RDEs

(1) Rough exponential:

prop. $\beta \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{X} \in \mathcal{C}^{\beta}([0,T], \mathbb{R}^1)$, s.t. $X_0 = 0$.

Then $V_t = e^{X_t - \frac{1}{2}[\mathbf{X}]_t}$ is unique solution

of $V_t = 1 + \int_0^t V_u \, d\mathbf{X}_u$. s.t. $(V, V) \in \mathcal{D}_X^{2\gamma}$.

Pf: i) By Itô formula. V_t satisfies the differential equation. $\Rightarrow$ existence ✓

ii) If $(\tilde{V}, \tilde{V}) \in \mathcal{D}_X^{2\beta}$ is another solution

Set $\gamma \in (\frac{1}{3}, \beta)$.

By results of stability:

$\|V - \tilde{V}\|_\gamma + \|R^V - R^{\tilde{V}}\|_{2\gamma} \underset{\gamma,T}{\lesssim} C(\|V - \tilde{V}\|_\gamma + \|R^V - R^{\tilde{V}}\|_{2\gamma}) \|\!|\!|X|\!|\!|_\gamma$.

But $\|\!|\!|X|\!|\!|_\gamma \underset{\gamma,T}{\lesssim} \|\!|\!|X|\!|\!|_\beta \, t^{\beta-\gamma} \xrightarrow[t \to 0]{} 0$

Choose $t_0 \in [0, T]$. s.t. $C \|\!|\!|X|\!|\!|_\gamma < 1$.

So: $V = \tilde{V}$, $R^V = R^{\tilde{V}}$ in $[0, t_0]$

repeat on $[k t_0, (k+1) t_0]$, $k \geq 1$. $\Rightarrow V \equiv \tilde{V}$.

cor. Additionally, $(K, K') \in \mathcal{D}_X^{2\gamma}$. Then: we have

$V_t = e^{\int_0^t K_u \, dX_u - \frac{1}{2} \int_0^t (K_u \otimes K_u) \, d[\mathbf{X}]_u}$ is the

unique solution of $V_t = 1 + \int_0^t V_u K_u \, d\mathbf{X}_u$

(2) <u>Young differential equations:</u>

<u>Lemma.</u> $\alpha \in (0,1]$. $f \in C_B^2$. Then: $\exists C = C(\alpha, T, \|f\|_{C^2})$.

s.t. $\|f(Y) - f(\tilde{Y})\|_\alpha \le C(1 + \|Y\|_\alpha + \|\tilde{Y}\|_\alpha)(|Y_0 - \tilde{Y}_0| + \|Y - \tilde{Y}\|_\alpha)$

for $\forall Y, \tilde{Y} \in C^\alpha$.

<u>Pf:</u> estimate $|f(X)_{st} - f(\tilde{X})_{st}|$

$= |\int_0^1 Df(\tilde{Y}_t + r(Y_t - \tilde{Y}_t))(Y_t - \tilde{Y}_t)dr - \cdots|$

by $|ab - \tilde{a}\tilde{b}| \le \square \cdots$

<u>Thm.</u> $\beta \in (\frac{1}{2}, 1]$, $X \in C^\beta$. $f \in C_B^2$. $\eta \in V$. Then:

$\exists$ unique path $Y \in C^\beta$. s.t. $Y_t = \eta + \int_0^t f(Y_s)dX_s$

<u>Pf:</u> 1) Set $\alpha \in (\frac{1}{2}, \beta)$. M_t: $C^\alpha \to C^\alpha$, $Y \mapsto \eta + \int_0 f(Y)dX$

$B_t = \{Y \in C^\alpha \mid Y_0 = \eta, \|Y\|_{\alpha,[0,t]} \le 1\}$.

$\|M_t(Y)\|_\alpha \overset{\text{Def of Young}}{\lesssim} (\|f(Y_0)\|_\infty \|X\|_\alpha + \|f(Y)\|_\alpha \|X\|_\alpha)$

$\overset{\text{Lem.}}{\lesssim} \|X\|_\alpha \lesssim \|X\|_\beta t^{\beta - \alpha}$. $\forall Y \in B_t$.

Choose $t = t_0 \in [0,T]$. st. $\|M_t(Y)\|_\alpha \le 1$.

So: $M_{t_0}: B_{t_0} \to B_{t_0}$

2) $\|M_{t_0}(Y) - M_{t_0}(\tilde{Y})\|_\alpha \lesssim \|f(Y) - f(\tilde{Y})\|_\alpha \|X\|_\alpha$

$\overset{\text{Lem}}{\lesssim} \|Y - \tilde{Y}\|_\alpha \|X\|_\alpha$

$\lesssim \|Y - \tilde{Y}\|_\alpha \|X\|_\beta t^{\beta-\alpha}$

Choose $t_1 \in (0, t_0)$. s.t. M_{t_1} is contraction.

$\Rightarrow$ By Banach fixed point Thm

And repeat it on $[kt_1, (k+1)t_1]$. We have

the unique and exist of Y.

3') Note $X \in C^\beta \Rightarrow Y \in C^\beta$. (of 2')).

So it satisfies the condition

<u>Thm.</u> (Stability).

$\beta \in (\frac{1}{2}, 1)$. $f \in C^2$. $X, \tilde{X} \in C^\beta$. $\eta, \tilde{\eta} \in V$. If

$Y, \tilde{Y}$ are unique solution of DE above

w.r.t. (η, X). $(\tilde{\eta}, \tilde{X})$. Then $\exists\, C = C(\alpha, T, \|X\|_\beta, \|\tilde{X}\|_\beta, \|f\|_{C^2})$. s.t. $\|Y - \tilde{Y}\|_\alpha \le C(|\eta - \tilde{\eta}| + \|X - \tilde{X}\|_\alpha)$. for $\forall\, \alpha \in (\frac{1}{2}, \beta)$.

<u>Pf:</u> $\|Y - \tilde{Y}\|_{\alpha, [0,t]} \lesssim (|\eta - \tilde{\eta}| + \|Y - \tilde{Y}\|_\alpha)\|X\|_\alpha + \|X - \tilde{X}\|_\alpha$

Choose t small enough. $(\|X\|_\alpha \le t^{\beta-\alpha}\|X\|_\beta)$

$\Rightarrow \|Y - \tilde{Y}\|_{\alpha, [0,t]} \lesssim |\eta - \tilde{\eta}| + \|X - X\|_\alpha$.

Cover $[0, T]$. with small intervals.

(3) <u>Rough Diff. equation:</u>

Next, we want to investigate RDE:

$$dY_t = f(Y_t)\, d\mathbf{X}_t. \qquad (*)$$

Step 1.: Show control rough path is stable:

$Y \in D_X^{2\gamma}$, F regular $\Rightarrow$ $F(Y) \in D_X^{2\alpha}$

Step 2.: Define $\int Z\, dX$, $Z \in D_X^{2\alpha}$, to construct contraction map, where $Z = F(Y)$.

Step 3.: Interpret $(*)$ as integral and use Banach fixed point argument.

<u>RMK</u>: It's similar as the case of Young DEs.

① <u>Estimations:</u>

<u>Lemma.</u> $\gamma \in (\frac{1}{3}, \frac{1}{2}]$, $X \in C^\alpha$, $f \in C_B^2$, $\forall (Y, Y') \in D_X^{2\gamma}$

$\Rightarrow (f(Y), Df(Y)Y') \in D_X^{2\gamma}$. Besides:

$$\|Df(Y)Y'\|_\alpha \le A(1+\|X\|_\alpha), \quad \|R^{f(Y)}\|_{2\gamma} \le A(1+\|X\|_\alpha)^2$$

$$A = C(\alpha, T, \|f\|_{C^2})\,(1+|Y_0'|+\|Y'\|_\alpha+\|R^Y\|_{2\gamma})^2.$$

<u>Pf</u>: Replace $Y = \cdots + R^Y$.

Recall: $f(Y)_{s,t} - Df(Y_s)Y_{st} = \int_0^1\int_0^1 D^2 f(Y_s + r r' Y_{st})\, Y_{st}^{\otimes 2}\, r\, dr\, dr'$.

Lemma (Stability)

$\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $X, \tilde{X} \in \mathcal{C}^\alpha$. $(Y, Y'), (\tilde{Y}, \tilde{Y}') \in D_X^{2\alpha}, D_{\tilde{X}}^{2\alpha}$.

$f \in C_B^3$. Then: Set $M \geq \|X\|_\alpha \vee \|\tilde{X}\|_\alpha \vee \|Y\|_{D_X^{2\alpha}} \vee \|\tilde{Y}\|_{D_{\tilde{X}}^{2\alpha}}$

We have:
$$\|Df(Y)Y' - Df(\tilde{Y})\tilde{Y}'\|_\alpha \vee \|R^{f(Y)} - R^{f(\tilde{Y})}\|_{2\alpha} \underset{M,\alpha,T}{\lesssim} |Y_0 - \tilde{Y}_0| + |Y_0' - \tilde{Y}_0'| + \|Y' - \tilde{Y}'\|_\alpha + \|X - \tilde{X}\|_\alpha + \|R^Y - R^{\tilde{Y}}\|_{2\alpha}$$

Pf: Routine as above, with Lemma in (2).

Lemma $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{X} \in \mathcal{C}^\alpha$. $f \in C_B^2$. $(Y, Y') \in D_X^{2\alpha}$.

$\Rightarrow (\int_0^\cdot f(Y_u)\, d\mathbf{X}_u, f(Y_\cdot)) \in D_X^{2\alpha}$. with:

$$\|f(Y)\|_\alpha \underset{\substack{\alpha,T \\ \|f\|_\infty}}{\lesssim} (|Y_0'| + \|Y'\|_\alpha)\|X\|_\alpha + \|R^Y\|_{2\alpha} T^\alpha.$$

$$\|R^{\int f(Y)\,d\mathbf{X}}\|_{2\alpha} \underset{f,T}{\lesssim} (1 + |Y_0'| + \|Y'\|_\alpha + \|R^Y\|_{2\alpha})^2 (1 + \|X\|_\alpha)^2 \vert\vert\vert \mathbf{X} \vert\vert\vert_\alpha$$

Pf: Recall the approxi. of $\int_0^t f(Y_u)\, d\mathbf{X}_u$.

apply the estimation above.

Lemma. (Stability)

Use the same notations as above. we have:

$$\|f(Y) - f(\tilde{Y})\|_\alpha \underset{M,T,\alpha}{\lesssim} (|Y_0' - \tilde{Y}_0'| + |Y_0 - \tilde{Y}_0| + \|\tilde{Y}' - Y'\|_\alpha)\|X\|_\alpha + \|R^Y - R^{\tilde{Y}}\|_{2\alpha} T^{2\alpha} + \|X - \tilde{X}\|_\alpha$$

$$\|R^{\int f(Y)\,d\mathbf{X}} - R^{\int f(\tilde{Y})\,d\tilde{\mathbf{X}}}\|_{2\alpha} \underset{\substack{M,T \\ \alpha}}{\lesssim} (\vert\vert\vert Y, \tilde{Y} \vert\vert\vert_{D_{X,\tilde{X}}^{2\alpha}} + \|X - \tilde{X}\|_\alpha) \vert\vert\vert \mathbf{X} \vert\vert\vert_\alpha + \|\mathbf{X}, \tilde{\mathbf{X}}\|_\alpha.$$

Rmk: We complete the estimation:

$$Y \to f(Y) \to \int f(Y) \quad \text{on } \mathbb{R}^{\square}$$

and stability.

② Diff. equations:

Thm. $\beta \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{X} \in \mathscr{C}^{\beta}([0,T], V)$. $f \in C_B^3(W, \mathcal{L}(V, W))$

and $\eta \in V$. Then $\exists$ unique $(Y, Y') \in D_X^{2\beta}$. s.t.

$Y' = f(Y)$. $Y_t = \eta + \int_0^t f(Y_s)\, d\mathbf{X}_s$. $\forall t \in [0,T]$.

Pf: Define $M_t(Y, Y') = (\eta + \int_0^{\cdot} f(Y_s)\, d\mathbf{X}_s, f(Y))$

and $B_t = \{ (Y, Y') \in D_X^{2\alpha} \mid Y_0 = \eta, Y_0' = f(\eta), \|Y, Y'\| \leq 1\}$

as in the case of Young DEs.

We can check $(Y, Y') \in D_X^{2\alpha}$. $\alpha \in (\frac{1}{3}, \beta)$.

is solution of it. $\underset{X \in \mathscr{C}^{\beta}}{\Rightarrow}$ $(Y, Y') \in D_X^{2\beta}$.

Rmk: We have proved well-def. of Itô-Lyon's map.

Thm. (Continuity of Lyon's map)

$\beta \in (\frac{1}{3}, \frac{1}{2}]$. $\mathbf{X}, \widetilde{\mathbf{X}} \in \mathscr{C}^{\beta}$. $f \in C_b^3$. $\eta, \widetilde{\eta} \in V$. $M = |||\mathbf{X}|||_{\beta}$

$\vee |||\widetilde{\mathbf{X}}|||_{\beta}$. Then for $(Y, Y') \in D_X^{2\alpha}$. $(\widetilde{Y}, \widetilde{Y}') \in D_{\widetilde{X}}^{2\alpha}$

are solutions of RDE above w.r.t. $(\eta, \mathbf{X})$. $(\widetilde{\eta}, \widetilde{\mathbf{X}})$.

We have: $\forall \gamma \in (\frac{1}{3}, \beta)$.

$$\|Y-\tilde{Y}\|_\gamma + \|Y'-\tilde{Y}'\|_\gamma + \|R^Y - R^{\tilde{Y}}\|_{2\gamma} \lesssim_{M,T} |\eta-\tilde{\eta}| + \|X-\tilde{X}\|_\gamma.$$

Pf: As before. We process on $[s,t] \subset [0,T]$.

s.t. $|s-t|$ is enough short. s.t. $\|Y, Y'\|_\gamma \le 1 \ldots$

Apply Lemma in ①.

③ low regularity:

Next, we consider low regularity case: $\gamma \le \frac{1}{3}$

1) For $F \in C_B^3$. consider $\gamma \in (\frac{1}{4}, \frac{1}{3}]$ as example:

Set $Z = F(Y)$. $Y = (Y^0, Y^1, Y^2) \in D_X^{3\gamma}(\mathbb{R})$. $\mathbf{X} \in \mathscr{C}_g^\gamma$.

Note $Z^0_{st} = DF(Y_s^0) Y_{st} + \frac{1}{2} D^2F(Y_s^0) Y_{st}^{0\otimes 2} + r$.

$\Rightarrow$ $Z_t^1 = DF(Y_s^0) Y_t^1$ and set $P(\sigma_1 \otimes \sigma_2) = \sigma_2 \otimes \sigma_1$.

$Z_t^2 = DF(Y_1^0) Y_s^2 + \frac{1}{2} D^2F(Y_s^0)(Y_s^1 \otimes Y_s^1)(id + P)$.

follows from $Y_{st}^0 = Y_s^1 X_{st}^1 + Y_s^2 X_{st}^2 + RY_{st}^0$

Lemma. $\|RZ_{st}^1\|_{2\gamma} < \infty$.

$\Rightarrow$ $Z \in D_X^{3\gamma}$. still is controlled path w.r.t X

and similar argument for $F(Y) \in D_X^{N\gamma}$. $\gamma \le \frac{1}{3}$

2) Def $\int_0^t Z\, d\mathbf{X} = \lim_{|\pi|\to 0} \sum \sum_{k=1}^{N-1} Z_s^k X_{st}^{k+1}$

3) Consider $F \in C^{N_\alpha+1}(U, \mathcal{L}(V, U))$, $\mathbf{X} \in \mathcal{C}_g^\alpha(V)$.

Def: $Y_0 \in U$, $\mathbf{Y} = (Y^0, Y^1, \dots, Y^{N_\alpha-1}) \in \mathcal{D}_X^{N_\alpha}(U)$

is solution of $d\mathbf{Y} = F(\mathbf{Y})\, d\mathbf{X}$

if $Y_t^0 = Y_0 + \left(\int_0^t F(\mathbf{Y})\, d\mathbf{X}\right)^0$ and

$Y_t^k = \left(\int_0^t F(\mathbf{Y})\, d\mathbf{X}\right)^k$, $\forall k \le N-1$.

Rmk: If $\mathbf{Y}$ is solution of RDE.

$\Rightarrow$ Y_t^k, $1 \le k \le N_\alpha$ are uniquely

determined by Y_t^0.

Thm (Universal limit Thm)

i) $\forall Y_0 \in U$, $\exists$ unique solution $\mathbf{Y} \in \mathcal{D}_X^{N_\alpha}(U)$

of RDE in Def above

ii) For $\mathbf{X}, \tilde{\mathbf{X}} \in \mathcal{C}_g^\alpha$, $Y_0, \tilde{Y}_0 \in U$, $\mathbf{Y}, \tilde{\mathbf{Y}}$

are solutions of RDE above w.r.t

$(\mathbf{X}, Y_0)$, $(\tilde{\mathbf{X}}, \tilde{Y}_0)$. Then:

$$\sum_0^{N-1} \| R^i Y - R^i \tilde{Y} \|_{(N-i)\alpha} \lesssim_{\substack{\alpha, T \\ \|X\|_\alpha \\ \|\tilde{X}\|_\alpha}} \| X, \tilde{X} \|_\alpha + |\tilde{Y}_0 - Y_0|.$$