Optimal Stopping

(1) <u>Setting:</u>

① <u>Time-homo case:</u>

X_t is Itô diffusion on $\mathbb{R}^n$. $g \in C^{\geq 0}(\mathbb{R}^n)$ is given reward function.

Find stopping time $\tau^* = \tau^*(x, w)$. for X_t. s.t.

$$\mathbb{E}^x(g(X_{\tau^*})) = \sup_{\substack{\tau \text{ is } X_t\text{-} \\ \text{stopping time}}} \mathbb{E}^x(g(X_\tau)). \quad \forall x \in \mathbb{R}^n.$$

<u>Def:</u> i) $f: \mathbb{R}^n \to \mathbb{R}^{\geq 0}$ measurable is called supermeanvalued w.r.t. X_t if:

$f(x) \geq \mathbb{E}^x(f(X_\tau))$. $\forall x$. $\forall$ stopping time τ.

ii) If in additional, f is l.s.c. then it's superharmonic.

<u>Lemma</u> If f is superharmonic $(\tau_k) \to 0$. Seq of stopping times w.r.t. X_t. Then:

$$f(x) = \lim_{k \to \infty} \mathbb{E}^x(f(X_{\tau_k})).$$

<u>Pf:</u> $f(x) = \mathbb{E}^x(f(x)) \overset{\text{l.s.c}}{\leq} \mathbb{E}^x(\varliminf_k f(X_{\tau_k}))$

$\overset{\text{Fatou's}}{\leq} \varliminf_k \mathbb{E}^x(f(X_{\tau_k}))$

Lemma. $f \in C^2(\mathbb{R}^n)$. Then f is superharmonic w.r.t. X_t $\Leftrightarrow$ $Af \leq 0$.

Pf: By Dynkin's formula.

Lemma. i) Superharmonic / meanvalued function space is LS.

ii) $\{f_j\}_{j \in J}$ is family of supermeanvalue $\Rightarrow$ $\inf_J \{f_j\}$. is supermeanvalued.

iii) (f_k) seq of supermeanvalued / harmonic. $\nearrow f$. $\Rightarrow$ f is also supermeanvalued / harmonic.

iv) If f is supermeanvalued. Then: $f(X_t)$ is supermartingale.

v) If f is supermeanvalued. $H \in B(\mathbb{R}^n)$. Then $\tilde{f} =: \mathbb{E}^x(f(X_{\tau_H}))$ is supermeanvalued.

Pf: i), ii), iii) are trivial.

iv) $\mathbb{E}^x(f(X_t) \mid \mathcal{F}_s) = \mathbb{E}^{X_s}(f(X_{t-s}))$

$\leq f(X_s)$

v) $\mathbb{E}^x(\tilde{f}(X_\alpha)) \overset{mp.}{=} \mathbb{E}^x(\theta_\alpha f(X_{\tau_H}))$

$= \mathbb{E}^x(f(X_{\tau_H^\alpha}))$

$\overset{iv)}{\leq} \mathbb{E}^x(f(X_{\tau_H}))$.

Def: $h: \mathbb{R}^n \to \mathbb{R}^1$. measurable.

i) f is supermeanvalued / harmonic and $f \geq h$. We say f is a supermeanvalued / harmonic majorant of h.

ii) $\bar{h}(x) = \inf\{f \mid f$ is supermeanvalued majorant of $h\}$. is call the least ~

Rmk: $\bar{h}$ is supermeanvalued if it's measurable.

iii) For $\hat{h}$ is superharmonic majorant of h and $\hat{h} \leq f$. for $\forall$ other superharmonic majorant of h. We call it the least ~.

Rmk: easy to check: $\hat{h}(x) \geq h^*(x)$

$$=: \sup_{\tau} \mathbb{E}^x(h(X_\tau)).$$

iv) $f: \mathbb{R}^n \to \mathbb{R}^{\geq 0}$. l.s.c. is called excess if $\mathbb{E}^x(f(X_s)) \leq f(x)$. $\forall s \geq 0$. $x \in \mathbb{R}^n$.

Thm. f is excess $\Leftrightarrow$ f is superharmonic.

Pf: ($\Rightarrow$) By Dynkin's formula.

consider $\tau \wedge t$. let $t \to \infty$.

Thm (Construction)

$g = g_0$. l.s.c. on $\mathbb{R}^n$. Set $g_n(x) = \sup_{t \in S_n} \mathbb{E}^x(g(X_t))$. $S_n = \{ k/2^n : 0 \le k \le 4^n \}$.

Then: $g_n \uparrow \hat{g}$. where $\hat{g}$ is the least superharmonic majorant of g. and $\hat{g} = \bar{g}$.

Pf: Set $\check{g}(x) = \lim_{n \to \infty} g_n(x)$.

1) $\check{g}(x) \ge g_n(x) \ge \mathbb{E}^x(g_{n-1}(X_t))$

$\xrightarrow{MCT} \mathbb{E}^x(\check{g}(X_t))$. $\forall t \in S =: \bigcup S_n$.

Approxi. $\forall t \in \mathbb{R}^1$ by $(t_n) \subset S$. and apply Fatou's Lemma. $\Rightarrow \check{g}$ is excess.

2) On the other hand. if f is super-meanvalue. majorant of g.

By induct: $f(x) \ge \mathbb{E}^x(f(X_t))$

$\ge \mathbb{E}^x(g_n(X_t))$

$\Rightarrow f(x) \ge g_{n+1}(x)$. $\forall n$. $\Rightarrow f(x) \ge \check{g}$.

So: $\check{g} = \bar{g} = \hat{g}$.

Cor. $h_0 = g$. $h_n(x) =: \sup_{t \ge 0} \mathbb{E}^x(h_{n-1}(X_t))$.

Then: $h_n \uparrow \hat{g}$.

Pf: Set $h = \lim h_n \ge \hat{g}$.

converse by induct: $\hat{g} \ge h_n$. $\forall n$

Since $\hat{g}(x) \ge \sup \mathbb{E}^x(\hat{g}(X_t))$

Thm. (Existence of optimal stopping)

For g l.s.c. lower bdd. rewarding function. Then:

i) $g^* = \hat{g}$.

ii) For $D_\varepsilon = \{x \mid g(x) < \hat{g}(x) - \varepsilon\}$. $\tau_\varepsilon =: \tau_{D_\varepsilon}$.

If g is bdd. Then: for $\forall x$.

$|g^*(x) - \mathbb{E}^x(g(X_{\tau_\varepsilon}))| \leq \varepsilon$.

iii) $D =: \{g < g^*\}$. set $g_N = g \wedge N$ and

$D_N =: \{g_N < \hat{g}_N\}$. $\sigma_N =: \tau_{D_N}$. Then:

$D_N \subset D_{N+1}$. $D_N \subset g^{-1}[0, \ ) \cap D$. $D = \bigcup D_N$.

Besides, if $\sigma_N \overset{a.s.}{\underset{\forall N}{<}} \infty$. Then $g^*(x) = \lim_{N\to\infty} \mathbb{E}^x(g(X_{\sigma_N}))$

Rmk: If $\sigma_N \overset{a.s.}{\underset{\forall N}{<}} \infty$ $(g(X_{\sigma_N}))_N$ is u.i.

Then $g^*(x) = \mathbb{E}^x(g(X_{\tau_D}))$. $\tau^* = \tau_D$.

Pf: 1°) For g is bdd. set $\tilde{g}_\varepsilon(x) = \mathbb{E}^x(\hat{g}(X_{\tau_\varepsilon}))$.

which's also supermeanvalued.

We prove: $g(x) \leq \tilde{g}_\varepsilon(x)$. $\forall x$.

Note: $\hat{g}(x) \leq \tilde{g}_\varepsilon(x) + \sup_x \{g(x) - \tilde{g}_\varepsilon(x)\}$

since the latter is supermartingale.

So: $\hat{g}(x) \leq \tilde{g}_\varepsilon(x) \leq \mathbb{E}^x(g(X_{\tau_\varepsilon}) + \varepsilon) \underset{\leq}{\xrightarrow{\varepsilon \to 0}} g^*(x)$.

$\Rightarrow \hat{g} = g^*$.

2) Replace g by $g_N = g \wedge N$. if g isn't bdd. $g^* \geq g^*_N = \hat{g}_N \uparrow h \geq \hat{g}$.

So we proved i) and ii).

For iii): if g is bdd. then by BDT and l.s.c of g. $\Rightarrow \mathbb{E}^x(g(X_{\tau_\varepsilon})) \to \mathbb{E}^x(g(X_{\tau_0})) = g^*(x)$

if g isn't bdd. $h =: \lim_{N\to\infty} \hat{g}_N$.

So: $h \leq \hat{g}$. and $\hat{g}_N \leq h$. $\Rightarrow h = \hat{g}$.

$g^*(x) = \lim_{N\to\infty} \hat{g}_N(x) \overset{\text{bdd}}{=} \lim_{N\to\infty} \mathbb{E}^x(g_N(X_{\tau_N})) \leq g^*(x)$

$\Rightarrow h = \hat{g} = g^*$.

Next, it's easy to check other statements.

<u>Cor.</u> If $\exists$ Borel set H. s.t. $\tilde{g}_H(x) =: \mathbb{E}^x(g(X_{\tau_H}))$ is supermeanvalued majorant of g $\Rightarrow g^* = \tilde{g}_H$. $\tau^* = \tau_H$.

<u>Pf:</u> $\hat{g}(x) \leq \tilde{g}_H(x) \leq \sup_\tau \mathbb{E}^x(g(X_\tau)) = g^*(x)$.

<u>Cor.</u> $D =: \{g < \hat{g}\}$. If $\tilde{g}_D \geq g$. Then: $\tilde{g}_D = g^*$.

<u>Pf:</u> Note $\tilde{g}_D(X_{\tau_D}) = \hat{g}(X_{\tau_D})$.

$\Rightarrow \tilde{g}_D(x) = \mathbb{E}^x(\hat{g}(X_{\tau_D}))$ which is supermeanvalued. By cor. above.

Rmk: The Thm above gives some sufficient cond. that τ^* exists.

τ^* needn't exist generally

e.g. $X_t = t$. $g(\xi) = \xi^2 / 1+\xi^2$.

$\Rightarrow \tau^* = \infty$.

Thm: (Unique for optimal stopping)

$D =: \{g < g^*\}$. If $\exists\, \tau^*$ is optimal stopping time. Then $\tau^* \overset{\forall x \in D}{\geq} \tau_D$. a.s. $g^*(x) = \mathbb{E}^x(g(X_{\tau_D}))$.

Pf: 1) If $\mathbb{P}_x(\tau < \tau_D) > 0$.

Note $g(X_\tau) < g^*(X_\tau)$. since $\tau < \tau_D$

So $\mathbb{E}^x(g(X_\tau)) = \int_{\tau < \tau_D} + \int_{\tau_D \geq \tau} < \mathbb{E}^x(g^*(X_\tau))$

$\leq g^*(x)$.

2) For $x \in D$. using $\mathbb{E}^x(\hat{g}(X_t))$ is supermart.

$g^*(x) = \mathbb{E}^x(g(X_{\tau^*})) \leq \mathbb{E}^x(\hat{g}(X_{\tau_D}))$

$\overset{x \in D}{=} \mathbb{E}^x(g(X_{\tau_D})) \leq g^*(x)$

3) For $x \in \partial D$. irregular. $\Rightarrow \tau_D > 0$. $\mathbb{P}_x$-a.s.

Choose (α_k). stopping time $\downarrow 0$. $\alpha_k < \tau_D$.

$\Rightarrow \mathbb{E}^x(g(X_{\tau_D})) = \mathbb{E}^x(\theta_{\alpha_k}(g(X_{\tau_D})))$

$\overset{mp.}{=} \mathbb{E}^x(g^*(X_{\alpha_k}))$

By Fatou's: $g^*(x) \leq \mathbb{E}^x(\underline{\lim}\, g^*(X_{\tau_k}))$

$\leq \cdots \leq \mathbb{E}^x(g(X_{\tau_D}))$

4) For $x \in \partial D$ regular / $x \notin \bar{D}$.

Then $\tau_D = 0$ a.s. $\mathbb{P}^x$. $g^*(x) = \mathbb{E}^x(g(X_{\tau_D}))$

Rmk: For A characteristic operator of X_t.

It's easy to check: $U = \{Ag > 0\} \subset D$.

We can test whether $U = \mathbb{R}^n$ first.

i.e. whether τ^* exists.

prop. $d \leq 2$. $X_t = B_t$ is d-dim SBM. Then

$g^*(x) = \|g\|_\infty$ the optimal. w.r.t. (X_t).

Pf: Lemma. $d \leq 2$. the only supermartingale functions on $\mathbb{R}^d$ is const.

Pf: If $\exists\, x, y \in \mathbb{R}^n$. s.t. $u(x) > u(y)$.

Set $\tau_n \downarrow T_{\{x\}} \underset{\text{recur.}}{\overset{\text{a.s.}}{<}} \infty$. s.t.:

$u(y) \geq \mathbb{E}(u(X_{\tau_n}))$

$\to \mathbb{E}^y(u(x)) = u(x)$.

Contradiction!

Rmk: When $d \geq 3$. the nonconst. supermartingale func. exists.

prop. $g^\Delta(x) = \sup\{\mathbb{E}^x(g(X_\tau)) \mid \tau$ is stopping time st. $\mathbb{E}^x(\tau) < \infty\} = g^*(x)$ if g is l.s.c.

Pf: Note $\mathbb{E}^x(g(X_\tau) I_{\{\tau<\infty\}}) \le$

$\mathbb{E}^x(\underline{\lim}_{k\to\infty} g(X_{\tau\wedge k})) \overset{\text{Fatou}}{\le}$

$\underline{\lim}_{k} \mathbb{E}^x(\cdots) \le g^0(x) \le g^*(x)$

for $\forall$ stopping time τ.

② Time-inhomo case:

For rewarding function $g \in C(\mathbb{R}^1 \times \mathbb{R}^n)$.

find $g_0(x)$ and τ^*. s.t.

$g_0(x) = \mathbb{E}^x(g(\tau^*, X_{\tau^*})) = \sup_\tau \mathbb{E}^x(g(\tau, X_\tau))$

Def: $Y_t = Y_t^{s,x} = \begin{pmatrix} s+t \\ X_t^x \end{pmatrix}$, $t \ge 0$. on $\mathbb{R}^1 \times \mathbb{R}^n$.

So: $dY_t = \begin{pmatrix} 1 \\ b(X_t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma(X_t) \end{pmatrix} dB_t$

$= \hat{b}(Y_t)dt + \hat{\sigma}(Y_t)dB_t$. where

$\hat{b}(t,\xi) = \begin{pmatrix} 1 \\ b(\xi) \end{pmatrix}$. $\hat{\sigma}(t,\xi) = \begin{pmatrix} 0 \\ \sigma(\xi) \end{pmatrix}$.

$\Rightarrow g_0(x) = g^*(0,x) = \sup_\tau \mathbb{E}^{(0,x)}(g(Y_\tau)) = \mathbb{E}^{(0,x)}(g(Y_{\tau^*}))$

which transforms to the case ①.

Rmk: $A_Y = \frac{\partial}{\partial t} + A_X$. characteristic operator.

(3) Involving integral:

For $g \in C^{\geq 0}(\mathbb{R}^k)$. $f: \mathbb{R}^k \to \mathbb{R}^{\geq 0}$. Lipschitz conti.

consider find $\phi(x)$. and τ^*. s.t.

$$\phi(x) = \sup_\tau \mathbb{E}^x\left(\int_0^\tau f(X_t)\,dt + g(X_\tau)\right)$$

$$= \mathbb{E}^x\left(\int_0^{\tau^*} f(X_t)\,dt + g(X_{\tau^*})\right).$$

Def: $dW_t = \begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = \begin{pmatrix} b(X_t) \\ f(X_t) \end{pmatrix} dt + \begin{pmatrix} \sigma(X_t) \\ 0 \end{pmatrix} dB_t$

So: $\phi(\eta) = \sup_\tau \mathbb{E}^{(x,0)}(Y_t + g(X_t))$

$$= \sup_\tau \mathbb{E}^{(x,0)}(\tilde{g}(W_t)). \quad \tilde{g}(x,\eta) = g(x) + \eta.$$

$$A_W \phi(x,\eta) = A_X \phi(x,\eta) + f(x) \frac{\partial}{\partial \eta} \phi(x,\eta).$$

which transf. to case (1).

(2) Variational inequi.

Fix $h \subset \mathbb{R}^k$. $dY_t = b(Y_t)dt + \sigma(Y_t)dB_t$. $Y_0 = \eta$.

Assume: $f \in C(\mathbb{R}^k)$, $g \in C(\mathbb{R}^k)$. s.t.

i) $\mathbb{E}^\eta\left(\int_0^{\tau_h} f(X_t)\,dt\right) < \infty$. $\forall \eta \in h$.

ii) $(g^-(X_\tau))_{\tau \leq \tau_h}$ is u.i.

Set $\mathcal{J} = \{\tau \leq \tau_h \mid \tau$ is Y_t-stopping time$\}$.

Find τ^*, $\Phi(\eta)$. s.t. $\Phi(\eta) = \sup_{\tau \in \mathcal{J}} J^\tau(\eta) = J^{\tau^*}(\eta)$.

where $J^{\tau}(y) =: E^{y}(\int_0^{\tau} f(Y_t)dt + g(Y_\tau))$

Thm If we can find $\phi: \bar{G} \to \mathbb{R}'$. s.t.

i) $\phi \in C'(G) \cap C(\bar{G})$. ii) $\phi \geq g$. $\lim_{t \to \tau_G} \phi(Y_t) = g(Y_{\tau_G}) I_{\tau_G < \infty}$.

and $D =: \{x \in G \mid \phi(x) > g(x)\}$. assume:

iii) $E^{y}(\int_0^{\tau_G} I_{\{Y_t \in \partial D\}} dt) = 0$. iv) ∂D is Lipschitz.

v) $\phi \in C^2(G/\partial D)$. $\partial^2 \phi$ is locally bdd near ∂D.

vi) $L\phi + f \leq 0$ on G/D. Then: $\phi \geq \Phi$ on G.

In addition, if vii) $L\phi + f = 0$ on D.

viii) $\tau_D < \infty$ a.s. ix). $(\phi(Y_\tau))_{\tau \leq \tau_D}$ is u.i. $\forall y$

Then: $\phi = \Phi$ on G. $\tau_D = \tau^*$.

Rmk: It's a sufficient criterion to test whether ϕ is optimal.

Pf: i) Using Dynkin's formula and Fatou's.

approxi. $\tau = \tau_G$. by τ_{G_n} a.e. $< \infty$.

2) By u.i. $\Rightarrow$ the equation holds.