Stochastic MFE.

(1) Limit Thm:

Consider n-player stochastic diff. game:

$dX_t^i = b(X_t^i, \hat{\mu}_t^n, \alpha_t^i)dt + \sigma(X_t^i, \hat{\mu}_t^n, \alpha_t^i)dW_t^i$.

where $\hat{\mu}_t^n \triangleq \sum_1^n \delta_{X_t^k}/n$. $(X_0^i) \overset{i.i.d}{\sim} \lambda_0 \in \mathcal{P}(\mathbb{R}^d)$.

$b: \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times A \to \mathbb{R}^d$. W_t is m-dim.

$\sigma: \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \times A \to \mathbb{R}^{d\times m}$.

Denote: Objective $J_i^n(\vec{\alpha}) := \mathbb{E}\Big(\int_0^T$

$f(X_t^i, \hat{\mu}_t^n, \alpha_t^i)dt + g(X_T^i, \hat{\mu}_T^n)\Big)$

Next, we want to describe Nash equi for large n. and get MFG limit.

Def: $\Phi: C([0,T], \mathcal{P}(\mathbb{R}^d)) \to C([0,T], \mathcal{P}(\mathbb{R}^d))$

is defined as follow:

Fix $\mu = (\mu_t) \in C([0,T], \mathcal{P}(\mathbb{R}^d))$. And α^* solve control problem uniquely:

$$\begin{cases} \sup_\alpha \mathbb{E}\left(\int_0^T f(X_t^{\mu,\alpha}, \mu_t, \alpha_t)\,dt + g(X_T^{\mu,\alpha}, \alpha_T)\right) \\ dX_t^{\mu,\alpha} = b(X_t^{\mu,\alpha}, \mu_t, \alpha_t)\,dt + \sigma(\cdots)\,dW_t. \end{cases}$$

Set $\phi(\mu) \triangleq (\mathcal{L}(X_t^{\mu,\alpha^*}))_{t\in[0,T]}$.

We say μ is MFE if $\phi(\mu) = \mu$.

Rmk: If α^* isn't unique, we can define:

$$\phi(\mu) := \{(\mathcal{L}(X_t^*))_t \mid X_t^* \text{ is optimal}\}$$

ii) Mean field control problem is:

$$\begin{cases} dX_t^\alpha = b(X_t^\alpha, \mu_t^\alpha, \alpha_t)\,dt + \sigma(X_t^\alpha, \mu_t^\alpha, \alpha_t)\,dW_t \\ \sup_\alpha \mathbb{E}\left(\int_0^T f(X_t^\alpha, \mu_t^\alpha, \alpha_t)\,dt + g(X_T^\alpha, \alpha_T)\right). \end{cases}$$

where $\mu_t^\alpha := \mathcal{L}(X_t^\alpha)$.

Rmk: If $\alpha^* = \tau$. then μ^{α^*} is MFE.

The difference between i) is :

μ is require to match the

law of X_t before optimization.

Thm. If $(b, \sigma, f, g) \in C_B$. (b, σ) are Lip.

with $(\|\cdot\|_{\mathbb{R}^2}, W_2)$. X_0^i . i.i.d $\in L^2$.

And assume (μ, τ^*) is MFE. S.t.

τ^* is Lipschitz. Then $\vec{\tau}_n := (\alpha^{n,1}, \ldots$

$\tau^{n,n})$ is open loop ε_n - Nash equi.

st. $\varepsilon_n \to 0$. where $\alpha_t^{n,i} := \alpha^*(t, X_t^i)$.

$$dX_t^i = b(X_t^i, \mu_t^n, \tau^*(t, X_t^i))dt + \sigma(\cdots)dW_t$$

Rmk: $\vec{\alpha}_n$ can also be shown it's a

close loop (Markovian)

Pf: 1) Consider $\bar{X}_t$ satisfies :

$d\bar{X}_t = b(\bar{X}_t, \mu_t, \alpha^*(\cdot\cdot))dt + \cdots$, $\mu_t = \mathcal{L}(\bar{X}_t)$.

As before, we have: $\forall i$.

$\lim_{n\to\infty} \mathbb{E}(\sup_t |X_t^i - \bar{X}_t|^2 + \sup_t W_1^2(\hat{\mu}_t, \mu_t)) = 0$.

$\Rightarrow \lim_{n\to\infty} J_i^n(\vec{\alpha}_n) \overset{DCT}{=} \mathbb{E}(\int_0^T f(\bar{X}_t, \mu_t, \alpha^*(t, \bar{X}_t))dt + \cdots)$

$\overset{\Delta}{=} J^\infty(\alpha^*)$. for $\forall i$

2) Consider player 1. (By sym.)

Next, we prove: $\varepsilon_n \overset{\Delta}{=} \sup_\beta J_1^n(\vec{\alpha}_n^{-1}, \beta) - J_1^n(\vec{\alpha}_n)$
$\longrightarrow 0$

Note $J^\infty(\alpha^*) \overset{Def}{\geq} J^\infty(\beta)$. $\forall \beta \in A$.

Where $J^\infty(\beta) = \mathbb{E}(\int_0^T f(Y_t^\beta, \mu_t, \beta_t) \cdots)$

$dY_t^\beta = b(Y_t^\beta, \mu_t, \beta_t)dt + \square$. $Y_0^\beta = X_0^1$.

We only need to prove: uniformly in β:

$J_1^n(\vec{\alpha}_n^{-1}, \beta) = \mathbb{E}(\int_0^T f(Y_t^1, \nu_t^n, \beta_t) + \cdots) \xrightarrow{n\to\infty} J^\infty(\beta)$.

where $\begin{cases} dY_t^i = b(Y_t^i, \nu_t^n, \alpha_t^i)dt + \cdots \\ dY_t^1 = b(Y_t^1, \nu_t^n, \beta)dt + \cdots \end{cases}$ $\nu_t^n = \sum_i^n \delta_{Y_t^i} / n$.

$\Longleftrightarrow$ prove $(Y^{\prime}, V^{n}) \to (Y^{\beta}, \mu)$. uniformly

3') First. show $V^{n} \to \mu$, uniformly in β.

Set $\mu_s^{n-1} := \frac{1}{n-1} \sum_{2}^{n} \delta_{Y_s^k}$.

$\Rightarrow \mathbb{E}\left(\sup_{t \in [0,T]} W_1(\mu_t^{n-1}, \mu_t^{n}) \right) \leq \frac{C}{n}$. $(\star)$

$\mathbb{E}\left(\sup_{s \in [0,t]} W_1^2(\mu_s^{n-1}, V_s^{n}) \right)$

i) $\begin{cases} \leq \frac{C}{n} + \frac{1}{n-1} \mathbb{E}\left(\sup_{s \in [0,t]} \sum_{2}^{n} |X_s^k - Y_s^k|^2 \right) \\ \text{Lip.} \\ \underset{(\star)}{\leq} \frac{C}{n} + \frac{C}{n-1} \sum_{2}^{n} \mathbb{E}\left(\int_0^t \left(\sup_{r \in [0,s]} |X_r^k - Y_r^k|^2 + \sup_{r \in [0,s]} W_1(\mu_r^{n-1}, V_r^{n}) \right) ds \right) \end{cases}$

write in SDE. $\downarrow$

Apply Gronwall's on i) :

$\mathbb{E}\left(\sup_{[0,t]} \frac{1}{n-1} \sum_{2}^{n} |X_s^k - Y_s^k|^2 \right) \leq \frac{C}{n} + C\, \mathbb{E}\left(\int_0^t \sup_{r \in [0,s]} W_1(\mu_r^{n-1}, V_r^{n}) ds \right)$

$\xRightarrow[\text{back}]{\text{replace}} \mathbb{E}\left(\sup W_1^2(\mu_s^{n-1}, V_s^{n}) \right) \leq C/n$.

S_0: $\mathbb{E}(\sup_{[0,T]} W_1^2(\nu_t^n, \mu_t)) \le c/n$.

4') $|Y_t^1 - Y_t^\beta| \overset{\text{Lip.}}{\lesssim} \int_0^t (|Y_t^1 - Y_t^\beta| + W_1(\nu_t^n, \mu_t))$

By Gronwall's inequi.:

$$\mathbb{E}(\sup_{[0,T]} |Y_t^1 - Y_t^\beta|) \le C\,\mathbb{E}(\sup_{[0,T]} W_1(\nu_t^n, \mu_t))$$

$$\le \frac{c}{n} + \mathbb{E}(\sup W_1(\mu_t^n, \mu_t))$$

$$\sim O(\tfrac{1}{n}) \to 0.$$

$$\Rightarrow (Y^1, \nu^n) \xrightarrow{n\to\infty} (Y^\beta, \mu). \text{ uniformly.}$$

(2) <u>Example: Semilinear MFG:</u>

Next, we consider $f, g \in C(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d); \mathbb{R}^d)$.

$$(P_\mu)\begin{cases} \sup_\alpha \mathbb{E}\left(\int_0^T f(X_t^\alpha, \mu_t) - \frac{1}{2}|\alpha_t|^2 dt + g(X_T^\alpha, \mu_T)\right) \\ dX_t^\alpha = \alpha_t dt + dW_t. \quad X_0^\alpha \sim \lambda_0 \in \mathcal{P}^2(\mathbb{R}^d) \end{cases}$$

where α is adapted and $\mathbb{E}(\int_0^T |\alpha_t|^2) < \infty$.

Define $\Phi(\mu) = \mathcal{L}(X^{T^*})$. where α^* is the optimal control.

<u>Thm.</u> (Schauder's fixed point)

K is cpt convex subset of a t.v.s.

$\Psi : K \to K$ is conti. $\Rightarrow$ Ψ has a fix point

<u>Thm.</u> Using $(\|\cdot\|_{\mathbb{R}^d}, W_1)$ metric. If (f, g) $\in C^{2,1}(\mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d))$. and their derivatives are uniformly bdd.

Then. there exists a MFE. And if $T > 0$ is small enough. (f, g) and their derivatives are lip. then: the MFE is unique.

Pf: 1) Solve optimal problem $c(\mu) \iff$

Solve $H(x,y) = \sup_{a \in \mathbb{R}^k} (a^T y - \frac{1}{2}|a|^2) + f$

$\Rightarrow \hat{\alpha} = y$. We have HJB equation:

$\partial_t V(t,x) + \frac{1}{2}|\nabla V(t,x)|^2 + f + \frac{1}{2}\Delta V = 0$

$V(T,x) = g(x,\mu_T)$

Apply Hopf-Cole transf.:

Let $u(t,x) = \exp(V(t,x))$. Then:

$\partial_t u(t,x) + \underbrace{\frac{1}{2}\Delta u(t,x)}_{\text{generator of BM.}} + f \cdot u = 0.$

Feynman-Kac formula $\Rightarrow$ $u(t,x) = \mathbb{E}\left(e^{g(W_T,\mu_T) + \int_t^T f(W_s,\mu_s)ds} \mid W_t = x\right)$

So, we have: $\alpha_\mu(t,x) = \hat{\alpha}(x,\nabla V)$

$= \nabla V = \frac{\nabla u}{u}.$

2) Next, establish conti. prop. of α_μ:

i) It's uniformly bdd since

$\alpha_\mu = \mathbb{E}\left(e^{g(\cdots)+\int_t^T f(\cdots)} (\partial_x g(\cdots) + \int_t^T f(\cdots))\right) / \mathbb{E}(\cdots)$

ii) And it's also jointly conti. on

$[0,T] \times \mathbb{R}^d \times \mathcal{P}^1(\mathbb{C}^d)$.

iii) $|\partial_j \gamma_\mu^i(t,x)|$ is also uniformly

bdd on (t,x,μ,i,j)

iv) (f,g) and their derivatives

are Lip. on m-var. $\Rightarrow \gamma_\mu(t,x)$

is W_1-Lip on μ. $\forall t,x$.

(since $x \mapsto e^x$ is Lip.).

3) Establish conti. property of ϕ:

First, Note $|X_t^\mu| \leq Ct + |W_t| + |X_0|$

$$\Rightarrow \sup_{\mu \in \mathcal{P}'} \int_{\mathbb{C}^d} \|x\|_T^2 \, \phi(\mu)(dx) \leq C(T, X_0, W).$$

So: $\phi(\mu) \in \mathcal{P}^2(\mathbb{C}^d) \subset \mathcal{P}^1(\mathbb{C}^d)$.

i.e. $\phi(\mathcal{P}'(\mathbb{C}^d)) \subset \mathcal{P}'(\mathbb{C}^d)$.

Next, $W_1(\phi(\mu), \phi(\nu)) \leq \mathbb{E}(\|X^\mu - X^\nu\|)$.

So we estimate $|X_t^\mu - X_t^\nu|$

$\overset{\text{Lip. of } a_\mu}{\leq} C\int_0^t |X_s^\mu - X_s^\nu| ds + \int_0^t |\alpha_\mu(s, X_s^\nu) - \alpha_\nu(s, X_s^\nu)| ds$

$\overset{\text{Gronwall}}{\Longrightarrow} \|X^\mu - X^\nu\|_T \leq C\int_0^t |\alpha_\mu(s, X_s^\nu) - \alpha_\nu(s, X_s^\nu)|$

So: if $\mu^n \to \mu$ in W_1.

By conti. of a_μ. we have:

$W_1(\phi(\mu^n), \phi(\mu)) \to 0$.

Additionally. if Lip. prop. iv) in 2) holds. then: $W_1(\phi(\mu), \phi(\nu)) \lesssim_T W_1(\mu, \nu)$.

When $T > 0$ small enough. $\Rightarrow \phi$ is a contraction. So MFE is unique.

Whatever. $\phi \in C(\mathcal{P}^1(C^d); \mathcal{P}^1(C^d))$.

4) Next, we want to use the Schauder's fix point Thm. to find

the fix point (MFE.)

Recall $M := \sup_{\mu \in P'(C)} \int_{C^d} \|x\|_T^2 \, \phi(\mu)(dx) < \infty$.

<u>Lemma.</u> (Aldous's criterion)

$(X_t^n) \subset D(\mathbb{R}^+; S) := \{$ càdlàg func. $: \mathbb{R}^+ \to S$. S is polish$\}$ is tight

if $\lim_{h \to 0} \overline{\lim_{n\to\infty}} \sup_{\sigma, \tau} \mathbb{E}(d(X_\sigma^n, X_\tau^n) \wedge 1) = 0$

where $\sigma \le \tau \le \sigma + h$. τ, σ. stopping times

Note $\mathbb{E}(|X^{\mu^n}_{T\wedge(\tau_n+\delta_n)} - X^{\mu^n}_{\tau_n}|) \overset{SDE}{\le}$

$\mathbb{E}\left(\int_{\tau_n}^{T\wedge(\tau_n+\delta_n)} |\alpha_{\mu^n}(t, X_t^{\mu^n})| dt + \|\sigma\|_\infty |W_{T\wedge(\cdot\cdot)} - W_{\tau_n}| \right)$

$\le C \delta_n \|\alpha\|_\infty + \|\sigma\|_\infty \sqrt{\delta_n}$. $\forall \tau_n$.

So: $\lim_{n\to\infty} \mathbb{E} |X^{\mu^n}_{\delta_n+\tau_n} - X^{\mu^n}_{\tau_n}| \wedge 1 = 0$

$\Rightarrow \phi(P'(C^d))$ is tight. (Prokhorov's)

Set $K := \{ \mathbb{P} \in \mathcal{P}^1(C^d) \mid \mathbb{E}^{\mathbb{P}}(\|X\|_T^2) \le m,\ \mathbb{E}^{\mathbb{P}}(|X_{t+\delta\wedge T} - X_t|) \le \delta\|\sigma\|_\infty + \sqrt{\delta}\|\sigma\|_\infty) \}$

$\Rightarrow K$ is convex. tight.

$\overset{\text{Prokhov's}}{\Rightarrow}$ $\bar{K}$ $\xrightarrow{} \mathcal{P}(\mathcal{P}^1(C^d))$ is convex. cpt. in $\mathcal{P}^1(C^d)$.

Note ϕ is W_1-conti. on $\mathcal{P}^1(C^d)$.

So it's also weakly conti. on $\bar{K}$.

$\Rightarrow$ Apply Schauder's fix point Thm

Rmk: Procedure:

HJB Analysis $\Rightarrow$ establish conti. prop. of optimal control $\Rightarrow$ Argue the law of optimal process can be confine in cpt set of $\mathcal{P}(C^d)$.

$\Rightarrow$ Apply Schauder / Kakutani's Thm

(2) <u>Uniqueness:</u>

<u>Thm.</u> If $f(x,m,\alpha) = f_1(x,\alpha) + f_2(x,m)$.

and monotonicity condition:

$$\int_{\mathbb{R}^d} (g(x,m_1) - g(x,m_2))(m_1 - m_2)(dx) \leq 0$$

$$\int_{\mathbb{R}^d} (f_2(x,m_1) - f_2(x,m_2))(m_1 - m_2)(dx) \leq 0$$

holds. Then: there's at most one MFE

<u>Pf:</u> By contradiction:

if $\mu \neq \nu$ are 2 MFEs.

then $\exists\ \alpha \neq \beta$. as optimal control of (P_μ). (P_ν).

$\Rightarrow$ Obtain 2 inequi. and add them up to get contra.

(3) MFG with different type agents:

Denote Θ is a polish space. And each agent is assigned a type parameter $\theta_i \in \Theta$. Consider

$$dX_t^i = b(X_t^i, \mu_t^n, \alpha_t^i, \theta_i)dt + \sigma(\cdots)dW_t.$$

$$X_0^i = x_0^i$$

where $\mu_t^n := \frac{1}{n}\sum_1^n \delta_{(X_t^k, \theta_k)}$

Set objective function for agent i is:

$$J_i(\vec{\alpha}) = \mathbb{E}\left(\int_0^T f(X_t^i, \mu_t^n, \alpha_t^i, \theta_i)dt + g(X_T^i, \mu_T^n, \theta_i)\right)$$

Def: Consider $(\xi, \theta) \sim M_0 \in \mathcal{P}(\mathbb{R}^d \times \Theta)$.

given initial dist. Set Φ as follow:

i) Fix $\mu \in \mathcal{P}(C^d \times \Theta)$. Solve

$$\begin{cases} \sup_\alpha \mathbb{E}\left(\int_0^T f(X_t^{\mu,\alpha}, \mu_t, \alpha_t, \theta)dt + g(X_T, \mu_T, \theta)\right) \\ dX_t^{\mu,\alpha} = b(X_t^{\mu,\alpha}, \mu_t, \alpha_t, \theta)dt + \square\, dW_t. \\ X_0 = \xi. \end{cases}$$

ii) α^* is optimal control. which is assume unique.

then let $\Phi(\mu) = \mathcal{L}(X^{\mu,\alpha^*}, \theta)$.

if $\Phi(\mu) = \mu$. $\Rightarrow$ we say μ is MFE

Def: (Equivalent MFE.)

Given $\mu_0 \in \mathcal{P}(C^d \times \Theta)$. set Φ:

$\mathcal{P}(C^d \times \Theta) \to \mathcal{P}(C^d \times \Theta)$ as follow

i) Fix $\mu \in \mathcal{P}(C^d)$. $(x_0, \theta) \in \mathbb{R}^d \times \Theta$.

Solve control problem :

$$\begin{cases} \sup_{\tau} \mathbb{E}\left(\int_0^T f(X_t^{\mu,\tau}, \mu_t, \tau_t, \theta)\,dt + g(X_T, \mu_T, \theta)\right) \\ dX_t^{\mu,\tau} = b(X_t^{\mu,\tau}, \mu_t, \tau_t, \theta)\,dt + \sigma\,dW_t. \quad X_0 = x_0 \end{cases}$$

ii) α^* is the optimal control.

which's assumed to be unique.

iii) Set $V_{x_0,\theta} \triangleq f(X^{\mu,\alpha^*})$. and define

$$V = \int_{\mathbb{R}^d \times \Theta} V_{x_0,\theta} \, \mu_0(dx_0, d\theta).$$

Let $\Phi(\mu) = V$. μ is MFE if

$\Phi(\mu) = \mu$.

<u>Rmk</u>: We can solve each optimal control under deterministic initial state (x_0, θ) and then integrate.