Saving Mark Watney: The Feasibility of the Space Travel Depicted in The Martian

Max Garner

Department of Mathematical Sciences
Durham University

Declaration

'This piece of work is a result of my own work and I have complied with the Department's guidance on multiple submission and on the use of AI tools. Material from the work of others not involved in the project has been acknowledged, quotations and paraphrases suitably indicated, and all uses of AI tools have been declared.'

Abstract

This report explores the scientific accuracy and real world feasibility of the space travel and trajectories depicted in The Martian. We focus on modelling maneuvers and analysing their underlying mathematical concepts, including gravity turn, interplanetary travel, gravity assist, and space rendezvous. We will compare the similarities and differences between current technological capabilities and space travel depicted in The Martian.

Contents

Chapter 1

Introduction to Space Travel and Orbital Dynamics

1.1 Introduction and Motivation

Space exploration has captivated the human imagination for centuries, and as a result has inspired countless pieces of science fiction. These works delve into the possibilities and challenges presented by exploring our solar system and beyond. A novel among these which gathered notable attention is Andy Weir's 'The Martian' [1], which as a result was adapted into a film by Ridley Scott [2].

The story begins on Mars, where the Hermes (interplanetary spacecraft) crew are gathering resources to test on the planet surface. They are currently on sol (solar day on Mars) 18 of a 30 sol mission, when disaster strikes. A storm causes a piece of debris to impale Mark Watney (protagonist), and the crew leave Mars in their MAV (Mars Ascent Vehicle) having presumed Mark as dead. On sol 54, NASA scientist Mindy Park discovers concrete evidence that Mark is in fact alive and trying to survive, and as a result a our overarching problem prevails. How can we keep Mark alive, and get him back to Earth as quickly as possible?

Initially, the director of NASA calls for supplies to be sent to Mark as soon as possible so that he does not have to rely on his amateur potato farming skills. The rocket carrying Mark's supplies however has an unsuccessful launch. Enter Rich Purnell, astrophysicist for NASA. He suggests a gravity assist manoeuvre whereby Hermes slingshots around Earth picking up supplies on the way, whilst redirecting course to Mars to then slingshot back to Earth (picking up Mark in the process). During the Hermes slingshot around Earth, a Chinese organisation offers the services of their spacecraft named the Taiyang Shen. The Taiyang Shen resupplies the Hermes with what they need for the extended length of their journey.

The compelling story supposedly intertwines scientific accuracy with outstanding storytelling. This report will delve into the objective part of this statement, by exploring the mathematical aspects of the space travel depicted in the film adaptation. We will analyse the film's portrayal of orbital mechanics and trajectory planning, with the aim of assessing whether the methods of travel might be feasible in the real world.

1.2 The Method of Patched Conics

In order to quantify the trips through space, we need to outline how this will be done. As a spacecraft travels through space, it will be acted upon by various gravitational fields. Taking into account the all of the gravitational sources of large masses in the solar system quickly becomes incredibly complicated. Instead, a common method is to patch together several conic trajectories in order to calculate the entire trajectory of interest [3].

The simplest possible path using the patched conic approximation considers departure, transfer and arrival. In travelling from the original planet to our target planet, this represents our spacecraft being primarily under the influence of three different gravitational sources. In the departure phase, the focus of the trajectory is the initial planet. In the transfer phase, the focus of the trajectory is the sun. Finally, in the arrival phase, the focus of the trajectory is the target planet.

In particular, we will split The Martian trajectory into four parts:

- Gravity Turn
- Interplanetary Journey
- Gravity Assist
- Space Rendezvous

1.3 Reference Frames

To be able to derive our three dimensional equations, we need to outline the reference frames that will be used. This section will describe how two masses interact with each other in firstly an inertial frame, and then a co-moving frame. This will lead us to discover the second order differential equations of motion which will be used in derivations later on. This section is based on [3].

1.3.1 Equations of Motion in the Inertial Frame

We start by picking some convenient point in space to make our fixed origin. Consider two point masses, m_1 and m_2. With respect to our fixed origin, they have position vectors $\mathbf{R_1}$

and $\mathbf{R_2}$ given by

$$\begin{aligned}\mathbf{R}_1 &= X_1\widehat{\mathbf{I}} + Y_1\widehat{\mathbf{J}} + Z_1\widehat{\mathbf{K}},\\ \mathbf{R}_2 &= X_2\widehat{\mathbf{I}} + Y_2\widehat{\mathbf{J}} + Z_2\widehat{\mathbf{K}}.\end{aligned} \tag{1.1}$$

The vector from $\mathbf{R}_1$ to $\mathbf{R}_2$ we will call $\mathbf{r}$,

$$\begin{aligned}\mathbf{r} &= \mathbf{R}_2 - \mathbf{R}_1,\\ \mathbf{r} &= (X_2 - X_1)\widehat{\mathbf{I}} + (Y_2 - Y_1)\widehat{\mathbf{J}} + (Z_2 - Z_1)\widehat{\mathbf{K}}.\end{aligned} \tag{1.2}$$

Then where r is the magnitude of $\mathbf{r}$, we can define the unit vector from m_1 toward m_2,

$$\widehat{\mathbf{u}}_r = \frac{\mathbf{r}}{r}. \tag{1.3}$$

Now employ Newton's law of gravitation,

$$F = G\frac{m_1 m_2}{r^2}, \tag{1.4}$$

in order to quantify the force $\mathbf{F}_{12}$ exerted on m_1 by m_2 and $\mathbf{F}_{21}$, the force exerted on m_2 by m_1. We are able to do this as gravitational attraction is the only force in this system. Combining this concept with Newton's third law about equal and opposite forces, we obtain

$$\begin{aligned}\mathbf{F}_{12} &= \frac{Gm_1m_2}{r^2}\widehat{\mathbf{u}}_r,\\ \mathbf{F}_{21} &= -\frac{Gm_1m_2}{r^2}\widehat{\mathbf{u}}_r.\end{aligned} \tag{1.5}$$

Newton's second law states that force is equal to mass multiplied by acceleration, which means we can write

$$\begin{aligned}m_1\ddot{\mathbf{R}}_1 &= \frac{Gm_1m_2}{r^2}\widehat{\mathbf{u}}_r,\\ m_2\ddot{\mathbf{R}}_2 &= -\frac{Gm_1m_2}{r^2}\widehat{\mathbf{u}}_r.\end{aligned} \tag{1.6}$$

Applying the definition of $\widehat{\mathbf{u}}_r$ leaves us with the two-body inertial equations of motion,

$$\begin{aligned}\ddot{\mathbf{R}}_1 &= Gm_2\frac{\mathbf{r}}{r^3},\\ \ddot{\mathbf{R}}_2 &= -Gm_1\frac{\mathbf{r}}{r^3}.\end{aligned} \tag{1.7}$$

1.3.2 Equation of Relative Motion

The equation of relative motion is crucial in orbital mechanics and creates a start point for finding the orbit equation. By differentiating (1.2) twice and plugging in (1.7),

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1, \\ \ddot{\mathbf{r}} &= -Gm_1\frac{\mathbf{r}}{r^3} - -Gm_2\frac{\mathbf{r}}{r^3}, \\ \ddot{\mathbf{r}} &= -G(m_1+m_2)\frac{\mathbf{r}}{r^3}.\end{aligned} \tag{1.8}$$

Lastly introduce the standard gravitational parameter, $\mu = G(m_1 + m_2)$, to obtain

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}. \tag{1.9}$$

Note that the standard gravitational parameter is introduced in cases where $m_1 \gg m_2$, so that $\mu \approx Gm_1$.

1.4 Rocket Equation

The very first step of describing the mathematics of space travel begins with the rocket equation (derivation based on [4]). In order to create a change in velocity, the spacecraft ejects burned fuel gases. Since the expelled gases have mass, it's non-zero momentum causes the spacecraft to accelerate in the opposite direction. The fuel is ejected at a constant rate and the force on the rocket is therefore also constant.

We will describe m_0 as the initial combined mass of fuel and the empty ship, m as the instantaneous mass (current mass of fuel and the mass of empty ship), and $\vec{v}$ as the instantaneous velocity. Seeing as the force exerted on the spacecraft is constant, the equation $F = ma$ tells us that as m is continually decreasing due to gas expulsion, our acceleration continuously increases.

We have $\mathbf{v} = v\mathbf{j}$ is the rocket velocity, $\mathbf{u} = -u\mathbf{j}$ is our fuel velocity and dm_g is the infinitesimal mass of gas expelled during an infinitesimal amount of time. We will say that dm_g causes a velocity increase for the rocket of $dv\mathbf{j}$. From the inertial frame of the earth, we see that the ejected gas has velocity $(v-u)\mathbf{j}$. Using all of this information, we form an equation which describes the momentum after an infinitesimal amount of time,

$$\begin{aligned}\mathbf{p}_f &= \mathbf{p}_{rocket} + \mathbf{p}_{gas}, \\ &= (m - dm_g)(v + dv)\mathbf{j} + dm_g(v-u)\mathbf{j}.\end{aligned} \tag{1.10}$$

The gravitational effect on the rocket is $\mathbf{F} = -mg\mathbf{j}$. Seeing as the impulse equation is $\Delta p = F\Delta t$, we can equate the changes in momentum,

$$\begin{aligned}\mathbf{p_f} - \mathbf{p_i} &= -mgdt\mathbf{j}, \\ [(m - dm_g)(v+dv) + dm_g(v-u) - mv]\mathbf{j} &= -mgdt\mathbf{j}.\end{aligned} \tag{1.11}$$

From this equation we can see that the term $dvdm_g$ is the product of two infinitesimal numbers, meaning we can disregard it as being negligible in comparison with the other terms. The equation expands out to become

$$mdv - udm_g = -mgdt. \tag{1.12}$$

Next we must recognise the fact that any increase in ejected gas mass is equal to the decrease in rocket mass. So by using $dm_g = -dm$, we obtain

$$\begin{aligned} mdv + dmu &= -mgdt, \\ dv &= -u\frac{dm}{m} - gdt. \end{aligned} \tag{1.13}$$

Lastly, integration is used from the start to finish of the infinitesimal amount of time,

$$\begin{aligned} \int_{v_i}^{v_f} dv &= -u\int_{m_i}^{m} \frac{1}{m}dm - g\int_{t_i}^{t_f} dt, \\ \Delta v &= u \ln\frac{m_i}{m} - g\Delta t. \end{aligned} \tag{1.14}$$

The rocket equation tells us that an increase in the ratio $\frac{m_i}{m}$ will increase our change in speed, and it also tells us that decreasing the burn time of the fuel, Δv, leads to a greater (more efficient) change in velocity. These key takeaways relate back to our real life practicality, and will help us in our validation of science fiction scientific accuracy later on.

1.5 The Orbit Equation

The orbit equation is used frequently within orbital mechanics, and will be key when building our patched conic approximations. Any type of flight path that is affected by the gravity of a significant mass can be modelled by the orbital equation. This section will describe the different types of orbital motion and how they are described by the parameters, which will lay the foundations for use in building our flight paths. It will be based on 'The Orbit Equation' chapter of [3].

In order to arrive at the orbit equation we begin with equation 1.9. We would like to transform this equation into a scalar one, to be able to work with it analytically. This will become particularly useful when looking at Lambert's problem, and aids us with deriving the Lagrange coefficients; the key to interplanetary travel.

We will first take the cross product of 1.9 and the angular momentum, $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$,

$$\ddot{\mathbf{r}} \times \mathbf{h} = -(\frac{\mu}{r^3})\mathbf{r} \times \mathbf{h}. \tag{1.15}$$

In order to make this equation friendlier to integrate, we will pull a d/dt out. We obtain,

$$\frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \ddot{\mathbf{r}} \times \mathbf{h} + \dot{\mathbf{r}} \times \dot{\mathbf{h}}, \tag{1.16}$$

and here we can notice that $\dot{\mathbf{h}} = \mathbf{0}$ and so our equation simplifies to

$$\frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) = \ddot{\mathbf{r}} \times \mathbf{h} = -\left(\frac{\mu}{r^3}\right) \mathbf{r} \times \mathbf{h}. \tag{1.17}$$

It is sensible to adjust the right hand side so that it also has a d/dt pulled out, and the trick is to use the following vector algebra identity [5],

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}. \tag{1.18}$$

Before applying this formula, we note that since $\mathbf{r} \cdot \mathbf{r} = r^2$, we can derive another identity,

$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) &= \frac{d}{dt}\left(r^2\right), \\ 2\mathbf{r} \cdot \dot{\mathbf{r}} &= 2r\dot{r}, \\ \mathbf{r} \cdot \dot{\mathbf{r}} &= r\dot{r}. \end{aligned} \tag{1.19}$$

We then combine the equations 1.17, 1.18 and 1.19 as follows,

$$\begin{aligned} \frac{d}{dt}(\dot{\mathbf{r}} \times \mathbf{h}) &= -\left(\frac{\mu}{r^3}\right) \mathbf{r} \times \mathbf{h}, \\ &= -\frac{\mu}{r^3}(\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})), \\ &= -\frac{\mu}{r^3}((\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}}), \\ &= -\frac{\mu}{r^3}(r\dot{r}\mathbf{r} - r^2\dot{\mathbf{r}}), \\ &= \mu\left(\frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}\mathbf{r}}{r^2}\right), \\ &= \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right). \end{aligned} \tag{1.20}$$

By integrating and rearranging we are left with the following,

$$\dot{\mathbf{r}} \times \mathbf{h} - \mu\frac{\mathbf{r}}{r} = \mathbf{B}, \tag{1.21}$$

for the Laplace vector $\mathbf{B} \in \mathbb{R}^3$ which is independent of time [6]. This tells us that interestingly, the left hand side of (1.21) is conserved.

To finish up, we need to first note the following vector algebra identity,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}), \tag{1.22}$$

and then have our equation formatted as,

$$\dot{\mathbf{r}} \times \mathbf{h} = \mu\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right), \tag{1.23}$$

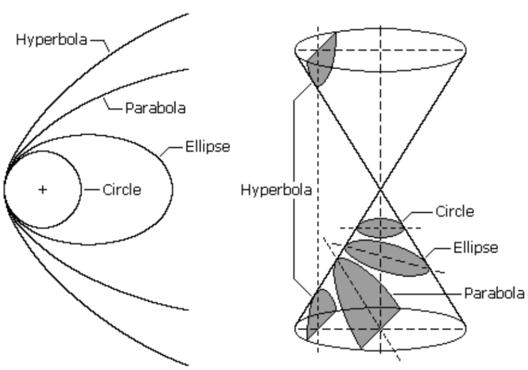
where $\mathbf{e} = \frac{\mathbf{B}}{\mu}$ is the eccentricity vector [3]. Then we take the scalar product of both sides with $\mathbf{r}$,

$$\begin{aligned} \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) &= \mu(\frac{\mathbf{r} \cdot \mathbf{r}}{r} + \mathbf{e} \cdot \mathbf{r}), \\ \mathbf{h} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) &= \mu(r + re\cos\nu), \\ \mathbf{h} \cdot \mathbf{h} &= \mu(r + re\cos\nu), \\ h^2 &= \mu\left(r + re\cos\nu\right), \end{aligned} \tag{1.24}$$

and a simple shuffle of the equation leaves us with the orbit equation,

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\nu}. \tag{1.25}$$

The shape of an orbit is determined by it's eccentricity, e. (1.1b) shows how the magnitude of eccentricity changes the type of orbit. As seen in (1.1a), these have different qualitative properties, as defined in [7]. If the plane cuts across one half-cone, it is an ellipse. A circle is an ellipse where the plane is parallel to the base of the cone. A parabola is the instance in which the plane is parallel to a line in the surface of the cone, and it only cuts across one half-cone. If the plane cuts both half-cones, this is a hyperbola. If we



(a)

Conic Section	Eccentricity *(e)*
circle	$e = 0$
ellipse	$0 < e < 1$
parabola	$e = 1$
hyperbola	$e > 1$

(b)

Figure 1.1: a) The conic sections [8] b) Qualitative meanings of eccentricity values

are to now think of (1.1) in terms of space travel and orbits rather than shapes, beginning with the ellipse, we can understand how this might look in space. For an object in orbit around another, nearly all of the time this will be an ellipse. However once $e = 1$, the orbit equation becomes

$$r = \frac{h^2}{\mu}\frac{1}{1 + \cos\nu}, \tag{1.26}$$

and when $\nu \to \pi$, $\cos\nu \to -1$. Therefore as $\nu \to \pi$, $r \to \infty$ [3]. And so, if we want to send a spacecraft from one orbit to another, we must send it on a hyperbolic trajectory.

This is because the hyperbolic trajectory has a non-zero velocity at infinite radius, unlike a parabolic trajectory.

To understand this, consider the total energy E of a spacecraft, $E = K + U$, where K is it's kinetic energy and U is it's gravitational potential energy. We have

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}. \tag{1.27}$$

If the spacecraft comes to a standstill at the furthest point from the central body, it has no kinetic energy. Then $E = -\frac{GMm}{r} \leq 0$, but seeing as E can only take values greater than or equal to 0, the spacecraft can only come to a standstill when $r \to \infty$ and $E = 0$. Therefore a parabolic trajectory has no velocity at the edge of the sphere of influence ($r = \infty$) and beyond. On the other hand, if the spacecraft does not come to a standstill, it will always have a positive value for E. If we rearrange the case where $E > 0$, we obtain $v > \sqrt{\frac{2GM}{r}}$. Therefore we see hyperbolic trajectories will be of use for exiting a gravitational pull with a positive velocity, as given $r = \infty$ we see that we still have a velocity v that is greater than 0.

1.6 Sphere of Influence

The method of patched conics is based upon the idea of spheres of influence. The sphere of influence describes the region of space around a celestial body where it's gravitational influence is stronger than any other nearby body [3]. Within a body's sphere of influence, the method of patched conics will take this as the singular gravitational field acting upon the spacecraft in question whilst neglecting other perturbing forces. As a result, each part of the method of patched conics is reduced to a two body problem. This greatly simplifies our trajectory approximations within each sphere. This is particularly useful in the case of the full trajectory described in The Martian, where there are eight notable conic sections we can split the journey into.

1.7 Orbital Anomalies

In orbital mechanics, there are three different angular parameters that we can consider in our calculations. They are depicted in figure (1.2).

- The mean anomaly, M, is the angle measured since periapsis of the hypothetical circular orbit that has assumed the actual orbit's periapsis as it's radius, and has the same period as the actual orbit. The vertex of the angle is the centre of the hypothetical circular orbit.
- The eccentric anomaly, E, assumes the same hypothetical orbit, and then marks the angle to the point on the circular orbit perpendicular to the actual position of the spacecraft. The vertex of the angle is the centre of the hypothetical circular orbit.

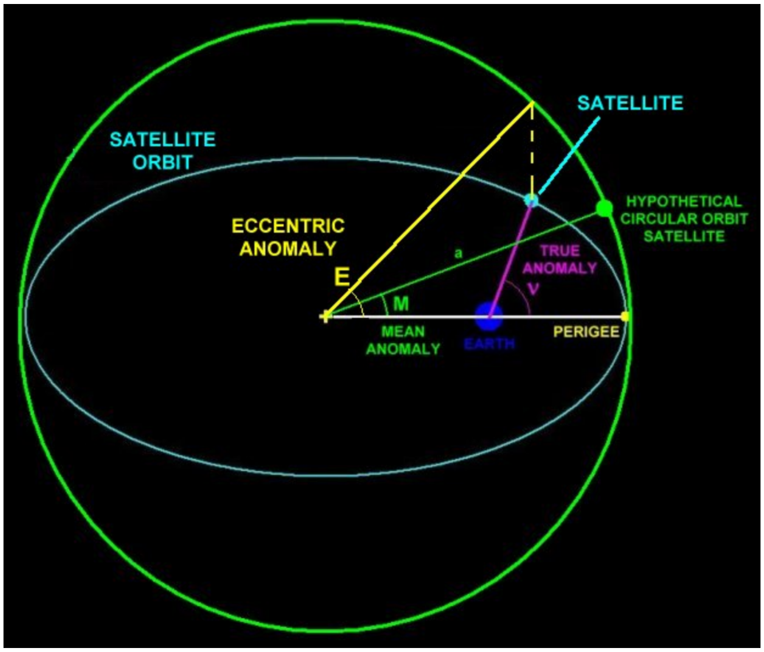


Figure 1.2: Orbital anomalies diagram [9].

- The true anomaly, ν, is the angle measured from periapsis to the position of the satellite. The vertex of the angle is the centre of mass of the body being orbited.

Chapter 2

Lift Off

2.1 Gravity Turn

So far, we have constructed an equation to describe the launch a spacecraft from a planet, and we have constructed an equation to describe the orbital flight pattern of planets and spacecraft. In this chapter, we will describe the middle point between these two equations.

Gravity turn is a method of launch trajectory which utilises the gravitational field of the celestial body it is departing from. During a gravity turn, the rocket starts with a near-vertical ascent and progressively leans towards the horizontal direction as gravity pulls it this way.

We can use this method to model what happens at three separate parts of The Martian; the MAV leaving Mars with everyone bar Watney at the beginning of the film, the Taiyang Shen resupplying the Hermes halfway through the film and the other MAV leaving Mars with just Watney at the end of the film.

2.2 Equations of Motion

In order to show gravity turn graphically, we will use the equations of motion presented by [10] which were adapted for computational use by [11]. The five parameters are: m - mass of the vehicle, H - altitude of the vehicle, X - distance downrange (horizontal) of the vehicle, V - velocity of the vehicle and γ - flight path angle of the vehicle. The equations of motion can be written as a series of first order differential equations, which we will form now (as laid out in [11]).

By resolving the velocity vector as shown in figure (2.1), we have the following equations,

$$\frac{dX}{dt} = V\cos\gamma, \tag{2.1}$$

$$\frac{dH}{dt} = V\sin\gamma. \tag{2.2}$$

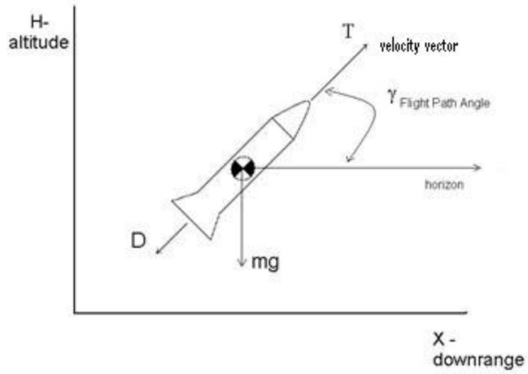


Figure 2.1: Gravity turn vectors [11].

Next we resolve our forces in the direction of the velocity, using Newton's second law: $\Sigma F = ma$. We have the following:

$$ma = \Sigma F, \tag{2.3}$$

$$= T - D - mg\sin\gamma + \frac{m\dot{X}^2}{(R_b + H)}\sin\gamma, \tag{2.4}$$

and using the fact that $a = \dot{V}$ this reduces further to

$$m\frac{dV}{dt} = T - D - \left(mg - \frac{m\dot{X}^2}{(R_b + H)}\right)\sin\gamma. \tag{2.5}$$

Here, R_b is the radius of the celestial body's surface. Similarly, we now resolve the forces parallel to the downrange distance to get

$$ma = \Sigma F, \tag{2.6}$$

$$= -mg\cos\gamma + \frac{m\dot{X}^2}{(R_b + H)}\cos\gamma. \tag{2.7}$$

In this direction we have $a = V\dot{\gamma}$ and so our equation becomes

$$mV\frac{d\gamma}{dt} = -\left(mg - \frac{m\dot{X}^2}{(R_b + H)}\right)\cos\gamma. \tag{2.8}$$

The change in mass is simply defined as

$$\dot{m} = \frac{\text{total mass of fuel}}{\text{burn time}}. \tag{2.9}$$

Then our final set of five differential equations is as follows:

$$\begin{aligned}
\frac{dX}{dt} &= V\cos\gamma, \\
\frac{dH}{dt} &= V\sin\gamma, \\
m\frac{dV}{dt} &= T - D - \left(mg - \frac{m\dot{X}^2}{(R_b+H)}\right)\sin\gamma, \\
mV\frac{d\gamma}{dt} &= -\left(mg - \frac{m\dot{X}^2}{(R_b+H)}\right)\cos\gamma, \\
\frac{dm}{dt} &= \frac{\text{total mass of fuel}}{\text{burn time}}.
\end{aligned} \tag{2.10}$$

2.3 Python Results

If we are to relate this back to our overarching problem, we can see that it would be desirable to match a gravity turn launch trajectory to a spacecraft in a circular orbit 155 - 21,000 miles above the surface of Mars [12], which would be where the Hermes is located. For initial conditions, we have limited information provided by the science fiction to describe the gravity turn situations. In the first scenario (first MAV launch), Rick Martinez tells us that the initial launch angle is 12.9 degrees, and in the third situation (second MAV launch) we are told that the MAV thrusts for 12 minutes and coasts for 52 minutes [2]. We might then use a range of possible initial conditions to better describe these journeys graphically.

As we have limited information regarding the MAV and it's launch from Mars, we will instead turn our attention to the launch of the Taiyang Shen which we can model significantly more accurately. Many missions have taken place in the past in order to resupply space stations, including the use of SpaceX's Falcon 9 by NASA to resupply the ISS (International Space Station) [13]. Seeing as the fictional Taiyang Shen serves the same purpose as the non-fictional Falcon 9, we will model a flight path of the Falcon 9 and assume that the Taiyang Shen might have a similar build.

2.3.1 Falcon 9 Model Trajectory

Falcon 9 is a reusable, two-stage rocket designed and manufactured by SpaceX for the reliable and safe transport of people and payloads into Earth orbit and beyond [14]. A two-stage rocket fires two impulses. Following the first impulse it will have used up all of it's stage 1 propellant, and will then eject the part of the rocket that houses the stage 1 propellant and thrusters. It will then coast for an amount of time without exerting any force before entering the second stage. Once the second stage thruster has ejected all of it's burnt fuel the part of the rocket that carried the stage 2 thruster and propellant is ejected, leaving the payload to continue flying towards it's destination.

Falcon 9 carries a payload called Dragon, and Dragon is capable of carrying up to 7 people and/or cargo in the spacecraft's pressurized section. In addition, Dragon can carry cargo in the spacecraft's unpressurised trunk, which can also accommodate secondary payloads [15].

The spacecraft has the following specifications [16][13]:

- Stage 1 empty mass of 25,600kg
- Stage 1 propellant mass of 395,700kg
- Stage 1 propellant burn time of 161 seconds
- Stage 1 thrust of 7,607kN at sea level and 8,227kN in vacuum
- Stage 2 empty mass of 3,900kg
- Stage 2 propellant mass of 92,670kg
- Stage 2 propellant burn time of 397 seconds
- Stage 2 thrust of 981kN in vacuum
- Carries up to 22,800kg of payload into low earth orbit

We will assume the following: a) that thrust inside Earth's atmosphere is constant across all altitudes (e.g. can use sea level thrust value while inside Earth's atmosphere), and b) that Falcon 9 has a small initial velocity to avoid the breakdown of the differential equations. For assumption a), we will use the boundary used by NASA mission control as the point of reentry and at which atmospheric drag becomes noticeable, which is an altitude of 122km [17].

Figure (2.2) shows us the resulting trajectory of the Falcon 9 (inspired by the MATLAB code presented by Callaway [11]), which would look similar to that of the Taiyang Shen. The graph shows a coast duration of 50 seconds after the first stage, and a coast of 100 seconds after the second stage. The results tell us that at the end of our first thrust stage Falcon 9 is at an altitude of 113.3km (<122km), and after our first coast stage it is at an altitude of 196.4km (>122km). Therefore according to our assumptions, in the first stage it is viable to use our value for sea level thrust (7,607kN) and disregard the first stage vacuum thrust value (8,227kN). Likewise, in the second stage we use the value for thrust in a vacuum of 981kN.

This code can be edited to increase the number of impulses to $n > 2$ to replicate the precise fuel burns a real rendezvous might undergo once the Falcon 9 payload gets close

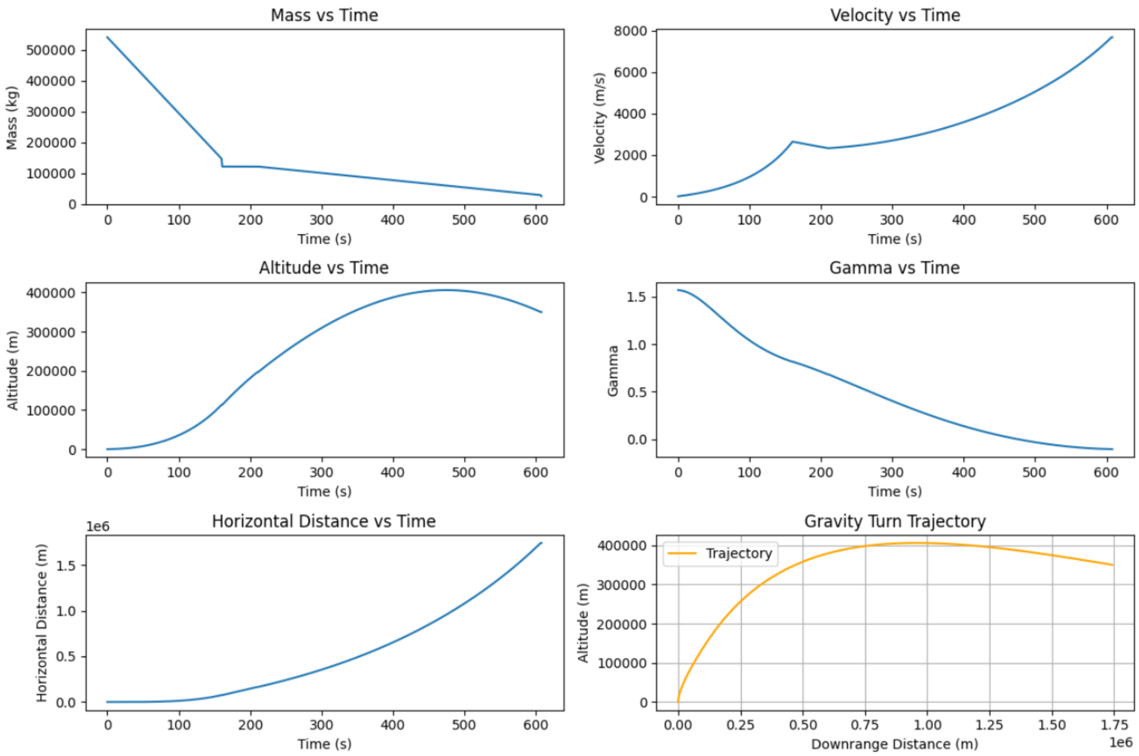


Figure 2.2: Gravity turn depiction, trajectory of Falcon 9.

to the ISS (or when the Taiyang Shen payload gets closer to the Hermes). Having created a model for a spacecraft of similar purpose to the Taiyang Shen, it is clear that we can reach the altitude required in order to resupply the Hermes. However in section 4.4.1 we will discover that the velocity that our Falcon 9 model has reached is not high enough. At the end of the second stage coast, the velocity of the Falcon 9 is $7.68kms^{-1}$, but we will see that the Hermes gravity assist takes place at speeds greater than $30kms^{-1}$. Applying this to our model, if we change only first stage thrust, then we see that the Falcon 9 begins to reach the required speeds with a first stage thrust that is around 64,000N (over 8 times more powerful). If the Taiyang Shen has over 72 stage 1 thrusters identical to those used by Falcon 9 (which uses 9 stage 1 thrusters [14]), then we see that this rendezvous becomes feasible.

Chapter 3

Lambert's Problem

3.1 Applications and Motivations

In orbital mechanics, Lambert's problem concerns the determination of an orbit given a starting position vector, a final position vector and the total flight time. It has important applications in space rendezvous and orbital determination. Whenever a spacecraft is sent to an orbit, planet or asteroid, Lambert's problem will need to be solved in order to calculate the required trajectory.

In The Martian, solving the Lambert problem would be a pivotal element in JPL (Jet Propulsion Laboratories) and NASA's calculations to help Watney leave the desolate surface of Mars. Rich Purnell along with the other scientists are tasked with finding a viable trajectory to bring Watney and the rest of the Hermes crew home, together.

3.2 Lagrange Coefficients

The Lagrange coefficients are incredibly powerful as they simplify difficult orbital equations. Scientists can use them to design space missions, understand behaviour of space craft and celestial bodies in different gravitational fields, and plan trajectories.

The position and velocity vectors given in components are

$$\begin{aligned} \mathbf{r} &= x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z, \\ \mathbf{v} &= \dot{x}\mathbf{i}_x + \dot{y}\mathbf{i}_y + \dot{z}\mathbf{i}_z, \end{aligned} \tag{3.1}$$

and on both of the bodies orbits, we have

$$\begin{aligned} \mathbf{r} &= F\mathbf{r_0} + G\mathbf{v_0}, \\ \mathbf{v} &= \dot{F}\mathbf{r_0} + \dot{G}\mathbf{v_0}, \end{aligned} \tag{3.2}$$

where $(\mathbf{r_0}, \mathbf{v_0})$ are the position and velocity vectors at time t_0, while $(\mathbf{r}, \mathbf{v})$ are the corresponding vectors at another time t [18]. Our aim is to rewrite (3.2) in terms of x, y, $\dot{x}$, $\dot{x}$, x_0, y_0, $\dot{x}_0$ and $\dot{y}_0$. Here the components at time $t = 0$ are defined as constant.

Now we will define our initial conditions,

$$\begin{aligned} \mathbf{r_0} &= x_0\hat{\mathbf{p}} + y_0\hat{\mathbf{q}}, \\ \mathbf{v_0} &= \dot{x}_0\hat{\mathbf{p}} + \dot{y}_0\hat{\mathbf{q}}, \end{aligned} \tag{3.3}$$

in which we have created unknowns $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}$. We proceed by finding the angular momentum,

$$\begin{aligned} \mathbf{h} &= \mathbf{r_0} \times \mathbf{v_0}, \\ &= \hat{\mathbf{w}}(x_0\dot{y}_0 - y_0\dot{x}_0), \end{aligned} \tag{3.4}$$

and by using the fact that $\hat{\mathbf{w}}$ is a unit vector we know that the right hand side of (3.4) is the magnitude of the angular momentum [3]. We can rewrite our unknowns, using the magnitude of angular momentum, like so

$$\begin{aligned} \hat{\mathbf{p}} &= \frac{\dot{y}_0}{h}\mathbf{r_0} - \frac{y_0}{h}\mathbf{v_0}, \\ \hat{\mathbf{q}} &= -\frac{\dot{x}_0}{h}\mathbf{r_0} - \frac{x_0}{h}\mathbf{v_0}, \end{aligned} \tag{3.5}$$

Substituting these unit vectors into the equations for general position and velocity, we obtain,

$$\begin{aligned} \mathbf{r} &= \frac{x\dot{y}_0 - y\dot{x}_0}{h}\mathbf{r_0} + \frac{-xy_0 + yx_0}{h}\mathbf{v_0}, \\ \mathbf{v} &= \frac{\dot{x}\dot{y}_0 - \dot{y}\dot{x}_0}{h}\mathbf{r_0} + \frac{-\dot{x}y_0 + \dot{y}x_0}{h}\mathbf{v_0}. \end{aligned} \tag{3.6}$$

We can then compare the coefficients of (3.6) and (3.2) in order to realise our lagrange coefficient equations in terms of initial and final position:

$$\begin{aligned} f &= \frac{x\dot{y}_0 - y\dot{x}_0}{h}, \\ \dot{f} &= \frac{\dot{x}\dot{y}_0 - \dot{y}\dot{x}_0}{h}, \\ g &= \frac{-xy_0 + yx_0}{h}, \\ \dot{g} &= \frac{-\dot{x}y_0 + \dot{y}x_0}{h}. \end{aligned} \tag{3.7}$$

Note these equations are formulated such that

$$f\dot{g} - g\dot{f} = 1, \tag{3.8}$$

which implies that given any three of these functions we can solve for the remaining one.

3.3 Lagrange Coefficients in Terms of the True Anomaly

Equation (3.7) lays the foundation to be able to formulate the Lambert problem with the universal variable. However an often overlooked step is the the one detailing the transition from (3.7) to the following equations in terms of the change in true anomaly ($\Delta\nu = \nu - \nu_0$),

$$\begin{aligned} f &= 1 - \frac{\mu r}{h^2}(1 - \cos\Delta\nu), \\ \dot{f} &= \frac{\mu}{h}\frac{1 - \cos\Delta\nu}{\sin\Delta\nu}\left[\frac{\mu}{h^2}(1 - \cos\Delta\nu) - \frac{1}{r_0} - \frac{1}{r}\right], \\ g &= \frac{r r_0}{h}\sin\Delta\nu, \\ \dot{g} &= 1 - \frac{\mu r_0}{h^2}(1 - \cos\Delta\nu). \end{aligned} \tag{3.9}$$

Before proceeding we should outline number of mathematical tools that will be used. This includes a useful format of the orbit equation (1.25),

$$e\cos\nu = \frac{h^2}{\mu r} - 1, \tag{3.10}$$

as well as the definitions for x, y, $\dot{x}$, $\dot{y}$ [19],

$$\begin{aligned} x &= r\cos\nu, \\ \dot{x} &= -\frac{\mu}{h}\sin\nu, \\ y &= r\sin\nu, \\ \dot{y} &= \frac{\mu}{h}(e + \cos\nu). \end{aligned} \tag{3.11}$$

The following trigonometric identities in terms of the true anomaly are crucial for the derivation:

$$\begin{aligned} \sin\Delta\nu &= \sin\nu\cos\nu_0 - \cos\nu\sin\nu_0, \\ \cos\Delta\nu &= \cos\nu\cos\nu_0 + \sin\nu\sin\nu_0. \end{aligned} \tag{3.12}$$

We will also use the fact that eccentricity, e, is identical for arbitrary points on an orbit. Mathematically,

$$\frac{h^2}{\mu r_a}\frac{1}{1 + e\cos\nu_a} = \frac{h^2}{\mu r_b}\frac{1}{1 + e\cos\nu_b}. \tag{3.13}$$

Finally, we must note that for the initial conditions ($t = 0$), we will denote all values with a subscript 0. For example,

$$\begin{aligned} y_0 &= r_0\sin\nu_0, \\ \dot{y}_0 &= \frac{\mu}{h}(e_0 + \cos\nu_0). \end{aligned} \tag{3.14}$$

This also means that $e = e_0$, however e is comprised from r and ν, and e_0 is comprised from r_0 and ν_0. Remember, all initial conditions are considered constants. We will later see how this is put in to practice when the Lambert problem is solved using python.

3.3.1 f Derivation

To find f in terms of the change in true anomaly, we want to show that

$$\frac{x\dot{y}_0 - y\dot{x}_0}{h} = 1 - \frac{\mu r}{h^2}(1 - \cos\Delta\nu). \tag{3.15}$$

We substitute (3.11) into the left hand side of the equation like so,

$$\begin{aligned} \frac{x\dot{y}_0 - y\dot{x}_0}{h} &= \frac{1}{h}\left[r\cos\nu\left(\frac{\mu}{h}(e_0 + \cos\nu_0)\right) - r\sin\nu\left(-\frac{mu}{h}\sin\nu_0\right)\right], \\ &= \frac{r\mu}{h^2}\cos\nu(e_0 + \cos\nu_0) + \frac{r\mu}{h^2}\sin\nu\sin\nu_0, \\ &= \frac{r\mu}{h^2}e\cos\nu + \frac{r\mu}{h^2}\cos\nu\cos\nu_0 + \frac{r\mu}{h^2}\sin\nu\sin\nu_0, \end{aligned} \tag{3.16}$$

where in the last line we have replaced e_0 with e for upcoming simplifications. Next we take the useful form of the orbit equation (3.10) and apply it to the first term, and we will group the second and third terms like so,

$$\frac{x\dot{y}_0 - y\dot{x}_0}{h} = \frac{r\mu}{h^2}\left(\frac{h^2}{\mu r} - 1\right) + \frac{r\mu}{h^2}(\cos\nu\cos\nu_0 + \sin\nu\sin\nu_0). \tag{3.17}$$

When we multiply out and apply the trigonometric identity (3.12), we are left with

$$\begin{aligned} \frac{x\dot{y}_0 - y\dot{x}_0}{h} &= 1 - \frac{\mu r}{h^2} + \frac{\mu r}{h^2}(\cos\Delta\nu), \\ &= 1 - \frac{\mu r}{h^2}(1 - \cos\Delta\nu), \end{aligned} \tag{3.18}$$

and our derivation of the first Lagrange coefficient is complete.

3.3.2 $\dot{\mathbf{f}}$ Derivation

The derivation of $\dot{f}$ is by far the longest of the four, and as a result requires slightly more critical thinking. We want to show that

$$\frac{\dot{x}\dot{y}_0 - \dot{y}\dot{x}_0}{h} = \frac{\mu}{h}\frac{1 - \cos\Delta\nu}{\sin\Delta\nu}\left[\frac{\mu}{h^2}(1 - \cos\Delta\nu) - \frac{1}{r_0} - \frac{1}{r}\right]. \tag{3.19}$$

We begin by substituting in our equations (3.11) directly,

$$\begin{aligned}\dot{f} &= \frac{1}{h}\left(-\frac{\mu}{h}\sin\nu\left(\frac{\mu}{h}(e_0+\cos\nu_0)\right)+\frac{\mu}{h}(e+\cos\nu)\frac{\mu}{h}\sin\nu_0\right),\\ &= \frac{\mu^2}{h^3}(-\sin\nu(e_0+\cos\nu_0)+(e+\cos\nu)\sin\nu_0),\\ &= \frac{\mu^2}{h^3}(-e_0\sin\nu+e\sin\nu_0+\cos\nu\sin\nu_0-\sin\nu\cos\nu_0),\\ &= \frac{\mu^2}{h^3}(-e_0\sin\nu+e\sin\nu_0-\sin\Delta\nu),\end{aligned} \tag{3.20}$$

where in the last line we have used our $\sin\Delta\nu$ identity for the first time.

By having a look at our target equation, we see that it might beneficial to remove a factor of $\frac{1-\cos\Delta\nu}{\sin\Delta\nu}$ from what we currently have. First, we will explore the consequence this has on the third term, $-\sin\Delta\nu$:

$$\begin{aligned}-\sin\Delta\nu &= \frac{1-\cos\Delta\nu}{\sin\Delta\nu}\frac{\sin\Delta\nu}{1-\cos\Delta\nu}\cdot-\sin\Delta\nu.\\ &= \frac{1-\cos\Delta\nu}{\sin\Delta\nu}\left[\frac{-\sin^2\Delta\nu}{1-\cos\Delta\nu}\right],\\ &= \frac{1-\cos\Delta\nu}{\sin\Delta\nu}\left[-\frac{1-\cos^2\Delta\nu}{1-\cos\Delta\nu}\right].\end{aligned} \tag{3.21}$$

Now we can spot that the numerator can be expanded using the difference of squares principle:

$$\begin{aligned}-\sin\Delta\nu &= \frac{1-\cos\Delta\nu}{\sin\Delta\nu}\left[-\frac{(1+\cos\Delta\nu)(1-\cos\Delta\nu)}{1-\cos\Delta\nu}\right],\\ &= \frac{1-\cos\Delta\nu}{\sin\Delta\nu}[-1-\cos\Delta\nu].\end{aligned} \tag{3.22}$$

Switching our focus back out to the entire derivation, we will substitute in the term we have just found for $-\sin\Delta\nu$:

$$\begin{aligned}\dot{f} &= \frac{\mu^2}{h^3}(-e_0\sin\nu+e\sin\nu_0-\sin\Delta\nu),\\ &= \frac{\mu^2}{h^3}\left(\frac{1-\cos\Delta\nu}{\sin\Delta\nu}[-1-\cos\Delta\nu]-e_0\sin\nu+e\sin\nu_0\right),\\ &= \frac{\mu}{h}\left[\frac{1-\cos\Delta\nu}{\sin\Delta\nu}\right]\left[\frac{\mu}{h^2}(-1-\cos\Delta\nu)+\frac{\mu}{h^2}\frac{\sin\Delta\nu}{1-\cos\Delta\nu}(-e_0\sin\nu+e\sin\nu_0)\right].\end{aligned} \tag{3.23}$$

The first term of this equation is close to what we require, and we can rearrange it slightly into the format we require,

$$\dot{f} = \frac{\mu}{h}\left[\frac{1-\cos\Delta\nu}{\sin\Delta\nu}\right]\left[\frac{\mu}{h^2}(1-\cos\Delta\nu)-\frac{2\mu}{h^2}+\frac{\mu}{h^2}\frac{\sin\Delta\nu}{1-\cos\Delta\nu}(-e_0\sin\nu+e\sin\nu_0)\right]. \tag{3.24}$$

We have now successfully managed to obtain the correct first term of $\dot{f}$. Unfortunately this is as far as is reasonable to show. Beyond this point, excessive amounts of algebra are required to show that the following identity holds:

$$-\frac{2\mu}{h^2}+\frac{\mu}{h^2}\frac{\sin\Delta\nu}{1-\cos\Delta\nu}(-e_0\sin\nu+e\sin\nu_0)=-\frac{1}{r}-\frac{1}{r_0}. \tag{3.25}$$

For mathematical efficiency, we assume this identity to be true. This completes our derivation, as incorporating this identity gives us that

$$\dot{f}=\frac{\mu}{h}\frac{1-\cos\Delta\nu}{\sin\Delta\nu}\left[\frac{\mu}{h^2}(1-\cos\Delta\nu)-\frac{1}{r_0}-\frac{1}{r}\right]. \tag{3.26}$$

3.3.3 g Derivation

To show the third Lagrange coefficient, we want to show that

$$\frac{-xy_0+yx_0}{h}=\frac{rr_0}{h}\sin\Delta\nu, \tag{3.27}$$

and this will be the most simple conclusion to come to out of the four derivations. We have

$$\begin{aligned}\frac{-xy_0+yx_0}{h}&=-r\cos\nu\cdot r_0\sin\nu_0+r\sin\nu\cdot r_0\cos\nu_0,\\&=-rr_0\cos\nu\sin\nu_0+rr_0\sin\nu\cos\nu_0,\\&=rr_0(\sin\nu\cos\nu_0-\cos\nu\sin\nu_0),\end{aligned} \tag{3.28}$$

and the derivation is complete once we use our trigonometric identity (3.12),

$$\frac{-xy_0+yx_0}{h}=rr_0\sin\Delta\nu. \tag{3.29}$$

3.3.4 ġ Derivation

The derivation of $\dot{g}$ is incredibly similar to that of f. We know that to prove $\dot{g}$, we want to show

$$\frac{-\dot{x}y_0+\dot{y}x_0}{h}=1-\frac{\mu r_0}{h^2}(1-\cos\Delta\nu). \tag{3.30}$$

As before, we substitute in the corresponding values from (3.11) to get

$$\begin{aligned}\frac{-\dot{x}y_0+\dot{y}x_0}{h}&=\frac{1}{h}\left(\frac{\mu}{h}\sin\nu\cdot r_0\sin\nu_0+\frac{\mu}{h}(e+\cos\nu)\cdot r_0\cos\nu_0\right),\\&=\frac{\mu r_0}{h^2}\sin\nu\sin\nu_0+\frac{\mu r_0}{h^2}e\cos\nu_0+\frac{\mu r_0}{h^2}\cos\nu\cos\nu_0.\end{aligned} \tag{3.31}$$

Now we will use the same eccentricity substitution concept as in the derivation of f. We will swap e for e_0 and we will use the trigonometric identity (3.12) like so,

$$\begin{aligned}\frac{-\dot{x}y_0 + \dot{y}x_0}{h} &= \frac{\mu r_0}{h^2}e_0\cos\nu_0 + \frac{\mu r_0}{h^2}(\sin\nu\sin\nu_0 + \cos\nu\cos\nu_0),\\ &= \frac{\mu r_0}{h^2}\left(\frac{h^2}{\mu r_0} - 1\right) + \frac{\mu r_0}{h^2}\cos\Delta\nu,\\ &= 1 - \frac{\mu r_0}{h^2} + \frac{\mu r_0}{h^2}\cos\Delta\nu,\\ &= 1 - \frac{\mu r_0}{h^2}(1 - \cos\Delta\nu).\end{aligned} \tag{3.32}$$

3.4 Universal Formulation of the Two Body Problem

Our goal is now to unify the formulations for various two-body orbits in a format that can work for any eccentricity. This is because the efficient solving of the Lambert problem utilises a universal variable [18]. This will include two methodologies; a time transformation formula, and a family of transcendental functions [18].

3.4.1 Time Transformation Formula

We have four types of orbits: circular, elliptic, parabolic and hyperbolic. For each, the rate of change of true anomaly differs, and so we require a time transformation formula to account for this. The formula will allow for the conversion of time into an angular parameter.

We will introduce a new independent variable, χ, which describes elliptic, parabolic and hyperbolic orbits respectively like so:

$$\chi = \begin{cases}\sqrt{a}(E - E_0) & \text{if } e < 1,\\ \sqrt{a}\left(\tan\frac{1}{2}\nu - \tan\frac{1}{2}\nu_0\right) & \text{if } e = 1,\\ \sqrt{-a}(H - H_0) & \text{if } e > 1.\end{cases}$$

E is the elliptic eccentric anomaly, ν is the true anomaly and H is the hyperbolic anomaly. A subscript of 0 represents the anomaly at time t_0. χ is to be regarded as a generalised anomaly.

We will use Sundman's time transformation, which is defined by:

$$\sqrt{\mu}\frac{dt}{d\chi} = r. \tag{3.33}$$

When χ is used as the independent variable instead of the time t, then the nonlinear equations of motion can be converted into linear constant-coefficient differential equations [19]. This will allow us to solve Lambert's problem numerically.

3.4.2 Transcendental Function Family

Traditional methods for solving the Lambert problem utilise iterative techniques. However these methods can converge slowly for some trickier orbits, and so we will introduce a family of transcendental functions to aid computational efficiency [18]. This provides an alternative approach with improved capabilities.

We will consider the following family of transcendental functions known as Stumpff functions,

$$C_n(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{(2k+n)!}, \tag{3.34}$$

for integer values of n. In particular, we are interested in C_2 and C_3 which will be used when numerically solving the Lambert problem. We have [20]

$$\begin{aligned} C_2(z) &= \frac{1}{2!} - \frac{z}{4!} + \frac{z^2}{6!} - \dots, \\ &= \begin{cases} \frac{1-\cos\sqrt{z}}{z} & \text{if } z > 0, \\ \frac{1}{4} & \text{if } z = 0, \\ \frac{\cosh\sqrt{-z}-1}{-z} & \text{if } z < 0. \end{cases} \end{aligned} \tag{3.35}$$

in the case of $n = 2$, and when $n = 3$,

$$\begin{aligned} C_2(z) &= \frac{1}{3!} - \frac{z}{5!} + \frac{z^2}{7!} - \dots, \\ &= \begin{cases} \frac{\sqrt{z}-\sin\sqrt{z}}{(\sqrt{z})^3} & \text{if } z > 0, \\ \frac{1}{6} & \text{if } z = 0, \\ \frac{\sinh\sqrt{-z}-\sqrt{-z}}{(\sqrt{-z})^3} & \text{if } z < 0. \end{cases} \end{aligned} \tag{3.36}$$

Next, note that the total energy constant is $-\mu/2a$ [3], where a is the semi-major axis of the orbit. We will define (for convenience) α to be the following,

$$\alpha = \alpha_0 = \frac{1}{a} = \frac{2}{r_0} - \frac{v_0^2}{\mu}, \tag{3.37}$$

where v_0 is the magnitude of the velocity vector $\mathbf{v_0}$. We can then summarise the Lagrange coefficients (3.7) as the following in terms of χ, α, C_2 and C_3 amongst other already defined variables [20],

$$
\begin{aligned}
f &= 1 - \frac{\chi^2}{r_0} C_2\left(\alpha_0 \chi^2\right), \\
g &= t - t_0 - \frac{\chi^3}{\sqrt{\mu}} C_3\left(\alpha_0 \chi^2\right), \\
\dot{f} &= \frac{\sqrt{\mu}}{r r_0} \chi\left[\alpha_0 \chi^2 C_3\left(\alpha_0 \chi^2\right) - 1\right], \\
\dot{g} &= 1 - \frac{\chi^2}{r} C_2\left(\alpha_0 \chi^2\right).
\end{aligned} \tag{3.38}
$$

This set of equations is valid for any type of conic motion.

3.5 Universal Lambert Problem

In order to solve Lambert's problem, we must use a numerical method; namely a bisection algorithm. Bisection algorithms are used to find the roots of a function. The bisection algorithm in the context of the Lambert problem will be used to iteratively narrow down the possible parameter range until we find the output of a final and initial velocity, to a given tolerance. The algorithm is perfect due to it's guaranteed convergence given a reasonable starting range.

3.5.1 Algorithm Setup

From this point, we will take C_2 to mean $C_2(\alpha_0\chi^2)$ and likewise $C_3 = C_3(\alpha_0\chi^2)$. We are aiming to set up some equations using f, $\dot{f}$, g and $\dot{g}$. If we equate our expressions for f from (3.7) and (3.38), we have

$$
\begin{aligned}
1 - \frac{\mu r}{h^2}[1 - \cos\Delta\nu] &= 1 - \frac{\chi^2}{r_0} C_2, \\
\frac{\chi^2 C_2}{r_0} &= \frac{\mu r}{h^2}[1 - \cos\Delta\nu], \\
\chi^2 &= \frac{r r_0 \mu}{h^2 C_2}[1 - \cos\Delta\nu], \\
\chi &= \sqrt{\frac{r r_0 \mu}{h^2 C_2}[1 - \cos\Delta\nu]}.
\end{aligned} \tag{3.39}
$$

Next, we will introduce the simplification that $p = h^2/mu$. Then we substitute (3.39) into $\dot{f}$ from (3.38), to obtain

$$\begin{aligned}\sqrt{\frac{\mu}{p}}\left[\frac{1-\cos\Delta\nu}{\sin\Delta\nu}\right]\left[\frac{1}{p}(1-\cos\Delta\nu)-\frac{1}{r_0}-\frac{1}{r}\right] &= \frac{\sqrt{\mu}}{rr_0}\sqrt{\frac{rr_0}{pC_2}(1-\cos\Delta\nu)}\left[\alpha_0\chi^2C_3-1\right],\\ \left[\frac{1-\cos\Delta\nu}{\sin\Delta\nu}\right]\left[\frac{1}{p}(1-\cos\Delta\nu)-\frac{1}{r_0}-\frac{1}{r}\right] &= \frac{1}{rr_0}\sqrt{\frac{rr_0}{C_2}(1-\cos\Delta\nu)}\left[\alpha_0\chi^2C_3-1\right].\end{aligned} \tag{3.40}$$

The next stage of simplification involves multiplying both sides by $\frac{rr_0\sin\Delta\nu}{1-\cos\Delta\nu}$ [20] and rearranging to leave

$$\frac{rr_0}{p}(1-\cos\Delta\nu) = r_0 + r - \sin\Delta\nu\sqrt{\frac{rr_0}{1-\cos\Delta\nu}}\left(\frac{1-\alpha_0\chi^2C_3}{\sqrt{C_2}}\right). \tag{3.41}$$

This can also be written

$$B = r_0 + r + \frac{A(\psi C_3 - 1)}{\sqrt{C_2}}, \tag{3.42}$$

where we have

$$\begin{aligned}A &= \sin\Delta\nu\sqrt{\frac{rr_0}{1-\cos\Delta\nu}},\\ B &= \frac{rr_0}{p}(1-\cos\Delta\nu),\\ \psi &= \alpha_0\chi^2.\end{aligned} \tag{3.43}$$

Then to round up the setup of the algorithm, the unknown velocity vectors $\mathbf{v}$ and $\mathbf{v_0}$ are found [18] using

$$\begin{aligned}\mathbf{v_0} &= \frac{1}{g}(\mathbf{r} - f\mathbf{r_0}),\\ \mathbf{v} &= \frac{1}{g}(\dot{g}\mathbf{r} - \mathbf{r_0}),\end{aligned} \tag{3.44}$$

where our equations (3.7) can be written using (3.43),

$$\begin{aligned}f &= 1 - \frac{B}{r_0},\\ g &= A\sqrt{\frac{B}{\mu}},\\ \dot{g} &= 1 - \frac{B}{r}.\end{aligned} \tag{3.45}$$

3.5.2 Bisection Algorithm

Bate, Mueller, and White [7] introduced a Newton iterative scheme for finding the parameter ψ, which is effective for most problems but struggles with converging on challenging hyperbolic orbits. To address this issue, Vallado and McClain [21] proposed a bisection technique that performs well across all orbit types, albeit approximately 5% slower. In this approach, the objective is to determine the value of ψ corresponding to a given change in time (Δt). This is achieved by establishing bounds for the correct value of ψ and initially selecting a value halfway between these bounds. Through successive iterations, the upper and lower bounds are refined until the interval for ψ is sufficiently narrow to pinpoint the correct value (within specified tolerance). We will use computational algorithm 3 as outlined by Sharaf, Saad and Nouh [18], which has output $\mathbf{v_0}$ and $\mathbf{v}$. We have the following inputs:

- $\mathbf{r_o} = (x_0, y_0, z_0)$
- $\mathbf{r} = (x, y, z)$
- $\Delta t = t - t_0$
- ψ_0 (initial value), ψ_U (upper value), ψ_L (lower value)
- $t_m (t_m = 1$ for short way transfers, $t_m = -1$ for long way transfers)
- M (large positive integer for number of iterations)
- Tol (specified tolerance)
- μ

We must also note that the spacecraft can take the long way round the celestial body it is orbiting ($\Delta\nu > \pi$), or the short way ($\Delta\nu < \pi$). Then the solution to a given Lambert problem is given by the bisection algorithm (1).

3.6 Lambert Problem Applications

The Lambert problem is a fundamental tool in mission design, allowing for precise orbit determination. Utilizing Lambert's problems allows mission designers to optimize trajectories, minimize mission duration and achieve any other specific mission objectives.

In order to get an idea of how mission planners use Lambert's problem, we will use the ephemeris data supplied by the Jet Propulsion Laboratory [22]. We will use the concepts introduced in the 'Orbital Mechanics with Python' series by Alfonso Gonzalez [23]. We select an initial planet and a target planet, as well as a launch date and an arrival date. The ephemeris data contains the positions of the planets in the solar system from 2000 to

Algorithm 1 Universal Lambert Problem

1: **procedure** BISECTION($\mathbf{r_0}, \mathbf{r}, \psi_0, \psi_U, \psi_L, t_m, M, Tol, \mu$)
2: $\quad r_0 = \mod \mathbf{r_0}$
3: $\quad r = \mod \mathbf{r}$
4: $\quad \gamma = (\mathbf{r_0} \cdot \mathbf{r})/(r * r_0)$
5: $\quad \beta = t_m \sqrt{(1 - \gamma^2)}$
6: $\quad A = t_m \sqrt{r r_0 (1 + \gamma)}$
7: $\quad$ **if** $A = 0$ **then**
8: $\qquad$ we cannot calculate orbit; go to step 11
9: $\quad$ **end if**
10: $\quad$ **for all** $i \in [1, ..., M]$ **do**
11: $\qquad$ compute $C_2(\psi)$ and $C_3(\psi)$
12: $\qquad B = r_0 + r + \frac{1}{C_2}[A(\psi C_2 - 1)]$
13: $\qquad$ **if** $A > 0$ and $B < 0$ **then**
14: $\qquad\quad$ readjust ψ_L until $B > 0$
15: $\qquad$ **end if**
16: $\qquad \chi = \sqrt{\frac{B}{C_2}}$
17: $\qquad \Delta\widetilde{t} = \frac{1}{\sqrt{\mu}}\left(\chi^3 C_3 + A\sqrt{B}\right)$
18: $\qquad$ **if** $|\Delta t - \Delta\widetilde{t}| < Tol$ **then**
19: $\qquad\quad$ go to step
20: $\qquad$ **end if**
21: $\qquad$ **if** $\Delta\widetilde{t} \leq \Delta t$ **then**
22: $\qquad\quad \psi_L = \psi$
23: $\qquad\quad \psi_1 = \frac{1}{2}(\psi_U + \psi_L)$
24: $\qquad\quad \psi_0 = \psi_1$
25: $\qquad\quad$ End i
26: $\qquad$ **end if**
27: $\qquad \psi_U = \psi$
28: $\quad$ **end for**
29: $\quad$ compute f, g, $\dot{g}$ from (3.45)
30: $\quad \dot{x}_0 = \frac{1}{g}(x - f x_0)$, $\dot{y}_0 = \frac{1}{g}(y - f y_0)$, $\dot{z}_0 = \frac{1}{g}(z - f z_0)$
31: $\quad \dot{x} = \frac{1}{g}(\dot{g}x - x_0)$, $\dot{y} = \frac{1}{g}(\dot{g}y - y_0)$, $\dot{z} = \frac{1}{g}(\dot{g}z - z_0)$
32: **end procedure**

2050, meaning that with an input of initial planet, target planet, launch date and arrival date we can extract the data necessary to solve Lambert's problem; $\mathbf{r}$, $\mathbf{r_0}$, t and t_0. By coding algorithm (1), we are then supplied with our output values $\mathbf{v}$ and $\mathbf{v_0}$. We then use a differential equation solver on python to solve for the position of the spacecraft at each time $T \in [t_0, t]$ for which we also have the position of the initial and target planet (extracted from ephemeris data). We can choose any small enough time step to give us a smooth plot.

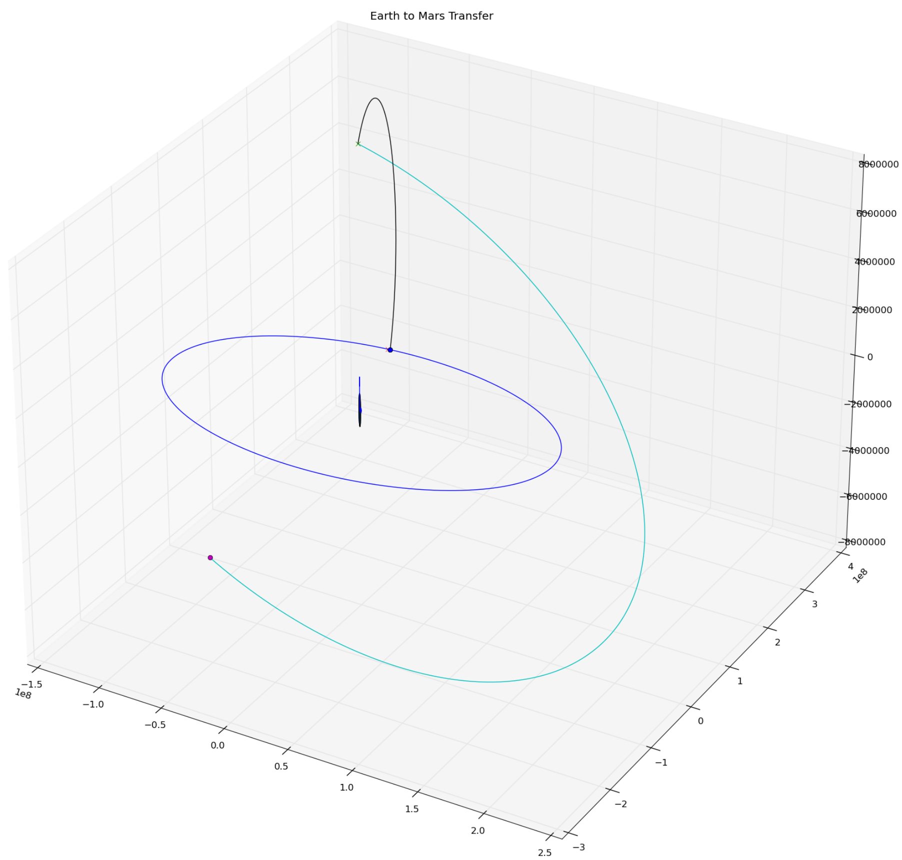


Figure 3.1: Earth to Mars transfer, 01/01/2024 - 01/01/2025.

Figure (3.1) is a trajectory given a departure date of January 1st 2024 and an arrival date of January 1st 2025. The blue and magenta dots show the initial points of Earth and Mars respectively, and in this case $\mathbf{r_0}$ is the position of Earth on January 1st 2024 (purple dot). The black line shows the trajectory the spacecraft takes under the influence of the Sun's gravity (centre), given $\mathbf{v_0}$, where $\mathbf{v_0}$ has been propagated from January 1st 2024 to January 1st 2025.

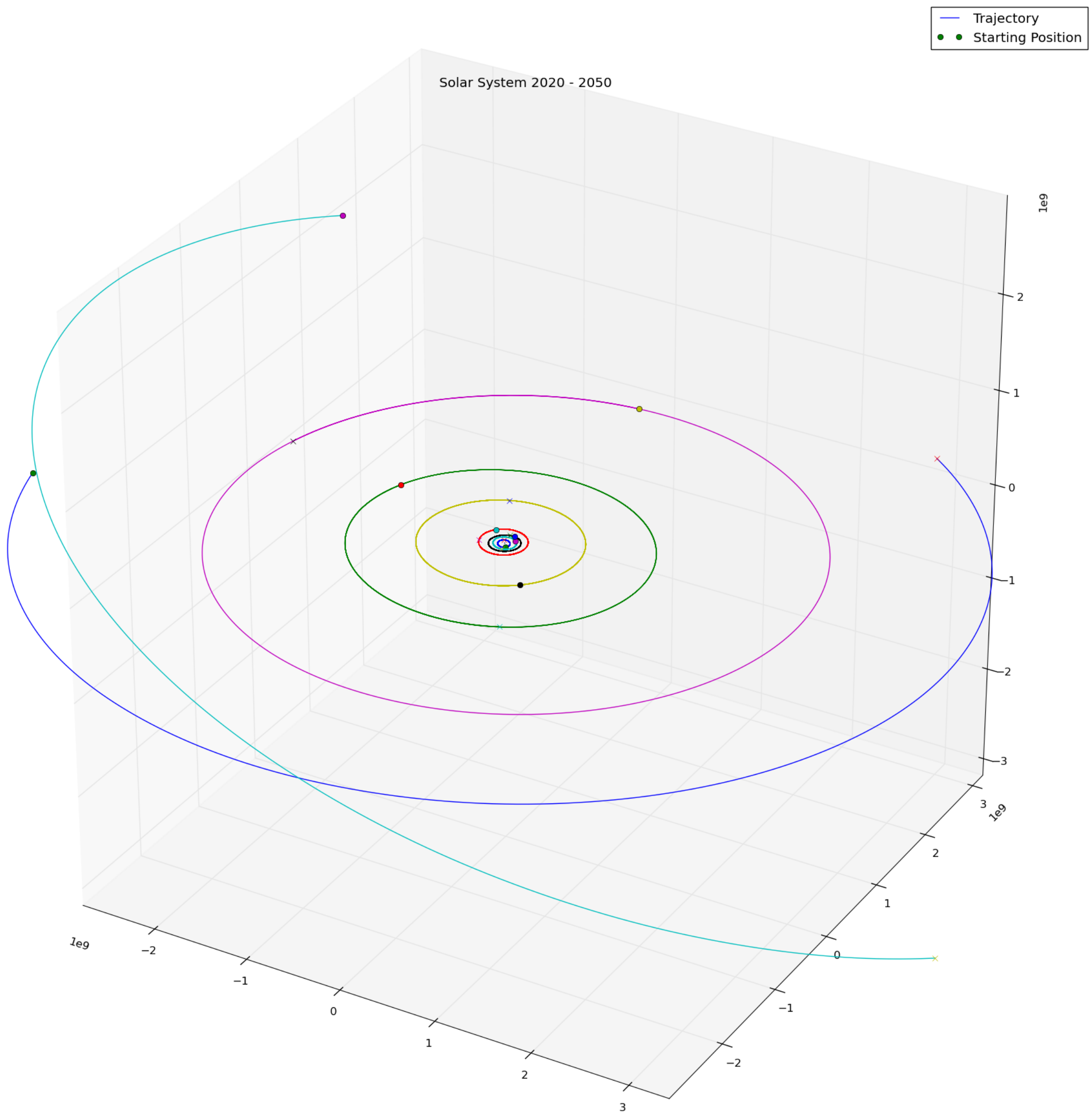


Figure 3.2: Solar system planet positions (including Pluto), 2000 - 2050 [22].

3.6.1 Building a Trajectory

The Martian film gives us sufficient information to look at the application of the Lambert problem. The timeline is as follows:

- Sol 18 - Hermes leaves Mars, without Watney
- Sol 227 - Hermes performs gravity assist around Earth, picking up supplies from Taiyang Shen
- Sol 561 - Hermes performs slingshot around Mars, picking up Watney
- Sol 772 - Hermes arrives back on Earth

We can look into greater detail at the feasibility of these journeys by picking an arbitrary date for sol 1. We already know the time taken for the transfer, so by picking an arbitrary date we take the ephemeris data provided by real world NASA in order to check how the journey parameters would look.

A reasonable date to pick for sol 1 is the date that NASA are hopeful to send a real-life MAV to Mars, which is June 2028 [24]. If successful, the Mars Sample Return Program will be the first successful launch of a rocket from the surface of a planet other than Earth. NASA, JPL and ESA (European Space Agency) are working in conjunction to bring samples currently being collected by the Mars Perseverance Rover back to Earth in the early 2030's.

The Martian gives us a timeline to work with, but we cannot forget to convert the Mars solar days to our 24-hour clock convention. A Mars solar day has a mean period of 24 hours 39 minutes 35.244 seconds, and is referred to as a 'sol' in order to distinguish this from the roughly 3% shorter solar day on Earth [25]. Now, we can pick a date for sol 18 in order to convert the supplied timeline into four separate conventional epochs (dates). By picking the Hermes leave time as the 1st of June 2028, our converted timeline is:

- Launch from Mars - 1st June 2028, 00:00
- Earth flyby - 1st January 2029, 17:54
- Mars flyby - 10th December 2029, 22:16
- Earth arrival - 15th July 2030, 17:29

Thanks to our ephemeris data [22] and each of the four epochs given in our converted timeline, we now have the tools to build a full trajectory. We will call 'Leg i' the journey from the position at epoch i to the position at epoch $i+1$. Then, for Leg i we have the following input data; $\mathbf{r_i}$, $\mathbf{r_{i+1}}$, t_i and t_{i+1}. This allows us to solve three consecutive Lambert problems, with $i \in \{1, 2, 3\}$. We then plot the resulting trajectories in figure (3.3), and we have created a candidate trajectory for the Hermes in The Martian film.

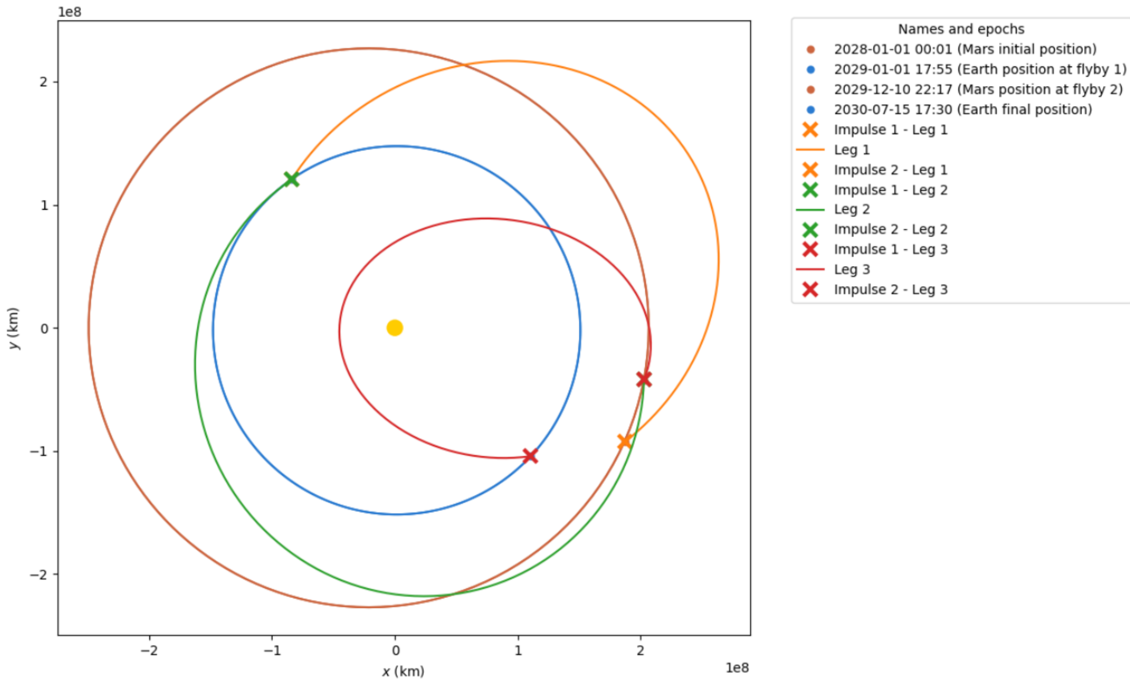


Figure 3.3: Hermes trajectory, with launch date 1st June 2028.

To comment initially on figure (3.3) it looks as though we have attained a reasonable trajectory. There are no velocity changes depicted at each of the flybys where the angle of flight is changed drastically, which suggests obtainable results. The ability to obtain desired gravity assist results will be explored in the next chapter.

To understand why a drastic angle change is more difficult to obtain, consider two two-dimensional velocity vectors both with speed $1ms^{-1}$ (unit circle). Let $\mathbf{v_1} = (1, 0)$ and $\mathbf{v_2} = (\mathbf{x}, \mathbf{y})$, where $\sqrt{x^2 + y^2} = 1$. To get from $\mathbf{v_1}$ to $\mathbf{v_2}$ requires $\boldsymbol{\Delta}\mathbf{v}$, such that $\mathbf{v_1} + \boldsymbol{\Delta}\mathbf{v} = \mathbf{v_2}$. The largest possible value for $|\boldsymbol{\Delta}\mathbf{v}|$ is 2, when $\mathbf{v_2} = (-1, 0)$. This coincides with the largest possible angle change of $180°$.

3.6.2 Porkchop Plots

To fully utilize Lambert's problem in a practical setting, we must explore combinations of launch dates and arrival dates. This involves analyzing the trade-offs between various launch windows and arrival opportunities to optimize mission trajectories for space exploration. One effective tool for visualizing these trade-offs is through the use of porkchop plots. Porkchop plots, also known as Lambert targeter plots, are graphical representations that display contours of key mission parameters.

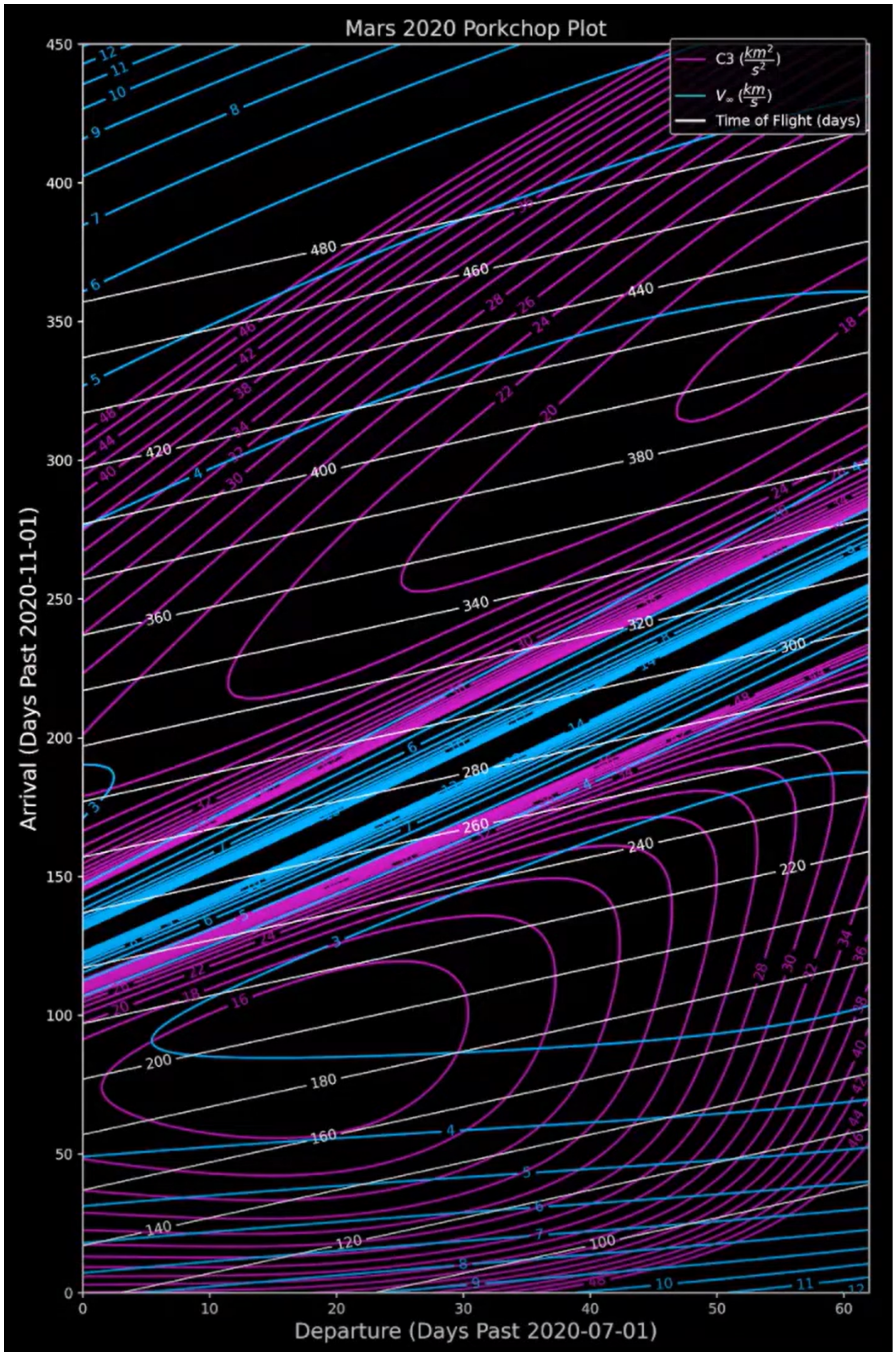


Figure 3.4: Porkchop plot for the Mars 2020 mission [26], by Alfonso Gonzalez [23].

C_3 is the characteristic energy of an interplanetary trajectory. It is the energy of the spacecraft during departure calculated using the velocity of the spacecraft from the reference frame of the body it is leaving [27]. Formally, the magnitude of the difference in velocity required by the spacecraft at departure and the velocity of the planet it is leaving, squared. v_∞ is the orbital velocity of arrival. It is the velocity of the spacecraft during arrival from the reference frame of the body it is arriving at [27]. Mathematically, the magnitude of the difference in velocity of the spacecraft at arrival and the velocity of the planet it is arriving at. C_3 and v_∞ are related with the following equation,

$$C_3 = v_\infty^2. \tag{3.46}$$

By examining the results of the pork-chop plots given by a launch window and an arrival window, we can look at these values to find out at what times we can feasibly launch a spacecraft, and at what particular dates the most economical interplanetary transfer is possible (least C_3 required at departure, smallest v_∞ at arrival).

For example, we can look at figure (3.4) to find the optimal launch and arrival window for the given dates. The smallest possible characteristic energy (pink contours) on this graph is $16km^2s^{-2}$ (bottom left), meaning that we would need to input $16km^2s^{-2}$ excess specific energy to a rocket to escape Earth at any point on this contour. The smallest possible V_∞ (blue contours) on this graph is $3kms^{-1}$ (bottom half), meaning that we would need to slow our spacecraft down by $3kms^{-1}$ to land on Mars at any point on this contour. Looking at the intersection of these contours provides us with an approximate departure date range of 10th July 2020 to 25th July 2020, and an approximate arrival date range of 1st February 2021 to 10th February 2021. This is precisely the thinking that mission designers undergo, as if we cross reference these dates with the Mission Concept: Mars 2020 document published in 2013 by NASA [26], we can see that the planned launch was July/August 2020 from Cape Canaveral Air Force Station and that the planned arrival was February 2021 at a yet to be determined site.

Porkchop Plot Application

We will use a porkchop plot to explore further the feasibility of the date we picked for launch in the previous subsection, 1st June 2028. Looking at figure (3.5) we can see that there are no ΔV contours where 1st June 2028 (x-axis) crosses 1st January 2029 (y-axis). This means that the total ΔV required to make this interplanetary transfer is greater than $50kms^{-1}$, where we have assumed a ΔV greater than this is too large to consider sensible.

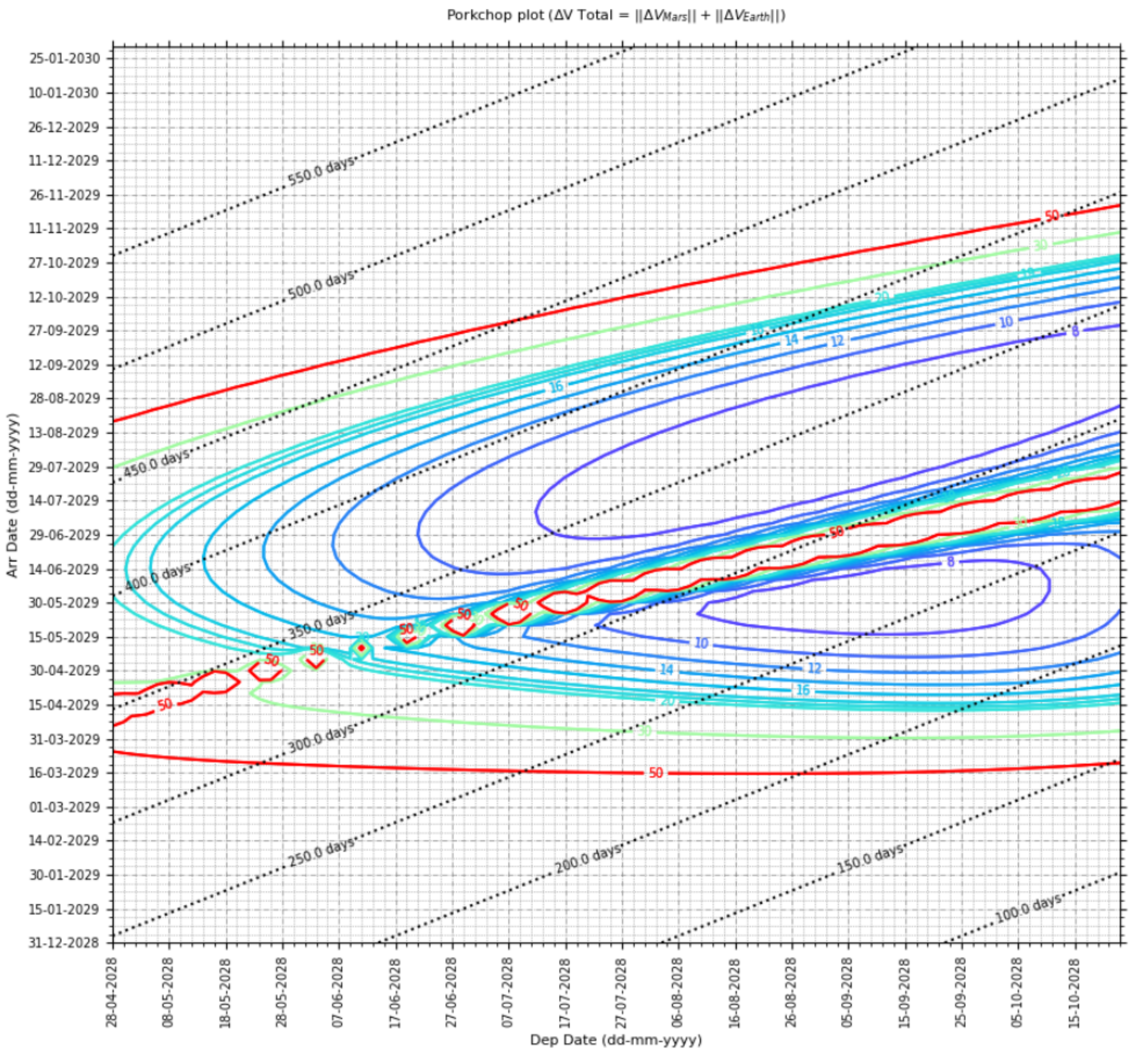


Figure 3.5: Porkchop plot for 2028 launch from Mars to Earth, information from [28],[29].

We shall reconsider an initial launch date, sticking to the parameters given to us by The Martian film. This means we are looking for a journey that takes 214.75 days (209 sols), and so we look at the dotted black lines on figure (3.5) to find the journey of smallest ΔV (for economical benefit). This leads us to an x-axis value of 15th October 2028, and a y-axis value of 17th May 2029 (date of first flyby), which shows a ΔV of approximately $10kms^{-1}$. So, by waiting 10.5 months to launch our rocket we have a journey where we need to accelerate the rocket over 5 times less (less than 20% of the fuel). For this improved initial launch, we have the second flyby occurring on 23rd April 2030, and the Earth landing on 25th November 2030.

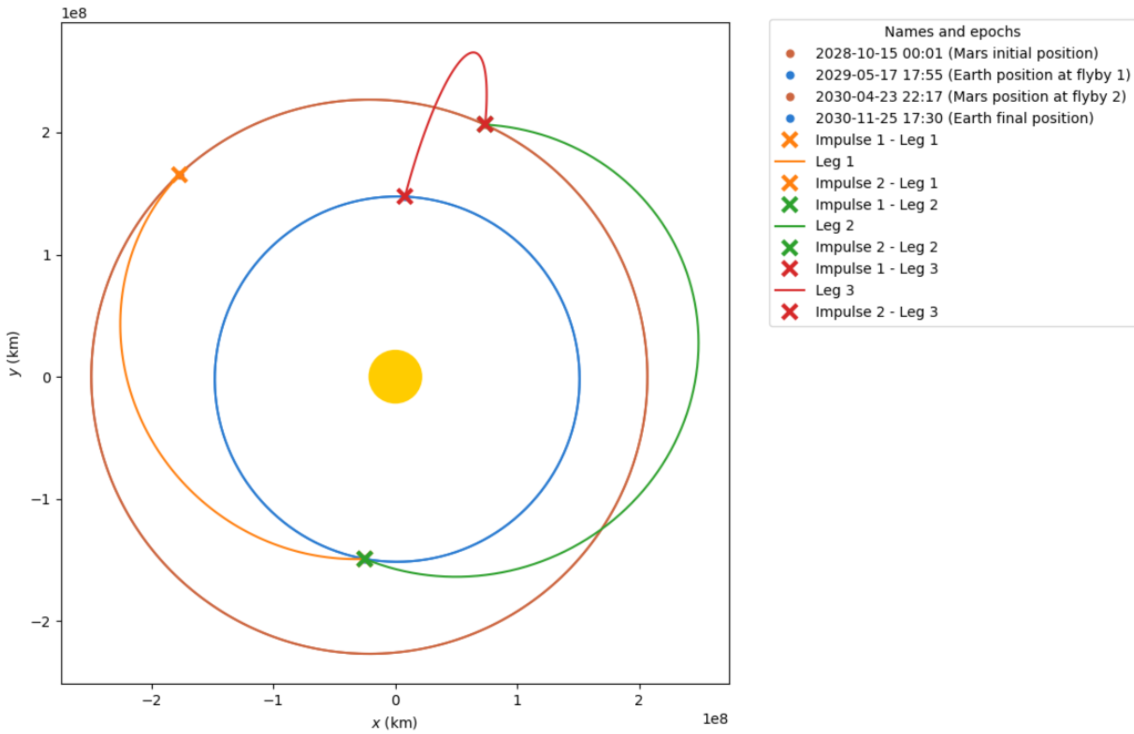


Figure 3.6: Hermes trajectory, with launch date 15th October 2028.

Figure (3.6) shows us how the trajectory would look with an initial date of 15th October 2028. Once again, legs 1 and 2 look reasonable at first glance. However, leg 3 shows a journey which would require a large flyby impulse to be imparted by Mars, and then again a large impulse to land on Earth due to the difference in flight path angles. So, we shall look at leg 3 using a porkchop plot.

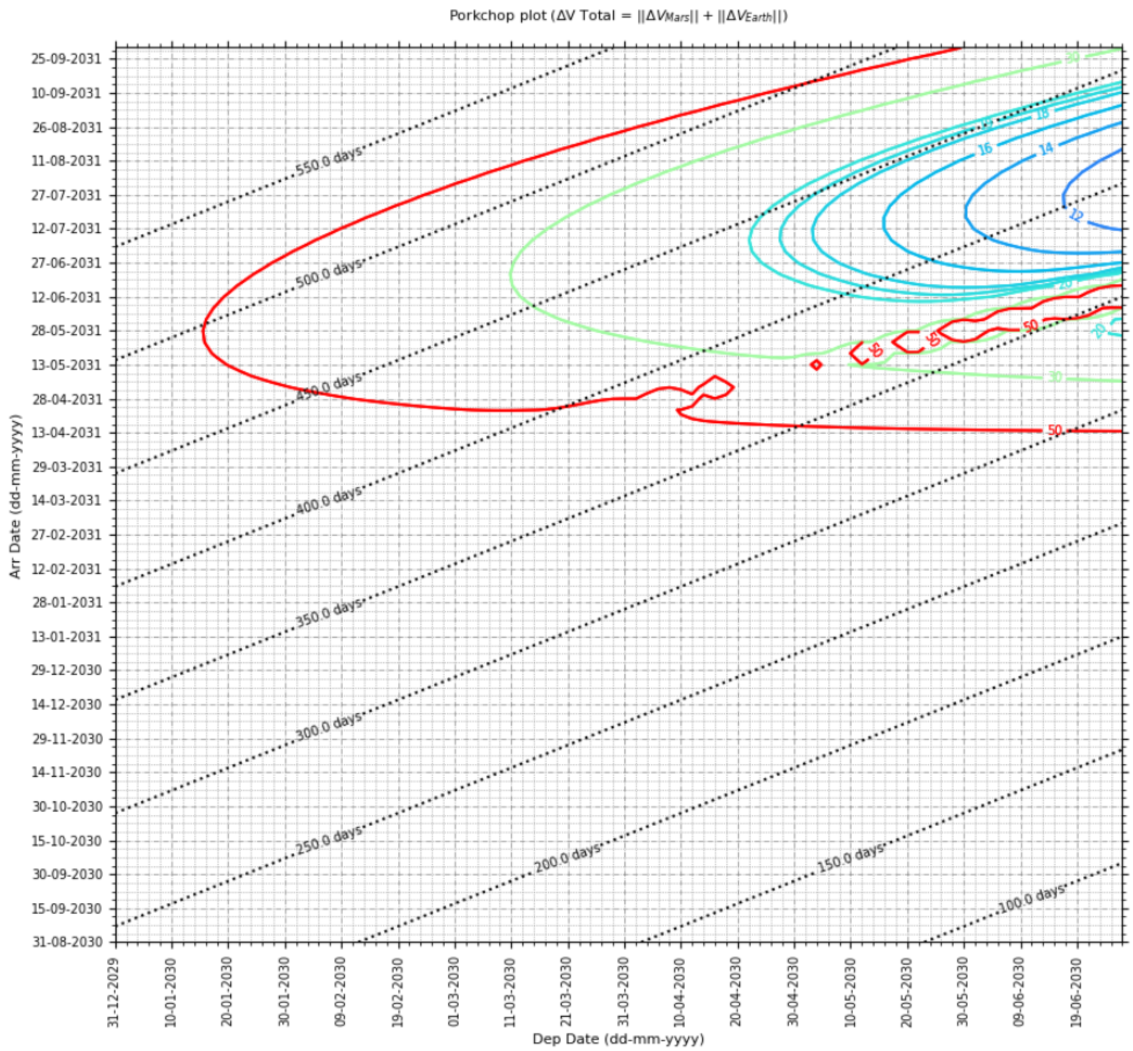


Figure 3.7: Porkchop plot for 2030 flyby from Mars to Earth, information from [28],[29].

Examining figure (3.7) shows us that our initial guess that leg 3 is unrealistic was correct, and so 15th October 2028 would not work as day 1 in The Martian film. The porkchop plot does show us however that a longer flight time for leg 3 is still feasible with a flyby date around Mars of 23rd April 2030. Tracking vertically upwards from this date on figure (3.7) shows us that if the Taiyang Shen stocks the Hermes with about an extra years worth of supplies, then this would be a feasible journey where the ΔV required is instead approximately $20kms^{-1}$. Leg 3 would take about 435 days instead of 217 days.

Chapter 4

Gravity Assist

In The Martian, the concept of gravity assist is one of the central factors of Mark Watney's survival. Gravity assist, also known as a slingshot maneuver, involves using the gravitational pull of a celestial body to alter the velocity of a spacecraft to a more desirable value [30]. Viewers are introduced to a potential real world application of gravity assist, as we see that it is a method that may be used to travel extensive distances with limited fuel resources and time constraints.

Originally, the plan is to deliver Watney the relevant supplies by sol 868 which is fine considering he could plant enough potatoes to last him until sol 912. However on sol 134, Watney's crops are destroyed, and he is no longer able to farm. His rations as of sol 134 are enough to last him to sol 409, and with potatoes he can survive until sol 609. This means NASA are tasked with shaving down their time taken to save Watney from 734 days to just 475. As a new plan, NASA hurries the probe launch by removing safety inspections. This turns out to be a huge mistake, because (in true science fiction fashion) the probe blows up. Luckily for Watney, some NASA executives hold a secret meeting named 'Project Elrond', where Rich Purnell makes the bold claim that he can get the Hermes back to Mars on sol 561 using a gravity assist manoeuvre.

4.1 Gravity Assist Key Concepts

The basic principles of gravity assist [30] are

- momentum conservation - the planet with significantly more mass imparts force on spacecraft with negligible change in it's own velocity
- energy conservation - gravity's pull rotates the spacecraft velocity in the reference frame of the planet, without a change in magnitude

4.1.1 Momentum Conservation

Inspired by R.C. Johnson [30], we label the spacecraft as having mass m, and the planet having mass M, with velocity vectors $\mathbf{v}$ and $\mathbf{V}$ respectively. By the law of conservation of momentum, we have

$$m\mathbf{v}_i + M\mathbf{V}_i = m\mathbf{v}_f + M\mathbf{V}_f, \tag{4.1}$$

$$\mathbf{V}_f - \mathbf{V}_i = \frac{m}{M}(\mathbf{v}_i - \mathbf{v}_f). \tag{4.2}$$

Let's assume we have a spacecraft of mass approximately $10^3 kg$, and a planet of mass in the region $[10^{24}, 10^{27}]kg$. Then clearly $m/M \to 0$, and so we have $\mathbf{V}_f = \mathbf{V}_i$. From now we will consider $\mathbf{V}$, where $\mathbf{V}_i = \mathbf{V}_f = \mathbf{V}$. This tells us that the change in velocity of the planet is negligible. For example, when Galileo travelled to Jupiter it slowed Venus by 1.6 inches per billion years and slowed Earth by 5.2 inches per billion years [31].

4.1.2 Energy Conservation

Energy conservation applies in the reference frame of the planet [30]. The approach velocity of the spacecraft from the point of view of the planet is $\mathbf{u}_i = \mathbf{v}_i - \mathbf{V}$. Similarly, the departure velocity is $\mathbf{u}_f = \mathbf{v}_f - \mathbf{V}$. The departure velocity, $\mathbf{u}_f$, has the same magnitude: $|\mathbf{u}_f| = |\mathbf{u}_i|$. Therefore the kinetic energy in the system is conserved.

4.2 Gravity Assist - Stationary

It helps to begin our considerations by looking at how a spacecraft might be acted upon by a planet that is stationary. In figure (4.1), we simplify the scenario to a two-dimensional representation. In the lower right section of the diagram, we see the magnitude and direction of the spacecraft's initial velocity. Moving to the upper left section, we observe the effect Jupiter's gravitational force has on the spacecraft's velocity. While the direction of the spacecraft's velocity is significantly altered by Jupiter's gravitational pull, the magnitude remains unchanged. In the middle of the diagram, a long arrow indicates a temporary but substantial increase in magnitude of the spacecraft's velocity. It's important to note that these velocities are all relative to Jupiter [32].

In order to plot accurate two-dimensional results, we will use numerical integration [33]. We want to turn our differential equations,

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{v}, \\ \frac{d\mathbf{v}}{dt} &= \mathbf{a} = -\frac{\mu}{r^3}\mathbf{x}, \end{aligned} \tag{4.3}$$

into versions that can be calculated using python. As usual, $\mathbf{x}$, $\mathbf{v}$ and $\mathbf{a}$ are the position, velocity and acceleration vectors respectively. We have taken (1.9) to be our acceleration

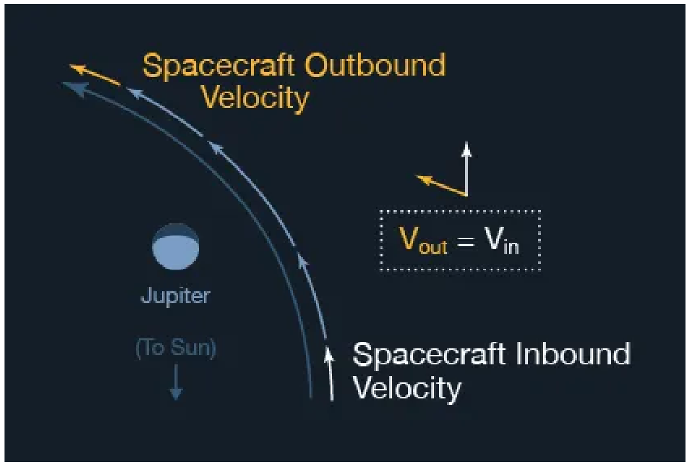


Figure 4.1: Gravity slingshot around a stationary Jupiter [32].

in order to satisfy the semi-implicit Euler-Cromer method [33], which requires $\mathbf{v} = \mathbf{f}(t, v)$ and $\mathbf{a} = \mathbf{g}(t, x)$. We must split our two dimensional differential equations into two sets of one dimensional differential equations like so:

$$\begin{aligned} \frac{dx}{dt} &= v_x, \\ \frac{dv_x}{dt} &= -\frac{\mu}{r^3}x, \\ \frac{dy}{dt} &= v_y, \\ \frac{dv_y}{dt} &= -\frac{\mu}{r^3}y. \end{aligned} \tag{4.4}$$

The semi-implicit Euler-Cromer method has formula

$$\begin{aligned} v_{n+1} &= v_n + g(t_n, x_n)\Delta t, \\ x_{n+1} &= x_n + f(t_n, v_{n+1})\Delta t, \end{aligned} \tag{4.5}$$

into which we plug our set of four differential equations to gain

$$\begin{aligned} v_{n+1} &= v_n - \frac{\mu}{r^3}x_n\Delta t, \\ x_{n+1} &= x_n + v_{n+1}\Delta t. \end{aligned} \tag{4.6}$$

This recursive relation has been solved numerically in order to show the gravity assist trajectory around a stationary Earth in figure (4.2).

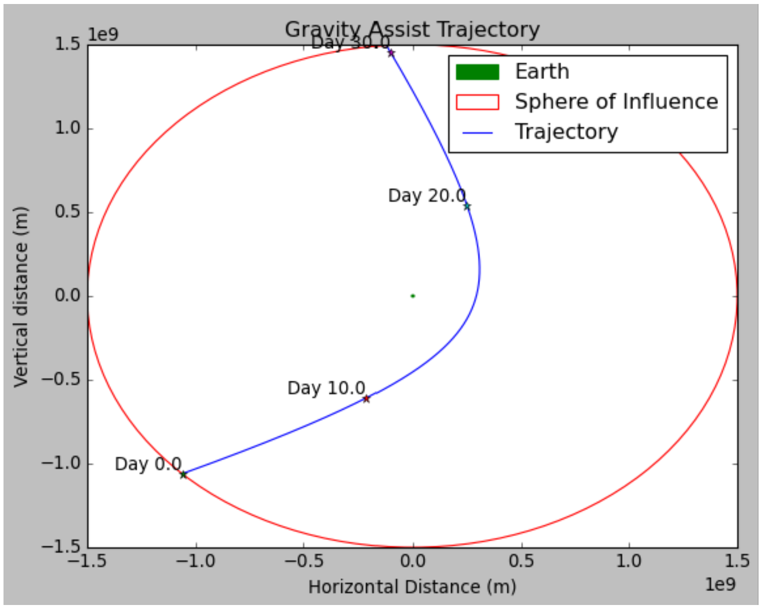


Figure 4.2: Gravity assist trajectory around a stationary Earth, initial velocity of $1000ms^{-1}$ and 25° angle of approach, to scale.

4.3 Gravity Assist - Moving

When a spacecraft undergoes a gravity assist in reality, it's not just interacting with a stationary planet, as previously discussed. Jupiter possesses significant angular momentum as it orbits the Sun. In figure (4.3), we illustrate Jupiter's motion along its solar orbit with a green vector (though it's simplified; Jupiter's orbit is actually an arc, not a straight line). The Sun is located below the bottom of the diagram. During the spacecraft's interaction with Jupiter, it acquires this Sun-relative vector, or a significant portion of it. The addition of this vector to the spacecraft's velocity vectors v_{in} and v_{out} is depicted. This resultant vector shows how the spacecraft's velocity, relative to the Sun, receives a substantial boost from Jupiter. Notably, the rotation of the vector from v_{in} and v_{out}, representing the bending of the spacecraft's path by Jupiter's gravity, contributes to this increase.

In order to see how this works in reality, we can use the same semi-implicit Euler-Cromer method. However this time we must also include an array to model the straight line movement of the celestial body providing the assist. Figure (4.4) shows a gravitational assist trajectory whereby the aim was to reduce the speed of the spacecraft. The spacecraft enters the sphere of influence of Earth at a speed greater than that of the planet, and so the gravitational force of the Earth reduces the speed.

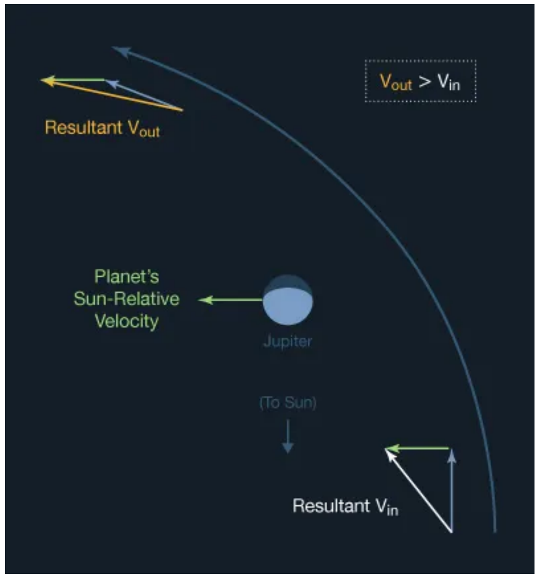


Figure 4.3: Gravity slingshot around a moving Jupiter [32].

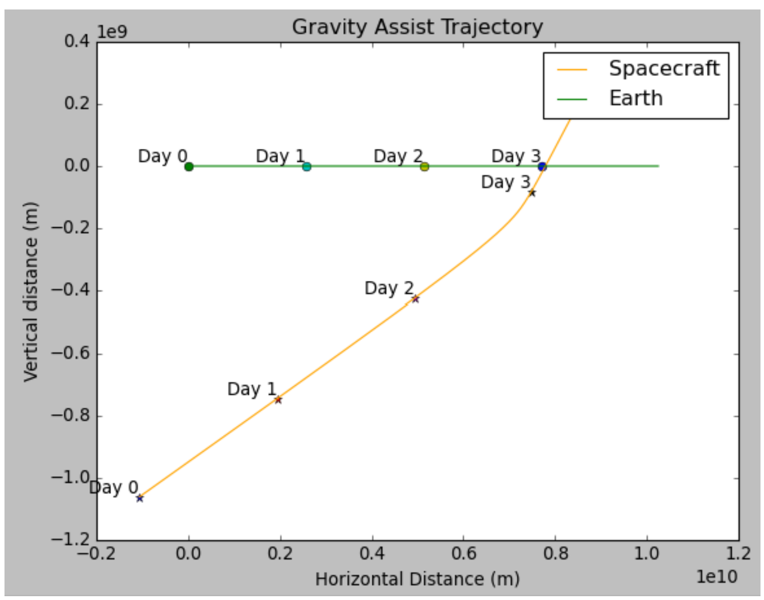


Figure 4.4: Gravity assist trajectory around a stationary Earth, initial speed of $35000ms^{-1}$ and final speed of $15666ms^{-1}$ to scale.

4.4 Gravity Assist Scan

Once we know the angle of approach as well as approach velocity, as given by the Lambert problem, we can perform a scan across a range of gravity assist angles in order to find which will give us the greatest benefit. We can alter our objectives easily in python, and (4.1) shows an example where we are purely looking for the largest speed boost. The results are two windows of angles that obtain the required speed boost, and the angles in between correspond to angles where the spacecraft crashes into Earth.

Table 4.1: Optimal speed increase gravity assist scan around Earth: Initial spacecraft speed = 35000.72km/s, sphere of influence entry point = 225°. Shows speed increases of 5x and above.

ΔV **multiple**	**Angle of Approach**
6.189	3.72
8.028	3.73
11.079	3.74
17.119	3.75
19.064	3.79
12.470	3.80
9.209	3.81
7.262	3.82
5.968	3.83
5.044	3.84

4.4.1 Gravity Assist Scan Applied

As promised at the end of section 3.6.1, we will now discuss the methodology of obtaining desired gravity assist results. First our assumptions, of which there are two. Assumption number one is that our rocket knows the velocity and sphere of influence entry point it must reach at the time specified by the Lambert problem, and that it can fire thrusters before sphere of influence entry to adjust it's entry angle without compromising the speed it is travelling at. Secondly, we will assume that upon leaving the sphere of influence, the rocket can fire thrusters in the direction parallel to it's velocity, to speed up or slow down to the velocity required by the full Hermes trajectory.

To continue from the point of having made these assumptions, we need to obtain four values given by the Hermes trajectory. For this example, we will use the Hermes trajectory described by the 1st June 2028 launch date, and look at how a flyby around Earth would be used to patch together leg 1 and leg 2. The values, and how to obtain them, are:

- Angle of Approach (22.858°) - we use the formula $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ where $\mathbf{a}$ is the velocity of Earth at the time of flyby and **b** is the velocity of the spacecraft at the end of leg 1

- Departure Angle (2.786°) - we use the formula $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ where **a** is the velocity of Earth at the time of flyby and **b** is the velocity of the spacecraft at the beginning of leg 2

- Initial Speed ($33.435kms^{-1}$) - this is the speed of the spacecraft upon entering Earth's sphere of influence, and is the norm of the velocity at the end of leg 1

- Final Speed ($33.059kms^{-1}$) - this is the speed of the spacecraft upon leaving Earth's sphere of influence, and is the norm of the velocity at the beginning of leg 2

Now we will determine the sphere of influence entry point. Projected on to two-dimensions, this is given by $(soi\cos\theta,\ soi\sin\theta)$ where soi is the radius of the projected circle. θ is the the angle of approach.

As previously stated, it is easy to alter the objectives of our code in python. Due to our assumptions, we are now looking for an angle of approach which will give us the desired departure angle of 2.786°. To do this, we simply look at where the departure angle given by the flyby is within a certain tolerance (0.01 degrees in this case). We change the last lines of the gravity assist scan code to look like the following:

```
    depart = np.arctan((y[-1]-y_p[-1])/(x[-1]-x_p[-1]))*180/np.pi
    if depart<dep_angle+tol and depart>dep_angle-tol:
        if distance(x[-1],y[-1]) > Re + safe :
            deltav = mod_v1 - v_final
            print('Must increase/(decrease) velocity by =',deltav,'Angle of approach =',i*0.01,'Departure angle =',depart)
```

Running the scan gives us just one option for sphere of influence entry angle. We must manufacture an entry angle of 42.55° which gives us a departure angle of 2.790°. Upon departure of the sphere of influence, now we only require minor adjustments; we must adjust the flight path angle by 0.004°, and slow down the spacecraft by $378.80ms^{-1}$.

Chapter 5

Saving Mark Watney

5.1 Space Rendezvous

In order for two spacecraft to meet in the vast expanse we call outer space, we require precise calculations and intricate maneuvers to perform space rendezvous. This chapter delves into the mathematics underpinning this concept, and we will comment on it's potential applications to the rendezvous scenarios portrayed in The Martian.

Space rendezvous has been crucial in humanity's ongoing quest to venture beyond the confines of Earth's surface. A notable real world example is the Apollo Lunar Module's rendezvous with the Command Module during NASA's Apollo program [34]. The mission crescendoed during the Apollo 11 lunar landing, and afterwards the Lunar Module had a rendezvous with the orbiting Command Module following its descent and subsequent ascent from the moon. This allowed for the successful return of Neil Armstrong and his crew, having marked a 'giant leap for mankind'.

Building upon this historical significance, space rendezvous plays a central role in space missions to this day. Technology advancements have allowed for the increase precision of these maneuvers since, and the reliability has increased as a result.

5.1.1 Developing Useful Equations

We can develop Newton's equation, $F = ma$, by splitting the force into two relevant components. This is the gravitational force exerted upon the spacecraft by the celestial body it is orbiting, and the force exerted on the spacecraft by other external forces such as its thrusters and friction. Mathematically [35] this can be expressed

$$m\ddot{\mathbf{r}} = -\frac{\mu\hat{\mathbf{r}}}{r^2} + \mathbf{F}(t). \tag{5.1}$$

By splitting r into it's x and y components we obtain the following equations

$$\begin{aligned} m\ddot{x} &= -\frac{GM_p m}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} + F_x(t), \\ m\ddot{y} &= -\frac{GM_p m}{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}} + F_y(t), \end{aligned} \tag{5.2}$$

where G is Newton's constant, M_p and m are the masses of the planet and the rocket respectively and the F_x and F_y are other time dependent forces.

These equations are simpler to solve and analyse in terms of polar coordinates, so we will use the substitutions

$$\begin{aligned} x &= r\sin\theta, \\ y &= r\cos\theta. \end{aligned} \tag{5.3}$$

We want to show that equations (5.2) can be rewritten

$$\begin{aligned} \ddot{r} &= \frac{-GM_p}{r^2} + r\dot{\theta}^2 + \frac{F_r}{m}, \\ \ddot{\theta} &= -\frac{2\dot{\theta}\dot{r}}{r} + \frac{F_\theta}{m}, \end{aligned} \tag{5.4}$$

where we have

$$\begin{aligned} F_r &= F_x\sin\theta + F_y\cos\theta, \\ F_\theta &= F_x\cos\theta - F_y\sin\theta. \end{aligned} \tag{5.5}$$

To begin, we want to find the relevant equations for $\ddot{x}$ and $\ddot{y}$ to substitute in to (5.2). Our time derivatives for x can be found using the knowledge that r and θ are both functions of time,

$$\begin{aligned} x &= r\sin\theta, \\ \dot{x} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta, \\ \ddot{x} &= \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\cos\theta - r\dot{\theta}^2\sin\theta, \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\cos\theta. \end{aligned} \tag{5.6}$$

Similarly for the time derivatives of y, we have

$$\begin{aligned} y &= r\cos\theta, \\ \dot{y} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \\ \ddot{y} &= \ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\sin\theta - r\dot{\theta}^2\cos\theta, \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\sin\theta. \end{aligned} \tag{5.7}$$

We should also note the following handy relation,

$$x^2 + y^2 = r^2 \sin^2 \theta + r^2 \cos^2 \theta. \tag{5.8}$$

For the derivation, it suffices to substitute the corresponding time derivatives into the first equation from (5.2), and rearrange into our equation for $\ddot{r}$ by assuming we already know our equation for $\ddot{\theta}$. First, we have

$$\begin{aligned} m\ddot{x} &= -\frac{GM_p m}{x^2 + y^2} \frac{x}{\sqrt{x^2 + y^2}} + F_x(t), \\ m\left(\left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\cos\theta\right) &= -\frac{GM_p m}{r^2} \frac{r \sin\theta}{r} + F_x(t), \\ \left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\cos\theta &= -\frac{GM_p \sin\theta}{r^2} + \frac{F_x(t)}{m}. \end{aligned} \tag{5.9}$$

Now, we assume the second equation from (5.4) to be true, and then we can simplify the second bracket on the left hand side of (5.9),

$$\begin{aligned} 2\dot{r}\dot{\theta} + r\left(-\frac{2\dot{r}\dot{\theta}}{r} + \frac{F_\theta}{mr}\right) &= 2\dot{r}\dot{\theta} - 2\dot{r}\dot{\theta} + \frac{F_\theta}{m}, \\ &= \frac{F_\theta}{m}, \end{aligned} \tag{5.10}$$

and so we are left with

$$\left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \frac{F_\theta \cos\theta}{m} = -\frac{GM_p \sin\theta}{r^2} + \frac{F_x}{m}. \tag{5.11}$$

We should take note that in the following steps our aim is to have F_r as a variable, rather than F_θ and F_x. We rearrange our equation to get closer to the desired format,

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 + \frac{F_\theta \cos\theta}{m \sin\theta} &= -\frac{GM_p}{r^2} + \frac{F_x}{m \sin\theta}, \\ \ddot{r} &= -\frac{GM_p}{r^2} + r\dot{\theta}^2 + \frac{F_x}{m \sin\theta} - \frac{F_\theta \cos\theta}{m \sin\theta}. \end{aligned} \tag{5.12}$$

Now all that is left to show is that

$$\begin{aligned} \frac{F_x}{m \sin\theta} - \frac{F_\theta \cos\theta}{m \sin\theta} &= \frac{F_r}{m}, \\ F_x - F_\theta \cos\theta &= F_r \sin\theta, \\ F_x &= F_r \sin\theta + F_\theta \cos\theta, \end{aligned} \tag{5.13}$$

into which we can substitute in our equations for F_r and F_θ,

$$\begin{aligned} F_x &= \sin\theta(F_x\sin\theta + F_y\cos\theta) + \cos\theta(F_x\cos\theta - F_y\sin\theta), \\ &= F_x\sin^2\theta + F_y\cos\theta\sin\theta + F_x\cos^2\theta - F_y\sin\theta\cos\theta, \\ &= F_x. \end{aligned} \tag{5.14}$$

It then follows that

$$\ddot{r} = \frac{-GM_p}{r^2} + r\dot{\theta}^2 + \frac{F_r}{m}, \tag{5.15}$$

holds, and as a result also confirms our assumption of $\ddot{\theta}$.

5.1.2 Simple Solution

In order to keep our initial problem simple, we will assume the spacecraft we are looking to rendezvous with (Mark Watney and his MAV) is moving in a circular orbit around our celestial body (Mars). If we consider this circular orbit without any external forces, then we can derive the format of our simplest solution. A circular orbit with no external forces has the following properties,

$$\begin{aligned} F_\theta &= 0, \\ F_r &= 0, \\ r &= r_0, \\ \dot{r} &= 0, \\ \ddot{r} &= 0. \end{aligned} \tag{5.16}$$

By substituting equations (5.16) into (5.4) we are left with

$$\begin{aligned} 0 &= -\frac{GM_p}{r_0^2} + r\dot{\theta}^2, \\ 0 &= \ddot{\theta}. \end{aligned} \tag{5.17}$$

As the angular acceleration is equal to 0 then we can integrate this once to obtain the angular velocity which must be equal to a constant,

$$\dot{\theta} = \omega_0. \tag{5.18}$$

We integrate this once more to obtain our angle equation,

$$\theta = \omega_0 t + \theta_0. \tag{5.19}$$

By substituting this into equations (5.17), we have

$$r_0\dot{\theta}^2 = \frac{GM_p}{r_0^2}, \tag{5.20}$$

$$\omega_0^2 = \frac{GM_p}{r_0^3}, \tag{5.21}$$

$$\omega_0 = \sqrt{\frac{GM_p}{r_0^3}}. \tag{5.22}$$

So given that we have a circular orbit with no external forces, we see that our solution is of the form

$$r = r_0, \tag{5.23}$$

$$\theta = \omega_0 t + \theta_0, \tag{5.24}$$

which works for any initial angle θ_0. Note that our angular velocity is constantly $\sqrt{\frac{GM_p}{r_0^3}}$.

5.1.3 Numerical Computation Setup

For numerical computation, as in [35], we will use the target trajectory as described above ($r = r_0$ and $\theta = \omega_0 t$). As well as this we introduce relative coordinates for the spacecraft we have control over,

$$\begin{aligned} r &= r_0 + z(t), \\ \theta &= \omega_0 t + \phi(t), \end{aligned} \tag{5.25}$$

where z and ϕ represent the relative altitude and angle respectively. By substituting into (5.4) we obtain

$$\begin{aligned} \ddot{z} &= -\frac{GM_p}{(r_0+z)^2} + (r_0+z)(\omega_0+\dot{\phi})^2 + \frac{F_r}{m}, \\ \ddot{\phi} &= -\frac{2(\omega_0+\dot{\phi})\dot{z}}{(r_0+z)} + \frac{F_\theta}{m(r_0+z)}. \end{aligned} \tag{5.26}$$

To be able to solve on python, we require a set of first order differential equations (e.g. gravity turn (2.10)). We introduce new variables, such that $\dot{z} = b$ and $\dot{\phi} = c$. Then our set of two second order differential equations (5.4) can be transformed into the set of four

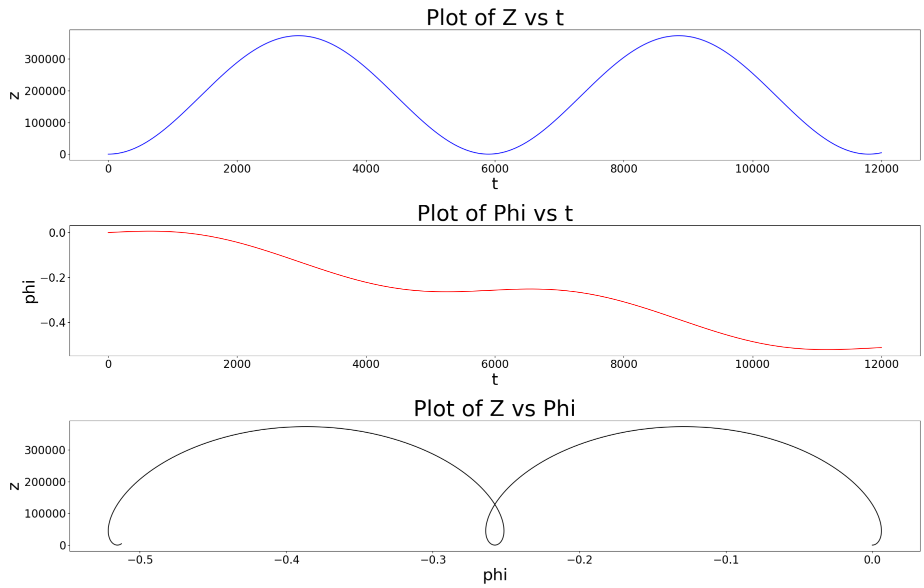


Figure 5.1: Two full cycles of equations (5.27).

first order differential equations,

$$\begin{aligned}\frac{dz}{dt} &= b,\\ \frac{db}{dt} &= -\frac{GM_p}{(r_0+z)^2} + (r_0+z)(\omega_0+c)^2 + \frac{F_r}{m},\\ \frac{d\phi}{dt} &= c,\\ \frac{dc}{dt} &= \frac{2(\omega_0+c)b}{(r_0+z)} + \frac{F_\theta}{m(r_0+z)}.\end{aligned} \tag{5.27}$$

Figure (5.1) shows how our values for z and ϕ change over time given initial conditions $z(0) = \phi = \dot{z} = 0$, and $\dot{\phi} = 100/(r_0 + z_0)$. This corresponds to an elliptical orbit where the initial condition is the perigee [35].

5.1.4 Simple Solution Speed

In terms of practicality of the application to real life, it is advantageous to also derive an expression for the speed of the spacecraft moving in circular orbit. The speed is written

$$|\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2}. \tag{5.28}$$

We notice that we have already obtained these derivatives, in (5.6) and (5.7):

$$\dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta, \tag{5.29}$$

$$\dot{y} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta. \tag{5.30}$$

Then we simply substitute these equations into (5.28),

$$\begin{aligned}
|\mathbf{v}| &= \sqrt{\dot{x}^2 + \dot{y}^2}, \\
&= \sqrt{(\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 + (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2}, \\
&= \sqrt{(\dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\cos^2\theta) + (\dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\sin^2\theta)}, \\
&= \sqrt{\dot{r}^2(\sin^2\theta + \cos^2\theta) + r^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta)}, \\
&= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}.
\end{aligned} \tag{5.31}$$

We can find the exact value of this velocity for a spacecraft in our previously mentioned environment (no external forces, circular orbit). In that case, we have $r = r_0$, $\dot{r} = 0$ and $\dot{\theta} = \omega_0 = \sqrt{\frac{GM_p}{r_0^3}}$. We have

$$\begin{aligned}
|\mathbf{v}| &= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}, \\
&= \sqrt{0 + r_0^2\frac{GM_p}{r_0^3}}, \\
&= \sqrt{\frac{GM_p}{r_0}}.
\end{aligned} \tag{5.32}$$

5.1.5 Simple Solution Displacement

The final important quantity we must derive is the displacement between our target and our spacecraft, which we are aiming to reduce to zero. In figure (5.2) the distance between our target and the spacecraft is denoted by s. We can find this quantity by simply splitting

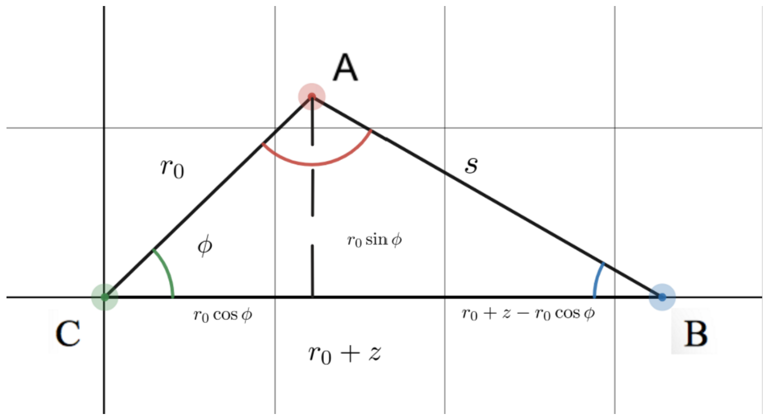


Figure 5.2: Rendezvous depiction. A is the location of our target, B is the location of our spacecraft and C is the location of the celestial body that the target and spacecraft are orbiting.

CB into $r_0 \cos\phi$ and $r_0 + z - r_0 \cos\phi$ and applying Pythagoras' theorem,

$$\begin{aligned}
s^2 &= (r_0 \sin\phi)^2 + (r_0 + z - r_0 \cos\phi)^2, \\
s &= \sqrt{(r_0 \sin\phi)^2 + (r_0(1 - \cos\phi) + z)^2}, \\
&= \sqrt{r_0^2 \sin^2\phi + r_0^2(1 - \cos\phi)^2 + 2r_0 z(1 - \cos\phi) + x^2}, \\
&= \sqrt{r_0^2 \sin^2\phi + r_0^2 - 2r_0^2 \cos\phi + r_0^2 \cos^2\phi + 2r_0 z(1 - \cos\phi) + z^2}, \\
&= \sqrt{r_0^2(\sin^2\phi + 1 - 2\cos\phi + \cos^2\phi) + 2r_0 z(1 - \cos\phi) + z^2}, \\
&= \sqrt{2r_0^2(1 - \cos\phi) + 2r_0 z(1 - \cos\phi) + z^2}, \\
&= \sqrt{(2r_0^2 + 2r_0 z)(1 - \cos\phi) + z^2}, \\
&= \sqrt{2r_0(r_0 + z)(1 - \cos\phi) + z^2}.
\end{aligned} \tag{5.33}$$

Using our definition for r, we can also write this as

$$s = \sqrt{2rr_0(1 - \cos\phi) + z^2}. \tag{5.34}$$

5.1.6 Tying Together the Rendezvous

To use this mathematics in a practical setting requires numerical computation, and we will use the format laid out in [35]. We will assume our target and spacecraft are in the same plane.

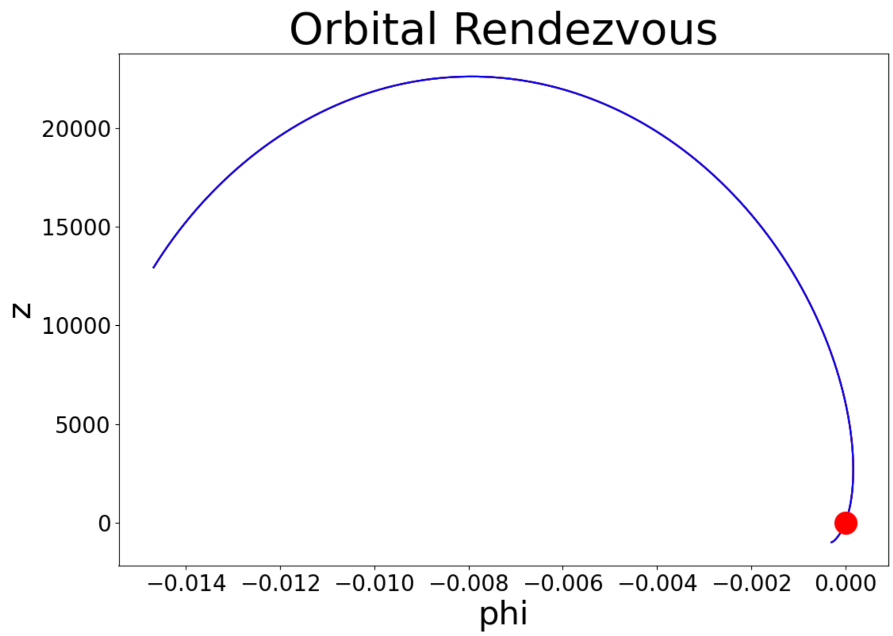


Figure 5.3: Rendezvous depiction. $F_r = 32N$, $F_\theta = 100N$ and $t_{thrust} = 266s$. Origin (rendezvous point) marked with circle.

First, we shall consider a spacecraft which can thrust in 4 separate directions. Two thrusters push parallel to the orbit direction; the positive F_θ direction (forwards) and the negative F_θ direction (backwards). Similarly, the other two thrusters will push the spacecraft radially; the positive F_r direction (to a higher orbit) and the negative F_r direction (to a lower orbit). In the first instance, we will look at a spacecraft that has initial conditions that describe a circular orbit. A spacecraft on a circular orbit 1km lower than the target and 2km behind has the initial conditions

$$\begin{aligned} z &= 1000, \\ \dot{z} &= 0, \\ \phi &= -\frac{2000}{r_0}, \\ \dot{\phi} &= \sqrt{\frac{GM_p}{(r_0+z)^3}} - \omega_0. \end{aligned} \tag{5.35}$$

We will assume the thrusters are fired from $t = 0$ up until t_{thrust}. The total impulse available to be created by the spacecraft is $(|F_r| + |F_\theta|)t_{thrust}$. Figure (5.3) shows the z and ϕ parameters given $F_r = 32N$, $F_\theta = 100N$ and $t_{thrust} = 266s$. The red circle is the rendezvous target, which stays constant as z and ϕ are relative coordinates. This computes a minimum distance of 0.85m at 377.0s, which would be close enough to bring Mark Watney on board.

5.1.7 Hyperbolic Rendezvous

In our science fiction setting, there is a factor making it completely different to what we have so far described. This is the fact that our spacecraft is not in fact on a familiar and friendly circular orbit, but a hyperbolic one. The spacecraft is undergoing it's second gravitational assist around Mars during the rendezvous, meaning that it will be leaving the sphere of influence of Mars at the end of the trajectory. It is therefore important that we consider a more general approach to the initial conditions, so that we can solve any given rendezvous problem.

New Relative Coordinates

Recall our relative coordinates, $r = r_0 + z$ and $\theta = \omega_0 t + \phi$. We are aiming to find equations for r, $\dot{r}$, ϕ and $\dot{\phi}$ that can work for any given orbit.

By recalling the orbit equation (1.25),

$$r = \frac{h^2}{\mu} \frac{1}{1 + e \cos \nu}, \tag{5.36}$$

we can see that the format we should take for z is

$$z = \frac{h^2}{\mu} \frac{1}{1 + e \cos \nu} - r_0. \tag{5.37}$$

For the time derivative of this equation, and noting that r_0 is a constant, [3] tells us that the velocity in the radial direction is given by

$$\dot{z} = v_r = \dot{r} = \frac{\mu}{h} e \sin \nu. \tag{5.38}$$

A similar approach is taken for the relative angular coordinates. The angle we want θ to be equal to in the relative coordinate system is the true anomaly, ν. Then it is clear that we should take

$$\phi = \nu - \omega_0 t. \tag{5.39}$$

Using the time derivative in this direction from [3], we have

$$\frac{d\nu}{dt} = \frac{h}{r} = \frac{\mu}{h}(1 + e \cos \nu). \tag{5.40}$$

Therefore the time derivative of ϕ is

$$\dot{\phi} = \frac{\mu}{h}(1 + e \cos \nu) - t. \tag{5.41}$$

Bringing everything together, our new set of initial conditions is

$$\begin{aligned}
z &= \frac{h^2}{\mu}\frac{1}{1+e\cos\nu} - r_0, \\
\dot{z} &= \frac{\mu}{h} e \sin\nu, \\
\phi &= \nu - \omega_0 t, \\
\dot{\phi} &= \frac{\mu}{h}(1+e\cos\nu) - \omega_0.
\end{aligned} \tag{5.42}$$

If we were to obtain the relevant parameters for The Martian problem, they can be plugged into this set of equations. From there, we need to know the maximum possible thrust attainable by the Hermes. The final step is adjusting the F_r, F_θ and t_{thrust} in order to reduce the minimum displacement between Mark Watney and the Hermes to a value where capture is possible.

5.1.8 Mark Watney Application

In The Martian, Vincent Kapoor (NASA) is visually upset when JPL describes the adjustments that should be made by the MAV in order to rendezvous with the Hermes. The Hermes is unable to enter Mars' orbit, or it will not have enough fuel for the return leg to Earth (so it must perform a flyby, not entering a circular orbit at any point). On the other hand, the MAV was only designed to reach a low Mars orbit. As a result JPL say the MAV must lose $5000kg$ in weight by losing the following items: martian soils and samples, life support, the 5 unused seats, control panels, secondary and tertiary communication systems, the nose air lock, a hull panel and it's windows. We are not equipped with the mass of the MAV or it's thrust capabilities as initial conditions, but we have the equations to which we can input values.

Once Watney is close to the Hermes, Commander Lewis is able to thrust in any direction in a 214 meter radius around the Hermes as the crew have tied together all of their tethers. On top of this, it is stated that a relative velocity of $10m/s$ is 'like jumping on a moving train' and is the maximum relative velocity that a rendezvous can be performed with. We can implement these constraints to test the feasibility of the rendezvous presented in the film.

Chapter 6

Conclusion

To conclude this report, the way that science fiction (in particular The Martian) explores the complexities and possibilities of space travel offers a compelling area of study. We have gained insight into the intricacies that are orbital dynamics, gravity turn, Lambert's problem, gravity assist and space rendezvous, as well as their applications.

The space travel depicted in The Martian uses the following conic sections:

1. MAV lift-off from Mars.
2. MAV and Hermes space rendezvous.
3. Hermes interplanetary travel; Mars to Earth.
4. Hermes gravity assist around Earth (simultaneous to Taiyang Shen rendezvous).
5. Hermes interplanetary travel; Earth to Mars.
6. Hermes gravity assist around Mars (simultaneous to Mark Watney rendezvous).
7. Hermes interplanetary travel; Mars to Earth.
8. Hermes Earth landing.

Summing these parts (excluding landing) as one intertwines the four chapters of this dissertation nicely. For example, for any arbitrary sol 1, we could take the following approach.

1. Use gravity turn to describe the required MAV path that will intercept the Hermes, which is in low Mars orbit. Use the initial angle of the MAV which is forced sideways due to the storm.
2. Use Lambert's problem to find the initial velocity that the Hermes must attain to reach Earth at the desired point in time.

3. Once Watney is found to be alive, use the Lambert problem. Use the arrival date as when we must reach Watney by (given how long he can survive with his rations supply), and use the departure date as when the spacecraft will reach Earth as a result of stage 2. This will output a desired velocity which we want the Hermes to be travelling at following the gravity assist around Earth.

4. Use a gravity assist scan to find the desired angle and speed of approach to the Earth. Thrusters can be fired accordingly by the Hermes to reach the necessary velocity. Use the gravity assist results in order to calculate the gravity turn path the Taiyang Shen must take to successfully rendezvous with the Hermes for resupply.

5. Once again use the Lambert problem. Use the arrival date as when the desired Hermes return to Earth date is, and use the departure date as when the spacecraft should have reached Watney by.

6. Using the stage 5 output of a desired initial velocity directly after the gravity assist around Mars, we can perform a gravity assist scan to discover how we will reach this velocity. When this path has been confirmed, match it with a gravity turn trajectory of the MAV towards the Hermes, which can be fine tuned using the concept of space rendezvous once near.

We can also note that the space rendezvous can be aided, as in the fiction, by the piercing of Watney's suit. By using the rocket equation (1.14) laid out in our introduction, we can find out what mass of gas he would need to eject from his suit to reach the required Δv. At this moment he is outside the sphere of influence of Mars, and can be thought of as having zero gravity acting upon him. Then the change in velocity he can reach is given by $\Delta v = u\frac{m_i}{m}$, where u might be derived from the changing pressure in his suit as a function of time.

In summary, we have detailed each conic section of the space travel journey described in The Martian, and explored how we might build a full trajectory using them. Despite some concepts seeming fantastical, it is clear that their origins are rooted in real-word mathematics. Andy Weir has pushed the boundaries of our imagination while allowing room for scientific enquiry. As portrayed by the plans for the first ever launch of a spacecraft from a foreign planet in the early 2030's by NASA [24], while humanity continues to unravel the mathematical mysteries of the universe we should have no doubt that many science fiction dreams will continue to become reality.

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Appendix A

Gravity Turn - Taiyang Shen Trajectory

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint

#Global variables
Re = 6378135 #earth radius, m
ge = 9.81 #gravity

#Simulation parameters
tstep = 1
tfirst_thrust = 161
tfirst_coast = 50
tsecond_thrust = 397
tsecond_coast = 100

t1 = tfirst_thrust
t2 = t1 + tfirst_coast
t3 = t2 + tsecond_thrust
t4 = t3 + tsecond_coast

#Knowledge of rocket
T_s = 7607000 #thrust at sea level, N
T_v = 981000 #thrust in vacuum, N
mass_s1 = 25600 #stage 1 empty mass, kg
mass_p1 = 395700 #stage 1 propellant mass, kg
mass_s2 = 3900 #stage 2 empty mass, kg
mass_p2 = 92670 #stage 2 propellant mass, kg
mass_payload = 22800 #payload mass, kg

mdot1 = mass_p1/tfirst_thrust #mass flow of first thrust, kg/s
mdot2 = mass_p2/tsecond_thrust #mass flow of first thrust, kg/s
```

```
#Initial conditions
mnot= mass_s1 + mass_p1 + mass_s2 + mass_p2 + mass_payload #total mass in
    kg
gammanot=89.94*np.pi/180 #gamma must be in radians
Vnot=18; #Non-zero initial velocity, m/s
Hnot=0; #Initial Altitude - Earth's surface, m
Xnot=0; #Initial X position, m

#Equations of motion for thrust at sea level
def thrusteoms1(y, t):
    global Re, ge, mdot1, T_s
    ydot = np.zeros_like(y)

    ydot[0] = -mdot1
    ydot[1] = y[3] * np.sin(y[4])
    ydot[2] = y[3] * np.cos(y[4])
    ydot[3] = (1/y[0]) * (T_s - (y[0] * ge - y[0] * ((y[3] * np.cos(y[4]))
    **2) / (Re + y[1])) * np.sin(y[4]))
    ydot[4] = -(1/y[0]) * (1/y[3]) * (y[0] * ge - y[0] * ((y[3] * np.cos(y
    [4]))**2) / (Re + y[1])) * np.cos(y[4])

    return ydot

#Equations of motion for thrust in vacuum
def thrusteoms2(y, t):
    global Re, ge, mdot2, T_v
    ydot = np.zeros_like(y)

    ydot[0] = -mdot2
    ydot[1] = y[3] * np.sin(y[4])
    ydot[2] = y[3] * np.cos(y[4])
    ydot[3] = (1/y[0]) * (T_v - (y[0] * ge - y[0] * ((y[3] * np.cos(y[4]))
    **2) / (Re + y[1])) * np.sin(y[4]))
    ydot[4] = -(1/y[0]) * (1/y[3]) * (y[0] * ge - y[0] * ((y[3] * np.cos(y
    [4]))**2) / (Re + y[1])) * np.cos(y[4])

    return ydot

#Equations of motion for coast stages (T=0, mdot = 0)
def coasteoms(yc, t):
    global Re, ge
    ycdot = np.zeros_like(yc)

    ycdot[0] = 0
    ycdot[1] = yc[3] * np.sin(yc[4])
    ycdot[2] = yc[3] * np.cos(yc[4])
    ycdot[3] = (1/yc[0]) * ( - (yc[0] * ge - yc[0] * ((yc[3] * np.cos(yc
    [4]))**2) / (Re + yc[1])) * np.sin(yc[4]))
    ycdot[4] = -(1/yc[0]) * (1/yc[3]) * (yc[0] * ge - yc[0] * ((yc[3] * np.
```

```
    cos(yc[4]))**2) / (Re + yc[1])) * np.cos(yc[4])

    return ycdot

#Time arrays
t_initial_launch = np.arange(0, t1, tstep)
t_first_stage_coast = np.arange(t1, t2, tstep)
t_second_thrust = np.arange(t2, t3, tstep)
t_second_stage_coast = np.arange(t3, t4, tstep)

#First thrust stage
y0 = [mnot, Hnot, Xnot, Vnot, gammanot] #Initial conditions
y = odeint(thrusteoms1, y0,t_initial_launch, atol=1e-8, rtol=1e-8) #Solve
    ODE

#Coast stage
yc0 = [y[-1,0] - mass_s1, y[-1,1], y[-1,2], y[-1,3], y[-1,4]] #Initial
    conditions
yc = odeint(coasteoms, yc0,t_first_stage_coast, atol=1e-8, rtol=1e-8) #
    Solve ODE

#Second thrust stage
y2_0 = [yc[-1,0], yc[-1,1], yc[-1,2], yc[-1,3], yc[-1,4]] #Initial
    conditions
y2 = odeint(thrusteoms2, y2_0,t_second_thrust, atol=1e-8, rtol=1e-8) #Solve
     ODE

#Second coast stage
yc2_0 = [y2[-1,0] - mass_s2, y2[-1,1], y2[-1,2], y2[-1,3], y2[-1,4]] #
    Initial conditions
yc2 = odeint(coasteoms, yc2_0,t_second_stage_coast, atol=1e-8, rtol=1e-8) #
    Solve ODE

#Combine results
t_combined = np.concatenate((t_initial_launch, t_first_stage_coast,
    t_second_thrust, t_second_stage_coast))
y_combined = np.concatenate((y, yc, y2, yc2))

#Plot results
plt.figure(figsize=(12, 8))

plt.subplot(3, 2, 1)
plt.plot(t_combined, y_combined[:, 0])
plt.title('Mass vs Time')
plt.xlabel('Time (s)')
plt.ylabel('Mass (kg)')

plt.subplot(3, 2, 3)
plt.plot(t_combined, y_combined[:, 1])
plt.title('Altitude vs Time')
```

```
plt.xlabel('Time (s)')
plt.ylabel('Altitude (m)')

plt.subplot(3, 2, 5)
plt.plot(t_combined, y_combined[:, 2])
plt.title('Horizontal Distance vs Time')
plt.xlabel('Time (s)')
plt.ylabel('Horizontal Distance (m)')

plt.subplot(3, 2, 2)
plt.plot(t_combined, y_combined[:, 3])
plt.title('Velocity vs Time')
plt.xlabel('Time (s)')
plt.ylabel('Velocity (m/s)')

plt.subplot(3, 2, 4)
plt.plot(t_combined, y_combined[:, 4])
plt.title('Gamma vs Time')
plt.xlabel('Time (s)')
plt.ylabel('Gamma')

downrange_distance = y_combined[:, 2]
altitude = y_combined[:, 1]

plt.subplot(3, 2, 6)
plt.plot(downrange_distance, altitude, label='Trajectory', color = 'orange'
    )
plt.title('Gravity Turn Trajectory')
plt.xlabel('Downrange Distance (m)')
plt.ylabel('Altitude (m)')
plt.legend()
plt.grid(True)

plt.tight_layout()
plt.show()
```

Appendix B

Lambert Problem Application - Hermes Trajectory

```python
from astropy import units as u
from astropy.time import Time
from astropy.coordinates import solar_system_ephemeris

solar_system_ephemeris.set("jpl")

import numpy as np
from matplotlib import pyplot as plt

from poliastro.bodies import Sun, Earth, Mars
from poliastro.ephem import Ephem
from poliastro.maneuver import Maneuver
from poliastro.plotting import StaticOrbitPlotter
from poliastro.twobody import Orbit
from poliastro.util import norm, time_range

#Main dates
date_launch = Time("2028-01-01 00:00", scale="utc").tdb
date_flyby_1 = Time("2029-01-01 17:54", scale="utc").tdb
date_flyby_2 = Time("2029-12-10 22:16", scale="utc").tdb
date_arrival = Time("2030-07-15 17:29", scale="utc").tdb

#Mars' ephemeris data
mars = Ephem.from_body(Mars, time_range(date_launch, end=date_arrival,
    periods=500))

#Earth's ephemeris data
earth = Ephem.from_body(Earth, time_range(date_launch, end=date_arrival))

#Starting position and velocity of Mars
r_m0, v_m0 = mars.rv(date_launch)
```

```

#Create orbits
ss_m0 = Orbit.from_ephem(Sun, mars, date_launch)
ss_efly = Orbit.from_ephem(Sun, earth, date_flyby_1)
ss_mfly = Orbit.from_ephem(Sun, mars, date_flyby_2)
ss_e1 = Orbit.from_ephem(Sun, earth, date_arrival)

#Solve for maneuver to Earth, apply maneuver, propagate until flyby
man_earth_1 = Maneuver.lambert(ss_m0,ss_efly)
ic1, ss__target1 = ss_m0.apply_maneuver(man_earth_1, intermediate=True)
ic1_end = ic1.propagate(date_flyby_1)

#Solve for maneuver to Mars, apply maneuver, propagate until flyby
man_mars = Maneuver.lambert(ss_efly,ss_mfly)
ic2, ss__target2 = ss_efly.apply_maneuver(man_mars, intermediate=True)
ic2_end = ic2.propagate(date_flyby_2)

#Solve for maneuver to back to Earth, apply maneuver, propagate until arrival
man_earth_2 = Maneuver.lambert(ss_mfly,ss_e1)
ic3, ss__target3 = ss_mfly.apply_maneuver(man_earth_2, intermediate=True)
ic3_end = ic3.propagate(date_arrival)

#Plot trajectory
plotter = StaticOrbitPlotter()

plotter.plot_body_orbit(Mars, ss_m0.epoch, label="Mars initial position")
plotter.plot_body_orbit(Earth, ss_efly.epoch, label="Earth position at flyby 1")
plotter.plot_body_orbit(Mars, ss_mfly.epoch, label="Mars position at flyby 2")
plotter.plot_body_orbit(Earth, ss_e1.epoch, label="Earth final position")

plotter.plot_maneuver(ss_m0, man_earth_1, label="Leg 1", color="C1")
plotter.plot_maneuver(ss_efly, man_mars, label="Leg 2", color="C2")
plotter.plot_maneuver(ss_mfly, man_earth_2, label="Leg 3", color="C3")

plt.show()
```

Appendix C

Gravity Assist Scan - Trajectory Optimisation

```python
import matplotlib.pyplot as plt
import numpy as np

Re = 6378135
mu = 3.986004418e14

#Choose velocity, angle of approach and sphere of influence entry point
mod_v = 35000.72

#Sphere of influence radius and entry point
soi = 1500000000
soi_entry = [-(soi/np.sqrt(2)-1), -(soi/np.sqrt(2))-1]

#Define delta t
tstep = 1

#Planet start position and velocity
x_p = [0]
y_p = [0]
v_planet = 29780

#Initial conditions for rocket
x = [soi_entry[0]]
y = [soi_entry[1]]

def distance(x_1, y_1):
    return ( (x[-1] - x_p[-1])**2 + (y[-1] - y_p[-1])**2 )**0.5

def velocity(v, x_rocket, x_p, y_p):
    global tstep, mu
    return v[-1] - mu * x_rocket[-1] * tstep / (distance(x[-1], y[-1]))**3
```

```

def position(v, x):
    global tstep
    return x[-1] + v[-1] * tstep

#Create delatv_max variable
deltav_max = 0

#Desired deltav target , as a fraction: final/initial velocity
desire = 3

#Safe height above earth
safe = 1000

#Iterate between angles 3.65 and 3.90 (degrees)
for i in range(365,390):
    gamma = i*0.01*np.pi/180
    x = [soi_entry[0]]
    y = [soi_entry[1]]
    v_x = [mod_v*np.cos(gamma)]
    v_y = [mod_v*np.sin(gamma)]
    x_p = [0]
    y_p = [0]
    while Re + safe < distance(x[-1], y[-1]) < soi:
        v_x.append(velocity(v_x, x, x_p, y_p))
        v_y.append(velocity(v_y, y, x_p, y_p))

        x.append(position(v_x, x))
        y.append(position(v_y, y))

        x_p.append(tstep*v_planet + x_p[-1])
        y_p.append(0)

    v_init = ((v_x[0]**2) + (v_y[0]**2))**0.5
    v_final = ((v_x[-1]**2) + (v_y[-1]**2))**0.5
    if v_final/v_init > desire:
        if distance(x[-1],y[-1]) > Re + safe :
            deltav_max = v_final/v_init
            print('Delta V multiple =',deltav_max,'Final velocity =',
    v_final, 'Angle of approach =',i*0.01)
```