LECTURE 1

RUNLIN ZHANG

For some general references, scroll down to the end.

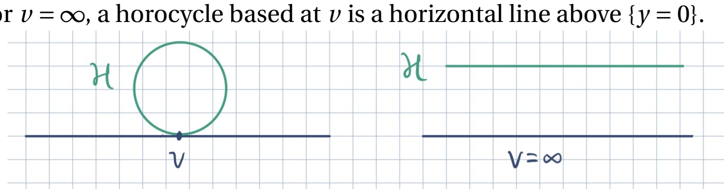
1. MOTIVATIONS/APPLICATIONS

A few sample results that turn out to be related to homogeneous dynamics are listed.

1.1. **Horocycles on constant negative curvature surfaces.**

Equip $\mathbb{H}^2 := \{x + iy \in \mathbb{C}, y > 0\}$ with the metric $\frac{\mathrm{dx}^2+\mathrm{dy}^2}{y^2}$. Let $\Gamma \leq \mathrm{Isom}(\mathbb{H}^2)$ be a discrete (torsion free) subgroup such that $\mathbb{H}^2/\Gamma$ is compact (such a subgroup is called a uniform lattice). Then $\mathbb{H}^2/\Gamma$ is a compact surface of constant negative curvature. Conversely, every surface with constant negative curvature arises this way. Let $\pi : \mathbb{H}^2 \to \mathbb{H}^2/\Gamma = M$ be the quotient map.

Consider a horocycle $\mathscr{H}$ in $\mathbb{H}^2$. Explicitly, for each $v \in \{x + iy, y = 0\}$, a horocycle based at v is a circle (with respect to the Euclidean metric) in $\mathbb{H}^2$ tangent to $\{y = 0\}$ at v. For $v = \infty$, a horocycle based at v is a horizontal line above $\{y = 0\}$.



Now we take the image of $\mathscr{H}$ under the projection π.

Theorem 1.1 ([Hed36])**.** *For every $\mathscr{H}$, $\pi(\mathscr{H})$ is dense in M.*

If $M = \mathbb{H}^2/\Gamma$ ($\Gamma \leq \mathrm{Isom}(\mathbb{H}^2)$ still discrete) is just of finite volume, then

Theorem 1.2.
1. *$\pi(\mathscr{H})$ is either closed or dense in M.*
2. *Let $\pi(\mathscr{H}_i)$ be a sequence of closed horocycles, then as the length goes to infinity, $\pi(\mathscr{H}_i)$ becomes dense in M.*

[Reference missing]

Remark 1.3. *Assume $M = \mathbb{H}^2/\Gamma$ has finite volume. Then there exists closed $\pi(\mathscr{H})$ iff M is non-compact.*

By comparison, the image under π of a geodesic is very different. The image could be closed, dense, or in between. And closed geodesics do not necessarily equidistribute towards the volume measure (though on average they do equidistribute).

1.2. **Isometric immersion of hyperbolic spaces.**

Let $\mathbb{H}^3$ be the three dimensional hyperbolic space $\{(x + iy, z) \in \mathbb{C} \times \mathbb{R}, z > 0\}$ equipped with the metric $\frac{1}{z^2}(\mathrm{dx}^2 + \mathrm{dy}^2 + \mathrm{dz}^2)$. Let $\Gamma \leq \mathbb{H}^3$ be a discrete (torsion free) subgroup, such that $\mathbb{H}^3$ is compact (finite volume suffices). Consider an isometric embedding $\iota : \mathbb{H}^2 \to \mathbb{H}^3$. The image of ι can be explicitly described. There are two cases:

Date: 2022.2.

1. given a circle on $\{z=0\}$, then there exists a unique half-sphere in $\mathbb{H}^3$ whose boundary is this given circle;
2. given a line on $\{z=0\}$, then there exists a unique half-plane in $\mathbb{H}^3$ whose boundary is this given line.

Then $\iota(\mathbb{H}^2)$ is either a half-sphere or a half-plane described above. Similarly, we consider the image of $\iota(\mathbb{H}^2)$ under $\pi : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma =: M$,

Theorem 1.4.
1. *$\pi(\iota(\mathbb{H}^2))$ is either closed or dense in M;*
2. *Given an infinite sequence of distinct closed $\pi(\iota_i(\mathbb{H}^2))$, then $\lim_i \pi(\iota_i(\mathbb{H}^2))$ is dense in M.*

Remark 1.5. *That the volume of $\pi(\iota_i(\mathbb{H}^2))$ would go to infinity is automatic.*

1.3. **Oppenheim conjecture/Margulis theorem.** Consider a non-degenerate real quadratic form in three (larger than 3 also ok) variables, viewed as a function $Q : \mathbb{R}^3 \to \mathbb{R}$. Assume it is indefinite. Note that if Q is a quadratic form with rational coefficients or proportional to such a form, then $Q(\mathbb{Z}^3)$ is discrete in $\mathbb{R}$.

Theorem 1.6. *If Q is NOT proportional to a quadratic form with rational coefficients, then $Q(\mathbb{Z}^3)$ is dense in $\mathbb{R}$.*

Remark 1.7. *It is also true replacing $\mathbb{Z}^3$ by primitive vectors. The proof is a bit harder.*

Remark 1.8. *This is false if Q has two variables.*

Proof. There are badly approximable numbers. Such a number α would satisfy the following. There exists a positive number $c > 0$ such that for all $(m,n) \in \mathbb{Z}^2$ ($(m,n) \neq (0,0)$), $|m| \cdot |m\alpha + n| > c$. We can find such an α s.t. $\alpha^2 \notin \mathbb{Q}$, e.g. $\alpha = \sqrt{2}+1$. Now we just take $Q(x_1,x_2) := x_1^2 - \alpha^2 x_2^2 = (x_1 - \alpha x_2)(x_1 + \alpha x_2)$. Indeed if you want $Q(x_1,x_2)$ to be small, then at least one of $x_1 \pm \alpha x_2$ has to be small. But then the other one is comparable with x_2. Hence $|Q(x_1,x_2)| >> c$. □

The above theorem admits a quantitative version in certain cases. Let Q be a (nondegenerate) quadratic form in 4 variables of signature $(3,1)$ (what follows does not apply to signature $(2,2)$, $(1,2)$). Assume Q is irrational as above.

Theorem 1.9. *There exists $\lambda_Q > 0$ such that for every $a < b \in \mathbb{R}$,*

$$\#\left\{x \in \mathbb{Z}^4 \,\middle|\, Q(x) \in (a,b),\ \|x\| \le T\right\} \sim \mathrm{Vol}\left\{x \in \mathbb{R}^4 \,\middle|\, Q(x) \in (a,b),\ \|x\| \le T\right\} \sim \lambda_Q (b-a) T^2$$

Remark 1.10. *There was some discussion about why the exponent of T is only 2 even when the dimension of the relevant set is the "bounded neighborhood" of something of dimension 3. As remarked by Jinpeng An, the shape of the region, as T gets large, tend to collapse in certain region (maybe we will discuss this more precisely later in the course). Actually, the growth rate resembles the growth rate of the level set of the quadratic form with respect to the Haar measure there. These two facts are related when asking (a,b) to shrink to a point. See* [BR95].

1.4. **Littlewood conjecture.** Let $\alpha \in \mathbb{R}$ (assume everything is irrational just in case of some trivialities). By pigeon-hole principle(?), one can show that

$$\inf_{(m\neq 0,n)\in\mathbb{Z}^2} |m| \cdot |m\alpha + n| \le 1.$$

On the other hand there exists α ("badly approximable numbers") such that

$$\inf_{(m\neq 0,n)\in\mathbb{Z}^2} |m|\cdot|m\alpha+n|>0.$$

The Littlewood conjecture is

Conjecture 1.11. *For every pair $(\alpha,\beta)\in\mathbb{R}^2$ irrational,*

$$\inf_{(m\neq 0,n_1,n_2)\in\mathbb{Z}^3} |m|\cdot|m\alpha+n_1|\cdot|m\beta+n_2|=0.$$

To make it look closer to the Oppenheim conjecture, you may write $l_\alpha(x,y,z):=\alpha x+y$, $l_\beta(x,y):=\beta x+z$ and $\varphi(x,y,z):=x$. Let $L(x,y,z):=\varphi\cdot l_\alpha\cdot l_\beta$. Then the conjecture asserts that when L is "irrational", then $\inf_{(x,y,z)\in\mathbb{Z}^3}|L(x,y,z)|$ is dense at 0. By comparison, the Oppenheim conjecture is equivalent to $Q(\mathbb{Z}^3)$ being dense at 0.

Our current knowledge is

Theorem 1.12. *The set*

$$\left\{(\alpha,\beta)\in\mathbb{R}^2,\ \textit{that fails this conjecture}\right\}$$

has Hausdorff dimension 0.

Remark 1.13. *For every $\delta>0$, there exists (α,β) such that*

$$\liminf_{n\in\mathbb{Z}} n^{1+\delta}\,\|n\alpha\|\,\left\|n\beta\right\|>0.$$

According to [Gal62]*, this was done in* [Spe42]*. So the exponent on n is the best one can hope for. On the other hand, maybe one can improve n by* $\log n$ *(see* [Gal62] *for some restrictions though).*

A more recent survey is "AROUND THE LITTLEWOOD CONJECTURE IN DIOPHANTINE APPROXIMATION", Bugeaud.

link=https://pmb.centre-mersenne.org/item/10.5802/pmb.1.pdf

1.5. **Quantum unique ergodicity.** Let (M,d) be a closed hyperbolic surface of constant negative curvature. Let Δ be the Laplacian operator $-y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$.

Fact 1. Eigenvalues of Δ are non-negative and discrete in $\mathbb{R}$, say enumerated as

$$0=\lambda_1<\lambda_2<\ldots$$

Fact 2. For each λ_i, the eigenspace E_{λ_i} consists of smooth functions and has finite dimension;

Fact 3. Different eigenspaces are mutually orthogonal and $L^2(M)$ is spanned by them.

(see e.g. Thm 3.2.1, Jost, Riemannian Geometry; Thm 4.43, Gallot, Hulin, Lafontaine.)

Now take $f_i\in E_{\lambda_i}$. We are interested in the limiting behavior of the sequence of measures $\{|f_i|^2\,\mathrm{Vol}\}$, normalized to be probability measures.

A theorem (quantum ergodicity) of Snirelman says that there exists a density one subsequence n_i such that $\lim|f_{n_i}|^2\,\mathrm{Vol}=\mathrm{Vol}$ (suitably normalized) in the weak* topology (this theorem holds for more general compact Riemannian manifold, as long as the geodesic flow is ergodic, a property that holds for every negatively curved surface).

Conjecture 1.14 (Quantum unique ergodicity)**.** $\lim|f_n|^2\,\mathrm{Vol}=\mathrm{Vol}$ *holds without passing to any subsequence.*

This is still open. Progress is made when the fundamental group is a "congruence subgroup" where there is an additional supply of operators, called Hecke operators, that commute with the Laplacian.

Theorem 1.15. *Assume $\{f_i\}$ is a sequence of Hecke-Laplacian eigenfunctions. Then*

$$\lim |f_n|^2 \mathrm{Vol} = \mathrm{Vol}$$

in the weak topology.*

In the non-compact congruence case, this also holds for Hecke-Maass forms whose proof requires one more step to guarantee non-divergence.

2. Measure rigidity

2.1. **Unipotent flows.** Consider $\mathrm{SL}_2(\mathbb{R})$ and a discrete subgroup Γ. Equip $\mathrm{SL}_2(\mathbb{R})$ with a right invariant Riemannian metric. Then the volume measure m_X on $\mathrm{SL}_2(\mathbb{R})/\Gamma$ is left invariant under $\mathrm{SL}_2(\mathbb{R})$. We normalize it to be a probability measure.

Consider the subgroup

$$\left\{ u_s := \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \ s \in \mathbb{R} \right\}$$

Theorem 2.1. *Assume $\mathrm{SL}_2(\mathbb{R})/\Gamma$ is compact. Then m_X is the unique $\{u_s\}$-invariant probability measure.*

This would immediately imply the denseness result above.

Theorem 2.2. *Assume $\mathrm{SL}_2(\mathbb{R})/\Gamma$ has finite volume. Then each $\{u_s\}$-invariant probability measure is a convex combination (possible in the form of an integral) of the following*

1. m_X*;*
2. *the $\{u_s\}$-invariant measure supported on a closed (and compact) orbit of $\{u_s\}$.*

The implication to orbit closure requires an analysis on this convex combination.

In general, Ratner's measure classification on **ergodic** invariant measures for Ad-unipotent flows roughly reads as follows.

Theorem 2.3 (Measure rigidity theorem)**.** *Assume the following*

- *a connected Lie group (2nd countable) G together with a discrete subgroup Γ;*
- *a one-parameter **Ad-unipotent** subgroup $U = \{u_s\}_{s\in\mathbb{R}}$ of G.*

*Then every U-invariant ergodic probability measure μ on G/Γ is **homogeneous**.*

Ad-unipotent means that the image of $\{u_s\}$ under the Adjoint representation in $GL(\mathfrak{g})$ consist of unipotent matrices.

For a measure μ on G/Γ define the closed subgroup of G by

$$H := G_\mu := \{g \in G, \ g_*\mu = \mu\}.$$

We say that a probability measure μ is homogeneous if there exists $x \in X = G/\Gamma$ such that $\mu(Hx) = 1$.

Remark 2.4. *When G is a semisimple closed subgroup of SL_n, Ad-unipotent is the same as being unipotent in SL_n.*

Remark 2.5. *Various "connected" assumptions may be dropped with similar conclusions. E.g. one may consider $\{u_s\}_{s\in\mathbb{Z}}$.*

Remark 2.6. *Let H, x be as in the theorem and the definition above. Then Hx is closed in G/Γ. This is proved in [Raghunathan, 1.13] assuming G/Γ admits a finite G-invariant measure (i.e., Γ is a lattice in G), but the proof carries through without this assumption.*

Remark 2.7. *Let H, x be as in the theorem and the definition above. Then by modifying x, one can show that $Hx = H^\circ x$ and U is contained in H°.*

Theorem 2.8 (Equidistribution and topological rigidity I)**.** *Further assume that* Γ *is a lattice in* G*. Then for every* x*, there exists* $\{u_s\} \leq H \leq G$ *closed connected subgroup such that*

1. Hx *is closed and supports an* H*-invariant probability measure* μ_H*;*
2. *for every bounded continuous function* $f : G/\Gamma \to \mathbb{R}$*,*
$$\lim_{T\to\infty}\int_0^T f(u_t\cdot x)\,\mathrm{dt} \text{ exists and is equal to } \int f(x)\mu_H(x).$$
3. $U \curvearrowright \mu_H$ *is ergodic;*
4. $U\cdot x$ *is dense in* $H\cdot x$*.*

The logic of Ratner is

$$\text{Measure rigidity} \implies \text{Equidistribution} \implies \text{Topological rigidity.}$$

Nevertheless, there is a different (potential) approach by deducing the topological rigidity bypassing ergodic theory.

The topological rigidity is the original Raghunathan's conjecture.

Theorem 2.9 (Topological rigidity II)**.** *Let* G, Γ *be as in the last theorem. Let* $L \leq G$ *be a Lie subgroup generated by one-parameter Ad-unipotent subgroups. Then for every* $x \in G/\Gamma$*, there exists* $L \leq H \leq G$ *and* $V \leq L$ *some one-parameter Ad-unipotent subgroup (of G) such that*

1. Hx *is closed and supports an* H*-invariant probability measure;*
2. $\overline{Lx} = \overline{Vx} = Hx$*;*
3. $V \curvearrowright \mu_H$ *is ergodic.*

2.2. **Higher rank diagonalizable action.** Fact: Let

$$a_t := \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix}.$$

Then the $\{a_t\}_{t\in\mathbb{R}}$ action on $\mathrm{SL}_2(\mathbb{R})/\Gamma$ admits many invariant probability measures/closed sets and they are not easy to classify. The conjecture is that the situation would become better in higher rank.

Let

$$A := \left\{ \begin{bmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3} \end{bmatrix} . t_i \in \mathbb{R}, \sum t_i = 0 \right\} \cong \mathbb{R}^2.$$

Consider the $A \curvearrowright \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$.

Conjecture 2.10.
- *Every ergodic invariant probability measure is homogeneous;*
- *Every bounded (in the unbounded case, statements need to be modified) orbit of* A *is homogeneous.*

Of course one can propose similar (but necessarily more complicated) conjectures for other (semisimple) Lie groups G and other A's.

Theorem 2.11. *Let* $G := \mathrm{SL}_3(\mathbb{R})$, $\Gamma = \mathrm{SL}_3(\mathbb{Z})$ *and* A *same as above. Let* μ *be an* A*-invariant ergodic probability measure on* G/Γ*. Assume for some* $a \in A$, $h_\mu(a) > 0$*. Then* μ *is the* G*-invariant probability measure on* G/Γ*.*

The topological implication is that

Theorem 2.12. *The Hausdorff dimension of*

$$\{x \in G/\Gamma,\ Ax \text{ is bounded}\}$$

is 2.

Note that the union of compact A-orbits is a countable union, hence also has Hausdorff dimension 2.

A theorem of slightly different flavor, related to the AQUE theorem above, is

Theorem 2.13. *Let* $G := \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, $\Gamma \leq G$ *irreducible (e.g.* $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$*). Let* $H := \{e\} \times \mathrm{SL}_2(\mathbb{R})$*. And*

$$A := \{(a_t, e),\ t \in \mathbb{R}\}$$

Let μ *be an* A*-invariant probability measure such that*

- $h(a, \nu) > 0$ *for every ergodic component* ν *of* μ*;*
- μ *is* H*-recurrent (some assumption weaker than* H*-invariant),*

then μ *is the* G*-invariant probability measure.*

Remark 2.14. *Same conclusion holds replacing the 2nd factor* $\mathrm{SL}_2(\mathbb{R})$ *by* $\mathrm{SL}_2(\mathbb{Q}_p)$*. This* p*-adic version is what is required for the AQUE theorem.*

Remark 2.15. *This theorem is not easily reduced to the ergodic case due to the recurrence condition.*

Remark 2.16. *Whether one can eliminate the entropy assumption remains an open problem.*

3. Further reading

Here are some general references.

[BM00] is a nice introduction to homogeneous dynamics including a proof of Oppenheim conjecture in the last chapter.

Einsiedler and Ward have a (ongoing) book project on homogeneous dynamics available on the authors' homepages.

What we plan to cover in this course (and almost everything I write here) can be found in the monograph [EEE+10].

References

[BM00] M. Bachir Bekka and Matthias Mayer, *Ergodic theory and topological dynamics of group actions on homogeneous spaces*, London Mathematical Society Lecture Note Series, vol. 269, Cambridge University Press, Cambridge, 2000. MR 1781937

[BR95] Mikhail Borovoi and Zeév Rudnick, *Hardy-Littlewood varieties and semisimple groups*, Invent. Math. **119** (1995), no. 1, 37–66. MR 1309971

[EEE+10] Manfred Leopold Einsiedler, David Alexandre Ellwood, Alex Eskin, Dmitry Kleinbock, Elon Lindenstrauss, Gregory Margulis, Stefano Marmi, and Jean-Christophe Yoccoz (eds.), *Homogeneous flows, moduli spaces and arithmetic*, Clay Mathematics Proceedings, vol. 10, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2010. MR 2572399

[Gal62] P. Gallagher, *Metric simultaneous diophantine approximation*, J. London Math. Soc. **37** (1962), 387–390. MR 157939

[Hed36] Gustav A. Hedlund, *Fuchsian groups and transitive horocycles*, Duke Math. J. **2** (1936), no. 3, 530–542. MR 1545946

[Spe42] D. C. Spencer, *The lattice points of tetrahedra*, J. Math. Phys. Mass. Inst. Tech. **21** (1942), 189–197. MR 7767