0, \mathrm{Y}_1)$.)

3. TOTALLY GEODESIC HYPERBOLIC PLANES IN $\mathbb{H}^3$, I

We apply ideas in Lec.4 to yet another example. Our ultimate goal (hopefully achieved in the next set of exercises) is to show that the image of a totally geodesic immersion of a hyperbolic plane in a closed hyperbolic three manifold is either closed or dense.

Notations:

- $\mathrm{G} := \mathrm{SL}_2(\mathbb{C})$, $\mathrm{H} := \mathrm{SL}_2(\mathbb{R})$ and Γ is a cocompact lattice in G;
- $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ and $\mathfrak{h} := \mathrm{Lie}(\mathrm{H})$;
- $A := \left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, t \in \mathbb{R} \right\} = \{\boldsymbol{a}_t, t \in \mathbb{R}\}$;
- $U := \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, t \in \mathbb{R} \right\} = \{\boldsymbol{u}_t, t \in \mathbb{R}\}$;
- $V := \left\{ \begin{bmatrix} 1 & it \\ 0 & 1 \end{bmatrix}, t \in \mathbb{R} \right\} = \{\boldsymbol{v}_t, t \in \mathbb{R}\}$;
- $V^+ := \{\boldsymbol{v}_t, t \geq 0\}$, $V^- := \{\boldsymbol{v}_t, t \leq 0\}$;
- $W := AUV$, $W^+ := AUV^+$, $W^- := AUV^-$;

Exercise 3.1. *Let* $\mathfrak{h}^\perp := \{i \cdot X, X \in \mathfrak{sl}_2(\mathbb{R})\}$. *Show that* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. *Moreover, this decomposition is preserved by the* $\mathrm{Ad}(\mathrm{H})$*-action.*

Exercise 3.2. *Let* $\mathrm{H}.\Lambda_0$ *be a non-closed* H*-orbit in* G/Γ. *Show that* $\mathrm{Y}_0 := \overline{\mathrm{H}.\Lambda_0}$ *contains a* W^+ *or a* W^-*-orbit.*

4. NONDIVERGENCE IN RANK 1, A NUMBER FIELD EXAMPLE

In these set of exercises, it is more convenient to write $\mathbb{R}^4$ as $\mathbb{R}^2 \oplus \mathbb{R}^2$.

Exercise 4.1. *Show that* $\mathbb{Z}[\sqrt{2}]$ *is a principal ideal domain.*

Thus every torsion free (finitely generated) $\mathbb{Z}[\sqrt{2}]$-module is free.

Fix an embedding of $\mathbb{Q}(\sqrt{2})$ in $\mathbb{R}$. Let σ be the other embedding of $\mathbb{Q}(\sqrt{2})$ in $\mathbb{R}$. Consider the action of $\mathbb{Q}(\sqrt{2})$ on $\mathbb{R}^2 \oplus \mathbb{R}^2$ given by

$$x.(v, w) := (x.v, \sigma(x).w).$$

Exercise 4.2. *This is a linear action. Write down the matrix representation of this action. Namely, for every* $x = a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$, *write down a* 4*-by-*4 *matrix representing the action of* x *on* $\mathbb{R}^2 \oplus \mathbb{R}^2$ *with respect to the standard basis.*

Let Δ be a rank-1 $\mathbb{Z}[\sqrt{2}]$-submodule in $\mathbb{R}^2 \oplus \mathbb{R}^2$. We may write $\Delta = \mathbb{Z}[\sqrt{2}].(v, w)$. Let $\|\Delta\| := \|v\| \cdot \|w\|$.

Exercise 4.3. *Show that* $\|\Delta\|$ *is independent of the choice of generator for the* $\mathbb{Z}[\sqrt{2}]$*-module* Δ.

Define

$$\mathrm{X}_4'(\mathbb{Z}[\sqrt{2}]) := \left\{ \Lambda \leq \mathbb{R}^2 \oplus \mathbb{R}^2 \text{ lattice }, \Lambda \text{ is preserved by } \mathbb{Z}[\sqrt{2}] \right\}.$$

Exercise 4.4. *Show that such a lattice is a rank-*2 $\mathbb{Z}[\sqrt{2}]$*-module.*

Thus for $\Lambda \in \mathrm{X}_4'(\mathbb{Z}[\sqrt{2}])$, we can find a $\mathbb{Z}[\sqrt{2}]$-basis (v_1, w_1) and (v_2, w_2) in $\mathbb{R}^2 \oplus \mathbb{R}^2$. Define $\|\Lambda\| := \|v_1 \wedge v_2\| \cdot \|w_1 \wedge w_2\|$. Define $\det(\Lambda) := (v_1 \wedge v_2, w_1 \wedge w_2) \in (\mathbb{R} \oplus \mathbb{R})/\mathbb{Z}[\sqrt{2}]^\times$. Here $\mathbb{Z}[\sqrt{2}]^\times$ denotes the invertible elements in this ring $\mathbb{Z}[\sqrt{2}]$.

Exercise 4.5. *Show that indeed, the value of* $\det(\Lambda)$ *in* $(\mathbb{R} \oplus \mathbb{R})/\mathbb{Z}[\sqrt{2}]^\times$ *is independent of the choice of bases. Thus* $\|\Lambda\|$ *is also independent of the choice of bases.*

Exercise 4.6. *Find the relation between this newly defined* $\|\Lambda\|$ *and the old* $\|\Lambda\|_{\mathrm{old}}$ *defined as the volume of* $\mathbb{R}^4/\Lambda$.

Define

$$\mathrm{X}_4(\mathbb{Z}[\sqrt{2}]) := \left\{\Lambda \in \mathrm{X}_4'(\mathbb{Z}[\sqrt{2}]) \,\middle|\, \det \Lambda = 1\right\}.$$

Here "1" is the image of $(1,1)$ in $(\mathbb{R} \oplus \mathbb{R})/\mathbb{Z}[\sqrt{2}]^\times$. Equip $\mathrm{X}_4(\mathbb{Z}[\sqrt{2}])$ with the Chabauty topology, viewing it as a collection of closed subgroups of $\mathbb{R}^2 \oplus \mathbb{R}^2$.

Exercise 4.7. *Show that the free* $\mathbb{Z}[\sqrt{2}]$*-module with basis* $\{(e_1, e_1), (e_2, e_2)\}$ *(denote this module as* Λ_0*) belongs to* $\mathrm{X}_4(\mathbb{Z}[\sqrt{2}])$ *and that* $g \mapsto g.\Lambda_0$ *induces a homeomorphism*

$$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}]) \cong \mathrm{X}_4(\mathbb{Z}[\sqrt{2}]).$$

For $\Lambda \in \mathrm{X}_4(\mathbb{Z}[\sqrt{2}])$, define

$$\mathrm{sys}_{\mathbb{Z}[\sqrt{2}]}(\Lambda) := \inf_{\Delta \leq \Lambda} \|\Delta\|$$

where Δ varies over all rank-1 $\mathbb{Z}[\sqrt{2}]$-submodule of Λ. For every $\varepsilon > 0$, let

$$\mathscr{C}_\varepsilon := \left\{\Lambda \in \mathrm{X}_4(\mathbb{Z}[\sqrt{2}]) \,\middle|\, \mathrm{sys}_{\mathbb{Z}[\sqrt{2}]}(\Lambda) \geq \varepsilon\right\}.$$

Exercise 4.8. *For every* $\varepsilon > 0$, $\mathscr{C}_\varepsilon$ *is a compact subset of* $\mathrm{X}_4(\mathbb{Z}[\sqrt{2}])$.

Exercise 4.9. *Conversely, every compact subset of* $\mathrm{X}_4(\mathbb{Z}[\sqrt{2}])$ *is contained in* $\mathscr{C}_\varepsilon$ *for some* $\varepsilon > 0$.

Exercise 4.10. *For* $\varepsilon > 0$ *small enough, for every* $\Lambda \in \mathrm{X}_4(\mathbb{Z}[\sqrt{2}])$, *the set*

$$\{(v, w) \in \Lambda \mid \|v\| \, \|w\| < \varepsilon\}$$

is either $\{0\}$ *or generates a rank-*1 $\mathbb{Z}[\sqrt{2}]$*-submodule of* Λ.

Let $\boldsymbol{u}_t := \left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right)$ and $U := \{\boldsymbol{u}_t,\ t \in \mathbb{R}\}$.

Exercise 4.11. *Prove the following. For every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for every* $\Lambda \in \mathrm{X}_4(\mathbb{Z}[\sqrt{2}])$,

- *either* Λ *contains a* $\mathbb{Z}[\sqrt{2}]$*-submodule preserved by* U *with norm smaller than* ε,
- *or*

$$\limsup_{T \to +\infty} \frac{1}{T} \mathrm{Leb}\{t \in [0, T] \mid \boldsymbol{u}_t.\Lambda \notin \mathscr{C}_\delta\} \leq \varepsilon.$$

5. A more geometric take on horocycles

I assume you have some familiarity with geometry on the upper half space in this section.

Notations:

- $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2,\ y > 0\}$ equipped with the metric $\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$ and the left action of $\mathrm{SL}_2(\mathbb{R})$ via fractional linear transformations;
- $T^1(\mathbb{H}^2)$ is the unit tangent bundle of $\mathbb{H}^2$;

- $\partial\mathbb{H}^2 := \{(x,0) \in \mathbb{R}^2,\ x \in \mathbb{R}\} \sqcup \{\infty\}$ be the boundary of $\mathbb{H}^2$; The topology on $\{(x,y),\ x \in \mathbb{R},\ y \geq 0\}$ is the natural topology and the topology on $\overline{\mathbb{H}^2} := \mathbb{H}^2 \sqcup \partial\mathbb{H}^2$ is the one-point compactification topology. The action of $\mathrm{SL}_2(\mathbb{R})$ extends continuously to $\overline{\mathbb{H}^2}$;
- Let Γ_0 be a discrete subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma_0\backslash\mathbb{H}^2$ is a closed surface of genus $g \geq 2$;
- Let $\Gamma_0' := [\Gamma_0, \Gamma_0]$, recall that Γ_0' is a normal subgroup of Γ_0 and $\Gamma_0/\Gamma_0' \cong \mathbb{Z}^{2g}$;
- For $x \in \mathbb{H}^2$ and a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, define the limit set $\mathrm{Limit}_x(\Gamma) := \overline{\Gamma.x} \setminus \Gamma.x$ in $\overline{\mathbb{H}^2}$.

Exercise 5.1. $\mathrm{Limit}_x(\Gamma) \subset \partial\mathbb{H}^2$ *for every discrete subgroup* Γ *of* $\mathrm{SL}_2(\mathbb{R})$ *and every* $x \in \mathbb{H}^2$.

Exercise 5.2. *For every* $x, y \in \mathbb{H}^2$ *and discrete subgroup* Γ *of* $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{Limit}_x(\Gamma) = \mathrm{Limit}_y(\Gamma)$.

Thus the limit set is independent of the choice of base point and we henceforth denote it by $\mathrm{Limit}(\Gamma)$.

Exercise 5.3. *Let* Γ *be a discrete subgroup of* $\mathrm{SL}_2(\mathbb{R})$. *Show that* $\mathrm{Limit}(\Gamma)$ *is a* Γ*-minimal set.*

(A Γ-set is said to be Γ-minimal iff either it is empty or for every x in this set, $\Gamma.x$ is dense in this set. Actually $\mathrm{Limit}(\Gamma)$, if infinite, is the unique nonempty Γ-minimal set)

Recall that for every geodesic Y (or closed convex subset) on $\mathbb{H}^2$ and every $x \in \mathbb{H}^2$, there is a unique point, denoted as $\pi_Y(x)$, in Y such that

$$\mathrm{dist}(x, Y) = \mathrm{dist}(x, \pi_Y(x)).$$

For every $x \in T^1(\mathbb{H}^2)$, let $x^+ := \lim_{t\to+\infty} g_t.x$ and $x^- := \lim_{t\to-\infty} g_t.x$ where g_t denotes the geodesic flow. Let $\widehat{x^-x^+}$ be the unique geodesic in $T^1\mathbb{H}^2$ connecting x^- and x^+. By abuse of notation we also let $\widehat{x^-x^+}$ denote its projection to $\mathbb{H}^2$. Fix some point $o \in \mathbb{H}^2$ (say, take $o = (0,1)$), and $x \in T^1\mathbb{H}^2$, let $t = t_o(x)$ be the unique real number such that

$$x = g_t.\pi_{\widehat{x^-x^+}}(o).$$

(a priori, $\pi_{\widehat{x^-x^+}}(o)$ is just an element in $\mathbb{H}^2$ but we identify it with the unique element on $\widehat{x^-x^+} \subset T^1\mathbb{H}^2$ whose projection to $\mathbb{H}^2$ is $\pi_{\widehat{x^-x^+}}(o)$)

Exercise 5.4. *The map* $\Phi_o : T^1\mathbb{H}^2 \to (\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \setminus \Delta\partial\mathbb{H}^2) \times \mathbb{R}$ *defined by*

$$x \mapsto \Phi_o(x) := (x^-, x^+, t_o(x))$$

is a homeomorphism.

This is the so-called Hopf coordinate.

Exercise 5.5. *Check that* $\Phi_o(g_t.x) = (x^-, x^+, t_o(x) + t)$.

Exercise 5.6. *Check that for* $\gamma \in \mathrm{SL}_2(\mathbb{R})$, $\Phi_o(\gamma.x) = (\gamma.x^-, \gamma.x^+, *)$ *for some real number* $*$.

Thus the orbits of Γ on $T^1\mathbb{H}^2/\{g_t\}_{t\in\mathbb{R}}$ corresponds to the orbits of Γ on $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \setminus \Delta\partial\mathbb{H}^2$.

Exercise 5.7. *Let* Γ *be a discrete subgroup of* $\mathrm{SL}_2(\mathbb{R})$. *Using the fact that* g_t*-action on* $\Gamma\backslash T^1\mathbb{H}^2$ *is not minimal, show that the action of* Γ *on* $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \setminus \Delta\partial\mathbb{H}^2$ *is not minimal.*

This action is still quite chaotic, at least when Γ is a lattice, but if we take one step further, it becomes totally discontinuous.

Let $\mathrm{FAT}\Delta$ be the "fat diagonal" in $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \times \partial\mathbb{H}^2$, i.e.

$$\mathrm{FAT}\Delta := \left\{(x_1, x_2, x_3) \in (\partial\mathbb{H}^2)^3,\ x_i = x_j,\ \exists i \neq j\right\}.$$

Exercise 5.8. *Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{R})$. Show that the diagonal Γ-action on $(\partial\mathbb{H}^2)^3\setminus$ FATΔ is conjugate to the Γ-action on $\mathbb{H}^2$.*

Now turn to the special Γ_0, Γ_0' we defined. Recall in Lec 2 we have shown that $\mathrm{Limit}(\Gamma_0)$ is the full $\partial\mathbb{H}^2$. Show that also

Exercise 5.9. $\mathrm{Limit}(\Gamma_0') = \partial\mathbb{H}^2$.

(Hint: use Exer 5.2 and the fact that Γ_0' is a normal subgroup)

Exercise 5.10. *Use this and the "thin" property of hyperbolic space to show that closed geodesics are dense in $\Gamma_0'\backslash T^1\mathbb{H}^2$.*

(In Lec.3 we established denseness of closed geodesics on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ by constructing an explicit one and then considering commensurable lattices)

For a point v on $\partial\mathbb{H}^2$ and $x \in \mathbb{H}^2$, let $\mathscr{H}_v(x)$ be the unique horocycle – the unique Euclidean circle tangent to $\partial\mathbb{H}^2$ at $v(\neq\infty)$ and passing through x (when $v=\infty$, $\mathscr{H}_v(x)$ is a horizontal line passing through x). We shall think of $\mathscr{H}_v(x)$ as a subset of $T^1\mathbb{H}^2$ by equipping every point $\mathscr{H}_v(x)$ with the unique unit tangent vector that is orthogonal to $\mathscr{H}_v(x)$ and pointing towards v.

In Lec.2 we have shown that the projection of every horocycle is dense in $\Gamma_0\backslash T^1\mathbb{H}^2$. Here is a more geometric approach following Hedlund's paper.

Exercise 5.11. *Show that for every nonempty open interval $I\subset\partial\mathbb{H}^2$ and $x\in\mathbb{H}^2$, the set*

$$\bigcup_{v\in I}\mathscr{H}_v(x)$$

is dense in $\Gamma_0'\backslash T^1\mathbb{H}^2$.

(Hint: use Exer.5.10)

Exercise 5.12. *Let $v\in\partial\mathbb{H}^2$, show that if there exists $x\in\mathbb{H}^2$ such that $\mathscr{H}_v(x)$ is dense in $\Gamma_0'\backslash T^1\mathbb{H}^2$, then $\mathscr{H}_v(y)$ is dense in $\Gamma_0'\backslash T^1\mathbb{H}^2$ for every $y\in\mathbb{H}^2$.*

Exercise 5.13. *The set of v such that $\mathscr{H}_v(x)$ is dense in $\Gamma_0'\backslash T^1\mathbb{H}^2$ is dense in $\partial\mathbb{H}^2$.*

Let $\mathscr{D}$ be a Dirichlet fundamental domain for Γ_0'. Accept the fact that if Γ_0' were finitely generated, then $\mathscr{D}$ would have only finitely many sides.

Exercise 5.14. *Show that Γ_0' is not finitely generated.*

Exercise 5.15. *Let $v\in\partial\mathbb{H}^2\cap\overline{\mathscr{D}}$, then $\mathscr{H}_v(x)$ is not dense in $\Gamma_0'\backslash T^1\mathbb{H}^2$.*

(Hint: without loss of generality assume $v=\infty$, argue that, fixing a base point o, there is an upper bound for the y-coordinate of $\gamma.o$ as γ varies in Γ_0'.)

Since $\mathscr{H}_v(x)$ is not compact in $\Gamma_0'\backslash T^1\mathbb{H}^2$, we have demonstrated an orbit of the horocycle flow that is neither dense nor compact.

Exercise 5.16. *Take some $y\in T^1\mathbb{H}^2$ such that $\{g_t.y\}$ is compact in $\Gamma_0\backslash T^1\mathbb{H}^2$. Show that $\mathscr{H}_{y^+}(x)$ is dense in $\Gamma_0\backslash T^1\mathbb{H}^2$.*

(Hint: approximate some dense horocycle in $T^1\mathbb{H}^2$)

Exercise 5.17. *Let $v\in\partial\mathbb{H}^2$ and fix some $x\in\mathbb{H}^2$. Suppose the Euclidean radius of $\gamma.\mathscr{H}_v(x)$ can be arbitrarily large as γ varies in Γ_0. Then $\mathscr{H}_v(x)$ is dense in $\Gamma_0\backslash T^1\mathbb{H}^2$.*

(When the horocycle is based at infinity, by saying the Euclidean radius is large, we mean that the horocycle could be very low) (Hint: show that you can approximate every periodic geodesic)

Exercise 5.18. *Show that indeed, since* $\Gamma_0 \backslash T^1\mathbb{H}^2$ *is compact, that the Euclidean radius of* $\gamma.\mathscr{H}_v(x)$ *can be arbitrarily large as* γ *varies in* Γ_0 *for every pair* $v \in \partial\mathbb{H}^2$ *and* $x \in \mathbb{H}^2$*.*

(Hint: use the fact that the some (well, in the current case, every) geodesic stemming from v is bounded in $\Gamma_0 \backslash T^1\mathbb{H}^2$)