Evaluation of Reproducibility When The Data Are Curves

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Abstract

Evaluation of reproducibility is important in assessing whether a new method or instrument can reproduce the results from a traditional golden standard approach. In this paper, we propose a measure to assess measurement agreement for curve data which are frequently encountered in medical research and many other research fields. Statistical inference procedures for this measure are developed. Furthermore, consistency and asymptotic normality of the proposed estimator are established. Formulae to compute the standard error of the proposed estimator and confidence intervals for the proposed measure are derived. The coverage probability of the confidence intervals are empirically tested for small to moderate sample size via Monte Carlo simulations. An example in physiology study is used to illustrate the proposed statistical inference procedures.

Key Words and Phrases: Concordance correlation coefficient, curve data, image data, kappa coefficient.

1 Introduction

Evaluation of reproducibility is needed for many scientific research problems. For example, when a new instrument is developed, it is of interest to assess whether the new instrument can reproduce the results obtained by using a traditional golden standard criterion. Indeed, the need to quantify agreement arises in many research fields when two approaches or two raters simultaneously evaluate a response. There are some traditional criteria for measuring agreement between two rating approaches, such as Pearson's correlation coefficient and paired t-test when the responses are continuous. Even though these criteria had been used in practice, they fail to detect poor agreement in some situations (see, for example, Lin, 1989). Thus, the topic of assessing agreement for measurements by two approaches has become an interesting research topic. Some measures to evaluate reproducibility include intraclass correlation (Fleiss, 1986, Quan and Shih, 1996) and within-subject coefficient of variation (Lee, Koh and Ong, 1989). Lin(1989) introduced the concordance correlation coefficient which assesses the linear relationship between two variables under the constraint that the intercept is zero and the slope is one. This measure is more appropriate for assessing reproducibility of continuous outcomes. The kappa coefficient (Cohen, 1960) and weighted kappa coefficient (Cohen, 1968) are the most popular indices for measuring agreement for nominal outcomes. It can be shown that the concordance correlation is analogous to Cohen's weighted kappa coefficient for nominal outcomes.

The concordance correlation coefficient has been widely applied to various research fields since its introduction. Several extensions have been proposed to address different problems recently. Extending Lin's ideas, Chinchilli, Martel, Kumanyika and Lloyd (1996) suggested a weighted concordance correlation coefficient for repeated measures design. Vonesh, Chinchilli and Pu (1996) used the concordance correlation coefficient to assess goodness-of-fit for generalized nonlinear mixed-effects models. King and Chinchilli (1999) developed a robust version of the concordance correlation coefficient, and a generalized concordance coefficient for continuous and categorical data was proposed in King and Chinchilli (2001). To accommodate covariate adjustment, Barnhart and Williamson (2001) proposed a generalized estimating equations approach to model the concordance correlation coefficient via three sets of estimating equations.

This paper deals with the problem when data are curves and proposes a measure to evaluate reproducibility of paired curve data. Curve data are also called functional data. Image data and growth curve data are special cases thereof. Analysis of functional data is becoming an important topic in the statistical literature. Many interesting applications can be found in the excellent book by Ramsey and Silverman (1997) and references therein. The method in this paper is developed to analyze an actual data set collected in the Noll Physiological Research Center at The Pennsylvania State University (courtesy of Dr. W. L. Kenney). The study was designed to explore the validity of using the polar heart rate monitor which would provide an unencumbered measure of heart rate, and its purpose was to compare the performance of the polar heart rate monitor with that of the ECG, known as a *golden standard approach*, and to validate the polar heart rate monitor for future use. During this study, the heart rates by the polar heart rate monitor and by the golden standard approach were recorded every minute over an experimental period for each individual. Plots of typical sample curves are displayed in Figures 2 and 3. The Pearson correlation coefficient and the concordance correlation coefficient proposed by Lin (1989) cannot be directly applied for this case because the observations are curves over a time interval. The weighted concordance correlation coefficient for repeated measures may not be appropriate for this situation because an observation for each subject is a pair of curves over time rather than a multi-dimensional vector or a set of repeated measurements. In this paper, we develop a concordance correlation coefficient for curve data and propose an estimator for it. The consistency and asymptotic normality of the proposed estimator are established. Based on the asymptotic normality, we provide a formula to compute the standard error of the resulting estimate. The small sample performance of the proposed standard error formula is investigated via Monte Carlo simulation, and it is accurate for practical use. Statistical inferences on the concordance correlation coefficient are also discussed. The physiological data set is used to illustrate the proposed methodology.

This paper is organized as follows. In Section 2, we give the motivations and introduce a concordance correlation coefficient for curve data. We then present characteristics of the concordance correlation coefficient. Furthermore, we propose an estimator for the coefficient. The consistency and asymptotic normality of the proposed estimator are established. Statistical inference procedure for the proposed coefficient is also derived. Since correlation coefficient is a quantity often of interest to researchers, it is important to define and make statistical inference for the correlation coefficient for curve data. In Section 2, we also define a correlation coefficient for curve data and discuss how to make statistical inferences on the correlation coefficient. Section 3 contains simulation results and applications of the proposed concordance correlation coefficient to the physiology data set aforementioned. In Section 4, we present an extension of the concordance correlation coefficient for image data. Conclusions are given in Section 5. The proofs of the asymptotic normality in Theorems 2.1 and 4.1 involve some technical arguments and are given in the Appendix.

2 Concordance correlation coefficient for curve data

Let x and y denote scores from two raters or measurements from two instruments. Let us first consider that both x and y are univariate. Suppose that $(x_1, y_1), \cdots, (x_n, y_n)$ are independent and identically paired observations from (x, y). Denote $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$. It is said that $\mathbf{x}$ and $\mathbf{y}$ are in perfect agreement if $x_i = y_i$ for $i = 1, \cdots, n$. Therefore, if $\mathbf{x}$ and $\mathbf{y}$ are in perfect agreement, then the angle between $\mathbf{x}$ and $\mathbf{y}$, denoted by $\theta(\mathbf{x}, \mathbf{y})$, is 0, which implies that $\cos\theta = 1$. It is well known that

$$\cos\theta = \frac{<\mathbf{x}, \mathbf{y}>}{\|\mathbf{x}\|\|\mathbf{y}\|},$$

where $<\cdot, \cdot>$ denotes the inner product, and $\|\cdot\|$ is the Euclidean norm in R^n. In other words, $<\mathbf{x}, \mathbf{y}> = \sum_{i=1}^n x_i y_i$, and $\|\mathbf{x}\|^2 = \sum_{i=1}^n x_i^2$ for any n-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$. However, $\cos\theta$ cannot be a good measure for agreement because for any positive constant c,

$$\cos\theta(c\mathbf{x}, \mathbf{y}) = \cos\theta(\mathbf{x}, c\mathbf{y}) = \cos\theta(\mathbf{x}, \mathbf{y}).$$

This implies that $\cos\theta$ cannot detect a scale change on either $\mathbf{x}$ or $\mathbf{y}$.

The sample correlation coefficient between $\mathbf{x}$ and $\mathbf{y}$ can be written as

$$\rho(\mathbf{x}, \mathbf{y}) = \frac{<\mathbf{x} - \bar{x}, \mathbf{y} - \bar{y}>}{\|\mathbf{x} - \bar{x}\|\|\mathbf{y} - \bar{y}\|},$$

where $\bar{x}$ and $\bar{y}$ are the sample means of $\mathbf{x}$ and $\mathbf{y}$, respectively. If $\mathbf{x}$ and $\mathbf{y}$ are in perfect agreement, then $\rho(\mathbf{x}, \mathbf{y}) = 1$. However, it cannot detect a location shift on either $\mathbf{x}$ or $\mathbf{y}$ and a scale change on either $\mathbf{x}$ or $\mathbf{y}$ because for any constants $a > 0$ and b

$$\rho(a\mathbf{x} + b, \mathbf{y}) = \rho(\mathbf{x}, a\mathbf{y} + b) = \rho(\mathbf{x}, \mathbf{y}).$$

The degree of concordance between x and y can be characterized by the expected value of the squared difference $E(x - y)^2$. Using this characterization, Lin (1989) proposed the concordance correlation coefficient

$$\rho_c(x, y) = \frac{2\text{cov}(x, y)}{\text{var}(x) + \text{var}(y) + \{E(x) - E(y)\}^2}.$$

He also suggested to use the sample counterparts to estimate the ρ_c:

$$\widehat{\rho}_c = \frac{2\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^n (y_i - \bar{y})^2 + n(\bar{x} - \bar{y})^2}.$$

The concordance correlation coefficient is effective for assessing agreement between two random variables x and y, and it has been applied to problems from various research fields.

Evaluation of reproducibility for curve data is a frequently encountered practical problem. In this section we propose a concordance correlation coefficient for curve data to address the problem. To get more insights into the concordance correlation coefficient and for the sake of ease of presentation, we first concentrate on the case when data are collected over an interval of a real line. In Section 4, we will extend the ideas to image data and show that the approach we propose here can handle very general cases.

Suppose that $X(t)$ and $Y(t)$, $t \in \mathcal{I}$, a finite closed real interval, are responses of two instruments. Regard $X(\cdot)$ and $Y(\cdot)$ as two random elements in some probability functional space $\mathcal{S}$. For the probability functional space $\mathcal{S}$, define an inner product

$$< X(\cdot), Y(\cdot) >= E \int_{\mathcal{I}} X(t) Y(t)\, w(t)\, dt,$$

where $w(\cdot)$ is a weight function and takes non-negative values over $\mathcal{I}$.

Using the notion of inner product, define the correlation coefficient for two random elements X and Y in $\mathcal{S}$ as:

$$\rho(X, Y) = \frac{< X - E(X), Y - E(Y) >}{\|X - E(X)\| \|Y - E(Y)\|}, \tag{2.1}$$

where $\|X\| = \sqrt{< X, X >}$. Thus when X and Y is univariate, $\rho(X, Y)$ coincides with the Pearson correlation coefficient when $w(\cdot)$ equals to a constant for all $t \in \mathcal{I}$.

For any two random elements X and Y in $\mathcal{S}$, the degree of agreement can be characterized by the expected value of the squared difference. That is

$$\|X - Y\|^2 = \|E(X) - E(Y)\|^2 + \|X - E(X)\|^2 + \|Y - E(Y)\|^2 - 2 < X - E(X), Y - E(Y) >$$

if all terms in the above equation exist and are finite.

A concordance correlation coefficient for $X(\cdot)$ and $Y(\cdot)$ is defined as

$$\rho_c(X, Y) = 1 - \frac{\|X - Y\|^2}{\|E(X) - E(Y)\|^2 + \|X - E(X)\|^2 + \|Y - E(Y)\|^2}, \tag{2.2}$$

which equals to

$$\rho_c(X,Y) = \frac{2 < X - E(X), Y - E(Y) >}{\|E(X) - E(Y)\|^2 + \|X - E(X)\|^2 + \|Y - E(Y)\|^2}.$$

Both ρ and ρ_c depend on the weight function $w(\cdot)$. The weight function allows one to assign different importance to different parts of t. No matter what the weight function is, the concordance correlation coefficient possesses the following characteristics, same as those of two random variables (see Lin, 1989):

(a) $|\rho_c| \le |\rho| \le 1$. Also, ρ_c and ρ have the same sign.

(b) $\rho_c = \rho$ if and only if $\|E(X) - E(Y)\| = 0$ and $\|X - E(X)\| = \|Y - E(Y)\|$.

(c) $\rho_c = 0$ if and only if $\rho = 0$.

(d) $\rho_c = \pm 1$ if and only if $\rho = \pm 1$, $\|X - E(X)\| = \|Y - E(Y)\|$, and $\|E(X) - E(Y)\| = 0$.

Using Cauchy-Schwarz inequality and the definition of ρ_c, the proofs of these characteristics are straightforward and are omitted here.

2.1 Statistical inferences

In this section, we propose statistical inference procedures for ρ and ρ_c defined in (2.1) and (2.2). Without loss of generality, we assume that the interval $\mathcal{I} = [0,1]$. For the sake of easy presentation, suppose that for subject i, $i = 1, \cdots, n$, $(X_i(t), Y_i(t))$ was observed at $t = t_j$, $j = 1, \cdots, N$ with $0 \le t_1 < \cdots < t_N \le 1$. This implies that all subjects were observed at the same t_j, $j = 1, \cdots, N$. When different subjects were observed at different t, the proposed estimation procedure is still applicable by some minor modifications without extra difficulty. Using the sample counterparts to estimate ρ and ρ_c, we have

$$\widehat{\rho} = \frac{\frac{1}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N \{X_i(t_j) - \overline{X}(t_j)\}\{Y_i(t_j) - \overline{Y}(t_j)\} w(t_j)}{\{\frac{1}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N (X_i(t_j) - \overline{X}(t_j))^2 w(t_j)\}^{1/2}\{\frac{1}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N (Y_i(t_j) - \overline{Y}(t_j))^2 w(t_j)\}^{1/2}}, \quad (2.3)$$

and

$$\widehat{\rho}_c = \frac{\frac{2}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N \{X_i(t_j) - \overline{X}(t_j)\}\{Y_i(t_j) - \overline{Y}(t_j)\} w(t_j)}{\frac{1}{N}\sum_{j=1}^N \{\overline{X}(t_j) - \overline{Y}(t_j)\}^2 w(t_j) + \frac{1}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N \left[\{X_i(t_j) - \overline{X}(t_j)\}^2 + \{Y_i(t_j) - \overline{Y}(t_j)\}^2\right] w(t_j)}, \quad (2.4)$$

where $\overline{X}(t_j)$ and $\overline{Y}(t_j)$ are the sample means of $X(t_j)$ and $Y(t_j)$, respectively.

Theorem 2.1 *Suppose that* $\{(X_i(t_1), Y_i(t_1)), \cdots, (X_i(t_N), Y_i(t_N))\}, i = 1, \cdots, n$, *is a random sample from* $\{(X(t_1), Y(t_1)), \cdots, (X(t_N), Y(t_N))\}$. *If Conditions (A)–(D) given in Appendix A.1 hold, then*

(a) $\widehat{\rho}$ *is a consistent estimator for* ρ, *and* $\sqrt{n}(\widehat{\rho} - \rho)$ *has an asymptotic normal distribution with zero mean and variance* $\sigma_\rho^2 = a^T \Sigma_1 a$, *where*

$$a = (1/\|X - E(X)\|\|Y - E(Y)\|, -\frac{1}{2}\rho/\|X - E(X)\|^2, -\frac{1}{2}\rho/\|Y - E(Y)\|^2)^T, \tag{2.5}$$

and

$$\Sigma_1 = cov\left\{\begin{pmatrix} \int_{\mathcal{I}}\{X(t) - EX(t)\}\{Y(t) - EY(t)\}w(t)\,dt \\ \int_{\mathcal{I}}\{X(t) - EX(t)\}^2 w(t)\,dt \\ \int_{\mathcal{I}}\{Y(t) - EY(t)\}^2 w(t)\,dt \end{pmatrix}\right\}. \tag{2.6}$$

Equivalently,

$$\sigma_\rho^2 = E\left[\frac{1}{2}\int_{\mathcal{I}}\{\rho {Z_1}^2(t) - Z_1(t)Z_2(t) + \rho Z_2^2(t)\}w(t)\,dt\right]^2, \tag{2.7}$$

where $Z_1(t)$ *and* $Z_2(t)$ *are the standardized variable of* $X(t)$ *and* $Y(t)$*:*

$$Z_1(t) = \frac{X(t) - EX(t)}{\|X(t) - EX(t)\|},$$

and

$$Z_2(t) = \frac{Y(t) - EY(t)}{\|Y(t) - EY(t)\|};$$

(b) $\widehat{\rho}_c$ *is a consistent estimator for* ρ_c, *and* $\sqrt{n}(\widehat{\rho}_c - \rho_c)$ *has an asymptotic normal distribution with zero mean and variance* $\sigma_{\rho_c}^2 = b^T \Sigma_2 b$, *where*

$$b = \frac{(2, -\rho_c, -\rho_c, 2\rho_c)^T}{E\int_{\mathcal{I}} X^2(t)w(t)\,dt + E\int_{\mathcal{I}} Y^2(t)w(t)\,dt - 2E\int_{\mathcal{I}} X(t)Y(t)w(t)\,dt}, \tag{2.8}$$

and

$$\Sigma_2 = cov\left\{\begin{pmatrix} \int_{\mathcal{I}}\{X(t) - EX(t)\}\{Y(t) - EY(t)\}w(t)\,dt \\ \int_{\mathcal{I}} X^2(t)w(t)\,dt \\ \int_{\mathcal{I}} Y^2(t)w(t)\,dt \\ \int_{\mathcal{I}}\{(X(t) - EX(t))EY(t) + Y(t)EX(t)\}w(t)\,dt \end{pmatrix}\right\}. \tag{2.9}$$

To estimate standard errors of $\widehat{\rho}$ and $\widehat{\rho}_c$, we estimate σ_ρ^2 and $\sigma_{\rho_c}^2$ by their sample counterparts:

$$\widehat{\sigma}_\rho^2 = \widehat{a}^T \widehat{\Sigma}_1 \widehat{a} \quad \text{and} \quad \widehat{\sigma}_{\rho_c}^2 = \widehat{b}^T \widehat{\Sigma}_2 \widehat{b},$$

where $\widehat{a}, \widehat{b}, \widehat{\Sigma}_1$ and $\widehat{\Sigma}_2$ are the corresponding sample counterparts. Thus, using Theorem 2.1, standard error formulae for $\widehat{\rho}$ and $\widehat{\rho}_c$ are

$$\mathrm{SE}(\widehat{\rho}) = \widehat{\sigma}_\rho/\sqrt{n-2}, \quad \text{and} \quad \mathrm{SE}(\widehat{\rho}_c) = \widehat{\sigma}_{\rho_c}/\sqrt{n-2}. \tag{2.10}$$

We use the factor $1/\sqrt{n-2}$ rather than $1/\sqrt{n}$ because two degrees of freedom are lost in estimating the means of X and Y. The accuracy of these two standard error formulae for small to moderate sample sizes will be tested in Section 3, and we find that they perform well even for small sample size.

Asymptotic $100(1-\alpha)\%$ confidence intervals for ρ and ρ_c are

$$\widehat{\rho} \pm t_{n-2}(1-\alpha/2)\mathrm{SE}(\widehat{\rho}), \quad \text{and} \quad \widehat{\rho}_c \pm t_{n-2}(1-\alpha/2)\mathrm{SE}(\widehat{\rho}_c), \tag{2.11}$$

where $t_{n-2}(\cdot)$ is the inverse function of the cumulative distribution function of t distribution with $n-2$ degrees of freedom. In Section 3, we will demonstrate that these two confidence intervals have coverage probability close to $1-\alpha$ even when sample size n is small.

Since the range of both ρ and ρ_c is $[-1,1]$, one can improve upon the normal approximation by using Fisher's Z-transformation. That is,

$$\widehat{Z} = \frac{1}{2}\ln\frac{1+\widehat{\rho}}{1-\widehat{\rho}},$$

and

$$\widehat{Z}_c = \frac{1}{2}\ln\frac{1+\widehat{\rho}_c}{1-\widehat{\rho}_c}.$$

Using the δ-method, it can be shown that $\sqrt{n}(\widehat{Z} - \frac{1}{2}\ln\frac{1+\rho}{1-\rho})$ and $\sqrt{n}(\widehat{Z}_c - \frac{1}{2}\ln\frac{1+\rho_c}{1-\rho_c})$ have asymptotic normal distributions with zero means and variances

$$\sigma_z^2 = \sigma_\rho^2/(1-\rho^2)^2, \quad \text{and} \quad \sigma_{z_c}^2 = \sigma_{\rho_c}^2/(1-\rho_c^2)^2,$$

respectively. Thus,

$$\widehat{\sigma}_z^2 = \widehat{\sigma}_\rho^2/(1-\widehat{\rho}^2)^2, \quad \text{and} \quad \widehat{\sigma}_{z_c}^2 = \widehat{\sigma}_{\rho_c}^2/(1-\widehat{\rho}_c^2)^2,$$

respectively. Furthermore, $100(1-\alpha)\%$ confidence intervals for $Z = 2^{-1}\ln\{(1+\rho)/(1-\rho)\}$ and $Z_c = 2^{-1}\ln\{(1+\rho_c)/(1-\rho_c)\}$ are

$$\widehat{Z} \pm t_{n-2}(1-\alpha/2)\widehat{\sigma}_z/\sqrt{n-2}, \quad \text{and} \quad \widehat{Z}_c \pm t_{n-2}(1-\alpha/2)\widehat{\sigma}_{z_c}/\sqrt{n-2}. \tag{2.12}$$

Using (2.12), we may construct asymmetric confidence intervals for ρ and ρ_c. The performance of these two standard error formulae will be examined in Section 3.

2.2 Weighted concordance correlation coefficient

Chinchilli, Martel, Kumanyika and Lloyd (1996) proposed a weighted concordance correlation coefficient for repeated measurement designs. To compare the weighted concordance correlation coefficient with the concordance correlation coefficient for curve data, $\widehat{\rho}_c$ defined in (2.4), we first find the relation between $\widehat{\rho}_c$ and the componentwise concordance correlation coefficients.

Denote

$$
\begin{aligned}
s_x^2(t) &= \frac{1}{n}\sum_{i=1}^{n}\{X_i(t)-\overline{X}(t)\}^2,\\
s_y^2(t) &= \frac{1}{n}\sum_{i=1}^{n}\{Y_i(t)-\overline{Y}(t)\}^2,\\
r(t) &= \frac{\frac{1}{n}\sum_{i=1}^{n}\{X_i(t)-\overline{X}(t)\}\{Y_i(t)-\overline{Y}(t)\}}{s_x(t)s_y(t)},\\
r_c(t) &= \frac{\frac{2}{n}\sum_{i=1}^{n}\{X_i(t)-\overline{X}(t)\}\{Y_i(t)-\overline{Y}(t)\}}{s_x^2(t)+s_y^2(t)+(\overline{X}(t)-\overline{Y}(t))^2},
\end{aligned}
$$

where $r(t)$ and $r_c(t)$ are the Pearson correlation coefficient and Lin's concordance correlation coefficient at t, respectively. It can be shown that $\widehat{\rho}$ and $\widehat{\rho}_c$ can be rewritten as

$$
\widehat{\rho} = \frac{\sum_{j=1}^{N} r(t_j)s_x(t_j)s_y(t_j)w(t_j)}{\sqrt{\sum_{j=1}^{N} s_x^2(t_j)w(t_j)}\sqrt{\sum_{j=1}^{N} s_y^2(t_j)w(t_j)}}
$$

and

$$
\begin{aligned}
\widehat{\rho}_c &= \frac{\sum_{j=1}^{N} r_c(t_j)[s_x^2(t_j)+s_y^2(t_j)+\{\overline{X}(t_j)-\overline{Y}(t_j)\}^2]w(t_j)}{\sum_{j=1}^{N} s_x^2(t_j)w(t_j)+\sum_{j=1}^{N} s_y^2(t_j)w(t_j)+\sum_{j=1}^{N}\{\overline{X}(t_j)-\overline{Y}(t_j)\}^2w(t_j)}\\
&= \sum_{j=1}^{N} w_j r_c(t_j),
\end{aligned}
$$

where

$$
w_j = \frac{(s_x^2(t_j)+s_y^2(t_j)+(\overline{X}(t_j)-\overline{Y}(t_j))^2)w(t_j)}{\sum_{j=1}^{N} s_x^2(t_j)w(t_j)+\sum_{j=1}^{N} s_y^2(t_j)w(t_j)+\sum_{j=1}^{N}(\overline{X}(t_j)-\overline{Y}(t_j))^2w(t_j)}.
$$

Therefore, $\widehat{\rho}_c$ is a weighted average of the componentwise concordance correlation coefficient $r_c(t_j)$ over t_j. But this is different from the one proposed by Chinchilli, *et al* (1996), which is a weighted average over sample units, see equation (14) in their paper. The newly proposed $\widehat{\rho}_c$ is derived from the point of view of functional data analysis and the motivation in this paper is also different from theirs.

2.3 Sample size determination

Since the standard deviation of $\widehat{Z}_c$ involves unknown parameters, a pilot study is necessary in order to determine a sample size such that the type I and II errors satisfy certain predetermined levels. After the collection of pilot study data, one can compute the least acceptable ρ_c, namely $\rho_{c,a}$. If the $100(1-\alpha)$ lower confidence bound is greater than or equal to $\rho_{c,a}$, then we would accept the agreement of scores evaluated by the two different approaches. This is parallel to the concept of testing the null hypothesis $H_0 : \rho_c \leq \rho_{c,a}$ against the one-sided alternative hypothesis $H_1 : \rho_c > \rho_{c,a}$. See Lin (1992) for details. Thus if the power $1-\beta$ at the alternative $\rho_c = \rho_{c,b}$ is required, then by Theorem 2.1, the sample size is given by the formula

$$n = 2 + \left[\frac{t_{n-2}(1-\alpha)\sigma_{\rho_c,a} - t_{n-2}(\beta)\sigma_{\rho_c,b}}{\rho_{c,a} - \rho_{c,b}} \right]^2 ,$$

where $\sigma_{\rho_c,a}$ and $\sigma_{\rho_c,b}$ are the values of σ_{ρ_c}, defined in Theorem 2.1, when $\rho_c = \rho_{c,a}$ and $\rho_{c,b}$, respectively. They may be estimated by their sample counterparts. An alternative approach to determine the sample size can be derived by using the asymptotic normality of $\widehat{Z}_c$ defined in Section 2.1.

3 Simulation study and applications

In this section, we investigate the small sample performance of the proposed estimators and test the accuracy of proposed standard error formulae in (2.10) when sample size is small. We also investigate the performance of the confidence intervals in (2.11) and (2.12) in terms of coverage probability via Monte Carlo simulations. We then illustrate the proposed ideas by the data set of heart rates aforementioned in Section 1.

3.1 Simulation study

To assess the performance of proposed estimators in (2.3) and (2.4) and their standard error formulae, a Monte Carlo simulation was conducted for four underlying K-dependent gaussian processes $(X(t), Y(t))$ with mean $(\mu_x(t), \mu_y(t))$, variance $(\sigma_x^2(t), \sigma_y^2(t))$ and covariance $\mathrm{cov}(X(t), Y(t)) = \sigma_{xy}(t)$. For each guassian process, we take $K = 20, 40$ and set sample size $n = 10, 20, 50$. We generate $N = 50, 100$ sample points over a period of time $[0, 1]$. Thus, for each case of $48 (= 4 \times 2 \times 3 \times 2)$ situations, we conduct 1000 Monte Carlo simulations using MATLAB. In our simulations, the weight function involved in the definition of ρ and ρ_c is taken to be 1. The mechanism of generating simulated data is given in Section A.3 in the Appendix.

Case 1. In this case, $\mu_x(t) = \mu_y(t) = 0$, $\sigma_x^2(t) = \sigma_y^2(t) = 1$ and $\sigma_{xy} = 0.95$. In this setting, $\rho = \rho_c = 0.95$ with no difference in location and scale parameters.

Case 2. We take $\mu_x(t) = -\sqrt{0.05t}$, $\mu_y(t) = \sqrt{0.05t}$, $\sigma_x^2 = \sigma_y^2(t) = 1$ and $\sigma_{xy}(t) = 0.95$. The two mean functions are depicted in Figure 1 (a). In this case, $\rho = 0.95$ and $\rho_c = 0.9048$ with slight location shift. The pointwise correlation coefficient function and the concordance correlation coefficient function are depicted in Figure 1(b), from which we can see that both the correlation coefficient and concordance correlation coefficient function are close to 1.

Case 3. In this example, we set $\mu_x(t) = -\sqrt{0.1}/2$, $\mu_y(t) = \sqrt{0.1}/2$, $\sigma_x^2(t) = 1.1^2$, $\sigma_y^2(t) = 0.9^2$, $\sigma_{xy}(t) = 1.1 \times 0.9 \times \{\sin(2\pi t) + 1\}/2$. The covariance function is depicted in Figure 1(a). In this example, $\rho = 0.5$ and $\rho_c = 0.4670$ with a slight constant differences in both locations and variance and a varying covariance function. The pointwise correlation coefficient and the concordance correlation coefficient are displayed in Figure 1(c), from which it can be seen that the pointwise correlation coefficient and the pointwise concordance correlation coefficient vary between 0 to 1.

Case 4. Let $\mu_x(t) = -\sqrt{0.05t}$, $\mu_y(t) = \sqrt{0.05t}$, $\sigma_x^2(t) = 1.1^2$, $\sigma_y^2(t) = 0.9^2$, and $\sigma_{xy} = 1.1 \times 0.9 \times \{\sin(2\pi t) + 3\}/4$. The covariance function is depicted in Figure 1(a). In this case, the difference in location and covariance between $X(t)$ and $Y(t)$ are varying in t, but the difference in variance is a constant. In the example, we have $\rho = 0.75$ and $\rho_c = 0.7005$. The pointwise correlation coefficient and the pointwise concordance correlation coefficient vary between 0.5 and 1.

Simulation results are summarized in Tables 1 to 4. In these Tables, the column labeled "mean(std)" presents the mean and standard deviation of the 1000 estimates of ρ and ρ_c in the 1000 simulations. The column labeled "SE(std)" lists the mean and standard deviation of the 1000 estimates of standard errors defined in (2.10). CP_1 and CP_2 stand for the coverage probabilities of the confidence intervals in (2.11) and (2.12) at the significance level $\alpha = 0.05$, respectively.

From Tables 1 to 4, it can be seen that the averages of estimates of $\widehat{\rho}$ and $\widehat{\rho}_c$ are very close to the true values no matter what the sample size n is. The average of estimated standard errors

(SE) is very close to the standard deviation of 1000 estimates, which can be regarded as the true value of standard deviation of $\widehat{\rho}$ and $\widehat{\rho}_c$. The difference between the average of estimated standard errors and the true value is less than one standard deviation of the estimated standard errors. This implies that the standard error formulae proposed in (2.10) are surprisingly accurate even for small to moderate sample size. Comparing the two columns labeled CP_1 and CP_2 of these tables, we found that the Fisher Z-transform makes the normal approximation for $\widehat{\rho}$ better, but not for $\widehat{\rho}_c$. This is because the Fisher Z-transform actually is a variance-stable transform for the sample correlation of bivariate normal sample (see, for example, Muirhead, 1982, page 159), but not for the concordance correlation coefficient. However, the Z-transform is useful for constructing an asymmetric confidence interval of ρ_c. Note that the Monte Carlo standard error for the coverage probability is 0.0069. It can be seen from the last two columns of these tables that most of the coverage probabilities are very close to the true probability 0.95. This implies that the proposed confidence interval formulae in (2.11) and (2.12) work well even for small sample size.

3.2 Applications

We demonstrate in this section the proposed statistical inference procedure via an application to an actual data set collected in the Noll Physiological Research Center at The Pennsylvania State University. The purpose to this study was to evaluate the performance of the polar heart rate monitor and to validate the the monitor for future such use. One of advantages of using the polar heart rate monitor is that it would provide an unencumbered measure of heart rate. In other words, the subject can move about freely without the wire connections necessary to record an ECG. One use of such data is in industrial settings where work is done in hot environments or while wearing protective clothing which inhibit the evaporation of sweat. During this study, heart rates measured by the polar heart rate monitor and by a golden standard approach were recorded every minute over 75 minutes experiment period for each individual. We apply the proposed statistical procedure in Section 2 for this data set.

In this study, experiments were conducted under two different heating conditions: *passive heating* and *active heating*. There are 7 subjects in the passive heating group, and 6 subjects in the active heating group. For each group, sample curves of two typical subjects are depicted in Figures 2, 3 (a) and (b). The sample mean and standard deviation curves are also depicted in Figures 2 , 3 (c) and (d). From Figures 2 and 3, the heart rates measured by polar heart rate monitor are quite consistent with those by the golden standard approach. For the passive heating group, we obtain that $\widehat{\rho} = 0.9495$ with standard error 0.0290, and $\widehat{\rho}_c = 0.9412$ with standard error 0.0338. The high correlation coefficient and concordance correlation coefficient indicates that the results yielded by the polar heart rate monitor agree well with the results yielded by the golden standard approach. The values of $\widehat{\rho}$ and $\widehat{\rho}_c$ are very close. This implies that there is no location shift and no variance change. For the active heating group, the resulting estimates are $\widehat{\rho} = 0.9633$ with standard error 0.0261, and $\widehat{\rho}_c = 0.9262$ with standard error 0.0500. For this group, $\widehat{\rho}_c$ is smaller than $\widehat{\rho}$. From Figure 3, it can be seen that there is a slight lag shift between the two mean functions. This indicates that ρ_c is effective in detecting location shifts whereas ρ is not. In general, ρ_c is a better measure for reproducibility of curve data than ρ.

4 Concordance correlation coefficient for image data

A concordance correlation coefficient for curve data has been derived in Section 2. Image data are commonplace in medical research. It is of interest to assess agreement of image data. In this section, we extend the idea in Section 2 and derive a concordance correlation coefficient for image data.

Suppose that $X(\mathbf{t})$ and $Y(\mathbf{t})$, $\mathbf{t} \in \mathbf{I}$, a closed rectangle of the plane R^2, are image data produced by two instruments. Let $X(\mathbf{t})$ and $Y(\mathbf{t})$ be two elements on a probability functional space $\mathcal{S}$, and define an inner product for $\mathcal{S}$ as:

$$< X, Y >= E \int_{\mathbf{I}} X(\mathbf{t})Y(\mathbf{t})\, w(t) d\mathbf{t}.$$

Thus we can define a correlation coefficient and a concordance correlation coefficient in the same ways as (2.1) and (2.2).

$$\rho(X, Y) = \frac{< X - E(X), Y - E(Y) >}{\|X - E(X)\| \|Y - E(Y)\|}, \tag{4.1}$$

and

$$\rho_c(X, Y) = \frac{2 < X - E(X), Y - E(Y) >}{\|X - E(X)\|^2 + \|Y - E(Y)\|^2 + \|E(X) - E(Y)\|^2}. \tag{4.2}$$

One can check that the characteristics of ρ_c in Section 2 are still valid.

Here we focus on the estimation of ρ and ρ_c. Without loss of generality, it is assumed that the rectangle $\mathbf{I} = [0, 1]^2$. Suppose that $X_i(\mathbf{t}), Y_i(\mathbf{t})$ was observed at $\mathbf{t}_{kl} = (t_{1k}, t_{2l})$, $k = 1, \cdots, K$ and

$l = 1, \cdots, L$ with $0 \le t_{11} < \cdots < t_{1K} \le 1$ and $0 \le t_{21} < \cdots < t_{2L} \le 1$. Denote $N = KL$. Using the sample counterparts to estimate ρ and ρ_c, we have

$$\widehat{\rho} = \frac{\frac{1}{nN}\sum_{i,k,l}\{X_i(\mathbf{t}_{kl}) - \overline{X}(\mathbf{t}_{kl})\}\{Y_i(\mathbf{t}_{kl}) - \overline{Y}(\mathbf{t}_{kl})\}w(\mathbf{t}_{kl})}{\{\frac{1}{nN}\sum_{i,k,l}(X_i(\mathbf{t}_{kl}) - \overline{X}(\mathbf{t}_{kl}))^2 w(\mathbf{t}_{kl})\}^{1/2}\{\frac{1}{nN}\sum_{i,k,l}(Y_i(\mathbf{t}_{kl}) - \overline{Y}(\mathbf{t}_{kl}))^2 w(\mathbf{t}_{kl})\}^{1/2}}, \tag{4.3}$$

and

$$\widehat{\rho}_c = \frac{\frac{2}{nN}\sum_{i,k,l}\{X_i(\mathbf{t}_{kl}) - \overline{X}(\mathbf{t}_{kl})\}\{Y_i(\mathbf{t}_{kl}) - \overline{Y}(\mathbf{t}_{kl})\}w(\mathbf{t}_{kl})}{\frac{1}{nN}\sum_{i,k,l}\left[\{X_i(\mathbf{t}_{kl}) - \overline{X}(\mathbf{t}_{kl})\}^2 + \{Y_i(\mathbf{t}_{kl}) - \overline{Y}(\mathbf{t}_{kl})\}^2\right]w(\mathbf{t}_{kl}) + \sum_{k,l}\{\overline{X}(\mathbf{t}_{kl}) - \overline{Y}(\mathbf{t}_{kl})\}^2 w(\mathbf{t}_{kl})}. \tag{4.4}$$

Computer codes for computing $\widehat{\rho}$ and $\widehat{\rho}_c$ defined in (2.3) and (2.4) can be used directly to compute $\widehat{\rho}$ and $\widehat{\rho}_c$ in (4.3) and (4.4) provided that we vectorize the image data matrix $(X(t_{ij}))_{K\times L}$ and $(Y(t_{ij}))_{K\times L}$ first. Similar to Theorem 2.1, both $\widehat{\rho}$ and $\widehat{\rho}_c$ are consistent and have asymptotic normal distributions.

Theorem 4.1 *Suppose that* $\{(X_i(\mathbf{t}_{kl}), Y_i(\mathbf{t}_{kl}))\}, i = 1, \cdots, n, \ k = 1, \cdots, K, l = 1, \cdots, L$ *is a random sample from* $\{(X(\mathbf{t}_{kl}), Y(\mathbf{t}_{kl}))\}$. *If Conditions (A), (B), (C') and (D') given in Appendix A.2 hold, then*

(a) the $\widehat{\rho}$*, given by (4.3), is a consistent estimator for* ρ*, and* $\sqrt{n}(\widehat{\rho} - \rho)$ *has an asymptotic normal distribution with zero mean and variance* $\sigma_\rho^2 = a^T\Sigma_1 a$*, where*

$$a = (1/\|X - E(X)\|\|Y - E(Y)\|, -\frac{1}{2}\rho/\|X - E(X)\|^2, -\frac{1}{2}\rho/\|Y - E(Y)\|^2)^T, \tag{4.5}$$

and

$$\Sigma_1 = cov\left\{\begin{pmatrix} \int_{\mathbf{I}}\{X(\mathbf{t}) - EX(\mathbf{t})\}\{Y(\mathbf{t}) - EY(\mathbf{t})\}w(\mathbf{t})\,d\mathbf{t} \\ \int_{\mathbf{I}}\{X(\mathbf{t}) - EX(\mathbf{t})\}^2 w(\mathbf{t})\,d\mathbf{t} \\ \int_{\mathbf{I}}\{Y(\mathbf{t}) - EY(\mathbf{t})\}^2 w(\mathbf{t})\,d\mathbf{t} \end{pmatrix}\right\}; \tag{4.6}$$

(b) the $\widehat{\rho}_c$*, defined in (4.4), is a consistent estimator for* ρ_c*, and* $\sqrt{n}(\widehat{\rho}_c - \rho)$ *has an asymptotic normal distribution with zero mean and variance* $\sigma_{\rho_c}^2 = b^T\Sigma_2 b$*, where*

$$b = \frac{(2, -\rho_c, -\rho_c, 2\rho_c)^T}{E\int_{\mathbf{I}} X^2(\mathbf{t})w(\mathbf{t})\,dt + E\int_{\mathbf{I}} Y^2(\mathbf{t})w(\mathbf{t})\,d\mathbf{t} - 2E\int_{\mathbf{I}} X(\mathbf{t})Y(\mathbf{t})w(\mathbf{t})\,d\mathbf{t}}, \tag{4.7}$$

and

$$\Sigma_2 = cov\left\{\begin{pmatrix} \int_{\mathbf{I}}\{X(\mathbf{t}) - EX(\mathbf{t})\}\{Y(\mathbf{t}) - EY(\mathbf{t})\}w(\mathbf{t})\,d\mathbf{t} \\ \int_{\mathbf{I}} X^2(\mathbf{t})w(\mathbf{t})\,d\mathbf{t} \\ \int_{\mathbf{I}} Y^2(\mathbf{t})w(\mathbf{t})\,d\mathbf{t} \\ \int_{\mathbf{I}}\{(X(\mathbf{t}) - EX(\mathbf{t}))EY(\mathbf{t}) + Y(\mathbf{t})EX(\mathbf{t})\}w(\mathbf{t})\,d\mathbf{t} \end{pmatrix}\right\}. \tag{4.8}$$

As discussed in Section 2, we may apply the Fisher Z-transformation for $\widehat{\rho}$ and $\widehat{\rho}_c$ to obtain a better normal approximation. Furthermore, one may construct an asymptotic confidence interval using the asymptotic properties in Theorem 4.1. The results in Theorem 4.1 can be extended to the general cases in which the argument $\mathbf{t}$ is multidimensional. However, this extension is less significant for practical use and we do not pursue the issue further.

5 Conclusions

In this paper, we proposed a concordance correlation coefficient for curve data and image data. Its characteristics have been investigated. We proposed an estimator for the concordance correlation coefficient and established the asymptotic normality of the proposed estimator. A standard error formula for the resulting estimate is derived and empirically tested. An actual data set is illustrated the proposed methodology.

Appendix

In this section, we first present detailed proofs of Theorems 2.1 and 4.1. Then we describe the mechanism of generating simulated data in the Monte Carlo simulation in Section 3.

A.1 Proof of Theorem 2.1

Definition A.1. Let $0 = t_0 < \cdots < t_l = 1$ be any partition $\mathcal{P}$ of $[0,1]$, and $f(t)$ be a real function defined on $[0,1]$. If the variation

$$V_{\mathcal{P}} = \sum_{l=1}^{l} |f(t_{j+1}) - f(t_j)|$$

has an upper bound which is independent of the choice of $\mathcal{P}$, then f is called a function of bounded variation. The least upper bound of $V_{\mathcal{P}}$ is called the total variation of f and is denoted by $V(f)$.

For a function of bounded variation, it has been shown that

$$\left| \int_0^1 f(t)dt - \frac{1}{N}\sum_{j=1}^{N} f(t_j) \right| \leq V(f) \max_{0\leq j\leq N-1} |t_{j+1} - t_j|. \tag{A.1}$$

See, for example, Theorem 5.3 of Hua and Wang (1981). Particularly, for the evenly partitioned intervals $t_j = j/N$, $j = 0, \cdots, N$, it follows that

$$\left| \int_0^1 f(t)dt - \frac{1}{N}\sum_{j=0}^{N} f(t_j) \right| \leq \frac{1}{N} V(f).$$

The equation (A.1) tells us how to impose assumptions on $V(f)$ and $\max_{0\leq j\leq N-1} |t_{j+1} - t_j|$ in a natural way. Let $\mathcal{X}$ and $\mathcal{Y}$ denote the space consisting of paths $X(\cdot)$ and $Y(\cdot)$. Further, define

$$\begin{aligned} \mathcal{Z}_1 &= \{X(\cdot)Y(\cdot) : X \in \mathcal{X} \quad \text{and} \quad Y \in \mathcal{Y}\}, \\ \mathcal{Z}_2 &= \{X^2(\cdot) : X \in \mathcal{X}\}, \\ \mathcal{Z}_3 &= \{Y^2(\cdot) : Y \in \mathcal{Y}\}. \end{aligned}$$

Conditions

(A) For $\mathcal{F} = \mathcal{X}, \mathcal{Y}, \mathcal{Z}_i$, $i = 1, 2$ and 3,

$$\sup_{f\in\mathcal{F}} V(f) < \infty \qquad \text{a.s.} \tag{A.2}$$

(B) The weight function $w(t)$ is a function of bounded variation.

(C) There exists a constant Δ such that

$$\max_{0\leq j\leq N-1} |t_{j+1} - t_j| \leq \frac{\Delta}{N}$$

and $\sqrt{n}/N \to 0$ as $n \to \infty$.

(D)

$$\begin{aligned} &E[\int_{\mathcal{I}} X^2(t)w(t)\, dt]^2 < \infty, \\ &E[\int_{\mathcal{I}} Y^2(t)w(t)\, dt]^2 < \infty, \\ &E[\int_{\mathcal{I}} X(t)Y(t)w(t)\, dt]^2 < \infty. \end{aligned}$$

Lemma A.1 *Under the conditions of Theorem 2.1, we have*

$$\frac{1}{n}\sum_{i=1}^{n} \frac{1}{N}\sum_{j=1}^{N} X_i(t_j)Y_i(t_j)w(t_j) = \frac{1}{n}\sum_{i=1}^{n} \int_{\mathcal{I}} X_i(t)Y_i(t)w(t)\, dt + O_P(\frac{1}{N}). \tag{A.3}$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\sum_{j=1}^{N}X_i^2(t_j)w(t_j) = \frac{1}{n}\sum_{i=1}^{n}\int_{\mathcal{I}}X_i^2(t)w(t)\,dt + O_P(\frac{1}{N}), \tag{A.4}$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\sum_{j=1}^{N}Y_i^2(t_j)w(t_j) = \frac{1}{n}\sum_{i=1}^{n}\int_{\mathcal{I}}Y_i^2(t)w(t)\,dt + O_P(\frac{1}{N}), \tag{A.5}$$

$$\frac{1}{N}\sum_{j=1}^{N}\overline{X}^2(t_j)w(t_j) = \int_{\mathcal{I}}\overline{X}^2(t)w(t)\,dt + O_P(\frac{1}{N}), \tag{A.6}$$

$$\frac{1}{N}\sum_{j=1}^{N}\overline{Y}^2(t_j)w(t_j) = \int_{\mathcal{I}}\overline{Y}^2(t)w(t)\,dt + O_P(\frac{1}{N}), \tag{A.7}$$

$$\frac{1}{N}\sum_{j=1}^{N}\overline{X}(t_j)\overline{Y}(t_j)w(t_j) = \int_{\mathcal{I}}\overline{X}(t)\overline{Y}(t)w(t)\,dt + O_P(\frac{1}{N}). \tag{A.8}$$

Proof. Denote

$$\mathcal{G} = \{X(\cdot)Y(\cdot)w(\cdot) : X \in \mathcal{X}, Y \in \mathcal{Y}\}.$$

Since $w(t)$ is a function of bounded variation, using (A.2), it can be shown that

$$\sup_{f\in\mathcal{G}} V(f) < \infty, \qquad \text{a.s.}$$

Thus, by Condition (C), it follows that

$$\begin{aligned}
&|\frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\sum_{j=1}^{N}X_i(t_j)Y_i(t_j)w(t_j) - \frac{1}{n}\sum_{i=1}^{n}\int_{\mathcal{I}}X_i(t)Y_i(t)w(t)\,dt| \\
\le\ & \frac{1}{n}\sum_{i=1}^{n}|\frac{1}{N}\sum_{j=1}^{N}X_i(t_j)Y_i(t_j)w(t_j) - \int_{\mathcal{I}}X_i(t)Y_i(t)w(t)\,dt| \\
\le\ & \frac{1}{n}\sum_{i=1}^{n}V(X_iY_iw)\max_{0\le j\le N-1}|t_{j+1}-t_j| \\
\le\ & \sup_{f\in\mathcal{G}}V(f)\frac{\Delta}{N}.
\end{aligned}$$

So (A.3) follows. Along the same lines as the proof of (A.3), it can be shown that (A.4) to (A.8) hold. This completes the proof of Lemma A.1.

Proof of Theorem 2.1. Denote

$$A_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{N}\sum_{j=1}^{N}\{X_i(t_j) - \overline{X}(t_j)\}\{Y_i(t_j) - \overline{Y}(t_j)\}w(t_j).$$

Using Lemma A.1, it follows that

$$\begin{aligned} A_n &= \frac{1}{n}\sum_{i=1}^n \int_{\mathcal{I}} X_i(t)Y_i(t)w(t)\,dt - \int_{\mathcal{I}} \overline{X}(t)\overline{Y}(t)w(t)\,dt + O_P(\frac{1}{N}) \\ &= \frac{1}{n}\sum_{i=1}^n \int_{\mathcal{I}} \{X_i(t) - EX(t)\}\{Y_i(t) - EY(t)\}w(t)\,dt \\ &\quad - \int_{\mathcal{I}} \{\overline{X}(t) - EX(t)\}\{\overline{Y}(t) - EY(t)\}w(t)\,dt + O_P(\frac{1}{N}). \end{aligned}$$

Since

$$E\int_{\mathcal{I}} \{\overline{X}(t) - EX(t)\}\{\overline{Y}(t) - EY(t)\}w(t)\,dt = O_P(\frac{1}{n}),$$

and

$$\mathrm{var}\{\int_{\mathcal{I}} \{\overline{X}(t) - EX(t)\}\{\overline{Y}(t) - EY(t)\}w(t)\,dt\} = O_P(\frac{1}{n^2}),$$

we have

$$A_n = \frac{1}{n}\sum_{i=1}^n \int_{\mathcal{I}} \{X_i(t) - EX(t)\}\{Y_i(t) - EY(t)\}w(t)\,dt + O_P(\frac{1}{n}) + O_P(\frac{1}{N}) \tag{A.9}$$

which tends to $E\int_{\mathcal{I}}\{X(t) - EX(t)\}\{Y(t) - EY(t)\}w(t)\,dt$ in probability by the weak law of large number and $N \to \infty$. Similarly, we can show that all other terms involved in $\widehat{\rho}$ and $\widehat{\rho}_c$ are convergent in probability, and therefore $\widehat{\rho}$ and $\widehat{\rho}_c$ are consistent.

Now we establish the asymptotic normality of $\widehat{\rho}$ and $\widehat{\rho}_c$. Denote

$$B_n = \frac{1}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N \{X_i(t_j) - \overline{X}(t_j)\}^2 w(t_j),$$

and

$$C_n = \frac{1}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N \{Y_i(t_j) - \overline{Y}(t_j)\}^2 w(t_j).$$

Using similar arguments to those for A_n, it can be shown that

$$B_n = \frac{1}{n}\sum_{i=1}^n \int_{\mathcal{I}} \{X_i(t) - EX(t)\}^2 w(t)\,dt + O_P(\frac{1}{n}) + O_P(\frac{1}{N}),$$

and

$$C_n = \frac{1}{n}\sum_{i=1}^n \int_{\mathcal{I}} \{Y_i(t) - EY(t)\}^2 w(t)\,dt + O_P(\frac{1}{n}) + O_P(\frac{1}{N}).$$

Note that $U_n = (A_n, B_n, C_n)^T$ and

$$U = \begin{pmatrix} E\int_{\mathcal{I}}\{X(t) - EX(t)\}\{Y(t) - EY(t)\}w(t)\,dt \\ E\int_{\mathcal{I}}\{X(t) - EX(t)\}^2 w(t)\,dt \\ E\int_{\mathcal{I}}\{Y(t) - EY(t)\}^2 w(t)\,dt \end{pmatrix}.$$

Thus, by the multivariate central limit theorem, the Slutsky theorem, and the assumption $\sqrt{n}/N \to 0$, it follows that $\sqrt{n}(U_n - U)$ has an asymptotic normal distribution with mean zero and variance Σ_1 given in (2.6)

Define $g(u_1, u_2, u_3) = u_1/\sqrt{u_2 u_3}$. Thus, $\widehat{\rho} = g(A_n, B_n, C_n)$. Using the delta method, it follows that $\sqrt{n}(\widehat{\rho} - \rho)$ has an asymptotic normal distribution with mean zero and variance $a^T \Sigma_1 a$, where a is given in (2.5). By some algebraic calculations, (2.7) follows.

To establish the asymptotic normality of $\widehat{\rho}_c$, we rewrite $\widehat{\rho}_c$ as

$$\widehat{\rho}_c = \frac{\frac{2}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N \{X_t(t_j) - \overline{X}(t_j)\}\{Y_i(t_j) - \overline{Y}(t_j)\}w(t_j)}{\frac{1}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N X_i^2(t_j)w(t_j) + \frac{1}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N Y_i^2(t_j)w(t_j) - \frac{2}{N}\sum_{j=1}^N \overline{X}(t_j)\overline{Y}(t_j)w(t_j)}.$$

Denote

$$\begin{aligned} D_n &= \frac{1}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N X_i^2(t_j)w(t_j), \\ E_n &= \frac{1}{n}\sum_{i=1}^n \frac{1}{N}\sum_{j=1}^N Y_i^2(t_j)w(t_j), \end{aligned}$$

and

$$F_n = \frac{1}{N}\sum_{j=1}^N \overline{X}(t_j)\overline{Y}(t_j)w(t_j).$$

By straightforward calculation, we have

$$\begin{aligned} D_n &= \frac{1}{n}\sum_{i=1}^n \int_{\mathcal{I}} X_i^2(t)w(t)\,dt + O_P(\frac{1}{N}), \\ E_n &= \frac{1}{n}\sum_{i=1}^n \int_{\mathcal{I}} Y_i^2(t)w(t)\,dt + O_P(\frac{1}{N}). \end{aligned}$$

Next we deal with F_n. It is not difficult to show that

$$F_n = \int_{\mathcal{I}} \overline{X}(t)\overline{Y}(t)w(t)\,dt + O_P(\frac{1}{N})$$

which equals to

$$\int_{\mathcal{I}} \{\overline{X}(t) - EX(t)\}\{\overline{Y}(t) - EY(t)\}w(t)\,dt + \int_{\mathcal{I}} \{(\overline{X}(t) - EX(t))EY(t) + EX(t)\overline{Y}(t)\}w(t)\,dt + O_P(\frac{1}{N}).$$

This is equal to

$$F_n = \frac{1}{n}\sum_{i=1}^n \int_{\mathcal{I}} \{(X_i(t) - EX(t))EY(t) + EX(t)Y_i(t)\}w(t)\,dt + O_P(\frac{1}{n}) + O_P(\frac{1}{N})$$

Define $V_n = (A_n, D_n, E_n, F_n)^T$ and

$$V = \begin{pmatrix} E\int_{\mathcal{I}}\{X(t) - EX(t)\}\{Y(t) - EY(t)\}w(t)\,dt \\ E\int_{\mathcal{I}} X^2(t)w(t)\,dt \\ E\int_{\mathcal{I}} Y^2(t)w(t)\,dt \\ \int_{\mathcal{I}} EY(t)EX(t)\}w(t)\,dt \end{pmatrix}.$$

By Condition (C), the multivariate central limit theorem and the Slutsky theorem, $\sqrt{n}(V_n - V)$ has an asymptotic normal distribution with mean zero and variance Σ_2 given by (2.9).

Define $h(v_1, v_2, v_3, v_4) = 2v_1/(v_2 + v_3 - 2v_4)$. Thus, $\widehat{\rho}_c = h(A_n, D_n, E_n, F_n)$. Using the delta method, $\sqrt{n}(\widehat{\rho}_c - \rho_c)$ has an asymptotic normal distribution with mean zero and variance $b^T\Sigma_2 b$, where b is defined in (2.8). This completes the proof of Theorem 2.1.

A.2 Proof of Theorem 4.1

Definition A.2 Let $0 = t_{10} < \cdots < t_{1m_1} = 1$ and $0 = t_{20} < \cdots < t_{2m_2} = 1$ be any partition $\mathcal{P}$ of $[0,1]^2$, and $f(t_1, t_2)$ be a real function defined on $[0,1]^2$. Define

$$\Delta_{10} f(t_{1i}, t_2) = f(t_{1(i+1)}, t_2) - f(t_{1i}, t_2),$$
$$\Delta_{01} f(t_1, t_{2j}) = f(t_1, t_{2(j+1)}) - f(t_1, t_{2j}),$$
$$\Delta_{11} f(t_{1i}, t_{2j}) = f(t_{1i}, t_{2j}) - f(t_{1(i+1)}, t_{2j}) - f(t_{1i}, t_{2(j+1)}) + f(t_{1(i+1)}, t_{2(j+1)}).$$

If the variation

$$V_{\mathcal{P}} = \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1} |\Delta_{11} f(t_{1i}, t_{2j})| + \sum_{i=0}^{m_1-1} |\Delta_{10} f(t_{1i}, 1)| + \sum_{j=0}^{m_2-1} |\Delta_{01} f(1, t_{2j})|$$

has an upper bound which is independent of the choice $\mathcal{P}$, then f is called a function of bounded variation in the sense of Harday and Krause. The least upper bound of $V_{\mathcal{P}}$ is called the total variation of f and is denoted by $V(f)$.

Condition

(C') There exists Δ_1 and Δ_2 such that

$$\max_{0\le k\le K} |t_{1(k+1)} - t_{1k}| \le \frac{\Delta_1}{K},$$

and

$$\max_{0\le l\le L} |t_{2(l+1)} - t_{2l}| \le \frac{\Delta_2}{L}.$$

Further, $\sqrt{n}/N \to 0$ as $n \to \infty$, where $N = KL$.

(D')

$$E[\int_{\mathbf{I}} X^2(\mathbf{t})w(\mathbf{t})\,d\mathbf{t}]^2 < \infty,$$
$$E[\int_{\mathbf{I}} Y^2(\mathbf{t})w(\mathbf{t})\,d\mathbf{t}]^2 < \infty,$$
$$E[\int_{\mathbf{I}} X(\mathbf{t})Y(\mathbf{t})w(\mathbf{t})\,dt]^2 < \infty.$$

Lemma A.2 *Under the conditions of Theorem 4.1, we have*

$$\frac{1}{nN}\sum_{i,k,l} X_i(\mathbf{t}_{kl})Y_i(\mathbf{t}_{kl})w(\mathbf{t}_{kl}) = \frac{1}{n}\sum_{i=1}^{n}\int_{\mathbf{I}} X_i(\mathbf{t})Y_i(\mathbf{t})w(\mathbf{t})\,dt + O_P(\frac{1}{N}). \tag{A.10}$$
$$\frac{1}{nN}\sum_{i,k,l} X_i^2(\mathbf{t}_{kl})w(\mathbf{t}_{kl}) = \frac{1}{n}\sum_{i=1}^{n}\int_{\mathbf{I}} X_i^2(\mathbf{t})w(\mathbf{t})\,dt + O_P(\frac{1}{N}), \tag{A.11}$$
$$\frac{1}{nN}\sum_{i,k,l} Y_i^2(\mathbf{t}_{kl})w(\mathbf{t}_{kl}) = \frac{1}{n}\sum_{i=1}^{n}\int_{\mathbf{I}} Y_i^2(\mathbf{t})w(\mathbf{t})\,dt + O_P(\frac{1}{N}), \tag{A.12}$$
$$\frac{1}{N}\sum_{k,l} \overline{X}^2(\mathbf{t}_{kl})w(\mathbf{t}_{kl}) = \int_{\mathbf{I}} \overline{X}^2(\mathbf{t})w(\mathbf{t})\,dt + O_P(\frac{1}{N}), \tag{A.13}$$
$$\frac{1}{N}\sum_{k,l} \overline{Y}^2(\mathbf{t}_{kl})w(\mathbf{t}_{kl}) = \int_{\mathbf{I}} \overline{Y}^2(\mathbf{t})w(\mathbf{t})\,d\mathbf{t} + O_P(\frac{1}{N}), \tag{A.14}$$
$$\frac{1}{N}\sum_{k,l} \overline{X}(\mathbf{t}_{kl})\overline{Y}(\mathbf{t}_{kl})w(\mathbf{t}_{kl}) = \int_{\mathbf{I}} \overline{X}(\mathbf{t})\overline{Y}(\mathbf{t})w(\mathbf{t})\,d\mathbf{t} + O_P(\frac{1}{N}). \tag{A.15}$$

Proof. Denote

$$\mathcal{H} = \{X(\cdot)Y(\cdot)w(\cdot) : X \in \mathcal{X}, Y \in \mathcal{Y}\}.$$

Since $w(t)$ is a function of bounded variation, using (A.2), it can be shown that

$$\sup_{f\in\mathcal{H}} V(f) < \infty, \qquad \text{a.s.}$$

Using Theorem 5.3 of Hua and Wang (1981), it follows,

$$\left|\int_{[0,1]^2} f(\mathbf{t})\,dt - \frac{1}{N}\sum_{k,l} f(\mathbf{t}_{kl})\right| \le V(f) \max_{0\le k\le K-1} |t_{1(j+1)} - t_{1j}| \max_{0\le l\le L-1} |t_{2(l+1)} - t_{2l}|. \tag{A.16}$$

Thus, by Condition (C) and A.16), it follows that

$$|\frac{1}{nN}\sum_{i,k,l} X_i(t_j)Y_i(t_j)w(t_j) - \frac{1}{n}\sum_{i=1}^{n}\int_{\mathcal{I}} X_i(t)Y_i(t)w(t)\,dt|$$

$$
\begin{aligned}
&\le \frac{1}{n}\sum_{i=1}^{n}|\frac{1}{N}\sum_{k,l}X_i(\mathbf{t}_{kl})Y_i(\mathbf{t}_{kl})w(\mathbf{t}_{kl}) - \int_{\mathbf{I}} X_i(\mathbf{t})Y_i(\mathbf{t})w(\mathbf{t})\,d\mathbf{t}| \\
&\le \frac{1}{n}\sum_{i=1}^{n} V(X_iY_iw)\max_{0\le j\le K-1}|t_{1(k+1)} - t_{1k}|\max_{0\le l\le L-1}|t_{2(l+1)} - t_{2l}| \\
&\le \sup_{f\in\mathcal{G}} V(f)\frac{\Delta_1\Delta_2}{N}.
\end{aligned}
$$

So (A.10) follows. Along the same lines as the proof of (A.10), it can be shown that (A.11)—(A.15) hold. This completes the proof of Lemma A.1.

Proof of Theorem 4.1 Using Lemma A.2, Theorem 4.1 follows by similar arguments in the proof of Theorem 2.1.

A.3 Mechanism of generating simulated data

In this section, we describe how to generate a random sample $(X(t), Y(t))$ used in our Monte Carlo simulation in Section 3. The algorithm is stated as follows.

Algorithm

Step 1 (*Generate moving average with order K (MA(K)) process*)

For given N and K, generate $\varepsilon_1, \cdots, \varepsilon_{N+K}$ iid $N(0,1)$ and $\eta_1, \cdots, \eta_{N+K}$ iid $N(0,1)$. Furthermore, ε_i's and η_j's are independent. Let $t_j = j/(N+1)$, $j = 1, \cdots, N$. For each j,

$$
X_0(t_j) = \frac{1}{\sqrt{K}}\sum_{k=1}^{K}\varepsilon_{j+k}, \quad \text{and} \quad Y_0(t_j) = \frac{1}{\sqrt{K}}\sum_{k=1}^{K}\eta_{j+k} \quad \text{for} \quad j = 1, \cdots, N.
$$

It is easily verified that both $X_0(t_j)$ and $Y_0(t)$ are MA(K) processes with autocorrelation $1 - j/K$ for $|j| \le K$ and 0 for $|j| > K$. The variances of $X_0(t_j)$ and $Y_0(t_j)$ are equal to 1. Furthermore, X_0 and Y_0 are independent.

Step 2 (*Generate a K-dependent gaussian process*)

For given mean function $(\mu_x(t), \mu_y(t))$ and covariance matrix $\Sigma(t) = \text{cov}\{(X(t), Y(t))\}$

$$
(X(t), Y(t)) = (\mu_x(t), \mu_y(t)) + (X_0(t), Y_0(t))\Sigma^{1/2}(t)
$$

for $t = t_j$ with $j = 1, \cdots, N$. Thus, it is easily verified that the generated data have desired mean function and covariance structure.

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Table 1: Simulation Results of Case 1 ($\rho = \rho_c = 0.95$)

(N,k)		mean(std)	SE(std)	CP_1	CP_2
		$n = 10$			
(50,20)	$\widehat{\rho}$	0.9495 (0.0168)	0.0146 (0.0063)	0.899	0.933
	$\widehat{\rho}_c$	0.9432 (0.0181)	0.0161 (0.0069)	0.945	0.934
(100,20)	$\widehat{\rho}$	0.9497 (0.0120)	0.0108 (0.0037)	0.932	0.944
	$\widehat{\rho}_c$	0.9442 (0.0127)	0.0120 (0.0040)	0.958	0.938
(50,40)	$\widehat{\rho}$	0.9478 (0.0232)	0.0192 (0.0102)	0.890	0.920
	$\widehat{\rho}_c$	0.9406 (0.0253)	0.0214 (0.0111)	0.918	0.917
(100,40)	$\widehat{\rho}$	0.9490 (0.0172)	0.0145 (0.0061)	0.908	0.927
	$\widehat{\rho}_c$	0.9428 (0.0186)	0.0160 (0.0066)	0.950	0.925
		$n = 20$			
(50,20)	$\widehat{\rho}$	0.9493 (0.0115)	0.0107 (0.0031)	0.922	0.942
	$\widehat{\rho}_c$	0.9464 (0.0118)	0.0112 (0.0032)	0.950	0.946
(100,20)	$\widehat{\rho}$	0.9496 (0.0085)	0.0078 (0.0019)	0.927	0.933
	$\widehat{\rho}_c$	0.9469 (0.0087)	0.0082 (0.0020)	0.947	0.936
(50,40)	$\widehat{\rho}$	0.9490 (0.0155)	0.0138 (0.0049)	0.903	0.923
	$\widehat{\rho}_c$	0.9456 (0.0163)	0.0145 (0.0051)	0.928	0.914
(100,40)	$\widehat{\rho}$	0.9493 (0.0108)	0.0105 (0.0030)	0.933	0.941
	$\widehat{\rho}_c$	0.9463 (0.0111)	0.0110 (0.0031)	0.948	0.940
		$n = 50$			
(50,20)	$\widehat{\rho}$	0.9496 (0.0068)	0.0068 (0.0012)	0.950	0.954
	$\widehat{\rho}_c$	0.9484 (0.0069)	0.0070 (0.0012)	0.959	0.951
(100,20)	$\widehat{\rho}$	0.9499 (0.0050)	0.0050 (0.0008)	0.943	0.943
	$\widehat{\rho}_c$	0.9488 (0.0051)	0.0051 (0.0008)	0.958	0.944
(50,40)	$\widehat{\rho}$	0.9493 (0.0094)	0.0090 (0.0020)	0.930	0.932
	$\widehat{\rho}_c$	0.9479 (0.0096)	0.0092 (0.0021)	0.940	0.933
(100,40)	$\widehat{\rho}$	0.9498 (0.0072)	0.0068 (0.0013)	0.927	0.935
	$\widehat{\rho}_c$	0.9486 (0.0073)	0.0069 (0.0013)	0.938	0.930

Table 2: Simulation Results of Case 2 ($\rho = 0.95$ and $\rho_c = 0.9048$)

(N,k)		mean(std)	SE(std(SE))	CP_1	CP_2
		$n = 10$			
(50,20)	$\widehat{\rho}$	0.9495 (0.0168)	0.0146 (0.0063)	0.899	0.933
	$\widehat{\rho}_c$	0.8993 (0.0300)	0.0274 (0.0102)	0.932	0.929
(100,20)	$\widehat{\rho}$	0.9497 (0.0120)	0.0108 (0.0037)	0.932	0.944
	$\widehat{\rho}_c$	0.9006 (0.0219)	0.0207 (0.0064)	0.950	0.950
(50,40)	$\widehat{\rho}$	0.9478 (0.0232)	0.0192 (0.0102)	0.890	0.920
	$\widehat{\rho}_c$	0.8935 (0.0404)	0.0362 (0.0154)	0.928	0.930
(100,40)	$\widehat{\rho}$	0.9490 (0.0172)	0.0145 (0.0061)	0.908	0.927
	$\widehat{\rho}_c$	0.8977 (0.0319)	0.0277 (0.0101)	0.922	0.920
		$n = 20$			
(50,20)	$\widehat{\rho}$	0.9493 (0.0115)	0.0107 (0.0031)	0.922	0.942
	$\widehat{\rho}_c$	0.9033 (0.0199)	0.0192 (0.0048)	0.935	0.943
(100,20)	$\widehat{\rho}$	0.9496 (0.0085)	0.0078 (0.0019)	0.927	0.933
	$\widehat{\rho}_c$	0.9039 (0.0151)	0.0143 (0.0032)	0.935	0.935
(50,40)	$\widehat{\rho}$	0.9490 (0.0155)	0.0138 (0.0049)	0.903	0.923
	$\widehat{\rho}_c$	0.9001 (0.0277)	0.0251 (0.0075)	0.922	0.919
(100,40)	$\widehat{\rho}$	0.9493 (0.0108)	0.0105 (0.0030)	0.933	0.941
	$\widehat{\rho}_c$	0.9011 (0.0197)	0.0194 (0.0048)	0.957	0.954
		$n = 50$			
(50,20)	$\widehat{\rho}$	0.9496 (0.0068)	0.0068 (0.0012)	0.950	0.954
	$\widehat{\rho}_c$	0.9046 (0.0120)	0.0122 (0.0020)	0.947	0.951
(100,20)	$\widehat{\rho}$	0.9499 (0.0050)	0.0050 (0.0008)	0.943	0.943
	$\widehat{\rho}_c$	0.9053 (0.0090)	0.0090 (0.0012)	0.940	0.946
(50,40)	$\widehat{\rho}$	0.9493 (0.0094)	0.0090 (0.0020)	0.930	0.932
	$\widehat{\rho}_c$	0.9037 (0.0164)	0.0159 (0.0032)	0.947	0.945
(100,40)	$\widehat{\rho}$	0.9498 (0.0072)	0.0068 (0.0013)	0.927	0.935
	$\widehat{\rho}_c$	0.9043 (0.0129)	0.0121 (0.0020)	0.932	0.937

Table 3: Simulation Results of Case 3 ($\rho = 0.5$ and $\rho_c = 0.4670$)

(N, k)		mean(std)	SE(std)	CP_1	CP_2
		$n = 10$			
(50,20)	$\widehat{\rho}$	0.5006 (0.1387)	0.1168 (0.0398)	0.896	0.921
	$\widehat{\rho}_c$	0.4448 (0.1287)	0.1117 (0.0356)	0.909	0.917
(100,20)	$\widehat{\rho}$	0.5005 (0.1017)	0.0911 (0.0271)	0.925	0.933
	$\widehat{\rho}_c$	0.4457 (0.0953)	0.0875 (0.0244)	0.935	0.937
(50,40)	$\widehat{\rho}$	0.4912 (0.1738)	0.1451 (0.0514)	0.888	0.912
	$\widehat{\rho}_c$	0.4346 (0.1623)	0.1378 (0.0457)	0.898	0.907
(100,40)	$\widehat{\rho}$	0.4962 (0.1387)	0.1157 (0.0385)	0.908	0.920
	$\widehat{\rho}_c$	0.4412 (0.1313)	0.1112 (0.0340)	0.914	0.922
		$n = 20$			
(50,20)	$\widehat{\rho}$	0.4975 (0.0939)	0.0890 (0.0226)	0.934	0.936
	$\widehat{\rho}_c$	0.4553 (0.0894)	0.0854 (0.0206)	0.939	0.940
(100,20)	$\widehat{\rho}$	0.4970 (0.0730)	0.0666 (0.0143)	0.932	0.934
	$\widehat{\rho}_c$	0.4541 (0.0696)	0.0641 (0.0133)	0.934	0.933
(50,40)	$\widehat{\rho}$	0.4970 (0.1229)	0.1077 (0.0278)	0.897	0.914
	$\widehat{\rho}_c$	0.4524 (0.1173)	0.1041 (0.0254)	0.903	0.917
(100,40)	$\widehat{\rho}$	0.4974 (0.0892)	0.0862 (0.0206)	0.931	0.939
	$\widehat{\rho}_c$	0.4531 (0.0855)	0.0832 (0.0186)	0.938	0.943
		$n = 50$			
(50,20)	$\widehat{\rho}$	0.4968 (0.0597)	0.0577 (0.0090)	0.948	0.946
	$\widehat{\rho}_c$	0.4599 (0.0567)	0.0557 (0.0084)	0.951	0.955
(100,20)	$\widehat{\rho}$	0.4994 (0.0430)	0.0431 (0.0060)	0.945	0.943
	$\widehat{\rho}_c$	0.4625 (0.0419)	0.0417 (0.0057)	0.949	0.945
(50,40)	$\widehat{\rho}$	0.4958 (0.0770)	0.0724 (0.0128)	0.919	0.931
	$\widehat{\rho}_c$	0.4589 (0.0738)	0.0699 (0.0116)	0.930	0.931
(100,40)	$\widehat{\rho}$	0.4983 (0.0614)	0.0573 (0.0091)	0.934	0.928
	$\widehat{\rho}_c$	0.4610 (0.0592)	0.0552 (0.0084)	0.931	0.931

Table 4: Simulation Results of Case 4 ($\rho = 0.75$ and $\rho_c = 0.0.7005$)

(N, k)		mean(std)	SE(std)	CP_1	CP_2
			$n = 10$		
(50,20)	$\widehat{\rho}$	0.7483 (0.0824)	0.0689 (0.0284)	0.893	0.918
	$\widehat{\rho}_c$	0.6804 (0.0874)	0.0770 (0.0278)	0.924	0.917
(100,20)	$\widehat{\rho}$	0.7493 (0.0600)	0.0534 (0.0184)	0.919	0.934
	$\widehat{\rho}_c$	0.6824 (0.0648)	0.0599 (0.0187)	0.943	0.938
(50,40)	$\widehat{\rho}$	0.7419 (0.1045)	0.0873 (0.039)	0.888	0.910
	$\widehat{\rho}_c$	0.6714 (0.1098)	0.0964 (0.0370)	0.915	0.915
(100,40)	$\widehat{\rho}$	0.7462 (0.0827)	0.0683 (0.0277)	0.894	0.927
	$\widehat{\rho}_c$	0.6778 (0.0902)	0.0773 (0.0271)	0.914	0.922
			$n = 20$		
(50,20)	$\widehat{\rho}$	0.7476 (0.0556)	0.0522 (0.0157)	0.927	0.934
	$\widehat{\rho}_c$	0.6917 (0.0593)	0.0567 (0.0152)	0.937	0.944
(100,20)	$\widehat{\rho}$	0.7477 (0.0425)	0.0388 (0.0094)	0.928	0.934
	$\widehat{\rho}_c$	0.6911 (0.0457)	0.0426 (0.0097)	0.946	0.940
(50,40)	$\widehat{\rho}$	0.7468 (0.0730)	0.0635 (0.0204)	0.900	0.907
	$\widehat{\rho}_c$	0.6875 (0.0780)	0.0700 (0.0199)	0.914	0.918
(100,40)	$\widehat{\rho}$	0.7477 (0.0521)	0.0504 (0.0140)	0.921	0.943
	$\widehat{\rho}_c$	0.6885 (0.0570)	0.0558 (0.0139)	0.940	0.948
			$n = 50$		
(50,20)	$\widehat{\rho}$	0.7478 (0.0348)	0.0336 (0.0062)	0.938	0.949
	$\widehat{\rho}_c$	0.6956 (0.0366)	0.0364 (0.0062)	0.955	0.949
(100,20)	$\widehat{\rho}$	0.7496 (0.0249)	0.0250 (0.0039)	0.955	0.949
	$\widehat{\rho}_c$	0.6976 (0.0275)	0.0272 (0.004)	0.943	0.942
(50,40)	$\widehat{\rho}$	0.7470 (0.0451)	0.0423 (0.009)	0.918	0.928
	$\widehat{\rho}_c$	0.6946 (0.0483)	0.0457 (0.0087)	0.932	0.931
(100,40)	$\widehat{\rho}$	0.7486 (0.0360)	0.0333 (0.0062)	0.931	0.936
	$\widehat{\rho}_c$	0.6959 (0.0386)	0.0360 (0.0062)	0.933	0.930

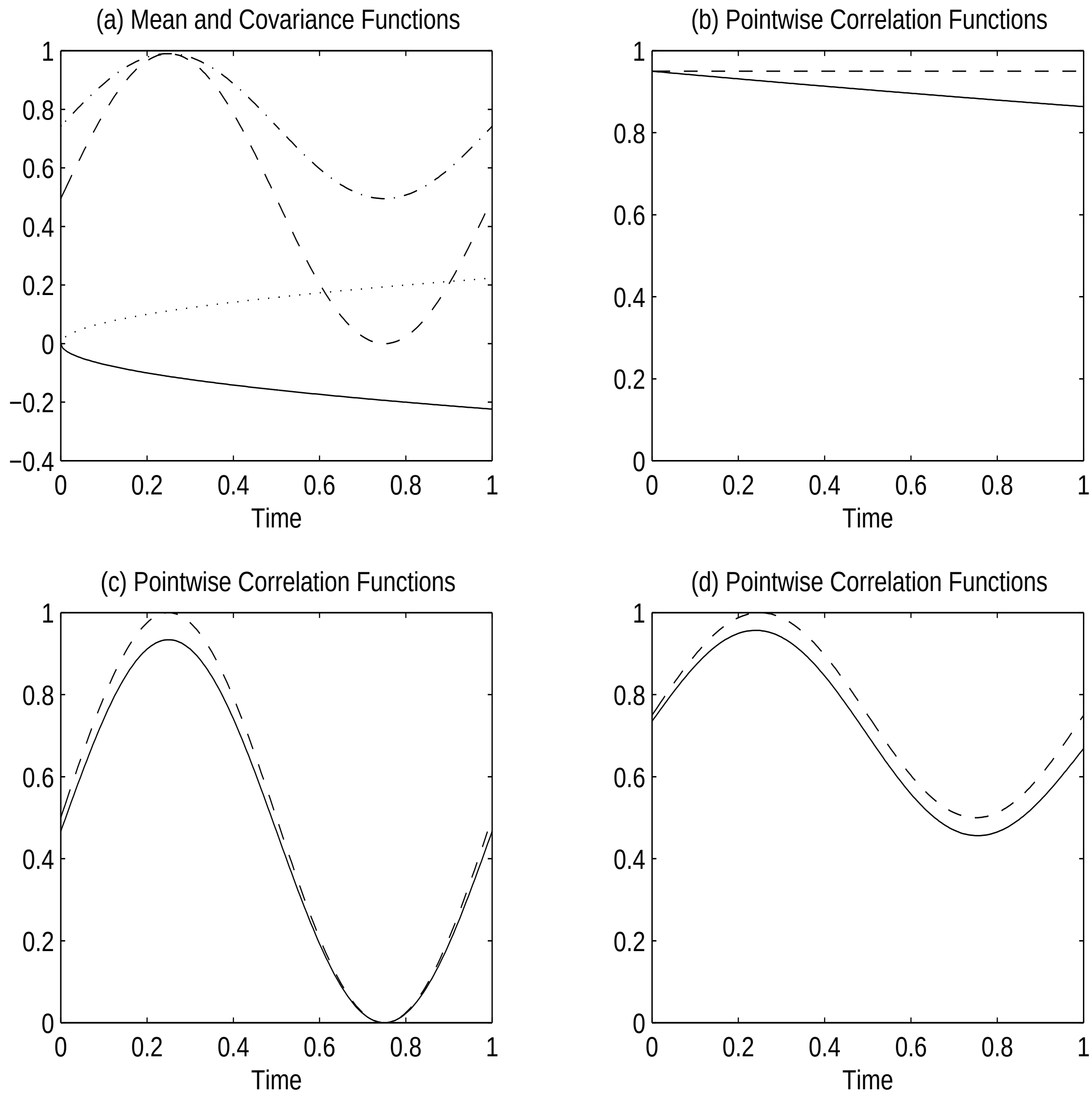

Figure 1: *Plots of mean, covariance and correlation coefficient functions. In (a), solid and dotted lines stand for the mean functions* $-\sqrt{0.05t}$ *and* $\sqrt{0.05t}$*, respectively; dashed and dash-dotted lines stands for the covariance functions* $1.1 \times 0.9 \times \{sin(2\pi t) + 1\}/2$ *and* $1.1 \times 0.9 \times \{sin(2\pi t) + 3\}/4$*, respectively. (b)—(d) are correlation coefficient function* $\rho(t)$ *and Lin's concordance correlation coefficient function* $\rho_c(t)$ *evaluated at time* t*. In (b)—(d), solid line stands for the concordance correlation coefficient and the dashed line for the correlation coefficient.*

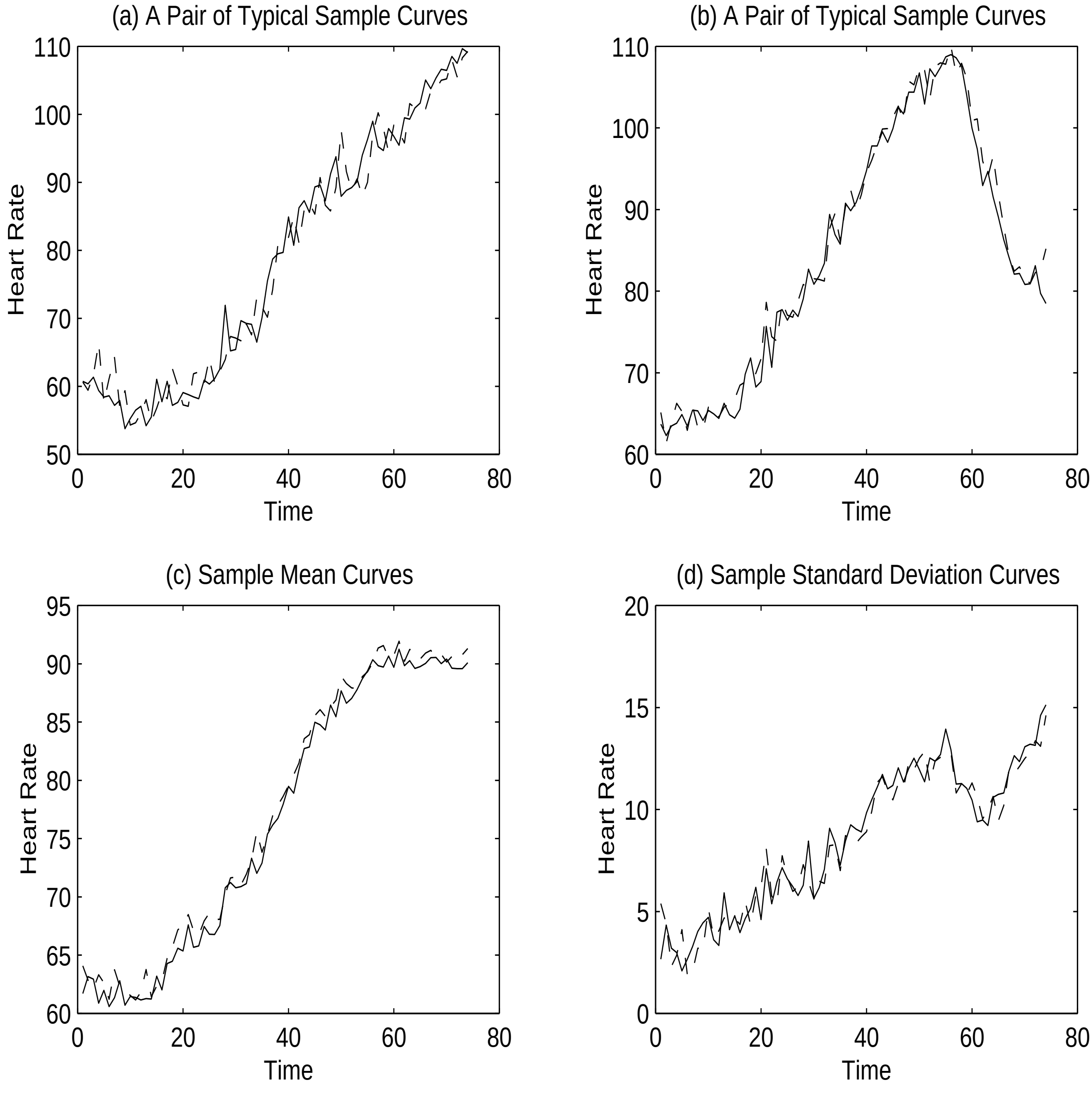

Figure 2: *Plots of typical sample, sample mean and standard deviation curves of the passive heating group. (a) and (b) are two typical sample curves. (c) and (d) are the sample mean and standard deviation curves. In (a)—(d), solid line stands for the golden standard approach, and dashed line for the polar heart rate monitor.*

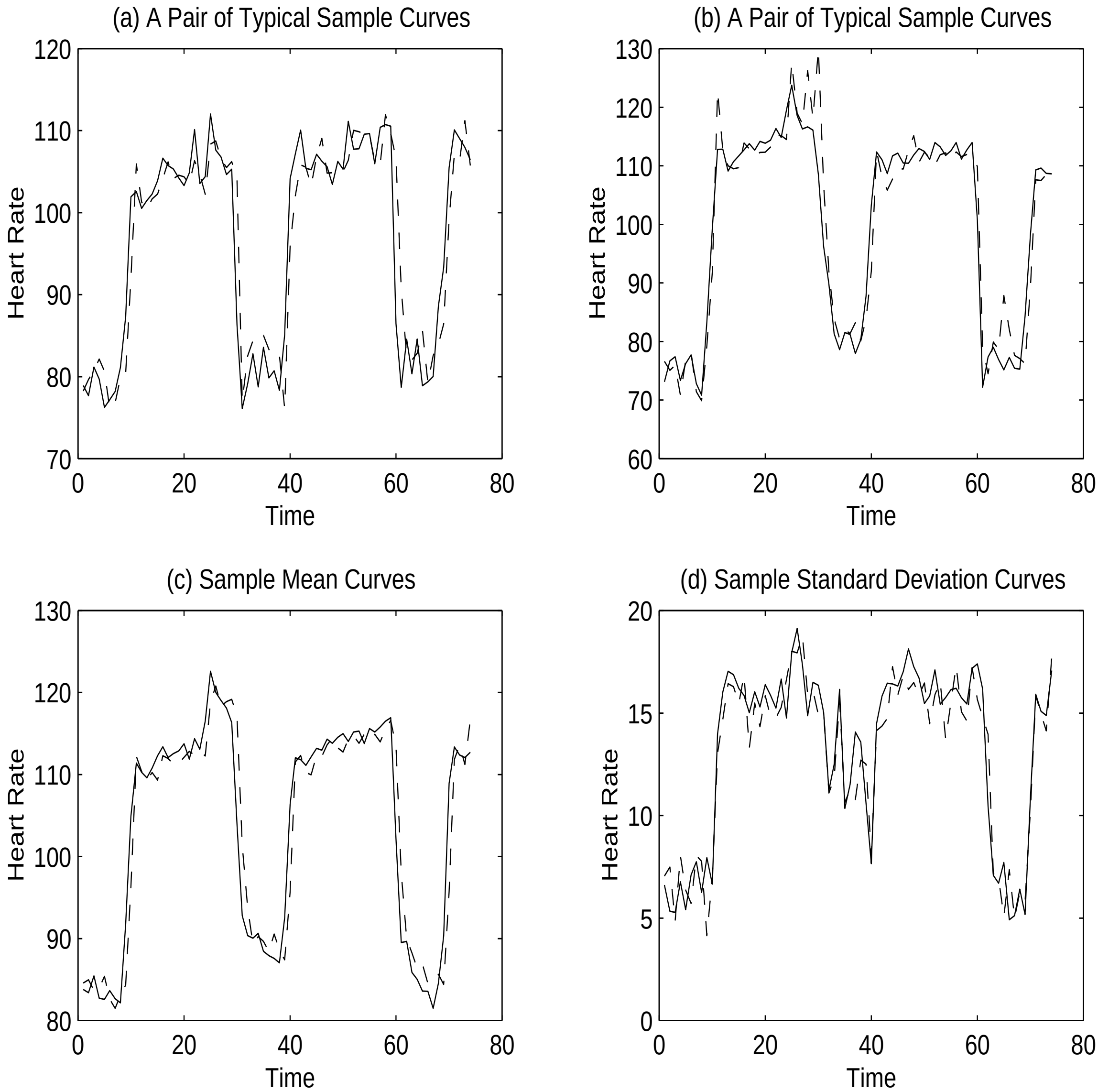

Figure 3: *Plots of typical sample, sample mean and standard deviation curves of the active heating group. Captions are the same as those in Figure 2.*