Smooth of Proba. Laws

Next, we denote $\mathcal{L}_d$ is Lebesgue measure in $\mathbb{R}^d$.

(1) <u>Absolutely Conti.</u> :

Set $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathcal{F} = \mathcal{F}_W$. Fix H-isonormal W.

① <u>Dimension = 1</u>:

<u>Prop.</u> $X \in \mathbb{D}^{1,2}$ and $\|DX\|_H \neq 0$ a.s. Satisfies:

$DX / \|DX\|_H^2 \in D(\delta)$. Then: law of $X \ll \mathcal{L}_1$ and its density $p(t) = \mathbb{E}(I_{\{X>t\}} \delta(\frac{DX}{\|DX\|_H^2}))$

<u>Pf</u>: For $\psi \in C_c(\mathbb{R}^1)$. $\varphi(t) = \int_{-\infty}^t \psi(s) ds$.

$\Rightarrow \varphi(X) \in \mathbb{D}^{1,2}$. $D\varphi(X) = \psi(X) DX$.

So: $\psi(X) = \langle D\varphi(X), \frac{DX}{\|DX\|_H^2} \rangle$.

i.e. $\mathbb{E}(\psi(X)) = \mathbb{E}(\varphi(X)\, \delta(\frac{DX}{\|DX\|^2}))$

Set $\psi = I_{(a,b)}$, and use Fubini.

<u>Thm.</u> (Weaker conditions)

$X \in \mathbb{D}^{1,1}_{loc}$ s.t. $\|DX\|_H \neq 0$ a.s. Then the law of $X \ll \mathcal{L}_1$.

<u>Pf</u>: <u>Lemma.</u> M is polish. μ is finite on $\mathcal{B}_M$.

Then $\forall f : M \to \mathbb{R}^1$. bdd. measurable.

$\exists (f_n) \subset C_B(M) \xrightarrow{n \to \infty} f$ a.e. w.r.t μ.

Rmk: Recall $\forall$ finite measure μ on polish space is regular.

pf: By Lusin's Thm, $\exists K_n \underset{cpt}{\subset} \mu$. $\mu(K_n^c) \le 1/n$. $f|_{K_n}$ is conti.

Use Tietze Thm $\Rightarrow f_n = f|_{K_n}$.

$\|f_n\|_\infty \le \|f\|_\infty$. $f_n \to f$ a.e.

Return to the pf: prove $I_E(X)=0$. $\mathcal{L}_1(E)=0$.

By localization. Set $f \in \mathbb{D}^{1,1}$.

Consider $E \subset (-1,1)$. $\mathcal{L}_1(E)=0$. Note $\exists f_n \in C_c[-1,1]$.

St. $f_n \to I_E$. $\mu + dx$ - a.s. Set $\varphi_n(t) = \int_{-1}^t f_n$.

$D\varphi_n(X) = f_n(X) DX \to I_E(X) DX$ in L^1 and a.s.

$\varphi_n(X) \to \int_{-1}^X I_E\, dt = 0$. in L^1 and a.s

So: $I_E(X) \|DX\|_H = 0$ a.s. since D is closed

② **Dimension ≥ 2:**

Def: For $X = (X_1 \cdots X_d) : \Omega \to \mathbb{R}^d$. $X_i \in \mathbb{D}^{1,1}_{loc}$.

The Malliavin matrix $\gamma =: (\langle DX_i, DX_j \rangle_H)_{n\times n}$.

prop. $X : \Omega \to \mathbb{R}^n$. St. $X_j \in \mathbb{D}^{1,2}$. $\forall\, 1 \le j \le n$. and.

i) γ is a.s. invertible ii) $(\gamma^{-1})_{ij} DX_j \in D(\delta)$.

$\Rightarrow$ density $p(\cdot)$ of $X \ll \mathcal{L}_d$ in $\mathbb{R}^d$.

Pf: For $\varphi \in C_b^\infty(\mathbb{R}^d)$. $\Rightarrow \varphi(X) \in \mathbb{D}^{1,2}$

$D\varphi(X) = \sum_1^d \partial_j \varphi(X) DX_j$. $\langle D\varphi(X), DX_i \rangle = \sum \gamma_{ij} \cdot \partial_j \varphi(X)$.

We can solve: $\partial_j \varphi(X) = \sum_1^d (\gamma^{-1})_{jk} \langle D\varphi(X), DX_k \rangle$

So: $\mathbb{E}(\partial_j \varphi(X)) = \mathbb{E}(\varphi(X)\, \delta(\sum_1^d (\gamma^{-1})_{jk} \langle D\varphi(X), DX_k \rangle)$

The conclusion follows from:

Lemma. μ is finite on $\mathbb{R}^d$. If $\exists C \geq 0$ s.t.

$\forall \varphi \in C_b^\infty(\mathbb{R}^d)$. $|\int_{\mathbb{R}^d} \partial_j \varphi \, d\mu| \leq C \|\varphi\|_\infty$

Then $\mu \ll \mathcal{L}_d$ and the density of $\mu \in L^{d/(d-1)}$.

Thm. (Bouleau. Hirsch)

$X: \Omega \to \mathbb{R}^d$. $X_j \in \mathbb{D}^{1,p}_{loc}$. for some $p > 1$. If γ is a.s. invertible. Then: the law of $X \ll \mathcal{L}_d$.

(2) Smoothness:

Next, we want to find explicit form of density of random vector X as in dim $=1$.

Pf: For V. Hilbert. $\mathbb{D}^\infty(V) := \bigcap_{k \geq 1} \bigcap_{p \geq 1} \mathbb{D}^{k,p}(V)$. $\mathbb{D}^\infty = \mathbb{D}^\infty(\mathbb{R}^1)$

Rmk: i) $\mathcal{J}(V) \subset \mathbb{D}^\infty(V)$.

ii) Note: If $X_j \in \mathbb{D}^\infty$. $\varphi \in C_p^\infty(\mathbb{R}^n)$. Then:

$\varphi(X) \in \mathbb{D}^\infty$. $D\varphi(X) = \sum \partial_j \varphi(X) DX_j$.

Set $\varphi = xy \Rightarrow$ So: $\mathbb{D}^\infty$ is an algebra.

Def: i) Random matrix M is nondegenerated if $M_{ij} \in \mathbb{D}^\infty$, $\forall i, j$. M is a.s. invertible and $|M^{-1}| \in L^p(\Omega)$, $\forall p \geq 1$.

ii) Random vector X is nondegenerated if $X_j \in \mathbb{D}^\infty$ and its Malliavin matrix γ is nondegenerated.

Lemma: If M is nondegenerate. Then $\forall i, j \leq n$, $(M^{-1})_{ij} \in \mathbb{D}^\infty$ and $D(M^{-1})_{ij} = -\sum_{k,l}^{n} (M^{-1})_{ik} (M^{-1})_{lj} DM_{kl}$.

Pf: By Cramer, $|M| \cdot M^{-1} \in \mathbb{D}^\infty$.

Since its entries are polynomial of M_{ij}.

Next, we prove: $|M|^{-1} \in \mathbb{D}^\infty$.

1) Prove: $|M| > 0$ a.s. or $|M| < 0$ a.s.

Set $\psi_n = n I_{(0, 1/n)}$, $\varphi_n = \int_{-\infty}^t \psi_n \, dx$.

$\Rightarrow \| D\varphi_n(x) \|_H \leq \| D(|M|) \|_H^2 / |M|^2$

Besides, $\varphi_n(|M|) \xrightarrow{L^2} I_{(0,\infty)}(|M|)$

So, $I_{(0,\infty)}(|M|) \in \mathbb{D}^{1,2}$.

2) WLOG. Set $|M| > 0$ a.s. $\varphi_n(t) = 1/(t+n^{-1})$, $t > 0$ and extend to $C_p^\infty(\mathbb{R}^1)$.

$\Rightarrow \varphi_n(|M|) \in \mathbb{D}^\infty \to |M|^{-1}$ in L^p and a.s.

$D\varphi_n(|M|) \xrightarrow{p} -|M|^{-1} \cdot D(|M|)$

By closeness of D $\Rightarrow |M|^{-1} \in \mathbb{D}^{1,p}$, $\forall p \geq 1$.

similarly. by induction. $D^k \varphi(X_n) \xrightarrow{p} \square$

So: $|M|^{-1} \in \mathbb{D}^{k,p}$. $\forall p \geq 1$. $\Rightarrow |M|^{-1} \in \mathbb{D}^\infty$.

Thm. If $u \in \mathbb{D}^\infty(H)$. Then: $u \in D(\delta)$ and $\delta(u) \in \mathbb{D}^\infty$.

prop. If X is degenerate. Then $\forall Y \in \mathbb{D}^\infty$. $\exists Z_\alpha = Z_\alpha(X, Y) \in \mathbb{D}^\infty$. st. $\forall \varphi \in C_p^\infty(\mathbb{R}^d)$. we have:

$\mathbb{E}(\partial_\alpha \varphi(X) Y) = \mathbb{E}(\varphi(X) Z_\alpha)$. Besides.

$Z_{e_j} = \delta(\sum_1^d (Y\gamma^{-1})_{jk} DX_k)$. $Z_{\alpha+e_j}(X, Y) = Z_{e_j}(X, Z_\alpha)$

Pf: For $\alpha = e_j$. we have obtain it in (1) ②.

By induction: $\mathbb{E}(\partial_{\alpha+e_j} \varphi(X) \cdot Y) =$

$\mathbb{E}(\partial_{e_j} \varphi(X) Z_\alpha(X, Y))$

Thm. X is degenerated. Then law of X has density $p(\cdot) \in \mathcal{S}(\mathbb{R}^d)$ w.r.t. $\mathcal{L}^d$.

Pf: In (1) ②. we has $p(\cdot) \ll \mathcal{L}^d$.

set $u(\xi) := \widehat{p(\cdot)}(\xi) =: \mathbb{E}(\varphi_\xi(X))$

1) $|\xi|^{2k} u(\xi) = \mathbb{E}(\Delta^k \varphi_\xi(X)) \overset{\text{prop}}{=} \mathbb{E}(\varphi_\xi(X) Z_k)$

$\Rightarrow u(\xi)$ is rapidly decreasing.

2) $|\xi|^{2k} \partial_\alpha u(\xi) = \mathbb{E}(\Delta^k \varphi_\xi(X) \cdot (iX)^\alpha)$

$\overset{\text{prop.}}{=} \mathbb{E}(\varphi_\xi(X) Z_k(X, (iX)^\alpha))$

So $\partial_\alpha u(\xi)$ is also rapid decreasing.

$\Rightarrow u \in \mathcal{S}(\mathbb{R}^d) \Rightarrow \check{u} = \rho(t) \in \mathcal{S}(\mathbb{R}^d)$

($\wedge : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$. bijection)

(3) Stochastic Diff Equations:

Fix W on $H = L^2(\mathbb{R}^{\geq 0}; \mathbb{R}^d)$. $\vec{B}_t = (B_t^1 \cdots B_t^d)$

$\mathcal{F}_t := \sigma(B_s^i,\ s \le t,\ i \le d)$. right-conti.

Consider : $X(t) = (X_1(t) \cdots X_m(t))$.

$$\begin{cases} dX(t) = \beta(t, X(t))\,dt + \sum_{j=1}^{d} V_j(t, X(t))\,dB_t^j \\ X(0) = x_0 \in \mathbb{R}^m. \end{cases} \quad (*)$$

To guarantee it has unique conti. adapted L^p-integrable solution. We assume :

i) $\beta : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$, $V_j : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$, $j \le d$ are measurable. $|\beta(\cdot,0)|$, $|V_j(\cdot,0)|$ are bdd.

ii) $\exists L > 0$. s.t. $|\beta(s,x) - \beta(s,y)| + \sum_1^d |V_j(s,x) - V_j(s,y)| \le L|x-y|$. for $\forall s \in [0,T]$ and $x, y \in \mathbb{R}^m$.

prop. If $\exists p > 1$, $\varepsilon > 0$. s.t. $X \in \mathbb{D}^{1,1+\varepsilon} \cap L^p(\Omega)$ and $DX \in L^p(\Omega; H)$. Then : $X \in \mathbb{D}^{1,p}$

Rmk: It's intuitive since $\|X\|_{1,p} < \infty$

Thm. $(X_t)_{t\le 2}$ is unique solution of $(*)$

Then $X_j(t) \in \mathbb{D}^{1,\infty} := \bigcap_{p\ge 1} \mathbb{D}^{1,p}$, $\forall j \le m$

and we have $\sup_{0\le r\le t} \mathbb{E}(\sup_{r\le s\le 2} |D_r X_j(s)|^p)$

$< \infty$ for $\forall p$, and $j = 1, \cdots m$.

Besides, $\exists (a_{k\ell}), (b_k)$, uniformly bdd

adapted m-dimensional, s.t. $1 \le j \le d$.

$$D_r^j X_t = V_j(r, X_r) + \sum_{k=1}^{m} \int_r^t b_k(s) D_r^j X_k(s)\, ds$$

$$+ \sum_{k=1}^{m} \sum_{\ell=1}^{d} \int_r^t a_{k\ell}(s) D_r^j X_k(s)\, dB_s^\ell .$$

$(*)$: $D_r X_k$ is $\mathbb{R}^d$-valued vector. $D_r^j X_k$ is its j^{th} component.

When $V_j, B \in C^1$. Then: $b_k(s) = \partial_k B(s, X_s)$

$a_{k\ell}(s) = \partial_k V_\ell(s, X_s)$.

Pf: Consider (X_n) is Picard iteration of $(*)$

$$X_{n+1} = \Phi(X_n) := X_0 + \int_0^t B(s, X_n(s))\, ds + \sum \int_0^t V_j(s, X_n(s))\, dB_s$$

$$\psi_n(t) := \sup_{0\le r\le t} \mathbb{E}(\sup_{r\le s\le t} |D_r X_n(s)|^p)$$

Next, we use induction (on n) to prove:

i) Prove: $X_n \in \mathbb{D}^{1,\infty}$, $\forall n \ge 1$, and $t \in (0, 2)$.

Since B, V_j are Lipschitz.

So, $B(s, X_n(s)), V_j(s, X_n(s)) \in \mathbb{D}^{1,\infty}$, and

$\exists b_{n,i,k}, a_{n,i,j,k} \le L$, s.t.

$$D_r B^i(s, X_n) = \sum_k b_{n,i,k} D_r X_n(s) \qquad \in L^2.$$

$$D_r V^i(s, X_n) = \sum_k a_{n,i,j,k} D_r X_n(s)$$

$$\Rightarrow D_r B^i, D_r V^i \in \mathbb{D}^{1,2}.$$

Besides $D_r \int_0^t V_j(s, X_n(s))\, dB_s^j = V_j(r, X_n(r))$

$$+ \int_r^t D_r V_j(s, X_n(s))\, dB_s^j.$$

By bdd assumption on V_j and BDG inequality apply on 2^{nd} term in RHS

$\Rightarrow D_r \int_0^t V_j\, dB_s^j \in L^p$, $\forall p \geq 1$.

So with prop. above $\int_0^t V_j\, dB_s^j \in \mathbb{D}^{1,\infty}$.

it also holds for $\int_0^t B_j\, dt \Rightarrow X_{n+1} \in \mathbb{D}^{1,\infty}$.

2) Prove: $\psi_n(t) < \infty$ and $\exists c_1, c_2$ s.t.

$$\psi_{n+1}(t) \leq c_1 + c_2 \int_0^t \psi_n(s)\, ds.$$

By the assumption i), ii). we have:

$$\mathbb{E}\left(\sup_{r \leq s \leq t} |D_r X_{n+1}(s)|^p\right) \leq c_p\left(\gamma_p + T^{p-1} L^p \int_0^t \mathbb{E}|D_r X_n|\, ds\right)$$

where $\gamma_p := \sup_{n,j} \mathbb{E} \sup_{(0,T)} |V_j(s, X_n(s))|^p < \infty$

Since (X_n) is bdd in L^p. V_j is Lipschitz.

3°) Inductively deduced from 2):

$$\psi_n(t) \leq c_1 \sum_{j=0}^{n} (c_2 t)^j / j! \leq c_1 e^{c_2 t}.$$

$\Rightarrow \psi_n$ is uniformly bdd.

Combine with $X_n \xrightarrow{L^p} X$. So $X \in \mathbb{D}^{1,\infty}$.

<u>Rmk</u>: If $B, V_j \in C^\infty$, and their derivatives are bdd. Then we can iterate the pf:

Thm Under (*), if $B, V_j \in C^\infty$ and all derivatives w.r.t x are bdd. Then the unique solution X_t for (*) has $X_j(t) \in \mathbb{D}^\infty$, $\forall j \le m$.

(4) Hörmander Thm:

We still consider SDE (*) with B, V_j are indept of time t, belong to C^∞ and has bdd derivatives.

Def: i) Consider smooth vector field on $\mathbb{R}^m$: $u \in C^\infty(\mathbb{R}^m; \mathbb{R}^m)$. $[u, v]_0 := Du \cdot v - Dv \cdot u$.
where $(Du(x))_{ij} = \partial_j u_i(x)$.

ii) Set $S_0 := \{V_1, \cdots V_d\}$. $V_0 := B - \frac{1}{2}\sum_1^d DV_j \cdot V_j$

$S_{k+1} := S_k \cup \{[u, V_j]_0 : u \in S_k, 0 \le j \le d\}$.

$V_k(x) := \text{span}\{u(x) \mid u \in S_k\}$.

We say $\{B, V_1 \cdots V_d\}$ satisfy Hörmander condition if $\bigcup_{k \ge 0} V_k(x) = \mathbb{R}^m$, $\forall x \in \mathbb{R}^m$.

Thm (Proba. version of Hörmander Thm)

For $X_0 \in \mathbb{R}^m$, $B, V_1 \cdots V_d$ satisfies Hörmander condition. $\Rightarrow$ law of $X(t)$, solution for (*) has density $p_t(x) \in \mathcal{S}(\mathbb{R}^m)$ w.r.t $\mathcal{L}^m$.

Pf: prove: X is degenerate.

Lemma: $M \geq 0$ random sym matrix. $M_{ij} \in L^p(\Omega)$ for $\forall p \geq 1$. If for $\forall p \geq 2$. $\exists C_p, \varepsilon_p > 0$. s.t. $\forall 0 < \varepsilon < \varepsilon_p$. we have:

$\sup_{|x|=1} \mathbb{P}(x^T M x \leq \varepsilon) \leq C_p \varepsilon^p$. Then:

$|M|^{-1} \in \cap_{p \geq 1} L^p(\Omega)$.

Pf: Set $\lambda = \inf Ev(M) \Rightarrow \lambda^m \leq |M|$.

Next prove: $\mathbb{E}(\lambda^{-q}) < \infty$. $\forall q \geq 2$.

$\mathbb{E}(\lambda^{-q}) = \int_0^\infty q t^{q-1} \mathbb{P}(\lambda < t^{-1}) \, dt$.

Find $(x_n)_1^N \subset \{|x|=1\} \subset \bigcup_1^N B(x_n, \varepsilon^2)$

$\overset{\exists k}{\Rightarrow} x^T M x \geq x_k^T M x_k - 2\|M\| \varepsilon^2$.

So: $\mathbb{P}(\lambda \leq \varepsilon) = \mathbb{P}(\exists |x|=1. \; x^T M x \leq \varepsilon)$

$\leq \mathbb{P}(\|M\| \geq \varepsilon^{-1}) + \sum_1^N \mathbb{P}(x_k^T M x_k \leq 3\varepsilon)$

$\leq \varepsilon^p \mathbb{E}(\|M\|^p) + N \sup_{|x|=1} \mathbb{P}(x^T M x \leq 3\varepsilon)$

With condition $\Rightarrow \leq \varepsilon^p$. for ε small.

Set $Y(t) = I + \int_0^t B'(X(s)) Y(s) \, ds + \sum_1^d \int_0^t V_j'(X(s)) Y_s \, dB_s^j$

Rmk: It can be seen as the derivative of solution X_t for $(*)$.

1) Prove $Y(t)$ is a.s. invertible:

By Itô's formula applied on $f(A) = A^{-1}$ if $|A| \neq 0$

Note $f'(A)H = -A^{-1}HA^{-1}$, $f''(A)\langle H, \tilde{H}\rangle = 2A^{-1}HA^{-1}\tilde{H}A^{-1}$

$\Rightarrow Y^{-1}(t) = I^{-1} - \int_0^t Y(s)^{-1} B'(X(s)) Y(s) Y(s)^{-1} ds - \sum_1^d$

if Y^{-1} exists

$\int_0^t Y(s)^{-1} V_j'(X(s)) Y(s) Y(s)^{-1} dB_s^j + \sum_1^d \int_0^t$

$Y(s)^{-1} V_j'(X(s)) Y(s) Y(s)^{-1} V_j'(X(s)) Y(s) Y(s)^{-1} ds$

Set $Z(t) = I - \int_0^t (Z(s) B'(X(s)) - \sum_1^d Z(s) V_j'(X(s))^2) ds$

$- \int_0^t \sum_1^d Z(s) V_j'(X(s)) dB_s^j$.

We expect $Z(t) = Y(t)^{-1}$.

Check $Z(t)Y(t) = I$ which follows from Itô's formula applied on $f(A,B) = AB$.

2°) Prove: $D_r^j X(t) = Y(t) Y(r)^{-1} V_j(X(r))$

$\Leftarrow$ check RHS satisfies the SDE in Thm in (3).

3°) So we have: $\gamma_t = \sum_{j=1}^d \int_0^t (D_r^j X(t))(D_r^j X(t))^T dt$

$= Y(t) \sum_1^d \int_0^t Y(r)^{-1} V_j(X(r)) V_j(X(r))^T (Y(r)^{-1})^T dr \cdot Y(t)^T$

$=: Y(t) C(t) Y(t)^T$ from 2)

Since $Y(t)$ has a.s. inversion $Y(t)^{-1}$, and $Y(t)^{-1} \in L^p(\Omega)$, $\forall p \geq 1$ follows from basic SDE Thm.

So: we only need to check on $C(t)$:

i.e. $|Y_t|^{-1} \in \bigcap_{p\geq 1} L^p \iff |C(t)|^{-1} \in \bigcap_{p\geq 1} L^p$.

By Lemma. we only need to check: $C(t)$ satisfies its conditions

4) Fix $|x|=1$. Set $Z_u(t) := x^T Y(t)^{-1} u(X(s))$ for u smooth vector field.

Apply Itô's on $Y(t)^{-1} u(X(t))$, we have:

$$Z_u(t) = x^T u(x_0) + \int_0^t Z_{[V_0,u]_0 + \frac{1}{2}\sum_1^d [V_j,[V_j,u]_0]_0}(s)ds$$
$$+ \sum_1^d \int_0^t Z_{[V_j,u]}(s)\, dB_s^j. \qquad (\Delta)$$

Set $S_0' = S_0$. $S_{k+1}' = \{ [V_j, u]_0,\ 1\leq j\leq d,\ u\in S_k';$
$[V_0,u]_0 + \frac{1}{2}\sum_1^d [V_j,[V_j,u]_0]_0,\ u\in S_k' \} \cup S_k'$

$V_k'(x) = \text{span}\{u(x) \mid u\in S_k'\}$.

Rmk: i) It's better to use (S_k') rather than (S_k) due to (Δ) above.

ii) Note that $\bigcup_{k\geq 0} V_k'(x) = \bigcup_{k\geq 0} V_k(x)$.

5) Prove: $x^T C(t) x \neq 0$, $\forall |x|=1$.

Note $x^T C(t) x = \sum_1^d \int_0^t |Z_{V_j}(r)|^2 dr$

By contradiction: $|Z_{V_j}| = 0$, $\forall j$.

$\Rightarrow Z_{[V_0,V_k]_0 + \frac{1}{2}\sum [V_j,[V_j,V_k]_0]_0} = Z_{[V_j,V_k]_0} \equiv 0$.

Inductively. $Z_u \equiv 0$. $\forall u \in S_k$. $\forall k \geq 0$.

$\Rightarrow$ $0 = Z_u(0) = x^T u(x_0)$. $\forall u \in \bigcup V_k'(x_0)$

By Hörmander's condition $\overset{\text{contrad.}}{\Rightarrow}$ $x = 0$. $\forall |x| = 1$

b) <u>Def</u>: i) $A = (A_\varepsilon)_{\varepsilon \in (0,1]}$ events. We say A is almost false if $\forall p \geq 1$. $\exists C_p$. s.t. $\mathbb{P}(A_\varepsilon) \leq C_p \varepsilon^p$. $\forall \varepsilon \in (0,1]$.

ii) $A \Rightarrow_\varepsilon B$ denote: $A / B \overset{\Delta}{=} (A_\varepsilon / B_\varepsilon)$ is almost false.

<u>Lemma</u>[2] $f \in C'([0,1]; \mathbb{R}')$, $\alpha \in (0,1]$. If f is α-Hölder conti. Then: $\|f'\|_\infty = \|f\|_1 \leq 4 \|f\|_\infty^{\alpha/(1+\alpha)} \cdot \|f'\|_\alpha^{1/(1+\alpha)}$.

<u>Pf</u>: Fix x_0. st. $\|f'\|_\infty = |f'(x_0)|$.

Note: $\forall |x - x_0| \leq (\|f'\|_\infty / 2\|f\|_\alpha)^{1/\alpha}$

$\Rightarrow |f'(x)| \geq \frac{1}{2} \|f'\|_\infty$. by Hölder conti.

So: $\exists x_1 \in I$. st. $|f(x_1)| \geq A|I|/2$.

where I is interval above

<u>Lemma</u>[3] (Bernstein's inequi)

For M is c.l.m. st. $M_0 = 0$. Then:

$\mathbb{P}(M_T^* \geq x, \langle M \rangle_T \leq y) \leq 2e^{-x^2/2y}$. for $\forall$ stopping time T.

Lemma 4 For a, b are $\mathbb{R}^d$-valued adapted

$$Z(t) = Z_0 + \int_0^t a(s)\,ds + \sum_1^d \int_0^t b_j(s)\,dB_j^s$$

If $\|a\|_{\frac{1}{3}}, \|b\|_{\frac{1}{3}} \in \bigcap_{p \geq 1} L^p(\Omega)$ Then:

$\exists r \in (0,1)$. st. $\forall \varepsilon$. we have:

$\{\|Z\|_\infty \leq \varepsilon\} \Rightarrow_\varepsilon \{\|a\|_\infty \leq \varepsilon^r\} \cup \{\|b\|_\infty \leq \varepsilon^r\}$.

Pf: // Set $M_t = \sum_1^d \int_0^t b_j(s)\,dB_j$ in Lem 3.

$$\Rightarrow \mathbb{P}\left(\left(\int_0^\cdot b(s)\,dB_s\right)_T^* \geq \varepsilon^p,\ \|b\|_\infty \leq \varepsilon \right)$$

$$\leq 2e^{-\varepsilon^{r}2/2\varepsilon^2 T},\quad a \in (0,1).$$

since $\langle M \rangle_t = \int_0^t \|b\|_\infty^2\,ds$.

$S_0 := \{\|b\|_\infty \leq \varepsilon\} \Rightarrow_\varepsilon \{\|\int_0^\cdot b\,dB_s\|_\infty \leq \varepsilon^a\}$.

// Apply Itô's formula on Z^2:

$$Z(t)^2 = Z_0^2 + 2\int_0^t Z(s)a(s)\,ds + 2\int_0^t Z(s) b(s)\,dB_s + \int_0^t |b(s)|^2\,ds$$

$\{\|a\|_\infty \leq \varepsilon^{-1/4}\}$ is allmost true by Chebyshev.

replace in $Z_t^2 = \cdots$ and iterate:

$$\{\|Z\|_\infty < \varepsilon\} \Rightarrow_\varepsilon \{\|\int_0^\cdot Z_s a_s\,ds\|_\infty \leq \varepsilon^{3/4}\}$$

$$\Rightarrow_\varepsilon \{\|\int_0^\cdot Z_s b_s\,dB_s\|_\infty \leq \varepsilon^{1/2}\} \Rightarrow_\varepsilon \{\int_0^T |b_s|^2\,ds \leq \varepsilon^{\frac{1}{2}}\}$$

$$\overset{\text{Cauchy}}{\Rightarrow_\varepsilon} \{\int_0^T |b_s|\,ds \leq \varepsilon^{\frac{1}{4}}\}$$

use Lemma 2: $\alpha = \frac{1}{3}$, we have:

$$\|b\|_\infty \leq 4\left(\int_0^T |b(s)|\,ds\right)^{\frac{1}{4}} \|b\|_{\frac{1}{3}}^{\frac{3}{4}}$$

By Chebyshev again: $\{\|b\|_{\frac{1}{2}} \le \varepsilon^{-2}\}$ $\forall \varepsilon \in (0,1)$

is almost true. So:

$\{\|z\|_\infty < \varepsilon\} \Rightarrow_\varepsilon \{\|b\|_\infty \le \varepsilon^s\}$. $\forall s < \frac{1}{16}$.

replace in $z^2 = \square$ and iterate it similarly.

we can obtain: $\{\|z\|_\infty < \varepsilon\} \Rightarrow_\varepsilon \{\|a\|_\infty < \varepsilon^{1/80}\}$.

7') Pf of Thm:

Note $x^T C(T) x = \sum_1^n \int_0^T |z v_i(r)|^2 dr$

$\overset{\text{Cauchy}}{\geq} \frac{1}{T} \sum_1^n \left(\int_0^T |z v_i(r)| dr \right)^2$

$\underset{\tau = \frac{1}{2}}{\overset{\text{Lem}^2}{\geq}} \sum_1^n c(b, T) \|z v_i\|_\infty^8$

So: $\{X^T C(T) X \le \varepsilon\} \Rightarrow_\varepsilon \{\|z v_i\|_\infty \le \varepsilon^{\frac{1}{8}}\}$. $\forall i$.

$\Rightarrow_\varepsilon \bigcap_{v \in V_k} \{\|z v\|_\infty \le \varepsilon^{2k}\}$. $\forall k \geq 1$.

by using (D) in 4) and Lemma^4 iteratively

By Hörmander condition $V_k'(x) = \mathbb{R}^m$ if k

is large enough. Pick $v \in V_k'(x)$. $z v(0) = 1$.

So: $\{X^T C(T) x \le \varepsilon\} \Rightarrow_\varepsilon X$. satisfies Lemma'