

Lecture 2

Spans and Matrices

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Based on Dr. Ralph Chikhany's Slide

Reminders

- Get access to Gradescope, Campuswire.
- Obtain the textbook.
- Problem Set 1 due by 11.59 pm on Friday (NY time). *Next week*
 - ✓ Late work policy applies.
- Recap Quiz 1 due by 11.59 pm on Sunday (NY time). *This Friday on Gradescope*,
 - ❖ Late work policy does not apply.
- Recap Quiz is timed.
 - Once you start, you have 60 minutes to finish it (even if you close the tab)

Cheat Sheet

- Cheat Sheet overleaf: <https://www.overleaf.com/read/jjbswyqvzdx#8803d5>

Recap

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be a set of vectors in \mathbb{R}^n . We define

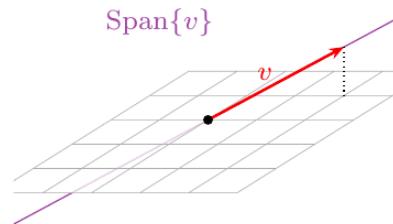
$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ = set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$

Recap

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be a set of vectors in \mathbb{R}^n . We define

$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ = set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$

$$u, v, w \in \mathbb{R}^3$$



Recap

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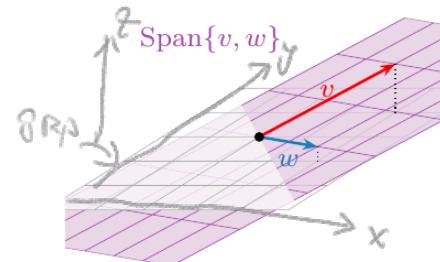
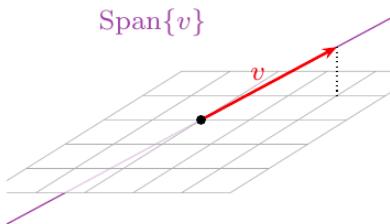
$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ = set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$

$$\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_m \vec{v}_m$$

\vec{w} is not parallel / colinear with \vec{v}

Then $\text{span}\{\vec{v}, \vec{w}\}$ is a plane in \mathbb{R}^3

Case 1 →



← purple plane here is the span

↳
Case 2

\vec{w} is colinear / parallel with \vec{v}

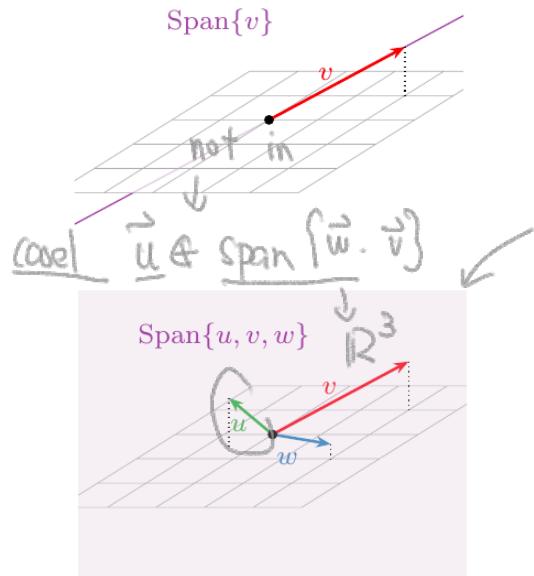
$\text{span}\{\vec{v}, \vec{w}\} = \text{span}\{\vec{v}\}$ is still a line.

Recap

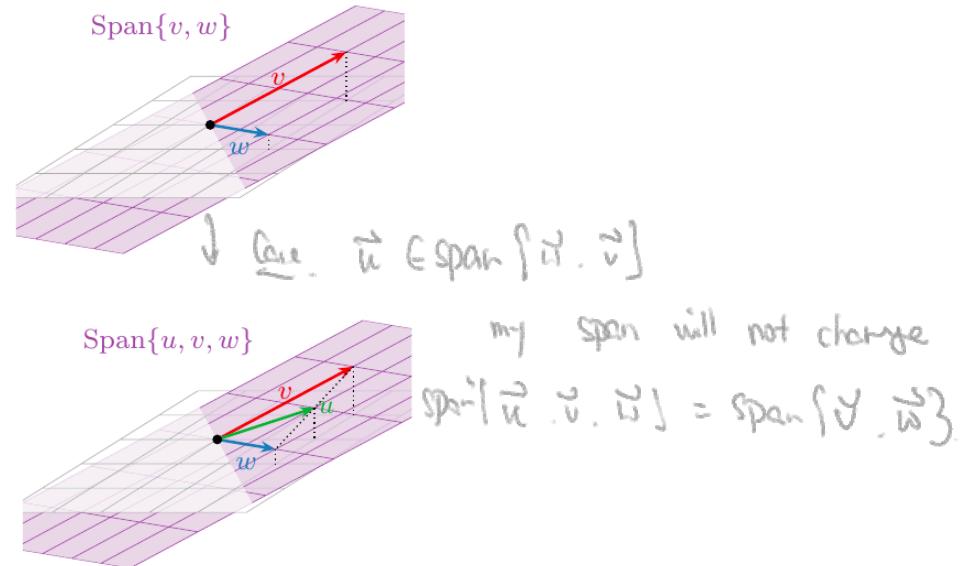
Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be a set of vectors in \mathbb{R}^n . We define

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} = \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$$

3-Vector



2-Vector





NYU

Strang Section 1.3 - Matrices

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed),
N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by
Margalit and Rabinoff, in addition to our text

Matrices

An $m \times n$ matrix A is a rectangular array of (real) numbers a_{ij} with m rows and n columns, where

m : # rows

n : # columns

$m \times n$ matrix as $\mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

a_{ij} → column index
↓ row index

A matrix is called square if it is $n \times n$, i.e., it has the same number of rows and columns.

square $\mathbb{R}^{n \times n}$

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

is 2×3 matrix

$A \in \mathbb{R}^{2 \times 3}$

$a_{12} = 2$

$a_{23} = 6$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the i th row and the j th column. It is called the **ij th entry** of the matrix.

The entries $a_{11}, a_{22}, a_{33}, \dots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

A **diagonal matrix** is a **square** matrix whose only nonzero entries are on the main diagonal.

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for all v in \mathbf{R}^n .

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

*j*th column *i*th row

$$\left(\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array} \right) \quad \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array} \right)$$

diag elements of matrix
are a_{ii} $i = 1, \dots, \min\{m, n\}$

the smaller one
of m or n

$$\left(\begin{array}{ccc} a_{11} & 0 & a_{12} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{array} \right)$$

← diag matrix $\in \mathbf{R}^{n \times n}$
only a_{ii} can be non-zero

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

← later

$|a_{ij}=0 \text{ if } i \neq j|$

Example: $\begin{pmatrix} a_{12} \\ 0 \end{pmatrix} \quad 1 \neq 2$

$a_{13} \quad 1 \neq 3$

Matrices

The zero matrix (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A . In other words, the ij entry of A^T is a_{ji} .

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix} \quad i\text{-th row}$$

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{i1} & \cdots \\ a_{12} & a_{22} & \cdots & a_{i2} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ a_{ij} & a_{ij} & \cdots & a_{ij} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix} \quad \begin{matrix} i\text{-th column} \\ j\text{-th column} \end{matrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \xrightarrow{\text{flip}} A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$$\Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Example. $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a vector $\mathbb{R}^{3 \times 1}$

$$A^T = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \quad \mathbb{R}^{1 \times 3}$$

3-row
1-column

1-row
2-column

Column vs. Row Representation of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

↑ vector 1 ↑ Vector 2 ↑ vector n

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$\in \mathbb{R}^{2 \times 3}$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \in \mathbb{R}^2$$

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$\vec{v}_3 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$$

each column of A is a vector $\in \mathbb{R}^m$

$$\vec{v}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \dots \vec{v}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix}$$

$$\text{then } A = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_m]$$

Column vs. Row Representation of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$[a_{11} \ \dots \ a_{1j} \ \dots \ a_{1n}] = \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \end{bmatrix}^T \quad \vec{r}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \end{pmatrix} \in \mathbb{R}^n$$

$$\vec{r}_2 = \begin{pmatrix} a_{21} \\ \vdots \\ a_{2n} \end{pmatrix} \in \mathbb{R}^n$$

Then $A = \begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix}$

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in \mathbb{R}^{2 \times 3}$

$$\vec{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \vec{r}_1^T = (1 \ 2 \ 3)$$

$$\vec{r}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \quad \vec{r}_2^T = (4 \ 5 \ 6)$$

$$A = \begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \end{bmatrix}$$

m - Row vector
each row vector $\in \mathbb{R}^n$

Linear Combination in Matrix Notation

* Multiply a matrix with a vector

{ Column Understanding
Row Understanding }

A linear combination of n vectors, $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, in \mathbb{R}^m is given by

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n$$

a vector in \mathbb{R}^m
↑ will give

where $x_1, x_2, \dots, x_n \in \mathbb{R}$.

$A \in \mathbb{R}^{m \times n}$ can only multiply with a vector $\vec{x} \in \mathbb{R}^n$

This can be expressed as an $m \times n$ matrix A multiplying a vector $\vec{x} \in \mathbb{R}^n$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{bmatrix} A = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \vec{x} \in \mathbb{R}^n = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_n \vec{v}_n$$

↓
 $\vec{v}_i \in \mathbb{R}^m$

!! n is the # row of vector

You can understand a vector in \mathbb{R}^m as a $\mathbb{R}^{m \times 1}$ matrix

Pool $\{Ax \mid x \in \mathbb{R}^n\}$ gives me the span.

$$A \in \mathbb{R}^{m \times n} \quad x \in \mathbb{R}^n$$

$$A = [\vec{v}_1 \dots \dots \vec{v}_n]$$

Ax lie in the span of the column vectors of matrix A

$$\frac{Ax \in \text{span}\{\vec{v}_1 \dots \vec{v}_n\}}{\downarrow} \text{ Yes!}$$

$$Ax \in \mathbb{R}^m$$

is L.C. of $\vec{v}_1 \dots \vec{v}_n$

What is the size of matrix A

For all the vector v in the span of the column vectors of matrix A , we can find a vector x , such that $Ax = v$ Yes!

if $\vec{v} \in \text{span}\{\vec{v}_1 \dots \vec{v}_n\}$

can we find a vector x s.t. $Ax = \vec{v}$

$$\vec{v} \text{ is L.C. of } \{\vec{v}_1 \dots \vec{v}_n\} \text{ means } \vec{v} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n. \text{ Then } A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{v}$$

Dot product as matrix vector multiplication

$x \cdot y$ is $x^T y$

↑
here
is a dot

$$x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \stackrel{\text{def}}{=} x_1 y_1 + x_2 y_2 + \cdots + x_n y_n. \rightarrow \text{is a real number}$$

understand $\vec{x} \cdot \vec{y}$ as a L.C.

Transpose of \vec{y}

Then

$$\begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}$$

by definition

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$$\vec{x} \cdot \vec{y} = x^T y$$

$$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

↓
weights / coef.

a vector in \mathbb{R}^m
is a $\mathbb{R}^{m \times 1}$ matrix

$$\vec{y} \cdot \vec{x} = \begin{pmatrix} y^T \\ \downarrow \\ \mathbb{R}^m \\ \downarrow \\ \mathbb{R}^{1 \times n} \end{pmatrix} \times \begin{pmatrix} x^T \\ \downarrow \\ \mathbb{R}^n \end{pmatrix} \rightarrow (\mathbb{R}) \text{ is real number}$$

$\mathbb{R}^{m \times n}$ can multiply \mathbb{R}^n

identity matrix

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for all v in \mathbf{R}^n .

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

↑ identity Matrix

↳ all other are zero

main diag element are 1

$$I_3 \cdot \vec{x} = \vec{x}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\textcircled{1} \quad I_3 = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \parallel & \parallel & \parallel \\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$I_3 \vec{x} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{x}$$

② Row

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$\begin{pmatrix} (1, 0, 0) \cdot (x_1, x_2, x_3) \\ (0, 1, 0) \cdot (x_1, x_2, x_3) \\ (0, 0, 1) \cdot (x_1, x_2, x_3) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Linear Combination in Matrix Notation

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Compute $A\vec{x}$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A\vec{x} = x_1 \cdot \vec{v}_1 + x_2 \cdot \vec{v}_2 + x_3 \cdot \vec{v}_3$$

$$= x_1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 0 \cdot x_2 + 0 \cdot x_3 \\ -x_1 + x_2 + 0 \cdot x_3 \\ 0 \cdot x_1 - x_2 + x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

Dot Product with Rows

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1, 0, 0) \cdot (x_1, x_2, x_3) \\ (-1, 1, 0) \cdot (x_1, x_2, x_3) \\ (0, -1, 1) \cdot (x_1, x_2, x_3) \end{bmatrix}.$$

They are the same for

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1, 0, 0) \cdot (x_1, x_2, x_3) \\ (-1, 1, 0) \cdot (x_1, x_2, x_3) \\ (0, -1, 1) \cdot (x_1, x_2, x_3) \end{bmatrix}.$$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1, 0, 0) \cdot (x_1, x_2, x_3) \\ (-1, 1, 0) \cdot (x_1, x_2, x_3) \\ (0, -1, 1) \cdot (x_1, x_2, x_3) \end{bmatrix}.$$

Matrix times vector

Matrix times vector

$$Ax = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\vec{v}_1 + d\vec{v}_2 + e\vec{v}_3. \quad (3)$$

\uparrow
 x

Dot Product View:

$$A\vec{x} = \begin{bmatrix} \vec{r}_1^T \\ \vec{r}_2^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$$

← easier one to do
calculation.

$$A \in \mathbb{R}^{m \times n}$$

$$\text{then } \vec{r}_i \in \mathbb{R}^n \quad \vec{x} \in \mathbb{R}^n$$

The same size ← They can do dot product.

each row do } $\rightarrow \mathbb{R}^m$
and dot product } vertical

$$A \in \mathbb{R}^{m \times n}, \quad x \in \mathbb{R}^n \rightarrow Ax \in \mathbb{R}^m$$

Examples

Let v_1, v_2, v_3 be vectors in \mathbf{R}^3 . How can you write the vector equation

$$2\vec{v}_1 + 3\vec{v}_2 - 4\vec{v}_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

use the vector here as column vector

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

The system $A\vec{x} = \vec{b}$



The result of $A\vec{x}$, where A is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$ is a vector $\vec{b} \in \mathbb{R}^m$, where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

coef unknown

Then $A \vec{x} = \vec{b}$ ← right hand side
by defn ||

$x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n$

If A is a square matrix, i.e., A is $n \times n$, and $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} = \vec{b} \in \mathbb{R}^n$.

Linear System \Leftrightarrow L.C.

$$b_1 = x_1 \cdot a_{11} + x_2 \cdot a_{12} + \cdots + x_n \cdot a_{1n}$$

$$b_2 = x_1 \cdot a_{21} + x_2 \cdot a_{22} + \cdots + x_n \cdot a_{2n}$$

$$\vdots$$

$$b_m = x_1 \cdot a_{m1} + x_2 \cdot a_{m2} + \cdots + x_n \cdot a_{mn}$$

$$\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

is L.C. of $\vec{v}_1 \dots \vec{v}_n$

$$\text{if } \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = x_1 \cdot \vec{v}_1 + \cdots + x_n \cdot \vec{v}_n$$

The system $A\vec{x} = \vec{b}$: What if \vec{x} is unknown?

The result of $A\vec{x}$, where A is an $m \times n$ matrix and $\vec{x} \in \mathbb{R}^n$ is a vector
 $\vec{b} \in \mathbb{R}^m$, where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

When A and \vec{x} are given, computing \vec{b} is straight forward. However, the reverse is not always true (or even possible). That is, if A and \vec{b} are given, it is not always possible to find \vec{x} .

If A is a square matrix, i.e., A is $n \times n$, and $\vec{x} \in \mathbb{R}^n$, then $A\vec{x} = \vec{b} \in \mathbb{R}^n$.

Next lecture is to solve $A\vec{x} = \vec{b}$
square
left unknown right hand side.

Why diag matrix is easy
 $A = \begin{pmatrix} a_{11} & & & \\ & \ddots & & \\ & & a_{nn} & \\ & & & \text{are zero} \end{pmatrix}$

$$Ax = b \quad \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{nn}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{array} \right. \quad \Rightarrow$$

Examples

Consider the system $A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}$.

Suppose that b_1, b_2 , and b_3 are given, and you want to compute x_1, x_2 , and x_3 in terms of the components of \vec{b} .

"Lower Triangular"

Coef

$$x_1 = b_1$$

$$-x_1 + x_2 = b_2$$

$$-x_1 + x_3 = b_3$$

$$\rightarrow x_1 = b_1$$

$$\rightarrow x_2 = b_1 + b_2$$

$$\rightarrow x_3 = b_1 + b_2 + b_3$$

easy to solve because we can solve one by one.

Examples

Consider the system $A\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}$.

Suppose that b_1, b_2 , and b_3 are given, and you want to compute x_1, x_2 , and x_3 in terms of the components of \vec{b} .

inverse of matrix

Equation

$$A \vec{x} = \vec{b}$$

↑
coef unknown Right hand side

$$\begin{aligned} x_1 &= b_1 \\ -x_1 + x_2 &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned}$$

Solution

$$\begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3. \end{aligned}$$

solution

$$\vec{x} = A^{-1} \vec{b}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Not every matrix have an inverse

Cyclic

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b.$$

Two ways to calculate the matrix vector multiplication
Linear combination

Next time.

"Not every matrx have a inverse"

Dot product

Not every matrix have an inverse

Cyclic

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b.$$

$$b = (1, 3, 5)$$

Not every matrix have an inverse

Cyclic $Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = b.$

$$b = (0,0,0)$$

Review

- Two ways to calculate the Matrix-vector multiplication

Row Representation

$$A = \begin{bmatrix} \vec{r}_1^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$\vec{r} \in \mathbb{R}^n \quad x \in \mathbb{R}^n$$

$$Ax = \begin{bmatrix} \vec{r}_1 \cdot x \\ \vdots \\ \vec{r}_m \cdot x \end{bmatrix} \in \mathbb{R}^m$$

Linear system

$$\begin{cases} 2x + y = 1 \\ 3x + 2y = 1 \end{cases}$$

Column Representation

$$A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$\vec{v}_i \in \mathbb{R}^m$$

$$Ax = x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n \in \mathbb{R}^m$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

a LC.

$$A(\vec{y}) = (\vec{l})$$

Get $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ unknown R.H.S.



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Questions?