LECTURE 11, ERGODIC DECOMPOSITION OF UNIPOTENT INVARIANT PROBABILITY MEASURES

RUNLIN ZHANG

CONTENTS

The main reference of this lecture is [Sha91] and [MS95, Section 2].
Notations

- $X := \mathrm{G}/\Gamma$ with $\mathrm{G} := \mathrm{SL}_n(\mathbb{R})$ and $\Gamma := \mathrm{SL}_n(\mathbb{Z})$;
- $U = \{\mathbf{u}_s\}$ is a one-parameter unipotent subgroup of G.

1. U-ERGODIC MEASURES

The following is the description of ergodic U-invariant probability measures due to Ratner [Ra91].

Theorem 1.1. *Let μ be a U-invariant ergodic probability measure on X, then there exists $x \in X$ and a closed connected subgroup $H \leq \mathrm{G}$ containing U such that*

1. *$H.x$ is closed and supports an H-invariant probability measure $\widehat{\mathrm{m}}_{H.x}$;*
2. *$\mu = \widehat{\mathrm{m}}_{H.x}$.*

By writing $x = [g]_\Gamma$ and replacing H by $g^{-1}Hg$, we may rephrase the above theorem as

Theorem 1.2. *Let μ be a U-invariant ergodic probability measure on X, then there exists $g \in \mathrm{G}$ and a closed connected subgroup $H \leq \mathrm{G}$ containing $g^{-1}Ug$ such that*

1. *$[H]_\Gamma := H\Gamma/\Gamma$ is closed and supports an H-invariant probability measure $\widehat{\mathrm{m}}_{[H]_\Gamma}$;*
2. *$\mu = g_* \widehat{\mathrm{m}}_{[H]_\Gamma}$.*

In particular, $\mathrm{supp}(\mu) = g[H]_\Gamma$.

Example 1.3. *If $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$ and $U := \{\mathbf{u}_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, s \in \mathbb{R}\}$, then candidates of H are $\{U, \mathrm{G}\}$.*

Example 1.4. *If $\mathrm{G} = \mathrm{SL}_2(\mathbb{C})$ ($\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$) and $U := \{\mathbf{u}_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, s \in \mathbb{R}\}$, then candidates of H are $\{U, V, \mathrm{SL}_2(\mathbb{R}), \mathrm{G}\}$, where $V = \{\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, s \in \mathbb{C}\}$.*

Date: 2022.04.

We attempt to define where the pair (g,H) is supposed to live in.

Definition 1.5. *For two subgroups A,B of* G*, define*

$$N(A,B) := \left\{ g\in \mathrm{G} \,\middle|\, gAg^{-1}\supset B\right\}.$$

In this terminology, g as above belongs to $N(H,U)$. When $H=U$, $N(U,U)$ is just the normalizer of U in G. When $H=\mathrm{G}$, $N(\mathrm{G},U)=\mathrm{G}$. Where is H? (We do not want U to appear in the definition of this space.)

Definition 1.6. *Let $\mathscr{H}$ be the collection of subgroup L of* G *satisfying*

1. *L is a connected and closed subgroup;*
2. *$[L]_\Gamma$ is closed and supports an L-invariant probability measure $\widehat{\mathrm{m}}_{[L]_\Gamma}$;*
3. *some one-parameter unipotent subgroup of L acts ergodically on $\widehat{\mathrm{m}}_{[L]_\Gamma}$.*

Thus H as above belongs to $\mathscr{H}$.

Thus from μ we get a pair (g,H). To recover μ, we take the unique probability Haar measure supported on the $gH\Gamma/\Gamma$.

Lemma 1.7. *Let H_1,H_2 be two connected closed subgroups of* G *such that $H_i\Gamma$ (i=1,2) are both closed. Let $g_1,g_2\in\mathrm{G}$. Then $g_1H_1\Gamma=g_2H_2\Gamma$ iff there exist $h_2\in H_2$ and $\gamma_2\in\Gamma$ such that*

$$g_2h_2\gamma_2=g_1,\ \gamma_2^{-1}H_2\gamma_2=H_1.$$

Proof. It only suffices to prove the " $\Longrightarrow$ " direction. The other direction follows directly.

So assume $g_1H_1\Gamma=g_2H_2\Gamma$. Then

$$H_1\Gamma=g_3H_2\Gamma,\ g_3:=g_1^{-1}g_2.$$

Thus $\mathrm{id}\in g_3H_2\Gamma$ and

$$1=g_3h_2\gamma_2,\ \exists h_2\in H_2,\gamma_2\in\Gamma_2.$$

This already implies that

$$g_2h_2\gamma_2=g_2g_3^{-1}=g_1.$$

Now we have

$$H_1\Gamma=\gamma_2^{-1}h_2^{-1}H_2\Gamma=\gamma_2^{-1}H_2\gamma_2\Gamma.$$

By inspecting a small neighborhood of $[\mathrm{id}]_\Gamma$ and use the fact that $[H_1]_\Gamma$ and $[\gamma_2^{-1}H_2\gamma_2]_\Gamma$ are both embedded submanifolds, we see that $H_1=\gamma_2^{-1}H_2\gamma_2$. □

However, not every such pair give one U-ergodic measure.

2. Candidates of homogeneous orbit closure

Now we take some $x_0\in X$. Eventually, we would know that the closure of $U.x_0$ is homogeneous and the homogeneous measure is finite and U-ergodic. But this does not follow immediately from the Thm. 1.1 above. Nevertheless, we can say something (without appealing to Thm. 1.1 above).

Definition 2.1. *Let $\mathscr{A}$ (depending on $x_0\in X$ and U) be the collection of subgroup L of* G *satisfying*

1. *L is a connected closed subgroup of* G *containing U;*
2. *$L.x_0$ is closed.*

Lemma 2.2. *The collection $\mathscr{A}$ has a smallest element. Indeed, if $L_1,L_2\in\mathscr{A}$ then $(L_1\cap L_2)^\circ\in\mathscr{A}$.*

Proof. First we remark that for a closed subgroup $L \le \mathrm{G}$, if $L\Gamma$ is closed then $L^\circ\Gamma$ is also closed. Indeed, one shows that every orbit of L° on $L/L\cap\Gamma$, which is homeomorphic to $L\Gamma/\Gamma$ (Exercise, use Baire Category theorem), is open and hence closed. So it suffices to show that if $L_3 := L_1 \cap L_2$, then $L_3\Gamma$ is closed. This follows from a similar reasoning. Indeed, every orbit of L_3 on $[L_1]_\Gamma \cap [L_2]_\Gamma$ is open and hence closed. To see why every orbit is open, one may take a local neighborhood. □

Take $g_0 \in \mathrm{G}$ such that $x_0 = [g_0]_\Gamma$.

Theorem 2.3. *Let $H := H_{\mathscr{A}}$ be this smallest element in $\mathscr{A}$, then*

1. *the closed set $H.x_0$ supports a finite H-invariant measure $\mathrm{m}_{H.x_0}$;*
2. *the measure $\mathrm{m}_{H.x_0}$ is U-ergodic;*
3. *there exists a $\mathbb{Q}$-algebraic subgroup $\mathbf{H}'$ of SL_n such that $g_0^{-1}Hg_0 = \mathbf{H}'(\mathbb{R})^\circ$, more precisely, $\mathbf{H}'$ is the smallest $\mathbb{Q}$-algebraic subgroup containing $g_0^{-1}Ug_0$. In particular, H is algebraic.*

[You may ignore the last statement if you are allergic to algebraic groups.]

Before the proof, note that there is a locally finite measure $\mathrm{m}_{H.x_0}$ that is only "quasi-invariant under H" (for instance, the one induced from a right invariant Riemannian metric). A priori, it is not clear why it is H-invariant. But one can still talk about ergodicity and the associated unitary representation (with suitably twisted action). You may ignore this issue by pretending $\mathrm{m}_{H.x_0}$ is H-invariant from the start.

Here is a sketch of proof.

Step 1. By Mautner's phenomenon (see [Moo80, Theorem 1.1] and some supplementary arguments in [Sha91, Proposition 2.7]), there exists a closed normal subgroup $F \lhd H$ containing U such that for every unitary representation of H, every U-fixed vector is F-fixed. Thus to show U-ergodicity, suffices to show F-ergodicity.

Step 2. Let Γ_H be the stabilizer of x_0 in H. Explicitly, $\Gamma_H = H \cap g_0\Gamma g_0^{-1}$. Define

$$F' := \overline{F\cdot\Gamma_H}.$$

Since F is normal, F' is a closed subgroup of H. Since F' is right invariant under Γ_H, $F'\Gamma_H/\Gamma_H$ is closed in H/Γ_H. Thus $F'.x_0$ is closed. And F' contains U. By minimality of H, $F' = H$.

Step 3. Now we show F-ergodicity of $\mathrm{m}_{H.x_0}$. Let Ω be a F-invariant measurable set of H/Γ_H. Assume $\mathrm{m}_{H.x_0}(\Omega) > 0$, we want to show its complement has zero measure. Since F is normal, we see that its preimage $\widetilde{\Omega}$ right invariant under the group $F\cdot\Gamma_H$. Let m_H be the right H-invariant Haar measure on H. Then $\mu := \mathrm{m}_H|_{\widetilde{\Omega}}$ is right $F\cdot\Gamma_H$-invariant. Since μ is a locally finite measure, by continuity, the stabilizer of μ in H (under the action from the right) is a closed subgroup. Thus μ is right F'-invariant, hence H-invariant. By uniqueness of Haar measure, $\mu = \mathrm{m}_H$ (up to a scalar, which has to be 1). In particular, the complement of $\widetilde{\Omega}$ has zero measure. This implies that the complement of Ω also has zero measure.

Step 4. It remains to show that $\mathrm{m}_{H.x_0}$ is a finite measure. In fact every U-ergodic locally finite measure ν is finite. By pointwise ergodic theorem (see [Wal82, Theorem 1.14, Section 1.6]), for every $f \in L^1(\nu)$, for ν-almost every x,

$$f^*(x) := \lim_{S\to+\infty} \frac{1}{S}\int_0^S f(\mathbf{u}_s.x)\,\mathrm{d}s \text{ exists.}$$

Moreover $f^* \in L^1(\nu)$ and is U-invariant. By ergodicity, f^* is a constant, which has to be 0 if ν is an infinite measure.

On the other hand, by non-divergence of unipotent flow, there exists a compact set C such that if f is the indicator function of C, then $f^* \neq 0$.

Thus ν has to be finite. This finishes the proof of 1 and 2 of Thm.2.3.

Step 5. To save notation, we assume $g_0 = \mathrm{id}$ here.

Let $\mathbf{L}$ be the smallest $\mathbb{Q}$-algebraic subgroup of SL_n containing U. Let $\pi_1 : \mathbf{L} \to \mathbf{T}$ be the maximal quotient (algebraic) torus of $\mathbf{L}$. π_1 is defined over $\mathbb{Q}$. Since U is unipotent and π_1 preserves this property, the image of $\pi_1(U)$ consists of unipotent elements. But torus $\mathbf{T}$ only contains semisimple elements. Thus U is contained in the kernel of π_1, which is in the form of a semisimple (algebraic) group semidirect product with a unipotent (algebraic) group. In particular, $\mathbf{L}$ admits no nontrivial characters (:=algebraic group morphisms to $\mathbb{C}^\times = \mathrm{GL}_1$). By a theorem of Borel–Harish-Chandra (see for instance [Bor19, Corollary 13.2]), $\mathrm{L} \cap \Gamma$ ($\mathrm{L} := \mathbf{L}(\mathbb{R})^\circ$) is a lattice in L and in particular (by Exer. 6.2 from Exercise Sheet 3.), $\mathrm{L}\Gamma/\Gamma$ is closed. By minimality of H, $H \subset \mathrm{L}$. Our goal is to show $H = \mathrm{L}$ (this is what we mean by saying H is "algebraic"). We do know that the Zariski closure of H is equal to $\mathbf{L}$.

Step 6. Let $\mathfrak{h}$ be the Lie algebra of H. By Levi's decomposition (reference? probably Bourbaki's book?), there exists a semisimple $\mathfrak{m}$ and solvable ideal $\mathfrak{r}$ of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{m} \ltimes \mathfrak{r}$. By [Bor91, ChII, Corollary 7.9], since $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}]$, $\mathfrak{m}$ is already algebraic (the corresponding Lie subgroup M is algebraic). Let $\mathbf{M}$ be the corresponding $\mathbb{R}$-algebraic subgroup. We seek to show that $\mathfrak{r}$ consists of nilpotent matrices and hence is algebraic. By [Bor91, ChII, Corollary 7.7], this shows that H is algebraic. Since H normalizes $\mathfrak{r}$ and "normalizing $\mathfrak{r}$" is an algebraic condition, we have that $\mathbf{L}$ normalizes $\mathfrak{r}$. Thus $\mathfrak{r}$ is an ideal of $\mathfrak{l}$. Let $\pi_2 : \mathfrak{l} \to \mathfrak{l}/\mathfrak{u} \cong \mathfrak{m}$ (here $\mathfrak{u}$ is the Lie algebra of the unipotent radical of $\mathbf{L}$), then $\pi_2(\mathfrak{r})$ is an ideal of $\mathfrak{l}/\mathfrak{u}$. But every non-zero ideal of a semisimple Lie algebra is semisimple and can not be solvable. Thus $\pi_2(\mathfrak{r}) = 0$, or $\mathfrak{r} \subset \mathfrak{u}$, which consists of nilpotent matrices. The rest of the claim in 3. of Thm. 2.3 also follows by Borel density theorem.

3. Tubes

Assume U acts on $g_* \mathrm{m}_{[H]}$ ergodically, it is still possible for some $h \in H$, $U[gh]_\Gamma$ is trapped in a closed homogeneous set of smaller dimension.

Definition 3.1. *For $H \in \mathscr{H}$, define*

$$\mathrm{Sing}(H,U) := \bigcup_{L \in \mathscr{H}, L \subsetneq H} N(L,U);$$
$$NS(H,U) := N(H,U) \setminus \mathrm{Sing}(H,U);$$
$$T(H,U) := NS(H,U)\Gamma/\Gamma.$$

Lemma 3.2. *Let $H_1, H_2 \in \mathscr{H}$. If $NS(H_1,U)\Gamma \cap NS(H_2,U)\Gamma \neq \emptyset$, then H_1 is Γ-conjugate to H_2 and $NS(H_1,U)\Gamma = NS(H_2,U)\Gamma$.*

Proof. So assume $NS(H_1,U)\Gamma \cap NS(H_2,U)\Gamma \neq \emptyset$, which means that we can find $g_1 \in NS(H_1,U)$ and $\gamma_1 \in \Gamma$ such that $g_1\gamma_1 \in NS(H_2,U)$. By definition, we have

$$g_1^{-1} U g_1 \subset H_1 \cap \gamma_1 H_2 \gamma_1^{-1}.$$

We know (the connected component of) $H' := H_1 \cap \gamma_1 H_2 \gamma_2^{-1}$ has a closed orbit based at $[\mathrm{id}]_\Gamma$. But we do not know whether it supports a finite H'-invariant measure. This is where we apply Thm.2.3 (to the unipotent group $g_1^{-1}Ug_1$ and $x_0 = [\mathrm{id}]_\Gamma$) to conclude that there exists $L \subset H'$, $L \in \mathscr{H}$ such that $g_1^{-1}Ug_1 \subset L$, or $g_1 \in N(L,U)$.

Therefore $H_1 = \gamma_1 H_2 \gamma_1^{-1}$ for otherwise L will be strictly contained in at least one of H_1 or $\gamma_1 H_2 \gamma_1^{-1}$ and this would imply $g_1 \notin NS(H_1,U)$ or $g_1\gamma_1 \notin NS(H_2,U)$, contradicting against our assumption. $NS(H_1,U)\Gamma = NS(H_2,U)\Gamma$ follows immediately. □

Since U acts ergodically on m_X, we have $G \in \mathscr{H}$ and

$$X = \bigsqcup_{[H]\in\mathscr{H}/\sim_\Gamma} T(H,U)$$

thanks to the Lem.3.2.

Definition 3.3. *For $[H] \in \mathscr{H}/\sim_\Gamma$, let*

$$\mu^{[H]} := \mu|_{T(H,U)}.$$

Example 3.4. *If $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ and $U = \{\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, s \in \mathbb{R}\}$, then $\mathscr{H}/\sim_\Gamma = \{U, \mathrm{SL}_2(\mathbb{R})\}$ (if you pass to a smaller subgroup of $\mathrm{SL}_2(\mathbb{Z})$ then this set has other $\mathrm{SL}_2(\mathbb{Q})$-conjugates of U that are not conjugate over Γ). And $T(U,U)$ consists of compact orbits of U, $T(\mathrm{G},U)$ is the complement of $T(U,U)$.*

We have proved

Theorem 3.5. *For a U-invariant probability measure μ,*

$$\mu = \sum_{[H]\in\mathscr{H}/\sim_\Gamma} \mu^{[H]}$$

and each $\mu^{[H]}$ is U-invariant.

We have not used Thm.1.1 yet. For a finite positive measure μ on X, let $\widehat{\mu} := \mu/\mu(X)$ be the unique probability measure proportional to μ.

Theorem 3.6. *Assume $\mu^{[H]} \neq 0$. For almost every U-ergodic component ν of $\widehat{\mu^{[H]}}$, there exists $g_\nu \in N(H,U)$ such that $\nu = (g_\nu)_* \mathrm{m}_{[H]_\Gamma}$.*

Proof. First we have the (abstract) ergodic decomposition

$$\widehat{\mu^{[H]}} = \int_{\mathrm{Prob}(X)^{U,\mathrm{Erg}}} \nu\, \lambda(\nu).$$

Thus for almost every ν, $\nu(T(H,U)) = 1$. Take such a ν, by Thm.1.2, there exists $H_1 \in \mathscr{H}$ and $g_1 \in N(H_1,U)$ such that $\nu = (g_1)_*\widehat{\mathrm{m}}_{[H_1]_\Gamma}$. By pointwise ergodic theorem, we can find a full measure set of $h_1 \in H_1$ such that

$$\lim_{S\to+\infty} \frac{1}{S}\int_0^S (\mathbf{u}_s)_*\delta_{[g_1h_1]_\Gamma}\,\mathrm{d}s = (g_1)_*\widehat{\mathrm{m}}_{[H_1]_\Gamma}.$$

In particular, $\overline{U.[g_1h_1]_\Gamma} = g_1[H_1]_\Gamma$. One sees that $g_1h_1 \in N(H_1,U)$ and we claim that $g_1h_1 \in NS(H_1,U)$. Otherwise, there exists $L \subsetneqq H_1$ with $L \in \mathscr{H}$ such that $g_1h_1 \in N(L,U)$. This implies that $\overline{U.[g_1h_1]_\Gamma} \subset g_1h_1[L]_\Gamma$. Since $\dim L$ is strictly smaller than $\dim H_1$, we have a contradiction.

So now $[g_1h_1]_\Gamma \in T(H_1,U)$, moreover, all such h_1 is of full measure in H_1 and consequently $\nu(T(H_1,U)) = (g_1)_*\widehat{\mathrm{m}}_{[H_1]_\Gamma}(T(H_1,U)) = 1$. But $\nu(T(H,U)) = 1$. Thus $T(H_1,U)$ and $T(H,U)$ have nontrivial intersection. By Lem.3.2, for some $\gamma_1 \in \Gamma$, $H_1 = \gamma_1 H \gamma_1^{-1}$ and $T(H_1,U) = T(H,U)$. Hence $[g_1H_1]_\Gamma = g_1\gamma_1[H]_\Gamma$. Let $g_\nu := g_1\gamma_1$. One can check that $g_\nu \in N(H,U)$ and $\nu = (g_\nu)_*\widehat{\mathrm{m}}_{[H]_\Gamma}$ □

REFERENCES

[Bor91] Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR 1102012

[Bor19] ______, *Introduction to arithmetic groups*, University Lecture Series, vol. 73, American Mathematical Society, Providence, RI, 2019, Translated from the 1969 French original [MR0244260] by Lam Laurent Pham, Edited and with a preface by Dave Witte Morris. MR 3970984

[Moo80] Calvin C. Moore, *The Mautner phenomenon for general unitary representations*, Pacific J. Math. **86** (1980), no. 1, 155–169. MR 586875

[MS95] Shahar Mozes and Nimish Shah, *On the space of ergodic invariant measures of unipotent flows*, Ergodic Theory Dynam. Systems **15** (1995), no. 1, 149–159. MR 1314973

[Ra91] Marina Ratner, *On Raghunathan's measure conjecture*, Ann. of Math. (2) **134** (1991), no. 3, 545–607. MR 1135878

[Sha91] Nimish A. Shah, *Uniformly distributed orbits of certain flows on homogeneous spaces*, Math. Ann. **289** (1991), no. 2, 315–334. MR 1092178

[Wal82] Peter Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982. MR 648108