EXERCISE 1

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截止日期：最迟在 3.25 提交作业。

评分标准：取 sup-norm ——只要做对一小道题，就能得到满分。

Exercise 0.1. *Let* $\mathrm{G} := \mathrm{SL}_2(\mathbb{R})$ *act on* $X := \mathbb{R}^n$ *continuously, that is to say, we have a continuous map* $\mathrm{G} \times X \to X$ *satisfying suitable compatibility conditions. For every* $x \in X$*, let* G_x *be the stabilizer of* x *in* G*. By assumption* G_x *is closed in* G*. Show that the orbit map*

$$\begin{aligned} \mathrm{G}/\mathrm{G}_x &\to \mathrm{G}.x \\ [g] &\mapsto g.x \end{aligned}$$

is a homeomorphism if the orbit is open in its closure. Here G/G_x *is equipped with the quotient topology and* $\mathrm{G}.x$ *is equipped with the subspace topology.*

Remark 0.1. *Hint: Apply Baire's category theorem to* X *and then make use of the group action. Once you finish proving this exercise, it should be clear to you that the statement holds for more general* G *and* X*.*

Exercise 0.2. *Let* $\mathrm{G} := \mathrm{SL}_2(\mathbb{R})$ *and*

$$U := \left\{ \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} s \in \mathbb{R} \right\}.$$

Let Γ *be a discrete subgroup of* G*. Assume the above exercise. Show that* $Ug\Gamma/\Gamma$ *is dense in* G/Γ *iff* $\Gamma.e_1$ *is dense in* $\mathbb{R}^2 \setminus \{0\}$ *where* $e_1 := (1,0) \in \mathbb{R}^2$*.*

Exercise 0.3. *Consider the action of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathscr{H}^2 := \{z \in \mathbb{C}, Im(z) > 0\}$ *defined by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z := \frac{az+b}{cz+d}.$$

Show that the map $g \mapsto g.i$ *induces a homeomorphism* $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cong \mathscr{H}^2$*.*

Exercise 0.4. *Let* $\rho : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{GL}(V)$ *be a faithful (namely,* ρ *is injective) finite dimensional real representation of* $\mathrm{SL}_2(\mathbb{R})$*. Show that there does not exist an* $\mathrm{SL}_2(\mathbb{R})$*-invariant Euclidean metric on* V*.*

Exercise 0.5. *Show that there does not exist a Riemannian metric (that is to say, a smooth metric) on* $\mathrm{SL}_2(\mathbb{R})$ *that is both left and right* $\mathrm{SL}_2(\mathbb{R})$*-invariant.*

Remark 0.2. *Consider the conjugate action of* $\mathrm{SL}_2(\mathbb{R})$ *at the identity and use the exercise above.*

Definition 0.3. *Recall that a discrete subgroup* Γ *is said to be a lattice in* G *iff* G/Γ *admits a finite* G*-invariant measure.*

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Exercise 0.6. *Let Γ be a lattice in $\mathrm{SL}_2(\mathbb{R})$, and assume Γ is not cocompact in $\mathrm{SL}_2(\mathbb{R})$. Let $X := \mathrm{SL}_2(\mathbb{R})/\Gamma$. Let d be a right invariant Riemannian metric on $\mathrm{SL}_2(\mathbb{R})$, which induces a quotient Riemannian metric d_X on X, from which we can define a (volume) measure on X. Accept the fact that such a measure is necessarily the $\mathrm{SL}_2(\mathbb{R})$-invariant finite measure on X. Show that a sequence $(x_n) \subset X$ goes to ∞ iff $\mathrm{InjRad}(x_n) \to 0$ as $n \to \infty$.*

Exercise 0.7. *Assume the notations and the conclusion of the exercise above. Show that $(g_n\Gamma/\Gamma) \subset X$ goes to ∞ iff there exists $\gamma_n \in \Gamma$ such that $\mathrm{dist}(\mathrm{id}, g_n\gamma_n g_n^{-1}) \to 0$.*

Exercise 0.8. *For a matrix $X = (x_{i,j})$, let $\|X\|_{\sup} := \sup_{i,j} |x_{i,j}|$. By a direct computation, show that there exists a constant $C > 0$, such that for every $\varepsilon > 0$ and $X, Y \in \mathrm{SL}_2(\mathbb{R})$ with $\|\mathrm{id} - X\| \leq \varepsilon$ and $\|\mathrm{id} - Y\| \leq \varepsilon$, we have that*

$$\left\|\mathrm{id} - XYX^{-1}Y^{-1}\right\| \leq C \cdot \varepsilon^2.$$

Exercise 0.9. *Notations as in the exercise above. Show that there exists a neighborhood $\mathcal{N}$ of id in $\mathrm{SL}_2(\mathbb{R})$ such that for every discrete subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$, $\Gamma \cap \mathcal{N}$ generates an abelian group.*

Exercise 0.10. *Notations as in the exercise above. Show that there exists a neighborhood $\mathcal{N}'$ of id in $\mathrm{SL}_2(\mathbb{R})$ such that for every discrete subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$, there exists $g \in \mathrm{SL}_2(\mathbb{R})$ such that $g\Gamma g^{-1} \cap \mathcal{N}' = \{\mathrm{id}\}$.*

Exercise 0.11. *Let Γ in $\mathrm{SL}_2(\mathbb{R})$ be a lattice. Use previous exercises to show that Γ is not cocompact iff it contains non-identity unipotent matrices.*

Remark 0.4. *The "if" direction is proved in the class. This is a special instance of Kazhdan–Margulis theorem.*

Exercise 0.12. *Let $a_t := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ and $u_s := \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$. In the class we have seen that for a discrete subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$, if $x \in \mathrm{SL}_2(\mathbb{R})/\Gamma$ belongs to a compact u_s-orbit, then $a_t.x$ diverges as t goes to $-\infty$. Now assume Γ is a lattice. Show that the converse holds. Namely, if $a_t.x$ diverges as t goes to $-\infty$, then $\{u_s.x\}_{s\in\mathbb{R}}$ is compact.*

We say that a matrix $g \in \mathrm{SL}_2(\mathbb{R})$ is $\mathbb{R}$-*diagonalizable* iff there exists $h \in \mathrm{SL}_2(\mathbb{R})$ such that hgh^{-1} is a diagonal matrix. Note that for a matrix $X_{\neq \pm \mathrm{id}} \in \mathrm{SL}_2(\mathbb{R})$, being $\mathbb{R}$-*diagonalizable* is equivalent to being hyperbolic in the sense that $\mathrm{trace}(X) > 2$. Fix a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, an $\mathbb{R}$-diagonalizable matrix $\gamma \in \Gamma$ is said to be *primitive* iff it can not be written as $(\gamma')^n$ for some $n \in \mathbb{Z}$, $n \neq \pm 1$ and some other $\gamma' \in \Gamma$ that is $\mathbb{R}$-diagonalizable. By definition $\pm\mathrm{id}$ is never primitive. Let

$$\mathrm{Prim}(\Gamma) := \left\{\gamma \text{ is } \mathbb{R}\text{-diagonalizable and primitive}\right\}.$$

Exercise 0.13. *Assume $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup containing $\{\pm\mathrm{id}\}$. Find a bijection between*

$$\left\{ \textit{compact } \{a_t\}\textit{-orbits} \right\} \cong \mathrm{Prim}(\Gamma)/\sim_\Gamma$$

where $\sim_\Gamma$ is the equivalence relation defined by $g \sim_\Gamma h$ iff $g = \gamma h \gamma^{-1}$ for some $\gamma \in \Gamma$.

Exercise 0.14. *Classify all compact $\{a_t\}_{t\in\mathbb{R}}$-orbits on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$.*

Exercise 0.15. *Classify all divergent $\{a_t\}_{t\in\mathbb{R}}$-orbits on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$.*

Recall that an orbit $\{a_t.x\}$ is said to be divergent iff for every compact set in $C \subset \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ there exists $t_0 > 0$ such that for all $|t| > t_0$, we have $a_t.x \notin C$.