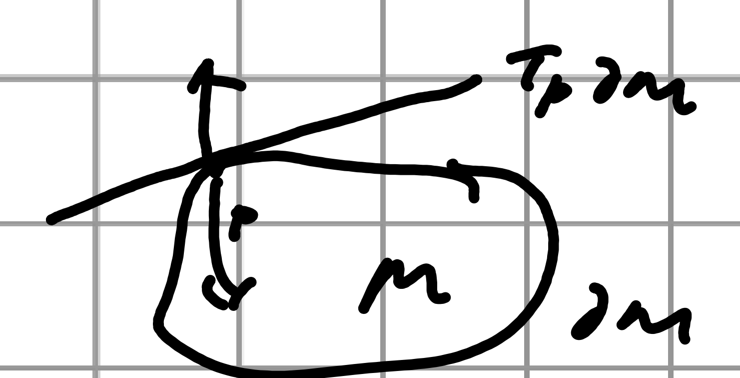
Stokes THM

(1) Ori. mfd with bdy:

Next, we want to endow ∂M an orientation.

Note $T_p \partial M \subset T_p M$ is hyperplane, cutting $T_p M$ into inward, outward part.



If $(v, v_1 \cdots v_{m-1})$ is oriented basis for $T_p M$, $(v_i) \subset T_p \partial M$. v is outward-pointing.

$\Rightarrow (v_1 \cdots v_{m-1})$ is ori. basis for $T_p \partial M$.

Rmk: We can also take volume α of M. And pick $X \in \mathfrak{X}(M)$ is outward pting along ∂M. Set $i_X(M)$ as ori. of ∂M.

(2) Stokes Thm:

Thm. M^m is oriented mfd with boundary and ω is $(m-1)$-form on M with cpt support $\Rightarrow \int_M d\omega = \int_{\partial M} \omega$.

<u>Rmk</u>: i) The integrand of $\int_{\partial M} \omega$ is actually $\omega|_{\partial M} = i^* \omega$. $i: \partial M \hookrightarrow M$.

ii) $\partial M = \emptyset \Rightarrow \int_M d\omega = 0$. (e.g. $\partial\partial M = \emptyset$)

iii) In some special case, eg. Green formula ... we see div, grad ... as kind of exterior derivative.

<u>Pf</u>: For $\{f_\alpha\}$ is PoU sub. to $\{(U_\alpha, \varphi_\alpha)\}$

$$d(\sum f_\alpha \omega) = \sum f_\alpha d\omega + d(\overset{\equiv 1}{\sum f_\alpha})\omega$$

$$= \sum f_\alpha d\omega.$$

So: we only consider on (U, φ). s.t.

i) $\varphi(U) = (0,1)^m$ or ii) $\varphi(U) = (0,1] \times (0,1)^{m-1}$

locally. $(\varphi^{-1})^* \omega \triangleq \tilde{\omega} = \sum_i^m (-1)^{j-1} \omega^i dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx_m$

$$\Rightarrow d\tilde{\omega} = \sum \frac{d\omega^j}{dx^j} dx^1 \wedge \cdots dx^m.$$

$$\text{So}: \int_{\varphi(U)} d\tilde{\omega} = \int_{\varphi(U)} \sum \frac{\partial \omega^j}{\partial x^j} dx_1 \cdots dx^m.$$

For i): $\tilde{\omega}$ cpt / supp in $(0,1)^m$.

By fundamental calculus. LHS = RHS = 0

For ii): $\text{LHS} = \int_{\square} \omega^1(1, x_2 \cdots x_m) dx_2 \cdots dx_m$

$= \text{RHS}$. $(\partial\varphi(U) = \{1\} \times (0,1)^{m-1})$.

(3) De-Rham Coho:

Def: i) $Z^k(M) := \{w \in \Omega^k(M) \mid dw = 0\}$ closed form

ii) $B^k(M) := \{w \in \Omega^k(M) \mid \exists \eta \in \Omega^{k-1}(M)$, s.t. $d\eta = w\}$. exact form.

Rmk: i) Note $B^k(M) \subset Z^k(M)$. We want to measure the elements in Z^k, but not in B^k. Set $H^k(M) = Z^k(M)/B^k(M)$ de Rham cohomology.

ii) $Z := \bigoplus_0^m Z^i = \ker d$. $B := \bigoplus_0^m B^i = \operatorname{Im} d$.

$H := Z/B \simeq \bigoplus_0^m H^i$. is an algebra with product "$\wedge$":

$(d\eta + w) \wedge (d\eta' + w') = d(\eta \wedge w' + \eta \wedge d\eta') + w \wedge (w' + d\eta') \in H$.

Thm. M^m is oriented closed mfd with n components. Then $H^0(M) \cong \mathbb{R}^n$.

Pf: $B^0 = 0 \Rightarrow Z^0 = H^0$. which's full of functions with zero derivative.

$\Rightarrow \forall f \in H^0$. $f \equiv$ const on each component.

<u>Cor.</u> Under cond. above. $H^m(M) \simeq \mathbb{R}^n$.

<u>Pf:</u> We only prove it has at least n dimensions:

$\phi : \omega \mapsto (\int_{M_1} \omega, \cdots \int_{M_n} \omega)$.

(M_k) is n components of M

$\mathrm{Im}\,\phi = \mathbb{R}^n$. (Let ω = d_i volume)

And $B^m < \ker \phi$.

<u>prop.</u> $F: M \to N$. diffeo. $\Rightarrow H^k(M) \simeq H^k(N)$

in sense of v.s. isomorphism. $\forall k$.

<u>Pf:</u> $F^*: \Omega^k(N) \longrightarrow \Omega^k(M)$ is pull-back

(satisfies: $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$. $id^* = id$.)

Since $d(F^*\omega) = F^* d\omega$. So it

also map $Z^k(N)$. $B^k(N)$ to $Z^k(M)$

. $B^k(M)$. Since F is diffeo. So

F^* is a v.s. isomorphism.