LECTURE 10, EQUIDISTRIBUTION OF UNIPOTENT FLOWS ON FINITE-VOLUME QUOTIENT OF $SL_2(\mathbb{R})$

RUNLIN ZHANG

CONTENTS

Notations

- $\mathrm{X}_2 := \mathrm{SL}_2(\mathbb{R}) / \mathrm{SL}_2(\mathbb{Z})$ and $U := \{\mathbf{u}_s,\ s \in \mathbb{R}\}$;
- $\mathbf{u}_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$, $\mathbf{a}_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$;
- $\widehat{\mathrm{m}}_{\mathrm{X}_2}$ is the $\mathrm{SL}_2(\mathbb{R})$-invariant measure on X_2, normalized to be a probability measure;
- $\mathrm{Prim}(\Lambda)$ be the set of non-zero primitive vectors in Λ for $\Lambda \leq \mathbb{R}^2$ discrete;
- $\mathrm{Prim}^1(\Lambda)$ be the set of rank-1 primitive subgroups of Λ.

1. Equidistribution on the modular surface

We illustrate the idea of [DS84] in the case X_2.

Theorem 1.1. *Let $\Lambda_0 \in \mathrm{X}_2$ be such that $U.\Lambda_0$ is not compact. Then*

$$\lim_{S \to +\infty} \mu_S := \lim_{S \to +\infty} \int_0^S (\mathbf{u}_s)_* \delta_{x_0}\, \mathrm{ds} = \widehat{\mathrm{m}}_{\mathrm{X}_2}.$$

This is also true for other non-cocompact lattices.

Consider

$$\mathscr{T} := \left\{ \Lambda \in \mathrm{X}_2 \,\middle|\, U.\Lambda \text{ is compact} \right\}.$$

Lemma 1.2. *The set of compact U-orbits is a tube: $\mathscr{T} = \{\mathbf{a}_t \mathbf{u}_s \cdot \mathbb{Z}^2,\ t \in \mathbb{R},\ s \in \mathbb{R}/\mathbb{Z}\}$. And $U.\Lambda$ is compact iff Λ contains a non-zero horizontal vector.*

We have proved this in previous sections.

Our proof of Thm.1.1 decomposes as:

Step 1. Passing to a subsequence, assume the limit of $(\mu_S)_S$ exists and call it μ. Thanks to the non-divergence theorem, we also know μ is a probability measure;

Step 1.5 Also μ is readily seen to be U-invariant since it comes from an averaging process;

Step 2. Show $\mu(\mathscr{T}) = 0$;

Date: 2022.04.

Step 3. Use the ergodic decomposition to conclude.

Step 1 should be clear. Let us take up Step 2.

Proof of Step 2. Fix $t_1 < t_2$, let

$$\mathcal{T}_{[t_1,t_2]} := \left\{ \mathbf{a}_t \mathbf{u}_s.\mathbb{Z}^2,\ t \in [t_1,t_2],\ s \in \mathbb{R}/\mathbb{Z} \right\}.$$

Thus it suffices to show that $\mu(\mathcal{T}_{[t_1,t_2]}) = 0$ for all $-\infty < t_1 < t_2 < +\infty$. By the definition of weak$*$ convergence, it suffices to find an open neighborhood $\mathcal{N}_\varepsilon$, for every $\varepsilon > 0$, of $\mathcal{T}_{[t_1,t_2]}$ such that $\limsup \mu_S(\mathcal{N}_\varepsilon) \leq \varepsilon$. Letting $\varepsilon \to 0$ the finishes the proof.

It only remains to prove Thm.1 below. □

Note that $\mathbf{u}_s \Lambda_0$ being close to $\mathcal{T}_{[t_1,t_2]}$ means that, for certain $v \in \mathrm{Prim}(\Lambda_0)$, we have $\mathbf{u}_s.v$ is close to

$$A_{[t_1,t_2]} := \{ \mathbf{a}_t \mathbf{u}_s.e_1 \mid t \in [t_1,t_2],\ t \in \mathbb{R} \} = [e^{t_1}, e^{t_2}] \times \{0\}.$$

For $C, \delta > 0$, consider the box

$$\mathrm{Box}_{C,\delta} := [-C, C] \times [-\delta, \delta].$$

Define

$$I(C,\delta) := \left\{ s \geq 0 \,\middle|\, \mathrm{Prim}(\mathbf{u}_s.\Lambda_0) \cap \mathrm{Box}_{C,\delta} \neq \emptyset \right\}.$$

For $\mathbb{Z}.v \in \mathrm{Prim}^1(\Lambda_0)$, consider

$$I(C,\delta,v) := \left\{ s \geq 0 \,\middle|\, \mathbf{u}_s.v \in \mathrm{Box}_{C,\delta} \right\}.$$

Since $I(C,\delta,v) = I(C,\delta,-v)$, this is independent of the choice of the generator of $\mathbb{Z}v$. Thus from the definition

$$I(C,\delta) = \bigcup_{\mathbb{Z}v \in \mathrm{Prim}^1(\Lambda_0)} I(C,\delta,v). \tag{1}$$

The key fact is that

Lemma 1.3. *Assume $\delta \cdot C \leq 0.1$. Then for two $\mathbb{Z}v \neq \mathbb{Z}w \in \mathrm{Prim}^1(\Lambda_0)$, $I(C,\delta,v) \cap I(C,\delta,w) = \emptyset$. In other words, Equa.1 above is a disjoint union when $\delta \cdot C \leq 0.1$.*

Proof. Otherwise the lattice $\mathbf{u}_s.\Lambda_0$ would contain two linearly independent vectors v, w in $[-C,C] \times [-\delta,\delta]$. Thus the triangle spanned by v, w is also contained in $[-C,C] \times [-\delta,\delta]$, implying $\|v \wedge w\| \leq 2(4C\delta) < 1$. This contradicts against the assumption Λ_0 is unimodular. □

For $\varepsilon > 0$, define

$$C_1(\varepsilon) := \varepsilon^{-1}, \quad \delta_1(\varepsilon) := 0.1\varepsilon.$$

For every $\mathbb{Z}v \in \mathrm{Prim}^1(\Lambda_0)$, there are three cases

Case 1. $I(C_1(\varepsilon), \delta_1(\varepsilon), v) = \emptyset$;

Case 2. $I(C_1(\varepsilon), \delta_1(\varepsilon), v) \neq \emptyset$ and $v \in \mathbb{R}e_1$; in this case $I(C_1(\varepsilon), \delta_1(\varepsilon), v) = \mathbb{R}_{\geq 0}$;

Case 3. $I(C_1(\varepsilon), \delta_1(\varepsilon), v) \neq \emptyset$ and $v \notin \mathbb{R}e_1$; in this case $I(C_1(\varepsilon), \delta_1(\varepsilon), v)$ is a closed interval of the form $[a_v, b_v]$.

Case 2 is impossible since Λ_0 contains no non-zero horizontal vector by assumption (see Lem.1.2).

Now take $S > 0$, there are sub-cases for case 3:

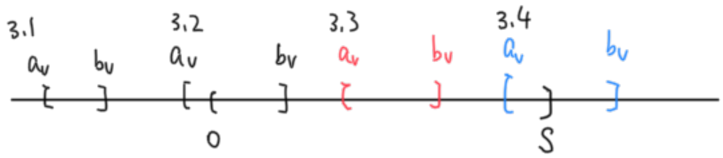
3.1 $S < a_v$ or $b_v < 0$; in this case $[0,S] \cap I(C_1(\varepsilon), \delta_1(\varepsilon), v) = \emptyset$;

3.2 $a_v \leq 0 \leq b_v \leq S$; in this case $[0,S] \cap I(C_1(\varepsilon), \delta_1(\varepsilon), v) = [0, b_v]$;

3.3 $0 < a_v \leq b_v < S$; in this case $[0,S] \cap I(C_1(\varepsilon), \delta_1(\varepsilon), v) = [a_v, b_v]$;

3.4 $0 \leq a_v \leq S \leq b_v$; in this case $[0,S] \cap I(C_1(\varepsilon), \delta_1(\varepsilon), v) = [a_v, S]$;

3.5 $[0,S] \subset [a_v, b_v]$.



Proposition 1.4. *Take C_2 satisfying $1 < C_2 < 0.5C_1(\varepsilon) = 0.5\varepsilon^{-1}$. Then*

$$\limsup_{S\to+\infty} \frac{1}{S} \operatorname{Leb}\left(I(C_2, \delta_1(\varepsilon)) \cap [0,S]\right) \le 4C_2\varepsilon.$$

From the proof it will be clear that the inequality holds for S large enough.

Only case 3.2, 3.3 and 3.4 above will contribute, for which we have three lemmas Lem.2.2,2.1,2.3 (see next section).

Proof. If every $v \in \operatorname{Prim}(\Lambda)$ falls in case 1 or case 3.1 (for every $S > 0$), then LHS in Prop.1.4 is zero and the inequality trivially holds. Otherwise, find $S > 0$ large enough such that no vector is in case 3.5.

$$\begin{aligned}\text{Numerator of LHS} &= \left| \bigsqcup_{v\in\text{case }3} [0,S] \cap I(C_2, \delta_1(\varepsilon), v) \right| \\ &\le 4C_2\varepsilon \cdot \sum |[0,S] \cap I(C_1(\varepsilon), \delta_1(\varepsilon), v)| \le 4C_2\varepsilon \cdot S\end{aligned}$$

$\square$

Now we take $C_2 > 1$, depending on t_1, t_2, such that $\operatorname{Box}_{C_2,\delta_1(\varepsilon)}$ contains $[e^{t_1}, e^{t_2}] \times \{0\}$. When $\varepsilon > 0$ is small enough, $C_2 < 0.5\varepsilon^{-1}$.

Theorem 1.5. *For every $\varepsilon > 0$, we can find a neighborhood $\mathcal{N}_\varepsilon$ of $\mathcal{T}_{t_1,t_2}$ such that*

$$\limsup_{S\to+\infty} \mu_S(\mathcal{N}_\varepsilon) \le 4C_2\varepsilon.$$

Consequently for every limit point μ of (μ_S), $\mu(\mathcal{T}_{t_1,t_2}) = 0$.

Proof. Just define $\mathcal{N}_\varepsilon$ to be those lattices whose primitive vectors intersect non-trivially with $\operatorname{Box}_{C_2,\delta_1(\varepsilon)}$. Then apply Prop.1.4. $\square$

Thus we have completed step 2.

Proof of Step 3. So we have a U-invariant probability measure μ with $\mu(\mathcal{T}) = 0$. By classification of ergodic U-invariant probability measures ν on X, either ν is supported on $\mathcal{T}$ or $\nu = \widehat{\mathrm{m}}_{\mathrm{X}_2}$. Let

$$\mu = \int_{\operatorname{Prob}(\mathrm{X}_2)^{U,Erg}} \nu\, \lambda(\nu)$$

be the ergodic decomposition of μ, then

$$0 = \mu(\mathcal{T}) = \int \nu(\mathcal{T})\, \lambda(\nu).$$

Thus λ-almost every ν, $\nu(\mathcal{T}) = 0 \implies \nu = \widehat{\mathrm{m}}_{\mathrm{X}_2}$. So $\mu = \widehat{\mathrm{m}}_{\mathrm{X}_2}$. $\square$

2. Proof of Lemmas

Lemma 2.1. *[Case 3.3] Assume* $\Lambda_0 \cap \mathbb{R}e_1 = \{0\}$*, then*

$$|I(C_2, \delta_1(\varepsilon), v)| \le C_2\varepsilon \cdot |I(C_1(\varepsilon), \delta_1(\varepsilon), v)|.$$

Proof. If the LHS is 0, then nothing needs to be done. Otherwise, wlog, assume $v = (v_1, v_2)$ with $v_2 > 0$. Then

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + s v_2 \\ v_2 \end{pmatrix}.$$

And

$$I(C, \delta_1(\varepsilon), v) = \frac{1}{v_2}[-v_1 - C, -v_1 + C].$$

Thus

$$|I(C_2, \delta_1(\varepsilon), v)| = \frac{2C}{v_2} = C\varepsilon \cdot \frac{2\varepsilon^{-1}}{v_2} = C\varepsilon \cdot |I(C_1(\varepsilon), \delta_1(\varepsilon), v)|.$$

□

Lemma 2.2. *[Case 3.2] Assume* $\Lambda_0 \cap \mathbb{R}e_1 = \{0\}$*,* $\mathbb{Z}v \in \mathrm{Prim}^1(\Lambda)$ *and* $S > 0$ *satisfy case 3.2 above. Also assume* $C_2 \le 0.5\varepsilon^{-1}$*. Then*

$$|[0, S] \cap I(C_2, \delta_1(\varepsilon), v)| \le 4C_2\varepsilon \cdot |[0, S] \cap I(C_1(\varepsilon), \delta_1(\varepsilon), v)|.$$

Proof. In this case

$$[0, S] \cap I(C_1(\varepsilon), \delta_1(\varepsilon), v) = [0, b_v].$$

If $[0, S] \cap I(C_2, \delta_1(\varepsilon), v)$ is empty nothing needs to be done. Otherwise

$$0 < -v_1 + C_2 \implies v_1 < C_2.$$

Then

$$\begin{aligned} |[0, S] \cap I(C_2, \delta_1(\varepsilon), v)| \le \frac{2C_2}{v_2} &= \frac{2C_2}{-v_1 + \varepsilon^{-1}} \cdot \frac{-v_1 + \varepsilon^{-1}}{v_2} \\ &= \frac{2C_2}{-v_1 + \varepsilon^{-1}} \cdot |[0, S] \cap I(C_1(\varepsilon), \delta_1(\varepsilon), v)| \end{aligned}$$

It remains to observe

$$\frac{2C_2}{-v_1 + \varepsilon^{-1}} \le \frac{2C_2}{-C_2 + \varepsilon^{-1}} \le \frac{2C_2}{-0.5\varepsilon^{-1} + \varepsilon^{-1}} = 4C_2\varepsilon.$$

□

Lemma 2.3. *[Case 3.4] Assume* $\Lambda_0 \cap \mathbb{R}e_1 = \{0\}$*,* $\mathbb{Z}v \in \mathrm{Prim}^1(\Lambda)$ *and* $S > 0$ *satisfy case 3.4 above. Also assume* $\varepsilon^{-1} \ge 2C_2$*. Then*

$$|[0, S] \cap I(C_2, \delta_1(\varepsilon), v)| \le 4C_2\varepsilon \cdot |[0, S] \cap I(C_1(\varepsilon), \delta_1(\varepsilon), v)|.$$

Proof. In this case

$$[0, S] \cap I(C_1(\varepsilon), \delta_1(\varepsilon), v) = [a_v, S].$$

If $[0, S] \cap I(C_2, \delta_1(\varepsilon), v)$ is empty nothing needs to be done. Otherwise

$$-v_1 - C_2 \le S \implies v_1 + S \ge -C_2.$$

Under this condition we have

$$|[0, S] \cap I(C_2, \delta_1(\varepsilon), v)| \le \frac{2C_2}{v_2}$$

and

$$|[0,S]\cap I(C_1(\varepsilon),\delta_1(\varepsilon),v)| = \frac{S-(-v_1-\varepsilon^{-1})}{v_2} = \frac{\varepsilon^{-1}+S+v_1}{v_2} \geq \frac{\varepsilon^{-1}-C_2}{v_2} \geq \frac{0.5\varepsilon^{-1}}{v_2}$$

Thus,

$$|[0,S]\cap I(C_2,\delta_1(\varepsilon),v)| \leq \frac{2C_2}{0.5\varepsilon^{-1}}|[0,S]\cap I(C_1(\varepsilon),\delta_1(\varepsilon),v)|.$$

Note that $\frac{2C_2}{0.5\varepsilon^{-1}} = 4C_2\varepsilon$. □

3. Other non-cocompact lattices

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R}) =: \mathrm{G}$ be a lattice. Let $X := \mathrm{G}/\Gamma$. If you are not familiar with hyperbolic geometry, you are welcome to take $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Main ideas are preserved in this case. We are going to take a more "geometric approach" in this section.

First we have the non-divergence theorem.

Theorem 3.1. *For every $\varepsilon > 0$, there exists a compact subset of $C \subset X$ such that for every $x \in X$, either*

$$\limsup \frac{1}{S}\mathrm{Leb}\{s \in [0,S],\ \mathbf{u}_s.x \notin C\} \leq \varepsilon$$

or $U.x$ is compact.

[If you did not know how to prove this, arguments below provide a proof]

Let

$$\mathscr{T} := \left\{x \in X \,\middle|\, U.x \text{ is compact}\right\}.$$

Using hyperbolic geometry, you can show that

Theorem 3.2. *There exist finitely many points $y_1, \ldots, y_l$ in X with compact U-orbits such that*

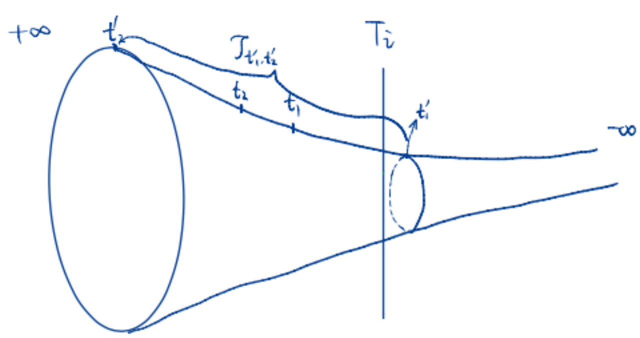
$$\mathscr{T}_i := AU.y_i$$

are mutually disjoint, and $\mathscr{T} = \sqcup_{i=1,\ldots,l}\mathscr{T}_i$.

Fix $x_0 \notin \mathscr{T}$, define μ_S and μ as in the class. Let us explain why $\mu(\mathscr{T}) = 0$. Define, for $-\infty \leq t_1 < t_2 \leq +\infty$,

$$\mathscr{T}_{t_1,t_2,i} := \left\{\mathbf{a}_t\mathbf{u}_s.y_i \,\middle|\, t_1 < t < t_2,\ s \in \mathbb{R}\right\}$$
$$\widetilde{\mathscr{T}}_{t_1,t_2,i} := \left\{\mathbf{a}_t\mathbf{u}_s.\widetilde{y}_i \,\middle|\, t_1 < t < t_2,\ s \in \mathbb{R}\right\},\ \widetilde{\mathscr{T}}_i := AU.\widetilde{y}_i.$$

where $\widetilde{y}_i$ is some fixed lift of y_i in $\mathrm{G}/\Gamma \cap \pm 1U$.



Theorem 3.3. *Fix some $-\infty < t_1 < t_2 < +\infty$ and some $i = 1, ..., l$. For every $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}_\varepsilon$ of $\mathcal{T}_{t_1,t_2,i}$ such that*

$$\limsup \frac{1}{S} \operatorname{Leb}\{s \in [0,S] \mid \mathbf{u}_s.x_0 \in \mathcal{N}_\varepsilon\} \leq \varepsilon.$$

To prove this statement, without loss of generality, we may and do assume that $y_i = [\mathrm{id}]_\Gamma$ ($[\bullet]_\Gamma$ stands for the image of $\bullet$ in the quotient by Γ).

In light of the case of X_2, we are going to find two nbhd $\mathcal{N}_\varepsilon \subset \mathcal{N}'_\varepsilon$ such that the time a noncompact U-orbit spends in $\mathcal{N}_\varepsilon$ is much smaller than that in $\mathcal{N}'_\varepsilon$.

Consider the natural projection $p : \mathrm{G}/\pm 1U \cap \Gamma \to \mathrm{G}/\Gamma$. It is an injection restricted to $\widetilde{\mathcal{T}_i}$ and is a closed embedding when further restricted to (the closure of) $\widetilde{\mathcal{T}}_{t'_1,t'_2,i}$ (for some $t'_1 < t_1$ very small, $t'_2 = t'_2(\varepsilon)$ very large, to be determined). Thus we can find an open neighborhood $\widetilde{\Omega}_\varepsilon$ of $\widetilde{\mathcal{T}}_{t'_1,t'_2,i}$ in $\mathrm{G}/\pm 1U \cap \Gamma$ such that π is injective (actually homeomorphism onto its image) on $\widetilde{\Omega}_\varepsilon$.

In a different vein, by hyperbolic geometry, there exists a compact set of C of G/Γ such that its complement consists of disjoint union of "cusps", each of which is isometric to

$$\left\{x + iy \,\middle|\, x \in [-a,a],\, y > b\right\} / -a + iy \sim a + iy$$

for some $a, b > 0$ with the standard hyperbolic metric $\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$. Moreover, the number of cusps is exactly l and we can enumerate them as $(\mathrm{cusp}_i)_{i=1,...,l}$ such that for some $T_i \in \mathbb{R}$,

$$\mathrm{cusp}_i = \left\{k\mathbf{a}_t\mathbf{u}_s.y_i \,\middle|\, k \in \mathrm{SO}_2(\mathbb{R}),\, t < T_i,\, s \in \mathbb{R}\right\} = \mathrm{SO}_2(\mathbb{R})\mathcal{T}_{-\infty,T_i,i}.$$

Thus the preimage of cusp_i under p is

$$\widetilde{\mathrm{cusp}}_i := \left\{k\mathbf{a}_t\mathbf{u}_s.\widetilde{y}_i \,\middle|\, k \in \mathrm{SO}_2(\mathbb{R}),\, t < T_i,\, s \in \mathbb{R}\right\} = \mathrm{SO}_2(\mathbb{R})\widetilde{\mathcal{T}}_{-\infty,T_i,i}.$$

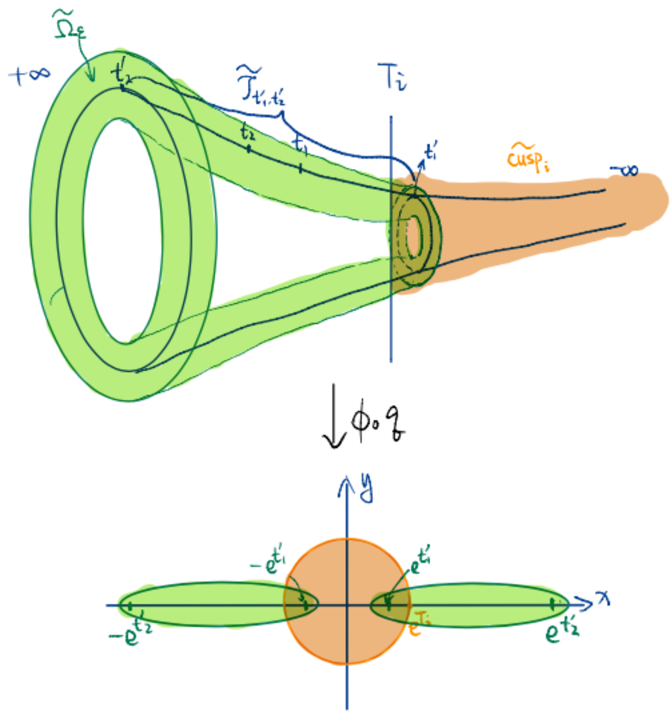
[this is because $\Gamma \cap \mathrm{SO}_2(\mathbb{R}) \cdot \{\mathbf{a}_t\ t < T_i\} \cdot U$ is contained in $\pm 1U$.] We choose $t'_1 := T_i - 1$ (or any number smaller than T_i).

Let us define q to be the natural quotient $\mathrm{G}/\Gamma \cap \pm U \to \mathrm{G}/\pm U$ and $\phi : \mathrm{G}/\pm U \to \mathbb{R}^2/\pm 1$ by $\phi(g) := g.e_1/\pm 1$. For notational convenience, we will be working with $\mathbb{R}^2$ rather than $\mathbb{R}^2/\pm 1$. Here is a diagram.

$$\begin{tikzcd}
& \mathrm{G}/\Gamma \cap \pm U \arrow[dl, "p"'] \arrow[dr, "q"] & & \\
\mathrm{G}/\Gamma & & \mathrm{G}/\pm U \arrow[r, hook, "\phi"] & \mathbb{R}^2/\pm 1
\end{tikzcd}$$

The $\widetilde{\mathrm{cusp}}_i$ is already q-saturated: $q^{-1}q(\widetilde{\mathrm{cusp}}_i) = \widetilde{\mathrm{cusp}}_i$. More concretely,

$$\phi \circ q(\widetilde{\mathrm{cusp}}_i) = \left\{v_{\neq 0} \in \mathbb{R}^2 \,\middle|\, \|v\| < e^{T_i}\right\} / \pm 1.$$



[In the picture above, $\widetilde{\Omega}_\varepsilon$ should have been $\widetilde{\Omega}_i$.]

$\widetilde{\Omega}_i$ may not be q-saturated. However, its image is an open neighborhood of the image of

$$\phi\circ q(\widetilde{\mathscr{T}}_{t_1',t_2',i}) = (e^{t_1'}, e^{t_2'}) \times \{0\}/\pm 1.$$

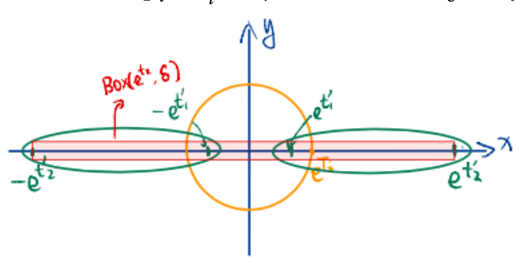
Then one can show that there exists a smaller open nbhd Ω' of $q(\widetilde{\mathscr{T}}_{t_1',t_2',i})$ such that its preimage under q is contained in $\widetilde{\Omega}_i$. Thus we can choose $\delta = \delta(\varepsilon) > 0$ small enough such that

$$\widetilde{\Omega}_i' := (\phi\circ q)^{-1}\left((e^{t_1'}, e^{t_2'}) \times (-\delta, \delta)\right)/\pm 1.$$

is contained in $\widetilde{\Omega}_i$. To combine $\widetilde{\mathrm{cusp}}_i$ with $\widetilde{\Omega}_i'$, choose a even smaller δ such that

$$\widetilde{\mathscr{N}_\varepsilon'} := (\phi\circ q)^{-1}\left(\mathrm{Box}(e^{t_2'}, \delta)\right)/\pm 1.$$

is contained in $\widetilde{\mathrm{cusp}}_i \cup \widetilde{\Omega}_i'$. So p restricted to $\widetilde{\mathscr{N}_\varepsilon'}$ is injective.



Also let

$$\widetilde{\mathscr{N}_\varepsilon} := (\phi\circ q)^{-1}\left(\mathrm{Box}(e^{t_2+1}, \delta)\right)/\pm 1.$$

Let $\mathscr{N}_\varepsilon' := p(\widetilde{\mathscr{N}_\varepsilon'})$ and $\mathscr{N}_\varepsilon := p(\widetilde{\mathscr{N}_\varepsilon})$. They are open neighborhoods of $\mathscr{T}_{t_1,t_2,i}$.

At this point, one can adapt the strategy of previous sections to prove Thm.3.3 and hence analogues of Thm.1.1 for other lattices.

REFERENCES

[DS84] S. G. Dani and John Smillie, *Uniform distribution of horocycle orbits for Fuchsian groups*, Duke Math. J. **51** (1984), no. 1, 185–194. MR 744294