# LECTURE 1

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## Notation

The set of positive integers is denoted by $\mathbb{Z}^+$. For a real number $x$, let $\langle x \rangle$ denote the distance to the nearest integer, namely, $\langle x \rangle = \inf_{n\in\mathbb{Z}} |x-n|$. Leb denotes the standard Lebesgue measure on $\mathbb{R}^n$ where the $n$ is understood from the context.

For $x, y \in \mathbb{Z}$ non-zero, we let $\gcd(x,y) \in \mathbb{Z}^+$ to be the greatest common divisor of $|x|$ and $|y|$. If one of them is zero but the other is not, we set $\gcd(x,y)$ to be the absolute value of the non-zero one. Also, $\gcd(0,0) := 0$. Two integers are said to be coprime iff $\gcd(x,y) = 1$.

Abbreviate "infinitely many" as "i.m."; " almost every " as "a.e.".

## 1. Lecture 1, Dirichlet's theorem, badly approximable numbers and Khintchin's zero-one law

References: I am mostly following [Zaf17, Cas50]. One may also consult the survey [BRV16] (available on arxiv).

1.1. **Foreword.** Number theory provides a huge amount of interesting problems. Besides "elementary" methods, tools from different branches of math are introduced to solve them. Assuming Galois theory and ring theory, one can give an introduction to number fields. Assuming complex analysis, one can study Riemann zeta functions, Dirichlet L functions or modular forms.

This course is concerned with so-called "Diophantine approximation" problems, which are concerned with approximating real numbers by rational numbers. Actually, we will focus on a specific (still unsolved!) problem: Littlewood conjecture. We will present the work of Einsiedler–Katok–Lindenstrauss on this conjecture, showing the exception set has dimension zero. They use tools coming from dynamics, which will be introduced later.

As this course is supposed to be introductory, we will start with some basics before discussing the deep work of EKL.

1.2. **The beginning.** The starting point of Diophantine approximation is the following:

**Fact 1.1.** *The set of rational numbers $\mathbb{Q}$ is dense in real numbers $\mathbb{R}$. In other words, for every $x \in \mathbb{R}$ and $\varepsilon > 0$, there exist two integers $(p, q)$ with $q > 0$ such that $\left|x - \frac{p}{q}\right| < \varepsilon$.*

To have a better approximation of $x \in \mathbb{R}$, one should use rational numbers with large denominators. How large it has to be?

**Theorem 1.2** (Dirichlet)**.** *For every $x \in \mathbb{R}$ and $N \in \mathbb{Z}^+$, there exists $(p, q) \in \mathbb{Z}^2$ with $0 < q \leq N$ such that*

$$\left|x - \frac{p}{q}\right| < \frac{1}{Nq}.$$

The proof is based on "drawer's principle" (or pigeon-hole principle).

*Proof.* For $k = 1, 2, ..., N$ find $n_k \in \mathbb{Z}$ such that $kx - n_k \in [0, 1)$. Write

$$\{kx - n_k,\ k = 1, ..., N\} = \{x_1 \leq x_2 \leq ... \leq x_N\}.$$

Thus, one of the numbers

$$\{x_1, x_2 - x_1, x_3 - x_2, ..., x_N - x_{N-1}, 1 - x_N\}$$

has to be strictly smaller than $\frac{1}{N}$. Say $x_{i_0} - x_{i_0-1} < \frac{1}{N}$. By convention, $x_0 := 0 = 0 \cdot x$ and $x_{N+1} := 1 = 0 \cdot x + 1$. Therefore, for some integers $\{k_1 < k_2\} \subset \{0, ..., N\}$, one has for some $p \in \mathbb{Z}$,

$$|k_2 x - k_1 x - p| < \frac{1}{N}$$

Let $q := k_2 - k_1$, then

$$\left|x - \frac{p}{q}\right| < \frac{1}{Nq},$$

proving the assertion. $\square$

1.3. **Badly approximable numbers.** As a corollary of the above theorem, one gets

**Corollary 1.3.** *For every $x \in \mathbb{R}$, there exist infinitely many pairs of integers $(p, q)$ such that*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2}.$$

*In other words, $\langle qx \rangle < \frac{1}{q}$ for infinitely many $q \in \mathbb{Z}^+$.*

**Definition 1.4.** *A real number $x$ is said to be* ***badly approximable*** *iff there exists $c > 0$ such that*

$$\left|x - \frac{p}{q}\right| > \frac{c}{q^2}, \quad \forall\, (p, q) \in \mathbb{Z}^2,\ q > 0.$$

*Or in other words, $\langle qx \rangle q > c$ for all $q \in \mathbb{Z}^+$. We will let* **BAD** *denote the set of badly approximable numbers. If an irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$ is not badly approximable, we say that $x$ is* ***well-approximable****.*

This definition is non-trivial in the sense that there are badly approximable numbers as well as well-approximable numbers.

**Example 1.5.** *$\sqrt{2}$ is badly approximable.*

*Proof.* Take $\varepsilon \in (0, 1)$. Assume that there are integers $p, q$ with $q > 0$ such that

$$q\left|q\sqrt{2} - p\right| < \varepsilon.$$

Thus

$$\left|q\sqrt{2} + p\right| < \varepsilon/q + 2q\sqrt{2} < 4q.$$

Multiplying the above two together gives

$$q\left|2q^2 - p^2\right| < 4q\varepsilon \implies \left|2q^2 - p^2\right| < 4\varepsilon.$$

But $2q^2 - p^2$ is a non-zero integer, so $\left|2q^2 - p^2\right| \geq 1$. Thus $1 < 4\varepsilon$. This finishes the proof, showing $q\langle q\sqrt{2}\rangle \geq \frac{1}{4}$ for every $q \in \mathbb{Z}^+$. $\square$

**Conjecture 1.6.** *Algebraic numbers that are not contained in a quadratic number field are not bad.*

So far no single example seems known about this conjecture. For instance, it is unknown whether $\sqrt[3]{2}$ is badly approximable or not.

**Remark 1.7.** *However, for every $\varepsilon > 0$ and irrational algebraic number $x$, there exists $c = c(x, \varepsilon) > 0$ such that $q^{1+\varepsilon}\langle qx\rangle > c(x, \varepsilon)$ for every $q \in \mathbb{Z}^+$. This is a theorem of Roth. Its generalization by Schmidt, known as subspace theorem, has applications to other problems in number theory.*

**Example 1.8.** 0.10100001000000001... *( the $n$-th group of 0's consists of $n + m$ consecutive zeros if there are $m$ digits in front of it) is well-approximable.*

1.4. **Littlewood conjecture.**

**Conjecture 1.9** (Littlewood)**.** *For every pair $(x, y)$ of real numbers, for every $\varepsilon > 0$, there exists $q \in \mathbb{Z}^+$ such that*

$$q\langle qx\rangle\langle qy\rangle < \varepsilon.$$

*Equivalently,*

$$\inf_{q\in\mathbb{Z}^+} q\langle qx\rangle\langle qy\rangle = 0. \tag{1}$$

**Remark 1.10.** *Note that if one of $x$ or $y$ does not belong to* **BAD***, then Equa.(1) holds.*

**Theorem 1.11** (Einsiedler–Katok–Lindenstrauss [EKL06])**.**

$$\dim\{(x, y) \in \mathbb{R}^2,\ \textit{Equa.}(1)\ \textit{fails}\ \} = 0.$$

This will be proved in later lectures. We will soon prove that

**Theorem 1.12.** $\mathrm{Leb}(\mathbf{BAD}) = 0$. *Consequently,*

$$\mathrm{Leb}\{(x, y) \in \mathbb{R}^2,\ \textit{Equa.}(1)\ \textit{fails}\ \} = 0.$$

The following two theorems will not be proved, but I find it healthy to compare them with Littlewood conjecture and EKL's work.

**Theorem 1.13** (Gallagher)**.**

$$\mathrm{Leb}\{(x, y) \in \mathbb{R}^2,\ \inf_{q\in\mathbb{Z}^+} q\langle qx\rangle\langle qy\rangle \cdot (\log q)^2 = 0\} = 0.$$

**Theorem 1.14** (Badziahin [Bad13])**.**

$$\dim\{(x, y) \in \mathbb{R}^2,\ \inf_{q\in\mathbb{Z}^+} q\langle qx\rangle\langle qy\rangle \cdot \log q \log\log q = 0\} = 2.$$

We will sometimes restrict our attention to numbers in the interval $[0, 1)$ without loss of generality.

1.5. **Khintchine's zero-one law.** Let $\psi : \mathbb{Z}^+ \to \mathbb{R}^+$ be a sequence of positive real numbers (for instance $\psi(q) := q^{-1}$). Define

$$\begin{aligned} W(\psi) &:= \left\{x \in [0, 1),\ \left|x - \frac{p}{q}\right| < q^{-1}\psi(q) \text{ for i.m. } q \in \mathbb{Z}^+, p \in \mathbb{Z}\right\} \\ &= \left\{x \in [0, 1),\ \langle qx\rangle < \psi(q) \text{ for i.m. } q \in \mathbb{Z}^+\right\} \end{aligned} \tag{2}$$

**Theorem 1.15** (Khintchin)**.** *Assume $\psi$ is non-increasing. Then,*

$$\mathrm{Leb}(W(\psi)) = \begin{cases} 0 & \textit{if } \sum \psi(n) < +\infty \\ 1 & \textit{if } \sum \psi(n) = +\infty \end{cases}.$$

**Remark 1.16.** *The assumption that $\psi$ is non-increasing is necessary.*

That $\mathrm{Leb}(\mathbf{BAD}) = 0$ follows directly from this theorem.

*Proof of Theorem 1.12 assuming Theorem 1.15.* For every $c > 0$ and $q \in \mathbb{Z}^+$, let $\psi_c(q) := cq^{-1}$. Then, **BAD** is the complement in $[0, 1)$ of the union of $W(\psi_{n^{-1}})$ as $n$ ranges over positive integers. Thus it suffices to show that $\mathrm{Leb}(W(\psi_c)) = 1$ for every $c > 0$. By Theorem 1.15, this follows from the fact that $\sum_{n\in\mathbb{Z}^+} cn^{-1} = +\infty$. ∎

1.6. **Proof of the convergence part.** In this subsection, we explain the convergence part of Theorem 1.15. Namely, we assume $\sum \psi(n) < +\infty$ and prove $\mathrm{Leb}(W(\psi)) = 0$. For this one uses the Borel–Cantelli lemma:

**Lemma 1.17.** *Let $(E_n)_{n\in\mathbb{Z}^+}$ be a sequence of measurable subsets of $[0,1)$ such that $\sum \mathrm{Leb}(E_n) < +\infty$. Then*

$$\mathrm{Leb}\left(\{x \in E_n \text{ for i.m. } n\}\right) = 0.$$

**Remark 1.18.** *The set $\{x \in E_n \text{ for i.m. } n\}$ is sometimes written as $\limsup E_n$.*

Let

$$W_n(\psi) := \{x \in [0,1) \mid \langle nx \rangle < \psi(n)\}.$$

In light of Lemma 1.17, it suffices to show that $\sum \mathrm{Leb}(W_n(\psi)) < \infty$. Indeed, for $n$ large enough (such that $\psi(n) < 0.5$),

$$\begin{aligned}
W_n(\psi) &= \bigsqcup_{i=0,1,\ldots,n-1} \{x \in W_n(\psi) \mid nx - i \in [0,1)\} \\
&= \bigsqcup_{i=0,1,\ldots,n-1} \left\{ x \in [\frac{i}{n}, \frac{i+1}{n}),\ nx \in [i, i+\psi(n)) \cup ((i+1)-\psi(n), i+1) \right\} \\
&= \bigsqcup_{i=0,1,\ldots,n-1} [\frac{i}{n}, \frac{i}{n} + \frac{\psi(n)}{n}) \cup (\frac{i+1}{n} - \frac{\psi(n)}{n}, \frac{i+1}{n}).
\end{aligned}$$

Hence,

$$\mathrm{Leb}(W_n(\psi)) = \sum_{i=0,1,\ldots,n-1} \frac{2\psi(n)}{n} = 2\psi(n).$$

Thus the divergence of $\sum \mathrm{Leb}(W_n(\psi))$ follows.

1.7. **Proof of the divergence part.** From now on assume $\sum \psi(n) = +\infty$ and we wish to show $\mathrm{Leb}(W(\psi)) = 1$. The proof will consist of two steps: $\mathrm{Leb}(W(\psi)) > 0$ and $\mathrm{Leb}(W(\psi)) > 0 \implies \mathrm{Leb}(W(\psi)) = 1$.

1.8. **Cassels' zero-one law.** In this subsection we prove

**Theorem 1.19.** $\mathrm{Leb}(W(\psi)) = 0$ *or* $1$.

Though we use the non-increasing feature of $\psi$ below, this assumption can be removed without much effort.

Choose a bijection $n \mapsto \lambda_n$ from $\mathbb{Z}^+$ to $\mathbb{Q}$. For a rational number $x$, find coprime integers $p, q$ with $q > 0$ (if $p = 0$, we set $q := 1$) such that $x = \frac{p}{q}$ and define $\Psi_{\mathrm{red}}(x) := q^{-1}\psi(q)$. One can check that

$$W(\psi) = W_{\mathrm{red}}(\psi) := \{x \in [0,1) \mid |x - \lambda_n| < \Psi_{\mathrm{red}}(\lambda_n) \text{ for i.m. } n\}.$$

For $k, N \in \mathbb{Z}^+$, let

$$\begin{aligned}
E_k &:= \left\{ x \in [0,1) \,\middle|\, |x - \lambda_n| < \frac{1}{k}\Psi_{\mathrm{red}}(\lambda_n) \text{ for i.m. } n \right\}, \\
E_k^N &:= \left\{ x \in [0,1) \,\middle|\, |x - \lambda_n| < \frac{1}{k}\Psi_{\mathrm{red}}(\lambda_n) \text{ for some } n > N \right\}.
\end{aligned}$$

Also let $E_\infty := \bigcap E_k$. So $E_1 = W_{\mathrm{red}}(\psi)$ and $E_k = \bigcap_{N=1}^{\infty} E_k^N$. Theorem 1.19 would follow from the following three lemmas.

For a positive integer $n$ and $x \in [0,1)$, define $T_n(x)$ to be the unique element in $[0,1)$ such that $T_n(x) - nx \in \mathbb{Z}$.

**Lemma 1.20.** *For every $k \in \mathbb{Z}^+$, $T_k(E_\infty) \subset E_1$.*

*Proof.* For $x \in E_k$ with $|x - \lambda_n| < k^{-1}\Psi_{\mathrm{red}}(\lambda_n)$, then $|kx - k\lambda_n| < \Psi_{\mathrm{red}}(\lambda_n) \leq \Psi_{\mathrm{red}}(k\lambda_n)$. Thus $kx \in E_1$. □

**Lemma 1.21.** *For every measurable set $E \subset [0,1)$ with $\mathrm{Leb}(E) > 0$, for every $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that*

$$\mathrm{Leb}\left(T_N(E)\right) > 1 - \varepsilon.$$

Let $\mathrm{I}(x,\delta) := (x-\delta, x+\delta)$. We will need Lebesgue's density theorem (see e.g. chapter 3, Theorem 1.4 of Stein's book "real analysis") for characteristic functions of Borel subsets.

**Theorem 1.22** (Lebesgue density theorem)**.** *Let $f$ be an integrable function on $[0,1)$. Then for Lebesgue almost every $x \in [0,1)$, one has*

$$\lim_{\varepsilon \to)^+} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(t) \operatorname{Leb}(t) = f(x).$$

*Proof.* By Lebesgue density theorem, find $\theta^* \in E$ such that for any $\varepsilon > 0$, there exists $\eta^* > 0$ such that for every $0 < \eta < \eta^*$,

$$\frac{\operatorname{Leb}(E \cap \mathrm{I}(\theta^*, \eta))}{2\eta} > 1 - \varepsilon.$$

Now choose $\eta = \frac{1}{2N}$ for $N \in \mathbb{Z}^+$ large such that the above inequality holds. Then

$$\operatorname{Leb}\left( NE \cap (N\theta^* - \frac{1}{2}, N\theta^* + \frac{1}{2}) \right) > 1 - \varepsilon.$$

By reducing modulo $\mathbb{Z}$, we get $\operatorname{Leb}(T_N(E)) > 1 - \varepsilon$. □

**Lemma 1.23.** *For every $k \in \mathbb{Z}^+$, $\operatorname{Leb}(E_1 \setminus E_k) = 0$. Consequently, $\operatorname{Leb}(E_\infty) > 0$ if $\operatorname{Leb}(E_1) > 0$.*

*Proof.* To save notation write $\alpha_n := \Phi_{\mathrm{red}}(\lambda_n)$ in the proof. Assume $\operatorname{Leb}(E_1 \setminus E_k) > 0$, find $N \in \mathbb{Z}^+$ large enough such that $\operatorname{Leb}(E_1 \setminus E_k^N) > 0$.

By Lebesgue density theorem again, we find $\theta^*_{\neq 0} \in E_1$ such that for every $\varepsilon > 0$, there exists $\eta^*(\varepsilon) > 0$ such that for every $0 < \eta < \eta^*(\varepsilon)$, one has

$$\operatorname{Leb}\left( \mathrm{I}(\theta^*, \eta) \cap E_k^N \right) < \varepsilon \operatorname{Leb}\left( \mathrm{I}(\theta^*, \eta) \right). \tag{3}$$

We take $\varepsilon := \frac{1}{2(k+1)}$ and write $\eta^* := \eta^*(\varepsilon)$. Take $n$ sufficiently large (that is, $n > N$ and $2\alpha_n < \eta^*$) such that

$$|\theta^* - \lambda_n| < \alpha_n.$$

By definition, one has $\mathrm{I}\left(\lambda_n, \frac{1}{k}\alpha_n\right) \subset E_k^N$. Let $\eta := |\theta^* - \lambda_n| + \frac{1}{k}\alpha_n$, which is smaller than $\eta^*$. Also, $\mathrm{I}(\lambda_n, \frac{1}{k}\alpha_n) \subset \mathrm{I}(\theta^*, \eta)$. Hence,

$$\begin{aligned} &\frac{\operatorname{Leb}(\mathrm{I}(\lambda_n, \frac{\alpha_n}{k}))}{\operatorname{Leb}(\mathrm{I}(\theta^*, \eta))} = \frac{\alpha_n/k}{\eta} > \frac{\alpha_n/k}{\alpha_n + \alpha_n/k} = \frac{1}{1+k} \\ \implies &\operatorname{Leb}\left( \mathrm{I}(\theta^*, \eta) \cap E_k^N \right) > \frac{1}{1+k} \operatorname{Leb}(\mathrm{I}(\theta^*, \eta)), \end{aligned}$$

which is a contradiction against Equa.(3). □

*Proof of Theorem 1.19.* Assume $\operatorname{Leb}(E_1) > 0$ and want to show $\operatorname{Leb}(E_1) = 1$. By Lemma 1.23, $\operatorname{Leb}(E_\infty) > 0$. Apply Lemma 1.21 to $E = E_\infty$, we get $\operatorname{Leb}(\bigcup_{n \in \mathbb{Z}^+} T_n(E_\infty)) = 1$. But this set is contained in $E_1$ by Lemma 1.20. So we obtain $\operatorname{Leb}(E_1) = 1$ and we are done. □

1.9. **Partial converse to Borel–Cantelli.** The proof of the divergence part is more difficult partly because the converse to the Borel–Cantelli lemma is not true. However, we do have a partial converse assuming certain independence properties for the sequence of sets $(E_n)$.

**Lemma 1.24.** *Let $(E_n)$ be a sequence of measurable subsets of $[0,1)$. Then for every pair of integers $0 < m < n$, we have*

$$\operatorname{Leb}(\bigcup_{i=m}^{n} E_i) \geq \frac{\left( \sum_{i=m}^{n} \operatorname{Leb}(E_i) \right)^2}{\sum_{i=m}^{n} \sum_{j=m}^{n} \operatorname{Leb}(E_i \cap E_j)}. \tag{4}$$

*Proof.* This is a consequence of Cauchy-Schwarz.

$$\left( \int_0^1 \left( \sum_{i=m}^{n} \mathbf{1}_{E_i} \right) \cdot (\mathbf{1}_{\cup_{i=m}^{n} E_i}) \operatorname{Leb} \right)^2 \leq \int_0^1 \left( \sum_{i=m}^{n} \mathbf{1}_{E_i} \right)^2 \operatorname{Leb} \cdot \int_0^1 \mathbf{1}^2_{\cup_{i=m}^{n} E_i} \operatorname{Leb}$$

For the left hand side one has:

$$\left(\int_0^1 \left(\sum_{i=m}^n \mathbf{1}_{E_i}\right) \cdot (\mathbf{1}_{\cup_{i=m}^n E_i}) \,\mathrm{Leb}\right)^2 = \left(\sum_{i=m}^n \mathrm{Leb}(E_i)\right)^2,$$

and for the right hand side:

$$\int_0^1 \left(\sum_{i=m}^n \mathbf{1}_{E_i}\right)^2 \mathrm{Leb} \cdot \int_0^1 \mathbf{1}^2_{\cup_{i=m}^n E_i} \,\mathrm{Leb} = \sum_{i,j=m}^n \mathrm{Leb}(E_i \cap E_j) \cdot \mathrm{Leb}\left(\bigcup_{i=m}^n E_i\right).$$

Putting them together finishes the proof. □

From this lemma, one can easily prove the following converse to Borel–Cantelli.

**Lemma 1.25.** *Let $(E_n)_{n\in\mathbb{Z}^+}$ be a sequence of measurable subsets of $[0,1)$ such that $\sum \mathrm{Leb}(E_n) < \infty$. Assume furthermore that $\mathrm{Leb}(E_i \cap E_j) = \mathrm{Leb}(E_i)\,\mathrm{Leb}(E_j)$ for every $i \neq j$ (namely, $E_n$'s are independent from each other). Then*

$$\mathrm{Leb}\left(\{x \in E_n \text{ for i.m. } n\}\right) = 1.$$

However, our sets are not independent. Nevertheless, we will be able to find a lower bound for the RHS of Equa.(4), which shows that $W(\psi)$ has positive Lebesgue measure. Our proof is then complete by invoking Theorem 1.19.

1.10. **A reduction.** Let $\psi_1(n) := \min\{\psi(n), \frac{1}{n}\}$. As $W(\psi_1) \subset W(\psi)$, it suffices to show that $\mathrm{Leb}(W(\psi_1)) = 1$.

**Lemma 1.26.** $\sum \psi_1(n) = +\infty$.

**Remark 1.27.** *It is not true in general that for two non-increasing sequence $(a_n)$ and $(b_n)$ of positive real numbers, $\sum a_n = \sum b_n = +\infty$ would imply $\sum \min\{a_n, b_n\} = +\infty$.*

*Proof.* Assuming $\sum \psi_1(n) < +\infty$, we will show that $\sum \psi(n) < +\infty$, which is a contradiction.

Decompose $\mathbb{Z}^+ \setminus \{1\} = I \sqcup J$ such that

$$i \in I \iff \psi(n) \leq \frac{1}{n}, \quad i \in J \iff \psi(n) > \frac{1}{n}.$$

Thus $\sum_I \psi(n) < +\infty$ and $\sum_J \frac{1}{n} < +\infty$. Decompose $J = \bigsqcup_{i\in\mathbb{Z}^+} J_i$ where $J_i = \{a_i, a_i + 1, ..., b_i\}$ and $b_i + 1 < a_{i+1}$. Therefore

$$+\infty > \sum_{n\in J} \frac{1}{n} > \sum_{i\in\mathbb{Z}^+} \int_{a_i}^{b_i+1} \frac{1}{x}\mathrm{dx} = \log\left(\frac{b_i+1}{a_i}\right).$$

On the other hand,

$$\sum_{n\in J} \psi(n) = \sum_i \sum_{j\in J_i} \psi(n) \leq \sum_i \sum_{j\in J_i} \psi(a_i - 1) = \sum_i \sum_{j\in J_i} \frac{1}{a_i - 1} = \sum_i \frac{b_i - (a_i - 1)}{a_i - 1}.$$

Define $\lambda_i := \frac{b_i}{a_i - 1} - 1 > 0$ for every $i \in \mathbb{Z}^+$, then $\sum_{n\in J} \psi(n) \leq \sum \lambda_i$, which will be shown to be convergent.

Note that (for $a_i > 1$)

$$\frac{b_i + 1}{a_i} - 1 > \frac{1}{2}\left(\frac{b_i}{a_i - 1} - 1\right).$$

Indeed for $\frac{p}{q} > 1$ with $q > 1$, one has $\frac{p+1}{q+1} - 1 > \frac{1}{2}(\frac{p}{q} - 1)$. Since this, after the denominators are cleared, is equivalent to $(q-1)(p+q) > 0$.

So we have $\sum \log(1 + \frac{1}{2}\lambda_i)$ is convergent. This implies that $\sum \lambda_i$ is convergent by Lemma 1.28. □

**Lemma 1.28.** *Let $(\lambda_n)$ be a sequence of non-negative real numbers, one has that*

$$\sum \lambda_n < +\infty \iff \sum \ln(1 + \lambda_n) < +\infty.$$

*Proof.* Note that we may assume that $(\lambda_n)$ tends to 0 for otherwise both sides are divergent. For $x \geq 0$, $\ln(1+x) \leq x$. Conversely, for $x$ sufficiently small, $\ln(1+x) > \frac{1}{2}x$. So we are done. □

In light of Lemma 1.26, we will assume $\psi(n) \leq \frac{1}{2n}$ in the next subsection.

1.11. **Quasi-independence.** For this subsection, define

$$E_n := \bigcup_{q=2^{n-1}}^{2^n-1} \bigcup_{p\in\{1,\dots,q\},\gcd(p,q)=1} \mathrm{I}\left(\frac{p}{q}, \frac{\psi(2^n)}{2^n}\right).$$

As $\psi(n) \le \frac{1}{2n}$, one can check that for every two distinct indices $(p,q),(p',q')$ appearing above,

$$\mathrm{I}\left(\frac{p}{q}, \frac{\psi(2^n)}{2^n}\right) \cap \mathrm{I}\left(\frac{p'}{q'}, \frac{\psi(2^n)}{2^n}\right) = \emptyset.$$

Also, since $\frac{\psi(2^n)}{2^n} \le \frac{\psi(q)}{q}$, the set $E_n$ is contained in

$$W_n(\psi) := \left\{ x \in [0,1) \;\middle|\; \left| x - \frac{p}{q} \right| < \Psi_{\mathrm{red}}(\frac{p}{q}) \text{ for some } 2^{n-1} \le q \le 2^n - 1 \right\}.$$

Thus, if $x$ belongs to $E_n$ infinitely many $n$'s, then $x$ belongs to $W(\psi)$. Therefore, it suffices to prove that

- RHS of Equa.(4) for such $(E_i)$ has a lower bound independent of $M$ and for $N$ large enough, i.e., there exists $C > 0$ such that for every $M$, for $N$ large enough
$$\sum_{i,j=M}^{N} \mathrm{Leb}(E_i \cap E_j) \le C \left(\sum_{i=M}^{N} \mathrm{Leb}(E_i)\right)^2;$$
- $\sum \mathrm{Leb}(E_n) = +\infty$.

Let $\phi$ be Euler's totient function. Namely, for a positive integer $N$, $\phi(N) := |(\mathbb{Z}/N\mathbb{Z})^\times|$ is the number of integers in $\{1,\dots,N\}$ that are coprime to $N$. Firstly we have

$$\mathrm{Leb}(E_n) = \left(2 \cdot \frac{\psi(2^n)}{2^n}\right) \cdot \sum_{q=2^{n-1}}^{2^n-1} \phi(q). \tag{5}$$

Then estimate the Lebesgue measure of $E_m \cap E_n$ for $m < n$. For $(a,b)$ (resp. $(c,d)$) appearing in the index of $E_m$ (resp. $E_n$), one has

$$\mathrm{Leb}\left(\mathrm{I}(\frac{a}{b}, \frac{\psi(2^m)}{2^m}) \cap \mathrm{I}(\frac{c}{d}, \frac{\psi(2^n)}{2^n})\right) \le \mathrm{Leb}\left(\mathrm{I}(\frac{c}{d}, \frac{\psi(2^n)}{2^n})\right) = 2\frac{\psi(2^n)}{2^n}.$$

For distinct $(c_1,d_1),(c_2,d_2)$ appearing in the index of $E_n$, one has

$$\left|\frac{c_1}{d_1} - \frac{c_2}{d_2}\right| = \left|\frac{c_1d_2 - c_2d_1}{d_1d_2}\right| \ge \frac{1}{d_1d_2} \ge \frac{1}{2^{2n}}.$$

Thus, for every fixed $(a,b)$ appearing in the index of $E_m$, the number of $(c,d)$ appearing in the index of $E_n$ such that $\mathrm{I}(\frac{a}{b}, \frac{\psi(2^m)}{2^m}) \cap \mathrm{I}(\frac{c}{d}, \frac{\psi(2^n)}{2^n}) \ne \emptyset$ is at most

$$\frac{2\frac{\psi(2^m)}{2^m}}{\frac{1}{2}\frac{1}{2^{2n}}} + 2 = 4 \cdot 2^{2n} \cdot \frac{\psi(2^m)}{2^m} + 2.$$

Therefore,

$$\mathrm{Leb}(E_m \cap E_n) \le \left(2\frac{\psi(2^n)}{2^n}\right) \cdot \left(4 \cdot 2^{2n} \cdot \frac{\psi(2^m)}{2^m} + 2\right) \cdot \left(\sum_{q=2^{m-1}}^{2^m-1} \phi(q)\right)$$

Combining with Equa.(5), one has (for $m < n$)

$$\mathrm{Leb}(E_m \cap E_n) \le 2 \cdot \frac{2^{2n}}{\sum_{q=2^{n-1}}^{2^n-1} \phi(q)} \cdot \mathrm{Leb}(E_n)\,\mathrm{Leb}(E_m) + \left(4\frac{\psi(2^n)}{2^n}\right) \cdot \left(\sum_{q=2^{m-1}}^{2^m-1} \phi(q)\right). \tag{6}$$

Before proceeding further, note two consequences of Lemma 1.29 to be presented in the next subsection. There exists a constant $C > 0$ such that for all positive integers $k$,

$$2^{2k} \le C \cdot \sum_{q=2^{k-1}}^{2^k-1} \phi(q);$$

$$\sum_{q=1}^{2^{k-1}-1} \phi(q) \le C \cdot \sum_{q=2^{k-1}}^{2^k-1} \phi(q).$$

The first inequality and Equa.(5) imply that

$$\sum_{n=1}^{N}\mathrm{Leb}(E_n)\geq\sum_{n=1}^{N}2C^{-1}2^n\psi(2^n)\geq 2C^{-1}\sum_{n=1}^{N}\sum_{q=2^n}^{2^{n+1}-1}\psi(q)=\sum_{q=2}^{2^{N+1}-1}\psi(q)$$

which diverges to $+\infty$.

Take two positive integers $M<N$.

Now we go back to Equa.(6) and sum over $m<n, m,n=M,...,N$. The first summand in Equa.(6) is bounded from above by

$$\sum_{m<n,m,n=M,...,N}2C\,\mathrm{Leb}(E_m)\,\mathrm{Leb}(E_n)$$

whereas the second summand is

$$\sum_{n=M}^{N}4\frac{\psi(2^n)}{2^n}\sum_{q=2^{M-1}}^{2^{n-1}-1}\phi(q)\leq\sum_{n=M}^{N}C\,\mathrm{Leb}(E_n).$$

Consequently,

$$\begin{aligned}\sum_{m,n=M,...,N}\mathrm{Leb}(E_m\cap E_n)&\leq 2C\sum_{m,n=M,...,N}\mathrm{Leb}(E_m)\,\mathrm{Leb}(E_n)+3C\sum_{n=M}^{N}\mathrm{Leb}(E_n)\\&=2C\left(\sum_{n=M}^{N}\mathrm{Leb}(E_n)\right)^2+3C\sum_{n=M}^{N}\mathrm{Leb}(E_n).\end{aligned}$$

Since $\sum\mathrm{Leb}(E_n)$ diverges, there exists $C'>0$ (independent of $M$) such that

$$\sum_{n=M}^{N}\mathrm{Leb}(E_n)<C'\left(\sum_{n=M}^{N}\mathrm{Leb}(E_n)\right)^2$$

for all $N$ large enough. Thus

$$\sum_{m,n=M,...,N}\mathrm{Leb}(E_m\cap E_n)\leq C''\left(\sum_{n=M}^{N}\mathrm{Leb}(E_n)\right)^2$$

for some $C''>0$ and $N$ large enough, completing the proof.

### 1.12. **Average of Euler's totient function.**

**Lemma 1.29.** *For any integer $N\geq e$, one has*

$$\left|\sum_{n=1}^{N}\phi(n)-\frac{1}{2\zeta(2)}N^2\right|\leq 5N\ln N$$

*where $\zeta(s):=\sum_{n\in\mathbb{Z}^+}\frac{1}{n^s}$ is the usual Riemann zeta function.*

One may note that $\sum\phi(n)$ is counting primitive integral vectors in a cone.

The proof is based on "Fubini", "change of variable" and the Mobius function:

**Definition 1.30.** *Decompose a positive integer $n\neq 1$ into products of distinct prime numbers $n=\prod_{i=1}^{k}p_i^{d_i}$ with $d_i\in\mathbb{Z}^+$ and $k\in\mathbb{Z}^+$. Define the Mobius function $\mu:\mathbb{Z}^+\to\{-1,0,1\}$ by*

$$\mu(n)=\begin{cases}(-1)^k & \text{if } n\neq 1,\ d_i=1 \text{ for every } i;\\ 1 & \text{if } n=1;\\ 0 & \text{otherwise.}\end{cases}$$

**Lemma 1.31.** *For $n\in\mathbb{Z}^+$, one has*

$$\sum_{d|n}\mu(n)=\begin{cases}0 & n\neq 1\\ 1 & n=1\end{cases}$$

*Proof.* $0=(1-1)^n=\sum\binom{n}{j}(-1)^j=\sum_{d|n}\mu(n)$. ∎

**Lemma 1.32.**

$$\sum_{d\in\mathbb{Z}^+}\frac{\mu(d)}{d^2}\cdot\sum_{n\in\mathbb{Z}^+}\frac{1}{n^2}=1,\ \textit{or equivalently,}\ \sum_{d\in\mathbb{Z}^+}\frac{\mu(d)}{d^2}=\zeta(2)^{-1}.$$

*Proof.* Expand the product and apply the lemma above. □

*Proof of Lemma 1.29.*

$$\begin{aligned}\sum_{n=1}^{N}\phi(n)&=\sum_{n=1}^{N}\sum_{m=1,\ldots,n;\,(m,n)=1}1=\sum_{n=1}^{N}\sum_{m=1}^{n}\sum_{d|(m,n)}\mu(d)\\&=\sum_{n=1}^{N}\sum_{d|n}\sum_{m=1,\ldots,n;\,d|m}\mu(d)=\sum_{n=1}^{N}\sum_{d|n}\frac{n}{d}\mu(d)\\&=\sum_{\{(m,d),\ md\le N\}}m\mu(d)=\sum_{d=1}^{N}\mu(d)\sum_{m=1}^{\lfloor\frac{N}{d}\rfloor}m\\&=\sum_{d=1}^{N}\mu(d)\left(\frac{1}{2}\frac{N^2}{d^2}+\mathrm{error}_1(d)\right)\end{aligned}$$

where

$$|\mathrm{error}_1(d)|=\left|\int_0^{\lfloor\frac{N}{d}\rfloor}x\mathrm{dx}+\int_{\lfloor\frac{N}{d}\rfloor}^{\frac{N}{d}}x\mathrm{dx}-\sum_{m=1}^{\lfloor\frac{N}{d}\rfloor}m\right|\le\frac{1}{2}\frac{N}{d}+\frac{N}{d}\le 2\frac{N}{d}.$$

So if $N\ge e$,

$$\left|\sum_{d=1}^{N}\mathrm{error}_1(d)\right|\le\sum_{d=1}^{N}2\frac{N}{d}\le 2N\left(1+\sum_{d=2}^{N}\frac{1}{d}-\int_1^N\frac{1}{x}\mathrm{dx}\right)\le 2N(\ln(N)+1)\le 4N\ln(N).$$

Therefore,

$$\sum_{n=1}^{N}\phi(n)=\frac{N^2}{2}\sum_{d=1}^{+\infty}\frac{\mu(d)}{d^2}+\mathrm{error}_2(d)$$

with

$$|\mathrm{error}_2(d)|\le 4N\ln(N)+\frac{N^2}{2}\sum_{d=N+1}^{\infty}\frac{1}{d^2}\le 4N\ln(N)+\frac{N^2}{2}\int_N^\infty\frac{1}{x^2}\mathrm{dx}\le 5N\ln(N)$$

if $N\ge e$.

□

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