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TESTING MULTIVARIATE UNIFORMITY AND ITS APPLICATIONS

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ABSTRACT. Some new statistics are proposed to test the uniformity of random samples in the multidimensional unit cube $[0,1]^d$ $(d \geq 2)$. These statistics are derived from number-theoretic or quasi-Monte Carlo methods for measuring the discrepancy of points in $[0,1]^d$. Under the null hypothesis that the samples are independent and identically distributed (i.i.d.) with a uniform distribution in $[0,1]^d$, we obtain some asymptotic properties of the new statistics. By Monte Carlo simulation, it is found that the finite-sample distributions of the new statistics are well approximated by the standard normal distribution, $N(0,1)$, or the chi-squared distribution, $\chi^2(2)$. A power study is performed, and possible applications of the new statistics to testing general multivariate goodness-of-fit problems are discussed.

1. INTRODUCTION

Testing uniformity in the unit interval $[0,1]$ has been studied by many authors. Some early work in this area is [Ney37], [Pea39] and [AD54]. Quesenberry and Miller [QJ77, JQ79] made a thorough Monte Carlo simulation to compare a number of existing statistics for testing uniformity in $[0,1]$ and recommended Watson's U^2-test [Wat62] and Neyman's smooth test [Ney37] as general choices for testing uniformity in $[0,1]$. D'Agostino and Stephens [DS86, Chapter 6] gave a comprehensive review on tests for uniformity in $[0,1]$.

Testing uniformity of random samples in the multidimensional unit cube $(d \geq 2)$,

$$\bar{C}^d = [0,1]^d = \{\boldsymbol{x} = (x_1, \dots, x_d)' \in R^d : \ 0 \leq x_i \leq 1, \ i = 1, \dots, d\},$$

seems to have received less attention in the literature. Two well-known quantities are the Kolmogorov-Smirnov type statistic,

$$KS_n = \sup_{\boldsymbol{x} \in R^d} |F_n(\boldsymbol{x}) - F(\boldsymbol{x})|, \tag{1.1}$$

and the Cramér-von Mises type statistic,

$$CM_n = n \int_{R^d} [F_n(\boldsymbol{x}) - F(\boldsymbol{x})]^2 \psi(\boldsymbol{x}) \, d\boldsymbol{x}. \tag{1.2}$$

Here, $F(\boldsymbol{x})$ is the null distribution function (d.f.), $F_n(\boldsymbol{x})$ denotes the empirical distribution function (e.d.f.) based on n i.i.d. samples, and $\psi(\boldsymbol{x}) \geq 0$ in (1.2) is a

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suitable weight function. Unfortunately, the Kolmogorov-Smirnov type statistic is difficult to compute for large d.

In the literature of number-theoretic methods or quasi-Monte Carlo methods [Nie92, FW94, SJ94], there are a number of criteria for measuring whether a set of points is uniformly scattered in the unit cube $\bar{C}^d$. These criteria are called *discrepancies*, and they arise in the error analysis of of quasi-Monte Carlo methods for evaluating multiple integrals. Given an integral over the unit cube,

$$I(f) = \int_{\bar{C}^d} f(\boldsymbol{x})\, d\boldsymbol{x}, \tag{1.3}$$

quasi-Monte Carlo methods approximate this integral by the sample mean,

$$Q(f) = \frac{1}{n}\sum_{i=1}^{n} f(\boldsymbol{z}_i), \tag{1.4}$$

over a set of uniformly scattered sample points, $\mathcal{P} = \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_n\} \subset \bar{C}^d$. Examples of good sets for quasi-Monte Carlo integration are given in [HW81, Nie92, SJ94, Tez95] and the references therein. The worst-case quadrature error of a quasi-Monte Carlo method is bounded by a generalized Koksma-Hlawka inequality,

$$|I(f) - Q(f)| \le D(\mathcal{P})V(f), \tag{1.5}$$

where $V(f)$ is a measure of the variation or fluctuation of the integrand, and the discrepancy, $D(\mathcal{P})$, is a measure of the quality of the quadrature rule, or equivalently, of the set of points, $\mathcal{P}$. A smaller discrepancy implies a better set of points.

The precise definitions of the discrepancy and the variation depend on the space of integrands. In the original Koksma-Hlawka inequality [Nie92, Chap. 2], $V(f)$ is the variation of the integrand in the sense of Hardy and Krause, and the discrepancy is the star discrepancy, defined as follows:

$$D^*(\mathcal{P}) = \sup_{\boldsymbol{x}\in\bar{C}^d} \left| \frac{|\mathcal{P}\cap[\boldsymbol{0},\boldsymbol{x}]|}{n} - \mathrm{Vol}([\boldsymbol{0},\boldsymbol{x}]) \right|, \tag{1.6}$$

where $|\quad|$ denotes the number of points in a set, $\mathrm{Vol}([\boldsymbol{0},\boldsymbol{x}])$ denotes the volume of the hypercube $[\boldsymbol{0},\boldsymbol{x}]$ ($\boldsymbol{x} \in R^d$). Taking the null distribution in (1.1) to be the uniform distribution, $F(\boldsymbol{x}) = \mathrm{Vol}([\boldsymbol{0},\boldsymbol{x}])$, and noting that the e.d.f. is $F_n(\boldsymbol{x}) = |\mathcal{P}\cap[\boldsymbol{0},\boldsymbol{x}]|/n$, the star discrepancy, (1.6), is a special case of the Kolmogorov-Smirnov type statistic, (1.1).

This relationship between discrepancies arising in quasi-Monte Carlo quadrature error bounds and goodness-of-fit statistics is rather general and has been discussed in [Hic98b, Hic99]. If the space of integrands is a reproducing kernel Hilbert space, then one may obtain a computationally simple form for the discrepancy. A special case is the $\mathcal{L}^2$-star discrepancy, which corresponds to the Cramér-von Mises statistic, (1.2), for the uniform distribution with weight function $\psi(\boldsymbol{x}) = 1$.

Hickernell [Hic98a] proposed a generalized discrepancy based on reproducing kernels. This discrepancy has a computationally simple formula. Three special cases, the *symmetric discrepancy*, the *centered discrepancy* and the *star discrepancy*, have interesting geometrical interpretations. Thus, all three of them may give useful statistics for testing multivariate uniformity of a set of points.

However, to perform a statistical test, one must know the probability distribution of the test statistic under the assumption of i.i.d. uniform random points. This problem is the focus of this article. Section 2 defines the new test statistics and describes their asymptotic behavior. For statistical reasons the discrepancy itself is not the best goodness-of-fit statistic, but useful statistics are derived from the discrepancy. Section 3 presents simulation results on how well the distributions of the statistics are approximated by their asymptotic limits and on the power performance of the new statistics. Some applications are also discussed.

2. The new statistics and their asymptotic properties

2.1. **Some $\mathcal{L}^2$-type discrepancies.** The generalized $\mathcal{L}^2$-type discrepancy proposed in [Hic98a] is as follows:

$$[D(\mathcal{P})]^2 = M^d - \frac{2}{n}\sum_{k=1}^{n}\prod_{j=1}^{d}[M + \beta^2\mu(z_{kj})] \tag{2.1}$$

$$+\frac{1}{n^2}\sum_{k,l=1}^{n}\prod_{j=1}^{d}\left(M + \beta^2\left[\mu(z_{kj}) + \mu(z_{lj}) + \frac{1}{2}B_2(\{z_{kj} - z_{lj}\}) + B_1(z_{kj})B_1(z_{lj})\right]\right),$$

where, the notation $\{\ \}$ denotes the fractional part of a real number or vector, β is an arbitrarily given positive constant, and $\mu(\cdot)$ is an arbitrary function satisfying

$$\mu \in \left\{ f : \frac{df}{dx} \in \mathcal{L}^\infty([0,1]) \text{ and } \int_0^1 f(x)\,dx = 0 \right\}.$$

The constant M is determined in terms of β and μ as follows:

$$M = 1 + \beta^2 \int_0^1 \left(\frac{d\mu}{dx}\right)^2 dx. \tag{2.2}$$

The $B_1(\cdot)$ and $B_2(\cdot)$ in (2.1) are the first and the second degree Bernoulli polynomials, respectively:

$$B_1(x) = x - \frac{1}{2} \quad \text{and} \quad B_2(x) = x^2 - x + \frac{1}{6}.$$

For any $z_1,\ z_2 \in [0,1]$, it is true that

$$B_2(\{z_1 - z_2\}) = B_2(|z_1 - z_2|) = |z_1 - z_2|^2 - |z_1 - z_2| + \frac{1}{6}.$$

The three special cases of $[D(\mathcal{P})]^2$ (denoted by $D_s(\mathcal{P})^2$, $D_c(\mathcal{P})^2$ and $D_*(\mathcal{P})^2$, respectively) given in [Hic98a] are derived by taking three different choices of the function $\mu(\cdot)$ and the constant β in (2.2):

1) the symmetric discrepancy:

$$\mu(x) = -\frac{1}{2}B_2(x) = -\frac{1}{2}(x^2 - x + \frac{1}{6}), \qquad \beta^{-1} = \frac{1}{2}, \qquad M = \frac{4}{3},$$

$$D_s(\mathcal{P})^2 = \left(\frac{4}{3}\right)^d - \frac{2}{n}\sum_{k=1}^{n}\prod_{j=1}^{d}(1 + 2z_{kj} - 2z_{kj}^2) + \frac{2^d}{n^2}\sum_{k,l=1}^{n}\prod_{j=1}^{d}(1 - |z_{kj} - z_{lj}|); \tag{2.3}$$

2) the centered discrepancy:

$$\mu(x) = -\frac{1}{2}B_2\left(\left\{x-\frac{1}{2}\right\}\right) = -\frac{1}{2}\left(\left|x-\frac{1}{2}\right|^2 - \left|x-\frac{1}{2}\right| + \frac{1}{6}\right), \quad \beta^{-1}=1, \quad M=\frac{13}{12},$$

$$\begin{aligned}(2.4)\quad D_c(\mathcal{P})^2 = \left(\frac{13}{12}\right)^d &- \frac{2}{n}\sum_{k=1}^{n}\prod_{j=1}^{d}\left(1+\frac{1}{2}\left|z_{kj}-\frac{1}{2}\right| - \frac{1}{2}\left|z_{kj}-\frac{1}{2}\right|^2\right)\\ &+ \frac{1}{n^2}\sum_{k,l=1}^{n}\prod_{j=1}^{d}\left[1+\frac{1}{2}\left|z_{kj}-\frac{1}{2}\right| + \frac{1}{2}\left|z_{lj}-\frac{1}{2}\right| - \frac{1}{2}\left|z_{kj}-z_{lj}\right|\right];\end{aligned}$$

3) the star discrepancy:

$$\mu(x) = \frac{1}{6}-\frac{x^2}{2}, \qquad \beta^{-1}=1, \qquad M=\frac{4}{3},$$

$$(2.5)\quad D_*(\mathcal{P})^2 = \left(\frac{4}{3}\right)^d - \frac{2}{n}\sum_{k=1}^{n}\prod_{j=1}^{d}\left(\frac{3-z_{kj}^2}{2}\right) + \frac{1}{n^2}\sum_{k,l=1}^{n}\prod_{j=1}^{d}[2-\max(z_{kj},z_{lj})].$$

2.2. **Asymptotic properties of the discrepancies.** The null hypothesis for testing the uniformity of random samples $\mathcal{P}=\{\boldsymbol{z}_1,\dots,\boldsymbol{z}_n\}\subset\bar{C}^d$ can be stated as

$$(2.6)\qquad H_0: \ \boldsymbol{z}_1,\dots,\boldsymbol{z}_n \text{ are uniformly distributed in } \bar{C}^d.$$

The alternative hypothesis H_1 implies rejection of H_0 in (2.6). A test for (2.6), that is, a *test of multivariate uniformity*, can be performed by determining whether the value of a test statistic is unlikely under the null hypothesis. If one wishes to use one of the discrepancies described above as a test statistic, then its probability distribution under the null hypothesis must be calculated. Although, this distribution is too complicated to describe for finite sample size, it can be characterized rather simply in the limit of infinite sample size.

The main results on the asymptotic properties of the discrepancies are contained in Theorems 2.1 and 2.3 below. Their proofs rely on the theory of U-type statistics [Ser80, Chapter 5].

Theorem 2.1. *Under the null hypothesis* (2.6), *the statistic* $[D(\mathcal{P})]^2$ *given by* (2.1) *has the asymptotic property*

$$(2.7)\qquad [D(\mathcal{P})]^2 \overset{a.s.}{\to} 0 \quad (n\to\infty),$$

for arbitrary function $\mu(\cdot)$ *and arbitrary constant* β, *where* "$\overset{a.s.}{\to}$" *means "converges almost surely".*

Proof. For $\mathcal{P}=\{\boldsymbol{z}_1,\dots,\boldsymbol{z}_n\}$ with $\boldsymbol{z}_k=(z_{k1},\dots,z_{kd})'$ $(k=1,\dots,n)$, under the null hypothesis (2.6), the random variables z_{kj} $(k=1,\dots,n,\ j=1,\dots,d)$ have a uniform distribution $U[0,1]$. Let

$$(2.8)\qquad g_1(\boldsymbol{z}_k) = \prod_{j=1}^{d}[M+\beta^2\mu(z_{kj})],$$

and

$$h(\boldsymbol{z}_k, \boldsymbol{z}_l) = \prod_{j=1}^{d} \Big(M + \beta^2 \Big[\mu(z_{kj}) + \mu(z_{lj}) + \frac{1}{2} B_2(|z_{kj} - z_{lj}|) + B_1(z_{kj}) B_1(z_{lj}) \Big] \Big) \tag{2.9}$$

for $k, l = 1, \ldots, n$. Then the square discrepancy given by (2.1) can be written as

$$\begin{aligned} [D(\mathcal{P})]^2 &= M^d - \frac{2}{n} \sum_{k=1}^{n} g_1(\boldsymbol{z}_k) + \frac{1}{n^2} \sum_{k,l=1}^{n} h(\boldsymbol{z}_k, \boldsymbol{z}_l) \\ &= M^d - \frac{2}{n} \sum_{k=1}^{n} g_1(\boldsymbol{z}_k) + \frac{1}{n^2} \left[2 \sum_{k<l}^{n} h(\boldsymbol{z}_k, \boldsymbol{z}_l) + \sum_{k=1}^{n} h(\boldsymbol{z}_k, \boldsymbol{z}_k) \right] \\ &= M^d - \frac{2}{n} \sum_{k=1}^{n} g_1(\boldsymbol{z}_k) + \frac{n-1}{n} \cdot \frac{2}{n(n-1)} \sum_{k<l}^{n} h(\boldsymbol{z}_k, \boldsymbol{z}_l) + \frac{1}{n^2} \sum_{k=1}^{n} g_2(\boldsymbol{z}_k), \end{aligned} \tag{2.10}$$

where

$$g_2(\boldsymbol{z}_k) = \prod_{j=1}^{d} \left(M + \beta^2 \left[\frac{1}{12} + 2\mu(z_{kj}) + B_1(z_{kj})^2 \right] \right). \tag{2.11}$$

Note that both $\{g_1(\boldsymbol{z}_k)\}_{k=1}^n$ and $\{g_2(\boldsymbol{z}_k)\}_{k=1}^n$ are sequences of i.i.d. random variables. By the strong law of large numbers,

$$U_1 = \frac{1}{n} \sum_{k=1}^{n} g_1(\boldsymbol{z}_k) \overset{\text{a.s.}}{\to} E[g_1(\boldsymbol{z}_1)] = M^d, \tag{2.12}$$

and $\frac{1}{n} \sum_{k=1}^{n} g_2(\boldsymbol{z}_k) \overset{\text{a.s.}}{\to} E[g_2(\boldsymbol{z}_1)] < \infty$. It follows that

$$\frac{1}{n^2} \sum_{k=1}^{n} g_2(\boldsymbol{z}_k) \overset{\text{a.s.}}{\to} 0. \tag{2.13}$$

By the theory of the U-type statistics [Ser80, Chapter 5],

$$U_2 = \frac{2}{n(n-1)} \sum_{k<l}^{n} h(\boldsymbol{z}_k, \boldsymbol{z}_l) \tag{2.14}$$

is a second-order U-statistic. By the strong law of large numbers for general U-statistics,

$$U_2 \overset{\text{a.s.}}{\to} E[h(\boldsymbol{z}_1, \boldsymbol{z}_2)] = M^d \quad (n \to \infty).$$

Therefore, by (2.10), as $n \to \infty$, $[D(\mathcal{P})]^2 \overset{\text{a.s.}}{\to} M^d - 2M^d + M^d = 0$. □

Corollary 2.2. *Under the null hypothesis* (2.6)*, it is true that*

$$D_s(\mathcal{P})^2 \overset{a.s.}{\to} 0, \qquad D_c(\mathcal{P})^2 \overset{a.s.}{\to} 0, \qquad D_*(\mathcal{P})^2 \overset{a.s.}{\to} 0,$$

as $n \to \infty$*, where* $D_s(\mathcal{P})^2$*,* $D_c(\mathcal{P})^2$ *and* $D_*(\mathcal{P})^2$ *are given by* (2.3)-(2.5)*, respectively.*

The following theorem and corollaries consider pieces of the discrepancy defined in (2.1). These pieces are then re-combined to give new statistics for testing multivariate normality.

Theorem 2.3. *Let U_1 and U_2 be given by* (2.12) *and* (2.14)*, respectively. Then, under the null hypothesis* (2.6)*,*

$$\sqrt{n}\begin{pmatrix} U_1 - M^d \\ U_2 - M^d \end{pmatrix} \xrightarrow{\mathcal{D}} N_2(\mathbf{0}, \mathbf{\Sigma}) \quad (n \to \infty),$$

where "$\xrightarrow{\mathcal{D}}$" means "converges in distribution", and $\mathbf{\Sigma}$ is a singular covariance matrix:

$$\mathbf{\Sigma} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \zeta_1, \tag{2.15}$$

where $\zeta_1 = (M^2 + \beta^4 c^2)^d - M^{2d}$ and $c^2 = \int_0^1 \mu(x)^2\, dx$.

Proof. Since the random variable U_1 given by (2.12) is a first-order U-statistic and the random variable U_2 given by (2.14) is a second-order U-statistic, by the central limit theorem for U-statistics, we have

$$\sqrt{n}(U_1 - EU_1)\Big/\sqrt{\operatorname{var}\left(\sqrt{n}U_1\right)} \xrightarrow{\mathcal{D}} N(0,1), \tag{2.16}$$

where $EU_1 = M^d$ by the proof of Theorem 2.1. It is easy to calculate

$$\operatorname{var}\left(\sqrt{n}U_1\right) = (M^2 + \beta^4 c^2)^d - M^{2d}.$$

By Lemma A of [Ser80, p. 183], we obtain the variance of U_2:

$$\operatorname{var}(U_2) = \frac{4(n-2)}{n(n-1)}\zeta_1 + \frac{2}{n(n-1)}\zeta_2, \tag{2.17}$$

where ζ_1 is given by (2.15) and $\zeta_2 = [M^2 + \beta^4(2c^2 + 1/90)]^d - M^{2d}$. By the theorem in [Ser80, p. 189], U_2 can be written as

$$U_2(n) = \hat{U}_2(n) + R_n, \tag{2.18}$$

where $\hat{U}_2(n)$ is a random variable that can be written as a sum of i.i.d. random variables as follows:

$$\hat{U}_2(n) - EU_2(n) = \frac{2}{n}\sum_{i=1}^{n} h_1(\boldsymbol{z}_i), \tag{2.19}$$

for some function $h_1(\cdot)$ [Ser80, p. 188, equation (2)] and $EU_2(n) = M^d$. Formula (2.19) can be written as

$$\hat{U}_2(n) = \frac{1}{n}\sum_{i=1}^{n} h_2(\boldsymbol{z}_i), \tag{2.20}$$

where $h_2(\cdot) = 2h_1(\cdot) + M^d$. The R_n in (2.18) is a residual term, $R_n = o_p\left(n^{-1}(\log n)^\delta\right)$ $(n \to \infty)$, which implies that R_n tends to zeros in probability, where $\delta > 1/v$ $(v > 0)$ if $E\{h(\boldsymbol{z}_1, \boldsymbol{z}_2)\}^v < \infty$. $h(\cdot,\cdot)$ is defined by (2.9). Under the null hypothesis (2.6), $E\{h(\boldsymbol{z}_1, \boldsymbol{z}_2)\}^v < \infty$ for any $v > 0$. Combining (2.18) and (2.20), we can write

$$U_2 = \frac{1}{n}\sum_{i=1}^{n} h_2(\boldsymbol{z}_i) + R_n.$$

Then

$$\sqrt{n}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \sqrt{n}\begin{pmatrix} \frac{1}{n}\sum_{k=1}^n g_1(\boldsymbol{z}_k) \\ \frac{1}{n}\sum_{k=1}^n h_2(\boldsymbol{z}_k) + R_n \end{pmatrix} = \frac{1}{\sqrt{n}}\sum_{k=1}^n \begin{pmatrix} g_1(\boldsymbol{z}_k) \\ h_2(\boldsymbol{z}_k) \end{pmatrix} + \begin{pmatrix} 0 \\ \sqrt{n}R_n \end{pmatrix}. \tag{2.21}$$

Under the null hypothesis (2.6), by the multivariate central limit theorem, we have

$$\sqrt{n}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \xrightarrow{\mathcal{D}} \text{2-dimensional normal distribution}$$

because of the independence of the $\boldsymbol{z}_k$ $(k = 1, \ldots, n)$ and the fact that

$$\begin{pmatrix} 0 \\ \sqrt{n}R_n \end{pmatrix} \xrightarrow{\mathcal{P}} \mathbf{0} \quad (n \to \infty),$$

where "$\xrightarrow{\mathcal{P}}$" means "converges in probability". It is easy to obtain the covariance between $\sqrt{n}U_1$ and $\sqrt{n}U_2$:

$$\operatorname{cov}(\sqrt{n}U_1, \sqrt{n}U_2) = 2[(M^2 + \beta^4 c^2)^d - M^{2d}] = 2\zeta_1,$$

where ζ_1 is the same as that in (2.15). Then we have

$$\sqrt{n}\begin{pmatrix} U_1 - M^d \\ U_2 - M^d \end{pmatrix} \xrightarrow{\mathcal{D}} N_2(\mathbf{0}, \mathbf{\Sigma}),$$

with $\mathbf{\Sigma}$ given by (2.15). This completes the proof. □

Theorem 2.3 implies that the asymptotic distribution of $(\sqrt{n}U_1, \sqrt{n}U_2)$ is a singular (degenerate) normal distribution. This property results in the following corollary which shows why the discrepancy itself is not necessarily a suitable goodness-of-fit statistic.

Corollary 2.4. *Under the null hypothesis* (2.6), *the generalized $\mathcal{L}^2$-type discrepancy $[D(\mathcal{P})]^2$ given by* (2.1) *has a further asymptotic property:*

$$\sqrt{n}[D(\mathcal{P})]^2 \xrightarrow{\mathcal{P}} 0 \quad (n \to \infty).$$

Proof. By the proof of Theorem 2.1, $[D(\mathcal{P})]^2$ can be written as

$$[D(\mathcal{P})]^2 = M^d - 2U_1 + \frac{n-1}{n}U_2 + \frac{1}{n^2}\sum_{k=1}^n g_2(\boldsymbol{z}_k),$$

where $g_2(\cdot)$ is given by (2.11). By (2.13) and (2.14), we can write

$$\frac{1}{n^2}\sum_{k=1}^n g_2(\boldsymbol{z}_k) = O_p\left(\frac{1}{n}\right),$$

where the notation "$f(n) = O_p(\frac{1}{n})$" means $nf(n) \xrightarrow{\mathcal{P}}$ a constant. Then we have

$$\sqrt{n}[D(\mathcal{P})]^2 = -2\sqrt{n}(U_1 - M^d) + \sqrt{n}(U_2 - M^d) - \frac{1}{\sqrt{n}}U_2 + O_p\left(\frac{1}{\sqrt{n}}\right). \tag{2.22}$$

Since $U_2 \stackrel{\text{a.s.}}{\to} M^d < \infty$ $(n \to \infty)$ by Theorem 2.1, we have $(1/\sqrt{n})U_2 = O_p(1/\sqrt{n})$. Then we can write (2.22) as

$$\begin{aligned}\sqrt{n}[D(\mathcal{P})]^2 &= -2\sqrt{n}(U_1 - M^d) + \sqrt{n}(U_2 - M^d) + O_p\left(\frac{1}{\sqrt{n}}\right) \\ &= \sqrt{n}(-2,1)\begin{pmatrix} U_1 - M^d \\ U_2 - M^d \end{pmatrix} + O_p\left(\frac{1}{\sqrt{n}}\right).\end{aligned} \tag{2.23}$$

By Theorem 2.3, we obtain

$$\sqrt{n}[D(\mathcal{P})]^2 \stackrel{\mathcal{D}}{\to} N(0, \boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a}),$$

where $\boldsymbol{a}' = (-2,1)$ and $\boldsymbol{\Sigma}$ is given by (2.15). It turns out that $\boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a} = 0$. Therefore, $\sqrt{n}[D(\mathcal{P})]^2 \stackrel{\mathcal{D}}{\to} 0$. This implies $\sqrt{n}[D(\mathcal{P})]^2 \stackrel{\mathcal{P}}{\to} 0$. □

A better goodness-of-fit statistic than the discrepancy can be obtained by a linear combination of U_1 and U_2 that is not a degenerate normal distribution. This is the idea behind the statistic A_n defined below.

Corollary 2.5. *Under the null hypothesis* (2.6)*, the statistic*

$$A_n = \sqrt{n}[(U_1 - M^d) + 2(U_2 - M^d)]/(5\sqrt{\zeta_1}) \stackrel{\mathcal{D}}{\to} N(0,1) \quad (n \to \infty), \tag{2.24}$$

where U_1 and U_2 are defined by (2.12) *and* (2.14)*, respectively, and ζ_1 is given by* (2.15)*.*

Proof. It was noted in the proof of Corollary 2.4 that $\boldsymbol{a}' = (-2,1)$ is an eigenvector associated with the zero-eigenvalue of $\boldsymbol{\Sigma}$ given by (2.15). On the other hand, $\boldsymbol{b} = (1,2)'$ $(\boldsymbol{a}'\boldsymbol{b} = 0)$ is an eigenvector associated with the eigenvalue $5\zeta_1$ of $\boldsymbol{\Sigma}$. By Theorem 2.3, $E(A_n) = 0$ and the variance

$$\text{var}(A_n) = \boldsymbol{b}'\boldsymbol{\Sigma}_n\boldsymbol{b}/(25\zeta_1) \to \boldsymbol{b}'\boldsymbol{\Sigma}\boldsymbol{b}/(25\zeta_1) = 1, \quad (n \to \infty)$$

under the null hypothesis (2.6), where $\boldsymbol{\Sigma}_n$ is the covariance matrix of $(\sqrt{n}U_1, \sqrt{n}U_2)$, which turns out to be

$$\boldsymbol{\Sigma}_n = \begin{pmatrix} \zeta_1 & 2\zeta_1 \\ 2\zeta_1 & \frac{4(n-2)}{n-1}\zeta_1 + \frac{2}{n-1}\zeta_2 \end{pmatrix}, \tag{2.25}$$

and ζ_2 is the same as in (2.17). Assertion (2.24) holds as a result of Theorem 2.3. □

The statistic A_n can be employed for testing hypothesis (2.6). Larger values of $|A_n|$ imply rejection for the null hypothesis (2.6). It can be verified that $\boldsymbol{\Sigma}_n$ given by (2.25) tends to singularity very slowly. For example, under the symmetric discrepancy, when $n = 1000$, the condition number of $\boldsymbol{\Sigma}_n = 1467$ $(d = 2)$, 1196 $(d = 5)$ and 841 $(d = 10)$. Therefore, for finite sample size n (e.g., $n \leq 1000$), we recommend to use the normal distribution $N_2(\mathbf{0}, \boldsymbol{\Sigma}_n)$ as the approximate joint distribution of $(\sqrt{n}U_1, \sqrt{n}U_2)$. Based on this idea, we propose the following χ^2-type statistic for testing the null hypothesis (2.6):

$$T_n = n[(U_1 - M^d), (U_2 - M^d)]\boldsymbol{\Sigma}_n^{-1}[(U_1 - M^d), (U_2 - M^d)]'. \tag{2.26}$$

The approximate null distribution of T_n can be taken as the chi-squared distribution $\chi^2(2)$. Larger values of T_n imply rejection for the null hypothesis (2.6). The convergence rate of $\sqrt{n}[D(\mathcal{P})]^2 \xrightarrow{\mathcal{P}} 0$ in Corollary 2.4 is also very slow. For example, for the symmetric discrepancy, we can write

$$\sqrt{n}D_s(\mathcal{P})^2 = \left(-2, \frac{2^d(n-1)}{n}\right)[\sqrt{n}(U_1 - EU_1), \sqrt{n}(U_2 - EU_2)]' + R_n. \tag{2.27}$$

The residual term $R_n = [2^d - (\frac{4}{3})^d]/\sqrt{n}$ in (2.27) tends to zero very slowly. When $n = 10,000$, $R_n = 0.0222$ ($d = 2$), 0.2779 ($d = 5$) and 10.0624 ($d = 10$).

For the three special cases of the generalized $\mathcal{L}^2$-type discrepancy, we can easily obtain the parameters needed to define the statistics A_n and T_n in (2.24) and (2.26):

1) the symmetric discrepancy:

$$U_1 = \frac{1}{n}\sum_{k=1}^{n}\prod_{j=1}^{d}(1 + 2z_{kj} - 2z_{kj}^2),$$

$$U_2 = \frac{2^{d+1}}{n(n-1)}\sum_{k<l}^{n}\prod_{j=1}^{d}(1 - |z_{kj} - z_{lj}|).$$

$M = 4/3$, $\zeta_1 = (9/5)^d - (6/9)^d$ and $\zeta_2 = 2^d - (16/9)^d$;

2) the centered discrepancy:

$$U_1 = \frac{1}{n}\sum_{k=1}^{n}\prod_{j=1}^{d}(1 + \frac{1}{2}|z_{kj} - \frac{1}{2}| - \frac{1}{2}|z_{kj} - \frac{1}{2}|^2),$$

$$U_2 = \frac{2}{n(n-1)}\sum_{k<l}^{n}\prod_{j=1}^{d}\left[1 + \frac{1}{2}|z_{kj} - \frac{1}{2}| + \frac{1}{2}|z_{lj} - \frac{1}{2}| - \frac{1}{2}|z_{kj} - z_{lj}|)\right].$$

$M = 13/12$, $\zeta_1 = (47/40)^d - (13/12)^{2d}$ and $\zeta_2 = (57/48)^d - (13/12)^{2d}$;

3) the star discrepancy:

$$U_1 = \frac{1}{n}\sum_{k=1}^{n}\prod_{j=1}^{d}\left(\frac{3 - z_{kj}}{2}\right),$$

$$U_2 = \frac{2}{n(n-1)}\sum_{k<l}^{n}\prod_{j=1}^{d}[2 - \max(z_{kj}, z_{lj})].$$

$M = 4/3$, $\zeta_1 = (9/5)^d - (16/9)^d$ and $\zeta_2 = (11/6)^d - (16/9)^d$.

3. Monte Carlo study and applications

The exact finite-sample distributions of the statistics A_n (given by (2.24)) and T_n (given by (2.26)) under the null hypothesis (2.6) are not readily obtained. However, the effectiveness of the approximation of their finite-sample distributions by their asymptotic distributions can be studied by Monte Carlo simulation. The approximation of the distribution of T_n by $\chi^2(2)$ is not only influenced by the convergence of U_1 and U_2 but by the convergence of $\mathbf{\Sigma}_n$ as well, while the convergence of A_n to normal $N(0,1)$ is only influenced by the convergence of U_1 and U_2. Therefore, it is expected that the approximation of the distribution of A_n by $N(0,1)$ is better than that of T_n by $\chi^2(2)$.

3.1. **Numerical comparisons between the finite-sample distributions of A_n and $N(0,1)$, and T_n and $\chi^2(2)$.** In the simulation, for each sample size n ($n = 25,\ 50,\ 100,\ 200$), we generate 10,000 uniform samples $\boldsymbol{Z} = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n)'$. The elements of $\boldsymbol{Z}$ are i.i.d. $U(0,1)$-variates. Then we obtain 10,000 values of A_n and T_n under the three (symmetric, centered and star) discrepancies, respectively. These values are sorted and the ordered samples of A_n and T_n are obtained. The empirical quantiles of A_n and T_n, respectively, are calculated from 10,000 order statistics of A_n and T_n. Tables 1, 2, and 3 list the numerical comparisons between some selected $100(1-\alpha)$-percentiles ($\alpha = 1\%$, 5% and 10%) of A_n and T_n and the $100(1-\alpha)$-percentiles of $N(0,1)$ and $\chi^2(2)$. Since a test using A_n is two-sided, and a test using T_n is one-sided, we list the upper (U) and lower (L) percentiles of both A_n and $N(0,1)$, and only the upper percentiles of both T_n and $\chi^2(2)$.

The closer the percentiles of A_n and the percentiles of $N(0,1)$ are, the better approximation we obtain by using $N(0,1)$ as the approximate finite-sample distribution of A_n. The same is true for the numerical comparison between the statistic T_n and $\chi^2(2)$. Several empirical conclusions can be summarized from the numerical results in Tables 1–3:

a) the standard normal $N(0,1)$ approximates the finite-sample distribution of A_n better than the $\chi^2(2)$ approximates the finite-sample distribution of T_n;
b) the approximation of the finite-sample distribution of A_n by $N(0,1)$, and the approximation of the finite-sample distribution of T_n by $\chi^2(2)$, appear to be the best for the symmetric discrepancy; and
c) the approximation of the percentiles of the finite-sample distribution of A_n, and the approximation of the percentiles of the finite-sample distribution of T_n for $\alpha = 5\%$ and $\alpha = 10\%$ are much better than for $\alpha = 1\%$; and
d) the approximation for the case $n = 25$ is almost as good as those cases for $n = 200$.

3.2. **Type I error rates.** Based on the numerical comparisons shown in Tables 1–3, we perform a simulation of the empirical type I error rates of the two statistics A_n and T_n under the three discrepancies. For convenience, we choose the null distribution of the random vectors $\boldsymbol{z}_i$'s to be composed of i.i.d. $U(0,1)$ marginals. In the simulation, we generate 2,000 sets of $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n\}$ for each n with the components of $\boldsymbol{z}_i$ consisting of i.i.d. $U(0,1)$ variates. Tables 4, 5, and 6 summarize the simulation results on the type I error rates of A_n and T_n under the three discrepancies, where the percentiles of A_n are chosen as those of $N(0,1)$ and the percentiles of T_n as those of $\chi^2(2)$. It shows that for $\alpha = 5\%$ and $\alpha = 10\%$, the type I error rates are better controlled for the statistics A_n and T_n, while the type I error rates for $\alpha = 1\%$ tend to be large by using the percentiles of $N(0,1)$ for A_n and the percentiles of $\chi^2(2)$ for T_n in most cases.

3.3. **Power study.** Now we turn to study the power of A_n and T_n in testing hypothesis (2.6). The alternative distributions are chosen to be *meta-type uniform distributions*. The theoretical background of some meta-type multivariate distributions is given in [KS91] and [FFK97]. The idea for constructing the alternative distributions is as follows. Let random vector $\boldsymbol{x} = (X_1, \ldots, X_d)$ have a d.f. $F(\boldsymbol{x})$ ($\boldsymbol{x} \in R^d$). Denote by $F_i(x_i)$ the marginal d.f. of X_i ($i = 1, \ldots, d$), which is assumed to be continuous. Define the random vector $\boldsymbol{u} = (U_1, \ldots, U_d)$ by

$$U_i = F_i(X_i), \quad i = 1, \ldots, d. \tag{3.1}$$

TABLE 1. Comparisons between the empirical percentiles of A_n, T_n and the percentiles of $N(0,1)$ and $\chi^2(2)$ for the symmetric discrepancy (U=upper, L=lower)

	A_n				T_n		
α	1%	5%	10%	α	1%	5%	10%
$N(0,1)$ (U)	2.5758	1.9600	1.6449	$\chi^2(2)$ (U)	9.2103	5.9915	4.6052
(L)	-2.5758	-1.9600	-1.6449				
$d=2$							
$n=25$(U)	2.9759	2.1836	1.7969	$n=25$	13.7011	6.7329	4.3474
(L)	-2.6414	-1.9566	-1.6775				
50(U)	2.6455	2.0585	1.7237	50	13.5989	6.3139	4.3748
(L)	-2.4507	-1.9969	-1.6847				
100(U)	2.6058	2.0078	1.6935	100	15.2375	6.2400	4.3013
(L)	-2.4772	-1.9102	-1.6327				
200(U)	2.5923	1.9897	1.6601	200	15.2146	6.7055	4.3251
(L)	-2.6379	-1.9279	-1.6301				
$d=5$							
$n=25$(U)	2.9520	2.1698	1.8390	$n=25$	10.6401	6.0031	4.2624
(L)	-2.4989	-1.9750	-1.7051				
50(U)	2.7856	2.0814	1.7396	50	11.4226	6.1015	4.4108
(L)	-2.3915	-1.9143	-1.6484				
100(U)	2.6219	2.0271	1.6849	100	11.4069	6.0066	4.2354
(L)	-2.5679	-1.9372	-1.6497				
200(U)	2.6534	2.0671	1.7026	200	12.7450	6.0279	4.4207
(L)	-2.5295	-1.9199	-1.6241				
$d=10$							
$n=25$(U)	3.2566	2.3549	1.9631	$n=25$	11.1983	6.3805	4.6083
(L)	-2.4658	-1.9266	-1.6559				
50(U)	2.9148	2.0965	1.7282	50	10.6401	5.7312	4.2321
(L)	-2.5183	-1.8729	-1.5886				
100(U)	2.7635	2.0861	1.7492	100	10.0934	5.7531	4.3382
(L)	-2.4521	-1.8981	-1.6210				
200(U)	2.8986	2.0855	1.6790	200	10.7926	6.3940	4.6259
(L)	-2.5858	-1.9743	-1.6473				

Table 2. Comparisons between the empirical percentiles of A_n, T_n and the percentiles of $N(0,1)$ and $\chi^2(2)$ for the centered discrepancy (U=upper, L=lower)

	A_n				T_n		
α	1%	5%	10%	α	1%	5%	10%
$N(0,1)$ (U)	2.5758	1.9600	1.6449	$\chi^2(2)$ (U)	9.2103	5.9915	4.6052
(L)	-2.5758	-1.9600	-1.6449				
$d=2$							
$n=25$(U)	3.0050	2.1787	1.8135	$n=25$	14.7350	6.6225	4.4123
(L)	-2.5991	-1.9844	-1.6711				
50(U)	2.8674	2.1098	1.8002	50	13.2973	6.4543	4.4933
(L)	-2.4697	-1.9843	-1.6570				
100(U)	2.7243	2.0026	1.6594	100	15.1737	6.4344	4.2733
(L)	-2.4646	-1.9056	-1.6336				
200(U)	2.6659	2.0101	1.6705	200	15.9685	6.5391	4.4612
(L)	-2.4212	-1.9303	-1.6483				
$d=5$							
$n=25$(U)	2.9690	2.1944	1.8036	$n=25$	12.0965	6.2195	4.2602
(L)	-2.6154	-1.9494	-1.6214				
50(U)	2.7896	2.0864	1.7549	50	12.3615	6.3261	4.3271
(L)	-2.5491	-1.9160	-1.6482				
100(U)	2.7273	2.0579	1.6885	100	12.0372	6.1285	4.3838
(L)	-2.4673	-1.9303	-1.6325				
200(U)	2.6119	1.9727	1.6686	200	14.1589	6.3164	4.3577
(L)	-2.4947	-1.9312	-1.6596				
$d=10$							
$n=25$(U)	3.0643	2.1765	1.7999	$n=25$	11.8336	6.2523	4.4722
(L)	-2.5317	-1.9699	-1.7057				
50(U)	2.7659	2.0547	1.7327	50	10.3546	5.8925	4.4789
(L)	-2.4328	-1.9181	-1.6338				
100(U)	2.7047	1.9789	1.6812	100	10.6855	5.9538	4.3766
(L)	-2.5490	-1.9115	-1.6321				
200(U)	2.6627	2.0203	1.7219	200	11.4535	6.3443	4.4515
(L)	-2.6466	-1.9017	-1.5883				

TABLE 3. Comparisons between the empirical percentiles of A_n, T_n and the percentiles of $N(0,1)$ and $\chi^2(2)$ for the star discrepancy (U=upper, L=lower)

	A_n				T_n		
α	1%	5%	10%	α	1%	5%	10%
$N(0,1)$ (U)	2.5758	1.9600	1.6449	$\chi^2(2)$ (U)	9.2103	5.9915	4.6052
(L)	-2.5758	-1.9600	-1.6449				
$d=2$							
$n=25$(U)	2.7798	2.1064	1.7502	$n=25$	18.1738	7.4153	4.2674
(L)	-2.4210	-1.8260	-1.5894				
50(U)	2.6203	2.0494	1.7041	50	17.7953	7.3728	4.3048
(L)	-2.4262	-1.8957	-1.6033				
100(U)	2.7023	2.0041	1.6674	100	20.2587	7.0995	4.2731
(L)	-2.5098	-1.9783	-1.6809				
200(U)	2.7373	2.0500	1.7079	200	21.0088	7.6134	4.3452
(L)	-2.5060	-1.9245	-1.6407				
$d=5$							
$n=25$(U)	2.9385	2.0791	1.7058	$n=25$	14.3399	6.3745	4.0116
(L)	-2.2055	-1.7930	-1.5380				
50(U)	2.8831	2.0920	1.7123	50	16.6810	6.7896	4.1579
(L)	-2.4423	-1.8550	-1.5617				
100(U)	2.7979	2.1148	1.7166	100	14.6865	6.7067	4.1141
(L)	-2.3267	-1.8339	-1.6227				
200(U)	2.8597	2.0473	1.6776	200	18.5595	6.6619	4.1685
(L)	-2.4238	-1.8852	-1.5846				
$d=10$							
$n=25$(U)	2.9127	2.1215	1.7109	$n=25$	14.8453	6.4506	4.1141
(L)	-2.1692	-1.8174	-1.5444				
50(U)	2.9363	2.1483	1.7379	50	15.4874	6.7275	4.1528
(L)	-2.3410	-1.8569	-1.5453				
100(U)	2.7731	2.0901	1.7077	100	15.2233	6.4890	4.0312
(L)	-2.4133	-1.8551	-1.5742				
200(U)	2.8309	2.0886	1.6720	200	16.2075	6.8058	4.1076
(L)	-2.4539	-1.8862	-1.5914				

TABLE 4. Empirical type I error rates of A_n, T_n under the symmetric discrepancy

	A_n			T_n		
α	1%	5%	10%	1%	5%	10%
			$d = 2$			
$n = 25$	0.0165	0.0660	0.1265	0.0245	0.0550	0.0905
50	0.0130	0.0575	0.1095	0.0265	0.0615	0.0915
100	0.0150	0.0615	0.1120	0.0255	0.0595	0.0860
200	0.0095	0.0585	0.1010	0.0270	0.0580	0.0920
			$d = 5$			
$n = 25$	0.0160	0.0695	0.1315	0.0195	0.0525	0.1035
50	0.0140	0.0605	0.1095	0.0205	0.0545	0.0970
100	0.0150	0.0525	0.1060	0.0190	0.0515	0.0815
200	0.0125	0.0540	0.0925	0.0175	0.0535	0.0930
			$d = 10$			
$n = 25$	0.0235	0.0650	0.1245	0.0240	0.0620	0.1005
50	0.0105	0.0550	0.1095	0.0105	0.0435	0.0860
100	0.0145	0.0595	0.1145	0.0155	0.0550	0.1010
200	0.0075	0.0410	0.0835	0.0135	0.0420	0.0825

TABLE 5. Empirical type I error rates of A_n, T_n under the centered discrepancy

	A_n			T_n		
α	1%	5%	10%	1%	5%	10%
			$d = 2$			
$n = 25$	0.0180	0.0655	0.1240	0.0275	0.0610	0.0950
50	0.0120	0.0565	0.1020	0.0270	0.0640	0.0940
100	0.0170	0.0640	0.1210	0.0260	0.0595	0.0950
200	0.0090	0.0495	0.0955	0.0280	0.0525	0.0890
			$d = 5$			
$n = 25$	0.0135	0.0665	0.1195	0.0200	0.0575	0.0915
50	0.0135	0.0625	0.1235	0.0210	0.0570	0.0960
100	0.0095	0.0530	0.1070	0.0185	0.0515	0.0895
200	0.0130	0.0475	0.1050	0.0145	0.0575	0.0910
			$d = 10$			
$n = 25$	0.0150	0.0690	0.1325	0.0230	0.0650	0.1095
50	0.0100	0.0460	0.1045	0.0125	0.0410	0.0835
100	0.0105	0.0495	0.1125	0.0145	0.0555	0.1045
200	0.0100	0.0455	0.0970	0.0170	0.0500	0.0850

It is obvious that each U_i in (3.1) has a uniform distribution $U(0,1)$ but the joint distribution of $\boldsymbol{u} = (U_1, \ldots, U_d)$ may be quite different from the uniform distribution in $\bar{C}^d$. If the joint d.f. $F(\boldsymbol{x})$ of $\boldsymbol{x} = (X_1, \ldots, X_d)$ possesses a density function $f(\boldsymbol{x}) = f(x_1, \ldots, x_d)$, where $f_i(x_i)$ denotes the marginal density function of X_i, then the joint density function of $\boldsymbol{u} = (U_1, \ldots, U_d)$ can be obtained by a direct

TABLE 6. Empirical type I error rates of A_n, T_n under the star discrepancy

	A_n			T_n		
α	1%	5%	10%	1%	5%	10%
			$d=2$			
$n=25$	0.0105	0.0470	0.0955	0.0325	0.0630	0.0845
50	0.0105	0.0450	0.0895	0.0385	0.0610	0.0860
100	0.0075	0.0465	0.0930	0.0300	0.0550	0.0800
200	0.0100	0.0510	0.1030	0.0360	0.0630	0.0920
			$d=5$			
$n=25$	0.0100	0.0560	0.1165	0.0260	0.0665	0.0975
50	0.0050	0.0530	0.0980	0.0255	0.0665	0.0920
100	0.0105	0.0575	0.1090	0.0325	0.0675	0.0940
200	0.0120	0.0555	0.0985	0.0290	0.0585	0.0835
			$d=10$			
$n=25$	0.0115	0.0460	0.0940	0.0245	0.0630	0.0890
50	0.0200	0.0525	0.1060	0.0275	0.0605	0.0890
100	0.0120	0.0525	0.0895	0.0265	0.0550	0.0810
200	0.0110	0.0470	0.0945	0.0300	0.0545	0.0780

calculation:

$$p(u_1,\ldots,u_d) = f(F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d))\Big/\prod_{i=1}^{d} f_i(F_i^{-1}(u_i)), \tag{3.2}$$

where $(u_1,\ldots,u_d)\in \bar{C}^d$ and $F_i^{-1}(\cdot)$ denotes the inverse function of $F_i(\cdot)$. It is clear that the complexity of (3.2) is determined by the joint distribution of $\boldsymbol{x}=(X_1,\ldots,X_d)$ in (3.1). In particular, if the random variables X_i's in (3.1) are independent, then the U_i's given by (3.1) will be i.i.d. $U(0,1)$ variates. In this case, the random vector $\boldsymbol{u}=(U_1,\ldots,U_d)$ is uniformly distributed in $\bar{C}^d$. This can be seen from (3.2).

In the simulation, we choose the random vector $\boldsymbol{x}=(X_1,\ldots,X_d)$ in (3.1) to have a joint distribution belonging to the subclasses of elliptical distributions [FKN90, Chapter 3], where the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are chosen as $\boldsymbol{\mu}=\boldsymbol{0}$ and $\boldsymbol{\Sigma}=(\sigma_{ij})$, where $\sigma_{ii}=1$ and $\sigma_{ij}=\sigma_{ji}=\rho=0.5$ for $1\le i\ne j\le d$. Except the normal distribution $N_d(\boldsymbol{\mu},\boldsymbol{\Sigma})$, we give general expressions for the density functions of the selected subclasses of elliptical distributions below:

1) the multivariate t-distribution $\boldsymbol{x}\sim Mt_d(m,\boldsymbol{\mu},\boldsymbol{\Sigma})$, its density function is given by

$$C|\boldsymbol{\Sigma}|^{-1/2}\left(1+m^{-1}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)^{-(d+m)/2}, \quad m>0,\ \boldsymbol{x}\in R^d,$$

where C is a normalizing constant (the following C's have a similar meaning, but its value may be different);

2) the Kotz type distribution, its density function is given by

$$C|\boldsymbol{\Sigma}|^{-1/2}\left[(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]^{N-1}\exp\left\{-r[(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})]^s\right\}, \quad r,s>0,\ 2N+d>2,\ \boldsymbol{x}\in R^d;$$

3) the Pearson type VII distribution, its density function is given by

$$C|\mathbf{\Sigma}|^{-1/2}\left(1+m^{-1}(\boldsymbol{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)^{-N}, \quad N>d/2,\ m>0,\ \boldsymbol{x}\in R^d;$$

4) the Pearson type II distribution, its density function is given by

$$C|\mathbf{\Sigma}|^{-1/2}\left(1-(\boldsymbol{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)^{m}, \quad m>-1,\ (\boldsymbol{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})<1;$$

5) the multivariate Cauchy distribution $MC_d(\boldsymbol{\mu},\mathbf{\Sigma})$, its density function is given by

$$\frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}\left(1+(\boldsymbol{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)^{-(d+1)/2}, \quad \boldsymbol{x}\in R^d.$$

When the random vector $\boldsymbol{x}=(X_1,\dots,X_d)'$ is generated by one of the elliptical distributions above, then the random vector $\boldsymbol{u}=(U_1,\dots,U_d)'$ given by (3.1) is considered to have a meta-type uniform distribution denoted as follows:

0) $\boldsymbol{u}\sim MNU$ when $\boldsymbol{x}\sim N_d(\mathbf{0},\mathbf{\Sigma})$;
1) $\boldsymbol{u}\sim MTU$ when $\boldsymbol{x}$ has a multivariate t-distribution with $m=5$;
2) $\boldsymbol{u}\sim MKU$ when $\boldsymbol{x}$ has a Kotz type distribution with $N=2$, $r=1$ and $s=0.5$;
3) $\boldsymbol{u}\sim MPVIIU$ when $\boldsymbol{x}$ has a Pearson type VII distribution with $N=10$ and $m=2$;
4) $\boldsymbol{u}\sim MPIIU$ when $\boldsymbol{x}$ has a Pearson type II distribution with $m=3/2$;
5) $\boldsymbol{u}\sim MCU$ when $\boldsymbol{x}$ has a Cauchy distribution.

The power of the multivariate test for uniformity is the probability that the statistical test correctly identifies a sample coming from one of the above distributions as being non-uniform. Table 7 summarizes the simulation results on the power of A_n and T_n, where the simulation is done with 2,000 replications, the critical points of both A_n and T_n are chosen as those of $N(0,1)$ and $\chi^2(2)$, respectively, and the empirical samples from the elliptical distributions are generated by the *TFWW algorithm* [Tas77] and [FW94, pp. 160-170]. It shows that the two statistics A_n and T_n under the three discrepancies are powerful for testing uniformity in $\bar{C}^d$ in most cases. The χ^2-type statistic T_n seems to be more powerful than the normal-type statistic A_n. For both A_n and T_n, they seem to be more powerful in the higher dimensional case ($d=10$) than in lower dimensional case ($d=5$). It is also noticed that the two statistics A_n and T_n under the symmetric discrepancy seem to be more powerful than under the centered discrepancy and the star discrepancy in most cases.

A power comparison between the two statistics A_n, T_n and some existing statistics, such as the Kolmogorov-Smirnov type statistic (1.1), seems to be infeasible in high dimensions because of computational difficulties. Thus, we have not performed such comparisons.

3.4. **Applications.** By using the Rosenblatt transformation [Ros52], we can transfer a test for the simple hypothesis

$$H_0:\ \boldsymbol{x}_1,\dots,\boldsymbol{x}_n \text{ have a known d.f. } F(\boldsymbol{x}), \tag{3.3}$$

TABLE 7. Empirical power of A_n, T_n for testing multi-dimensional uniformity against the meta-type uniform distributions ($\alpha = 5\%$)

	discrepancy	n	MNU	MTU	MKU	MPVIIU	MPIIU	MCU
					$d = 5$			
A_n	symmetric	25	0.3815	0.5070	0.9280	0.0680	1.0000	0.6690
		50	0.6720	0.7400	0.9970	0.0695	1.0000	0.8720
		100	0.9480	0.9470	0.9235	0.0865	1.0000	0.9765
	centered	25	0.2170	0.2515	0.8570	0.5195	1.0000	0.3325
		50	0.2470	0.2845	0.9710	0.6980	1.0000	0.3825
		100	0.3195	0.3705	0.9215	0.8895	1.0000	0.4515
	star	25	0.3675	0.3585	0.4300	0.3600	1.0000	0.3450
		50	0.4565	0.4590	0.5870	0.4000	1.0000	0.4380
		100	0.6430	0.6220	0.7300	0.5275	1.0000	0.6185
T_n	symmetric	25	0.7305	0.7845	0.9775	0.7315	1.0000	0.8175
		50	0.9965	0.9950	1.0000	0.9835	1.0000	0.9955
		100	1.0000	1.0000	0.9235	1.0000	1.0000	1.0000
	centered	25	0.2945	0.3165	0.9590	0.5055	1.0000	0.3695
		50	0.6700	0.6425	1.0000	0.8195	1.0000	0.6680
		100	0.9945	0.9965	0.9230	0.9960	1.0000	0.9830
	star	25	0.3945	0.3810	0.5265	0.4270	1.0000	0.3775
		50	0.6095	0.6035	0.8660	0.6765	1.0000	0.5675
		100	0.9125	0.8960	0.9230	0.9485	1.0000	0.8820
					$d = 10$			
A_n	symmetric	25	0.9910	0.9830	0.9435	0.1960	1.0000	0.9940
		50	1.0000	0.9995	0.9980	0.1860	1.0000	1.0000
		100	1.0000	1.0000	1.0000	0.2075	1.0000	1.0000
	centered	25	0.5215	0.5650	0.5105	0.9915	1.0000	0.6555
		50	0.6965	0.7170	0.5440	1.0000	1.0000	0.7880
		100	0.8605	0.8745	0.5290	1.0000	1.0000	0.9205
	star	25	0.7770	0.7895	0.7790	0.7170	1.0000	0.7670
		50	0.9265	0.9265	0.9125	0.8805	1.0000	0.9095
		100	0.9965	0.9930	0.9925	0.9770	1.0000	0.9860
T_n	symmetric	25	0.9980	0.9975	0.9940	0.9885	1.0000	0.9975
		50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
		100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	centered	25	0.8915	0.8780	0.9990	0.9890	1.0000	0.8675
		50	1.0000	0.9985	1.0000	1.0000	1.0000	0.9960
		100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	star	25	0.8250	0.8320	0.8520	0.9015	1.0000	0.8120
		50	0.9820	0.9790	0.9930	0.9985	1.0000	0.9720
		100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

to a test for the uniformity of random points in $\bar{C}^d$ and then apply the two statistics A_n and T_n to test for uniformity of the transformed variates in $\bar{C}^d$. Denote by

$$\begin{aligned} F_1(x_1) &= \text{the marginal d.f. of } X_1, \\ F_{2|1}(x_2|x_1) &= \text{the conditional d.f. of } X_2 \text{ given } X_1 = x_1, \\ &\vdots \qquad\qquad \vdots \\ F_{k+1|(1,\ldots,k)}(x_{k+1}|x_1,\ldots,x_k) &= \text{the conditional d.f. of } X_{k+1} \\ &\qquad \text{given } (X_1,\ldots,X_k) = (x_1,\ldots,x_k), \end{aligned} \tag{3.4}$$

where $k = 1, \ldots, d-1$. Perform the following series of Rosenblatt transformations on each observation $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{id})$ $(i = 1, \ldots, n)$:

$$\begin{aligned} U_{i1} &= F_1(x_{i1}) \\ U_{i2} &= F_{2|1}(x_{i2}|x_{i1}) \\ &\vdots \qquad\qquad \vdots \\ U_{i,k+1} &= F_{k+1|(1,\ldots,k)}(x_{i,k+1}|(x_{i1},\ldots,x_{ik})) \end{aligned} \tag{3.5}$$

where $i = 1, \ldots, n$ and $k = 1, \ldots, d-1$. If the null hypothesis (3.3) is true, the random vectors $\boldsymbol{u}_i = (U_{i1}, \ldots, U_{id})$ $(i = 1, \ldots, n)$ are i.i.d. and the components of $\boldsymbol{u}_i$ have a uniform distribution $U[0,1]$.

The more frequent cases in applications are those hypotheses which involve unknown parameters in the null distributions. Justel, Peña and Zamar [JPZ97] proposed a multivariate version of the Kolmogorov-Smirnov type statistic (1.1) for testing the simple hypothesis (3.3). Their statistics are difficult to compute for large dimensions ($d \geq 3$) and always require estimating unknown parameters in the null distribution. The two statistics A_n and T_n developed in this paper are easy to compute in arbitrary dimension and estimation of unknown parameters can be avoided when the null distribution belongs to the class of the multivariate normal distributions, the spherically symmetric distributions, the l_1-norm symmetric distributions [FKN90, Chapter 5], the l_p-norm symmetric distributions [YM95], or the L_p-norm spherical distributions [GS97].

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