Diffusion Process Representations for a Scalar-Field Schrödinger Equation Solution in Rotating Coordinates

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Abstract A particular class of Schrödinger initial value problems is considered, wherein a particle moves in a scalar field centered at the origin, and more specifically, the distribution associated to the solution of the Schrödinger equation has negligible mass in the neighborhood of the origin. The Schrödinger equation is converted to the dequantized form, and a non-inertial frame centered along the trajectory of a classical particle is employed. A solution approximation as a series expansion in a small parameter is obtained through the use of complex-valued diffusion-process representations, where under a smoothness assumption, the expansion converges to the true solution. The computations required for solution up to a finite order are purely analytical.

1 Introduction

Diffusion representations have long been a useful tool in solution of second-order Hamilton-Jacobi partial differential equations (HJ PDEs), cf. [7, 10] among many others. The bulk of such results apply to real-valued HJ PDEs, that is, to HJ PDEs where the coefficients and solutions are real-valued. The Schrödinger equation is complex-valued, although generally defined over a real-valued space domain, which presents difficulties for the development of stochastic control representations. In [17, 18], a representation for the solution of a Schrödinger-equation initial value problem over a scalar field was obtained as a stationary value for a complex-valued diffusion process control problem. Although there is substantial existing work on the relation of stochastic processes to the Schrödinger equation (cf. [9, 14, 21, 22, 27]), the approach considered in [17, 18] is along a slightly different path, closer

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to [2, 3, 4, 6, 13, 16]. However, the representation in [17, 18] employs stationarity of the payoff [19] rather than optimization of the payoff, where stationarity can be used to overcome the limited-duration constraints of methods that use optimization of the payoff.

Here we discuss a particular problem class, and use diffusion representations as a tool for approximate solution of the Schrödinger equation. We will consider a specific type of weak field problem. Suppose we have a particle in a scalar field centered at the origin, but in the special case where the particle is sufficiently far from the origin that the distribution associated to the corresponding Schrödinger equation has negligible density near the origin. More specifically, let the particle mass be denoted by m, and let $\hbar$ denote Planck's constant. The simplest scalar-field example, which can be instructive if only purely academic, is the quadratic-field case, generating the quantum harmonic oscillator. Of somewhat more interest is the case where one has the potential energy generated by the field interacting with the particle taking the form $\bar{V}(x) = -\bar{c}/|x|$. Let the solution of the Schrödinger equation at time, t, and position, x, be denoted by $\psi(t,x)$, and consider the associated distribution given by $\tilde{P}(t,x) \doteq [\psi^*\psi](t,x)$. Formally speaking, when $\hbar/m$ is sufficiently small, one expects that $\tilde{P}(t,\cdot)$ can be approximated in some sense by a Dirac-delta function centered at $\xi(t)$, where $m\ddot{\xi}(t) = -\nabla_x\bar{V}(\xi(t))$. We will consider a non-inertial frame where the origin will be centered at $\xi(t)$ for all t. In particular, we consider a case where $\xi(t)$ follows a circular orbit with constant angular velocity. That is, we consider $\xi(t) = \hat{\delta}\big(\cos(\omega t), \sin(\omega t)\big)$ where $\hat{\delta} \in (0,\infty)$. (In the interests of space and reduction of clutter, where it will not lead to confusion, we will often write (x_1,x_2) in place of $(x_1,x_2)^T$, etc.) Although such motion can be generated by a two-dimensional harmonic oscillator, we will focus mainly on the $\bar{V}(x) = -\bar{c}/|x|$ class, in which case $\omega \doteq [\bar{c}/(m\hat{\delta}^3)]^{1/2}$. We suppose that $\hat{\delta}$ is sufficiently large such that $\tilde{P}(t,x) \ll 1$ for $|x| < \hat{\delta}/2$, and thus that one may approximate $\bar{V}$ in the vicinity of $\xi(t)$ by a finite number of terms in a power series expansion centered at $\xi(t)$. We will use a set of complex-valued diffusion representations to obtain an approximation to the resulting Schrödinger equation solution. If the solution is holomorphic in x and a small parameter, then the approximate solution converges as the number of terms in the set of diffusion representations approaches infinity.

The analysis will be carried out only in the case of a holomorphic field approximation. As our motivation is the case where $\hat{\delta}$ is large relative to the associated position distribution, one expects that the case of a $-\bar{c}/|x|$ potential may be sufficiently well-modeled by a finite number of terms in a power series expansion. However, an analysis of the errors induced by such an approximation to a $-\bar{c}/|x|$ potential is beyond the scope of this already long paper, and may be addressed in a later effort; the focus here is restricted to the diffusion-representation based method of solution approximation method *given* such an approximation to the potential. We remark that in the case of a quadratic potential, we recover the quantum harmonic oscillator solution. Also, in the case of $\bar{V}(x) = -\bar{c}/|x|$, as $\hat{\delta} \to \infty$, the solution approaches that of the free particle case. The computations required for solution up to any finite polynomial-in-space order may be performed analytically.

In Section 2, we review the Schrödinger initial value problem, and the dequantized form of the problem. The solution to the dequantized form of the problem will be approximated through the use of diffusion representations; the solution to the originating Schrödinger initial value problem is recovered by a simple transformation. As it is used in Section 2, we briefly recall the stat operator in Section 3.1. In Section 3.2, the dequantized form will be converted into a form over a rotating and translating reference frame centered at the position of a classical particle following a circular trajectory generated by the central field. Then, in Section 3.3, we discuss equivalent forms over a complex space domain, and over a double-dimension real-valued domain. Classical existence, uniqueness and smoothness results will be applied to the problem in this last form. These will then be transferred to the original form as a complex-valued solution over a real space domain. In Section 4, we indicate the expansion of the solution in a small parameter related to the inverse of the distance to the origin of the field. A power series representation will be used, where this will be over both space and the small parameter. In particular, we will assume that at each time, the solution will be holomorphic over space and the small parameter. The functions in the expansion are solutions to corresponding HJ PDEs, where these are also indicated here. The HJ PDE for the first term, say $k=0$, has a closed-form solution, and this is given in Section 5. Then, in Section 6, it is shown that for $k\geq 1$, given the solutions to the preceding terms, the HJ PDE for the k^{th} term takes a linear parabolic form, with a corresponding diffusion representation. It is shown that diffusion representation may be used to obtain the solution of the $k+1$ HJ PDE given the solutions to the k-and-lower HJ PDE solutions. The required computations may be performed analytically. In Section 7, this method is applied to obtain the next term in the expansion in the case of a cubic approximation of the classic $1/r$ type of potential, and additional terms may be obtained similarly.

2 Dequantization

We recall the Schrödinger initial value problem, given as

$$0 = i\hbar\psi_t(s,x) + \frac{\hbar^2}{2m}\Delta_x\psi(s,x) - \psi(s,x)\bar{V}(x), \quad (s,x)\in\mathscr{D}, \tag{1}$$

$$\psi(0,x) = \psi_0(x), \quad x\in\mathbb{R}^n, \tag{2}$$

where initial condition ψ_0 takes values in $\mathbb{C}$, Δ_x denotes the Laplacian with respect to the space (second) variable, $\mathscr{D} \doteq (0,t)\times\mathbb{R}^n$, and subscript t will denote the derivative with respect to the time variable (the first argument of ψ here) regardless of the symbol being used for time in the argument list. We also let $\overline{\mathscr{D}} \doteq [0,t)\times\mathbb{R}^n$. We consider the Maslov dequantization of the solution of the Schrödinger equation (cf. [15]), which similar to a standard log transform, is $S:\overline{\mathscr{D}}\to\mathbb{C}$ given by $\psi(s,x) = \exp\{\frac{i}{\hbar}S(s,x)\}$. Note that $\psi_t = \frac{i}{\hbar}\psi S_t$, $\psi_x = \frac{i}{\hbar}\psi S_x$ and $\Delta_x\psi = \frac{i}{\hbar}\psi\Delta_x S - \frac{1}{\hbar^2}\psi|S_x|_c^2$ where for $y\in\mathbb{C}^n$, $|y|_c^2 \doteq \sum_{j=1}^n y_j^2$. (We remark that notation $|\cdot|_c^2$ is not intended to indicate a squared norm; the range is complex.) We find that (1)–(2) become

$$0 = -S_t(s,x) + \tfrac{i\hbar}{2m}\Delta_x S(s,x) + H^0(x, S_x(s,x)), \quad (s,x)\in\mathscr{D}, \tag{3}$$
$$S(0,x) = \bar{\phi}(x), \quad x\in\mathbb{R}^n, \tag{4}$$

where $H:\mathbb{R}^n\times\mathbb{C}^n\to\mathbb{C}$ is the Hamiltonian given by

$$H^0(x,p) = -\left[\tfrac{1}{2m}|p|_c^2 + \bar{V}(x)\right] = \operatorname*{stat}_{v\in\mathbb{C}^n}\left\{v\cdot p + \tfrac{m}{2}|v|_c^2 - \bar{V}(x)\right\}, \tag{5}$$

and stat is defined in Section 3.1. We look for solutions in the space

$$\mathscr{S} \doteq \{S:\overline{\mathscr{D}}\to\mathbb{C}\,|\,S\in C_p^{1,2}(\mathscr{D})\cap C(\overline{\mathscr{D}})\}, \tag{6}$$

where $C_p^{1,2}$ denotes the space of functions which are continuously differentiable once in time and twice in space, and which satisfy a polynomial-growth bound.

3 Preliminaries

In this section, we collect condensed discussions of relevant classical material as well as some recently obtained results and definitions.

3.1 Stationarity definitions

Recall that classical systems obey the stationary action principle, where the path taken by the system is that which is a stationary point of the action functional. For this and other reasons, as in the definition of the Hamiltonian given in (5), we find it useful to develop additional notation and nomenclature. Specifically, we will refer to the search for stationary points more succinctly as *staticization*, and we make the following definitions. Suppose $(\mathscr{Y},|\cdot|)$ is a generic normed vector space over $\mathbb{C}$ with $\mathscr{G}\subseteq\mathscr{Y}$, and suppose $F:\mathscr{G}\to\mathbb{C}$. We say $\bar{y}\in\operatorname{argstat}\{F(y)\,|\,y\in\mathscr{G}\}$ if $\bar{y}\in\mathscr{G}$ and either $\limsup_{y\to\bar{y},y\in\mathscr{G}\setminus\{\bar{y}\}}|F(y)-F(\bar{y})|/|y-\bar{y}|=0$, or there exists $\delta>0$ such that $\mathscr{G}\cap B_\delta(\bar{y})=\{\bar{y}\}$ (where $B_\delta(\bar{y})$ denotes the ball of radius δ around $\bar{y}$). If $\operatorname{argstat}\{F(y)\,|\,y\in\mathscr{G}\}\neq\emptyset$, we define the possibly set-valued stat^s operator by

$$\operatorname*{stat}^s_{y\in\mathscr{G}} F(y) \doteq \operatorname{stat}^s\{F(y)\,|\,y\in\mathscr{G}\} \doteq \left\{F(\bar{y})\,\middle|\,\bar{y}\in\operatorname{argstat}\{F(y)\,|\,y\in\mathscr{G}\}\right\}.$$

If $\operatorname{argstat}\{F(y)\,|\,y\in\mathscr{G}\}=\emptyset$, $\operatorname{stat}^s_{y\in\mathscr{G}}F(y)$ is undefined. We will also be interested in a single-valued stat operation. In particular, if there exists $a\in\mathbb{C}$ such that $\operatorname{stat}^s_{y\in\mathscr{G}}F(y)=\{a\}$, then $\operatorname{stat}_{y\in\mathscr{G}}F(y)\doteq a$; otherwise, $\operatorname{stat}_{y\in\mathscr{G}}F(y)$ is undefined. At times, we may abuse notation by writing $\bar{y}=\operatorname{argstat}\{F(y)\,|\,y\in\mathscr{G}\}$ in the event that the argstat is the set $\{\bar{y}\}$. For further discussion, we refer the reader to [19]. The following is immediate from the above definitions.

Lemma 1. *Suppose $\mathscr{Y}$ is a Hilbert space, with open set $\mathscr{G} \subseteq \mathscr{Y}$, and that $F : \mathscr{G} \to \mathbb{C}$ is Fréchet differentiable at $\bar{y} \in \mathscr{G}$ with Riesz representation $F_y(\bar{y}) \in \mathscr{Y}$ Then, $\bar{y} \in$* $\operatorname{argstat}\{F(y) \,|\, y \in \mathscr{G}\}$ *if and only if $F_y(\bar{y}) = 0$.*

3.2 The non-inertial frame

As noted in the introduction, we suppose a central scalar field such that a particular solution for the motion of a classical particle in the field takes the form $\xi(t) = \hat{\delta}\big(\cos(\omega t), \sin(\omega t)\big)$ where $\hat{\delta}, \omega \in (0,\infty)$. In particular, we concentrate on the potential $\bar{V}(x) = -\bar{c}/|x|$, in which case $\omega \doteq [\bar{c}/(m\hat{\delta}^3)]^{1/2}$. We consider a two-dimensional space model and a non-inertial frame centered at $\xi(t)$ for all $t \in (0,\infty)$, with the first basis axis in the positive radial direction and the second basis vector in the direction of the velocity of the particle. Let positions in the non-inertial frame be denoted by $z \in \mathbb{R}^2$, where the transformation between frames at time $t \in \mathbb{R}$ is given by

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = G_{\omega t} x - \begin{pmatrix} \hat{\delta} \\ 0 \end{pmatrix} \doteq \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \hat{\delta} \\ 0 \end{pmatrix}. \tag{7}$$

We will denote this transformation as $z = z^*(x)$, with its inverse denoted similarly as $x = x^*(z)$, where $x^*(z) = (G_{\omega t})^T (z + (\hat{\delta}, 0)^T)$.

For $z \in \mathbb{R}^2$, define $V(z) \doteq \bar{V}(x^*(z))$ and $\phi(z) \doteq \bar{\phi}(x^*(z))$. Then, $\tilde{S}^f : \overline{\mathscr{D}} \to \mathbb{C}$ defined by $\tilde{S}^f(s,z) \doteq \hat{\hat{S}}^f(s, x^*(z))$ is a solution of the forward-time dequantized HJ PDE problem given by

$$0 = -S_t(s,z) + \tfrac{i\hbar}{2m}\Delta_z S(s,z) - (A_0 z + b_0)^T S_z(s,z) - \tfrac{1}{2m}|S_z(s,z)|_c^2 - V(z), \;\; (s,z) \in \mathscr{D}, \tag{8}$$

$$S(0,z) = \phi(z), \quad z \in \mathbb{R}^2, \quad \text{where} \quad A_0 \doteq \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } b_0 \doteq -\omega\hat{\delta}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{9}$$

if and only is $\hat{\hat{S}}^f$ is a solution of (3)–(4). (We remark that one may see [25] for further discussion of non-inertial frames in the context of the Schrödinger equation.) In order to apply the diffusion representations as an aid in solution, we will find it helpful to reverse the time variable, and hence we look instead, and equivalently, at the Hamilton-Jacobi partial differential equation (HJ PDE) problem given by

$$0 = S_t(s,z) + \tfrac{i\hbar}{2m}\Delta_z S(s,x) - (A_0 z + b_0)^T S_z(s,z) - \tfrac{1}{2m}|S_z(s,z)|_c^2 - V(z), \quad (s,z) \in \mathscr{D}, \tag{10}$$

$$S(t,z) = \phi(z), \quad z \in \mathbb{R}^n. \tag{11}$$

In this last form, we will fix $t \in (0,\infty)$, and allow s to vary in $(0,t]$.

3.3 Extensions to the complex domain

Various details of extensions to the complex domain must be considered prior to the development of the representation. This material is rather standard, but it is required for the main development. Models (1)–(2), (3)–(4) and (10)–(11) are typically given as HJ PDE problems over real space domains. However, as in Doss et al. [1, 2, 3], we will find it convenient to change the domain to one where the space components lie over the complex field. We also extend the domain of the potential to $\mathbb{C}^2$, i.e., $V:\mathbb{C}^2\to\mathbb{C}$, and we will abuse notation by employing the same symbol for the extended-domain functions. Throughout, for $k\in\mathbb{N}$, and $z\in\mathbb{C}^k$ or $z\in\mathbb{R}^k$, we let $|z|$ denote the Euclidean norm. Let $\mathscr{D}_\mathbb{C}\doteq(0,t)\times\mathbb{C}^2$ and $\overline{\mathscr{D}}_\mathbb{C}=(0,t]\times\mathbb{C}^2$, and define

$$\begin{aligned}\mathscr{S}_\mathbb{C}\doteq\{S:\overline{\mathscr{D}}_\mathbb{C}\to\mathbb{C}\,|\,&S \text{ is continuous on } \overline{\mathscr{D}}_\mathbb{C}, \text{ continuously differentiable in time on}\\ &\mathscr{D}_\mathbb{C}, \text{ and holomorphic on } \mathbb{C}^2 \text{ for all } r\in(0,t]\,\}, \qquad (12)\\ \mathscr{S}^p_\mathbb{C}\doteq\{S\in\mathscr{S}_\mathbb{C}\,|\,&S \text{ satisfies a polynomial growth condition in space,}\\ &\text{uniformly on } (0,t]\,\}. \qquad (13)\end{aligned}$$

The extended-domain form of problem (10)–(11) is

$$0=S_t(s,z)+\tfrac{i\hbar}{2m}\Delta_z S(s,z)-(A_0z+b_0)^T S_z(s,z)-\tfrac{1}{2m}|S_z(s,z)|_c^2-V(z),\;\;(s,z)\in\mathscr{D}_\mathbb{C}, \qquad (14)$$

$$S(t,z)=\phi(z),\quad z\in\mathbb{C}^2. \qquad (15)$$

Remark 1. We remark that a holomorphic function on $\mathbb{C}^2$ is uniquely defined by its values on the real part of its domain. In particular, $\tilde{S}:\overline{\mathscr{D}}\to\mathbb{C}$ uniquely defines its extension to a time-indexed holomorphic function over complex space, say $\bar{S}:\overline{\mathscr{D}}_\mathbb{C}\to\mathbb{C}$, if the latter exists. Consequently, although (10)–(11) form an HJ PDE problem for a complex-valued solution over real time and real space, (14)–(15) is an equivalent formulation, under the assumptions that a holomorphic solution exists and one has uniqueness for both.

Throughout the remainder, we will assume the following.

$$V,\phi:\mathbb{C}^2\to\mathbb{C} \text{ are holomorphic on } \mathbb{C}^2. \qquad (A.1)$$

Remark 2. The assumption on V requires a remark. Recall that we are interested here in a class of problems where $\hat{\delta}$ is large in the sense that the distribution associated to the solution of the Schrödinger initial value problem has only very small probability mass outside a ball of radius less than $\hat{\delta}$. If $\bar{V}$ is of the $\bar{c}/|x|$ form, one would use only a finite number of terms in the power series expansion around $z=0$. The focus here is on a diffusion-representation based method for approximate solution of the Schrödinger initial value problem given a holomorphic potential. The errors introduced by the use of a truncated power series for a $\bar{c}/|x|$-type potential for large $\hat{\delta}$ are outside the scope of the discussion.

We will refer to a linear space over the complex [real] field as a complex [real] space. Although (14)–(15) form an HJ PDE problem for a complex-valued solution over real time and complex space, there is an equivalent formulation as a real-valued solution over real time and a double-dimension real space. We will find such formulations to be helpful in the analysis to follow. Further, although it is natural to work with complex-valued state processes in this problem domain, in order to easily apply many of the existing results regarding existence, uniqueness and moments, we will also find it handy to use a "vectorized" real-valued representation for the complex-valued state processes. We begin from the standard mapping of the complex plane into $\mathbb{R}^2$, denoted here by $\mathscr{V}_{00}:\mathbb{C}\to\mathbb{R}^2$, with $\mathscr{V}_{00}(z)\doteq(x,y)^T$, where $x=\mathbf{Re}(z)$ and $y=\mathbf{Im}(z)$. This immediately yields the mapping $\mathscr{V}_0:\mathbb{C}^2\to\mathbb{R}^{2n}$ given by $\mathscr{V}_0(x+iy)\doteq(x^T,y^T)^T$, where component-wise, $(x_j,y_j)^T=\mathscr{V}_{00}(x_j)$ for all $j\in]1,n[$, where throughout, for integer $a\le b$, we define $]a,b[=\{a,a+1,\dots b\}$. Also in the interests of a reduction of cumbersome notation, we will henceforth frequently abuse notation by writing (x,y) in place of $(x^T,y^T)^T$ when the meaning is clear. Lastly, we may decompose any function in $\mathscr{S}_{\mathbb{C}}$, say $F\in\mathscr{S}_{\mathbb{C}}$, as

$$(\bar{R}(r,\mathscr{V}_0(z)),\bar{T}(r,\mathscr{V}_0(z)))^T\doteq\mathscr{V}_{00}(F(r,z)), \tag{16}$$

where $\bar{R},\bar{T}:\overline{\mathscr{D}_2}\doteq(0,t]\times\mathbb{R}^{2n}\to\mathbb{R}$, and we also let $\mathscr{D}_2\doteq(0,t)\times\mathbb{R}^{2n}$. For later reference, it will be helpful to recall some standard relations between derivative components, which are induced by the Cauchy-Riemann equations. For all $(r,z)=(r,x+iy)\in(0,t)\times\mathbb{C}^2$ and all $j,k,\ell\in]1,n[$, and suppressing the arguments for reasons of space we have

$$\mathbf{Re}[F_{z_j,z_k}]=\bar{R}_{x_j,x_k}=-\bar{R}_{y_j,y_k}=\bar{T}_{y_j,x_k}=\bar{T}_{x_j,y_k}, \tag{17}$$

$$\mathbf{Im}[F_{z_j,z_k}]=-\bar{R}_{x_j,y_k}=-\bar{R}_{y_j,x_k}=-\bar{T}_{y_j,y_k}=\bar{T}_{x_j,x_k}. \tag{18}$$

4 An expansion

We now reduce our problem class to the two-dimensional space case (i.e., $n=2$). We will expand the desired solutions of our problems, and use these expansions as a means for approximation of the solution. First, we consider holomorphic V in the form of a finite or infinite power series. In the simple example case where $\bar{V}$ generates the quantum harmonic oscillator, one may take $\bar{V}(x)=\frac{\bar{c}^q}{2}[x_1^2+x_2^2]$, in which case

$$V(z)=\frac{\bar{c}^q}{2}\hat{\delta}^2+\bar{c}^q\hat{\delta}z_1+\frac{\bar{c}^q}{2}[z_1^2+z_2^2].$$

The scalar field of most interest takes the form $-\bar{V}(x)=\bar{c}/|x|$, yielding $-V(z)=\bar{c}/|z+(\hat{\delta},0)|$. In this case, recalling that this effort focuses on the case where $\hat{\delta}$ is large relative to the radius of the "non-negligible" portion of the probability distribution associated to the solution, we consider only a truncated power series, and let $\check{V}^K(z)$ denote the partial sum containing only terms up to order $K+2<\infty$ in z. We will be interested in the dependence of the potential and the resulting solutions in the parameter $\hat{\varepsilon} \doteq 1/\hat{\delta}$. We also recall from Section 3.2 that $\omega \doteq [\bar{c}/(m\hat{\delta}^3)]^{1/2}$, or $\bar{c} = m\omega^2\hat{\delta}^3$. We explicitly indicate the expansion up to the fourth-order term in z and the form of higher-order terms. One finds,

$$
\begin{aligned}
&-\breve{V}^2(z) = -\sum_{k=0}^{2} \hat{\varepsilon}^k \hat{V}^k(z), && (19)\\
&-\hat{V}^0(z) = m\omega^2\big[\hat{\delta}^2 - \hat{\delta} z_1 + (z_1^2 - z_2^2/2)\big],\\
&-\hat{V}^1(z) = m\omega^2\big[-z_1^3 + 3z_1 z_2^2/2\big], \quad -\hat{V}^2(z) = m\omega^2\big[z_1^4 - 3z_1^2 z_2^2 + 3z_2^4/8\big],
\end{aligned}
$$

and more generally, for $k > 1$, $-\hat{V}^k(z) = m\omega^2\Big[\sum_{j=0}^{k+2} c^V_{k+2,j} z_1^j z_2^{k-j}\Big]$, for proper choice of coefficients $c^V_{k,j}$.

Here, we find it helpful to explicitly consider the dependence of $\tilde{S}$ and $\bar{S}$ (solutions of (10)–(11) and (14)–(15), respectively) on $\hat{\varepsilon}$, where for convenience of exposition, we also allow $\hat{\varepsilon}$ to take complex values. Abusing notation, we let $\tilde{S} : \overline{\mathscr{D}} \times \mathbb{C} \to \mathbb{C}$ and $\bar{S} : \overline{\mathscr{D}}_{\mathbb{C}} \times \mathbb{C} \to \mathbb{C}$, and denote the dependence on their arguments as $\tilde{S}(s,z,\hat{\varepsilon})$ and $\bar{S}(s,z,\hat{\varepsilon})$. We let $\breve{\mathscr{D}} \doteq \mathscr{D} \times \mathbb{C}$, $\bar{\breve{\mathscr{D}}} \doteq \overline{\mathscr{D}} \times \mathbb{C}$, $\breve{\mathscr{D}}_{\mathbb{C}} \doteq \mathscr{D}_{\mathbb{C}} \times \mathbb{C}$ and $\bar{\breve{\mathscr{D}}}_{\mathbb{C}} \doteq \overline{\mathscr{D}}_{\mathbb{C}} \times \mathbb{C}$, where we recall that the physical-space components are now restricted to the two-dimensional case. We also let

$$
\begin{aligned}
\breve{\mathscr{S}}_{\mathbb{C}} \doteq \{S : \bar{\breve{\mathscr{D}}}_{\mathbb{C}} \to \mathbb{C} \,|\, &S \text{ is continuous on } \bar{\breve{\mathscr{D}}}_{\mathbb{C}}, \text{ continuously differentiable in time on}\\
&\breve{\mathscr{D}}_{\mathbb{C}}, \text{ and } S(r,\cdot,\cdot) \text{ is holomorphic on } \mathbb{C}^2 \times \mathbb{C} \text{ for all } r \in (0,t]\,\}, \quad (20)\\
\breve{\mathscr{S}}^p_{\mathbb{C}} \doteq \{S \in \breve{\mathscr{S}}_{\mathbb{C}} \,|\, &S \text{ satisfies a polynomial growth condition in space,}\\
&\text{uniformly on } (0,t]\,\}. \quad (21)
\end{aligned}
$$

We will make the following assumption throughout the sequel.

$$
\text{There exists a unique solution, } \bar{S} \in \breve{\mathscr{S}}_{\mathbb{C}}, \text{ to (14)–(15).} \qquad (A.2)
$$

We also let the power series expansion for ϕ be arranged as

$$
\phi(z) = \sum_{k=0}^{\infty} \hat{\varepsilon}^k \phi^k(z) \doteq \phi^0(z) + \sum_{k=1}^{\infty} \hat{\varepsilon}^k \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} b^{\phi}_{k+2,l,j} z_1^j z_2^{l-j}, \qquad (22)
$$

where $\phi^0(z)$ is quadratic in z. We consider the following terminal value problems. The zeroth-order problem is

$$
\begin{aligned}
&0 = S^0_t + \tfrac{i\hbar}{2m}\Delta_z S^0 - \big(A_0 z + b_0\big)^T S^0_z - \tfrac{1}{2m}|S^0_z|^2_c - \hat{V}^0, \qquad (s,z) \in \mathscr{D}_{\mathbb{C}}, && (23)\\
&S^0(t,z) = \phi^0(z), \qquad z \in \mathbb{C}^2. && (24)
\end{aligned}
$$

For $k \geq 1$, the k^{th} terminal value problem is

$$0 = S_t^k + \frac{i\hbar}{2m}\Delta_z S^k - \left(A_0 z + b_0 + \tfrac{1}{m}S_z^0\right)^T S_z^k - \frac{1}{2m}\sum_{\kappa=1}^{k-1}\left(S_z^\kappa\right)^T S_z^{k-\kappa} - \hat{V}^k, \quad (s,z)\in\mathscr{D}_{\mathbb{C}}, \tag{25}$$

$$S^k(t,z) = \phi^k(z), \qquad z\in\mathbb{C}^2. \tag{26}$$

Note that for $k\geq 1$, given the $\hat{S}^\kappa$ for $\kappa<k$, (25) is a linear, parabolic, second-order PDE, while zeroth-order case (23) is a nonlinear, parabolic, second-order PDE. Also note that (23) is (25) in the case of $k=0$, but as its form is different, it is worth breaking it out separately. It is also worth noting here that if the S^k are all polynomial in z of order up to k, then the right-hand side of (25) is polynomial in z of order up to k, as is the right-hand side of (26).

Theorem 1. *Assume there exists a unique solution, $\hat{S}^0$, in $\breve{\mathscr{S}}_{\mathbb{C}}$ to (23)–(24), and that for each $k\geq 1$, there exists a unique solution, $\hat{S}^k$, in $\breve{\mathscr{S}}_{\mathbb{C}}$ to (25)–(26). Then, $\bar{S} = \sum_{k=0}^\infty \hat{\varepsilon}^k\hat{S}^k$.*

Remark 3. It is worth noting here that if the S^k are all polynomial in z of order up to $k+2$, then for each k, the right-hand side of (25) is polynomial in z of order up to $k+2$, as is the right-hand side of (26). That is, with the expansion in powers of $\hat{\varepsilon} = \hat{\delta}^{-1}$, the resulting constituent HJ PDE problems indexed by k are such that one might hope for polynomial-in-z solutions of order $k+2$, and this hope will be realized further below.

Proof. Let $\bar{\mathbb{N}} \doteq \mathbb{N}\cup\{0\}$. By Assumption $(A.2)$, $\bar{S}$ has a unique power series expansion on $\bar{\bar{\mathscr{D}}}_{\mathbb{C}}$, which we denote by

$$\bar{S}(s,z,\hat{\varepsilon}) = \sum_{k=0}^\infty \hat{\varepsilon}^k\tilde{c}^k(s,z) \doteq \sum_{k=0}^\infty \hat{\varepsilon}^k\sum_{l=0}^\infty\sum_{j=0}^\infty \tilde{\tilde{c}}_{k,l,j}(s) z_1^j z_2^{l-j},$$

where the $\tilde{\tilde{c}}_{k,l,j}(\cdot):(0,t]\to\mathbb{C}$ form a time-indexed set of coefficients, and obviously, the $\tilde{c}^k(\cdot,\cdot):\overline{\mathscr{D}}_{\mathbb{C}}\to\mathbb{C}$ are given by $\tilde{c}^k(s,z) = \sum_{l=0}^\infty\sum_{j=0}^\infty \tilde{\tilde{c}}_{k,l,j}(s)z_1^j z_2^{l-j}$ for all $k\in\bar{\mathbb{N}}$. For all $k\in\bar{\mathbb{N}}$, define the notation $\tilde{c}^{-k}(\cdot,\cdot)\doteq\sum_{j=k+1}^\infty \hat{\varepsilon}^{j-(k+1)}\tilde{c}^j(\cdot,\cdot)$. Also define $V^{-k}\doteq\hat{\varepsilon}^{-(k+1)}\left[V-\sum_{j=0}^k\hat{\varepsilon}^j\hat{V}^j\right]$ and $\phi^{-k}\doteq\sum_{j=k+1}^\infty\hat{\varepsilon}^{j-(k+1)}\phi^j = \hat{\varepsilon}^{-(k+1)}\left[\phi - \sum_{j=0}^k\hat{\varepsilon}^j\phi^j\right]$ for all $k\in\bar{\mathbb{N}}$. Recall that $\bar{S}$ is the unique solution in $\breve{\mathscr{S}}_{\mathbb{C}}$ of (14)–(15). By (15),

$$\tilde{c}^k(t,z) = \phi^k(z) \text{ and } \tilde{c}^{-k}(t,z) = \phi^{-k}(z) \quad \forall z\in\mathbb{C}^2. \tag{27}$$

Separating the $\tilde{c}^0$ and $\tilde{c}^{-0}$ components of $\bar{S}$ in (14) yields

$$\begin{aligned} 0 =& \tilde{c}_t^0 + \tfrac{i\hbar}{2m}\Delta_z\tilde{c}^0 - (A_0z+b_0)^T\tilde{c}_z^0 - \tfrac{1}{2m}|\tilde{c}_z^0|_c^2 - \hat{V}^0 \\ &+ \hat{\varepsilon}\left\{\tilde{c}_t^{-0} + \tfrac{i\hbar}{2m}\Delta_z\tilde{c}^{-0} - (A_0z+b_0+\tfrac{1}{m}\tilde{c}_z^0)^T\tilde{c}_z^{-0} - \tfrac{\hat{\varepsilon}}{2m}|\tilde{c}_z^{-0}|_c^2 - V^{-0}\right\}. \end{aligned} \tag{28}$$

Now, note that as $\bar{S}(s,\cdot,\cdot)$ is holomorphic for all $s\in(0,t]$, we have $\bar{S}_z(s,\cdot,\cdot)$ and $\Delta_z\bar{S}(s,\cdot,\cdot)$ holomorphic for all $s\in(0,t]$. Further, by standard results on the composition of holomorphic mappings, noting that $g:\mathbb{C}^2\to\mathbb{C}$ given by $g(z)\doteq|z|_c^2 = z^Tz$ is holomorphic, we see that $|\tilde{S}_z(s,\cdot,\cdot)|_c^2 = g\big(\tilde{S}_z(s,\cdot,\cdot)\big)$ is holomorphic for all $s \in (0,t]$. Combining these insights, we see that with $S = \bar{S}$ all terms on the right-hand side of (14), with the exception of S_t are holomorphic in $(z,\hat{\varepsilon})$, which implies that $\bar{S}_t(s,\cdot,\cdot)$ is holomorphic for all $s \in (0,t]$. Consequently, for any $s \in (0,t]$, the right-hand side of (14) with $S = \bar{S}$ has a unique power series expansion. This implies that, as (28) is satisfied for all $\hat{\varepsilon} \in \mathbb{C}$, we must have

$$0 = \tilde{c}_t^0 + \tfrac{i\hbar}{2m}\Delta_z \tilde{c}^0 - (A_0 z + b_0)^T \tilde{c}_z^0 - \tfrac{1}{2m}|\tilde{c}_z^0|_c^2 - \hat{V}^0, \tag{29}$$
$$0 = \tilde{c}_t^{-0} + \tfrac{i\hbar}{2m}\Delta_z \tilde{c}^{-0} - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T \tilde{c}_z^{-0} - \tfrac{\hat{\varepsilon}}{2m}|\tilde{c}_z^{-0}|_c^2 - V^{-0}. \tag{30}$$

By (27), (29) and the assumptions, $\tilde{c}^0 = \hat{S}^0$.

Next, separating the $\tilde{c}^1$ and $\tilde{c}^{-1}$ components, (30) implies

$$\begin{aligned} 0 =& \tilde{c}_t^1 + \tfrac{i\hbar}{2m}\Delta_z \tilde{c}^1 - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T \tilde{c}_z^1 - \hat{V}^1 \\ &+ \hat{\varepsilon}\Big\{\tilde{c}_t^{-1} + \tfrac{i\hbar}{2m}\Delta_z \tilde{c}^{-1} - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T \tilde{c}_z^{-1} - V^{-1} - \tfrac{\hat{\varepsilon}}{2m}|\tilde{c}_z^{-1}|_c^2\Big\}. \end{aligned} \tag{31}$$

Similar to the $k = 0$ case, as (31) is satisfied for all $\hat{\varepsilon} \in \mathbb{C}$, we have

$$0 = \tilde{c}_t^1 + \tfrac{i\hbar}{2m}\Delta_z \tilde{c}^1 - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T \tilde{c}_z^1 - \hat{V}^1, \tag{32}$$
$$0 = \tilde{c}_t^{-1} + \tfrac{i\hbar}{2m}\Delta_z \tilde{c}^{-1} - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T \tilde{c}_z^{-1} - V^{-1} - \tfrac{1}{2m}\sum_{\kappa=1}^{0}\big(\tilde{c}_z^\kappa\big)^T \tilde{c}_z^{k-\kappa} - \tfrac{\hat{\varepsilon}}{2m}|\tilde{c}_z^{-1}|_c^2, \tag{33}$$

where the zero-valued penultimate term on the right-hand side of (33) is included because analogous terms will appear with non-zero value in higher-order expansion equations. By (27), (32) and the assumptions, $\tilde{c}^1 = \hat{S}^1$. Proceeding inductively, one finds $\tilde{c}^k = \hat{S}^k$ for all $k \in \bar{\mathbb{N}}$, which yields the assertion. □

4.1 An alternate assumption

It may be worth noting the following reformulation and assumption. Let $\tilde{g}_{\hat{\delta}} : \mathbb{C}^2 \to \mathbb{C}^2$ and $\hat{g}_{\hat{\delta}} : \mathbb{R} \to \mathbb{R}$ be given by $\tilde{g}_{\hat{\delta}}(z) \doteq (1/\hat{\delta})z$ and $\hat{g}_{\hat{\delta}}(s) \doteq s/\hat{\delta}^2$. Let $\tilde{s} \doteq \hat{g}_{\hat{\delta}}(s) = s/\hat{\delta}^2$ and $\tilde{z} \doteq \tilde{g}_{\hat{\delta}}(z) = (1/\hat{\delta})z$. Note that under this change of variables, the angular rate becomes $\hat{\omega} = \frac{d\theta}{d\tilde{s}} = \frac{d\theta}{ds}\frac{ds}{d\tilde{s}} = \hat{\delta}^2\omega$, and where the units of $\hbar$ are such that the resulting scaling is the identity. Let $\tilde{\bar{S}}(\tilde{s},\tilde{z}) \doteq \bar{S}(\hat{g}_{\hat{\delta}}^{-1}(\tilde{s}), \tilde{g}_{\hat{\delta}}^{-1}(\tilde{z})) = \bar{S}(\hat{g}_{\hat{\delta}}^{-1}(\tilde{s}), \tilde{g}_{\hat{\delta}}^{-1}(\tilde{z}), \hat{\varepsilon})$ for all $(s,z) \in \overline{\mathscr{D}}$, where we recall the abuse of notation regarding explicit inclusion of the third argument in $\bar{S}$. Note that $\tilde{\bar{S}}_{\tilde{s}}(\tilde{s},\tilde{z}) = \bar{S}_s(\hat{g}_{\hat{\delta}}^{-1}(\tilde{s}), \tilde{g}_{\hat{\delta}}^{-1}(\tilde{z}))\frac{\hat{g}_{\hat{\delta}}^{-1}(\tilde{s})}{d\tilde{s}} = \hat{\delta}^2\bar{S}_s(s,z)$, with similar expressions for the space derivatives. The HJ PDE problem for $\tilde{\bar{S}}$, corresponding to (14)–(15) for $\bar{S}$, is

$$
\begin{aligned}
0 = &S_{\tilde{s}}(\tilde{s},\tilde{z}) + \tfrac{i\hbar}{2m}\Delta_{\tilde{z}} S(\tilde{s},\tilde{z}) - \hat{\omega}(\bar{A}_0\tilde{z}+\bar{b}_0)^T S_{\tilde{z}}(\tilde{s},\tilde{z}) - \tfrac{1}{2m}|S_{\tilde{z}}(\tilde{s},\tilde{z})|_c^2 \\
&- \tilde{V}(\tilde{z}), \qquad (\tilde{s},\tilde{z}) \in (0,\tilde{t}) \times \mathbb{C}^2,
\end{aligned} \tag{34}
$$

$$
S(\tilde{t},\tilde{z}) = \tilde{\phi}(\tilde{z}), \quad \tilde{z} \in \mathbb{C}^2, \qquad \bar{A}_0 \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \bar{b}_0 \doteq -\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{35}
$$

$\tilde{t} = t/\hat{\delta}^2$, $\tilde{\phi}(\tilde{z}) \doteq \phi(\tilde{g}_{\hat{\delta}}^{-1}(\tilde{z}))$ and $\frac{1}{\hat{\delta}^2}\tilde{V}(\tilde{z}) \doteq V(\tilde{g}_{\hat{\delta}}^{-1}(\tilde{z})) = V(z)$.

Note that in the case of a truncated expansion of a potential of form $-\bar{V}(x) = \bar{c}/|x|$, one obtains $-\tilde{V}(\tilde{z}) = -\sum_{k=0}^{K}\hat{\tilde{V}}^k(\tilde{z})$ $-\hat{\tilde{V}}^0(\tilde{z}) = m\hat{\omega}^2\left[1 - \tilde{z}_1 + (\tilde{z}_1^2 - \tilde{z}_2^2/2)\right]$ and

$$
-\hat{\tilde{V}}^k(\tilde{z}) = m\hat{\omega}^2 \sum_{j=0}^{k+2} c_{k+2,j}^V \tilde{z}_1^j \tilde{z}_2^{k+2-j} \quad \text{for } k \geq 1.
$$

In particular, one should note that the change of variables leads to a lack of $\hat{\varepsilon}^k$ in the expansion of the potential. With this reformulation in hand, consider the following assumption, where we note that $\breve{\mathscr{S}}_{\mathbb{C}}$ in $(A.2)$ is replaced by $\mathscr{S}_{\mathbb{C}}$ in $(A.2')$.

There exists a unique solution, $\tilde{\bar{S}} \in \mathscr{S}_{\mathbb{C}}$ to (34)–(35). $\qquad (A.2')$

Corollary 1. *Assume* $(A.2')$ *in place of* $(A.2)$. *Assume there exists a unique solution,* $\hat{S}^0$, *in* $\breve{\mathscr{S}}_{\mathbb{C}}$ *to (23)–(24), and that for each* $k \geq 1$, *there exists a unique solution,* $\hat{S}^k$, *in* $\breve{\mathscr{S}}_{\mathbb{C}}$ *to (25)–(26). Then,* $\bar{S} = \sum_{k=0}^{\infty} \hat{\varepsilon}^k \hat{S}^k$.

Proof. Let $\tilde{\bar{S}}$ satisfy $(A.2')$. Fix an arbitrary $\tilde{s} \in (0,\tilde{t})$, and let $D > 0$. Let $\mathscr{P}(D)$ denote the polydisc in $\mathbb{C}^2$ of multiradius $\bar{D} \doteq (D,D)$. By standard results (cf.[24]), for all $\tilde{z} \in \mathscr{P}(D)$,

$$
\tilde{\bar{S}}(\tilde{s},\tilde{z}) = \sum_{l=0}^{\infty}\sum_{j=0}^{l} \frac{\tilde{\bar{S}}_{z_1^j z_2^{l-j}}(\tilde{s},0)}{j!(l-j)!}\tilde{z}_1^j \tilde{z}_2^{l-j},
$$

which through application of the Cauchy integral formula,

$$
= \sum_{l=0}^{\infty}\sum_{j=0}^{l} \frac{1}{(2\pi i)^2} \int_{\partial\mathscr{P}(D)} \frac{\tilde{\bar{S}}(\tilde{s},\zeta_1,\zeta_2)}{\zeta_1^{j+1}\zeta_2^{l-j+1}}\, d\zeta_1\, d\zeta_2 \tilde{z}_1^j \tilde{z}_2^{l-j} \quad \forall (\tilde{s},\tilde{z}) \in (0,\infty) \times \mathbb{C}^2, \tag{36}
$$

where $\partial\mathscr{P}(D) \doteq \{\zeta \in \mathbb{C}^2 \,|\, |\zeta_1| = D, |\zeta_2| = D\}$. For each $\tilde{s} \in (0,\tilde{t})$, we may express the Taylor series representation for $\tilde{\bar{S}}$ as $\tilde{\bar{S}}(\tilde{s},\tilde{z}) = \sum_{l=0}^{\infty}\sum_{j=0}^{l} \tilde{\bar{c}}_{l,j}(\tilde{s})\tilde{z}_1^j \tilde{z}_2^{l-j}$ for all $\tilde{z} \in \mathbb{C}^2$. Let $0 \leq j \leq l < \infty$. Then, by (36) and the uniqueness of the Taylor expansion, we see that

$$
\tilde{\bar{c}}_{l,j}(\tilde{s}) = \frac{1}{(2\pi i)^2} \int_{\partial\mathscr{P}(D)} \frac{\tilde{\bar{S}}(\tilde{s},\zeta_1,\zeta_2)}{\zeta_1^{j+1}\zeta_2^{l-j+1}}\, d\zeta_1\, d\zeta_2,
$$

and the right-hand side is independent of $D \in (0,\infty)$. Further, letting $\zeta_\kappa = De^{i\theta_k}$ for $\kappa \in \{1,2\}$ and $\zeta \in \partial\mathscr{P}(D)$, this becomes

$$
\tilde{\bar{c}}_{l,j}(\tilde{s}) = \frac{1}{(2\pi i)^2} \int_0^{2\pi}\int_0^{2\pi} \frac{-\tilde{\bar{S}}(\tilde{s},De^{i\theta_1},De^{i\theta_2})}{D^l \exp\{i[(j+1)\theta_1 + (l-j+1)\theta_2]\}}\, d\theta_1\, d\theta_2.
$$

Let $\{\tilde{s}_n\} \subset (0,\tilde{t})$ be a sequence such that $\tilde{s}_n \to \tilde{s} \in (0,\tilde{t})$. By the Bounded Converegence Theorem, for any $0 \le j \le l < \infty$,

$$\lim_{n\to\infty} \tilde{\bar{c}}_{l,j}(\tilde{s}_n) = \frac{1}{(2\pi i)^2} \lim_{n\to\infty} \int_0^{2\pi}\int_0^{2\pi} \frac{-\tilde{\bar{S}}(\tilde{s}_n, De^{i\theta_1}, De^{i\theta_2})}{D^l \exp\{i[(j+1)\theta_1 + (l-j+1)\theta_2]\}}\, d\theta_1\, d\theta_2$$
$$= \frac{1}{(2\pi i)^2} \int_0^{2\pi}\int_0^{2\pi} \frac{-\tilde{\bar{S}}(\tilde{s}, De^{i\theta_1}, De^{i\theta_2})}{D^l \exp\{i[(j+1)\theta_1 + (l-j+1)\theta_2]\}}\, d\theta_1\, d\theta_2 = \tilde{\bar{c}}_{l,j}(\tilde{s}),$$

and we see that each $\tilde{\bar{c}}_{l,j}(\cdot)$ is continuous.

Similarly, for $0 \le j \le l < \infty$,

$$\lim_{h\to 0} \frac{\tilde{\bar{c}}_{l,j}(\tilde{s}+h) - \tilde{\bar{c}}_{l,j}(\tilde{s})}{h} = \lim_{h\to 0} \frac{1}{(2\pi i)^2} \int_0^{2\pi}\int_0^{2\pi} \frac{-1}{D^l \exp\{i[(j+1)\theta_1 + (l-j+1)\theta_2]\}}$$
$$\cdot \frac{\tilde{\bar{S}}(\tilde{s}+h, De^{i\theta_1}, De^{i\theta_2}) - \tilde{\bar{S}}(\tilde{s}, De^{i\theta_1}, De^{i\theta_2})}{h}\, d\theta_1\, d\theta_2.$$

Recalling that $\tilde{\bar{S}}$ is continuously differentiable on $(0,\tilde{t})$, we find that the integrand is bounded, and another application of the Bounded Convergence Theorem yields

$$\lim_{h\to 0} \frac{\tilde{\bar{c}}_{l,j}(\tilde{s}+h) - \tilde{\bar{c}}_{l,j}(\tilde{s})}{h} = \frac{1}{(2\pi i)^2} \int_0^{2\pi}\int_0^{2\pi} \frac{-\tilde{\bar{S}}_t(\tilde{s}, De^{i\theta_1}, De^{i\theta_2})}{D^l e^{i[(j+1)\theta_1 + (l-j+1)\theta_2]}}\, d\theta_1\, d\theta_2,$$

and we see that $\tilde{\bar{c}}_{l,j} \in C^1(0,\tilde{t})$.

Now, let $\bar{S}(s,z) \doteq \tilde{\bar{S}}(\hat{g}_{\hat{\delta}}(s), \tilde{g}_{\hat{\delta}}(z))$ for all $(s,z) \in (0,t] \times \mathbb{C}^2$, and let $\hat{\varepsilon} = 1/\hat{\delta}$. By Theorem 1, it is sufficient to show that $\bar{S}$ satisfies Assumption $(A.2)$. We have $\bar{S}(s,z) = \sum_{l=0}^{\infty}\sum_{j=0}^{l} \tilde{\bar{c}}_{l,j}(s)\hat{\varepsilon}^l z_1^j z_2^{l-j}$ for all $(s,z) \in (0,t] \times \mathbb{C}^2$. Letting $\hat{S}^l(s,z) \doteq \sum_{j=0}^{l} \tilde{\bar{c}}_{l,j}(s) z_1^j z_2^{l-j}$ for $l \in \mathbb{N}$, the smoothness assertions of the corollary then follow directly from the above and the composition of analytic functions. The existence and uniqueness are also easily demonstrated, and the steps are omitted. $\square$

5 Periodic $\hat{S}^0$ solutions

In order to begin computation of the terms in the expansion of Theorem 1, we must obtain a solution of the complex-valued, second-order, nonlinear HJ PDE problem given by (23)–(24). We note that we continue to work with the case of dimension $n = 2$ here. We will choose the initial condition, ϕ^0, such that the resulting solution will be periodic with frequency that is an integer multiple of ω, where we include the case where the multiple is zero (i.e., the steady-state case). We also note that we are seeking periodic solutions, $\hat{S}^0$ that are themselves clearly physically meaningful.

Recall that the original, forward-time solution, $\tilde{S}^f$, of (8)-(9) is a solution of the dequantized version of the original Schrödinger equation. Let $\tilde{\psi}^f(s,z) \doteq \exp\left\{\frac{i}{\hbar}\tilde{S}^f\right\}$

for all $(s,z) \in \overline{\mathscr{D}}^f \doteq [0,t) \times \mathbb{R}^2$. Recall also that for physically meaningful solutions, at each $s \in [0,t)$, $\tilde{P}^f(s,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ given by $\tilde{P}(s,\cdot) \doteq \left[\psi^* \psi\right](s,\cdot)$ represents an unnormalized density associated to the particle at time s. Let $\tilde{R}^f, \tilde{T}^f : \overline{\mathscr{D}}^f \to \mathbb{R}$ be given by $\tilde{R}^f(s,z) \doteq \mathbf{Re}[\tilde{S}^f(s,z)]$ and $\tilde{T}^f(s,z) \doteq \mathbf{Im}[\tilde{S}^f(s,z)]$ for all $(s,z) \in \overline{\mathscr{D}}^f$. Then, $\tilde{P}^f(s,z) = \exp\left\{\frac{-2}{\hbar}\tilde{T}^f(s,z)\right\}$ for all $(s,z) \in \overline{\mathscr{D}}^f$. This suggests that we should seek $\tilde{S}^f$ such that $\exp\left\{\frac{-2}{\hbar}\tilde{T}^f(s,\cdot)\right\}$ represents an unnormalized probability density for all $s \in [0,t)$.

Although the goal in this section is to generate a set of physically meaningful periodic solutions to the zeroth-order term, we do not attempt a full catalog of all possible such solutions. Let $\hat{S}^{0,f}(s,z) \doteq \hat{S}^0(t-s,z)$ for all $(s,z) \in \overline{\mathscr{D}}^f$. As we seek $\hat{S}^0(t-s,\cdot)$ that are quadratic, we let the resulting time-dependent coefficients be defined by

$$\hat{S}^{0,f}(s,z) = \tfrac{1}{2} z^T Q(s) z + \Lambda^T(s) z + \rho(s). \tag{37}$$

It should be noted here that the condition that $\exp\left\{\frac{-\hbar}{2}\tilde{T}^f(s,\cdot)\right\}$ represent an unnormalized density implies that the imaginary part of $Q(s)$ should be nonnegative definite for all $s \in [0,t)$, which is a significant restriction on the set of allowable solutions.

As $\hat{S}^{0,f}(s,\cdot)$ is quadratic, its values over $\mathbb{C}^2$ are fully defined by its values over $\mathbb{R}^2$, and hence it is sufficient to solve the problem on the real domain. The forward-time version of (23)–(24), with domain restricted to $\overline{\mathscr{D}}^f$ is

$$0 = -S_t^{0,f} + \tfrac{i\hbar}{2m}\Delta_z S^{0,f} - \left(A_0 z + b_0\right)^T S_z^{0,f} - \tfrac{1}{2m}|S_z^{0,f}|_c^2 - \hat{V}^0, \quad (s,z) \in (0,t) \times \mathbb{R}^2, \tag{38}$$

$$S^{0,f}(0,z) = \phi^0(z) \quad \forall z \in \mathbb{R}^2. \tag{39}$$

Remark 4. It is worth noting that any solution of form (37) to (38)–(39) is the unique solution in $\mathscr{S}_{\mathbb{C}}^p$, and in particular, where this uniqueness is obtained through a controlled-diffusion representation [17, 18].

Substituting form (37) into (38), and collecting terms, yields the system of ordinary differential equations (ODEs) given as

$$\frac{d}{ds}Q(s) = -\left(A_0^T Q(s) + Q(s) A_0\right) - \tfrac{1}{m} Q^2(s) + m\omega^2 T^V, \tag{40}$$

$$\frac{d}{ds}\Lambda(s) = -\left(A_0^T + \tfrac{1}{m}Q(s)\right)\Lambda + \omega\hat{\delta}Q(s)u^2 - m\omega^2\hat{\delta}u^1, \tag{41}$$

$$\frac{d}{ds}\rho(s) = \tfrac{i\hbar}{2m}\,\mathrm{tr}[Q(s)] + \omega\hat{\delta}(u^2)^T\Lambda(s) - \tfrac{1}{2m}\Lambda^T(s)\Lambda(s) + m\omega^2\hat{\delta}^2, \tag{42}$$

$$T^V = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad u^1 \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u^2 \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{43}$$

where $Q : [0,t) \to \mathbb{C}^{2\times 2}$, $\Lambda : [0,t) \to \mathbb{C}^2$ and $\rho : [0,t) \to \mathbb{C}$. Throughout, we assume that $Q(s)$ is symmetric for all $s \in [0,t)$. Note that if $Q(s)$ is nonsingular for all $s \in [0,t)$, then (37) may also be written as

$$\hat{S}^{0,f}(s,z) = \tfrac{1}{2}\left(z+Q^{-1}(s)\Lambda(s)\right)^T Q(s)\left(z+Q^{-1}(s)\Lambda(s)\right) + \rho(s) - \Lambda^T(s)Q^{-1}(s)\Lambda(s),$$

where we see that $-Q^{-1}(s)\Lambda(s)$ may be interpreted as a mean of the associated distribution at each time s. Consequently, we look for solutions with $-Q^{-1}(s)\Lambda(s) \in \mathbb{R}^2$ for all s.

One may use a Bernoulli-type substitution as a means for seeking solutions of (40). That is, suppose $Q(s) = W(s)U^{-1}(s)$, where $U(s)$ is nonsingular for all $s \in [0,t)$. Then, without loss of generality, we may take $W(0) = Q(0)$, $U(0) = \mathscr{I}_{2\times 2}$. The resulting system of ODEs is

$$\frac{d}{ds}\begin{pmatrix} U \\ W \end{pmatrix} = \mathscr{B}\begin{pmatrix} U \\ W \end{pmatrix}, \qquad \mathscr{B} \doteq \begin{bmatrix} 0 & \omega & 1/m & 0 \\ -\omega & 0 & 0 & 1/m \\ 2m\omega^2 & 0 & 0 & \omega \\ 0 & -m\omega^2 & -\omega & 0 \end{bmatrix}.$$

Employing the Jordan canonical form, one obtains the solution as

$$\left(U(s)^T, W(s)^T\right)^T = RPe^{J\omega s}P^{-1}R^{-1}\left(\mathscr{I}_{2\times 2}, Q(0)\right)^T, \tag{44}$$

where

$$P = \begin{bmatrix} 0 & 2 & -i & i \\ -3 & 0 & 2 & 2 \\ 3 & 0 & -1 & -1 \\ 0 & -1 & i & -i \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 1 \\ -i/2 & 1/2 & 1/2 & -i \\ i/2 & 1/2 & 1/2 & i \end{bmatrix},$$
$$e^{J\omega s} = \begin{bmatrix} 1 & \omega s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \exp\{i\omega s\} & 0 \\ 0 & 0 & 0 & \exp\{-i\omega s\} \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m\omega & 0 \\ 0 & 0 & 0 & m\omega \end{bmatrix}.$$

We remark that, as our goal here is the generation of periodic solutions that may be used as a basis for the expansion to follow, and as this work is already of substantial length, we will not discuss the question of stability of the above solutions, to perturbations within the class of physically meaningful $Q = U^{-1}W$.

Note that we seek solutions that generate periodic densities $\tilde{P}^f(s,\cdot)$, and that the $(1,2)$ entry of $e^{J\omega s}$ has secular behavior. Examining (44), we see that a sufficient condition for avoidance of secular growth/decay of Q, is that entries in the second row of $P^{-1}R^{-1}\left(\mathscr{I}_{2\times 2}, Q(0)\right)^T$ be zero. One easily sees that this corresponds to $Q_{2,1}(0) = -m\omega$ and $Q_{2,2}(0) = 0$, and considering here only symmetric Q, we take $Q_{1,2}(0) = -m\omega$. That is, we have

$$Q(0) = \begin{bmatrix} \bar{k}_0 i m\omega & -m\omega \\ -m\omega & 0 \end{bmatrix}, \tag{45}$$

for some $\bar{k}_0 \in \mathbb{C}$. Propagating the resulting solutions, we find that the imaginary part of $\bar{k}_0$ being nonnegative is necessary and sufficient for satisfaction of the condition that $\mathbf{Im}[Q(s)]$ be nonnegative-definite for all s. We also note that with such initial condition, $Q_{1,2}, Q_{2,1}, Q_{2,2}$ remain constant for all s, while the real and imaginary parts of $Q_{1,1}$ are periodic. That is, $Q(s)$ takes the form

$$Q(s) = \begin{bmatrix} im\omega p(s) & -m\omega \\ -m\omega & 0 \end{bmatrix} \quad \forall s \in [0,t), \tag{46}$$

where $p(s) = [\bar{k}_1^+ e^{2i\omega s} + \bar{k}_1^-]/[\bar{k}_1^+ e^{2i\omega s} - \bar{k}_1^-]$ with $\bar{k}_1^+ \doteq \bar{k}_0 + 1$ and $\bar{k}_1^- \doteq \bar{k}_0 - 1$.

One may seek steady-state solutions by substitution of form (45) into the right-hand side of (40), and setting this to be zero. One easily finds that the unique steady state solution (among those with $\mathbf{Im}[\bar{k}_0] \geq 0$) is

$$Q(s) = \bar{Q}^0 \doteq \begin{bmatrix} im\omega & -m\omega \\ -m\omega & 0 \end{bmatrix} \quad \forall s \in [0,t). \tag{47}$$

Next we consider the linear term in $\hat{S}^{0,f}$, where this satisfies (41). We focus on the steady-state Q case of (47). Substituting (47) into (41) yields

$$\dot{\Lambda} = \begin{bmatrix} -i\omega & 2\omega \\ 0 & 0 \end{bmatrix} \Lambda - 2m\omega^2 \hat{\delta} u^1.$$

This has a steady-state solution in the case that $-i\Lambda_1(0) + 2\Lambda(0) = 2m\omega\hat{\delta}$, or equivalently, the one-parameter set of steady-state solutions given by $\Lambda(s) = \bar{\Lambda}^0 \doteq \big(id, m\omega\hat{\delta} - d/2\big)^T$ for $d \in \mathbb{C}$. This includes, in particular, the cases $\bar{\Lambda}^0 = (0, m\omega\hat{\delta})^T$ (i.e., $d = 0$) and $\bar{\Lambda}^0 = m\omega\hat{\delta}(-2i, 2)^T$ (i.e., $d = -2m\omega\hat{\delta}$), where this latter case is obtained if one requires $(\bar{Q}^0)^{-1}\bar{\Lambda}^0$ to be real valued. We also remark that more generally, the solution is given for all $s \in [0,t)$ by

$$\Lambda(s) = \begin{bmatrix} -i\exp\{-i\omega s\} & 2i[\exp\{-i\omega s\} - 1] \\ 0 & 0 \end{bmatrix} \Lambda(0) + 2i[1 - \exp\{-i\omega s\}] m\omega\hat{\delta} u^1.$$

Lastly, we turn to the zeroth-order term. Note that the one may allow secular growth in the real part of $\rho(\cdot)$ with no effect on the associated probability distribution, as is standard in solutions of the quantum harmonic oscillator. Continuing to focus on the steady-state solution, but allowing a real-valued secular zeroth-order term, we substitute the above steady-state quadratic and linear coefficients into (42). This yields

$$\dot{\rho} = \frac{-\hbar\omega}{2} + m\omega^2 \left[\frac{3\hat{\delta}^2}{2} + \frac{3d^2}{4(m\omega)^2} \right] \doteq \bar{c}_1(d),$$

and we see that this is purely real if and only if $d \in \mathbb{R}$, and we have

$$\Lambda(s) = \bar{\Lambda}^0 \doteq \big(id, m\omega\hat{\delta} - d/2\big)^T, \qquad \rho^0(s) = \rho^0(0) + \bar{c}_1(d)s \quad \forall s \in [0,t). \tag{48}$$

We will restrict ourselves to the simple, steady-state case (modulo the real part of ρ^0) given by (47),(48) with $\bar{k}_0 = 1$, $d = 0$, for our actual computations of succeeding terms in the expansion. However, the theory will be sufficiently general to encompass the periodic case as well.

6 Diffusion representations for succeeding terms

As noted above, we will use diffusion representations to obtain the solutions to the HJ PDEs (25)–(26) that define the succeeding terms in the expansion, i.e., to obtain the $\hat{S}^k$ for $k \in \mathbb{N}$. In order to achieve this goal, we need to define the complex-valued diffusion dynamics and the expected payoffs that will yield the $\hat{S}^k$. The representation result naturally employs the Itô integral rule. As the dynamics are complex-valued, we need an extension of the Itô rule to that process domain. In a similar fashion to that of Section 3.3, we use the Itô rule for the double-dimension real case to obtain the rule for the complex case. Once the Itô rule is established, the proof of the representation is straightforward. However, additional effort is require to generate the machinery by which the the actual solutions are computed, where the machinery relies mainly on computation of moments for the diffusion process.

6.1 The underlying stochastic dynamics

We let $(\Omega, \mathscr{F}, P)$ be a probability triple, where Ω denotes a sample space, $\mathscr{F}$ denotes a σ-algebra on Ω, and P denotes a probability measure on $(\Omega, \mathscr{F})$. Let $\{\mathscr{F}_s \,|\, s \in [0,t]\}$ denote a filtration on $(\Omega, \mathscr{F}, P)$, and let $B_\cdot$ denote an $\mathscr{F}_\cdot$-adapted Brownian motion taking values in $\mathbb{R}^n$. We will be interested in diffusion processes given by the linear stochastic differential equation (SDE) in integral form

$$\begin{aligned}
\zeta_r = \zeta_r^{(s,z)} &= z + \int_s^r -\big(A_0\zeta_\rho + b_0 + \tfrac{1}{m}\hat{S}^0_z(\rho,\zeta_\rho)\big)\,d\rho + \sqrt{\tfrac{\hbar}{m}}\,\tfrac{1+i}{\sqrt{2}}\int_s^r dB_\rho \\
&\doteq z + \int_s^r \lambda(\rho,\zeta_\rho)\,d\rho + \sqrt{\tfrac{\hbar}{m}}\,\tfrac{1+i}{\sqrt{2}}\,B_r^\Delta \quad \forall r \in [s,t],
\end{aligned} \tag{49}$$

where $z \in \mathbb{C}^2$, $s \in [0,t)$, $B_r^\Delta \doteq B_r - B_s$ for $r \in [s,t)$, and

$$\begin{aligned}
\lambda(\rho,z) &\doteq -[A_0 z + b_0 + \tfrac{1}{m}S^0_z(\rho,z)] = -[A_0 z + b_0 + \tfrac{1}{m}Q(\rho)z + \tfrac{1}{m}\Lambda(\rho)] \\
&\doteq -A_{>0}(\rho)z - b_{>0}(\rho).
\end{aligned} \tag{50}$$

Let $\bar{f} : [0,t] \times \mathbb{C}^2 \to \mathbb{C}^2$, and suppose there exists $K_{\bar{f}} < \infty$ such that $|\bar{f}(s,z^1) - \bar{f}(s,z^2)| \le K_{\bar{f}}|z^1 - z^2|$ for all $(s,z^1),(s,z^2) \in \overline{\mathscr{D}}_{\mathbb{C}}$. For $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$, consider the complex-valued diffusion, $\zeta_\cdot \in \mathscr{X}_s$, given by

$$\zeta_r = \zeta_r^{(s,z)} = z + \int_s^r \bar{f}(\rho, \zeta_\rho)\, d\rho + \int_s^r \tfrac{1+i}{\sqrt{2}} \sigma\, dB_\rho, \tag{51}$$

where $\sigma \in \mathbb{R}^{n\times n}$, and note that this is a slight generalization of (49). For $s \in (0,t]$, let

$$\mathscr{X}_s \doteq \{\zeta : [s,t] \times \Omega \to \mathbb{C}^2 \,|\, \zeta \text{ is } \mathscr{F}_\cdot\text{-adapted, right-continuous and such that } \mathbb{E} \sup_{r\in[s,t]} |\zeta_r|^m < \infty\; \forall m \in \mathbb{N}\}. \tag{52}$$

We supply $\mathscr{X}_s$ with the norm $\|\zeta\|_{\mathscr{X}_s} \doteq \max_{m\in]1,\bar{M}[} \left[\mathbb{E} \sup_{r\in[s,t]} |\zeta_r|^m\right]^{1/m}$. It is important to note here that complex-valued diffusions have been discussed elsewhere in the literature; see for example, [26] and the references therein.

We also define the isometric isomorphism, $\mathscr{V} : \mathscr{X}_s \to \mathscr{X}_s^v$ by $[\mathscr{V}(\zeta)]_r \doteq [\mathscr{V}(\xi + i\nu)]_r \doteq (\xi_r^T, \nu_r^T)^T$ for all $r \in [s,t]$ and $\omega \in \Omega$, where

$$\mathscr{X}_s^v \doteq \{(\xi,\nu) : [s,t] \times \Omega \to \mathbb{R}^{2n} \,|\, (\xi,\nu) \text{ is } \mathscr{F}_\cdot\text{-adapted, right-continuous and such that } \mathbb{E} \sup_{r\in[s,t]} [|\xi_r|^m + |\nu_r|^m] < \infty\; \forall m \in \mathbb{N}\}, \tag{53}$$

$$\|(\xi,\nu)\|_{\mathscr{X}_s^v} \doteq \max_{m\in]1,\bar{M}[} \left[\mathbb{E} \sup_{r\in[s,t]} (|\xi_r|^m + |\nu_r|^m)\right]^{1/m}. \tag{54}$$

Under transformation by $\mathscr{V}$, (51) becomes

$$\begin{pmatrix}\xi_r\\ \nu_r\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix} + \int_s^r \hat{f}(\rho, \xi_\rho, \nu_\rho)\, d\rho + \int_s^r \tfrac{1}{\sqrt{2}} \hat{\sigma}\, dB_\rho \quad \forall r \in [s,t], \tag{55}$$

where $\hat{f}(\rho,\xi_\rho,\nu_\rho) \doteq \left((\mathbf{Re}[\bar{f}(\rho,\xi_\rho + i\nu_\rho)])^T, (\mathbf{Im}[\bar{f}(\rho,\xi_\rho + i\nu_\rho)])^T\right)^T$ and $\hat{\sigma} \doteq (1,1)^T$. Throughout, concerning both real and complex stochastic differential equations, typically given in integral form such as in (56), *solution* refers to a strong solution, unless specifically cited as a weak solution. The following are easily obtained from existing results; see [18, 23].

Lemma 2. *Let $s \in [0,t)$, $z \in \mathbb{C}^2$ and $(x,y) = \mathscr{V}_0(z)$. There exists a unique solution, $(\xi,\nu) \in \mathscr{X}_s^v$, to (55).*

Lemma 3. *Let $s \in [0,t)$, $z \in \mathbb{C}^2$ and $(x,y) = \mathscr{V}_0(z)$. $\zeta \in \mathscr{X}_s$ is a solution of (51) if and only if $\mathscr{V}(\zeta) \in \mathscr{X}_s^v$ is a solution of (55).*

Lemma 4. *Let $s \in [0,t)$ and $z \in \mathbb{C}^2$. There exists a unique solution, $\zeta \in \mathscr{X}_s$, to (51).*

We remark that one may apply Lemmas 2–4 to the specific case of (49) in order to establish existence and uniqueness. In particular, for the dynamics of (49), the corresponding process $(\xi,\nu) = \mathscr{V}(\zeta)$ satisfies

$$\begin{pmatrix}\xi_r\\ \nu_r\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix} + \int_s^r - \left[\begin{pmatrix}A^r_{>0}(\rho) & -A^i_{>0}(\rho)\\ A^i_{>0}(\rho) & A^r_{>0}(\rho)\end{pmatrix}\begin{pmatrix}\xi_r\\ \nu_r\end{pmatrix} + \begin{pmatrix}b^r_{>0}(\rho)\\ b^i_{>0}(\rho)\end{pmatrix}\right] d\rho$$

$$+\sqrt{\tfrac{\hbar}{2m}}\begin{pmatrix} I_{n\times n}\\ I_{n\times n}\end{pmatrix} B_r^\Delta$$

$$\doteq \begin{pmatrix} x\\ y\end{pmatrix} + \int_s^r -\bar{A}_{>0}(\rho)\begin{pmatrix} \xi_r\\ \nu_r\end{pmatrix} - \bar{b}_{>0}(\rho)\,d\rho + \sqrt{\tfrac{\hbar}{2m}}\,\bar{\mathscr{I}}\,B_r^\Delta \quad \forall r\in[s,t], \quad (56)$$

where $A^r_{>0}(\rho)\doteq \mathbf{Re}(A_{>0}(\rho)), A^i_{>0}(\rho)\doteq \mathbf{Im}(A_{>0}(\rho))$, $\big((b^r_{>0}(\rho))^T,(b^i_{>0}(\rho))^T\big)^T \doteq \mathscr{V}_0(b_{>0}(\rho))$ for all $\rho\in[0,t)$.

6.2 Itô's rule

The representation results will rely on a minor generalization of Itô's rule to the specific complex-diffusion dynamics of interest here. It might be worthwhile to note that the complex-valued diffusions considered here belong to a very small subclass of complex-valued diffusions, and this is somehow related to the specific nature of the complex aspect of the Schrödinger equation. The following complex-case Itô rule is similar to existing results (cf., [26]).

Lemma 5. *Let $\bar{g}\in\mathscr{S}_{\mathbb{C}}$ and $(s,z)\in\overline{\mathscr{D}}_{\mathbb{C}}$, and suppose diffusion process $\zeta_\cdot$ is given by (51). Then, for all $r\in[s,t]$,*

$$\begin{aligned}\bar{g}(r,\zeta_r) =& \bar{g}(s,z) + \int_s^r \bar{g}_t(\rho,\zeta_\rho) + \bar{g}_z^T(\rho,\zeta_\rho)\bar{f}(\rho,\zeta_\rho)\,d\rho + \int_s^r \tfrac{1+i}{\sqrt{2}}\bar{g}_z^T(\rho,\zeta_\rho)\sigma\,dB_\rho\\ &+ \tfrac{1}{2}\int_s^r \mathrm{tr}\left[\bar{g}_{zz}(\rho,\zeta_\rho)(\sigma\sigma^T)\right]d\rho. \end{aligned} \quad (57)$$

Proof. Let $(g^r(s,x,y),g^i(s,x,y))\doteq \mathscr{V}_{00}\big(\bar{g}(s,\mathscr{V}_0^{-1}(x,y))\big)$, $(f^r(s,x,y),f^i(s,x,y))\doteq \mathscr{V}_0\big(\bar{f}(s,\mathscr{V}_0^{-1}(x,y))\big)$ for all $(s,x,y)\in\overline{\mathscr{D}}_2$, and note that it is trivial to show that $\bar{g}_t(r,z)=g^r_t(r,x,y)+ig^i_t(r,x,y)$, for all $(x,y)=\mathscr{V}_0(z)$, $(r,z)\in\mathscr{D}_{\mathbb{C}}$. Also, using the Cauchy-Riemann equations,

$$\bar{g}_z^T(r,z)\bar{f}(r,z) = \left[(g^r_x)^T f^r + (g^r_y)^T f^i\right](r,x,y) + i\left[(g^i_x)^T f^r + (g^i_y)^T f^i\right](r,x,y),$$

for all $(x,y)=\mathscr{V}_0(z)$, $(r,z)\in\overline{\mathscr{D}}_{\mathbb{C}}$. Defining the derivative notation $g^r_{x^2}(s,x,y)\doteq \big((g^r_x)^T,(g^r_y)^T\big)^T(r,x,y)$ and vector notation $\hat{f}(r,x,z)\doteq\big((f^r)^T,(f^i)^T\big)(r,x,y)$ for all $(r,x,y)\in\overline{\mathscr{D}}_2$, this becomes

$$\bar{g}_z^T(r,z)\bar{f}(r,z) = \big(g^r_{x^2}(r,x,y)\big)^T\hat{f}(r,x,y) + i\big(g^i_{x^2}(r,x,y)\big)^T\hat{f}(r,x,y), \quad (58)$$

for all $(x,y)=\mathscr{V}_0(z)$, $(r,z)\in\overline{\mathscr{D}}_{\mathbb{C}}$. Similarly, letting $\hat{\sigma}\doteq\big(\sigma^T,\sigma^T\big)^T$,

$$\bar{g}_z^T(r,z)\tfrac{1+i}{\sqrt{2}}\sigma = \tfrac{1}{\sqrt{2}}\left[\big(g^r_{x^2}(r,x,y)\big)^T\hat{\sigma} + i\big(g^i_{x^2}(r,x,y)\big)^T\hat{\sigma}\right], \quad (59)$$

for all $(x,y)=\mathscr{V}_0(z)$, $(r,z)\in\overline{\mathscr{D}}_{\mathbb{C}}$.

Next, let $\bar{a} \doteq \left(\frac{1+i}{\sqrt{2}}\right)^2 \sigma\sigma^T$ and $(a^r_{j,l}, a^i_{j,l}) \doteq \mathscr{V}_{00}(\bar{a}_{j,l})$ for all $j,l \in]1,n[$. Using (17),(18), we find

$$\bar{g}_{z_j,z_l}\bar{a}_{j,l} = g^i_{x_j,y_l}a^r_{j,l} + g^r_{x_j,y_l}a^i_{j,l} + i\big[-g^r_{x_j,y_l}a^r_{j,l} + g^i_{x_j,y_l}a^i_{j,l}\big] \quad \forall j,l \in]1,n[. \tag{60}$$

Also, by the definition of $\bar{a}$, we see that $a^r = 0$ and $a^i = \sigma\sigma^T$. Applying these in (60) yields

$$\bar{g}_{z_j,z_l}\bar{a}_{j,l} = g^r_{x_j,y_l}[\sigma\sigma^T]_{j,l} + ig^i_{x_j,y_l}[\sigma\sigma^T]_{j,l} \quad \forall j,l \in]1,n[. \tag{61}$$

Considering (58), (59) and (61), and letting $(\xi_r, \nu_r) \doteq \mathscr{V}_0(\zeta_r)$ for all $r \in (0,t]$, a.e. $\omega \in \Omega$ we see that (57) is equivalent to a pair of equations for the real and imaginary parts, where the real-part equation is

$$\begin{aligned} g^r(r,\xi_r,\nu_r) = & g^r(s,x,y) + \int_s^r g^r_t(\rho,\xi_\rho,\nu_\rho) + (g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{f}(\rho,\xi_\rho,\nu_\rho) \\ & + \tfrac{1}{2}\sum_{j,l=1}^n g^r_{x_j,y_l}(\rho,\xi_\rho,\nu_\rho)[\sigma\sigma^T]_{j,l}\,d\rho + \tfrac{1}{\sqrt{2}}\int_s^r (g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{\sigma}\,dB_\rho, \end{aligned} \tag{62}$$

with an analogous equation corresponding to the imaginary part.

On the other hand, applying Itô's rule to real functions g^r and g^i separately, and then applying (17), (18), we find

$$\begin{aligned} g^r(r,\xi_r,\nu_r) = & g^r(s,x,y) + \int_s^r g^r_t(\rho,\xi_\rho,\nu_\rho) + (g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{f}(\rho,\xi_\rho,\nu_\rho) \\ & + \tfrac{1}{4}\sum_{j,l=1}^n \big[g^r_{x_j,x_l} + g^r_{x_j,y_l} + g^r_{y_j,x_l} + g^r_{y_j,y_l}\big](\rho,\xi_\rho,\nu_\rho)[\sigma\sigma^T]_{j,l}\,d\rho \\ & + \tfrac{1}{\sqrt{2}}\int_s^r (g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{\sigma}\,dB_\rho, \\ = & g^r(s,x,y) + \int_s^r g^r_t(\rho,\xi_\rho,\nu_\rho) + (g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{f}(\rho,\xi_\rho,\nu_\rho) \\ & + \tfrac{1}{2}\sum_{j,l=1}^n g^r_{x_j,y_l}(\rho,\xi_\rho,\nu_\rho)[\sigma\sigma^T]_{j,l}\,d\rho + \tfrac{1}{\sqrt{2}}\int_s^r (g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{\sigma}\,dB_\rho, \end{aligned} \tag{63}$$

with a similar equation for the imaginary part. Comparing (63) with (62), and similarly for the imaginary parts, one obtains the result. □

We apply this result to the particular case of interest here.

Lemma 6. *Let $\hat{S} \in \mathscr{S}_{\mathbb{C}}$ and $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$, and suppose ζ satisfies (49). Then, for all $r \in (s,t]$,*

$$\begin{aligned} \hat{S}(r,\zeta_r) = & \hat{S}(s,z) + \int_s^r \hat{S}_t(\rho,\zeta_\rho) - \hat{S}^T_z(\rho,\zeta_\rho)\big[A_{>0}(\rho)\zeta_\rho + b_{>0}(\rho)\big] + \tfrac{i\hbar}{2m}\Delta_z\hat{S}(\rho,\zeta_\rho)\,d\rho \\ & + \sqrt{\tfrac{\hbar}{m}}\,\tfrac{1+i}{\sqrt{2}}\int_s^r \hat{S}^T_z(\rho,\zeta_\rho)\,dB_\rho. \end{aligned} \tag{64}$$

Proof. Dynamics (49) have form (51) with $f(r,z) = A_{>0}(r)z + b_{>0}(r)$ and $\sigma = \sqrt{\frac{\hbar}{m}}\mathscr{I}_{n\times n}$. In this case, $\frac{1}{2}\,\mathrm{tr}\big[\hat{S}_{zz}(r,z)(\sigma\sigma^T)\big] = \frac{i\hbar}{2m}\Delta_z\hat{S}(r,z)$ for all $(r,z)\in\mathscr{S}_{\mathbb{C}}$, which yields the result. □

Theorem 2. *Let $k\in\mathbb{N}$. Let $\hat{S}^\kappa \in \mathscr{S}^p_{\mathbb{C}}$ satisfy (25)–(26) for all $\kappa \in]1,k[$. Let $(s,z)\in\overline{\mathscr{D}}_{\mathbb{C}}$, and let $\zeta\in\mathscr{X}_s$ satisfy (49). Then,*

$$\hat{S}^k(s,z) = \mathbb{E}\bigg\{\int_s^t -\tfrac{1}{2m}\sum_{\kappa=1}^{k-1}\big[S_z^\kappa(r,\zeta_r)\big]^T S_z^{k-\kappa}(r,\zeta_r) - \hat{V}^k(\zeta_r)\,dr + \phi^k(\zeta_t)\bigg\}.$$

Proof. Taking expectations in (64), and using the martingale property (cf., [5, 8]), we have

$$\begin{aligned}\hat{S}^k(s,z) = \mathbb{E}\bigg\{ -\int_s^t \hat{S}_t(r,\zeta_r) - \hat{S}_z^T(r,\zeta_r)\big[A_{>0}(r)\zeta_r + b_{>0}(r)\big] + \tfrac{i\hbar}{2m}\Delta_z\hat{S}(r,\zeta_r)\,dr \\ + \hat{S}^k(t,\zeta_t)\bigg\}.\end{aligned}$$

Combining this with (25)–(26) yields the result. □

6.3 Moments and Iteration

Note that Theorem 2 yields an expression for the k^{th} term in our expansion for $\bar{S}$, $\hat{S}^k$, from the previous terms, $\hat{S}^\kappa$ for $\kappa<k$. We now examine how this generates a computationally tractable scheme. It is heuristically helpful to examine the first two iterates. For $(s,z)\in\overline{\mathscr{D}}_{\mathbb{C}}$, we have

$$\begin{aligned}\hat{S}^1(s,z) &= \mathbb{E}\bigg\{\int_s^t -\hat{V}^1(\zeta_r)\,dr + \phi^1(\zeta_t)\bigg\} \\ &= \mathbb{E}\bigg\{\int_s^t m\omega^2\Big(-[\zeta_r]_1^3 + (3/2)[\zeta_r]_1[\zeta_r]_2^2\Big)\,dr + \sum_{l=0}^{3}\sum_{j=0}^{l} b^\phi_{3,l,j}[\zeta_t]_1^j[\zeta_t]_2^{l-j}\bigg\},\end{aligned} \tag{65}$$

$$\begin{aligned}\hat{S}^2(s,z) &= \mathbb{E}\bigg\{\int_s^t -\tfrac{1}{2m}\big|\hat{S}_z^1(r,\zeta_r)\big|_c^2 - \hat{V}^2(\zeta_r)\,dr + \phi^2(\zeta_t)\bigg\} \\ &= \mathbb{E}\bigg\{\int_s^t -\tfrac{1}{2m}\big|\hat{S}_z^1(r,\zeta_r)\big|_c^2 + m\omega^2\Big([\zeta_r]_1^4 - 3[\zeta_r]_1^2[\zeta_r]_2^2 + (3/8)[\zeta_r]_2^4\Big)\,dr \\ &\qquad + \sum_{l=0}^{4}\sum_{j=0}^{l} b^\phi_{4,l,j}[\zeta_t]_1^j[\zeta_t]_2^{l-j}\bigg\}.\end{aligned} \tag{66}$$

Note that the right-hand side of (65) consists of an expectation of a polynomial in ζ_t and an integral of a polynomial in $\zeta_\cdot$, and further, that the dynamics of ζ

are linear in the state variable. Thus, we may anticipate that $\hat{S}^1(s,\cdot)$ may also be polynomial. Applying this anticipated form on the right-hand side of (66) suggests that the polynomial form will be inherited in each $\hat{S}^k$. This will form the basis of our computational scheme.

The computation of the expectations that generate the $\hat{S}^k$ for $k \geq 1$ will be obtained through the moments of the underlying diffusion process. Thus, the first step is solution of (49). We let the state transition matrices for deterministic linear systems $\dot{y}_r = -A_{>0}(r)y_r$ and $\dot{y}_r^{(2)} = -\bar{A}_{>0}(r)y_r^{(2)}$ be denoted by $\Phi(r,s)$ and $\Phi^{(2)}(r,s)$, respectively. More specifically, with initial (or terminal) conditions, $y_s = \bar{y}$ and $y_s^{(2)} = \bar{y}^{(2)}$, the solutions at time r are given by $y_r = \Phi(r,s)\bar{y}$ and $y_r^{(2)} = \Phi^{(2)}(r,s)\bar{y}^{(2)}$, respectively. The solutions of our SDEs are given by the following.

Lemma 7. *Linear SDE (49) has solution given by $\zeta_r = \mu_r + \Delta_r$, where*

$$\mu_r = \Phi(r,s)z + \int_s^r \Phi(r,\rho)\big(-b_{>0}(\rho)\big)\,d\rho, \quad \Delta_r = \sqrt{\tfrac{\hbar}{m}}\,\tfrac{1+i}{\sqrt{2}}\int_s^r \Phi(r,\rho)\,dB_\rho$$

for all $r \in [s,t]$. Linear SDE (56) has solution given by $X_r^{(2)} = \mu_r^{(2)} + \Delta_r^{(2)}$, where

$$\mu_r^{(2)} = \Phi^{(2)}(r,s)x^{(2)} + \int_s^r \Phi^{(2)}(r,\rho)\big(-\bar{b}_{>0}(\rho)\big)\,d\rho,$$
$$\Delta_r^{(2)} = \sqrt{\tfrac{\hbar}{2m}}\int_s^r \Phi^{(2)}(r,\rho)\bar{\mathscr{I}}\,dB_\rho$$

for all $r \in [s,t]$, where $x^{(2)} \doteq (x^T, y^T)^T$.

Proof. The case of (56) is standard,cf., [12]. We sketch the proof in the minor variant case of (49), where this uses the Itô-rule approach, but for the complex-valued diffusion case. For $0 \leq s \leq r \leq t$, let $\alpha_r \doteq \Phi(s,r)\zeta_r = \Phi^{-1}(r,s)\zeta_r$. By Lemma 5,

$$\alpha_r = \int_s^r \Phi^{-1}(\rho,s)[-b_{>0}(\rho)]\,d\rho + \sqrt{\tfrac{\hbar}{m}}\,\tfrac{1+i}{\sqrt{2}}\int_s^r \Phi^{-1}(\rho,s)\,dB_\rho,$$

which implies $\zeta_r = \int_s^r \Phi(r,\rho)[-b_{>0}(\rho)]\,d\rho + \sqrt{\tfrac{\hbar}{m}}\,\tfrac{1+i}{\sqrt{2}}\int_s^r \Phi(r,\rho)\,dB_\rho$. □

Lemma 8. *For all $r \in [s,t]$, $X_r^{(2)}$ and ζ_r have normal distributions.*

Proof. The case of $X_r^{(2)}$ is standard, cf. [11], and then one notes $\zeta_r = \mathscr{V}_0(X_r^{(2)})$. □

Lemma 9. *For all $r \in [s,t]$, μ_r is the mean of ζ_r, and Δ_r is a zero-mean normal random variable with covariance given by $\mathbb{E}\big[\Delta_r\Delta_r^T\big] = \frac{i\hbar}{m}\int_s^r \Phi(r,\rho)\Phi^T(r,\rho)\,d\rho$, where further, $\mathbb{E}\big[(\zeta_r - \mu_r)(\zeta_r - \mu_r)^T\big] = \mathbb{E}\big[\Delta_r\Delta_r^T\big]$.*

Proof. That Δ_r has zero mean is immediate from its definition. Given Lemmas 7 and 8, it is sufficient to obtain the expression for $\mathbb{E}\big[\Delta_r\Delta_r^T\big]$. By Lemma 7,

$$\mathbb{E}\big[\Delta_r\Delta_r^T\big] = \tfrac{i\hbar}{m}\mathbb{E}\Big\{\big[\textstyle\int_s^r \Phi(r,\rho)\,dB_\rho\big]\big[\int_s^r \Phi(r,\rho)\,dB_\rho\big]^T\Big\},$$

where the term inside the expectation is purely real, and consequently by standard results (cf., [11]), one obtains the asserted representation. □

As noted above, we will perform the computations mainly in the simpler, steady-state case of $\bar{k}_0 = 1$. In this case, we have

$$-A_{>0} = \omega \begin{pmatrix} -i & 0 \\ 2 & 0 \end{pmatrix}, \quad \text{and} \quad -b_{>0} = \tfrac{d}{2m} \begin{pmatrix} -2i \\ 1 \end{pmatrix}. \tag{67}$$

In the case $d = 0$, we have $-b_{>0} = 0$, while in the case $d = -2m\omega\hat{\delta}$, we have $-b_{>0} = \omega\hat{\delta}(2i, -1)^T$.

Theorem 3. *In the case $\bar{k}_0 = 1$, for all $r \in [s,t]$, ζ_r is a normal random variable with mean and covariance given by, with $\hat{d} \doteq d/(m\omega)$,*

$$\mu_r = \begin{pmatrix} \mu_r^1 \\ \mu_r^2 \end{pmatrix} \quad \textit{and} \quad \tilde{\Sigma}_r \doteq \begin{pmatrix} \tilde{\Sigma}_r^{1,1} & \tilde{\Sigma}_r^{1,2} \\ \tilde{\Sigma}_r^{2,1} & \tilde{\Sigma}_r^{2,2} \end{pmatrix}, \qquad \textit{where}$$
$$\mu_r^1 = e^{-i\omega(r-s)} z_1 + \hat{d}(e^{-i\omega(r-s)} - 1),$$
$$\mu_r^2 = 2i[e^{-i\omega(r-s)} - 1] z_1 + z_2 + \hat{d}\big[2i((e^{-i\omega(r-s)} - 1) - 3\omega(r-s)/2\big],$$
$$\tilde{\Sigma}_r^{1,1} = \tfrac{\hbar}{m\omega} \tfrac{1}{2} \big(1 - e^{-2i\omega(r-s)}\big),$$
$$\tilde{\Sigma}_r^{1,2} = \tilde{\Sigma}_r^{2,1} = \tfrac{\hbar}{m\omega} i \big[2\big(e^{-i\omega(r-s)} - 1\big) - \big(e^{-2i\omega(r-s)} - 1\big)\big],$$
$$\tilde{\Sigma}_r^{2,2} = \tfrac{\hbar}{m\omega} \big[2\big(e^{-2i\omega(r-s)} - 1\big) - 8\big(e^{-i\omega(r-s)} - 1\big) - 3i\omega(r-s)\big].$$

Proof. The expression for μ_r is immediate from Lemma 7. To obtain the expression for the covariance, we evaluate the integral in Lemma 9. Letting $\tilde{\Sigma}_r \doteq \mathbb{E}\big[\Delta_r \Delta_r^T\big]$, component-wise, that integral is

$$\tilde{\Sigma}_r^{1,1} = \tfrac{i\hbar}{m} \int_s^r e^{-2i\omega(r-\rho)}\, d\rho, \qquad \tilde{\Sigma}_r^{1,2} = \tilde{\Sigma}_r^{2,1} = \tfrac{i\hbar}{m} \int_s^r 2i\big[e^{-2i\omega(r-\rho)} - e^{-i\omega(r-\rho)}\big]\, d\rho,$$
$$\tilde{\Sigma}_r^{2,2} = \tfrac{i\hbar}{m} \int_s^r -4\big[e^{-2i\omega(r-\rho)} - e^{-i\omega(r-\rho)}\big]^2 + 1\, d\rho.$$

Evaluating these, one obtains the asserted expression for the covariance. □

Theorem 4. *For $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$, $\hat{S}^1(s,z) = \sum_{l=0}^{3} \sum_{j=0}^{l} \hat{c}_{l,j}^1(s) z_1^j z_2^{l-j}$, where the time-indexed coefficients, $\hat{c}_{l,j}^1(\cdot)$ are obtained by the evaluation of linear combinations of moments of up to third-order of the normal random variables ζ_r and closed-form time-integrals. For $k > 1$ and $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$, the $\hat{S}^k$ also take the similar forms, $\hat{S}^k(s,z) = \sum_{l=0}^{k+2} \sum_{j=0}^{l} \hat{c}_{l,j}^k(s) z_1^j z_2^{l-j}$. Given the coefficient functions $\hat{c}_{l,j}^{\kappa}(s)$ for $\kappa < k$, the time-indexed coefficients $\hat{c}_{l,j}^k(s)$ are obtained by the evaluation of linear combinations of moments of up to $(k+2)^{th}$-order of the normal random variables ζ_r and closed-form time-integrals.*

Proof. By Fubini's Theorem and Theorem 2,

$$\hat{S}^k(s,z) = \int_s^t -\frac{1}{2m}\sum_{\kappa=1}^{k-1} \mathbb{E}\Big\{\big[S_z^\kappa(r,\zeta_r)\big]^T S_z^{k-\kappa}(r,\zeta_r)\Big\} \qquad (68)$$

$$+ m\omega^2 \sum_{j=0}^{k+2} c_{k+2,j}^V \mathbb{E}\big\{[\zeta_r]_1^j[\zeta_r]_2^{k+2-j}\big\}\,dr + \sum_{l=0}^{k+2}\sum_{j=0}^{l} b_{k+2,l,j}^\phi \mathbb{E}\big\{[\zeta_t]_1^j[\zeta_t]_2^{l-j}\big\}.$$

In particular,

$$\hat{S}^1(s,z) = \int_s^t m\omega^2\Big[\mathbb{E}\big\{-[\zeta_r]_1^3\big\} + \tfrac{3}{2}\mathbb{E}\big\{[\zeta_r]_1[\zeta_r]_2^2\big\}\Big]\,dr + \sum_{l=0}^{3}\sum_{j=0}^{l} b_{3,l,j}^\phi \mathbb{E}\big\{[\zeta_t]_1^j[\zeta_t]_2^{l-j}\big\}. \qquad (69)$$

We see that (69) immediately yields the assertions regarding $\hat{S}^1$. If for $\kappa < k$, the $\hat{S}^\kappa(s,z)$ are polynomials in z of order at most $\kappa+2$, then the products-of-derivatives, $[S_z^\kappa(r,\zeta_r)]^T S_z^{k-\kappa}(r,\zeta_r)$, in (68) are of order at most $k+2$ in ζ_r, and the asserted form follows. □

7 The $\hat{S}^1$ term

In Section 5, steady-state and periodic solutions for the zeroth-order term in the expansion were computed. Here, we proceed an additional step, computing $\check{S}^1 \doteq \hat{S}^0 + \frac{1}{\hat{\delta}}\hat{S}^1$. We perform the actual computations for $\hat{S}^1$ only in the steady-state case of $\bar{k}_0 = 1$. For $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$, we may obtain $\hat{S}^1(s,z)$ from (69), using the expressions for the mean and variance of normal ζ_r given in Theorem 3. We see that we must evaluate integrals of the moments $\mathbb{E}\big\{[\zeta_r]_1^3\big\}$ and $\mathbb{E}\big\{[\zeta_r]_1[\zeta_r]_2^2\big\}$ as well as the general moments $\mathbb{E}\big\{[\zeta_t]_1^j[\zeta_t]_2^{l-j}\big\}$ for $j \in]0,l[$, $l \in]0,3[$. There are well-known expressions for all moments of normal random variables. In particular,

$$\mathbb{E}\big\{[\zeta_r]_1^3\big\} = [\mu_r]_1^3 + 3[\mu_r]_1\tilde{\Sigma}_r^{1,1},$$
$$\mathbb{E}\big\{[\zeta_r]_1[\zeta_r]_2^2\big\} = [\mu_r]_1[\mu_r]_2^2 + [\mu_r]_1\tilde{\Sigma}_r^{2,2} + 2[\mu_r]_2\tilde{\Sigma}_r^{1,2}.$$

This implies that for the integral term in (69), we must evaluate the integrals of moments given by $\int_s^t[\mu_r]_1^3\,dr$, $\int_s^t[\mu_r]_1\tilde{\Sigma}_r^{1,1}\,dr$, $\int_s^t[\mu_r]_1[\mu_r]_2^2\,dr$, $\int_s^t[\mu_r]_1\tilde{\Sigma}_r^{2,2}\,dr$, and $\int_s^t[\mu_r]_2\tilde{\Sigma}_r^{1,2}\,dr$. We note that, as our interest is in the solution of the original forward-time problem, it is sufficient to take $s=0$. Further, as our interest will be in periodic-plus-drift solutions, we take $t=\tau \doteq 2\pi/\omega$. With assiduous effort, one eventually finds

$$\mathbb{E}\Big\{\int_0^\tau -\hat{V}^1(\zeta_r)\,dr\Big\} = \int_0^\tau m\omega^2\Big[\mathbb{E}\big\{-[\zeta_r]_1^3\big\} + \tfrac{3}{2}\mathbb{E}\big\{[\zeta_r]_1[\zeta_r]_2^2\big\}\Big]\,dr$$
$$= m\omega^2\Big\{\tfrac{3\pi\hat{d}}{\omega}[z_1^2 + iz_1z_2 - z_2^2] + c_1(\tau)(1,2i)z + c_2(\tau)\Big\}, \qquad (70)$$

where

$$c_1(\tau) = (3\pi/\omega)\left[\hat{d}^2(1-3i\pi)/2 - \hbar/(m\omega)\right],$$
$$c_2(\tau) = \tfrac{\pi\hat{d}\hbar}{m\omega^2}(18i\pi - 9/2) + \tfrac{3\pi\hat{d}^3}{2\omega}\left((1/3) - 3i\pi - 6\pi^2\right).$$

From (70), we see that the expected value, $\mathbb{E}\int_0^\tau -\hat{V}^1(\zeta_r)\,dr$ has at most quadratic terms in z. (In contrast, for typical $t \neq \tau$, this integral is cubic in z.) Consequently, it may be of interest to take terminal cost, ϕ^1 to be quadratic rather than the more general hypothesized cubic form. Suppose we specifically take

$$\phi^1(z) \doteq \tfrac{1}{2} z^T Q^1 z, \tag{71}$$

where Q^1 has components $Q^1_{j,k}$. Noting that we are seeking a solution of form $\breve{S}^1 = \hat{S}^0 + \frac{1}{\delta}\hat{S}^1$, we find it helpful to now allow general $d \in \mathbb{C}$ with corresponding $\bar{\Lambda}^0$ given by (48). Also, note from Theorem 3 that

$$\mu_\tau = z - \left(0, \tfrac{3\pi d}{m\omega}\right)^T, \qquad \tilde{\Sigma}^{1,1}_\tau = \tilde{\Sigma}^{1,1}_\tau = 0, \qquad \tilde{\Sigma}^{2,2}_\tau = \tfrac{-6i\pi\hbar}{m\omega}. \tag{72}$$

Combining (69) and (70)–(72), we find

$$\hat{S}^1(\tau,z) = \tfrac{1}{2} z^T (Q^1 + Q^\Delta) z + b^T z + \rho^1(\tau), \tag{73}$$

where

$$Q^\Delta = 6\pi d \begin{pmatrix} 1 & i/2 \\ i/2 & -1 \end{pmatrix}, \qquad b = \left[\tilde{k}_1\left(Q^1 + Q^\Delta\right) + \tilde{k}_2 Q^\Delta\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{74}$$

$$\rho^1(\tau) = \tilde{k}_1\left[\tfrac{\tilde{k}_1}{2} + \tfrac{i\hbar}{d}\right] Q^1_{2,2} + \tfrac{\pi d\hbar}{m\omega}(18i\pi - 9/2) + \tfrac{3\pi d^3}{2m^2\omega^2}\left((1/3) - 3i\pi - 6\pi^2\right), \tag{75}$$

$$\tilde{k}_1 = \tfrac{-3\pi d}{m\omega}, \qquad \tilde{k}_2 = \tfrac{i\hbar}{d} + \tfrac{d}{2m\omega}(3\pi - i). \tag{76}$$

Recalling that $\hat{S}^0(\tau,z) = \frac{1}{2} z^T \bar{Q}^0 z + (\bar{\Lambda}^0)^T z + \rho^0(\tau)$, we find

$$\begin{aligned}\breve{S}^1(\tau,z) &= \hat{S}^0(\tau,z) + \tfrac{1}{\delta}\hat{S}^1(\tau,z) \\ &= \tfrac{1}{2} z^T\left[\bar{Q}^0 + \tfrac{1}{\delta}\left(Q^1 + Q^\Delta\right)\right] z + \left[\bar{\Lambda}^0 + \tfrac{1}{\delta} b\right]^T z + \rho^0(\tau) + \tfrac{1}{\delta}\rho^1(\tau)\end{aligned} \tag{77}$$

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