Low Regularity

Note Young integral copes with the case $\alpha \in (\frac{1}{2}, 1]$ and we consider $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ above. Next, we cover all the $(0, 1]$.

(1) <u>Signature:</u>

<u>Def:</u> $\mathbb{X}_{\cdot,\cdot} = (1, \mathbb{X}^1_{\cdot,\cdot}, \cdots \mathbb{X}^{N_\alpha}_{\cdot,\cdot}) : \Delta_T \to T_1^{N_\alpha}(V)$

is a conti. map. $\alpha \in (0, 1]$. $N_\alpha = \lfloor 1/\alpha \rfloor$

it's α-Hölder rough path if:

i) $\|\mathbb{X}^n\|_{n\alpha} < \infty$. $\forall\ n \leq N_\alpha$

ii) satisfies Chen's relation $\mathbb{X}_{s,u} = \mathbb{X}_{s,t} \otimes \mathbb{X}_{t,u}$ $\forall\ s \leq u \leq t$.

<u>Rmk:</u> We can set $\mathbb{X}_t \triangleq \mathbb{X}_{0,t}$ as a actual path and conversely set $\mathbb{X}_{t,s} \triangleq \mathbb{X}_t^{-1} \otimes \mathbb{X}_s$.

<u>e.g.</u> $x : [0, T] \to V$. Smooth. path. $\alpha \in (0, 1]$.

Set $\Delta^k_{[s,t]} = \{ s \leq r_1 \leq \cdots \leq r_n \leq t \}$

We can lift x as rough path:

$$\mathbb{X}^n_{s,t} \overset{\Delta}{=} \int_{\Delta^n_{[s,t]}} dx_{r_1} \otimes dx_{r_2} \cdots \otimes dx_{r_n}. \quad \mathbb{X} := (1, \cdots \mathbb{X}^{N_\alpha})$$

<u>Def</u>: $S(X)_{s,t} := (1, \mathbb{X}^1_{s,t}, \cdots \mathbb{X}^n_{s,t} \cdots) \in T^{\infty}(V)$ is signature of X over $[s,t]$.

<u>Rmk</u>: i) Note $\|\mathbb{X}^n_{s,t}\| \le \frac{\|X'\|^n_\infty}{n!} |t-s|^n$. and it satisfies Chen's. $\Rightarrow$ it's rough path.

ii) Note $\Delta^n_{[s,t]} = \bigcup_{j=0}^n A^n_j$. $A^n_j = \Delta^j_{[s,u]} \times \Delta^{n-j}_{[u,t]}$.

$$\Rightarrow \mathbb{X}^n_{s,t} = \sum_{j=0}^n \mathbb{X}^j_{s,u} \otimes \mathbb{X}^{n-j}_{u,t}$$

iii) Note if $X \in C^\alpha$. then $\|\mathbb{X}^n_{s,t}\| \lesssim_n |t-s|^{n\alpha}$ as well. So it's necessary!

<u>Cor</u>. $S(X)_{s,u} \otimes S(X)_{u,t} = S(X)_{s,t}$

<u>Def</u>: $\|\mathbf{X}; \widetilde{\mathbf{X}}\|_{(\alpha, N_\alpha)} \overset{\Delta}{=} \sum_1^{N_\alpha} \|\mathbb{X}^n - \widetilde{\mathbb{X}}^n\|_{n\alpha}$ distance.

(2) Geometric Rough Path:

① <u>Def</u>: α-rough path $\mathbf{X}$ is geometric if $\exists\, x^{(n)} \in C^\infty$. s.t. $\|\mathbf{X}; \mathbf{X}^{(n)}\|_{(\alpha, N_\alpha)} \to 0$. where $\mathbf{X}^{(n)}$ is canonical lift of $x^{(n)}$.

Lemma (Shuffle product formula)

For $\mathbf{X}$: α-Hölder geometric rough path

$$\Rightarrow X^m_{st} \otimes X^n_{st} = \sum_{\sigma \in S(m,n)} P_\sigma(X^{m+n}_{st}).$$

$S(m,n)$ is set of (m,n)-shuffle σ. s.t.

$\sigma(1) < \cdots < \sigma(m)$, $\sigma(m+1) < \cdots < \sigma(m+n)$.

Pf: By def, consider $X \in C^\infty$ and use the canonical lift. then apply approxi.

$$LHS = \int_{\substack{s<t_1<\cdots<t_m<t \\ s<t_{m+1}<\cdots<t_{m+n}<t}} dX_{t_1} \otimes \cdots \otimes dX_{t_{m+n}}$$

$$= \sum_{\sigma \in S(m,n)} \int_{s<\sigma^{-1}(1)<\cdots<\sigma^{-1}(m+n)} dX_{t_1} \otimes \cdots \otimes dX_{t_{m+n}}$$

$$= RHS$$

Def: i) N^{th} order free nilpotent group over V is $G^N(V) \triangleq \{ g \in T_1^N(V) \mid g_m \otimes g_n = \sum_{\sigma \in S(m,n)} P_\sigma(g_{m+n}).\ \forall m,n \} \subset T_1^N(V)$.

ii) α-Hölder rough path is weakly geometric if it takes value in $G^N(V)$.

Rmk: i) $G^N(V)$ is exponential of Lie poly's so it's actual group.

ii) As before, if $\dim V < \infty$, then the weaker one is equi. with this one.

② Characterization:

Def: Denote $T(V) := T^{\infty}(V)$, $T_1(V) := T^{\infty}_1(V)$

and $T_0(V) := \{ \xi \in T(V) \mid \xi_0 = 0 \}$.

i) $\pi_n : T(V) \to V^{\otimes n}$. canonical proj.

ii) $\exp : T_0(V) \to T_1(V)$ $\quad \exp(\xi) = \sum_{n \geq 0} \xi^{\otimes n}/n!$

$\log : T_1(V) \to T_0(V)$ $\quad \log(\xi) = \sum_{n \geq 1} (-1)^{n-1} \frac{(\xi - 1)^{\otimes n}}{n}$

Rmk: $\exp \circ \log = id_{T_1(V)}$. $\log \circ \exp = id_{T_0(V)}$

iii) $\xi \in T_1(V)$ is group-like, if it satisfies:

$\xi_m \otimes \xi_n = \sum_{\sigma \in S(m,n)} P_\sigma(\xi_{m+n})$. $\forall m, n \geq 0$.

iv) Set Lie bracket: $[\xi, \eta] = \xi \otimes \eta - \eta \otimes \xi$.

in $T_0(V)$. $\mathcal{L}^1(V) = V$. $\mathcal{L}^n(V) = [V, \mathcal{L}^{n-1}(V)]$.

Rmk: i) $[\ ,\ ]$ satisfies Jacobi's indentity:

$\sum_{cyc} [\xi, [\gamma, \eta]] = 0 \quad \forall \xi, \eta, \gamma \in T(V)$.

ii) $\forall \xi \in \mathcal{L}^n(V)$ are called homogenous Lie poly. of degree n, which can be written in: $\xi = [v_1, [v_2, \cdots [v_{n-1}, v_n] \cdots]$.

v) Lie series is $\xi = (0, \xi_1, \cdots \xi_n \cdots) \in T_0(V)$.

s.t. $\xi_n \in \mathcal{L}_n$. $\forall n \geq 1$. We denote such

span by $\mathcal{L}(V)$.

Thm. For $\xi \in T_1(V)$. ξ is group-like $\Leftrightarrow$ $\log \xi$ is

a Lie series. $\overset{\text{if } \dim V < \infty}{\Leftrightarrow}$ $\exists (V_i)_1^n$. s.t. $\xi = e^{V_1} \otimes \cdots \otimes e^{V_n}$.

Thm. (Lyons and Victoir's extension)

$\forall \mathbb{X}_t \in T_1^{N_\alpha}(V)$. α-Hölder rough path.

there exists unique extension $\mathbb{X}_t = ($

$\mathbb{X}_t$. $\mathbb{X}_t^{N_\alpha+1} \cdots) \in T_1(V)$. satisfies Chen's

relation and α-Hölder conti.

Besides, $\mathbb{X}$ is weakly geometric $\Rightarrow$ $\mathbb{X}_{s,t}$

is group-like.

Cor. $\mathbb{X}$ is weakly geometric $\Rightarrow$ Lyons'

extension $\mathbb{X}$ is Lie series.

Cor. $\forall v \in V$. $w \in V$. $\Rightarrow$ $\log(e^v \otimes e^w)$ is Lie series.

Pf: Note $e^v \otimes e^w$ is canonical lifting

of $x_t = \begin{cases} tv. & 0 \leq t \leq 1 \\ v + (t-1)w. & 1 \leq t \leq 2 \end{cases}$ on

$[0, 2]$. $\Rightarrow$ group-like.

(3) Controlled Rough Path:

Def: i) $Y_t = (Y_t^0, \dots Y_t^{N-1})$, $Y_t^0 \in U$. $Y_t^k \in \mathcal{L}(V^{\otimes k}, U)$.

Y is α-Hölder rough path controlled by $\mathbb{X}$. α-Hölder rough path if:

$$R Y_{s,t}^i := \begin{cases} Y_t^i - Y_s^i - \sum_{j}^{N_\alpha - 1 - i} Y_s^{i+j} X_{s,t}^j & 0 \le i \le N_\alpha - 2. \\ Y_t^{N_\alpha - 1} - Y_s^{N_\alpha - 1}. & i = N_\alpha - 1. \end{cases}$$

Satisfies: $\| R Y^i \|_{(N_\alpha - i)\alpha} < \infty$. $\forall i \le N_\alpha - 1$.

ii) Denote space of such rough path by $D_{X,\alpha}(U)$.

equipped with $\| Y \|_{X,\alpha} = \sum_0^{N-1} \| R^i Y \|_{(N-i)\alpha}$. and

norm $|||Y|||_{X,\alpha} \triangleq \sum_0^{N-1} | Y_0^i | + \| Y \|_{X,\alpha}$.

Rmk: Y^i can be seen as i^{th} "derivative" of Y. And RY^i is reminder of Taylor expansion of Y w.r.t $\mathbb{X}$.

Thm. Y^i are unique if $\mathbb{X}$ is truly α-rough.

i.e. $\overline{\lim_{t \downarrow s}} \frac{\| X_{s,t}^i \|}{|t-s|^{(i+1)\alpha}} = \infty$. $\forall i \le N_\alpha - 1$

prop. $Y \in D_{X,\alpha}(U) \Rightarrow Y^i \in \alpha$-Hölder conti.

Cor. $(D_{X,\alpha}(U), |||\cdot|||_{X,\alpha})$ is Banach space.