NYU

Linear Algebra

Lecture 5

Matrix Operations and Inverse Matrix

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Matrix Operations

Matrix multiplication

$\vec{r}_i^T \in \mathbb{R}^n$ $\quad$ $\vec{v}_j \in \mathbb{R}^n$

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \text{ith row} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

$\mathbb{R}^{m\times n}$ $\qquad$ jth column $\mathbb{R}^{n\times p}$ $\qquad$ ij entry $\mathbb{R}^{m\times p}$

$$c_{ij} = \vec{r}_i^T v_j = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

Question : *A* is a matrix, write a matrix-vector multiplication as the first column

$A = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$. find a vector $\vec{x}$ such that $A x = \underbrace{\vec{v}_1}_{\mathbb{R}^m}$

$A \in \mathbb{R}^{m\times n}$ $\qquad$ $\vec{x} \in \mathbb{R}^n$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad A\vec{x} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n$$

I want $Ax = \vec{v}_1$. then $x_1 = 1$. $x_2 = \cdots = x_n = 0$

$$\Rightarrow \quad x = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Follow Up Question,

Matrix $I_n = \begin{pmatrix} 1 & & 0 \\ 0 & 1 & \\ \vdots & & \ddots \\ 0 & 0 & & 1 \end{pmatrix}$. what is $A I_n$ $\qquad$ $\mathbb{R}^{m\times n}$ $\mathbb{R}^{n\times n} \to \mathbb{R}^{m\times n}$

① first Column vector of AI_n

$$A[v_1' \ v_2' \cdots v_n'] = [Av_1' \ \ Av_2' \ \ \cdots \ \ Av_n']$$

Column vectors

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$A I_n = A \Leftarrow [\ v_1 \quad v_2 \quad \cdots \quad v_n\ ]$$

Matrix multiplication

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \begin{array}{c} \\ \\ i\text{th row} \\ \\ \\ \end{array} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

jth column

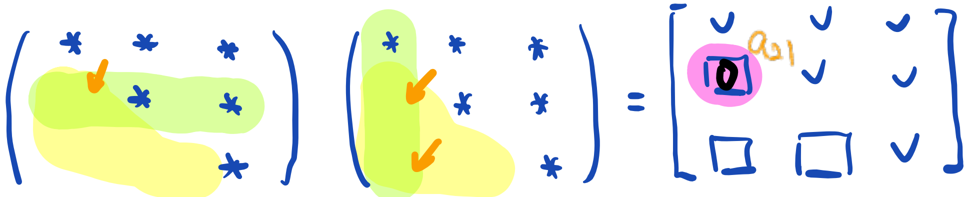
ij entry

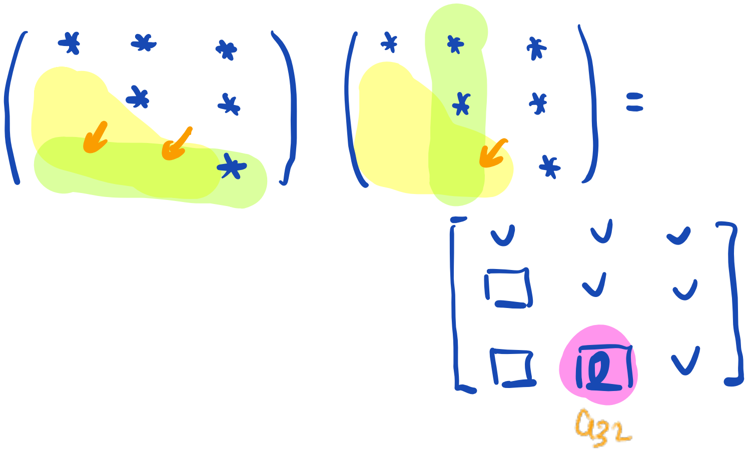
Lower Triangular Matrix × Lower Triangular Matrix → Lower Triangular Matrix

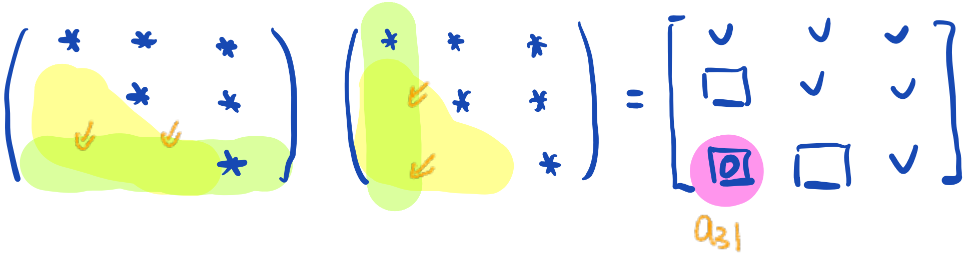
Question : *A* is a matrix, write a matrix-vector multiplication as the first column

Is upper triangular matrix times a upper triangular matrix still upper triangular

Yes!







Diagonal Matrix

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix,

Square Matrix

What is Dx?

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

(☆).

$D \cdot x = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_{11} \cdot x_1 \\ d_{22} \cdot x_2 \\ \vdots \\ d_{nn} \cdot x_n \end{bmatrix}$

Inverse Matrix

linear System

Defn I

- Inverse Matrix $Ax = b \Leftrightarrow x = A^{-1}b$ is the only solution for all vector b

Suppose A is an $n \times n$ matrix (square matrix), then A is invertible if there exists a matrix A^{-1} such that

check

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

← another definition.

We can only talk about an inverse of a square matrix, but not all square matrices are invertible. We will discuss such restrictions in future lectures.

$A^{-1}A$ is I_n s.t.

$A^{-1}A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ = using defn! $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← Solve the linear system $Ax = A\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

the first column of $A^{-1}A$

the only solution is $A^{-1}\left[A\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right]$

there is a solution $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ "

$A^{-1}A \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ = using defn I $\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

the second column of $A^{-1}A$

Example

Example $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

I claim $B = A^{-1}$. Check:

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This also means

$$\begin{cases} 2x + y = a \\ x + y = b \end{cases}$$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

The solution is

$$\begin{cases} x = a - b \\ y = -a + 2b \end{cases}$$

$$B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = B \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} a \\ b \end{pmatrix}$$

Inverse of a Diagonal Matrix

$$D \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} d_{11} x_1 \\ \vdots \\ d_{nn} x_n \end{pmatrix}$$

Let $D = \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$ be an $n \times n$ diagonal matrix, then

$$\begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix}$$

$D^{-1} = \begin{bmatrix} 1/d_{11} & & & \\ & 1/d_{22} & & \\ & & \ddots & \\ & & & 1/d_{nn} \end{bmatrix}$ provided that $d_{ii} \neq 0$.

Recall

Suppose we are given a system of m equations in n unknowns:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

This system can be written in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

in augmented form $\longrightarrow$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$[A|b] \to \underbrace{[I|\vec{x}]}_{Ax = b}$

Two Operations

- Linear combine two rows 'ellimination'

  Replace Row (i) with $\underbrace{* \cdot (j) + (i)}_{\text{L.C. of (j) and (i)}}$

Ellimination Matrix

$$[\, A \mid b \,] \xrightarrow{\text{Replace Row (i) with } * \cdot (j) + (i)} [\, \underbrace{E_{ij} A}_{\text{Matrix} \times \text{Matrix}} \mid E_{ij} b \,]$$

- Permutation

  Switch Two Rows (i) and (j)

Permutation Matrix

$$[\, A \mid b \,] \xrightarrow{\text{Switch Two Rows (i) and (j)}} [\, P_{ij} A \mid P_{ij} b \,]$$

Permutation Matrices

Recall

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{I\,\vec{x}} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Permutation Matrices

$$P_{ij} = \begin{array}{c} \\ \left[\begin{array}{cccccccc} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & \overset{a_{ii}}{0} & \dots & 1 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0^{\;a_{ij}} & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{array}\right] \\ \qquad\qquad \uparrow \qquad\qquad \uparrow \\ \qquad \text{i-th Column} \quad \text{j-th Column} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow \text{i-th row} \\ \\ \leftarrow \text{j-th row} \\ \\ \\ \end{array}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

$a_{ii} = a_{jj} = 0$ $\qquad$ $a_{ij} = a_{ji} = 1$

$P_{ij} A \rightsquigarrow$ switch i-th row and j-th row of A

Permutation Matrices

$$P_{ij} = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots \\ \color{blue}0 & \color{blue}0 & \color{blue}\dots & \color{blue}0 & \color{blue}\dots & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ \vdots \\ \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}1 & \color{red}\dots & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

$$P_{ij} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \color{blue}x_j \\ \vdots \\ \color{red}x_i \\ \vdots \\ x_n \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ 0 & 1 & 0 & \dots & 0 \\ \color{red}1 & \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{P_{31}\,\vec{x}} \begin{bmatrix} \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}\dots & \color{blue}0 \\ 0 & 1 & 0 & \dots & 0 \\ \color{red}1 & \color{red}0 & \color{red}0 & \color{red}\dots & \color{red}0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

Elimination Matrices

$$E_{ji} = \begin{array}{cl} \begin{array}{cccccccc} & & & \text{Col } i & & \text{Col } j & & \end{array} & \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \cdots & \underset{a_{ji} = *}{\star} & \cdots & 1 & \cdots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{array} \end{array}$$

$\Rightarrow$

change my Row (j)

with $(*) \cdot$ Row (i)

$+$ Row (j)

Elimination Matrices

$$E_{ji} = \begin{array}{cc} \begin{array}{cccccccc} & & & \text{Col } i & & \text{Col } j & & \end{array} & \\ \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & & \ddots & & & & & \\ 0 & 0 & \dots & 1 & \dots & 0 & \dots & 0 \\ \vdots & & & & \ddots & & & \\ 0 & 0 & \dots & \star & \dots & 1 & \dots & 0 \\ \vdots & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{array} \end{array}$$

$$E_{ji} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j + (\star \cdot x_i) \\ \vdots \\ x_n \end{bmatrix}$$

Elimination Matrices

What does the matrix $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$ when it acts on it?

a_{21}

$E_{21}x$ Replace Row (2) with $-2 *$ Row (1) + Row (2)

$$E_{21}\vec{x} = \begin{bmatrix} 2 \\ 8 + (-2) \times 2 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

Elimination Matrices

What does the matrix $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ do to the vector $\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$ when it acts on it?

a_{31}

Replace Row (3) with Row (1) + Row (3)

$$\begin{bmatrix} 2 \\ 4 \\ 10+2=12 \end{bmatrix}$$

Solving Linear Systems

Elimination

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

$$[A \mid b] \longrightarrow [I \mid \vec{x}]$$

Column by Column

Elimination – Summary of the previous example

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

First Glum

$$\left(\begin{array}{ccc|c} 1 & \boxed{2} & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & \boxed{-5} & -10 & -20 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that would produce ugly fractions.

Let's swap the last two rows first.

$$\xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

Second Glum

Elimination

$$\left(\begin{array}{ccc|c} 1 & 0 & \boxed{-1} & -2 \\ 0 & 1 & \boxed{2} & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{R_3 = R_3 \div 10} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We want these to be zero.

Let's make this a 1 first.

$$\xrightarrow{R_1 = R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Third Column

$$\xrightarrow{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{translates into}} \begin{aligned} x \quad\quad &= 1 \\ y \quad &= -2 \\ z &= 3 \end{aligned}$$

Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \xrightarrow{\text{substitute solution}} \begin{aligned} 1 + 2\cdot(-2) + 3\cdot 3 &= 6 \\ 2\cdot 1 - 3\cdot(-2) + 2\cdot 3 &= 14 \\ 3\cdot 1 + (-2) - 3 &= -2 \end{aligned} \checkmark$$

The idea of Inverse Matrices

Consider the following system:

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 2 \\ 4x_1 + 9x_2 - 3x_3 &= 8 \\ -2x_1 - 3x_2 + 7x_3 &= 10 \end{aligned}$$

Our goal is to find x_1, x_2, and x_3. In matrix form, this system is:

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$$A \qquad \vec{x} = \vec{b}$$

$[A \mid \vec{b}] \leftarrow [A \mid I]$

$$\xrightarrow{\text{idea}} \qquad \vec{x} = A^{-1} \vec{b}$$

$[I \mid A^{-1}\vec{b}] \leftarrow [I \mid A^{-1} I]$

$= [I \mid A^{-1}$

The idea of Inverse Matrices

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_{\vec{b}}$$

$$\Longleftrightarrow \underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_{\vec{b}}$$

$$\xrightarrow{\text{idea}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}^{-1}}_{A^{-1}} \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_{\vec{b}}$$

The idea of Inverse Matrices

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

$$\Downarrow \textbf{elimination}$$

Two operation

Column by column.

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Example

Example: Find the inverse of $A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 4 & 9 & -3 & 0 & 1 & 0 \\ -2 & -3 & 7 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 & 1 \end{array}\right] R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -1 & 1 \end{array}\right] R_3 \div 4$$

$$\left[\begin{array}{ccc|ccc} 2 & 4 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right] R_1 - 4R_2$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -6 & 9 & -4 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right] \begin{array}{l} R_1 + 6R_3 \\ R_2 - R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 27/2 & -11/2 & 3/2 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right] R_1/2$$

Example

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 27/4 & -11/4 & 3/4 \\ 0 & 1 & 0 & -11/4 & 5/4 & -1/4 \\ 0 & 0 & 1 & 3/4 & -1/4 & 1/4 \end{array}\right]$$

Inver Matrix

Solution

Linear System

$$\begin{cases} 2x_1 + 4x_2 - 2x_3 = b_1 \\ 4x_1 + 9x_2 - 3x_3 = b_2 \\ -2x_1 - 3x_2 + 7x_3 = b_3 \end{cases}$$

$$A\vec{x} = \vec{b}$$

$$\begin{cases} b_1 = 27/4\, x_1 - 11/4\, x_2 + 3/4\, x_3 \\ b_2 = -11/4\, x_1 + 5/4\, x_2 - 1/4\, x_3 \\ b_3 = 3/4\, x_1 - 1/4\, x_2 + 1/4\, x_3 \end{cases}$$

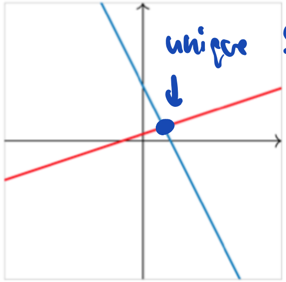
$$\vec{b} = A^{-1}\vec{x}$$

Three Cases

Not all the Matrix have inverse.

$x - 3y = -3$

$2x + y = 8$



unique Solution

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & 1 & 8 \end{array}\right) \quad R2 - 2*R1$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 7 & 14 \end{array}\right) \begin{array}{l} R1' \\ R2' \end{array}$$

← upper Triangular Form

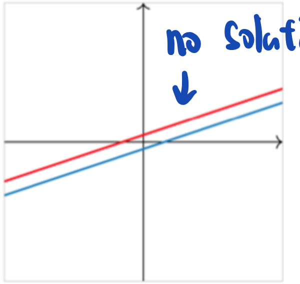
Two pivot!!

using $R2'$ → $x_2 = 2$

using $x_2 = 2$ and $R1'$ → x_1

$x - 3y = -3$

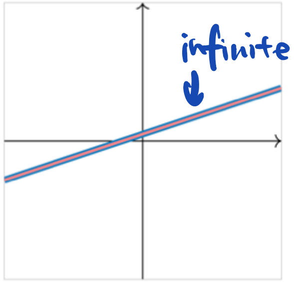
$x - 3y = 3$



no Solution

$x - 3y = -3$

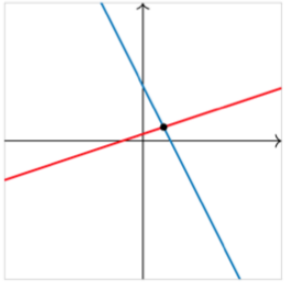
$2x - 6y = -6$



infinite Solution.

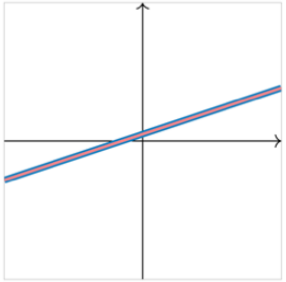
Three Cases

$x - 3y = -3$
$2x + y = 8$



$x - 3y = -3$
$x - 3y = 3$

$x - 3y = -3$
$2x - 6y = -6$



$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 1 & -3 & 3 \end{array}\right) \quad R2-R1$$

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & \boxed{0} & 6 \end{array}\right)$$

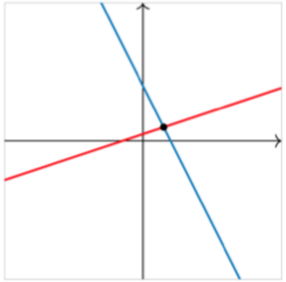
x

only one pivot

it should be zero! No solution

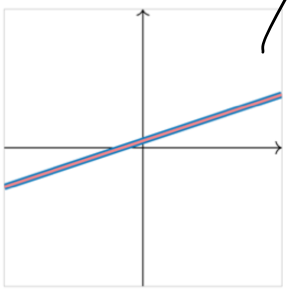
Three Cases

$x - 3y = -3$
$2x + y = 8$



$x - 3y = -3$
$x - 3y = 3$

$x - 3y = -3$
$2x - 6y = -6$



$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -6 & -6 \end{array}\right) \quad R2 - 2R1$$

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 0 & 0 \end{array}\right) \quad R1' \text{ 1 pivot}$$

x_2 can be any value

using $R1'$ to solve x_1

$$\begin{cases} x_2 = x_2 \\ x_1 = -3 + 3x_2 \end{cases}$$

Example

(1) Using Elimination to change Linear System to Upper Trianglar Form

$$\left(\begin{array}{cc|c} 1 & * & * \\ 0 & * & * \end{array}\right)$$

→ using this to determin x_1

→ using this to determine x_2

Can be zero .

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \end{array}\right)$$

→ means $0 \cdot x_1 + 0\, x_2 = 0$ ⇒ x_2 can be any things

Note on Infinite Solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

← $x_1 = 4 - 3x_3$ → $x_1 = 4 - 3\times\left(\frac{1}{2} - x_2\right) = 3x_2 + \frac{5}{2}$

← $x_2 = 1 + 2x_3$ → $x_3 = \frac{1}{2} - x_2$

← x_3 can be any vale.

if it's not 0, no solution

Equal to
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$
← This is all my solution!

Example

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

→ $x_1 = 4 - x_2 - x_3 - 0 \cdot x_4 = 4 - x_2 - (1 - 2x_4) = 3 - x_2 + 2x_4$

4 unknowns 3 Eq.

→ x_2 can be any vale

→ $x_3 = 1 - 2x_4$

→ x_4 can be any vale

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$
← This is all solutions!

Block Matrix*

Block Matrices

4 by 6 matrix
2 by 2 blocks

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}.$$

Block multiplication If the cuts between columns of A match the cuts between rows of B, then block multiplication of AB is allowed:

$$\begin{array}{cc} & \begin{matrix} n_1 & \quad n_2 \end{matrix} \\ \begin{matrix} m_1 \\ m_2 \end{matrix} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \begin{array}{cc} \begin{matrix} k_1 & \quad \end{matrix} & \\ \begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} & \begin{matrix} n_1 \\ n_2 \end{matrix} \end{array} = \begin{array}{cc} \begin{matrix} m_1 \\ m_2 \end{matrix} & \begin{bmatrix} \overset{n_1}{A_{11}}\overset{k_1}{B_{11}}{}^{n_1} + \overset{n_2}{A_{12}}\overset{k_1}{B_{21}}{}^{n_2} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix} \end{array}. \tag{1}$$

Example 3 **(Important special case)** Let the blocks of A be its n columns. Let the blocks of B be its n rows. Then block multiplication AB adds up ***columns times rows***:

Columns times rows

$$\begin{bmatrix} | & & | \\ \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} — & \boldsymbol{b}_1 & — \\ & \vdots & \\ — & \boldsymbol{b}_n & — \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 + \cdots + \boldsymbol{a}_n\boldsymbol{b}_n \end{bmatrix}. \tag{2}$$

$$AB = \begin{pmatrix} — r_1 — \\ \vdots \\ — r_m — \end{pmatrix} \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} r_1c_1 & r_1c_2 & \cdots & r_1c_p \\ r_2c_1 & r_2c_2 & \cdots & r_2c_p \\ \vdots & \vdots & & \vdots \\ r_mc_1 & r_mc_2 & \cdots & r_mc_p \end{pmatrix}$$

Elimination by Block

One at a time $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

Block elimination

$$\left[\begin{array}{c|c} I & \mathbf{0} \\ \hline -CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & D - CA^{-1}B \end{array}\right].$$