Finance in Discrete Time.

(1) **Settings:**

$(\Omega, \mathcal{F}, \mathbb{P})$ prob. space with $|\Omega| < \infty$. and $P(\{\omega\}) > 0$ for $\forall \omega \in \Omega$. N is expire time. Fix filtration $(\mathcal{F}_n)_0^N$. st. $\mathcal{F}_0 = \{\phi, \Omega\}$. $\mathcal{F}_N = P(\Omega)$. $\mathcal{F}_k \subset \mathcal{F}_{k+1}$. $\forall k$.

Def: i) The financial market contains $d+1$ assets:

S^0 : riskless one. $S^1 \sim S^d$: risky assets.

Set S_n^i is price of n^{th} asset at $t=n$.

Rmk: Denote r is bank interest rate

$S_0 : (1+r)^n$ is discount factor at $t=n$.

Next. we set $S_0^0 = 1$

ii) The trading strategy H is vector stoc. process $(H_n)_{n=0}^N := ((H_n^0 \cdots H_n^d))_0^N$. st. $\forall i, n$. $H_n^i \in \mathcal{F}_{n-1}$. (predictable.)

iii) The value of portfolio H is $V_n(H)$ $:= H_n \cdot S_n = \sum_{i=0}^d H_n^i S_n^i$.

Set $\widetilde{V}_n(H) := \beta_n (H_n \cdot S_n) = H_n \cdot \widetilde{S}_n$. the discounted value. where $\beta_n = 1/S_n^0$. and $\widetilde{S}_n := (1, \beta_n S_n^1, \cdots \beta_n S_n^d)$.

Rmk: We normalize the bank holding as an unit to obtain the "true prize" (value) $\tilde{S}_n$.

iv) H is self-financing (SF). if $H_n \cdot S_n = H_{n+1} \cdot S_n$.

Rmk: It means the total asset we hold doesn't change — There's no other money in or out.

And Note that $V_{n+1}(H) - V_n(H) = H_{n+1} \cdot (S_{n+1} - S_n)$. Change of V only from the change of S. the stock price.

$\Rightarrow$ $V_n(H) = V_0(H) + \sum_1^n H_j \cdot \Delta S_j$.

$V_n(H)$ is mart. transf. of S by H.

prop. For H is predictable. $V_0 \in \mathcal{F}_0$. There's unique pred. process $(H_n^0)_{n=0}^N$. st. H is SF with V_0.

Pf: $\tilde{V}_n(H) = H_n^0 + \sum_{i=1}^{d} H_n^i \tilde{S}_n^i \overset{SF}{=} V_0 + \sum_1^n H_j \cdot \Delta \tilde{S}_j$

$\Rightarrow$ Solve H_n^0. which is $\mathcal{F}_{n-1}$-measurable.

Ref: Numéraire is the unit we reckon the value in. always positive.

prop. SF strategies is invariant under change of numéraire

(2) Arbitrage:

<u>Def</u>: i) A strategy H is admissible if it's SF and $V_n(H) \geq 0$. $\forall n \leq N$.

ii) It's arbitrage if $V_0(H) = 0$. $\mathbb{P}(V_N(H) > 0) > 0$.

iii) A market is viable if it's NA.

<u>Thm.</u> (NA $\Leftrightarrow$ EMMs exist)

The market is viable $\Leftrightarrow$ $\exists$ p.m. $\mathbb{P}^*$ on Ω. s.t. $\mathbb{P}^* \sim \mathbb{P}$. and $\tilde{S}_n$ is $\mathbb{P}^*$-mart. (EMM).

<u>Pf</u>: ($\Leftarrow$). $\tilde{V}_n(H)$ is also $\mathbb{P}^*$-mart.

follows from mart. transf.

So: $\mathbb{E}^*(\tilde{V}_N(H)) = \tilde{V}_0(H)$.

Then by contradiction!

($\Rightarrow$). Set $\Gamma = \text{Cone}(\{X > 0. \text{ r.v.}\})$.

i.e. $\forall X. Y > 0$. r.v.'s. $\forall c > 0$. const.

$X + Y \in \Gamma$. $cX \in \Gamma$.

So: $V_0(H) = 0 \underset{\text{ad.}}{\overset{H \text{ is}}{\Rightarrow}} \tilde{V}_N(H) \notin \Gamma$. by NA

Set $\tilde{G}_n(H) := \sum_1^{n} (H_j^1 \Delta \tilde{S}_j^1 + \cdots H_j^d \Delta \tilde{S}_j^d)$

Choose H^0. st. (H) is SF.

So: $\tilde{G}_N(H) = 0$ by NA. $\underset{\text{is pd.}}{\overset{\text{if } H}{\Rightarrow}}$ $\tilde{G}_N(H) \notin \Gamma$.

Set: $V := \{ \tilde{G}_N(H) \mid (H^1, \dots H^d) \text{ is pred.} \}$.

$K := \{ X \in \Gamma \mid \sum_{w \in \Omega} X(w) = 1 \}$.

By Hahn-Banach to separate V, K:

$\exists\ \lambda = (\lambda(w))_{w \in \Omega}$ s.t. $\begin{cases} \sum \lambda(w) X(w) > 0. & \forall X \in K. \\ \sum \lambda(w) \tilde{G}_N(H)(w) = 0. & \forall H \dots \end{cases}$

choose $X(\tilde{w}) = 1$. $\Rightarrow$ $\lambda(\tilde{w}) > 0$. $\forall \tilde{w} \in \Omega$.

So: set $\mathbb{P}^*(\{w\}) = \lambda(w) / \sum \lambda(w) \sim \mathbb{P}$.

$\Rightarrow \mathbb{E}(\sum_1^{\tilde{N}} H_j \cdot \Delta \tilde{S}_j) = 0$. $\underset{H}{\overset{\text{choose}}{\Rightarrow}}$ $\mathbb{E}(\sum_{j=1}^{n} H_j^i \Delta \tilde{S}_j^i) = 0$. $n \le N$

By mart. transf. $(H_j^i)^{-1} \cdot \Delta(\sum_1^n H_j^i \Delta \tilde{S}_j^i) = \tilde{S}_n^i$

is also mart. under $\mathbb{P}^*$.

(3) <u>Completeness:</u>

<u>Def:</u> i) A contingent claim is a payoff func. $h \geq 0$. and $h \in \mathcal{F}_N$.

ii) h is attainable if $\exists\ (H)$ admissible portfolio replicates h at $t = N$.

iii) A market is complete if $\forall$ contingent claim is attainable.

<u>Lemma</u> In viable market. If attainable h can be replicated by SF. (H). Then H is also admissible.

Rmk: It means $\forall$ attainable h'' can be extended to SF strategies.

Pf: $\mathbb{P}^*$ is EMM on N. $\Rightarrow \widetilde{V}_n(H) = \mathbb{E}_*(\widetilde{V}_N(H) \mid \mathcal{F}_n)$.

where H replicates h at $t=N$.

So: $\widetilde{V}_n(H) = \mathbb{E}_*(h/S_N^0 \mid \mathcal{F}_n) \geq 0$. $\forall n \leq N$.

Thm A viable market is complete $\Leftrightarrow$ EMM $\mathbb{P}^*$ exists and is unique.

Pf: $(\Rightarrow)$. Exist is from NA. If $\mathbb{P}_1$, $\mathbb{P}_2$ are EMMs. Then $\forall h \geq 0 \in \mathcal{F}_N$,

$\exists H$. admissible. s.t. $h/S_N^0 = \widetilde{V}_N(H)$

So: $\mathbb{E}_1(h/S_N^0) = \mathbb{E}_1(\widetilde{V}_N(H))$

$= \widetilde{V}_0(H) = \mathbb{E}_2(\widetilde{V}_N(H))$

$= \mathbb{E}_2(h/S_N^0)$

$\Rightarrow \forall f \in \mathcal{F}_N$. $\mathbb{E}_1(f) = \mathbb{E}_2(f)$.

$(\Leftarrow)$. Set $\widetilde{V} := \{h_0 + \sum_1^N H_n \cdot \Delta\widetilde{S}_n \mid h_0 \in \mathcal{F}_0$. H is predictable$\}$.

If $\exists h \in \mathcal{F}_N$. ≥ 0. $h/S_N^0 \notin \widetilde{V}$.

Then $\widetilde{V} \subsetneq \mathbb{R}^n$. $\exists X \in \mathbb{R}^n$. $X \perp \widetilde{V}$.

For EMM $\mathbb{P}^*$. Set $\mathbb{P}^{**}(\{w\}) :=$

$(\frac{1}{2} + \frac{X(w)}{2\mathbb{E}_*(X)})\, \mathbb{P}^*(\{w\})$. $\sim \mathbb{P}^*$.

(If $\mathbb{E}^*(X)=0$. we set $\mathbb{P}^{**} = (1+\frac{X(\omega)}{\|X\|_\infty})\mathbb{P}^*$)

$\Rightarrow \mathbb{P}^{**} \underset{X\neq 0}{\neq} \mathbb{P}^*$. but it's EMM. contradict!

Rmk: Under the unique EMM $\mathbb{P}^*$ (we call it risk-neutral measure).

$\forall$ contingent claim h can be replicated in mart. transf. of mart. $(\tilde{S}_n)$. by H. (so $\tilde{V}_n(H)$ is $\mathbb{P}^*$-mart.) $\Rightarrow$ h has mart.-represent.

Thm. (Risk-Neutral Valuation Formula)

In complete viable market. the arbitrage-free price of asserts with payoff h is:

$V_n(H) = (1+r)^{n-N}\mathbb{E}^*(V_N(H)|\mathcal{F}_n) = \mathbb{E}^*(h|\mathcal{F}_n)/(1+r)^{N-n}$.

Pf: Note: $V_n(H)/S_n^0 = \mathbb{E}^*(h/S_N^0|\mathcal{F}_n)$.

Rmk: To calculate the arbitrage price $V_n(H)$.

We only need to know: i) r. ii) $\mathcal{F}_n$ iii) $\mathbb{P}^*$.

We don't need to know underlying p.m. $\mathbb{P}$.

(4) Europe Options:

Consider $d=1$. price vector $(S_n^0, S_n) = ((1+r)^n, S_n)$

where $S_{n+1} = \begin{cases} (1+a) S_n & \text{with prob. } p \\ (1+b) S_n & \text{with prob. } 1-p \end{cases}$

with $-1 < a < b$. $S_0 > 0$.

Set $\Omega = \{1+a, 1+b\}^N$. $\mathcal{F}_n = \sigma(T_1 \cdots T_n)$.

where $T_{n+1} = S_{n+1} / S_n$. N is expire time.

<u>Lemma.</u> The market is viable $\Leftrightarrow$ $r \in (a, b)$

<u>Pf:</u> $(\tilde{S}_n)$ is $\mathbb{P}^*$ - mart. where $\mathbb{P}^*$ correspd to prob. $p^* \in (0, 1)$. EMM exists $\Leftrightarrow$ $\mathbb{E}^*(\tilde{S}_{n+1} \mid \mathcal{F}_n) = \tilde{S}_n$. $\Leftrightarrow$ $\mathbb{E}^*(T_{n+1} | \mathcal{F}_n) = 1 + r$.

$\Leftrightarrow$ $(1+a) p^* + (1+b)(1-p^*) = 1+r$. p^* exists.

<u>Cor.</u> EMM exists unique given by $p^* = \frac{b-r}{b-a}$. So the market is complete.

<u>prop.</u> (Discrete Black-Scholes Formula)

The perfect hedging strategy H_n replicates the European call option is $H_n(S_{n-1}) :=$

$$\frac{c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a))}{(b-a) S_{n-1}}$$. where $c(n, x) :=$

$$\sum_{j=0}^{N-n} \binom{N-n}{j} p^{*j} (1-p^*)^{N-n-j} (x (1+a)^j (1+b)^{N-n-j} - K)_+ .$$

K is striking price. p^* corspd. unique EMM.

Pf: By FTAP. the price C_n of the call option with striking-price K is:

$$C_n := (1+r)^{-(N-n)} \mathbb{E}^*((S_n \prod_{n+1}^{N} T_k - K)_+ \mid \mathcal{F}_n)$$

$$= C(n, S_n) \quad (\text{Discrete BS Formula})$$

For H replicates the claim: it satisfies:

$$(1+r)^n H_n^0 + H_n S_n = C(n, S_n).$$

So:
$$\begin{cases} (1+r)^n H_n^0 + (1+a) H_n S_{n-1} = C(n, (1+a) S_{n-1}) \\ (1+r)^n H_n^0 + (1+b) H_n S_{n-1} = C(n, (1+b) S_{n-1}) \end{cases}$$

Subtract the equations $\Rightarrow$ obtain H_n

Then also solve H_n^0. from above.

Rmk: Note $C(n, x) \uparrow$ on x. So:

$H_n \geq 0$. $\Rightarrow$ The replicate strategy doesn't involve short-selling.

Cor. If payoff function $h(\cdot) \uparrow$. Then the perfect hedging strategy doesn't involve short-selling on risky assets.

Pf: general $C(n, x)$ is:

$$\Sigma \binom{N-n}{j} \cdots \quad h(x). \uparrow \text{ on } x.$$

① Next, we consider the conti. limit of this binomial model so to generate conti. BS formula.

<u>Denote</u>: $c \geq 0$ is instantaneous interest rate. $\sigma > 0$ volatility.

<u>Rmk</u>: When separating $[0,T]$ into N intervals

$$\Rightarrow r = cT/N. \quad e^{cT} := \lim_{N\to\infty} (1+\frac{cT}{N})^N$$

Set $\log((1+a)/(1+r)) = -\sigma/\sqrt{N}$. $\log((1+b)/(1+r)) = \sigma/\sqrt{N}$.

<u>Lemma</u> (Put-Call Parity)

For European put and call option P_t, C_t, with expire time T and striking price K.

$\Rightarrow S_t + P_t - C_t = K e^{-c(T-t)}$. if NA.

<u>Pf</u>: Denote $\Pi_t =: S_t + P_t - C_t$.

i) $\Pi_t < K e^{-c(T-t)}$. We borrow $K e^{-c(T-t)}$.

buy one Π_t and reserve $K e^{-c(T-t)} - \Pi_t$.

At $t = T$: $\begin{cases} \Pi_T = S + 0 - (S-K) = K. & \text{if } S \geq K. \\ \Pi_T = S + (K-S) - 0 = K. & \text{if } S < K. \end{cases}$

We return back K to bank. for debt.

but profit risklessly $\Rightarrow$ contradict!

ii) $\Pi_t > K e^{c(T-t)}$. Short the portfolio in i)

$\Rightarrow$ Argue similarly.

<u>Thm</u> (BS formula for calls)

The price of European call option is:

$C_t := S_t \phi(d_+) - K e^{-r(T-t)} \phi(d_-)$. where

ϕ is df. of $N(0,1)$. $d_\pm \triangleq \dfrac{\log(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

<u>Pf</u>: WLOG. set $t=0$. (Markov prop.)

1) $C_0^{(N)} = (1+\frac{rT}{N})^{-N} \mathbb{E}_*(S_0 \prod_1^N T_i - K)_+$

$= \mathbb{E}_*(S_0 e^{Y_N} - (1+r)^{-N} K)_+$.

where $Y_N = \sum_1^N X_n^N$. $X_n^N = \log(T_n^N/(1+r))$.

its mean $M_N = (1-2p^*)\sigma/\sqrt{N}$.

and note: $1-2p^* = 1 - 2(b-r)/(b-a)$

$= 1 - 2 \dfrac{e^{\sigma/\sqrt{N}} - 1}{e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}}}$.

$\Rightarrow 1-2p^* = -\frac{1}{2}\sigma/\sqrt{N} + O(N^{-1})$ by Taylor.

So: $M_N = \frac{\sigma}{\sqrt{N}}(-\frac{1}{2}\sigma/\sqrt{N} + O(N^{-1}))$.

with var. $\sigma_N^2 = \sigma^2/N$.

By CLT: $C_0^{(N)} \to C_0 = \mathbb{E}(S_0 e^Y - e^{-rT} K)_+$.

where $Y \sim N(-\frac{1}{2}\sigma^2 T, \sigma^2 T)$.

2) By routine calculation $\Rightarrow$ obtain it!

Cor. (BS formula for puts)

The price of its put option is:

$$Ke^{-r(T-t)}\Phi(-d_-) - S_t\Phi(-d_+).$$

Pf: By Put-call parity.

Rmk: Note that BS formula doesn't involve the mean rate of return μ — but only r and σ.

② About Volatility:

it can be estimated by: i) Historic Volatility.

ii) Implied volatility — Solve for σ from BS formula.

Rmk: We introduce "The Greeks", which are the partial derivative of option price w.r.t input para. e.g. $\Delta = \partial C/\partial S$. delta.

We can check in European option:

Vega $= \partial C/\partial\sigma > 0$. i.e. C_t is increasing function of volatility σ.

Rmk: Greeks have interpretion for sensitivities

View: Volatility is caused by future uncertainty and shows the market reaction to new info.

but trading itself is also one of main cause of volatility.

Besides. option like volatility ($C_t \uparrow$ if $\sigma \uparrow$).

Rmk: If $\sigma \to 0$. Then $d_1 \to \infty$. $\Phi(d_1) \to 1$.

So: $C_t = S_t - Ke^{-r(T-t)}$. (riskless)

when $S_t \to \infty$. it also has same phenomenon. $\Rightarrow$ The option is more likely to be exercised.

(5) American Option:

We denote $\tilde{C}_t$. $\tilde{P}_t$ are call and put option

Rmk: $\tilde{C}_t \geq (S_t - Ke^{-r(T-t)})_+$. $\tilde{P}_t \geq (Ke^{-r(T-t)} - S_t)_+$.

Pf: Consider long one $\tilde{C}_t$ and $Ke^{-r(T-t)}$ in cash. with put-call parity.

(it also holds by Merton's Thm below)

Thm. (Merton's)

American call option is equi. with European call option. i.e. $\tilde{C} = C$.

Pf: consider $\begin{cases} \text{one } \tilde{C}_t \text{ and } Ke^{-r(T-t)} \text{ in cash} & (\text{I}) \\ \text{one share } S_t. & (\text{II}) \end{cases}$

the value of (I) is k at $t = T$.

If we exercise it at $t_0 < T$.

$\Rightarrow$ value of (I) : $S_{t_0} - k + k e^{-r(T-t_0)}$

$<$ value of (II) $= S_{t_0}$.

But if $t_0 = T$. $(I) = S_T \vee k \geq (I)$

So we would not like to exercise it early. which's not optimal.

Rmk: i) The option insures the investor against the fall in S_t. So if he exercises early, then he loses such insurance.

ii) Early exercise lose the bank interest between t and T.

iii) Sometimes, early exercise also makes sense. — when keeping some main bussiness activities is more important than little interest.

For American put option:

Build snell envelop by working backward from $t = T < \infty$. — dynamic program. to find $\tilde{P}_t$.