LECTURE 12, QUANTITATIVE OPPENHEIM I

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Contents

Main reference: [EMM98, Section 3].
Notations

- $Q_0(x_1, x_2, x_3, x_4) := 2x_1x_4 + x_2^2 + x_3^2$ real quadratic form of signature $(3,1)$ on $\mathbb{R}^4$.
- Let $(\mathbf{e}_1, ..., \mathbf{e}_4)$ be the standard basis of $\mathbb{R}^4$; and for a vector v, define its coefficients by $v = \sum (v)_i \mathbf{e}_i$ and we also write $v = ((v)_1, ..., (v)_4)$.
- Let $(\mathbf{f}_1, ..., \mathbf{f}_4)$ be another ONB(=orthogonal normal basis) defined by $\mathbf{f}_2 = \mathbf{e}_2, \mathbf{f}_3 = \mathbf{e}_3$ and $\mathbf{f}_1 = \frac{\mathbf{e}_1+\mathbf{e}_4}{\sqrt{2}}$, $\mathbf{f}_4 = \frac{\mathbf{e}_1-\mathbf{e}_4}{\sqrt{2}}$. If $v = \sum a_i \mathbf{f}_i$, we also write $v = (a_1, ..., a_4)_{\mathbf{f}}$.
- One can verify that $Q_0((x_1, ..., x_4)_{\mathbf{f}}) = x_1^2 + x_2^2 + x_3^2 - x_4^2$.
- $\mathrm{K} := \mathrm{SO}_{Q_0}(\mathbb{R}) \cap \mathrm{SO}_4(\mathbb{R})$.
- $\mathbf{a}_t := \mathrm{diag}(e^{-t}, 1, 1, e^t)$, contained in $\mathrm{SO}_{Q_0}(\mathbb{R})$.

1. Detect points by probabilistic methods

Assume $Q_0 \circ g_0$ is irrational. Define

$$V_{(a,b)}(\mathbb{Z}) := \left\{ \mathbf{v} \in g_0.\mathbb{Z}^4 \,\middle|\, Q_0(\mathbf{v}) \in (a,b) \right\},$$

$$N_T := \# V_{a,b}(\mathbb{Z}, T),\ V_{a,b}(\mathbb{Z}, T) := \left\{ \mathbf{v} \in V_{(a,b)}(\mathbb{Z}) \,\middle|\, \|\mathbf{v}\| \le T \right\}.$$

Consider the function

$$1_{\square}(x, y) := 1_{(1,2]}(x) \cdot 1_{(a,b)}(y).$$

Hence

$$N_{2T} - N_T = \sum_{\mathbf{v} \in g_0.\mathbb{Z}^4} 1_{\square}\left(\frac{\|\mathbf{v}\|}{T}, Q_0(\mathbf{v}) \right).$$

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Find a compactly supported continuous function h approximating $1_{\square}$ from above. Then one can find some (non-negative) $f \in C_c(\mathbb{R}_{>0} \times \mathbb{R}^3)$ such that

$$h(x,y) = \frac{1}{x^2} \int f(x, w_2, w_3, y')\,|\mathrm{dw}_2 \wedge \mathrm{dw}_3| \tag{1}$$

where $y' := \frac{y - w_2^2 - w_3^2}{2x}$.

1.1. **A coarse upper bound.** By abbreviating $V_{a,b}(\mathbb{Z}, 2T - T) := V_{a,b}(\mathbb{Z}, 2T) \setminus V_{a,b}(\mathbb{Z}, T)$, we have

$$\begin{aligned} N_{2T} - N_T &\leq \sum_{\mathbf{v} \in g_0.\mathbb{Z}^4,\, \mathbf{v} \in V_{a,b}(\mathbb{Z}, 2T-T)} h\left(\frac{\|\mathbf{v}\|}{T}, Q_0(\mathbf{v})\right) \\ &= \sum_{\mathbf{v} \in g_0.\mathbb{Z}^4,\, \mathbf{v} \in V_{a,b}(\mathbb{Z}, 2T-T)} \frac{T^2}{\|\mathbf{v}\|^2} \int f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, \frac{Q_0(\mathbf{v}) - w_2^2 - w_3^2}{2\,\|\mathbf{v}\|\, T^{-1}}\right) |\mathrm{dw}_2 \wedge \mathrm{dw}_3| \end{aligned} \tag{2}$$

Each summand here is either 0 or ≥ 1 since we are keeping the index $\mathbf{v} \in V_{a,b}(\mathbb{Z}, 2T - T)$.

Now we need the following lemma, to be proved later (see Lem.2.10 where this is proved).

Lemma 1.1. *Given $f \in C_c(\mathbb{R}_{>0} \times \mathbb{R}^3)$ and $\varepsilon \in (0,1)$, there exists $T_0 = T_0(f, \varepsilon) > 0$ such that for every $T > T_0$, for every $\mathbf{v} \in \mathbb{R}^4$ we have*

$$\left| \frac{1}{2C_4} T^2 \int f(\mathbf{a}_{\ln T} k.\mathbf{v})\,\widehat{\mathrm{m}}_{\mathrm{K}}(k) - \frac{T^2}{\|\mathbf{v}\|^2} \int f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, w_4\right) |\mathrm{dw}_2 \wedge \mathrm{dw}_3| \right| < \varepsilon$$

where

$$w_4 := \frac{Q_0(\mathbf{v}) - w_2^2 - w_3^2}{2\,\|\mathbf{v}\|\, T^{-1}}$$

is a function in (w_2, w_3), for every fixed $\mathbf{v}$ and T.

Apply Lem.1.1 with some $\varepsilon < 0.5$, then for T sufficiently large, each $\mathbf{v} \in V_{a,b}(\mathbb{Z}, 2T - T)$, either

$$\frac{T^2}{\|\mathbf{v}\|^2} \int f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, \frac{Q_0(\mathbf{v}) - w_2^2 - w_3^2}{2\,\|\mathbf{v}\|\, T^{-1}}\right) = \frac{1}{2C_4} T^2 \int 2 f(\mathbf{a}_t k.\mathbf{v})\,\widehat{\mathrm{m}}_{\mathrm{K}}(k) = 0$$

or ≥ 0.5 .

Therefore

$$\begin{aligned} N_{2T} - N_T &\leq \sum_{\mathbf{v} \in g_0.\mathbb{Z}^4, \mathbf{v} \in V_{a,b}(\mathbb{Z}, 2T-T)} 2 \cdot \frac{1}{2C_4} T^2 \int f(\mathbf{a}_t k.\mathbf{v})\,\widehat{\mathrm{m}}_{\mathrm{K}}(k) \\ &\leq \sum_{\mathbf{v} \in g_0.\mathbb{Z}^4} 2 \frac{1}{2C_4} T^2 \int f(\mathbf{a}_t k.\mathbf{v})\,\widehat{\mathrm{m}}_{\mathrm{K}}(k) = 2 \frac{1}{2C_4} T^2 \int \widetilde{f}(\mathbf{a}_t k g_0.\mathbb{Z}^4)\,\widehat{\mathrm{m}}_{\mathrm{K}}(k). \end{aligned} \tag{3}$$

where

$$\widetilde{f} : \mathrm{X}_4 \to \mathbb{R} \text{ defined by } \widetilde{f}(\Lambda) := \sum_{\mathbf{v} \in \Lambda} f(\mathbf{v}).$$

If $\widetilde{f}$ were a bounded function, then immediately we see that for some constant $C = C(f) > 0$,

$$N_{2T} - N_T \leq T^2 C \implies N_{2^n T_0} \leq T_0^2 C(1 + 4^1 + \ldots + 4^{n-1}) + N_{T_0} = \frac{1 - 4^n}{1 - 4} T_0^2 C + N_{T_0} \leq (2^n T_0)^2 C + N_{T_0}.$$

This shows that for T large,

$$N_T \leq 2CT^2.$$

Unfortunately our $\widetilde{f}$ is not bounded. Nevertheless we still have

Theorem 1.2. *There exists a constant $C = C(f) > 0$ such that*

$$\int \widetilde{f}(\mathbf{a}_t k g_0.\mathbb{Z}^4)\widehat{\mathrm{m}}_{\mathrm{K}}(k) \leq C$$

for all $t > 0$.

By arguments outlined above and Thm.1.2 we get

Theorem 1.3. *There exists a constant $C > 0$ such that $N_T \leq CT^2$ for T sufficiently large.*

1.2. **The exact upper/lower bound.** Equipped with Thm.1.3, let us revisit Equa.(2):

$$\begin{aligned}\frac{N_{2T}-N_T}{T^2} &\leq T^{-2}\sum_{\mathbf{v}\in g_0.\mathbb{Z}^4,\mathbf{v}\in V_{a,b}(\mathbb{Z},2T)} h\left(\frac{\|\mathbf{v}\|}{T}, Q_0(\mathbf{v})\right) \\ &= T^{-2}\sum_{\mathbf{v}\in g_0.\mathbb{Z}^4,\mathbf{v}\in V_{a,b}(\mathbb{Z},2T)} \frac{T^2}{\|\mathbf{v}\|^2}\int f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, Q_0(\mathbf{v})\right)|\mathrm{d}w_2\wedge \mathrm{d}w_3|.\end{aligned} \tag{4}$$

Fix an $\varepsilon > 0$, the range of T such that Lem.1.1 is not applicable is bounded. Thus

$$\frac{N_{2T}-N_T}{T^2} \leq T^{-2}\sum_{\mathbf{v}\in g_0.\mathbb{Z}^4,\mathbf{v}\in V_{a,b}(\mathbb{Z},2T)}\left(\frac{1}{2C_4}T^2\int f(\mathbf{a}_t k.\mathbf{v})\widehat{\mathrm{m}}_{\mathrm{K}}(k) + O(\varepsilon)\right) + O_\varepsilon(T^{-2}). \tag{5}$$

By Thm.1.3, the number of indices is bounded by $C(2T)^2$, hence

$$\begin{aligned}\frac{N_{2T}-N_T}{T^2} &\leq T^{-2}\left(\sum_{\mathbf{v}\in g_0.\mathbb{Z}^4,\mathbf{v}\in V_{a,b}(\mathbb{Z},2T)}\frac{1}{2C_4}T^2\int f(\mathbf{a}_t k.\mathbf{v})\widehat{\mathrm{m}}_{\mathrm{K}}(k)\right) + O_\varepsilon(T^{-2}) + O(\varepsilon) \\ &\leq \frac{1}{2C_4}\left(\sum_{\mathbf{v}\in g_0.\mathbb{Z}^4}\int f(\mathbf{a}_t k.\mathbf{v})\widehat{\mathrm{m}}_{\mathrm{K}}(k)\right) + O_\varepsilon(T^{-2}) + O(\varepsilon) \\ &= \frac{1}{2C_4}\int \widetilde{f}(\mathbf{a}_t k g_0\mathbb{Z}^4)\widehat{\mathrm{m}}_{\mathrm{K}}(k) + O_\varepsilon(T^{-2}) + O(\varepsilon)\end{aligned} \tag{6}$$

Hence (let $\varepsilon \to 0$ after taking the limit $\lim_T$)

$$\limsup_{T\to+\infty}\frac{N_{2T}-N_T}{T^2} \leq \lim_{t\to+\infty}\int \frac{1}{2C_4}\widetilde{f}(\mathbf{a}_t k g_0\mathbb{Z}^4)\widehat{\mathrm{m}}_{\mathrm{K}}(k).$$

That the RHS is a true limit is justified below.
The exact lower bound is proved similarly.

Theorem 1.4. *Assume $Q_0\circ g_0$ is not rational, then for every $f \in C_c(\mathbb{R}^4)$,*

$$\lim_{t\to+\infty}\int \widetilde{f}(\mathbf{a}_t k g_0\mathbb{Z}^4)\widehat{\mathrm{m}}_{\mathrm{K}}(k) = \int_{\mathrm{X}_4}\widetilde{f}(x)\widehat{\mathrm{m}}_{\mathrm{X}_4}(x) = C_6\int_{\mathbb{R}^4} f(\mathbf{v})\,\mathrm{d}\mathrm{v}$$

where $C_6 > 0$ depending only on the dimension.

Let us evaluate $\int_{\mathbb{R}^4} f(\mathbf{v})\,\mathrm{dv}$ for our f. By change of variables $y' =: \frac{y-w_2^2-w_3^2}{2x}$,

$$\int f(\mathbf{v})\,\mathrm{dv} = \int f(x,w_2,w_3,y')\,\mathrm{d}x\,\mathrm{d}y'\,\mathrm{d}w_2\,\mathrm{d}w_3 = \int \frac{1}{2x}f(x,w_2,w_3,\frac{y-w_2^2-w_3^2}{2x})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}w_2\,\mathrm{d}w_3.$$

where we have used

$$\mathrm{d}y' = \frac{\mathrm{d}y - 2w_2\,\mathrm{d}w_2 - 2w_3\,\mathrm{d}w_3}{2x} - \frac{\mathrm{d}x}{2x^2}(y - w_2^2 - w_3^2).$$

Recall Equa.(1), we have

$$\int f(\mathbf{v})\,\mathrm{dv} = \int \frac{x}{2} h(x,y)\,\mathrm{d}x\mathrm{d}y.$$

As $h(x,y)$ approximates $1_{\square}$ we get

$$\int \frac{x}{2} h(x,y)\,\mathrm{d}x\mathrm{d}y \to \int_{y=a}^{b}\int_{x=1}^{2} \frac{x}{2}\,\mathrm{d}x\mathrm{d}y = \frac{2^2-1}{4}(b-a).$$

Thus, by collecting the constants $C_7 := \frac{1}{2C_4} C_6 \frac{2^2-1}{4}$,

$$\lim_{T\to+\infty} \frac{N_{2T}-N_T}{T^2} = C_7(b-a).$$

Now a geometric series argument shows that

Corollary 1.5.

$$\lim_{T\to+\infty} \frac{N_T}{T^2} = \frac{1}{3} C_7(b-a).$$

2. Proof of the Lemma

2.1. Nontrivial contribution to the integral.

Definition 2.1. *For $(x,y,z)\in\mathbb{R}^3$ with $x\neq 0$ and $a\in\mathbb{R}$, we let*

$$\phi_a(x,y,z) := \frac{a-y^2-z^2}{2x},$$

in other words, $\phi_a(x,y,z)$ is the unique real number such that

$$Q_0\left(x,y,z,\phi_a(x,y,z)\right) = a.$$

Definition 2.2. *Given $f\in C_c(\mathbb{R}_{>0}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$, we fix $C_1 = C_1(f) > 1$ such that*

$$\mathrm{Supp}(f) \subset (C_1^{-1}, C_1)\times(-C_1, C_1)^3.$$

Also fix $C_2 > |a_0|, |b_0|$.

The following two are something directly following from the definition.

Lemma 2.3. *Let $\mathbf{v}_{\neq 0}\in\mathbb{R}^4$ and $T>1$. Let $(w_2,w_3)\in\mathbb{R}^2$ be such that*

$$f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, \phi_{Q_0(\mathbf{v})}(\frac{\|\mathbf{v}\|}{T}, w_2, w_3)\right) \neq 0,$$

then

1. $C_1^{-1}T \le \|\mathbf{v}\| \le C_1 T$ *and* $|w_2|, |w_3| \le C_1$*;*
2. $\left|\phi_{Q_0(\mathbf{v})}(\frac{\|\mathbf{v}\|}{T}, w_2, w_3)\right| \le C_1.$*;*
3. $|Q_0(\mathbf{v})| \le 4C_1^2$.

For a vector $\mathbf{w}$, $\mathbf{w}(i)\in\mathbb{R}$ is defined by $\mathbf{w} = \sum \mathbf{w}(i)\mathbf{e}_i$.

Lemma 2.4. *Let $\mathbf{v}_{\neq 0}\in\mathbb{R}^4$ and $T>1$. Let $\mathbf{w}\in K.\mathbf{v}$. If $f(\mathbf{a}_{\ln T}.\mathbf{w})\neq 0$, then*

1. $C_1^{-1}T \le \mathbf{w}(1) \le C_1 T$, $|\mathbf{w}(2)|, |\mathbf{w}(3)| \le C_1$ *and* $|\mathbf{w}(4)| \le C_1 T^{-1}$*;*
2. $\|\mathbf{v}\| \le 2C_1 T$*;*
3. $\|\mathbf{v}\| \ge C_1^{-1} T$*;*
4. $|Q_0(\mathbf{v})| \le 4C_1^2$.

Proof. For item 3, $\|\mathbf{v}\| = \|\mathbf{w}\| \ge \mathbf{w}(1) \ge C_1^{-1}T$.

For item 4, $Q_0(\mathbf{v}) = Q_0(\mathbf{w}) = \mathbf{w}(1)\mathbf{w}(4) + \mathbf{w}(2)^2 + \mathbf{w}(3)^2 \le 2C_1^2 + C_1^2 + C_1^2 = 4C_1^2$. □

2.2. **Representative in a K-orbit.** By working with the basis $\mathbf{f}$, one sees that for every $\mathbf{v} \in \mathbb{R}^4$, there exists $k_{\mathbf{v}} \in K$ such that

$$k_{\mathbf{v}}.\mathbf{v} = (u_1, 0, 0, u_4)_{\mathbf{f}} \text{ for some } u_1, u_4 \geq 0.$$

Indeed, if we set

$$r_1(\mathbf{v}) := \frac{\|\mathbf{v}\| + Q_0(\mathbf{v})}{2},\ r_2(\mathbf{v}) := \frac{\|\mathbf{v}\| - Q_0(\mathbf{v})}{2}$$

or equivalently,

$$r_1(\mathbf{v}) := \mathbf{v_f}(1)^2 + \mathbf{v_f}(2)^2 + \mathbf{v_f}(3)^2,\ r_2(\mathbf{v}) := \mathbf{v_f}(4)^2$$

where we assume $\mathbf{v} = (\mathbf{v_f}(1), \ldots, \mathbf{v_f}(4))_{\mathbf{f}}$. Then there exists $k \in K$ such that

$$k.\mathbf{v} = (\sqrt{r_1}, 0, 0, \sqrt{r_2})_{\mathbf{f}} =: \mathbf{v}^*.$$

To summarize the discussion in the basis $\mathbf{e}$:

Lemma 2.5. *For every $\mathbf{v} \in R^4$ there exists a unique $\mathbf{v}^* \in \mathbb{R}^4$ satisfying*

1. $Q_0(\mathbf{v}^*) = Q_0(\mathbf{v})$*;*
2. $\|\mathbf{v}^*\| = \|\mathbf{v}\|$*;*
3. $\mathbf{v}^*(1) \geq |\mathbf{v}^*(4)|$ *and* $\mathbf{v}^*(2) = \mathbf{v}^*(3) = 0$.

Also $\mathbf{v}^* \in K.\mathbf{v}$.

What we are going to need is the following slightly perturbed version.

Lemma 2.6. *Let $\mathbf{v} \in \mathbb{R}^4$ and $(w_2, w_3) \in \mathbb{R}$ satisfying $|w_2|, |w_3| \leq C_1$. Assume $\|\mathbf{v}\|^2 \geq Q_0(\mathbf{v}) + 4C_1^2$. Then there exists a unique $\mathbf{v}^*(w_2, w_3) = \mathbf{w} \in \mathbb{R}^4$ such that*

1. $Q_0(\mathbf{w}) = Q_0(\mathbf{v})$*;*
2. $\|\mathbf{w}\| = \|\mathbf{v}\|$*;*
3. $\mathbf{w}(1) \geq |\mathbf{w}(4)|$ *and* $\mathbf{w}(2) = w_2, \mathbf{w}(3) = w_3$.

Also $\mathbf{w} \in K.\mathbf{v}$.

Sketch of proof. Indeed under the assumption above

$$\left| \frac{Q_0(\mathbf{v}) - w_2^2 - w_3^2}{2} \right| \leq \frac{1}{2}(Q_0(\mathbf{v}) + 2C_1^2)$$

and

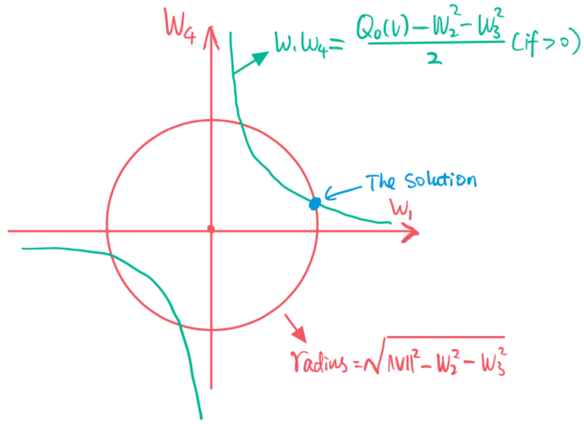
$$\|\mathbf{v}\|^2 - w_2^2 - w_3^2 \geq Q_0(\mathbf{v}) + 4C_1^2 - C_1^2 - C_1^2 = Q_0(\mathbf{v}) + 2C_1^2.$$

Hence the equation

$$\begin{cases} xy = \frac{Q_0(\mathbf{v}) - w_2^2 - w_3^2}{2} \\ x^2 + y^2 = \|\mathbf{v}\|^2 - w_2^2 - w_3^2 \end{cases}$$

admits a unique solution with $x \geq |y|$.

Here is a picture ($x = w_1,\ y = w_4$)



□

2.3. **Approximates I, the points.**

Lemma 2.7. *Assumption as in Lem.2.3. Further assume $T \geq 8C_1^3$ and $T^2 \geq 16C_1^4$. Define $\mathbf{w} = \mathbf{v}^*(w_2, w_3)$ as in Lem.2.6. Then for $C_3 = 46C_1^7$,*

$$\operatorname{dist}_\infty\left(\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, \phi_{Q_0(\mathbf{v})}(\frac{\|\mathbf{v}\|}{T}, w_2, w_3)\right), \mathbf{a}_{\ln T}.\mathbf{w}\right) \leq C_3 T^{-2}.$$

Note that $T \geq 8C_1^3 \implies \|\mathbf{v}\| \geq 4C_1^2 + 4C_1^2 \geq Q_0(\mathbf{v}) + 4C_1^2$ by Lem.2.3. Thus Lem.2.6 is applicable.

Proof. First we have

$$|\mathbf{w}(4)|^2 \leq |\mathbf{w}(1)|\,|\mathbf{w}(4)| = \left|Q_0(\mathbf{v}) - w_2^2 - w_3^2\right| \leq 4C_1^2 + 2C_1^2 = 6C_1^2.$$

Hence the difference of the first coordinate:

$$\begin{aligned}
&\left|\|\mathbf{v}\|^2 - \mathbf{w}(1)^2\right| = \mathbf{w}(2)^2 + \mathbf{w}(3)^2 + \mathbf{w}(4)^2 \leq 8C_1^2 \\
\implies &\left|T^{-1}\|\mathbf{v}\| - T^{-1}\mathbf{w}(1)\right| \leq T^{-1}\frac{8C_1^2}{\|\mathbf{v}\| + \mathbf{w}(1)} \leq T^{-1}\frac{8C_1^2}{\|\mathbf{v}\|} \leq 8C_1^3 T^{-2} \leq C_3 T^{-2}.
\end{aligned}$$

From here we also see that

$$|\mathbf{w}(1)| \geq \|\mathbf{v}\| - 8C_1^3 T^{-1} \geq \frac{1}{2}C_1^{-1}T + (\frac{1}{2}C_1^{-1}T - 8C_1^3 T^{-1}) \geq \frac{1}{2}C_1^{-1}T.$$

Here we are using the assumption $T^2 \geq 16C_1^4 \implies \frac{1}{2}C_1^{-1}T - 8C_1^3 T^{-1} \geq 0$.

Now the difference of the last coordinate (note that $w_2 = \mathbf{w}(2)$ and $w_3 = \mathbf{w}(3)$ from Lem.2.6)

$$\begin{aligned}
&\left|\frac{Q_0(\mathbf{v}) - \mathbf{w}(2)^2 - \mathbf{w}(3)^2}{2\|\mathbf{v}\|T^{-1}} - \frac{Q_0(\mathbf{v}) - \mathbf{w}(2)^2 - \mathbf{w}(3)^2}{2\mathbf{w}(1)T^{-1}}\right| \\
\leq &\frac{1}{2}(6C_1^2)T\left|\frac{1}{\|\mathbf{v}\|} - \frac{1}{\mathbf{w}(1)}\right| = \frac{(6C_1^2)T}{2}\frac{|\|\mathbf{v}\| - \mathbf{w}(1)|}{\|\mathbf{v}\|\mathbf{w}(1)} \\
\leq &\frac{(6C_1^2)T}{2}\frac{8C_1^3T^{-1}}{1/2C_1^{-2}T^2} = 48C_1^7 T^{-2} \leq C_3 T^{-2}.
\end{aligned}$$

□

Lemma 2.8. *Assumption as in Lem.2.4. Define $w_2 := \mathbf{w}(2)$ and $w_3 := \mathbf{w}(3)$. Then for $C_3 = 48C_1^7$,*

$$\mathrm{dist}_\infty\left(\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, \phi_{Q_0(\mathbf{v})}(\frac{\|\mathbf{v}\|}{T}, w_2, w_3)\right), \mathbf{a}_{\ln T}.\mathbf{w}\right) \leq C_3 T^{-2}.$$

Proof. The difference of the first coordinate:

$$\left|\|\mathbf{v}\|^2 - \mathbf{w}(1)^2\right| = \mathbf{w}(2)^2 + \mathbf{w}(3)^2 + \mathbf{w}(4)^2 \leq 3C_1^2$$

$$\Longrightarrow \left|T^{-1}\|\mathbf{v}\| - T^{-1}\mathbf{w}(1)\right| \leq T^{-1}\frac{3C_1^2}{\mathbf{w}(1)} \leq 3C_1^3 T^{-2} \leq C_3 T^{-2}.$$

And the difference of the last coordinate

$$\begin{aligned}&\left|\frac{Q_0(\mathbf{v}) - \mathbf{w}(2)^2 - \mathbf{w}(3)^2}{2\|\mathbf{v}\|T^{-1}} - \frac{Q_0(\mathbf{v}) - \mathbf{w}(2)^2 - \mathbf{w}(3)^2}{2\mathbf{w}(1)T^{-1}}\right|\\ \leq&\frac{T}{2}(6C_1^2)\left|\frac{1}{\|\mathbf{v}\|} - \frac{1}{\mathbf{w}(1)}\right| = \frac{(6C_1^2)T}{2}\frac{|\|\mathbf{v}\| - \mathbf{w}(1)|}{\|\mathbf{v}\|\mathbf{w}(1)}\\ \leq&\frac{(6C_1^2)T}{2}\frac{3C_1^3T^{-1}}{C_1^{-2}T^2} = 9C_1^7T^{-2} \leq C_3T^{-2}.\end{aligned}$$

□

2.4. **Approximates II, the measures.** Let $S(r)$ be the sphere of radius r in $\mathbb{R}^3$ centered at the origin. Let $\widehat{\mathrm{m}}_{S(r)}$ be the normalized (to be a probability measure) volume measure on $S(r)$.

Assume $r_1(\mathbf{v}) \geq 2C_1^2$. For $(x_2, x_3) \in \mathbb{R}^2$ with $|x_2|, |x_3| \leq C_1$, there exists a unique $x_1 > 0$ such that

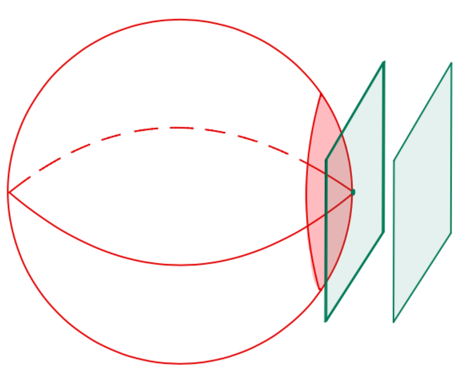
$$x_1^2 + x_2^2 + x_3^2 = r_1(\mathbf{v}).$$

Let $\mathrm{D}(C_1)$ be the image of $\{(x_2, x_3),\ |x_2|, |x_3| \leq C_1\}$ in $S(\sqrt{r_1})$ thus defined. And identify $|\mathrm{dx}_2 \wedge \mathrm{dx}_3||_{|x_i|\leq C_1}$ as a measure on $\mathrm{D}(C_1) \subset S(\sqrt{r_1})$ by this. Equivalently one may first restrict the differential form $\mathrm{dx}_2 \wedge \mathrm{dx}_3$ to $\mathrm{D}(C_1)$ and then take the measure associated with it.

Lemma 2.9. *Let $\mathbf{v} \in \mathbb{R}^4$ be satisfying $\frac{\|\mathbf{v}\|^2}{2} \geq |Q_0(\mathbf{v})|$ and $\|\mathbf{v}\|^2 \geq 16C_1^2$.*

$$\left\|\|\mathbf{v}\|^2\widehat{\mathrm{m}}_{S(\sqrt{r_1})} - 2C_4|\mathrm{dw}_2 \wedge \mathrm{dw}_3|\right\|_{\mathrm{D}(C_1)} \leq \frac{1}{\|\mathbf{v}\|^2}(C_5Q_0(\mathbf{v}) + C_5)$$

where $C_4 > 0$ is a constant depending only on the dimension and $C_5 > 1$ depends on C_1. See Equa.(7), (8) below.



Note that our assumption implies that $r_1(\mathbf{v}) = 1/2(\|\mathbf{v}\|^2 + Q_0(\mathbf{v})) \geq 4C_1^2$. Thus the paragraph above the proposition makes sense.

Proof. First let us write $\widehat{\mathrm{m}}_{S(\sqrt{r_1})}$ in terms of differential forms. By taking the differential

$$x_1^2 + x_2^2 + x_3^2 = r^2 \implies 2x_1\,\mathrm{dx}_1 + 2x_2\,\mathrm{dx}_2 + 2x_3\,\mathrm{dx}_3 = 2r\,\mathrm{dr}.$$

Thus

$$\mathrm{dx}_1 \wedge \mathrm{dx}_2 \wedge \mathrm{dx}_3 = \frac{r\,\mathrm{dx}_2 \wedge \mathrm{dx}_3}{x_1} \wedge \mathrm{dr}$$

So up to constant (depending possibly on r), the spherical measure can be induced from $\frac{r\,\mathrm{dx}_2\wedge\mathrm{dx}_3}{x_1}$. To make it have total mass independent of r, we consider

$$\mathrm{dx}_1 \wedge \mathrm{dx}_2 \wedge \mathrm{dx}_3 = \frac{\mathrm{dx}_2 \wedge \mathrm{dx}_3}{r x_1} \wedge r^2\,\mathrm{dr}.$$

Since the volume of ball of radius R is some constant multiple of $R^3/3 = \int_0^R r^2\,\mathrm{dr}$, there exists some constant $C_4 > 0$ depending only on the dimension such that

$$\widehat{\mathrm{m}}_{S(\sqrt{r_1})} = C_4 \frac{\mathrm{dx}_2 \wedge \mathrm{dx}_3}{\sqrt{r_1}\,x_1}. \tag{7}$$

By assumption,

$$2r_1 = \|\mathbf{v}\|^2 + Q_0(\mathbf{v}) \geq \|\mathbf{v}\|^2 - |Q_0(\mathbf{v})| \geq \frac{1}{2}\|\mathbf{v}\|^2 \implies r_1 \geq 4C_1^2.$$

Thus for $(x_1, x_2, x_3) \in S(\sqrt{r_1})$,

$$2\sqrt{r_1}x_1 = 2\sqrt{r_1}\sqrt{r_1 - x_2^2 - x_3^2} \geq \|\mathbf{v}\|\sqrt{r_1 - 2C_1^2} \geq \|\mathbf{v}\|\sqrt{\frac{1}{2}r_1} \geq \frac{\|\mathbf{v}\|^2}{8}.$$

On the other hand

$$\left|\|\mathbf{v}\|^2 - 2r_1\right| = |Q_0(\mathbf{v})|$$

and

$$\left|2r_1 - 2\sqrt{r_1}x_1\right| = 2\sqrt{r_1}\left|\frac{r_1 - (r_1 - x_2^2 - x_3^2)}{\sqrt{r_1} + \sqrt{r_1 - x_2^2 - x_3^2}}\right| \leq 2\left|x_2^2 + x_3^2\right| \leq 4C_1^2.$$

Therefore, when restricted to $\mathrm{D}(C_1)$, we have

$$\begin{aligned}\left|\|\mathbf{v}\|^2\,\widehat{\mathrm{m}}_{S(\sqrt{r_1})} - 2C_4\,|\mathrm{dx}_2 \wedge \mathrm{dx}_3|\right| &= 2C_4\left|\frac{\|\mathbf{v}\|^2}{2\sqrt{r_1}x_1} - 1\right||\mathrm{dx}_2 \wedge \mathrm{dx}_3| \\ &= 2C_4\left|\frac{\|\mathbf{v}\|^2 - 2\sqrt{r_1}x_1}{2\sqrt{r_1}x_1}\right||\mathrm{dx}_2 \wedge \mathrm{dx}_3| \\ &\leq 2C_4\left|\frac{|Q_0(\mathbf{v})| + 4C_1^2}{\frac{1}{4}\|\mathbf{v}\|^2}\right||\mathrm{dx}_2 \wedge \mathrm{dx}_3|\end{aligned}$$

Thus if integrating a function taking value in $[-M, M]$, the difference is at most

$$2C_4\left|\frac{Q_0(\mathbf{v}) + 4C_1^2}{\frac{1}{8}\|\mathbf{v}\|^2}\right|(2C_1)^2 \cdot M = \|\mathbf{v}\|^{-2} \cdot \left|64C_4C_1^2(|Q_0(\mathbf{v})| + 4C_1^2)\right| \cdot M.$$

Taking

$$C_5 := 256C_4C_1^4 \tag{8}$$

completes the proof. □

2.5. **Proof of the lemma.** Fix $\mathbf{v} \in \mathbb{R}^4$, we identify $S(r)$ with a subset of $\mathbb{R}^4$ by embedding

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, \sqrt{r_2(\mathbf{v})})_{\mathbf{f}}.$$

Let us state Lem.1.1 again:

Lemma 2.10. *Given $f \in C_c(\mathbb{R}_{>0} \times \mathbb{R}^3)$ and $\varepsilon \in (0,1)$, there exists $T_0 = T_0(f, \varepsilon) > 0$ such that for every $T > T_0$, for every $\mathbf{v} \in \mathbb{R}^4$ we have*

$$\left| \frac{1}{2C_4} T^2 \int f(\mathbf{a}_{\ln T} k.\mathbf{v}) \widehat{\mathrm{m}}_{\mathrm{K}}(k) - \frac{T^2}{\|\mathbf{v}\|^2} \int f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, w_4\right) |\mathrm{d}w_2 \wedge \mathrm{d}w_3| \right| < \varepsilon$$

where

$$w_4 := \frac{Q_0(\mathbf{v}) - w_2^2 - w_3^2}{2\|\mathbf{v}\| T^{-1}}$$

is a function in (w_2, w_3), for every fixed $\mathbf{v}$ and T.

Proof. We are going to choose some $T_0 \geq 10C_1^3$.

Rewrite

$$\frac{1}{2C_4} T^2 \int f(\mathbf{a}_{\ln T} k.\mathbf{v}) \widehat{\mathrm{m}}_{\mathrm{K}}(k) = \frac{1}{2C_4} T^2 \int f(\mathbf{a}_{\ln T}.\mathbf{w}) \widehat{\mathrm{m}}_{\mathrm{K}.\mathbf{v}}(\mathbf{w})$$

By Lem.2.4, 2.6, if $T \geq T_0$, by change of variable $\mathbf{w} \mapsto (w_2, w_3) := (\mathbf{w}(2), \mathbf{w}(3))$:

$$\begin{aligned}\frac{1}{2C_4} T^2 \int f(\mathbf{a}_{\ln T}.\mathbf{w}) \widehat{\mathrm{m}}_{\mathrm{K}.\mathbf{v}}(\mathbf{w}) =& \frac{1}{2C_4} T^2 \int_{f(\mathbf{a}_{\ln T}.\mathbf{w}) \neq 0} f(\mathbf{a}_{\ln T}.\mathbf{w}) \widehat{\mathrm{m}}_{\mathrm{K}.\mathbf{v}}(\mathbf{w}) \\ =& \frac{1}{2C_4} T^2 \int_{\mathrm{D}(C_1)} f(\mathbf{a}_{\ln T}.\mathbf{v}^*(w_2, w_3)) \widehat{\mathrm{m}}_{S(\sqrt{r_1})}(w_2, w_3).\end{aligned}$$

Note that when $f(\mathbf{a}_{\ln T} k.\mathbf{v}) \neq 0$ for some $k \in K$, $T \geq 10C_1^3 \implies \|\mathbf{v}\|^2 \geq Q_0(\mathbf{v}) + 4C_1^2$ by Lem.2.4. So Lem.2.6 is applicable to $\mathbf{v}$ and $(w_2, w_3) := (\mathbf{w}(2), \mathbf{w}(3))$. Moreover, Lem.2.6 implies that $\mathbf{w} = \mathbf{v}^*(\mathbf{w}(2), \mathbf{w}(3))$.

By Lem.2.3, the RHS is equal to

$$\frac{T^2}{\|\mathbf{v}\|^2} \int f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, w_4\right) |\mathrm{d}w_2 \wedge \mathrm{d}w_3| = \frac{T^2}{\|\mathbf{v}\|^2} \int_{\mathrm{D}(C_1)} f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, w_4\right) |\mathrm{d}w_2 \wedge \mathrm{d}w_3|$$

Recall from Lem.2.3 and 2.4 that when $f(\mathbf{a}_{\ln T}.\mathbf{w}) \neq 0$ or when $f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, w_4\right) \neq 0$, we always have

$$\frac{1}{C_1} T \leq \|\mathbf{v}\| \leq 2C_1 T$$

and

$$|Q_0(\mathbf{v})| \leq 4C_1^2. \tag{9}$$

Now it suffices to show that

$$\left| \|\mathbf{v}\|^2 \int_{\mathrm{D}(C_1)} f(\mathbf{a}_{\ln T}.\mathbf{w}) \widehat{\mathrm{m}}_{S(\sqrt{r_1})}(w_2, w_3) - 2C_4 \int_{\mathrm{D}(C_1)} f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, w_4\right) |\mathrm{d}w_2 \wedge \mathrm{d}w_3| \right| < \varepsilon.$$

By Lem.2.7 and 2.8, for T large enough,

$$\left| 2C_4 \int_{\mathrm{D}(C_1)} f(\mathbf{a}_{\ln T}.\mathbf{w}) |\mathrm{d}w_2 \wedge \mathrm{d}w_3| - 2C_4 \int_{\mathrm{D}(C_1)} f\left(\frac{\|\mathbf{v}\|}{T}, w_2, w_3, w_4\right) |\mathrm{d}w_2 \wedge \mathrm{d}w_3| \right| < 0.5\varepsilon.$$

By Lem.2.9 and Equa.(9), for T large enough,

$$\left| \|\mathbf{v}\|^2 \int_{\mathrm{D}(C_1)} f(\mathbf{a}_{\ln T}.\mathbf{w}) \widehat{\mathrm{m}}_{S(\sqrt{r_1})}(w_2, w_3) - 2C_4 \int_{\mathrm{D}(C_1)} f(\mathbf{a}_{\ln T}.\mathbf{w}) |\mathrm{d}w_2 \wedge \mathrm{d}w_3| \right| < 0.5\varepsilon.$$

Combining these two, we are done. $\square$

References

[EMM98] Alex Eskin, Gregory Margulis, and Shahar Mozes, *Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture*, Ann. of Math. (2) **147** (1998), no. 1, 93–141. MR 1609447