COUNT LIFTS OF NON-MAXIMAL CLOSED HOROCYCLES ON $SL_N(\mathbb{Z})\backslash SL_N(\mathbb{R})/SO_N(\mathbb{R})$

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Abstract. A closed horocycle $\mathcal{U}$ on $SL_N(\mathbb{Z})\backslash SL_N(\mathbb{R})/SO_N(\mathbb{R})$ has many lifts to the universal cover $SL_N(\mathbb{R})/SO_N(\mathbb{R})$. Under some conditions on the horocycle, we give a precise asymptotic count of its lifts of bounded distance away from a given base point in the universal cover. This partially generalizes previous work of Mohammadi–Golsefidy.

Contents

1. Introduction

Let X be a locally symmetric space and Y be a closed submanifold (or suborbifold) of X. Let $\pi : \widetilde{X} \to X$ be the Riemannian universal covering. Then $\pi(Y)$ forms a locally finite family of lifts of Y with the fundamental group of X acting transitively on different lifts. A natural question is to count asymptotically how many lifts of Y has distance less than R to a fixed point as R tends to $+\infty$.

In the present paper, we specialize to the situation where the locally symmetric space takes the form $X = \Gamma\backslash \mathrm{SL}_N(\mathbb{R})/\mathrm{SO}_N(\mathbb{R})$ with Γ commensurable with

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$\mathrm{SL}_N(\mathbb{Z})$ and the closed submanifold takes the form $Y = \Gamma\backslash\Gamma U \,\mathrm{SO}_N(\mathbb{R})/\,\mathrm{SO}_N(\mathbb{R})$ for some horocyclic subgroup U. Then $\widetilde{X} = \mathrm{SL}_N(\mathbb{R})/\,\mathrm{SO}_N(\mathbb{R})$ and each lift of Y is a *horocycle* in $\widetilde{X}$. Horocycles are interesting from the point view of geometry (see [KLP17, 2.10]) and topology (see [Sch10, 12.3]).

Two examples of our main theorem are presented below and the more general statement comes later as Theorem 4.6. Let $o \in \mathrm{SL}_N(\mathbb{R})/\,\mathrm{SO}_N(\mathbb{R})$ be the identity coset. All distances below are induced from the standard trace form

$$(X, Y) \mapsto \mathrm{Tr}(XY^{tr})$$

on the Lie algebra. To save notation, write $\mathrm{K} := \mathrm{SO}_N(\mathbb{R})$.

Example 1. Let $N = 3$ and

$$U := \left\{ \begin{bmatrix} 1 & *_1 & *_2 \\ & 1 & *_3 \\ & & 1 \end{bmatrix},\ *_i \in \mathbb{R},\ i = 1, 2, 3 \right\}$$

Then as $R \to +\infty$

$$\#\left\{\gamma U\mathrm{K}/\mathrm{K} \mid \mathrm{dist}(\gamma U\mathrm{K}/\mathrm{K}, o) \leq R,\ \gamma \in \mathrm{SL}_N(\mathbb{Z})\right\} \sim R^{\frac{1}{2}} e^{2\sqrt{2}R} \cdot \frac{\pi^{\frac{1}{2}} \cdot 3 \cdot 2^{\frac{1}{4}}}{7 \cdot \xi(2)\xi(3)}$$

where $\xi(s)$ is the completed Riemann zeta function. The notation $\sim$ means the ratio between the left hand side and the right hand side goes to 1 as $R \to +\infty$.

This example is also a special case of [MG14, Theorem 3] with a different constant. We suspect that this is due to firstly in the middle of page 1328 of loc. cit., the authors erroneously stated "the stabilizer of $\mathcal{U}$ is U" (compare Lemma 4.1). And secondly, at the last row, where there is a curly arrow, of the multi-lined equations on page 1331 of loc. cit., the Haar measures are off by a constant. The next example is new.

Example 2. Let $N = 3$ and

$$U := \left\{ \begin{bmatrix} 1 & 0 & *_1 \\ & 1 & *_2 \\ & & 1 \end{bmatrix},\ *_i \in \mathbb{R},\ i = 1, 2 \right\}$$

Then as $R \to +\infty$

$$\#\left\{\gamma U\mathrm{K}/\mathrm{K} \mid \mathrm{dist}(\gamma U\mathrm{K}/\mathrm{K}, o) \leq R,\ \gamma \in \mathrm{SL}_N(\mathbb{Z})\right\} \sim R^{\frac{1}{2}} e^{2\sqrt{2}R} \cdot \frac{\pi^{\frac{3}{2}}}{2^{\frac{1}{4}} \cdot \xi(2)\xi(3)}.$$

Our counting result Theorem 4.6 has some limitations. It only treats certain special closed horocycles. This is because of the corresponding restriction put on the dynamical Theorem 2.1. The proof of this uses the equidistribution theorem of [EMS96] and the non-divergence criterion in [Zha20], both of which requires the homogeneous measure to be "defined over $\mathbb{Q}$". But as our homogeneous measures have a compact part, Borel density lemma does not apply to show, and it is not true that, they are always defined over $\mathbb{Q}$. That being said, this should only be a technical matter. Indeed, the work of [EMS96] essentially relies on the unipotent measure rigidity theorem of Ratner [Ra91] and the linearization technique of [DM91]. The work of [Zha20] relies on [KM98] (and [DGUL19] beyond the case of SL_N). None of which (except for [DGUL19]) requires the homogeneous measure to be "defined over $\mathbb{Q}$". Thus it is possible to argue without this assumption. We hope to return to this point in the future. Also, the effective aspects of the counting problem is not treated here. The reader is referred to [DKL16] for discussions.

Our main dynamical result Theorem 2.1 is stated in Section 2 and the proof is given in the section after that. Our main counting result Theorem 4.6 is in Section 4 where a large portion of work is devoted to the volume computations. Justifications of Examples above are given after Corollary 4.8.

2. Preliminaries and statement of the main results

2.1. **Preparations.** Let $e_1, ..., e_N$ be the standard basis of $\mathbb{R}^N$ and $\|\cdot\|$ denote the standard Euclidean norm. For $I \subset \{1, ..., N\}$, write $e_I := \wedge_{i\in I} e_i$, defined up to ± 1. Let $\mathbb{R}^I$ be the subspace of $\mathbb{R}^N$ defined by

$$(x_1, ..., x_N) \in \mathbb{R}^I \iff x_i = 0,\ \forall i \notin I.$$

Let $\mathscr{I}_0 := \{I_1, ..., I_{k_0}\}$ be an unordered partition of $\{1, ..., N\}$, namely,

$$\{1, ..., N\} = I_1 \sqcup I_2 \sqcup ... \sqcup I_{k_0}.$$

Without loss of generality, assume that for every pair $k_1 < k_2$ and $i_1 \in I_{k_1}$, $i_2 \in I_{k_2}$ we have $i_1 < i_2$. An equivalence relation on $\{1, ..., N\}$ is defined by

$$i \sim_{\mathscr{I}_0} j \iff i, j \in I_k,\ \exists\, k \in \{1, ..., k_0\}.$$

Let

$$\begin{aligned}
\mathbf{P}_{\mathscr{I}_0} &:= \left\{ g \in \mathrm{SL}_N \;\middle|\; g\mathbb{R}^{I_k} \subset \mathbb{R}^{\cup_{i\leq k} I_i},\ \forall k \right\}\\
\mathbf{U}_{\mathscr{I}_0} &:= \left\{ g \in \mathrm{SL}_N \;\middle|\; (id - g)\mathbb{R}^{I_k} \subset \mathbb{R}^{\cup_{i< k} I_i},\ \forall k \right\}\\
\mathbf{L}_{\mathscr{I}_0} &:= \left\{ g \in \mathrm{SL}_N \;\middle|\; g\mathbb{R}^{I_k} \subset \mathbb{R}^{I_k},\ \forall k \right\}\\
\mathbf{M}_{\mathscr{I}_0} &:= \{ g \in \mathbf{L}_{\mathscr{I}_0} \mid \det(g|_{\mathbb{R}^{I_k}}) = 1,\ \forall k \}\\
\mathbf{A}_{\mathscr{I}_0} &:= \{ g \in \mathbf{L}_{\mathscr{I}_0} \mid g|_{\mathbb{R}^{I_k}} = d_k\, id,\ \forall k,\ \exists d_k > 0 \}\\
\mathbf{K}_{\mathscr{I}_0} &:= \mathbf{L}_{\mathscr{I}_0} \cap \mathrm{SO}_N = \mathbf{P}_{\mathscr{I}_0} \cap \mathrm{SO}_N
\end{aligned}$$

where by convention $\mathbb{R}^{\cup_{i<1} I_i} := \{0\}$. As $\mathscr{I}_0$ varies, $\{\mathbf{P}_{\mathscr{I}_0}\}$ is a set of standard parabolic subgroups. $\mathbf{U}_{\mathscr{I}_0}$ is the unipotent radical of $\mathbf{P}_{\mathscr{I}_0}$ and $\mathbf{L}_{\mathscr{I}_0}$ is the unique Levi subgroup of $\mathbf{P}_{\mathscr{I}_0}$ stable under transpose inverse. One also sees that $\mathbf{M}_{\mathscr{I}_0} = \oplus_{k=1,...,k_0} \mathbf{M}_{I_k}$, a direct sum of $\mathrm{SL}_{|I_k|}$'s, and $(\mathbf{K}_{\mathscr{I}_0})^\circ = \oplus_{k=1,...,k_0} \mathbf{K}_{I_k}$, a direct sum of $\mathrm{SO}_{|I_k|}$'s, where

$$\begin{aligned}
\mathbf{M}_{I_k} &:= \left\{ g \in \mathbf{M}_{\mathscr{I}_0} \;\middle|\; g|_{\mathbb{R}^{I_j}} = id,\ \forall j \neq k \right\},\\
\mathbf{K}_{I_k} &:= \left\{ g \in \mathbf{K}_{\mathscr{I}_0} \;\middle|\; g|_{\mathbb{R}^{I_j}} = id,\ \forall j \neq k \right\}.
\end{aligned}$$

We shall use the Roman letter for the analytic identity components of the real points of these algebraic groups. For instance,

$$\begin{aligned}
\mathrm{A}_{\mathscr{I}_0} &= \{ \mathbf{a} = \mathrm{diag}(a_1, ..., a_N) \mid a_i = a_j,\ \forall i \sim_{\mathscr{I}_0} j;\ a_k > 0,\ \forall k \}\\
\mathrm{K}_{\mathscr{I}_0} &= \left\{ g \in \mathrm{SL}_N(\mathbb{R}) \;\middle|\; g|_{\mathbb{R}^I_k} \in \mathrm{SO}_{|I_k|}(\mathbb{R}),\ \forall k \right\}.
\end{aligned}$$

Let Γ be a lattice in $\mathrm{SL}_N(\mathbb{R})$ commensurable with $\mathrm{SL}_N(\mathbb{Z})$. There are two types of measures attached to $\mathscr{I}_0$:

Type 1: the unique probability Haar measure supported on $\mathrm{M}_{\mathscr{I}_0}\mathrm{U}_{\mathscr{I}_0}\Gamma/\Gamma$, call it $\mu_{\mathscr{I}_0}$;

Type 2: the unique probability Haar measure supported on $\mathrm{K}_{\mathscr{I}_0}\mathrm{U}_{\mathscr{I}_0}\Gamma/\Gamma$, call it $\nu_{\mathscr{I}_0}$.

Translates of measures of Type 1 is the subject of [MG14], where it is called "horospherical". Though their quotients in $\Gamma\backslash\mathrm{SL}_N(\mathbb{R})/\mathrm{SO}_N(\mathbb{R})$ are supported on quotients of horospheres (they are of codimension 1!) defined by level sets of Busemann functions (see [Ebe96, 1.10.1]) only when $\mathscr{I}_0$ consists only two parts, namely when $\mathrm{P}_{\mathscr{I}_0}$ is a proper maximal parabolic. These are most relevant for the study of counting rational points on flag varieties. We call measures of Type 2 **horocyclic** which is the subject of the current paper. These measures are supported on closed horocycles (see [KLP17] for a geometric definition for their lifts). As Type 1 and 2 has a nontrivial intersection when $\mathscr{I}_0$ is most refined, i.e. $|I_k|=1$ for all k, the count of *maximal* horocycles is also carried out in [MG14]. The novelty of this paper is the treatment of non-maximal cases.

As usual, given a sequence (g_n) in $\mathrm{SL}_N(\mathbb{R})$, the study of limiting measures of $(g_n\nu_{\mathscr{I}_0})$ (or $(g_n\mathrm{m}_{\mathscr{I}_0})$) consists of two steps:

Step 1. determine when is the sequence of the measures non-divergent;

Step 2. determine the limiting homogeneous measure in the non-divergent case.

For the first step we use the nondivergence criterion in [Zha20] (note that the group here may not be generated by $\mathbb{R}$-unipotents, so the work of [DM91] is not sufficient) whereas for the second step we use [EMS96].

2.2. **Statement of the main results.** One has a natural bijection

$$\mathrm{SL}_N(\mathbb{R})=\mathrm{SO}_N(\mathbb{R})\mathrm{M}_{\mathscr{I}_0}\times\mathrm{A}_{\mathscr{I}_0}\times\mathrm{U}_{\mathscr{I}_0}=\mathrm{SO}_N(\mathbb{R})(\oplus_k\mathrm{M}_{I_k})\times\mathrm{A}_{\mathscr{I}_0}\times\mathrm{U}_{\mathscr{I}_0}.$$

We may further decompose

$$\oplus_k\mathrm{M}_{I_k}=\oplus_k\mathrm{K}_{I_k}\mathrm{A}^+_{\mathrm{M}_{I_k}}\mathrm{K}_{I_k},$$

where

$$\mathrm{A}_{\mathrm{M}_{I_k}}:=\{\mathbf{a}=\mathrm{diag}(a_1,...,a_N)\in\mathrm{SL}_N(\mathbb{R})\mid a_i=1,\ \forall i\notin I_k;\ a_i>0,\ \forall i\in I_k\},$$
$$\mathrm{A}^+_{\mathrm{M}_{I_k}}:=\left\{\mathbf{a}=\mathrm{diag}(a_1,...,a_N)\in\mathrm{A}_{\mathrm{M}_{I_k}}\ \middle|\ a_{i+1}/a_i\geq 1,\ \forall i,i+1\in I_k\right\}.$$

Now we have a surjection

$$\mathrm{SO}_N(\mathbb{R})\times\oplus_k\mathrm{A}^+_{\mathrm{M}_{I_k}}\times(\oplus_k\mathrm{K}_{I_k})\times\mathrm{A}_{\mathscr{I}_0}\times\mathrm{U}_{\mathscr{I}_0}\to\mathrm{SL}_N(\mathbb{R}).$$

Thus we may write

$$g_n=k_n\cdot\oplus a_n^k\cdot c_n\cdot b_n\cdot u_n.$$

And hence

$$g_n\cdot\nu_{\mathscr{I}_0}=k_n\oplus a_n^k b_n\cdot\nu_{\mathscr{I}_0}.$$

As (k_n) is a bounded sequence, it suffices to understand the limit behaviour of $(\oplus a_n^k b_n\cdot\nu_{\mathscr{I}_0})$. By passing to a subsequence and modifying from left by a bounded sequence we may assume that the sequence $(\oplus a_n^k b_n)$ is **clean** in the following sense. First $(\oplus a_n^k)$ is clean:

1. for each $k\in\{1,...,k_0\}$, either (a_n^k) is unbounded or $a_n^k\equiv id$ for all n;

in which case we let

$$\mathscr{I}(a_n,\infty):=\left\{I_k\ \middle|\ (a_n^k)\text{ is unbounded }\right\},$$
$$\mathscr{I}(a_n,1):=\left\{I_k\ \middle|\ a_n^k\equiv id\right\}.$$

Second (b_n) is clean:

2. for each $k\in\{1,...,k_0\}$, $\lambda_{I_1\cup...\cup I_k}(b_n)$ diverges to $+\infty$, remains constantly equal to 1, or converges to 0.

where for a diagonal matrix $\mathbf{a} = \operatorname{diag}(a_1, ..., a_N)$ and $I \subset \{1, ..., N\}$, we denote $\lambda_I(a) := \prod_{i\in I} a_i$. In this case define

$$\mathscr{I}(b_n, 0) := \{I_k \mid \lambda_{I_1\cup...\cup I_k}(b_n) \text{ converges to } 0\}$$
$$\{k_1 < k_2 < ... < k_{l_0} = k_0\} := \{k = 1, ..., k_0 \mid \lambda_{I_1\cup...\cup I_k}(b_n) \text{ equals to } 1\}$$

and

$$\mathscr{I}_1(b_n) := \left\{I_1 \cup ... \cup I_{k_1}, I_{k_1+1} \cup ... \cup I_{k_2}, ..., I_{k_{l_0-1}+1} \cup ... \cup I_{k_{l_0}}\right\},$$

which is a partition coarser than $\mathscr{I}_0$. From the definition, (b_n) is contained in $\mathrm{M}_{\mathscr{I}_1(b_n)}$.

For simplicity define

$$\mathscr{I}_1(\text{new}) := \mathscr{I}_1(b_n) \setminus \mathscr{I}_0,\ \mathscr{I}_1(\text{old}, \infty) := \mathscr{I}_1(b_n) \cap \mathscr{I}_0 \cap \mathscr{I}(a_n, \infty),$$
$$\mathscr{I}_1(\infty) := \mathscr{I}_1(\text{new}) \sqcup \mathscr{I}_1(\text{old}, \infty),$$
$$\mathscr{I}_1(0) := \mathscr{I}_1(b_n) \cap \mathscr{I}_0 \cap \mathscr{I}(a_n, 0).$$

Thus for a clean sequence,

$$\mathscr{I}_1 := \mathscr{I}_1(b_n) = \mathscr{I}_1(\text{new}) \sqcup \mathscr{I}_1(\text{old}, \infty) \sqcup \mathscr{I}_1(0) = \mathscr{I}_1(\infty) \sqcup \mathscr{I}_1(0).$$

Now we have all the necessary terminologies to state our theorem.

Theorem 2.1. *Assume $(\oplus a_n^k b_n)$ is clean. Then*

1. *$(\oplus a_n^k b_n \cdot \nu_{\mathscr{I}_0})$ is non-divergent iff $\mathscr{I}(b_n, 0) = \emptyset$;*
2. *if $\mathscr{I}(b_n, 0) = \emptyset$, then $(\oplus a_n^k b_n \cdot \nu_{\mathscr{I}_0})$ converges to the unique probability Haar measure supported on*

$$(\oplus_{I\in\mathscr{I}_1(0)} \mathrm{K}_I)(\oplus_{I\in\mathscr{I}_1(\infty)} \mathrm{M}_I)\mathrm{U}_{\mathscr{I}_1}\Gamma/\Gamma.$$

A sequence (μ_n) of probability measures is said to be *non-divergent* iff for every $\varepsilon > 0$ there exists a bounded set B such that $\mu_n(B) > 1 - \varepsilon$ for all n.

By comparison, it follows from [MG14, Theorem 1] that (in this case $(\oplus a_n^k)$ stabilizes the measure and plays no role)

Theorem 2.2. *Assume (b_n) is clean. Then*

1. *$(\oplus b_n \cdot \mu_{\mathscr{I}_0})$ is non-divergent iff $\mathscr{I}(b_n, 0) = \emptyset$;*
2. *if $\mathscr{I}(b_n, 0) = \emptyset$, then $(b_n \cdot \mu_{\mathscr{I}_0})$ converges to the unique probability Haar measure supported on*

$$(\oplus_{I\in\mathscr{I}_1} \mathrm{M}_I)\mathrm{U}_{\mathscr{I}_1}\Gamma/\Gamma.$$

3. Translates of horocyclic measures

3.1. **Proof of Theorem 2.1, nondivergence.** Notations same as section 2. By [Zha20, Theorem 1.3] (see also section 4 therein),

Theorem 3.1. *Assume $(\oplus a_n^k b_n)$ is clean. The sequence of probability measures $(\oplus a_n^k b_n \cdot \nu_{\mathscr{I}_0})$ is non-divergent iff there exists $\varepsilon > 0$ such that for every $\mathbb{Q}$-subsapce W stabilized by $\mathbf{K}_{\mathscr{I}_0}\mathbf{U}_{\mathscr{I}_0}$,*

$$\left\|\oplus a_n^k b_n \cdot (W \cap \mathbb{Z}^N)\right\| \geq \varepsilon$$

where $\|\cdot\|$ denotes the covolume of a discrete subgroup in the $\mathbb{R}$-linear subspace spanned by it.

So we need to classify all $\mathbf{K}_{\mathscr{I}_0}\mathbf{U}_{\mathscr{I}_0}$-stable subspaces.

Lemma 3.2. *A proper subspace W of $\mathbb{R}^N$ is $\mathbf{U}_{\mathscr{I}_0}$-stable iff there exists j in $\{1, ..., k_0 - 1\}$ such that*

$$\mathbb{R}^{I_1 \cup ... \cup I_j} \subset W \subsetneqq \mathbb{R}^{I_1 \cup ... \cup I_{j+1}}.$$

Proof. That if W satisfies the sandwich condition then W is $\mathbf{U}_{\mathscr{I}_0}$-stable follows directly from definition. Assume W is $\mathbf{U}_{\mathscr{I}_0}$-stable, find the largest k such that there exists $w \in W \setminus \mathbb{R}^{I_1 \cup ... \cup I_k}$. Then $w \in \mathbb{R}^{I_1 \cup ... \cup I_{k+1}} \setminus \mathbb{R}^{I_1 \cup ... \cup I_k}$ and from the definition of $\mathrm{U}_{\mathscr{I}_0}$, the map $u \mapsto uw - w$ is a surjection from $\mathrm{U}_{\mathscr{I}_0}$ to $\mathbb{R}^{I_1 \cup ... \cup I_k}$. Hence

$$\mathbb{R}^{I_1 \cup ... \cup I_k} \subset W \subset \mathbb{R}^{I_1 \cup ... \cup I_{k+1}}.$$

Now taking $j := k$ or $k + 1$ completes the proof. ◻

Now since for each k, K_{I_k} acts on $\mathbb{R}^{I_k}$ irreducibly, we conclude that

Lemma 3.3. *A proper subspace W of $\mathbb{R}^N$ is $\mathbf{K}_{\mathscr{I}_0}\mathbf{U}_{\mathscr{I}_0}$-stable iff there exists $j \in \{1, ..., k_0 - 1\}$ such that*

$$W = \mathbb{R}^{I_1 \cup ... \cup I_j}.$$

Now we can finish the proof of Theorem 2.1, part 1. For each n and $j \in \{1, ..., k_0 - 1\}$,

$$\left\| \oplus a_n^k b_n \cdot \mathbb{Z}^{I_1 \cup ... \cup I_j} \right\| = \left\| b_n \cdot \mathbb{Z}^{I_1 \cup ... \cup I_j} \right\| = \lambda_{I_1 \cup ... \cup I_j}(b_n).$$

So we are done by combining Theorem 3.1 and the lemma just above.

3.2. **Proof of Theorem 2.1, equidistribution.** Same notations as in Theorem 2.1. Assume part 1 holds. Note that for every n,

$$\oplus a_n^k b_n (\mathrm{K}_{\mathscr{I}_0} \mathrm{U}_{\mathscr{I}_0}) \subset (\oplus_{I \in \mathscr{I}_1(0)} \mathrm{K}_I)(\oplus_{I \in \mathscr{I}_1(\infty)} \mathrm{M}_I) \mathrm{U}_{\mathscr{I}_1}.$$

Hence it suffices to work inside the latter, denoted by $\mathrm{F}_{\mathscr{I}_1}$ ($\mathbf{F}_{\mathscr{I}_1}$ the corresponding connected $\mathbb{Q}$-subgroup) for simplicity. By nondivergence, there exist sequences (k_n) in $\mathrm{K}_{\mathscr{I}_0}$, (u_n) in $\mathrm{U}_{\mathscr{I}_0}$, bounded (δ_n) in $\mathrm{F}_{\mathscr{I}_1}$ and (γ_n) in $\mathrm{F}_{\mathscr{I}_1} \cap \Gamma$, such that

$$\oplus a_n^k b_n k_n u_n = \delta_n \gamma_n.$$

By passing to a finite cover, we assume

$$\gamma_n = \oplus_{I \in \mathscr{I}_1(\infty)} \gamma_n^I \cdot \gamma_{\mathrm{U},n}$$

for some $\gamma_n^I \in \mathrm{M}_I \cap \Gamma$ and $\gamma_{\mathrm{U},n} \in \mathrm{U}_{\mathscr{I}_1} \cap \Gamma$.

By [EMS96, Theorem 2.1] and passing to a subsequence if necessary, to prove Theorem 2.1, it suffices to show that the only connected $\mathbb{Q}$-subgroup of $\mathbf{F}_{\mathscr{I}_1}$ containing

$$\gamma_n \mathbf{K}_{\mathscr{I}_0} \mathbf{U}_{\mathscr{I}_0} \gamma_n^{-1} \tag{1}$$

is $\mathbf{F}_{\mathscr{I}_1}$. Since $\mathbf{U}_{\mathscr{I}_1}$ is contained in $\mathbf{U}_{\mathscr{I}_0}$ and $\mathbf{U}_{\mathscr{I}_0}$ is normalized by $\mathbf{F}_{\mathscr{I}_1}$, we have

$$\mathbf{U}_{\mathscr{I}_0} \subset \gamma_n \mathbf{K}_{\mathscr{I}_0} \mathbf{U}_{\mathscr{I}_0} \gamma_n^{-1}$$

for all n. On the other hand, because of the special form γ_n takes,

$$\oplus_{I \in \mathscr{I}_1(0)} \mathbf{K}_I \subset \gamma_n \mathbf{K}_{\mathscr{I}_0} \mathbf{U}_{\mathscr{I}_0} \gamma_n^{-1}$$

for all n. Using the special form of γ_n again, to prove our theorem, it suffices to show that for every $J \in \mathscr{I}_1(\infty)$, the union of

$$\gamma_n^J (\oplus_{I \in \mathscr{I}_0, I \subset J} \mathbf{K}_I)(\mathbf{U}_{\mathscr{I}_0} \cap \mathbf{M}_J)(\gamma_n^J)^{-1}$$

is $\mathbb{Q}$-Zariski dense in $\mathbf{M}_J$. By definition of $\mathscr{I}_1$, $a_n \in \mathrm{M}_{\mathscr{I}_1}$. Hence we can write $a_n = \oplus_{J\in\mathscr{I}_1} a_n^J$. By repeating some arguments above if possible, we may assume for each $J \in \mathscr{I}_1(\infty)$,

$$(\oplus_{I\in\mathscr{I}_0, I\subset J} a_n^I) b_n^J k_n^J u_n^J = \delta_n^J \gamma_n^J$$

for some sequences (k_n^J) in $\oplus_{I\in\mathscr{I}_0,I\subset J}\mathbf{K}_I(=: \mathbf{K}_{\mathscr{I}_0,J})$, (u_n^J) in $\mathbf{U}_{\mathscr{I}_0}\cap\mathbf{M}_J$ and bounded (δ_n^J) in M_J.

There are two cases to consider: $J \in \mathscr{I}_1(\text{old}, \infty)$ or $J \in \mathscr{I}_1(\text{new})$. The first case is simpler. Indeed, in this case $J \in \mathscr{I}_0$ and hence $(b_n^J) = (id)$, $\mathbf{U}_{\mathscr{I}_0} \cap \mathbf{M}_J = \{id\}$. So we are considering the limit of (note that the condition in Equa.(1) is not only sufficient, but also necessary)

$$a_n^J \mathrm{m}_{\mathrm{K}_{\mathscr{I}_0,J}} \text{ in } \mathrm{M}_J/\mathrm{M}_J\cap\Gamma$$

where $\mathrm{m}_{\mathrm{K}_{\mathscr{I}_0,J}}$ is the unique $\mathrm{K}_{\mathscr{I}_0,J}$-invariant probability measure supported the closed orbit of $\mathrm{K}_{\mathscr{I}_0,J}$ passing through the identity coset. Hence it converges to the M_J-invariant Haar measure as (a_n^J) is unbounded (see [EM93]).

Now let us assume $J \in \mathscr{I}_1(\text{new})$. Identify $\mathrm{M}_J = \mathrm{SL}_{|J|}(\mathbb{R})$ and $\mathscr{I}_0|_J = \mathscr{I}_0' = \{I_1', ..., I_{k_0'}'\}$, a partition of $\{1, ..., |J|\}$. By assumption, $k_0' \geq 2$. Under this identification, $\mathrm{M}_J \cap \Gamma$ is some lattice Γ_J in $\mathrm{SL}_{|J|}(\mathbb{R})$ commensurable with $\mathrm{SL}_{|J|}(\mathbb{Z})$, $\mathrm{U}_{\mathscr{I}_0} \cap \mathrm{M}_J = \mathrm{U}_{\mathscr{I}_0'}$ and $\mathrm{K}_{\mathscr{I}_0,J}$ becomes $\mathrm{K}_{\mathscr{I}_0'}$.

The sequence $(\oplus_{I\in\mathscr{I}_0,I\subset J} a_n^I)$ becomes $(\oplus_{I\in\mathscr{I}_0'} a_n^I)$, contained in $\oplus_{I\in\mathscr{I}_0'}\mathrm{A}_I^+$. And b_n^J is in $\mathrm{A}_{\mathscr{I}_0'}$. By the definition of $\mathscr{I}_1(\text{new})$ we have

$$\lambda_{I_1'\cup...\cup I_k'}(b_n^J) \to +\infty, \quad \forall\, k = 1, ..., k_0' - 1.$$

In particular, for all $l = 1, ..., k_0' - 1$,

$$\lambda_{\{1,...,l\}}(b_n^J) \to +\infty.$$

Also recall

$$\oplus_{I\in\mathscr{I}_0'} a_n^I b_n^J k_n^J u_n^J = \delta_n^J \gamma_n^J.$$

We would like to show that the only $\mathbb{Q}$-subgroup containing

$$\gamma_n^J \mathbf{K}_{\mathscr{I}_0'} \mathbf{U}_{\mathscr{I}_0'} (\gamma_n^J)^{-1}$$

is $\mathrm{SL}_{|J|}$. To do this, it suffices to show that for every nontrivial irreducible $\mathbb{Q}$-representation (ρ, V) of $\mathrm{SL}_{|J|}$, rational vectors fixed by $\gamma_n^J \mathbf{K}_{\mathscr{I}_0'} \mathbf{U}_{\mathscr{I}_0'} (\gamma_n^J)^{-1}$ for all n must be zero. Now assume otherwise and we seek to derive a contradiction.

Let $\mathrm{U}_{\min}$ be the group of all upper triangular unipotent matrices in $\mathrm{SL}_{|J|}(\mathbb{R})$. Let $\mathbf{D}$ be the full diagonal torus in $\mathrm{SL}_{|J|}$. Let $\psi^+ \in X^*(\mathbf{D})$ (the character group of $\mathbf{D}$) be the highest weight appearing in (ρ, V). The "highest" is with respect to the partial order compatible with $\mathrm{U}_{\min}$.

Let $\mathcal{W}_{\mathscr{I}_0'}$ be the subgroup of the Weyl group generated by the reflection about the roots

$$\{\lambda_{i,j} \mid i \sim_{\mathscr{I}_0'} j\}$$

where $\lambda_{i,j}$ is defined by

$$\lambda_{i,j}(\mathrm{diag}(d_1, ..., d_{|J|})) := d_i/d_j.$$

Note that $\mathcal{W}_{\mathscr{I}_0'}$ admits a set of representatives

$$\{w_1, ..., w_{s_1}\} \subset \mathrm{K}_{\mathscr{I}_0'}.$$

Let $\mathcal{C}_{\mathscr{I}_0'}$ be the weights appearing in (ρ, V) that lie in the convex hull of

$$\{w \cdot \psi^+ \mid w \in \mathcal{W}_{\mathscr{I}_0'}\}.$$

Then $\oplus_{\alpha \in \mathcal{C}_{\mathscr{I}_0'}} V_\alpha$ is fixed by $\mathbf{U}_{\mathscr{I}_0'}$. Indeed

Lemma 3.4. *We have* $\oplus_{\alpha \in \mathcal{C}_{\mathscr{I}_0'}} V_\alpha = V^{\mathbf{U}_{\mathscr{I}_0'}}$.

Proof. This has essentially been done in [Shi19]. Let us briefly recall how. Both $\oplus_{\alpha \in \mathcal{C}_{\mathscr{I}_0'}} V_\alpha$ and $V^{\mathbf{U}_{\mathscr{I}_0'}}$ are $\mathbf{M}_I$-stable. Thus we can find a $\mathbf{M}_I$-stable complement W of $\oplus_{\alpha \in \mathcal{C}_{\mathscr{I}_0'}} V_\alpha$ in $V^{\mathbf{U}_{\mathscr{I}_0'}}$. Hence W contains a vector fixed by $\mathbf{U}_{\min}$, which is a contradiction. □

We do not plan to characterize $V^{\mathbf{K}_{\mathscr{I}_0'}\mathbf{U}_{\mathscr{I}_0'}}$. Rather we need the following

Lemma 3.5. *There exists $C_2 > 0$ such that for all $v \in V^{\mathbf{K}_{\mathscr{I}_0'}\mathbf{U}_{\mathscr{I}_0'}}$ and $d \in \mathrm{A}^+_{\mathscr{I}_0'}$, we have*

$$\|\rho(d)v\| \geq C_2 \|v\|.$$

Proof. Without loss of generality assume different weight spaces are orthogonal to each other.

Decompose $\mathcal{C}_{\mathscr{I}_0'} = \mathcal{C}_1 \sqcup ... \sqcup \mathcal{C}_r$ into disjoint $\mathcal{W}_{\mathscr{I}_0'}$-orbits. For each $i = 1, .., r$, let ψ_i^+ be the unique element in $\mathcal{C}_i$ that are in the positive Weyl chamber.

Take a vector $v \in V^{\mathbf{K}_{\mathscr{I}_0'}\mathbf{U}_{\mathscr{I}_0'}}$. Write $v = v^1 \oplus v^2 \oplus ... \oplus v^r$ such that each

$$v^i = \oplus_{\psi \in \mathcal{C}_i} v^i_\psi \in \oplus_{\psi \in \mathcal{C}_i} V_\psi.$$

Because v is fixed by $\mathbf{K}_{\mathscr{I}_0'}$, $\{w_1, ..., w_{s_1}\}$ transitively permutes $\{v^i_\psi\}_{\psi \in \mathcal{C}_i}$. Thus there exists a constant $C_1 > 0$ such that

$$\left\|v^i_\psi\right\| \geq C_1 \left\|v^i\right\|, \ \forall i = 1, ..., r, \ \forall \psi \in \mathcal{C}_i.$$

Foe every i, ψ_i^+ is a positive linear combinations of $\lambda_{\{1\}}, ..., \lambda_{\{1,...,|J|\}}$. As there are only finitely many such, we can find $C_1' > 0$ such that for all $d \in \mathrm{A}^+_{\mathscr{I}_0'}$,

$$|\psi_i^+(d)| \geq C_1'.$$

Now

$$\begin{aligned}
\|\rho(d)v\|^2 &= \sum_{i=1}^r \left\|\rho(d) \oplus_{\psi \in \mathcal{C}_i} v^i_\psi\right\|^2 \\
&\geq \sum_{i=1}^r \left\|\rho(d) v^i_{\psi_i^+}\right\|^2 = \sum_{i=1}^r |\psi_i^+(d)|^2 \left\|v^i_{\psi_i^+}\right\|^2 \\
&\geq C_1'^2 \sum_{i=1}^r C_1^2 \left\|v^i\right\|^2 = (C_1 C_1')^2 \|v\|.
\end{aligned}$$

Taking $C_2 := (C_1 C_1')^2$ completes the proof. □

Now start the proof. Let v be a rational vector fixed by $\gamma_n^J \mathbf{K}_{\mathscr{I}_0'} \mathbf{U}_{\mathscr{I}_0'} (\gamma_n^J)^{-1}$. Because $v_n := \rho(\gamma_n^J)^{-1} v$ have bounded denominators, there exists $C_3 > 0$ such that

$$\|v_n\| \geq C_3, \ \forall n.$$

Also v_n and hence $\rho(b_n^J)v_n$ are fixed by $\mathbf{K}_{\mathscr{I}_0'}\mathbf{U}_{\mathscr{I}_0'}$. Thus (for simplicity ρ is dropped from the notations below)

$$\left\|\delta_n^J \cdot v\right\| = \left\|\delta_n^J\gamma_n^J \cdot v_n\right\| = \left\|\oplus a_n^I b_n^J (k_n^J u_n^J)\cdot v_n\right\| = \left\|\oplus a_n^I b_n^J \cdot v_n\right\| \geq C_2 \left\|b_n^J \cdot v_n\right\|.$$

For every $\psi \in \mathcal{C}_{\mathscr{I}_0'}$, its restriction to $\mathrm{A}_{\mathscr{I}_0'}$ is the same as the restriction of ψ^+. But $\psi^+(b_n^J) \to +\infty$ by assumption. Hence

$$\left\|b_n^J \cdot v_n\right\| = \psi^+(b_n^J)\left\|v_n\right\| \geq \psi^+(b_n^J)C_3 \to +\infty.$$

This is a contradiction since $\left\|\delta_n^J \cdot v\right\|$ is bounded.

4. Volume computation and count

In this section we apply equidistribution of homogeneous measures to count lifts of certain horocycles. Notations are inherited from section 2. Write $\mathrm{G} := \mathrm{SL}_N(\mathbb{R})$ and $\mathrm{K} := \mathrm{SO}_N(\mathbb{R})$. Gothic letters are used to denote Lie algebras. Let $\Gamma \leq \mathrm{SL}_N(\mathbb{R})$ be a lattice commensurable with $\mathrm{SL}_N(\mathbb{Z})$.

Sometimes to simplify notations, we use $[x]_H$ to denote the equivalence class of x in some space X with a group H acting from the right. E.g. for $g \in \mathrm{G}$, $[g]_\Gamma$ is its image in G/Γ.

4.1. Statement of the theorem with proof outlined.

4.1. Statement of the theorem with proof outlined. Proofs of lemmas in this subsection are often delayed to a later subsection. Links are provided to facilitate readers of the digital version to navigate back and forth.

Let $\Gamma\backslash\Gamma\mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}$ be a closed horocycle in the locally symmetric space $\Gamma\backslash\mathrm{G}/\mathrm{K}$ for some partition $\mathcal{I}_0$ of $\{1,...,N\}$. Let G_{hor} be the stabilizer in G of the (lifted) horocycle $\mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}$ in G/K. Thus the set of all lifts of $\Gamma\backslash\Gamma\mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}$ in G/K is parametrized by $\Gamma/\Gamma\cap G_{\mathrm{hor}}$ via

$$\gamma \mapsto \gamma\mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}.$$

Lemma 4.1. *We have that* $G_{\mathrm{hor}} = N_{\mathrm{K}}(\mathrm{U}_{\mathscr{I}_0})\cdot\mathrm{U}_{\mathscr{I}_0}$ *and hence* $G^{\circ}_{\mathrm{hor}} = \mathrm{K}_{\mathscr{I}_0}\cdot\mathrm{U}_{\mathscr{I}_0}$.

See 4.4.1 for the proof.

Now choose Γ' to be a *neat* finite index subgroup of Γ. Being neat means that for every $\gamma \in \Gamma'$, no eigenvalue of γ is a root of unity except for 1. Such a choice of Γ' always exists ([Bor19, 17.4]). There exists $\{q_1,...,q_{r_0}\} \subset \Gamma$ such that

$$\Gamma\mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K} = \bigsqcup_{i=1}^{r_0}\Gamma' q_i\mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K} = \bigsqcup_{i=1}^{r_0} q_i\Gamma_i\mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}$$

where $\Gamma_i := q_i^{-1}\Gamma' q_i$. Note that Γ_i's are also neat.

Lemma 4.2. *For each* $i = 1,...,r_0$, $\Gamma_i \cap G_{\mathrm{hor}} = \Gamma_i \cap \mathrm{U}_{\mathscr{I}_0}$.

See 4.4.2 for the proof.

Thus the set of lifts of $\Gamma\backslash\Gamma\mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}$ is parametrized by

$$\bigsqcup_{i=1,\dots,r_0} q_i\Gamma_i/\mathrm{U}_{\mathscr{I}_0}\cap\Gamma_i.$$

To count these lifts, we consider the function $\mathrm{ht} : \mathrm{G}/G_{\mathrm{hor}} \to \mathbb{R}$ defined as

$$\mathrm{ht}([g]_{G_{\mathrm{hor}}}) := \mathrm{dist}(g\mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}, [id]_{\mathrm{K}}).$$

With respect to this function, define $B_R \subset \mathrm{G}/G_{\mathrm{hor}}$ as

$$B_R := \{[g]_{G_{\mathrm{hor}}} \mid \mathrm{ht}([g]) \leq R\}.$$

We will actually use $\widetilde{B}_R$, the preimage of B_R in $\mathrm{G}/G^{\circ}_{\mathrm{hor}}$, later.

For each $i = 1, ..., r_0$ and $R > 0$, consider the function $\varphi^i_R : \mathrm{G}/\Gamma_i \to \mathbb{R}$ defined by

$$\begin{aligned}\varphi^i_R([g]_\Gamma) :=& \frac{1}{C^i_R} \sum_{\gamma \in \Gamma_i/\mathrm{U}_{\mathscr{I}_0}\cap\Gamma_i} 1_{B_R}([g\gamma]_{G_{\mathrm{hor}}})\\ =& \frac{1}{C^i_R} \sum_{\gamma \in \Gamma_i/\mathrm{U}_{\mathscr{I}_0}\cap\Gamma_i} 1_{\widetilde{B}_R}([g\gamma]_{G^{\circ}_{\mathrm{hor}}})\end{aligned}$$

for certain sequence (C^i_R) to be defined soon (see Equa.(2)).

From the definition, if as R tends to infinity, $\varphi^i_R([q_i])$ converges to 1 for each i then we could conclude that

$$\#\left\{\gamma \mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K} \mid \mathrm{dist}(\gamma \mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}, [id]_{\mathrm{K}}) \leq R\right\} \sim \sum_{i=1,...,r_0} C^i_R.$$

For simplicity write

$$\mu_{\mathrm{A}} := \prod_{i<j, i\sim_{\mathscr{I}_0} j} \frac{\lambda_{ij}(a) - \lambda_{ji}(a)}{2} \lambda_{\mathscr{I}_0}(b) \cdot \mathrm{m}_{\mathrm{A}^+_{\mathrm{M}_{\mathscr{I}_0}}}(a) \otimes \mathrm{m}_{\mathrm{A}_{\mathscr{I}_0}}(b).$$

Recall for $a = \mathrm{diag}(a_1, ..., a_N)$, $\lambda_{ij}(a) := a_i/a_j$ and $\lambda_{\mathscr{I}_0}(a) := \prod_{i<j, i\not\sim_{\mathscr{I}_0} j} \lambda_{ij}(a)$. By abuse of notation we think of μ_A also as a measure on $\mathfrak{a}^+_{\mathrm{M}_{\mathscr{I}_0}} \oplus \mathfrak{a}_{\mathscr{I}_0}$ where $\mathfrak{a}^+_{\mathrm{M}_{\mathscr{I}_0}} := \log(\mathrm{A}^+_{\mathrm{M}_{\mathscr{I}_0}})$.

Lemma 4.3. *For*

$$C_7 := \mathrm{Vol}(\mathrm{K}) \cdot 2^{-\frac{\sum_{i<j}|I_i||I_j|}{2}}$$

the surjective map

$$\Phi_7 : \mathrm{K} \times \mathrm{A}^+_{\mathrm{M}_{\mathscr{I}_0}} \times \mathrm{A}_{\mathscr{I}_0} \to \mathrm{G}/\mathrm{G}^{\circ}_{\mathrm{hor}}$$

defined by group multiplication induces

$$(\Phi_7)_* \left(C_7 \cdot \widehat{\mathrm{m}}_{\mathrm{K}} \otimes \mu_{\mathrm{A}}\right) = \mathrm{m}_{\mathrm{G}/\mathrm{G}^{\circ}_{\mathrm{hor}}}.$$

See 4.4.3 for the proof and the definition of $\mathrm{m}_{\mathrm{G}/G^{\circ}_{\mathrm{hor}}}$. For $R > 0$, write

$$\begin{aligned}B_{\mathfrak{a}}(R) &:= \left\{(x, y) \in \mathfrak{a}^+_{\mathrm{M}_{\mathscr{I}_0}} \oplus \mathfrak{a}_{\mathscr{I}_0} \,\middle|\, \|x + y\| \leq R\right\},\\ B^+_{\mathfrak{a}}(R) &:= \left\{(x, y) \in \mathfrak{a}^+_{\mathrm{M}_{\mathscr{I}_0}} \oplus \mathfrak{a}^+_{\mathscr{I}_0} \,\middle|\, \|x + y\| \leq R\right\}.\end{aligned}$$

Here

$$\mathfrak{a}^+_{\mathscr{I}_0} := \left\{\mathbf{a} \in \mathfrak{a}_{\mathscr{I}_0} \,\middle|\, \sum_{i=1}^{k} a_i|I_i| \geq 0,\ \forall k = 1, ..., k_0 - 1;\ \sum_{i=1}^{k_0} a_i|I_i| = 0\right\}$$

where we have written

$$\mathbf{a} = \mathrm{diag}(a_1 id_{|I_1|}, ..., a_{k_0} id_{|I_{k_0}|}) \in \mathfrak{a}_{\mathscr{I}_0}.$$

Lemma 4.4. *The height ball pulls back to*

$$(\Phi_7)^{-1}(\widetilde{B}_R) = \mathrm{K} \times \exp(B_{\mathfrak{a}}(R)).$$

See 4.4.4 for the proof. Now we define

$$C_R^i := C_7 \mu_{\mathrm{A}} \left(B_{\mathfrak{a}}^+(R) \right) \frac{\mathrm{Vol}(G_{\mathrm{hor}}^{\circ}/\mathrm{U}_{\mathscr{I}_0} \cap \Gamma_i)}{\mathrm{Vol}(\mathrm{G}/\Gamma_i)}. \tag{2}$$

Let $\mathbf{v}_0^+$ be the sum of positive roots. Via the trace form, it is identified with $\mathbf{v}_0$ in the proof of Lemma below (see 4.4.5).

Lemma 4.5. *As R tends to $+\infty$, we have*

$$C_R^i \sim \mathrm{Vol}(\mathrm{K}) \left(\frac{1}{2}\right)^{\frac{N(N-1)}{2}} \left(\frac{2\pi R}{\|\mathbf{v}_0^+\|}\right)^{\frac{N-2}{2}} e^{\|\mathbf{v}_0^+\| R} \cdot \frac{\mathrm{Vol}(G_{\mathrm{hor}}^{\circ}/\mathrm{U}_{\mathscr{I}_0} \cap \Gamma_i)}{\mathrm{Vol}(\mathrm{G}/\Gamma_i)}.$$

See 4.4.5 for the proof. Now we can state the counting theorem

Theorem 4.6. *The asymptotic count of lifts of a closed horocycle is given by*

$$\begin{aligned} &\# \left\{ \gamma \mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K} \mid \mathrm{dist}(\gamma \mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}, [id]_{\mathrm{K}}) \le R,\ \gamma \in \Gamma \right\} \\ &\sim \left(\frac{1}{2}\right)^{\frac{N(N-1)}{2}} \left(\frac{2\pi R}{\|\mathbf{v}_0^+\|}\right)^{\frac{N-2}{2}} e^{\|\mathbf{v}_0^+\| R} \cdot \frac{\mathrm{Vol}(G_{\mathrm{hor}}/G_{\mathrm{hor}} \cap \Gamma)/(2^{k_0}-1)}{\mathrm{Vol}(\mathrm{K}\backslash \mathrm{G}/\Gamma)}. \end{aligned}$$

And if Γ is neat, then it can be rewritten as

$$\begin{aligned} &\# \left\{ \gamma \mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K} \mid \mathrm{dist}(\gamma \mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}, [id]_{\mathrm{K}}) \le R,\ \gamma \in \Gamma \right\} \\ &\sim \left(\frac{1}{2}\right)^{\frac{N(N-1)}{2}} \left(\frac{2\pi R}{\|\mathbf{v}_0^+\|}\right)^{\frac{N-2}{2}} e^{\|\mathbf{v}_0^+\| R} \cdot \frac{\mathrm{Vol}(\mathrm{U}_{\mathscr{I}_0}/\mathrm{U}_{\mathscr{I}_0} \cap \Gamma)\, \mathrm{Vol}(\mathrm{K}_{\mathscr{I}_0})}{\mathrm{Vol}(\mathrm{K}\backslash \mathrm{G}/\Gamma)}. \end{aligned}$$

Proof assuming Lemma 4.9 and Proposition 4.10 . Indeed it follows from Lemma 4.9 and Proposition 4.10 in the next section that

$$\# \left\{ \gamma \mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K} \mid \mathrm{dist}(\gamma \mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}, [id]_{\mathrm{K}}) \le R,\ \gamma \in \Gamma \right\} \sim \sum_{i=1,\dots,r_0} C_R^i.$$

It remains to sum C_R^i's together. Call $C_{R,0}$ the terms in Lemma 4.5 that are independent of i, i.e.

$$C_R^i = C_{R,0} \cdot \frac{\mathrm{Vol}(G_{\mathrm{hor}}^{\circ}/\mathrm{U}_{\mathscr{I}_0} \cap \Gamma_i)}{\mathrm{Vol}(\mathrm{G}/\Gamma_i)}.$$

Recall $\Gamma_i = q_i^{-1}\Gamma' q_i$ for some $q_i \in \Gamma$ such that

$$\Gamma = \bigsqcup_{i=1}^{r_0} \Gamma' q_i (G_{\mathrm{hor}} \cap \Gamma).$$

Regarding this as a decomposition of disjoint $G_{\mathrm{hor}} \cap \Gamma$ -orbits on $\Gamma'\backslash\Gamma$, we have for every $j = 1, \dots, r_0$,

$$[\Gamma : \Gamma_j] = [\Gamma : \Gamma'] = \sum_{i=1}^{r_0} [G_{\mathrm{hor}} \cap \Gamma : G_{\mathrm{hor}} \cap \Gamma_i] = \sum_{i=1}^{r_0} [G_{\mathrm{hor}} \cap \Gamma : \mathrm{U}_{\mathscr{I}_0} \cap \Gamma_i].$$

Let π_0 be the set of connected components, then by Lemma 4.2

$$\begin{aligned}\sum_{i=1}^{r_0} C_R^i =& \sum_{i=1}^{r_0} C_{R,0}\cdot\frac{\mathrm{Vol}(G_{\mathrm{hor}}/\mathrm{U}_{\mathscr{I}_0}\cap\Gamma_i)}{|\pi_0(G_{\mathrm{hor}})|\,\mathrm{Vol}(\mathrm{G}/\Gamma_i)}\\ =& \sum_{i=1}^{r_0} C_{R,0}\cdot\frac{\mathrm{Vol}(G_{\mathrm{hor}}/G_{\mathrm{hor}}\cap\Gamma)[G_{\mathrm{hor}}\cap\Gamma:\mathrm{U}_{\mathscr{I}_0}\cap\Gamma_i]}{|\pi_0(G_{\mathrm{hor}})|\,\mathrm{Vol}(\mathrm{G}/\Gamma)[\Gamma:\Gamma_i]}\\ =& C_{R,0}\cdot\frac{\mathrm{Vol}(G_{\mathrm{hor}}/G_{\mathrm{hor}}\cap\Gamma)}{|\pi_0(G_{\mathrm{hor}})|\,\mathrm{Vol}(\mathrm{G}/\Gamma)}.\end{aligned}$$

In our case, $\pi_0(G_{\mathrm{hor}})=\pi_0(N_{\mathrm{G}}\mathrm{U}_{\mathscr{I}_0}\cap\mathrm{K})$ and the latter of which can be identified as

$$\begin{aligned}&S(O_{|I_1|}(\mathbb{R})\times\ldots\times O_{|I_{k_0}|}(\mathbb{R}))\\ &:=\left\{(M_1,\ldots,M_{k_0})\in O_{|I_1|}(\mathbb{R})\times\ldots\times O_{|I_{k_0}|}(\mathbb{R})\;\middle|\;\prod_{i=1}^{k_0}\det(M_i)=1\right\}.\end{aligned}$$

Hence $|\pi_0(G_{\mathrm{hor}})|=2^{k_0}-1$. ◻

Note that (see 4.4.5 for the explicit expression of $\mathbf{v}_0$)

$$\left\|\mathbf{v}_0^+\right\|=\|\mathbf{v}_0\|=\sqrt{\sum_{i=1}^{N}(N-2i+1)^2}.$$

admits an explicit expression. Call this P_N for simplicity.

Lemma 4.7. *For $N\in\mathbb{Z}_{>0}$, we have the identity*

$$P_N^2=\sum_{i=1}^{N}(N-2i+1)^2=\frac{1}{3}N(N-1)(N+1).$$

Proof. By induction, for $k\in\mathbb{Z}_{>0}$,

$$\sum_{l=1}^{k} l^2=\frac{1}{6}k(k+1)(2k+1).$$

When $N=2n$ is even,

$$\begin{aligned}&\sum_{i=1}^{N}(N-2i+1)^2=2\sum_{i=1}^{n}(2i-1)^2=2\left(-\sum_{i=1}^{n}(2i)^2+\sum_{i=1}^{2n}i^2\right)\\ &=\frac{2}{6}\left(-4n(n+1)(2n+1)+2n(2n+1)(4n+1)\right)\\ &=\frac{1}{3}2n\left(-(2n+2)(2n+1)+(2n+1)(4n+1)\right)=\frac{1}{3}2n(2n+1)(2n-1).\end{aligned}$$

When $N=2n+1$ is odd,

$$\sum_{i=1}^{N}(N-2i+1)^2=2\sum_{i=1}^{n}(2i)^2=\frac{8}{6}n(n+1)(2n+1)=\frac{1}{3}2n(2n+2)(2n+1).$$

So the proof completes. ◻

In the case $\Gamma=\mathrm{SL}_N(\mathbb{Z})$, the rest of the expressions can also be worked out explicitly. Let $n_k:=|I_k|$.

Corollary 4.8. *We have*

$$\begin{aligned}&\#\left\{\gamma \mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K} \mid \mathrm{dist}(\gamma \mathrm{U}_{\mathscr{I}_0}\mathrm{K}/\mathrm{K}, [id]_{\mathrm{K}}) \leq R,\ \gamma \in \mathrm{SL}_N(\mathbb{Z})\right\}\\ \sim &\left(\frac{1}{2}\right)^{\frac{N(N-1)}{2}} \left(\frac{2\pi R}{P_N}\right)^{\frac{N-2}{2}} e^{P_N R} \frac{1}{2^{k_0}-1} \prod_{k=1}^{k_0} \frac{\mathrm{Vol}(\mathrm{SO}_{n_k}(\mathbb{R}))}{n_k! 2^{n_k-1}} \frac{\mathrm{Vol}(\mathrm{SO}_N(\mathbb{R}))}{\mathrm{Vol}(\mathrm{SL}_N(\mathbb{R})/\mathrm{SL}_N(\mathbb{Z}))}.\end{aligned}$$

Here P_N is as in Lemma 4.7, $\mathrm{Vol}(\mathrm{SO}_n(\mathbb{R}))$ is as in Lemma 4.15, and

$$\mathrm{Vol}(\mathrm{SL}_N(\mathbb{R})/\mathrm{SL}_N(\mathbb{Z})) = \zeta(2)\zeta(3) \cdot \ldots \cdot \zeta(N)$$

where ζ is the Riemann zeta function.

If one employs the completed Riemann zeta function

$$\xi(s) := \frac{1}{2}s(1-s)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s),$$

one can write

$$\frac{\mathrm{Vol}(\mathrm{SO}_N(\mathbb{R}))}{\mathrm{Vol}(\mathrm{SL}_N(\mathbb{R})/\mathrm{SL}_N(\mathbb{Z}))} = \frac{2^{\frac{N(N-1)}{4}} N!(N-1)!}{\xi(2)\cdot\xi(3)\cdot\ldots\cdot\xi(N)}.$$

Proof. Note that $\mathrm{SL}_N(\mathbb{Z})$ intersect nontrivially with every connected component of $N_{\mathrm{K}}(\mathrm{U}_{\mathscr{I}_0})$, thus

$$\mathrm{Vol}(G_{\mathrm{hor}}/G_{\mathrm{hor}} \cap \mathrm{SL}_N(\mathbb{Z})) = \mathrm{Vol}(G^{\circ}_{\mathrm{hor}}/G^{\circ}_{\mathrm{hor}} \cap \mathrm{SL}_N(\mathbb{Z})).$$

Also one can check that

$$G^{\circ}_{\mathrm{hor}} \cap \mathrm{SL}_N(\mathbb{Z})) = (\mathrm{K}_{\mathscr{I}_0} \cap \mathrm{SL}_N(\mathbb{Z})) \ltimes (\mathrm{U}_{\mathscr{I}_0} \cap \mathrm{SL}_N(\mathbb{Z}))$$

and

$$\#\mathrm{K}_{\mathscr{I}_0} \cap \mathrm{SL}_N(\mathbb{Z}) = \prod_{k=1}^{k_0} \#\,\mathrm{SO}_{n_k}(\mathbb{Z}) = \prod_{k=1}^{k_0} n_k! 2^{n_k-1}.$$

Thus

$$\mathrm{Vol}(G^{\circ}_{\mathrm{hor}}/G^{\circ}_{\mathrm{hor}} \cap \mathrm{SL}_N(\mathbb{Z})) = \frac{\mathrm{Vol}(\mathrm{K}_{\mathscr{I}_0})}{\prod_{k=1}^{k_0} n_k! \cdot 2^{n_k-1}} \cdot \mathrm{Vol}(\mathrm{U}_{\mathscr{I}_0}/\mathrm{U}_{\mathscr{I}_0} \cap \mathrm{SL}_N(\mathbb{Z}))$$

As $(E_{ij},\ i<j,\ i \not\sim_{\mathscr{I}_0} j)$ forms an ortho-normal basis of $\mathfrak{u}_{\mathscr{I}_0}$,

$$\mathrm{U}_{\mathscr{I}_0}/\mathrm{U}_{\mathscr{I}_0} \cap \mathrm{SL}_N(\mathbb{Z}) \xrightarrow[\cong]{\text{isometry}} \mathbb{R}^{\dim \mathrm{U}_{\mathscr{I}_0}}/\mathbb{Z}^{\dim \mathrm{U}_{\mathscr{I}_0}}.$$

So it has volume one and

$$\mathrm{Vol}(G^{\circ}_{\mathrm{hor}}/G^{\circ}_{\mathrm{hor}} \cap \mathrm{SL}_N(\mathbb{Z})) = \frac{\mathrm{Vol}(\mathrm{K}_{\mathscr{I}_0})}{\prod_{k=1}^{k_0} n_k! 2^{n_k-1}} = \frac{\prod_{k=1}^{k_0} \mathrm{Vol}(\mathrm{SO}_{n_k}(\mathbb{R}))}{\prod_{k=1}^{k_0} n_k! 2^{n_k-1}}.$$

Finally the identity

$$\mathrm{Vol}(\mathrm{SL}_N(\mathbb{R})/\mathrm{SL}_N(\mathbb{Z})) = \zeta(2)\zeta(3) \cdot \ldots \cdot \zeta(N)$$

is classical. One may see [Lan66], noting that the standard *Chevalley basis* (see [Hum72, 25.2]) forms an ortho-normal basis under the trace form and that SL_N is a simply connected algebraic group. $\square$

Proof of Example 1(1) and 2(1). For these two cases, $N = 3$.

$$P_3^2 = \frac{1}{3}3(3-1)(3+1) = 8.$$

The asymptotic in Coro.4.8 excluding the terms concerning $\mathscr{I}_0$ is

$$R^{\frac{1}{2}} e^{2\sqrt{2}R} \cdot 2^{-3(3-1)/2} \cdot \pi^{\frac{1}{2}} 2^{-\frac{1}{4}} \cdot \frac{2^{\frac{3(3-1)}{4}} 3!(3-1)!}{\xi(2)\xi(3)} = R^{\frac{1}{2}} e^{2\sqrt{2}R} \frac{\pi^{\frac{1}{2}} \cdot 3 \cdot 2^{\frac{1}{4}}}{\xi(2)\xi(3)}.$$

Now we calculate the term depending on $\mathscr{I}_0$. For example 1, $\mathscr{I}_0 = \{\{1\},\{2\},\{3\}\}$. Thus $k_0 = 3$ and all n_k's are 1 and the term is just $1/(2^3-1) = 1/7$. For example 2, $\mathscr{I}_0 = \{\{1,2\},\{3\}\}$. Thus $k_0 = 2$ and $n_1 = 2$, $n_2 = 1$. So the term is

$$\frac{1}{2^2-1} \cdot \frac{2\sqrt{2}\pi}{2^2} = \frac{\pi}{3 \cdot 2^{\frac{1}{2}}}.$$

□

4.2. **Completion of the proof.** As usual the pointwise convergence is deduced from a weak convergence plus the following

Lemma 4.9. *The family of sets $(B_R)_{R\geq 1}$ is well-rounded in the sense of* [EM93, Proposition 1.3].

For the precise definition of well-roundedness and the proof, see 4.4.6.

The weak convergence is the following

Proposition 4.10. *For every compactly supported function ψ on G/Γ_i,*

$$\lim_{R\to+\infty} \int \varphi_R^i(x)\psi(x)\mathrm{m}_{\mathrm{G}/\Gamma_i}(x) = \int \psi(x)\mathrm{m}_{\mathrm{G}/\Gamma_i}(x). \tag{3}$$

The rest of this section is devoted to the proof of this proposition.

For $R, \varepsilon > 0$, define

$$B_{\mathfrak{a}}^+(R,\varepsilon) := B_{\mathfrak{a}}^+(R) \setminus B_{\mathfrak{a}}^+(\varepsilon R).$$

Lemma 4.11. *Let Λ be a lattice commensurable with $\mathrm{SL}_N(\mathbb{Z})$. For every $\delta > 0$, every sequence (R_n) tending to infinity and (g_n) with $g_n \in \mathrm{K} \times \exp(B_{\mathfrak{a}}^+(R_n,\delta))$,*

$$\lim_{n\to\infty} (g_n)_* \widehat{\nu}_{[\mathscr{I}_0]} = \widehat{\mathrm{m}}_{\mathrm{G}/\Lambda}.$$

Proof. Follows directly from Theorem 2.1. □

In a different vein, for $C \in \mathbb{R}$, define

$$\mathcal{C}_C := \left\{ \mathbf{a},\ a_i - a_j \geq \max\{0, C\},\ \forall k,\ i<j \in I_k;\ \sum_{i\in I_1\cup\ldots\cup I_k} a_i \geq C,\ \forall 1 \leq k < k_0 \right\}$$

where $\mathbf{a} = \mathrm{diag}(a_1,...,a_N)$ for an element $\mathbf{a} \in \mathfrak{a}^+_{\mathrm{M}_{\mathscr{I}_0}} \oplus \mathfrak{a}_{\mathscr{I}_0}$. Thus $\mathcal{C}_0$ is nothing but $\mathfrak{a}^+_{\mathrm{M}_{\mathscr{I}_0}} \oplus \mathfrak{a}^+_{\mathscr{I}_0}$.

Lemma 4.12. *For every i and every compact set B of G/Γ_i, there exists $C = C_B^i < 0$ such that if*

$$(k,a,b) \notin \mathrm{K} \times \exp(\mathcal{C}_C)$$

then

$$[kabG_{\mathrm{hor}}]_{\Gamma_i} \cap B = \emptyset.$$

See 4.4.7 for a sketch of proof. Define for $C \in \mathbb{R}$ and $R > 0$,

$$B_{\mathfrak{a}}^{C,+}(R) := B_{\mathfrak{a}}(R) \cap \mathcal{C}_C.$$

Asymptotically the measures of $B_{\mathfrak{a}}^{C,+}(R)$, $B_{\mathfrak{a}}^{+}(R)$, $B_{\mathfrak{a}}^{+}(R,\varepsilon)$ are not so different from each other.

Lemma 4.13. *For every $C < 0$,*

$$\lim_{R\to+\infty} \frac{\mu_{\mathrm{A}}\left(B_{\mathfrak{a}}^{C,+}(R)\right)}{\mu_{\mathrm{A}}\left(B_{\mathfrak{a}}^{+}(R)\right)} = 1.$$

For every $\varepsilon \in (0,1)$, there exists $\delta_\varepsilon > 0$ such that

$$\limsup_{R\to+\infty} \frac{\mu_{\mathrm{A}}\left(B_{\mathfrak{a}}^{+}(R)\right)}{\mu_{\mathrm{A}}\left(B_{\mathfrak{a}}^{+}(R,\delta_\varepsilon)\right)} \leq \max\{1-\varepsilon, \frac{1}{1+\varepsilon}\}.$$

See 4.4.8 for the proof. Now fix ψ, a nonzero compactly supported continuous function on G/Γ_i. To establish Equa.(3), it suffices to show that for every $\varepsilon > 0$,

$$\begin{aligned}
&\liminf_{R\to+\infty} \int \varphi_R^i(x)\psi(x)\mathrm{m}_{\mathrm{G}/\Gamma_i}(x) \geq (1-\varepsilon)\int \psi(x)\mathrm{m}_{\mathrm{G}/\Gamma_i}(x) - \varepsilon \\
&\limsup_{R\to+\infty} \int \varphi_R^i(x)\psi(x)\mathrm{m}_{\mathrm{G}/\Gamma_i}(x) \leq (1+\varepsilon)\int \psi(x)\mathrm{m}_{\mathrm{G}/\Gamma_i}(x) + \varepsilon.
\end{aligned} \tag{4}$$

Let $\|\psi\|_{\sup} := \sup\{|\psi(x)|,\, x \in \mathrm{G}/\Gamma_i\}$. Choose

$$\delta := \delta_{\varepsilon'} \text{ with } \varepsilon' := \frac{\varepsilon}{\|\psi\|_{\sup}\mathrm{Vol}(\mathrm{G}/\Gamma_i)}$$

according to Lemma 4.13. Choose $C := C_B^i$ with $B := \mathrm{supp}(\psi)$ according to Lemma 4.12. For simplicity write $\Gamma_U := \mathrm{U}_{\mathscr{I}_0} \cap \Gamma_i$. Also define the map p, q by natural projections:

$$\begin{tikzcd}
 & \mathrm{G}/\Gamma_U \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathrm{G}/\Gamma_i & & \mathrm{G}/G_{\mathrm{hor}}^{\circ}.
\end{tikzcd}$$

Now we can start the proof. Let LHS be the left hand side of Equa.(3). For each $R > 0$,

$$\begin{aligned}
\mathrm{LHS} =& \frac{1}{C_R^i}\int 1_{\widetilde{B}_R}([g]_{G_{\mathrm{hor}}^{\circ}})\psi\circ p([g]_{\Gamma_U})\mathrm{m}_{[\mathrm{G}]_{\Gamma_U}}([g]) \\
=& \frac{1}{C_R^i}\int_{\widetilde{B}_R}\left(\psi\circ p(g[h]_{\Gamma_U})\mathrm{m}_{[G_{\mathrm{hor}}^{\circ}]_{\Gamma_U}}([h])\right)\mathrm{m}_{\mathrm{G}/G_{\mathrm{hor}}^{\circ}}([g]) \\
=& \frac{1}{C_R^i}\langle\psi, \int_{\widetilde{B}_R}\left(g_*\mathrm{m}_{[\mathrm{G}_{\mathrm{hor}}^{\circ}]_{\Gamma_i}}\right)\mathrm{m}_{\mathrm{G}/G_{\mathrm{hor}}^{\circ}}\rangle \\
=& \frac{1}{C_R^i}\langle\psi, \int_{\mathrm{K}\times\exp(B_{\mathfrak{a}}(R))}\left((ka)_*\mathrm{m}_{[\mathrm{G}_{\mathrm{hor}}^{\circ}]_{\Gamma_i}}\right)C_7\widehat{\mathrm{m}}_{\mathrm{K}}(k)\otimes\mu_A(a)\rangle \\
=& \frac{1}{C_R^i}\langle\psi, \int_{\mathrm{K}\times\exp(B_{\mathfrak{a}}^{C,+}(R))}\left((ka)_*\mathrm{m}_{[\mathrm{G}_{\mathrm{hor}}^{\circ}]_{\Gamma_i}}\right)C_7\widehat{\mathrm{m}}_{\mathrm{K}}(k)\otimes\mu_A(a)\rangle.
\end{aligned}$$

Lemma 4.12 is applied in the last step.

Now take the limit. We will treat the case of lim sup only. The other one lim inf is similar. By applying Lemma 4.13,

$$\begin{aligned}
&\limsup_{R\to\infty} \frac{1}{C^i_R}\langle \psi, \int_{\mathrm{K}\times\exp(B^{C,+}_{\mathfrak{a}}(R))} \left((ka)_*\mathrm{m}_{[\mathrm{G}^\circ_{\mathrm{hor}}]_{\Gamma_i}}\right) C_7 \widehat{\mathrm{m}}_{\mathrm{K}}(k)\otimes\mu_A(a)\rangle \\
\leq &\limsup_{R\to\infty} \frac{1}{C^i_R}\langle \psi, \int_{\mathrm{K}\times\exp(B^{+}_{\mathfrak{a}}(R,\delta))} \left((ka)_*\mathrm{m}_{[\mathrm{G}^\circ_{\mathrm{hor}}]_{\Gamma_i}}\right) C_7 \widehat{\mathrm{m}}_{\mathrm{K}}(k)\otimes\mu_A(a)\rangle + \varepsilon \\
\leq &\limsup_{R\to\infty} \langle \psi, \frac{\mathrm{Vol}(\mathrm{G}/\Gamma_i)}{\mu_A(B^{+}_{\mathfrak{a}}(R,\delta))} \int_{\mathrm{K}\times\exp(B^{+}_{\mathfrak{a}}(R,\delta))} \left((ka)_*\widehat{\mathrm{m}}_{[\mathrm{G}^\circ_{\mathrm{hor}}]_{\Gamma_i}}\right) \widehat{\mathrm{m}}_{\mathrm{K}}(k)\otimes\mu_A(a)\rangle + \varepsilon \\
= &\langle \psi, \mathrm{m}_{\mathrm{G}/\Gamma_i}\rangle + \varepsilon.
\end{aligned}$$

Now the proof is complete.

4.3. **Decomposition of Haar measures.** In this section we collect some more-or-less standard facts on expressions of Haar measures in various coordinates. Some arguments are provided when we fail to identify a precise reference. General references include [Hel00, Ch.I, Sec.5] and [Kna02, Ch.VIII]

Recall that the measure m_H for a closed subgroup H of G refers to the measure induced from the induced Riemannian metric, which is induced from the trace form. And for a finite measure μ, define a probability measure $\widehat{\mu} := \mu/|\mu|$.

We have the bijection

$$\Phi_4 : (\mathrm{KM}_{\mathscr{I}_0}) \times \mathrm{A}_{\mathscr{I}_0} \times \mathrm{U}_{\mathscr{I}_0} \cong \mathrm{G},$$

and the surjection

$$\Phi_5 : \mathrm{K} \times \mathrm{M}_{\mathscr{I}_0} \times \mathrm{A}_{\mathscr{I}_0} \times \mathrm{U}_{\mathscr{I}_0} \twoheadrightarrow (\mathrm{KM}_{\mathscr{I}_0}) \times \mathrm{A}_{\mathscr{I}_0} \times \mathrm{U}_{\mathscr{I}_0}.$$

By [Kna02, Proposition 8.44], there exists a constant $C_4 > 0$ such that

$$(\Phi_4\circ\Phi_5)_*\left(C_4\cdot \widehat{\mathrm{m}}_{\mathrm{K}} \otimes \mathrm{m}_{\mathrm{M}_{\mathscr{I}_0}} \otimes \lambda_{\mathscr{I}_0}(a)\mathrm{m}_{\mathrm{A}_{\mathscr{I}_0}}(a)\otimes \mathrm{m}_{\mathrm{U}_{\mathscr{I}_0}}\right) = \mathrm{m}_{\mathrm{G}}. \tag{5}$$

Now we seek to determine the constant C_4 (see Lemma 4.16 below).

On $\mathrm{KM}_{\mathscr{I}_0}$, there are two natural measures. One is obtained from the push forward of $\mathrm{m}_{\mathrm{K}}\otimes\mathrm{m}_{\mathrm{M}_{\mathscr{I}_0}}$ called μ. The other one is to regard $\mathrm{KM}_{\mathscr{I}_0}$ as a closed submanifold of G and obtain one from the induced Riemannian metric called ν.

Lemma 4.14. *Notations as above,*

$$\mu = \mathrm{Vol}(\mathrm{K}_{\mathscr{I}_0})\nu. \tag{6}$$

Before the proof, recall some general constructions on principal bundles (see [Tu17, 27.1]). Let K_0 be a compact connected Lie group equipped with a top-degree nonzero K_0-invariant differential form ω_{K_0}. Let

$$\begin{tikzcd} E \arrow[loop right, "K_0"] \arrow[d, "\pi"] \\ B \end{tikzcd}$$

be a principal K_0-bundle where K_0 acts from the right. Take a local trivialization $\{(U_i, \phi_{U_i})\}$,

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\cong", "\phi_{U_i}"'] \arrow[d, "\pi"] & U_i \times K_0 \arrow[dl, "p_i"] \\ U_i. \end{tikzcd}$$

For an index i and $u \in U_i$, define $\iota_{i,u}(k) := (u,k)$ from K_0 to $U_i \times K_0$. And let $q_i : U_i \times K_0 \to K_0$ be the natural projection. Thus the above diagram can be completed as

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\cong", "\phi_{U_i}"'] \arrow[d, "\pi"] & U_i \times K_0 \arrow[dl, "p_i"] \arrow[d, "q_i"] & K_0 \arrow[l, "\iota_{i,u}"'] \\
U_i & K_0 &
\end{tikzcd}.$$

Note that transition maps between different charts take the form

$$\begin{tikzcd}
 & \pi_{-1}(U_i \cap U_j) \arrow[dl, "\phi_{U_i}"'] \arrow[dr, "\phi_{U_j}"] & \\
(U_i \cap U_j) \times K_0 \arrow[rr, "(u,k)\mapsto(u,k_{ji}(u)k)", "\phi_{ji}"'] & & (U_i \cap U_j) \times K_0
\end{tikzcd}$$

for some $k_{ji}(u) \in K_0$ depending on i, j, u.

Define a (in general incompatible) system of degree $k_0 := \dim K_0$ differential forms $(\pi^{-1}(U_i), \omega_i^{\perp})$ by

$$\omega_i^{\perp} := \phi_{U_i}^*(q_i^* \omega_{K_0}).$$

One can check that (for every i and $u \in U_i$)

$$\iota_{i,u}^*(q_i^* \omega_{K_0}) = \omega_{K_0}. \tag{7}$$

Let ω_B be a top-degree differential form on the base B. One can check that the system of differential forms

$$(\pi^{-1}(U_i), \pi^* \omega_B \wedge \omega_i^{\perp})$$

is compatible and hence glue together to a global top-degree differential form $\widetilde{\omega}_B$ on E. Moreover, for any other system of differential forms $(\pi^{-1}(U_i), \phi_{U_i}^* \eta_i)$, if

$$(\pi^{-1}(U_i), \pi^* \omega_B \wedge \phi_{U_i}^* \eta_i)$$

is compatible and if $\iota_{i,u}^*(\eta_i) = \omega_{K_0}$ for all indices i and $u \in U_i$, then for all i,

$$\pi^* \omega_B \wedge \omega_i^{\perp} = \pi^* \omega_B \wedge \phi_{U_i}^* \eta_i. \tag{8}$$

Let ω'_B be a top-degree differential form of compact support on B and define $\widetilde{\omega}'_B$ in the same way as $\widetilde{\omega}_B$ above. Using local coordinates and partition of unity, one can check that

$$\int_E \widetilde{\omega}'_B = \int_B \omega'_B \cdot \int_{K_0} \omega_{K_0}. \tag{9}$$

Now suppose that both the total space and the base admit an action (by diffeomorphisms) of a(n abstract) group H commuting with the action of K_0 and that π is H-equivariant. Then one can check, using Equa.(8), that

$$\omega_B \text{ is } H\text{-invariant} \implies \widetilde{\omega}_B \text{ is } H\text{-invariant}.$$

Proof. Take $B := \mathrm{KM}_{\mathscr{I}_0}$, $E := \mathrm{K} \times \mathrm{M}_{\mathscr{I}_0}$ and $\pi : E \to B$ be the group multiplication. Let $K_0 := \mathrm{K}_{\mathscr{I}_0}$ acting (from right) on E by

$$R_x \cdot (k, m) := (kx, x^{-1}m).$$

Then this gives a (trivial) principal $\mathrm{K}_{\mathscr{I}_0}$-bundle. Indeed, one has the following commutative diagrams

$$\begin{tikzcd} \mathrm{K}\times\mathrm{M}_{\mathscr{I}_0} \arrow[r, "f_1", "\cong"'] \arrow[d, "\pi"] & \mathrm{K}\times\mathrm{K}_{\mathscr{I}_0}\times\exp(\mathfrak{p}\cap\mathfrak{m}_{\mathscr{I}_0}) \arrow[d] \\ \mathrm{K}\mathrm{M}_{\mathscr{I}_0} \arrow[r, "f_2", "\cong"'] & \mathrm{K}\times\exp(\mathfrak{p}\cap\mathfrak{m}_{\mathscr{I}_0}) \end{tikzcd}$$

where $\mathfrak{p}$ is the (-1)-eigenspace in $\mathfrak{g}$ with respect to the Cartan involution associated with K. Also, in the above diagram, the right vertical arrow is the natural projection, the left vertical arrow and f_2^{-1} are just multiplication, and f_1^{-1} is defined by $(k,x,m)\mapsto(kx,x^{-1}m)$.

Let $H:=\mathrm{K}\times\mathrm{M}_{\mathscr{I}_0}$ act from left by

$$(a,b)\cdot(k,m):=(ak,mb^{-1}).$$

Now we take ω_B (resp. ω_E) be the H-invariant volume form on B (resp. E) inducing the measure ν (resp. μ). To prove the lemma, it suffice to show that for a compactly supported smooth function f on B,

$$\int_X(\pi^*f)\omega_E=\mathrm{Vol}(\mathrm{K}_{\mathscr{I}_0})\cdot\int_B f\omega_B.$$

By discussion above, $\widetilde{\omega}_B$ is H-invariant and hence there exists $C>0$ such that

$$\widetilde{\omega}_B=\pm C\cdot\omega_E.$$

To identify C, it suffices to compare their values at the identity.

Let $l_0:=\dim\mathrm{K}_{\mathscr{I}_0}$, $l_1:=\dim\mathrm{K}$ and $l_2:=\dim\mathfrak{p}\cap\mathfrak{m}_{\mathscr{I}_0}$. Let

$$\begin{aligned}&\mathrm{ONB}_{\mathfrak{k}_{\mathscr{I}_0}}:=\{v_1,...,v_{l_0}\}, \quad \mathrm{ONB}_{\mathfrak{k}}:=\{v_1,...,v_{l_0},...,v_{l_1}\},\\ &\mathrm{ONB}_{\mathfrak{p}\cap\mathfrak{m}_{\mathscr{I}_0}}:=\{w_1,...,w_{l_2}\}\end{aligned}$$

be ortho-normal bases for $\mathfrak{k}_{\mathscr{I}_0}$, $\mathfrak{k}$ and $\mathfrak{p}\cap\mathfrak{m}_{\mathscr{I}_0}$ respectively. Then it follows that

$$\begin{aligned}\mathscr{B}_1:=\Big(&\sqrt{2}^{-1}(v_1,-v_1),...,\sqrt{2}^{-1}(v_{l_0},-v_{l_0}),\sqrt{2}^{-1}(v_1,v_1),...,\\ &\sqrt{2}^{-1}(v_{l_0},v_{l_0}),(v_{l_0+1},0)...,(v_{l_1},0),(0,w_1),...,(0,w_{l_2})\Big)\end{aligned}$$

is a set of ortho-normal basis for $\mathfrak{k}\oplus\mathfrak{m}_{\mathscr{I}_0}$ and their projections to $\mathfrak{k}+\mathfrak{m}_{\mathscr{I}_0}\subset\mathfrak{g}$ under π_* become

$$\mathscr{B}_2:=\big(0,...,0,\sqrt{2}v_1,...,\sqrt{2}v_{l_0},v_{l_0+1},...,v_{l_1},w_1,...,w_{l_2}\big).$$

For $i=1,2$, let θ_i be obtained by wedging nonzero vectors from ONB_i together. Let

$$\theta_3:=\frac{1}{\sqrt{2}}v_1\wedge...\wedge\frac{1}{\sqrt{2}}v_{l_0}\in\wedge^{l_0}\mathfrak{k}_{\mathscr{I}_0},$$

whose push-forward to $\mathfrak{k}\oplus\mathfrak{m}$ under the differential at identity of $x\mapsto(x,x^{-1})$ gives the first l_0 vectors in $\mathscr{B}_1$.

By definition $\langle\omega_E,\theta_1\rangle=\pm1$. By tracing the definition of $\widetilde{\omega}_B$ and using Equa.(7), one shows that

$$\langle\widetilde{\omega}_B,\theta_1\rangle=\pm\langle\omega_B,\theta_2\rangle\cdot\langle\omega_{\mathrm{K}_{\mathscr{I}_0}},\theta_3\rangle=\pm1.$$

Thus $\widetilde{\omega}_B=\pm\omega_E$.

Therefore by Equa.(9)

$$\int_E \pi^* f\, \omega_E = \pm \int_E \pi^* f\, \widetilde{\omega}_B = \pm \int_B f \omega_B \cdot \mathrm{Vol}(\mathrm{K}_{\mathscr{I}_0})$$

and we are done. $\square$

These constructions also yield a formula for $\mathrm{Vol}(\mathrm{SO}_n(\mathbb{R}))$, making corollary 4.8 more explicit, which should be well-known. A proof is provided for the sake of completeness.

Lemma 4.15. *Let* $\mathrm{Vol}(\mathrm{SO}_n(\mathbb{R}))$ *be induced from the trace form. Then*

$$\mathrm{Vol}(\mathrm{SO}_n(\mathbb{R})) = \mathrm{Vol}(\mathrm{SO}_{n-1}(\mathbb{R})) \cdot \mathrm{Vol}(S^{n-1}) \cdot 2^{\frac{n-1}{2}}$$

where S^{n-1} *is the standard unit sphere on* $\mathbb{R}^n$ *and* $\mathrm{Vol}(S^{n-1})$ *is taken with respect to the standard Euclidean metric on* $\mathbb{R}^n$*. Combining with the formula for area of the unit sphere we get*

$$\mathrm{Vol}(\mathrm{SO}_n(\mathbb{R})) = 2^{\frac{n(n-1)}{4}} \prod_{k=2}^{n} \mathrm{Vol}(S^{k-1}) = 2^{\frac{n(n-1)}{4}} \prod_{k=2}^{n} \frac{2\pi^{k/2}}{\Gamma(\frac{k}{2})}.$$

Proof. It suffices to prove the first recursive formula. For simplicity we write $\mathrm{K}_n := \mathrm{SO}_n(\mathbb{R})$ and $\mathfrak{k}_n$ for its Lie algebra. Embed K_{n-1} in K_n at the lower right block. Let $(e_1, ..., e_n)$ be the standard basis for $\mathbb{R}^n$. Then K_{n-1} is precisely the stabilizer of e_1 in K_n. Then

$$\mathrm{ONB}_n := \left\{ \sqrt{2}^{-1}(E_{ij} - E_{ji}) \;\middle|\; 1 \leq i < j \leq n \right\}$$

forms an ortho-normal basis of $\mathfrak{k}_n$. Identify the tangent space of $\mathrm{K}_n/\mathrm{K}_{n-1}$ at the identity coset as $\mathfrak{k}_{n-1}^{\perp}$, the $\mathbb{R}$-subspace of $\mathfrak{k}_n$ spanned by

$$\mathrm{ONB}_{n-1}^{\perp} := \left\{ \sqrt{2}^{-1}(E_{1j} - E_{j1}) \;\middle|\; j = 2, ..., n \right\}$$

Let $\omega_{\mathrm{K}_n/\mathrm{K}_{n-1}}$ be the invariant volume form on $\mathrm{K}_n/\mathrm{K}_{n-1}$ whose value at identity coset is given by the dual of $\mathrm{ONB}_{n-1}^{\perp}$. Let $E := \mathrm{K}_n$, $B = \mathrm{K}_n/\mathrm{K}_{n-1}$ and $\pi : E \to B$ be the natural projection. Then they give a K_{n-1}-principal bundle over B. Also let $H := \mathrm{K}_n$ naturally act from the left. Let ω_{K_n} be the volume form deduced from the Trace form. So we have a lift $\widetilde{\omega}_{\mathrm{K}_n/\mathrm{K}_{n-1}}$ on E and

$$\int_E \widetilde{\omega}_{\mathrm{K}_n/\mathrm{K}_{n-1}} = \pm \int_B \omega_{\mathrm{K}_n/\mathrm{K}_{n-1}} \cdot \mathrm{Vol}(\mathrm{K}_{n-1}).$$

Since $\widetilde{\omega}_{\mathrm{K}_n/\mathrm{K}_{n-1}}$ is H-invariant, a computation at tangent space at identity shows that

$$\widetilde{\omega}_{\mathrm{K}_n/\mathrm{K}_{n-1}} = \omega_{\mathrm{K}_n}.$$

Thus

$$\mathrm{Vol}(\mathrm{K}_n) = \pm \mathrm{Vol}(\mathrm{K}_{n-1}) \cdot \int_B \omega_{\mathrm{K}_n/\mathrm{K}_{n-1}}.$$

Consider the diffeomorphism $\varphi : [k]_{\mathrm{K}_{n-1}} \to k \cdot e_1$ from $\mathrm{K}_n/\mathrm{K}_{n-1} \to S^{n-1}$. Then the measure induced from $\omega_{\mathrm{K}_n/\mathrm{K}_{n-1}}$ is pushed to a K_n-invariant measure on S^{n-1} and thus is proportional to the Vol measure on S^{n-1}. To see the proportional scalar, we compute the differential at the identity coset:

$$\mathrm{d}\, \varphi_{[id]_{\mathrm{K}_{n-2}}}(E_{1j} - E_{j1}) = (E_{1j} - E_{j1}) \cdot e_1 = -e_j.$$

Thus the Jacobian is $2^{-(n-1)/2}$ and

$$\int_B \omega_{\mathrm{K}_n/\mathrm{K}_{n-1}} = 2^{\frac{n-1}{2}} \operatorname{Vol}(S^{n-1}).$$

So we are done. □

Lemma 4.16. *Let C_4 be as in Equa.(5), then*

$$C_4 = \frac{\operatorname{Vol}(\mathrm{K})}{\operatorname{Vol}(\mathrm{K}_{\mathscr{I}_0})} \cdot 2^{\frac{\sum_{i<j}|I_i||I_j|}{2}}.$$

Proof. By Equa.(6) above,

$$\begin{aligned}&(\Phi_5)_*(\frac{\operatorname{Vol}(\mathrm{K})}{\operatorname{Vol}(\mathrm{K}_{\mathscr{I}_0})}\widehat{\mathrm{m}}_{\mathrm{K}} \otimes \mathrm{m}_{\mathrm{M}_{\mathscr{I}_0}} \otimes \lambda_{\mathscr{I}_0}(a)\mathrm{m}_{\mathrm{A}_{\mathscr{I}_0}}(a) \otimes \mathrm{m}_{\mathrm{U}_{\mathscr{I}_0}})\\ &= \nu \otimes \mathrm{m}_{\mathrm{M}_{\mathscr{I}_0}} \otimes \lambda_{\mathscr{I}_0}(a)\mathrm{m}_{\mathrm{A}_{\mathscr{I}_0}}(a) \otimes \mathrm{m}_{\mathrm{U}_{\mathscr{I}_0}}.\end{aligned}$$

It remains to consider the constant brought by $(\Phi_4)_*$, which is a computation of the Jacobian of the differential at the identity.

Let $E_{i,j}$ (or E_{ij} if no confusion might arise) be the matrix whose (i,j)-th entry is 1 and is equal to 0 elsewhere. Then

$$\operatorname{Tr}(E_{i,j}E_{s,t}) = \begin{cases} 1 & i=t,\, j=s; \\ 0 & \text{otherwise.}\end{cases}$$

Thus

$$\mathrm{ONB}_{\mathrm{U}_{\mathscr{I}_0}} := \{E_{i,j} \mid i<j,\, i \not\sim_{\mathscr{I}_0} j\}$$

forms an ortho-normal basis (to be abbreviated as ONB below) of $\mathfrak{u}_{\mathscr{I}_0}$. Also,

$$\mathcal{W} := \left\{ \frac{1}{\sqrt{2}}(E_{i,j}+E_{j,i}) \,\middle|\, i<j,\, i \not\sim_{\mathscr{I}_0} j \right\}$$

is a set of ONB for the subspace spanned by it. Fix $\mathrm{ONB}_{\mathscr{I}_0} := \{v_1, ..., v_l\}$, some set of ONB for $\mathfrak{k} + \mathfrak{m}_{\mathscr{I}_0} + \mathfrak{a}_{\mathscr{I}_0}$. Then $\mathrm{ONB}_{\mathscr{I}_0} \sqcup \mathcal{W}$ is a set of ONB for $\mathfrak{g}$ and $\mathrm{ONB}_{\mathscr{I}_0} \sqcup \mathrm{ONB}_{\mathrm{U}_{\mathscr{I}_0}}$ is a ONB for $(\mathfrak{k}+\mathfrak{m}_{\mathscr{I}_0}) \oplus \mathfrak{a}_{\mathscr{I}_0} \oplus \mathfrak{u}_{\mathscr{I}_0}$.

By comparing $\mathrm{ONB}_{\mathscr{I}_0} \sqcup \mathrm{ONB}_{\mathrm{U}_{\mathscr{I}_0}}$ against $\mathrm{ONB}_{\mathscr{I}_0} \sqcup \mathcal{W}$ under $\mathrm{d}\,\Phi_4|_{id}$ and noting that

$$E_{ij} \xrightarrow{\mathrm{d}\,\Phi_4|_{id}} \sqrt{2}^{-1}\frac{E_{ij}+E_{ji}}{\sqrt{2}} + \frac{E_{ii}-E_{ji}}{2} \in \sqrt{2}^{-1}\frac{E_{ij}+E_{ji}}{\sqrt{2}} + \mathfrak{k},$$

we get

$$|\det(\mathrm{d}\,\Phi_4|_{id})| = 2^{-\frac{\sum_{i<j}|I_i||I_j|}{2}}.$$

Therefore

$$(\Phi_4)_*\left(2^{-\frac{\sum_{i<j}|I_i||I_j|}{2}} \cdot \nu \otimes \lambda_{\mathscr{I}_0}(a)\mathrm{m}_{\mathrm{A}_{\mathscr{I}_0}}(a) \otimes \mathrm{m}_{\mathrm{U}_{\mathscr{I}_0}}\right) = \mathrm{m}_{\mathrm{G}}.$$

And we are done. □

Next we turn to the Cartan decomposition, by [Hel00, Theorem 5.8], the map

$$\Phi_6 : \mathrm{K}_{\mathscr{I}_0} \times \mathrm{A}^+_{\mathrm{M}_{\mathscr{I}_0}} \times \mathrm{K}_{\mathscr{I}_0} \to \mathrm{M}_{\mathscr{I}_0}$$

induces

$$(\Phi_6)_*(C_6\widehat{\mathrm{m}}_{\mathrm{K}_{\mathscr{I}_0}} \otimes \prod_{i<j, i\sim_{\mathscr{I}_0} j} \frac{\lambda_{ij}(a)-\lambda_{ji}(a)}{2}\mathrm{m}_{\mathrm{A}^+_{\mathrm{M}_{\mathscr{I}_0}}}(a) \otimes \widehat{\mathrm{m}}_{\mathrm{K}_{\mathscr{I}_0}}) = \mathrm{m}_{\mathrm{M}_{\mathscr{I}_0}}$$

for some constant $C_6 > 0$.

Lemma 4.17. *We have that*

$$C_6 = \mathrm{Vol}(\mathrm{K}_{\mathscr{I}_0})^2.$$

Proof. It suffices to do this for $\Phi_6' : \mathrm{SO}_n(\mathbb{R}) \times \mathrm{A}^+ \times \mathrm{SO}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R})$ where A^+ is the subgroup of $\mathrm{SL}_n(\mathbb{R})$ defined by

$$\{a = \mathrm{diag}(a_1, ..., a_n) \mid \lambda_{ij}(a) = a_i/a_j \geq 1,\ \forall i < j;\ a_i > 0,\ \forall i\}.$$

Take a_0 in the interior of A^+. It suffices to show that the determinant of the differential of

$$\Phi : (k_1, a, k_2) \mapsto k_1 a k_2 a_0^{-1}$$

at (id, a_0, id) from $\mathfrak{k}_n \oplus \mathfrak{a} \oplus \mathfrak{k}_n$ to $\mathfrak{sl}_n$ is equal to ± 1. Here $\mathfrak{k}_n$ denotes the Lie algebra of $\mathrm{SO}_n(\mathbb{R})$.

From the definition we see that

$$\begin{aligned} \mathrm{d}\,\Phi|_{(id,a_0,id)}(v_1, v_2, 0) &= v_1 + v_2; \\ \mathrm{d}\,\Phi|_{(id,a_0,id)}(0, 0, w) &= \mathrm{Ad}(a_0)w. \end{aligned}$$

Let

$$\mathrm{ONB}_{\mathfrak{k}_n} := \left\{ \frac{1}{\sqrt{2}}(E_{ij} - E_{ji}) \;\middle|\; i < j \right\}$$

be an ortho-normal basis on $\mathfrak{k}_n$. Then for $i < j$,

$$\mathrm{d}\,\Phi|_{(id,a_0,id)}(0, 0, \frac{1}{\sqrt{2}}(E_{ij} - E_{ji})) \in \left(\frac{\lambda_{ij}(a_0) - \lambda_{ji}(a_0)}{2} \cdot \frac{1}{\sqrt{2}}(E_{ij} + E_{ji}) \right) + \mathfrak{k}_n.$$

By taking the wedge of them, we are done. $\square$

Combining Lemma 4.16 and 4.17, we have shown

Lemma 4.18. *Let C_7 be as in Lemma 4.3, then*

$$C_7 = \mathrm{Vol}(\mathrm{K}) \cdot 2^{-\frac{\sum_{i<j} |I_i||I_j|}{2}}.$$

The natural map defined by taking multiplication

$$\Phi_7 : \mathrm{K} \times \mathrm{A}^+_{\mathrm{M}_{\mathscr{I}_0}} \times \mathrm{A}_{\mathscr{I}_0} \times \mathrm{K}_{\mathscr{I}_0} \times \mathrm{U}_{\mathscr{U}_{\mathscr{I}_0}} \to \mathrm{G}$$

induces

$$(\Phi_7)_* \left(C_7 \cdot \rho_{\mathscr{I}_0}(a, b) \cdot \widehat{\mathrm{m}}_{\mathrm{K}} \otimes \mathrm{m}_{\mathrm{A}^+_{\mathrm{M}_{\mathscr{I}_0}}}(a) \otimes \mathrm{m}_{A_{\mathscr{I}_0}}(b) \otimes \mathrm{m}_{\mathrm{K}_{\mathscr{I}_0}} \otimes \mathrm{m}_{\mathrm{U}_{\mathscr{I}_0}} \right) = \mathrm{m}_{\mathrm{G}}.$$

where we have used the shorthand

$$\rho_{\mathscr{I}_0}(a, b) := \lambda_{\mathscr{I}_0}(b) \cdot \prod_{i<j, i \sim_{\mathscr{I}_0} j} \frac{\lambda_{ij}(a) - \lambda_{ji}(a)}{2}.$$

4.4. **Proof of the lemmas.**

4.4.1. *Proof of Lemma 4.1.*

Proof. So take $g \in \mathrm{G}$ with $g\mathrm{U}_{\mathscr{I}_0}\mathrm{K} = \mathrm{U}_{\mathscr{I}_0}\mathrm{K}$. In particular $g \in \mathrm{U}_{\mathscr{I}_0}\mathrm{K}$, so there exists $u_g \in \mathrm{U}_{\mathscr{I}_0}$, $k_g \in \mathrm{K}$ such that

$$g = u_g k_g.$$

Then

$$u_g k_g \mathrm{U}_{\mathscr{I}_0} \subset \mathrm{U}_{\mathscr{I}_0}\mathrm{K} \implies k_g \mathrm{U}_{\mathscr{I}_0} k_g^{-1} \subset \mathrm{U}_{\mathscr{I}_0}\mathrm{K} \implies k_g \mathrm{U}_{\mathscr{I}_0} k_g^{-1} \subset \mathrm{K}\mathrm{U}_{\mathscr{I}_0}.$$

Take a representation (ρ, V) and a vector $v \in V$ such that $\mathbf{U}_{\mathscr{I}_0}$ is equal to the stabilizer of v. Hence

$$k_g \mathrm{U}_{\mathscr{I}_0} k_g^{-1} \cdot v \subset \mathrm{K}\mathrm{U}_{\mathscr{I}_0} \cdot v$$

is bounded. By the feature of polynomials, we must have

$$k_g \mathrm{U}_{\mathscr{I}_0} k_g^{-1} \cdot v = v.$$

In particular k_g normalize $\mathbf{U}_{\mathscr{I}_0}$. On the other hand, it is direct to check that $N_{\mathrm{K}}(\mathrm{U}_{\mathscr{I}_0})\mathrm{U}_{\mathscr{I}_0} \subset G_{\mathrm{hor}}$. So we are done. □

4.4.2. *Proof of Lemma 4.2.*

Proof. The nontrivial direction is $\Gamma_i \cap G_{\mathrm{hor}} \subset \Gamma_i \cap \mathrm{U}_{\mathscr{I}_0}$. So take $\gamma \in \Gamma_i \cap G_{\mathrm{hor}}$. By Lemma 4.1, $G_{\mathrm{hor}} = N_{\mathrm{K}}(\mathrm{U}_{\mathscr{I}_0})\mathrm{U}_{\mathscr{I}_0}$ and we may write

$$\gamma = k_\gamma u_\gamma.$$

As in the proof of last lemma, take a $\mathbb{Q}$-representation (ρ, V) and a vector $v \in V(\mathbb{Q})$ such that $\mathbf{U}_{\mathscr{I}_0}$ is equal to the stabilizer of v. Consider $(\gamma^n \cdot v)$. On the one hand, this is a discrete set. On the other hand

$$\gamma^n \cdot v = k_\gamma^n u_n' \cdot v = k_\gamma^n \cdot v$$

for some $u_n' \in \mathrm{U}_{\mathscr{I}_0}$. And hence $(\gamma^n \cdot v)$ is bounded. Thus there exists $m < n$ such that

$$\gamma^n \cdot v = \gamma^m \cdot v \implies \gamma^{n-m} \cdot v = v.$$

Hence γ^{n-m} is a unipotent matrix in $\mathrm{U}_{\mathscr{I}_0}$. As Γ_i is assumed to be neat, we conclude that γ is a unipotent matrix. Hence by considering its logarithm, we see that $\gamma \in \mathrm{U}_{\mathscr{I}_0}$. □

4.4.3. *Proof of Lemma 4.3.*

Proof. By definition, $\mathrm{m}_{\mathrm{G}/\mathrm{G}^\circ_{\mathrm{hor}}}$ is the unique G-invariant measure on $\mathrm{G}/\mathrm{G}^\circ_{\mathrm{hor}}$ such that the fibre integration formula

$$\int_{\mathrm{G}/\mathrm{G}^\circ_{\mathrm{hor}}} \int_{\mathrm{G}^\circ_{\mathrm{hor}}} f(gh) \mathrm{m}_{\mathrm{G}^\circ_{\mathrm{hor}}}(h) \mathrm{m}_{\mathrm{G}/\mathrm{G}^\circ_{\mathrm{hor}}}([g]) = \int_{\mathrm{G}} f(x) \mathrm{m}_{\mathrm{G}}(x)$$

holds for all compactly supported function f on G. Thus we are done by Lemma 4.18. □

4.4.4. *Proof of Lemma 4.4.*

Proof. This follows in the same way as in [MG14, Lemma 29]. □

4.4.5. *Proof of Lemma 4.5.*

Proof. This follows by applying [MG14, Lemma 25,26]. So we need to feed it with an Euclidean space, a vector $\mathbf{v}_0$, a cone $\mathcal{C}$ with $\mathbf{v}_0 \in \mathcal{C}$. We identify $\mathfrak{a}$, the Lie algebra of the full diagonal group in G, with $\mathbb{R}^{N-1}$ such that the trace form goes to the standard Euclidean metric. Note that $\mathfrak{a}_{\mathrm{M}_{\mathscr{I}_0}} \oplus \mathfrak{a}_{\mathscr{I}_0}$ is naturally identified with $\mathfrak{a}$. Recall

$$\mathcal{C}_C := \left\{ \mathbf{a},\ a_i - a_j \geq \max\{0, C\},\ \forall k,\ i<j \in I_k;\ \sum_{i \in I_1 \cup \ldots \cup I_k} a_i \geq C,\ \forall 1 \leq k < k_0 \right\}.$$

where $\mathbf{a} = \mathrm{diag}(a_1, ..., a_N)$ for an element $\mathbf{a} \in \mathfrak{a}^+_{\mathrm{M}_{\mathscr{I}_0}} \oplus \mathfrak{a}_{\mathscr{I}_0}$. Thus $\mathcal{C}_0$ is a cone and $\mathcal{C}_C = \mathcal{C}_0 + x$ for some $x \in \mathfrak{a}$. Let

$$\mathbf{v}_0 := \sum_{1 \leq i < j \leq N} E_{ii} - E_{jj} = \mathrm{diag}(N-1, N-3, ..., -N+1)$$

where E_{ii} is the matrix whose (i,i)-th entry is 1 and is 0 elsewhere. One can also check that $\mathbf{v}_0 \in \mathcal{C}_0$. Then [MG14, Lemma 25,26] implies that for every $C \in \mathbb{R}$,

$$\int_{\|\mathbf{y}\| \leq R, \mathbf{y} \in \mathcal{C}_C} e^{\mathrm{Tr}(\mathbf{v}_0 \mathbf{y})}\, \mathrm{d}\mathbf{y} \sim \left(\frac{2\pi R}{\|\mathbf{v}_0\|} \right)^{\frac{N-2}{2}} \cdot e^{\|\mathbf{v}_0\| R},$$

indenpendent of C. Now it suffices to show that

$$\mu_A(B^+_{\mathfrak{a}}(R)) \sim \left(\frac{1}{2}\right)^{\sum_{k=1}^{k_0} \frac{|I_k|(|I_k|-1)}{2}} \int_{\|\mathbf{y}\| \leq R, \mathbf{y} \in \mathcal{C}_C} e^{\mathrm{Tr}(\mathbf{v}_0 \mathbf{y})}\, \mathrm{d}\mathbf{y}\,. \tag{10}$$

By unfolding the definitions,

$$\begin{aligned} &\left(\frac{1}{2}\right)^{\sum_{k=1}^{k_0} \frac{|I_k|(|I_k|-1)}{2}} \int_{\|\mathbf{y}\| \leq R, \mathbf{y} \in \mathcal{C}_C} e^{\mathrm{Tr}(\mathbf{v}_0 \mathbf{y})}\, \mathrm{d}\mathbf{y} \\ =& \int_{\|\mathbf{y}\| \leq R, \mathbf{y} \in \mathcal{C}_C} \exp\left(\sum_{1 \leq s < t \leq k_0} \sum_{i \in I_s, j \in I_t} y_i - y_j \right) \cdot \prod_{k=1}^{k_0} \prod_{i<j \in I_k} \frac{1}{2} e^{y_i - y_j}\, \mathrm{d}\mathbf{y}\,. \end{aligned}$$

By comparison, from the definition of μ_A, we have

$$\begin{aligned} \mu_A(B^+_{\mathfrak{a}}(R)) = &\int_{\|\mathbf{y}\| \leq R, \mathbf{y} \in \mathcal{C}_0} \exp\left(\sum_{1 \leq s < t \leq k_0} \sum_{i \in I_s, j \in I_t} y_i - y_j \right) \\ &\cdot \prod_{k=1}^{k_0} \prod_{i<j \in I_k} \frac{1}{2} e^{y_i - y_j} \left(1 - e^{-2(y_i - y_j)}\right) \mathrm{d}\mathbf{y} \end{aligned}$$

where $\mathbf{y} = \mathrm{diag}(y_1, ..., y_N)$. Thus it suffices to show that the contribution of the factors

$$1 - e^{-2(y_i - y_j)}$$

are negligible. First note that on $\mathcal{C}_0$, $0 \leq 1 - e^{-2(y_i - y_j)} \leq 1$, thus the LHS of Equa.(10) is no larger than the RHS.

Recall $B^{C,+}_{\mathfrak{a}}(R) = B_{\mathfrak{a}}(R) \cap \mathcal{C}_C$. For every $C, R > 0$,

$$\mu_A(B^+_{\mathfrak{a}}(R)) \geq \mu_A(B^{C,+}_{\mathfrak{a}}(R)).$$

For any $\varepsilon \in (0,1)$, there exists $C_\varepsilon > 0$ such that

$$1 - e^{-2(y_i - y_j)} \in [1-\varepsilon, 1]$$

for all $k = 1, ..., k_0$, $i < j \in I_k$ and $\mathbf{y} \in \mathcal{C}_{C_\varepsilon}$. Thus

$$\mu_A(B_{\mathfrak{a}}^{C_\varepsilon,+}(R)) \geq \left((1-\varepsilon)\cdot\frac{1}{2}\right)^{\sum_{k=1}^{k_0}\frac{|I_k|(|I_k|-1)}{2}} \int_{\|\mathbf{y}\|\leq R, \mathbf{y}\in\mathcal{C}_{C_\varepsilon}} e^{\mathrm{Tr}(\mathbf{v}_0\mathbf{y})}\,\mathrm{d}\mathbf{y}\,.$$

Letting $R \to +\infty$ and then $\varepsilon \to 0$ concludes the proof. □

4.4.6. *Proof of Lemma 4.9.*

Proof. By an equivalent definition of well-rounded ([EM93, Proposition 1.3]), we need to show that for every $\varepsilon \in (0,1)$, there exists a neighborhood Ω of identity in G such that for all $R \geq 1$,

$$(1-\varepsilon)\mathrm{m}_{\mathrm{G}/G_{\mathrm{hor}}}(\bigcup_{\omega\in\Omega}\omega B_R) < \mathrm{m}_{\mathrm{G}/G_{\mathrm{hor}}}(B_R) < (1+\varepsilon)\mathrm{m}_{\mathrm{G}/G_{\mathrm{hor}}}(\bigcap_{\omega\in\Omega}\omega B_R).$$

For every $\delta \in (0,1)$, choose a neighborhood Ω_δ of identity in G satisfying

$$\mathrm{dist}([\omega^{-1}]_{\mathrm{K}}, [id]_{\mathrm{K}}) \leq \delta, \quad \forall \omega \in \Omega_\delta.$$

From definition of B_R, for all $R \geq 1$ and $\omega \in \Omega_\delta$,

$$\begin{aligned} &B_{R-\delta} \subset \omega B_R \subset B_{R+\delta} \\ \implies &\bigcup_{\omega\in\Omega}\omega B_R \subset B_{R+\delta}, \bigcap_{\omega\in\Omega}\omega B_R \supset B_{R-\delta}. \end{aligned}$$

From the description of Haar measures in Lemma 4.5, for every $\varepsilon \in (0,1)$ we can find $\delta_\varepsilon > 0$ such that for all $R \geq 1$,

$$\begin{aligned} \mathrm{m}_{\mathrm{G}/G_{\mathrm{hor}}}(B_{R+\delta_\varepsilon}) &< \frac{1}{1-\varepsilon}\mathrm{m}_{\mathrm{G}/G_{\mathrm{hor}}}(B_R) \\ \mathrm{m}_{\mathrm{G}/G_{\mathrm{hor}}}(B_{R-\delta_\varepsilon}) &> \frac{1}{1+\varepsilon}\mathrm{m}_{\mathrm{G}/G_{\mathrm{hor}}}(B_R). \end{aligned}$$

Thus by choosing $\Omega := \Omega_{\delta_\varepsilon}$, one can verify the well-roundedness. □

4.4.7. *Proof of Lemma 4.12.*

Proof. Indeed, for every $\varepsilon > 0$ and for $C \in \mathbb{R}$ sufficiently small, for every $(k,a,b) \notin \mathrm{K} \times \exp(\mathcal{C}_C)$, one can find an integral vector v in $\wedge^i\mathbb{Z}^N$ for some i such that

$$\|kab \cdot v\| \leq \varepsilon.$$

The proof is complete by Mahler's criterion. □

4.4.8. *Proof of Lemma 4.13.*

Proof. From the definition of μ_A, one sees that there exists a constant $C' > 0$ depending on C such that

$$\mu_A|_{\mathcal{C}_C\setminus\mathcal{C}_0} \leq C'\,\mathrm{Leb}$$

where Leb is some Lebesgue measure on the linear space $\mathfrak{a}$. Thus

$$\limsup_{R\to+\infty}\frac{\mu_A(B_{\mathfrak{a}}^{C,+}(R)\setminus B_{\mathfrak{a}}^{+}(R))}{\mathrm{Leb}(B_{\mathfrak{a}}^{+}(R))} < \infty.$$

But one also sees from 4.4.5 that

$$\lim_{R\to+\infty} \frac{\mathrm{Leb}(B^+_{\mathfrak{a}}(R))}{\mu_A(B^+_{\mathfrak{a}}(R))} = 0.$$

So this proves the first part. The second part follows from Lemma 4.5. □

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