LECTURE 7

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Contents

Notation

1. Lecture 7, dimension and entropy

For more on dimension of metric spaces, see Mattilde's book [Mat95]. Hochman's notes[1] are recommended as an introduction to entropy in dynamics.

From this lecture on, we will loosely follow the EKL paper [EKL06] and EL's Pisa notes[2].

1.1. **Upper Minkowski dimension.**

Define a metric d on $[0,1)^2$ by

$$d(\mathbf{x},\mathbf{y}) := \inf\{\|\mathbf{x}-\mathbf{y}-\mathbf{v}\|_{\sup},\ \mathbf{v}\in\mathbb{Z}^2\}.$$

Replacing sup-norm by the usual Euclidean norm has no effect the definition of dimension below. But we find it slightly more convenient to work with the sup-norm. This metric is compatible with the topology defined by identifying $[0,1)^2$ with $\mathbb{R}^2/\mathbb{Z}^2$.

For a subset $E\subset[0,1)^2$, define for $s>0,\varepsilon>0$,

$$\mathcal{H}^s_\varepsilon(E) := \inf\left\{\sum \operatorname{diam}(B_i)^s \;\middle|\; (B_i) \text{ countable open balls covering } E \text{ of diameter } <\varepsilon\right\}$$
$$\mathcal{H}^s(E) := \lim_{\varepsilon\to 0}\mathcal{H}^s_\varepsilon(E).$$

The **Hausdorff dimension** of E is defined by

$$\dim_{\mathrm{H}}(E) := \inf\{s>0 \mid \mathcal{H}^s(E)=0\} = \inf\{s>0\mid \mathcal{H}^s(E)<+\infty\}.$$

What is more relevant to us is the notion of upper Minkowski dimension (also called box dimension), which is larger than the Hausdorff dimension.

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[1] Available here: http://math.huji.ac.il/~mhochman/courses/dynamics2014/notes.5.pdf
[2] Available here: https://people.math.ethz.ch/~einsiedl/Pisa-Ein-Lin.pdf.

Definition 1.1. *Given a compact metric space (X, d) and $\varepsilon > 0$, a subset $S \subset X$ is said to be an ε-**separating set** iff*

$$x, y \in S \text{ and } x \neq y \implies d(x, y) > \varepsilon.$$

*Let E be a subset of X. Let $\mathrm{Sep}(E, \varepsilon)$ denote the largest size of ε-separating sets contained in E. Then the **upper Minkowski dimension** is*

$$\dim_{\Box}(E) := \limsup_{\varepsilon \to 0} \frac{\log(\mathrm{Sep}(E, \varepsilon))}{\log(\varepsilon^{-1})}$$

if $\mathrm{Sep}(E, \varepsilon)$ is finite for ε small enough. Otherwise $\dim_{\Box}(E) := +\infty$.

Lemma 1.2. *Hausdorff dimension of a set is no greater than its upper Minkowski dimension.*

Proof. By definition, it suffices to show that

$$\begin{aligned} &\forall\, s > \dim_{\Box}(E),\ \exists\, C > 0,\ \forall\, \varepsilon > 0, \\ &\exists \text{ covering of } E \text{ by countably many balls of radius} < \varepsilon \\ &\text{such that } \sum \mathrm{diam}(B_i)^s < C. \end{aligned}$$

Take such an s. By definition, for ε small enough, one has

$$\frac{\log \mathrm{Sep}(E, \varepsilon)}{\log(\varepsilon^{-1})} < s, \text{ equivalently, } \mathrm{Sep}(E, \varepsilon) < \varepsilon^{-s}.$$

Let $S = \{s_1, ..., s_l\} \subset E$ be a ε-separated set with $l = \mathrm{Sep}(E, \varepsilon)$. Then it is a maximal ε-separating set. Let $B_i := B_{2\varepsilon}(s_i)$, the ball of radius 2ε centered at s_i. Then (B_i) forms a covering of E by balls of diameter 4ε. And

$$\sum \mathrm{diam}(B_i)^s = 4^s \varepsilon^s l < 4^s.$$

As RHS is independent of ε, we are done. ☐

1.2. **Measure entropy of finite partitions.** Let $(X, \mathscr{B})$ be a set equipped with a σ-algebra and μ be a probability measure on $(X, \mathscr{B})$. The triple $(X, \mathscr{B}, \mu)$ is often referred to as a **probability space**.

In our examples, X is often the underlying set of a compact metrizable topological space and $\mathscr{B}$ is the Borel σ-algebra: the smallest σ-algebra containing all open and closed subsets of X. In this case the triple $(X, \mathscr{B}, \mu)$ is referred to as a Borel probability space.

A **finite measurable partition**[3] is a set of measurable subsets $\mathcal{P} = \{P_1, ..., P_l\} \subset \mathscr{B}$ of X such that

$$X = \bigsqcup_{i=1}^{l} P_i.$$

We define the **entropy** of a partition $\mathcal{P} = \{P_i\}$ by

$$H_\mu(\mathcal{P}) := \sum_{i=1}^{l} -\mu(P_i) \log(\mu(P_i)).$$

where by convention,

$$-0 \cdot \log(0) := 0.$$

If $\phi(x) := -x \log(x)$ defined on $[0, 1]$, then ϕ is strictly convex/concave in the sense that for every $\sum_{i=1}^{l} \lambda_i = 1$ with $\lambda_i > 0$, one has

$$\sum_{i=1}^{l} \lambda_i \phi(x_i) \leq \phi(\sum_{i=1}^{l} \lambda_i x_i), \quad \forall\, x_1, ..., x_l \in [0, 1]$$
$$\text{and `` $=$'' holds iff } x_1 = x_2 = ... = x_l.$$

Entropy of a partition is a non-negative number and it is zero iff the partition consists of null ($\mu(P_i) = 0$) and co-null ($\mu(P_i^c) = 0$) sets.

Lemma 1.3. *Let $\mathcal{P} = (P_i)_{i=1}^{d}$ be a finite measurable partition, then*

- $H_\mu(\mathcal{P}) \leq \log d$;
- $H_\mu(\mathcal{P}) = \log d$ *iff* $\mu(P_i) = d^{-1}$ *for every* $i = 1, ..., d$.

[3] Sometimes the word "measurable" is omitted.

Proof. This is a consequence of the convexity/concavity of $x \mapsto -x\log(x)$. $\square$

Given two partitions $\mathcal{P} = (P_i)$ and $\mathcal{Q} = (Q_j)$, let $\mathcal{P} \vee \mathcal{Q}$ be the partition consisting of $\{P_i \cap Q_j\}$ as i, j vary. We define the **entropy of $\mathcal{Q}$ conditional on $\mathcal{P}$** by

$$H_\mu(\mathcal{Q}|\mathcal{P}) := \sum_{i,j} -\mu(P_i \cap Q_j)\log\frac{\mu(P_i \cap Q_j)}{\mu(P_i)}.$$

If we let $\mu_i^{\mathcal{P}}$ denote the probability measure $\frac{1}{\mu(P_i)}\mu|_{P_i}$[4] whenever $\mu(P_i) \neq 0$, then

$$H_\mu(\mathcal{Q}|\mathcal{P}) = \sum_i \mu(P_i) H_{\mu_i^{\mathcal{P}}}(\mathcal{Q}).$$

Lemma 1.4. *Let $\mathcal{P} = (P_i)$ and $\mathcal{Q} = (Q_j)$ be two finite partitions. Then*

$$\max\{H_\mu(\mathcal{P}), H_\mu(\mathcal{Q})\} \leq H_\mu(\mathcal{P} \vee \mathcal{Q}) \leq H_\mu(\mathcal{P}) + H_\mu(\mathcal{Q}).$$

Actually, we have

$$\begin{gathered} H_\mu(\mathcal{P} \vee \mathcal{Q}) = H_\mu(\mathcal{P}) + H_\mu(\mathcal{Q}|\mathcal{P}) = H_\mu(\mathcal{Q}) + H_\mu(\mathcal{P}|\mathcal{Q}) \\ 0 \leq H_\mu(\mathcal{Q}|\mathcal{P}) \leq H_\mu(\mathcal{Q}),\ 0 \leq H_\mu(\mathcal{P}|\mathcal{Q}) \leq H_\mu(\mathcal{P}). \end{gathered}$$

Proof. Firstly, a direct computation shows that

$$\sum_{i,j} -\mu(P_i \cap Q_j)\log(\mu(P_i \cap Q_j)) = \sum_{i,j} -\mu(P_i \cap Q_j)\log(\frac{\mu(P_i \cap Q_j)}{\mu(P_i)}) + \sum_i \mu(P_i)\log(\mu(P_i)).$$

So $H_\mu(\mathcal{P} \vee \mathcal{Q}) = H_\mu(\mathcal{P}) + H_\mu(\mathcal{Q}|\mathcal{P})$. That $0 \leq H_\mu(\mathcal{Q}|\mathcal{P})$ follows from the definition. It only remains to show that $H_\mu(\mathcal{Q}|\mathcal{P}) \leq H_\mu(\mathcal{Q})$. By the convexity/concavity of $-x\log(x)$, we have for each fixed j,

$$\begin{aligned} &\sum_i \mu(P_i)\left(-\frac{\mu(P_i \cap Q_j)}{\mu(P_i)}\log\left(\frac{\mu(P_i \cap Q_j)}{\mu(P_i)}\right)\right) \\ \leq &-\left(\sum_i \mu(P_i)\frac{\mu(P_i \cap Q_j)}{\mu(P_i)}\right)\cdot\log\left(\sum_i \mu(P_i)\frac{\mu(P_i \cap Q_j)}{\mu(P_i)}\right) \\ = &-\mu(Q_j)\log(\mu(Q_j)). \end{aligned}$$

Summing over j completes the proof. $\square$

1.3. **Dynamical entropy.** Let $T : (X, \mathscr{B}) \to (X, \mathscr{B})$ be a measurable map ($T^{-1}\mathscr{B} \subset \mathscr{B}$) preserving the measure μ. For a finite partition $\mathcal{P}$, define

$$h_\mu(T, \mathcal{P}) := \lim_{n\to+\infty}\frac{1}{n}H_\mu(\mathcal{P} \vee T^{-1}\mathcal{P} \vee ... \vee T^{-(n-1)}\mathcal{P}).$$

Lemma 1.5. *The limit indeed exists in $[0, +\infty]$. Also,*

$$h_\mu(T, \mathcal{P}) = \inf_{n\in\mathbb{Z}^+}\frac{1}{n}H_\mu(\mathcal{P} \vee T^{-1}\mathcal{P} \vee ... \vee T^{-(n-1)}\mathcal{P}).$$

Proof. Fix $\mathcal{P}$, let $a_n := H_\mu(\mathcal{P} \vee T^{-1}\mathcal{P} \vee ... \vee T^{-(n-1)}\mathcal{P})$. Then the sequence (a_n) is non-negative and satisfies

$$a_{n+m} \leq a_n + a_m.$$

For any such sequence, similar conclusion holds. Indeed, we show that for every fixed $n \in \mathbb{Z}^+$ and $\varepsilon > 0$, there exists N_0 such that for every $N > N_0$,

$$\frac{a_N}{N} < \frac{a_n}{n} + \varepsilon.$$

Let $C_n := \max a_1, ..., a_n$. Write $N = dn + r$ with $d \in \mathbb{Z}_{\geq 0}$ and $r \in \{0, 1, ..., n-1\}$. Then $a_N \leq da_n + a_r$ and

$$\frac{a_N}{N} \leq \frac{da_n}{dn + r} + \frac{a_r}{N} \leq \frac{a_n}{n} + \frac{c_n}{N}.$$

So taking N_0 such that $c_n < \varepsilon N_0$ suffices. $\square$

Define the **measure entropy** of T with respect to μ as

$$h_\mu(T) := \sup\{h_\mu(T, \mathcal{P}) \mid \mathcal{P} \text{ is a finite partition }\}.$$

[4]The notation $\mu|_{P_i}$ means the restriction of μ to P_i, namely, $\mu|_{P_i}(E) := \mu(P_i \cap E)$.

1.4. **Main theorem.** Recall:

$$\alpha_t := \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} \in \mathrm{A}^+ := \left\{ \begin{bmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{bmatrix} \,\middle|\, \sum t_i = 0,\ t_1, t_2 > 0 \right\}$$

and for $\alpha, \beta \in [0,1)$,

$$\Lambda_{\alpha,\beta} := \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} .\mathbb{Z}^3 \in \mathrm{X}_3, \quad \mathbf{u}^+_{\alpha,\beta} := \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}.$$

The map from $\mathbb{R}^2/\mathbb{Z}^2$ to X_3 induced by $(\alpha, \beta) \mapsto \Lambda_{\alpha,\beta}$ is continuous.

The main purpose of this lecture is to explain:

Theorem 1.6. *Let $\mathscr{C}$ be a compact subset of X_3. Let*

$$E_{\mathscr{C}} := \left\{ (\alpha, \beta) \in [0,1)^2 \,\middle|\, \mathrm{A}^+.\Lambda_{\alpha,\beta} \subset \mathscr{C} \right\}.$$

If $E_{\mathscr{C}}$ has positive upper Minkowski dimension, then $\mathscr{C}$ supports an A-invariant measure ν with $h_\nu(\alpha_1) > 0$.

Later it will be shown that such a measure can not exist, from which we deduce that $E_{\mathscr{C}}$ has zero upper Minkowski dimension and hence zero Hausdorff dimension. Consequently, the exceptional set to Littlewood conjecture is a countable union of sets with box dimension zero. In particular, it has Hausdorff dimension zero.

1.5. **Outline of the proof.** The proof of Theorem 1.6 consists of two steps

Step 1. Construct an α_1-invariant measure with positive entropy;

Step 2. Use an average process to promote it to an A-invariant measure. The point is that the entropy does not decrease when passing to the limit.

From now on, we fix such a compact subset as in Theorem 1.6 and call it $\mathscr{C}_1$ till the end of the proof. Also

1. fix some $\delta_1 \in (0,1)$ such that $\dim_{\square}(E_{\mathscr{C}_1}) > \delta_1$;
2. fix some $\delta_2 \in (0,1)$ such that $\mathrm{InjRad}(x) > \delta_2$ for every $x \in \mathscr{C}_1$.

Furthermore, choose $\delta_3, \delta_4 \in (0,1)$ such that $B(\delta_4) \subset \mathcal{O}_{e^{-3}\delta_3} \subset \mathcal{O}_{\delta_3} \subset B(\delta_2)$. Consequently,

3. $e^{-3}\delta_3 \leq \|(s_1, s_2)\| \leq \delta_3 \implies \delta_4 < d^{\mathrm{X}_3}(\mathbf{u}^+_{s_1,s_2}.x, x) < \delta_2$ for every $x \in \mathscr{C}_1$.

By making $\delta_4 > 0$ even smaller, we assume

$$d(\mathbf{s}, \mathbf{t}) > e^{-3}\delta_3 \implies d(\Lambda_{\mathbf{s}}, \Lambda_{\mathbf{t}}) > \delta_4, \ \forall \mathbf{s}, \mathbf{t} \in [0,1)^2 \cong \mathbb{R}^2/\mathbb{Z}^2.$$

Also we decompose $[0,1)^2 = \bigcup_{i=1}^{l_0} \square_i$ into union of subsets of diameter smaller than δ_3. Hence for $\mathbf{s}, \mathbf{t}$ contained in the same $\square_i$, one has $d^{\mathrm{X}_3}(\Lambda_{\mathbf{s}}, \Lambda_{\mathbf{t}}) < \delta_2$.

1.6. **Step 1, construction of the measure.** By assumption, we can find a sequence of positive numbers (ε_n) decreasing to 0 such that

$$\frac{\log\left(\mathrm{Sep}(E_{\mathscr{C}_1}, \varepsilon_n)\right)}{\log(\varepsilon_n^{-1})} > \delta_1,$$

or equivalently,

$$\mathrm{Sep}(E_{\mathscr{C}_1}, \varepsilon_n) > \left(\frac{1}{\varepsilon_n}\right)^{\delta_1}.$$

Let $\mathcal{S}_n$ be an ε_n-separating set for $(E_{\mathscr{C}_1}, d^{\mathrm{X}_3})$ contained in some $\square_i$ whose size is at least $l_0^{-1}\,\mathrm{Sep}(E_{\mathscr{C}_1}, \varepsilon_n)$.

For a non-empty finite subset $F \subset \mathrm{X}_3$, let m_F denote the uniform probability measure supported on F, namely,

$$\mathrm{m}_F(E) := \frac{\#F \cap E}{\#F}.$$

For n large enough such that $\varepsilon_n < e^{-3}\delta_3$, choose $d_n \in \mathbb{Z}^+$ such that $\delta_3 < e^{3d_n}\varepsilon_n \leq e^3\delta_3$. Let

$$\mu_n := \frac{1}{d_n}\sum_{i=0}^{d_n-1} (\alpha_1)^i_* \mathrm{m}_{S_n} = \frac{1}{d_n}\sum_{i=0}^{d_n-1} (\alpha_i)_* \mathrm{m}_{S_n}.$$

By assumption, (μ_n) is a sequence of probability measures supported on $\mathscr{C}_1$. By the "diagonal argument", we can select a convergent subsequence (μ_{n_k}) under the weak$*$ topology. Let μ denote the limit measure.

Lemma 1.7. *The limit measure μ is α_1-invariant.*

Proof. Indeed, as $n \to \infty$,

$$(\alpha_1)_*\mu_n - \mu_n = \frac{1}{d_n}\left((\alpha_{d_n})_*\mathrm{m}_{S_n} - \mathrm{m}_{S_n}\right)$$

converges to 0. □

1.7. **Separation properties under iterations.**

Lemma 1.8. *For every pair of distinct points $\mathbf{s}, \mathbf{t} \in \mathcal{S}_n$, there exists $j \in \{0, 1, ..., d_n - 1\}$ such that*

$$d(\alpha_j.\Lambda_{\mathbf{s}}, \alpha_j.\Lambda_{\mathbf{t}}) \geq \delta_4.$$

Proof. When $d(\mathbf{s}, \mathbf{t}) > e^{-3}\delta_3$, then the conclusion holds for $j = 0$.

Now assume $d(\mathbf{s}, \mathbf{t}) \leq e^{-3}\delta_3$ and let $\mathbf{t}' \in \mathbf{t} + \mathbb{Z}^2$ be such that $d(\mathbf{s}, \mathbf{t}) = \|\mathbf{s} - \mathbf{t}'\|_{\sup}$. By our choice of d_n, there exists $j \in \{0, 1, ..., d_n - 1\}$ such that

$$\left\|e^{3j}(\mathbf{s} - \mathbf{t}')\right\|_{\sup} = e^{3j}\left\|\mathbf{s} - \mathbf{t}'\right\|_{\sup} > e^{-3}\delta_3.$$

We choose j to be the smallest one with this property. Then

$$e^{-3}\delta_3 < \left\|e^{3j}\mathbf{s} - e^{3j}\mathbf{t}'\right\|_{\sup} \leq \delta_3, \text{ which implies } d(\mathbf{u}_{e^{3j}\mathbf{s} - e^{3j}\mathbf{t}'}.x, x) > \delta_4, \ \forall\, x \in \mathscr{C}_1. \quad (1)$$

Since

$$\alpha_j.\Lambda_{\mathbf{s}} = \mathbf{u}^+_{e^{3j}\mathbf{s} - e^{3j}\mathbf{t}'}.\alpha_j.\Lambda_{\mathbf{t}'} = \mathbf{u}^+_{e^{3j}\mathbf{s} - e^{3j}\mathbf{t}'}\alpha_j.\Lambda_{\mathbf{t}}$$

and

$$\alpha_j.\Lambda_{\mathbf{t}} \in \mathscr{C}_1,$$

we have by Equa.(1)

$$d\left(\alpha_j.\Lambda_{\mathbf{s}}, \alpha_j.\Lambda_{\mathbf{t}}\right) > \delta_4.$$

□

1.8. **Test partitions.** Let X be a compact metrizable space. For a subset $E \subset X$, let $\mathrm{Int}(E)$ be its interior points, $\overline{E}$ its closure, E^c its complement and ∂E its boundary.

Lemma 1.9. *For every $\varepsilon > 0$, there exists a finite measurable partition $\mathcal{P}$ of $\mathscr{C}_1$ such that $\mu(\partial P) = 0$ and $\mathrm{diam}(P) < \varepsilon$ for every $P \in \mathcal{P}$.*

Proof. For every $x \in \mathscr{C}_1$, find $0 < r_x < 0.5\varepsilon$ such that $\mu(\partial B_x(r_x)) = 0$. Indeed, the sets

$$\partial B_x(r), \quad 0 < r < 0.5\varepsilon$$

form an uncountable family of disjoint measurable subsets. Thus one of them must have zero μ-measure. By compactness, we find $x_1, ..., x_k \in \mathscr{C}_1$ such that

$$\mathscr{C}_1 \subset \bigcup_{i=1}^{k} B_{x_i}(r_{x_i}).$$

Define

$$P_1 := B_{x_1}(r_{x_1}),\ P_2 := B_{x_2}(r_{x_2}) \setminus B_{x_1}(r_{x_1}),\ P_3 := B_{x_3}(r_{x_3}) \setminus \left(B_{x_1}(r_{x_1}) \cup B_{x_2}(r_{x_2})\right), ...$$

Note that $\partial(A \cap B) \subset \partial A \cup \partial B$ and $\partial(A^c) = \partial(A)$. Then

$$\partial P_j \subset \bigcup_{i \leq j} \partial B_{x_i}(r_{x_i})$$

has μ-measure zero. Thus $\mathcal{P} := (P_1, P_2, ..., P_k)$ is a desired partition. □

Lemma 1.10. *Let (ν_n) be a sequence of Borel probability measures converging to ν in weak$*$ topology, then for every Borel measurable subset $E \subset X$ with $\nu(\partial E) = 0$, one has $\nu(E) = \lim_{n \to \infty} \nu_n(E)$.*

Proof. Without loss of generality, we assume E is bounded. Take an open bounded set F containing $\overline{E}$.

Choose a sequence of continuous functions (f_k) (resp. (g_k)) such that $f_k \leq \mathbf{1}_{\mathrm{Int}(E)}$ (resp. $g_k \leq \mathbf{1}_{F\setminus\overline{E}}$) and (f_k) converges to $\mathbf{1}_{\mathrm{Int}(E)}$ (resp. (g_k) converges to $\mathbf{1}_{F\setminus\overline{E}}$). Then

$$\begin{aligned}\nu(\mathrm{Int}(E)) &= \lim_{k\to\infty}\int f_k(x)\nu(x)\\ &= \lim_{k\to\infty}\lim_{n\to\infty}\int f_k(x)\nu_n(x)\\ &\leq \liminf_{n\to\infty}\nu_n(\mathrm{Int}(E)).\end{aligned}$$

Let

$$F_k(x) := \begin{cases} 1-g_k(x) & x\in F\\ 0 & x\notin F\end{cases}.$$

Then (F_k) is a sequence of continuous functions such that $\mathbf{1}_{\overline{E}} \leq F_k \leq \mathbf{1}_F$ for every k and converges to $\mathbf{1}_{\overline{E}}$ pointwise. Therefore,

$$\begin{aligned}\nu(\overline{E}) &= \lim_{k\to\infty}\int F_k(x)\nu(x)\\ &= \lim_{k\to\infty}\lim_{n\to\infty}\int F_k(x)\nu_n(x)\\ &\geq \limsup_{n\to\infty}\nu_n(\overline{E}).\end{aligned}$$

Putting together we have

$$\nu(\mathrm{Int}(E)) \leq \liminf_{n\to\infty}\nu_n(\mathrm{Int}(E)) \leq \limsup_{n\to\infty}\nu_n(\overline{E}) \leq \nu(\overline{E}).$$

But $\nu(\partial E)=0$, so the above inequalities are all equalities and we are done. □

1.9. **Completion of step 1.** In this subsection we complete step one, namely, we show

Lemma 1.11. *Let μ be as constructed in Section 1.6. Then $h_\mu(\alpha_1) \geq 3\delta_1$.*

We fix a finite measurable partition $\mathcal{P}$ as in Lemma 1.9 with $\varepsilon=\delta_4$. For every k and $P\in\mathcal{P}\vee\alpha_1^{-1}\mathcal{P}\vee ...\vee\alpha_1^{-(k-1)}\mathcal{P}$,

$$\partial P\subset\partial P_{i_0}\cup\partial\alpha_1^{-1}(P_{i_1})\cup...\cup\partial\alpha_1^{-(k-1)}(P_{i_{k-1}}) = \partial P_{i_0}\cup\alpha_1^{-1}(\partial P_{i_1})\cup...\cup\alpha_1^{-(k-1)}(\partial P_{i_{k-1}})$$

has μ-measure zero since μ is α_1-invariant by Lemma 1.7. It is sufficient to show that $h_\mu(T,\mathcal{P})\geq 3\delta_1$.

For two integers $i<j$, abbreviate

$$\mathcal{P}_i^j := \alpha_1^{-i}\mathcal{P}\vee\alpha_1^{-(i+1)}\mathcal{P}\vee ...\vee\alpha_1^{-j}\mathcal{P}.$$

By Lemma 1.10, for each fixed k,

$$\frac{1}{k}H_\mu(\mathcal{P}_0^{k-1}) = \frac{1}{k}\lim_{n\to\infty}H_{\mu_n}(\mathcal{P}_0^{k-1}). \tag{2}$$

Lemma 1.12. *Let ν_1,ν_2 be two probability measures, $\lambda\in[0,1]$ and $\mathcal{Q}=(Q_i)$ be a finite measurable partition. Then*

$$H_{\lambda\nu_1+(1-\lambda)\nu_2}(\mathcal{Q}) \geq \lambda H_{\nu_1}(\mathcal{Q}) + (1-\lambda)H_{\nu_2}(\mathcal{Q}).$$

Proof. This follows from the convexity/concavity of $-x\log(x)$. □

By applying this to $\mu_n = \frac{1}{d_n}\sum_{j=0}^{d_n-1}(\alpha_j)_*\mathrm{m}_{\mathcal{S}_n}$, we get

$$H_{\mu_n}(\mathcal{P}_0^{k-1}) \geq \frac{1}{d_n}\sum H_{(\alpha_j)_*\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_0^{k-1}) = \frac{1}{d_n}\sum_{j=0}^{d_n-1}H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_j^{j+(k-1)}). \tag{3}$$

Let $l_n\in\mathbb{Z}_{\geq 0}$ be defined by

$$l_nk \leq d_n-1 < (l_n+1)k.$$

By Lemma 1.4,

$$\sum_{j=0,k,...,l_nk} H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_j^{j+(k-1)}) \geq H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_0^{l_nk+k-1}).$$

In general, for every $r = 0, 1, ..., k-1$, let $l_n(r) \in \mathbb{Z}_{\geq 0}$ (so $l_n(0) = l_n$) be defined by

$$l_n(r)k + r \leq d_n - 1 < (l_n(r)+1)k + r.$$

By Lemma 1.4,

$$\begin{aligned}\sum_{j=r,r+,...,r+l_n(r)k} H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_j^{j+(k-1)}) &\geq H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_r^{r+l_n(r)k+k-1}) \\ &\geq H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_0^{r+l_n(r)k+k-1}) - H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_0^{r-1}).\end{aligned} \tag{4}$$

By Lemma 1.8, for every pair $\mathbf{s}_1 \neq \mathbf{s}_2$ in $\mathcal{S}_n$, there exists $0 \leq j \leq d_n - 1$ such that $d(\alpha_j.\Lambda_{\mathbf{s}_1}, \alpha_j.\Lambda_{\mathbf{s}_2}) > \delta_4$. Since $\mathrm{diam}(P) < \delta_4$ for every $P \in \mathcal{P}$, we conclude that $\Lambda_{\mathbf{s}_1}$ and $\Lambda_{\mathbf{s}_2}$ can not lie in the same element of the partition $\alpha_j^{-1}(\mathcal{P})$ and in particular $\mathcal{P}_0^{l_n(r)+r+k-1}$.

So we conclude that for every $r = 0, 1, ..., k-1$,

$$H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_0^{l_n(r)k+r+k-1}) = \log(\#\mathcal{S}_n).$$

Combined with Equa.(3,4), we get

$$H_{\mu_n}(\mathcal{P}_0^{k-1}) \geq \frac{k}{d_n}\log(\#\mathcal{S}_n) - \frac{k}{d_n} H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_0^{k-2}).$$

By the definition of $\mathcal{S}_n$ (as in Sect.1.6), $\log\#\mathcal{S}_n > \delta_1\log(\varepsilon_n^{-1}) - \log(l_0)$. Therefore,

$$H_{\mu_n}(\mathcal{P}_0^{k-1}) > \frac{k}{d_n}\delta_1\log(\varepsilon_n^{-1}) - \frac{k}{d_n}\left(H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_0^{k-2}) + \log(l_0)\right).$$

By the choice of d_n (see Section 1.6), we have for any $\varepsilon > 0$ and n large enough,

$$\frac{\log(\varepsilon_n^{-1})}{d_n} \geq 3 - \frac{3+\log(\delta_3)}{d_n} \geq 3 - \varepsilon.$$

Hence,

$$H_{\mu_n}(\mathcal{P}_0^{k-1}) \geq (3-\varepsilon)k\delta_1 - \frac{k}{d_n}\left(H_{\mathrm{m}_{\mathcal{S}_n}}(\mathcal{P}_0^{k-2}) + \log(l_0)\right).$$

Combined with Equa.(2), we get

$$H_\mu(T,\mathcal{P}) = \lim_{k\to+\infty}\frac{1}{k}H_\mu(\mathcal{P}_0^{k-1}) = \lim_{k\to+\infty}\frac{1}{k}\lim_{n\to+\infty}H_{\mu_n}(\mathcal{P}_0^{k-1}) \geq (3-\varepsilon)\delta_1.$$

Letting $\varepsilon \to 0$ finishes the proof.

1.10. **Conditional entropy.** We need the general notion of conditional entropy for step two. Let $(X, \mathscr{B}_X)$ be a compact metrizable space together with its Borel σ-algebra.

Definition 1.13. *Let $\mathcal{P}$ be a finite measurable partition and $\mathscr{A}$ be a σ-subalgebra. Let $(\mu_x^{\mathscr{A}})_{x\in X'}$ be a family of conditional measures where $X' \in \mathscr{A}$ is a co-null set in X. Note that the map $x \mapsto H_{\mu_x^{\mathscr{A}}}(\mathcal{P})$ is measurable and non-negative. Define the* ***conditional entropy of*** $\mathcal{P}$ ***given*** $\mathscr{A}$ *by*

$$H_\mu(\mathcal{P}|\mathscr{A}) := \int_{X'} H_{\mu_x^{\mathscr{A}}}(\mathcal{P})\,\mu(x).$$

Note that when $\mathscr{A}$ is the σ-subalgebra generated by a finite measurable partition $\mathcal{Q}$, then $H_\mu(\mathcal{P}|\mathscr{A})$ coincides with the $H_\mu(\mathcal{P}|\mathcal{Q})$ defined previously.

Lemma 1.14. *Let $\mathscr{A}_1 \subset \mathscr{A}_2 \subset ...$ be a sequence of σ-subalgebras and $\mathscr{A}_\infty$ be the smallest σ-subalgebra containing them. Let $\mathcal{P}$ be a finite measurable partition. Then*

$$\lim_{n\to\infty} H_\mu(\mathcal{P}|\mathscr{A}_n) = H_\mu(\mathcal{P}|\mathscr{A}).$$

Proof. By the theorem on conditional measures, for each $P \in \mathcal{P}$,

$$\mu_x^{\mathscr{A}_n}(P) \text{ converges to } \mu_x^{\mathscr{A}_\infty}(P) \text{ for almost every } x.$$

Thus

$$H_{\mu_x^{\mathscr{A}_n}}(\mathcal{P}) \text{ converges to } H_{\mu_x^{\mathscr{A}_\infty}}(\mathcal{P}) \text{ for almost every } x.$$

Also $H_{\mu_x^{\mathscr{A}_n}}(\mathcal{P})$ is bounded by $\log\#\mathcal{P}$. So the conclusion follows from the dominated/bounded convergence theorem. □

A useful observation is that

Lemma 1.15. *Let $\mathscr{A}$ be a countably generated σ-subalgebra and X' be a full measure subset, If for every $x \in X'$, there exists $P \in \mathcal{P}$ with*

$$[x]_{\mathscr{A}} \cap X' \subset P,$$

then $H_\mu(\mathcal{P}|\mathscr{A}) = 0$.

Proof. Indeed, for almost every x, $\mu_x^{\mathscr{A}}(P \cap X') = \mu_x^{\mathscr{A}}(P \cap X' \cap [x]_{\mathscr{A}})$ is equal to 0 or 1. Moreover, there exists a full measure subset such that for every x in this subset,

$$\mu_x^{\mathscr{A}}(X' \cap P) = \mu_x^{\mathscr{A}}(P), \quad \forall\, P \in \mathcal{P}.$$

Hence $H_{\mu_x^{\mathscr{A}}}(\mathcal{P})$ is equal to zero μ-a.e.,which implies the claim. □

If the condition as in the lemma is satisfied, we say that $\mathcal{P}$ is **μ-essentially contained in $\mathscr{A}$**.

1.11. **Step 2, construction of the measure.** The construction is just performing average along A^+. Let

$$\mathbf{a}_{s,t} := \begin{bmatrix} e^s & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-s-t} \end{bmatrix}$$

and

$$\nu_T := \frac{1}{T^2} \int_0^T \int_0^T (\mathbf{a}_{s,t})_* \mu \,\mathrm{dsdt}.$$

which is supported on $\mathscr{C}_1$ by assumption.

Find a convergent subsequence $\nu := \lim_n \nu_{T_n}$. As before, we can show that

Lemma 1.16. *ν is A-invariant.*

What is less trivial is

Lemma 1.17. *$h_\nu(\alpha_1) \geq h_\mu(\alpha_1)$.*

Thus the proof of Theorem 1.6 is complete modulo this lemma.

Note that for each finite measurable partition $\mathcal{P}$, we have

$$h_\nu(\alpha_1, \mathcal{P}) \geq \limsup h_{\nu_n}(\alpha_1, \mathcal{P}).$$

whenever ν_n converges to ν (under weak$*$ topology) and boundary of each element in $\mathcal{P}$ has vanishing ν-measure. So the real task is to find a "generating partition" that works for all ν_n.

1.12. **Dynamical entropy as conditional entropy.** Assume $(X, \mathscr{B}, \mu)$ is a probability space and $T : X \to X$ is an invertible measure preserving map (by which I mean the inverse of T is also measurable).

Lemma 1.18. *Let $\mathcal{P}_1^\infty$ be the smallest σ-subalgebra containing all $\mathcal{P}_1^n$. Then $h_\mu(T, \mathcal{P}) = H_\mu(\mathcal{P}|\mathcal{P}_1^\infty) = H_\mu(\mathcal{P}|\mathcal{P}_{-\infty}^{-1}) = h_\mu(T^{-1}, \mathcal{P})$.*

Remark 1.19. *If one does not assume the knowledge of conditional measures, especially the "martingale convergence theorem", then the proof below shows that $h_\mu(T, \mathcal{P}) = \lim_{n\to\infty} H_\mu(\mathcal{P}|\mathcal{P}_1^n)$. Similar remarks apply to the the lemma below.*

Proof.

$$\begin{aligned} &H_\mu(\mathcal{P} \vee T^{-1}\mathcal{P} \vee ... \vee T^{-(n-1)}\mathcal{P}) \\ = &H_\mu(T^{-(n-1)}\mathcal{P}) + H_\mu(T^{-(n-2)\mathcal{P}}|T^{-(n-1)}\mathcal{P}) + ... + H_\mu(\mathcal{P}|\mathcal{P}_1^{n-1}) \\ = &H_\mu(\mathcal{P}) + H_\mu(\mathcal{P}|T^{-1}\mathcal{P}) + H_\mu(\mathcal{P}|T^{-1}\mathcal{P} \vee T^{-2}\mathcal{P}) + ... + H_\mu(\mathcal{P}|\mathcal{P}_1^{n-1}). \end{aligned}$$

As the sequence $(H_\mu(\mathcal{P}|\mathcal{P}_1^{n-1}))$ converges to $H_\mu(\mathcal{P}|\mathcal{P}_1^\infty)$, it converges to the same limit on average. So we are done by invoking the definition of $h_\mu(T, \mathcal{P})$.

The rest of the equalities follow if $h_\mu(T, \mathcal{P}) = h_\mu(T^{-1}, \mathcal{P})$, which is true since

$$H_\mu(\mathcal{P} \vee T^{-1}\mathcal{P} \vee ... \vee T^{-(n-1)}\mathcal{P}) = H_\mu(T^{n-1}\mathcal{P} \vee T^{n-2}\mathcal{P} \vee ... \vee \mathcal{P}).$$

□

More generally, we have

Lemma 1.20. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two finite measurable partitions, then*

$$h_\mu(T,\mathcal{P}) \le h_\mu(T,\mathcal{P}\vee\mathcal{Q}) = h_\mu(T,\mathcal{Q}) + H_\mu(\mathcal{P}|\mathcal{P}_1^\infty \vee \mathcal{Q}_{-\infty}^{+\infty}).$$

Proof. The proof is similar. We present the first two steps in a more explicit way.

$$\begin{aligned} &H_\mu(\mathcal{P}\vee T^{-1}\mathcal{P}\vee\mathcal{Q}\vee T^{-1}\mathcal{Q}) \\ =&H_\mu(T^{-1}\mathcal{Q}) + H_\mu(\mathcal{Q}|T^{-1}\mathcal{Q}) + H_\mu(T^{-1}\mathcal{P}|\mathcal{Q}\vee T^{-1}\mathcal{Q}) + H_\mu(\mathcal{P}|T^{-1}\mathcal{P}\vee\mathcal{Q}\vee T^{-1}\mathcal{Q}) \\ =&H_\mu(\mathcal{Q}) + H_\mu(\mathcal{Q}|\mathcal{Q}_1^1) + H_\mu(\mathcal{P}|\mathcal{Q}_{-1}^0) + H_\mu(\mathcal{P}|\mathcal{P}_1^1\vee\mathcal{Q}_0^1) \end{aligned}$$

$$\begin{aligned} &H_\mu(\mathcal{P}\vee T^{-1}\mathcal{P}\vee T^{-2}\mathcal{P}\vee\mathcal{Q}\vee T^{-1}\mathcal{Q}\vee T^{-2}\mathcal{Q}) \\ =&H_\mu(\mathcal{Q}) + H_\mu(\mathcal{Q}|\mathcal{Q}_1^1) + H_\mu(\mathcal{Q}|\mathcal{Q}_1^2) + H_\mu(\mathcal{P}|\mathcal{Q}_{-2}^0) + H_\mu(\mathcal{P}|\mathcal{P}_1^1\vee\mathcal{Q}_{-1}^1) + H_\mu(\mathcal{P}|\mathcal{P}_1^2\vee\mathcal{Q}_0^2) \end{aligned}$$

The general formula goes as[5]

$$\begin{aligned} &H_\mu(\mathcal{P}_0^n\vee\mathcal{Q}_0^n) \\ =&\sum_{k=0}^n H_\mu(\mathcal{Q}|\mathcal{Q}_1^k) + \sum_{k=0}^n H_\mu\left(T^{-(n-k)}\mathcal{P}\,\middle|\,\bigvee_{i=0}^{k-1} T^{-(n-i)}\mathcal{P}\vee\bigvee_{i=1}^{n} T^{-i}\mathcal{Q}\right) \\ =&\sum_{k=0}^n H_\mu(\mathcal{P}|\mathcal{Q}_1^k) + \sum_{k=0}^n H_\mu\left(\mathcal{P}\,\middle|\,\bigvee_{i=0}^{k-1} T^{-(k-i)}\mathcal{P}\vee\bigvee_{i=-(n-k)+1}^{k} T^{-i}\mathcal{Q}\right) \\ =&\sum_{k=0}^n H_\mu(\mathcal{Q}|\mathcal{Q}_1^k) + \sum_{k=0}^n H_\mu(\mathcal{P}|\mathcal{P}_1^k\vee\mathcal{Q}_{-n+k+1}^k) \end{aligned}$$

Left hand side divided by n converges to $H_\mu(T,\mathcal{P}\vee\mathcal{Q})$. Moreover, $H_\mu(\mathcal{Q}|\mathcal{Q}_1^k)$ as $k\to+\infty$ converges to $H_\mu(\mathcal{Q}|\mathcal{Q}_1^\infty)$, and $H_\mu(\mathcal{P}|\mathcal{P}_1^a\vee\mathcal{Q}_{-b}^c)$ as $a,b,c\to+\infty$ converges to $H_\mu(\mathcal{P}|\mathcal{P}_1^\infty\vee\mathcal{Q}_{-\infty}^{+\infty})$. We are done by taking the limit of their averages. □

1.13. **Generating partition computes the entropy.**

Corollary 1.21. *Let $\mathcal{Q}$ be a finite measurable partition. If for every finite measurable partition $\mathcal{P}$ satisfying $\mu(\partial P)=0$ for all $P\in\mathcal{P}$, one has $\mathcal{P}$ is μ-essentially contained in $\mathcal{P}_1^\infty\vee\mathcal{Q}_{-\infty}^{+\infty}$, then*

$$h_\mu(T) = h_\mu(T,\mathcal{Q}).$$

Proof. By Lemma 1.9, there exists an increasing sequence of finite measurable partitions $(\mathcal{P}_n)$ with diameter decreasing to 0 and μ-trivial boundary. So for each n,

$$h_\mu(T,\mathcal{P}_n) \le h_\mu(T,\mathcal{Q}) + H_\mu(\mathcal{P}_n|(\mathcal{P}_n)_1^\infty\vee\mathcal{Q}_{-\infty}^{+\infty}) = h_\mu(T,\mathcal{Q})$$

by Lemma 1.15 and 1.20. It remains to show that $h_\mu(T)=\lim h_\mu(T,\mathcal{P}_n)$.

Take another finite partition $\mathcal{P}$. Let $\mathcal{P}_\infty$ be the smallest σ-subalgebra containing all $\mathcal{P}_n$'s. Then $\mathcal{P}_\infty$ is countably generated and every atom consists of one single point. So $H_\mu(\mathcal{P}|\mathcal{P}_\infty)=0$ by Lemma 1.15 and

$$\inf H_\mu(\mathcal{P}|\mathcal{P}_1^\infty\vee(\mathcal{P}_n)_{-\infty}^{+\infty}) \le \lim H_\mu(\mathcal{P}|\mathcal{P}_n) = H_\mu(\mathcal{P}|\mathcal{P}_\infty) = 0.$$

Invoking Lemma 1.20,

$$\lim_{n\to\infty} h_\mu(T,\mathcal{P}) - h_\mu(T,\mathcal{P}_n) \le \lim H_\mu(\mathcal{P}|\mathcal{P}_1^\infty\vee(\mathcal{P}_n)_{-\infty}^{+\infty}) = 0.$$

So we are done.

□

1.14. **Expansiveness modulo centralizer.** Now return to our specific dynamics. As a first step, we note that

Lemma 1.22. *If $y\in[x]_{\mathcal{Q}_{-\infty}^{+\infty}}$, then $y=z.x$ for some $z\in\mathrm{Z}(\alpha_1)\cap B(\delta_4)$.*

Here and below, $\mathrm{Z}(\alpha_1)$ denotes the centralizer of α_1 in $\mathbf{SL}_3(\mathbb{R})$.

[5] By convention $\mathcal{Q}_1^0$ is the trivial partition consisting of one element.

Proof. Since $y \in [x]_{\mathcal{Q}}$, we have $d(x,y) < \delta_4 < \mathrm{InjRad}(x)$. Thus, there exists a unique $g_y \in \mathbf{SL}_3(\mathbb{R})$ such that

$$y = g_y.x, \quad d(x,y) = d(I_3, g_y) < \delta_4.$$

Thus $g_y \in B(\delta_4)$. It remains to show that $g_y \in \mathrm{Z}(\alpha_1)$. If not, $\left\|\alpha_1^k g_y \alpha_1^{-k}\right\|$ can be arbitrarily large as k varies in $\mathbb{Z}$. We will see that this is not true.

Similarly, since $y \in [x]_{\mathcal{Q}_{-\infty}^{+\infty}}$, for every $k \in \mathbb{Z}$, there exists a unique $g_y^k \in \mathbf{SL}_3(\mathbb{R})$ such that

$$\alpha_1^k.y = g_y^k \alpha_1^k.x, \quad d(\alpha_1^k.x, \alpha_1^k.y) = d(I_3, g_y^k) < \delta_4.$$

On the other hand

$$\alpha_1^k.y = \alpha_1^k g_y.x = \alpha_1^k g_y \alpha_1^{-k}.(\alpha_1^k.x).$$

We claim that $g_y^k = \alpha_1^k g_y \alpha_1^{-k}$ for all $k \in \mathbb{Z}$, which would imply that $\alpha_1^k g_y \alpha_1^{-k}$ is bounded and force $g_y \in \mathrm{Z}(\alpha_1)$.

If the claim were not true, we find $k_0 \in \mathbb{Z}$ such that $g_y^{k_0} = \alpha_1^{k_0} g_y \alpha_1^{-k_0}$ but

$$g_y^{k_0+1} \neq \alpha_1^{k_0+1} g_y \alpha_1^{-k_0-1}, \text{ or } g_y^{k_0-1} \neq \alpha_1^{k_0-1} g_y \alpha_1^{-k_0+1}.$$

Let us treat the former case.

Recall that $\delta_2 < \mathrm{InjRad}(\alpha_1^k.x)$ for each $k \in \mathbb{Z}$ and hence there exists at most one element $g \in B(\delta_2)$ with $\alpha_1^k.y = g\alpha_1^k.x$. Therefore,

$$d(g_y^{k_0}, I_3) < \delta_4, \quad d(\alpha_1 g_y^{k_0} \alpha_1^{-1}, I_3) \geq \delta_2.$$

But this is not the case, as

$$g_y^{k_0} \in B(\delta_4) \subset \mathcal{O}_{e^{-3}\delta_3} \implies \alpha_1 g_y^{k_0} \alpha_1^{-1} \in \mathcal{O}_{\delta_3} \subset B(\delta_2).$$

□

1.15. **Poincare recurrence.**

Slogan: If something happens once, then it should happen infinitely many times.

Lemma 1.23. *Let $(X, \mathscr{B}, \lambda)$ be a probability space and $T : X \to X$ is a measurable map preserving λ. For $E \in \mathscr{B}$, define*

$$E' := \left\{x \in E \;\middle|\; \alpha_1^n.x \in E \text{ for infinitely many } n \in \mathbb{Z}^+\right\}.$$

Then $\lambda(E \setminus E') = 0$.

Proof. Let

$$E_N := \bigcup_{i=N}^{\infty} T^{-i}(E).$$

Then

$$E_0 \supset E_1 = T^{-1}(E_0) \supset \ldots \supset E_N = T^{-N}(E_0).$$

But T preserves λ. So

$$\lambda(E_0) = \lambda(E_1) = \ldots. = \lambda(E_N) \implies \lambda(E_0 \setminus E_N) = 0 \implies \lambda(E \setminus E_N) = 0.$$

□

1.16. **Generating partition.**

Lemma 1.24. *Let λ be an α_1-invariant probability measure supported on $\mathscr{C}_1$. Fix a finite measurable partition $\mathcal{Q}$ (of $\mathscr{C}_1$) with λ-trivial boundary and* $\mathrm{diam}(\mathcal{Q}) < \delta_4$*. Then $\mathcal{Q}$ satisfies the condition of Corollary 1.21, hence $h_\lambda(T) = h_\lambda(\alpha_1, \mathcal{Q})$.*

For every $z \in \mathrm{Z}(\alpha_1) \cap B(\delta_4)$, let

$$S_z := \{x \in X \mid x, z.x \text{ lie in the same } P \in \mathcal{P}\} = \bigcup_{P \in \mathcal{P}} (P \cap z^{-1}P)$$

$$D_z := \{x \in X \mid x, z.x \text{ lie in different } P \neq P' \in \mathcal{P}\} = \bigcup_{P \in \mathcal{P}} (P \setminus z^{-1}P)$$

and the associated recurrence sets

$$\mathcal{R}S_z := \{x \in S_z,\ \alpha_1^n.x \in S_z \text{ for infinitely many } n \in \mathbb{Z}^+\},$$
$$\mathcal{R}D_z := \{x \in D_z,\ \alpha_1^n.x \in D_z \text{ for infinitely many } n \in \mathbb{Z}^+\},$$
$$\mathcal{R}_z := \mathcal{R}S_z \sqcup \mathcal{R}D_z$$

By Poincare recurrence, $\mu(\mathcal{R}_z) = 1$.

For $x \in \bigcap_z \mathcal{R}_z$, we have

$$[x]_{\mathcal{P}_1^\infty \vee \mathcal{Q}_{-\infty}^{+\infty}} \subset [x]_{\mathcal{P}},$$

where for a finite partition $\mathcal{P}$, $[x]_{\mathcal{P}} := [x]_{\sigma(\mathcal{P})}$ is the unique element $P \in \mathcal{P}$ containing x.

Take $y \in [x]_{\mathcal{P}_1^\infty \vee \mathcal{Q}_{-\infty}^{+\infty}}$. By Lemma 1.22, $y = z.x$ for some $z \in \mathrm{Z}(\alpha_1) \cap B(\delta_4)$. Since $y \in [x]_{\mathcal{P}_1^\infty}$,

$$\begin{aligned}
&[\alpha_1^n.x]_{\mathcal{P}} = [\alpha_1^n z.x]_{\mathcal{P}} = [z\alpha_1^n.x]_{\mathcal{P}}, \quad \forall\, n \in \mathbb{Z}^+ \\
\implies &\alpha_1^n.x \in S_z, \quad \forall\, n \in \mathbb{Z}^+ \\
\implies &x \in \mathcal{R}S_z \implies y \in [x]_{\mathcal{P}}.
\end{aligned}$$

Unfortunately, $\bigcap_z \mathcal{R}_z$, being an uncountable intersection, may not have full measure (even not clear if it is measurable). So an approximation argument is needed and will be presented in the next subsection.

1.17. **Proof of Lemma 1.24.** For $n \in \mathbb{Z}^+$, let

$$\begin{aligned}
P(\frac{1}{n}) &:= \left\{ x \;\middle|\; d(x, P^c) > \frac{1}{n} \right\}, \\
S_{z,\frac{1}{n}} &:= \left\{ x \in X \;\middle|\; x, z.x \text{ lie in the same } P(\frac{1}{n}),\, \exists\, P \in \mathcal{P} \right\}, \\
D_{z,\frac{1}{n}} &:= \left\{ x \in X \;\middle|\; x, z.x \text{ lie in different } P(\frac{1}{n}),\, P'(\frac{1}{n}),\, \exists\, P \neq P' \in \mathcal{P} \right\}.
\end{aligned}$$

Let $\mathcal{R}S_{z,\frac{1}{n}}$ and $\mathcal{R}D_{z,\frac{1}{n}}$ denote the corresponding recurrence sets and

$$\mathcal{R}_{z,\frac{1}{n}} := \mathcal{R}S_{z,\frac{1}{n}} \sqcup \mathcal{R}D_{z,\frac{1}{n}}.$$

By Poincare recurrence, $\mathcal{R}_{z,\frac{1}{n}}$ is of full measure in $\mathscr{C}_1(\frac{1}{n}) := \bigsqcup_{P \in \mathcal{P}} P(\frac{1}{n})$.

Fix a countable dense subset

$$\mathrm{CZ} \subset \mathrm{Z}(\alpha_1) \cap B(\delta_4)$$

Define

$$\mathscr{C}_1' := \bigcup_n \bigcap_{z \in \mathrm{CZ}} \mathcal{R}_{z,\frac{1}{n}} \subset \bigsqcup_{P \in \mathcal{P}} \mathrm{Int}(P)$$

Then $\mathscr{C}_1'$ is of full measure in $\mathscr{C}_1$. For $x \in \mathscr{C}_1'$, we show that $[x]_{\mathcal{P}_1^\infty \vee \mathcal{Q}_{-\infty}^{+\infty}} \subset [x]_{\mathcal{P}}$.

Fix n such that $x \in \bigcap_{z \in \mathrm{CZ}} \mathcal{R}_{z,\frac{1}{n}}$ and take $y \in [x]_{\mathcal{P}_1^\infty \vee \mathcal{Q}_{-\infty}^{+\infty}}$.

By Lemma 1.22, $y \in [x]_{\mathcal{Q}_{-\infty}^{+\infty}} \implies y = z_y.x$ for some $z_y \in \mathrm{Z}(\alpha_1) \cap B(\delta_4)$. Choose $z \in \mathrm{CZ}$ sufficiently close to z_y such that

$$z_y z^{-1}.P(\frac{1}{n}) \subset P(\frac{1}{n+1}),\ \forall\, P \in \mathcal{P}.$$

Since $x \in \mathcal{R}_{z,\frac{1}{n}} = \mathcal{R}D_{z,\frac{1}{n}} \bigsqcup \mathcal{R}S_{z,\frac{1}{n}}$. If $x \in \mathcal{R}D_{z,\frac{1}{n}}$, then there are $P \neq P' \in \mathcal{P}$ such that

$$\alpha_1.x \in P(\frac{1}{n}),\ z\alpha_1.x \in P'(\frac{1}{n}).$$

Hence

$$\alpha_1.y = z_y z^{-1} z\alpha_1.x \in P'(\frac{1}{n+1}).$$

It follows that $y \notin [x]_{\alpha_1^{-1}\mathcal{P}}$, a contradiction. Therefore, $x \in \mathcal{R}S_{z,\frac{1}{n}} \subset S_{z,\frac{1}{n}}$. For some $P \in \mathcal{P}$,

$$x, z.x \in P(\frac{1}{n}) \implies y = z_y z^{-1} z.x \in P(\frac{1}{n+1}).$$

In particular, $y \in [x]_{\mathcal{P}}$.

1.18. **Conclusion.**

Lemma 1.25. *For every $T > 0$ and finite partition $\mathcal{P}$ of $\mathscr{C}_1$,*

$$H_{\nu_T}(\mathcal{P}) \geq \frac{1}{T^2}\int_0^T \int_0^T H_{(\mathbf{a}_{s,t})_*\mu}(\mathcal{P})\mathrm{dsdt}$$

In fact, $h_{\nu_T}(\alpha_1) = h_\mu(\alpha_1)$, but it requires more work.

Proof. Let $Y := \mathscr{C}_1 \times [0,T]^2$, a compact metrizable space. Define a Borel probability measure λ on Y by

$$\lambda(E \times F) := \frac{1}{T^2}\int_{(s,t)\in F} (\mathbf{a}_{s,t})_*\mu(E)\mathrm{dsdt}.$$

Define a σ-subalgebra

$$\mathscr{A}_1 := \left\{\mathscr{C}_1 \times F \,\middle|\, F \text{ is a Borel measurable subset of } [0,T]^2\right\}$$

and a finite partition

$$\mathcal{P}_1 := \left\{P \times [0,T]^2 \,\middle|\, P \in \mathcal{P}\right\}.$$

One can immediately check that if points $x \in Y$ are written as $(\Lambda_x, (s_x, t_x))$, then conditional measures $\lambda_x^{\mathscr{A}_1}$ can be chosen to be $((\mathbf{a}_{s_x,t_x})_*\mu) \otimes \delta_{(s_x,t_x)}$. Hence

$$H_\lambda(\mathcal{P}_1|\mathscr{A}_1) = \int H_{\lambda_x^{\mathscr{A}_1}}(\mathcal{P}_1)\,\lambda(x) = \frac{1}{T^2}\int_0^T \int_0^T H_{(\mathbf{a}_{s,t})_*\mu}(\mathcal{P})\mathrm{dsdt}.$$

On the other hand,

$$H_\lambda(\mathcal{P}_1) = H_{\nu_T}(\mathcal{P}).$$

So it remains to prove that $H_\mu(\mathcal{P}_1|\mathscr{A}_1) \leq H_\mu(\mathcal{P}_1)$. Choose an increasing sequence of finite σ-subalgebras $\mathscr{B}_i$ converging to $\mathscr{A}_1$. By Lemma 1.14,

$$H_\lambda(\mathcal{P}_1|\mathscr{B}_i) \to H_\lambda(\mathcal{P}_1|\mathscr{A}_1).$$

But each $H_\lambda(\mathcal{P}_1|\mathscr{B}_i) \leq H_\lambda(\mathcal{P}_1)$. So we are done.

□

Now take a finite partition $\mathcal{Q}$ with trivial boundary w.r.t. ν and ν_{T_n} for all n and diameter $< \delta_4$. So $\mathcal{Q}$ has trivial boundary w.r.t. $(\mathbf{a}_{s,t})_*\mu$ for Lebesgue-almost all (s,t). In particular, by Lemma 1.24, $h_{(\mathbf{a}_{s,t})_*\mu}(\alpha_1, \mathcal{Q}) = h_{(\mathbf{a}_{s,t})_*\mu}(\alpha_1)$ for almost all (s,t). Now

$$\begin{aligned}
h_\nu(\alpha_1, \mathcal{Q}) &= \inf_k \frac{1}{k} H_\nu(\mathcal{Q}_0^{k-1}) = \inf_k \lim_n \frac{1}{k} H_{\nu_{T_n}}(\mathcal{Q}_0^{k-1}) \\
&\geq \inf_k \limsup_n \frac{1}{T_n^2}\int_0^{T_n}\int_0^{T_n} \frac{H_{(\mathbf{a}_{s,t})_*\mu}(\mathcal{Q}_0^{k-1})}{k}\,\mathrm{dsdt} \\
&\geq \limsup_n \frac{1}{T_n^2}\int_0^{T_n}\int_0^{T_n} h_{(\mathbf{a}_{s,t})_*\mu}(\alpha_1, \mathcal{Q})\,\mathrm{dsdt} \\
&= \limsup_n \frac{1}{T_n^2}\int_0^{T_n}\int_0^{T_n} h_{(\mathbf{a}_{s,t})_*\mu}(\alpha_1)\,\mathrm{dsdt} \\
&= \limsup_n \frac{1}{T_n^2}\int_0^{T_n}\int_0^{T_n} h_\mu(\alpha_1) = h_\mu(\alpha_1).
\end{aligned}$$

So finally, the proof of Lemma 1.17 is complete.

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