EXERCISE SHEET 4

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截止日期：并没有，选做；如需反馈可以发给我。

评分标准：取 sup-norm ——只要做对一小道题，就能得到满分。当然，你也可以尝试说明题目出错了。

提示：你可以自由使用序号靠前习题的结果来解答序号靠后的习题。

如对习题（陈述，定义等）有任何的疑问，请联系我。

CONTENTS

1. AN EXAMPLE OF EQUIDISTRIBUTION OF UNIPOTENT FLOWS

Notations

- $G = \mathrm{SL}_2(\mathbb{C})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$ and $X := G/\Gamma$;
- $U = \left\{ \mathbf{u}_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\}$ and $x_0 = [g_0] \in G/\Gamma$.

Let (S_n) be a sequence of positive real numbers tending to $+\infty$ such that the following limit exists:

$$\mu := \lim_{S_n \to +\infty} \frac{1}{S_n} \int_0^{S_n} (\mathbf{u}_s)_* \delta_{[g_0]} \, \mathrm{d}s .$$

Assume the fact that such a μ belongs to $\mathrm{Prob}(X)^U$.

Recall the definitions of $\mathscr{H}$, $T(H,U)$,... (see Lec.11, Def.1.6, Def.3.1). And V_H, v_H same as in Lec.12.

Exercise 1.1. *Let $H \in \mathscr{H}$, $H \neq G$. Show that if $\mu(T(H,U)) > 0$, then there exists a bounded set $\Phi \subset V_H$ and a sequence $(\gamma_n) \subset \Gamma$ such that*

$$\mathbf{u}_{[0,S_n]} g_0 \gamma_n . v_H \subset \Phi.$$

Exercise 1.2. *Same notations as the exercise above. Conclude that there exists $\gamma \in \Gamma$ such that*

$$\mathbf{u}_{[0,+\infty)} g_0 \gamma . v_H \subset \Phi.$$

Exercise 1.3. *Same notations as the exercise above. Conclude that $g_0^{-1} U g_0 \subset N_G(\gamma H \gamma^{-1})^{(1)}$.*

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Exercise 1.4. *Use exercises above to show that if* $x_0 = [g_0] \notin [\mathrm{Sing}(G,U)]_\Gamma$*, then*

$$\lim_{S_n \to +\infty} \frac{1}{S_n} \int_0^{S_n} (\mathbf{u}_s)_* \delta_{[g_0]} \, \mathrm{ds} = \widehat{\mathrm{m}}_{G/\Gamma}.$$

[Hint: use Lec.11, Thm.2.3 if it helps.]

Exercise 1.5. *Conclude that if* $x_0 = [g_0] \notin [\mathrm{Sing}(G,U)]_\Gamma$*, then* $U.x_0$ *is dense in* G/Γ*.*

2. Homogeneous sets of bounded volume

Notations

- $G := \mathrm{SL}_N(\mathbb{R})$ and $\Gamma := \mathrm{SL}_N(\mathbb{Z})$.
- Fix a right G-invariant Riemannian metric on G, which induces Riemannian metrics on G/Γ and also on immersed submanifolds. Volumes below are all induced from this.

For $C > 0$, let

$$\mathscr{A} := \left\{ H \leq G \,\middle|\, H \text{ is a closed connected subgroup of } G, \mathrm{Vol}(H/H \cap \Gamma) < \infty. \right\}$$

$$\mathscr{A}_C := \left\{ H \leq G \,\middle|\, H \text{ is a closed connected subgroup of } G, \mathrm{Vol}(H/H \cap \Gamma) < C. \right\}$$

Definition 2.1. *Given a sequence* (H_n) *of closed subgroups of* G*, we say that* (H_n) ***converges*** *iff for every (infinite) subsequence* (n_k) *and* $h_{n_k} \in H_{n_k}$ *such that* $\lim_k h_{n_k}$ *exists, there exists* $h'_n \in H_n$ *for each* n*, such that*

$$\lim_k h_{n_k} = \lim_n h'_n.$$

Exercise 2.1. *Given a sequence* (H_n) *of closed subgroups of* G*, there exists a subsequence that converges.*

From now on we fix a convergent sequence (H_n). And assume each H_n is connected. Let

$$L := \left\{ g \in G \,\middle|\, g = \lim_n h_n, \, \exists h_n \in H_n \right\}$$

Exercise 2.2. *Show that* L *is a closed subgroup.*

Exercise 2.3. *There exists a subsequence* n_k *such that* $(\mathfrak{h}_{n_k})$ *(the Lie algebra of* H_{n_k}*) converges.*

From now on we assume $(\mathfrak{h}_n)$ converges to $\mathfrak{h}_\infty$.

Exercise 2.4. *Find an example of* (H_n) *such that* $\mathfrak{h}_\infty$ *is not the Lie algebra of* L*.*

Now we further assume that $\{H_n\} \subset \mathscr{A}_{C_0}$ for some $C_0 > 0$.

Exercise 2.5. *Show that under the assumption above,* $\mathfrak{h}_\infty = \mathrm{Lie}(L)$.

Exercise 2.6. *Show that* $(H_n \cap \Gamma)$ *converges and its limit is given by*

$$\Gamma_\infty := \left\{ \gamma \in \Gamma \,\middle|\, \exists n_0, \, \forall n > n_0, \, \gamma \in H_n \cap \Gamma \right\}.$$

Exercise 2.7. *Show that* Vol_{H_n} *converges to* Vol_L *in the weak$*$ topology.*

Exercise 2.8. *Show that* Γ_∞ *is a lattice in* L*. Indeed show that*

$$\mathrm{Vol}(L/\Gamma_\infty) \leq \limsup \mathrm{Vol}(H_n/H_n \cap \Gamma).$$

[Hint, consider compact parts of a fundamental domain]

It is a fact that once you know Γ_∞ is a lattice in L, then it is finitely generated.

Exercise 2.9. *Assume the fact above. Show that there exists n_0 such that for all $n > n_0$, $\Gamma \cap H_n \supset \Gamma_\infty$.*

Continuing this way, using more inputs from the theory of algebraic groups, one can show that

Theorem 2.2 (Dani–Margulis)**.** *We have that*

$$\#\left\{H \cap \Gamma \,\middle|\, H \in \mathscr{A}_{C_0}\right\} < \infty.$$

Exercise 2.10. *For $H \in \mathscr{A}$ and $g \in G$, show that*

$$\mathrm{Vol}(gH\Gamma/\Gamma) = \frac{\left\|\mathrm{Ad}(g).v_H\right\|}{\|v_H\|}\mathrm{Vol}(H\Gamma/\Gamma).$$

Here v_H is a vector in $\wedge^{\dim H}\mathfrak{sl}_n$ defined by $v_1 \wedge \ldots \wedge v_{\dim H}$ where $(v_1, \ldots, v_{\dim H})$ is a basis for $\mathfrak{h}$, the Lie algebra of H.

Exercise 2.11. *Assume the theorem above, show that $\Gamma.v_H$ is a discrete subset of $\wedge^{\dim H}\mathfrak{sl}_n$.*

3. Orbit counting and equidistribution

Notations

- $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $H = \left\{\begin{bmatrix} x & 2y \\ y & x \end{bmatrix} \,\middle|\, x^2 - 2y^2 = 1\right\}$;
- $V := \{\text{2-by-2 real matrices with trace } 0\}$;
- $V(\mathbb{Z}) := \{\text{2-by-2 integer matrices with trace } 0\}$
- $M_0 := \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ and $p_0(x) := x^2 - 2$;
- for a matrix M, its characteristic polynomial is denoted by $\mathrm{char}_M(x) := \det(xI - M) = x^2 - \mathrm{Tr}(M)x + \det(M)$;
- $X_{p_0}(\mathbb{R}) := \{M \in V, \mathrm{char}_M(x) = p_0(x)\}$, $X_{p_0}(\mathbb{Z}) := \{M \in V(\mathbb{Z}), \mathrm{char}_M(x) = p_0(x)\}$;
- for a 2-by-2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define $\mathrm{ht}(M) := \sqrt{a^2 + b^2 + c^2 + d^2}$;
- $B_R := \left\{M \in X_{p_0}(\mathbb{R}) \,\middle|\, \mathrm{ht}(M) \leq R\right\}$.

Exercise 3.1. *Show that every pair of matrices $M_1, M_2 \in X_{p_0}(\mathbb{R})$, there exists $g \in G$ such that $gM_1g^{-1} = M_2$.*

Let G acts on $X_{p_0}(\mathbb{R})$ by $g.M := gMg^{-1}$. The above exercise shows that this action is transitive.

Exercise 3.2. *The stabilizer of M_0 in G is equal to H.*

Exercise 3.3. *$H \cap \Gamma$ is a lattice in H.*

Exercise 3.4. *Show that the action of Γ on $X_{p_0}(\mathbb{Z})$ is transitive.*

[Hint: $\mathbb{Z}[\sqrt{2}]$ is a PID]

Further notations

- $\mathrm{m}_{G/H}$ is a G-invariant locally finite measure on G/H;
- similarly, m_G and m_H denote Haar measures on G and H respectively.

Note that G and H are unimodular: left Haar measures are the same as right Haar measures.

Definition 3.1. *We say that a triple* $(\mathrm{m}_G, \mathrm{m}_H, \mathrm{m}_{G/H})$ *is compatible iff for every compactly supported function* $f \in C_c(G)$*, we have*

$$\int_{G/H}\int_H f(gh)\mathrm{m}_H(h)\mathrm{m}_{G/H}([g]) = \int_G f(g)\mathrm{m}_G([g]). \tag{1}$$

Exercise 3.5. *Show that for every triple of Haar measures* $(\mathrm{m}_G, \mathrm{m}_H, \mathrm{m}_{G/H})$*, there exists a constant* $c > 0$ *such that for every* $f \in C_c(G)$*,*

$$\int_{G/H}\int_H f(gh)\mathrm{m}_H(h)\mathrm{m}_{G/H}([g]) = c\cdot\int_G f(g)\mathrm{m}_G([g]).$$

From now on we fix the unique triple $(\mathrm{m}_G, \mathrm{m}_H, \mathrm{m}_{G/H})$ satisfying

1. $(\mathrm{m}_G, \delta_\Gamma, \widehat{\mathrm{m}}_{G/\Gamma})$ and $(\mathrm{m}_H, \delta_{H\cap\Gamma}, \widehat{\mathrm{m}}_{H/H\cap\Gamma})$ are compatible. Here δ_Γ (resp. $\delta_{H\cap\Gamma}$) denotes the counting measure on Γ (resp. $H\cap\Gamma$).
2. $(\mathrm{m}_G, \mathrm{m}_H, \mathrm{m}_{G/H})$ is compatible.

Its existence is guaranteed by the Exer.3.5 above.

Exercise 3.6. *Find the asymptotics of*

$$\mathrm{m}_{G/H}(B_R) := \mathrm{m}_{G/H}\left(\left\{[g]\in G/H \,\middle|\, \mathrm{ht}(g.M_0)\le R\right\}\right).$$

Definition 3.2. *Define* $\varphi_R : G/\Gamma \to \mathbb{R}$ *by*

$$\varphi_R([g]) := \#\left(g\Gamma.M_0\cap B_R\right).$$

We say that $\frac{1}{\mathrm{m}_{G/H}(B_R)}\varphi_R$ *converges to* 1 *weakly iff for all* $\psi\in C_c(G/\Gamma)$*,*

$$\lim_{R\to+\infty}\frac{1}{\mathrm{m}_{G/H}(B_R)}\int_{G/\Gamma}\varphi_R([g])\psi([g])\widehat{\mathrm{m}}_{G/\Gamma}([g]) = \int\psi([g])\widehat{\mathrm{m}}_{G/\Gamma}([g]). \tag{2}$$

Exercise 3.7. *Show that if* $\frac{1}{\mathrm{m}_{G/H}(B_R)}\varphi_R$ *converges to* 1 *weakly then for every* $[g]\in G/\Gamma$*,*

$$\lim_{R\to+\infty}\frac{1}{\mathrm{m}_{G/H}(B_R)}\varphi_R([g]) = 1.$$

In particular, in light of Exer.3.4,

$$\#X_{p_0}(\mathbb{Z})\cap B_R \sim \mathrm{m}_{G/H}(B_R).$$

[Hint: use Exer.3.6].

Exercise 3.8. *Show that the left hand side of Equa.(2) (excluding the limit) is equal to*

$$\frac{1}{\mathrm{m}_{G/H}(B_R)}\int_{\{g.M_0\in B_R\}}\left(\int\psi(x)\,g_*\widehat{\mathrm{m}}_{H\Gamma/\Gamma}(x)\right)\mathrm{m}_{G/H}([g])$$

Exercise 3.9. *Use "linearization technique" to show that for every sequence* (g_n) *such that* $([g_n])$ *diverges in* G/H*, we have*

$$\lim_{n\to+\infty}(g_n)_*\widehat{\mathrm{m}}_{H\Gamma/\Gamma} = \widehat{\mathrm{m}}_{G/\Gamma}.$$

Exercise 3.10. *Use Exer.3.9 to conclude that* $\frac{1}{\mathrm{m}_{G/H}(B_R)}\varphi_R$ *converges to* 1 *weakly.*