LECTURE 9.5

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CONTENTS

1. STEP 2

$X = \mathrm{SL}_2(\mathbb{R})/\Gamma$ with Γ discrete. Let B be the subgroup $AU = \{\boldsymbol{a}_t\boldsymbol{u}_s,\ t,s \in \mathbb{R}\}$ and let $V = \{\boldsymbol{v}_s,\ s \in \mathbb{R}\}$ be the lower triangular one-parameter unipotent flow.

Now we come to Step 2.

Lemma 1.1. *If B-invariant, U-ergodic probability measures μ on $\mathrm{SL}_2(\mathbb{R})/\Gamma$ exist then Γ is a lattice and μ is equal to (normalized) m_X.*

By the proof from Lec.8, we have the following

Lemma 1.2. *Same assumption. The measure μ is ergodic (actually mixing) with respect to $a^{\mathbb{Z}}$-action for every $a_{\neq \mathrm{id}} \in A$.*

Let μ be the B-invariant, a-ergodic measure. Here a is an element of A such that $a^n v a^{-n} \to id$ as $n \to +\infty$ for $v \in V$. Want to show μ coincides with the m_X (up to a scalar) and in particular, m_X is finite.

Fix some o in the support of μ. Choose neighborhoods of identity in B, V that are very small compared to the injectivity radius at o. "local B, V orbit" means with respect to these neighborhoods. Then choose $\delta > 0$ very small compared to these neighborhoods.

Consider $B_\delta(o)$. Let $\mathrm{Gene}(f,\mu)$ be those $x \in X$ such that

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} f(a^n x) = \int f(x)\mu(x).$$

Note that this set is $V \cdot A$-stable. Let E_f be its intersection with $B_\delta(o)$.

Consider the σ-algebra $\mathscr{A}$ on $B_\delta(o)$ defined by $x \sim y$ iff x and y are locally on the same B-orbit. Let $E'_f \subset E_f$ be those x such that the conditional measure $\mu_x^{\mathscr{A}}$ is the restriction of some (left-)B-invariant Haar measure (when we identify $[x]^{\mathscr{A}}$ as a subset of B via the orbit map). Then μ being B-invariant, E'_f is a conull set in E_f (use the uniqueness of

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conditional measure). Then let $\widetilde{E}_f$ be the subset of $B_{\delta,x}$ that is on the local V-orbit of some element in E'_f. Thus $\widetilde{E}_f$ is conull in B_δ with respect to μ and m_X.

1.1. **Conclude the proof from here.** First assume $\mathrm{m}_X < \infty$. Every point $x \in \widetilde{E}_f$ is generic for μ. But since the $a^{\mathbb{Z}}$-action on m_X is also ergodic, and $\mathrm{m}_X(\widetilde{E}_f) > 0$, we can find a point $x \in \widetilde{E}_f$ generic for m_X. Thus $\int f(x)\mu(x) = \int f(x)\mathrm{m}_X(x)$. Since f is arbitrary we are done.

Now assume $\mathrm{m}_X = \infty$. Then the associated unitary representation is absence of constants. Thus by mixing, for every (real-valued) $\phi, \psi \in L^2(X, \mathrm{m}_X)$, we have

$$\lim_{n\to\infty} \int \phi(a^n.x)\psi(x)\mathrm{m}_X(x) = 0.$$

Take $\phi = f$ and $\psi = 1_{\widetilde{E}_f}$, then

$$\lim_{n\to\infty} \int_{\widetilde{E}_f} f(a^n.x)\mathrm{m}_X(x) = \lim_{n\to\infty} \int f(a^n.x)1_{\widetilde{E}_f}(x)\mathrm{m}_X(x) = 0.$$

Let us compute

$$\begin{aligned}\mathrm{m}_X(\widetilde{E}_f)\int f(x)\mu(x) &= \int_{\widetilde{E}_f} \left(\lim_N \frac{1}{N}\sum_{n=0}^{N-1} f(a^n.x)\right)\mathrm{m}_X(x)\\ \text{(bounded convergence thm)} &= \lim_N \frac{1}{N}\sum_{n=0}^{N-1}\left(\int_{\widetilde{E}_f} f(a^n.x)\mathrm{m}_X(x)\right) = 0,\end{aligned}$$

which is impossible if $f > 0$. Hence $\mathrm{m}_X = \infty$ leads to a contradiction.

1.2. **How to conclude proof as in** [Ra92]**.** See [Ra92, Page 27,28] Let $\Omega_f := a^{\mathbb{Z}}\widetilde{E}_f$ and $\Omega := a^{\mathbb{Z}}B_\delta(o)$. Note that Ω_f is conull in Ω w.r.t. both μ and m_X.

Consider the following

$$\begin{aligned}\mathrm{m}_X(\Omega_f)\cdot\langle f,\mu\rangle &= \int_{x\in\Omega_f}\left(\lim\sum\frac{1}{N}\sum f(a^n.x)\right)\mathrm{m}_X(x)\\ &\overset{?}{=} \lim\left(\int_{x\in\Omega_f}\sum\frac{1}{N}\sum f(a^n.x)\mathrm{m}_X(x)\right)\\ (\Omega_f \text{ and } \mathrm{m}_X \text{ are } a\text{-stable}) &= \lim\left(\int_{x\in\Omega_f} f(x)\mathrm{m}_X(x)\right) = \langle f, \mathrm{m}_X\rangle.\end{aligned}$$

The $\overset{?}{=}$ would become a true $=$ if $\mathrm{m}_X|_\Omega$ were known to be finite. Assume f is non-negative. Replacing Ω by a subset B with finite volume so that the equality goes through and then we take sup over all such B's. This proves that $\overset{?}{=}$ may be replaced by $\leq$ (when f is non-negative). But this implies that

$$\mathrm{m}_X(\Omega_f) \leq \frac{\langle f, \mathrm{m}_X\rangle}{\langle f,\mu\rangle} < \infty$$

with appropriate choice of f. Now we can go back to $\overset{?}{=}$ above and claim that it is a true equality, which implies that

$$\mathrm{m}_X(\Omega_f) = \frac{\langle f,\mathrm{m}_X\rangle}{\langle f,\mu\rangle} \implies \frac{1}{\mathrm{m}_X(\Omega_f)}\cdot\mathrm{m}_X|_{\Omega_f} = \mu \implies \frac{1}{\mathrm{m}_X(\Omega)}\cdot\mathrm{m}_X|_\Omega = \mu..$$

If m_X is finite, then we can show that a is ergodic w.r.t. m_X and the proof ends here. Otherwise, we are not far away.

Let $C := \mathrm{m}_X(\Omega) = |\mu|$. Write $\Omega = \Omega_o$ (also depend on δ), then $\mathrm{m}_X(\cup\Omega_o) = C$ as o ranges over support of μ. Let $\cup\Omega_o =: \Omega_\mu$, then every $x \in X$ shares the same V-orbit with some $y \in \Omega_\mu$ or $w.x \in \Omega_\mu$ where w is the nontrivial Weyl. Indeed, $\mathrm{SL}_2(\mathbb{R}) = VB \cup VwB = VB \cup wB$. So it suffice to show that $\mathrm{m}_X(v\Omega) = C$ for all $v \in V$ (then $|\mathrm{m}_X| \leq 2C$).

($\Omega := \Omega_\mu$ below)

Note that $V \cdot \Omega = X$. As Ω is open, it suffices to show that for every $v \in V$, $\mathrm{m}_X(v.\Omega \cup \Omega) = \mathrm{m}_X(\Omega)$.

Consider the $E_f(\Omega)$ be the (a,μ)-generic points in Ω, which is co-null with respect to μ. Similarly define $\widetilde{E_f}(\Omega)$ which is conull w.r.t. μ and m_X. Now we consider $v \cdot \widetilde{E_f}(\Omega)$ for a $v \in V$, which is conull in $v.\Omega$ (w.r.t. m_X). Let $\Omega_v := a^{\mathbb{Z}} \cdot v.\widetilde{E_f}(\Omega)$. Same argument as before now shows that

$$\frac{1}{\mathrm{m}_X(\Omega_v)} \cdot \mathrm{m}_X|_{\Omega_v} = \mu.$$

This implies that

$$\mathrm{m}_X(v.\widetilde{E_f}(\Omega) \cup \Omega) = \mathrm{m}_X(v.\Omega \cup \Omega) = \mathrm{m}_X(\Omega) = C.$$

So we are done.

2. Conditional expectations

As a reference, see [EW11, Ch.5] and [Cou16, Part IV and Ch.17].

Let X be a nice space and $\mathscr{B}_X$ its Borel σ-algebra. Let $\mu \in \mathrm{Prob}(X)$. Let $\mathscr{A}$ be a countably generated (equal to the smallest sub-σ-algebra containing certain countable collection of measurable sets, say $\mathscr{A}_0 := \{A_i\}$) sub-σ-algebra of $\mathscr{B}_X$. For convenience, assume the complement of every A_i is also contained in $\mathscr{A}_0$. For $x \in X$, let the **atom** containing x be $[x]^{\mathscr{A}} := \cap_{x\in A_i} A_i$.

Theorem 2.1. *(Conditional Expectations) Let $(X, \mathscr{B}_X, \mu)$ and $\mathscr{A}$ as above.*

1. Existence. *There exists $X' \in \mathscr{A}$ of full measure such that we have a measurable map $X' \to \mathrm{Prob}(X)$ denoted as $x \mapsto \mu_x^{\mathscr{A}}$ such that $\mu_x^{\mathscr{A}}([x]^{\mathscr{A}}) = 1$ and*
$$\int_A \int f(y)\mu_x^{\mathscr{A}}(y)\mu(x) = \int f(x)\mu(x) \tag{1}$$
for every $A \in \mathscr{A}$ and $f \in L^1(X, \mathscr{B}_X, \mu)$.
2. Uniqueness. *If $x \mapsto \nu_x^{\mathscr{A}}$ is another measurable map from a possibly different full measure set X'' to $\mathrm{Prob}(X)$ satisfying Equa.1 for every compactly supp. cont. function f, then for some full measure set $X''' \subset X' \cap X''$ we have $\mu_x^{\mathscr{A}} = \nu_x^{\mathscr{A}}$ for $x \in X'''$.*

Example 2.2. *Let $\mathscr{A} = \mathscr{B}_X$. Then $[x]^{\mathscr{A}} = \{x\}$ and $\mu_x^{\mathscr{A}} = \delta_x$.*

Example 2.3. *Let $\mathscr{A}$ be the sigma algebra generated by a finite partition $\{P_1, \ldots, P_l\} \subset \mathscr{B}_X$ of X, then for $x \in P_i$, $[x]^{\mathscr{A}} = P_i$ and $\mu_x^{\mathscr{A}} = (\mu(P_i))^{-1}\mu|_{P_i}$.*

Example 2.4. *Let $X = [0,1] \times [0,1]$ and $\mu = \mathrm{Leb}$ be the standard Lebesgue measure defined by $|\mathrm{dx} \wedge \mathrm{dy}|$. Let $\mathscr{A} := \{\{x\} \times [0,1] \mid x \in [0,1]\}$. Then $[(x,y)]_{(x,y)}^{\mathscr{A}} = \{x\} \times [0,1]$ and $\mu_{(x,y)}^{\mathscr{A}}$ is induced by $|\mathrm{dy}|$.*

This example can be generalized to foliations on manifolds where X is a box where one has a local chart.

Example 2.5. *Everything same as in the last example except let μ be the standard Lebesgue measure supported on $\mathrm{diag}(X) := \{(x,x), x \in X\}$. Then $[(x,y)]_{(x,y)}^{\mathscr{A}} = \{x\} \times [0,1]$ and $\mu_{(x,y)}^{\mathscr{A}} = \delta_y$.*

Example 2.6. *If you have a measurable measure preserving $\pi : (X, \mathscr{B}_X, \mu) \to (Y, \mathscr{B}_Y, \nu)$ with X, Y nice. Let $\mathscr{A} := \pi^{-1}\mathscr{B}_Y$. And Equa. 1 can be viewed as a fibre integration formula (you can replace the μ on the LHS by ν). Then atoms are fibres of π. In some sense, in general all $\mathscr{A}$ arises from such a π.*

3. More details on local invariant measures

Assume G is a Lie group. Let $U \subset G$ be an open subset. We say a measure μ on U is locally invariant under G iff for every measurable subset $A \subset U$ and $g \in G$ such that $gA \subset U$, we have $\mu(gA) = \mu(A)$.

Lemma 3.1. *μ is the restriction of some left Haar measure on G.*

Proof. Fix a countable set $(g_i)_{i\in\mathbb{Z}_{\geq 0}}$ in G such that $G = \cup g_i U$. Assume $g_0 = id$. Let $A_0 := U$, $A_1 := g_1 U \setminus U$, $A_2 := g_2 U \setminus (U \cup g_1 U)$.... Then $G = \sqcup A_i$. Define an extension of μ' by

$$\mu'(E) := \sum_{i\geq 0} \mu(g_i^{-1}(E \cap A_i)).$$

Then one can prove that μ' is left G-invariant. □

To check local-invariant, it is helpful to know

Lemma 3.2. *Assume U is connected and $\delta > 0$. And μ is locally invariant only for $g \in B_\delta(id) \subset G$. Then U is locally invariant.*

Proof. For every g, consider all possible finite words $(g_i)_{i=1}^n$ in $B_\delta(id)$ such that $g = g_n \cdot ... \cdot g_1$. For every $x \in U$, consider such words further satisfying $g_k \cdot ... \cdot g_1.x \in U$ for all $k = 1, .., n$. This should solve the problem. □

References

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