LECTURE 2

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1. MINIMALITY OF HOROCYCLE FLOW ON COMPACT SURFACES OF CONSTANT NEGATIVE CURVATURE

Write
$$u_s := \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}; \quad a_t := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}; \quad X := \mathrm{SL}_2(\mathbb{R})/\Gamma.$$

Let X be equipped with the quotient topology. Sometimes for $g \in \mathrm{SL}_2(\mathbb{R})$, we write $[g]_\Gamma$ for its image in X. The first example of unipotent rigidity is perhaps the following:

Theorem 1.1 ([Hed36])**.** *Let Γ be a discrete and cocompact subgroup of $\mathrm{SL}_2(\mathbb{R})$. Then the action of $U := \{u_s\}_{s\geq 0}$ on $\mathrm{SL}_2(\mathbb{R})/\Gamma$ is* ***minimal****, that is to say, for every $x \in \mathrm{SL}_2(\mathbb{R})/\Gamma$, the set*
$$\left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \cdot x \,\middle|\, s \geq 0 \right\}$$
is dense in $\mathrm{SL}_2(\mathbb{R})/\Gamma$.

In a dual formulation, this says that for every nonzero vector $v \in \mathbb{R}^2$, $\Gamma.v$ is dense in $\mathbb{R}^2$.

Remark 1.2. *The proof below applies equally well to the case $\{u_s\}_{s\in\mathbb{Z}}$ with the same conclusion. Namely, for every $x \in X$, $\{u_s.x\}_{s\in\mathbb{Z},s\geq 0}$ is dense in X. However, whether $\{u_{s^2}.x\}_{s\in\mathbb{Z}}$ is dense in X seems unknown (reference??).*

Remark 1.3. *When identifying $\mathrm{SL}_2(\mathbb{R})/\Gamma$ with the unit tangent bundle of a hyperbolic manifold/orbifold, the orbits of $\{u_s\}_{s\in\mathbb{R}}$ are identified with horocycles.*

We fix some right invariant metric $d(\cdot,\cdot)$ on $\mathrm{SL}_2(\mathbb{R})$, compatible with the topology. We will not be bothered about the explicit form of the metric. So just take its existence as a fact. Assuming this, define the quotient metric on $\mathrm{SL}_2(\mathbb{R})/\Gamma$ by
$$d([g]_\Gamma, [h]_\Gamma) := \inf_{\gamma\in\Gamma} d(g\gamma, h) = \inf_{\gamma_1,\gamma_2\in\Gamma} d(g\gamma_1, h\gamma_2).$$

Fix such a metric, we can define injectivity radius at a point $x \in X$ by
$$\mathrm{InjRad}(x) := \inf\left\{\delta > 0 \,\middle|\, g \mapsto g.x \text{ is injective on } d(g,e) < \delta\right\}.$$
Note that InjRad is continuous and since Γ is discrete, $\mathrm{InjRad}(x) > 0$ for all $x \in X$. Therefore if Γ is a cocompact lattice, there exists (and we fix such an) $r_X > 0$ such that $\mathrm{InjRad}(x) \geq r_X$ for all $x \in X$.

Also, one can check that for $d(g_i, e) < \frac{r_x}{4}$ for $i = 1,2$, we have $d(g_1.x, g_2.x) = d(g_1, g_2)$.

Lemma 1.4. *For every $x \in X$, the orbit $\{u_s.x,\ s \geq 0\}$ is not periodic, that is, for every $s \geq 0$, $u_s.x = x \implies s = 0$. As every unipotent matrix in $\mathrm{SL}_2(\mathbb{R})$ is conjugate to an element of U, this implies that Γ contains no (nontrivial) unipotent matrices.*

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Proof. Assume otherwise, then we can find $g_0 \in \mathrm{SL}_2(\mathbb{R})$ such that

$$s_0 := \inf\{s > 0 \,|\, u_s.g_0\Gamma = g_0\Gamma\} > 0.$$

In the current case inf is actually achieved at s_0. Consider

$$\begin{aligned} & a_{-t}u_{s_0}g_0\Gamma = a_{-t}g_0\Gamma \\ \Longrightarrow & \begin{bmatrix} 1 & e^{-2t}s_0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} g_0\Gamma = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} g_0\Gamma. \end{aligned}$$

As $t \to +\infty$, this implies the existence of compact orbit of U of arbitrarily small period, which is impossible due to the fact $r_X > 0$. More explicitly, for t large enough such that

$$d\left(id, \begin{bmatrix} 1 & e^{-2t}s_0 \\ 0 & 1 \end{bmatrix}\right) < r_X,$$

One has, by the definition of r_X, that $u_{e^{-2t}s_0} = id$, or in other words, $s_0 = 0$. □

Corollary 1.5. *Keep the assumption in the theorem, and pick some $x \in X$.*

1. *The map $t \mapsto u_t.x$ from $\mathbb{R}_{\geq 0}$ to X is injective.*
2. *There exists $t_n, s_n \to +\infty$ with $|t_n - s_n| \to \infty$ such that $d(x_n, y_n) \to 0$. (let $x_n := u_{t_n}.x$ and $y_n := u_{s_n}.x$)*

Proof. 1. is straightforward. For 2., use pigeon-hold principle. □

Now we start to prove the theorem. The crucial notion here is

Definition 1.6. *Say we have a (semi)group G acting on a topological space W by homeomorphisms. A subset V of W is said to be G-**minimal** iff it is closed, G-stable and no proper closed G-stable subset.*

Let Y denote the orbit closure $\overline{\{u_s \cdot x_0\}_{s\geq 0}}$. Without loss of generality (by Zorn's lemma) we assume that Y is a $\{u_s\}_{s\geq 0}$-minimal set. Our strategy is to find some $y \in Y$ and a larger group whose orbit based at y is contained in Y.

Proof of Theorem??, Step 1. Keep notations as in the corollary above. Write (for large enough n) $y_n = A_n x_n$ for some $d(A_n, \mathrm{id}) \leq r_X/4$. Write

$$A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \quad \text{with } a_n, d_n \to 1,\ b_n, c_n \to 0.$$

The key calculation is:

$$\begin{aligned} u_s A_n u_s^{-1} &= \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}\begin{bmatrix} 1 & -s \\ & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_n & b_n - sa_n \\ c_n & d_n - sc_n \end{bmatrix} \\ &= \begin{bmatrix} a_n + sc_n & b_n + s(d_n - a_n) - s^2 c_n \\ c_n & d_n - sc_n \end{bmatrix}. \end{aligned} \tag{1}$$

Case I, $c_n = 0$ for infinitely many n.

Case II. $c_n \neq 0$ for n large enough.

Equa.?? above suggests that the upper right corner dominates when s is large (this is called "shearing phenomenon", we will return to this point later).

[pictures]

We will kill the upper right corner according to the following computation

$$u_t(u_sA_nu_s^{-1}) = \begin{bmatrix} a_n+(s+t)c_n & b_n+s(d_n-a_n)-s^2c_n+t(d_n-sc_n) \\ c_n & d_n-sc_n \end{bmatrix}. \tag{2}$$

Define $t=t(s)$ by imposing the following equality

$$\begin{aligned} & b_n+s(d_n-a_n)-s^2c_n+t(d_n-sc_n)=0 \\ \Longleftrightarrow\ & t=-\frac{b_n+s(d_n-a_n)-s^2c_n}{d_n-sc_n}=-\frac{b_n-sa_n}{d_n-sc_n}-s \end{aligned} \tag{3}$$

The range of s for which the $t(s)$ is ill-defined will be excluded from the discussion (see $s=s_{\delta,n}$ below, where one has $d_n-sc_n=1\pm\delta$ with δ small). With this choice of $t=t(s)$,

$$u_t(u_sA_nu_s^{-1}) = \begin{bmatrix} (d_n-sc_n)^{-1} & 0 \\ c_n & d_n-sc_n \end{bmatrix}. \tag{4}$$

Now for $\delta>0$ (we will let $\delta\to 0$ in a moment), choose $s=s_{\delta,n}\geq 0$ such that either $d_n-sc_n=1+\delta$ or $1-\delta$, depending on the signature of c_n. So

$$s_{\delta,n}=\frac{d_n-1-\delta}{c_n} \text{ or } \frac{d_n-1+\delta}{c_n},$$

whichever is positive.

Define (Insert pictures here!!)

$$y'_{\delta,n}:=u_{t(s)}u_s.y_n,\quad x'_{\delta,n}:=u_s.x_n,\quad \text{where } s=s_{\delta,n}.$$

Then by definition

$$\begin{aligned} y'_{\delta,n} &= u_{t(s)}u_sA_nu_s^{-1}u_s.x_n = u_{t(s)}u_sA_nu_s^{-1}.x'_{\delta,n} \\ &= \begin{bmatrix} (1\pm\delta)^{-1} & \\ & (1\pm\delta) \end{bmatrix}.x'_{\delta,n} \end{aligned} \tag{5}$$

Fix δ, let n vary. By passing to a subsequence n_k, assume that $y'_{\delta,n}$ and $x'_{\delta,n}$ converge to, say, $y_{\delta,\infty}$ and $x_{\delta,\infty}$ respectively. Hence

$$y'_{\delta,\infty}=\begin{bmatrix} (1+\delta)^{-1} & \\ & (1+\delta) \end{bmatrix}\cdot x'_{\delta,\infty} \text{ or } \begin{bmatrix} (1-\delta)^{-1} & \\ & (1-\delta) \end{bmatrix}\cdot x'_{\delta,\infty}.$$

Without loss of generality, assume that the first case happens for infinitely many $\delta>0$ converging to 0. It looks like we are not making any progress except that the "transverse difference" is now in the direction of the diagonal, which normalizes U. So it is time to invoke the following general fact, which is why we introduced the notion of minimal set.

Lemma 1.7. *Let $\Gamma \curvearrowright Z$ by homeomorphisms. Γ is a semi-group and Z a topological space. Assume that V is a Γ-minimal set and W is a Γ-invariant closed set. If $\phi\in Homeo(X)$ normalizes (the image of) Γ and there exist $v_0\in V$ and $w_0\in W$ with $\phi(v_0)=w_0$. Then $\phi(V)$ is contained in W.*

Proof of the Lemma.

$$\phi(V)=\phi(\overline{\Gamma.v_0})=\overline{\phi(\Gamma.v_0)}=\overline{\Gamma.w_0}\subset W.$$

□

From the lemma (applied to $V=W=Y$, $Z=X$), we see that for a set of δ converging to 0 and for every $y\in Y$,

$$\begin{bmatrix} (1+\delta)^{-1} & \\ & (1+\delta) \end{bmatrix}\cdot y\in Y.$$

Thus

$$\left\{\begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix}\cdot y\right\}_{t\geq 0}$$

is contained in Y for every y. Of course the orbit of the semi-group $\{u_s y\}_{s\geq 0}$ is also contained in Y.

But if you think about it the orbit of the full group $\{u_s y\}_{s\in\mathbb{R}}$ is also in Y (well, a priori, $u_s Y \subset Y$ but since Y is minimal $u_s Y = Y$).

[Do not need this if you had started with the full group U] Fix some y, and take a limit point y' of $a_t y$ as $t \to +\infty$. We see that the orbit of the full group $\{a_t y'\}_{t\in\mathbb{R}}$ (though now y' may not be an arbitrary element in Y) is also guaranteed in Y.

In summary we have found some point $y' \in Y$ such that $\{a_t u_s \cdot y'\}_{t,s\in\mathbb{R}}$ is contained in Y. Thus we are done modulo the following lemma.

□

Let $B^+ := \{a_t u_s\}_{t,s\in\mathbb{R}}$ and $B := \{(\pm 1) a_t u_s\}_{t,s\in\mathbb{R}}$. B^+ is the identity component of B and $B = B^+ \sqcup (-1)B^+$ where we have abbreviated the matrix $\begin{bmatrix} -1 & \\ & -1 \end{bmatrix}$ as "-1".

Lemma 1.8. *The action of B^+ on X is minimal.*

Exercise 1. *The lemma also holds when only assuming Γ to be discrete and of finite co-volume (referred to as a* ***lattice****).*

Question 1.9. *Does it hold for geometrically finite Γ? And again, how about ∞-genus case?*

Proof. We are going to show that the B-action is minimal first and then explain why this is sufficient.

An equivalent formulation is that the Γ-action on $\mathrm{SL}_2(\mathbb{R})/B$ is minimal. To prove this, we will take a geometric point of view.

Recall that $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half space $\mathscr{H}^2 := \{z = x + iy \mid x \in \mathbb{R},\, y > 0\}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot z := \frac{az+b}{cz+d}.$$

This action preserves the Riemannian metric (referred to as the hyperbolic metric)

$$(\mathrm{dx}^2 + \mathrm{dy}^2)/y^2.$$

Geodesics under the hyperbolic metrics are (Euclidean) circles perpendicular to the x-axis together with all the vertical lines.

Another important point is that as the y-coordinate approaches 0, the (hyperbolic) distance between two point of (Euclidean) distance $\asymp 1$ actually goes to ∞. The $\mathrm{SL}_2(\mathbb{R})$-action extends continuous to the "boundary" defined by

$$\partial\mathscr{H}^2 := \{(x,y),\, y = 0\} \sqcup \{\infty\}.$$

where the topology near ∞ is coming from the "one-point compactification". Thus topologically the boundary is a circle. (somehow it is more intuitive to use the disk model, but I forgot the formula on how the group acts and the metric). In particular the action at ∞ is given as follows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\infty = \frac{a\infty+b}{c\infty+d} = \begin{cases} a/c, & \text{if } c \neq 0 \\ \infty, & \text{otherwise} \end{cases}.$$

Why care? Note that the stabilizer of ∞ is exactly B and the action is transitive on $\partial\mathscr{H}^2$ (Exercise: convince yourself that this gives a topological homeomorphism $\mathrm{SL}_2(\mathbb{R})/B \cong \mathscr{H}^2$! I am not sure how trivial/hard this is, but if you get worried, maybe consult this lemma???). Thus it suffices to show the action of Γ on $\partial\mathscr{H}^2$ is minimal.

Claim 1.1. *For every $z \in \mathscr{H}^2$, the orbit closure $\overline{\Gamma \cdot z} \supset \partial\mathscr{H}^2$.*

Assuming the claim, let W be a closed Γ-invariant set on $\partial\mathscr{H}^2 \cong S^1$. Thus its complement consists of disjoint union of open intervals (labelled as I_i's). Take such an interval I_0 with endpoints w_1, w_2. We argue that $\Gamma \cdot \widehat{w_1 w_2}$ (the unique geodesic connecting w_1 and w_2) never contains I_0 in its closure. Indeed, Γ translates of $\widehat{w_1 w_2}$ are just geodesics with endpoints outside the region between $\widehat{w_1 w_2}$ and I_0 (see figure??). Hence we are done.

Proof of the Claim. By co-compactness, we can find a bounded region $\mathscr{B} \subset \mathscr{H}^2$ (whose diameter under hyperbolic distance is denoted by $\mathrm{diam}(\mathscr{B})$) such that $\Gamma \cdot \mathscr{B} = \mathscr{H}^2$. For every $z \in \partial\mathscr{H}^2$ and a neighborhood $\mathscr{N}_{z,r_0}$ of radius r_0 (in the Euclidean metric) of z, we are going to show that some $\gamma \cdot \mathscr{B}$ is contained in $\mathscr{N}_{z,r_0}$. Indeed we can find $\gamma b \in \mathscr{N}_{z,r_0/2}$ for some $\gamma \in \Gamma$, $b \in \mathscr{B}$. When r_0 is sufficiently small one can show that

$$d_{\text{Hyperbolic}}(z', \gamma d) \le \mathrm{diam}(\mathscr{B}) \implies d_{\text{Euclidean}}(z', \gamma d) \le r_0/2.$$

This finishes the proof. □

Finally, as promised, we explain how to get the minimality of B^+ from that of B. So take $x_0 \in X$ and we know

$$\overline{Bx_0} = \overline{B^+x_0 \cup B^+(-1)x_0} = X.$$

As $\overline{B^+x_0} \cup \overline{B^+(-1)x_0}$ is B-invariant and closed, hence it is actually also equal to X. As X is connected, their intersection $\overline{B^+x_0} \cap \overline{B^+(-1)x_0}$ is non-empty. But this again, is a B-invariant closed set, so has to be the full X. In particular $\overline{B^+x_0} = X$. And the proof completes. □

Exercise 2. *Prove this for the strong unstable foliations of Anosov diffeomorphisms/flows (on compact manifolds). (Marcus?)*

REFERENCES

[Hed36] Gustav A. Hedlund, *Fuchsian groups and transitive horocycles*, Duke Math. J. **2** (1936), no. 3, 530–542. MR 1545946