LECTURE 9, CLASSIFICATION OF FINITE INVARIANT MEASURES UNDER UNIPOTENT FLOWS, SL(2,R)

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CONTENTS

1. ERGODICITY AND EXTREMALITY

As last time, unless otherwise specified, we assume G and X are nice. So G is a locally compact (I forgot to add this condition last time) and σ-compact metrizable group and X is a σ-compact metrizable space. The induced $\mathrm{Prob}(X)$ with the weak$*$ topology is not necessarily compact (unless X is compact) but the $\mathrm{Meas}(X)^{\leq 1}$ is. And the induced action of G on $\mathrm{Meas}(X)^{\leq 1}$, $\mathrm{Prob}(X)$ and $\mathrm{LFM}(X)$ (decompose X into countable union of compact pieces and consider probability or finite measures supported on finite unions of them) are also nice.

Lemma 1.1. *A G-invariant probability measure μ is ergodic iff it is extremal in the space of G-invariant probability measures. Or more succinctly,* $\mathrm{Prob}(X)^{G,erg} = \mathrm{Extre}(\mathrm{Prob}(X)^G)$.

Being **extremal** means that μ can not be written as convex combination of different invariant probability measures. That is to say, if $\mu = a\nu_1 + (1-a)\nu_2$ for some $a \in (0,1)$ and $\nu_i \in \mathrm{Prob}(X)^G$, then $\nu_1 = \nu_2 = \mu$. In particular, two different ergodic μ_1, μ_2 must be singular w.r.t. each other. Namely, we may partition $X = A \sqcup B$ into two parts such that $\mu_1(B) = 0$ and $\mu_2(A) = 0$.

Sketch of Proof. If μ is not ergodic, then we can pick two complementary invariant measurable sets. Then μ is the sum of the restriction of μ to these two sets. Conversely, if $\mu = a\nu_1 + (1-a)\nu_2$ then ν_1 and ν_2 are absolutely continuous w.r.t. μ. So we find two invariant $L^1(\mu)$-functions which are forced to be constants unless μ is not ergodic. □

By general facts from functional analysis (Hahn–Banach theorem), the convex combinations of $\mathrm{Extre}(\mathrm{Prob}(X)^G)$ are dense in $\mathrm{Prob}(X)^G$ (pretend X to be compact first and then do the general case). A theorem of Choquet says that more precisely (See Thm.4.8 and 8.20 of the book of Einsiedler–Ward [EW11]),

Theorem 1.2 (Ergodic decomposition)**.** *For every $\mu \in \mathrm{Prob}(X)^G$ there exists a unique Borel probability measure $\lambda \in \mathrm{Prob}(\mathrm{Prob}(X)^G)$ such that*

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- $\lambda(\mathrm{Prob}(X)^{G,erg}) = 1$;
- $\mu = \int_{\nu \in \mathrm{Prob}(X)^{G,erg}} \nu \, \lambda(\nu)$.

Let me add that $\mathrm{Prob}(X)^{G,erg}$ is not closed in general (Exercise: find such an example) but in the world of unipotent flows, this is closed due to a theorem of Mozes–Shah.

For this reason to classify invariant probability measures, we often start with ergodic ones.

2. Pointwise ergodic theorem for a flow

We can construct a new invariant probability measure from known ones by convex combination. But how to get one to start with? Well, in general such a measure may not exist (say, the $\mathrm{SL}_2(\mathbb{R})$-action on the space of lines of $\mathbb{R}^2$). But for a flow, namely a continuous $\mathbb{R}$-action (denote the action $\mathbb{R} \times X \to X$ by $(t,x) \mapsto T_t.x$) on a nice X, we can consider

$$\frac{1}{T}\int_0^T (T_t)_* \delta_x \,\mathrm{dt} = \frac{1}{T}\int_0^T \delta_{T_t.x} \,\mathrm{dt}$$

as $T \to +\infty$. You can replace the δ-measure supported on $\{x\}$ by any other probability measure. Using this construction, one shows that

Lemma 2.1. *Let $(T_t)_{t\in\mathbb{R}}$ be a flow on X. If further assume X is compact, then there exists a (T_t)-invariant probability measures.*

Conversely, every ergodic flow-invariant probability measure may be constructed this way from a delta measure. Actually, more is true. This is the pointwise ergodic theorem.

Theorem 2.2. *Let T_t denotes the action of $\mathbb{R}$ on a nice space X. Let μ be an ergodic Borel probability (T_t)-invariant measure on X. Then for every $f \in L^1(X, \mathscr{B}_X, \mu)$ there exists a measurable set E_f of full measure ($\mu(E_f) = 1$) such that for every $x \in E_f$ we have*

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(T_t.x)\,\mathrm{dt} = \int f(x)\mu(x). \tag{1}$$

Using the fact that $C_c(X)$ admits a countable dense subset for a nice X, a diagonal argument shows that

Corollary 2.3. *Assumption as in the above theorem. There exists a full measure set E such that for every $x \in E$,*

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T (T_t)_* \delta_x \,\mathrm{dt} = \mu \tag{2}$$

where the limit is taken with respect to the weak-$*$ topology.

There is no such general ergodic theorem beyond the world of amenable groups.

A point x satisfying Equa.1 (or 2) is sometimes called f-**generic** (or generic). To emphasize the group action and the invariant measure, we may also say (T_t,μ)-generic points. In general, it may be very difficult to describe the set of generic points. One beauty of unipotent flows is that you do have an explicit description of generic points in this case.

3. Ergodic measures for unipotent flows

Let us start with the easiest case.

Definition 3.1. *Given a continuous action of G on X. We say that the action is* ***uniquely ergodic*** *iff the action admits a unique invariant probability measure.*

Theorem 3.2. *Let Γ be a discrete subgroup of $\mathrm{G} := \mathrm{SL}_2(\mathbb{R})$, then there exists a unique up-to-a-scalar G-invariant locally finite measure $\mathrm{m_X}$ on $\mathrm{X} := \mathrm{G}/\Gamma$.*

A reference is Raghunathan's book [Rag72]. For existence and uniqueness of invariant (Haar) measures on a (nice) topological group, one may consult [DE14].

Thus when such a measure is finite, we get an example of unique ergodic action.

Lemma 3.3. *Assume a flow T_t on a compact space X is uniquely ergodic with the unique invariant probability measure denoted by μ, then for every $x \in X$, Equa.2 holds.*

You can not drop the compactness assumption.

Theorem 3.4. *Assume Γ is a cocompact discrete subgroup of $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$. Then the U-action is uniquely ergodic.*

The existence is guaranteed. One needs to prove the uniqueness. The result is due to Furstenberg [Fur73].

A more general result is

Theorem 3.5. *Let Γ be a discrete subgroup of G. Then every $\mu \in \mathrm{Prob(X)}^{U,erg}$ is one of the following:*

1. *supported on a closed (necc. compact) U-orbit;*
2. $\mathrm{m}_X/|\mathrm{m}_X|$ *with* $|\mathrm{m}_X| < \infty$.

In particular, if X has no compact U-orbit and $|\mathrm{m}_X| = \mathrm{m}_X(X)$ is not finite, then there is no finite U-invariant measure. Though this does not prevent the existence of dynamically interesting infinite U-invariant measures.

4. Outline of the proof and step 1

The proof to be presented here consists of two parts

Step 1. Upgrade from U-invariance to B-invariance if the measure is not supported on a compact U-orbit;

Step 2. Show that the action of B is uniquely ergodic unless m_X is infinite.

The first step is essentially achieved by a combination of ideas from Lec.2 and pointwise ergodic theorem. For the second step we will follow Ratner's paper[Ra92]. It might be possible to do the second step by a duality argument in the style of Lec.2.

Compared to Lec.2 we will do the following adjustment

$$\text{compact topological spaces} \longrightarrow \text{probability invariant measures}$$
$$\text{minimal set} \longrightarrow \text{generic points}$$

We shall actually use compact subsets of generic points so that we can take limits.

Lemma 4.1. *Let μ be an ergodic U-invariant probability measure on $\mathrm{SL}_2(\mathbb{R})/\Gamma$ where Γ is a discrete subgroup, then*

1. *either μ is supported on a closed U-orbit;*
2. *or μ is B-invariant.*

Before the proof we make the following observation

Lemma 4.2. *If x, y are both (U,μ) generic points and $y = g.x$ with $g \in \mathrm{G}$ normalizing U, then $g_*\mu = \mu$.*

Proof. Since $g \in \mathrm{G}$ normalizes U, we find some constant $c_g > 0$ such that $g\boldsymbol{u}_t g^{-1} = \boldsymbol{u}_{c_g t}$. By definition of genericity we have

$$\begin{aligned} g_*\mu = & g_* \lim_{T\to+\infty} \frac{1}{T}\int_{t=0}^{T} (\boldsymbol{u}_t)_*\delta_x \,\mathrm{dt} = \lim_{T\to+\infty} \frac{1}{T}\int_{t=0}^{T} g_*(\boldsymbol{u}_t)_*\delta_x \,\mathrm{dt} \\ = & \lim_{T\to+\infty} \frac{1}{T}\int_{t=0}^{T} (\boldsymbol{u}_{c_g\cdot t})_* g_*\delta_x \,\mathrm{dt} = \lim_{T\to+\infty} \frac{1}{T}\int_{t=0}^{T} (\boldsymbol{u}_{c_g\cdot t})_*\delta_y \,\mathrm{dt} = \mu. \end{aligned}$$

□

Proof of Lemma 4.1. Without loss of generality assume μ is not supported on a closed U-orbit. In light of Lem.4.2 above, we hope to find a pair x, y that are both (U,μ)-generic and $y = \boldsymbol{a}_t.x$ for $t \neq 0$ arbitrarily close to 0.

Recall the argument from Lec.2 basically goes like:

Step 1. find two sequences (x_n) and (y_n) with $d(x_n, y_n) \to 0$ and for each n, x_n and y_n are not on the same local U-orbit;

Step 2. if for infinitely many n, x_n and y_n are on the same B-orbit, then we are done;

Step 3. otherwise, depending on $\delta > 0$, we find s_n, t_n such that every limit pair (x_∞, y_∞) of $x'_n := \boldsymbol{u}_{t_n} x_n$ and $y'_n := \boldsymbol{u}_{s_n} y_n$ are differed by some $\boldsymbol{a}_t$ with $t \in [\delta/C, C\delta]$ for some constant $C > 1$;

Step 4. as a complement to Step 3, it should be noted that the choice of s_n is determined by t_n and the choice of t_n has the freedom of multiplying by a (multiplicatively) bounded number. This havs the effect of changing the C in step 3 by another C';

Step 5. so far we have demonstrated $\boldsymbol{a}_t$ with $|t| \to 0$, $t \neq 0$ with $\boldsymbol{a}_t \in \mathrm{G}_\mu$, the stabilizer of μ in G. Since G_μ is a closed subgroup, Step 4 implies $A \subset \mathrm{G}_\mu$.

Below is a detailed account of carrying out the above strategy in the measure theoretic setting. You may try to figure out how by yourself.

We need to guarantee the limits (x_∞, y_∞) to be generic. Since the set of generic points is usually not closed, we define

$$E_{U,\mu} := \left\{ (U,\mu)\text{-generic points} \right\}$$

and take E to be a compact subset of $E_{U,\mu}$ such that $\mu(E) > 0.9$.

Take T_0 large enough such that the following set

$$F := \left\{ x \in X \,\middle|\, \frac{1}{T}\mathrm{Leb}\{t \in [0,T], \boldsymbol{u}_t.x \in E\} \geq 0.9,\ \forall T \geq T_0 \right\} \tag{3}$$

has $\mu(F) > 0.9$. (how? First by ptws ergodic theorem applied to the indicator function of E, we see that

$$\left\{ x \in X \,\middle|\, \lim_{T\to+\infty} \frac{1}{T}\mathrm{Leb}\{t \in [0,T], \boldsymbol{u}_t.x \in E\} = \mu(E) > 0.9 \right\}$$

has full measure 1. Thus as S varies over positive integers, the countable union of the following set

$$F_S := \left\{ x \in X \,\middle|\, \frac{1}{T}\mathrm{Leb}\{t \in [0,T], \boldsymbol{u}_t.x \in E\} > 0.9,\ \forall T \geq S \right\}$$

has measure 1. Thus we can find some T_0 such that F_{T_0} has measure at least 0.9.)

We claim that there exist pairs (x, y) in F arbitrarily close to each other and yet not on the same local U-orbit (unless μ is supported on a compact U-orbit, which by assumption does not happen).

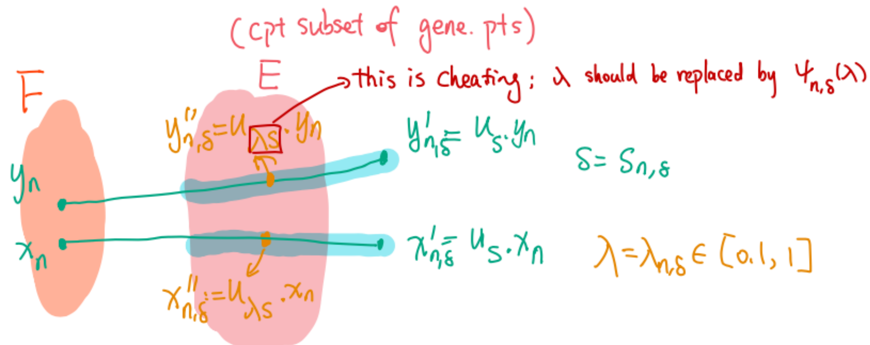
To be precise, two points x, y are said to be on the same *local* U-orbit if $x = \boldsymbol{u}_s.y$ for some $s \in (-1, 1)$.

If the claim were not true, then there exists $\varepsilon > 0$ such that if $x, y \in F$ and $d(x, y) < \varepsilon$ then $x = \boldsymbol{u}_s y$ for some $|s| < 1$. Cover F by countably many measurable sets $\{B_i\}$ of diameter smaller than ε. Then $B_i \cap F \subset \boldsymbol{u}_{(-1,1)}.x_i$ for some x_i. So $F \subset \bigcup_i \boldsymbol{u}_{(-1,1)}.x_i$. Thus for some x_i,

$$\mu(\{\boldsymbol{u}_s.x_i \,|\, |s| < 1\}) > 0.$$

By ergodicity this implies that $U.x$ has to close up (you can cite ptws ergodic theorem to prove this but you do not have to) and that μ is the U-invariant measure supported on this orbit. Contradiction.

Here is a summary-by-picture:



By def. of F, we can choose $\lambda=\lambda_{n,\delta} \in [0.1, 1]$ s.t. $x''_{n,\delta}, y''_{n,\delta}$ lie in E, a compact subset of generic points.

rmk: you may think of F as contained in E if you like

Recall calculation from Lec.2,

$$u_t A_n u_s^{-1} = \begin{bmatrix} a_n + (s+t)c_n & b_n + s(d_n - a_n) - s^2 c_n + (t-s)(1 + d_n - sc_n) \\ c_n & 1 + d_n - sc_n \end{bmatrix} \tag{4}$$

for

$$A_n = \begin{bmatrix} 1 + a_n & b_n \\ c_n & 1 + d_n \end{bmatrix} \quad \text{with } a_n, b_n, c_n, d_n \to 0.$$

(We have secretly changed a_n and d_n. Also, $t + s$ is replaced by t.) Compared to Lec.2, let us make a little adjustment on the choice of $s_{n,\delta}$ and particularly $t_{n,\delta}$ to simplify matters. Assume $c_n \neq 0$. For a small number $\delta > 0$, choose $s_{n,\delta}$ as before, namely,

$$s_{n,\delta} := \frac{d_n + \delta}{c_n}$$

(choosing $s = (d_n - \delta)/c_n$ is also ok). We also need an additional parameter $\lambda = \lambda_{n,\delta} \in (0.1, 1)$ to be determined in a moment and let $s'_{n,\delta} := \lambda_{n,\delta} s_{n,\delta}$. Choose $t'_{n,\delta} := \phi_{n,\delta}(\lambda_{n,\delta}) \cdot$

$s'_{n,\delta}$ where

$$\phi_{n,\delta}(\lambda_{n,\delta}) := \frac{a_n - (1-\lambda_{n,\delta})d_n - \lambda_{n,\delta}\delta}{1+(1-\lambda_{n,\delta})d_n - \lambda_{n,\delta}\delta} + 1.$$

This choice is such that the upper right corner of Equa.4 converges to 0 asymptotically. Indeed with $\lambda = \lambda_{n,\delta}$, $s = s'_{n,\delta}$ and $t = t'_{n,\delta}$ we have

$$\begin{aligned} & b_n + s(d_n - a_n) - s^2 c_n + (t-s)(1+d_n - sc_n) \\ =& b_n + s(d_n - a_n) - s\lambda(d_n+\delta) + (t-s)(1+d_n) - (t-s)\lambda(d_n+\delta) \\ =& b_n + s(-a_n + d_n - \lambda d_n - \lambda\delta) + (t-s)(1+d_n - \lambda d_n - \lambda\delta) = b_n \to 0. \end{aligned}$$

Let us firstly cheat by assuming that $\phi(\lambda) \equiv 1$. We have

$$s'_{n,\delta} = t'_{n,\delta} = \lambda_{n,\delta} s_{n,\delta}.$$

For every $\delta > 0$ and n large enough such that $s_{n,\delta} > T_0$. Then by the definition of F (Equa.3),

$$\begin{aligned} &\mathrm{Leb}\left(\left\{\lambda \in (0.1,1) \,\middle|\, \boldsymbol{u}_{\lambda s_{n,\delta}} x \in E\right\}\right) > 0.9 - 0.1 = 0.8; \\ &\mathrm{Leb}\left(\left\{\lambda \in (0.1,1) \,\middle|\, \boldsymbol{u}_{\lambda t_{n,\delta}} y \in E\right\}\right) > 0.8. \end{aligned} \tag{5}$$

So these two sets share a common element, which we choose to be the $\lambda_{n,\delta}$. Define $x''_{n,\delta} := \boldsymbol{u}_{s'_{n,\delta}}.x_n$ and $y''_{n,\delta} := \boldsymbol{u}_{s'_{n,\delta}}.y_n$, then $x''_{n,\delta}, y''_{n,\delta} \in E$. By letting $n \to +\infty$ (pass to a subsequence if necessary) and by Equa.4 above, we get

$$y_{\infty,\delta} = \begin{bmatrix} (1-\lambda_{\infty,\delta}\delta)^{-1} & 0 \\ 0 & 1-\lambda_{\infty,\delta}\delta \end{bmatrix} x_{\infty,\delta}$$

where $x_{\infty,\delta} := \lim x''_{n,\delta} \in E$, $y_{\infty,\delta} := \lim y''_{n,\delta} \in E$ and $\lambda_{\infty,\delta} := \lim \lambda_{n,\delta} \in [0.1,1]$. So we get a sequence of non-identity elements in A converging to id that maps some generic point to another one. By Lem.4.2, they are contained in G_μ, which is a closed subgroup. Thus A is contained in G_μ and the proof completes.

□

To avoid cheating... As functions on $[0,2]$ indexed by n,δ, we can check that as $n \to \infty$ and $\delta \to 0$, the function $\phi_{n,\delta}$ converges to the constant 1 uniformly. Thus for n sufficiently large and δ sufficiently small, we may and do assume that

$$\phi_{n,\delta}(\lambda) \in [0.99, 1.01], \quad \phi'_{n,\delta}(\lambda) \in [-0.01, 0.01], \quad \forall \lambda \in [0,2].$$

Let $\psi_{n,\delta}(\lambda) := \phi_{n,\delta}(\lambda)\cdot\lambda$ and so $\psi_{n,\delta}(0) = 0$ (for n large and δ small such that everything is well-defined). Thus

$$\psi'_{n,\delta}(\lambda) \in [0.98, 1.02], \quad \forall \lambda \in [0,2]. \tag{6}$$

So $\psi = \psi_{n,\delta}$ defines a diffeomorphism from $[0.1,1] \to \psi([0.1,1])$. Note that

$$[0.15, 0.95] \subset \psi([0.1,1]) \subset [0.05, 1.05].$$

Let (abbr. $\psi := \psi_{n,\delta}$ and $s := s_{n,\delta}$)

$$\begin{aligned} A &:= \left\{\lambda \in [0.1,1] \,\middle|\, \boldsymbol{u}_{\psi(\lambda)s}.y \in E\right\}; \\ B := \psi(A) &= \left\{\lambda \in \psi([0.1,1]) \,\middle|\, \boldsymbol{u}_{\lambda s}.y \in E\right\}. \end{aligned}$$

Equa.3 and 6 imply that $\mathrm{Leb}(B) \geq 0.9 - 0.2 = 0.7$. Thus

$$\begin{aligned} \mathrm{Leb}(A) &= \int 1_A(x)\,\mathrm{d}x = \int 1_A(\psi^{-1}y)\left|(\psi^{-1})'(y)\right|\mathrm{d}y \\ &= \int 1_B(y)\left|\psi'(\psi^{-1}y)\right|^{-1}\mathrm{d}y \geq 0.7\cdot 0.98 \geq 0.6. \end{aligned}$$

Thus combined with Equa.5 and arguments following that equation we complete the proof without cheating. □

We will do step 2 the next time.

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