Space of Rough Paths.

(1) Definitions:

Def: $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $\mathscr{C}^\alpha([0,T], V) \triangleq \{ (X, \mathbb{X}) \in C^\alpha \times C_2^{2\alpha} \mid$ it satisfies Chen's relation:

i.e. $\mathbb{X}_{s,t} = \mathbb{X}_{s,u} + \mathbb{X}_{u,t} + X_{s,u} \otimes X_{u,t} \}$. is

space of α-Hölder rough paths on V.

Rmk: i) $\mathbb{X}$ is enhancement of X. postulated

as the value of $\int_s^t X_{s,r} \otimes dX_r$.

If $X \in C^\infty$. we can simply define

$\mathbb{X}_{s,t} := \int_s^t X_{s,r} \otimes dX_r$. (Canonical lift)

ii) Actually, $t \mapsto (X_{0,t}, \mathbb{X}_{0,t})$ is

enough to determine $\mathbb{X}$.

since by chen's relation, $\mathbb{X}_{t,t} = 0$. $\forall t$

$\mathbb{X}_{u,t} = \mathbb{X}_{0,t} - \mathbb{X}_{0,u} - X_{s,t} \otimes X_{s,t}$. $\forall s \le t$.

iii) $\mathscr{C}^\alpha$ is closed subset of Banach

space $C^\alpha \otimes C_2^{2\alpha}$ with norm $\|\cdot\|_\alpha + \|\cdot\|_{2\alpha}$. But it's not LS. (by Chen's)

Def: Distance in $\mathscr{C}^\alpha$ is defined by $\|\mathbf{X};\tilde{\mathbf{X}}\|_\alpha$

$\triangleq \|X-\tilde{X}\|_\alpha + \|\mathbb{X}-\tilde{\mathbb{X}}\|_{2\alpha}$

Rmk: i) It's pseudometric. But $|X_0-\tilde{X}_0|$

$+ \|\cdot;\cdot\|_\alpha$ is real metric.

ii) The enhancement $\mathbb{X}$ is never unique. Note for $F \in C^{2\alpha}$.

set $\tilde{\mathbb{X}}_{s,t} = \mathbb{X}_{s,t} + F_{s,t}$ generate a new lift satisfying Chen's.

Conversely, for $(X,\mathbb{X})$ and $(X,\tilde{\mathbb{X}})$.

set $G = \mathbb{X}_{s,t} - \tilde{\mathbb{X}}_{s,t}$. $\overset{\text{Chen's}}{\Rightarrow}$ $G_{s,t} = G_{s,u} + G_{u,t}$

So G is a real path (increment).

iii) The non-uniqueness is intuitive if we think different setting of $\int_s^t X_{s,u}\, dX_u$, e.g. Itô, Stratonovich.

Thm. (Lyons-Victoir extension)

For $X \in \mathscr{C}^\alpha$, $\alpha \le 1$. $\exists\, \mathbb{X} \in C_2^{2\alpha}$ st. Chen's holds.

Rmk: If $\alpha > \frac{1}{2}$. Then $\mathbb{X}$ is uniquely defined by Young integral. ($\alpha = \frac{1}{2}$ will fail)

(2) Geometric Rough path:

Def: i) $\mathscr{C}_g^{\alpha} := \{ (X, \mathbb{X}) \in \mathscr{C}^{\alpha} \mid (e_i^* \otimes e_j^*)(\mathbb{X}_{s,t}) + (e_j^* \otimes e_i^*)(\mathbb{X}_{s,t}) = e_i^*(X_{st})\, e_j^*(X_{st}) \}$.

the space of α-Hölder weakly GRPs.

Rmk: i) Actually. RHS $= \int_s^t e_i^*(X_{s,r})\, d e_j^*(X_r) + \int_s^t e_j^*(X_{s,r})\, d e_i^*(X_r)$

ii) Set $\mathrm{sym}(\mathbb{X}_{st}) := \big((e_i^* \otimes e_j^*)(\cdots) + (e_j^* \otimes e_i^*)(\cdots) \big)/2$ in sense of "matrix". $\Rightarrow \mathrm{sym}(\mathbb{X}_{s,t}) = \frac{X_{st} \otimes X_{st}}{2}$.

ii) $\mathscr{C}_g^{0,\alpha}$ is space of α-Hölder GRPs. defined by

$$\mathscr{C}_g^{0,\alpha} := \overline{\{ (X, \mathbb{X}) \mid X \in C^\infty,\ \mathbb{X} \text{ is canonical} \}}^{\|\cdot\|_\alpha}$$

Rmk: i) $\mathscr{C}_g^{0,\alpha} \subsetneq \mathscr{C}_g^{\alpha}$. actually $\mathscr{C}_g^{0,\alpha}$ is separable if V is. But $\mathscr{C}_g^{\alpha}$ isn't

ii) $\forall\ \frac{1}{3} < \beta < \alpha \leq \frac{1}{2}$. if $\dim V < \infty$. then:

$$\mathscr{C}_g^{\alpha} \subset \mathscr{C}_g^{0,\beta}.$$

prop. (Geodesic Approxi.)

$\forall\ \mathbf{X} \in \mathscr{C}_g^{\beta}([0,T], \mathbb{R}^d)$. $\exists\ X^n \in C^\infty$. s.t.

$(X^n, \mathbb{X}^n) := (X^n, \int X^n_{0,t} \, dX^n_t) \xrightarrow{n} (X, \mathbb{X}) = \mathbf{X}$

on $[0,T]$. Besides, $\sup_n \| (X^n, \mathbb{X}^n) \|_\beta \lesssim \| X, \mathbb{X} \|_\beta$.

(3) Control Paths:

Recall we have $\int_0^T f(X_s) \, dX_s = \lim_{\pi \to 0} \sum f(X_s) X_{st} + Df(X_s) \mathbb{X}_{s,t}$ for sufficient smooth func. f.

But it can't be extended for general one

Def: i) For $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $X \in C^\alpha$. We say (Y, Y') is control path w.r.t X if $Y \in C^\alpha([0,T], V)$, $Y' \in C^\alpha([0,T], \mathcal{L}(V, W))$, $R^Y \in C_2^{2\alpha}([0,T], W)$ st. $Y_{st} = Y'_s X_{st} + R^Y_{st}$.

ii) Denote the space of it by $\mathcal{D}_X^{2\alpha}([0,T], W)$.

iii) Equip $\mathcal{D}_X^{2\alpha}$ by the norm $\| Y, Y' \|_{\mathcal{D}_X^{2\alpha}} \triangleq |Y_0| + |Y'_0| + \| Y' \|_\alpha + \| R^Y \|_{2\alpha}$

Rmk: Y' won't be unique generally. e.g. $X, Y \in C^{2\alpha}$ and then $\forall Y' \in C$ satisfies the condition!

Too much regularity on X, Y lead to less information of Y'.

① **Prop.** $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. $X \in C^\alpha$. $f \in C^2$. Then:

$(f(X), Df(X)) \in \mathcal{D}_X^{2\alpha}$.

Pf: $|R_{s,t}^{f(X)}| = |f(X_t) - f(X_s) - Df(X_s) X_{st}|$

$= |\int_0^1 (Df(X_s + r X_{st}) - Df(X_s)) X_{st} dr|$

$\leq \|f\|_{C^2} \|X\|_\alpha^2 |t-s|^{2\alpha}$.

Prop. $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, $X \in C^\alpha$. (Y, Y'), $(Z, Z') \in \mathcal{D}_X^{2\alpha}$

$\Rightarrow YZ \in \mathcal{D}_X^{2\alpha}$ with $(YZ)' = YZ' + Y'Z$.

Pf: $R_{s,t}^{YZ} := (YZ)_{st} - (YZ)'_s X_{st}$.

$= Y_s R_{st}^Z + R_{st}^Y Z_s + Y_{st} Z_{st}$

Rmk: $\mathcal{D}_X^{2\alpha}$ is algebra with the norm $\|\cdot\|_{\mathcal{D}_X^{2\alpha}}$

② Doob-Meyer Decomp.:

Next we consider the uniqueness of the derivative Y'.

Note that if we set $V = \mathbb{R}^1$, then:

$$Y'_s = \frac{Y_{st}}{X_{st}} - \frac{R_{st}^Y}{|s-t|^{2\alpha}} \frac{|t-s|^{2\alpha}}{X_{st}}.$$

So: $\lim_{t \downarrow s} \frac{Y_{st}}{X_{st}} = Y'_s$ if $\overline{\lim_{t \downarrow s}} \frac{|X_{st}|}{|s-t|^{2\alpha}} = \infty$.

Def: i) $X \in \mathcal{C}^\alpha$ is rough at time $s \in [0,T]$ if

$\forall v^* \in V^* \setminus \{0\}$. we have $\overline{\lim_{t \downarrow s}} \frac{|v^*(X_{st})|}{|t-s|^{2\alpha}} = \infty$.

ii) We call X is truly rough if its rough time is dense in $[0,T]$.

Prop. $(X, \mathbb{X}) \in \mathcal{C}^\alpha$. $(Y, Y') \in \mathcal{D}_X^{2\alpha}$. If X is rough at time s. Then: $Y_{st} = O(|t-s|^{2\alpha})$ $(t \to s)$ and $Y_s' = 0$.

Cor. X is truly rough $\Rightarrow \forall (Y, Y'), (Y, \tilde{Y}') \in \mathcal{D}_X^{2\alpha}$, we have $Y' \equiv \tilde{Y}'$.

Pf: Note we have: $O(|t-s|^{2\alpha}) = (Y_s' - \tilde{Y}_s') X_{st}$

Pf: Note $\frac{Y_s' X_{st}}{|t-s|^{2\alpha}} = \frac{Y_{st}}{|t-s|^{2\alpha}} + O(1) = O(1)$.

$\forall w^* \in u^*$. Set $v^*(v) \triangleq w^*(Y_s' v)$. $v \in V$.

So: $\frac{|v^*(X_{st})|}{|s-t|^{2\alpha}} = O(1) \nrightarrow 0$.

$\Rightarrow v^* \equiv 0$. $\Rightarrow Y_s' v \equiv 0$. $\forall v \Rightarrow Y_s' \equiv 0$.

Thm. (Doob-Meyer Decomp.).

$\bar{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha$. $(Y, Y') \in \mathcal{D}_X^{2\alpha}$. If X is rough at time s. Then: $\int_s^t Y \, d\bar{X} = O(|t-s|^{2\alpha})$ and $Y_s = 0$.

Pf: $(\int Y d\mathbf{X}, Y) \in \mathcal{D}_X^{2\alpha}$. Apply prop. above.

Cor. If X is truly rough. $(\tilde{Y}, \tilde{Y}') \in \mathcal{D}_X^{2\alpha}$

and $Z, \tilde{Z} \in C([0,T], U)$. Satisfy:

$$\int_0^\cdot Y d\mathbf{X} + \int_0^\cdot Z dt = \int_0^\cdot \tilde{Y} d\mathbf{X} + \int_0^\cdot \tilde{Z} dt.$$

Then: $(Y, Y') \equiv (\tilde{Y}, \tilde{Y}')$. $Z \equiv \tilde{Z}$.

Rmk: It's alike the case of stochastic integration.

Thm. $\forall$ Brownian motion on $V = \mathbb{R}^d$ is truly rough w.r.t Hölder exponent $\alpha \in (\frac{1}{4}, \frac{1}{2})$ a.s.

Pf: It follows from law of iterated log.

(4) BM as rough paths:

Thm. (Kolomogorov Criteria).

$(X, \mathbb{X}): \Omega \times [0,T] \to \mathbb{R}^d \times \mathbb{R}^{d\times d}$ measurable stochastic and satisfies Chen's relation a.s.

For $q \geq 2$, $\beta > 1/q$. If $\exists C > 0$, $\forall (s,t) \in \Delta_{[0,T]}$

s.t. $\|X_{s,t}\|_{L^q} \leq C|t-s|^\beta$. $\|\mathbb{X}_{s,t}\|_{L^{q/2}} \leq C|t-s|^{2\beta}$.

Then. $\forall \alpha \in [0, \beta - \frac{1}{q})$. $\exists$ modification $(\tilde{X}, \tilde{\mathbb{X}})$

and r.v. $K_\tau \in L^2$. $\mathbb{K}_\tau \in L^{2/2}$. s.t. $\forall (s,t)$.

$|\tilde{X}_{st}| \leq K_\tau |s-t|^\tau$. $|\tilde{\mathbb{X}}_{st}| \leq \mathbb{K}_\tau |s-t|^{2\tau}$.

Pf: It's identical with classical one.

Rmk: We can consider to apply classical KC on $\mathbf{X} = (X, \mathbb{X})$ but not on $\mathbb{X}$. Then, we can only get $\mathbb{X} \in C^\tau$. Since we lose Chen's relation!

① Itô BMs:

Lemma (Discrete BDG inequi.).

For $\forall$ discrete time c.l.m. (X_n).

QV of X is define by $[X]_n :=$ $\sum_{k=1}^n (\delta X)_k^2$. where $\delta X_k \triangleq X_k - X_{k-1}$.

Then X also satisfies BDG inequi.

We enhance B by $\mathbb{B}^{Itô}_{s,t} := \int_s^t B_{s,r} \otimes dB_r$.

in sense of Itô integral. which satisfies Chen's

prop. $\forall \alpha \in (\frac{1}{3}, \frac{1}{2})$. $\mathbf{B} = (B, \mathbb{B}^{Itô}) \in \mathscr{C}^\alpha([0,T], \mathbb{R}^d)$.

Pf: By scaling: $\|B_{st}\|_{L^2} = \|B_1\|_{L^2} |t-s|^{\frac{1}{2}}$

$$\mathbb{E} |\mathbb{B}^{2\pi}_{s,t}|^{\frac{2}{2}} = \mathbb{E} |\int_s^t B_{s,r} \otimes dB_r|^{\frac{2}{2}}$$

$$\overset{BDG}{\leq} C_2 \, \mathbb{E} |\int_s^t B^2_{s,r} dr|^{\frac{2}{4}}$$

$$\leq C_2 \, \mathbb{E} \sup |B_{s,r}|^{\frac{2}{2}} |t-s|^{\frac{2}{4}}$$

$$\overset{BDG}{\leq} C_2^2 \, |t-s|^{2/2}.$$

Apply Kolmogorov criteria on $\mathbb{B}$.

Rmk: Note $\mathrm{Sym}(\mathbb{B}^{2\pi}_{s,t}) = \frac{1}{2}(\int_s^t B^i_{s,t} dB^j_t + \int_s^t B^j_{s,t} dB^i_t)_{d\times d}$

$$\overset{2\pi}{=} \frac{1}{2}(B^i_{st} B^j_{st} - \langle B^i, B^j\rangle_{s,t})_{d\times d}$$

$$= \frac{1}{2}(B_{st} \otimes B_{st} - (t-s) I).$$

$\Rightarrow \mathbb{B} \notin \mathcal{C}^{\alpha}_{g}([0,T], \mathbb{R}^d)$!

② Stratonovich BM:

Recall $\int_0^t Y_s \circ dX_s = \int_0^t Y_s dX_s + \frac{1}{2}\langle Y, X\rangle_t$

Rmk: The advantage of using Stratonovich integral is that it obeys first order calculus.

i.e. $X_t Y_t = X_0 Y_0 + \int X_t \circ dY_t + \int Y_t \circ dX_t$

$$f(X_t) = f(X_0) + \int_0^t Df(X_s) \circ dX_s$$

Set $\mathbb{B}^{strat}_{s,t} \overset{\Delta}{=} \int_s^t B_{sr} \otimes \circ dB_r$

$= \mathbb{B}^{Itô}_{s,t} + \frac{1}{2}(t-s)I.$

$\Rightarrow \mathrm{sym}(\mathbb{B}^{strat}_{s,t}) = B_{s,t} \otimes B_{s,t}/2$

<u>Thm.</u> $\forall \alpha \in (\frac{1}{3}, \frac{1}{2})$. $(B, \mathbb{B}^{strat}) \in \mathscr{C}_g^{0,\alpha}([0,T], \mathbb{R}^d)$.

<u>Pf:</u> Note $(t-s)I \in C^1$.

So: $(B, \mathbb{B}^{strat}) \in \mathscr{C}_g^{\beta} \subset \mathscr{C}_g^{0,\alpha}$.

$\forall \beta \in (\alpha, \frac{1}{2})$. for every fix α.

<u>Rmk:</u> We define antisymmetric part of $(\mathbb{X}_{s,t})$

by $\mathrm{Anti}(\mathbb{X}_{s,t})_{ij} = \frac{1}{2}(\int_s^t X^i_{st} dX^j_t + \int_s^t X^j_{st} dX^i_t)$

under the case of $V = \mathbb{R}^d$.

If $d=1 \Rightarrow \mathbb{X}_{s,t} = X^2_{s,t}/2$ is a simple lift.

The new info is encoded in anti-part in fact.

<u>Prop.</u> Set $B^{(n)}_t = B_t$ if $t = \frac{kT}{2^n}$ and linearly

intepolate $B^{(n)}_t$ on $[kT/2^n, (k+1)T/2^n]$.

Then: $(B^{(n)}, \int_0 B^{(n)} \otimes dB^{(n)}) \xrightarrow[\mathscr{C}_g^{\alpha}]{a.s.} (B, \mathbb{B}^{strat})$

<u>Pf:</u> Note $B^{(n)} = E(B \mid \sigma(B_{kT/2^n}, 0 \le k \le 2^n))$

(Check indept. by cov. and use Markov)

Apply Levy's Thm on $(B^{(n)})$, and Doob's