NYU

Linear Algebra

# Lecture 2
# Vectors and Spans

**Yiping Lu**

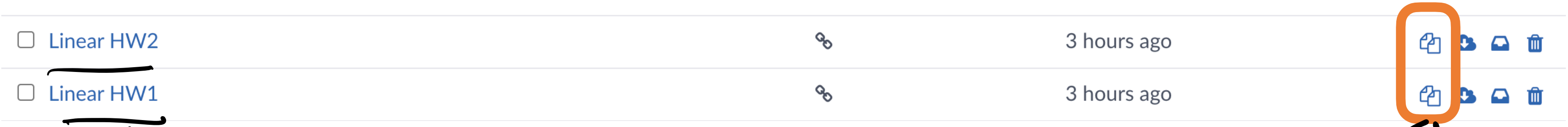
**Based on Dr. Ralph Chikhany's Slide**

# Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time). Two weeks
  - ✓ Late work policy applies.
- Recap Quiz 1 due by 11.59 pm on Sunday (NY time). Next week.
  - ❖ Late work policy does not apply.
- Recap Quiz is timed.
  - ❑ Once you start, you have 60 minutes to finish it (even if you close the tab)

Latex -> Overleaf -> Copy (Not required)

remember to copy




homepage

You can put what you want to recap in the (anonymous) form.

# ReCap

Course notes adapted from *Introduction to Linear Algebra* by Strang ($5^{th}$ ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

# Vector Addition

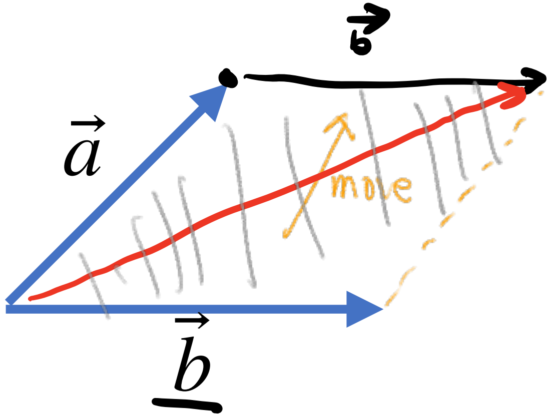
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}$$

we can't add 2-dimension with 3-dimensional vector

can just add two same size vector

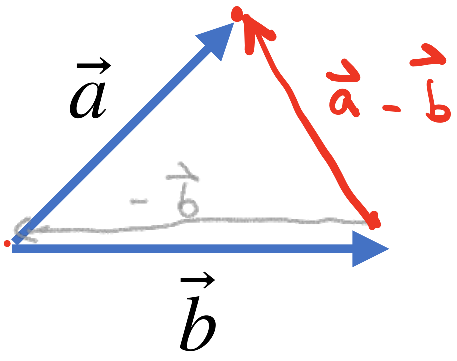
① $\vec{x} \in \mathbb{R}^n$ $\vec{y} \in \mathbb{R}^n$ $x+y$ ✓

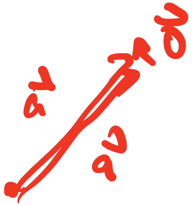
② $\vec{x} \in \mathbb{R}^n$ $\vec{y} \in \mathbb{R}^m$, $n \neq m$ $x+y$ ✗




parallelogram Rule.

$$\vec{a} + \vec{b} = ?$$




$$\vec{a} - \vec{b} = ?$$

$$\vec{a} + (-\vec{b})$$

$$\vec{a} - \vec{a} = ? \ \vec{0}$$

$\vec{0}$ is not a zero.

# Scalar vector multiplication

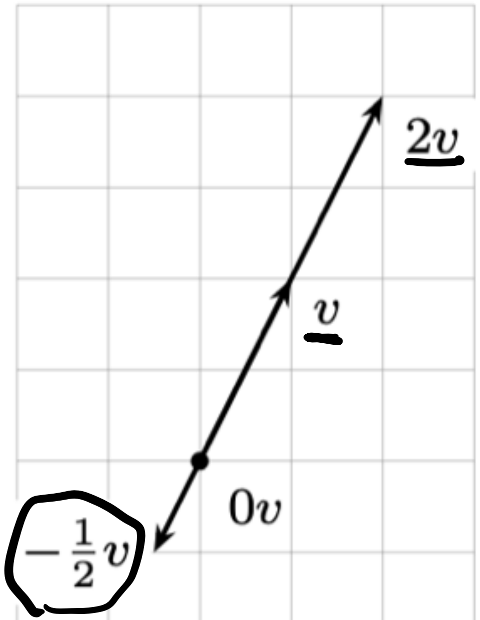
$$c\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}$$

① don't change the direction, change the length.

② $0 \cdot \vec{v} = \vec{0} \neq 0$.

③ $c \cdot (\vec{x} + \vec{y}) = c \cdot \vec{x} + c \cdot \vec{y}$

Some multiples of $v$.




negative scalar.

reverse my vector.

# Dot Product

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz \;. \neq \begin{pmatrix} a \cdot x \\ b \cdot y \\ c \cdot z \end{pmatrix}$$

is not a vector but a scalar.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$$

a.b.c coef.

x. y z. ← linear Combination .

Dot product is a linear combination

Lecture 3

# Dot Product

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz .$$

**Example 3** Dot products enter in economics and business. We have three goods to buy and sell. Their prices are $(p_1, p_2, p_3)$ for each unit—this is the "price vector" $\boldsymbol{p}$. The quantities we buy or sell are $(q_1, q_2, q_3)$—positive when we sell, negative when we buy. *Selling $q_1$ units at the price $p_1$ brings in $q_1 p_1$.* The total income (quantities $q$ times prices $p$) is ***the dot product $\boldsymbol{q} \cdot \boldsymbol{p}$ in three dimensions***:

$$\textbf{\textit{Income}} = (q_1, q_2, q_3) \cdot (p_1, p_2, p_3) = \underline{q_1}\,\underline{p_1} + \underline{q_2}\,\underline{p_2} + \underline{q_3}\,\underline{p_3} = \textbf{\textit{dot product.}}$$

price vector quantity vector.

# Dot Product

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$$

$$\left\| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\| = \sqrt{a^2 + b^2 + c^2}$$

$\rightarrow \|\vec{a} + \vec{b}\|$ vector $\rightarrow |\vec{a} \cdot \vec{b}|$ scalar

**Length** $\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}}$

define.

$c \cdot \vec{v}$ vector

$\|c \cdot \vec{v}\| = |c| \cdot \|\vec{v}\|$

c: scalar

scalar

absolute value of c

length of $\vec{v}$

$\vec{v}$ vector $\Leftarrow \|\cdot\|$

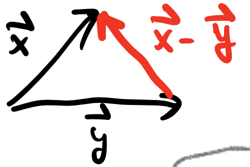
$c$ scalar $\Leftarrow |\cdot|$

length

$\|\vec{v}\|$ And $|c|$

absolute value

**Distance** $\text{dist}(x, y) = \|\vec{x} - \vec{y}\|$.




**Unit Vector:** $\|\vec{v}\| = 1$

$\boxed{\dfrac{\vec{x}}{\|\vec{x}\|}}$ is the same direction as $\vec{x}$

What is the unit vector of $(1,1)$?

the direction

$\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \sqrt{2}$

$\Rightarrow \dfrac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

# Dot Product

Communicative $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

$\begin{pmatrix} a \\ b \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \left( \begin{matrix} ax \\ by \end{matrix} \right) = \begin{pmatrix} xa \\ yb \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix}$

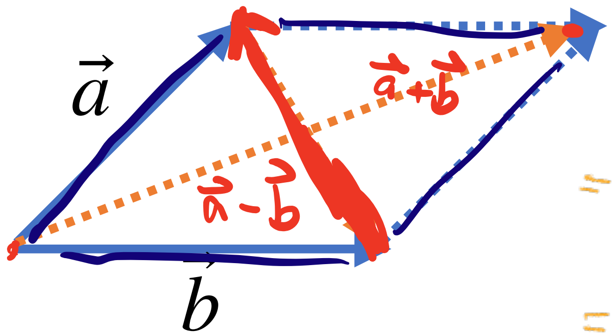
Distributive $(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$

L C & (dot product)

L.C. $(c_1\vec{a}+c_2\vec{b})\cdot\vec{c} = \underbrace{(c_1\vec{a})\cdot\vec{c}}_{c_1\cdot(\vec{a}\cdot\vec{c})} + \underbrace{(c_2\vec{b})\cdot\vec{c}}_{c_2\cdot(\vec{b}\cdot\vec{c})} = c_1(\vec{a}\cdot\vec{c}) + c_2(\vec{b}\cdot\vec{c})$

(LC) dot product (vector)

**Example** $\|\vec{a}+\vec{b}\|^2 + \|\vec{a}-\vec{b}\|^2 =$

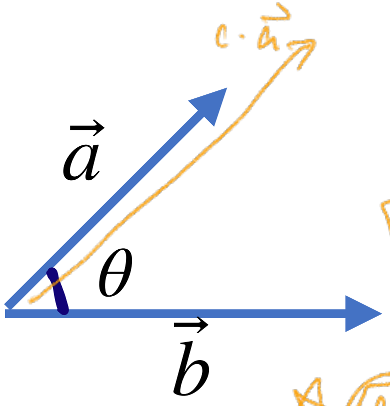



$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) + (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$= \|\vec{a}\|^2 + \cancel{2\vec{a}\cdot\vec{b}} + \|\vec{b}\|^2 + \|\vec{a}\| - \cancel{2\vec{a}\cdot\vec{b}} + \|\vec{b}\|^2$

$= 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2$

# Angle



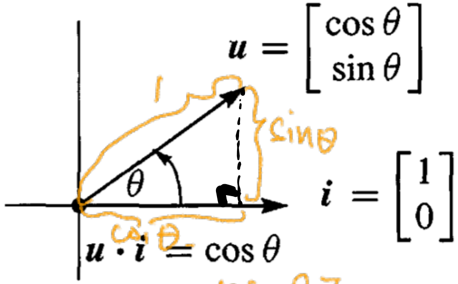

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\cdot\|\vec{b}\|}$$

$c>0$

$$= \frac{(c\vec{a})\cdot\vec{b}}{\|c\vec{a}\|\cdot\|\vec{b}\|} \Rightarrow = \frac{c\cdot(\vec{a}\cdot\vec{b})}{c\cdot\|\vec{a}\|\cdot\|\vec{b}\|}$$

$c>0$ scalar multiplication don't change the angle.

$\|\vec{u}\|=1$



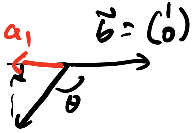

$\vec{u} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$

$\vec{i} = \begin{bmatrix}1\\ 0\end{bmatrix}$

$\vec{u}\cdot\vec{i} = \cos\theta.$

How to decide whether $\theta$ is larger than $\frac{\pi}{2}$

$\vec{a}\cdot\vec{b} < 0 \iff \cos\theta < 0.$




$a_1 < 0$ $\Downarrow$ $\vec{a}\cdot\vec{b} < 0$

$\vec{a} = \binom{a_1}{a_2}$ $\quad \vec{a}\cdot\vec{b} = a_1$

$\vec{x}\cdot\vec{y} = 0 \iff$ orthogonal

What is the unit vector parallel/orthogonal to $(4,3)$?

easy

$\left\|\binom{4}{3}\right\| = 5$

unit vector $\frac{1}{5}\binom{3}{4} = \binom{3/5}{4/5}$

harder one

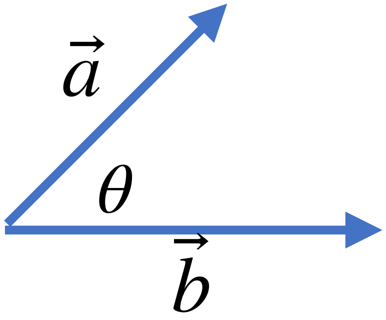
$\binom{-3/5}{4/5}$ unit

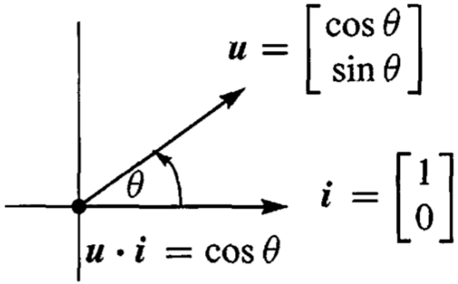
check orthogonal

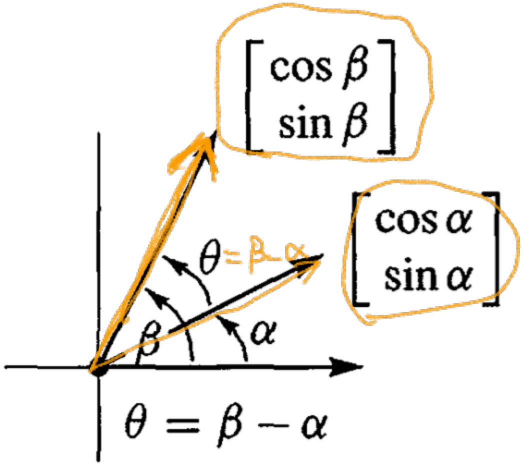
$\binom{-3/5}{4/5}\cdot\binom{4}{3} = -3/5\times 4 + 4/5\times 3$

$= 0$

# Angle




calculate $\cos(\beta - \alpha)$

diffrent

$$\cos(\beta - \alpha) = \frac{\begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}}{\underbrace{\left\| \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix} \right\|}_{\text{unit vector}} \cdot \underbrace{\left\| \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \right\|}_{\text{unit vector}}}$$

$= \cos\beta \cdot \cos\alpha + \sin\beta \sin\alpha$

$= 1$

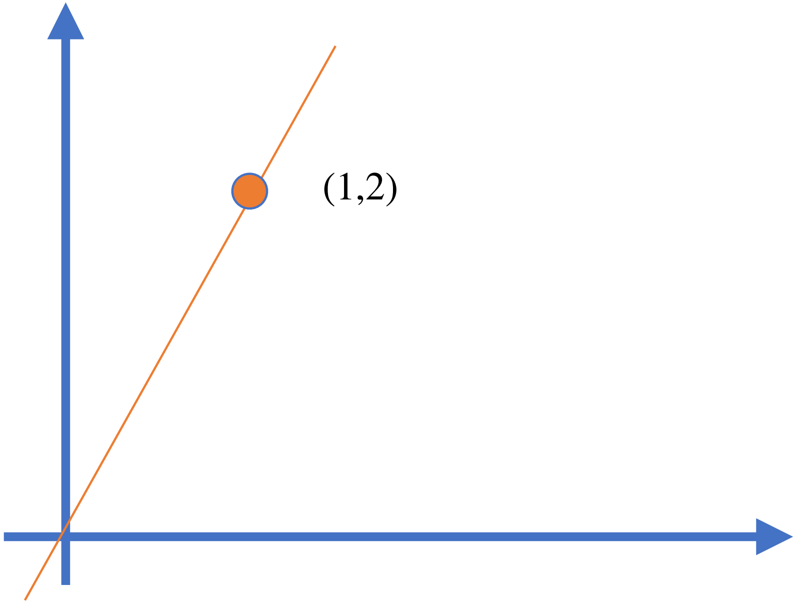
$= \cos\beta \cos\alpha + \sin\beta \sin\alpha$

# Motivation: Best fit of linear equation

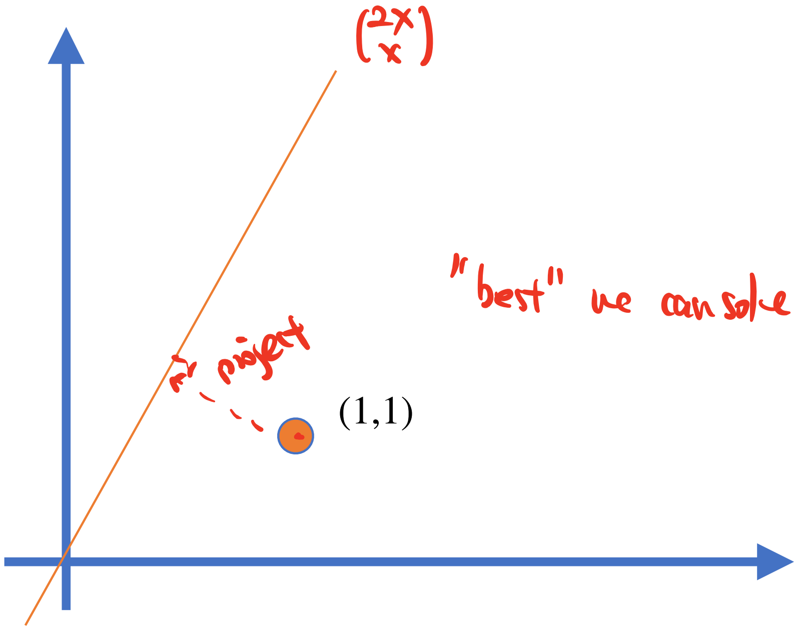
Not Required

overdetermined linear system

$$2x = 2$$
$$x = 1$$




$$\begin{pmatrix}2x\\x\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$




"best" we can solve.

# Example

**1.2 C** Find a vector $\boldsymbol{x} = (c, d)$ that has dot products $\boldsymbol{x} \cdot \boldsymbol{r} = 1$ and $\boldsymbol{x} \cdot \boldsymbol{s} = 0$ with the given vectors $\boldsymbol{r} = (2, -1)$ and $\boldsymbol{s} = (-1, 2)$.

How is this question related to Example **1.1 C**, which solved $c\boldsymbol{v} + d\boldsymbol{w} = \boldsymbol{b} = (1, 0)$?

$\vec{x} \cdot \vec{r} = 1$ means $2c - d = 1$

$\vec{x} \cdot \vec{s} = 0$ means $-c + 2d = 0$

$\Rightarrow$ Solving Eq: $\begin{cases} 2c - d = 1 \\ -c + 2d = 0 \end{cases}$

$\downarrow$

$c \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} + d \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

**1.1 C** Find two equations for the unknowns $c$ and $d$ so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals the vector $\boldsymbol{b}$:

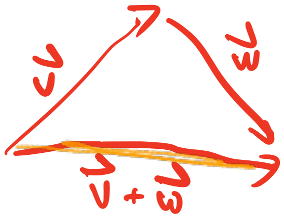
$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Equal to.

# Inequalities

**SCHWARZ INEQUALITY** $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\| \, \|\boldsymbol{w}\|$ $|\cos\theta| < 1$ $|\cos\theta| = \dfrac{|\vec{v} \cdot \vec{w}|}{\|\vec{v}\| \cdot \|\vec{w}\|}$

**TRIANGLE INEQUALITY** $\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$




**Example 6** The dot product of $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (b, a)$ is $2ab$. Both lengths are $\sqrt{a^2 + b^2}$. The Schwarz inequality in this case says that $2ab \leq a^2 + b^2$.

# Reminder: Linear Combination

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

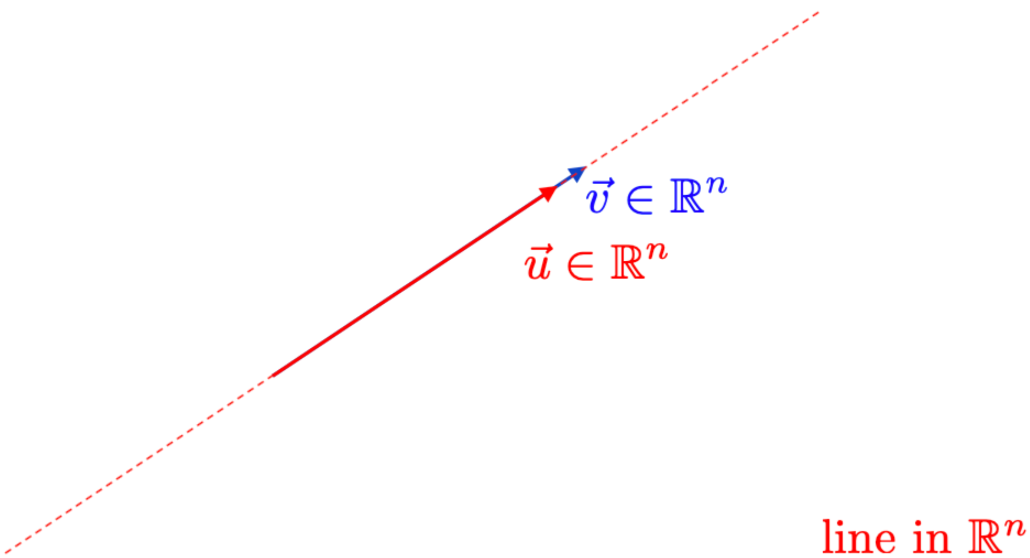
where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and $w$ is a vector in $\mathbf{R}^n$.

Definition

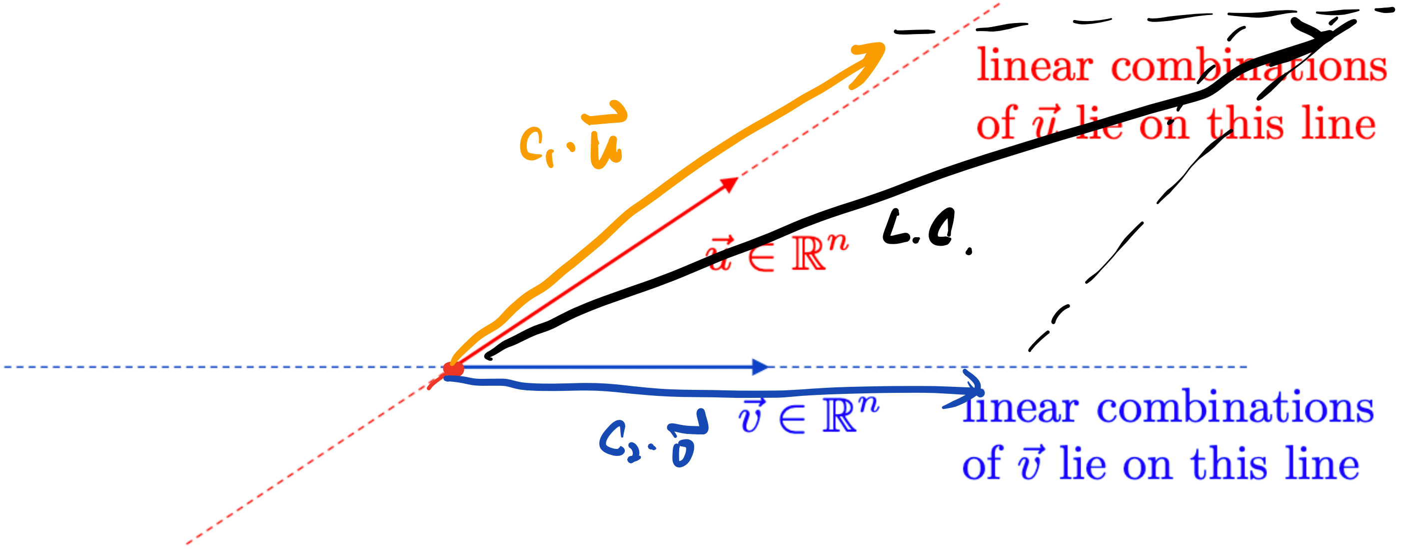
We call $w$ a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

# Geometric Interpretation of Linear Combinations

$\vec{u} \parallel \vec{v}$



# Geometric Interpretation of Linear Combinations




linear combinations of $\vec{u}$ and $\vec{v}$ lie on a plane in $\mathbb{R}^2$ ← any vector in $\mathbb{R}^2$.

# Transfer Linear Equation to a Linear Combination Problem

$$\begin{cases} 2x + y = 1 \\ x + y = 1 \end{cases} \iff \text{is the vector } \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

lies in the L.C. of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x$ and $y$ are coef .

# Spans

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

# Reminder: Linear Combination

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and $w$ is a vector in $\mathbf{R}^n$.

Definition

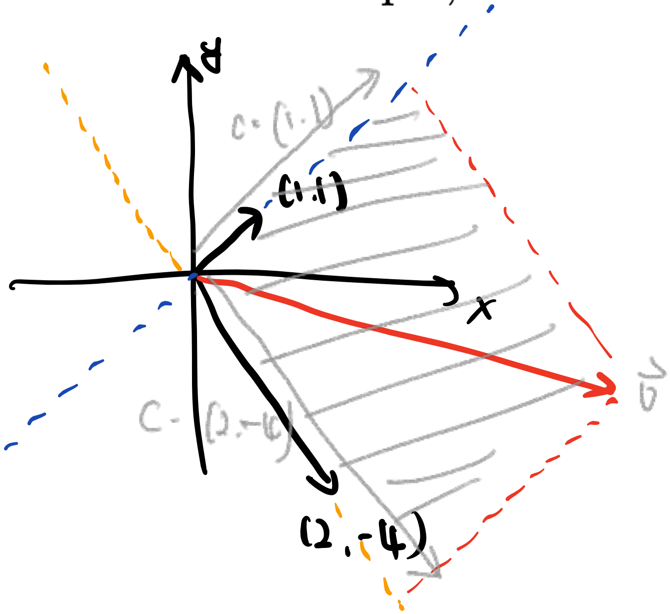
We call $w$ a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

# Span

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$$

For example, what is the span of $(2, -4)$ and $(1, 1)$?




any vector in $\mathbb{R}^2$.

# Span

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$$

exist

$\exists\, c_1$

Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in the span of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$?

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ← Find $c_1$

$c_1 = 1/2$ $\quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in the span of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ ← can't Find $c_1$ satisfy.

$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is not in the span of $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

# Span

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be a set of vectors in $\mathbb{R}^n$. We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$$

Is $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ in the span of $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$? Yes!

span of $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is all in $\mathbb{R}^2$.

Algeb.

$$\begin{bmatrix} 4 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \text{Find } c_1 \text{ and } c_2.$$

How to solve? Next week.

$c_1 = 1$ $\quad$ $c_2 = 2$

# More Precise Definition

"the set of"  "such that"

**Definition**

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

$$\boxed{\mathrm{Span}\{v_1, v_2, \ldots, v_p\} = \{x_1v_1 + x_2v_2 + \cdots + x_pv_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbf{R}\}.}$$

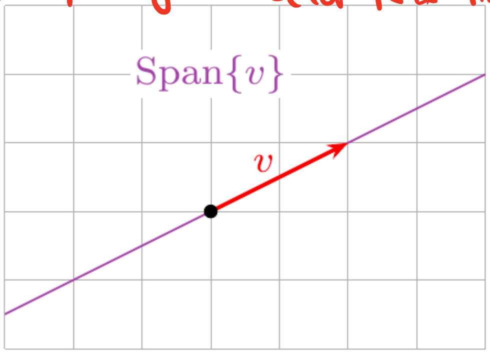
Synonyms: $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

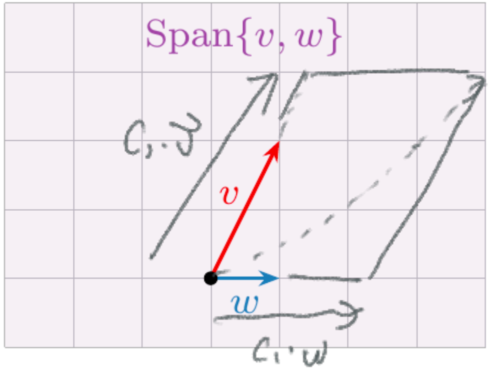
This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

# Span in $\mathbb{R}^2$

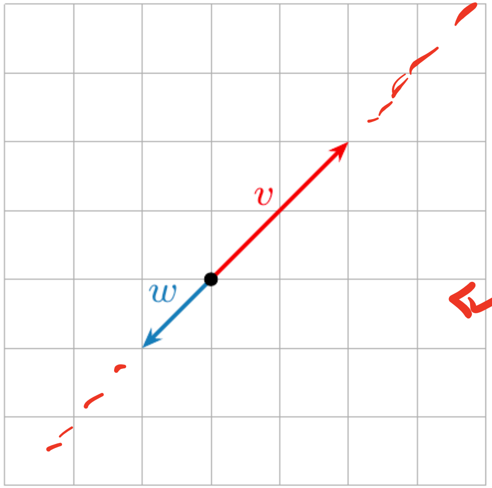
Drawing a picture of Span$\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.

span of a single vector is a line.



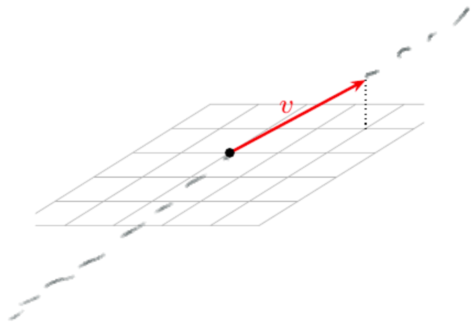

if $\vec{v}$ and $\vec{w}$ are not parallel then the span is whole $\mathbb{R}^2$.



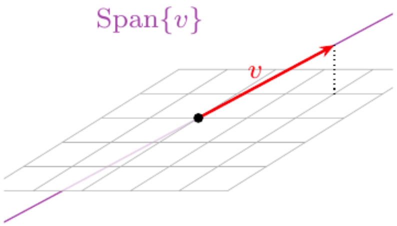

if $\vec{v}$ and $\vec{w}$ are parallel.

then the span is a line

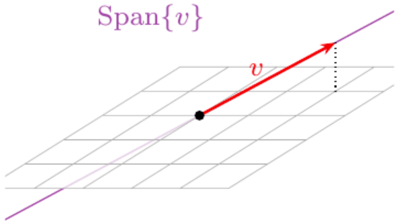
# Span in $\mathbb{R}^3$

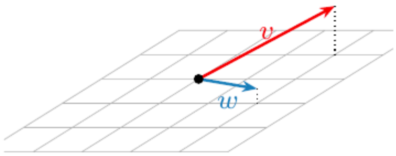


# Span in $\mathbb{R}^3$

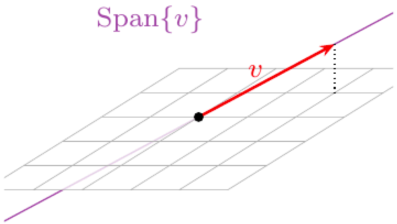


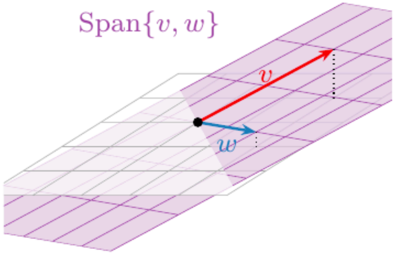
# Span in $\mathbb{R}^3$



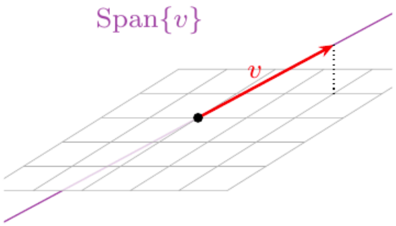



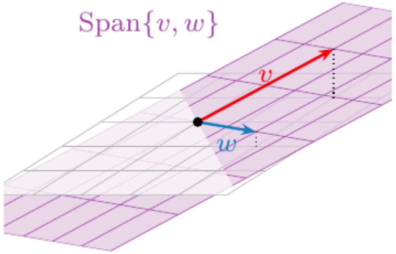
# Span in $\mathbb{R}^3$

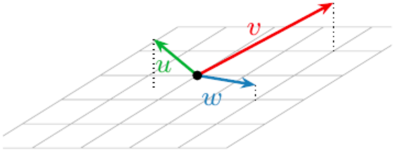


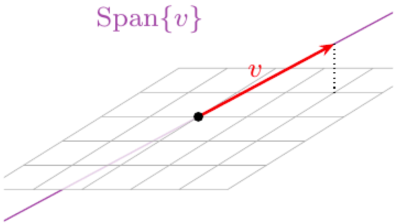
# Span in $\mathbb{R}^3$

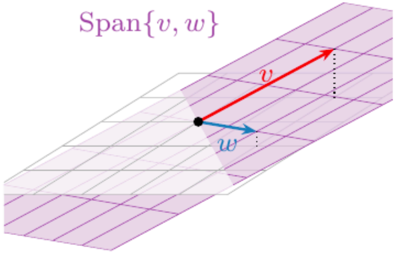


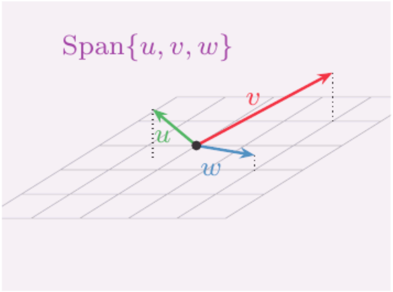



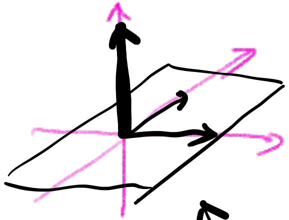
# Span in $\mathbb{R}^3$



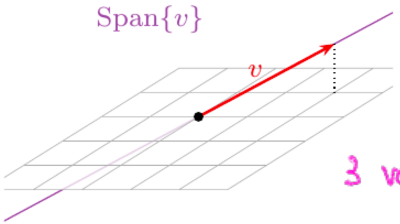



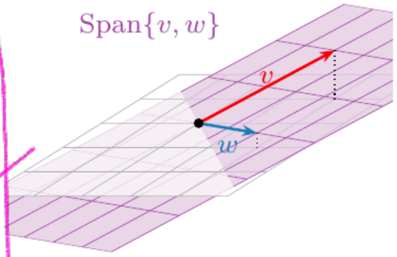
# Span in $\mathbb{R}^3$



1 vector



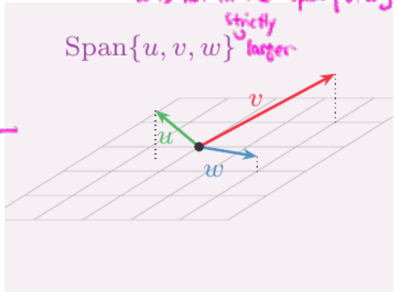

2 vectors



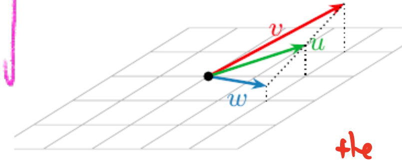

3 vector

$\vec{u}$ is not in the span$\{\vec{v}, \vec{w}\}$

strictly larger




whole $\mathbb{R}^3$ space




if the third vector $\vec{v}$ is a L.C. of $\vec{u}$, $\vec{w}$

$\vec{v}$ is not helpful

the span is still a plane.

Then $\vec{v} = a_1\vec{u} + a_2\vec{w}$

then $\text{span}\{\vec{v}, \vec{u}, \vec{w}\} = \text{span}\{\vec{u}, \vec{w}\}$

if $\vec{x} \in \text{span}\{\vec{v}, \vec{u}, \vec{w}\}$

$\vec{x} = c_1\vec{u} + c_2\vec{w} + c_3\vec{v}$

$= c_1\vec{u} + c_2\vec{w} + c_3(a_1\vec{u} + a_2\vec{w})$

$= (c_1 + c_3a_1)\vec{u} + (c_2 + c_3a_2)\vec{w}$

if $\vec{x} \in \text{span}\{\vec{u}, \vec{w}\}$

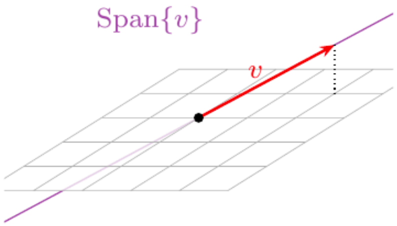
$\vec{x} = c_1 \cdot \vec{u} + c_2 \cdot \vec{w} = \vec{0} + \dots = 0 \cdot \vec{v} + c_1 \cdot \vec{u} + c_2 \cdot \vec{w}$

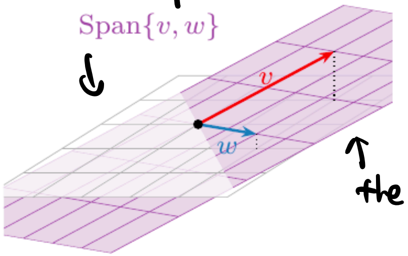
$\Rightarrow \vec{x} \in \text{span}\{\vec{v}, \vec{u}, \vec{w}\}$

# Span in $\mathbb{R}^3$

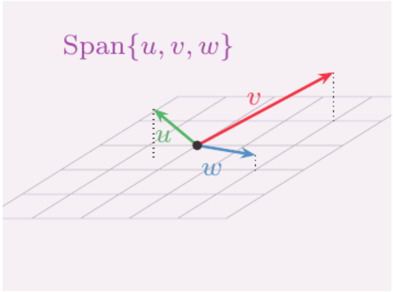
x-axis y-axis

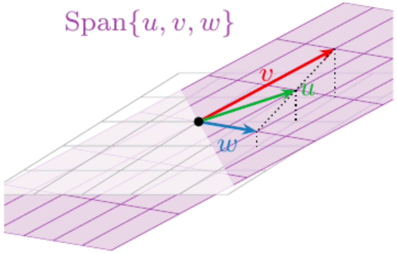
white space is not the span




the purple part is the span



# Questions?