

Diffusion Theory

To describe motion of a small particle suspended in a moving liquid, subject to random molecular bombardments. If $b(t, x)$ is velocity of fluid at time t and point x . Establish SDE:

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t) dB_t.$$

Interpret in Itô:

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dB_t$$

Remk: b is drift coefficient and σ is diffusion coefficient

Def: (Time-homogeneous) Itô diffusion is a stochastic process $X_t(\omega) = (\xi, \omega) \times \mathbb{R} \rightarrow \mathbb{R}$.

satisfies: $dX_t = b(X_t) dt + \sigma(X_t) dB_t, t \geq s$
 $\text{and } X_s = x.$

where \vec{B}_t is m-dim Bm. $b: \mathbb{R}^n \rightarrow \mathbb{R}^n$.

$\sigma: \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ satisfies Lipschitz condition:

$$|b(x_1 - y_1) - b(y_1)| + |\sigma(x_1 - y_1)| \leq D|x - y|.$$

Rmk: i) In fact, we can conclude X_t is

strong Feller process by Lipschitz.

ii) Note it satisfies E & u Thm.

for any bdd interval. Denote the

unique solution by $X_t = X_t^{s,x}$, $t \geq s$.

iii) Replace $\sigma(X_t)$, $b(X_t)$ by $\sigma(X_{t,s})$, $b(X_{t,s})$.

it becomes inhomogeneous case.

(1) Properties:

① Time-homogeneous:

prop. $(X_t)_{t \geq 0}$ is time-homogeneous.

$$\begin{aligned} \text{Pf: } X_{s+h}^{s,x} &= x + \int_s^{s+h} b(X_u^{s,x}) du + \int_s^{s+h} \sigma(X_u^{s,x}) dB_u \\ &\stackrel{u=s+v}{=} x + \int_0^h b(X_{s+v}^{s,x}) dv + \int_0^h \sigma(X_{s+v}^{s,x}) dB_v \end{aligned}$$

where $\tilde{B}_v = B_{s+v} - B_s$.

$$X_h^{0,x} = x + \int_0^h b(X_v^{0,x}) dv + \int_0^h \sigma(X_v^{0,x}) d\tilde{B}_v$$

also satisfies identical SDE.

By strong Uniqueness :

$$(X_{s+h}^{s,x})_{t \geq 0} \xrightarrow{\Delta} (X_h^{0,x})_{t \geq 0}$$

Rmk: Moreover, we have $(X_{s+h}^{s,x})_{t \geq 0}$ indep^t

with $\sigma(B_r, r \leq s)$.

② Markov Property

Remark: i) α^x is law of $(X_t)_{t \geq 0}$.

ii) $\mathcal{G}_t^{(m)} = \sigma(X_s, s \leq t), M_t = \sigma(X_s, s \leq t)$

Rank: $M_t \subset \mathcal{G}_t^{(m)}$. by E & n. Then.

Thm. (Simple Markov)

f is bnd Borel $: \mathbb{R}^n \rightarrow \mathbb{R}'$. Then $\forall t, h \geq 0$

$$\mathbb{E}^x \circ f(X_{t+h}) | \mathcal{G}_t^{(m)}(w) = \mathbb{E}^{X_t(w)} \circ f(x_h)$$

Pf: $F(x, t, r, w) \stackrel{\Delta}{=} X_r(w), r \geq t$. incept of $\mathcal{G}_t^{(m)}$

$$\Rightarrow X_r(w) = F(x_t, t, r, w)$$

$$\text{prove: } \mathbb{E} \circ f(F(x_t, t, t+h, w)) | \mathcal{G}_t^{(m)} =$$

$$\mathbb{E} \circ f(F(x, 0, h, w)) |_{x=X_t(w)}$$

$$\text{Set } g(x, w) = f \circ F(x, t, t+h, w)$$

$$\exists \sum \phi_k(w) \varphi_k(x) \nearrow g(x, w).$$

$$\mathbb{E} \circ \sum \phi_k(w) \varphi_k(x_t) | \mathcal{G}_t^{(m)} =$$

$$\sum \varphi_k(x_t) \mathbb{E} \circ \phi_k(w) | \mathcal{G}_t^{(m)} =$$

$$\sum \varphi_k(\eta) \mathbb{E} \circ \phi_k(w) | \mathcal{G}_t^{(m)} |_{\eta=x_t}$$

$$\Rightarrow \mathbb{E} \circ g(x_t, w) | \mathcal{G}_t^{(m)} = \mathbb{E} \circ g(\eta, w) | \mathcal{G}_t^{(m)} |_{\eta=x_t}$$

$$= \mathbb{E} \circ g(\eta, w) |_{\eta=x_t} \text{. by incept!}$$

Finally, apply time-homogeneous of (X_t) .

Rank: Since $M_t \subset \mathcal{F}_t^{(m)}$. So (X_t) is also
 M_t -Markov process.

Thm (Strong Markov)

$f: \mathbb{R}^n \rightarrow \mathbb{R}^n$. bdd. Borel. τ is stopping time

w.r.t $\mathcal{F}_t^{(m)}$. s.t. $\tau < \infty$. a.s. Then:

$$\mathbb{E}^x \circ f \circ X_{\tau+h} \mid \mathcal{F}_\tau^{(m)} = \mathbb{E}^{X_\tau^{(m)}} (f(X_h))$$

Pf: Set $F(x, t, r, w) = X_r^{(w)}$. $r \geq t$. Again,

Argue as in Time-homo. by strong unique:

$F(x, \tau, \tau+h, w)$ indept with $\mathcal{F}_\tau^{(m)}$.

$$\text{prove: } \mathbb{E}^x \circ f \circ F(x, 0, \tau+h, w) \mid \mathcal{F}_\tau^{(m)} = \mathbb{E}^x \circ f \circ F(x, 0, h, w) \mid \mathcal{F}_\tau^{(m)}$$

Similar as before: $\sum p_k(x) \mathbb{P}(x, t, r, w) \neq \mathbb{P}(x, t, r, w)$

$$\therefore f \circ F(x, t, r, w).$$

Cor. (f_k) are bdd. Borel on \mathbb{R} . Then:

$$\mathbb{E}^x \circ \prod_i f_i \circ X_{\tau+h_i} \mid \mathcal{F}_\tau^{(m)} = \mathbb{E}^x \circ \prod_{k=1}^n f_k \circ X_{h_k})$$

Pf: By induction. Directly.

Cor. $\forall n \in \mathbb{N}_0$. bdd. Then:

$$\mathbb{E}^x \circ \theta_{\tau, n} \mid \mathcal{F}_\tau^{(m)} = \mathbb{E}^x \circ n$$

Pf: Approx by cor. above.

Rmk: In inhomogeneous case. If

$$i) |a(x, t) - a(\eta, t)| \vee |b(x, t) - b(\eta, t)| \leq k_1|x - \eta|$$

$$ii) |a(x, t)| \vee |b(x, t)| \leq k_2(|x| + 1)$$

Then (X_t) is strong Markov process.

③ Mean Value Property:

Defn: $H \in B_r \mathbb{R}^n$. $\bar{\tau}_H = \inf \{t > 0 \mid X_t \notin H\}$.

where τ is another stopping time.

$$g \in C_B(\mathbb{R}^n), \eta = g(X_{\bar{\tau}_H}) I_{\{\bar{\tau}_H < \infty\}}$$

$$\text{Lemma. } \theta_\alpha \eta I_{\{\tau < \infty\}} = g(X_{\bar{\tau}_H}) I_{\{\bar{\tau}_H < \infty\}}.$$

Pf: consider $\sum f(X_{t_j}) \chi_{[t_i, t_{i+1}) \cap \{\bar{\tau}_H\}} \rightarrow \eta$.

Cor. For $h \subset H$. $h \in B_r \mathbb{R}^n$. If $\bar{\tau}_H < \infty$. a.s

$$\text{Then } \theta_{\bar{\tau}_H} g(X_{\bar{\tau}_H}) = g(X_{\bar{\tau}_H}).$$

Pf: $Z_H^{\bar{\tau}_H} = Z_H$. since $\bar{\tau}_H < \infty$. a.s.

$$\text{cor. } \mathbb{E}^x [f(X_{\bar{\tau}_H})] = \int_{\partial H} \mathbb{E}^{\eta} [f(X_{\bar{\tau}_H})] \alpha^x(X_{\bar{\tau}_H}) d\eta$$

for $f \in C_B(\mathbb{R}^n)$. $h \subset H$. measurable.

Pf: By cor. above. apply Markov property.

Def: Uniform measure of X on $\partial\mathcal{H}$ is $M_{\mathcal{H}}$.

Defined by $M_{\mathcal{H}}^x(F) = \mathbb{E}^x(X_{2n} \in F), F \subset \partial\mathcal{H}$.

Prop: By above, $\varphi(x) = \mathbb{E}^x(f(X_{2n}))$ satisfies mean value property:

$$\varphi(x) = \int_{\partial\mathcal{H}} \varphi(y) dM_{\mathcal{H}}(y), \quad \forall x \in \mathcal{H} \subset \mathbb{R}^n. \text{ Borel.}$$

(It also holds for all time-homogeneous strong Markov process.)

(2) Generator:

Def: generator of X_t is defined by:

$$Af(x) = \lim_{t \rightarrow 0} \frac{\mathbb{E}^x(f(X_t)) - f(x)}{t} \quad \text{with } D(A) =$$

{ $f: \mathbb{R}^n \rightarrow \mathbb{R}^m | Af \text{ exists } \forall x$ }.

Lemma. $Y_t = Y_t^x(w) = x + \int_0^t u(s,w) ds + \int_0^t v(s,w) dB_s$
is Itô process on \mathbb{R}^m . B_s is m-dim BM.

If $f \in C_c^2(\mathbb{R}^n)$. τ is stopping time w.r.t

$\mathcal{F}_t^{(\tau)}$. s.t. $\mathbb{E}^x(\tau) < \infty$. u, v are bdd s.t. Y

$\in \text{Supp}(f)$. Then:

$$\begin{aligned} \mathbb{E}^x(f(Y_\tau)) &= f(x) + \mathbb{E}^x \int_0^\tau \left(\sum_i u_i \frac{\partial f}{\partial x_i}(Y_s) + \frac{1}{2} \sum_{i,j} \mathbb{E}^x(v_i v_j^\top)_{ij} \right. \\ &\quad \left. \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_s) \right) ds \end{aligned}$$

Pf: Apply Itô's Formula on $f(Y_\tau)$.

And replace z by $zk = z \wedge k$ first.

Then. Let $k \rightarrow \infty$. ζ converges in L^2 .

Thm. For Itô Diffusion $X_t = b(x_t)dt + \sigma(x_t)dB_t$.

If $f \in C_0^\infty(\mathbb{R}^n)$. Then $f \in D(A)$. St.

$$Af(x) = \sum b_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Pf: By Lemma. directly check.

Cor. $A = \frac{1}{2} A$. for n -dim B_m : (\vec{B}_t) .

Thm. (Dynkin's Formula)

If $f \in C_0^\infty(\mathbb{R}^n)$. τ is stopping time. St. $\mathbb{E}^x(\tau) < \infty$.

Then: $\mathbb{E}^x(f(X_\tau)) = f(x) + \mathbb{E}^x \left(\int_0^\tau Af(x_s) ds \right)$

Rmk: If τ is exit time of Borel set.

St. $\mathbb{E}^x(\tau) < \infty$. Then it holds for $f \in C^2$.

e.g. i) For n -dim B_m : (\vec{B}_t) . start at \vec{n} .

τ_k is exist time of $k = \{ |x| < R \}$

Consider $f(x) = |x|^2$. first set $\tilde{\sigma}_k = \tau_k \wedge n$

$$\begin{aligned} \text{So: } \mathbb{E}^{\vec{n}}(f(B_{\tilde{\sigma}_k})) &= f(\vec{n}) + \mathbb{E}^{\vec{n}} \left(\int_0^{\tilde{\sigma}_k} \frac{1}{2} Af(B_s) ds \right) \\ &= |\vec{n}|^2 + n \mathbb{E}^{\vec{n}}(\tilde{\sigma}_k). \end{aligned}$$

Set $n \rightarrow \infty$. LHS $\rightarrow \mathbb{E}^n (f(B_{2^n})) = R^2$

$$\Rightarrow \mathbb{E}^n (\tau_k) = \frac{1}{n} (R^2 - n^2)$$

ii) For n -dim BM (B_t). $n \geq 2$. $|b| > R$. Next.

We will find prob. of B_t start at b .
ever hits $k = \{x \mid |x| \leq R\}$.

Denote: ζ_k is exit time of $A_k = \{R < |x| \leq 2^k R\}$.

$$\tau_k = \inf \{t > 0 \mid B_t \in k\}.$$

$$\text{Set } f(x) = \begin{cases} -\log|x|, & n=2 \\ |x|^{2-n}, & n>2. \end{cases} \quad f_{nk} \in C_c^{\infty}(C^2(R)).$$

So. $f_{nk} = f$ on A_k . So $A f_{nk} = A f = 0$ on A_k .

Denote $p_k = P^b (B_{\tau_k} = R)$. $\varrho_k = P^b (B_{\tau_k} = 2^k R)$

By Dynkin's: $\mathbb{E}^b (f(B_{\tau_k})) = f(b)$. $\forall k$.

(Apply on f_{nk} . Note $B_{\tau_k} \in A_k$).

1) $n=2$:

$$-\log R \cdot p_k - \log 2^k R \varrho_k = -\log |b|.$$

So $\varrho_k \rightarrow 0$. as $k \rightarrow \infty$.

$$\text{i.e. } P^b (\tau_k < \infty) = 1$$

2) $n \geq 3$:

$$p_k \cdot R^{2-n} + \varrho_k (2^k R)^{2-n} = |b|^{2-n}.$$

$$\text{Set } k \rightarrow \infty. \therefore P^b (\tau_k < \infty) = \left(\frac{|b|}{R}\right)^{2-n}$$

\therefore BM is transient $\Leftrightarrow n \geq 3$.

Def: i) Characteristic operator A of Itô diffusion (X_t) is defined by:

$$Af(x) = \lim_{k \rightarrow \infty} \frac{\mathbb{E}^x [f(X_{n_k})] - f(x)}{\mathbb{E}^x [Z_{n_k}]}$$

where $(n_k) \downarrow x$. open set. Z is

exist time. If $\mathbb{E}^x [Z_{n_k}] = \infty \forall k$.

Then define $Af(x) = 0$.

$D_A = \{f \mid Af(x) \text{ exists. } \forall x. \exists (n_k) \downarrow x\}$.

Rmk: We have $D_A \subseteq D_A$ and $Af =$

Af on $\{f \in D_A\}$.

Note that C^2 is dense in L^1 . Next,

We only need to prove it for C^2 .

ii) $x \in \mathbb{R}^n$ is trap for (X_t) if

$$\mathbb{P}^x \{ \exists t \geq 0. X_t = x \} = 1.$$

Lemma: If x is not a trap for (X_t) . Then:

$$\exists u \ni x. \text{open. st. } \mathbb{E}^x [Z_u] < \infty.$$

$$\underline{\text{pf:}} \quad \exists t. \exists \epsilon. \alpha^x (\{X_t = x\}) < 1.$$

\Rightarrow By anti. of X . $\exists u$ nbd of x .

$$\text{s.t. } \alpha^x (\{X_t \in U\}) \stackrel{\Delta}{=} p > 0.$$

discreten Z_n . $\mathbb{E}^x [Z_n]$ will be dominated

by expectation of α^x .

Thm $f \in C^2 \Rightarrow f \in D_A$. and satisfies:

$$Af = \sum b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum (\sigma \sigma^T)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Pf: 1) x is trap for (X_t) .

Then $Af(x) = 0$.

choose V open, bdd. $\forall x \in V$. Modify f to
fn. outside V . s.t. $f \in C_0^\infty(\mathbb{R}^n)$.

Note: $\mathbb{E}^{x_0} f(X_{t+1}) = f(x_0) \Rightarrow Af(x_0) = 0$.

2) X is not a trap for (X_t) .

$\exists n$. nbd of x . s.t. $\mathbb{E}^{x_0} Z_n < \infty$.

WLOG. consider $U_k \subset U$. $Z_k := Z_{U_k}$.

$$\left| \frac{\mathbb{E}^{x_0} f(X_{Z_k}) - f(x_0)}{\mathbb{E}^{x_0} Z_k} - Af(x_0) \right| =$$

$$\left| \mathbb{E}^{x_0} \int_0^{Z_k} Af(X_s) - Af(x_0) ds \right| / \mathbb{E}^{x_0} Z_k$$

$$\leq \sup_{y \in U_k} |Af(y) - Af(x_0)| \xrightarrow{k \rightarrow \infty} 0$$

Since $Af \in C_0^\infty(\mathbb{R}^n)$.

Rmk: So we prove: $D(A) \subset D(A) \cap C^2$. So:

An Itô diffusion is a diffusion in
sense of Dynkin's. Defind by anti.

(X_t) with A generator. If $f \in C_0^\infty$. s.t.

$$\mathbb{E}^{x_0} f(X_{t+s}) = f(x_0) + \mathbb{E}^{x_0} \int_0^s Af(X_u) du$$

(3) Kolmogorov's Backward Equation:

Thm. For $f \in C_c^{\infty}(\mathbb{R}^n)$. Define: $u(t, x) = \mathbb{E}^x f(X_t)$

Then $u(t, \cdot) \in D_A$, $\forall t$, and satisfies:

$$\begin{cases} \frac{d u}{d t} = A u, & t > 0, x \in \mathbb{R}^n \\ u(0, x) = f(x), & x \in \mathbb{R}^n \end{cases}$$

Moreover if $w(t, x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n)$ then satisfies equation above. Then $u = w$. n.s

Pf: $t \mapsto u(t, x)$ is differentiable. Directly check.

For uniqueness:

If $\tilde{A}w = Aw - \frac{d w}{d t} = 0$. $w(0, x) = f(x)$.

Define $Y_t(w) = (s-t, X_t^{s,x})$. fix (s, x) .

which has generator \tilde{A} .

Apply Dynkin's: $\mathbb{E}^{s,x} (w(Y_t)) = w(s, x)$. $\forall t \geq 0$

Rmk: Define Semigroup $\mathcal{Q}_t: f \mapsto \mathbb{E}^x (f(X_t))$

Then resolution R_α is: $R_\alpha g(x) =$

$$\int_0^\infty e^{-\alpha t} \mathcal{Q}_t g(x) dt \stackrel{\text{Dynkin}}{=} \mathbb{E}^x \left[\int_0^\infty e^{-\alpha t} g(X_t) dt \right]$$

for $\alpha > 0$. $g \in C_b$. (since $\|R_\alpha g\| \leq 1$).

Lemma: $f \geq 0$. measurable. on \mathbb{R}^n . Def: $u(x) = \mathcal{Q}_t f(x)$.

i) f is l.s.c $\Rightarrow u$ is l.s.c

ii) $f \in C_0 \Rightarrow u$ is conti.

Pf: By Pf of Uniqueness part of E and u Thm:

$$\mathbb{E} (|X_t^x - X_t^y|^2) \leq C(t) |x - y|^2. \quad (\text{From Brownian})$$

S₁: We can find say $(X_n) \rightarrow x$.

$$\text{S.t. } X_t^{x_n} \rightarrow X_t^x \text{ a.s.}$$

$$\text{i) } u(x) = \mathbb{E}(g(X_t^x)) \stackrel{\text{l.s.c}}{\leq} \mathbb{E}(\liminf_n g(X_t^{x_n}))$$

$$\stackrel{\text{Fatou's}}{\leq} \liminf_n \mathbb{E}(g(X_t^{x_n})) = \liminf_n u(x_n)$$

ii) Apply i) on g and $-g$.

(4) Feynman - Kac Formula:

Thm. $f \in C_c^2(\mathbb{R}^n)$, $z \in C(\mathbb{R}^n)$, $z \geq 0$. A is generator of (X_t) .

If $v(t, x) = \mathbb{E}_x^x e^{-\int_0^t z(X_s) ds} f(x_t)$ Then:

$$\begin{cases} \frac{\partial v}{\partial t} = Av - zu & t > 0, x \in \mathbb{R}^n \\ v(0, x) = f(x) & x \in \mathbb{R}^n. \end{cases}$$

Moreover, if $w(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ and w satisfies the equation above. then $w = v$.

Pf: Set $Y_t = f(x_t)$, $Z_t = e^{-\int_0^t z(X_s) ds}$

$$\lambda Z_t = -Z_t \lambda f(x_t) \lambda t, \quad \lambda Y_t = \dots (Ito).$$

$$\lambda(Y_t Z_t) = Y_t \lambda Z_t + Z_t \lambda Y_t. \quad (\lambda Z_t \lambda Y_t = 0)$$

$\Rightarrow Y_t Z_t$ is Ito process.

5) $V(t, x) = \mathbb{E}^x [Y_t | Z_t]$ is differentiable at t .

$$\frac{1}{r} (V(t+r, x_r) - V(t, x))$$

$$= \frac{1}{r} (\mathbb{E}^x [\mathbb{E}^{x_r} [Z_t f(x_{t+r})]] - \mathbb{E}^x [Z_t f(x_t)])$$

$$\text{Markov} \quad = \frac{1}{r} (\mathbb{E}^x [\mathbb{E}^x [f(x_{t+r}) e^{-\int_0^t Z_s ds}]] - \mathbb{E}^x [Z_t f(x_t)])$$

$$= \frac{1}{r} \mathbb{E}^x [Z_{t+r} f(x_{t+r}) e^{\int_0^t Z_s ds} - Z_t f(x_t)]$$

$$= \frac{1}{r} \mathbb{E}^x [f(x_{t+r}) Z_{t+r} - f(x_t) Z_t] + \frac{1}{r} \mathbb{E}^x \dots$$

$$\rightarrow \frac{\partial V}{\partial t} + Z_t(x) V(t, x).$$

For uniqueness:

$$\text{Set } H_t = (s-t, X_t^{s,x}, Z_t), \quad Z_t = Z + \int_0^t Z_s ds.$$

$$\text{fix } (s, x, z). \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$$

$$\text{If } w \text{ satisfies: } \hat{A}w = -\frac{\partial w}{\partial t} + Aw - Zw = 0$$

$$\text{Note } H \text{ has generator } A_H = -\frac{\partial}{\partial s} + A + Zw \frac{\partial}{\partial z}$$

Argue as before:

$$\text{Set } \phi(s, x, z) = w(s, x) e^{-z}. \quad A \phi = 0.$$

Apply Dynkin's on $\phi(H_{t+2R})$. Set $R \rightarrow \infty$.

Remark: $\langle \cdot \rangle$ kill \sim Diffusion

Note for $dX_t = b(X_t)dt + \sigma(X_t)dB_t$.

(X_t) has generator A :

$$Af(x) = \sum b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum (\sigma \sigma^T)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}$$

It's natural to ask:

If we can find processes whose generator

$$\text{has form: } \tilde{A}f = \sum b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} - cf.$$

where $c(x) \in C_b(\mathbb{R}^n)$.

Def: A killing time kills X_t is a random time ς .

S.t. if $s < \varsigma$, $\tilde{X}_t = X_t$ for $t < \varsigma$. $\tilde{X}_t = \emptyset \notin \mathbb{R}^n$.

a coffin state. $\Rightarrow (\tilde{X}_t)$ is strong Markov process

$$\text{and } \mathbb{E}^{x_0} f(\tilde{X}_{\varsigma}) = \mathbb{E}^{x_0} f(X_s) \chi_{\{\varsigma > s\}} = \mathbb{E}^{x_0} f(X_s) e^{-\int_s^{\varsigma} c(X_u) du},$$

for $\forall f \in C_b(\mathbb{R}^n)$ for some $C(s) \geq 0$

Rmk: i) If $c \geq 0$. Then killing time ς associated with $c(x)$. always exists.

S_1 : (\tilde{X}_t) is the process with generator

$$\tilde{A} = A - c(x). \text{ by FK formula.}$$

ii) $c(x)$ is interpreted as killing rate:

$$c(x) = \lim_{t \downarrow 0} \frac{1}{t} Q^{x_0} X_0 \text{ is killed in time interval } (0, t] \text{ ("killed" = "in coffin")}$$

iii) If $c \geq 0$. Then ς can be constructed explicitly.

(5) Mart. Problem:

① Thm. If X_0 is Itô diffusion with generator A .

Then $\forall f \in C_c^2(\mathbb{R}^n)$. $M_t = f(X_t) - \int_0^t Af(X_s) ds$ is a mart. w.r.t. (\mathcal{M}_t) .

Pf. If $\lambda X_t = b(X_t) \lambda t + r(X_t) \lambda B_t$.

Then : $M_t = f(x_0) + \int_0^t \nabla f^T(x_r) \sigma(x_r) dB_r$.

by Itô formula. (The integral is mart.)

Def. L is semi-elliptic differential operator of

$$\text{form} : L = \sum b_i \frac{\partial}{\partial x_i} + \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \quad b_i, a_{ij}$$

are locally bdd Borel. on \mathbb{R}^n .

\tilde{P}^x on $C(\mathbb{R}^n)^{B_{\mathbb{R}^n}}$, B) solves mart problem

for L (stnrnt nt x) if process

$$\begin{cases} M_t = f(X_t^x) - \int_0^t L f(X_r^x) dr & \text{is } \tilde{P}^x\text{-mart} \\ \mu_r = f(x_0) \end{cases}$$

w.r.t $B_{\mathbb{R}^n}^{C_0}$ for $f \in C_0(\mathbb{R}^n)$.

where $X_0^{(0,0)} = w_0$. We n. convenient process.

The mart problem is well-posed if \tilde{P}^x is unique p.m. to solve it.

Thm. If $\tilde{\alpha}^x$ is p.m. on $\text{cn. m.} = C(\mathbb{R}^n)^{B_{\mathbb{R}^n}}, B$)

induced by the law α^x of Itô diffusion

$(X_t)_{t \geq 0}$ with generator A . Then $\tilde{\alpha}^x$ solves mart. problem for operator A .

Rmk. It also holds for $(X_t)_t$ is even the

weak solution of $\lambda X_t = \sigma(X_t) \lambda B_t + b(X_t) \lambda t$

Cor. If \tilde{P}^x solves the mart. problem for

$$L = \sum b_i \frac{\partial}{\partial x_i} + \sum \frac{1}{2} (\sigma \sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \text{ Then:}$$

there exists a weak solution (X_t) of

$$\lambda X_t = b(X_t) \lambda t + \sigma(X_t) \lambda B_t.$$

Moreover, the mart. problem for L is

well-posed $\Leftrightarrow (X_t)_{t \geq 0}$ is Markov process.

Rmk: If b, σ satisfies Lipschitz condition

then, \tilde{P}^x induced by law of $X_t = \alpha t$
is the unique solution for mart. problem.

(But it's not necessary condition.)

Thm. (Stroock - Varadhan)

$L = \sum b_i \frac{\partial}{\partial x_i} + \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ has unique solution

for mart. problem if (a_{ij}) is positive definite, $a_{ij}(x)$ is conti. b(x) measurable. and $\exists D$.

$$\text{s.t. } |b(x)| + \|a(x)\|^{\frac{1}{2}} \leq D(1+|x|).$$

② Note that $(X_t)_{t \geq 0}$ is Itô process $\Rightarrow (\phi(X_t))_{t \geq 0}$ is Itô process as well. for $\phi \in C^2_c(\mathbb{R}^n)$

Next. we will find conditions. i.e. (X_t) is Itô diffusion.

$\Rightarrow \phi(X_t)$ is Itô diffusion as well.

$$\text{Thm. } \lambda X_t = b(X_t)X_t + \sigma(X_t) \lambda B_t, \quad b \in \mathbb{R}^n, \sigma \in \mathbb{R}^{n \times n}, X_0 = x.$$

$$\lambda Y_t = u(t, w) \lambda t + v(t, w) \lambda B_t, \quad u \in \mathbb{R}^n, v \in \mathbb{R}^{n \times n}, Y_0 = x.$$

$$\text{Then } (X_t) \xrightarrow{\lambda} (Y_t). \quad (\Rightarrow) \quad VV^T(c_{t,w}) = \sigma \sigma^T(c_{Y_t}).$$

$$\text{and } \mathbb{E}^x(u(t, \cdot) | N_t) = b(Y_t). \quad \text{at } \times \lambda P \text{ a.s.}$$

$$\text{where } N_t = \sigma(Y_s, s \leq t).$$

Pf: (\Leftarrow) Suppose $A = \sum b_i \frac{\partial}{\partial x_i} + \frac{1}{2} \mathcal{I}(\sigma \sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$
is generator of (X_t) .

$$\text{Def: } H f_{(t,w)} = \sum u_{i(t,w)} \frac{\partial f}{\partial x_i}(Y_t) + \frac{1}{2} \sum (VV^T)_{ij}(t,w) \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_t) \text{ for } f \in C_c^2.$$

$$N_{t+s} = \mathbb{E}^x(f(Y_s) | N_t) = (I + \hat{A}) \text{ Formulas}$$

$$f(Y_t) + \mathbb{E}^x \int_t^s H f_{(r,w)} dr | N_t$$

$$= f(Y_t) + \mathbb{E}^x \int_t^s \mathbb{E}^x H f_{(r,w)} | N_r | N_t$$

$$\text{and. } = f(Y_t) + \mathbb{E}^x \int_t^s A f(Y_r) dr | N_t$$

Where \mathbb{E}^x is expectation under law of (Y_t)

$$\Rightarrow \mu_t = f(Y_t) - \int_0^t A f(Y_s) ds \text{ is mart.}$$

w.r.t. R^x . law of $(Y_t)_{t \geq 0}$

Since (X_t) is Markov \Rightarrow by uniqueness:

$$(X_t) \xrightarrow{\lambda} (Y_t). \quad i.e. \quad R^x = \alpha^x.$$

$$(\Rightarrow) \quad \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E}^x [f(Y_{t+h}) | N_t] - f(Y_t) =$$

$$\sum \mathbb{E}^x(u_{i(t,w)} | N_t) \frac{\partial f}{\partial x_i}(Y_t) + \frac{1}{2} \mathbb{E}^x \sum (VV^T)_{ij} | N_t | \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_t)$$

Besides, since $X_t \sim Y_t$, so Y_t is Markov.

$$\begin{aligned}
 LHS &= \lim_{h \downarrow 0} \frac{1}{h} (\mathbb{E}^{Y_t} [f(Y_{t+h})] - \mathbb{E}^{Y_t} [f(Y_t)]) \\
 &= \sum \mathbb{E}^{Y_t} [u_i(t, w_j) \frac{\partial f}{\partial x_i}(Y_t)] + \frac{1}{2} \mathbb{E}^{Y_t} [\langle VV^T(t, w_j) \rangle] \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_t) \\
 \Rightarrow & \left\{ \begin{array}{l} \mathbb{E}^X [u_i(t, w_j) | N_t] = \mathbb{E}^{Y_t} [u_i(t, w_j)] = b(Y_t) \\ \mathbb{E}^X [VV^T(t, w_j) | N_t] = \mathbb{E}^{Y_t} [VV^T(t, w_j)] = \sigma^2(Y_t) \end{array} \right.
 \end{aligned}$$

The conclusion follows from the next Lemma:

Lemma: There exists an N_t -adapted process $W(t, w)$ s.t. $VV^T(t, w) = W(t, w)$ a.s.

Pf: It's directly from Ito^* formula:

$$\begin{aligned}
 Y_i Y_j(t, w) &= x_i x_j + \int_0^t Y_i \lambda Y_j + \int_0^t Y_j \lambda Y_i + \\
 &\quad \int_0^t \langle VV^T \rangle_{ij} ds, \text{ a.s.}
 \end{aligned}$$

Rmk: $\lambda(t, \cdot)$ and $\sigma(t, \cdot)$ may not be N_t -adapted.

Cor. (Recognize a Bm)

A Ito^* process $\lambda Y_t = u(t, w) \lambda t + v(t, w) \lambda \beta t$

is Bm $\Leftrightarrow \mathbb{E}^X [u(t, \cdot) | N_t] = 0, \langle VV^T(t, w) \rangle = I_n$.

Thm. $\phi(X_t)$ image of Ito^* diffusion X_t by C^2 func.

\tilde{Z}_t (Ito^* diffusion) $\Leftrightarrow A(f, \phi) = \hat{A}(f) \circ \phi$

$A f = \sum a_i x_i + \sum c_{ij} x_i x_j$ (2^n -order poly's), where

A. \hat{A} are generators of X_t, Z_t .

(b) Random Time Change:

Def: i) For $c(t, w) \geq 0$. \mathcal{F}_t -adapted.

$\beta_t(w) = \int_0^t c(s, w) ds$. is said to be
a random time change with time
change rate $c(t, w)$.

ii) Right-inverse of β_t is $\alpha_t = \inf \{ s |$
 $\beta_s > t \}$.

Rmk: i) α_t is called right-inverse is from
 $\beta_t \nearrow \infty$. increasing. So $c(t, w)$ is
right-conti. Besides, $\beta(\alpha_t(w), w) = t$.

ii) If $c(t, w) > 0$. Then $\beta_t \nearrow \infty$ strictly.
 $t \mapsto \alpha_t(w)$ is anti. So it's also
left-inverse of β_t .

iii) α_t is a \mathcal{F}_t -stopping time:

$$\{\tau_t < s\} = \{t < \beta_s\} \in \mathcal{F}_s.$$

Question: For X_t , I_t^α diffusion. Y_t I_t^α process.

$$\begin{cases} dX_t = \sigma(X_t) dB_t + b(X_t) dt, & X_0 = x, \\ dY_t = u(t, w) dt + v(t, w) dB_t, & Y_0 = x. \end{cases}$$

When does there $\exists \beta_t$. st. $Y_{\beta_t} \sim X_t$?

Thm. If $\exists c(t, w) \geq 0$. \mathbb{F}_t -adapted. s.t. $dt \times dP$. a.s.

$$\mu(t, w) = c(t, w) b(Y_t). \quad VV^T(t, w) = c(t, w) \cdot \sigma \sigma^T(t, Y_t).$$

Then: $Y_{at} \sim X_t$.

Cor. For $c(t, w) \geq 1$. $Y_t = \int_0^t \overline{\log(w)} dB_s$.

where B_s is n-lim BM. Then:

Y_{at} is also n-lim BM.

Rmk: i) If $c(t, w) > 0$. Then $Y_t = \widehat{B}_{\beta t}$. where

\widehat{B}_t is n-lim BM (since βt is inverse)

ii) $c(t, w) = 1$. Then it's special case before.

Next. We want to prove: time change of a Itô integral is again a Itô integral is again a Itô integral. Ariven by n different BM. \widetilde{B}_t :

Lemmas. $s \mapsto \alpha(s, w)$ is conti. $\alpha(s_0, w) = 0$. Fix $t > 0$. s.t.

$$\beta t < \infty. \quad \mathbb{E}(Y_t) < \infty.$$

$$\text{Set } t_j = \begin{cases} j/2k & \text{if } j/2k \leq t \\ \alpha t & \text{if } j/2k > t \end{cases}$$

Chon r_j . s.t. $T_{r_j} = t_j$. If $f(s, w) \geq 0$. \mathbb{F}_s -adapted. bdd. s-conti. Then:

$$\lim_{k \rightarrow \infty} \mathbb{I} f(T_{r_j}, w) \Delta B_j = \int_0^{at} f(s, w) \Delta B_s. \text{ a.s.}$$

$$\text{where } \Delta B_j = B_{ar_j} - B_{Tr_j}.$$

Thm. c Time change Formula for Itô integral

$c(s, w), \tau(s, w)$ are S -cont. $\tau(0, w) = 0$. $E(\tau_t) < \infty$.

$c(s, w) > 0$. For B_s m -lim BM and $v(s, w) \in V_H^{nm}$ bdd. S -cont. Def:

$$\tilde{B}_t = \int_0^{\alpha t} \sqrt{c(s, w)} dB_s \text{ as in Lemma above.}$$

Then: \tilde{B}_t is $\mathcal{F}_{\tau_t}^{nm} - BM$ and $\int_0^{\tau_t} v(s, w) dB_s$
 $= \int_0^t v(\tau_r, w) \sqrt{c(\tau_r, w)} d\tilde{B}_r$. P-a.s.

Rmk: $N_{t,x} = \tau_i(w) = (\beta_i^{-1})' = 1/c(\tau_i, w)$.

$$\underline{Pf}: \int_0^{\tau_t} v(s, w) dB_s \stackrel{(Def)}{=} \lim_{k \rightarrow \infty} \sum v(\tau_j, w) A B_{\tau_j}.$$

$$= \lim_{k \rightarrow \infty} \sum v(\tau_j, w) \sqrt{1/c(\tau_j, w)} \Delta \tilde{B}_j$$

$$= \int_0^t v(\tau_r, w) \sqrt{1/c(\tau_r, w)} d\tilde{B}_r$$

Ex. 1. (Brownian Motion on unit sphere)

i) $n=2$:

Consider $g(t, x) = e^{ix} = (\cos x, \sin x)$.

Set $Y(t) = g(t, B_t) = (\cos B_t, \sin B_t)$

where B_t is 1-lim BM.

$Y(t) = (Y_1(t), Y_2(t))$ called BM on unit circle.

satisfies
$$\begin{cases} \lambda Y_1 = -\sin(B_t) \lambda B_t - \frac{1}{2} \cos(B_t) \lambda t \\ \lambda Y_2 = \cos(B_t) \lambda B_t - \frac{1}{2} \sin(B_t) \lambda t \end{cases}$$

i.e. $\begin{cases} \lambda Y_1 = -Y_2 \lambda B_2 - \frac{1}{2} Y_1 \lambda t \\ \lambda Y_2 = Y_1 \lambda B_1 + \frac{1}{2} Y_2 \lambda t. \end{cases}$

ii) $n=3$:

Consider $\phi(x) = x/|x|$, $x \in \mathbb{R}^n \setminus \{0\}$.

Set $Y_t = (Y_{1(t)}, \dots, Y_{n(t)}) = \phi(\vec{B}_t)$. \vec{B}_t is n -dim BM.

By Itô: $\lambda Y = \frac{1}{|B|} \cdot \sigma(Y) \lambda B + \frac{1}{|B|^2} b(Y) \lambda t$.

where $\sigma_{ij}(Y) = \delta_{ij} - Y_i Y_j$, $b(Y) = -\frac{n-1}{2} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$

Note $(\cos, w) = \sqrt{|B|} \cos |w|$. So: $\beta(t, w) = \int_0^t \sqrt{|B_s|^2} ds$.

Set $Z_t(w) = Y_{\alpha(z,w)}(w)$.

By Thm above. $\lambda Z_t = \sigma(Z) \lambda \vec{B} + b(Z) \lambda t$. It's diffusion

We call Z is BM on unit sphere S^n . Since it's invariant under orthogonal transformation.

Thm.

$\phi: \mathbb{R}^2 \rightarrow \mathbb{C}$. $B = (B_1, B_2)$ 2-dim BM. If $\Delta \phi = 0$.

$|\Delta u|^2 = |\Delta v|^2$, $\nabla u \cdot \nabla v = 0$. Then $\phi(B)$ is 2-dim times change BM. in the plane.

Pf: Set $Y = \phi(B) = (\phi(B_1, B_2), \psi(B_1, B_2))$

$$\lambda Y = \frac{1}{2} \left(\begin{matrix} \Delta u \\ \Delta v \end{matrix} \right) \lambda t + \left(\begin{matrix} u_1' & u_2' \\ v_1' & v_2' \end{matrix} \right) \lambda B$$

So: Y is a mart ($\Rightarrow \Delta \phi = 0$).

By Gr. above: $\phi(B_1, B_2) = (\tilde{B}_{p_1}^{(u)}, \tilde{B}_{p_2}^{(u)})$

where $\tilde{B}^{(u)}$, $\tilde{B}^{(v)}$ are two 1-dim BM.

$$\text{and } \beta_1(t, w) = \int_0^t |\nabla w|^2 (B_1, B_2) ds$$

$$\beta_2(t, w) = \int_0^t |\nabla v|^2 (B_1, B_2) ds$$

If $|\nabla u|^2 = |\nabla v|^2$, $\nabla u \cdot \nabla v = 0$. Then:

$$\sigma\sigma^T = |\nabla w|^2 (B_1, B_2) I \Rightarrow Y = \vec{B}_{\beta+}.$$

Rmk: Lévy have proved:

$$\begin{aligned}\phi(B_1, B_2) \text{ is time change 2-lim} \\ \beta_m \Leftrightarrow \Delta \phi = 0.\end{aligned}$$

(7) Girsanov Thm for Diffusion:

I^t claims: i) If we change drift coefficient of a given I^t process. Then: the law won't change dramatically
ii) Moreover, the law of new process will be absolutely conti. w.r.t the original process.

Lemma (Bayes' Rule)

For p.m. m, v on (Ω, \mathcal{G}) . st. $\mu(m)$ = f.o.w.s $\lambda(m)$ for some $f \in L'$. If X is r.v. st. $\mathbb{E}_v c(X) = \int |X| f o w s \lambda m < \infty$.

Then $\mathbb{E}_v c(X|N) \mathbb{E}_m c(f|N) = \mathbb{E}_m c(f|N)$ for $\forall N \subset \mathcal{G}$. sub σ -algebra.

Pf: i) $\forall n \in \mathbb{N} \quad \int_{\Omega} \mathbb{E}_n \circ \chi_{\{N\}} f d\lambda = \int_{\Omega} \mathbb{E}_m \circ f \chi_{\{N\}}$

ii) On the other hand:

$$\begin{aligned} LHS &= \mathbb{E}_m \circ \mathbb{E}_n \circ \chi_{\{N\}} f \cdot \chi_n \\ &= \mathbb{E}_m \circ \mathbb{E}_m \circ \mathbb{E}_n \circ \chi_{\{N\}} f \chi_{\{N\}} \chi_n \\ &= \mathbb{E}_m \circ \mathbb{E}_n \circ \chi_{\{N\}} \mathbb{E}_m \circ f \chi_{\{N\}} \chi_n \end{aligned}$$

Thm. (Hirsanoir Thm. Version I)

$Y_t(\omega)$ is Itô process : $dY_t(\omega) = \alpha(t, \omega) dt + \beta B_t$.

and $Y_0 = 0$. for T fixed $\leq \infty$. B_t is n -dim BM.

If $M_t = e^{-\int_0^t \alpha(s, \omega) dB_s - \frac{1}{2} \int_0^t \alpha^2(s, \omega) ds}$ is P -mart.

w.r.t. $(\mathcal{F}_t^{(n)})_{t \leq T}$ Def: $\mu(\omega) = M_T(\omega) \rho(\omega)$.

Then: α is p.m. on $\mathcal{F}_T^{(n)}$. st. Y_t is n -dim BM. w.r.t. α . $0 \leq t \leq T$.

Rmk: i) $P \rightarrow \alpha$ is called Hirsanoir trans. of measure.

ii) Reznikov condition: $\mathbb{E}_P \circ e^{\frac{1}{2} \int_0^T \alpha^2(s) ds} < \infty$ guarantees $(M_t)_{t \leq T}$ is P -mart.

iii) M_t is P -mart $\Rightarrow M_T \lambda_P = \mu \in \lambda_P$ on $\mathcal{F}_T^{(n)}$. $0 \leq t \leq T$.

Pf: Check $\mathbb{E}_P \circ f M_T = \mathbb{E}_P \circ f M_t$.

$\forall f \in \mathcal{F}_t$. b.s.

Pf: i) $\alpha(\omega) = \mathbb{E}_\alpha(\mathbf{1}) = \mathbb{E}_p(M_T) = \mathbb{E}_p(M_\infty) = 1$.

$\int_0^\cdot \alpha \text{ is a p.m.}$

2) Next, check living charac. on \tilde{Y}^t :

First, prove $Y_i(t)$ is \mathcal{Q} -mart. ~~W15i~~.

$$\text{Note set } R_t = - \int_0^t \alpha(s, \omega) dB_s - \frac{1}{2} \int_0^t \alpha^2 ds$$

$$\begin{aligned} \alpha M_t &= \alpha e^{R_t} = e^{R_t} \alpha R_t + \frac{e^{R_t}}{2} (\alpha R_t)^2 \\ &= M_t \sum_i (-\alpha_i(t) \beta_i(t)). \end{aligned}$$

$$\Rightarrow \alpha(Y_i(t) M_t) = M_t Y_i^{(i)}(t) \alpha \beta_i. \text{ P-mart.}$$

$$Y_i^{(i)} = \begin{cases} -Y_i(t) \alpha_j(t), & j \neq i \\ 1 - Y_i(t) \alpha_i(t), & j = i \end{cases}$$

By Bayes's Lemma:

$$\mathbb{E}_\alpha(Y_i(t) | \mathcal{F}_s) = \frac{\mathbb{E}_p(M_t Y_i | \mathcal{F}_s)}{\mathbb{E}_p(M_t | \mathcal{F}_s)} = Y_i(s)$$

Analogously, $(Y_i(t) Y_j(t) - \delta_{ij} t) M_t$ is \mathcal{P} -mart.

Rmk: i) Note $M_T > 0$. a.s. So $\alpha \sim \mathcal{P}$.

ii) Note we shift (B_t) by $\alpha(s, \omega)$. Then:

It's still B_M under \mathcal{Q} if same
mart. condition holds.

Thm. (Girsanov Thm. Version II).

$Y_t \in \mathbb{R}^n$. $dY_t = \beta(t, w) dt + \sigma(t, w) dB_t$. $t \leq T$.

Ito process. $\beta \in \mathbb{R}^n$. $\sigma \in \mathbb{R}^{n \times n}$. B_t is m-lim B_m .

If $\exists u(t, w) \in W_n^m$. $\gamma(t, w) \in W_n^m$. st.

$$\sigma(t, w) u(t, w) = \beta(t, w) - \gamma(t, w).$$

and $M_t(w) = \exp(-\int_0^t u(s, w) dB_s - \frac{1}{2} \int_0^t u^2(s, w) ds)$
 $t \leq T$. is a p-mart.

Then for $\lambda \alpha = M_T \lambda P$ on $\mathcal{F}_T^{(m)}$. α is p.m on
 $\mathcal{F}_T^{(m)}$. St. $\hat{B}^{(t)} = \int_0^t u(s, w) ds + B_t$. $t \leq T$ is
 B_m under α . And $dY_t = \gamma(t, w) dt + \sigma(t, w) \hat{B}^{(t)}$.

Pf: 1') $\hat{B}^{(t)}$ is B_m under α follows directly
from Version I.

2') The representation is easy to check.

Thm. (Girsanov Thm. Version III. for diffusion)

For X_t . $Y_t \in \mathbb{R}^n$. Ito diffusion and process.

St. $\begin{cases} dX_t = b(X_t) dt + \sigma(X_t) dB_t \\ dY_t = (Y_t, w) + b(Y_t) dt + \sigma(Y_t) dB_t. \end{cases}$

b, σ satisfies Lipschitz and. $Y \in W_n^m$.

If $\exists u(t, w) \in W_n^m$. st. $\sigma(Y_t) u(t, w) = Y_t, w$.

Suppose M_t . α . $\hat{B}^{(t)}$ as defined above.

and μ_t is P -mart. w.r.t. \mathcal{F}_t^m . Then:

α is p.m. on \mathcal{G}_T^m and $\lambda Y_t = b(Y_t) \lambda t + \sigma(Y_t) \lambda \hat{B}_t$, $t \leq T$.

So: α -law of Y_t^x = P -law of X_t^x for $t \leq T$.

Pf: It follows from weak uniqueness.

Rmk: It can be used to produce weak solution of SDE:

Suppose Y_t is (weak or strong) solution of $\lambda Y_t = b(Y_t) \lambda t + \sigma(Y_t) \lambda B_t$.

We wish to find weak solution for related SDE: $\lambda X_t = \alpha(X_t) \lambda t + \sigma(X_t) \lambda B_t$.

If $\exists \pi_0: \mathbb{R}^n \rightarrow \mathbb{R}^m$, s.t. $\sigma(\eta) \pi_0(\eta) = b(\eta) - \alpha(\eta)$,

and satisfies Novikov's condition.

Then: $\exists (\hat{B}_t, \alpha)$, s.t. $c(Y_t, \hat{B}_t)$ is its weak solution under α .