LECTURE 7, MORE ON NONDIVERGENCE OF UNIPOTENT FLOWS

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CONTENTS

Notations

- $X_N := \{\text{unimodular lattices in } \mathbb{R}^N\} \cong SL_N(\mathbb{R})/SL_N(\mathbb{Z})$;
- for a discrete subgroup Δ in $\mathbb{R}^N$, let $\|\Delta\| := \mathrm{Vol}(\Delta_{\mathbb{R}}/\Delta)$ where $\Delta_{\mathbb{R}}$ denotes the $\mathbb{R}$-linear span of Δ in $\mathbb{R}^N$;
- for $\Lambda \le \mathbb{R}^N$, $\mathrm{sys}(\Lambda) := \inf_{v \neq 0 \in \Lambda} \|v\|$;
- for $\delta > 0$, $\mathscr{C}_\delta := \{\Lambda \in X_N : \mathrm{sys}(\Lambda) \ge \delta\}$;
- $\mathrm{Prim}^k(\Lambda) := \{\text{primitive subgroups of } \Lambda \text{ of rank } k\}$;
- $\mathrm{Prim}(\Lambda) := \bigcup_{k=0}^{\mathrm{rank}(\Lambda)} \mathrm{Prim}^k(\Lambda)$.

1. SUMMARY AND DEFINITIONS

We would like to illustrate the main ideas behind [Kle10, Section 3] using X_4 as an example. The discussion can be generalized to X_N and even to $G(\mathbb{R})/G(\mathbb{Z})$ for other semisimple algebraic groups G. Warning: our presentation (and sometimes definitions!) differs from [Kle10, Section 3] and is "less careful" in many ways.

The discussion is useful beyond unipotent flows on X_N. We would like to mention [EMS97, MW02] here.

Definition 1.1. *Fix (C,α) two positive constants. A map $\phi : I \to SL_N(\mathbb{R})$ is said to be (C,α)-good at $\Lambda \in X_N$ if for every primitive subgroup Δ of Λ, every interval $J \subset I$, every $\rho \in (0,1)$ (the case $\rho \ge 1$ is rather trivial), define $M(J,\Delta) := \sup_{s\in J} \|\phi_s.\Delta\|$, then we have*

$$\frac{1}{|J|}\left|\left\{s \in J \,\middle|\, \|\phi_s.\Delta\| \le \rho \cdot M(J,\Delta)\right\}\right| \le C \cdot \rho^{\alpha}.$$

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The main examples for us are unipotent flows.

Lemma 1.2. *There are constants $C_N, \alpha_N > 0$, depending only on N such that for every nilpotent matrix u in $\mathfrak{sl}_N(\mathbb{R})$ and for every (finite or infinite) interval I in $\mathbb{R}$, $\phi(t) := \exp(t.u)$ is (C_N, α_N)-good at every $\Lambda \in \mathrm{X}_N$.*

Proof. Exercise or see [Kle10]. □

Theorem 1.3. *Fix $C, \alpha, \varepsilon, \delta$ positive constants. There exists a constant $\kappa = \kappa(C, \alpha, \varepsilon, \delta) > 0$ such that the following holds. Let $\Lambda \in \mathrm{X}_N$ and $\phi : I \to \mathrm{SL}_N(\mathbb{R})$. Assume*

- *ϕ is (C, α)-good at Λ;*
- *$\sup_{t\in I} \left\|\phi_t.\Delta\right\| \geq \delta$ for every $\Delta \in \mathrm{Prim}(\Lambda)$,*

then

$$\frac{1}{|I|}\,\mathrm{Leb}\left\{s \in I \,\middle|\, \phi_s.\Lambda \notin \mathscr{C}_\kappa\right\} \leq \varepsilon.$$

In the case of unipotent flows and infinite I, if the condition fails, then Λ contains a primitive subgroup fixed by the unipotent flow with small norm.

2. NONDIVERGENCE AND FLAGS

The key notion is being (δ, ρ)-protected, which provides a sufficient condition to guarantee non-divergence.

Definition 2.1.
A subset $\mathscr{F}$ of $\mathrm{Prim}(\Lambda)$ is said to be a ***flag*** *if for every two element Δ_1 and Δ_2 in $\mathscr{F}$, either $\Delta_1 \subset \Delta_2$ or $\Delta_1 \supset \Delta_2$. The* ***length*** *of a flag $\mathscr{F}$ is simply the cardinality of $\mathscr{F}$.*

Definition 2.2.
*Let $\delta, \rho \in (0,1)$. Let $\Lambda \in \mathrm{X}_N$ and $\mathscr{F} = \{\Delta_1 \leq \Delta_2 \leq ... \leq \Delta_l\}$ be a flag in $\mathrm{Prim}(\Lambda)$. We say that Λ is weakly (δ, ρ)-**protected** by $\mathscr{F}$ iff*

1. *$\rho \cdot \delta \leq \|\Delta_i\| \leq \delta$ for every $i = 1, .., l$;*
2. *$\|\Delta\| \geq 0.5\delta$ for every $\Delta \notin \mathscr{F}$* ***comparable*** *with $\mathscr{F}$, i.e. $\mathscr{F} \cup \{\Delta\}$ is still a flag.*

*Now given a map $\phi : I \to \mathrm{SL}_N(\mathbb{R})$. We say that $s \in I$ is weakly (δ, ρ)-**protected** by $\mathscr{F}$ iff*

1. *$\rho \cdot \delta \leq \left\|\phi_s.\Delta_i\right\| \leq \delta$ for every $i = 1, .., l$;*
2. *$\left\|\phi_s.\Delta\right\| \geq 0.5\delta$ for every $\Delta \notin \mathscr{F}$ comparable with $\mathscr{F}$.*

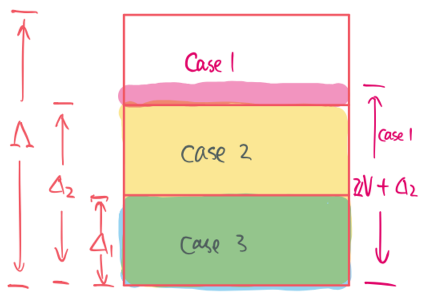
That is to say, $\phi_s.\Lambda$ is (δ, ρ)-protected by $\phi_s.\mathscr{F}$.

I shall drop the word "weakly" later. But keep in mind our definition is different from [Kle10] where 0.5δ is replaced by δ.

From the definition, such a flag is not allowed to contain $\{0\}$ or Λ. Thus the maximal possible length is $N - 1$.

One may wish to compare with the definition of Siegel sets.

Lemma 2.3 (Criterion of non-divergence in terms of flags)**.** *Fix $\delta, \rho \in (0,1)$. Assume for some reason that $\rho < 0.5$. Then there exists a constant $\theta = \theta(\delta, \rho) > 0$ (from the proof, can take $\theta = \rho^N \delta$) such that if $\Lambda \in \mathrm{X}_N$ is (δ, ρ)-protected by some flag $\mathscr{F}$ of $\mathrm{Prim}(\Lambda)$, then $\|\Delta\| \geq \theta$ for every primitive subgroup $\Delta \leq \Lambda$. In particular $\mathrm{sys}(\Lambda) \geq \theta$.*



Proof of a Special Case Done in The Class. Say $\mathscr{F} = \{\Delta_1 \leq \Delta_2\}$, which gives a filtration of Λ. For $v \in \Lambda$, there are three cases. We will show $\|v\| \geq \rho\delta$.
Case 1. $v \in \Lambda \setminus \Delta_2$.
Then $\Delta_3 + \mathbb{Z}.v$ is compatible with $\mathscr{F}$, though it may not be primitive. $((\Delta_3)_{\mathbb{R}} + \mathbb{R}.v) \cap \Lambda$ is a primitive subgroup compatible with $\mathscr{F}$ and contains $\Delta_2 + \mathbb{Z}.v$. Thus

$$\|\Delta_2 + \mathbb{Z}.v\| \geq \|((\Delta_2)_{\mathbb{R}} + \mathbb{R}.v) \cap \Lambda\| \geq 0.5\delta.$$

On the other hand

$$\|\Delta_2 + \mathbb{Z}.v\| \leq \|\Delta_2\| \cdot \|v\| \leq \delta \|v\|.$$

Combined together gives $\|v\| \geq 0.5$.
Case 2. $v \in \Delta_2 \setminus \Delta_1$.
Either $\Delta_1 + \mathbb{Z}.v$ has the same rank as Δ_2 or not. Anyway, we always have,

$$\|\Delta_1 + \mathbb{Z}.v\| \geq \min\{\rho\delta, 0.5\delta\} = \rho\delta.$$

On the other hand

$$\|\Delta_1 + \mathbb{Z}.v\| \leq \|\Delta_1\| \cdot \|v\| \leq \delta \|v\|.$$

Combined together gives $\|v\| \geq \rho$.
Case 3. $v \in \Delta_1$.
Then either $\mathbb{Z}.v$ has the same rank as Δ_1, in which case $\|\mathbb{Z}.v\| \geq \|\Delta_1\| \geq \rho\delta$ or $\mathbb{Z}.v$ has smaller rank than Δ_1, in which case $\|\mathbb{Z}.v\| \geq 0.5\delta \geq \rho\delta$.

□

Proof in General. [Read this only if you feel necessary!] Let $\mathscr{F} = \{\Delta_1 \leq \Delta_2 \leq \ldots \leq \Delta_l\}$ be the flag and Δ is a primitive subgroup of Λ. Let $V_k := \mathbb{R}^N / (\Delta_k)_{\mathbb{R}}$ and π_k be the natural quotient map $\mathbb{R}^N \to V_k$.

Note that if $\Delta' \leq \Lambda$ is contained in Δ_k for $k \in \{1, \ldots, l\}$, then

$$\left\|\pi_{k-1}(\Delta')\right\|_{V_{k-1}} = \left\|\pi_{k-1}(\Delta' + \Delta_{k-1})\right\|_{V_{k-1}} = \frac{\left\|\Delta' + \Delta_{k-1}\right\|}{\|\Delta_{k-1}\|} \geq \rho.$$

[rmk: actually, we see from the proof that unless $\Delta' + \Delta_{k-1} = \Delta_k$, we would have $\left\|\pi_{k-1}(\Delta')\right\|_{V_{k-1}} = 1$.] By convention $\|\{0\}\|_V = 1$ for all Euclidean spaces V.

$$\begin{aligned}
&\delta \left\|\pi_{k-1}(\Delta' + \Delta_{k-1})\right\|_{V_{k-1}} \geq \left\|\pi_{k-1}(\Delta' + \Delta_{k-1})\right\|_{V_{k-1}} \cdot \|\Delta_{k-1}\| = \left\|\Delta' + \Delta_{k-1}\right\| \geq \rho\delta \\
&\implies \left\|\pi_{k-1}(\Delta' + \Delta_{k-1})\right\|_{V_{k-1}} \geq \rho.
\end{aligned}$$

Let a be the largest index such that Δ_a is contained in Δ. By default, $\Delta_0 := \{0\}$ if $\Delta_1 \neq \{0\}$. If $a = l$, then we are done with $\theta = \rho\delta$. Assume otherwise.

$$\begin{aligned}
\|\Delta\| &= \|\pi_{a+1}(\Delta)\|_{V_{a+1}} \cdot \|\Delta \cap \Delta_{a+1}\| = \|\pi_{a+1}(\Delta + \Delta_{a+1})\|_{V_{a+1}} \cdot \|\pi_a(\Delta \cap \Delta_{a+1})\|_{V_a} \cdot \|\Delta_a\| \\
&= \|\pi_{a+2}(\Delta)\|_{V_{a+2}} \cdot \|\pi_{a+1}(\Delta + \Delta_{a+1}) \cap \pi_{a+1}(\Delta_{a+2})\|_{V_{a+1}} \cdot \|\pi_a(\Delta \cap \Delta_{a+1})\|_{V_a} \cdot \|\Delta_a\| \\
&= \|\pi_{a+2}(\Delta)\|_{V_{a+2}} \cdot \|\pi_{a+1}\left((\Delta + \Delta_{a+1}) \cap \Delta_{a+2}\right)\|_{V_{a+1}} \cdot \|\pi_a(\Delta \cap \Delta_{a+1})\|_{V_a} \cdot \|\Delta_a\| \\
&= \|\pi_{a+2}(\Delta)\|_{V_{a+2}} \cdot \|\pi_{a+1}\left(\Delta \cap \Delta_{a+2}\right)\|_{V_{a+1}} \cdot \|\pi_a(\Delta \cap \Delta_{a+1})\|_{V_a} \cdot \|\Delta_a\| \\
&\quad \ldots\ldots \\
&= \|\pi_{a+k-1}\left(\Delta \cap \Delta_{a+k}\right)\|_{V_{a+k-1}} \cdot \ldots \cdot \|\pi_{a+1}\left(\Delta \cap \Delta_{a+2}\right)\|_{V_{a+1}} \cdot \|\pi_a(\Delta \cap \Delta_{a+1})\|_{V_a} \cdot \|\Delta_a\| \\
&= \|\pi_{a+k-1}\left(\Delta \cap \Delta_{a+k}\right)\|_{V_{a+k-1}} \cdot \ldots \cdot \|\pi_{a+1}\left(\Delta \cap \Delta_{a+2}\right)\|_{V_{a+1}} \cdot \|\pi_a(\Delta \cap \Delta_{a+1})\|_{V_a} \cdot \|\Delta_a\|
\end{aligned}$$

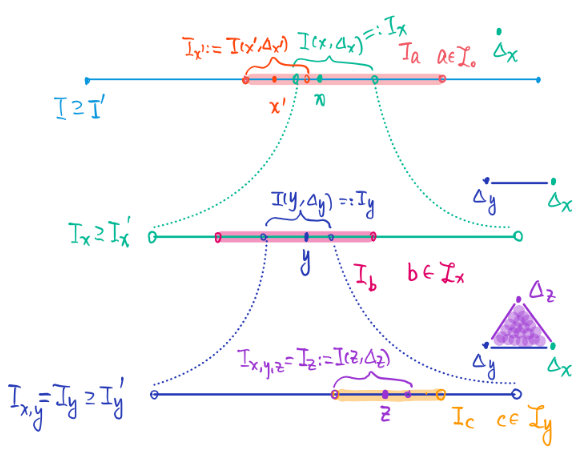
where k is the smallest positive integer such that Δ is contained in Δ_{a+k}. By invoking the key observation above we have

$$\|\Delta\| \geq \rho^k \delta.$$

So we are done by taking $\theta := \rho^N \delta$. □

3. The proof

Instead of proving by induction, we have decided to unfold this process. This makes the proof much longer but hopefully less mysterious. Here is a guide for Step 1-3.



Step 1. By assumption for every $\Delta \in \mathrm{Prim}(\Lambda)$,

$$\sup_{s \in I} \left\|\phi_s.\Delta\right\| \geq \delta.$$

Consider the open subset

$$I' := \left\{ s \in I \,\middle|\, \exists \Delta \in \mathrm{Prim}(\Lambda),\ \left\|\phi_s.\Delta\right\| < 0.9\delta \right\}$$

Write it as a disjoint union of open intervals

$$I' = \bigsqcup_{a \in \mathscr{I}_0} I_a.$$

Thus for every $\Delta \in \mathrm{Prim}(\Lambda)$,

$$\sup_{t \in I'} \left\|\phi_s.\Delta\right\| \geq 0.9\delta.$$

For $a \in \mathscr{I}_0$, consider (the 0.9 here is just to get a finite cover later, but it is not necessary to do so)

$$\mathscr{A}_a := \left\{(x,\Delta) \in I_a \times \mathrm{Prim}(\Lambda) \,\middle|\, \left\|\phi_x.\Delta\right\| < 0.9\delta\right\}.$$

For each $(x,\Delta) \in \mathscr{A}_a$, define

$$I(x,\Delta) := \text{ the connected component of } \left\{s \in I_a \,\middle|\, \left\|\phi_s.\Delta\right\| < \delta\right\} \text{ containing } x.$$

For every $x \in I_a$, pick some Δ_x such that $I_x := I(x,\Delta_x)$ is maximal among (the finitely many) $I(x,\Delta)$ as (x,Δ) varies in $\mathscr{A}_a$. By this choice, I_x and Δ_x satisfy

1. for every $\Delta \in \mathrm{Prim}(\Lambda)$, $\sup_{s\in I_x} \left\|\phi_s.\Delta\right\| \geq 0.9\delta$;
2. $\sup_{s\in I_x} \left\|\phi_s.\Delta_x\right\| \leq \delta$.

I_a admits a finite sub-covering by I_x's and by passing to a further sub-covering, we assume

$$I_a = \bigcup_{x\in\mathscr{I}_a} I_x \quad \text{with multiplicity } \leq 2$$

where $\mathscr{I}_a$ is certain finite subset of I_a (finiteness is not important, multi ≤ 2 is). Also define

$$\mathscr{P}_x := \left\{\Delta \in \mathrm{Prim}(\Lambda) \,\middle|\, \Delta \text{ is comparable to } \Delta_x\right\}.$$

Step 2. Consider the open subset of I_x:

$$I'_x := \left\{s \in I_x \,\middle|\, \exists\, \Delta \in \mathscr{P}_x,\ \left\|\phi_s.\Delta\right\| < 0.8\delta\right\}.$$

Write it as a disjoint union of open intervals

$$I'_x = \bigsqcup_{b\in\mathscr{I}_x} I_b.$$

For $b \in \mathscr{I}_x$, consider

$$\mathscr{A}_b := \left\{(y,\Delta) \in I_b \times \mathscr{P}_x \,\middle|\, \left\|\phi_y.\Delta\right\| < 0.8\delta\right\}.$$

For each $(y,\Delta) \in \mathscr{A}_b$, define

$$I(y,\Delta) := \text{ the connected component of } \left\{s \in I_b \,\middle|\, \left\|\phi_s.\Delta\right\| < 0.9\delta\right\} \text{ containing } y.$$

For every $y \in I_b$, pick some Δ_y such that $I_{x,y} := I(y,\Delta_y)$ is maximal among (the finitely many) $I(y,\Delta)$ as (y,Δ) varies in $\mathscr{A}_b$. By this choice, $I_{x,y}$ and Δ_y satisfy

1. for every $\Delta \in \mathscr{P}_x$, $\sup_{s\in I_{x,y}} \left\|\phi_s.\Delta\right\| \geq 0.8\delta$;
2. $\sup_{s\in I_{x,y}} \left\|\phi_s.\Delta_y\right\| \leq 0.9\delta$.

Similarly,

$$I_b = \bigcup_{y\in\mathscr{I}_b} I_{x,y} \quad \text{with multiplicity } \leq 2$$

where $\mathscr{I}_b$ is some finite subset of I_b. Also define

$$\mathscr{P}_{x,y} := \left\{\Delta \in \mathrm{Prim}(\Lambda) \,\middle|\, \Delta \text{ is comparable to } \{\Delta_x, \Delta_y\}\right\}.$$

Step 3. Consider the open subset of $I_{x,y}$:

$$I'_{x,y} := \left\{ s \in I_{x,y} \,\middle|\, \exists\, \Delta \in \mathscr{P}_{x,y},\ \left\| \phi_s.\Delta \right\| < 0.7\delta \right\}.$$

Write it as a disjoint union of open intervals

$$I'_{x,y} = \bigsqcup_{c \in \mathscr{I}_{x,y}} I_c.$$

For $c \in \mathscr{I}_{x,y}$, consider

$$\mathscr{A}_c := \left\{ (z,\Delta) \in I_c \times \mathscr{P}_{x,y} \,\middle|\, \left\| \phi_z.\Delta \right\| < 0.7\delta \right\}.$$

For each $(z,\Delta) \in \mathscr{A}_c$, define

$$I(z,\Delta) := \text{ the connected component of } \left\{ s \in I_c \,\middle|\, \left\| \phi_s.\Delta \right\| < 0.8\delta \right\} \text{ containing } z.$$

For every $z \in I_c$, pick some Δ_z such that $I_{x,y,z} := I(z,\Delta_z)$ is maximal among (the finitely many) $I(z,\Delta)$ as (z,Δ) varies in $\mathscr{A}_c$. By this choice, $I_{x,y,z}$ and Δ_z satisfy

1. for every $\Delta \in \mathscr{P}_{x,y}$, $\sup_{s \in I_{x,y,z}} \left\| \phi_s.\Delta \right\| \geq 0.7\delta$;
2. $\sup_{s \in I_{x,y,z}} \left\| \phi_s.\Delta_z \right\| \leq 0.8\delta$.

Similarly,

$$I_c = \bigcup_{z \in \mathscr{I}_c} I_{x,y,z} \quad \text{with multiplicity } \leq 2$$

where $\mathscr{I}_c$ is certain finite subset of I_c. Now $\{\Delta_x, \Delta_y, \Delta_z\}$ is already a complete flag modulo $\{0\}$ and Λ.

Good and bad points 1. For x, a, y, b, z, let

$$I_{x,y,z}(\text{good}) := \left\{ s \in I_{x,y,z} \,\middle|\, \left\| \phi_s.\Delta_z \right\| \geq \rho\delta \right\}, \quad I_{x,y,z}(\text{bad}) := I_{x,y,z} \setminus I_{x,y,z}(\text{good}).$$

By (C,α)-goodness, we choose $\rho \in (0,1)$ such that

$$\left| I_{x,y,z}(\text{bad}) \right| \leq (0.01\varepsilon) \left| I_{x,y,z} \right|.$$

Thus

$$\begin{aligned} \left| I'_{x,y}(\text{bad}) \right| &:= \left| \bigsqcup_{c \in \mathscr{I}_{x,y}} \bigcup_{z \in \mathscr{I}_c} I_{x,y,z}(\text{bad}) \right| \leq \sum_c \sum_z \left| I_{x,y,z}(\text{bad}) \right| \leq \sum_c \sum_z (0.01\varepsilon) \cdot \left| I_{x,y,z} \right| \\ &\leq \sum_c 2(0.01\varepsilon) \cdot |I_c| = (0.02\varepsilon) \cdot \left| I'_{x,y} \right|. \end{aligned}$$

Define $I'_{x,y}(\text{good}) := I'_{x,y} \setminus I'_{x,y}(\text{bad})$, so $I'_{x,y} = I'_{x,y}(\text{good}) \sqcup I'_{x,y}(\text{bad})$.

So far, we have the following regarding each $I_{x,y}$:

1. $s \in I_{x,y} \setminus I'_{x,y} \implies \left\| \phi_s.\Delta \right\| \geq 0.7\delta,\ \forall \Delta \in \mathscr{P}_{x,y}$;
2. $s \in I'_{x,y}(\text{good}) \implies \exists \Delta_z \in \mathscr{P}_{x,y},\ \rho\delta \leq \left\| \phi_s.\Delta_z \right\| \leq 0.8\delta$;
3. $\left| I'_{x,y}(\text{bad}) \right| \leq 2\delta \cdot \left| I'_{x,y} \right|$.

Good and bad points 2. Define

$$I_{x,y}(\text{good}) := \left\{ s \in I_{x,y} \,\middle|\, \left\| \phi_s.\Delta_y \right\| \geq \rho\delta \right\}, \quad I_{x,y}(\text{bad}) := I_{x,y} \setminus I_{x,y}(\text{good}).$$

And ρ is chosen such that

$$\left| I_{x,y}(\text{bad}) \right| \leq (0.01\varepsilon) \left| I_{x,y} \right|.$$

Thus,

$$\left|I'_x(\text{bad})\right| := \left|\bigsqcup_{b\in\mathscr{I}_x}\bigcup_{y\in\mathscr{I}_b} I_{x,y}(\text{bad})\right| \le \sum_b\sum_y \left|I_{x,y}(\text{bad})\right| \le \sum_b\sum_y (0.01\varepsilon)\cdot\left|I_{x,y}\right|$$
$$\le \sum_b 2(0.01\varepsilon)\cdot|I_b| = (0.02\varepsilon)\cdot\left|I'_x\right|.$$

Define $I'_x(\text{good})$ by imposing $I'_x = I'_x(\text{good}) \sqcup I'_x(\text{bad})$.
So far, regarding I_x we have:

1. $s \in I_x \setminus I'_x \implies \left\|\phi_s.\Delta\right\| \ge 0.8\delta,\ \forall\Delta\in\mathscr{P}_x$;
2. $s \in I'_x(\text{good}) \cap I_{x,y} \implies \rho\delta \le \left\|\phi_s.\Delta_y\right\| \le 0.9\delta$;
3. $\left|I'_x(\text{bad})\right| \le 2\delta\cdot\left|I'_x\right|$.

Good and bad points 3. Finally, define

$$I_x(\text{good}) := \left\{s\in I_x \,\middle|\, \left\|\phi_s.\Delta_x\right\| \ge \rho\delta\right\},\quad I_x(\text{bad}) := I_x\setminus I_x(\text{good}).$$

And ρ is chosen such that

$$|I_x(\text{bad})| \le 0.01\varepsilon\,|I_x|.$$

Thus,

$$\left|I'(\text{bad})\right| := \left|\bigsqcup_{a\in\mathscr{I}_0}\bigcup_{x\in\mathscr{I}_a} I_x(\text{bad})\right| \le \sum_a\sum_x |I_x(\text{bad})| \le \sum_a\sum_x 0.01\varepsilon\cdot|I_x|$$
$$\le \sum_a 2\cdot 0.01\varepsilon\cdot|I_a| = (0.02\varepsilon)\cdot\left|I'\right|.$$

Define $I'(\text{good})$ by imposing $I' = I'(\text{good})\sqcup I'(\text{bad})$. Here we have:

1. $s\in I\setminus I' \implies \left\|\phi_s.\Delta\right\| \ge 0.9\delta,\ \forall\Delta\in\operatorname{Prim}(\Lambda)$;
2. $s\in I'(\text{good})\cap I_x \implies \rho\delta\le\left\|\phi_s.\Delta_x\right\|\le\delta$;
3. $\left|I'(\text{bad})\right| \le 2\delta\cdot\left|I'\right|$.

Warp-up. Now we collect all the bad points together and let

$$I(\text{bad}) := I'(\text{bad}) \cup \left(\bigcup_{a\in\mathscr{I}_0, x\in\mathscr{I}_a} I'_x(\text{bad})\right) \cup \left(\bigcup_{a\in\mathscr{I}_0, x\in\mathscr{I}_a}\ \bigcup_{b\in\mathscr{I}_x, y\in\mathscr{I}_b} I'_{x,y}(\text{bad})\right)$$

We have

$$\left|\bigcup_{a,x,b,y} I'_{x,y}(\text{bad})\right| \le \sum_{a,x,b\in\mathscr{I}_x}\ \sum_{y\in\mathscr{I}_b}\left|I'_{x,y}(\text{bad})\right| \le (0.02\varepsilon)\cdot\sum_{a,x,b}\sum_y\left|I_{x,y}\right|$$
$$\le (0.04\varepsilon)\cdot\sum_{a,x,b\in\mathscr{I}_x}|I_b| \le (0.04\varepsilon)\cdot\sum_{a,x}|I_x|$$
$$\le (0.08\varepsilon)\cdot|I|$$

and

$$\left|\bigcup_{a\in\mathscr{I}_0,x\in\mathscr{I}_a} I'_x(\text{bad})\right| \le \sum_{a\in\mathscr{I}_0}\ \sum_{x\in\mathscr{I}_a}\left|I'_x(\text{bad})\right| \le (0.02\varepsilon)\cdot\sum_{a\in\mathscr{I}_0}\ \sum_{x\in\mathscr{I}_a}|I_x|$$
$$\le (0.04\varepsilon)\cdot\sum_{a\in\mathscr{I}_0}|I_a| \le (0.04\varepsilon)\cdot|I|.$$

Hence

$$|I(\text{bad})| \le (0.14\varepsilon)\cdot|I| < \varepsilon\,|I|. \tag{1}$$

Let $s\in I\setminus I(\text{bad})$.
Case 1. $s\in I\setminus I'$, then $\left\|\phi_s.\Delta\right\| \ge 0.9\delta > 0.5\delta,\ \forall\Delta\in\operatorname{Prim}(\Lambda)$ so it is (ρ,δ)-protected by the trivial flag.

Case 2. $s \in I' \setminus I(\text{bad}) = (\sqcup I_a) \setminus I(\text{bad}) = (\sqcup_a \cup_x I_x) \setminus I(\text{bad})$. Say $s \in I_x \setminus I(\text{bad})$. Then

$$\rho\delta \leq \left\|\phi_s.\Delta_x\right\| \leq \delta.$$

Case 2.1. $s \in I_x \setminus I'_x$. Then $\left\|\phi_s.\Delta\right\| \geq 0.8\delta > 0.5\delta$ for all $\Delta \in \mathscr{P}_x$. This means that s is (ρ,δ)-protected by $\{\Delta_x\}$.

Case 2.2. $s \in I'_x \setminus I(\text{bad}) = (\sqcup I_b) \setminus I(\text{bad}) = (\sqcup_b \cup_y I_{x,y}) \setminus I(\text{bad})$. Say $s \in I_{x,y} \setminus I(\text{bad})$. Then

$$\rho\delta \leq \left\|\phi_s.\Delta_y\right\| \leq \delta.$$

Case 2.2.1. $s \in I_{x,y} \setminus I'_{x,y}$. Then $\left\|\phi_s.\Delta\right\| \geq 0.7\delta > 0.5\delta$ for all $\Delta \in \mathscr{P}_{x,y}$. This means that s is (ρ,δ)-protected by $\{\Delta_x,\Delta_y\}$.

Case 2.2.2. $s \in I'_{x,y} \setminus I(\text{bad}) = \sqcup_c I_c \setminus I(\text{bad}) = \sqcup_c \cup_z I_{x,y,z} \setminus I(\text{bad})$. Say $x \in I_{x,y,z} \setminus I(\text{bad})$, then

$$\rho\delta \leq \left\|\phi_s.\Delta_z\right\| \leq \delta.$$

Thus s is (ρ,δ)-protected by $\{\Delta_x,\Delta_y,\Delta_z\}$.

Now every $s \in I \setminus I(\text{bad})$ falls into one of the cases 1, 2.1, 2.2.1 and 2.2.2, so it is (ρ,δ)-protected. Hence Lem.2.3 implies if $s \in I \setminus I(\text{bad})$ then $\phi_s.\Lambda \in \mathscr{C}_\theta$ with $\theta = \theta(\delta,\rho)$. Now we take $\kappa := \theta$. Combining with Equa.1, we are done.

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