LECTURE 12, LINEARIZATION TECHNIQUE

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Contents

The main reference of this lecture is [MS95, Section 3]. One may also consult Dani–Margulis [DM93], Shah [Sha91b], Ratner [Ra91a, Ra91b], Eskin–Mozes–Shah [EMS96], Eskin–Margulis–Mozes [EMM98, Section 4]. The references [Sha91a, Sha09] contain some examples. An effective treatment appears in [LMMS19].

1. A gap in the last lecture

In the last lecture we proved $X = \bigsqcup_{[H]\in\mathscr{H}/\Gamma} T(H,U)$ and conclude from here that $\mu = \sum \mu|_{T(H,U)}$. One additional argument is needed here: the index should be countable.

Lemma 1.1. *$\mathscr{H}$ is countable.*

Proof. For every $H\in\mathscr{H}$, $H\cap\Gamma$ is a lattice in H. Thus $H\cap\Gamma$ is finitely generated (note that this seems not obvious unless $H\cap\Gamma$ is cocompact. In the case at hand, H is algebraic by arguments from last lecture and $H\cap\Gamma$ is an arithmetic lattice, hence this follows from the theory of Siegel sets, see [Bor19]; in general probably one has to quote structure theorem of [GR70] in the rank-one case and use arithmeticity theorem of Margulis in higher-rank case. See also [Gel14, Lecture 3, Section 5] for another possibly more geometric proof). Hence the set $\{H\cap\Gamma,\ H\in\mathscr{H}\}$ is countable.

Date: 2022.04.

Since H can be recovered from $H\cap\Gamma$ by

$$H = \left(\overline{\overline{H\cap\Gamma}}\cap \mathrm{SL}_n(\mathbb{R})\right)^{\circ},$$

we are done. Here $\overline{\overline{H\cap\Gamma}}$ means the closure of $H\cap\Gamma$ in $\mathrm{SL}_n(\mathbb{C})$ with respect to the topology defined by polynomials. □

2. Two examples

Here we include two examples, a little bit beyond $\mathrm{SL}_2(\mathbb{R})$, to illustrate what kind of objects we are dealing with. You are welcome to test the general theory using these (still rather special) examples!

In both examples, set

- $G = \mathrm{SL}_2(\mathbb{C})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$, $U = \left\{ \mathbf{u}_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \middle| \, s\in\mathbb{R}\right\}$;
- $\mathbf{U}(\mathbb{C}) := \left\{ \mathbf{u}_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \middle| \, s\in\mathbb{C}\right\}$
- for $t\in\mathbb{C}^\times$, let $\mathbf{a}_t := \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$.

One can show that Γ is a lattice in G (using non-divergence of unipotent flows, for instance). And G/Γ can be embedded in $\mathrm{SL}_4(\mathbb{R})/\mathrm{SL}_4(\mathbb{Z})$.

2.1. Example 1.

[I made a mistake about this example in the class]

- $H := \left\{ \mathbf{u}_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \middle| \, s\in\mathbb{C}\right\}$, $\mathfrak{h}$ is the Lie algebra of H;
- $K_H := \{\mathbf{a}_z,\ z\in\mathbb{C}, |z|=1\}$.

Lemma 2.1. $N(H,U) = N_G(H) = \left\{\mathbf{a}_t\cdot\mathbf{u}_s,\ t\in\mathbb{C}^\times,\ s\in\mathbb{C}\right\} =: B$.

Proof. Let $g\in G$. Indeed, g belongs to $N(H,U)$ iff $\mathrm{Ad}(g)\cdot\mathfrak{h}$ contains $\mathfrak{u}$. By Bruhat decomposition (ref??),

$$G = BwB \sqcup B$$

where $w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. If $g\in B$, then $\mathrm{Ad}(g)\cdot\mathfrak{h} = \mathfrak{h}\supset\mathfrak{u}$. On the other hand, if $g = b_1 w b_2$ for $b_i\in B$ then

$$\mathrm{Ad}(g)\cdot\mathfrak{h} = Ad(b_1)Ad(w)\cdot\mathfrak{h} = Ad(b_1)\cdot\begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix} \implies \mathrm{Ad}(g)\cdot\mathfrak{h}\cap\mathfrak{h} = \{0\}.$$

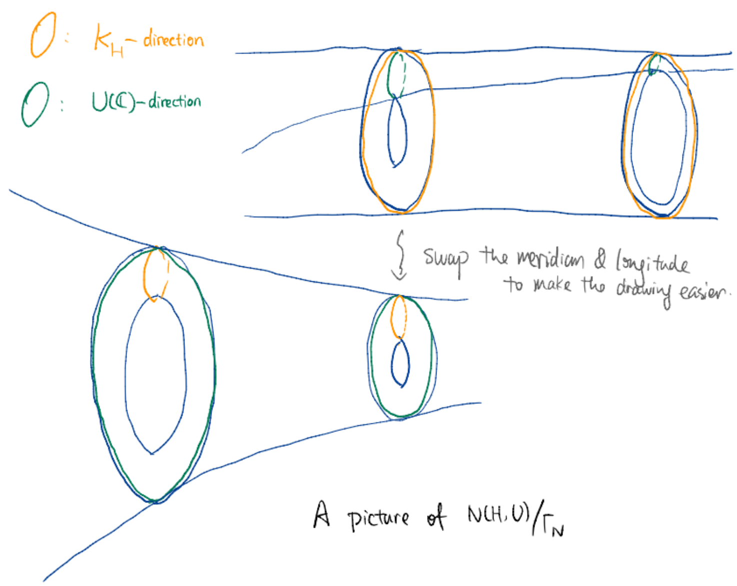
So we are done. □

The orbits of B on G/Γ are all dense, and hence not easy to draw. Since $N(H,U)$ ($=B$ here) is stable under right translation by $N_G(H)$ and therefore $N_G(H)\cap\Gamma$ (call it Γ_N for simplicity). Thus $N(H,U)$ being closed implies that $N(H,U)/\Gamma_N\subset G/\Gamma_N$ is closed. We will draw pictures for $N(H,U)/\Gamma_N$. (warning! pictures are just for illustration, they may be wrong in many aspects!)

By the way, a quick computations show that

$$\Gamma_N = \left\{ \begin{bmatrix} 1 & \mathbb{Z}[i] \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & \mathbb{Z}[i] \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} i & \mathbb{Z}[i] \\ 0 & -i \end{bmatrix} \right\}.$$

Here is a picture for $N(H,U)/\Gamma_N$ with $U\Gamma_N/\Gamma_N$ contained in here:

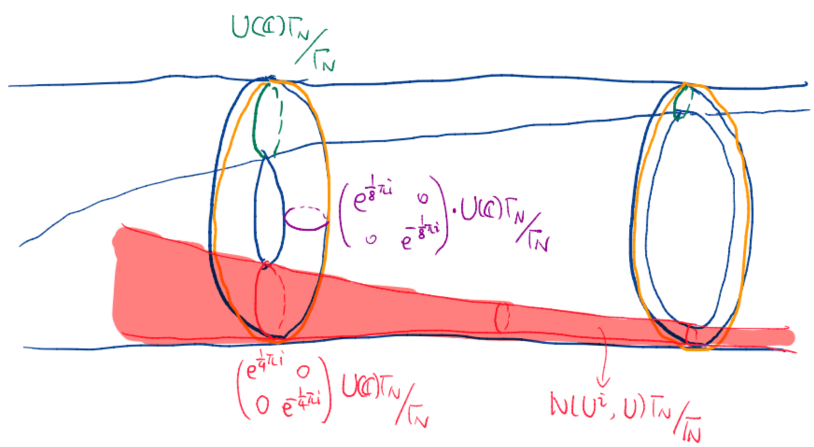


What about $\mathrm{Sing}(H,U)$? The possible $L \in \mathscr{H}$ and $L \subsetneq H$ are given as follows. For $z \in \mathbb{C}$, let $U^z := \left\{ \mathbf{u}_{s \cdot z} = \begin{bmatrix} 1 & sz \\ 0 & 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}$. Then every proper nontrivial connected subgroup of H is of this form. And $U^z \in \mathscr{H}$ iff $U^z \cap \Gamma$ is a lattice in U^z iff $\mathbb{R}.z \cap \mathbb{Z}[i] \leq \mathbb{R}.z$ is a lattice.

Note that

$$N(U^z, U) = \begin{bmatrix} \sqrt{z}^{-1} & 0 \\ 0 & \sqrt{z} \end{bmatrix} \cdot N_G(U^z) = \begin{bmatrix} \sqrt{z}^{-1} & 0 \\ 0 & \sqrt{z} \end{bmatrix} \cdot \{\mathbf{a}_t,\ t \in \mathbb{R}^\times\} \cdot \mathbf{U}(\mathbb{C})$$

And $\mathrm{Sing}(H,U)$ is the union of these $N(U^z, U)$ as z varies over $\mathbb{Z}[i]$.



2.2. **Example 2.**

- $H := \mathrm{SL}_2(\mathbb{R})$, $U^i := \{\mathbf{u}_{is},\ s \in \mathbb{R}\}$.

Lemma 2.2. $N(H,U) = U^i \cdot \mathrm{SL}_2(\mathbb{R}) \sqcup U^i \cdot \mathrm{SL}_2(\mathbb{R}) \cdot \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

Note that $U^i\cdot \mathrm{SL}_2(\mathbb{R})\cdot \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\cdot U^i\cdot \mathrm{SL}_2(\mathbb{R}) = \mathbf{U}(\mathbb{C})\cdot \mathrm{SL}_2(\mathbb{R})\cdot \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

Proof. As before, $N(H,U) = \{g\in G, \mathrm{Ad}(g)\cdot\mathfrak{h}\supset\mathfrak{u}\}$. It is direct to observe that RHS is a subset of LHS. It remains to do the converse.

Recall Bruhat decomposition again: $G = BwB\sqcup B$. If $g\in B$, then we are done since B is contained in the right hand side.

Now assume $g\in BwB$. Every element b of B can be written as $\mathbf{a}_t\mathbf{u}_s$ with $t\in\mathbb{C}^\times, s\in\mathbb{C}$. Since w normalizes $\{\mathbf{a}_t,\ t\in\mathbb{C}^\times\}$, we can write

$$g^{-1} = u_2\mathbf{a}_{t_1}wu_1,\quad \exists u_1,u_2\in\mathbf{U}(\mathbb{C}), t_1\in\mathbb{C}^\times.$$

Thus (to save notation we omit Ad in the following)

$$\begin{aligned} g^{-1}\cdot\mathfrak{u} =&(u_2\mathbf{a}_{t_1}wu_1)\cdot\mathfrak{u} = (u_2\mathbf{a}_{t_1})\cdot\begin{bmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{bmatrix} \\ =& u_2\cdot\begin{bmatrix} 0 & 0 \\ t_1^{-2}\mathbb{R} & 0 \end{bmatrix} = \begin{bmatrix} * & * \\ t_1^{-2}\mathbb{R} & * \end{bmatrix}\subset\mathfrak{h}=\mathfrak{sl}_2(\mathbb{R}). \end{aligned}$$

Thus $t_1^{-2}\mathbb{R}\subset\mathbb{R}\implies t_1\in\mathbb{R}\cup i\mathbb{R}$. In either case (write $u_1 = \mathbf{u}_{z_1}$ for some $z_1\in\mathbb{C}$),

$$g^{-1}\cdot\mathfrak{u} = u_2\cdot\begin{bmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{bmatrix} = \begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{bmatrix}\cdot\begin{bmatrix} 1 & -z_1 \\ 0 & 1 \end{bmatrix} = \mathbb{R}\cdot\begin{bmatrix} z_1 & -z_1^2 \\ 1 & -z_1 \end{bmatrix}\subset\mathfrak{sl}_2(\mathbb{R}).$$

Thus $z_1\in\mathbb{R}$. And the proof completes. □

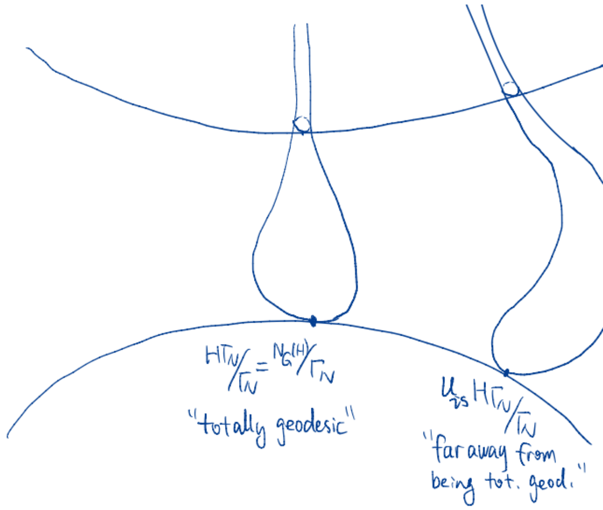
The above proof also shows that

Lemma 2.3. $N_G(H) = \mathrm{SL}_2(\mathbb{R})\sqcup\mathrm{SL}_2(\mathbb{R})\cdot\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

And hence one can check that

Lemma 2.4. $\Gamma_N := N_G(H)\cap\Gamma = \mathrm{SL}_2(\mathbb{Z})\sqcup\mathrm{SL}_2(\mathbb{Z})\cdot\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

Below is a picture of $N(H,U)/\Gamma_N$ sitting inside G/Γ_N as a closed subset. Note that its projection to G/Γ is dense (you probably saw this in the Exercise Sheet).

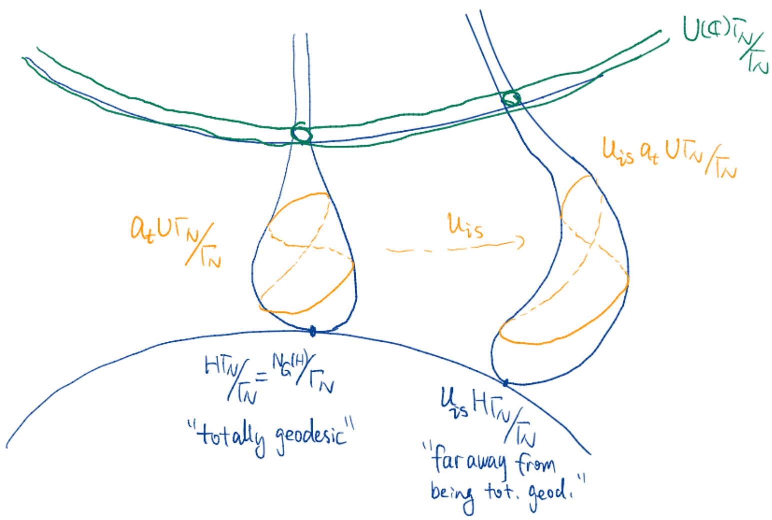


A picture of $N(H,U)/\Gamma_N$

Up to Γ_N-conjugacy, the only proper nontrivial connected subgroup of H containing U is just U itself. Thus $\mathrm{Sing}(H,U) = N(U,U)\Gamma_N$.

Lemma 2.5. *$N(U,U) = N_G(U)$ and is generated by* $\{\mathbf{a}_t \cdot \mathbf{u}_s,\ t \in \mathbb{R}^\times,\ s \in \mathbb{R}\} \cup \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

Thus the picture is not so new. Note that since U^i commutes with U, U^i translates of $U\Gamma_N/\Gamma_N$ does not "twist" the appearance of $U\Gamma_N/\Gamma_N$ (unlike U^i translates of $H\Gamma_N/\Gamma_N$).



Remark/Question: if you let $s,t \Rightarrow +\infty$ at the same time, it is unclear to me (asymptotically) how $u_{is} a_t U\Gamma_N/\Gamma_N$ look like relative to $u_{is} H\Gamma_N/\Gamma_N$

3. Revisit Oppenheim

Recall the notations when we discuss Oppenheim conjecture.

- $G = \mathrm{SL}_3(\mathbb{R})$, $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, $X := G/\Gamma$;
- $H_0 := \mathrm{SO}_{Q_0}(\mathbb{R})$ with $Q_0(x_1, x_2, x_3) = 2x_1x_3 - x_2^2$;
- $U = \left\{ \mathbf{u}_s := \exp\left(s \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right) \middle| s \in \mathbb{R} \right\} \subset H_0$;
- $A = \left\{ \mathbf{a}_t := \exp\left(t \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) \middle| t \in \mathbb{R} \right\} \subset H_0$;
- also we fix some $g_0 \in G$ and $x_0 := [g_0]_\Gamma := g_0\Gamma/\Gamma$.

As we explained, Oppenheim conjecture follows once we prove

Theorem 3.1. *If $Q := Q_0 \circ g_0$ is irrational, then $H_0 g_0 \Gamma/\Gamma$ is dense in X.*

Actually we proved something weaker in [Lec.6, Thm.1.2], which is sufficient. Now we would like to explain how to use Ratner's description of ergodic U-invariant probability measures to prove this stronger claim.

The idea is as follows. Take

$$K_0 := (H_0 \cap \mathrm{SO}_3(\mathbb{R}))^\circ,$$

a maximal connected compact subgroup of H_0 (I forgot whether H_0 is connected at this point...). Then $\mathbf{a}_t K_0.x_0 \subset H_0.x_0$ and we seek to show that as $t \to +\infty$ ($-\infty$ also ok), $\mathbf{a}_t K_0.x_0$ becomes dense in X. And this is achieved by the following equidistribution theorem

Theorem 3.2. *Let $\widehat{\mathrm{m}}_{K_0.x_0}$ be the unique K_0-invariant probability measure on $K_0.x_0$. Then*

$$\lim_{t \to +\infty} (\mathbf{a}_t)_* \widehat{\mathrm{m}}_{K_0.x_0} = \widehat{\mathrm{m}}_X$$

where $\widehat{\mathrm{m}}_X$ denotes the unique G-invariant probability measure on X and the convergence is w.r.t. the weak$*$ topology.

Remark 3.3. *From the proof, you will see that $\widehat{\mathrm{m}}_{K_0.x_0}$ can be replaced by any other probability measure that is absolutely continuous with respect to this one and does not affect the conclusion.*

Remark 3.4. *Instead of K_0, you can also use other subgroups of H_0 and prove analogues of the theorem above. Actually I lied a bit above, it would be easier if we replace K_0 by a bounded open subset of H_0. However, I prefer to do this in preparation for our later discussion on quantitative Oppenheim. This change only has an effect on Sec.3.10.*

3.1. **Step 1, nondivergence.** Let μ be a limit of $(\mu_t) := ((\mathbf{a}_t)_* \widehat{\mathrm{m}}_{K_0.x_0})$ as $t \to +\infty$.

Lemma 3.5. $\mu \in \mathrm{Prob}(X)$.

In other words, there is no escape of mass. This is a consequence of (C, α)-good property and a lemma in representation theory/linear algebra.

3.2. **Step 2, unipotent invariance.**

Lemma 3.6. *μ is U-invariant.*

Proof. Since $\widehat{\mathrm{m}}_{K_0.x_0}$ is K_0-invariant, μ_t is $\mathbf{a}_t K_0 \mathbf{a}_t$-invariant. Hence μ is invariant under the limit group, which turns out to be U.

More details: Let $\mathfrak{k}_0$ be the Lie algebra of K_0. Take $v_t \in \mathrm{Ad}(\mathbf{a}_t).\mathfrak{k}_0$, if $\lim v_t = v$, then by continuity of the induced map $G \times \mathrm{LFM}(X) \to \mathrm{LFM}(X)$, μ is $\exp(v)$-invariant.

Recall (see Lec.3) that the Lie algebra of H_0 is

$$\mathfrak{so}_{Q_0} = \left\{ \begin{bmatrix} x_{11} & x_{12} & 0 \\ x_{21} & 0 & x_{12} \\ 0 & x_{21} & -x_{11} \end{bmatrix} \right\}.$$

And the Lie algebra of $\mathrm{SO}_3(\mathbb{R})$ is given by anti-symmetric matrices. Thus by taking their intersection:

$$\mathfrak{k}_0 = \left\{ \begin{bmatrix} 0 & x_{12} & 0 \\ -x_{12} & 0 & x_{12} \\ 0 & -x_{12} & 0 \end{bmatrix} \right\}.$$

And

$$\mathrm{Ad}(\mathbf{a}_t)\mathfrak{k}_0 = \left\{ \begin{bmatrix} 0 & e^t x_{12} & 0 \\ -e^{-t} x_{12} & 0 & e^t x_{12} \\ 0 & -e^{-t} x_{12} & 0 \end{bmatrix} \right\}.$$

So depending on $s \in \mathbb{R}$, we take

$$v_t := \begin{bmatrix} 0 & s & 0 \\ -e^{-2t} s & 0 & s \\ 0 & -e^{-2t} s & 0 \end{bmatrix} \in \mathrm{Ad}(\mathbf{a}_t)\mathfrak{k}_0.$$

Then as s varies, $\lim v_t$ fills $\mathfrak{u}$, the Lie algebra of U. □

Thus by the first two steps we get (by passing to a subsequence)

$$\lim \mu_t = \mu \in \mathrm{Prob}(X)^U.$$

3.3. **Step 3, ergodic components and tubes.** By Theorem 3.5 and 3.6 from Lec.11, to show $\mu = \widehat{\mathrm{m}}_X$, it suffices to show that for every $H \in \mathscr{H}$, $H \neq G$, $\mu(T(H,U)) = 0$. (Note that Ratner's theorem [Lec 11,Theorem 1.1] is only used to go from $\mu(T(H,U)) = 0$ to $\mu = \widehat{\mathrm{m}}_X$. To show $\mu(T(H,U)) = 0$, we do not need it.) The way to achieve this is via:

Lemma 3.7. *For every compact subset E of $T(H,U)$ and $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}_\varepsilon$ of E such that*

$$\limsup_{t \to +\infty} \mu_t(\mathcal{N}_\varepsilon) \leq \varepsilon.$$

In view of Lec.10 (cf. [DS84]), we hope to find a bigger $\mathcal{N}_\varepsilon'$ such that

$$\mu_t(\mathcal{N}_\varepsilon) \leq \varepsilon \mu_t(\mathcal{N}_\varepsilon').$$

Since μ_t is a probability measure, this finishes the proof.

3.4. **Step 4, a lemma on linear representations.** Though we do not know how to find $\mathcal{N}_\varepsilon \subset \mathcal{N}_\varepsilon'$ at the moment, we do have something like this happening in a representation (rather than the complicated G/Γ) due to the (C,α)-good property. To give us more freedom (see below, the choice of Φ) for things to come, we need a slightly more flexible statement.

Definition 3.8. *Fix a non-empty connected bounded open set* $\mathrm{D} \subset \mathfrak{k}_0$, *let*

$$\psi_t : \mathrm{D} \to G, \quad x \mapsto \psi_t(x) := \mathbf{a}_t \exp(x).$$

Lemma 3.9. *Let V be a representation of G. Let W be a linear subspace of V. For every compact subset E of W and every $\varepsilon > 0$, there exists another compact set $F \subset W$ such that the following is true. For every open neighborhood Φ of F, there exists an open neighborhood Ψ of E such that for every $t \in \mathbb{R}$, $v \in V$, every ball $\mathrm{B} \subset \mathrm{D}$, at least one of the following is true*

1. $\overline{\psi_t(\mathrm{B}).v} \subset \Phi$;
2. $\mathrm{Leb}\left\{x \in \mathrm{B} \,\middle|\, \psi_t(x).v \in \Psi\right\} \leq \varepsilon\, \mathrm{Leb}\left\{x \in \mathrm{B} \,\middle|\, \psi_t(x).v \in \Phi\right\}$.

Remark 3.10. *The first possibility can often be excluded due to "algebraic" reasons (see Sec.3.10). And the second option is what we want.*

Remark 3.11. *You can replace* Leb *by any other measure equivalent to* Leb*, it is just that the choice of* Ψ *may depend on this measure. Actually in application we have in mind,* Leb *should be replaced by some measure which maps to* $\widehat{\mathrm{m}}_{K_0 x_0}$ *under the exponential and the orbit map. We are going to ignore this issue in the following.*

3.5. **Step 5, representation and dynamics, naive ideas.** Let $\Gamma_N := \Gamma \cap N_G(H)$.

Definition 3.12. *Let*

$$N_G(H)^{(1)} := \left\{g \in N_G(H) \,\middle|\, \det\left(\mathrm{Ad}(g), \mathfrak{h}\right) = \pm 1\right\}$$

Lemma 3.13. $\Gamma_N = \Gamma \cap N_G(H)^{(1)}$.

Take a representation V_H of G and a vector $v_H \in V_H$ such that the stabilizer of v_H (or just $\pm v_H$) in G is equal to $N_G(H)^{(1)}$. Moreover, we want V_H to be equipped with a $\mathbb{Q}$-structure (i.e., fix a copy of $\mathbb{Q}^{\dim V_H}$ in V_H, call it $V_H(\mathbb{Q})$) and $v_H \in V_H(\mathbb{Q})$. A priori, $N_G(H)^{(1)}$ is not known to be "observable" , the existence of such a pair (V_H, v_H) is not obvious. But one can take $V_H := \bigwedge^{\dim H} \mathfrak{sl}_n$ and $v_H := v_1 \wedge ... \wedge v_{\dim H}$ where $(v_1, ..., v_{\dim H})$ is a basis of $\mathfrak{h}$. For this specific choice of v_H, the stabilizer of $\pm v_H$ in G is equal to $N_G(H)^{(1)}$. You may also have other choices. For instance when $G = \mathrm{SL}_2(\mathbb{R})$ and H is equal to the upper triangular unipotent group, then V_H can be taken to be the standard representation $\mathbb{R}^2$ and $v_H = e_1$.

To go from the representation V_H to G/Γ, the following diagram is very natural.

$$\begin{tikzcd} & G/\Gamma_N \arrow[dl, "p"'] \arrow[dr, "q"] & & \\ G/\Gamma & & G/N_G(H)^{(1)} \arrow[r, hook, "\phi"] & V_H. \end{tikzcd}$$

Here p and q are natural projections and $\phi([g]) := g.v_H$. Strictly speaking ϕ may only be injective replacing V_H by $V_H/\pm 1$, but we will ignore this minor issue.

Here is something naive one can do at this stage. Recall $E \subset T(H,U)$ is a compact set.

1. Take a compact subset $\widetilde{E} \subset N(H,U)$ such that $E = [\widetilde{E}]_\Gamma$, the image of $\widetilde{E}$ in G/Γ;
2. Let $E^\vee := \phi \circ q(\widetilde{E}) = \widetilde{E}.v_H$;
3. Apply Lem.3.9 above to $E = E^\vee$ and W to be determined (you may take $W = V$ and see why it does NOT work). Then we get F (depending also on ε) by Lem.3.9, which asserts that for every open neighborhood Φ (we do not have a favorite Φ yet, so just fix some) there exists an open neighborhood Ψ of $E^\vee$ such that something holds.
4. We simply take $\mathcal{N}_\varepsilon := p\left((\phi \circ q)^{-1}\Psi\right)$ and $\mathcal{N}'_\varepsilon := p\left((\phi \circ q)^{-1}\Phi\right)$.

To simplify notations,

Definition 3.14. *For* $t \in \mathbb{R}$ *and* $[\gamma]_{\Gamma_N} \in \Gamma/\Gamma_N$,

$$\mathrm{D}_t(\mathcal{N}_\varepsilon) := \left\{y \in \mathrm{D} \,\middle|\, \psi_t(y).x_0 \in \mathcal{N}_\varepsilon\right\}$$
$$\mathrm{D}_t(\Psi, [\gamma]_{\Gamma_N}) := \left\{y \in \mathrm{D} \,\middle|\, \psi_t(y) g_0 \gamma.v_H \in \Psi\right\}.$$

And define $\mathrm{D}_t(\Psi) := \bigcup_{[\gamma] \in \Gamma/\Gamma_N} \mathrm{D}_t(\Psi, [\gamma])$. *Similarly define* $\mathrm{D}_t(\mathcal{N}'_\varepsilon)$, $\mathrm{D}_t(\Phi, [\gamma])$ *and* $\mathrm{D}_t(\Phi)$.

Thus from the definition (the naive definition of $\mathcal{N}_\varepsilon$ above, we will work with a different $\mathcal{N}_\varepsilon$ later in Sec.3.7)

$$\mathrm{D}_t(\mathcal{N}_\varepsilon) = \bigcup \mathrm{D}_t(\Psi, [\gamma]_{\Gamma_N}).$$

[The rest of this subsection may not make much sense. Please consider skipping to the next subsection.]

Let us see why this naive choice does not work.

Assume the first option of Lem.3.9 never happens. In particular, we have

$$\mathrm{Leb}\left(\mathrm{D}_t(\Psi, [\gamma]_{\Gamma_N})\right) \leq \varepsilon \,\mathrm{Leb}\left(\mathrm{D}_t(\Phi, [\gamma]_{\Gamma_N})\right)$$

for every $[\gamma]_{\Gamma_N}$.

A naive argument then goes like the following

$$\begin{aligned} \mu_t(\mathcal{N}_\varepsilon) = &\mathrm{Leb}(\mathrm{D}_t(\mathcal{N}_\varepsilon)) \leq \sum \mathrm{Leb}\left(\mathrm{D}_t(\Psi, [\gamma]_{\Gamma_N})\right) \\ \leq &\varepsilon \sum \mathrm{Leb}\left(\mathrm{D}_t(\Phi, [\gamma]_{\Gamma_N})\right) \end{aligned}$$

But we do not have a good control of the right hand side. We hope that $\mathrm{D}_t(\Phi, [\gamma]_{\Gamma_N})$, as $[\gamma]$ varies, would be a covering of $\mathrm{D}_t(\mathcal{N}'_\varepsilon)$ of multiplicity bounded by a constant C, independent of ε and t. If this were true, we would have

$$\mu_t(\mathcal{N}_\varepsilon) \leq (C\cdot\varepsilon)\,\mathrm{Leb}(\mathrm{D}_t(\mathcal{N}'_\varepsilon)) = (C\cdot\varepsilon)\mu_t(\mathcal{N}'_\varepsilon).$$

And we are done.

However, at least with these naive choices of $\mathcal{N}_\varepsilon$ and $\mathcal{N}'_\varepsilon$, $\mathrm{D}_t(\Phi, [\gamma]_{\Gamma_N})$ may not be covering of finite-index.

3.6. **Step 6, self-intersection.** Now we seek to refine the rather crude strategy proposed in Step 5 so that it would actually work.

First of all in general, unlike the $\mathrm{SL}_2(\mathbb{R})$-case, the projection $N(H,U)/\Gamma_N \to G/\Gamma$ is not injective (Example 1, see Sec.2.1, this is injective(at least almost), but example 2, see Sec.2.2, is not).

Lemma 3.15. *If $g \in G$ is such that for two different $[\gamma_1]_{\Gamma_N} \neq [\gamma_2]_{\Gamma_N} \in \Gamma/\Gamma_N$ we have $g\gamma_i \in N(H,U)$ for $i = 1,2$, then $g \in \mathrm{Sing}(H,U)\Gamma$.*

So ideally we would like to avoid $\mathrm{Sing}(H,U)$ (or its projection to G/Γ_N, or G/Γ) from our discussion. But this is impossible! Since usually $\mathrm{Sing}(H,U)$ is dense in $N(H,U)$ modulo Γ_N, every non-empty open set intersects non-trivially with it. Lucky for us, each time we only work with certain compact set F (to be found) in V_H (and we have the freedom of choosing its neighborhood). And the subset of $\mathrm{Sing}(H,U)$ that is "relevant to F" is indeed closed, see Lem.3.19.

To detect $N(H,U)$ inside V_H, it is convenient (though maybe not necessary) to have:

Definition 3.16. *Let W_H be the $\mathbb{R}$-linear subspace of V_H spanned by $N(H,U).v_H$.*

This W_H would be the W when we apply Lem.3.9 above.

Lemma 3.17. *We have*

$$(\phi\circ q)^{-1}(W_H) = N(H,U)/\Gamma_N.$$

Now is the important observation (the reader is reminded that being compact in V_H is not the same as being compact in $G/N_G(H)^{(1)}$, as the $\mathrm{SL}_2(\mathbb{R})$-case already told us, unless $G.v_H$ is closed in V_H, which is true if H is reductive by [Kem78] or if Γ is arithmetic and cocompact in G, for other reductive G's).

Definition 3.18. *Let F be a compact subset of W_H, let*

$$\mathrm{Sing}(F) := \left\{ g \in G \,\middle|\, g.v_H \in F,\ g\gamma.v_H \in F\ \exists \gamma \in \Gamma \setminus \Gamma_N \right\}$$

Thus $\mathrm{Sing}(F) \subset \mathrm{Sing}(H,U)$. The fact we need is that

Lemma 3.19. $\mathrm{Sing}(F)\Gamma$ *is closed.*

Sketch of proof. First note that $\Gamma.v_H$ is discrete in V_H. This is rather straight-forward since v_H is a rational vector and the image of Γ in $\mathrm{SL}(V_H)$ is commensurable with $\mathrm{SL}_N(\mathbb{Z})$ for $N = \dim V_H$. For non-arithmetic lattices, see [DM93] for a proof.

Therefore if (g_n) is bounded mod Γ and $(g_n.v_H)$ is bounded in V_H, then (g_n) is bounded modulo Γ_N. The conclusion follows quickly from here. □

Note that

$$\mathrm{Sing}(F)\Gamma := \left\{ g \in G \,\middle|\, g\gamma_1.v_H \in F,\ g\gamma_2.v_H \in F,\ \exists [\gamma_1] \neq [\gamma_2] \in \Gamma/\Gamma_N \right\}$$

Consequently, by a continuity argument and the discreteness of $\Gamma.v_H$,

Lemma 3.20. *Let E' be a compact set in $X \setminus [\mathrm{Sing}(F)]_\Gamma$. Then there exists an open neighborhood Φ of F such that for every $[g]_\Gamma \in E'$,*

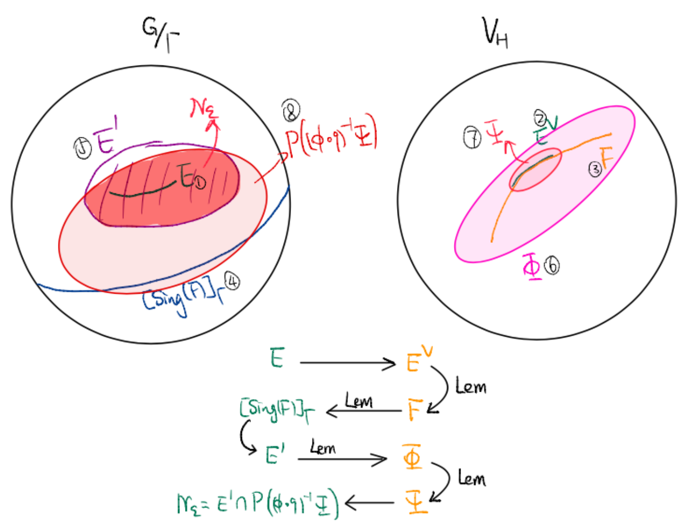
$$\#\left\{ [\gamma] \in \Gamma/\Gamma_N \,\middle|\, g\gamma.v_H \in \Phi \right\} \leq 1.$$

3.7. **Step 7, define the neighborhood.** Let us explain how to find $\mathcal{N}_\varepsilon$. Fix $E \subset T(H,U)$ and $\varepsilon > 0$.

Define $E^\vee$ as in Sec.3.5. By taking $W = W_H$ (see Def.3.16), Lem.3.9 offers some compact set F of W_H. By Lem.3.19, $[\mathrm{Sing}(F)]_\Gamma$ is closed and is contained $[\mathrm{Sing}(H,U)]_\Gamma$ by Lem.3.15.

Now we take E' to be any compact set away from $[\mathrm{Sing}(H,U)]_\Gamma$ whose interior contains E. We find an open neighborhood Φ of F such that the conclusion of Lem.3.20 holds. Then Ψ, an open neighborhood of E, is chosen according to Lem.3.9.

Just in case one gets confused, here is a diagram summarizing the logical dependence:



Now

$$\mathcal{N}_\varepsilon := \mathrm{Int}(E') \cap p\left((\phi \circ q)^{-1}\Psi\right)$$

3.8. **Step 8, a covering argument.** The proof will be concluded with the help of a covering argument, something we encountered when discussing nondivergence of unipotent flow on X_N. The argument here seems to differ from that of [EMS96].

Without loss of generality, assume D itself is a ball (the general case can be reduced to this one). The $\mathrm{D}^{(3)}(\bullet)$ is almost the same as $\mathrm{D}(\bullet)$ except that in Def.3.14, we replace D by the disk with the same center but whose radius is 3 times the radius of D (this is in order to apply Besicovitch's covering lemma, see Stein's book on real analysis, Chapter 3, Problem 3).

We further assume (this will be explained later in Sec.3.10)

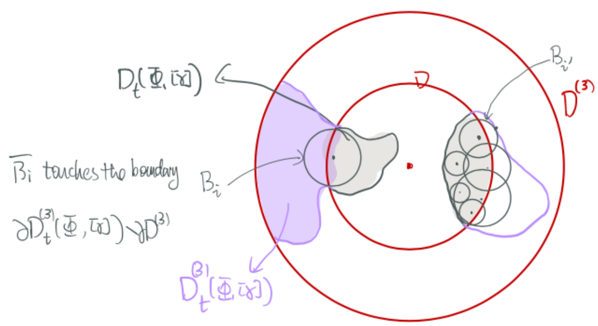
$$\text{for } t \text{ large enough}, \quad \psi_t(\mathrm{D})g_0\gamma.v_H \not\subseteq \Phi, \ \forall \gamma \in \Gamma. \tag{1}$$

Recall the definition $\mathrm{D}_t(\Phi)$ and $\mathrm{D}_t(\Phi,[\gamma])$ from Def.3.14.

For each $[\gamma]$ such that $\mathrm{D}_t(\Phi,[\gamma])$ is non-empty. Find balls $\{B_i\}_{i\in\mathscr{I}_{t,[\gamma]}} \subset \mathrm{D}_t^{(3)}(\Phi,[\gamma])$ whose **centers** cover $\mathrm{D}_t(\Phi,[\gamma])$. Here $\mathscr{I}_{t,[\gamma]}$ is some index set. We claim that we can find a covering such that for every $i \in \mathscr{I}_{t,[\gamma]}$, there exists $y \in \overline{B_i}$ such that

$$\psi_t(y)g_0\gamma.v_H \notin \Phi. \tag{2}$$

Indeed, for each $y \in \mathrm{D}_t(\Phi,[\gamma])$, take B_y to be the largest open ball centered at y. Then this collection would satisfy Equa.(2) by Equa.(1).



Let $\mathscr{I}_t := \bigsqcup_{[\gamma]\in\Gamma/\Gamma_N} \mathscr{I}_{t,[\gamma]}$. Then $\mathrm{D}_t(\Phi)$ is covered by the centers of $\{B_i\}_{i\in\mathscr{I}_t}$. By Besicovitch covering lemma, there exists a constant $C_0 > 0$, depending only on the dimension of $\mathfrak{k}_0$, and a subset $\mathscr{J}_t \subset \mathscr{I}_t$ such that $\{B_i\}_{i\in\mathscr{J}_t}$ is a covering of $\mathrm{D}_t(\Phi)$ of multiplicity (i.e., the maximal number of possible overlaps among B_i's) bounded by C_0.

Let $\mathscr{J}_{t,[\gamma]} := \mathscr{J}_t \cap \mathscr{I}_{t,[\gamma]}$.

Let me summarize the discussion in this subsection by the following lemma:

Lemma 3.21. *Take t such that Equa.(1) holds. There exists a covering of $\mathrm{D}_t(\Phi)$ by open balls $(B_j)_{j\in\mathscr{J}_t}$ together with a partition of the index set $\mathscr{J}_t = \sqcup_{[\gamma]\in\Gamma/\Gamma_N}\mathscr{J}_{t,[\gamma]}$ satisfying the following:*

1. *For $j \in \mathscr{J}_{t,[\gamma]}$, $B_j \subset \mathrm{D}^{(3)}(\Phi,[\gamma])$.*
2. *For $j \in \mathscr{J}_{t,[\gamma]}$, there exists $y \in \overline{B_j}$ such that*
$$\mathbf{a}_t \exp(y)g_0\gamma.v_H \notin \Phi.$$
3. *The multiplicity of the covering is at most C_0 for a constant $C_0 > 0$ depending only on the dimension of $\mathfrak{k}_0$. Or more formally,*
$$\sum_{j\in\mathscr{J}_t} 1_{B_j} \leq C_0.$$

3.9. Step 9, finish the proof under some assumption.

Lemma 3.22. *There is a constant $C_1 > 0$ such that for t satisfying Equa.(1)*

$$\mathrm{Leb}(\mathrm{D}_t(\mathcal{N}_\varepsilon)) \leq C_1 \varepsilon \,\mathrm{Leb}(\mathrm{D}).$$

Thus Lem.3.7 follows from this lemma provided Equa.(1) is verified.

Proof. Take $C_1 := 3^{\dim \mathfrak{k}_0} C_0$.

Take $y \in \mathrm{D}_t(\mathcal{N}_\varepsilon)$, then by Lem.3.20, there exists a unique $[\gamma_y] \in \Gamma/\Gamma_N$ such that

$$\psi_t(y) g_0 \gamma_y . v_H \in \Phi.$$

On the other hand, since $y \in \mathrm{D}_t(\Psi) \subset \mathrm{D}_t(\Phi)$, there exists $[\gamma] \in \Gamma/\Gamma_N$ and $j \in \mathscr{J}_{t,[\gamma]}$ such that

$$\psi_t(y) g_0 \gamma . v_H \in \Phi, \text{ and } y \in B_j.$$

By uniqueness, $[\gamma] = [\gamma_y]$. Let $\mathrm{D}_t(\mathcal{N}_\varepsilon, [\gamma]) := \mathrm{D}_t(\mathcal{N}_\varepsilon) \cap \mathrm{D}_t(\Psi, [\gamma])$.

We have proved that for every $[\gamma] \in \Gamma/\Gamma_N$,

$$\mathrm{D}_t(\mathcal{N}_\varepsilon, [\gamma]) = \bigcup_{j \in \mathscr{J}_{t,[\gamma]}} B_j \cap \mathrm{D}_t(\mathcal{N}_\varepsilon, [\gamma]). \tag{3}$$

By comparison, it may not be true that (even if you replace Φ by the smaller Ψ)

$$\mathrm{D}_t(\Phi, [\gamma]) = \bigcup_{j \in \mathscr{J}_{t,[\gamma]}} B_j \cap \mathrm{D}_t(\Phi, [\gamma]).$$

Now everything follows from this, the linear algebra lemma Lem.3.9 and the covering argument Lem.3.21. More details:

$$\begin{aligned}
\mathrm{Leb}(\mathrm{D}_t(\mathcal{N}_\varepsilon)) &= \sum_{[\gamma] \in \Gamma/\Gamma_N} \mathrm{Leb}(\mathrm{D}_t(\mathcal{N}_\varepsilon, [\gamma])) \\
(\text{ Equa.(3) }) &\leq \sum_{[\gamma] \in \Gamma/\Gamma_N} \sum_{j \in \mathscr{J}_{t,[\gamma]}} \mathrm{Leb}(\mathrm{D}_t(\mathcal{N}_\varepsilon, [\gamma]) \cap B_j) \\
&\leq \sum_{[\gamma] \in \Gamma/\Gamma_N} \sum_{j \in \mathscr{J}_{t,[\gamma]}} \mathrm{Leb}(\mathrm{D}_t(\Psi, [\gamma]) \cap B_j) \\
(\text{ Lem.3.9 and Equa.(2) }) &\leq \sum_{[\gamma] \in \Gamma/\Gamma_N} \sum_{j \in \mathscr{J}_{t,[\gamma]}} \varepsilon \,\mathrm{Leb}(B_j) \\
(\text{ Lem.3.21 }) &\leq \varepsilon C_0 \,\mathrm{Leb}(\mathrm{D}_t^{(3)}(\Phi)) \leq C_0 \varepsilon \,\mathrm{Leb}(\mathrm{D}^{(3)}) = C_1 \varepsilon \,\mathrm{Leb}(\mathrm{D}).
\end{aligned}$$

□

The promised $\mathcal{N}'_\varepsilon$ did not show up explicitly. You may take it to be $p\left((\phi \circ q)^{-1}\Phi\right)$ in light of the discussion above.

3.10. Step 10, linear expansion.

Note that the discussion so far only uses

- the limit measure μ is unipotent-invariant;
- (C, α)-good properties.

In particular, as long as μ can be shown to be unipotent invariant, the discussion above applies equally well if you replace $\mathbf{a}_{t_n}$ by any other sequences (g_n) in G and $\exp(\mathrm{D})$ by any other bounded smooth curve/manifold in G equipped with a smooth measure.

Now we explain why Equa.(1) holds, for our particular choice of $\mathbf{a}_t$ and $\exp(\mathrm{D})$.

Recall that we may think of (the connected component of) H_0 as the image of $\mathrm{SL}_2(\mathbb{R})$ under the Adjoint representation. And K_0 may be thought of as the image of $\mathrm{SO}_2(\mathbb{R})$, $\{\mathbf{a}_t\}$ the image of $\mathbf{b}_t := \mathrm{diag}(e^t, e^{-t})$.

Lemma 3.23. *Let V be an irreducible nontrivial representation of $\mathrm{SL}_2(\mathbb{R})$. Let Ω be a nonempty open subset of $\mathrm{SO}_2(\mathbb{R})$. Then for every constant $C>0$, there exists $T_0>0$ (depending on C, Ω, the choice of metric on V) such that for every $t>T_0$, every $v_{\neq}0\in V$*

$$\sup_{\omega\in\Omega}\|\mathbf{b}_t\omega.v\|\geq C\|v\|.$$

Remark 3.24. *After the proof is given, it should be clear that $\mathrm{SL}_2(\mathbb{R})$ can be replaced by any other simple Lie group, $\mathrm{SO}_2(\mathbb{R})$ replaced by a maximal compact subgroup, $\mathbf{b}_t$ replaced by any one-parameter diagonalizable subgroup that is stable under Cartan involution associated with this maximal compact subgroup. Moreover, once V is fixed, C can be taken to be $\kappa_1 e^{\kappa_2|t|}$ for some $\kappa_1,\kappa_2>0$ and the condition $t>T_0$ can be removed.*

Remark 3.25. *A weaker statement, with "for every $C>0$" replaced by "there exists some $c>0$" (and ignore the $t>T_0$ condition) holds in much greater generality, see* [RS18]. *And this condition is sufficient to conclude the limit measure supports on a unique tube (see* [RZ16]*).*

Proof of Equa.(1) assuming Lem.3.23. Assume otherwise, find some $\gamma_t\in\Gamma$ such that

$$\mathbf{a}_t\exp(\mathrm{D})g_0\gamma_t.v_H\subset\Phi$$

for t inside certain sequence tending to $+\infty$.

Decompose $V=V_1\bigoplus V_2$ in a H_0-equivariant way such that $V_1=V^{H_0}$, the vectors fixed by H_0. Write π_i for the projection $V\to V_i$ w.r.t. this decomposition. Without loss of generality we assume $V_1\perp V_2$ by changing the Euclidean metric. Thus for $t\in R$, $y\in\mathrm{D}$.

$$\begin{aligned}\mathbf{a}_t\exp(y)(g_0\gamma_t.v_H)&=\mathbf{a}_t\exp(y)(\pi_1(g_0\gamma_t.v_H)+\pi_2(g_0\gamma_t.v_H))\\&=\pi_1(g_0\gamma_t.v_H)+\mathbf{a}_t\exp(y)\pi_2(g_0\gamma_t.v_H)\\\implies\left\|\mathbf{a}_t\exp(y)g_0\gamma_t.v_H\right\|&=\left\|\pi_1(g_0\gamma_t.v_H)\right\|+\left\|\mathbf{a}_t\exp(y)\pi_2(g_0\gamma_t.v_H)\right\|.\end{aligned}$$

For the 2nd term, the above Lem.3.23 implies that for t large enough, for suitable choice of y_t,

$$\left\|\mathbf{a}_t\exp(y_t)\pi_2(g_0\gamma_t.v_H)\right\|\geq\left\|\pi_2(g_0\gamma_t.v_H)\right\|.$$

So $\mathbf{a}_t\exp(y_t)$ action does not decrease the norm of $g_0\gamma_t.v_H$. Since Φ is bounded, this implies that

$$(g_0\gamma_t.v_H)\text{ is bounded.}$$

But $\Gamma.v_H$, and hence $g_0\Gamma.v_H$ is discrete in V_H. A discrete, bounded set has no choice but being finite. After passing to a subsequence, we assume $\gamma_t=\gamma_1$ for all t (in some infinite subsequence tending to $+\infty$).

Now if $g_0\gamma_1.v_H\notin V_1$, then $\pi_2(g_0\gamma_1.v_H)\neq0$. Take $C_2>0$ such that every element in Φ has norm at most C_2. Apply Lem.3.23 to $C=1.1C_2\left\|\pi_2(g_0\gamma_1.v_H)\right\|^{-1}$, then we find y'_t, for t large enough, such that

$$\left\|\mathbf{a}_t\exp(y'_t)\pi_2(g_0\gamma_t.v_H)\right\|\geq1.1C_2\left\|\pi_2(g_0\gamma_1.v_H)\right\|^{-1}\left\|\pi_2(g_0\gamma_t.v_H)\right\|=1.1C_2.$$

So $\mathbf{a}_t\exp(y'_t)g_0\gamma_1.v_H$ can not live in Φ, a contradiction.

Thus $g_0\gamma_1.v_H\in V_1$, or in other words, $g_0\gamma_1.v_H$ is fixed by H_0. Recall the stabilizer of v_H in G is $N_G(H)^{(1)}$, thus, $g_0^{-1}H_0g_0\subset\gamma_1N_G(H)^{(1)}\gamma_1^{-1}\subset\gamma_1N_G(H)\gamma_1^{-1}$.

A Lie algebra computation shows that $\mathrm{Ad}(g_0)^{-1}\mathfrak{h}_0$ is a maximal proper Lie subalgebra. Actually, the only non-zero and non-full $\mathrm{Ad}(H_0)$-stable Lie subalgebra of $\mathfrak{sl}_3$ is $\mathfrak{h}_0$. Thus $\mathrm{Ad}(g_0^{-1})\mathfrak{h}_0=\mathrm{Ad}(\gamma_1)\mathfrak{h}$ and $g_0^{-1}H_0^\circ g_0=\gamma_1H\gamma_1^{-1}$. In particular $g_0^{-1}H_0^\circ g_0\cap\Gamma$ is a lattice in $g_0^{-1}H_0^\circ g_0$. This implies that $Q_0\circ g_0$ is proportional to a rational quadratic form, a contradiction.

□

Proof of Lem.3.23. Decompose V w.r.t. the $\mathbf{b}_t$ action

$$V = V^- \oplus V^0 \oplus V^+$$

into contracting/fixed/expanding subspaces. Namely, this decomposition is stable under $\mathbf{b}_t$ action. Moreover $V^0 = V^{\{\mathbf{b}_t\}}$ and for some $c_1, \kappa_1 > 0$,

$$\|\mathbf{b}_t.v\| \geq c_1 e^{\kappa_1 t} \|v\|, \forall v \in V^+;$$
$$\|\mathbf{b}_t.v\| \leq c_1^{-1} e^{-\kappa_1 t} \|v\|, \forall v \in V^-.$$

Let π^-, π^0, π^+ be the corresponding projections. We claim that there exists $c_2 > 0$ such that

$$\sup_{\omega\in\Omega} \left\|\pi^+(\omega.v)\right\| \geq c_2 \|v\|, \forall v \in V. \tag{4}$$

Once this is done, the proof completes. It suffices to verify Equa.(4) under the assumption $\|v\| = 1$. If not true, then we can find a sequence of unit vectors (v_n) such that

$$\sup_{\omega\in\Omega} \left\|\pi^+(\omega.v_n)\right\| \to 0.$$

Let v_∞ be any limit of (v_n). Since Ω is bounded, we have

$$\pi^+(\omega.v_\infty) = 0, \forall \omega \in \Omega.$$

In other words,

$$\Omega.v_\infty \subset V^- \oplus V^0.$$

Since this is a condition defined by vanishing of some polynomials and Ω is Zariski dense in $\mathrm{SO}_2(\mathbb{R})$, we have

$$\mathrm{SO}_2(\mathbb{R}).v_\infty \subset V^- \oplus V^0.$$

Since $w_0 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathrm{SO}_2(\mathbb{R})$ and $w_0 \mathbf{b}_t w_0^{-1} = \mathbf{b}_{-t}$, we see that $w_0 V^- = V^+$, $w_0 V^0 = V^0$ and $w_0 V^+ = V^-$. So

$$\mathrm{SO}_2(\mathbb{R}).v_\infty \subset V^0.$$

So every vector in V^0 is fixed by $\mathrm{SO}_2(\mathbb{R})$ and $\{\mathbf{b}_t\}$, which generate the full $\mathrm{SL}_2(\mathbb{R})$. This is a contradiction. □

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