McKean-Vlasov Equations

- Fix $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. $\xi^i \in \mathcal{F}_0$, i.i.d. $\mathbb{R}^d$-valued r.v.'s. (W^i) is indept. $\mathbb{F}$-m-dim-BM's
- Consider n-interacting particle system $(X_t^{n,k})_{k=1}^n$

: $dX_t^{n,i} = b(X_t^{n,i}, m_t^n)\,dt + \sigma(X_t^{n,i}, m_t^n)\,dW_t^i$ $(*)$

where $X_0^{n,i} = \xi^i$, $m_t^n = \frac{1}{n}\sum_1^n \delta_{X_t^{n,k}}$

$\Rightarrow$ Next consider its n-limit of (X^k)

Assumptions:

i) $\xi^i \in L^2$. iii) b, σ are bdd.

ii) $b : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \to \mathbb{R}^{d\times m}$

satisfy Lipschitz condition: $\exists L > 0$,

$|b(x,m) - b(x',m')|_{\mathbb{R}^d} + |\sigma(x,m) - \sigma(x',m')|_{\mathbb{R}^{d\times m}}$

$\leq L\,(|x - x'|_{\mathbb{R}^d} + W_2(m,m'))$

(1) McKean-Vlasov Limit:

Lemma. (*) has unique strong solution. $\forall n$.

Pf: Set $\vec{X}_t = (X_t^{n,1}, \cdots X_t^{n,n}) \in \mathbb{R}^{nd}$

$L_n(x_1 \cdots x_n) = \sum_1^n \delta x_k / n$. and

$B(\vec{x}) = (b(x_1, L_n(\vec{x})), \cdots b(x_n, \cdots))^T$

$\Sigma(\vec{x}) = \text{diag}[\sigma(x_1, L_n(\vec{x})) \cdots \sigma(x_n \cdots)]$.

$\Rightarrow d\vec{X}_t = B(\vec{X}_t)dt + \Sigma(\vec{\sigma}_t) d\vec{W}_t$.

Only need to check Lip. condition:

Note $W_2(L_n(\vec{x}), L_n(\vec{\eta})) \leq$ $\quad \pi = \frac{1}{n}\sum_1^n \delta(x_i, \eta_i)$

$\int_{\mathbb{R}^{d \times 2}} |u-v|^2 d\pi = \frac{1}{n}|\vec{x}-\vec{\eta}|^2$.

Motivation:

Note as $n \to \infty$. the interaction of μ_t^n.

become weaker since the contribution of particle i^{th} has order n^{-1}.

$\Rightarrow$ expect: $\hat{\mu}_t \to \mu_t$. (deter. measure flow)

It looks like: $dY_t^i = b(Y_t^i, \mu_t)dt + \sigma(Y_t^i, \mu_t)dW_t^i$

<u>Def</u>: i) $C^d := C([0,T], \mathbb{R}^d)$. $\|x\|_T := \sup_{[0,T]} |x_t|_{\mathbb{R}^d}$

ii) McKean-Vlasov equation is defined by:

$dY_t = b(Y_t, \mu_t)dt + \sigma(Y_t, \mu_t)dW_t$ with

$Y_0 = \xi$. $\mu = \mathcal{L}(Y) \in \mathcal{P}^2(C^d)$.

<u>Rmk</u>: i) μ is seen as C^d-valued r.v.

There's a canonical map:

$\mathcal{P}^2(C^d) \ni \mu \mapsto (\mu_t)_t \in C([0,T], \mathcal{P}^2(\mathbb{R}^d))$

ii) Note we also have:

$\sup_{[0,T]} W_{\mathbb{R}^d,2}(\mu_t, \nu_t) \le W_{C^d,2}(\mu, \nu)$.

(Use $\|\cdot\|_T$ on right side)

iii) We refer to μ rather (μ, Y) as solution since $\mu \mapsto Y$. uniquely

Thm. There's a unique strong solution of MV-equation (Def ii)). Besides, set $\mu^n = \sum_1^n \delta_{X^{n,k}}/n$. We have:

i) $\lim_{n\to\infty} \mathbb{E}(W^2_{C^d,2}(\mu^n, \mu)) = 0$

ii) $(X^{n,1}, \cdots X^{n,k}) \xrightarrow{W} (Y_1 \cdots Y_k)$. (copy of the solutions of MV-equation)

Rmk: i) $(X^{n,1}, \cdots X^{n,i})$ is asymptotically i.i.d.

ii) The kind of limits refer to propagation of chaos: For $\frac{1}{n}\sum_1^n \delta_{X_0^{n,i}} \to m_0 \in P^2(\mathbb{R}^d)$, we have $\frac{1}{n}\sum_1^n \delta_{X_t^{n,i}} \xrightarrow{pr} \mu_t \sim Y_t$ where (Y, μ) is solution and $\xi \sim m_0$. i.e. Chaoticity of initial dist. propagate to $t>0$.

Pf: i) U & E:

Denote: $d_t^2(\mu,\nu) = \inf_{X\sim\mu,\, Y\sim\nu} \mathbb{E}(\|X-Y\|_t^2)$

$dY_t^\mu = b(Y_t^\mu, \mu_t)dt + \sigma(Y_t^\mu, \mu_t)dW_t.$

has unique solution Y^μ.

Set map: $\Phi: \mathcal{P}^2(C^d) \to \mathcal{P}^2(C^d)$.

$\mu \mapsto \mathcal{L}(Y^\mu)$

Next we find $\Phi(\mu)=\mu$. So that

(Y^μ, μ) is solution of MV-equation

By Cauchy and AM-GM:

$$|Y_t^\mu - Y_t^\nu|^2 \le 2t\int_0^t |b(Y_t^\mu,\mu_r) - b(Y_t^\nu,\nu_r)|^2 dt + 2\left|\int_0^t \sigma(Y^\mu\cdots) - \sigma(Y^\nu\cdots)dW_r\right|^2.$$

$$\Rightarrow \mathbb{E}\|Y^\mu - Y^\nu\|_t^2 \overset{\text{Doob.}}{\le} 2t\,\mathbb{E}\left(\int_0^t |b\cdots - b\cdots|^2\right) + 8\,\mathbb{E}\left(\int_0^t |\sigma\cdots - \sigma\cdots|^2 dr\right)$$

$$\overset{\text{Lip.}}{\le} (8+2t)\,2L^2\,\mathbb{E}\left(\int_0^t \|Y^\mu - Y^\nu\|_r^2 + W_2^2(\mu_r,\nu_r)\right)$$

Apply Gronwall's inequality: we have,

$$\mathbb{E}\ \|Y^{\mu}-Y^{\nu}\|_t^2 \leq C_{L,T}\int_0^t W_2^2(\mu_r,\nu_r)\,dr.$$

$$\leq C\int_0^t \lambda_r^2(\mu,\nu)\,dr$$

$$\Rightarrow \lambda_t^2(\phi(\mu),\phi(\nu)) = LHS \leq C\,t\,\lambda_t^2(\mu,\nu).$$

Choose t small enough and use Fixed point Thm. set $[0,T]=\bigcup[t_k,t_{k+1})$. s.t. Δt is small enough.

2) Limit:

Set $\nu^n = \frac{1}{n}\sum_1^n \delta_{Y^i}$, $\mu^n = \frac{1}{n}\sum_1^n \delta_{X^{n,i}}$.

$Y_i \in L^2 \Rightarrow \nu^n \xrightarrow{W^2} \mu.\ (n\to\infty)$

$$\lambda_t^2(\nu^n,\mu^n) \overset{\pi=\frac{1}{n}\sum_1^n\delta(X^{n,i},Y^i)}{\leq} n^{-1}\cdot\sum_1^n \|X^{n,i}-Y^i\|_t^2.\ a.s.$$

Next, we estimate $\|X^{n,i}-Y^i\|_t^2$:

As above of 1): we have,

$$\mathbb{E}(\|X^{n,i}-Y^i\|_t^2) \leq C\,\mathbb{E}(\int_1^t \lambda_r^2(\mu^n,\mu)\,dr)$$

$\Rightarrow d_t^2(\mu^n, \nu^n) \leq C\,\mathbb{E}(\int_1^t d_r^2(\mu^n, \mu)dr)$

So: $\mathbb{E}(d_t^2(\mu^n, \mu)) \underset{\text{triangle}}{\leq} 2C\,\mathbb{E}(\int_1^t d_r^2(\mu^n, \mu)dr) +$

$$2\,\mathbb{E}(d_t^2(\nu^n, \mu))$$

Apply Gronwall's. Set $t = T$.

$$\mathbb{E}(W_2^2(\mu^n, \mu)) \leq e^{2CT}\,\mathbb{E}(W_2^2(\nu^n, \mu)) \xrightarrow{n\to\infty} 0$$

Besides. Note that

$$\mathbb{E}(\sup_{1\leq i\leq k} \|X^{n,i} - Y^i\|_t^2) \underset{C,k}{\lesssim} \mathbb{E}(W_2^2(\mu^n, \mu)) \to 0 \quad (n\to\infty)$$

(2) <u>PDEs from MV equation:</u>

① MV-equation admits a similar Kolmogorov forward equation. But the PDE is nonlinear and nonlocal.

Apply Itô's on $\varphi(Y_s)$. take expectation:

$$\frac{d}{dt}\mathbb{E}(\varphi(Y_t)) = \mathbb{E}(b(Y_t, \mu_t)\cdot\nabla\varphi(Y_t) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^T(Y_t, \mu_t)\nabla^2\varphi(Y_t))$$

Suppose μ_t has density $\mu(t,x)$.

$\Rightarrow \int_{\mathbb{R}^d} \varphi(x)\, \underbrace{\partial_t \mu(t,x)}\, dx \overset{\text{integrate by part}}{=}$

$$\int_{\mathbb{R}^d} \big(\underbrace{-\nabla_x (b(x,\mu_t)\mu(t,x)) + \tfrac{1}{2} Tr(\nabla^2 (\sigma\sigma^T(x,\mu_t)\mu(t,x))}\big)\varphi(x), \quad \forall \varphi \in C^\infty.$$

So we have the PDE w.r.t μ:

$$\partial_t \mu(t,x) = -\nabla_x (b(x,\mu_t)\mu(t,x)) + \cdots \quad (\star)$$

It's called F-P equation

Rmk: i) Nonlinear because of existence of $b(x,\cdot)$ and $\sigma\sigma^T(x,\cdot)$.

ii) Nonlocal because $b(x,\cdot)$ depend on $(\mu(t,x))_{x\in\mathbb{R}^d}$.

iii) Even if μ_t isn't smooth enough. We can consider it as weak solution of the PDE. (Actually it has unique one)

② Alternative for weak solution case:

1') Consider $\varphi \in C_c^\infty$. (L^n defined as before)

$$d\varphi(X_t^{n,i}) \overset{\text{Itô}}{=} L^{\mu_t^n}\varphi(X_t^{n,i})dt + \nabla\varphi(X_t^{n,i})\cdot\sigma(X_t^{n,i}, \mu_t^n)\, dW_t^i$$

Average

$$\Rightarrow d\langle \mu_t^n, \varphi\rangle \overset{\Delta}{=} \langle \mu_t^n, L^{\mu_t^n}\varphi\rangle dt + dM_t^n.$$

where $M_t^n \overset{\Delta}{=} \frac{1}{n}\sum_i \nabla\varphi(X_t^{n,i})\cdot\sigma(X_t^{n,i}, \mu_t^n)\, dW_t^i$

Note $\langle M^n\rangle_t = \frac{1}{n^2}\sum_1^n \int_0^t |\nabla\varphi(X_s^{n,i})\sigma^T(\dots)|^2 \sim O(\frac{t}{n})$

So: $\mathbb{E}((M_t^n)^2) \overset{BDG}{\leq} \frac{Ct}{n} \to 0 \quad (n\to\infty)$

2') Show $(\mu_t^n)_n$ is tight seq in $C(\mathbb{R}^d, \mathcal{P}^2)$.

$\Rightarrow \exists\, \mu^{n_k} \longrightarrow \mu$. So: $d\langle \mu^t, \varphi\rangle = \langle \mu^t, L_{\mu_t}\varphi\rangle$.

3') Show every subseq limit are identical.

(3) <u>Loss of Markov Property</u>:

It's essentially from the corresponding Fokker-Planck equation (✡) is nonlinear.

Def: Solutions of SDE is called a Markov family if:

$\mathbb{P}_x$ is law of solution starting at x. $\Rightarrow$ For the solution with random initial position $\sim m$. Its law $\mathbb{P}_m(A) = \int \mathbb{P}_x(A)\, dm(x)$.

Rmk: If we solve the SDE with every deterministic initial state then we can get the law of randomized initial one by integrating it.

But the solutions of MV-equation isn't Markov family though it has Markov property (By Lip. conditions)

(4) Common Noise:

consider the extension of main model:

$$dX_t^{n,i} = b(X_t^{n,i}, m_t^n)dt + \sigma(X_t^{n,i}, m_t^n)dW_t^i + \sigma_0(X_t^{n,i}, m_t^n)dB_t. \quad X_0^{n,i} = \xi^i.$$

Rmk: Note each $X_t^{n,i}$ is also driven by common BM: (B_t).

SLLN isn't appliable ← In this case, the asymptotic i.i.d. won't hold as before.

① Limit Thms:

consider conditional MV equation:

$$dY_t = b(Y_t, \tilde{m}_t)dt + \sigma(Y_t, \tilde{m}_t)dW_t + \sigma_0(\cdots)dB_t,$$

where $\tilde{m}_t := \mathcal{L}(Y \mid B)$ in sense of $\int \varphi \, d\tilde{m}_t = \mathbb{E}(\varphi(Y_t) \mid B)$. $\forall \varphi \in C^1$.

and $Y \in \mathbb{F}_0$. B_t is $\mathbb{F}_t$-BM.

Rmk: i) $\mathcal{L}(Y_t | B) = \mathcal{L}(Y_t | \mathcal{F}^B_{\leq t})$

ii) For b, σ, σ_0 are uniformly Lip.

$\Rightarrow$ E & U still hold

Thm. If Rmk ii) holds. Then:

i) There exists the unique strong solution for the cond. MV equa.

ii) $\exists \mu \in P^2$. s.t. $\lim_{n\to\infty} \mathbb{E}(W^2_{(2)}(\mu^n, \mu)) = 0$. and $\mathcal{L}(X^{n,1}, \ldots, X^{n,k}) \xrightarrow{w} \mu^{\otimes k}$.

Pf: Similar as the usual case

1) Estimate $\| X^{n,i} - Y^i \|_t$ where (Y^i) is copy of solution:

$dY^i_t = b(Y^i_t, \tilde{\mu}_t) dt + \sigma(\ldots) \ldots$

2) easy to see:

$W_2^2(\mathcal{L}(Z^1 | B), \mathcal{L}(Z^2 | B)) \leq \mathbb{E}(\| Z^1 - Z^2 \|^2 | B)$

⑧ PDEs:

set $L'_m \varphi(x) = b(x,m)\cdot\nabla\varphi(x) + \frac{1}{2}\mathrm{Tr}((\sigma\sigma^T + \sigma_0\sigma_0^T)(x,m)\nabla^2\varphi(x))$

Apply Itô's on $\varphi(X_t^{n,i})$ and average.

$\Rightarrow d\langle \mu_t^n, \varphi\rangle = \langle \mu_t^n, L'_{\mu_t^n}\varphi\rangle dt + dM_t^n + \langle \mu_t^n, \nabla\varphi(\cdot)^T\sigma(\cdot,\mu_t^n)\rangle dB_t$

where $M_t^n \triangleq \frac{1}{n}\int_0^t \sum_1^n \nabla\varphi(X_t^{n,i})\cdot\sigma(\cdot\cdot)dW_t$.

prove as before: (weak form)

$d\langle \mu_t, \varphi\rangle = \langle \mu_t, L'_{\mu_t}\varphi\rangle dt + \langle \mu_t, \nabla\varphi(\cdot)^T \cdot\sigma(\cdot,\mu_t)\rangle dB_t.$

If $d\mu_t(x) = \mu(t,x)dx$. then:

$$\partial_t \mu(t,x) = -\mathrm{div}_x(b(x,\mu_t)\mu(t,x)) + \frac{1}{2}\mathrm{Tr}(\nabla^2(\sigma\sigma^T + \sigma_0\sigma_0^T)(x,\mu_t)\mu(t,x)) - \sum_1^{m_0}\mathrm{div}_x(\mu(t,x)\sigma_0^j(t,x))dB_t^j$$

(ii) Long Behavior:

Next, we consider let $t \to \infty$.

eg. (Metastability)

$$dX_t^{n,i} = (\alpha(\bar{X}_t^n - X_t^{n,i}) + (X_t^{n,i} - |X_t^{n,i}|^3))dt + \sigma dW_t^i. \quad \bar{X}_t^n = \frac{1}{n}\sum_1^n X_t^{n,i}.$$

$$\Rightarrow \rho_m := \frac{1}{Z_m} e^{\frac{2}{\sigma^2}(\alpha m x - \frac{\alpha}{2}x^2 + \underbrace{\frac{1}{2}x^2 - \frac{1}{4}x^4}_{\triangleq V(x)})}$$

is stationary dist. ($\rho_m(\mathcal{L}^* f) = 0$)

if $m = \int y \, d\rho_m(y)$ holds.

Actually, there $\exists \sigma_c > 0$. if $\sigma \geq \sigma_c$. the only solution is $m = 0$; if $\sigma < \sigma_c$. there also $\exists m_1$. s.t. $\pm m_1$ are solutions. (tunneling phenomenon)

Rmk: When σ is large. the noise term dominates. When σ is small. the drift term dominates and most of particles will be near minizers ± 1 of $V(x)$.