# Staticization and Iterated Staticization *

**Abstract**

Conservative dynamical systems propagate as stationary points of the action functional. Using this representation, it has previously been demonstrated that one may obtain fundamental solutions for two-point boundary value problems for some classes of conservative systems via solution of an associated dynamic program. It is also known that the gravitational and Coulomb potentials may be represented as stationary points of cubicly-parameterized quadratic functionals. Hence, stationary points of the action functional may be represented via iterated staticization of polynomial functionals. This leads to representations through operations on sets of solutions of differential Riccati equations (DREs). A key step in this process is the reordering of staticization operations.

**Key words.** dynamic programming, stationary action, staticization, two-point boundary value problems, Schrödinger equation, conservative dynamical systems.

## 1 Introduction

Conservative dynamical systems propagate as stationary points of the action functional over the possible paths of the system. This viewpoint appears particularly useful in some applications in modern physics, including systems where relativistic effects are non-negligible and systems in the quantum domain, cf. [6, 7, 9, 21, 22]. The stationary-action formulation has also recently been found to be quite useful for generation of fundamental solutions to two-point boundary-value problems (TPBVPs) for conservative dynamical systems, cf. [3, 4, 17, 19].

To give a sense of this latter application domain, consider a finite-dimensional action-functional formulation of such a TPBVP. Let the path of the conservative system be denoted by $\xi_r$ for $r \in [0,t]$ with $\xi_0 = \bar{x}$, in which case the action functional, with an appended terminal cost, may take the form

$$J(\bar{x},t,u) \doteq \int_0^t T(u_r) - V(\xi_r)\,dr + \psi(\xi_t), \tag{1}$$

where $\dot{\xi} = u$, $u \in \mathcal{U} \doteq L_2(0,t)$, $T(\cdot)$ denotes the kinetic energy associated to the momentum (specifically taken to be $T(v) \doteq \frac{1}{2}v^T\mathcal{M}v$ throughout, with $\mathcal{M}$ positive-definite and symmetric), and $V(\cdot)$ denotes a potential energy field. If, for example, one takes $\psi(x) \doteq -\bar{v}^T\mathcal{M}x$, a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\dot{\xi}_t = \bar{v}$; if one takes $\psi$ to be a min-plus delta function centered at $z$, then a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\xi_t = z$, cf. [4]. In the early work of Hamilton, it was formulated as the least-action

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principle [10], which states that a conservative dynamical system follows the trajectory that minimizes the action functional. However, this is typically only the case for relatively short-duration cases, cf. [9] and the references therein. In such short-duration cases, optimization methods and semiconvex duality are quite useful [3, 4, 19]. However, in order to extend to longer-duration problems, it becomes necessary to apply concepts of stationarity [17, 18].

It is worth noting that if one defines $\operatorname{stat}_{x\in\mathcal{X}}\phi(x)$ to be the critical value of $\phi$ (defined rigorously in Section 2.1), then a gravitational potential given as $V(x)=-\mu/|x|$ for $x\neq 0$ and constant $\mu>0$, has the representation $V(x)=-(\frac{3}{2})^{3/2}\mu\operatorname{stat}_{\alpha>0}\{\alpha-\frac{\alpha^3|x|^2}{2}\}$, where we note that the argument of the stat operator is polynomial, [11, 19]. The Schrödinger equation in the context of a Coulomb potential may be similarly addressed. In that case, it is particularly helpful to consider an extension of the space variable to a vector space over the complex field, say $x\in\mathbb{C}^n$ rather than $x\in\mathbb{R}^n$. More specifically, for $x\in\mathbb{C}^n$, this representation takes a general form $V(x)=-(\frac{3}{2})^{3/2}\hat{\mu}\operatorname{stat}_{\alpha\in\mathcal{A}^R}\{\alpha-\frac{\alpha^3|x|_c^2}{2}\}$, where $\mathcal{A}^R\doteq\{\alpha=r[\cos(\theta)+i\sin(\theta)]\in\mathbb{C}\,|\,r\geq 0,\,\theta\in(\frac{-\pi}{2},\frac{\pi}{2})\}$, and for $x\in\mathbb{C}^n$, $|x|_c^2\doteq\sum_{j=1}^n x_j^2$, [14]. (We remark that notation $|\cdot|_c^2$ is not intended to indicate a squared norm; the range is complex.) In the simple one-dimensional case, the resulting function on $\mathbb{C}$ has a branch cut along the negative imaginary axis, and this generalizes to the higher-dimensional case in the natural way.

Although stationarity-based representations for gravitational and Coulomb potentials are inside the integral in (1), they may be moved outside through the introduction of $\alpha$-valued processes, cf. [11, 19]. In particular, not only does one seek the stationary path for action $J$, but the action functional itself can be given as a stationary value of an integral of a polynomial, leading to an iterated-stat problem formulation for such TPBVPs. This may be exploited in the solution of TPBVPs in such systems, cf. [11, 17, 19].

It has also been demonstrated that this stationary-action approach may be applied to TPBVPs for infinite-dimensional conservative systems described by classes of lossless wave equations, see for example [3, 4]. There, stat is used in the construction of fundamental solution groups for these wave equations by appealing to stationarity of action on longer horizons.

Lastly, it has recently been demonstrated that stationarity may be employed to obtain a Feynman-Kac type of representation for solutions of the Schrödinger initial value problem (IVP) for certain classes of initial conditions and potentials [16]. As with the conservative-system cases above, these representations are valid for indefinitely long duration problems, whereas with only the minimization operation, such representations are valid only on time intervals such that the action remains convex, which is always a bounded duration and potentially zero.

In all of these examples, one obtains the stationary value of an action functional where the action functional itself takes the form of a stationary value of a functional that is quadratic in the momentum (the $u.$ in put in (1)) and cubic in the newly introduced potential energy parameterization variable (a time-dependent form of the $\alpha$ parameter above). That is, the overall stationary value is obtained from iterated staticization operations, where the outer stat is over a variable in which the functional is quadratic. Thus, if one can invert the order of the of the stat operations, then the inner stat operation results in a functional that is obtained as a solution of a differential Riccati equation (DRE). (It should be noted that this DRE must typically be propagated past escape times, where this propagation may be efficiently performed through the use of what has been termed "stat duality", cf. [15].) Hence, after inversion of the order of the iterated stat operations, the problem may be reduced to a single stat operation such that the argument takes the form of a linear functional operating on a set of DRE solutions. Consequently, an issue of fundamental importance regards conditions under which one may invert the order of stat operations in an iterated staticization.

In Section 2, the staticization operator will be rigorously defined, and a general problem class along with some corresponding notation will be indicated. Then, in Section 3, a somewhat general condition will be indicated, and it will shown that one may invert the order of staticization operations under that condition. This will be demonstrated by obtaining an equivalence between iterated staticization and full staticization over both variables together. Section 4 will present several classes of problems for which the general condition of Section 3 holds. Finally, in Section 5, the stationary-action application above will be discussed. XXXXX Will we get to this last topic/section? XXXXX

# 2 Problem and Stationarity Definitions

Before the issue to be studied can be properly expressed, it is necessary to define stationarity and the staticization operator.

## 2.1 Stationarity definitions

As noted above, the motivation for this effort is the computation and propagation of stationary points of payoff functionals, which is unusual in comparison to the standard classes of problems in optimization (although one should note for example, [5]). In analogy with the language for minimization and maximization, we will refer to the search for stationary points as "staticization", with these points being statica, in analogy with minima/maxima, and a single such point being a staticum in analogy with minimum/maximum. One might note that here that the term staticization is being derived from a Latin root, staticus (presumably originating from the Greek, statikós), in analogy with the Latin root, maximus, of "maximization". We note that Ekeland [5] employed the term "extremization" for what is essentially the same notion that is being referred to here as staticization. We make the following definitions. Let $\mathcal{F}$ denote either the real or complex field. Suppose $\mathcal{U}$ is a normed vector space (over $\mathcal{F}$) with $\mathcal{A} \subseteq \mathcal{U}$, and suppose $G : \mathcal{A} \to \mathcal{F}$. We say $\bar{u} \in \operatorname{argstat}_{u\in\mathcal{A}} G(u) \doteq \operatorname{argstat}\{G(u)\,|\,u \in \mathcal{A}\}$ if $\bar{u} \in \mathcal{A}$ and either

$$\limsup_{u\to\bar{u},\,u\in\mathcal{A}\setminus\{\bar{u}\}} \frac{|G(u)-G(\bar{u})|}{|u-\bar{u}|} = 0, \tag{2}$$

or there exists $\delta > 0$ such that $\mathcal{A} \cap B_\delta(\bar{u}) = \{\bar{u}\}$ (where $B_\delta(\bar{u})$ denotes the ball of radius $\delta$ around $\bar{u}$). If $\operatorname{argstat}\{G(u)\,|\,u \in \mathcal{A}\} \neq \emptyset$, we define the possibly set-valued $\operatorname{stat}^s$ operation by

$$\operatorname*{stat}^s_{u\in\mathcal{A}} G(u) \doteq \operatorname{stat}^s\{G(u)\,|\,u\in\mathcal{A}\} \doteq \big\{G(\bar{u})\,\big|\,\bar{u} \in \operatorname{argstat}\{G(u)\,|\,u\in\mathcal{A}\}\big\}. \tag{3}$$

If $\operatorname{argstat}\{G(u)\,|\,u \in \mathcal{A}\} = \emptyset$, $\operatorname{stat}^s_{u\in\mathcal{A}} G(u)$ is undefined. Where applicable, we are also interested in a single-valued stat operation (note the absence of superscript $s$). In particular, if there exists $a \in \mathcal{F}$ such that $\operatorname{stat}^s_{u\in\mathcal{A}} G(u) = \{a\}$, then $\operatorname{stat}_{u\in\mathcal{A}} G(u) \doteq a$; otherwise, $\operatorname{stat}_{u\in\mathcal{A}} G(u)$ is undefined. At times, we may abuse notation by writing $\bar{u} = \operatorname{argstat}\{G(u)\,|\,u \in \mathcal{A}\}$ in the event that the argstat is the single point $\{\bar{u}\}$.

In the case where $\mathcal{U}$ is a Hilbert space, and $\mathcal{A} \subseteq \mathcal{U}$ is an open set, $G : \mathcal{A} \to \mathcal{F}$ is Fréchet differentiable at $\bar{u} \in \mathcal{A}$ with Riesz representation $DG(\bar{u}) \in \mathcal{U}$ if

$$\lim_{w\to 0,\, \bar{u}+w\in\mathcal{A}\setminus\{\bar{u}\}} \frac{|G(\bar{u}+w) - G(\bar{u}) - \langle DG(\bar{u}), w\rangle|}{|w|} = 0. \tag{4}$$

The following is immediate from the above definitions.

**Lemma 1** *Suppose $\mathcal{U}$ is a Hilbert space, with open set $\mathcal{A} \subseteq \mathcal{U}$, and that G is Fréchet differentiable at $\bar{u} \in \mathcal{A}$. Then, $\bar{u} \in \operatorname{argstat}\{G(y) \,|\, y \in \mathcal{A}\}$ if and only if $DG(\bar{u}) = 0$.*

### 2.2 Problem definition

Let $\mathcal{U}, \mathcal{V}$ be Hilbert spaces with inner products and norms on $\mathcal{U}$ denoted by $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ and $|\cdot|_{\mathcal{U}}$, and similarly for $\mathcal{V}$. Let the resulting inner product and norm on $\mathcal{U}\times\mathcal{V}$ be denoted by $\langle\cdot,\cdot\rangle_{\mathcal{U}\times\mathcal{V}}$ and $|\cdot|_{\mathcal{U}\times\mathcal{V}}$. Let $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \mathcal{V}$ be open. Throughout, we assume

$$G \in C^2(\mathcal{A}\times\mathcal{B};\mathcal{F}). \tag{A.1}$$

Then, for each $u \in \mathcal{A}$, let $g^{1,u} \in C^2(\mathcal{B};\mathcal{F})$ be given by $g^{1,u}(v) \doteq G(u,v)$ for all $v \in \mathcal{B}$. Similarly, for each $v \in \mathcal{B}$, let $g^{2,v} \in C^2(\mathcal{A};\mathcal{F})$ be given by $g^{2,v}(u) \doteq G(u,v)$ for all $u \in \mathcal{A}$. Further, let

$$\mathcal{A}_G \doteq \Big\{u \in \mathcal{A} \,\Big|\, \operatorname*{stat}_{v\in\mathcal{B}} g^{1,u}(v) \text{ exists}\Big\} \text{ and } \mathcal{B}_G \doteq \Big\{v \in \mathcal{B} \,\Big|\, \operatorname*{stat}_{u\in\mathcal{A}} g^{2,v}(u) \text{ exists}\Big\}. \tag{5}$$

Given $u \in \mathcal{A}_G$, let $\mathcal{M}^1(u) \doteq \operatorname{argstat}_{v\in\mathcal{B}} g^{1,u}(v)$. Similarly, given $v \in \mathcal{B}_G$, let $\mathcal{M}^2(v) \doteq \operatorname{argstat}_{u\in\mathcal{A}} g^{2,v}(u)$. Next, define $\bar{G}^1 : \mathcal{A}_G \to \mathcal{F}$ and $\bar{G}^2 : \mathcal{B}_G \to \mathcal{F}$ by

$$\bar{G}^1(u) \doteq \operatorname*{stat}_{v\in\mathcal{B}} g^{1,u}(v) \;\; \forall\, u \in \mathcal{A}_G \;\text{ and }\; \bar{G}^2(v) \doteq \operatorname*{stat}_{u\in\mathcal{A}} g^{2,v}(u) \;\; \forall\, v \in \mathcal{B}_G.$$

Finally, let

$$\hat{\mathcal{A}}_G \doteq \operatorname*{argstat}_{u\in\mathcal{A}_G} \bar{G}^1(u) \;\text{ and }\; \hat{\mathcal{B}}_G \doteq \operatorname*{argstat}_{v\in\mathcal{B}_G} \bar{G}^2(v).$$

We will discuss conditions under which

$$\operatorname*{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) = \operatorname*{stat}_{v\in\mathcal{B}_G} \bar{G}^2(v). \tag{6}$$

We will generally be concerned only with the left-hand equality in (6); obviously the right-hand equality would be obtained analogously. We refer to the left-hand object in (6) as iterated stat operations, while the center object will be referred to as a full stat operation. Although in some results, the existence of both the iterated and full stat operations are obtained, many of the results will assume the existence of one or both of these objects. We list the two potential assumptions below. In each result to follow, we will indicate when one or both of these is utilized. The full-stat assumption is as follows.

$$\text{Assume } \operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) \text{ exists, and let } (\bar{u},\bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v). \tag{A.2f}$$

Note that under Assumption $(A.2f)$,

$$\bar{u} \in \mathcal{A}_G, \;\; \bar{v} \in \mathcal{B}_G, \;\; \bar{v} \in \mathcal{M}^1(\bar{u}), \;\text{ and }\; \bar{u} \in \mathcal{M}^2(\bar{v}). \tag{7}$$

The iterated-stat assumption is as follows.

$$\text{Assume } \mathrm{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u) \text{ exists, and let } \bar{u}\in\hat{\mathcal{A}}_G. \tag{A.2i}$$

Note that under Assumption $(A.2i)$,

$$\exists\,\bar{v}\in\mathcal{M}^1(\bar{u}), \quad \text{and} \quad \underset{u\in\mathcal{A}_G}{\mathrm{stat}}\,\bar{G}^1(u) = g^{1,\bar{u}}(\bar{v}) = G(\bar{u},\bar{v}). \tag{8}$$

We will first obtain (6) under some general assumptions. After that, we will demonstrate that these assumptions are satisfied under each of other sets of assumptions, where the latter sets describe more commonly noted classes of systems (specifically, quadratic functions and Morse functions). Again, we mainly address only the left-hand equality of (6); the right-hand equality is handled similarly.

# 3 The General Case

Given $\mathcal{C}\subseteq\mathcal{V}$ and $\hat{v}\in\mathcal{V}$, we let $d(\hat{v},\mathcal{C})\doteq\inf_{v\in\mathcal{C}}|v-\hat{v}|$, and use this distance notation more generally throughout. In addition to $(A.1)$, we assume the following throughout this section.

There exist $\delta=\delta(\bar{u},\bar{v})>0$ and $K=K(\bar{u},\bar{v})<\infty$ such that $d(\bar{v},\mathcal{M}^1(u))\le K\,|\bar{u}-u|$ for all $u\in\mathcal{A}_G\cap B_\delta(\bar{u})$. $(A.3)$

We note that $(A.3)$ is trivially satisfied in the case that there exists $\delta>0$ such that $B_\delta(\bar{u})\cap\mathcal{A}_G=\emptyset$. It may be helpful to also note that $(A.3)$ is satisfied under the possibly more heuristically appealing, following assumption.

For every $\tilde{u}\in\mathcal{A}_G$ and every $\tilde{v}\in\mathcal{M}^1(\tilde{u})$, there exist $\delta=\delta(\tilde{u},\tilde{v})>0$ and $K=K(\tilde{u},\tilde{v})<\infty$ such that $d(\tilde{v},\mathcal{M}^1(u))\le K\,|\tilde{u}-u|$ for all $u\in\mathcal{A}_G\cap B_\delta(\tilde{u})$. $(A.3')$

XXXXX Probably Hölder with exponents $>1/2$ would work in the assumptions. Not sure it's worth the trouble given the specific cases to follow. XXXXX

**Lemma 2** *Assume $(A.2f)$. Then, $\bar{u}\in\hat{\mathcal{A}}_G$ and $G(\bar{u},\bar{v})\in\mathrm{stat}^s_{u\in\mathcal{A}_G}\bar{G}^1(u)$.*

*Proof:* Let $(\bar{u},\bar{v})$ be as in $(A.2f)$. Let $R\doteq d\big((\bar{u},\bar{v}),(\mathcal{A}\times\mathcal{B})^c\big)$. By Assumption $(A.3)$, there exist $\delta\in(0,R/2)$ and $K<\infty$ such that for all $u\in\mathcal{A}_G\cap B_\delta(\bar{u})$ and all $\epsilon\in(0,1)$, there exists $v\in\mathcal{M}^1(u)$ such that

$$|v-\bar{v}|\le(K+\epsilon)|u-\bar{u}|\le(K+\epsilon)\delta. \tag{9}$$

Let $\tilde{u}\in\mathcal{A}_G\cap B_{\delta/(K+1)}(\bar{u})$. By (7),

$$|\underset{v\in\mathcal{B}}{\mathrm{stat}}\,g^{1,\tilde{u}}(v)-\underset{v\in\mathcal{B}}{\mathrm{stat}}\,g^{1,\bar{u}}(v)| = |\underset{v\in\mathcal{B}}{\mathrm{stat}}\,g^{1,\tilde{u}}(v)-G(\bar{u},\bar{v})|,$$

and by (9), there exists $\tilde{v}\in B_\delta(\bar{v})$ such that this is

$$= |G(\tilde{u},\tilde{v})-G(\bar{u},\bar{v})|. \tag{10}$$

Let $f\in C^\infty\big((-3/2,3/2);\mathcal{A}\times\mathcal{B}\big)$ be given by $f(\lambda)=\big(\bar{u}+\lambda(\tilde{u}-\bar{u}),\bar{v}+\lambda(\tilde{v}-\bar{v})\big)$ for all $\lambda\in(-3/2,3/2)$. Define $W^0(\lambda)=[G\circ f](\lambda)$ for all $\lambda\in(-3/2,3/2)$, and note that by Assumption $(A.1)$ and standard results, $W^0\in C^2\big((-3/2,3/2);\mathcal{F}\big)$. Similarly, let $W^1(\lambda)=[(G_u,G_v)\circ f](\lambda)=\big(G_u(f(\lambda)),G_v(f(\lambda))\big)$. By Assumption $(A.1)$ and standard results, $W^1\in$

$C^1\big((-3/2,3/2);\mathcal{U}\times\mathcal{V}\big)$. Then, by the Mean Value Theorem (cf. [1, Th. 12.6]), there exists $\lambda_0\in(0,1)$ such that

$$|G(\tilde u,\tilde v)-G(\bar u,\bar v)| = |W^0(1)-W^0(0)| \le \Big|\frac{dG}{d(u,v)}(f(\lambda_0))\Big|\,\Big|\frac{df}{d\lambda}(\lambda_0)\Big|$$
$$= \big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)\big|\,\big|(\tilde u-\bar u,\tilde v-\bar v)\big|,$$

where $(u_0,v_0)\doteq f(\lambda_0)$, and which by (9),

$$\le \big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)\big|\,\sqrt{1+(K+1)^2}|\tilde u-\bar u|. \tag{11}$$

Similarly, there exists $\lambda_1\in(0,\lambda_0)$ such that

$$\big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)-\big(G_u(\bar u,\bar v),G_v(\bar u,\bar v)\big)\big| = |W^1(\lambda_0)-W^1(0)|$$
$$\le \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\,\Big|\frac{df}{d\lambda}(\lambda_1)\Big|\,|\lambda_1| \le \Big|\frac{d^2G}{d(u,v)^2}(u_1,v_1)\Big|\,\big|(u_1-\bar u,v_1-\bar v)\big|,$$

where $(u_1,v_1)\doteq f(\lambda_1)$, and this is

$$\le \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\sqrt{1+(K+1)^2}|\tilde u-\bar u|.$$

Recalling $(\bar u,\bar v)\in\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$, this implies

$$\big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)\big| \le \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\sqrt{1+(K+1)^2}|\tilde u-\bar u|. \tag{12}$$

Combining (11) and (12) yields

$$|G(\tilde u,\tilde v)-G(\bar u,\bar v)| \le \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\big[1+(K+1)^2\big]\,|\tilde u-\bar u|^2.$$

Let $K_1\doteq\big|\frac{d^2G}{d(u,v)^2}(\bar u,\bar v)\big|$. By $(A.1)$, there exists $\hat\delta\in(0,\delta/(K+1))$ such that for all $(u,v)\in B_{\hat\delta}(\bar u,\bar v)$, $\big|\frac{d^2G}{d(u,v)^2}(u,v)\big|\le K_1+1$. Hence, there exists $\bar C<\infty$ such that

$$|G(\tilde u,\tilde v)-G(\bar u,\bar v)| \le \bar C|\tilde u-\bar u|^2 \;\;\forall\,(\tilde u,\tilde v)\in\mathcal{A}_G\cap B_{\hat\delta/(K_1+1)}(\bar u). \tag{13}$$

Combining (10) and (13) one has $|\operatorname{stat}_{v\in\mathcal{B}} g^{1,\tilde u}(v)-\operatorname{stat}_{v\in\mathcal{B}} g^{1,\bar u}(v)|\le\bar C|\tilde u-\bar u|^2$, which upon recalling that $\tilde u\in\mathcal{A}_G\cap B_{\hat\delta/(K+1)}(\bar u)$ was arbitrary, yields the assertions. □

**Theorem 3** *Assume $(A.2f)$ and $(A.2i)$. Then*

$$\operatorname*{stat}_{u\in\mathcal{A}_G}\bar G^1(u) = G(\bar u,\bar v) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v).$$

*Proof:* The assertions follow directly from the assumption, $(A.2f)$ and Lemma 2. □

# 4 Some Specific Cases

We examine several classes of functionals that fit within the general class above.

## 4.1 The Quadratic Case

Here we take $\mathcal{A} = \mathcal{U}$ and $\mathcal{B} = \mathcal{V}$. We let

$$\begin{aligned} G(u,v) &= \tfrac{c}{2} + \langle w,u\rangle_{\mathcal{U}} + \langle y,v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_1 u,u\rangle_{\mathcal{U}} + \langle \bar{B}_2 v,u\rangle_{\mathcal{U}} + \tfrac{1}{2}\langle \bar{B}_3 v,v\rangle_{\mathcal{V}} \\ &= \tfrac{c}{2} + \langle w,u\rangle_{\mathcal{U}} + \langle y,v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_1 u,u\rangle_{\mathcal{U}} + \langle \bar{B}_2' u,v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_3 v,v\rangle_{\mathcal{V}}, \end{aligned} \tag{14}$$

for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$, where $\bar{B}_1 \in \mathcal{L}(\mathcal{U};\mathcal{U})$, $\bar{B}_2 \in \mathcal{L}(\mathcal{V};\mathcal{U})$, $\bar{B}_3 \in \mathcal{L}(\mathcal{V};\mathcal{V})$, $w \in \mathcal{U}$, $y \in \mathcal{V}$ and $c \in \mathcal{F}$, where $\mathcal{L}(\cdot,\cdot)$ generically denotes a space of bounded linear operators, and $\bar{B}_1, \bar{B}_3$ are self-adjoint. We present results under both the cases of $(A.2f)$ and $(A.2i)$.

### 4.1.1 When the full staticization is known to exist

We suppose $(A.2f)$. In [15], it is directly shown that under Assumption $(A.2f)$, in the case of (14), we have the following.

**Theorem 4** *Assume $(A.2f)$. Then, $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ exists, and*

$$\operatorname*{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v). \tag{15}$$

Here however, we show that Assumption $(A.3)$ is satisfied for $G$ given by (14), and hence that assertions (15) of Theorem 4 follow as a special case of Theorem 3. We begin by noting the following, which follows directly from (14) and Lemma 1.

**Lemma 5** *Let $\hat{u} \in \mathcal{A}$. Then $\hat{v} \in \mathcal{M}^1(\hat{u})$ if and only if $\bar{B}_2'\hat{u} + \bar{B}_3\hat{v} + y = 0$.*

We now indicate the key result of this section.

**Proposition 6** *Assumption $(A.3)$ is satisfied.*

*Proof:* We suppose $\mathcal{A}_G \neq \{\bar{u}\}$; otherwise the result is trivial. Let $\hat{u} \in \mathcal{A}_G \setminus \{\bar{u}\}$. By Lemma 5, $\hat{v} \in \mathcal{M}^1(\hat{u})$ if and only if $\bar{B}_2'\hat{u} + \bar{B}_3\hat{v} + y = 0$. However, by (7), $\bar{v} \in \mathcal{M}^1(\bar{u})$, and hence by Lemma 5, $\bar{B}_2'\bar{u} + \bar{B}_3\bar{v} + y = 0$. Combining these two inequalities, we see that $\hat{v} \in \mathcal{M}^1(\hat{u})$ if and only if $\bar{B}_2'(\hat{u}-\bar{u}) + \bar{B}_3(\hat{v}-\bar{v}) = 0$. We take $\hat{v} \doteq \bar{v} - \bar{B}_3^{\#}\bar{B}_2'(\hat{u}-\bar{u})$, where the # superscript indicates the Moore-Penrose pseudo-inverse, cf. [2]. Then, $\hat{v} \in \mathcal{M}^1(\hat{u})$ and $|\hat{v}-\bar{v}| \le |\bar{B}_3^{\#}||\bar{B}_2'||\hat{u}-\bar{u}|$, where the induced norms on the operators are employed, which yields the desired assertion. □

By Proposition 6, we may apply Theorem 3 to obtain the leftmost assertion of (15) in Theorem 4, if $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ exists. The existence of $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ given $(A.2f)$ is obtained in [15], and a proof is not repeated here.

### 4.1.2 When the iterated staticization is known to exist

We suppose $(A.2i)$. We will find that $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ exists, and obtain the equivalence between full and iterated staticization. We begin with a lemma (which is similar to Lemma 10 of [15]).

**Lemma 7** *Given any $\tilde{u} \in \mathcal{A}_G$, $\mathcal{M}^1(\tilde{u})$ is an affine subspace, and further, $G(\tilde{u}, \cdot)$ is constant on $\mathcal{M}^1(\tilde{u})$.*

*Proof:* Let $\tilde{u} \in \mathcal{A}_G$. By Lemma 5, $v \in \mathcal{M}^1(\tilde{u})$ if and only if $\bar{B}_3 v = -(\bar{B}_2' \tilde{u} + y)$, which yields the first assertion. Suppose $\tilde{v}, \hat{v} \in \mathcal{M}^1(\tilde{u})$. Then, using (8),

$$\begin{aligned} G(\tilde{u}, \tilde{v}) - G(\tilde{u}, \hat{v}) &= \langle \bar{B}_2' \tilde{u} + y, \tilde{v} - \hat{v} \rangle_{\mathcal{V}} + \tfrac{1}{2} \langle \bar{B}_3 \tilde{v}, \tilde{v} \rangle_{\mathcal{V}} - \tfrac{1}{2} \langle \bar{B}_3 \hat{v}, \hat{v} \rangle_{\mathcal{V}}, \\ &= \langle -\tfrac{1}{2}(\bar{B}_3 \tilde{v} + \bar{B}_3 \hat{v}), \tilde{v} - \hat{v} \rangle_{\mathcal{V}} + \tfrac{1}{2} \langle \bar{B}_3 \tilde{v}, \tilde{v} \rangle_{\mathcal{V}} - \tfrac{1}{2} \langle \bar{B}_3 \hat{v}, \hat{v} \rangle_{\mathcal{V}} = 0. \end{aligned}$$

□

**Theorem 8** *Assume $(A.2i)$, and let $\bar{v}$ be as given in (8). Then, $\operatorname{stat}_{(u,v) \in \mathcal{A} \times \mathcal{B}} G(u,v)$ exists, and $\operatorname{stat}_{(u,v) \in \mathcal{U} \times \mathcal{V}} G(u,v) = G(\bar{u}, \bar{v}) = \operatorname{stat}_{u \in \mathcal{A}_G} \bar{G}^1(u)$.*

*Proof:* Assume $(A.2i)$, and let $\bar{v}$ be as given in (8). By Lemma 5, $v \in \mathcal{M}^1(\bar{u})$ if and only if $\bar{B}_2' \bar{u} + \bar{B}_3 v + y = 0$. Let

$$\breve{v}(u) \doteq \bar{v} - \bar{B}_3^{\#} \big[ \bar{B}_2' u + y - (\bar{B}_2' \bar{u} + y) \big], \tag{16}$$

and note that

$$\breve{v}(\bar{u}) = \bar{v}. \tag{17}$$

Also note that, as $\bar{v}$ and $\tilde{v} \doteq -\bar{B}_3^{\#} [\bar{B}_2' \bar{u} + y]$ are both in $\mathcal{M}^1(\bar{u})$,

$$0 = \bar{B}_3 [\bar{v} - \tilde{v}] = \bar{B}_3 \big[ \bar{v} + \bar{B}_3^{\#} (\bar{B}_2' \bar{u} + y) \big]. \tag{18}$$

Then, using (16) and (18), we see

$$\begin{aligned} \bar{B}_3 \breve{v}(u) + \bar{B}_2' u + y &= \bar{B}_3 \big[ \bar{v} - \bar{B}_3^{\#} (\bar{B}_2' u - \bar{B}_2' \bar{u}) \big] + \bar{B}_2' u + y \\ &= \bar{B}_3 \big[ -\bar{B}_3^{\#} (\bar{B}_2' u + y) \big] + \bar{B}_2' u + y, \end{aligned}$$

which by definition of the pseudo-inverse

$$= 0.$$

Hence, $\breve{v}(u) \in \mathcal{M}^1(u) \ \forall\, u \in \mathcal{A}_G$, and consequently,

$$\bar{G}^1(u) = G(u, \breve{v}(u)) \ \forall\, u \in \mathcal{A}_G. \tag{19}$$

Then, by the choice of $\bar{u}$,

$$0 = \frac{d\bar{G}^1}{du}(\bar{u}),$$

which by (16), (19), $(A.1)$ and the chain rule,

$$= G_u(\bar{u}, \breve{v}(\bar{u})) + G_v(\bar{u}, \breve{v}(\bar{u})) \frac{d\breve{v}}{du}(\bar{u}),$$

which by (17) and then (8),

$$= G_u(\bar{u}, \bar{v}) + G_v(\bar{u}, \bar{v}) \frac{d\breve{v}}{du}(\bar{u}) = G_u(\bar{u}, \bar{v}).$$

From this and (8), we see that

$$(\bar{u},\bar{v}) \in \operatorname*{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) \quad \text{and} \quad G(\bar{u},\bar{v}) \in \operatorname*{stat}^s_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v). \tag{20}$$

Now suppose there exists $(\hat{u},\hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) \setminus \{(\bar{u},\bar{v})\}$. This implies

$$G_u(\hat{u},\hat{v}) = 0, \quad \text{and} \quad G_v(\hat{u},\hat{v}) = 0, \tag{21}$$

and consequently,

$$\hat{v} \in \mathcal{M}^1(\hat{u}), \quad \text{and} \quad \bar{G}^1(\hat{u}) = G(\hat{u},\hat{v}).$$

Let

$$\breve{v}'(u) \doteq \hat{v} - \bar{B}_3^{\#}\big[\bar{B}_2'u + y - (\bar{B}_2'\hat{u} + y)\big] \;\; \forall\, u \in \mathcal{A}_G, \tag{22}$$

and note that

$$\breve{v}'(\hat{u}) = \hat{v}. \tag{23}$$

Let $\hat{\hat{v}} \doteq -\bar{B}_3^{\#}(\bar{B}_2'\hat{u} + y)$, and note that $\hat{v},\hat{\hat{v}} \in \mathcal{M}^1(\hat{u})$. Consequently, by Lemma 7,

$$0 = \bar{B}_3(\hat{v} - \hat{\hat{v}}) = \bar{B}_3\big[\hat{v} + \bar{B}_3^{\#}(\bar{B}_2'\hat{u} + y)\big]. \tag{24}$$

Then, using (24) and the definition of the pseudo-inverse, we see that

$$\begin{aligned}
&\bar{B}_3\breve{v}'(u) + \bar{B}_2'u + y = \bar{B}_3\big[\hat{v} - \bar{B}_3^{\#}\big(\bar{B}_2'u + y - (\bar{B}_2'\hat{u} + y)\big)\big] + \bar{B}_2'u + y \\
&= \bar{B}_3\big[\hat{v} - \bar{B}_3^{\#}(\bar{B}_2'u + y\big] + \bar{B}_2'u + y = 0,
\end{aligned}$$

which implies that $\breve{v}'(u) \in \mathcal{M}^1(u)$ for all $u \in \mathcal{A}_G$. Hence,

$$\bar{G}^1(u) = G(u,\breve{v}'(u)) \;\; \forall\, u \in \mathcal{A}_G. \tag{25}$$

By (22), (25), $(A.1)$ and the chain rule,

$$\frac{d\bar{G}^1}{du}(\hat{u}) = G_u(\hat{u},\breve{v}'(\hat{u})) + G_v(\hat{u},\breve{v}'(\hat{u}))\frac{d\breve{v}'}{du}(\hat{u}),$$

which by (21) and (23),

$$= G_u(\hat{u},\hat{v}) + G_v(\hat{u},\hat{v})\frac{d\breve{v}'}{du}(\hat{u}) = 0,$$

which implies that $\hat{u} \in \hat{\mathcal{A}}_G$. Using this, (20) and $(A.2i)$, we see that $G(\hat{u},\hat{v}) = G(\bar{u},\bar{v})$. As $(\hat{u},\hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) \setminus \{(\bar{u},\bar{v})\}$ was arbitrary, we have the desired result. □

## 4.2 The Semi-Quadratic Case

Here we take $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} = \mathcal{V}$. We let

$$G(u,v) = f_1(u) + \langle f_2(u), v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_3(u)v, v\rangle_{\mathcal{V}}, \tag{26}$$

for all $u \in \mathcal{A}$ and $v \in \mathcal{V}$, where $f_1 \in C^2(\mathcal{A};\mathcal{F})$, $f_2 \in C^2(\mathcal{A};\mathcal{V})$ and $\bar{B}_3 \in C^2(\mathcal{A};\mathcal{L}(\mathcal{V},\mathcal{V}))$, and $f_2$, $\bar{B}_3$ are Fréchet differentiable with continuous Fréchet derivatives. In addition, suppose The Moore-Penrose pseudo-inverse $\bar{B}_3^{\#}(u)$ of $\bar{B}_3(u)$ is differentiable for all $u \in \mathcal{A}_G$, and there exists a constant $D > 0$ such that $|\bar{B}_3^{\#}(u)| \le D$ for all $u \in \mathcal{A}_G$. Similar to Lemma 5, the next lemma follows directly from (26) and Lemma 1.

**Lemma 9** *Let $\hat{u} \in \mathcal{A}$. Then $\hat{v} \in \mathcal{M}^1(\hat{u})$ if and only if $f_2(\hat{u}) + \bar{B}_3(\hat{u})\hat{v} = 0$.*

#### 4.2.1 When the full staticization is known to exist

**Lemma 10** *Assume* $(A.2f)$*. Then assumption* $(A.3)$ *is satisfied.*

*Proof:* The result is trivial for $\mathcal{A} = \{\bar{u}\}$. Suppose $\mathcal{A}_G \neq \{\bar{u}\}$. Let $\hat{u} \in \mathcal{A}_G \cap B_\delta(\bar{u})$, where $\delta > 0$, and $\hat{u} \neq \bar{u}$. Let $\hat{v} = \bar{v} - \bar{B}_3^{\#}(\hat{u}) f_2(\hat{u}) - \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}$. Note that

$$\begin{aligned}\bar{B}_3(\hat{u})\hat{v} + f_2(\hat{u}) &= \bar{B}_3(\hat{u})[\bar{v} - \bar{B}_3^{\#}(\hat{u}) f_2(\hat{u}) - \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}] + f_2(\hat{u}) \\ &= \bar{B}_3(\hat{u})\bar{v} - f_2(\hat{u}) - \bar{B}_3(\hat{u})\bar{v} + f_2(\hat{u}) = 0\end{aligned}$$

Therefore, $\hat{v} \in \mathcal{M}^1(\hat{u})$ by Lemma 9. Since $f_2$, $\bar{B}_3$ are Fréchet differentiable with continuous Fréchet derivatives, there exist constants $K_2 > 0$, $K_B > 0$, such that $|f_2(u_1) - f_2(u_2)| \leq K_2|u_1 - u_2|$, $|\bar{B}_3(u_1) - \bar{B}_3(u_2)| \leq K_B|u_1 - u_2|$ for all $u_1, u_2 \in B_\delta(\bar{u})$. We have

$$|\hat{v} - \bar{v}| = \big| - \{\bar{B}_3^{\#}(\hat{u}) f_2(\hat{u}) + \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}\}\big|$$

By Lemma 9, $\bar{B}_3(\bar{u})\bar{v} + f_2(\bar{u}) = 0$,

$$\begin{aligned}&= \big| - \bar{B}_3^{\#}(\hat{u})\{f_2(\hat{u}) - f_2(\bar{u}) - \bar{B}_3(\bar{u})\bar{v} + \bar{B}_3(\hat{u})\bar{v}\}\big| \\ &\leq |\bar{B}_3^{\#}(\bar{u})||f_2(\hat{u}) - f_2(\bar{u}) + (\bar{B}_3(\hat{u}) - \bar{B}(\bar{u}))\bar{v}|| \\ &\leq D\bigg[K_2|\hat{u} - \bar{u}| + K_B|\bar{v}||\hat{u} - \bar{u}|\bigg] \\ &\leq \bar{C}|\hat{u} - \bar{u}| \text{ for some } \bar{C} = \bar{C}(\bar{u}, \bar{v})\end{aligned}$$

which gives $(A.3)$. XXXX Comment: I think any $\delta > 0$ such that $\mathcal{A}_G \cap B_\delta(\bar{u}) \neq \emptyset$ works XXXX □

**Theorem 11** *Assume* $(A.2f)$*. Then* $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ *exists, and*

$$\operatorname*{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = G(\bar{u}, \bar{v}) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v).$$

*Proof:* $(A.3)$ is satisfied. This is a special case of Theorem 3.

#### 4.2.2 When the full staticization is known to exist

**Theorem 12** *Assume* $(A.2i)$*. Then* $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$ *exists, and*

$$\operatorname*{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = G(\bar{u}, \bar{v}) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v).$$

*Proof:* Assume $(A.2i)$, and let $\bar{v}$ be as given in (8). By Lemma 9, $v \in \mathcal{M}^1(u)$ if and only if $f_2(u) + \bar{B}_3(u)v = 0$. For $u \in \mathcal{A}_G$, let

$$\check{v}(u) \doteq \bar{v} - \bar{B}_3^{\#}(u) f_2(u) - \bar{B}_3^{\#}(u)\bar{B}_3(u)\bar{v}$$

Note that

$$\begin{aligned}\check{v}(\bar{u}) &= \bar{v} - \bar{B}_3^{\#}(\bar{u}) f_2(\bar{u}) - \bar{B}_3^{\#}(\bar{u})\bar{B}_3(\bar{u})\bar{v} \\ &= \bar{v} - \bar{B}_3^{\#}(\bar{u})[f_2(\bar{u}) + \bar{B}_3(\bar{u})\bar{v}]\end{aligned}$$

$$= \bar{v} - \bar{B}_3^{\#}(\bar{u})[0] = \bar{v}$$

We have

$$\begin{aligned}\bar{B}_3(u)\breve{v}(u) + f_2(u) &= \bar{B}_3(u)[\bar{v} - \bar{B}_3^{\#}(u)f_2(u) - \bar{B}_3^{\#}(u)\bar{B}_3(u)\bar{v}] + f_2(u)\\ &= \bar{B}_3(u)\bar{v} - \bar{B}_3(u)\bar{B}_3^{\#}(u)f_2(u) - \bar{B}_3(u)\bar{B}_3^{\#}(u)\bar{B}_3(u)\bar{v} + f_2(u)\\ &= \bar{B}_3(u)\bar{v} - f_2(u) - \bar{B}_3(u)\bar{v} + f_2(u) = 0\end{aligned}$$

Hence, $\breve{v}(u) \in \mathcal{M}^1(u)\ \forall u \in \mathcal{A}_G$, and

$$\bar{G}^1(u) = G(u, \breve{v}(u)) \quad \forall u \in \mathcal{A}_G$$

Then, by the choice of $\bar{u}$ and the assumption that $\bar{B}_3^{\#}(u)$ is differentiable for all $u \in \mathcal{A}_G$

$$\begin{aligned}0 &= \frac{d\bar{G}^1}{du}(\bar{u}),\\ &= G_u(\bar{u}, \breve{v}(\bar{u})) + G_v(\bar{u}, \breve{v}(\bar{u}))\frac{d\breve{v}}{du}(\bar{u}),\\ &= G_u(\bar{u}, \bar{v}) + G_v(\bar{u}, \bar{v})\frac{d\breve{v}}{du}(\bar{u}) = G_u(\bar{u}, \bar{v}).\end{aligned}$$

From this and (8), we see that

$$(\bar{u}, \bar{v}) \in \operatorname*{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \quad \text{and} \quad G(\bar{u}, \bar{v}) \in \operatorname*{stat^s}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v). \tag{27}$$

Now suppose there exists $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \setminus \{(\bar{u}, \bar{v})\}$. This implies

$$\begin{gathered}G_u(\hat{u}, \hat{v}) = 0, \qquad G_v(\hat{u}, \hat{v}) = 0\\ \hat{v} \in \mathcal{M}^1(\hat{u}), \quad \text{and} \quad \bar{G}^1(\hat{u}) = G(\hat{u}, \hat{v}).\end{gathered}$$

Let

$$\breve{v}\,'(u) \doteq \hat{v} - \bar{B}_3^{\#}(u)f_2(u) - \bar{B}_3^{\#}(u)\bar{B}_3(u)\hat{v} \ \ \forall\, u \in \mathcal{A}_G, \tag{28}$$

and note that

$$\breve{v}\,'(\hat{u}) = \hat{v}. \tag{29}$$

Using the definition of the pseudo-inverse, we see that

$$\begin{aligned}&\bar{B}_3(u)\breve{v}\,'(u) + f_2(u) = \bar{B}_3(u)\big[\hat{v} - \bar{B}_3^{\#}(u)f_2(u) - \bar{B}_3^{\#}(u)\bar{B}_3(u)\hat{v}] + f_2(u)\\ &\quad = \bar{B}_3(u)\hat{v} - f_2(u) - \bar{B}_3(u)\hat{v} + f_2(u) = 0,\end{aligned}$$

which implies that $\breve{v}\,'(u) \in \mathcal{M}^1(u)$ for all $u \in \mathcal{A}_G$. Hence,

$$\bar{G}^1(u) = G(u, \breve{v}\,'(u)) \ \ \forall\, u \in \mathcal{A}_G. \tag{30}$$

By (28), (30), and the assumption that $\bar{B}_3^{\#}(u)$ is differentiable for all $u \in \mathcal{A}_G$,

$$\begin{aligned}&\frac{d\bar{G}^1}{du}(\hat{u}) = G_u(\hat{u}, \breve{v}\,'(\hat{u})) + G_v(\hat{u}, \breve{v}\,'(\hat{u}))\frac{d\breve{v}\,'}{du}(\hat{u}),\\ &= G_u(\hat{u}, \hat{v}) + G_v(\hat{u}, \hat{v})\frac{d\breve{v}\,'}{du}(\hat{u}) = 0,\end{aligned}$$

which implies that $\hat{u} \in \hat{\mathcal{A}}_G$. Using this and $(A.2i)$, we see that $G(\hat{u}, \hat{v}) = G(\bar{u}, \bar{v})$. As $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \setminus \{(\bar{u}, \bar{v})\}$ was arbitrary, we have the desired result. □

### 4.3 The Uniformly Morse Case

Under condition $(A.1)$ and a condition that $G$ be uniformly Morse (see $(A.4)$ below), we will find that Assumption $(A.3')$ holds (and consequently, $(A.3)$ ). Hence, one may apply Theorem 3. More specifically, we assume the following.

$$\text{For all } (\hat{u},\hat{v}) \in \mathcal{A}\times\mathcal{B},\ G_{vv}(\hat{u},\hat{v}) \in \mathcal{L}(\mathcal{V};\mathcal{V}) \text{ is invertible.} \tag{A.4}$$

We present results under both the cases of $(A.2f)$ and $(A.2i)$.

#### 4.3.1 When the full staticization is known to exist

We suppose $(A.2f)$. We will find that $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ exists, and obtain the equivalence between full and iterated staticization.

**Theorem 13** *Assume* $(A.2f)$. *Then,* $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ *exists, and* $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} G(u,v)$.

The proof of this theorem is delayed until after the next lemmas. We note that the symmetric result in the case of $\operatorname{stat}_{v\in\mathcal{B}_G} \bar{G}^2(v)$ follows similarly under the analogous assumption.

**Lemma 14** *Assume* $(A.2f)$. *Then, there exists* $\epsilon > 0$ *and* $\breve{v} \in C^1(\mathcal{A}_G \cap B_\epsilon(\bar{u});\mathcal{B})$ *such that* $\breve{v}(\bar{u}) = \bar{v}$ *and* $G_v(u,\breve{v}(u)) = 0$ *for all* $u \in \mathcal{A}_G \cap B_\epsilon(\bar{u})$.

*Proof:* This is simply the implicit mapping theorem, cf. [13]. □

By Lemma 14 and the definition of $\mathcal{A}_G$,

$$\bar{G}^1(u) = \operatorname*{stat}_{v\in\mathcal{B}} g^{1,u}(v) = G(u,\breve{v}(u))\ \ \forall\, u \in \mathcal{A}_G \cap B_\epsilon(\bar{u}). \tag{31}$$

Then, by (31), the chain rule, $(A.1)$ and Lemma 14,

$$\bar{G}^1(\cdot) \in C^1(\mathcal{A}_G \cap B_\epsilon(\bar{u});\mathcal{F}). \tag{32}$$

**Lemma 15** *Assume* $(A.2f)$. *Then, there exists* $K < \infty$ *and* $\delta \in (0,\epsilon)$ *such that* $|\breve{v}(u) - \breve{v}(\bar{u})| = |\breve{v}(u) - \bar{v}| \le K|u-\bar{u}|$ *for all* $u \in \mathcal{A}_G \cap B_\delta(\bar{u})$.

*Proof:* By Lemma 14, $\frac{d\breve{v}}{du}(\cdot)$ is continuous on $\mathcal{A}_G \cap B_\epsilon(\bar{u})$. Let $K_0 \doteq \left|\frac{d\breve{v}}{du}(\bar{u})\right|$. By the continuity, there exists $\delta > 0$ such that $\left|\frac{d\breve{v}}{du}(u)\right| < K \doteq K_0 + 1$ for all $u \in \mathcal{A}_G \cap B_\delta(\bar{u})$. Hence, by the mean value theorem, we have the asserted bound. □

Note that Lemma 15 implies that Assumption $(A.3)$ is satisfied, and hence one may apply Theorem 3, which implies that the equivalence of stat and iterated stat holds under the assumption of existence of the latter.

**Lemma 16** *Assume* $(A.2f)$. *Then,* $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ *exists.*

*Proof:* Note first that by (31), (32) and the chain rule,

$$\frac{d}{du}\bar{G}^1(u)\big|_{u=\bar{u}} = \frac{d}{du}G(u,\breve{v}(u))\big|_{u=\bar{u}} = G_u(\bar{u},\breve{v}(\bar{u})) + G_v(\bar{u},\breve{v}(\bar{u}))\frac{d\breve{v}}{du}(\bar{u}),$$

which by $(A.2f)$ and Lemma 14,

$$= 0.$$

Consequently,

$$\bar{u} \in \operatorname*{argstat}_{u\in\mathcal{A}_G} \bar{G}^1(u) \quad \text{and} \quad \bar{G}^1(\bar{u}) \in \operatorname*{stat^s}_{u\in\mathcal{A}_G} \bar{G}^1(u). \tag{33}$$

Suppose $\hat{u} \neq \bar{u}$ is such that

$$\hat{u} \in \operatorname*{argstat}_{u\in\mathcal{A}_G} \bar{G}^1(u). \tag{34}$$

Then, by $(A.2f)$, there exists $\hat{v} \in \mathcal{M}^1(\hat{u})$. Noting $(A.4)$ and applying the implicit mapping theorem again, we find that there exists $\epsilon' > 0$ and $\breve{v}' \in C^1(\mathcal{A}_G \cap B_{\epsilon'}(\hat{u});\mathcal{B})$ such that

$$\breve{v}'(\hat{u}) = \hat{v} \quad \text{and} \quad G_v(u,\breve{v}'(u)) = 0 \;\; \forall\, u \in \mathcal{A}_G \cap B_{\epsilon'}(\hat{u}). \tag{35}$$

Then, by (34), another application of the chain rule and (35),

$$0 = \frac{d}{du}\bar{G}^1(u)\big|_{u=\hat{u}} = G_u(\hat{u},\breve{v}'(\hat{u})) + G_v(\hat{u},\breve{v}'(\hat{u}))\frac{d\breve{v}'}{du}(\hat{u}) = G_u(\hat{u},\hat{v}). \tag{36}$$

By (35) and (36), $(\hat{u},\hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$, and hence by $(A.2f)$,

$$G(\hat{u},\hat{v}) = G(\bar{u},\bar{v}). \tag{37}$$

By the fact that $\hat{v} \in \operatorname{argstat}_{v\in\mathcal{B}} g^{1,\hat{u}}(v)$ and (37), we have

$$\bar{G}^1(\hat{u}) = g^{1,\hat{u}}(\hat{v}) = G(\hat{u},\hat{v}) = G(\bar{u},\bar{v}).$$

As $\hat{u} \in \operatorname{argstat}_{u\in\mathcal{A}_G} \bar{G}^1(u) \setminus \{\bar{u}\}$ was arbitrary, we have the desired result. □

*Proof: (Proof of Theorem 13)* The assertion of the existence of $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ is simply Lemma 16. Then, noting that Lemma 15 implies that Assumption $(A.3)$ is satisfied, one may apply Theorem 3 to obtain the second assertion of the theorem. □

### 4.3.2 When the iterated staticization is known to exist

We suppose $(A.2i)$. We will find that $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ exists, and obtain the equivalence between full and iterated staticization.

**Lemma 17** *Assume* $(A.2i)$*. Then,* $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ *exists.*

*Proof:* By $(A.2i)$, (8) and the implicit mapping theorem, there exists $\delta > 0$ and $\breve{v} \in C^1(B_\delta(\bar{u})\cap \mathcal{A}_G;\mathcal{B})$ such that

$$\breve{v}(\bar{u}) = \bar{v} \quad \text{and} \quad G_v(u,\breve{v}(u)) = 0 \;\; \forall\, u \in B_\delta(\bar{u}) \cap \mathcal{A}_G. \tag{38}$$

By the differentiability of $\breve{v}$, $(A.1)$ and the chain rule,

$$\frac{d\bar{G}^1}{du}(\bar{u}) = G_u(\bar{u}, \breve{v}(\bar{u})) + G_v(\bar{u}, \breve{v}(\bar{u}))\frac{d\breve{v}}{du}(\bar{u}) = G_u(\bar{u}, \bar{v}) + G_v(\bar{u}, \bar{v})\frac{d\breve{v}}{du}(\bar{u}).$$

Using $(A.2i)$ and (8), this implies $0 = G_u(\bar{u}, \bar{v})$, and hence $(\bar{u}, \bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$.

Now suppose there exists $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \setminus \{(\bar{u}, \bar{v})\}$, which implies

$$G_u(\hat{u}, \hat{v}) = 0 \quad \text{and} \quad G_v(\hat{u}, \hat{v}) = 0. \tag{39}$$

By (39), $(A.1)$ and the implicit mapping theorem, there exists $\delta' > 0$ and $\breve{v}' \in C^1(B_\delta(\hat{u}) \cap \mathcal{A}_G; \mathcal{B})$ such that

$$\breve{v}'(\hat{u}) = \hat{v} \quad \text{and} \quad G_v(u, \breve{v}'(u)) = 0 \;\; \forall\, u \in B'_\delta(\hat{u}) \cap \mathcal{A}_G. \tag{40}$$

By the definition of $\mathcal{A}_G$ and (40),

$$\bar{G}^1(u) = \operatorname*{stat}_{v\in\mathcal{B}} g^{1,u}(v) = G(u, \breve{v}'(u)) \;\; \forall\, u \in \mathcal{A}_G \cap B_{\delta'}(\hat{u}). \tag{41}$$

By (40), (41), $(A.1)$ and the chain rule,

$$\frac{d\bar{G}^1(\hat{u})}{du}(\hat{u}) = G_u(\hat{u}, \breve{v}'(\hat{u})) + G_u(\hat{u}, \breve{v}'(\hat{u}))\frac{d\breve{v}'}{du}(\hat{u}),$$

which by (39),

$$= 0.$$

That is, $\hat{u} \in \operatorname{argstat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$, and using $(A.2i)$, this implies $\bar{G}^1(\hat{u}) = \bar{G}^1(\bar{u})$. Combining this with (40) and (41), one sees that

$$G(\hat{u}, \hat{v}) = \bar{G}^1(\hat{u}) = \bar{G}^1(\bar{u}).$$

and then by the definition of $\bar{G}^1$ and (8), this is

$$= g^{1,\bar{u}}(\bar{v}) = G(\bar{u}, \bar{v}).$$

As $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}$ was arbitrary, we see that $G(\hat{u}, \hat{v}) = G(\bar{u}, \bar{v})$ for all $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$. □

By Lemma 17 and Theorem 13 we have the following.

**Theorem 18** *Assume* $(A.2i)$*. Then,* $\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$ *exists, and* $\operatorname{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} G(u, v) = G(\bar{u}, \bar{v}) = \operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$.

# 5 A Stationary-Action Example

XXXXX The hope is that we will also be able to apply the strict concavity over $\alpha$ and the quadratic form over $u$ to obtain equality of the full- and iterated- staticization operations. Here, in the real-field gravitational case, the iterated staticization is roughly given by

$$\operatorname*{stat}_{u\in\mathcal{U}} \operatorname*{stat}_{\alpha\in\mathcal{A}} \int_0^t \tfrac{1}{2}|u_r|^2 + C\big[\alpha_r - \tfrac{\alpha_r^3|\xi_r|^2}{2}\big]\, dr$$

where $\xi_r \doteq x + \int_0^r u_\rho\, d\rho$ (with more definition detail in [19]). Might also want to include the complex-field case with diffusion dynamics for the Schrödinger equation. XXXXX

## 5.1 Deterministic Case

Let $0 \le s \le r \le t$. Let $\mathcal{U}$, $\mathcal{A}$ be subspaces of $L^2([s,t];\mathbb{R})$. Let $x \in \mathbb{R}$ be fixed and let $\xi_r \doteq x + \int_s^r u_\rho d\rho$. Let $\alpha \in \mathcal{A}$. The functional

$$J(u,\alpha) \doteq \int_s^t \frac{m}{2} u_r^2 dr + \int_s^t \left( \alpha_r - \frac{\alpha_r^3}{2} \xi_r^2 \right) dr$$

is twice Fréchet differentiable in $u$ and in $\alpha$, and the Riesz representations are (cf. Appendix)

$$\begin{aligned}
[\nabla_u J(u,\alpha)]_r &= m u_r - \int_r^t \alpha_\rho^3 \xi_\rho d\rho \\
[\nabla_\alpha J(u,\alpha)]_r &= 1 - \frac{3\alpha_r^2 \xi_r^2}{2} \\
[\nabla_u^2 J(u,\alpha)\delta]_r &= m\delta_r - \int_s^t \left( \int_{\rho \vee r}^t \alpha_\tau d\tau \right) \delta_\rho d\rho; \ \ \forall\, \delta \in \mathcal{U} \\
[\nabla_\alpha^2 J(u,\alpha)\gamma]_r &= -3\alpha_r \gamma \xi_r^2; \ \ \forall\, \gamma \in \mathcal{A}
\end{aligned}$$

**Theorem 19** $\operatorname{stat}_{u\in\mathcal{U}} \operatorname{stat}_{\alpha\in\mathcal{A}} J(u,\alpha) = \operatorname{stat}_{\alpha\in\mathcal{A}} \operatorname{stat}_{u\in\mathcal{U}} J(u,\alpha) = \operatorname{stat}_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} J(u,\alpha)$

*Proof:* $\operatorname{stat}_{u\in\mathcal{U}} \operatorname{stat}_{\alpha\in\mathcal{A}} J(u,\alpha)$ exists as this is just the stationary trajectory of a point mass $\xi$ with velocity $u$ in a gravitational field of the form $V(x) = \frac{1}{x}$. Therefore, $(A.2i)$ is satisfied. Since $J(u,\alpha)$ is semi-quadratic in $u$ and uniformly Morse in $\alpha$,the equalities follow. □

## 5.2 Stochastic Case

Let $(\Omega, P, \mathcal{F}_t)$ be a filtered probability spaces. Let $0 \le s \le r \le t$. Let $u \in \mathcal{U}$ be a stochastic process adapted to the filtration $\mathcal{F}_r$, where $\mathcal{U}$ is a subspace of $L^2(\Omega; L^2([s,t],\mathbb{R}))$, that is, $u(\omega) \in L^2([s,t],\mathbb{R})$ for each $\omega \in \Omega$. Let the diffusion process $\xi$ be given by

$$\begin{aligned}
d\xi_r &= u_r dr + \sigma dB_r \\
\xi_s &= x
\end{aligned}$$

where $x \in \mathbb{R}$ is fixed, $\sigma \in \mathbb{R}$ is a constant XXXX could make $\sigma$ an Itô integrable function XXXX, $B$ is a Brownian motion taking values in $\mathbb{R}$. Let $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is a subspace of $L^2(\Omega; L^2([s,t],\mathbb{R}))$, that is, $\alpha(\omega) \in L^2([s,t],\mathbb{R})$ for each $\omega \in \Omega$. The functional

$$J(u,\alpha) \doteq E^{s,x}\left[\int_s^t \frac{m}{2} u_r^2 dr + \int_s^t \left( \alpha_r - \frac{\alpha_r^3}{2} \xi_r^2 \right) dr\right]$$

is twice Fréchet differentiable in $u$ and in $\alpha$, and the Riesz representation are (cf. Appendix)

$$\begin{aligned}
[\nabla_u J(u,\alpha)]_r &= m u_r - \int_r^t \alpha_\rho^3 \xi_\rho d\rho \\
[\nabla_\alpha J(u,\alpha)]_r &= 1 - \frac{3\alpha_r^2 \xi_r^2}{2} \\
[\nabla_u^2 J(u,\alpha)\delta]_r &= m\delta_r - \int_s^t \left( \int_{\rho \vee r}^t \alpha_\tau d\tau \right) \delta_\rho d\rho; \ \ \forall\, \delta \in \mathcal{U} \\
[\nabla_\alpha^2 J(u,\alpha)\gamma]_r &= -3\alpha_r \gamma \xi_r^2; \ \ \forall\, \gamma \in \mathcal{A}
\end{aligned}$$

**Theorem 20** $\operatorname{stat}_{u\in\mathcal{U}} \operatorname{stat}_{\alpha\in\mathcal{A}} J(u,\alpha) = \operatorname{stat}_{\alpha\in\mathcal{A}} \operatorname{stat}_{u\in\mathcal{U}} J(u,\alpha) = \operatorname{stat}_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} J(u,\alpha)$

*Proof:* $\operatorname{stat}_{u\in\mathcal{U}} \operatorname{stat}_{\alpha\in\mathcal{A}} J(u,\alpha)$ exists as this is just the stationary trajectory of a point mass $\xi$ with velocity $u$ in a gravitational field of the form $V(x) = \frac{1}{x}$. Therefore, $(A.2i)$ is satisfied. Since $J(u,\alpha)$ is semi-quadratic in $u$ and uniformly Morse in $\alpha$,the equalities follow. $\square$

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# 6 Appendix: Calculation of Derivatives in Section 5.1, 5.2

## 6.1 Notations of Derivatives

Let $\mathcal{U}, Y$ be Hilbert spaces. We say a function $f \in C(\mathcal{U}; Y)$ is Fréchet differentiable at $u \in \mathcal{U}$ if there exists $Df : \mathcal{U} \to \mathcal{L}(\mathcal{U}; Y)$ such that

$$0 = \lim_{|\delta_u|_{\mathcal{U}} \to 0} \frac{|f(u + \delta_u) - f(u) - Df(u)\delta_u|_Y}{|\delta_u|_{\mathcal{U}}}$$

Now let $Y = \mathbb{R}$ and $\mathcal{A}$ be another Hilbert space. Let the functional $f : \mathcal{U} \times \mathcal{A} \to \mathbb{R}$ satisfy $f(u, \cdot) \in C^2(\mathcal{A}; \mathbb{R})$, $f(\cdot, \alpha) \in C^2(\mathcal{U}; \mathbb{R})$ for all $u \in \mathcal{U}$, $\alpha \in \mathcal{A}$. Let $D_u f : \mathcal{U} \times \mathcal{A} \to \mathcal{L}(\mathcal{U}; \mathbb{R})$, $D_\alpha f : \mathcal{U} \times \mathcal{A} \to \mathcal{L}(\mathcal{A}; \mathbb{R})$ be the Fréchet derivatives with respect to $u, \alpha$. Note that we have

$$[D_u f(u, \alpha)]\delta_u \in \mathbb{R}$$
$$[D_\alpha f(u, \alpha)]\delta_\alpha \in \mathbb{R}$$

By Riesz representation theorem, for each $\hat{u} \in \mathcal{U}$, $\hat{\alpha} \in \mathcal{A}$ there exists $\nabla_u f(\hat{u}, \hat{\alpha}) \in \mathcal{U}$ such that

$$D_u f(\hat{u}, \hat{\alpha})\delta_u =< \delta_u, \nabla_u f(\hat{u}, \hat{\alpha}) >_{\mathcal{U}}; \quad \forall \delta_u \in \mathcal{U}$$

and such $\nabla_u f(\hat{u}, \hat{\alpha})$ is unique for each pair $(\hat{u}, \hat{\alpha})$. Therefore, we can define the function $\nabla_u f : \mathcal{U} \times \mathcal{A} \to \mathcal{U}$. Similarly, we define $\nabla_\alpha f : \mathcal{U} \times \mathcal{A} \to \mathcal{A}$, where $D_\alpha f(\hat{u}, \hat{\alpha})\delta_\alpha =< \delta_\alpha, \nabla_\alpha f(\hat{u}, \hat{\alpha}) >$, for all $\delta_\alpha \in \mathcal{A}$, $(\hat{u}, \hat{\alpha}) \in \mathcal{U} \times \mathcal{A}$.

Let $D_u^2 f : \mathcal{U}\times\mathcal{A} \to \mathcal{L}(\mathcal{U}, \mathbb{R})$, $D_\alpha^2 f : \mathcal{U}\times\mathcal{A} \to \mathcal{L}(\mathcal{A}, \mathbb{R})$ denote the second Fréchet derivatives with respect to $u, \alpha$. That is,

$$0 = \lim_{|\delta_u|_{\mathcal{U}}\to 0} \frac{|D_u f(\hat{u}+\delta_u, \hat{\alpha}) - D_u f(\hat{u}, \hat{\alpha}) - D_u^2 f(\hat{u}, \hat{\alpha})\delta_u|_{\mathcal{L}(\mathcal{U};\mathbb{R})}}{|\delta_u|_{\mathcal{U}}}; \quad \forall \delta_u \in \mathcal{U}$$
$$0 = \lim_{|\delta_\alpha|_{\mathcal{A}}\to 0} \frac{|D_u f(\hat{u}, \hat{\alpha}+\delta_\alpha) - D_u f(\hat{u}, \hat{\alpha}) - D_u^2 f(\hat{u}, \hat{\alpha})\delta_\alpha|_{\mathcal{L}(\mathcal{A};\mathbb{R})}}{|\delta_\alpha|_{\mathcal{A}}}; \quad \forall \delta_\alpha \in \mathcal{A}$$

Note that for each $\delta_u \in \mathcal{U}$ and the pair $(\hat{u}, \hat{\alpha})$, we have $D_u^2 f(\hat{u}, \hat{\alpha})\delta_u \in \mathcal{L}(\mathcal{U};\mathbb{R})$ and $D_u^2 f(\hat{u}, \hat{\alpha})\delta_u : \mathcal{U} \to \mathbb{R}$. Again by Riesz representation theorem, there exists an unique $\eta_{(\hat{u},\hat{\alpha})}(\delta_u) \in \mathcal{U}$ such that

$$[D_u^2 f(\hat{u}, \hat{\alpha})\delta_u]\tilde{\delta}_u =< \tilde{\delta}_u, \eta_{(\hat{u},\hat{\alpha})}(\delta_u) >_{\mathcal{U}}; \ \forall\, \tilde{\delta}_u \in \mathcal{U}$$

We can define the mapping $\eta_{(\hat{u},\hat{\alpha})} : \mathcal{U} \to \mathcal{U}$ for all $(\hat{u}, \hat{\alpha}) \in \mathcal{U} \times \mathcal{A}$. The following results are standard. (cf. [1])

**Theorem 21** *For each $\delta_u \in \mathcal{U}$, the pair $(\hat{u}, \hat{\alpha})$, and any $\tilde{\delta}_u \in \mathcal{U}$, we have*

$$[D_u^2 f(\hat{u}, \hat{\alpha})\delta_u]\tilde{\delta}_u =< \tilde{\delta}_u, \eta_{(\hat{u},\hat{\alpha})}(\delta_u) >_{\mathcal{U}}=< \delta_u, \eta_{(\hat{u},\hat{\alpha})}(\tilde{\delta}_u) >_{\mathcal{U}}= [D_u^2 f(\hat{u}, \hat{\alpha})\tilde{\delta}_u]\delta_u$$

**Theorem 22** *There is a unique mapping $\nabla_u^2 f : \mathcal{U}\times\mathcal{A} \to \mathcal{L}(\mathcal{U})$, such that $\nabla_u^2 f(u, \alpha) \doteq \eta_{(u,\alpha)} \in \mathcal{L}(\mathcal{U})$, and*

$$D_u^2 f(u, \alpha)\delta_u =< \cdot, \nabla_u^2 f(u, \alpha)\delta_u >_{\mathcal{U}}; \ \forall\, \delta_u \in \mathcal{U}$$

*Proof:* This comes from Theorem 21 and the Riesz representation theorem. □

**Theorem 23** *Let $\nabla_u f : \mathcal{U} \times \mathcal{A} \to \mathcal{U}$ be as defined earlier and $\nabla_u^2 f$ be as in Theorem 22. Then*

$$\nabla_u^2 f = D\nabla_u f$$

## 6.2 Deterministic Case

Let $0 \le s \le r \le t$. Let $\mathcal{U}$, $\mathcal{A}$ be subspaces of $L^2([s, t]; \mathbb{R})$. Let $x \in \mathbb{R}$ be fixed and let $\xi_r \doteq x + \int_s^r u_\rho d\rho$.

**Theorem 24** *The functional*

$$J(u, \alpha) \doteq \int_s^t \frac{m}{2} u_r^2 dr + \int_s^t \left(\alpha_r - \frac{\alpha_r^3}{2}\xi_r^2\right) dr$$

*is Fréchet differentiable in $u, \alpha$, and the Fréchet derivatives have Riesz representation (cf. Section 6.1)*

$$[\nabla_u J(u, \alpha)]_r = m u_r - \int_r^t \alpha_\rho^3 \xi_\rho d\rho$$
$$[\nabla_\alpha J(u, \alpha)]_r = 1 - \frac{3\alpha_r^2 \xi_r^2}{2}$$

*Proof:* Let $\delta \in \mathcal{U}$, $F \doteq mu_r - \int_r^t \alpha_\rho^3 \xi_\rho d\rho$. We have

$$
\begin{aligned}
&|J(u+\delta,\alpha) - J(u,\alpha) - <F,\delta>| \\
=&\Big|\int_s^t \frac{m}{2}(u_r+\delta_r)^2 dr + \int_s^t \Big(\alpha_r - \frac{\alpha_r^3}{2}(x + \int_s^r u_\rho + \delta_\rho d\rho)^2\Big) dr - \int_s^t \frac{m}{2} u_r^2 dr + \int_s^t \Big(\alpha_r - \frac{\alpha_r^3}{2}\xi_r^2\Big) dr \\
&- \int_s^t \Big(mu_r - \int_r^t \alpha_\rho^3 \xi_\rho d\rho\Big)\delta_r dr\Big| \\
=&\Big|\int_s^t mu_r\delta_r dr - \int_s^t \alpha_r^3 (x + \int_s^r u_\rho d\rho)(\int_s^r \delta_\rho d\rho) dr - \int_s^t \frac{\alpha_r^3}{2}(\int_s^r \delta_\rho d\rho)^2 dr - \int_s^t \Big(mu_r - \int_r^t \alpha_\rho^3 \xi_\rho d\rho\Big)\delta_r dr\Big|
\end{aligned}
$$

Using integration by parts, we see that

$$
\begin{aligned}
\int_s^t \alpha_r^3 (x + \int_s^r u_\rho d\rho)(\int_s^r \delta_\rho d\rho) dr =& \Big(\int_s^t \alpha_r^3 (x + \int_s^r u_\rho d\rho) dr\Big)\Big(\int_s^t \delta_\rho d\rho\Big) \\
&- \int_s^t \Big(\int_s^r \alpha_\rho^3 (x + \int_s^\rho u_\tau d\tau) dr\Big)\delta_r dr \\
=& \int_s^t \Big(\int_s^t \alpha_r^3 (x + \int_s^r u_\rho d\rho) dr\Big)\delta_\rho d\rho \\
&- \int_s^t \Big(\int_s^r \alpha_\rho^3 (x + \int_s^\rho u_\tau d\tau) dr\Big)\delta_r dr \\
=& \int_s^t \Big(\int_r^t \alpha_\rho^3 (x + \int_s^\rho u_\tau d\tau) d\rho\Big)\delta_r dr \\
=& \int_s^t \Big(\int_r^t \alpha_\rho^3 \xi_\rho d\rho\Big)\delta_r dr \qquad (42)
\end{aligned}
$$

Therefore, using (42), we get

$$
\begin{aligned}
&|J(u+\delta,\alpha) - J(u,\alpha) - <F,\delta>| \\
=&\Big|\int_s^t mu_r\delta_r dr - \int_s^t \Big(\int_r^t \alpha_\rho^3 \xi_\rho d\rho\Big)\delta_r dr - \int_s^t \frac{\alpha_r^3}{2}(\int_s^r \delta_\rho d\rho)^2 dr - \int_s^t \Big(mu_r - \int_r^t \alpha_\rho^3 \xi_\rho d\rho\Big)\delta_r dr\Big| \\
=&\Big|\int_s^t \frac{\alpha_r^3}{2}(\int_s^r \delta_\rho d\rho)^2 dr\Big| = \mathcal{O}(|\delta|^2)
\end{aligned}
$$

Therefore, the Fréchet derivative $D_u J$ exists and has Riesz representation $[\nabla_u J(u,\alpha)]_r = F_r = mu_r - \int_r^t \alpha_\rho^3 \xi_\rho d\rho$. Similarly, let $\gamma \in \mathcal{A}$ and $G_r = 1 - \frac{3\alpha_r^2 \xi_r^2}{2}$. We have

$$
\begin{aligned}
&|J(u,\alpha+\gamma) - J(u,\alpha) - <G,\gamma>| \\
=&\Big|\int_s^t \frac{m}{2} u_r^2 dr + \int_s^t \Big((\alpha_r+\gamma_r) - \frac{(\alpha_r+\gamma_r)^3}{2}\xi_r^2\Big) dr - \Big(\int_s^t \frac{m}{2} u_r^2 dr + \int_s^t \Big(\alpha_r - \frac{\alpha_r^3}{2}\xi_r^2\Big) dr\Big) \\
&- \int_s^t (1 - \frac{3\alpha_r^2\xi_r^2}{2})\gamma_r dr\Big| \\
=&\Big|\int_s^t \Big(\gamma_r - \frac{3\alpha_r^2\gamma_r}{2}\xi_r^2\Big) dr - \int_s^t \frac{3\alpha_r\gamma_r^2 + \gamma_r^3}{2}\xi_r^2 dr - \int_s^t (1 - \frac{3\alpha_r^2\xi_r^2}{2})\gamma_r dr\Big| \\
=&\Big|\int_s^t \frac{3\alpha_r\gamma_r^2 + \gamma_r^3}{2}\xi_r^2 dr\Big| = \mathcal{O}(|\gamma|^2)
\end{aligned}
$$

Therefore, the Fréchet derivative $D_\alpha J$ exists and has Riesz representation $[\nabla_\alpha J(u,\alpha)]_r = G_r = 1 - \frac{3\alpha_r^2\xi_r^2}{2}$.

$\square$

**Theorem 25** *The 2nd order Fréchet derivatives $D_u^2 J$ and $D_\alpha^2 J$ exist, and we have the Riesz repressentations (cf. Section 6.1)*

$$\begin{aligned}
[\nabla_u^2 J(u,\alpha)\delta]_r &= m\delta_r - \int_s^t \left( \int_{\rho\vee r}^t \alpha_\tau d\tau \right) \delta_\rho d\rho; \ \ \forall\, \delta \in \mathcal{U} \\
[\nabla_\alpha^2 J(u,\alpha)\gamma]_r &= -3\alpha_r \gamma \xi_r^2; \ \ \forall\, \gamma \in \mathcal{A}
\end{aligned}$$

*Proof:* By Theorem 23, we just need to calculate the Fréchet derivatives of $\nabla_u J$ to find (the Riesz representation of) $D_u^2 J$. Let $\delta \in \mathcal{U}$. Let $H : \mathcal{U} \to \mathcal{U}$ be an operator and $[H\delta]_r \doteq m\delta_r - \int_s^t \left( \int_{\rho\vee r}^t \alpha_\tau d\tau \right) \delta_\rho d\rho$. We have

$$\begin{aligned}
&\left( \int_s^t |[\nabla_u J(u+\delta,\alpha)]_r - [\nabla_u J(u,\alpha)]_r - [H\delta]_r|^2 dr \right)^{\frac{1}{2}} \\
=&\left( \int_s^t \left| m(u_r+\delta_r) - \int_r^t \alpha_\rho^3 (x + \int_s^\rho u_\tau + \delta_\tau d\tau) d\rho - m u_r + \int_r^t \alpha_\rho^3 \xi_\rho d\rho - m\delta_r + \int_s^t \left( \int_{\rho\vee r}^t \alpha_\tau d\tau \right) \delta_\rho d\rho \right|^2 dr \right)^{\frac{1}{2}} \\
=&\left( \int_s^t \left| m\delta_r - \int_r^t \alpha_\rho^3 (\int_s^\rho \delta_\tau d\tau) d\rho - m\delta_r + \int_s^t \left( \int_{\rho\vee r}^t \alpha_\tau d\tau \right) \delta_\rho d\rho \right|^2 dr \right)^{\frac{1}{2}}
\end{aligned}$$

Using integration by parts, we see that

$$\begin{aligned}
\int_r^t \alpha_\rho^3 (\int_s^\rho \delta_\tau d\tau) d\rho &= \int_s^t \alpha_\rho^3 (\int_s^\rho \delta_\tau d\tau) d\rho - \int_s^r \alpha_\rho^3 (\int_s^\rho \delta_\tau d\tau) d\rho \\
&= \int_s^t \alpha_\rho^3 1_{\cdot \geq r}(\rho) (\int_s^\rho \delta_\tau d\tau) d\rho \\
&= \left( \int_s^t \alpha_\rho^3 1_{\cdot \geq r}(\rho) d\rho \right) \left( \int_s^t \delta_\rho d\rho \right) - \int_s^t \left( \int_s^\rho \alpha_\tau^3 1_{\cdot \geq r}(\tau) d\tau \right) \delta_\rho d\rho \\
&= \left( \int_s^t \left( \int_s^t \alpha_\rho^3 1_{\cdot \geq r}(\rho) d\rho \right) \delta_\rho d\rho \right) - \int_s^t \left( \int_s^\rho \alpha_\tau^3 1_{\cdot \geq r}(\tau) d\tau \right) \delta_\rho d\rho \\
&= \int_s^t \left( \int_{\rho\vee r}^t \alpha_\tau^3 d\tau \right) \delta_\rho d\rho
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left( \int_s^t |[\nabla_u J(u+\delta,\alpha)]_r - [\nabla_u J(u,\alpha)]_r - [H\delta]_r|^2 dr \right)^{\frac{1}{2}} \\
=&\left( \int_s^t \left| m\delta_r - \int_s^t \left( \int_{\rho\vee r}^t \alpha_\tau^3 d\tau \right) \delta_\rho d\rho - m\delta_r + \int_s^t \left( \int_{\rho\vee r}^t \alpha_\tau d\tau \right) \delta_\rho d\rho \right|^2 dr \right)^{\frac{1}{2}} \\
=&0
\end{aligned}$$

Therefore, $H = \nabla_u^2 J(u,\alpha)$.
Similarly, Let $\gamma \in \mathcal{A}$. Let $T : \mathcal{A} \to \mathcal{A}$ be an operator and $[T\gamma]_r \doteq -3\alpha_r \gamma \xi_r^2$. We have

$$\left( \int_s^t |[\nabla_\alpha J(u,\alpha+\gamma)]_r - [\nabla_\alpha J(u,\alpha)]_r - [T\gamma]_r|^2 dr \right)^{\frac{1}{2}}$$

$$=\left(\int_s^t \left|1-\frac{3(\alpha_r+\gamma_r)^2\xi_r^2}{2}-1+\frac{3\alpha_r^2\xi_r^2}{2}+3\alpha_r\gamma\xi_r^2\right|^2 dr\right)^{\frac{1}{2}}$$
$$=\left(\int_s^t \left|\frac{3\gamma_r^2\xi_r^2}{2}\right|^2 dr\right)^{\frac{1}{2}}=\mathcal{O}(|\gamma|^2)$$

Therefore, $T=\nabla_\alpha^2 J(u,\alpha)$. □

### 6.3 Stochastic Case

Let $(\Omega, P, \mathcal{F}_t)$ be a filtered probability spaces. Let $0\le s\le r\le t$. Let $u\in\mathcal{U}$ be a stochastic process adapted to the filtration $\mathcal{F}_r$, where $\mathcal{U}$ is a subspace of $L^2(\Omega; L^2([s,t],\mathbb{R}))$, that is, $u(\omega)\in L^2([s,t],\mathbb{R})$ for each $\omega\in\Omega$. Let the diffusion process $\xi$ be given by

$$\begin{aligned} d\xi_r &= u_r dr+\sigma dB_r \\ \xi_s &= x \end{aligned}$$

where $x\in\mathbb{R}$ is fixed, $\sigma\in\mathbb{R}$ is a constant XXXX could make $\sigma$ an Itô integrable function XXXX, $B$ is a Brownian motion taking values in $\mathbb{R}$. Let $\alpha\in\mathcal{A}$, where $\mathcal{A}$ is a subspace of $L^2(\Omega; L^2([s,t],\mathbb{R}))$, that is, $\alpha(\omega)\in L^2([s,t],\mathbb{R})$ for each $\omega\in\Omega$. Let

$$J(u,\alpha)\doteq E^{s,x}[\int_s^t \frac{m}{2}u_r^2 dr+\int_s^t\left(\alpha_r-\frac{\alpha_r^3}{2}\xi_r^2\right)dr]$$

**Lemma 26** *$L^2(\Omega; L^2([s,t],\mathbb{R}))$ is a Hilbert space with the inner product*

$$<f,g>\doteq E[\int_s^t [f(\omega)](r)[g(\omega)](r)dr]$$

*$L^2(\Omega; L^2([s,t],\mathbb{C}))$ is a Hilbert space with the inner product*

$$<f,g>\doteq E[\int_s^t [f(\omega)](r)\overline{[g(\omega)](r)}dr]$$

***Theorem 27*** *The functional $J(u,\alpha)$ is Fréchet differentiable in $u,\alpha$, and the Fréchet derivatives have Riesz representation*

$$\begin{aligned} [\nabla_u J(u,\alpha)]_r &= mu_r-\int_r^t \alpha_\rho^3\xi_\rho d\rho \\ [\nabla_\alpha J(u,\alpha)]_r &= 1-\frac{3\alpha_r^2\xi_r^2}{2} \end{aligned}$$

*Proof: Let $\delta\in\mathcal{U}$. Let $X_r\doteq\int_s^r \alpha_\rho^3\left(\int_s^\rho \sigma dB_\tau\right)d\rho$, $Y_r\doteq\int_s^r \delta_\rho d\rho$. By Itô's rule, we have*

$$\begin{aligned} d(XY) &= XdY+YdX+dX\cdot dY \\ dX_r &= \alpha_r^3\left(\int_s^r \sigma dB_\rho\right)dr=\alpha_r^3 B_{r-s}dr \\ dY_r &= \delta_r dr \end{aligned}$$

*Therefore, $dX \cdot dY = 0$, and*

$$\begin{aligned}
\int_s^t \alpha_r^3\Big(\int_s^r \sigma dB_\rho\Big)\Big(\int_s^r \delta_\rho d\rho\Big)dr &= \Big(\int_s^t \alpha_r^3 \int_s^r \sigma dB_\tau dr\Big)\Big(\int_s^t \delta_\rho d\rho\Big) - \int_s^t \Big(\int_s^r \alpha_\rho^3 \int_s^\rho \sigma dB_\tau d\rho\Big)\delta_r dr \\
&= \int_s^t \Big(\int_s^t \alpha_r^3 \int_s^r \sigma dB_\tau dr\Big)\delta_\rho d\rho - \int_s^t \Big(\int_s^r \alpha_\rho^3 \int_s^\rho \sigma dB_\tau d\rho\Big)\delta_r dr \\
&= \int_s^t \Big(\int_r^t \alpha_\rho^3 \int_s^\rho \sigma dB_\tau d\rho\Big)\delta_r dr \qquad (43)
\end{aligned}$$

*Let $F_r \doteq m u_r - \int_r^t \alpha_\rho \xi_\rho d\rho$, where $\xi_\rho \doteq x + \int_s^\rho u_\tau d\tau - \int_s^\rho \sigma dB_\rho$. We have*

$$\begin{aligned}
&|J(u+\delta,\alpha) - J(u,\alpha) - < F, \delta > | \\
=&\Big|E^{s,x}\Big[\int_s^t \frac{m}{2}(u_r+\delta_r)^2 dr + \int_s^t \Big(\alpha_r - \frac{\alpha_r^3}{2}(x + \int_s^r u_\rho + \delta_\rho d\rho + \int_s^r \sigma dB_\rho)^2\Big)dr\Big] \\
&- E^{s,x}\Big[\int_s^t \frac{m}{2}u_r^2 dr + \int_s^t \Big(\alpha_r - \frac{\alpha_r^3}{2}(x + \int_s^r u_\rho d\rho + \int_s^r \sigma dB_\rho)^2\Big)dr\Big] \\
&- E^{s,x}\Big[\int_s^t m u_r \delta_r dr - \int_s^t \Big(\int_r^t \alpha_\rho \xi_\rho d\rho\Big)\delta_r dr\Big]\Big| \\
=&\Big|E^{s,x}\Big[\int_s^t m u_r \delta_r dr - \int_s^t \alpha_r^3\Big(x + \int_s^r u_\rho d\rho\Big)\Big(\int_s^r \delta_\rho d\rho\Big)dr - \int_s^t \alpha_r^3\Big(\int_s^r \sigma dB_\rho\Big)\Big(\int_s^r \delta_\rho d\rho\Big)dr\Big] \\
&- E^{s,x}\Big[\frac{\alpha_r^3}{2}\Big(\int_s^r \delta_\rho d\rho\Big)^2\Big] - E^{s,x}\Big[\int_s^t m u_r \delta_r dr - \int_s^t \Big(\int_r^t \alpha_\rho \xi_\rho d\rho\Big)\delta_r dr\Big]\Big|
\end{aligned}$$

*By (42),(43), this is*

$$\begin{aligned}
=&\Big|E^{s,x}\Big[\int_s^t m u_r \delta_r dr - \int_s^t \Big(\int_r^t \alpha_\rho^3 (x + \int_s^\rho u_\tau d\tau) d\rho\Big)\delta_r dr - \int_s^t \Big(\int_r^t \alpha_\rho^3 \int_s^\rho \sigma dB_\tau d\rho\Big)\delta_r dr\Big] \\
&- E^{s,x}\Big[\frac{\alpha_r^3}{2}\Big(\int_s^r \delta_\rho d\rho\Big)^2\Big] - E^{s,x}\Big[\int_s^t m u_r \delta_r dr - \int_s^t \Big(\int_r^t \alpha_\rho \xi_\rho d\rho\Big)\delta_r dr\Big]\Big| \\
=&\Big|E^{s,x}\Big[\int_s^t m u_r \delta_r dr - \int_s^t \Big(\int_r^t \alpha_\rho^3 (x + \int_s^\rho u_\tau d\tau + \int_s^\rho \sigma dB_\tau) d\rho\Big)\delta_r dr\Big] \\
&- E^{s,x}\Big[\frac{\alpha_r^3}{2}\Big(\int_s^r \delta_\rho d\rho\Big)^2\Big] - E^{s,x}\Big[\int_s^t m u_r \delta_r dr - \int_s^t \Big(\int_r^t \alpha_\rho \xi_\rho d\rho\Big)\delta_r dr\Big]\Big| \\
=&\Big|E^{s,x}\Big[\frac{\alpha_r^3}{2}\Big(\int_s^r \delta_\rho d\rho\Big)^2\Big]\Big| = \mathcal{O}(|\delta|^2)
\end{aligned}$$

*Therefore, the Fréchet derivative $D_u J$ exists and has Riesz representation $[\nabla_u J(u,\alpha)]_r = F_r = m u_r - \int_r^t \alpha_\rho^3 \xi_\rho d\rho$. Similarly, let $\gamma \in \mathcal{A}$ and $G_r \doteq 1 - \frac{3\alpha_r^2 \xi_r^2}{2}$. We have*

$$\begin{aligned}
&|J(u,\alpha+\gamma) - J(u,\alpha) - < G, \gamma > | \\
=&\Big|E^{s,x}\Big[\int_s^t \frac{m}{2}u_r^2 dr + \int_s^t \Big((\alpha_r+\gamma_r) - \frac{(\alpha_r+\gamma_r)^3}{2}\xi_r^2\Big)dr\Big] - E^{s,x}\Big[\int_s^t \frac{m}{2}u_r^2 dr + \int_s^t \Big(\alpha_r - \frac{\alpha_r^3}{2}\xi_r^2\Big)dr\Big] \\
&- E^{s,x}\Big[\int_s^t (1 - \frac{3\alpha_r^2 \xi_r^2}{2})\gamma_r dr\Big]\Big|
\end{aligned}$$

$$=\left|E^{s,x}\left[\int_s^t\left(\gamma_r-\frac{3\alpha_r^2\gamma_r}{2}\xi_r^2\right)dr-\int_s^t\frac{3\alpha_r\gamma_r^2+\gamma_r^3}{2}\xi_r^2dr-\int_s^t(1-\frac{3\alpha_r^2\xi_r^2}{2})\gamma_r dr\right]\right|$$
$$=\left|E^{s,x}\left[\int_s^t\frac{3\alpha_r\gamma_r^2+\gamma_r^3}{2}\xi_r^2dr\right]\right|=\mathcal{O}(|\gamma|^2)$$

*Therefore, the Fréchet derivative $D_\alpha J$ exists and has Riesz representation $[\nabla_\alpha J(u,\alpha)]_r = G_r = 1-\frac{3\alpha_r^2\xi_r^2}{2}$.* □

***Theorem 28*** *The 2nd order Fréchet derivatives $D_u^2J$ and $D_\alpha^2J$ exist, and we have the Riesz repressentations*

$$[\nabla_u^2J(u,\alpha)\delta]_r=m\delta_r-\int_s^t\left(\int_{\rho\vee r}^t\alpha_\tau d\tau\right)\delta_\rho d\rho;\ \ \forall\,\delta\in\mathcal{U}$$
$$[\nabla_\alpha^2J(u,\alpha)\gamma]_r=-3\alpha_r\gamma\xi_r^2;\ \ \forall\,\gamma\in\mathcal{A}$$

*Proof: By Theorem 23, we just need to calculate the Fréchet derivatives of $\nabla_uJ$ to find (the Riesz representation of) $D_u^2J$. Let $\delta\in\mathcal{U}$. Let $H:\mathcal{U}\to\mathcal{U}$ be an operator and $[H\delta]_r\doteq m\delta_r-\int_s^t\left(\int_{\rho\vee r}^t\alpha_\tau d\tau\right)\delta_\rho d\rho$. We have*

$$\left(E^{s,x}\left[\int_s^t|[\nabla_uJ(u+\delta,\alpha)]_r-[\nabla_uJ(u,\alpha)]_r-[H\delta]_r|^2dr\right]\right)^{\frac12}$$
$$=\left(E^{s,x}\left[\int_s^t\left|m(u_r+\delta_r)-\int_r^t\alpha_\rho^3(x+\int_s^\rho u_\tau+\delta_\tau d\tau+\int_s^\rho\sigma dB_\tau)d\rho-mu_r+\int_r^t\alpha_\rho^3\xi_\rho d\rho\right.\right.\right.$$
$$\left.\left.\left.-m\delta_r+\int_s^t\left(\int_{\rho\vee r}^t\alpha_\tau d\tau\right)\delta_\rho d\rho\right|^2dr\right]\right)^{\frac12}$$
$$=\left(E^{s,x}\left[\int_s^t\left|m\delta_r-\int_r^t\alpha_\rho^3(\int_s^\rho\delta_\tau d\tau)d\rho-m\delta_r+\int_s^t\left(\int_{\rho\vee r}^t\alpha_\tau d\tau\right)\delta_\rho d\rho\right|^2dr\right]\right)^{\frac12}$$

*Using integration by parts, we see that for each fixed $\omega\in\Omega$*

$$\left[\int_r^t\alpha_\rho^3(\int_s^\rho\delta_\tau d\tau)d\rho\right](\omega)=\left[\int_s^t\alpha_\rho^3(\int_s^\rho\delta_\tau d\tau)d\rho-\int_s^r\alpha_\rho^3(\int_s^\rho\delta_\tau d\tau)d\rho\right](\omega)$$
$$=\left[\int_s^t\alpha_\rho^3 1_{\cdot\geq r}(\rho)(\int_s^\rho\delta_\tau d\tau)d\rho\right](\omega)$$
$$=\left[\left(\int_s^t\alpha_\rho^3 1_{\cdot\geq r}(\rho)d\rho\right)\left(\int_s^t\delta_\rho d\rho\right)\right](\omega)-\left[\int_s^t\left(\int_s^\rho\alpha_\tau^3 1_{\cdot\geq r}(\tau)d\tau\right)\delta_\rho d\rho\right](\omega)$$
$$=\left[\left(\int_s^t\left(\int_s^t\alpha_\rho^3 1_{\cdot\geq r}(\rho)d\rho\right)\delta_\rho d\rho\right)\right](\omega)-\left[\int_s^t\left(\int_s^\rho\alpha_\tau^3 1_{\cdot\geq r}(\tau)d\tau\right)\delta_\rho d\rho\right](\omega)$$
$$=\left[\int_s^t\left(\int_{\rho\vee r}^t\alpha_\tau^3 d\tau\right)\delta_\rho d\rho\right](\omega)$$

*Therefore,*

$$\left(E^{s,x}\left[\int_s^t|[\nabla_uJ(u+\delta,\alpha)]_r-[\nabla_uJ(u,\alpha)]_r-[H\delta]_r|^2dr\right]\right)^{\frac12}$$

$$
\begin{aligned}
=&\left(E^{s,x}\left[\int_s^t \left| m\delta_r - \int_s^t \left(\int_{\rho\vee r}^t \alpha_\tau^3 d\tau\right)\delta_\rho d\rho - m\delta_r + \int_s^t \left(\int_{\rho\vee r}^t \alpha_\tau d\tau\right)\delta_\rho d\rho\right|^2 dr\right]\right)^{\frac{1}{2}}\\
=&0
\end{aligned}
$$

*Therefore,* $H = \nabla_u^2 J(u,\alpha)$.
*Similarly, let* $\gamma \in \mathcal{A}$. *Let* $T : \mathcal{A} \to \mathcal{A}$ *be an operator and* $[T\gamma]_r \doteq -3\alpha_r\gamma\xi_r^2$.

$$
\begin{aligned}
&\left(E^{s,x}\left[\int_s^t |[\nabla_\alpha J(u,\alpha+\gamma)]_r - [\nabla_\alpha J(u,\alpha)]_r - [T\gamma]_r|^2 dr\right]\right)^{\frac{1}{2}}\\
=&\left(E^{s,x}\left[\int_s^t \left|1 - \frac{3(\alpha_r+\gamma_r)^2\xi_r^2}{2} - 1 + \frac{3\alpha_r^2\xi_r^2}{2} + 3\alpha_r\gamma\xi_r^2\right|^2 dr\right]\right)^{\frac{1}{2}}\\
=&\left(E^{s,x}\left[\int_s^t \left|\frac{3\gamma_r^2\xi_r^2}{2}\right|^2 dr\right]\right)^{\frac{1}{2}} = \mathcal{O}(|\gamma|^2)
\end{aligned}
$$

*Therefore,* $T = \nabla_\alpha^2 J(u,\alpha)$. $\square$