LECTURE 14, QUANTITATIVE OPPENHEIM II

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CONTENTS

Main reference: [EMM98].

If you are new to this circle of ideas, a first example to keep in mind maybe : $\mathbf{a}_t := \mathrm{diag}(e^t, e^{-t})$, $K := \mathrm{SO}_2(\mathbb{R})$, $X = \mathrm{X}_2$. Most arguments are trivialized here, yet you could see the main idea.

Notations

- $Q_0(x_1, x_2, x_3, x_4) := 2x_1x_4 + x_2^2 + x_3^2$ real quadratic form of signature $(3,1)$ on $\mathbb{R}^4$.
- Let $(\mathbf{e}_1, ..., \mathbf{e}_4)$ be the standard basis of $\mathbb{R}^4$; and for a vector v, define its coefficients by $v = \sum (v)_i \mathbf{e}_i$ and we also write $v = ((v)_1, ..., (v)_4)$.
- Let $(\mathbf{f}_1, ..., \mathbf{f}_4)$ be another ONB(=orthogonal normal basis) defined by $\mathbf{f}_2 = \mathbf{e}_2, \mathbf{f}_3 = \mathbf{e}_3$ and $\mathbf{f}_1 = \frac{\mathbf{e}_1+\mathbf{e}_4}{\sqrt{2}}$, $\mathbf{f}_4 = \frac{\mathbf{e}_1-\mathbf{e}_4}{\sqrt{2}}$. If $v = \sum a_i \mathbf{f}_i$, we also write $v = (a_1, ..., a_4)_{\mathbf{f}}$.
- One can verify that $Q_0((x_1, ..., x_4)_{\mathbf{f}}) = x_1^2 + x_2^2 + x_3^2 - x_4^2$.
- $\mathrm{K} := \mathrm{SO}_{Q_0}(\mathbb{R}) \cap \mathrm{SO}_4(\mathbb{R})$.
- $\mathbf{a}_t := \mathrm{diag}(e^{-t}, 1, 1, e^t)$, contained in $\mathrm{SO}_{Q_0}(\mathbb{R})$.

1. OUTLINE OF THE PROOF

Recall by last lecture, it remains to show the following

Theorem 1.1. *Let f be a compactly supported continuous function on $\mathbb{R}^4$ and let $\widetilde{f} : \mathrm{X}_4 \to \mathbb{R}$ be its Siegel transform. Let $g_0 \in G$ be such that $Q_0 \circ g_0$ is irrational. Then*

$$\lim_{t \to +\infty} \int_K \widetilde{f}(\mathbf{a}_t k g_0 \mathbb{Z}^4) \widehat{\mathrm{m}}_K(k) = \int \widetilde{f}(x) \widehat{\mathrm{m}}_{\mathrm{X}_4}(x).$$

As we explained, the difficulty here is that $\widetilde{f}$ is usually an integrable but unbounded function. And it suffices to show that the contribution of the part outside a large compact set is small. The following observation reduces the general task to a rather special function.

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Definition 1.2. *For a lattice $\Lambda \le \mathbb{R}^4$, let*

$$\mathrm{ht}_\infty(\Lambda) := \max_{i=1,\dots,3} \sup_{\Delta \in \mathrm{Prim}^i(\Lambda)} \frac{1}{\|\Delta\|} = \max_{i=1,\dots,3} (\mathrm{sys}^{(i)}(\Lambda))^{-1}.$$

Lemma 1.3. *Let f be a bounded, non-negative function with compact support on $\mathbb{R}^4$. Then there exists a constant $C_1 = C_1(f) > 1$ such that*

$$\widetilde{f}(\Lambda) \le C_1 \cdot \mathrm{ht}_\infty(\Lambda),\ \forall \Lambda \in \mathrm{X}_4.$$

Proof is left as an exercise.

Theorem 1.4. *For every $\varepsilon > 0$, there exists a compact set C_ε of X_4 such that for all $t > 0$,*

$$\int \left(\mathrm{ht}_\infty \cdot 1_{\mathrm{X}_4 \setminus C_\varepsilon}\right)(\mathbf{a}_t k g_0 \mathbb{Z}^4) \widehat{\mathrm{m}}_K(k) \le \varepsilon.$$

Proof of Thm.1.1 assuming Thm.1.4. Without loss of generality assume $f \ge 0$.

Fix $\varepsilon > 0$, choose $C_\varepsilon \subset \mathrm{X}_4$ as in Thm.1.4. Choose a compactly supported continuous function $1 \ge \varphi_\varepsilon \ge 1_{C_\varepsilon}$. Thus by equidistribution theorem obtained in Lec.12

$$\lim_{t\to+\infty} \int \left(\widetilde{f} \cdot \varphi_\varepsilon\right)(\mathbf{a}_t k g_0 \mathbb{Z}^4) \widehat{\mathrm{m}}_K(k) = \int \left(\widetilde{f} \cdot \varphi_\varepsilon\right)(x) \widehat{\mathrm{m}}_{\mathrm{X}_4}(x).$$

On the other hand by Thm.1.4 and Lem.1.3

$$\begin{aligned}\limsup_{t\to+\infty} \int \left(\widetilde{f} \cdot (1-\varphi_\varepsilon)\right)(\mathbf{a}_t k g_0 \mathbb{Z}^4) \widehat{\mathrm{m}}_K(k) &\le \limsup_{t\to+\infty} \int \left(C_1 \mathrm{ht}_\infty \cdot 1_{\mathrm{X}_4 \setminus C_\varepsilon}\right)(\mathbf{a}_t k g_0 \mathbb{Z}^4) \widehat{\mathrm{m}}_K(k) \\ &\le C_1 \varepsilon.\end{aligned}$$

Combining both and letting $\varepsilon \to 0$ we are done. □

In fact, something stronger than Thm.1.4 will be proved.

Proposition 1.5. *For $\delta \in (0,1)$ (we only need for some $\delta > 0$) and $\Lambda_0 \in \mathrm{X}_4$, there exists $C_2 = C_2(\delta, \Lambda_0) > 0$ such that for all $t > 0$*

$$\int \mathrm{ht}_\infty^{1+\delta}(\mathbf{a}_t k.\Lambda_0) \widehat{\mathrm{m}}_K(k) \le C_2.$$

This will be deduced from the following two propositions.

Proposition 1.6. *For every $\varepsilon > 0$, there exist $C_4(\varepsilon) > 1$ and $t_0(\varepsilon) > 0$ such that for all $\Lambda \in \mathrm{X}_4$ (this is important!), we have*

$$\int \mathrm{ht}_\delta^{\mathrm{new}}(\mathbf{a}_{t_0(\varepsilon)} k.\Lambda) \widehat{\mathrm{m}}_K(k) \le \varepsilon\, \mathrm{ht}_\delta^{\mathrm{new}}(\Lambda) + C_4(\varepsilon)$$

where $\alpha_\delta : \mathrm{X}_4 \to \mathbb{R}_{>0}$ is some function satisfying

$$C_5^{-1} \mathrm{ht}_\infty^{1+\delta} \le \mathrm{ht}_\delta^{\mathrm{new}} \le C_5 \mathrm{ht}_\infty^{1+\delta}.$$

Actually, we will find constants $c_0 > 0$ and $\kappa_i > 0$ for $i = 0,1,2,3,4$ such that

$$\mathrm{ht}_\delta^{\mathrm{new}}(\Lambda) = \sum_{i=1,2,3} c_0^{\kappa_i} (\mathrm{sys}^{(i)}(\Lambda))^{-1-\delta}.$$

To yield the result by applying this operator repeatedly, we need the following:

Proposition 1.7. *For every open neighborhood V of identity in H, there exists a neighborhood U of identity in K such that for all $t, s \ge 0$*

$$\mathbf{a}_t U \mathbf{a}_s \subset K \cdot V \cdot \mathbf{a}_{t+s} \cdot K.$$

Proof of Prop.1.5. From the description of α_δ as in Prop.1.6, we can find V_0, an open neighborhood of identity in H, such that

$$\frac{1}{2}\mathrm{ht}_\delta^{\mathrm{new}}(\Lambda) \le \mathrm{ht}_\delta^{\mathrm{new}}(v.\Lambda) \le 2\,\mathrm{ht}_\delta^{\mathrm{new}}(\Lambda),\ \forall v \in V_0,\ \Lambda \in \mathrm{X}_4.$$

Find U_0 by Prop.1.7. Let $\varepsilon := \frac{1}{4}\widehat{\mathrm{m}}_K(U_0)$. Applying Prop.1.6 we get some C_4, t_0. Let $C_6 := \frac{C_4}{\widehat{\mathrm{m}}_K(U_0)}$.

Fix $\Lambda_0 \in \mathrm{X}_4$, define a continuous function $\phi : G \to \mathbb{R}_{>0}$ by

$$\phi(g) := \int \mathrm{ht}_\delta^{\mathrm{new}}(gk.\Lambda_0)\widehat{\mathrm{m}}_K(k).$$

Thus it suffices to show that $\phi(\mathbf{a}_t)$, as t varies in $(0,+\infty)$, is bounded by Prop. 1.6.

The function ϕ enjoys the following properties

1. ϕ is bi-K-invariant;
2. for every $v \in V_0$ and $g \in G$, $\frac{1}{2}\phi(g) \le \phi(vg) \le 2\phi(g)$.

Combined with Prop.1.7, we see that for all $t \ge t_0$,

$$\phi(\mathbf{a}_{t_0}k\mathbf{a}_{t-t_0}) \ge \frac{1}{2}\phi(\mathbf{a}_t).$$

Also observe that

$$\begin{aligned}\frac{1}{\widehat{\mathrm{m}}_K(U_0)}\int_{U_0}\phi(\mathbf{a}_{t_0}kg)\widehat{\mathrm{m}}_K(k) &\le \frac{1}{\widehat{\mathrm{m}}_K(U_0)}\int_K \phi(\mathbf{a}_{t_0}kg)\widehat{\mathrm{m}}_K(k)\\ &\le \frac{1}{\widehat{\mathrm{m}}_K(U_0)}\cdot\left(\frac{1}{4}\widehat{\mathrm{m}}_K(U_0)\phi(g)+C_4\right)\\ &= \frac{1}{4}\phi(g)+C_6.\end{aligned}$$

Therefore, for $t > t_0$,

$$\begin{aligned}\phi(\mathbf{a}_t) &= \frac{1}{\widehat{\mathrm{m}}_K(U_0)}\int_{U_0}\phi(\mathbf{a}_t)\widehat{\mathrm{m}}_K(k)\\ &\le 2\frac{1}{\widehat{\mathrm{m}}_K(U_0)}\int_{U_0}\phi(\mathbf{a}_{t_0}k\mathbf{a}_{t-t_0})\widehat{\mathrm{m}}_K(k)\\ &\le \frac{1}{2}\phi(\mathbf{a}_{t-t_0})+C_6.\end{aligned}$$

Now, for $t > 0$, choose the unique $n_t \in \mathbb{Z}_{\ge 0}$ such that $t' := t - n_t t_0 \in (0, t_0]$. By applying the above inequality n_t times we get

$$\phi(\mathbf{a}_t) \le \frac{1}{2^{n_t}}\phi(\mathbf{a}_{t'}) + C_6(1+\frac{1}{2}+(\frac{1}{2})^2+\ldots)$$

Hence $\phi(\mathbf{a}_t)$, as t varies in $(0,+\infty)$, is bounded. □

2. WAVEFRONT LEMMA

We explain how Prop.1.7 is proved.

Proof. I am pretending $K = \mathrm{SO}_4(\mathbb{R})$ here. The justification of the arguments here without this false assumption is left to you.

Every matrix g of determinant one can be written as

$$g = k_1 d k_2,\ k_i \in \mathrm{SO}_n(\mathbb{R}),\ d \text{ is a diagonal matrix.}$$

The order of the diagonal entries of d can be permuted by changing k_1, k_2. The middle matrix is uniquely determined if we further assume

$$d = \mathrm{diag}(d_1, ..., d_n), \text{ with } d_1 \geq d_2 \geq ... \geq d_n > 0.$$

We let $\alpha_i(g) := d_1 \cdot ... \cdot d_i$. It suffices to show that, when $k \in K$ is close to identity, for every i, $\alpha_i(\mathbf{a}_{t_1} k \mathbf{a}_{t_2})$ is closed to $\alpha_i(\mathbf{a}_{t_1+t_2})$ multiplicatively.

To do this, note that

$$\alpha_i(g) = \sup_{\mathbf{v} \in \wedge^i \mathbb{R}^n,\, \|\mathbf{v}\|=1} \|g.\mathbf{v}\| = \sup_{\mathbf{v},\mathbf{w} \in \wedge^i \mathbb{R}^n,\, \|\mathbf{v}\|=\|\mathbf{w}\|=1} \left|\langle g.\mathbf{v}, \mathbf{w}\rangle\right|.$$

For $\varepsilon \in (0,1)$, choose $U = U(\varepsilon) \subset K$ such that for all i,

$$|\langle u.e_1 \wedge ... \wedge e_i, e_1 \wedge ... \wedge e_i\rangle| \geq \frac{1}{1+\varepsilon}.$$

Now take $u \in U$. On the one hand,

$$\begin{aligned} \left|\langle \mathbf{a}_{t_1} u \mathbf{a}_{t_2}.\mathbf{v}, \mathbf{w}\rangle\right| &= \left|\langle u \mathbf{a}_{t_2}.\mathbf{v}, \mathbf{a}_{t_1}.\mathbf{w}\rangle\right| \\ &\leq \left\|\mathbf{a}_{t_2}.\mathbf{v}\right\| \cdot \left\|\mathbf{a}_{t_1}.\mathbf{w}\right\| \leq \alpha_i(\mathbf{a}_{t_1+t_2}). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left|\langle \mathbf{a}_{t_1} u \mathbf{a}_{t_2}.e_1 \wedge ... \wedge e_i, e_1 \wedge ... \wedge e_i\rangle\right| \\ =&\alpha_i(\mathbf{a}_{t_1+t_2}) \,|\langle u.e_1 \wedge ... \wedge e_i, e_1 \wedge ... \wedge e_i\rangle| \geq \frac{1}{1+\varepsilon}\alpha_i(\mathbf{a}_{t_1+t_2}). \end{aligned}$$

So we are done.

□

3. THE HEIGHT FUNCTION

Prop.1.6 relies on the following proposition on representations. It is here that we are avoiding the case of signature $(2,1)$ and $(2,2)$.

Proposition 3.1. *For every $\varepsilon > 0$ there exists $t_1 = t_1(\varepsilon) > 0$ such that for all $t \geq t_1$, $\delta \in (0,1)$ and for all pure wedges $\mathbf{v}_{\neq 0} \in \wedge^i \mathbb{R}^n$ ($n = 4$ here), we have*

$$\int \|\mathbf{a}_t k.\mathbf{v}\|^{-1-\delta}\, \widehat{\mathrm{m}}_K(k) \leq \varepsilon \|\mathbf{v}\|^{-1-\delta}.$$

Proof. Omitted for now. □

A "pure wedge" (also called "decomposable vector") refers to a vector $\mathbf{v} \in \wedge^i \mathbb{R}^n$ that can be written as $v_1 \wedge ... \wedge v_k$ for some $v_i \in \mathbb{R}^n$.

3.1. **Preparations.** Fix $\varepsilon \in (0,1)$, find $t_1(\varepsilon)$ as in Prop.3.1. Find $C_7 = C_7(\varepsilon) > 1$ such that

$$C_7^{-1} \|\mathbf{v}\| \leq \left\|\mathbf{a}_{t_1}.\mathbf{v}\right\| \leq C_7 \|\mathbf{v}\|, \ \forall \mathbf{v} \in \sqcup \wedge^i \mathbb{R}^4.$$

Fix a strictly convex function $\kappa > 0$ on $[0,4]$. And find $C_8 > 1$ such that

$$\kappa_j \geq \frac{\kappa_{j-i} + \kappa_{j+i}}{2} + C_8^{-1}; \quad \kappa_0 = \kappa_4 = 1$$

for all $j \in \{1,2,3\}$ and $j \pm i \in \{0,1,2,3,4\}$.

Choose $c_0 \in (0,1)$ small enough, depending on ε,

$$c_0^{2C_8^{-1}} \leq C_7^2 c_0^{2C_8^{-1}} \leq (\varepsilon C_7^{-1})^{100}.$$

Define

$$\mathrm{ht}_\delta^{\mathrm{new}}(\Lambda) = \sum_{i=1,2,3} c_0^{\kappa_i} (\mathrm{sys}^{(i)}(\Lambda))^{-1-\delta}. \tag{1}$$

3.2. **The proof.** For each $l=1,2,3$ find $\Delta_1^{(l)} \in \mathrm{Prim}^l(\Lambda)$ such that $\mathrm{sys}^{(l)}(\Lambda) = \left\|\Delta_1^{(l)}\right\|$.

3.2.1. *Good indices.* We define $\mathrm{Good}(\Lambda) \subset \{1,2,3\}$ by

$$l \in \mathrm{Good}(\Lambda) \iff \forall \Delta \in \mathrm{Prim}^l(\Lambda) \setminus \Delta_1^{(l)},\ C_7^2 \|\Delta\|^{-1} < \mathrm{sys}^{(l)}(\Lambda)^{-1}. \tag{2}$$

Thus for $l \in \mathrm{Good}(\Lambda)$, $\Delta \in \mathrm{Prim}^l(\Lambda) \setminus \Delta_1^{(l)}$ and $k \in K$,

$$\begin{aligned}&\left\|\mathbf{a}_{t_1} k.\Delta\right\|^{-1-\delta} \le C_7^{1+\delta} \|\Delta\|^{-1-\delta} < C_7^{-1-\delta}\,\mathrm{sys}^{(l)}(\Lambda)^{-1-\delta} = C_7^{-1-\delta}\left\|\Delta_1^{(l)}\right\|^{-1-\delta} \le \left\|\mathbf{a}_{t_1} k.\Delta_1^{(l)}\right\|^{-1-\delta}\\ &\implies \forall k \in K,\ \mathrm{sys}^{(l)}(\mathbf{a}_{t_1} k.\Lambda)^{-1-\delta} = \left\|\mathbf{a}_{t_1} k.\Delta_1^{(l)}\right\|^{-1-\delta}.\end{aligned} \tag{3}$$

This implies that

$$\begin{aligned}\int c_0^{\kappa_l} \mathrm{sys}^{(l)}(\mathbf{a}_{t_1} k.\Lambda)^{-1-\delta}\, \widehat{\mathrm{m}}_K(k) &= \int c_0^{\kappa_l} \left\|\mathbf{a}_{t_1} k.\Delta_1^{(l)}\right\|^{-1-\delta} \widehat{\mathrm{m}}_K(k)\\ &\le \varepsilon c_0^{\kappa_l} \left\|\mathbf{a}_{t_1} k.\Delta_1^{(l)}\right\|^{-1-\delta}\\ &= \varepsilon \cdot c_0^{\kappa_l}\, \mathrm{sys}^{(l)}(\mathbf{a}_{t_1} k.\Lambda)^{-1-\delta}.\end{aligned} \tag{4}$$

3.2.2. *Bad indices.* $\mathrm{Bad}(\Lambda) := \{1,2,3\}\setminus\mathrm{Good}(\Lambda)$. In other words, we can find $\Delta_2^{(l)} \in \mathrm{Prim}^l(\Lambda)\setminus \Delta_1^{(l)}$ such that

$$C_7^2 \left\|\Delta_2^{(l)}\right\|^{-1} \ge \mathrm{sys}^{(l)}(\mathbf{a}_{t_1} k.\Lambda)^{-1}.$$

Recall the following inequalities

$$\left\|\Delta_1^{(l)}\right\| \cdot \left\|\Delta_2^{(l)}\right\| \ge \left\|\Delta_1^{(l)} \cap \Delta_2^{(l)}\right\| \cdot \left\|\Delta_1^{(l)} + \Delta_2^{(l)}\right\|,$$

from which we deduce that (let $a := \mathrm{rank}\,\Delta_1^{(l)} - \mathrm{rank}\,\Delta_1^{(l)} \cap \Delta_2^{(l)}$)

$$c_0^{2\kappa_l} \left\|\Delta_2^{(l)}\right\|^{-1-\delta} \left\|\Delta_2^{(l)}\right\|^{-1-\delta} \le \left(c_0^{\kappa_{l-a}} \left\|\Delta_1^{(l)} \cap \Delta_2^{(l)}\right\|^{-1-\delta}\right) \cdot \left(c_0^{\kappa_{l+a}} \left\|\Delta_1^{(l)} + \Delta_2^{(l)}\right\|^{-1-\delta}\right) \cdot c_0^{2\kappa_l - \kappa_{l-a} - \kappa_{l+a}}.$$

For the LHS we have

$$C_7^{-4}\left(c_0^{\kappa_l}\, \mathrm{sys}^{(l)}(\Lambda)^{-1-\delta}\right)^2 \le C_7^{-2(1+\delta)}\left(c_0^{\kappa_l}\, \mathrm{sys}^{(l)}(\Lambda)^{-1-\delta}\right)^2 \le \mathrm{LHS}$$

and for the RHS,

$$\mathrm{RHS} \le \left(c_0^{\kappa_{l-a}}\, \mathrm{sys}^{(l-a)}(\Lambda)^{-1-\delta}\right) \cdot \left(c_0^{\kappa_{l+a}}\, \mathrm{sys}^{(l+a)}(\Lambda)^{-1-\delta}\right) \cdot c_0^{C_8^{-1}}.$$

Since $c_0^{C_8^{-1}} \le \varepsilon^{50} C_7^{-50}$, by combining the above equations we get

$$\left(c_0^{\kappa_l}\, \mathrm{sys}^{(l)}(\Lambda)^{-1-\delta}\right)^2 \le \varepsilon^{50} C_7^{-46}\left(c_0^{\kappa_{l-a}}\, \mathrm{sys}^{(l-a)}(\Lambda)^{-1-\delta}\right) \cdot \left(c_0^{\kappa_{l+a}}\, \mathrm{sys}^{(l+a)}(\Lambda)^{-1-\delta}\right).$$

Thus

$$c_0^{\kappa_l}\, \mathrm{sys}^{(l)}(\Lambda)^{-1-\delta} \le \varepsilon^{20} C_7^{-23} \max_{l'=0,\dots,4}\left\{c_0^{\kappa_{l'}}\, \mathrm{sys}^{(l')}(\Lambda)^{-1-\delta}\right\}.$$

Now we choose $l_1 = l_1(\Lambda,\delta)$ such that the maximum of RHS is achieved. Then $l_1 \in \mathrm{Good}(\Lambda) \cup \{0,4\}$. Also take $l_0 \in \mathrm{Bad}(\Lambda)$. Then for every $k \in K$,

$$\begin{aligned}c_0^{\kappa_{l_0}}\, \mathrm{sys}^{(l_0)}(\mathbf{a}_{t_1} k.\Lambda)^{-1-\delta} &\le C_7^{1+\delta} c_0^{\kappa_{l_0}}\, \mathrm{sys}^{(l_0)}(\Lambda)^{-1-\delta} \le \varepsilon^{20} C_7^{-20} c_0^{\kappa_{l_1}}\, \mathrm{sys}^{(l_1)}(\Lambda)^{-1-\delta}\\ &\le \varepsilon^{20} C_7^{-18} c_0^{\kappa_{l_1}}\, \mathrm{sys}^{(l_1)}(\mathbf{a}_{t_1} k.\Lambda)^{-1-\delta}.\end{aligned}$$

3.2.3. *Wrap-up.* To save notation define

$$\alpha_l(\Lambda) := c_0^{\kappa_l}\,\mathrm{sys}^{(l)}(\Lambda)^{-1-\delta}.$$
$$\pi_*(\alpha_l)(\Lambda) := \int \alpha_l(\mathbf{a}_{t_1} k.\Lambda)\widehat{\mathrm{m}}_K(k).$$

So for $l \in \mathrm{Good}(\Lambda)$, we have

$$\pi_*(\alpha_l)(\Lambda) \leq \varepsilon\alpha_l(\Lambda).$$

For $l \in \mathrm{Bad}(\Lambda)$, we have ($l_1 = l_1(\Lambda)$ as above)

$$\pi_*(\alpha_l)(\Lambda) \leq \varepsilon^{20} C_7^{-18} \pi_*(\alpha_l)(\Lambda).$$

There are two cases.

Case I, $l_1 \in \{0, n\}$. In this case, for all l, $\alpha_l(\Lambda) \leq \max\{c_0^{\kappa_0}, c_0^{\kappa_n}\} = c_0$. Thus $\mathrm{ht}_\delta^{\mathrm{new}}(\Lambda) \leq 3c_0$. And

$$\pi_*(\mathrm{ht}_\delta^{\mathrm{new}})(\Lambda) \leq 3c_0 C_7^2.$$

Case II, $l_1 \in \mathrm{Good}(\Lambda)$.

$$\begin{aligned}
\pi_*(\mathrm{ht}_\delta^{\mathrm{new}})(\Lambda) &= \sum \pi_*(\alpha_l)(\Lambda) \\
&\leq \varepsilon \sum_{l\in\mathrm{Good}(\Lambda)} \alpha_l(\Lambda) + \varepsilon^{20} C_7^{-18} \pi_*(\alpha_{l_1})(\Lambda) \\
&\leq \varepsilon \sum_{l\in\mathrm{Good}(\Lambda)} \alpha_l(\Lambda) + \varepsilon^{21} C_7^{-18} \alpha_{l_1}(\Lambda) \\
&\leq 2\varepsilon \sum_{l\in\mathrm{Good}(\Lambda)} \alpha_l(\Lambda) \leq 2\varepsilon\,\mathrm{ht}_\delta^{\mathrm{new}}(\Lambda).
\end{aligned}$$

In either case, the following holds

$$\pi_*(\mathrm{ht}_\delta^{\mathrm{new}})(\Lambda) \leq 3c_0 C_7^2 + 2\varepsilon\,\mathrm{ht}_\delta^{\mathrm{new}}(\Lambda) \tag{5}$$

for all $\Lambda \in \mathrm{X}_4$. Recall c_0 and C_7 are only dependent on ε.

References

[EMM98] Alex Eskin, Gregory Margulis, and Shahar Mozes, *Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture*, Ann. of Math. (2) **147** (1998), no. 1, 93–141. MR 1609447