Seismic Wave Propagation

Abstract

We aim to generate brief perturbations at the Earth's surface through the study of seismic waves in a 2-dimensional model. We derive a model which maps the propagation of a seismic wave through various layers. We will derive a model to integrate the discrete version of the wave equations to create heat-map snapshots of the waves generated. We then use the experimental data provided in the module `s_wave_base` to calibrate our model. Now we will understand why seismic wave modelling is necessary.

1 Introduction

Seismic wave modelling is necessary to comprehend and predict earthquakes and tsunamis [1]. This is important because it can save many lives. According to the World Health Organization, between 1998-2017, more than 250 000 people died due to tsunamis and nearly 750,000 died due to earthquakes [2], [3]. They are also valuable as they provide evidence for the structure of the Earth's Interior. This means we can then understand how and why the Earth's climate has changed previously which we can use to fight climate change [4].

First, we will simulate wave propagation by proving the solutions to the 2 differential equations we form. Next, we will derive a method to solve these equations using the boundary conditions provided. Then we will find the partial derivatives of a function that can map on a 2-dimensional regular lattice. After we will do some calculations for a ray path going through various layers, which we will use to simulate brief excitations, measuring the signal received at 3 sensor locations. So what are seismic waves?

2 The Wave Equation

Seismic Waves are propagating vibrations in rocks. Vibrations in a solid are described by considering the local infinitesimal displacement of each point with respect to its rest position u_i [5]. Generally, this is a 3-component vector and the wave equation is then given in component notation by:

$$\rho\frac{\partial^2 v}{\partial t^2} = (\lambda+\mu)\sum_{j=1}^{3}\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu\sum_{j=1}^{3}\frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{1}$$

where ρ is the rock density and λ and μ are Lamé parameters. Lamé constants are two material-dependent values that appear in strain-stress relationships [6]. λ, μ and ρ all take positive values.

The wave equation in 1-dimension can be represented as: $u_{tt} = c^2 u_{xx}$. We are going to prove the solution is $u = f(x+ct) + g(x-ct)$. Start by setting $\xi = x + ct$ and $\eta = x - ct$ [7]. Our definitions of η and ξ imply that $x = \frac{1}{2}(\eta+\xi)$ and $ct = \frac{1}{2}(\eta-\xi)$. Via the Chain Rule, we get:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}.$$

The above simplifies to $u_\xi + u_\eta$ as $\frac{\partial \xi}{\partial x}$ and $\frac{\partial \eta}{\partial x}$ are 1. Now, we aim to define u_{xx}:

$$u_{xx} = (\partial_\xi + \partial_\eta)^2 u = u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}.$$

Using the same idea, we rewrite u_{tt}:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}.$$

This simplifies to $cu_\xi + cu_\eta$ as $\frac{\partial \xi}{\partial x}$ and $\frac{\partial \eta}{\partial x}$ are 1. Now we define u_{tt}:

$$u_{tt} = c^2(\partial_\xi - \partial_\eta)^2 u = c^2 u_{\xi\xi} - c^2 2u_{\eta\xi} + c^2 u_{\eta\eta}.$$

Hence, when we do $u_{tt} - c^2 u_{xx}$ we get the following:

$$u_{tt} - c^2 u_{xx} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) - c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = 0.$$

As we can see many terms cancel leaving:

$$\begin{aligned}
-4c^2 u_{\xi\eta} &= 0, \\
\frac{\partial}{\partial \eta}(\frac{\partial u}{\partial \xi}) &= 0, \\
\frac{\partial u}{\partial \xi} &= f'(\xi), \\
u &= f(\xi) + g(\eta), \\
u &= f(x+ct) + g(x-ct).
\end{aligned}$$

As required.

Note that $f(x+ct)$ represents a signal moving to the left with speed c and $g(x-ct)$ represents a signal moving to the right with speed c.

The coordinate system used will be: x_1 and x_2 coordinates run parallel to the surface while the x_3 coordinate points downwards with the origin being the surface of the Earth. Then we will assume $u_2 = 0$ and $\frac{\partial u_i}{\partial x_2} = 0$ so we can make this 3-dimensional problem into a 2-dimensional one.

Using the definitions $x = x_1$, $z = x_3$, $v = u_1$, $w = u_3$ and a change in the notation for partial derivatives, equation (1) becomes

$$\rho v_{tt} = (\lambda + \mu)(v_{xx} + w_{xz}) + \mu(v_{zz} + v_{xx}), \tag{2}$$

$$\rho w_{tt} = (\lambda + \mu)(v_{xz} + w_{zz}) + \mu(w_{xx} + w_{zz}), \tag{3}$$

which is a set of 2 linear partial differential equations. The coordinate system we use is the one geologists use (Figure 1): the z axis points downwards, $z = 0$ corresponding to the surface of the Earth.

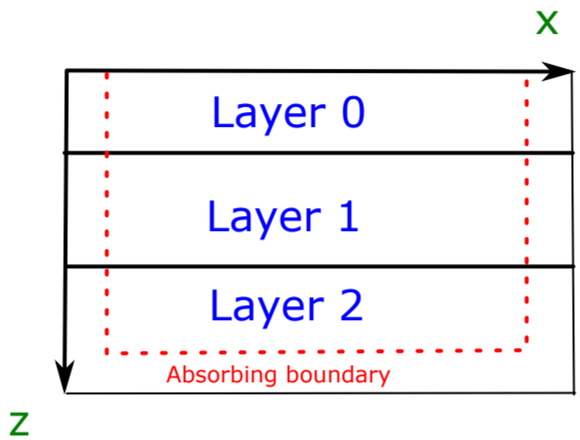

Figure 1: Graphical representation of the coordinate system, the layer structure, and the damping region.

The x axis runs horizontally to the right. Each rock layer has different values for the parameters ρ, λ, and μ and, for simplicity, we will assume that the layers are horizontal and have a constant thickness, making the 3 parameters functions of only the depth (z) and not the horizontal position (x). We will start by computing some simple analytic solutions to identify the 2 types of seismic waves. The aim of the next subsection is to prove solutions to the equations (2) and (3).

2.1 Plane Wave Solutions

Q1

The first solutions we want to prove are that $v(x,t) = A_P\sin(k_P x - \omega_P t)$ and $w = 0$. To make notation simpler, denote $v(x,t) = v$ and $w(x,t) = w$. As we have $w = 0$, w_{xz} is 0 too and $v_{zz} = 0$ by inspection as v has no z terms. Hence equation (2) simplifies to:

$$\rho v_{tt} = (\lambda + 2\mu)v_{xx}. \tag{4}$$

Now we need to work out v_{tt} and v_{xx}:

$$\begin{aligned} v_t &= -\omega_P A_P\cos(k_P x - \omega_P t), \\ v_{tt} &= -\omega_P^2 A_P\sin(k_P x - \omega_P t), && (5) \\ v_x &= k_P A_P\cos(k_P x - \omega_P t), \\ v_{xx} &= -k_P^2 A_P\sin(k_P x - \omega_P t). && (6) \end{aligned}$$

Using equations (4), (5), and (6), we get:

$$\begin{aligned} \rho(-\omega_P^2 A_P\sin(k_P x - \omega_P t)) &= (\lambda + 2\mu)(-k_P^2 A_P\sin(k_P x - \omega_P t)), \\ \rho\omega_P^2 &= (\lambda + 2\mu)k_P^2, \\ w_P^2 &= \frac{\lambda + 2\mu}{\rho}k_P^2, \\ w_P &= k_P\sqrt{\frac{\lambda + 2\mu}{\rho}}. && (7) \end{aligned}$$

Therefore v is a solution for equation (2) when w_P is in the above form. To get V_P , we divide ω_P by k_P.

$$\begin{aligned} V_P &= \frac{\omega_P}{k_P}, \\ V_P &= \sqrt{\frac{\lambda + 2\mu}{\rho}}. && (8) \end{aligned}$$

Now we need to show our v and w solve equation (3). w_{tt}, w_{zz} and w_{xx} are all 0 as $w = 0$, leaving $(\lambda + \mu)v_{xz} = 0$ which we know to be true as v has no z component. Hence, the solution satisfies equations (2) and (3).

Now we need to prove the following: $v(z,t) = A_S\sin(k_S z - \omega_S t)$ and $w = 0$ are solutions to equations (2) and (3). Let's re-define v as $v = v(z,t)$ and start with equation (2), so $w = 0$ meaning $w_{xz} = 0$ too and $v_{xx} = 0$ by inspection as v has no x terms. Hence equation (2) simplifies to:

$$\rho v_{tt} = \mu v_{zz}. \tag{9}$$

Now we need to work out v_{tt} and v_{zz}:

$$\begin{aligned} v_t &= -\omega_S A_S\cos(k_S z - \omega_S t), \\ v_{tt} &= -\omega_S^2 A_S\sin(k_S z - \omega_S t), && (10) \\ v_z &= k_S A_S\cos(k_S z - \omega_S t), \\ v_{zz} &= -k_S^2 A_S\sin(k_S z - \omega_S t), && (11) \end{aligned}$$

So using equations (9), (10), and (11) we get:

$$\begin{aligned} \rho(-\omega_S^2 A_S \sin(k_S z - \omega_S t)) &= \mu(-k_S^2 A_S \sin(k_S z - \omega_S t)), \\ \rho \omega_S^2 &= \mu k_S^2, \\ \omega_S &= k_S \sqrt{\frac{\mu}{\rho}}. \end{aligned} \tag{12}$$

Therefore v and w are solutions for equation (2) when ω_S is in the above form. We do the same as before to get V_S:

$$V_S = \frac{\omega_S}{k_S},$$
$$V_S = \sqrt{\frac{\mu}{\rho}}.$$

Like before, we need to show the solution solves equation (3). We get $(\lambda + \mu)v_{xz} = 0$ again as all the partial derivatives of w are 0 and this is true because v has no x component.

ω_P and k_P represent the angular frequency and wave number of their corresponding waves respectively. The formula for the speed of a wave is angular frequency divided by wave number, i.e. ω_P/k_P which is how we defined V_P [8]. The same idea will apply to ω_S, k_S and hence V_S. So V_P and V_S can be interpreted as the speed of the corresponding waves.

Note that the small displacements making a P wave (also known as a primary or pressure wave) are parallel to the travelling direction of the wave while for S waves (secondary or shear waves), they are perpendicular to it.

Now we need to show that:

$$\begin{aligned} v(x,z,t) &= \alpha f(x,z,t), \\ w(x,z,t) &= \beta f(x,z,t), \end{aligned} \tag{13}$$

where

$$\begin{aligned} f(x,z,t) &= A\sin(k(\alpha x + \beta z) - \omega t), \\ \alpha^2 + \beta^2 &= 1, \end{aligned} \tag{14}$$

is a solution to equations (2) and (3).

Q2

To simplify this, set $v = v(x,z,t)$, $f = f(x,z,t)$ and $w = w(x,z,t)$. Let us start by working out all the partial derivatives of f.

$$\begin{aligned} f &= A\sin(k(\alpha x + \beta z) - \omega t), \\ f_x &= \alpha k A\cos(k(\alpha x + \beta z) - \omega t), \\ f_{xx} &= -\alpha^2 k^2 A\sin(k(\alpha x + \beta z) - \omega t), \\ f_t &= -\omega A\cos(k(\alpha x + \beta z) - \omega t), \\ f_{tt} &= -\omega^2 A\sin(k(\alpha x + \beta z) - \omega t), \\ f_z &= \beta k A\cos(k(\alpha x + \beta z) - \omega t), \\ f_{zz} &= -\beta^2 k^2 A\sin(k(\alpha x + \beta z) - \omega t), \\ f_{xz} &= -\alpha\beta k^2 A\sin(k(\alpha x + \beta z) - \omega t). \end{aligned} \tag{15}$$

For future parts of this question, denote $A\sin(k(\alpha x + \beta z) - \omega t)$ as σ. Equation (2) in this case of v and w simplifies to:

$$\rho \alpha f_{tt} = (\lambda + \mu)(\alpha f_{xx} + \beta f_{xz}) + \mu(\alpha f_{zz} + \alpha f_{xx}). \tag{16}$$

Using equation (15) this becomes:

$$\begin{aligned}
-\rho\alpha\omega^2\sigma &= (\lambda+\mu)(-\alpha^3k^2\sigma - \alpha\beta^2k^2\sigma) + \alpha\mu(-\beta^2k^2\sigma - \alpha^2k^2\sigma), \\
\rho\alpha\omega^2 &= \alpha(\lambda+\mu)(\alpha^2k^2 + \beta^2k^2) + \alpha\mu(\beta^2k^2 + \alpha^2k^2), \\
\rho\omega^2 &= k^2(\lambda+2\mu)(\alpha^2+\beta^2), \\
\omega &= k\sqrt{\frac{(\lambda+2\mu)(\alpha^2+\beta^2)}{\rho}}. \qquad (17)
\end{aligned}$$

Remember $\alpha^2+\beta^2 = 1$, so:

$$\omega = k\sqrt{\frac{\lambda+2\mu}{\rho}}. \qquad (18)$$

This is of the same form as in equation (7) hence these values of v and w solve equation (2) if the above equation for ω is satisfied. Now we need to show that v and w solve equation (3). Equation (3) simplifies to:

$$\rho\beta f_{tt} = (\lambda+\mu)(\alpha f_{xz} + \beta f_{zz}) + \mu(\beta f_{xx} + \beta f_{zz}). \qquad (19)$$

Using equation (15) again, we get:

$$\begin{aligned}
-\rho\beta\omega^2\sigma &= (\lambda+\mu)(-\alpha^2\beta k^2\sigma - \beta^3k^2\sigma) + \mu(-\alpha^2\beta k^2\sigma - \beta^3k^2\sigma), \\
\rho\beta\omega^2 &= (\lambda+\mu)(\alpha^2\beta k^2 + \beta^3k^2) + \mu(\alpha^2\beta k^2 + \beta^3k^2), \\
\rho\omega^2 &= k^2(\lambda+2\mu)(\alpha^2+\beta^2), \\
\omega &= k\sqrt{\frac{(\lambda+2\mu)(\alpha^2+\beta^2)}{\rho}}. \qquad (20)
\end{aligned}$$

This is of the same form as the equation in (17) so equation (3) will be solved if ω is of the form as shown in equation (18). Therefore the values of v and w solve equation (2) and equation (3). In this case, the form of ω is the same as in equation (7) and this was for a P wave Hence, f is a P wave. As f is a P wave, its direction of propagation is parallel to the travelling direction of the wave.

Now we need to prove the same for $v(x,z,t) = \beta f(x,z,t)$ and $w(x,z,t) = -\alpha f(x,z,t)$. So when we substitute this into equation (2) we get:

$$\rho\beta f_{tt} = (\lambda+\mu)(\beta f_{xx} - \alpha f_{xz}) + \mu(\beta f_{zz} + \beta f_{xx}). \qquad (21)$$

Using (15) this becomes:

$$\begin{aligned}
-\rho\beta\omega^2\sigma &= (\lambda+\mu)(-\alpha^2\beta k^2\sigma - \beta^3k^2\sigma) + \mu(-\alpha^2\beta k^2\sigma - \beta^3k^2\sigma), \\
\rho\beta\omega^2 &= (\lambda+\mu)(\alpha^2\beta k^2 + \beta^3k^2) + \mu(\alpha^2\beta k^2 + \beta^3k^2), \\
\rho\omega^2 &= k^2(\lambda+2\mu)(\alpha^2+\beta^2). \qquad (22)
\end{aligned}$$

Hence, the same idea applies as in equation (17) so our new values of v and w satisfy equation (2). Now to test equation (3):

$$\begin{aligned}
-\rho\alpha\omega^2\sigma &= (\lambda+\mu)(-\alpha^3k^2\sigma - \alpha\beta^2k^2\sigma) + \alpha\mu(-\beta^2k^2\sigma - \alpha^2k^2\sigma), \\
\rho\alpha\omega^2 &= \alpha(\lambda+\mu)(\alpha^2k^2 + \beta^2k^2) + \alpha\mu(\beta^2k^2 + \alpha^2k^2), \\
\rho\omega^2 &= k^2(\lambda+2\mu)(\alpha^2+\beta^2), \\
\omega &= k\sqrt{\frac{(\lambda+2\mu)(\alpha^2+\beta^2)}{\rho}}. \qquad (23)
\end{aligned}$$

This is of the same form as equation (17), meaning equation (3) is satisfied too. Therefore, this v and w satisfy equations (2) and (3).

To fully encapsulate the problem, we must also add suitable boundary conditions. At the Earth-atmosphere interface, the condition is that there is no stress [5].

$$T_{ij} = \lambda \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu\left(\frac{\delta u_i}{\delta x_j} + \frac{\delta u_j}{\delta x_[i]}\right) = 0, \tag{24}$$

which in our 2-dimensional model becomes:

$$\begin{aligned} T_{13} = 0 = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\right), \\ T_{33} = 0 = \lambda \frac{\partial v}{\partial x} + (\lambda + 2\mu)\frac{\partial w}{\partial z}. \end{aligned} \tag{25}$$

For numerical solutions, we must use a finite domain. Unlike analytic solutions, where we can consider an infinite domain. We will use $x \in [0, X_{max}$, $z \in [0, Z_{max}]$, with $X_{max} = 4000$m and $Z_{max} = 2000$m. Note how the origin of our coordinate system corresponds to the top left of the domain.

We add some absorption for the other 3 sides of the domain by attenuating $\frac{dv}{dt}$ and $\frac{dw}{dt}$ on strips of with l_d on the edges of the domain (Figure 1) to avoid reflection of the waves on the boundaries. This will be further explained after equations (2) and (3) have been discretised.

Q3

To solve equations (2) and (3) numerically, we must first convert this system of second-order differential equations in time into 2 pairs of first-order differential equations with no parameters remaining on the left-hand side. To do this we need to use the boundary conditions given. These are:

$$\begin{aligned} T_{13} &= 0 = \mu(v_z + w_x), \\ T_{33} &= 0 = \lambda v_x + (\lambda + 2\mu) w_z. \end{aligned} \tag{26}$$

These can be simplified to:

$$v_z + w_x = 0, \tag{27}$$

$$\lambda v_x + (\lambda + 2\mu) w_z = 0. \tag{28}$$

We need to rearrange equation (27) and partially differentiate it with respect to x and z.

$$v_z = -w_x,$$

$$v_{zz} = -w_{xz}, \tag{29}$$

$$v_{xz} = -w_{xx}. \tag{30}$$

Now we should do the same as above for equation (28).

$$v_x = -\frac{\lambda + 2\mu}{\lambda} w_z, \tag{31}$$

$$v_{xx} = -\frac{\lambda + 2\mu}{\lambda} w_{xz}, \tag{32}$$

$$v_{xz} = -\frac{\lambda + 2\mu}{\lambda} w_{zz}. \tag{33}$$

Now, we will use equation (29) and equation (2) and aim to remove the w_{xz}.

$$\begin{aligned} \rho v_{tt} &= (\lambda + \mu)(v_{xx} - v_{zz}) + \mu(v_{xx} + v_{zz}), \\ \rho v_{tt} &= (\lambda + 2\mu) v_{xx} - \lambda v_{zz}. \end{aligned} \tag{34}$$

Now let us remove the v_{xx} using equations (32) and (29). Equation (32) can be re-written as:

$$v_{xx} = \frac{\lambda + 2\mu}{\lambda} v_{zz}. \tag{35}$$

Hence, equation (34) simplifies to:

$$\begin{aligned} \rho v_{tt} &= \frac{1}{\lambda}(\lambda + 2\mu)^2 v_{zz} - \lambda v_{zz}, \\ \rho v_{tt} &= \frac{1}{\lambda}(\lambda^2 + 4\mu\lambda + 4\mu^2) v_{zz} - \lambda v_{zz}, \\ v_{tt} &= \frac{4\mu}{\rho}(1 + \frac{\mu}{\lambda}) v_{zz}. \end{aligned} \tag{36}$$

This means the first pair of 1^{st} order differential equations that can be formed from equation (2) are:

$$\phi = v_t, \tag{37}$$

$$\phi_t = \frac{4\mu}{\rho}(1 + \frac{\mu}{\lambda}) v_{zz}. \tag{38}$$

Now for equation (3), we use the same idea as before. First, we aim to remove the v_{xz} using equation (30).

$$\begin{aligned} \rho w_{tt} &= (\lambda + \mu)(w_{zz} - w_{xx}) + \mu(w_{xx} + w_{zz}), \\ \rho w_{tt} &= (\lambda + 2\mu) w_{zz} - \lambda w_{xx}. \end{aligned} \tag{39}$$

Using equations (30) and (33) we can relate w_{xx} and w_{zz}.

$$w_{xx} = \frac{\lambda + 2\mu}{\lambda} w_{zz}.$$

Therefore equation (39) simplifies too:

$$w_{tt} = 0. \tag{40}$$

So, our second pair of 1^{st} order differential equations that can be formed from equation (3) are:

$$\psi = w_t, \tag{41}$$

$$\psi_t = 0. \tag{42}$$

Artificial Seismic waves are generated by geophysicists using vibroseis trucks as described in the video [9]. A vibroseis truck is a large truck that has a base plate in contact with the Earth [10]. A hydraulic jack and a heavy reaction mass are incorporated, along with the base plate, in each vibrator to impart vibrations in the Earth [10]. Figure (2) below shows a typical vibroseis source signature [10].

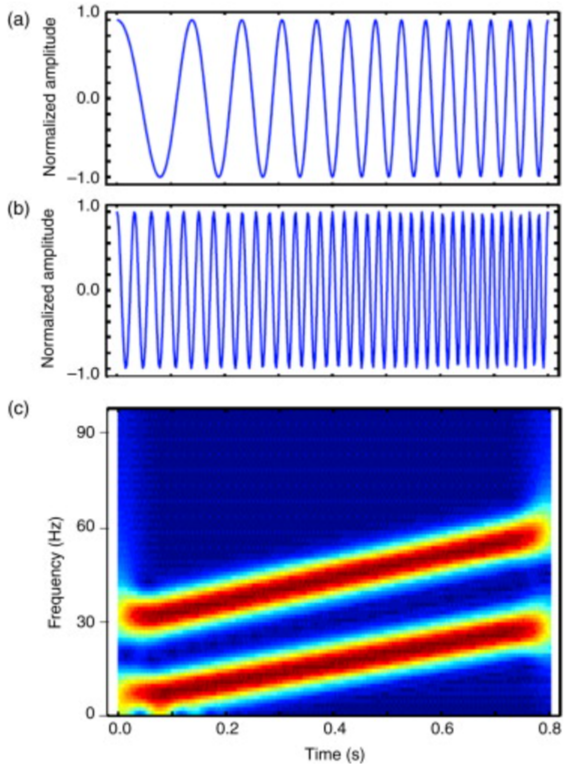

Figure 2: A typical vibroseis source signature [10]

We simulate these seismic waves by setting the displacement at the point $x = X_{max}/2$, $z = 0$, i.e. at the Earth's surface, to be

$$w(t, \frac{X_{max}}{2}, 0) = \sin(2\pi t\nu)\exp(-t\nu),\ t \in [0, 1/\nu], \tag{43}$$

where ν is set to 20Hz. **Q4**

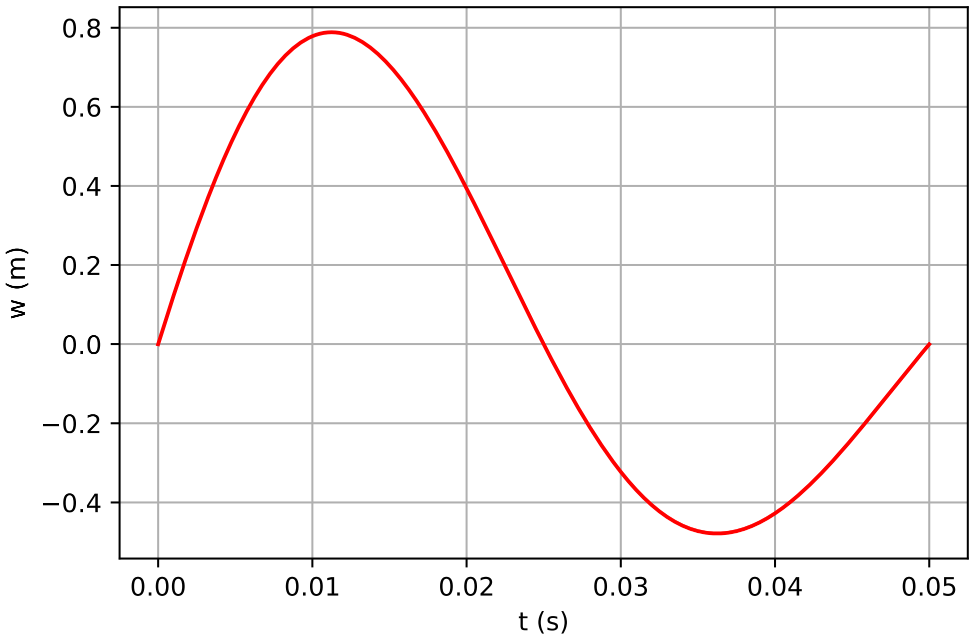

Figure 3: An illustration of the excitation for $\nu = 20$Hz

We will then consider seismic detectors which will capture the displacements v, w, and $\sqrt{v^2+w^2}$, the total displacement amplitude, at the Earth's surface. We will position them at $x=2000$, $x=2300$, and 2500, i.e. 0m, 300m, and 500m to the right of the source. In the next section, we will aim to solve equations (2) and (3) numerically but first, we have to discretise the equations.

3 Numerical Solutions

Given any function $f(x,z)$, we map it on a 2-dimensional regular lattice as follows:

$$f(x_i, z_j) = f_{i,j}, \tag{44}$$

where

$$x_i = x_0 + i\mathrm{dx}, \tag{45}$$
$$z_j = z_0 + j\mathrm{dz}, \tag{46}$$

with x_0 and z_0 representing the origin of the coordinate system. The number of lattice points in each direction will be $N_x = 1 + X_{max}/\mathrm{d}x$ and $N_z = 1 + Z_{max}/\mathrm{d}x$. (In the following calculations, x_0 and z_0 will be set to 0).

Q5

Now we need to show the following approximations are correct and determine the integer power m for each of them. Note: $O(\mathrm{dx}^m)$ means that the lowest power of dx in the remainder is m. For the forward finite difference, we consider the Taylor Series of the function $f(x_i, z_j)$ below. Remember we can denote as $f_{i,j}$ by equation (44).

$$\begin{aligned} f(x_i+\mathrm{dx}, z_j) &= f(x_i, z_j) + \sum_{n=1}^{\infty} \frac{\mathrm{dx}^n}{n!}\frac{\partial^n}{\partial x^n} f(x_i, z_j), \\ &= f_{i,j} + \mathrm{dx}\frac{\partial}{\partial x} f_{i,j} + \frac{\mathrm{dx}^2}{2!}\frac{\partial^2}{\partial x^2} f_{i,j} + \frac{\mathrm{dx}^3}{3!}\frac{\partial^3}{\partial x^3} f_{i,j} + ... \end{aligned} \tag{47}$$

Now we rearrange this for $\frac{\partial}{\partial x} f_{i,j}$.

$$\begin{aligned} \mathrm{dx}\frac{\partial}{\partial x} f_{i,j} &= f(x_i+\mathrm{dx}, z_j) - f_{i,j} - \frac{\mathrm{dx}^2}{2!}\frac{\partial^2}{\partial x^2} f_{i,j} - \frac{\mathrm{dx}^3}{3!}\frac{\partial^3}{\partial x^3} f_{i,j} - ..., \\ \frac{\partial}{\partial x} f_{i,j} &= \frac{f(x_i+\mathrm{dx}, z_j) - f_{i,j}}{\mathrm{dx}} - \frac{\mathrm{dx}}{2!}\frac{\partial^2}{\partial x^2} f_{i,j} - \frac{\mathrm{dx}^2}{3!}\frac{\partial^3}{\partial x^3} f_{i,j} - ... \end{aligned}$$

This can be further simplified to:

$$\frac{\partial}{\partial x} f_{i,j} = \frac{f(x_i+\mathrm{dx}, z_j) - f_{i,j}}{\mathrm{dx}} + O(\mathrm{dx}). \tag{48}$$

To simplify this we use equation (45). This equation implies that:

$$\begin{aligned} x_{i+1} &= x_0 + (i+1)\mathrm{dx}, \\ x_{i+1} &= x_0 + i\mathrm{dx} + \mathrm{dx}. \end{aligned} \tag{49}$$

We can rearrange equation (45) to remove the $i$dx:

$$i\mathrm{dx} = x_i - x_0. \tag{50}$$

Hence, equation (49) can be written as:

$$x_{i+1} = x_i + \mathrm{dx}. \tag{51}$$

So equation (48) can be simplified to:

$$\begin{aligned}\frac{\partial}{\partial x} f_{i,j} &= \frac{f(x_{i+1}, z_j) - f_{i,j}}{\mathrm{dx}} + O(\mathrm{dx}), \\ \frac{\partial f}{\partial x}(x_i, z_j) &= \frac{f_{i+1,j} - f_{i,j}}{\mathrm{dx}} + O(\mathrm{dx}). \end{aligned} \tag{52}$$

As required.

Now for the symmetric difference we consider equation (47) but we replace +dx with -dx so it becomes:

$$f(x_i - \mathrm{dx}, z_j) = f_{i,j} - \mathrm{dx}\frac{\partial}{\partial x} f_{i,j} + \frac{\mathrm{dx}^2}{2!}\frac{\partial^2}{\partial x^2} f_{i,j} - \frac{\mathrm{dx}^3}{3!}\frac{\partial^3}{\partial x^3} f_{i,j} + ... \tag{53}$$

To get the symmetric difference we subtract $f(x_i - \mathrm{dx}, z_j)$ from $f(x_i + \mathrm{dx}, z_j)$, which is defined in equation (47).

$$f(x_i + \mathrm{dx}, z_j) - f(x_i - \mathrm{dx}, z_j) = 2\mathrm{dx}\frac{\partial}{\partial x} f_{i,j} + \frac{2\mathrm{dx}^3}{3!}\frac{\partial^3}{\partial x^3} f_{i,j} + ...$$

Now we rearrange the above equation for $\frac{\partial}{\partial x} f_{i,j}$.

$$\begin{aligned} 2\mathrm{dx}\frac{\partial}{\partial x} f_{i,j} &= f(x_i + \mathrm{dx}, z_j) - f(x_i - \mathrm{dx}, z_j) - \frac{2\mathrm{dx}^3}{3!}\frac{\partial^3}{\partial x^3} f_{i,j} - ... \\ \frac{\partial}{\partial x} f_{i,j} &= \frac{f(x_i + \mathrm{dx}, z_j) - f(x_i - \mathrm{dx}, z_j)}{2\mathrm{dx}} - \frac{\mathrm{dx}^2}{3!}\frac{\partial^3}{\partial x^3} f_{i,j} - ... \end{aligned} \tag{54}$$

By equations (45) and (50), we can write:

$$\begin{aligned} x_{i-1} &= x_0 + (i-1)\mathrm{dx}, \\ x_{i-1} &= x_0 + i\mathrm{dx} - \mathrm{dx}, \\ x_{i-1} &= x_0 + x_i - x_0 - \mathrm{dx}, \\ x_{i-1} &= x_i - \mathrm{dx}. \end{aligned} \tag{55}$$

Therefore we can use equations (51) and (55) to simplify equation (53) to:

$$\frac{\partial}{\partial x} f_{i,j} = \frac{f(x_{i+1}, z_j) - f(x_{i-1}, z_j)}{2\mathrm{dx}} - \frac{\mathrm{dx}^2}{3!}\frac{\partial^3}{\partial x^3} f_{i,j} - ... \tag{56}$$

$$\frac{\partial f}{\partial x}(x_i, z_j) = \frac{f_{i+1,j} - f_{i-1,j}}{2\mathrm{dx}} + O(\mathrm{dx}^2). \tag{57}$$

As required.

Now we need to work out $\frac{\partial^2 f}{\partial x^2}(x_i, z_j)$. To start we extend the summation within equations (47) and (52) meaning we get the following 2 equations.

$$\begin{aligned} f(x_i + \mathrm{dx}, z_j) &= f_{i,j} + \mathrm{dx}\frac{\partial}{\partial x} f_{i,j} + \frac{\mathrm{dx}^2}{2!}\frac{\partial^2}{\partial x^2} f_{i,j} + \frac{\mathrm{dx}^3}{3!}\frac{\partial^3}{\partial x^3} f_{i,j} + \frac{\mathrm{dx}^4}{4!}\frac{\partial^4}{\partial x^4} f_{i,j} + ... \\ f(x_i - \mathrm{dx}, z_j) &= f_{i,j} - \mathrm{dx}\frac{\partial}{\partial x} f_{i,j} + \frac{\mathrm{dx}^2}{2!}\frac{\partial^2}{\partial x^2} f_{i,j} - \frac{\mathrm{dx}^3}{3!}\frac{\partial^3}{\partial x^3} f_{i,j} + \frac{\mathrm{dx}^4}{4!}\frac{\partial^4}{\partial x^4} f_{i,j} + ... \end{aligned}$$

Now we add the 2 equations together and aim to rearrange for $\frac{\partial^2}{\partial x^2} f_{i,j}$.

$$\begin{aligned} f(x_i + \mathrm{dx}, z_j) + f(x_i - \mathrm{dx}, z_j) &= 2f_{i,j} + 2\frac{\mathrm{dx}^2}{2!}\frac{\partial^2}{\partial x^2} f_{i,j} + 2\frac{\mathrm{dx}^4}{4!}\frac{\partial^4}{\partial x^4} f_{i,j} + ... \\ 2\frac{\mathrm{dx}^2}{2!}\frac{\partial^2}{\partial x^2} f_{i,j} + 2\frac{\mathrm{dx}^4}{4!}\frac{\partial^4}{\partial x^4} f_{i,j} + ... &= f(x_i + \mathrm{dx}, z_j) + f(x_i - \mathrm{dx}, z_j) - 2f_{i,j}, \\ \mathrm{dx}^2\frac{\partial^2}{\partial x^2} f_{i,j} &= f(x_i + \mathrm{dx}, z_j) + f(x_i - \mathrm{dx}, z_j) - 2f_{i,j} - 2\frac{\mathrm{dx}^4}{4!}\frac{\partial^4}{\partial x^4} f_{i,j} - ... \end{aligned}$$

Remember from equations (51) and (55) we can simplify $f(x_i + \mathrm{dx}, z_j)$ and $f(x_i - \mathrm{dx}, z_j)$. So

$$\begin{aligned}\frac{\partial^2}{\partial x^2} f_{i,j} &= \frac{f(x_i + \mathrm{dx}, z_j) + f(x_i - \mathrm{dx}, z_j) - 2f_{i,j}}{\mathrm{dx}^2} - 2\frac{\mathrm{dx}^2}{4!}\frac{\partial^4}{\partial x^4} f_{i,j} - \ldots \\ \frac{\partial^2 f}{\partial x^2}(x_i, z_j) &= \frac{f_{i+1,j} + f_{i-1,j} - 2f_{i,j}}{\mathrm{dx}^2} + O(\mathrm{dx}^2). \qquad (58)\end{aligned}$$

As required.

Finally, we need to work out $\frac{\partial^2 f}{\partial x \partial z}(x_i, z_j)$. To do this we must rewrite $\frac{\partial^2 f}{\partial x \partial z}(x_i, z_j)$ as $\frac{\partial}{\partial x}(\frac{\partial}{\partial z} f(x_i, z_j))$. To solve this, we need to derive $\frac{\partial}{\partial z} f(x_i, z_j)$ using symmetry in z and equation (57). Therefore

$$\frac{\partial}{\partial z} f(x_i, z_j) = \frac{f_{i,j+1} - f_{i,j-1}}{2\mathrm{dz}} + O(\mathrm{dz}^2).$$

Now, we rewrite $\frac{\partial}{\partial x}(\frac{\partial}{\partial z} f(x_i, z_j))$.

$$\begin{aligned}\frac{\partial}{\partial x}(\frac{\partial}{\partial z} f(x_i, z_j)) &= \frac{\partial}{\partial x}(\frac{f_{i,j+1} - f_{i,j-1}}{2\mathrm{dz}} + O(\mathrm{dz}^2)), \\ &= \frac{1}{2\mathrm{dz}}(\frac{\partial}{\partial x} f_{i,j+1} - \frac{\partial}{\partial x} f_{i,j-1}) + O(\mathrm{dz}^2). \qquad (59)\end{aligned}$$

We will use equation (57) to find $\frac{\partial}{\partial x} f_{i,j+1}$ and $\frac{\partial}{\partial x} f_{i,j-1}$ by simply replacing $_j$ in (57) with $_{j+1}$ and $_{j-1}$ respectively. Hence

$$\begin{aligned}\frac{\partial}{\partial x} f_{i,j+1} &= \frac{f_{i+1,j+1} - f_{i-1,j+1}}{2\mathrm{dx}} + O(\mathrm{dx}^2), \qquad (60) \\ \frac{\partial}{\partial x} f_{i,j-1} &= \frac{f_{i+1,j-1} - f_{i-1,j-1}}{2\mathrm{dx}} + O(\mathrm{dx}^2). \qquad (61)\end{aligned}$$

Using equations (60) and (61) we can simplify equation (59) to

$$\frac{\partial^2 f}{\partial x \partial z}(x_i, z_j) = \frac{f_{i+1,j+1} + f_{i-1,j-1} - f_{i-1,j+1} - f_{i+1,j-1}}{4\mathrm{dxdz}} + O(\mathrm{dx}^2\mathrm{dz}^2). \qquad (62)$$

Note that by symmetry we have

$$\begin{aligned}\frac{\partial f}{\partial z}(x_i, z_i) &\approx \frac{f_{i,j+1} - f_{i,j}}{\mathrm{dz}}, \\ \frac{\partial f}{\partial z}(x_i, z_i) &\approx \frac{f_{i,j+1} - f_{i,j-1}}{2\mathrm{dz}}, \\ \frac{\partial^2 f}{\partial z^2}(x_i, z_i) &\approx \frac{f_{i,j+1} - f_{i,j-1} - 2f_{i,j}}{\mathrm{dz}^2} \qquad (63)\end{aligned}$$

Assuming $\mathrm{dx} = \mathrm{dz}$, the boundary conditions (26) in discrete form become

$$\begin{aligned}0 &= v_{i,1} - v_{i,0} + \frac{1}{2}(w_{i+1,0} - w_{i-1,0}), \\ 0 &= \frac{\lambda}{2}(v_{i+1,0} - v_{i-1,0} + (\lambda + \mu)(w_{i,1} - w_{i,0}). \qquad (64)\end{aligned}$$

On the other edges, after each integration step, we damp the time derivative of v and w on a strip N_d points wide along the edges using

$$\begin{aligned}\frac{dv_{i,j}}{dt} &\to \frac{dv_{i,j}}{dt}(1 - \Gamma\frac{i}{N_d}) : \ i \in [0, N_d], j \in [0, N_z]), \\ \frac{dv_{i,j}}{dt} &\to \frac{dv_{i,j}}{dt}(1 - \Gamma\frac{N_x - i}{N_d}) : \ i \in [N_x - N_d, N_x), j \in [0, N_z]), \\ \frac{dv_{i,j}}{dt} &\to \frac{dv_{i,j}}{dt}(1 - \Gamma\frac{N_z - j}{N_d}) : \ j \in [N_z - N_d, N_z), i \in [0, N_x]), \qquad (65)\end{aligned}$$

where Γ is a parameter that is usually taken as 0.2. Now, we are getting ever closer to Numerical Simulations, where we will be able to generate nearly circular waves which will propagate through the rock layers by generating a brief excitation at 1 point on the surface. We will then record the displacements v and w measured on detectors at 3 different positions. However, interpreting the recorded signal will be difficult. This is where ray paths come in.

4 Ray Paths

We will use a simple model of ray paths to predict the expected travel time of waves that have undergone a specific succession of refractions and reflections to help us.

The whole point is to assume that, like light, the waves are propagating along straight lines but in all directions at once, and to follow each ray one by one. In the homogeneous rock layers, the rays are straight, but when they hit the boundary between 2 layers, the rays are reflected and refracted (with the amplitude of each relying on the rock properties and the angle of incidence, but we won't consider amplitudes). Each type of wave is partly reflected and partly refracted both as a wave of the same and the other type.
So at each interface, each ray generates 4 new rays, as illustrated in figure (4):

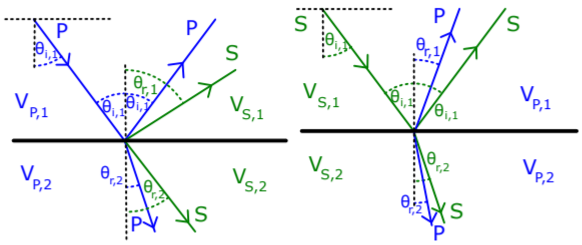

Figure 4: Reflection and transmission of wave on in-homogeneity surfaces.

In each layer, a ray is characterised by the incident angle ($< \pi/2$). The angle of incidence is the angle between the ray's direction and the vertical axis. As the interfaces between rock layers are all horizontal, the incident angle at the top of the layer is the same as at the bottom of the same layer. Snell's Law gives us the angles of refraction and reflection:

$$\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2}, \tag{66}$$

where v_1 is the speed of the incident wave and v_2 the speed of the refracted or reflected wave.

Snell's Law can be derived from various methods such as via the Maxwell Equations or Huygen's Principles. We are going to prove Snell's Law via Fermat's Principle [11]. All we need for this is Pythagoras' Theorem, the speed equation, and a diagram representing the path of a light ray.

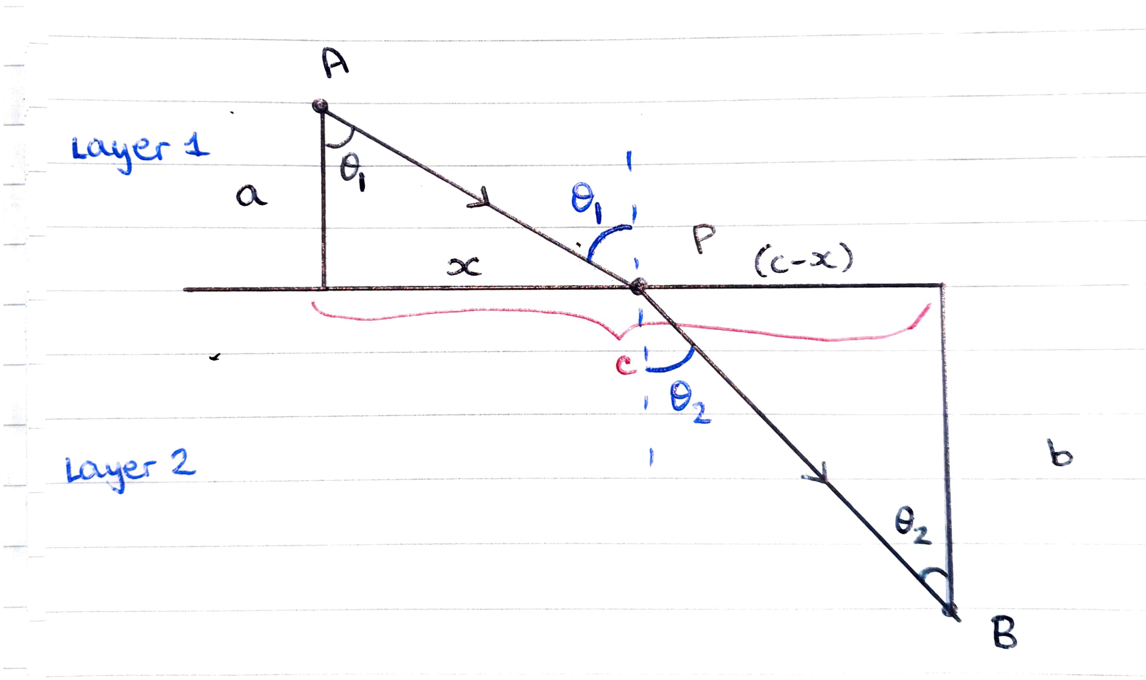

Figure 5: Light Movement Between 2 Layers

Our aim is to find the **shortest time taken** for the light path to travel from point A to point B. All the values denoted are arbitrary and will cancel nicely. First, let us denote the speeds as v_1 and v_2 in layers 1 and 2, respectively. Then, define the distances AP and PB using Pythagoras' Theorem. Hence $AP^2 = a^2 + x^2$ meaning $AP = \sqrt{a^2 + x^2}$. And, for PB, $PB^2 = b^2 + (c-x)^2$ meaning $PB = \sqrt{b^2 + (c-x)^2}$. Then we define $\sin(\theta_1)$ and $\sin(\theta_2)$:

$$\begin{aligned}
\sin(\theta_1) &= \frac{\text{opposite}}{\text{hypotenuse}}, \\
&= \frac{x}{\sqrt{a^2 + x^2}}, \\
\sin(\theta_2) &= \frac{c-x}{\sqrt{b^2 + (c-x)^2}}.
\end{aligned}$$

Then, we aim to find the time taken by the ray path to move the distance AB:

$$\text{time}_{AB} = \text{time}_{AP} + \text{time}_{PB}.$$

The time can be worked out using the speed equation: speed = distance/time. And, we rewrite this as time = distance/speed. Then we write this into the above.

$$\begin{aligned}
\text{time}_{AB} &= \frac{AP}{v_1} + \frac{PB}{v_2}, \\
\text{time}_{AB} &= \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (c-x)^2}}{v_2}.
\end{aligned}$$

Now, remember the aim, we need to minimise the time taken. So, we need the derivative of time_{AB} with respect to x, which is our variable, and set it equal to 0.

$$\begin{aligned}
\frac{d}{dx}\text{time}_{AB} &= \frac{1}{2}(2x)\frac{1}{v_1\sqrt{a^2 + x^2}} + \frac{1}{2}(-2(c-x))\frac{1}{v_2\sqrt{b^2 + (c-x)^2}}, \\
0 &= \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{c-x}{v_2\sqrt{b^2 + (c-x)^2}}, \\
\frac{x}{v_1\sqrt{a^2 + x^2}} &= \frac{c-x}{v_2\sqrt{b^2 + (c-x)^2}}.
\end{aligned}$$

The above expression contains our definitions of $\sin(\theta_1)$ and $\sin(\theta_2)$ meaning we get:

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}.$$

Note that for a wave reflected as a wave of the same type $V_1 = V_2$ and θ is unchanged, but if it is reflected as a wave of the other type then the incident angles differ. Note that we cannot always calculate θ_2, given θ_1, v_1, and v_2. In that case, we can take $\theta_2 = \pi/2$ which corresponds to surface waves - which we do not cover here. Surface waves are seismic waves that travel along the ground surface and they do the most damage after an earthquake. They come in 2 types: Love and Rayleigh Waves [12]. The whole idea of ray paths is that we will look at the wave trajectories.

4.1 Trajectories

As outlined in the introduction, when we simulate the propagation of the waves by solving differential equations, we will obtain some v and w displacements, which we call signals, at each of the detectors' positions. This will appear as a succession of *blips*. A *blip* is the original excitation after it has been reflected and refracted following a number of path segments. Each path, as shown in figure (6), will have different travel times and hence will correspond to a different delay in the measured signal.

Our aim is to determine which path each *blip* corresponds to. To do so we will consider some specific layers, solve the differential equation and generate plots of the signal obtained from the detectors. We will also determine the waves' travel time by considering the simplest types of paths and marking these times on the detector plots. We hope then to carry out our aim.

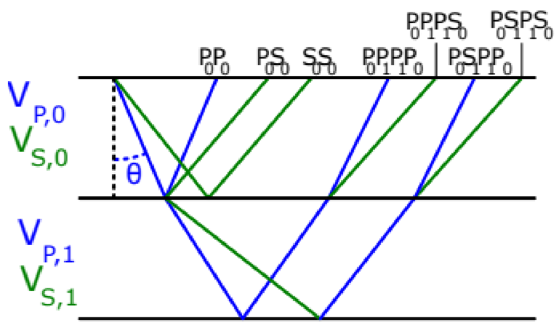

Figure 6: Some ray paths within the top 2 layers. W_i where $W = P$ or S refers to a W wave in layer i.

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Now we will consider layers of thickness D_i, speed $V_{P,i}$ and $V_{S,i}$ where i is the layer index starting with $i = 0$ at the top. We consider a path $W_{0,i_0}W_{1,i_1}W_{2,i_2}...W_{n,i_n}$ where W_{k,i_k} denotes the ray segment k of type W_k in layer i_k with incident angle θ_k. (For example, $P_0S_1P_1S_0$ is a P wave in layer 0, refracted as an S wave in layer 1, reflected as a P wave at the bottom of layer 1, and then refracted as an S wave in layer 0.) [see Figure 6].

A ray going through a layer of thickness D_i with an incident angle θ_i can be shown as:

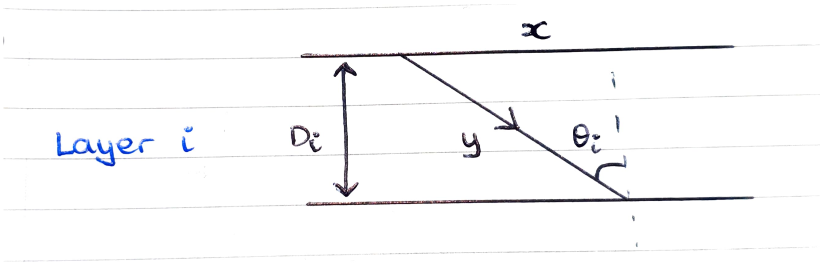


Figure 7: An illustration of the ray path going through the layer D_i

Using the picture, our aim is to work out x which is the horizontal distance covered by the ray when it reaches the next layer boundary.

$$\begin{aligned} \tan(\theta_i) &= \frac{x}{D_i}, \\ x &= D_i \tan(\theta_i). \end{aligned} \tag{67}$$

As required.

We also need to work out y which is the total distance covered inside that layer (the length of the ray segment) and the time needed by the wave to cover it.

$$\begin{aligned} \cos(\theta_i) &= \frac{D_i}{y}, \\ y &= \frac{D_i}{\cos(\theta_i)}. \end{aligned} \tag{68}$$

As required.

To work out the time, we use the equation for speed.

$$\text{speed} = \frac{\text{distance}}{\text{time}},$$

which implies that

$$\text{time} = \frac{\text{distance}}{\text{speed}}.$$

When subbing in for speed, use V_{W_k,i_k} to represent the wave's speed depending on whether it is a P or S wave and its layer.

$$\begin{aligned} \text{time} &= \frac{y}{V_{W_k,i_k}}, \\ \text{time} &= \frac{D_i}{\cos(\theta_k) V_{W_k,i_k}}. \end{aligned} \tag{69}$$

As required.

Now we aim to relate θ_k and θ_{k-1}. We can do this via Snell's Law except we replace θ_2 with θ_k and therefore we swap θ_1 with θ_{k-1}. Then we arrange this equation for θ_k. Hence

$$\begin{aligned} \frac{\sin(\theta_{k-1})}{\sin(\theta_k)} &= \frac{v_{k-1}}{v_k}, \\ \theta_k &= \arcsin(\frac{v_k}{v_{k-1}} \sin(\theta_{k-1})). \end{aligned} \tag{70}$$

As required.

Now we aim to use these results to write an expression for the total time taken by the wave to cover the full path, as a function of θ_0. First, we should think about how θ_i relates to θ_0 using equation (70). Let us see how θ_1, θ_2 and θ_3 relate to θ_0.

$$\begin{aligned}
\theta_1 &= \arcsin(\frac{v_1}{v_0}\sin(\theta_0)),\\
\theta_2 &= \arcsin(\frac{v_2}{v_1}\sin(\theta_1)),\\
&= \arcsin(\frac{v_2}{v_1}\frac{v_1}{v_0}\sin(\theta_0)),\\
&= \arcsin(\frac{v_2}{v_0}\sin(\theta_0)),\\
\theta_3 &= \arcsin(\frac{v_3}{v_2}\sin(\theta_2)),\\
&= \arcsin(\frac{v_3}{v_2}\frac{v_2}{v_0}\sin(\theta_0)),\\
&= \arcsin(\frac{v_3}{v_0}\sin(\theta_0)).
\end{aligned}$$

Hence, using our new values of speed:

$$\theta_k = \arcsin(\frac{V_{W_k,i_k}}{V_{W_0,i_0}}\sin(\theta_0)). \tag{71}$$

We will use the sum of equation (69), our definition for time, from $k = 0$ to $k = n$. So:

$$\text{total time} = \sum_{k=0}^{n} \frac{D_i}{\cos(\theta_k)V_{W_k,i_k}}. \tag{72}$$

And via equation (69) this simplifies too:

$$\text{total time} = \sum_{k=0}^{n} \frac{D_i}{\cos(\arcsin(\frac{V_{W_k,i_k}}{V_{W_0,i_0}}\sin(\theta_0)))V_{W_k,i_k}}. \tag{73}$$

Now we will consider paths made of segments crossing several layers for which we know the respective V_P and V_S, such as $P_0S_1P_1S_0$ for example. Notice that we can also consider multiple bounces such as a P wave in layer 1 is reflected as a P wave then reflected again at the surface as a P wave and reflected one last time as a P wave until it reaches the surface again ($P_0P_0P_0P_0$).

Given the angle of incidence for the first ray segment, Snell's Law allows us to determine the ray's incident angle after each subsequent reflection or transmission, and from then on we can compute the horizontal offset in each layer. Summing over these gives the offset where the wave hits the surface again. What we need to do though is consider the horizontal distance between the source and a detector and for each specific path to determine the incident angle of the first segment so that the wave hits the detector after following that path. We will do this easily using the bisection method. Now, we have enough background information to use Numerical Simulations to analyse the propagation of the seismic wave.

5 Numerical Simulations

We will consider the 4km wide and 2km deep domain described above and we place the source at the surface in the middle of the domain, $x = 2$km. The 3 detectors are then placed at

$x = 2000, x = 2300$, and $x = 2500$. We consider 3 layers of rock made out of 500m of granite, 300m of shale, and then 1200m of granite. The properties of the rocks are given in table 1 and the initial excitation is given by (43) [13], [14].

Rock	V_P (m/s)	V_S (m/s)	ρ kg/m^3
Granite	5980	3480	2660
Shale	2898	1290	2425

Table 1: Rock Properties

The program `run_G.py` creates heat-map snapshots of the waves generated by the excitation in a pure granite layer. The files it will generate are:

- Signal `TYPE` measured at the detector at `DIST`:
  `X4000_Z2000_N401_dDIST_f20_GSG_TYPE.pdf`
  where `DIST` is `0, 300` and `500` and `TYPE` is `v, w` or `Mod`.

- Snapshot of the wave in the granite medium at `t=N*0.00071786 s`:
  `X4000_Z2000_N401_f20_na30_g0.2_G_TYPE_N.pdf` where `TYPE` is `v` or `w` and `N` is an integer.

The program `run_GSG.py` creates heat-map snapshots of the waves generated by the excitation in the granite, shale, granite triple layer. It also generates the time signal measured at the 3 detectors with markers for the estimated propagation time for the following paths: P_0P_0, P_0S_0, S_0S_0, $P_0P_0P_0P_0$, $S_0P_0P_0P_0$, $P_0S_0P_0S_0$, $P_0P_1P_1P_0$, $P_0S_1P_1P_0$, $S_0P_1P_1P_0$. It will generate 3 files: 1 for v, 1 for w and a third one for the total displacement amplitude $\sqrt{v^2 + w^2}$.

Geophysicists have developed very sophisticated mathematical methods to extract the reflected waves as they do not know the underground layers from the measured signals which contain direct waves as well as noise. Instead of doing this, we will start by computing the propagation of the wave in a 2km deep single layer of granite. We then generate the same wave in the three-layer domain and we subtract from the measured signal at a detector, the signal we measured in the single-layer domain at the same detector. We call the resulting signal the reduced signal.

`run_GSG.py` generates the reduced signal measured at the 3 detectors for v, w as well as $\sqrt{v^2 + w^2}$. It also includes markers for the estimated propagation time for the same path as listed above.
The files it generates are:

- Signal `TYPE` measured at the detector at `DIST`:
  `X4000_Z2000_N401_dDIST_f20_GSG_TYPE.pdf`
  where `DIST` is `0, 300` and `500` and `TYPE` is `v, w` or `Mod`.

- Reduced signal `TYPE` measured at the detector at `DIST`:
  `X4000_Z2000_N401_dDIST_f20_GSG_TYPE.pdf`
  where `DIST` is `0, 300` and `500` and `TYPE` is `v, w` or `Mod`.

Examples of such figures, for a different layer structure, are shown in figure 8.

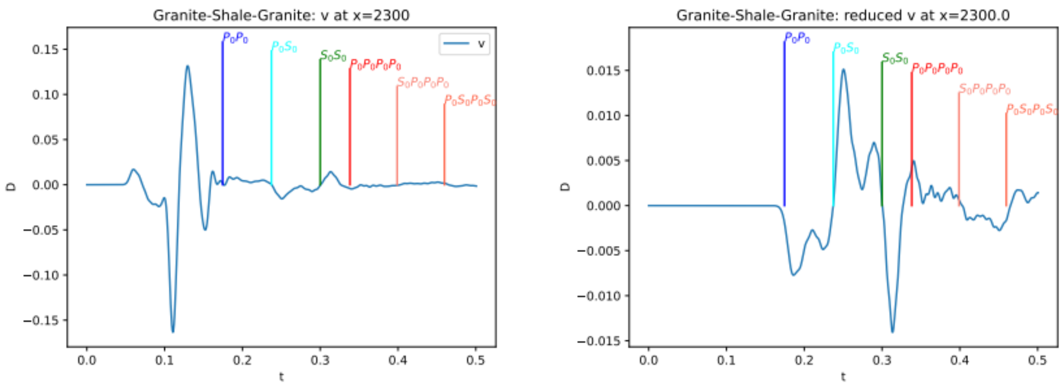


Figure 8: Example of signal measured at the detector at $d = 300$m a) full signal. b) reduced signal.

The aim of the coding task is to integrate the discrete version of the wave equations. The module `s_wave_base` defines the class `s_wave_base` which is itself a subclass of `PDE_2D_fast`.

- The `s_wave_base` class variable `v` contains the current value of the function $v, w, dv/dt$ and dw/dt as 3-dimensional arrays of shape `[Nx, Nz, 4]` with:
  - `v[:,:,0]` corresponding to v
  - `v[:,:,1]` to w
  - `v[:,:,2]` to dv/dt
  - `v[:,:,3]` to dw/dt
- The class variables `dx, dz, Nx` and `Nz` contain the corresponding values of the parameters.
- The class variable data contains a description of the layers as a list of the type `[[thickness1, Vp1, Vs1, rho1, col1], [thickness2, Vp2, Vs2, rho2, col2],...].`
- The class variables `lam, mu, rho` and `Gamma`, contain the value of λ, μ, ρ and Γ respectively on the grid as arrays of shape `[Nx,Nz]`.
- The class variable detectors contain the signal measured at the detector. It is a list of the type `[ [t0, lv0, lw0], [t1, lv1, lw1],...]` where `ti` is the time at which the signal was measured while `lvi` and `lwi` are each a list of values corresponding respectively to the v and w displacements at the detectors. (If there are 3 detectors, at 2000, 2300, and 2500, the lists `lv1` will have 3 elements each of the type `lvi = [v(2000), v(2300), v(2500)]` obtained at the time `i).`
- The class function `F(self, t, v_)` computes the right-hand side of the wave equation using loops to scan the grid.
- The class function `boundary(self, v_)` implements the boundary condition after each integration step.
- The class function `iterate(self, tmax, dt, fig_dt=-1, extra_dt = -1)` integrates the equation using the 4^{th} order Runge-Kutta method from `t=0` up to `tmax` using the time step `dt`.

- The class function `plot_detector_diff(self, d1, d2, dn, data_type )` generates figures of the reduced signals measured at the detectors. `d1` is the main signal, `d2` is the signal to subtract, `dn` is the index of the signal displacement (detector) and it uses the class function `eval_detector_diff(self, d1, d2, dn, data_type )`.

The python program `s_wave.py` imports the module `s_wave_base` and defines the class `s_wave` as a subclass of `s_wave_base`. The plots from the programs `run_GSG` and `run_G` are shown in the plots section below.

Q7

Using the programs, we can come to some vital conclusions. On the heat map for the wave in granite (see figures 12 and 13), the 2 types of waves cannot be told apart, as you cannot see the waves change and the heat maps are fairly symmetric.

The full signal measured at the detectors tells us that we get 2 successive *blips* within a short time except for Mod at $d = 300$m and $d = 500$m in the granite and granite-shale-granite layers. We can also observe that for the full signal measured at the detectors, in both the single granite layer and the triple granite shale granite layer, the *blips* occur roughly in the same place. For each Granite layer: at $d = 0$m, this is between 0.01 and 0.05 seconds, at $d = 300$m, this is between 0.11 and 0.15 seconds and at $d = 500$m, this is between 0.18 and 0.21 seconds (see figures 9, 10 and 11). For each granite shale granite layer, this is the same as its respective granite layer (see figures 14, 15 and 16). The *blips* in both graphs indicate when the wave reaches the surface in each layer. This implies that the granite-shale-granite layer is more sensitive to the waves as it is easier for them to get to the surface meaning the graph is more detailed and we can draw conclusions easier.

One difference between the granite shale granite plot and granite plot is that the pure granite plots have a clear *blip* and the rest of the graph has few excitations but the granite shale granite plots have a lot of smaller excitations after the large one.

The signal from which we have subtracted the pure granite signal is better to draw conclusions as it's more sensitive to the blips. The ray path approximation works well as all the path signals closely align with each blip. Hence, it allows us to interpret the result of the simulation as you have a clear match between the paths and the blips. In the v and w displacement, some of the *blips* in the signal are positive while others are negative. This comes from the ground oscillating to the left and right slightly within the short time span.

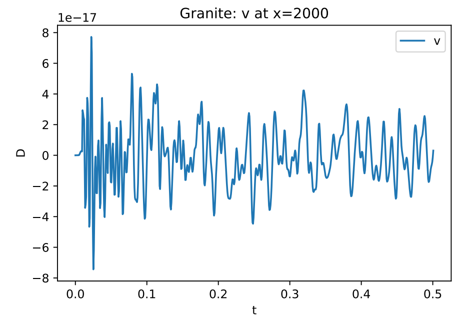
6 Conclusion

So what we can conclude is that the solutions for the 2D wave equation can be proven by using the angular frequency and wave number; solutions of this equation can also be related via the swapping and/or negating of simple constants. We can solve the 2D wave equation directly by solving the differential equation describing small vibrations inside solids by switching it to a first-order differential equation. Yet the easiest way to solve it is by using Taylor Series to work out the forward difference, symmetric difference, and second-order derivatives.

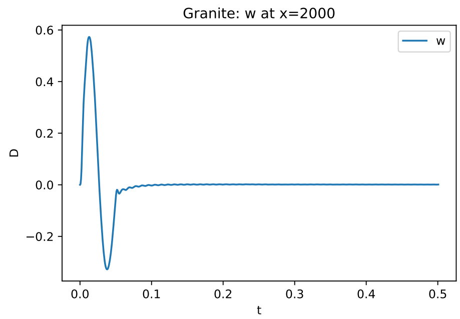
The most notable result from the ray path calculations was that you can calculate $\sin(\theta_k)$ from θ_0 using Snell's Law and by simply multiplying it by the speed in the k$^{\text{th}}$ layer over the speed in the 1$^{\text{st}}$ layer. This simplifies calculating the total time. From `run_GSG` and `run_G`, we can see that we cannot differentiate S and P Waves using heat maps due to symmetry. Ultimately though, the ray path approximates well as each *blip* is close to each path signal.

7 Plots

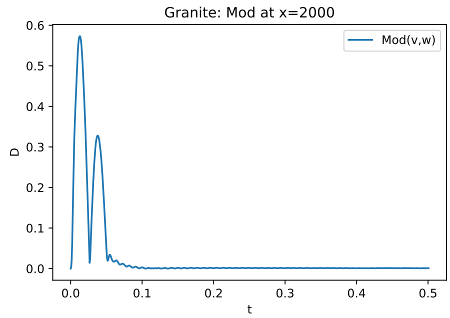
Here are all the plots from `run_GSG` and `run_G`:



(a) v

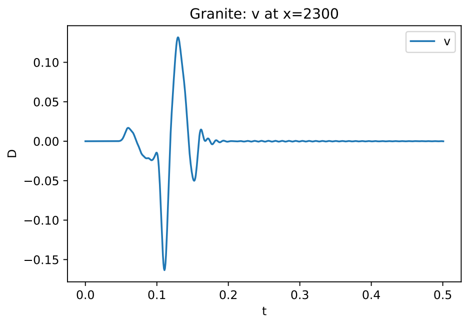


(b) w

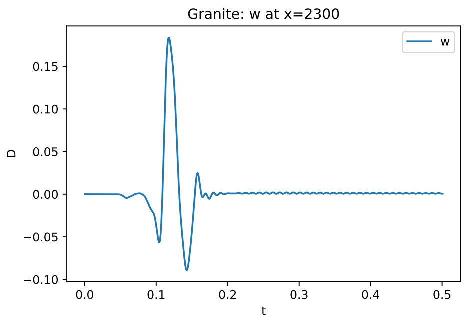


(c) Mod

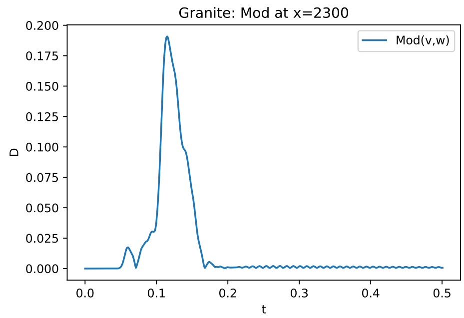
Figure 9: Granite signal measured at (the detector) at $d = 0$m



(a) v

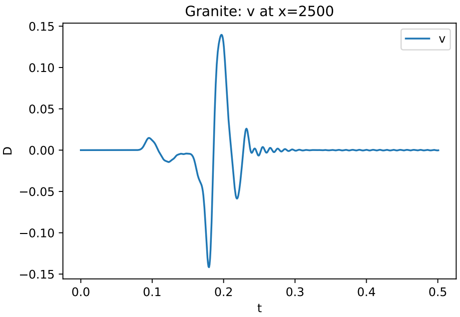


(b) w

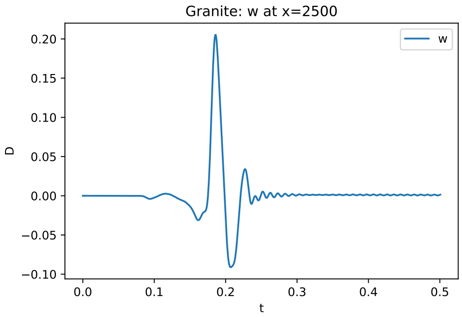


(c) Mod

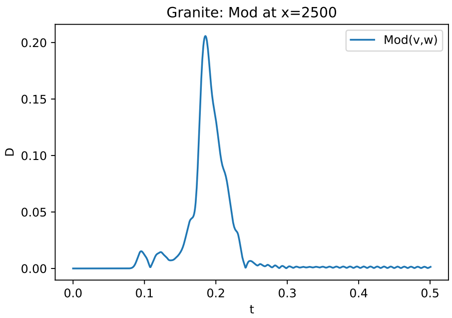
Figure 10: Granite signal measured at $d = 300$m



(a) v

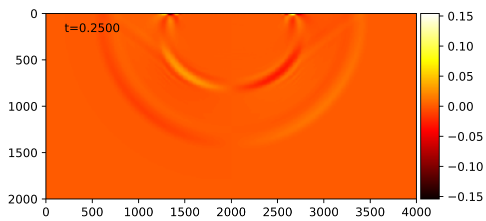


(b) w

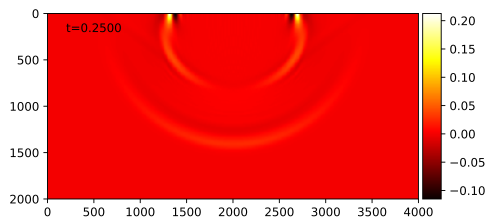


(c) Mod

Figure 11: Granite signal measured at $d = 500$m

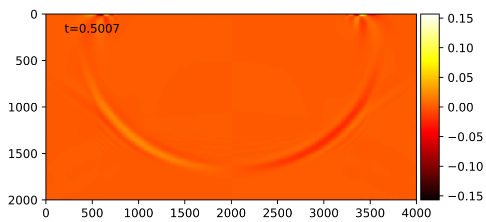


(a) v

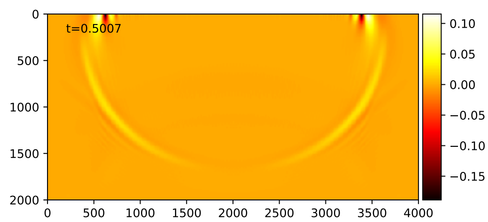


(b) w

Figure 12: Heatmaps for Granite at t = 0.2500

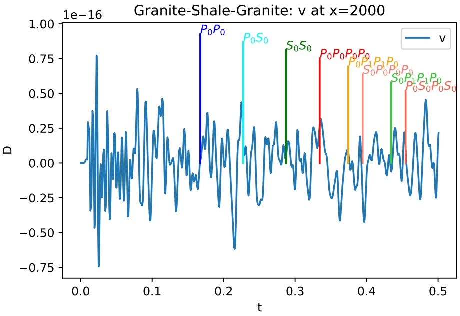


(a) v

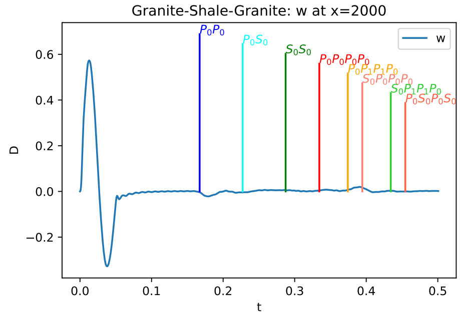


(b) w

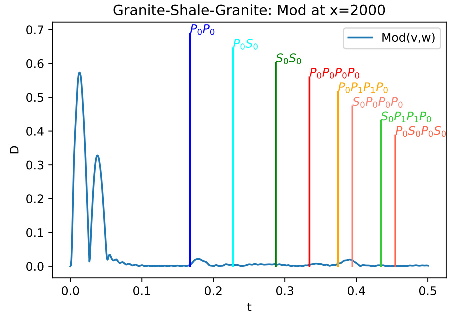
Figure 13: Heatmaps for Granite at t = 0.5007



(a) v

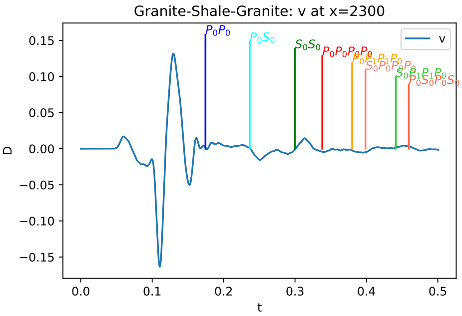


(b) w

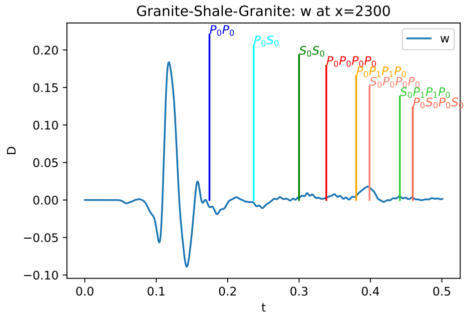


(c) Mod

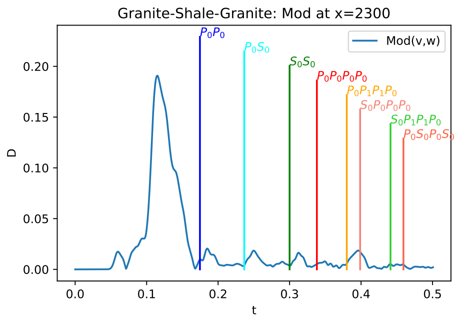
Figure 14: Granite-Shale-Granite (GSG) full signal measured at $d = 0$m



(a) v

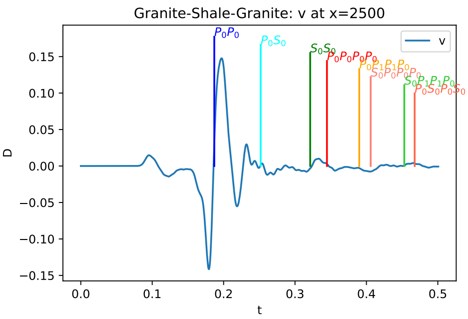


(b) w

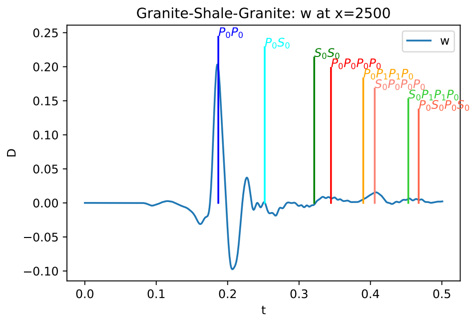


(c) Mod

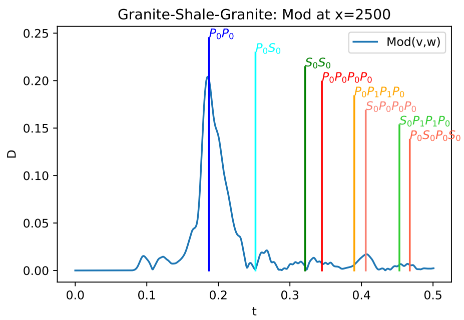
Figure 15: GSG full signal measured at $d = 300$m



(a) v

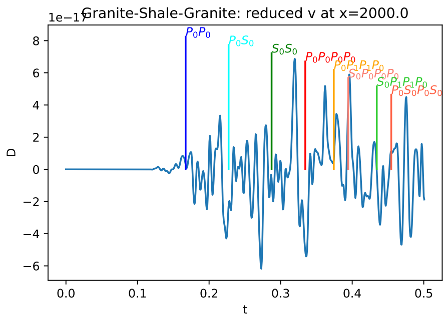


(b) w

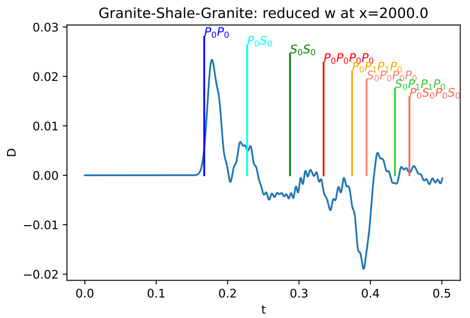


(c) Mod

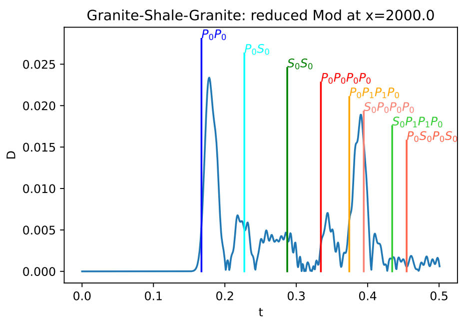
Figure 16: GSG full signal measured at $d = 500$m



(a) v

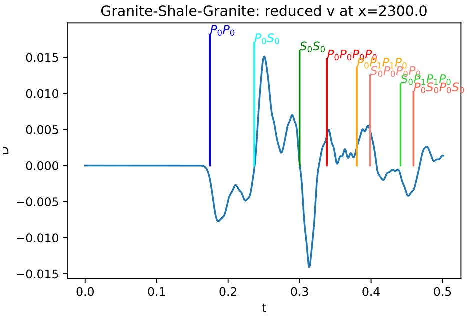


(b) w

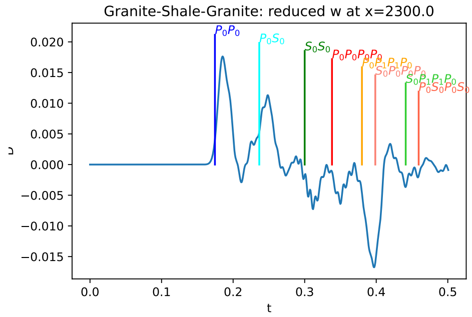


(c) Mod

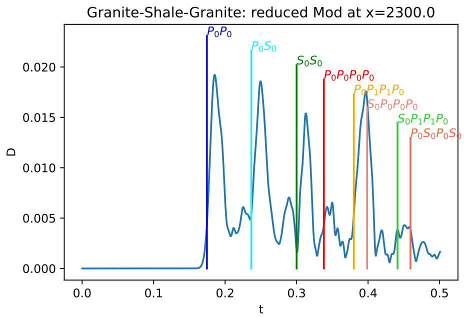
Figure 17: GSG reduced signal measured at $d = 0$m



(a) v

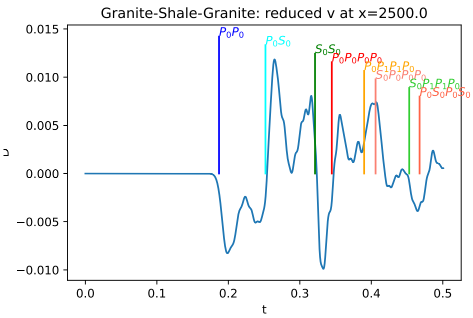


(b) w

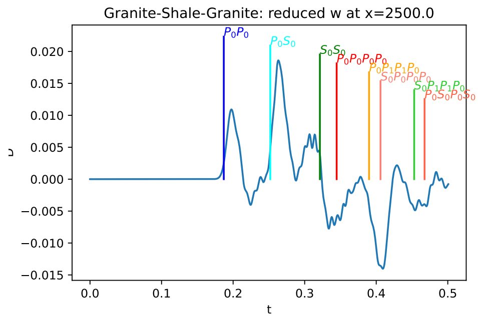


(c) Mod

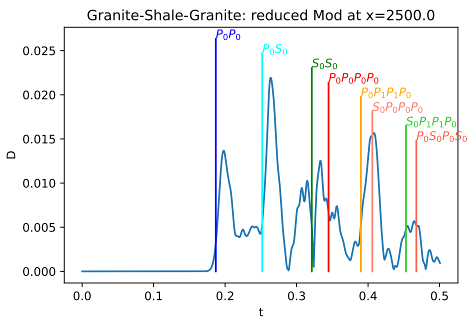
Figure 18: GSG reduced signal measured at $d = 300$m



(a) v

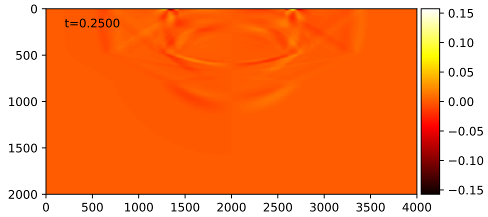


(b) w

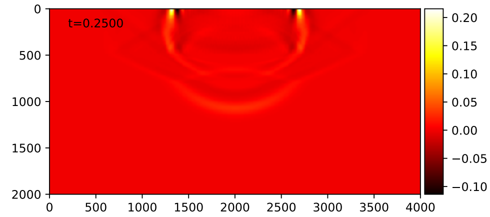


(c) Mod

Figure 19: GSG reduced signal measured at $d = 500$m

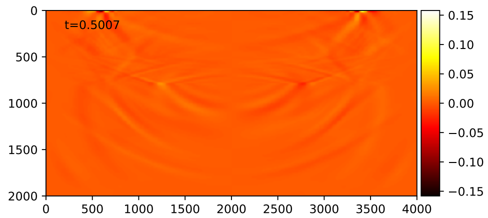


(a) v

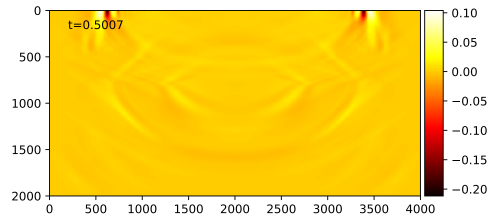


(b) w

Figure 20: Heatmaps for Granite-Shale-Granite at t = 0.2500



(a) v



(b) w

Figure 21: Heatmaps for Granite-Shale-Granite at t = 0.5007

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