LECTURE 8, ERGODICITY AND MIXING

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CONTENTS

1. BASIC CONSTRUCTIONS

For details the reader may consult Einsiedler–Ward's book on ergodic theory, GTM 259 (especially chapter 8 and appendices).

Let G be a "nice" (σ-compact metric) topological group and X a "nice" (σ-compact metric)) topological space. Assume G acts on X continuously, i.e. we have a continuous map $G \times X \to X$ satisfying some compatibility conditions.

Let $\mathscr{B}_X$ be the σ-algebra on X generated by open sets in X. This is termed the Borel σ-algebra. Then the G-action is also measurable with respect to $\mathscr{B}_X$. Thus G naturally acts on measures on $(X, \mathscr{B}_X)$.

Definition 1.1. *A measure μ on $\mathscr{B}_X$ is called a Borel measure. It is called a probability measure iff $\mu(X) = 1$. The collection of all probability measures is denoted as* $\mathrm{Prob}(X)$. *We view* $\mathrm{Prob}(X)$ *as a topological space equipped with the weak-$*$ topology.*

More precisely, we embed $\mathrm{Prob}(X)$ with the weakest topology such that

$$\mu \mapsto \int f(x)\mu(x)$$

is continuous for every

$$f \in C_c(X) := \{\text{compactly supported real-valued continuous functions on } X\}.$$

Being real-valued or complex-valued is not important.

Let

$$\mathrm{Meas}(X)^{\leq 1} := \{\text{finite measures } \mu \text{ on } X,\ \mu(X) \leq 1\},$$

also equipped with weak-$*$ topology. We also let

$$\mathrm{LFM}(X) := \{\text{locally finite measures on } X\},$$

be equipped with weak-$*$ topology. Note that $C_c(X)$ admits a countable dense subset.

Lemma 1.2. *With weak-$*$ topology,* $\mathrm{Meas}(X)^{\leq 1}$ *is a compact metrizable space. If X is compact, then so is* $\mathrm{Prob}(X)$.

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Remark 1.3. *If we forget about the topological structure on X, and take some probability measure μ, then up to completion, $(X, \mathscr{B}_X, \mu)$ is "isomorphic" to a convex combination of the natural measure on $[0,1]$ interval and atomic measures supported on single points. Thus the study of $(X, \mathscr{B}_X, \mu)$ is rather boring without a group action, unlike the case of topological X, when the classification of X is already a huge problem.*

We naturally has an action of G on $\mathrm{Prob}(X)$, $\mathrm{Meas}^{\leq 1}(X)$ and $\mathrm{LFM}(X)$ defined by

$$g \cdot \mu(E) := \mu(g^{-1}E)$$

for every measurable set E and measure μ.

Lemma 1.4. *The induced map $G \times \mathrm{LFM}(X) \to \mathrm{LFM}(X)$ is continuous.*

A measure μ is said to be G-invariant iff $g \cdot \mu = \mu$ for all $g \in G$. The collection of G-invariant probability measures is denoted as $\mathrm{Prob}(X)^G$. Similarly define $\mathrm{Meas}^{\leq 1}(X)^G$ and $\mathrm{LFM}(X)^G$.

To distinguish different p.m.p(= probability measure preserving) actions of G, a convenient functor is given by taking the associated unitary representation.

Take a $\mu \in \mathrm{LFM}(X)^G$. Then the associated unitary representation is given by

$$\begin{aligned} G \times L^2(X,\mu) &\to L^2(X,\mu) \\ (g,\phi) &\mapsto g \cdot \phi(x) := \phi(g^{-1}x) \end{aligned}$$

Lemma 1.5. *This is indeed a unitary representation:*

1. *for each $g \in G$, the action on $L^2(X,\mu)$ is a unitary;*
2. *the representation is continuous*

where $\mathscr{U}(L^2(X,\mu))$, the set of unitary operators on $L^2(X,\mu)$, is equipped with the strong operator topology.

In more concrete terms, using the following lemma, the continuity claim just asserts that if $g_n \to g$ in G and $\phi_n \to \phi$ in $L^2(X,\mu)$, then $g_n \cdot \phi_n \to g \cdot \phi$.

Lemma 1.6. *$L^2(X,\mu)$ admits a countable dense subset.*

Remark 1.7. *In most cases one should not expect the representation to be continuous with respect to the operator norm topology. And being continuous with respect to the weak operator topology is equivalent to being continuous w.r.t. strong operator topology in the current case (thanks Chengyang Wu and Hongrui Yuan for pointing this out).*

Thus for two p.m.p. G-actions to be isomorphic, it is necessary for the associated unitary representations to be isomorphic.

Properties of p.m.p. G-actions defined via the associated unitary representation are sometimes called "spectral properties".

2. Ergodicity and mixing

By convention, a nice p.m.p. action refers to those arising from a continuous action of G on a nice space X.

Definition 2.1. *A nice p.m.p. G-action on $(X, \mathscr{B}_X, \mu)$ is said to be ergodic iff every G-invariant measurable subset E of X is either μ-null ($\mu(E) = 0$) or μ-conull ($\mu(X \setminus E) = 0$).*

So ergodicity is something like irreducibility.

Lemma 2.2. *If a nice p.m.p. G-action on $(X, \mathscr{B}_X, \mu)$ is ergodic, then every μ-almost invariant measurable subset of X is either μ-null or μ-conull.*

A measurable subset $E \subset X$ is said to be μ-almost invariant iff for every $g \in G$,

$$\mu(gE\Delta E) = \mu((gE \setminus E) \cup (E \setminus gE)) = 0.$$

Since our group could be uncountable, this lemma is not so obvious. Using this lemma, one can show that

Lemma 2.3. *A nice p.m.p. G-action on $(X, \mathscr{B}_X, \mu)$ is ergodic iff the associated unitary representation has no fixed vector orthogonal to constants.*

Hint: Starting from a set E, one has the characteristic function 1_E. Starting from a function f, one considers its level sets.

Another spectral property we need is mixing.

Definition 2.4. *A nice p.m.p. G-action on $(X, \mathscr{B}_X, \mu)$ is said to be mixing iff for every two measurable subsets $E, F \subset X$ and every divergent sequence (g_n) in G, we have*

$$\lim_n \mu(g_n^{-1} E \cap F) = \mu(E)\mu(F).$$

This notion is useless for compact groups.

Lemma 2.5. *A nice p.m.p. G-action on $(X, \mathscr{B}_X, \mu)$ is mixing iff for every two $\phi, \psi \in L^2(X, \mu)$ orthogonal to constants and every divergent sequence (g_n) in G, we have*

$$\lim_n \langle g_n \cdot \phi, \psi \rangle = 0.$$

In this case we also say that the associated unitary representation on $L^2_0(X, \mu)$ (the subspace orthogonal to constants) is mixing.

Here $\langle \phi, \psi \rangle := \int \phi(x)\overline{\psi(x)}\mu(x)$.

3. Unitary representations of $\mathrm{SL}_2(\mathbb{R})$ are mixing

Notations

- $\mathrm{G} := \mathrm{SL}_2(\mathbb{R})$ and Γ is a discrete subgroup of G;
- $A := \left\{ \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \ t \in \mathbb{R} \right\} = \{\boldsymbol{a}_t, \ t \in \mathbb{R}\}$;
- $U := \left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \ s \in \mathbb{R} \right\} = \{\boldsymbol{u}_s, \ s \in \mathbb{R}\}$;
- $B := A \cdot U$.

Unitary representations, if containing no trivial factors, of $\mathrm{G} = \mathrm{SL}_2(\mathbb{R})$ are always mixing.

Theorem 3.1. *Let π be a unitary representation of* G *on a separable Hilbert space $\mathscr{H}$. Assume there is no non-zero* G*-fixed vectors. Show that π is mixing.*

Proof. By KAK-decomposition, it suffices to show that $\pi|_A$ is mixing. So take (a_n) to be a divergent sequence in A. By a diagonal argument, we find an infinite subsequence (a_{n_k}) such that for every $\phi, \psi \in \mathscr{H}$,

$$\lim_k \langle a_{n_k} \phi, \psi \rangle \text{ exists.}$$

This defines a linear map $E : \mathscr{H} \to \mathscr{H}$ such that the above limit is equal to $\langle E\phi, \psi \rangle$. One can check $\|E\|_{op} \leq 1$. So we wish to show that $E = 0$, which is going to be achieved by showing the image of E is invariant under G.

By passing to a further subsequence we assume either $(\log(a_{n_k}))_{1,1} \to +\infty$ or $-\infty$.

Define

$$U^- := \left\{ x \in \mathrm{G} \,\middle|\, \lim_{k\to+\infty} a_{n_k} x a_{n_k}^{-1} = 1 \right\},\; U^+ := \left\{ x \in \mathrm{G} \,\middle|\, \lim_{k\to+\infty} a_{n_k}^{-1} x a_{n_k} = 1 \right\}$$

Making use of these two groups, there are two things we firstly note. Let E^* be the adjoint of E.

1. $E \circ u = E$ for every $u \in U^-$. Indeed, for every pair ϕ, ψ in $\mathscr{H}$,

$$\begin{aligned} \langle Eu\phi, \psi\rangle &= \lim \langle a_{n_k} u\phi, \psi\rangle = \lim \langle a_{n_k} u a_{n_k}^{-1} a_{n_k} \phi, \psi\rangle \\ &= \lim \langle a_{n_k}\phi, a_{n_k} u^{-1} a_{n_k}^{-1} \psi\rangle = \langle E\phi, \psi\rangle. \end{aligned}$$

The last step is because $a_{n_k} u^{-1} a_{n_k}^{-1}\psi$ converges to ψ in norm. Hence $E \circ u = E$. By taking the adjoint, we get $u^{-1} \circ E^* = E^*$. Thus the image of E^* is fixed by U^-.

2. $u \circ E = E$ for every $u \in U^+$.

$$\begin{aligned} \langle uE\phi, \psi\rangle &= \langle E\phi, u^{-1}\psi\rangle = \lim \langle a_{n_k}\phi, u^{-1}\psi\rangle = \lim \langle a_{n_k} a_{n_k}^{-1} u a_{n_k} \phi, \psi\rangle \\ &= \lim \langle a_{n_k}\phi, \psi\rangle = \langle E\phi, \psi\rangle. \end{aligned}$$

Hence $u \circ E = E$. By taking the adjoint, we get $E^* \circ u^{-1} = E^*$. Thus the image of $1 - u$ is killed by E^* for every $u \in U^+$.

Next is the trick. Observe that $(a_{n_k}^{-1}) = (a_{n_k}^*)$ converges in W.O.T. to E^*. (Note that $*$ operation is continuous with respect to W.O.T. .)

3. $\ker E = \ker E^*$. Indeed,

$$\langle E\phi, E\phi\rangle = \lim_l \lim_k \langle a_{n_k}\phi, a_{n_l}\phi\rangle = \lim_l \lim_k \langle a_{n_l}^{-1}\phi, a_{n_k}^{-1}\phi\rangle = \langle E^*\phi, E^*\phi\rangle$$

(Exercise: show that in general $\ker E \neq \ker E^*$ for a bounded linear operator on a Hilbert space.)

Now we start to wrap up the proof. 1. and 3. imply that $E^*(1-u) = 0$ for every $u \in U^-$. Take $*$ of this, we get $E = u^{-1}E$. Thus the image of E is fixed by U^-. 2. asserts that image of E is fixed by U^+. Since U^- and U^+ generates G, we are done.

□

Let us quickly explain, using linear algebra, why you can write a matrix $g \in \mathrm{SL}_2(\mathbb{R})$ as $k_1 a k_2$ with k_i in $\mathrm{SO}_2(\mathbb{R})$ and a being diagonal. This fact was used to reducing the mixing in general to mixing of A. First we claim that we can write $g = k_1|g|$ where k_1 is orthogonal and $|g|$ is symmetric. Assuming the claim, since $|g|$ can be diagonalized under a orthogonal basis, we are done. Now let us prove the claim. The matrix gg^{tr} is symmetric and hence diagonalizable. Moreover it has positive eigenvalues. Hence makes sense to take $|g| := \sqrt{gg^{tr}}$. Then one defines $k_1 := g|g|^{-1}$ and it is direct to check that $\langle k_1 v, k_1 v\rangle = \langle v, v\rangle$ for every vector v. And we are done.

4. Examples

Example 4.1. *$G = \mathbb{Z}$ generated by $1 := R_\alpha$ acting on $\mathbb{R}/\mathbb{Z}$ by $R_\alpha \cdot x := x + \alpha$ for some real number α. Then this action preserves the natural Lebesgue measure m on $\mathbb{R}/\mathbb{Z}$. It is ergodic iff $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Moreover, this action is not mixing.*

Sketch of proof. There are two proofs. Assume $\alpha \notin \mathbb{Q}$.

Either you can argue that R_α generates a dense subgroup of $\mathbb{R}/\mathbb{Z}$ and then by continuity, m has to be invariant under the full $\mathbb{R}/\mathbb{Z}$. Then argue that m is the unique $\mathbb{R}/\mathbb{Z}$-invariant probability measure.

Or you can argue that there are no invariant L^2 functions by expanding them under the basis $\{x \mapsto e^{2\pi i n x}\}_{n\in\mathbb{Z}}$.

I leave it to you to show that R_α is not mixing. □

Example 4.2. *$G = \mathbb{Z}$ acts on $\mathbb{R}^2/\mathbb{Z}^2$ where the generator acts by $(x, y) \mapsto (x + y, x + 2y)$. Then G preserves the natural Lebesgue measure on $\mathbb{R}^2/\mathbb{Z}^2$ and the action is ergodic and mixing.*

Sketch of proof. Two ways: 1. Fourier analysis; 2. use the idea presented in last section (you need something contracted by the G action to make the argument work, what is this?). □

Let $M := \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. For $t \in \mathbb{R}$, let $M^t := \exp(t \cdot \log M)$. The above example is about the induced action of $M^{\mathbb{Z}}$ on $\mathbb{R}^2/\mathbb{Z}^2$. The reason why you have such an induced action is of course $\mathbb{Z}^2$ is preserved by $M^{\mathbb{Z}}$. For other t, this is not true. Nevertheless, each M^t defines a homeomorphism

$$\mathbb{R}^2/\mathbb{Z}^2 \mapsto \mathbb{R}^2/M^t \cdot \mathbb{Z}^2.$$

Let

$$X = \left\{(x, s) \,\middle|\, s \in \mathbb{R}/\mathbb{Z},\ x \in \mathbb{R}^2/M^t \cdot \mathbb{Z}^2\right\}.$$

Example 4.3. *Show that X has a natural measure m. Moreover, the action of $M^{\mathbb{R}}$ is ergodic but not mixing.*

This example tells you that in general an ergodic B-action (that is not extendable to an $\mathrm{SL}_2(\mathbb{R})$ p.m.p. action) may not be mixing. However, this example has the defect that the B-action is not "totally ergodic" in the sense that some infinite subgroup does not act ergodically. I do not know an example of totally ergodic B-action that is not mixing. Note that by argument from the last section, it must be A-mixing.