Lie Derivative

Recall we've defined Lie derivative for vector field X & Y. Next, we want to extend it on k-forms:

<u>Df</u>: X is vector field with flow φ_t. $\omega \in \Omega^k(M)$, $(L_X\omega)_p := \frac{d}{dt}\big|_{t=0} \varphi_t^* \omega_{\varphi_t(p)} \in \Lambda^k T_p^* M$.

<u>Rmk</u>: For $k=0$: $f \in C^\infty(M)$. $L_X f = Xf$.

<u>Prop.</u> i) L_X is a derivation: $L_X(\omega\wedge\eta) = L_X\omega\wedge\eta + \omega\wedge L_X\eta$.

ii) $L_X(d\omega) = d(L_X\omega)$.

iii) product formula: $L_X(\omega(Y_1, \cdots Y_k)) = L_X\omega(Y_1 \cdots Y_k) + \sum_1^k \omega(Y_1 \cdots L_X Y_j, \cdots Y_k)$

<u>Rmk</u>: From iii): $(L_X\omega)(Y_1 \cdots Y_k) = X(\omega(Y_1, \cdots Y_k)) - \sum_1^k \omega(Y_1 \cdots [X, Y_j], \cdots Y_k)$.

<u>Pf</u>: i) using $\varphi_t^*(\omega\wedge\eta) = \varphi_t^*\omega \wedge \varphi_t^*\eta$

Note: $d/dt\,|_{t=0} = \lim_{t\to 0} \square$. with bilinearity of "$\wedge$" product

ii) φ_t^* commutes with $d(\cdot)$ commutes with $d/dt\,|_{t=0}$. (by linearity)

iii) First for $k=1$:

$$L_X\, w(Y)_p \overset{\text{linear}}{=} \lim_{t\to 0} \frac{1}{t}(\varphi_t^* w_{\varphi_t p} - w_p)(\varphi_{-t*} Y_{\varphi_t p})$$

$$+ \lim_{t\to 0} w_p\left(\frac{1}{t}(\varphi_{-t}^* Y_{\varphi_t p} - Y_p)\right)$$

$$= L_X w_p(Y_p) + w_p(L_X Y_p)$$

For general k, similarly

$$L_X w(Y_1 \cdots Y_k) = \lim_{t\to 0} \frac{1}{t}(\varphi_t^* w_{\varphi_t p} - w_p)($$

$$\varphi_{-t*} Y^1_{\varphi_t p}, \cdots \varphi_{-t*} Y^k_{\varphi_t p}) + \sum_{j=0}^{k-1} \frac{1}{t}(w_p(Y_1 \cdots$$

$$Y_j, \varphi_{-t*} Y^{j+1}_{\varphi_t p} \cdots \varphi_{-t*} Y^k_{\varphi_t p}) - w_p(Y_1, \cdots Y_{j+1}$$

$$, \varphi_{-t*} Y^{j+2}_{\varphi_t p} \cdots \varphi_{-t*} Y^k_{\varphi_t p})) = RHS.$$

<u>prop.</u> (Cartan's formula)

For vector field X. $\Rightarrow L_X = d\,\iota_X + \iota_X d$.

<u>pf:</u> 1) RHS is derivation follows from d, ι_X are antideri.

2) Since LHS and RHS are both deri. We can use their locality: $\forall k$-form $w = \sum f_I\, dw_I$ on U. Since LHS & RHS both commutes

with L. We can only check they coincide on $C^\infty(M) = \Omega^0(M)$

$(\iota_X d + d\iota_X) f = \iota_X df = df(X) = Xf = L_X f$.

<u>prop.</u> $X, Y \in \mathfrak{X}(M)$. $\omega \in \Omega^1(M)$. Then:

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$$

<u>Pf:</u> LHS $= \iota_X d\omega(Y)$. So we can use Cartan's formula, together with prop. iii) above to find $L_X\omega(Y)$.

<u>Cor:</u> $\omega \in \Omega^k(M)$, $X_i \in \mathfrak{X}(M)$. We have:

$$d\omega(X_0 \cdots X_k) = \sum_0^k (-1)^i X_i\, \omega(X_0, \cdots \hat{X_i}, \cdots X_k)$$
$$+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0 \cdots \hat{X_i}, \cdots \hat{X_j}, \cdots X_k)$$

<u>Pf:</u> Induction on k by using Cartan formula: LHS $= L_{X_0}\omega(X_1 \cdots X_k) - (d\iota_{X_0}\omega)(X_1 \cdots X_k)$, with prop. iii).

<u>prop.</u> $L_X L_Y - L_Y L_X = L_{[X,Y]}$

<u>Pf:</u> Use prop. iii) above.