Lie Groups

(1) Definition:

Def: i) A Lie group is a smooth mfd G. st. G is group, $(g,h) \overset{\mu}{\mapsto} gh$ is smooth map.

Rmk: i) Any topo group whose underlying space is topo mfd is a Lie group. (Conti. struc $\Rightarrow$ smooth)

ii) Not any smooth mfd admits a Lie group structure. (e.g. for S^n. only work for $n=0,1,3$)

iii) 0-dim Lie group is countable group with discrete topo.

iv) struc. for product of 2 Lie groups : $(g,h)(g',h') = (gg', hh')$

ii) $l_g : h \mapsto gh$. $r_g : h \mapsto hg$. left / right translation from G to G.

Rmk: i) l_g, r_g are homeomorphisms

ii) Each connected components of G are homeomorphic by some r_g / l_g. And the component $G_0 \ni e$ is normal in G:

Since $g G_0 g^{-1} \ni e \Rightarrow g G_0 g^{-1} = G_0$. So: G/G_0 is 0-dim discrete Lie group.

e.g. $(\mathbb{R}^n, +)$, $T^n = \mathbb{R}^n / \mathbb{Z}^n$, $(SL(\mathbb{R}^n), \cdot)$ and $\mathrm{Aff}(\mathbb{R}^n) \cong \mathbb{R}^n \rtimes GL(\mathbb{R}^n)$ with $(v, A) \cdot (w, B) = (v + Aw, AB)$, where $(v, A) \in \mathrm{Aff}(\mathbb{R}^n)$ is $x \mapsto v + Ax$.

Rmk: Any cpt Lie group can be represented in matrix Lie group.

prop. G is m-dim Lie group. Then TG is trivial bundle. i.e. $TG \cong G \times \mathbb{R}^m$.

Pf: $\phi : G \times \underbrace{T_e G}_{\cong \mathbb{R}^m} \xrightarrow{\sim} TG$

$a, \xi \mapsto (a, (l_a)_* \xi)$.

Note $\phi^{-1}(a, \xi) = (a, (l_{a^{-1}})_* \xi)$

<u>prop.</u> $\mu : (g,h) \mapsto gh$. We have:

$$d\mu_{a,b}(X_a, Y_b) = (d r_b)_a (X_a) + (d l_a)_b (Y_b)$$

<u>Pf</u>: $RHS(f) = (X_a, Y_b)(f \circ \mu)$

$$= X_a(f \circ \mu \circ z_b) + Y_b(f \circ \mu \circ \tilde{z}_a)$$

$$= X_a(f \circ r_b) + Y_b(f \circ l_a)$$

where $z_b : g \mapsto (g, b)$

$\tilde{z}_a : g \mapsto (a, g)$

<u>Cor.</u> $i : G \to G$, $g \mapsto g^{-1}$ is smooth map.

<u>Def</u>: i) v.f. X on Lie group G is called left-invariant if $\forall g \in G$. we have $(l_g)_* X_h = X_{gh}$. $\forall h \in G$.

ii) $\mathfrak{g} := \{$ left-invariant v.f. on $G\}$.

<u>prop.</u> i) $\mathfrak{g}$ is LS. And $\dim \mathfrak{g} = \dim T_eG = \dim G$.

$\varphi : \mathfrak{g} \to T_eG$, $X \mapsto X_e$ is isomorphic

ii) $\forall X, Y \in \mathfrak{g}$. $\Rightarrow [X, Y] \in \mathfrak{g}$.

<u>Rmk</u>: So $\mathfrak{g}$ is a Lie algebra. We call $\mathfrak{g}$ is Lie algebra of G.

Pf: i) $\varphi^{-1}(X_e) = (d\,l_g)_e(X_e) = X_g$

easy to check $X \in \mathfrak{g}$

ii) Check $(l_g)_* [X,Y]_h = [X,Y]_{gh}$

$$X_{gh}(Yf) = (d\,l_g)_h X_h(Yf)$$
$$= X_h((Yf)\circ l_g)$$
$$\overset{(*)}{=} X_h Y(f\circ l_g)$$

$$(*)\ (Yf)\circ l_g(k) = Y_{gk} f$$
$$= (d\,l_g)_k Y_k f = Y_k(f\circ l_g)$$
$$= Y(f\circ l_g)(k)$$

Def: For $X \in \mathfrak{g}$. $\gamma_e(t)$ is integral curve of X through e. Set $\exp(tX_e) := \gamma_e(t)$

Rmk: i) $\exp: T_e G \to G$ isn't the Riemannian exponential func before! which is def by algebra structure rather Rieman metric

ii) $\mathfrak{g}$ is linear info. of G. Actually $\exp$ is a recovering from linear info. to nonlinear info in G.

prop. $\exp(tX_g) \overset{\Delta}{=} \gamma_g(t) = g\exp(tX_e)$

Pf: By uniqueness. check $g\gamma_e(t)$

is integral curve of X at g

$g\gamma_e(0) = g$. And $\frac{d}{dt}(g\gamma_e(t))$

$= (dL_g)_{\gamma_e(t)}(\gamma_e'(t))$

$= (dL_g)_{\gamma_e(t)} X_{\gamma_e(t)} = X_{g\gamma_e(t)}$.

Rmk: $\theta_t(g) = \gamma_g(t) = g\exp(tX_e)$. $\Rightarrow$

$\theta_t = r_{\exp(tX_e)}$

Cor. $\exp((s+t)X_e) = \exp(sX_e)\exp(tX_e)$

Pf: Set $g = \exp(sX_e)$.

Rmk: $\exp(tX)\exp(tY) \neq \exp(t(X+Y))$

Thm (BCH formula)

$\forall X, Y \in \mathfrak{g}$, $\exists Z: (-\varepsilon, \varepsilon) \to \mathfrak{g}$ smooth. s.t.

$\exp(tX)\exp(tY) = \exp(t(X+Y) + t^2 Z(t))$

for $\forall t \in (-\varepsilon, \varepsilon)$.

Rmk: $Z(t) = \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]]$

$- [Y, [Y, X]]) + \cdots$

prop. $(d\exp)_0 = \mathrm{id}_{\mathfrak{g}}$.

Pf: $X_e = \frac{d}{dt}\big|_{t=0} \varphi_e(t) = (d\exp)_0 \frac{d}{dt} tX$

$= (d\exp)_0 X$.

Cor. exp is diffeo. around 0. some nbd

Rmk: Even if G is connected. exp isn't always surjective.

e.g. $G = SL(\mathbb{R}^n)$. $A = \mathrm{diag}\{-1,-1\}$ isn't square of real matrix

By cor. above $\Rightarrow A \notin \mathrm{Im}(\exp)$

e.g. Consider $G = GL(\mathbb{R}^n) = \mathbb{R}^{n\times n}$.

i) Identify $T_e G \simeq \mathbb{R}^{n\times n}$. then

$\exp(X_e) = I + X_e + \cdots = \sum X_e^k / k!$

We can check it solves:

$\frac{d}{dt}(\exp(tX_e)) = L(\exp(tX_e))_* X_e$

$= X\exp(tX_e)$

ii) Claim $[X, Y]_e = [X_e, Y_e]$.

(The latter $[\,,\,]$ is commutator for $\mathbb{R}^{n\times n} \simeq \mathrm{End}(\mathbb{R}^n) \simeq T_e GL(\mathbb{R}^n)$ while the former is for endomorphism) identify $T_A G \cong \mathbb{R}^{n\times n}$. and set x_{ij}

is entry of matrix: $x_{ij}(A) = a_{ij}$.

And also def $X_A(x_{ij}) = \epsilon x_{ij}(X_A) = (X_A)_{ij}$ for $X_A \in T_A G$.

Next. check $[X, Y]_e x_{ij} = [X_e, Y_e]_{ij}$.

Note $(Y, e_{ij}) : A \longmapsto Y_A e_{ij} \overset{Y \in \mathfrak{g}}{=} (AY_e)_{ij}$

So: $X_e(Y e_{ij}) = (X_e Y_e)_{ij}$

<u>Rmk</u>: We can also check:

$$(L_X Y)_e = \frac{d}{dt}\Big|_{t=0} (\theta_{-t})_* Y_{\theta_t(e)}$$

$$= \frac{d}{dt}\Big|_{t=0} (\exp(tX_e) Y_e \exp(-tX_e))$$

$$\overset{\text{product}}{\underset{\text{rule}}{=}} X_e Y_e - Y_e X_e = [X_e, Y_e].$$

(2) <u>Lie homomor.</u>:

<u>Def</u>: i) Lie group homomor $\varphi : G \to H$ is a smooth group homo: $\varphi(gg') = \varphi(g)\varphi(g')$

ii) Lie algebra homomor $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a linear map. st.

$\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$. $\forall X, Y \in \mathcal{X}(M)$.

<u>prop.</u> Any Lie group homomor. $\varphi : G \to H$ has const. rank and induces a Lie alge. homomor. $\varphi_* : \mathfrak{g} \to \mathfrak{h}$.

<u>Pf:</u> i) $\varphi(gg') = \varphi(g)\varphi(g')$. i.e.

$\varphi \circ l_g = l_{\varphi(g)} \circ \varphi$. Take d_e:

$(d\varphi)_g \circ (dl_g)_e = (dl_{\varphi(g)})_e (d\varphi)_e$ (*)

$\Rightarrow r((d\varphi)_g) = r((d\varphi)_e)$. $\forall g$.

ii) By above: $(d\varphi)_g(X_g) = \square \circ l_{g*}(\cdot) \overset{(*)}{=}$

$(dl_{\varphi(g)})_e((d\varphi)_e(X_e)) = (d\varphi(X))_{\varphi(g)}$

$\Rightarrow X$ and $\varphi_*(X)$ are φ-related.

So: $[X, Y]$ and $[\varphi_*(X), \varphi_*(Y)]$ are φ-related.

And $[X, Y]$ and $\varphi_*([X, Y])$ are φ-related.

$\Rightarrow [\varphi_*(X), \varphi_*(Y)]_{\underset{\varphi(e)}{\overset{\shortparallel}{e}}} = (d\varphi)_e([X, Y]_e)$

$= \varphi_*([X, Y])_e$.

With left-invariant, replace e by g.

<u>Rmk</u>: $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$. $\Rightarrow$ isomor. of Lie group induce isomor. of Lie algebra.

(3) <u>Lie Subgroup</u>:

<u>Def</u>: i) $\mathfrak{h} \subset \mathfrak{g}$ Lie algebra is a linear subspace & $\forall X, Y \in \mathfrak{h} \Rightarrow [X, Y] \in \mathfrak{h}$.

We say $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$

ii) H is subgroup of Lie group G and also its immersed submfd. And:

$(h, \tilde{h}) \xmapsto{m|_H} h\tilde{h}$ is smooth in H.

we say H is Lie subgroup of G.

<u>Rmk</u>. i) Lie subgroup of a cpt subgroup may not be cpt.

ii) $\iota: H \hookrightarrow G$ is injective. $\Rightarrow$ $(\iota_*): \mathfrak{h} \to \mathfrak{g}$ is also injective since ι_* has const. rank.

$\Rightarrow \iota$ is injective immersion.

but ι isn't necessary to be a embed. which's motivation of Def ii).

Also: Lie group H with Lie algebra $\mathfrak{h}$. st. $\mathfrak{h}$ is Lie subalgebra of $\mathfrak{g}$ may not be embedded submfd of G.

Thm. (Cartan's closed subgroup)

$\iota: H \to G$ injective. If $\iota(H)$ is closed in G. then: $\iota(H)$ is embedded submfd. called Lie closed subgroup.)

Rmk: generally injective immersion with closed image may not be embedding

Cor. $\varphi: G \to H$ is Lie homo. $\Rightarrow$ $\ker\varphi$ is Lie closed subgroup of G.

Cor. $\varphi: G \to H$ is conti. Lie homo. $\Rightarrow$ φ is smooth.

(4) Left-invariant objects:

Def: left-invariant k-form ω satisfies:

$\omega_{g(h)}((L_g)_* V_h^1 \cdots (L_g)_* V_h^k) = \omega_h(V_h^1 \cdots V_h^k)$

Rmk: i.e. the value only depends on $\{V^i\}_1^k$ rather at which pt g.

Prop. ω is left-invariant one form. $\Rightarrow$

$d\omega(X, Y) = -\omega([X, Y])$.

Pf: Note $\omega(X, \cdot) \equiv$ const. $\Rightarrow Y\omega(X) = 0$.

Denote $(E_i) \subset T_e G$ a frame, extend them to left-invariant vector field.

Set $[E_i, E_j] = \sum_k c_{ij}^k E_k$. c_{ij}^k is struc const.

From antisym and Jacobi id. we have:

$c_{ij}^k + c_{ji}^k = 0$. $\sum_m (c_{ij}^m c_{mk}^l + c_{jk}^m c_{mi}^l + c_{ki}^m c_{mj}^l) = 0$

1) Set (θ^i) are dual frame. $\Rightarrow$ They're left-invari. one-form.

$\Rightarrow d\theta^i(E_j, E_k) = -\theta^i([E_j, E_k]) = -c_{jk}^i$.

And also $d\theta^i = -\sum_{j<k} c_{jk}^i \theta^j \wedge \theta^k$.

Eventually, we can define left-invari. one form from (θ^i).

Rmk: We can also def left-inva. $\mathfrak{g}$-value one-form $\omega_g(X_g) \triangleq (l_{g^{-1}})_* X_g$.

ii) Consider def connection ∇ by $\nabla_{E_i} E_j = 0$ and extend to TG. ((E_i) is left-invar.)

$\Rightarrow \forall X \in \mathfrak{g}$ is parallel with any curve.

So: $R \equiv 0$. We say it's a flat conn.

Rmk: $\forall$ holomy is trivial in this ∇.

And $P_\gamma^{t_0, t_1}$: $v^i E_i(\gamma(t_0)) \mapsto v^i E_i(\gamma(t_1))$.

iii) Put a inner product on $T_e G$ and let (E_i) is a o.n.b. $\Rightarrow$ Set $\langle E_i(e), E_j(e) \rangle_e = \langle E_i(g), E_j(g) \rangle_g$ to get left-invariant Riemann metric $\langle , \rangle$ on G.

i.e. $\langle X_g, Y_g \rangle_g = \langle (dl_h)_g X_g, (dl_h) Y_g \rangle_{hg}$.

Rmk: right-inva. is to replace l_h by r_h.

It's called bi-variant if both holds

$\Rightarrow \nabla$ in 2) is metric-compatible to this $\langle , \rangle$. But Torsion $= -[X,Y] \neq 0$

So: ∇ in 2) isn't LC connection.

Rmk: To get LC connection, we can see the Koszul formula:

$2\Gamma_{ijk} = C_{ijk} - C_{jki} - C_{kij}$. where

$\Gamma_{ijk} \overset{\Delta}{=} \langle \nabla_{E_i} E_j, E_k \rangle$. $C_{ijk} \overset{\Delta}{=} \langle [E_i, E_j], E_k \rangle$

prop. There's a bijection between inner product on $\mathfrak{g}$ and left-inv. metric on G.

Pf: Set $\langle X_g, Y_g \rangle^{\sim}_g \overset{\Delta}{=} \langle (dl_{g^{-1}})_g X_g, (dl_{g^{-1}})_g Y_g \rangle$

$\Rightarrow \langle , \rangle^{\sim}_g$ is left invariant.

Def: i) Conjugation by $g \in G$ is a automor. φ_g $: h \mapsto g h g^{-1}$, which gives a action $g \mapsto \varphi_g \in \mathrm{Aut}(G)$

ii) $\mathrm{Ad}_g \overset{\Delta}{=} (d\varphi_g)_e = (dl_g)_{g^{-1}} \circ (dr_{g^{-1}})_e$ on T_eG.

Rmk: i) $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{g})$ is a invertible linear transfer of T_eG, preserves $[,]$.

ii) For matrix Lie group. Ad_g : $X \mapsto gXg^{-1}$.

iii) We get adjoint represent of G : $Ad_\cdot : G \mapsto Aut(g)$.

iii) $ad_X := (D_e Ad_\cdot)(X)$. this's adjoint repre of Lie algebra g. i.e. $ad_\cdot$ is a homomor. from g to $End(g)$.

<u>prop.</u> $ad_X(Y) = [X, Y]$.

iv) Killing form on g over $\mathbb{F}$ is a $\mathbb{F}$-bilinear form $k(X,Y) := tr(ad_X \circ ad_Y)$.

<u>Rmk</u>: For matrix Lie algebra, since $tr(AB) = tr(BA)$. So we have:

$k(X,Y) = k(Y,X)$. and

$k(Ad_g X, Ad_g Y) = k(X,Y)$.

$\overset{\text{diff}}{\Longrightarrow}$ $k(ad_Z X, Y) = k(X, ad_Z Y)$

<u>prop</u>. i) $k(\ ,\ )$ is symmetric.

ii) $k([X,Y], Z) = k(X, [Y,Z])$

iii) $\forall \varphi \in \mathrm{Aut}(\mathfrak{g})$, $K(\varphi(X), \varphi(Y)) = K(X, Y)$. (e.g. let $\varphi = \mathrm{Ad}_g$)

__prop.__ There's a bijection between bi-variant metric on G and Ad-invariant inner product $\langle , \rangle$ on $\mathfrak{g}$. i.e. $\langle \mathrm{Ad}_g Y, \mathrm{Ad}_g Z \rangle = \langle Y, Z \rangle$. for $\forall Y, Z \in \mathfrak{g}$. $g \in G$.

__Rmk:__ G has bivariant metric $\Rightarrow$ Ad_g is an orthogonal transf. of $\mathfrak{g}$.

__prop.__ G is connected Lie group. Then $\langle , \rangle$ inner product in $\mathfrak{g}$ induces a bi-variant metric on G $\Rightarrow \langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$. $\forall X. Y. Z.$

__prop.__ left invar. metric on G is bi-variant $\Leftrightarrow$ $i: g \mapsto g^{-1}$ is an isometry.

__Thm.__ G admits bi-invar metric $\Leftrightarrow$ $G \cong K \times \mathbb{R}^m$. for some cpt group K. $m \geq 0$

__Rmk:__ Under bi-invariant metric: Riemann expon. map $\Leftrightarrow$ Lie group expon. map.

Thm G is cpt Lie group. Then, we have:
$K(\cdot,\cdot)$ is negative definite and $-K$
gives a bi-inver. Riemann metric on G.

Rmk: i) It can also be constructed by
Haar measure

ii) In this case, we have:

$R(X,Y) = \frac{1}{4} ad_{[X,Y]}$ and

$K(\Pi) = \frac{1}{4} \langle [X,Y], [X,Y] \rangle$.

Π is 2-plane spaned by X, Y.

So the sectional curva. ≥ 0.