Geometric Rough Paths

(1) Lyon's Lift:

Lemma. i) $g_1, g_2 \in T^{N+1}(\mathbb{R}^d)$. Then:

$$\pi_{0,N}(g_1) \otimes \pi_{0,N}(g_2) - \pi_{0,N}(g_1+g_2) =$$

$$g_1 \otimes g_2 - (g_1+g_2)$$

ii) $x^1, x^2 \in C^{1\text{-var}}([s,u], \mathbb{R}^d)$. s.t. $S_N(x^1)_{s,u} = S_N(x^2)_{s,u}$

Then. $\exists\, C = C(N) > 0$. $l = \int_s^u |dx^1| \vee \int_s^u |dx^2|$.

$$|S_{N+1}(x^1) - S_{N+1}(x^2)| \leq C\, l^{N+1}$$

Pf: i): is trivial. ii): $LHS = |\pi_{N+1}(S_{N+1}(\cdots) - \square)|$.

prop. For $x \in C^{1\text{-var}}([0,T], \mathbb{R}^d)$. Then: $\forall N \geq [p]$.

$\exists\, C = C(N,p) > 0$. right-conti. at p. s.t. $\forall s \leq t$.

$$\| S_N(x) \|_{p\text{-var}, [s,t]} \leq C \| S_{[p]}(x) \|_{p\text{-var}, [s,t]}$$

Pf: prove: $\forall N \geq [p]$. $\exists\, C_1(p)$ conti. (right) at p.

$$\| S_{N+1}(x) \|_{p\text{-var}, [s,t]} \leq \| S_N(x) \|_{p\text{-var}, [s,t]} \cdot C_1(p)$$

Set $\mathbf{X} = S_N(x)$. $\eta = S_{N+1}(x)$. $W(s,t) = \| \mathbf{X} \|^p_{p\text{-var}, [s,t]}$

Denote $x^{s,t}$ is geodesics of $\mathbf{X}_{s,t}$

Define $\Gamma_{s,t} = \eta_{s,t} - S_{N+1}(x^{s,t})_{s,t}$

1) Note $\eta_{s,t} \otimes \eta_{t,u} = \eta_{s,u}$ and

$\pi_{0,N}(\eta_{s,t}) = \pi_{0,N}(S_{N+1}(x^{s,t})_{s,t})$

Apply the Lemma i). we have:

$$I_{s,u} - (I_{s,t} + I_{t,u}) = S_{N+1}(x^{s,t} \sqcup x^{t,u})_{s,u}$$
$$- S_{N+1}(x^{s,u})_{s,u}$$

2) Apply Lemma ii). we have:

$$|I_{s,u} - (I_{s,t} + I_{t,u})| \leq C_2 w(s,u)^{\frac{N+1}{p}}$$

$$|I_{s,u}| \leq C_3 |x|^{N+1}_{1\text{-var},[s,t]}$$

Use Young-Loève bdd:

$$|\eta_{s,t} - S_{N+1}(x^{s,t})_{s,t}| \leq C_4 w(s,t)^{\frac{N+1}{p}} / (1-2^{1-\frac{N+1}{p}}).$$

$$S_0: |\pi_{N+1}(\eta_{s,t})|^{\frac{1}{N+1}} \leq C_{N,p} w(s,t)^{\frac{1}{p}}.$$

By equi. of homo. norm: $(\sup_k |\pi_k(\cdot)|^{\frac{1}{k}} \sim \|\cdot\|_{C\text{-}C})$

$$\|\eta_{s,t}\| \leq \widetilde{C}_{N,p} w(s,t)^{\frac{1}{p}}. \Rightarrow \|\eta\|_{p\text{-var}} \underset{p,N}{\lesssim} \|X\|_{p\text{-var}}$$

<u>Rmk:</u> $S_{N+1}(x^{s,t})$ is some kind of good approxi. (having shortest path) of $S_{N+1}(x)$.

<u>Def:</u> $\eta \in C_0^{p\text{-var}}([0,T], G_N(\mathbb{R}^d))$. $X \in C_0^{p\text{-var}}([0,T], G_{[p]})$. for $N \geq [p]$. η is p-Lyons lift of X of order N if $\pi_{0,[p]}(\eta) = X$.

Lemma. $x, \eta \in C_0([0,T], G^{N+1}(\mathbb{R}^d))$ s.t. $\pi_{0,N}(x) = \pi_{0,N}(\eta)$. Then: $h_t \triangleq \log(x_t^{-1} \otimes \eta_t) \in \mathcal{L}^N(\mathbb{R}^d)$ and $|h_{s,t}| \lesssim_N (\|x_{s,t}\| + \|\eta_{s,t}\|)^N$.

Pf: Lemma. i) $\|h\|_{\mathbb{R}^d} = \|e^h\|_{C-C}$. $\forall f \in \mathbb{R}^d$.

ii) $\|\cdot\|_{T^N} \sim \|\cdot\|_{C-C}$

$$|h_{s,t}| \overset{\text{homo}}{\underset{\text{equi}}{\leq}} C \|e^{h_{s,t}}\|^N$$

$$\leq C\|x_{s,t}^{-1} \otimes \eta_{s,t}\|^N \overset{\text{sym}}{\leq} C(\|x_{s,t}\| + \|\eta_{s,t}\|)^N.$$

Thm. For $N \geq [p] \geq 1$. $x \in C_0^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d))$

Then $\exists$ unique Lyon lift of x at order N.

denote by $S_N(x)$. in sense of:

$S_N: C_0^{p\text{-var}}([0,T], G^{[p]}) \longrightarrow C_0^{p\text{-var}}([0,T], G^N)$

is bijection of $\pi_{0,[p]}$ and BLF. s.t.

$\exists C_{N,p}$ conti. at p. $\|x\|_{p\text{-var}, [s,t]} \underset{C_{N,p}}{\lesssim} \|S_N(x)\|_{p\text{-var}, [s,t]} \underset{C_{N,p}}{\lesssim} \|x\|_{p\text{-var}, [s,t]}$

Rmk: For $N \leq [p]$. We can set $S_N = \pi_{0,N}$.

Pf: i) Existence:

By geodesics approxi. $\exists (x_n) \subset C^{1\text{-var}}$

s.t. $S_{[p]}(x_n) \xrightarrow{n} x$. $\sup_n \|S_{[p]}(x_n)\|_{p\text{-var}} < \infty$

For using Ascoli Lemma in G^n.

Consider in $\|\cdot\|_{p+\varepsilon\text{-var}}$. $\varepsilon > 0$.

s.t. $[p+\varepsilon] = [p]$. Apply prop. above.

$$\|S_N(X_n)_{s,t}\| \le C_{p+\varepsilon} \|S_{[p]}(X_n)\|_{(p+\varepsilon)\text{-var}, [s,t]} \quad (*)$$

$\Rightarrow (S_N(X_n))$ also satisfies equiconti.

and bdd. $\exists\, Z \in G_N(\mathbb{R}^d)$. s.t.

$S_N(X_n) \xrightarrow{n} Z$. $(n \to \infty)$. lift of X.

Besides. $\|Z_{s,t}\| \overset{(*)}{\le} C_{p+\varepsilon} \overline{\lim} \|S_{[p]}(X_n)\|_{(p+\varepsilon)\text{-var}, [s,t]}$

$\overset{\text{Ascoli}}{=} C_{p+\varepsilon} \|X\|_{(p+\varepsilon)\text{-var}, [s,t]}$.

$\longrightarrow C_p \|X\|_{p\text{-var}}$

$\|X\|_{p\text{-var}} \le \|Z\|_{p\text{-var}}$ is trivial by $\pi_{0,[p]}(Z) = X$

2°) Uniqueness :

If $Z_t, \tilde{Z}_t$ satisfy conditions. Set $h_t = \log(Z_t^{-1} \otimes \tilde{Z}_t)$. By induction on $M \ge [p]$.

where $Z_t, \tilde{Z}_t \in C^{p\text{-var}}([0,T], G^M(\mathbb{R}^d))$.

For $M+1$ case : Note $\pi_{0,M}(Z_t) = \pi_{0,M}(\tilde{Z}_t)$.

Apply the Lemma : $|h_{s,t}| \le C(\|Z_{s,t}\|^{M+1} + \|\tilde{Z}_{s,t}\|^{M+1})$.

$\le C\, \tilde{W}(s,t)^{(M+1)/p}$.

Since $\tilde{W}(s,t) \overset{\Delta}{=} \|\tilde{Z}_{t,s}\|_{p\text{-var}} + \|Z_{t,s}\|_{p\text{-var}} \in C^{1\text{-var}}$

$\Rightarrow h$ is $\frac{p}{M+1} < 1$ variation $\Rightarrow h \equiv 0$.

Thm. (Continue uniformly)

For $N \geq [p]$.

$S_N : C_0^{p\text{-var}}([0,T], G^{[p]}) \longrightarrow C_0^{p\text{-var}}([0,T], G^N)$ and

$S_N : C_0^{1/p\text{-Höl}}([0,T], G^{[p]}) \longrightarrow C_0^{1/p\text{-Höl}}([0,T], G^N)$ are

uniformly conti. in bdd set under $d_{p\text{-var}} / d_{1/p\text{-Höl}}$.

Rmk: Lyons lift has local Lipschitz property.

Rmk: For $N \geq [p]$. S_N and $\pi_{0,[p]}$ are bijection between

$\mathcal{N}([0,T], G^N(\mathbb{R}^d))$ and $\mathcal{N}([0,T], G^{[p]}(\mathbb{R}^d))$. where

$\mathcal{N} = C_0^{p\text{-var}}, C_0^{1/p\text{-Höl}}, C_0^{0,p\text{-var}}$ or $C_0^{0,\frac{1}{p}\text{-Höl}}$.

But $\mathcal{N}([0,T], G^N) \longrightarrow \pi_{0,[p]-1}(X) \in \mathcal{N}([0,T], G^{[p]-1})$

isn't bijection!

(2) **Def:** $C^{p\text{-var}}([0,T], G^{[p]}) / C^{1/p\text{-Höl}}([0,T], G^{[p]})$ are set of

weakly geometric p-var / $1/p$-Hölder rough path.

$C_0^{p\text{-var}}(\ldots) / C_0^{1/p\text{-Höl}}(\ldots)$ are geometric rough path.

Thm. (Invariance under Lip map)

If $\phi \in \mathrm{Lip}^{\gamma}_{loc}(\mathbb{R}^d, \mathbb{R}^e)$. $\gamma > p$. There exists unique

conti. and uniformly conti. on bdd set map ϕ^*:

$C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^d)) \longrightarrow C^{p\text{-var}}([0,T], G^{[p]}(\mathbb{R}^e))$.

Besides, $\phi_* \mathbf{X} = S_{[p]}(\phi \circ X)$. when $X \in C^{1\text{-var}}$.

Rmk: Note it's unlike the property of Semimart. M is invariant under C^2-map.

Pf: $\varphi \triangleq D\phi \in Lip_{loc}^{\gamma-1}$.

$J_\cdot = \int \phi_* X_\cdot \triangleq \int_0^\cdot \varphi(X_s) \, dX_s$ is a well-def rough integral of $X_s \in C(C^{p\text{-var}})$.

For $x_s \in C^{1\text{-var}}$. note it's just the common RS integral.