Background

(1) <u>Motivation</u>:

Consider $dY_t = f(Y_t)\,dX_t$. $(*)$.

where $X_t : [0,T] \to V$. $Y_t : [0,T] \to U$. and

$f : U \to L(V, U)$. next take $V = U = \mathbb{R}^d$.

The central question is:

How to interpret $(*)$ and define the integral

$Y_t = Y_0 + \int_0^t f(Y_s)\,dX_s$?

① One suggests simply take (X^n). (Y^n).

smooth enough and satisfy $X^n \to X$.

$Y^n \to Y$. in suitable topology.

Def: $\int_0^t f(Y_s)\,dX_s \stackrel{\Delta}{=} \lim_{n\to\infty} \int_0^t f(Y^n_s)\,dX^n_s$.

<u>Rmk</u>: It doesn't hold generally. e.g.:

take $X^n_t = -n^{-\frac{1}{3}}\cos(nt)$. $Y^n_t = n^{-\frac{1}{3}}\sin(nt)$.

under $\|\cdot\| = |\cdot|_\infty$. X^n_t. $Y^n_t \to 0$ $(n\to\infty)$.

But $\int_0^{2\pi} Y^n_t\,dX^n_t = n^{\frac{1}{3}}\pi \to \infty$.

And $(X, Y) \mapsto \int_0^t f(Y_s)\,dX_s$ isn't conti.

prop. $W = \{ X : [0,T] \to \mathbb{R}^d \mid X \text{ is conti.} \}$.

There doesn't exist norm $\|\cdot\|$ on

$E^\infty := \{ x \in C^\infty([0,T]; \mathbb{R}^d) \} \subseteq W$. s.t.

i) $(\overline{E^\infty}, \|\cdot\|)$. contain all BM paths a.s.

ii) $I_t : \quad E^\infty \times E^\infty \longrightarrow W$ can be

$\qquad (X, y). \longmapsto \int_0^t y_s dx_s$

extended conti. to $\overline{E^\infty} \times \overline{E^\infty}$ under $\|\cdot\|$.

Rmk: So the topology isn't even rich enough to cover the case of BM.

⊙ So we must consider another method. the main idea of rough theory is to "enhance" a path X with some additional data $\mathbb{X}$.

consider F is smooth function: By Taylor,

$$\int_s^t F(X_u) dX_s = F(X_s)(X_t - X_s) + \int_s^t (F(X_u) - F(X_s)) dX_u$$

$$= F(X_s) X_{t,s} + \int_s^t \int_s^u DF(X_v) dX_v \, dX_u.$$

$$= \square + DF(X_s) \int_s^t \int_s^u dX_v dX_u + \int_s^t \int_s^u (DF(X_v) - DF(X_s)) dX_v dX_u.$$

$$= \cdots = \sum D^k F(X_s) \int_s^t \int_s^{u_k} \cdots \int_s^{u_1} dX_{u_1} \cdots$$

$\int_0$ it will depend on $\mathbb{X} = (\mathbb{X}^1, \cdots \mathbb{X}^N, \cdots) :=$

$(X_{t,s}, \int_s^t \int_s^u dX_v \otimes dX_u, \int_s^t \int_s^u \int_s^v dX_r \otimes dX_u \otimes dX_v,$

$\cdots)$. where $\otimes$ is matrix tensor product.

<u>Rmk:</u> Depend on the roughness of path X_t.

We will define $\int_0 F(X_s) dX_s$ basing on

the first N terms of $\mathbb{X}$.

where N depends on regularity of X.

Actually, $\int_0^t F(\mathbb{X}_s) d\mathbb{X}_s := \lim_{\Pi \to 0} \sum_{\substack{[s,t] \\ \in \Pi}} \sum_{k=1}^N$

$F(X_s^k) \mathbb{X}_{s,t}^k$. ($N^{th}$ order).

For general $Y_t = (Y_t^0, \cdots Y_t^{N-1})$, we have:

$$\int_0^t Y_s dX_s := \lim_{\Pi \to 0} \sum_{[s,t]} (Y_s^0 \mathbb{X}_{s,t}^1 + Y_s^1 \mathbb{X}_{s,t}^2 + \cdots + Y_s^{N-1} \mathbb{X}_{s,t}^N)$$

$\Rightarrow$ it solves $dY = Y d\mathbb{X}$.

<u>Rmk:</u> In fact, if $X \in C^\alpha$, $\alpha > \frac{1}{2}$,

then: $\lim_{|\Pi| \to 0} \sum \int_s^t \int_s^u dX_v dX_u = 0$.

which is proved from Young.

<u>Def:</u> $X \mapsto (X, \mathbb{X}) \xrightarrow{\varphi} Y = y + \int_0^\cdot f(Y_s) dX_s$.

φ is called Itô - Lyon map.

(2) <u>Tensor product</u>:

<u>Def.</u> V is a LS. Family of admissible tensor norms on $(V^{\otimes n})$ is $\{|\cdot|_n\}$.

s.t. i) $|\xi \otimes \eta|_{m+n} \leq |\xi|_m |\eta|_n$. $\forall \xi \in V^m, \eta \in V^n$.

ii) $\forall \sigma \in S_n$ permutation. $P_\sigma : V^{\otimes n} \to V^{\otimes n}$, define by $P_\sigma(v_1 \otimes \cdots v_n) = v_{\sigma(1)} \otimes \cdots v_{\sigma(n)}$. We have:

$|P_\sigma(v)|_n = |v|_n$. $\forall v \in V^{\otimes n}$.

Next. we fix V and $(|\cdot|_n)$. on $(V^{\otimes n})$.

<u>Def</u>: i) $N \in \mathbb{N}$. N^{th}-truncated tensor algebra on V is $T^N(V) := \bigoplus_{n=0}^{N} V^{\otimes n} = \{\xi = (\xi_0, \xi_1 \cdots \xi_N) . \ \xi_k \in V^{\otimes k}\}$. $(V^{\otimes 0} = \mathbb{R}')$.

<u>Rmk</u>: $T^N(V)$ is LS.

ii) Set $(\xi \otimes \eta)_n \triangleq \sum_{k=0}^{n} \xi_k \otimes \eta_{n-k}$ for any $\xi, \eta \in T^N(V)$. $\Rightarrow \xi \otimes \eta \in T^N(V)$.

<u>Rmk</u>: i) $(T^N(V), +, \otimes)$ is an algebra with unit $\mathbb{1} = (1, 0 \cdots 0)$.

ii) $\forall \xi \in T^N(V)$. ξ has an inverse under $\otimes$. In particularly, for $\xi = (1, \xi_1 \cdots \xi_N)$,

$$\xi^{-1} = \sum_{n=0}^{N} (-1)^n (\xi - \mathbb{1})^{\otimes n}.$$

We denote $T_1^N(V) \triangleq \{ \xi \in T^N(V) . \xi_0 = 1 \}$.

<u>Pf:</u> Note that $\xi \otimes \eta = \sum_{\substack{i+j \le N \\ i,j \ge 0}} \xi^i \otimes \eta^j$. actually.

(3) Hölder Space:

① <u>Def</u>: $X \in C^\alpha([0,T], V)$ if $\sup_{\substack{[s,t] \in \\ [0,T]}} \frac{\| X_{s,t} \|}{|s-t|^\alpha} =: \| X \|_\alpha < \infty$

<u>Rmk</u>: i) $\| \cdot \|_\alpha$ is a seminorm. While $\widetilde{\| \cdot \|_\alpha} := |X_0| + \| \cdot \|_\alpha$ is a true norm.

ii) $C^\beta \subsetneq C^\alpha$. $\forall \beta > \alpha$. C^α isn't separable.

iii) $C^\alpha \subsetneq V^{1/\alpha}$ (e.g. step func.)

<u>Lemma</u>: i) (low-semi conti.) $X^n \in C^\alpha \to X$ pointwise.

Then: $\| X \|_\alpha \le \underline{\lim}_{n \to \infty} \| X^n \|_\alpha$.

ii) (Interpolation.) $\forall 0 < \alpha < \beta \le 1$. We have:

$$\| X \|_\alpha \le \| X \|_\beta^{\frac{\alpha}{\beta}} (\sup_{[s,t]} \| X_{s,t} \|)^{1 - \alpha/\beta}.$$

Cor. $\forall\, 0 < \alpha < \beta \leq 1$. $X \in C$. If $(X^n) \subset C^\beta$ $\xrightarrow{u} X$ and $\sup_n \|X^n\|_\beta < \infty$. Then: $X \in C^\beta$ and $\|X^n - X\|_\alpha \to 0$.

Pf: $\|X\|_\beta \leq \underline{\lim} \|X^n\|_\beta \leq \sup \|X^n\|_\beta$.

Besides, by interpolation, we have $X^n \to X$ under $\|\cdot\|_\alpha$.

Cor. $\forall\, 0 < \alpha < \beta \leq 1$. If $\sup_n (\|X_0^n\| + \|X^n\|_\beta) < \infty$. Then, $\exists\, (n_k)$ and $X \in C^\beta$ s.t. $X^{n_k} \to X$ under $\|\cdot\|_\alpha$.

Pf: By condition, it satisfies Ascoli Lemma. So with the cor. above, we have the result.

Rmk: It means: $C^\beta \subset\subset C^\alpha$.

Def: $C^{0,\alpha}([0,T], V) =$ closure of C^∞ under the norm $\|\cdot\|_\alpha$.

Thm. $\forall\, 0 < \alpha < 1$. $X \in C^{0,\alpha} \iff \lim_{\delta \to 0} \sup_{|s-t| < \delta} \frac{|X_{s,t}|}{|s-t|^\alpha} = 0$.

Pf: $(\Rightarrow)$. Let $X^n \xrightarrow{\alpha} X$. $X^n \in C^\infty$.

Note $\frac{\|X_{s,t}\|}{|s-t|^\alpha} \leq \frac{\|X_{s,t} - X^n_{s,t}\|}{|s-t|^\alpha} + \frac{\|X^n_{s,t}\|}{|s-t|^\alpha}$

Rmk: $C^{0,\gamma} \subsetneq C^{\gamma}$. since $X_t = t^{\gamma} \in C^{\gamma} / C^{0,\gamma}$

Cor. $\forall\ 0 < \gamma < \beta \le 1$. $C^{\beta} \subset C^{0,\gamma}$

Cor. $\overline{C(C^{\infty}, \|\cdot\|_{\gamma})}$ = space of conti. diff. paths

② **Two-parameter Func.**

Def: Set $\Delta_{[0,T]} = \{(s,t) \in [0,T]^2 \mid s \le t\}$.

$C_2 \stackrel{\Delta}{=} C_2(\Delta_{[0,T]}, V)$

$C_2^{\alpha} \stackrel{\Delta}{=} \{ A_{s,t} \in C_2 \mid \sup \frac{\|A_{s,t}\|}{|s-t|^{\alpha}} =: \|A\|_{\alpha} < \infty \}$.

Rmk: For $\alpha > 1$. C^{α} = const. func. But for C_2^{α}. it contains nontrivial. func. satisfying:

$\sum_{\pi} \|A_{s,t}\| \le \|A\|_{\alpha} |[0,T]| \, |\pi|^{\alpha-1} \to 0$