Iterated Staticization and Efficient Solution of Conservative and Quantum Systems*

Ruobing Zhao† William M. McEneaney‡

Abstract

Conservative dynamical systems propagate as stationary points of the action functional. Using this representation, it has previously been demonstrated that one may obtain fundamental solutions for two-point boundary value problems for some classes of conservative systems via solution of an associated dynamic program. It is also known that the gravitational and Coulomb potentials may be represented as stationary points of cubicly-parameterized quadratic functionals. Hence, stationary points of the action functional may be represented via iterated staticization of polynomial functionals. This leads to representations through operations on sets of solutions of differential Riccati equations (DREs). A key step in this process is the reordering of staticization operations.

1 Introduction

Conservative dynamical systems propagate as stationary points of the action functional over the possible paths of the system. This viewpoint appears particularly useful in some applications in modern physics, including systems where relativistic effects are non-negligible and systems in the quantum domain, cf. [4, 6, 20]. The stationary-action formulation has also recently been found to be quite useful for generation of fundamental solutions to two-point boundary-value problems (TPBVPs) for conservative dynamical systems, cf. [2, 16, 18]. In particular, one may address dynamical-systems questions such as these by control-theoretic methods.

To give a sense of this latter application domain, consider a finite-dimensional action-functional formulation of such a TPBVP. Let the path of the conservative system be denoted by ξ_r for $r \in [0,t]$ with $\xi_0 = \bar{x}$, in which case the action functional, with an appended terminal cost, may take the form

$$J(\bar{x},t,u) \doteq \int_0^t T(u_r) - V(\xi_r)\,dr + \psi(\xi_t), \tag{1.1}$$

where $\dot{\xi} = u$, $u \in \mathcal{U} \doteq L_2(0,t)$, $T(\cdot)$ denotes the kinetic energy associated to the momentum (specifically taken to be $T(v) \doteq \frac{1}{2}v^T\mathcal{M}v$ throughout, with $\mathcal{M}$ positive-definite and symmetric), and $V(\cdot)$ denotes a potential energy field. If, for example, one takes $\psi(x) \doteq -\bar{v}^T\mathcal{M}x$, a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\dot{\xi}_t = \bar{v}$; if one takes ψ to be a min-plus delta function centered at z, then a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\xi_t = z$, cf. [2]. In the early work of Hamilton, it was formulated as the least-action principle [7], which states that a conservative dynamical system follows the trajectory that minimizes the action functional. However, this is typically only the case for relatively short-duration cases, cf. [6, 2, 18] and the references therein. In order to extend to longer-duration problems, it becomes necessary to apply concepts of stationarity [16, 17].

It is worth noting that if one defines $\operatorname{stat}_{x\in\mathcal{X}}\phi(x)$ to be the critical value of ϕ (defined rigorously in Section 2.1), then a gravitational potential given as $V(x) = -\mu/|x|$ for $x \neq 0$ and constant $\mu > 0$, has the representation $V(x) = -(\frac{3}{2})^{3/2}\mu\operatorname{stat}_{\alpha>0}\{\alpha - \frac{\alpha^3|x|^2}{2}\}$, where we note that the argument of the stat operator is polynomial, [8, 18]. The Schrödinger equation in the context of a Coulomb potential may be similarly addressed. In that case, it is particularly helpful to consider an extension of the space variable to a vector space over the complex field, say $x \in \mathbb{C}^n$ rather than $x \in \mathbb{R}^n$. More specifically, for $x \in \mathbb{C}^n$, this representation takes a general form $V(x) = -(\frac{3}{2})^{3/2}\hat{\mu}\operatorname{stat}_{\alpha\in\mathcal{A}^R}\{\alpha - \frac{\alpha^3|x|_c^2}{2}\}$, where $\mathcal{A}^R \doteq \{\alpha = r[\cos(\theta) + i\sin(\theta)] \in \mathbb{C}\,|\, r \geq 0,\, \theta \in (\frac{-\pi}{2}, \frac{\pi}{2}]\}$, and for $x \in \mathbb{C}^n$, $|x|_c^2 \doteq \sum_{j=1}^n x_j^2$, [12]. (We remark that notation $|\cdot|_c^2$ is not intended to indicate a squared norm; the range is complex.) In the simple one-dimensional case, the resulting function on $\mathbb{C}$ has a branch cut along the negative imaginary axis, and this generalizes to the higher-dimensional case in the natural way.

Although stationarity-based representations for gravitational and Coulomb potentials are inside the integral in (1.1), they may be moved outside through the introduction of α-valued processes, cf. [8, 18]. In particular, not only does one seek the stationary path for

*Research partially supported by a grant from AFOSR.

†Department of Mechanical and Aerospace Engineering, University of California at San Diego, ruz015@eng.ucsd.edu.

‡Department of Mechanical and Aerospace Engineering, University of California at San Diego, wmceneaney@ucsd.edu.

action J, but the action functional itself can be given as a stationary value of an integral of a polynomial, leading to an iterated-stat problem formulation for such TPBVPs. This may be exploited in the solution of TPBVPs in such systems, cf. [8, 16, 18].

It has recently been demonstrated that stationarity may be employed to obtain a Feynman-Kac type of representation for solutions of the Schrödinger initial value problem (IVP) for certain classes of initial conditions and potentials [15]. As with the conservative-system cases above, these representations are valid for indefinitely long duration problems, whereas with only the minimization operation, such representations are valid only on time intervals such that the action remains convex.

In all of these examples, one obtains the stationary value of an action functional where the action functional itself takes the form of a stationary value of a functional that is quadratic in the momentum (the u. in put in (1.1)) and cubic in the newly introduced potential energy parameterization variable (a time-dependent form of the α parameter above). Thus, if one can invert the order of the of the stat operations, then the inner stat operation results in a functional that is obtained as a solution of a differential Riccati equation (DRE). (It should be noted that this DRE must typically be propagated past escape times, where this propagation may be efficiently performed through the use of what has been termed "stat duality", cf. [13].) Hence, after inversion of the order of the iterated stat operations, the problem may be reduced to a single stat operation such that the argument takes the form of a linear functional operating on a set of DRE solutions. Consequently, an issue of fundamental importance regards conditions under which one may invert the order of stat operations in an iterated staticization.

In Section 2, the staticization operator will be rigorously defined, and a general problem class along with some corresponding notation will be indicated. Then, in Section 3, a somewhat general condition will be indicated, and it will shown that one may invert the order of staticization operations under that condition. Section 4 will present two classes of problems for which the general condition of Section 3 holds. Finally, in Section 5, the stationary-action application above will be discussed.

2 Problem and Stationarity Definitions

2.1 Stationarity definitions As noted above, the motivation for this effort is the computation and propagation of stationary points of payoff functionals, which is unusual in comparison to the standard classes of problems in optimization (although one should note for example, [3]). In analogy with the language for minimization and maximization, we will refer to the search for stationary points as "staticization", with these points being statica, in analogy with minima/maxima, and a single such point being a staticum in analogy with minimum/maximum. We make the following definitions. Let $\mathcal{F}$ denote either the real or complex field. Suppose $\mathcal{U}$ is a normed vector space (over $\mathcal{F}$) with $\mathcal{A} \subseteq \mathcal{U}$, and suppose $G : \mathcal{A} \to \mathcal{F}$. We say $\bar{u} \in \operatorname{argstat}_{u\in\mathcal{A}} G(u) \doteq \operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\}$ if $\bar{u} \in \mathcal{A}$ and either

$$\limsup_{u\to\bar{u},u\in\mathcal{A}\setminus\{\bar{u}\}} \frac{|G(u) - G(\bar{u})|}{|u - \bar{u}|} = 0, \tag{2.2}$$

or there exists $\delta > 0$ such that $\mathcal{A} \cap B_\delta(\bar{u}) = \{\bar{u}\}$ (where $B_\delta(\bar{u})$ denotes the ball of radius δ around $\bar{u}$). If $\operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} \neq \emptyset$, we define the possibly set-valued stat^s operation by

$$\begin{aligned} \operatorname*{stat}^s_{u\in\mathcal{A}} G(u) &\doteq \operatorname{stat}^s\{G(u) \,|\, u \in \mathcal{A}\} \\ &\doteq \left\{ G(\bar{u}) \,\middle|\, \bar{u} \in \operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} \right\}. \end{aligned} \tag{2.3}$$

If $\operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} = \emptyset$, $\operatorname{stat}^s_{u\in\mathcal{A}} G(u)$ is undefined. Where applicable, we are also interested in a single-valued stat operation (note the absence of superscript s). In particular, if there exists $a \in \mathcal{F}$ such that $\operatorname{stat}^s_{u\in\mathcal{A}} G(u) = \{a\}$, then $\operatorname{stat}_{u\in\mathcal{A}} G(u) \doteq a$; otherwise, $\operatorname{stat}_{u\in\mathcal{A}} G(u)$ is undefined. At times, we may abuse notation by writing $\bar{u} = \operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\}$ in the event that the argstat is the single point $\{\bar{u}\}$.

In the case where $\mathcal{U}$ is a Hilbert space, and $\mathcal{A} \subseteq \mathcal{U}$ is an open set, $G : \mathcal{A} \to \mathcal{F}$ is Fréchet differentiable at $\bar{u} \in \mathcal{A}$ with Riesz representation $DG(\bar{u}) \in \mathcal{U}$ if

$$\lim_{w\to 0,\, \bar{u}+w\in\mathcal{A}\setminus\{\bar{u}\}} \frac{|G(\bar{u} + w) - G(\bar{u}) - \langle DG(\bar{u}), w\rangle|}{|w|} = 0. \tag{2.4}$$

The following is immediate from the above definitions.

Lemma 2.1. *Suppose $\mathcal{U}$ is a Hilbert space, with open set $\mathcal{A} \subseteq \mathcal{U}$, and that G is Fréchet differentiable at $\bar{u} \in \mathcal{A}$. Then, $\bar{u} \in \operatorname{argstat}\{G(y) \,|\, y \in \mathcal{A}\}$ if and only if $DG(\bar{u}) = 0$.*

2.2 Problem definition Let $\mathcal{U}, \mathcal{V}$ be Hilbert spaces with inner products and norms on $\mathcal{U}$ denoted by $\langle\cdot,\cdot\rangle_\mathcal{U}$ and $|\cdot|_\mathcal{U}$, and similarly for $\mathcal{V}$. Let the resulting inner product and norm on $\mathcal{U}\times\mathcal{V}$ be denoted by $\langle\cdot,\cdot\rangle_{\mathcal{U}\times\mathcal{V}}$ and $|\cdot|_{\mathcal{U}\times\mathcal{V}}$. Let $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \mathcal{V}$ be open. Throughout, we assume

$$G \in C^2(\mathcal{A} \times \mathcal{B}; \mathcal{F}). \tag{A.1}$$

Then, for each $u \in \mathcal{A}$, let $g^{1,u} \in C^2(\mathcal{B}; \mathcal{F})$ be given by $g^{1,u}(v) \doteq G(u, v)$ for all $v \in \mathcal{B}$. Similarly, for each

$v \in \mathcal{B}$, let $g^{2,v} \in C^2(\mathcal{A};\mathcal{F})$ be given by $g^{2,v}(u) \doteq G(u,v)$ for all $u \in \mathcal{A}$. Further, let

$$\mathcal{A}_G \doteq \{u \in \mathcal{A} \,|\, \operatorname*{stat}_{v\in\mathcal{B}} g^{1,u}(v) \text{ exists}\} \quad \text{and}$$
$$\mathcal{B}_G \doteq \{v \in \mathcal{B} \,|\, \operatorname*{stat}_{u\in\mathcal{A}} g^{2,v}(u) \text{ exists}\}. \tag{2.5}$$

Given $u \in \mathcal{A}_G$, let $\mathcal{M}^1(u) \doteq \operatorname{argstat}_{v\in\mathcal{B}} g^{1,u}(v)$. Similarly, given $v \in \mathcal{B}_G$, let $\mathcal{M}^2(v) \doteq \operatorname{argstat}_{u\in\mathcal{A}} g^{2,v}(u)$. Next, define $\bar{G}^1 : \mathcal{A}_G \to \mathcal{F}$ and $\bar{G}^2 : \mathcal{B}_G \to \mathcal{F}$ by

$$\bar{G}^1(u) \doteq \operatorname*{stat}_{v\in\mathcal{B}} g^{1,u}(v) \;\; \forall\, u \in \mathcal{A}_G \quad \text{and}$$
$$\bar{G}^2(v) \doteq \operatorname*{stat}_{u\in\mathcal{A}} g^{2,v}(u) \;\; \forall\, v \in \mathcal{B}_G.$$

Finally, let

$$\hat{\mathcal{A}}_G \doteq \operatorname*{argstat}_{u\in\mathcal{A}_G} \bar{G}^1(u) \quad \text{and} \quad \hat{\mathcal{B}}_G \doteq \operatorname*{argstat}_{v\in\mathcal{B}_G} \bar{G}^2(v).$$

We will discuss conditions under which

$$\operatorname*{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) = \operatorname*{stat}_{v\in\mathcal{B}_G} \bar{G}^2(v). \tag{2.6}$$

We will generally be concerned only with the left-hand equality in (2.6); obviously the right-hand equality would be obtained analogously. We refer to the left-hand object in (2.6) as iterated stat operations, while the center object will be referred to as a full stat operation. Although in some results, the existence of both the iterated and full stat operations are obtained, many of the results will assume the existence of one or both of these objects. We list the two potential assumptions below. In each result to follow, we will indicate when one or both of these is utilized. The full-stat assumption is as follows.

Assume $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ exists, and let $(\bar{u},\bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$. $\quad (A.2f)$

Note that under Assumption $(A.2f)$,

$$\bar{u} \in \mathcal{A}_G, \;\; \bar{v} \in \mathcal{B}_G, \;\; \bar{v} \in \mathcal{M}^1(\bar{u}), \;\; \text{and} \;\; \bar{u} \in \mathcal{M}^2(\bar{v}). \tag{2.7}$$

The iterated-stat assumption is as follows.

Assume $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ exists, and let $\bar{u} \in \hat{\mathcal{A}}_G$. $\quad (A.2i)$

Note that under Assumption $(A.2i)$,

$$\exists\, \bar{v} \in \mathcal{M}^1(\bar{u}), \quad \text{and} \quad \operatorname*{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = g^{1,\bar{u}}(\bar{v}) = G(\bar{u},\bar{v}). \tag{2.8}$$

We will first obtain (2.6) under some general assumptions.

3 The General Case

Given $\mathcal{C} \subseteq \mathcal{V}$ and $\hat{v} \in \mathcal{V}$, we let $d(\hat{v},\mathcal{C}) \doteq \inf_{v\in\mathcal{C}} |v - \hat{v}|$, and use this distance notation more generally throughout. In addition to $(A.1)$, we assume the following throughout this section.

$\exists\; \delta = \delta(\bar{u},\bar{v}) > 0$ and $K = K(\bar{u},\bar{v}) < \infty$ such that $d(\bar{v},\mathcal{M}^1(u)) \le K\,|\bar{u} - u|$ for all $u \in \mathcal{A}_G \cap B_\delta(\bar{u})$. $\quad (A.3)$

We note that $(A.3)$ is trivially satisfied in the case that there exists $\delta > 0$ such that $B_\delta(\bar{u}) \cap \mathcal{A}_G = \emptyset$. It may be helpful to also note that $(A.3)$ is satisfied under the possibly more heuristically appealing, following assumption.

For every $\tilde{u} \in \mathcal{A}_G$ and every $\tilde{v} \in \mathcal{M}^1(\tilde{u})$, $\exists\; \delta = \delta(\tilde{u},\tilde{v}) > 0$ and $K = K(\tilde{u},\tilde{v}) < \infty$ such that $d(\tilde{v},\mathcal{M}^1(u)) \le K\,|\tilde{u} - u|$ for all $u \in \mathcal{A}_G \cap B_\delta(\tilde{u})$. $\quad (A.3')$

Lemma 3.1. *Assume $(A.2f)$. Then, $\bar{u} \in \hat{\mathcal{A}}_G$ and $G(\bar{u},\bar{v}) \in \operatorname{stat}^s_{u\in\mathcal{A}_G} \bar{G}^1(u)$.*

Proof. Let $(\bar{u},\bar{v})$ be as in $(A.2f)$. Let $R \doteq d\big((\bar{u},\bar{v}),(\mathcal{A}\times\mathcal{B})^c\big)$. By Assumption $(A.3)$, there exist $\delta \in (0,R/2)$ and $K < \infty$ such that for all $u \in \mathcal{A}_G \cap B_\delta(\bar{u})$ and all $\epsilon \in (0,1)$, there exists $v \in \mathcal{M}^1(u)$ such that

$$|v - \bar{v}| \le (K+\epsilon)|u - \bar{u}| \le (K+\epsilon)\delta. \tag{3.9}$$

Let $\tilde{u} \in \mathcal{A}_G \cap B_{\delta/(K+1)}(\bar{u})$. By (2.7),

$$|\operatorname*{stat}_{v\in\mathcal{B}} g^{1,\tilde{u}}(v) - \operatorname*{stat}_{v\in\mathcal{B}} g^{1,\bar{u}}(v)| = |\operatorname*{stat}_{v\in\mathcal{B}} g^{1,\tilde{u}}(v) - G(\bar{u},\bar{v})|,$$

and by (3.9), there exists $\tilde{v} \in B_\delta(\bar{v})$ such that this is

$$= |G(\tilde{u},\tilde{v}) - G(\bar{u},\bar{v})|. \tag{3.10}$$

Let $f \in C^\infty\big((-3/2,3/2);\mathcal{A}\times\mathcal{B}\big)$ be given by $f(\lambda) = \big(\bar{u}+\lambda(\tilde{u}-\bar{u}), \bar{v}+\lambda(\tilde{v}-\bar{v})\big)$ for all $\lambda \in (-3/2,3/2)$. Define $W^0(\lambda) = [G \circ f](\lambda)$ for all $\lambda \in (-3/2,3/2)$, and note that by Assumption $(A.1)$ and standard results, $W^0 \in C^2\big((-3/2,3/2);\mathcal{F}\big)$. Similarly, let $W^1(\lambda) = [(G_u,G_v)\circ f](\lambda) = \big(G_u(f(\lambda)), G_v(f(\lambda))\big)$. By Assumption $(A.1)$ and standard results, $W^1 \in C^1\big((-3/2,3/2);\mathcal{U}\times\mathcal{V}\big)$. Then, by the Mean Value Theorem (cf. [1, Th. 12.6]), there exists $\lambda_0 \in (0,1)$ such that

$$\begin{aligned}
|G(\tilde{u},\tilde{v}) - G(\bar{u},\bar{v})| &= |W^0(1) - W^0(0)| \\
&\le \Big|\frac{dG}{d(u,v)}(f(\lambda_0))\Big|\,\Big|\frac{df}{d\lambda}(\lambda_0)\Big| \\
&= \big|\big(G_u(u_0,v_0), G_v(u_0,v_0)\big)\big|\,\big|(\tilde{u}-\bar{u}, \tilde{v}-\bar{v})\big|,
\end{aligned}$$

where $(u_0,v_0) \doteq f(\lambda_0)$, and which by (3.9),

$$\le \big|\big(G_u(u_0,v_0), G_v(u_0,v_0)\big)\big|\,\sqrt{1+(K+1)^2}\,|\tilde{u}-\bar{u}|. \tag{3.11}$$

Similarly, there exists $\lambda_1 \in (0,\lambda_0)$ such that

$$\begin{aligned}
&\big|\big(G_u(u_0,v_0), G_v(u_0,v_0)\big) - \big(G_u(\bar{u},\bar{v}), G_v(\bar{u},\bar{v})\big)\big| \\
&= |W^1(\lambda_0) - W^1(0)| \le \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\,\Big|\frac{df}{d\lambda}(\lambda_1)\Big|\,|\lambda_1|
\end{aligned}$$

$$\leq \Big| \frac{d^2G}{d(u,v)^2}(u_1,v_1) \Big| \big|(u_1-\bar{u}, v_1-\bar{v})\big|,$$

where $(u_1,v_1) \doteq f(\lambda_1)$, and this is

$$\leq \Big| \frac{d^2G}{d(u,v)^2}(f(\lambda_1)) \Big| \sqrt{1+(K+1)^2}|\tilde{u}-\bar{u}|.$$

Recalling $(\bar{u},\bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$, this implies

$$\begin{aligned} &\big|\big(G_u(u_0,v_0), G_v(u_0,v_0)\big)\big| \\ &\leq \Big| \frac{d^2G}{d(u,v)^2}(f(\lambda_1)) \Big| \sqrt{1+(K+1)^2}|\tilde{u}-\bar{u}|. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) yields

$$\begin{aligned} &|G(\tilde{u},\tilde{v}) - G(\bar{u},\bar{v})| \\ \leq &\Big| \frac{d^2G}{d(u,v)^2}(f(\lambda_1)) \Big| \left[1+(K+1)^2\right] |\tilde{u}-\bar{u}|^2. \end{aligned}$$

Let $K_1 \doteq \big|\frac{d^2G}{d(u,v)^2}(\bar{u},\bar{v})\big|$. By $(A.1)$, there exists $\hat{\delta} \in (0, \delta/(K+1))$ such that for all $(u,v) \in B_{\hat{\delta}}(\bar{u},\bar{v})$, $\big|\frac{d^2G}{d(u,v)^2}(u,v)\big| \leq K_1+1$. Hence, there exists $\bar{C} < \infty$ such that

$$|G(\tilde{u},\tilde{v}) - G(\bar{u},\bar{v})| \leq \bar{C}|\tilde{u}-\bar{u}|^2 \;\; \forall\, (\tilde{u},\tilde{v}) \in \mathcal{A}_G \cap B_{\hat{\delta}/(K_1+1)}(\bar{u}). \tag{3.13}$$

Combining (3.10) and (3.13) one has $|\operatorname{stat}_{v\in\mathcal{B}} g^{1,\tilde{u}}(v) - \operatorname{stat}_{v\in\mathcal{B}} g^{1,\bar{u}}(v)| \leq \bar{C}|\tilde{u}-\bar{u}|^2$, which upon recalling that $\tilde{u} \in \mathcal{A}_G \cap B_{\hat{\delta}/(K+1)}(\bar{u})$ was arbitrary, yields the assertions. ■

THEOREM 3.1. *Assume $(A.2f)$ and $(A.2i)$. Then*

$$\operatorname*{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v).$$

Proof. The assertions follow directly from the assumption, $(A.2f)$ and Lemma 3.1. ■

4 Some Specific Cases

4.1 The Semi-Quadratic Case

Throughout this section, we take $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} = \mathcal{V}$, and let

$$G(u,v) = f_1(u) + \langle f_2(u), v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_3(u)v, v\rangle_{\mathcal{V}}, \tag{4.14}$$

for all $u \in \mathcal{A}$ and $v \in \mathcal{V}$, where $f_1 \in C^2(\mathcal{A};\mathcal{F})$, $f_2 \in C^2(\mathcal{A};\mathcal{V})$ and $\bar{B}_3 \in C^2(\mathcal{A};\mathcal{L}(\mathcal{V},\mathcal{V}))$, and $\bar{B}_3(u)$ is self-adjoint for all $u \in \mathcal{A}$. For each $u \in \mathcal{A}$, let $\bar{B}_3^{\#}(u) \doteq [\bar{B}_3(u)]^{\#}$ denote the Moore-Penrose pseudo-inverse of $\bar{B}_3(u)$, and there exists a constant $D > 0$ such that $|\bar{B}_3^{\#}(u)| \leq D$ for all $u \in \mathcal{A}_G$. The next lemma follows directly from (4.14) and Lemma 2.1.

LEMMA 4.1. *Let $\hat{u} \in \mathcal{A}$. Then $\hat{v} \in \mathcal{M}^1(\hat{u})$ if and only if $f_2(\hat{u}) + \bar{B}_3(\hat{u})\hat{v} = 0$.*

4.1.1 When the full staticization is known to exist

LEMMA 4.2. *Assume $(A.2f)$. Then assumption $(A.3)$ is satisfied.*

Proof. The result is trivial for $\mathcal{A} = \{\bar{u}\}$. Suppose $\mathcal{A}_G \neq \{\bar{u}\}$. Choose an $\delta > 0$ such that $\mathcal{A}_G \cap (B_\delta(\bar{u})\backslash\{\bar{u}\}) \neq \emptyset$. Let $\hat{u} \in \mathcal{A}_G \cap B_\delta(\bar{u})$, and $\hat{u} \neq \bar{u}$. Let $\hat{v} = \bar{v} - \bar{B}_3^{\#}(\hat{u})f_2(\hat{u}) - \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}$. Note that $\bar{B}_3(\hat{u})\hat{v} + f_2(\hat{u}) = 0$. By Lemma 4.1, $\hat{v} \in \mathcal{M}^1(\hat{u})$. Since f_2, $\bar{B}_3$ are C^2, $\exists$ constants $K_2 > 0$, $K_B > 0$, such that $|f_2(u_1) - f_2(u_2)| \leq K_2|u_1-u_2|$, $|\bar{B}_3(u_1) - \bar{B}_3(u_2)| \leq K_B|u_1-u_2| \;\forall\, u_1,u_2 \in B_\delta(\bar{u})$. We have

$$|\hat{v}-\bar{v}| = \big| -\{\bar{B}_3^{\#}(\hat{u})f_2(\hat{u}) + \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}\}\big|$$

By Lemma 4.1, $\bar{B}_3(\bar{u})\bar{v} + f_2(\bar{u}) = 0$,

$$\begin{aligned} &= \big| -\bar{B}_3^{\#}(\hat{u})\{f_2(\hat{u}) - f_2(\bar{u}) - \bar{B}_3(\bar{u})\bar{v} + \bar{B}_3(\hat{u})\bar{v}\}\big| \\ &\leq D\Big[K_2|\hat{u}-\bar{u}| + K_B|\bar{v}||\hat{u}-\bar{u}|\Big] \\ &\leq \bar{C}|\hat{u}-\bar{u}| \text{ for some } \bar{C} = \bar{C}(\bar{u},\bar{v}) \end{aligned}$$

which gives $(A.3)$. ■

THEOREM 4.1. *Assume $(A.2f)$. Then $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ exists, and $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$.*

Proof. $(A.3)$ is satisfied. We may apply Theorem 3.1. ■

4.1.2 When the iterated staticization is known to exist

The case where the iterated staticization is known to exist appears to require additional assumptions.

THEOREM 4.2. *Assume $(A.2i)$. Assume that $\bar{B}_3(\cdot)$ and $f_2(\cdot)$ are locally Lipschitz. Assume that $f_2(u) \in \operatorname{Range}[\bar{B}_3(u)]$ for all $u \in \mathcal{A}_G$. Then $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ exists, and*

$$\operatorname*{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v).$$

Proof. Note that by assumption and Lemma 4.1, $\mathcal{A}_G = \mathcal{A}$. Let $\bar{u} \in \hat{\mathcal{A}}_G$, and let $\bar{v}$ be as in (2.8), which implies

$$G_v(\bar{u},\bar{v}) = 0. \tag{4.15}$$

Suppose $G_u(\bar{u},\bar{v}) \neq 0$. Then there exists $\epsilon > 0$ and $u_n \in \mathcal{A}\setminus\{\bar{u}\}$ with $u_n \to \bar{u}$ such that given $\epsilon > 0$, there exists $n_\delta \in I\!N$ such that

$$|G(u_{n_\delta},\bar{v}) - G(\bar{u},\bar{v})| > \epsilon|u_n-\bar{u}| \;\; \forall\, n \geq n_\delta. \tag{4.16}$$

Let

$$(4.17)\quad v_n \doteq \bar{v} - \bar{B}_3^{\#}(u_n)[f_2(u_n) + \bar{B}_3(u_n)\bar{v}] \;\; \forall n \in I\!N.$$

Note that using Lemma 4.1,

$$\begin{aligned}&|v_n - \bar{v}|\\ &\le |\bar{B}_3^{\#}(u_n)|\,\big|f_2(u_n) + \bar{B}_3(u_n)\bar{v} - f_2(\bar{u}) - \bar{B}_3(\bar{u})\bar{v}\big|,\end{aligned}$$

which by assumption,

$$\le D\big(|f_2(u_n) - f_2(\bar{u})| + |\bar{B}_3(u_n) - \bar{B}_3(\bar{u})||\bar{v}|\big),$$

and by the local Lipschitz assumption, there exists $K < \infty$ such that this is

$$(4.18)\quad \le DK(1 + |\bar{v}|)|u_n - \bar{u}| \;\; \forall n \in I\!N.$$

Also,

$$\begin{aligned}&\bar{B}_3(u_n)v_n + f_2(u_n)\\ &= \bar{B}_3(u_n)\big[\bar{v} - \bar{B}_3^{\#}(u_n)f_2(u_n) - \bar{B}_3^{\#}(u_n)\bar{B}_3(u_n)\bar{v}\big] + f_2(u_n),\end{aligned}$$

which by assumption and the properties of the pseudo-inverse,

$$(4.19)\quad = \bar{B}_3(u_n)\bar{v} - f_2(u_n) - \bar{B}_3(u_n)\bar{v} + f_2(u_n) = 0,$$

and hence, $v_n \in \mathcal{M}_1(u_n)$ for all $n \in I\!N$.

By $(A.2i)$, there exists $\bar{n} = \bar{n}(\epsilon)$ such that for all $n \ge \bar{n}$,

$$|G(u_n, v_n) - G(\bar{u}, \bar{v})| = |\bar{G}^1(u_n) - \bar{G}^1(\bar{u})| < \frac{\epsilon}{2}|u_n - \bar{u}|,$$

which implies

$$|G(u_n, v_n) - G(u_n, \bar{v}) + G(u_n, \bar{v}) - G(\bar{u}, \bar{v})| < \frac{\epsilon}{2}|u_n - \bar{u}|,$$

and hence $\forall\, n \ge \bar{n}$

$$(4.20)\quad |G(u_n, \bar{v}) - G(\bar{u}, \bar{v})| < \frac{\epsilon}{2}|u_n - \bar{u}| + |G(u_n, v_n) - G(u_n, \bar{v})|$$

Now, by (4.14), for any $v \in \mathcal{B}$,

$$\begin{aligned}&G(u_n, v) - G(u_n, v_n)\\ &= \langle f_2(u_n) + \tfrac{1}{2}\bar{B}_3(u_n)v, v\rangle - \langle f_2(u_n) + \tfrac{1}{2}\bar{B}_3(u_n)v_n, v_n\rangle,\end{aligned}$$

which by Lemma 4.1, the choice of v_n and the self-adjointness of $\bar{B}_3(u_n)$,

$$= \langle -\bar{B}_3(u_n)v_n + \tfrac{1}{2}\bar{B}_3(u_n)v, v\rangle + \tfrac{1}{2}\langle \bar{B}_3(u_n)v_n, v_n\rangle$$

$$(4.21)\quad = \tfrac{1}{2}\langle \bar{B}_3(u_n)(v - v_n), v - v_n\rangle.$$

Applying (4.18) in (4.21), we see that there exists $K_1 < \infty$ such that $|G(u_n, \bar{v}) - G(u_n, v_n)| \le K_1|u_n - \bar{u}|^2$ for all $n \in I\!N$, and consequently, there exists $\bar{n}_1 = \bar{n}_1(\epsilon) < \infty$ such that

$$(4.22)\quad |G(u_n, \bar{v}) - G(u_n, v_n)| < \frac{\epsilon}{2}|u_n - \bar{u}| \;\; \forall n \ge \bar{n}_1.$$

By (4.20) and (4.22),

$$(4.23)\quad |G(u_n, \bar{v}) - G(\bar{u}, \bar{v})| < \epsilon|u_n - \bar{u}| \;\; \forall n \ge \bar{n} \wedge \bar{n}_1.$$

However, (4.23) contradicts (4.16), and consequently,

$$(4.24)\quad G_u(\bar{u}, \bar{v}) = 0.$$

By (4.15) and (4.24),

$$\begin{aligned}&(\bar{u}, \bar{v}) \in \operatorname*{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \text{ and}\\ &G(\bar{u}, \bar{v}) \in \operatorname*{stat^s}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v).\end{aligned}$$

Now suppose $\exists (\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \setminus \{(\bar{u}, \bar{v})\}$. This implies

$$\begin{aligned}&G_u(\hat{u}, \hat{v}) = 0,\; G_v(\hat{u}, \hat{v}) = 0,\\ (4.25)\quad &\hat{v} \in \mathcal{M}^1(\hat{u}),\; \text{ and } \bar{G}^1(\hat{u}) = G(\hat{u}, \hat{v}).\end{aligned}$$

Similar to (4.17), let

$$\breve{v}(u) \doteq \hat{v} - \bar{B}_3^{\#}(u)f_2(u) - \bar{B}_3^{\#}(u)\bar{B}_3(u)\hat{v} \;\; \forall u \in \mathcal{A},$$

and note that $\breve{v}(\hat{u}) = \hat{v}$. Also, similar to (4.19), we see that $\bar{B}_3(u)\breve{v}(u) + f_2(u) = 0$, which implies that $\breve{v}(u) \in \mathcal{M}^1(u)$ for all $u \in \mathcal{A}$. Hence, $\bar{G}^1(u) = G(u, \breve{v}(u))$ for all $u \in \mathcal{A}$. Note that

$$\begin{aligned}&|\bar{G}^1(u) - \bar{G}^1(\hat{u})| = |G(u, \breve{v}(u)) - G(\hat{u}, \hat{v})|\\ &\le |G(u, \breve{v}(u)) - G(u, \hat{v})| + |G(u, \hat{v}) - G(\hat{u}, \hat{v})|,\end{aligned}$$

and note that by (4.25), given $\epsilon > 0$, there exists $\hat{\delta}_1 = \hat{\delta}_1(\epsilon) > 0$ such that for all $|u - \hat{u}| < \hat{\delta}_1$,

$$(4.26)\quad \le \frac{\epsilon}{2}|u - \hat{u}| + |G(u, \breve{v}(u)) - G(u, \hat{v})|.$$

Also, similar to the estimate (4.22), we also find that there exists $\hat{\delta}_2 = \hat{\delta}_2(\epsilon) > 0$ such that

$$|G(u, \breve{v}(u)) - G(u, \hat{v})| < \frac{\epsilon}{2}|u - \hat{u}| \;\; \forall\, |u - \hat{u}| < \hat{\delta}_2.$$

Using this in (4.26), we see that

$$(4.27)\quad |\bar{G}^1(u) - \bar{G}^1(\hat{u})| < \epsilon|u - \hat{u}| \;\; \forall\, |u - \hat{u}| < \hat{\delta}_1 \wedge \hat{\delta}_2.$$

Hence, $\frac{d\bar{G}^1}{du}(\hat{u}) = 0$, which implies that $\hat{u} \in \hat{\mathcal{A}}_G$. Using this and $(A.2i)$, we see that $G(\hat{u}, \hat{v}) = G(\bar{u}, \bar{v})$. As $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \setminus \{(\bar{u}, \bar{v})\}$ was arbitrary, we have the desired result. ■

4.2 The Uniformly Morse Case Under condition $(A.1)$ and condition $(A.4)$ below, we will find that Assumption $(A.3')$ holds (and consequently, $(A.3)$). Hence, one may apply Theorem 3.1.

For all $(\hat{u}, \hat{v}) \in \mathcal{A} \times \mathcal{B}$, $G_{vv}(\hat{u}, \hat{v}) \in \mathcal{L}(\mathcal{V}; \mathcal{V})$ is invertible.
$(A.4)$

4.2.1 When the full staticization is known to exist

We suppose $(A.2f)$. We will find that $\operatorname{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)$ exists, and obtain the equivalence between full and iterated staticization.

LEMMA 4.3. *Assume $(A.2f)$. Then, there exists $\epsilon>0$ and $\breve{v}\in C^1(\mathcal{A}_G\cap B_\epsilon(\bar{u});\mathcal{B})$ such that $\breve{v}(\bar{u})=\bar{v}$ and $G_v(u,\breve{v}(u))=0$ for all $u\in\mathcal{A}_G\cap B_\epsilon(\bar{u})$.*

LEMMA 4.4. *Assume $(A.2f)$. Then, there exists $K<\infty$ and $\delta\in(0,\epsilon)$ such that $|\breve{v}(u)-\breve{v}(\bar{u})|=|\breve{v}(u)-\bar{v}|\le K|u-\bar{u}|$ for all $u\in\mathcal{A}_G\cap B_\delta(\bar{u})$.*

LEMMA 4.5. *Assume $(A.2f)$. Then, $\operatorname{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)$ exists.*

THEOREM 4.3. *Assume $(A.2f)$. Then, $\operatorname{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)$ exists, and $\operatorname{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)=G(\bar{u},\bar{v})=\operatorname{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}}G(u,v)$.*

4.2.2 When the iterated staticization is known to exist

We suppose $(A.2i)$. We will find that $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}G(u,v)$ exists, and obtain the equivalence between full and iterated staticization.

LEMMA 4.6. *Assume $(A.2i)$. Then, $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}G(u,v)$ exists.*

THEOREM 4.4. *Assume $(A.2i)$. Then, $\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}G(u,v)$ exists, and $\operatorname{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}}G(u,v)=G(\bar{u},\bar{v})=\operatorname{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)$.*

5 Applications to Astrodynamics and the Schrödinger Equation

5.1 TPBVPs in Astrodynamics

One may obtain fundamental solutions to TPBVPs in astrodynamics through a stationary-action based approach [8, 9, 16, 18]. We briefly recall the case of the n-body problem. In this case, the action functional with an appended terminal cost (cf. [18]) takes the form indicated in (1.1), where now $x=\big((x^1)^T,(x^2)^T,\dots(x^n)^T\big)^T$, where the bodies are indexed by $j\in\mathcal{N}\doteq\{1,2,\dots n\}$, and $\xi_\cdot$, $u_\cdot$ are similarly constructed. The kinetic-energy term is $T(u_r)\doteq\frac{1}{2}\sum_{j=1}^n m_j|u_r^j|^2$, where m_j is the mass of the j^{th} body. The additive inverse of the potential at any $x\in\mathbb{R}^{3n}$ is given by

$$\begin{aligned}
-V(x)&=\sum_{(i,j)\in\mathcal{I}^\Delta}\frac{Gm_im_j}{|x^i-x^j|}\\
&=\max_{\alpha\in(0,\infty)^{M_n}}\sum_{(i,j)\in\mathcal{I}^\Delta}\left(\tfrac{3}{2}\right)^{3/2}G\,m_im_j\left[\alpha_{i,j}-\frac{\alpha_{i,j}^3|x^i-x^j|^2}{2}\right]\\
&\doteq\max_{\alpha\in(0,\infty)^{M_n}}\tilde{V}(x,\alpha)=\operatorname*{stat}_{\alpha\in(0,\infty)^{M_n}}\tilde{V}(x,\alpha),
\end{aligned}$$

where G is the universal gravitational constant, $\mathcal{I}^\Delta\doteq\{(i,j)\in\mathcal{N}^2\,|\,j>i\}$. and $M_n\doteq\#\mathcal{I}^\Delta$. Letting $\mathcal{U}_{(0,t)}\doteq L_2\big((0,t);\mathbb{R}^{3n}\big)$ and $\mathcal{A}_{(0,t)}\doteq L_2\big((0,t);(0,\infty)^{M_n}\big)$, one finds that the problem becomes that of finding the stationary-action value function given by

$$W(t,x)=\operatorname*{stat}_{u\in\mathcal{B}}\operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}_{(0,t)}}J(t,x,u,\tilde{\alpha}),\quad\text{where}$$

(5.28)
$$J(t,x,u,\tilde{\alpha})\doteq\left\{\int_0^t T(u_r)-\tilde{V}(\xi_r,\tilde{\alpha}_r)\,dr+\phi(\xi_t)\right\},$$

where $\mathcal{B}\subseteq\mathcal{U}_{(0,t)}$. If one is able to reorder the staticization operations, the result may be decomposed as

$$W(t,x)\doteq\operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}_{(0,t)}}\tilde{W}(t,x,\tilde{\alpha}),$$

(5.29)
$$\tilde{W}(t,x,\tilde{\alpha})\doteq\operatorname*{stat}_{u\in\mathcal{U}_{(0,t)}}\left\{\int_0^t T(u_r)-\tilde{V}(\xi_r,\tilde{\alpha}_r)\,dr+\phi(\xi_t)\right\}.$$

Suppose ϕ is a quadratic form, say $\phi(x;z)\doteq\frac{1}{2}(x-z)^TP_0(x-z)+\gamma_0$. (More generally, one may take ϕ to be a staticization of quadratic forms.) Then,

$$\tilde{W}(t,x,\tilde{\alpha})=\tfrac{1}{2}(x^TP_t^{\tilde{\alpha}}x+x^TQ_t^{\tilde{\alpha}}z+z^TQ_t^{\tilde{\alpha}}x+z^TR_t^{\tilde{\alpha}}z+\gamma_t^{\tilde{\alpha}}),$$

where $P_\cdot^{\tilde{\alpha}},Q_\cdot^{\tilde{\alpha}},R_\cdot^{\tilde{\alpha}}$ may be obtained from solution of $\tilde{\alpha}$-indexed DREs, and $\gamma_t^{\tilde{\alpha}}$ is simply an integral. The means by which this may be utilized for efficient generation of fundamental solutions is indicated in the above-noted references. The key is, of course, the ability to invert the order of the staticization operators.

Noting that J is semi-quadratic in u and uniformly Morse in $\tilde{\alpha}$, one obtains the following.

THEOREM 5.1. *Let $t\in(0,\infty)$ and $x\in\mathbb{R}^{3n}$. Suppose $\operatorname{stat}_{u\in\mathcal{B}}\operatorname{stat}_{\tilde{\alpha}\in\mathcal{A}_{(0,t)}}J(t,x,u,\tilde{\alpha})$ exists, where J is given by (5.28). Then,*

$$\begin{aligned}
\operatorname*{stat}_{u\in\mathcal{B}}\operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}_{(0,t)}}J(t,x,u,\tilde{\alpha})&=\operatorname*{stat}_{(u,\tilde{\alpha})\in\mathcal{B}\times\mathcal{A}_{(0,t)}}J(t,x,u,\tilde{\alpha})\\
&=\operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}_{(0,t)}}\operatorname*{stat}_{u\in\mathcal{B}}J(t,x,u,\tilde{\alpha})=W(t,x).
\end{aligned}$$

Proof. By assumption, $(A.2i)$ is satisfied. ■

5.2 Schrödinger IVPs

We indicate that application to Schrödinger IVPs. The general outline is similar to that of the previous subsection, but where the dynamics are now stochastic and complex-valued. In order to simplify matters, in this case we consider only the problem of a single particle in a central Coulomb field. The IVP is

$$\begin{aligned}
&0=i\hbar\psi_t(s,y)+\tfrac{\hbar^2}{2m}\Delta\psi(s,y)-\psi(s,y)V(y),\quad(s,y)\in\mathcal{D},\\
&\psi(0,y)=\psi_0(y),\quad y\in\mathbb{R}^n,
\end{aligned}$$

where $m \in (0,\infty)$ denotes particle mass, initial condition ψ_0 takes values in $\mathbb{C}$, V denotes the Coulomb potential function, Δ denotes the Laplacian with respect to the space (second) variable, $\mathcal{D} \doteq (0,t) \times \mathbb{R}^n$, and subscript t will denote the derivative with respect to the time variable (the first argument of ψ here) regardless of the symbol being used for time in the argument list. We consider what is sometimes referred to as the Maslov dequantization of the solution of the Schrödinger equation (cf. [11]), which as noted above, is $S : \bar{\mathcal{D}} \to \mathbb{C}$ given by $\psi(s,y) = \exp\{\frac{i}{\hbar} S(s,y)\}$. We also extend the space from $\mathbb{R}^n$ to $\mathbb{C}^n$, and reverse the time variable. The resulting transformed problem is given by [12, 14, 15]

$$0 = S_t(s,x) + \tfrac{i\hbar}{2m}\Delta S(s,x) + H(x, S_x(s,x)), \quad (s,x) \in \mathcal{D}_{\mathbb{C}} \doteq (0,t) \times \mathbb{C}^n, \tag{5.30}$$

$$S(t,x) = \phi(x), \quad x \in \mathbb{C}^n. \tag{5.31}$$

$$\begin{aligned} H(x,p) &\doteq -\left[\tfrac{1}{2m}|p|_c^2 + V(x)\right] \\ &= \operatorname*{stat}_{u^0 \in \mathbb{C}^n} \left\{ (u^0)^T p + \tfrac{m}{2}|u^0|_c^2 - V(x) \right\}, \end{aligned}$$

where for $x \in \mathbb{C}^n$, $|x|_c^2 \doteq \sum_{j=1}^n x_j^2$. (We remark that notation $|\cdot|_c^2$ is not intended to indicate a squared norm; the range is complex.) We fix $t \in (0,\infty)$, and allow s to vary in $(0,t]$.

Under certain conditions, the solution of this dequantized form of the Schrödinger IVP has a representation in the form of the value function of staticization controlled diffusion equation [15]. In particular, we suppose the solution satisfies $|S_{xx}| \le C(1+|x|^{2q})$ for some $q \in \mathbb{N}$. We let $(\Omega, \mathcal{F}, P)$ be a probability triple, where Ω denotes a sample space, $\mathcal{F}$ denotes a σ-algebra on Ω, and P denotes a probability measure on $(\Omega,\mathcal{F})$. Let $\{\mathcal{F}_s \,|\, s \in [0,t]\}$ denote a filtration on $(\Omega,\mathcal{F},P)$, and let $B_\cdot$ denote an $\mathcal{F}_\cdot$-adapted Brownian motion taking values in $\mathbb{R}^n$. For $s \in [0,t]$, let

$$\mathcal{U}_s \doteq \{u : [s,t] \times \Omega \to \mathbb{C}^n \,|\, u \text{ is } \mathcal{F}_\cdot\text{-adapted, right-continuous and such that } \mathbb{E}\textstyle\int_s^t |u_r|^m \, dr < \infty \ \forall m \in \mathbb{N}\}.$$

We supply $\mathcal{U}_s$ with the norm $\|u\|_{\mathcal{U}_s} \doteq \max_{m \in \{1,\ldots \bar{M}\}} \left[\mathbb{E}\int_s^t |u_r|^m \, dr\right]^{1/m}$, where $\bar{M} \ge 8q$. We will be interested in diffusion processes given by

$$\begin{aligned} \xi_r = \xi_r^{(s,x)} &= x + \int_s^r u_\rho \, d\rho + \sqrt{\tfrac{\hbar}{m}} \tfrac{1+i}{\sqrt{2}} \int_s^r dB_\rho \\ &\doteq x + \int_s^r u_\rho \, d\rho + \sqrt{\tfrac{\hbar}{m}} \tfrac{1+i}{\sqrt{2}} B_r^\Delta, \end{aligned}$$

where $x \in \mathbb{C}^n$, $s \in [0,t]$, $u \in \mathcal{U}_s$, and $B_r^\Delta \doteq B_r - B_s$ for $r \in [s,t]$. For $s \in (0,t)$ and $\hbar \in (0,1]$, we define payoff $J(s,\cdot,\cdot) : \mathbb{C}^n \times \mathcal{U}_s \to \mathbb{C}$ and stationary value, $\bar{S} : \mathcal{D}_{\mathbb{C}} \to \mathbb{C}$ by

$$\begin{aligned} J(s,x,u) &\doteq \mathbb{E}\left\{ \int_s^t \tfrac{m}{2}|u_r|_c^2 - V(\xi_r)\, dr + \phi(\xi_t) \right\}, \\ S(s,x) &\doteq \operatorname*{stat}_{u \in \mathcal{U}_s} J(s,x,u) \ \ \forall (s,x) \in \mathcal{D}_{\mathbb{C}}. \end{aligned}$$

The Coulomb potential generated by a point charge in the central field takes the form of $\bar{V}(x) = -\hat{\mu}/|x|, \ \forall x \in \mathbb{R}^n$, where $\hat{\mu}$ is a constant.

Lemma 5.1. *$\bar{V}(x) = -\hat{\mu}/|x|, \ \forall x \in \mathbb{R}^n$ extended to complex domain is $V(x) \doteq -\hat{\mu}/\sqrt{|x|_c^2}, \ \forall x \in \mathbb{C}^n$, where $|x|_c^2 \doteq \sum_{j=1}^n x_j^2$ for $x \in \mathbb{C}^n$, and we have*

$$\begin{aligned} V(x) &= -(\tfrac{3}{2})^{3/2}\hat{\mu} \operatorname*{stat}_{\bar{\alpha} \in \mathcal{A}^R} \{\alpha - \tfrac{\bar{\alpha}^3 |x|_c^2}{2}\} \\ &\doteq -\bar{c} \operatorname*{stat}_{\bar{\alpha} \in \mathcal{A}^R} \{\alpha - \tfrac{\bar{\alpha}^3 |x|_c^2}{2}\} \end{aligned}$$

where $\mathcal{A}^R \doteq \{\bar{\alpha} = r[\cos(\theta) + i\sin(\theta)] \in \mathbb{C} \,|\, r \ge 0, \, \theta \in (\frac{-\pi}{2}, \frac{\pi}{2}]\}$.

Lemma 5.2. *Let $\mathcal{A} \doteq L^2(\Omega; L^2([s,t]; \mathcal{A}^R))$. Then*

$$\begin{aligned} &\operatorname*{stat}_{\alpha \in \mathcal{A}} \mathbb{E}\left\{ \int_s^t \tfrac{m}{2}|u_r|_c^2 + \bar{c}[\alpha_r - \frac{\alpha_r^3 |\xi_2|_c^2}{2}]\, dr + \phi(\xi_t) \right\} \\ &= \mathbb{E}\left\{ \int_s^t \tfrac{m}{2}|u_r|_c^2 + \bar{c} \operatorname*{stat}_{\bar{\alpha} \in \mathcal{A}^R} [\bar{\alpha} - \frac{\bar{\alpha}^3 |\xi_2|_c^2}{2}]\, dr + \phi(\xi_t) \right\}. \end{aligned}$$

The problem of solving for $S(s,x)$ then becomes that of finding the stationary-action value function given by

$$S(s,x) \doteq \operatorname*{stat}_{u \in \mathcal{U}_s} \operatorname*{stat}_{\alpha \in \mathcal{A}} \mathbb{E}\left\{ \int_s^t \tfrac{m}{2}|u_r|_c^2 + \bar{c}[\alpha_r - \frac{\alpha_r^3 |\xi_2|_c^2}{2}]\, dr + \phi(\xi_t) \right\} \quad \forall (s,x) \in \mathcal{D}_{\mathbb{C}}.$$

which may be decomposed as

$$\begin{aligned} S(t,x) &\doteq \operatorname*{stat}_{\alpha \in \mathcal{A}} \tilde{S}(t,x,\tilde{\alpha}), \\ \tilde{S}(t,x,\alpha) &\doteq \operatorname*{stat}_{u \in \mathcal{U}_s} \mathbb{E}\left\{ \int_s^t \tfrac{m}{2}|u_r|_c^2 - \tilde{V}(\xi_r, \alpha_r)\, dr + \phi(\xi_t) \right\} \end{aligned}$$

where $\tilde{V}(\xi_r, \alpha_r) \doteq -\bar{c}[\alpha_r - \frac{\alpha_r^3 |\xi_2|_c^2}{2}]$.

Again, the problem becomes that of interchanging the order of the staticization operators, where we note that the functional is semi-quadratic in u and uniformly Morse in $\tilde{\alpha}$. Once that is achieved, the functional inside the staticization over $\tilde{\alpha}$ may be solved through DREs.

Theorem 5.2. $\mathbb{E}\left\{\int_s^t \frac{m}{2}|u_r|_c^2 + \bar{c}[\alpha_r - \frac{\alpha_r^3|\xi_2|_c^2}{2}]\, dr + \phi(\xi_t)\right\}$ *is twice Fréchet differentiable in α.*

For x and u taking values in $\mathbb{R}^3$, we have the following results.

THEOREM 5.3. *$\mathbb{E}\left\{\int_s^t \frac{m}{2}|u_r|^2 dr + \int_s^t \frac{\hat{\mu}}{\sqrt{|\xi_r|^2}} dr + \phi(\xi_t)\right\}$ is differentiable with respect to u everywhere.*

This theorem follows from the following lemmas.

LEMMA 5.3. *There exists a probability measure Q such that Q and P are mutually absolutely continuous, and $d\xi_s = \sqrt{\frac{\hbar}{m}}\frac{1+i}{\sqrt{2}} d\hat{B}_s$, where $\hat{B}.$ is a $Q-$Brownian motion.*

Proof. This follows from Girsanov Theorem. ■

LEMMA 5.4. *The function*

$$f(\bar{x}) \doteq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\frac{1}{\sqrt{(2\pi)^3|\sigma|}}\exp\left(-\frac{x^2+y^2+z^2}{2\sigma}\right)}{\sqrt{(x-\bar{x})^2+y^2+z^2}} dxdydz$$

is differentiable with respect to $\bar{x}$.

Proof. After changing to a spherical coordinate centered at $\bar{x}$, the result follows from applying Leibniz rule for Lebesgue integrals. ■

LEMMA 5.5. *For each $r \in [s,t]$, $f(r,u) \doteq E\left\{\frac{1}{\sqrt{|\xi_r|^2}}\right\}$ is differentiable with respect to u.*

Proof. Without loss of generality, we consider $x = (x,0,0) \in \mathbb{R}^3$. After a change of measure as in lemma 5.3, we have $f(r,u) = E^Q\{1/\sqrt{|x + \sqrt{\frac{\hbar}{m}}\frac{1+i}{\sqrt{2}}\hat{B}_r|^2}\}$. Let $\bar{x} \doteq -x$. By lemma 5.4, the chain rule, and note that $x + \int_s^r u_\rho d\rho$ is an affine functional of u, we have $f(r,u)$ is differentiable in u. ■

Note that so far we've only proved that $\mathbb{E}\left\{\int_s^t \frac{m}{2}|u_r|_c^2 + \bar{c}[\alpha_r - \frac{\alpha_r^3|\xi_2|_c^2}{2}]\, dr + \phi(\xi_t)\right\}$ is C^1 in u. Further work is needed in order to apply the results that follow from $(A.1)$, where the functional is C^2 in both variables.

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