Wiener Chaos

(1) Tensor product:

For H, V Hilbert spaces with o.n.b's: $(h_\alpha)_{\alpha\in A}$, $(V_\beta)_{\beta\in B}$.

Def: i) Tensor product $H \otimes V =: \{ \sum a_{\alpha,\beta} h_\alpha \otimes V_\beta \mid (a_{\alpha,\beta})_{A\times B} \in l^2(A\times B)\}$. endowed with inner product $\langle\ ,\ \rangle$:

$$\langle \sum a_{\alpha\beta} h_\alpha \otimes V_\beta, \sum b_{\alpha\beta} h_\alpha \otimes V_\beta \rangle = \sum a_{\alpha\beta} b_{\alpha\beta}$$

ii) $e: H\times V \to H\otimes V$. define by:

$e(\sum a_\alpha h_\alpha, \sum b_\beta V_\beta) =: \sum a_\alpha b_\beta h_\alpha \otimes V_\beta$.

where $(a_\alpha) \in l^2(A)$, $(b_\beta) \in l^2(B)$.

Rmk: e is conti. isometric. bilinear.

Lemma (Universal property).

U is Hilbert. $\eta: H\times V \to U$ is conti. bilinear form. Then: $\exists$ BLO. $T_\eta: H\otimes V \to U$. s.t. $\eta = T_\eta \circ e$.

$$\begin{tikzcd} H\times V \arrow[r, "e"] \arrow[dr, "\eta"'] & H\otimes V \arrow[d, "T_\eta"] \\ & U \end{tikzcd}$$

Pf: Set $T_\eta(h_\alpha \otimes V_\beta) = U_{\alpha\beta}$

$=: \eta(h_\alpha, V_\beta)$. Well-def

Since $(u_{\alpha\beta})$ is uniformly bdd.

<u>Cor.</u> $(\Omega_i, \mathcal{F}_i, \mu_i)$ σ-finite. Then we have:

$$L^2(\Omega_1, \mu_1) \otimes L^2(\Omega_2, \mu_2) \underset{iso}{\cong} L^2(\Omega_1 \times \Omega_2, \mu_1 \otimes \mu_2)$$

<u>Pf:</u> $\eta: \begin{array}{ccc} L^2(\Omega_1) \times L^2(\Omega_2) & \longrightarrow & L^2(\Omega_1 \times \Omega_2) \\ (f, g) & \longmapsto & ((x,y) \mapsto f(x)g(y)) \end{array}$

By Fubini. $\Rightarrow$ η is isometry.

Use universal prop. $\exists\, T_\eta$. BLO:

1) T_η is surjective. by Monotone class argument from indicators.

2) Check T_η is isometry.

<u>Def:</u> $\eta: H \times V \longrightarrow \mathcal{L}(H, V)$. $\eta(h, v) =: (g \longmapsto \langle g, h \rangle_H \cdot v)$

<u>Rmk:</u> i) η is rank-one operator.

ii) $H \times V \ncong \mathcal{L}(H, V)$

Otherwise $\forall\, T \in \mathcal{L}(H, V)$ can be approxi. by finite-rank op. $\Rightarrow$ cpt.

<u>Lemma</u> i) $H \otimes V \underset{iso}{\cong} \ell^2(A \times B)$.

ii) $H \otimes V \underset{iso}{\cong} \mathcal{L}_{HS}(H, V)$. for H, V. separable.

<u>Pf:</u> i) is trivial.

ii) $\exists$ BLO. T_η. st. $\eta = T_\eta \circ \iota$.

Assume (h_n). (v_n) are o.n.b's of H, V.

$\Rightarrow h_n \otimes v_m$ is o.n.b of $H \otimes V$.

Check : $\|T_\eta \xi\|_{HS}^2 = \|\xi\|_{H\otimes V}^2$. $\forall \xi \in H \otimes V$.

Besides. $\forall T \in \mathcal{L}_{HS}(H, V)$.

Note $Th_n = \sum_m a_{mn} v_m$. $a_{mn} = \langle Th_n, v_m \rangle$

Set $\xi =: \sum a_{mn} h_n \otimes v_m$. $\in H \otimes V$.

$\Rightarrow (T_\eta \xi)(h_k) = Th_k$. $\forall k$. So : $T_\eta \xi = T$.

i.e. T_η is surjective !

Rmk: i) We will write $T = \sum a_{mn} h_m \otimes v_n$.

$: \xi \in H \mapsto \sum a_{mn} \langle h_m, \xi \rangle v_n. \in V$.

ii) $\|\cdot\|_{HS}$ can be induced by inner product : $\langle T, S \rangle_{HS} = \sum a_{mn} b_{mn}$.

for $T = \sum a_{mn} h_m \otimes v_n$. $S = \sum b_{mn} h_m \otimes v_n$.

Besides $\langle T, S \rangle_{HS} = \sum \langle Te_n, Se_n \rangle$

$= \sum \langle S^* T e_n, e_n \rangle$

$= tr(S^* T)$.

Rmk: Actually $tr(S^*T)$ won't depend on the choice of o.n.b's.

Check by expanding (Te_n). (Se_n) into another basis (h_n).

(2) Decomposition :

Def. In real separable Hilbert space H with $\langle \cdot, \cdot \rangle$. H- isonormal Gaussian process is

$W = (W(h))_{h \in H}$. real-valued. r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$. st. $W(h)$ is Gaussian and $\mathbb{E}(W(h)W(k)) = \langle h, k \rangle$. $\forall h, k \in H$.

Rmk: It's generalization of white noise.

There always exists such process H in every H. (Constructed as Gaussian white noise)

Prop. i) $h \mapsto W(h)$ is linear isometry.

ii) W is Gaussian family.

Pf: i) Check $\mathbb{E}((W(\lambda h + \mu g) - \lambda W(h) - \mu W(g))^2) = 0$.

ii) By linearity of i).

e.g. (Fractional BM).

$\mathcal{E} =: \{ f \text{ is step func. on } [0, T] \}$. There exists $\langle , \rangle$ on $\mathcal{E}$. st. $\langle I_{[0,t]}, I_{[0,s]} \rangle = C_H(t, s)$ $:= \frac{1}{2}(t^{2H} + s^{2H} + |t-s|^{2H})$. $H \in (0, 1)$. Hurst. para.

Denote: V_H is completion of $\mathcal{E}$ under $\langle , \rangle$.

Then we call the V_H-isonormal Gaussian process by fractional BM with para. H.

Def: $n \in \mathbb{Z}^{\geq 0}$. Hermite polynomials (H_n) is def by:

$H_0 = 1$. $H_n = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}}) / n!$ $n \geq 1$.

Rmk: i) Recall it's basis of space of poly's.

ii) Set $F(t,x) = e^{tx - \frac{t^2}{2}}$. Then:

$$F(t,x) = e^{\frac{x^2}{2} - (x-t)^2/2}$$

$$= e^{\frac{x^2}{2}} \sum_{n \geq 0} \frac{t^n}{n!} \frac{d^n}{dt} e^{-\frac{(x-t)^2}{2}} \Big|_{t=0}$$

$$= \sum_{n \geq 0} t^n H_n(x)$$

i.e. H_n is coefficient of series expansion of $F(t,x)$ at $t=0$.

Lemma. i) $H_n'(x) = H_{n-1}(x)$. ii) $H_n(-x) = (-1)^n H_n(x)$

iii) $(n+1) H_{n+1}(x) = x H_n(x) - H_{n-1}(x)$.

Pf: i) $\frac{\partial}{\partial x} F(t,x) = \cdots$ ii) $F(-x,t) = F(x,-t)$.

iii) Consider $\frac{\partial}{\partial t} F(t,x) = \cdots$

Rmk: Another characterization: $H_n(x) = e^{-\partial x^2/2} x^n / n!$

Lemma $X, Y \sim N(0,1)$. For $n, m \geq 0$. We have:

$$\mathbb{E}(H_n(X) H_m(Y)) = \begin{cases} 0. & n \neq m \\ \mathbb{E}^n(XY)/n! & n = m. \end{cases}$$

Pf: Set $c := \mathbb{E}(XY)$. By charac.:

$$\mathbb{E}\left(e^{sX - s^2/2} e^{tY - \frac{t^2}{2}}\right) = e^{stc}.$$

Operate $\partial_x^n \partial_t^m |_{s=t=0}$ on both sides.

Def: The n^{th}-Wiener chaos is closure of

$LS\{H_n(W(h)) \mid h \in H, \|h\| = 1\}$ in $L^2(\Omega, \Sigma, \mathbb{P})$.

We denote it by $\mathcal{H}_n$.

Lemma. For $n \neq m$, $\mathcal{H}_n \perp \mathcal{H}_m$ in $L^2(\Omega, \mathbb{P})$.

Pf: Follows from the Lemma above.

Thm. (Decomposition)

$\Sigma_W \triangleq \sigma(W(h) \mid h \in H\} \subset \Sigma$. We have:

$$L^2(\Omega, \Sigma_W, \mathbb{P}) = \bigoplus_{n \geq 0} \mathcal{H}_n.$$

Pf: It remains to prove: $\forall X \in L^2(\Sigma_W)$,

$X \perp \bigoplus \mathcal{H}_n \Rightarrow X = 0$ a.s.

By density of Hermite poly's:

$\mathbb{E}(X\, p(W(h))) = 0$, $\forall p$ poly, $h \in H$.

Since it's subalgebra separating points.

$\Rightarrow \mathbb{E}(X I_B) = 0. \quad \forall B \in \Sigma_W. \Rightarrow X = 0 \text{ a.s.}$

Cr. Set $H = \mathbb{R}^1$. μ is law of $N(0,1)$.

$\Rightarrow ((n!)^{\frac{1}{2}} H_n)_{n \geq 0}$ is o.n.b. of $L^2(\mathbb{R}^1, \mathcal{B}_{\mathbb{R}^1}, \mu)$.

Pf: $W = \begin{matrix} \mathbb{R}^1 & \longrightarrow & L^2(\mathbb{R}^1, \mathcal{B}_{\mathbb{R}^1}, \mu) \\ h & \longmapsto & hx \end{matrix}$ where x

is Gaussian r.v. $\Rightarrow W$ is isonormal.

and note $H_n(x) = (-1)^n H_n(-x)$.

$\mathbb{R}^1$ only has $\{1\}$ or $\{-1\}$ as o.n.b.

$\Rightarrow$ dim $\mathcal{H}_n = 1$.

Besides. $\Sigma_W = \sigma(x) = \mathcal{B}_{\mathbb{R}^1}$.

Def: Set $\mathcal{P}_n^0 =: \{ p(W(h_1), W(h_2) \cdots W(h_k)) \mid k \leq n.$

$h_i \in H$, and p is polynomial with degree $\leq n\}$.

and $\mathcal{P}_n$ is closure of $\mathcal{P}_n^0$. $\mathcal{P}^0 = \bigcup_n \mathcal{P}_n^0$.

prop. $\mathcal{P}_n = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_n$

Pf: $\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_n \subset \mathcal{P}_n$ is trivial.

Conversely, we prove $\mathcal{P}_n \perp \mathcal{H}_m$. $\forall m > n$.

i.e. prove: $\mathbb{E}(p(W(h_1), \cdots W(h_k)) H_m(W(h))) = 0$.

Consider $\{h, e_1, \cdots e_j\}$ is o.n.b of $h \cup \{h_i\}_1^k$.

where here we set $\|h\| = 1$.

S. : $\sigma(W(e_1)\cdots W(e_j), W(h)) = \sigma(W(h_1)\cdots)$

Note $\mathbb{E}(W(e_1)^{\alpha_1}\cdots W(e_j)^{\alpha_j} W(h)^{\beta} H_m(W(h)))$

$= \mathbb{E}(W(e_1)^{\alpha_1}\cdots W(e_j)^{\alpha_j})\,\mathbb{E}(W(h)^{\beta} H_m(W(h)))$

$= 0$ since $\beta \le n < m$

<u>prop.</u> $\forall n \ge 0.$ $(\phi_\alpha)_{|\alpha|=n} := (\sqrt{\alpha!}\prod_{n\ge 0} H_{\alpha_n}(W(e_n)))_{|\alpha|=n}$

is o.n.b of H_n. where (e_n) is o.n.b of H.

<u>Pf:</u> 1) $\mathbb{E}(\phi_\alpha\phi_\beta) = \sqrt{\alpha!}\sqrt{\beta!}\prod \mathbb{E}(H_{\alpha_n}(W(e_n)) H_{\beta_n}(W(e_n)))$

$= \delta_{\alpha\beta} \Rightarrow \phi_\alpha \perp \phi_\beta.$

2) Check: $\mathcal{P}_n = \overline{\text{span}}\{(\phi_\alpha)_{|\alpha|\le n}\}$.

3) Note $H_n < \overline{\text{span}}\{(\phi_\alpha)_{|\alpha|=n}\}$. and

$\mathcal{P}_n = \overline{\text{span}}\{(\phi_\alpha)_{|\alpha|\le n}\} = \bigoplus_{k\le n} H_k$

<u>Denote:</u> $H_n(V) := \{\sum_1^m X_i v_i \mid X_i \in H_n.\ v_i \in V.\}$

<u>Rmk:</u> It's subspace of $L^2(\Omega; V)$ and can

be seen as $H_n \otimes V$.

<u>prop.</u> V is separable Hilbert. Then: $L^2(\Omega, \Sigma_W, \mathbb{P}; V)$

$= \bigoplus_{n\ge 0} H_n(V)$. Besides. if (v_k) is o.n.b. of V.

then $(\phi_\alpha \otimes v_k)_{|\alpha|=n,\ k\in\mathbb{Z}^{\ge 0}}$ is o.n.b of $H_n(V)$.