Programming 2021/22 — Assignment 2

December 18th 2021

1 About the assignment

You will write Python functions to make various calculations and draw various plots.

To support your programming, you will also need to do some mathematics.

Your mark for the assignment will be based on both the programming and mathematical parts. This is a programming module and so the programming is the main part of this assignment. Nonetheless, where there are relevant mathematical tasks, the idea is that your solution to a programming task should be based on them. Therefore, a student who submits only the programming part without corresponding maths may receive a mark of zero for any tasks for which a mathematical part was requested. Tasks where some maths is expected have "(Math)" after the task number.

It's important that you test your functions before submitting them. The only check that will be made at the point of submission to the Moodle server will be for the existence of functions with the right names. Therefore, pay close attention to examples given in relation to the earlier tasks and make sure that your functions work for those.

1.1 Rules

The assignment should be your own work without help from others. You may access all documentation that was permitted for the exam (see the information on Ultra). You may also view web pages referenced below (just click on the link which is formatted as underlined green text) and may look up online or in books the meaning of technical terms used in the assignment but should not seek further assistance from books or online sources.

You may import the `math` and `numpy` modules or individual items from those modules. You may also make the usual import of the `pyplot` part of `matplotlib` as `plt` and this is assumed in examples given below. Other imports are not allowed, specifically not other parts of `matplotlib` such as `matplotlib.patches` and `matplotlib.collections`.

Some marks will be reserved for the quality of plots produced, for example for sensible use of colour to distinguish different objects appearing in a plot. Marks will also be reserved for quality of code. In particular, there is a certain hierarchy to some of the tasks where a function written for an earlier task would naturally be called for a later task and doing so may be rewarded in marking.

Your submission will consist of two files:

- A single file of Python code which contains only imports, function definitions and comments. This should be submitted to the Moodle server. Functions may call each other and you may include additional functions to support the ones required as solutions to the assignment. Feel free to include comments and to leave commented-out code or commented-out print statements as examples of how your functions would be used. Do not call `plt.show()` in functions which are submitted as solutions to tasks.

- A single PDF file of your mathematical work, clearly identifying to which task each part of your file relates. This should be submitted to Gradescope having followed the link from Ultra.

It will probably not be possible to submit either file until after Christmas.

1.2 Difficulty of the tasks and time required

The tasks are ordered in a way that seems natural for the material. Tasks 1, 2, 3, 5 and 6 are deliberately clearly sign-posted. Tasks 4 and 7 are similar but you have to work out more for yourself.

Tasks 8 to 12 deal with the more challenging topic of simulating a individual particle bouncing off obstacles with 13 and 14 as small additional challenges. Tasks 15 and 16 deal with the extra challenge where the track of one particle becomes an obstacle for other particles.

Only you can decide how much time to spend on the assignment between now and the deadline. However, I do not expect you to spend days on this assignment or to make yourself miserable trying to answer questions for which you are not ready. The more challenging tasks are provided to stimulate and provide interest for students who have been enjoying programming.

I have not decided a mark scheme yet and will not finalise the number of marks per task until have done most of the marking. However, my inclination is to award about half the marks for the more straightforward tasks and to divide the remaining marks roughly equally between the more challenging tasks.

2 The assignment

2.1 Definitions of objects in the plane and representations in Python

As we saw during the practicals, a *point* in the plane can be represented in Python by a list of two numbers, the coordinates of the point. For example the point $(5, 4)$ would be represented by `[5, 4]` (or of course by various other forms such as `[5.0, 4.0]`).

An equation of the form $ax + by + c = 0$, where a, b and c are real numbers, defines a *line* in the plane. In Python, we can represent such an equation by a Python list containing a, b and c. For example, the equation $-3x + 2y + 5 = 0$ would be represented by `[-3, 2, 5]`. We can also understand this to represent the line defined by the equation (noting that infinitely many equations define the same line).

Any two distinct points in the plane define a *line segment* connecting the points. In Python, this can be represented as a list containing the two points, each point being represented as a list of two numbers. For example the line segment connecting $(1.5, 2.4)$ to $(5, 4)$ would be represented as `[[1.5, 2.4], [5, 4]]`.

A *directed line segment* is a line segment going from one point in the plane to another. This can be represented in the same way as a line segment but now the order of the points matters: `[[1, 3], [2, -1]]` is the same line segment as `[[2, -1], [1, 3]]` but they are different directed line segments. The first represents $(1, 3) \to (2, -1)$ whereas the second represents $(2, -1) \to (1, 3)$.

A *ray* is a part of a line that starts at a point in the plane and extends to infinity in a particular direction from that point. A ray can be represented in Python by a list with two elements: the first element is a list of two numbers representing a point and the second element is an angle (in radians) indicating the direction in the usual definition for polar coordinates. For example, `[[0, 0], 0]` denotes the ray starting at the origin and heading along the x-axis in the direction of increasing x, and `[[1.2, 2.1], math.pi/2]` denotes the ray starting at $(1.2, 2.1)$ and heading away parallel to the y-axis in the direction of increasing y.

A *path* in the plane is a sequence of directed line segments where the end point of one segment is the starting point of the next segment. A path can be represented in Python as a list of points. For example the path $(1, 2) \to (3, -1) \to (4.5, 6)$ consists of the directed line segments $(1, 2) \to (3, -1)$ and $(3, -1) \to (4.5, 6)$ and would be represented in Python by `[[1, 2], [3, -1], [4.5, 6]]`.

A rectangle in the plane with sides parallel to the coordinate axes is determined by a range of values for x and a range of values for y. A rectangle of this kind will be called a *window* in what follows. A window can be represented in Python as a list containing two lists each of which is a list of two numbers. For example `[[1, 3.4], [-2.1, 1]]` represents the window where x ranges from 1 to 3.4 and y ranges from -2.1 to 1.

2.2 Plotting objects in the plane

In what follows, we are going to find it useful to be able to add individual points, line segments etc to a plot. You will write functions that do this. It is important that these functions do not themselves call `plt.show` as we want to be able to add multiple things to a plot before displaying it on the screen using `plt.show()`. If you find this confusing, look at the example at the end of this section.

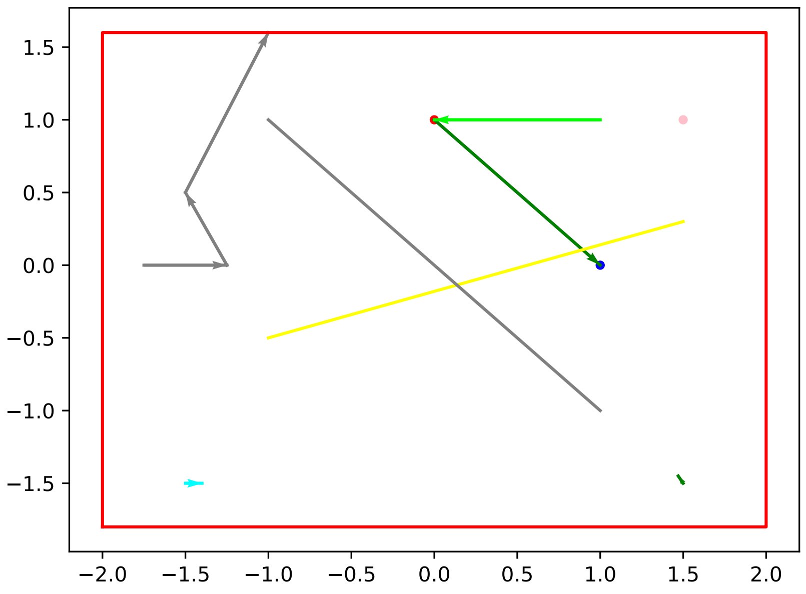
On Ultra, find `drawdirseg.py`. It defines a function to add a directed line segment to a plot in a specified colour. The function is `draw_directed_segment(dirseg, col)` and has two arguments: a directed line segment `dirseg` represented as described above and a colour `col` specified in a way acceptable to matplotlib (more on this below).

Have a look at the function. Note that it does not return anything. Instead it has an effect on the plot being produced by calling `plt.plot` and `plt.quiver`. In the latter call, the setting of `width=.004` works OK on my screen but you may find that you need to adjust it. Note also how the `col` argument is used to change to the colour in the calls to `plt.plot` and `plt.quiver`.

Task 1 (Python) Write the following Python functions based on on the Python representations of planar objects described in section 2.1:
`draw_point(pt, col)` to add the point specified by `pt` to a plot in the specified colour.
`draw_segment(seg, col)` that adds the line segment specified by `seg` to a plot in the specified colour.
`draw_path(path, col)` that adds the path specified by `path` to a plot as a sequence of directed line segments in the specified colour.
`draw_window(win, col)` to add the boundary of the window specified by `win` to a plot in the specified colour.

All of these can be implemented by using `plt.plot` or functions that are part of the assignment to do the actual work.

To test your code and the `draw_directed_segment` function defined above, find `example1.py` on Ultra which contains some code you can use to test your your four functions and my `draw_directed_segment`. When run, the result should be similar to:



In the code, note the use of two different ways to specify a colour to `matplotlib`. In fact a colour can be specified in several different ways described in the matplotlib tutorial on colors. The two easiest are by name (see the "xkcd color survey" link from the tutorial) or as an RGB color by providing a list of three numbers each between 0 and 1 (the amounts of red, green and blue light to include in the color). If you don't know what an RGB colour is, see the Wikipedia article about RGB colours.

In the rest of this assignment, functions written to draw things should make use of the functions defined in this section 2.2.

2.3 Intersections of lines and line segments

Task 2 (Math) Work out mathematically how to find where two lines in the plane cross: where they *intersect*. Assume that the lines are specified by equations of the the form described in section 2.1. Work out how to detect the case that the lines are parallel and do not cross.

Write a Python function `lines_intersect(line1, line2)` where `line1` and `line2` are lists of three numbers, each representing a line in the plane as described in section 2.1. If the lines cross, the function should compute the point of intersection and return it using the Python representation as a list of two numbers. In the case where the lines are parallel, the function should return the special Python value `None`. Make sure that your function works when either line is parallel to the x-axis or the y-axis, i.e. when the corresponding value of a or b is zero.

For example, `lines_intersect([1, 1, -3], [1, -1, 1])` should return `[1.0, 2.0]` and `lines_intersect([1, 1, 0], [2, 2, 3])` should return `None`

Task 3 (Math) The goal in this task is to write a function that computes the intersection between two line segments. The task is broken into sub-tasks to help you.

Task 3a For any line segment in the plane, there is a unique line to which it belongs. Write a Python function `line_from_segment(seg)` where the argument `seg` is a line segment represented as described in section 2.1. The function should return an equation for the line to which the segment belongs using the representation for lines described in section 2.1.

For example, `line_from_segment([[0,0], [1,1]])` might return `[1,-1, 0]` and `line_from_segment([[3,0], [3,2]])` might return `[2, 0, -6]` (remember that there are other correct answers).

Task 3b Given a line segment and a point, either the point lies on the line to which the segment belongs or it does not. This sub-task is concerned with the situation where the point does lie on that line. Then the point may or may not lie on the segment. Write a Python function `point_on_segment(point, seg)` that returns `True` if the point lies on the segment and `False` if does not. Your calculation should assume that the point does lie on the line to which the segment belongs (subject of course to the usual issues about accuracy of calculations using floating point numbers).

For example, `point_on_segment([0, -1], [[1,1], [3, 5]])` should return `False` and `point_on_segment([2, 3], [[1,1], [3, 5]])` should return `True`

Hint: if the point lies on the segment, the vectors formed by the difference between the point and the two ends of the segment point in opposite directions and their dot product will therefore be negative.

Task 3c Write a Python function `segments_intersect(seg1, seg2)` that computes the intersection between two line segments represented as described in section 2.1. If the segments cross, the function should return the point of intersection. Otherwise, it should return `None`, i.e. when there no intersection or when the segments overlap and the intersection consists of a line segment.

Hint: one way to compute the intersection of two line segments is to first find the intersection of the two lines two which the segments belong. You can then check if that point lies on both segments.

Task 4 (Math) Write a Python function `ray_segment_intersect(ray, seg)` that computes the intersection between a ray `ray` and a line segment `seg`. If the ray crosses the segment, the function should return the point of intersection. Otherwise, it should return `None`, i.e. when there is no intersection or when the the ray and segment overlap and the intersection consists of a line segment.

Hint: follow a similar approach to task 3.

2.4 Plotting lines and rays in windows

Lines and rays extend to infinity which is a nuisance when plotting them. Therefore, it can be useful to specify a window limiting the parts of lines and rays that should appear. The tasks in this section provide simple applications for some functions written for earlier tasks.

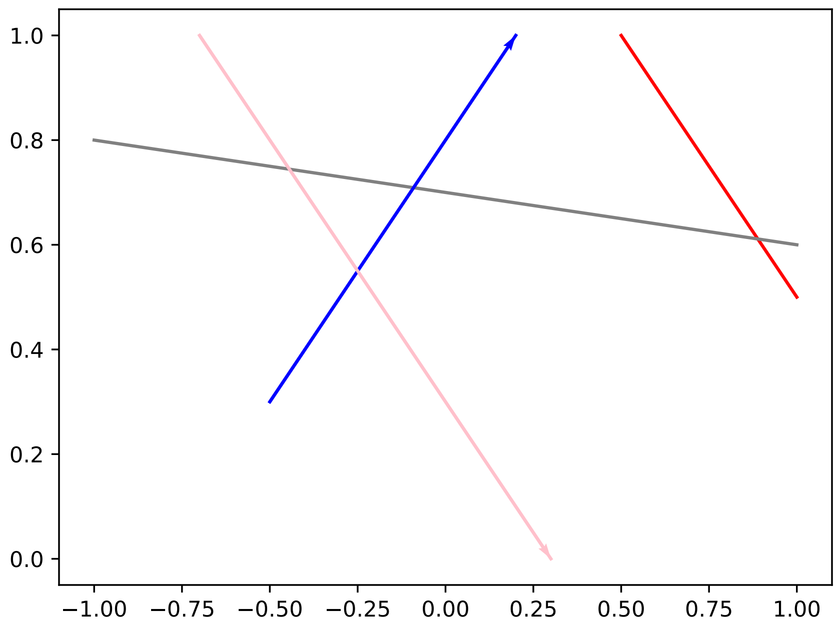
Task 5 Write `draw_line_in_window(line, window, col)` which adds to a plot the part of the line specified by `line` that lies in the window specified by `window`, drawn in colour `col`. This function should not draw the window, just the required line segment. If no part of the line lies in the window, the function should not add anything to the plot.

Hint: make use of functions written in section 2.3 to determine a line segment to draw. You may find it helpful first to write a functions to find the intersection between a line and a segment and to compute the boundary of a window a a list of line segments.

Task 6 Write a Python function `draw_lines_and_window(line1, line2, window)` which adds to a plot the parts of two lines that lie in the specified window and adds a markers at the point where the lines intersect if they intersect at a single point and that point lies in the window. It should also draw the boundary of the window. Make choices of colours that highlight the different objects.

Task 7 Write Python function `draw_ray_in_window(ray, window, col)` that adds to a plot the directed line segment that is the part of the ray `ray` that lies in the window `window`, drawn in colour `col`.

To help you test your functions for tasks 5 and 7, copy the code in `example2.py` on Ultra. When run, the result should be similar to:



2.5 Bouncing particle

Suppose that:

- a ray represents the track of a particle starting at a specified point in the plane and moving in a specified direction, when there is nothing with which the particle can collide;
- a line segment in the plane represents an obstacle with which the particle might collide;
- if the ray and line segment intersect, the particle collides with the obstacle and bounces off in the complementary direction as shown in the figure below, i.e. there is a new ray, representing the particle's track after the collision. The new ray starts from the point of collision and the direction of the new ray depends on the orientation of the line segment and the direction of the original ray.

You might recognise this as a model of what happens when (a photon of) light hits a mirror or a small hard ball collides with a hard surface.

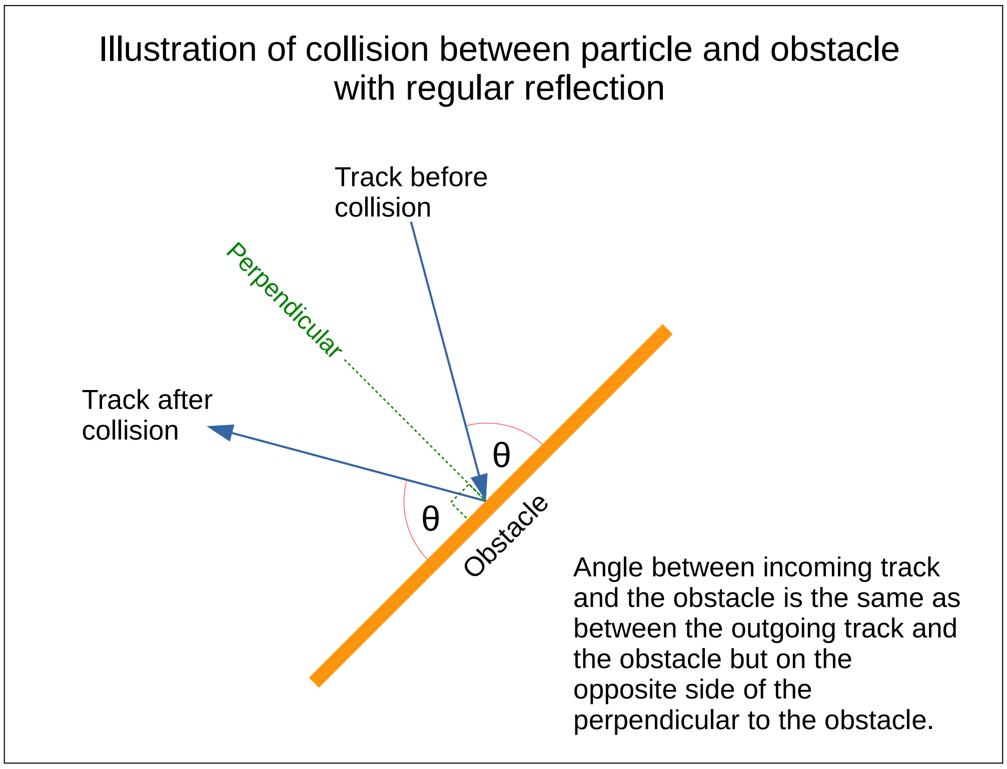


Illustration of collision between particle and obstacle with regular reflection
Track before collision
Perpendicular
Track after collision
θ
θ
Obstacle
Angle between incoming track and the obstacle is the same as between the outgoing track and the obstacle but on the opposite side of the perpendicular to the obstacle.

Task 8 (Math) Write Python function `one_bounce(ray, seg, win)` where `ray` represents the initial track of a particle and `seg` represents a single obstacle. The function should return a path representing the track of the particle until it leaves the window specified by `win`. This will be a path with a single segment if the particle does not collide with the obstacle and with two segments if it does. You may assume that the starting point of `ray` lies within the window and that the whole of the obstacle lies within the window.

For example, after `seg = [[0,3], [3,0]]` and `win = [[-1,4], [-1, 4]]`, `one_bounce([[1,1], math.pi/6], seg, win)` should return `[[1, 1], [1.634, 1.366], [0.268, -1.0]]` (rounding coordinates here to 3 decimal places just for readability) and `one_bounce([[0.75, 2], math.pi+0.1], seg, win)` should return `[[0.75, 2], [-1.0, 1.824]]`.

Hint for the Math part: consider three unit vectors v_{in}, v_{out} and s representing the direction of the inbound and outbound rays and the orientation of the segment. Let $s^{\dagger}$ be a unit vector orthogonal to s. From the geometry in the figure show that $v_{\text{in}}.s = v_{\text{out}}.s$ and $v_{\text{in}}.s^{\dagger} = -v_{\text{out}}.s^{\dagger}$ where . denotes the usual dot product for vectors. Hence express v_{out} in terms of v_{in} and s.

Hint for the Python part: read the help for the `atan2` function in the `math` module.

Task 9 Write `draw_one_bounce(path, seg, win)` that adds a depiction of the bouncing process to a plot. Here `path` would be a list of directed line segments (obtained from calling `one_bounce`) and `seg` and `win` would be the same values as passed to `one_bounce`).

For example, one might do

```
window = [[0,1],[0,1]]
obstacle = [[0.25,0.75], [0.75, 0.25]]
path = one_bounce([[0.1, 0.1], math.pi/4], obstacle, window)
draw_one_bounce(path, obstacle, window)
plt.show()
```

When writing `draw_one_bounce`, you should choose colors for the elements being added to the plot that distinguish the different types of object.

Task 10 Write Python function `multi_bounce(ray, seglist, win, maxpathlen)` where `ray` and `win` are as before, `seglist` is a list of obstacles and `maxpathlen` is a limit on the number of collisions to simulate. The function should return a path representing the track of the particle until either it leaves the window or the length ~~of~~ (number of segments) of the path is `maxpathlen`.

Task 11 Write `draw_multi_bounce(path, seglist, win`) that adds adds a depiction of the bouncing process to a plot. Here `path` would be a path (obtained from calling `multi_bounce`) and `seglist` and `win` would be the same values as passed to `multi_bounce`). Note that this function should also work if used to depict the bouncing ~~processes~~ process in the later ~~tasks 13and 14~~task 13.

Task 12 Write `draw_n_multi_bounce(pathlist, seglist, win`) that adds to a plot a depiction of the bouncing process for n particles starting with different rays but all in the same window and with the same obstacles. Here `pathlist` would be a list of length n, each element of which would be a path. The paths would be obtained by calling `multi_bounce` n times with different rays using the same values of `seglist` and `win` each time. Note that this function could also be used to depict the bouncing process for a single particle in the later task 14.

Task 13 One way of keeping the process alive while keeping it inside the window is for the particle to bounce when it meets the boundary of the window. However, if the particle hits a corner of the window, it should still escape to infinity. Write Python function `multi_bounce_reflect(ray, seglist, win, maxpathlen)` that returns a path representing the track of the particle until the length of the path is `maxpathlen` or the particle escapes to infinity.

Task 14 Another way of keeping the process alive but within the window is for the window to “wrap around”, i.e. when the particle passes through an edge of the window, it reappears at the corresponding point on the opposite edge and continues in the same direction as before. Write Python function
`multi_bounce_wrap(ray, seglist, window, pathlen)` that returns a ~~path~~ list of paths representing the track of the particle until the total length of the ~~path~~ paths in the list is `pathlen`. The first path in the list starts at the starting point and ends when the particle first hits an edge (or when the length of the path is `pathlen`~~.~~). The next path starts where the particle reappears on the opposite edge and so on.

2.6 Bouncing more than one particle

Now imagine that there are two or more particles on the move and that each particle, as well as bouncing off obstacles (and possibly the boundary of a window), also bounces off the tracks left by the other particles (it does not bounce off its own track). To make this more precise, we are going to assume that the particles start from their initial points at the same time and move at the same constant speed throughout so that distance travelled corresponds to time.

Task 15 Write Python function
`two_multi_bounce(ray1, ray2, segs, win, mode1, mode2, maxcolls)` where `ray1` and `ray2` represent the initial tracks of two particles, `segs` is a list of line segments representing obstacles with which the particles collide, `win` is the usual window and `mode1` and `mode2` indicate what should happen when the corresponding particle reaches the boundary of the window: 0 means that the particle heads off to

infinity but we don't see it any longer, 1 means that it bounces off the boundary and 2 means that it wraps around. The function should return a list with two elements, each of which is a ~~path~~ list of paths representing the track of a particle. `maxcolls` is the maximum total number of collisions to simulate~~.~~, i.e. the process stops at the time of that collision.

Task 16 Write Python function
`n_multi_bounce(rays, segs, win, modes, maxcolls)` where `rays` is a list of rays representing initial tracks of n particles and `modes` is a list of n modes for behaviour at the boundary of the window. Each of the particles bounces off the tracks left by the others but does not bounce off its own track. The function should return a list with n elements, each of which is a ~~path~~ list of paths representing the track of a particle. `maxcolls` is again the maximum total number of collisions to simulate.

You should be able to use your `draw_n_multi_bounce` function to visualise the results from these functions when you are testing them.