Employing the Staticization Operator in Conservative Dynamical Systems and the Schrödinger Equation

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***Abstract*— Conservative dynamical systems propagate as stationary points of the action functional. Using this representation, it has previously been demonstrated that one may obtain fundamental solutions for two-point boundary value problems for some classes of conservative systems via solution of an associated dynamic program. It is also known that the gravitational and Coulomb potentials may be represented as stationary points of cubicly-parameterized quadratic functionals. Hence, stationary points of the action functional may be represented via iterated staticization of polynomial functionals. This leads to representations through operations on sets of solutions of differential Riccati equations (DREs). A key step in this process is the reordering of staticization operations.**

Key words. dynamic programming, stationary action, staticization, two-point boundary value problems, Schrödinger equation, conservative dynamical systems.

I. Introduction

Conservative dynamical systems propagate as stationary points of the action functional over the possible paths of the system. This viewpoint appears particularly useful in certain domains of modern physics, including the areas of quantum and relativistic systems, cf. [4], [6], [20]. The stationary-action formulation has also recently been found to be quite useful for generation of fundamental solutions to two-point boundary-value problems (TPBVPs) for conservative dynamical systems, cf. [2], [16], [18].

To give a sense of this latter application domain, consider a finite-dimensional action-functional formulation of such a TPBVP. Let the path of the conservative system be denoted by ξ_r for $r \in [0,t]$ with $\xi_0 = \bar{x}$, in which case the action functional, with an appended terminal cost, may take the form

$$J(\bar{x},t,u) \doteq \int_0^t T(u_r) - V(\xi_r)\,dr + \psi(\xi_t), \qquad (1)$$

where $\dot{\xi} = u$, $u \in \mathcal{U} \doteq L_2(0,t)$, $T(\cdot)$ denotes the kinetic energy associated to the momentum (specifically taken to be $T(v) \doteq \frac{1}{2}v^T\mathcal{M}v$ throughout, with $\mathcal{M}$ positive-definite and symmetric), and $V(\cdot)$ denotes a potential energy field. If, for example, one takes $\psi(x) \doteq -\bar{v}^T\mathcal{M}x$, a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\dot{\xi}_t = \bar{v}$; if one takes ψ to be a min-plus delta function centered at z, then a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\xi_t = z$, cf. [2]. In the early work of Hamilton, it was formulated as the least-action principle [7], which states that a conservative dynamical system follows the trajectory that minimizes the action functional. However, this is typically only the case for relatively short-duration cases, cf. [6], [2], [18] and the references therein. In order to extend to longer-duration problems, it becomes necessary to apply concepts of stationarity [16], [17].

It is worth noting that if one defines $\mathrm{stat}_{x\in\mathcal{X}}\,\phi(x)$ to be the critical value of ϕ (defined rigorously in Section II-A), then a gravitational potential given as $V(x) = -\mu/|x|$ for $x \neq 0$ and constant $\mu > 0$, has the representation $V(x) = -(\frac{3}{2})^{3/2}\mu\,\mathrm{stat}_{\alpha>0}\{\alpha - \frac{\alpha^3|x|^2}{2}\}$, where we note that the argument of the stat operator is polynomial, [8], [18]. The Schrödinger equation in the context of a Coulomb potential may be similarly addressed. In that case, it is particularly helpful to consider an extension of the space variable to a vector space over the complex field, say $x \in \mathbb{C}^n$ rather than $x \in \mathbb{R}^n$. More specifically, for $x \in \mathbb{C}^n$, this representation takes a general form $V(x) = -(\frac{3}{2})^{3/2}\hat{\mu}\,\mathrm{stat}_{\alpha\in\mathcal{A}^R}\{\alpha - \frac{\alpha^3|x|_c^2}{2}\}$, where $\mathcal{A}^R \doteq \{\alpha = r[\cos(\theta) + i\sin(\theta)] \in \mathbb{C} \,|\, r \geq 0,\, \theta \in (\frac{-\pi}{2}, \frac{\pi}{2}]\}$, and for $x \in \mathbb{C}^n$, $|x|_c^2 \doteq \sum_{j=1}^n x_j^2$, [12]. (We remark that notation $|\cdot|_c^2$ is not intended to indicate a squared norm; the range is complex.) In the simple one-dimensional case, the resulting function on $\mathbb{C}$ has a branch cut along the negative imaginary axis, and this generalizes to the higher-dimensional case in the natural way.

Although stationarity-based representations for gravitational and Coulomb potentials are inside the integral in (1), they may be moved outside through the introduction of α-valued processes, cf. [8], [18]. In particular, not only does one seek the stationary path for action J, but the action functional itself can be given as a stationary value of an integral of a polynomial, leading to an iterated-stat problem formulation for such TPBVPs. This may be exploited in the solution of TPBVPs in such systems, cf. [8], [16], [18].

It has also been demonstrated that this stationary-action approach may be applied to TPBVPs for infinite-dimensional conservative systems described by classes of lossless wave equations, see for example [2]. There, stat is used in the construction of fundamental solution groups for these wave equations by appealing to stationarity of action on longer horizons.

Lastly, it has recently been demonstrated that stationarity may be employed to obtain a Feynman-Kac type of representation for solutions of the Schrödinger initial value problem (IVP) for certain classes of initial conditions and potentials [15]. As with the conservative-system cases above, these representations are valid for indefinitely long duration problems, whereas with only the minimization operation,

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such representations are valid only on time intervals such that the action remains convex.

In all of these examples, one obtains the stationary value of an action functional where the action functional itself takes the form of a stationary value of a functional that is quadratic in the momentum (the $u.$ in put in (1)) and cubic in the newly introduced potential energy parameterization variable (a time-dependent form of the α parameter above). That is, the overall stationary value is obtained from iterated staticization operations, where the outer stat is over a variable in which the functional is quadratic. Thus, if one can invert the order of the of the stat operations, then the inner stat operation results in a functional that is obtained as a solution of a differential Riccati equation (DRE). (It should be noted that this DRE must typically be propagated past escape times, where this propagation may be efficiently performed through the use of what has been termed "stat duality", cf. [13].) Hence, after inversion of the order of the iterated stat operations, the problem may be reduced to a single stat operation such that the argument takes the form of a linear functional operating on a set of DRE solutions. Consequently, an issue of fundamental importance regards conditions under which one may invert the order of stat operations in an iterated staticization.

In Section II, the staticization operator will be rigorously defined, and a general problem class along with some corresponding notation will be indicated. Then, in Section III, a somewhat general condition will be indicated, and it will shown that one may invert the order of staticization operations under that condition. Section IV will present several classes of problems for which the general condition of Section III holds. Finally, in Section V, the stationary-action and Schrödinger-equation applications will be discussed.

II. Problem and Stationarity Definitions

Before the issue to be studied can be properly expressed, it is necessary to define stationarity and the staticization operator.

A. Stationarity definitions

As noted above, the motivation for this effort is the computation and propagation of stationary points of payoff functionals, which is unusual in comparison to the standard classes of problems in optimization (although one should note for example, [3]). In analogy with the language for minimization and maximization, we will refer to the search for stationary points as "staticization", with these points being statica, in analogy with minima/maxima, and a single such point being a staticum in analogy with minimum/maximum. We note that Ekeland [3] employed the term "extremization" for what is essentially the same notion that is being referred to here as staticization. We make the following definitions. Let $\mathcal{F}$ denote either the real or complex field. Suppose $\mathcal{U}$ is a normed vector space (over $\mathcal{F}$) with $\mathcal{A} \subseteq \mathcal{U}$, and suppose $G : \mathcal{A} \to \mathcal{F}$. We say $\bar{u} \in \operatorname{argstat}_{u\in\mathcal{A}} G(u) \doteq \operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\}$ if $\bar{u} \in \mathcal{A}$ and either

$$\limsup_{u\to\bar{u},\, u\in\mathcal{A}\setminus\{\bar{u}\}} \frac{|G(u) - G(\bar{u})|}{|u - \bar{u}|} = 0, \qquad (2)$$

or $\bar{u}$ is an isolated point of $\mathcal{A}$. If $\operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} \neq \emptyset$, we define the possibly set-valued stat^s operation by

$$\begin{aligned} \operatorname*{stat}^s_{u\in\mathcal{A}} G(u) &\doteq \mathrm{stat}^s\{G(u) \,|\, u \in \mathcal{A}\} \\ &\doteq \big\{ G(\bar{u}) \,\big|\, \bar{u} \in \operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} \big\}. \end{aligned} \qquad (3)$$

If $\operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\} = \emptyset$, $\mathrm{stat}^s_{u\in\mathcal{A}} G(u)$ is undefined. Where applicable, we are also interested in a single-valued stat operation (note the absence of superscript s). In particular, if there exists $a \in \mathcal{F}$ such that $\mathrm{stat}^s_{u\in\mathcal{A}} G(u) = \{a\}$, then $\mathrm{stat}_{u\in\mathcal{A}} G(u) \doteq a$; otherwise, $\mathrm{stat}_{u\in\mathcal{A}} G(u)$ is undefined. At times, we may abuse notation by writing $\bar{u} = \operatorname{argstat}\{G(u) \,|\, u \in \mathcal{A}\}$ in the event that the $\operatorname{argstat}$ is the single point $\{\bar{u}\}$. The following is immediate from the above definitions.

Lemma 1: Suppose $\mathcal{U}$ is a Hilbert space, with open set $\mathcal{A} \subseteq \mathcal{U}$, and that G is Fréchet differentiable at $\bar{u} \in \mathcal{A}$. Then, $\bar{u} \in \operatorname{argstat}\{G(y) \,|\, y \in \mathcal{A}\}$ if and only if the Fréchet derivative is zero at $\bar{u}$.

B. Problem definition

Let $\mathcal{U}, \mathcal{V}$ be Hilbert spaces with inner products and norms on $\mathcal{U}$ denoted by $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ and $|\cdot|_{\mathcal{U}}$, and similarly for $\mathcal{V}$. Let the resulting inner product and norm on $\mathcal{U}\times\mathcal{V}$ be denoted by $\langle\cdot,\cdot\rangle_{\mathcal{U}\times\mathcal{V}}$ and $|\cdot|_{\mathcal{U}\times\mathcal{V}}$. Let $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \mathcal{V}$ be open. Throughout, we assume

$$G \in C^2(\mathcal{A}\times\mathcal{B};\mathcal{F}). \qquad (A.1)$$

Then, for each $u \in \mathcal{A}$, let $g^{1,u} \in C^2(\mathcal{B};\mathcal{F})$ be given by $g^{1,u}(v) \doteq G(u,v)$ for all $v \in \mathcal{B}$. Similarly, for each $v \in \mathcal{B}$, let $g^{2,v} \in C^2(\mathcal{A};\mathcal{F})$ be given by $g^{2,v}(u) \doteq G(u,v)$ for all $u \in \mathcal{A}$. Further, let

$$\begin{aligned} \mathcal{A}_G &\doteq \big\{ u \in \mathcal{A} \,\big|\, \operatorname*{stat}_{v\in\mathcal{B}} g^{1,u}(v) \text{ exists} \big\} \quad \text{and} \\ \mathcal{B}_G &\doteq \big\{ v \in \mathcal{B} \,\big|\, \operatorname*{stat}_{u\in\mathcal{A}} g^{2,v}(u) \text{ exists} \big\}. \end{aligned} \qquad (4)$$

Given $u \in \mathcal{A}_G$, let $\mathcal{M}^1(u) \doteq \operatorname{argstat}_{v\in\mathcal{B}} g^{1,u}(v)$. Similarly, given $v \in \mathcal{B}_G$, let $\mathcal{M}^2(v) \doteq \operatorname{argstat}_{u\in\mathcal{A}} g^{2,v}(u)$. Next, define $\bar{G}^1 : \mathcal{A}_G \to \mathcal{F}$ and $\bar{G}^2 : \mathcal{B}_G \to \mathcal{F}$ by

$$\bar{G}^1(u) \doteq \operatorname*{stat}_{v\in\mathcal{B}} g^{1,u}(v) \;\; \forall\, u \in \mathcal{A}_G \quad \text{and}$$

$$\bar{G}^2(v) \doteq \operatorname*{stat}_{u\in\mathcal{A}} g^{2,v}(u) \;\; \forall\, v \in \mathcal{B}_G.$$

Finally, let

$$\hat{\mathcal{A}}_G \doteq \operatorname*{argstat}_{u\in\mathcal{A}_G} \bar{G}^1(u) \quad \text{and} \quad \hat{\mathcal{B}}_G \doteq \operatorname*{argstat}_{v\in\mathcal{B}_G} \bar{G}^2(v).$$

We will discuss conditions under which

$$\operatorname*{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = \operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v) = \operatorname*{stat}_{v\in\mathcal{B}_G} \bar{G}^2(v). \qquad (5)$$

We will generally be concerned only with the left-hand equality in (5); obviously the right-hand equality would be obtained analogously. We refer to the left-hand object in (5) as iterated stat operations, while the center object will be referred to as a full stat operation. Although in some results, the existence of both the iterated and full stat operations are obtained, many of the results will assume the existence of one or both of these objects. We list the two potential assumptions below. In each result to follow, we will indicate when one or both of these is utilized. The full-stat assumption is as follows.

Assume $\mathrm{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ exists, and let $(\bar{u},\bar{v})\in \mathrm{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}\, G(u,v)$. $\quad (A.2f)$

Note that under Assumption $(A.2f)$,

$$\bar{u}\in\mathcal{A}_G,\ \ \bar{v}\in\mathcal{B}_G,\ \ \bar{v}\in\mathcal{M}^1(\bar{u}),\ \text{ and }\ \bar{u}\in\mathcal{M}^2(\bar{v}). \tag{6}$$

The iterated-stat assumption is as follows.

Assume $\mathrm{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)$ exists, and let $\bar{u}\in\hat{\mathcal{A}}_G$. $\quad (A.2i)$

Note that under Assumption $(A.2i)$,

$$\exists\,\bar{v}\in\mathcal{M}^1(\bar{u}),\ \text{ and }\ \operatorname*{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)=g^{1,\bar{u}}(\bar{v})=G(\bar{u},\bar{v}). \tag{7}$$

We will first obtain (5) under some general assumptions.

III. The General Case

Given $\mathcal{C}\subseteq\mathcal{V}$ and $\hat{v}\in\mathcal{V}$, we let $d(\hat{v},\mathcal{C})\doteq\inf_{v\in\mathcal{C}}|v-\hat{v}|$, and use this distance notation more generally throughout. In addition to $(A.1)$, we assume the following throughout this section.

There exist $\delta=\delta(\bar{u},\bar{v})>0$ and $K=K(\bar{u},\bar{v})<\infty$ such that $d(\bar{v},\mathcal{M}^1(u))\le K\,|\bar{u}-u|$ for all $u\in\mathcal{A}_G\cap B_\delta(\bar{u})$. $\quad (A.3)$

We note that $(A.3)$ is trivially satisfied in the case that there exists $\delta>0$ such that $B_\delta(\bar{u})\cap\mathcal{A}_G=\emptyset$. It may be helpful to also note that $(A.3)$ is satisfied under the possibly more heuristically appealing, following assumption.

For every $\tilde{u}\in\mathcal{A}_G$ and every $\tilde{v}\in\mathcal{M}^1(\tilde{u})$, there exist $\delta=\delta(\tilde{u},\tilde{v})>0$ and $K=K(\tilde{u},\tilde{v})<\infty$ such that $d(\tilde{v},\mathcal{M}^1(u))\le K\,|\tilde{u}-u|$ for all $u\in\mathcal{A}_G\cap B_\delta(\tilde{u})$. $\quad (A.3')$

Lemma 2: Assume $(A.2f)$. Then, $\bar{u}\in\hat{\mathcal{A}}_G$ and $G(\bar{u},\bar{v})\in\mathrm{stat}^s_{u\in\mathcal{A}_G}\,\bar{G}^1(u)$.

Proof: Let $(\bar{u},\bar{v})$ be as in $(A.2f)$. Let $R\doteq d\big((\bar{u},\bar{v}),(\mathcal{A}\times\mathcal{B})^c\big)$. By Assumption $(A.3)$, there exist $\delta\in(0,R/2)$ and $K<\infty$ such that for all $u\in\mathcal{A}_G\cap B_\delta(\bar{u})$ and all $\epsilon\in(0,1)$, there exists $v\in\mathcal{M}^1(u)$ such that

$$|v-\bar{v}|\le(K+\epsilon)|u-\bar{u}|\le(K+\epsilon)\delta. \tag{8}$$

Let $\tilde{u}\in\mathcal{A}_G\cap B_{\delta/(K+1)}(\bar{u})$. By (6),

$$|\operatorname*{stat}_{v\in\mathcal{B}}g^{1,\tilde{u}}(v)-\operatorname*{stat}_{v\in\mathcal{B}}g^{1,\bar{u}}(v)|=|\operatorname*{stat}_{v\in\mathcal{B}}g^{1,\tilde{u}}(v)-G(\bar{u},\bar{v})|,$$

and by (8), there exists $\tilde{v}\in B_\delta(\bar{v})$ such that this is

$$=|G(\tilde{u},\tilde{v})-G(\bar{u},\bar{v})|. \tag{9}$$

Let $f\in C^\infty\big((-3/2,3/2);\mathcal{A}\times\mathcal{B}\big)$ be given by $f(\lambda)=\big(\bar{u}+\lambda(\tilde{u}-\bar{u}),\bar{v}+\lambda(\tilde{v}-\bar{v})\big)$ for all $\lambda\in(-3/2,3/2)$. Define $W^0(\lambda)=[G\circ f](\lambda)$ for all $\lambda\in(-3/2,3/2)$, and note that by Assumption $(A.1)$ and standard results, $W^0\in C^2\big((-3/2,3/2);\mathcal{F}\big)$. Similarly, let $W^1(\lambda)=[(G_u,G_v)\circ f](\lambda)=\big(G_u(f(\lambda)),G_v(f(\lambda))\big)$. By Assumption $(A.1)$ and standard results, $W^1\in C^1\big((-3/2,3/2);\mathcal{U}\times\mathcal{V}\big)$. Then, by the Mean Value Theorem (cf. [1, Th. 12.6]), there exists $\lambda_0\in(0,1)$ such that

$$\begin{aligned}&|G(\tilde{u},\tilde{v})-G(\bar{u},\bar{v})|=|W^0(1)-W^0(0)|\\&\le\Big|\frac{dG}{d(u,v)}(f(\lambda_0))\Big|\,\Big|\frac{df}{d\lambda}(\lambda_0)\Big|\\&=\big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)\big|\,\big|(\tilde{u}-\bar{u},\tilde{v}-\bar{v})\big|,\end{aligned}$$

where $(u_0,v_0)\doteq f(\lambda_0)$, and which by (8),

$$\le\big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)\big|\,\sqrt{1+(K+1)^2}|\tilde{u}-\bar{u}|. \tag{10}$$

Similarly, there exists $\lambda_1\in(0,\lambda_0)$ such that

$$\begin{aligned}&\big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)-\big(G_u(\bar{u},\bar{v}),G_v(\bar{u},\bar{v})\big)\big|\\&=|W^1(\lambda_0)-W^1(0)|\le\Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\,\Big|\frac{df}{d\lambda}(\lambda_1)\Big|\,|\lambda_1|\\&\le\Big|\frac{d^2G}{d(u,v)^2}(u_1,v_1)\Big|\,\big|(u_1-\bar{u},v_1-\bar{v})\big|,\end{aligned}$$

where $(u_1,v_1)\doteq f(\lambda_1)$, and this is

$$\le\Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\sqrt{1+(K+1)^2}|\tilde{u}-\bar{u}|.$$

Recalling $(\bar{u},\bar{v})\in\mathrm{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}\,G(u,v)$, this implies

$$\begin{aligned}&\big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)\big|\\&\le\Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\sqrt{1+(K+1)^2}|\tilde{u}-\bar{u}|.\end{aligned} \tag{11}$$

Combining (10) and (11) yields

$$|G(\tilde{u},\tilde{v})-G(\bar{u},\bar{v})|\le\Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\big[1+(K+1)^2\big]\,|\tilde{u}-\bar{u}|^2.$$

Let $K_1\doteq\big|\frac{d^2G}{d(u,v)^2}(\bar{u},\bar{v})\big|$. By $(A.1)$, there exists $\hat{\delta}\in(0,\delta/(K+1))$ such that for all $(u,v)\in B_{\hat{\delta}}(\bar{u},\bar{v})$, $\big|\frac{d^2G}{d(u,v)^2}(u,v)\big|\le K_1+1$. Hence, there exists $\bar{C}<\infty$ such that

$$|G(\tilde{u},\tilde{v})-G(\bar{u},\bar{v})|\le\bar{C}|\tilde{u}-\bar{u}|^2\ \ \forall\,(\tilde{u},\tilde{v})\in\mathcal{A}_G\cap B_{\hat{\delta}/(K_1+1)}(\bar{u}). \tag{12}$$

Combining (9) and (12) one has $|\mathrm{stat}_{v\in\mathcal{B}}\,g^{1,\tilde{u}}(v)-\mathrm{stat}_{v\in\mathcal{B}}\,g^{1,\bar{u}}(v)|\le\bar{C}|\tilde{u}-\bar{u}|^2$, which upon recalling that $\tilde{u}\in\mathcal{A}_G\cap B_{\hat{\delta}/(K+1)}(\bar{u})$ was arbitrary, yields the assertions. ■

Theorem 3: Assume $(A.2f)$ and $(A.2i)$. Then

$$\operatorname*{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)=G(\bar{u},\bar{v})=\operatorname*{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}}G(u,v).$$

Proof: The assertions follow directly from the assumption, $(A.2f)$ and Lemma 2. ■

IV. Some Specific Cases

We examine several classes of functionals that fit within the general class above.

A. The Semi-Quadratic Case

Here we take $\mathcal{A} \subseteq \mathcal{U}$ and $\mathcal{B} = \mathcal{V}$. We let

$$G(u, v) = f_1(u) + \langle f_2(u), v\rangle_{\mathcal{V}} + \tfrac{1}{2}\langle \bar{B}_3(u)v, v\rangle_{\mathcal{V}}, \qquad (13)$$

for all $u \in \mathcal{A}$ and $v \in \mathcal{V}$, where $f_1 \in C^2(\mathcal{A}; \mathcal{F})$, $f_2 \in C^2(\mathcal{A}; \mathcal{V})$ and $\bar{B}_3 \in C^2(\mathcal{A}; \mathcal{L}(\mathcal{V}, \mathcal{V}))$, and f_2, $\bar{B}_3$ are Fréchet differentiable with continuous Fréchet derivatives. In addition, suppose The Moore-Penrose pseudo-inverse $\bar{B}_3^{\#}(u)$ of $\bar{B}_3(u)$ is differentiable for all $u \in \mathcal{A}_G$, and there exists a constant $D > 0$ such that $|\bar{B}_3^{\#}(u)| \leq D$ for all $u \in \mathcal{A}_G$. The next lemma follows directly from (13) and Lemma 1.

Lemma 4: Let $\hat{u} \in \mathcal{A}$. Then $\hat{v} \in \mathcal{M}^1(\hat{u})$ if and only if $f_2(\hat{u}) + \bar{B}_3(\hat{u})\hat{v} = 0$.

1) When the full staticization is known to exist:

Lemma 5: Assume $(A.2f)$. Then assumption $(A.3)$ is satisfied.

Proof: The result is trivial for $\mathcal{A} = \{\bar{u}\}$. Suppose $\mathcal{A}_G \neq \{\bar{u}\}$. Choose an $\delta > 0$ such that $\mathcal{A}_G \cap (B_\delta(\bar{u}) \backslash \{\bar{u}\}) \neq \emptyset$. Let $\hat{u} \in \mathcal{A}_G \cap B_\delta(\bar{u})$, and $\hat{u} \neq \bar{u}$. Let $\hat{v} = \bar{v} - \bar{B}_3^{\#}(\hat{u})f_2(\hat{u}) - \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}$. By properties of pseudoinverse, we have

$$\begin{aligned}
&\bar{B}_3(\hat{u})\hat{v} + f_2(\hat{u}) \\
&= \bar{B}_3(\hat{u})[\bar{v} - \bar{B}_3^{\#}(\hat{u})f_2(\hat{u}) - \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}] + f_2(\hat{u}) \\
&= \bar{B}_3(\hat{u})\bar{v} - f_2(\hat{u}) - \bar{B}_3(\hat{u})\bar{v} + f_2(\hat{u}) = 0
\end{aligned}$$

Therefore, $\hat{v} \in \mathcal{M}^1(\hat{u})$ by Lemma 4. Since f_2, $\bar{B}_3$ are Fréchet differentiable with continuous Fréchet derivatives, there exist constants $K_2 > 0$, $K_B > 0$, such that $|f_2(u_1) - f_2(u_2)| \leq K_2|u_1 - u_2|$, $|\bar{B}_3(u_1) - \bar{B}_3(u_2)| \leq K_B|u_1 - u_2|$ for all $u_1, u_2 \in B_\delta(\bar{u})$. We have

$$|\hat{v} - \bar{v}| = \big| - \{\bar{B}_3^{\#}(\hat{u})f_2(\hat{u}) + \bar{B}_3^{\#}(\hat{u})\bar{B}_3(\hat{u})\bar{v}\}\big|$$

By Lemma 4, $\bar{B}_3(\bar{u})\bar{v} + f_2(\bar{u}) = 0$,

$$\begin{aligned}
&= \big| - \bar{B}_3^{\#}(\hat{u})\{f_2(\hat{u}) - f_2(\bar{u}) - \bar{B}_3(\bar{u})\bar{v} + \bar{B}_3(\hat{u})\bar{v}\}\big| \\
&\leq |\bar{B}_3^{\#}(\bar{u})||f_2(\hat{u}) - f_2(\bar{u}) + (\bar{B}_3(\hat{u}) - \bar{B}(\bar{u}))\bar{v}\| \\
&\leq D\Big[K_2|\hat{u} - \bar{u}| + K_B|\bar{v}||\hat{u} - \bar{u}|\Big] \\
&\leq \bar{C}|\hat{u} - \bar{u}|, \text{ where } \bar{C} \doteq DK_2 + K_B|\bar{v}|
\end{aligned}$$

which gives $(A.3)$. ■

Theorem 6: Assume $(A.2f)$. Then $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u)$ exists, and $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = G(\bar{u}, \bar{v}) = \operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$.

Proof: $(A.3)$ is satisfied. This is a special case of Theorem 3. ■

2) When the iterated staticization is known to exist:

Theorem 7: Assume $(A.2i)$. Then $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$ exists, and $\operatorname{stat}_{u\in\mathcal{A}_G} \bar{G}^1(u) = G(\bar{u}, \bar{v}) = \operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$.

Proof: Assume $(A.2i)$, and let $\bar{v}$ be as given in (7). By Lemma 4, $v \in \mathcal{M}^1(u)$ if and only if $f_2(u) + \bar{B}_3(u)v = 0$. For $u \in \mathcal{A}_G$, let

$$\breve{v}(u) \doteq \bar{v} - \bar{B}_3^{\#}(u)f_2(u) - \bar{B}_3^{\#}(u)\bar{B}_3(u)\bar{v}$$

Note that $\breve{v}(\bar{u}) = \bar{v}$, and $\forall u \in \mathcal{A}_G$, we have

$$\begin{aligned}
&\bar{B}_3(u)\breve{v}(u) + f_2(u) \\
=&\bar{B}_3(u)[\bar{v} - \bar{B}_3^{\#}(u)f_2(u) - \bar{B}_3^{\#}(u)\bar{B}_3(u)\bar{v}] + f_2(u) \\
=&\bar{B}_3(u)\bar{v} - \bar{B}_3(u)\bar{B}_3^{\#}(u)f_2(u) - \bar{B}_3(u)\bar{B}_3^{\#}(u)\bar{B}_3(u)\bar{v} \\
&+ f_2(u) \\
=&\bar{B}_3(u)\bar{v} - f_2(u) - \bar{B}_3(u)\bar{v} + f_2(u) = 0
\end{aligned}$$

Hence, $\breve{v}(u) \in \mathcal{M}^1(u)$ $\forall u \in \mathcal{A}_G$, and $\bar{G}^1(u) = G(u, \breve{v}(u))$ $\forall u \in \mathcal{A}_G$. Then, by the choice of $\bar{u}$ and the assumption that $\bar{B}_3^{\#}(u)$ is differentiable for all $u \in \mathcal{A}_G$

$$\begin{aligned}
0 = \frac{d\bar{G}^1}{du}(\bar{u}) &= G_u(\bar{u}, \breve{v}(\bar{u})) + G_v(\bar{u}, \breve{v}(\bar{u}))\frac{d\breve{v}}{du}(\bar{u}), \\
&= G_u(\bar{u}, \bar{v}) + G_v(\bar{u}, \bar{v})\frac{d\breve{v}}{du}(\bar{u}) = G_u(\bar{u}, \bar{v}).
\end{aligned}$$

This and (7) imply that $(\bar{u}, \bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$ and $G(\bar{u}, \bar{v}) \in \operatorname{stat}^s{}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v)$.
Now suppose $\exists\, (\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \backslash \{(\bar{u}, \bar{v})\}$. This implies

$$\begin{gathered}
G_u(\hat{u}, \hat{v}) = 0, \qquad G_v(\hat{u}, \hat{v}) = 0 \\
\hat{v} \in \mathcal{M}^1(\hat{u}), \quad \text{and} \quad \bar{G}^1(\hat{u}) = G(\hat{u}, \hat{v}).
\end{gathered}$$

Let $\breve{v}\,'(u) \doteq \hat{v} - \bar{B}_3^{\#}(u)f_2(u) - \bar{B}_3^{\#}(u)\bar{B}_3(u)\hat{v}$, $\forall\, u \in \mathcal{A}_G$, and note that $\breve{v}\,'(\hat{u}) = \hat{v}$. Using the properties of the pseudo-inverse, we see that

$$\begin{aligned}
&\bar{B}_3(u)\breve{v}\,'(u) + f_2(u) \\
=&\bar{B}_3(u)\big[\hat{v} - \bar{B}_3^{\#}(u)f_2(u) - \bar{B}_3^{\#}(u)\bar{B}_3(u)\hat{v}\big] + f_2(u) \\
=&\bar{B}_3(u)\hat{v} - f_2(u) - \bar{B}_3(u)\hat{v} + f_2(u) = 0,
\end{aligned}$$

which implies that $\breve{v}\,'(u) \in \mathcal{M}^1(u)$ for all $u \in \mathcal{A}_G$. Hence,

$$\bar{G}^1(u) = G(u, \breve{v}\,'(u)) \;\; \forall\, u \in \mathcal{A}_G. \qquad (14)$$

By definition of $\breve{v}\,'(u)$ and (14), and the assumption that $\bar{B}_3^{\#}(u)$ is differentiable for all $u \in \mathcal{A}_G$,

$$\begin{aligned}
\frac{d\bar{G}^1}{du}(\hat{u}) &= G_u(\hat{u}, \breve{v}\,'(\hat{u})) + G_v(\hat{u}, \breve{v}\,'(\hat{u}))\frac{d\breve{v}\,'}{du}(\hat{u}), \\
&= G_u(\hat{u}, \hat{v}) + G_v(\hat{u}, \hat{v})\frac{d\breve{v}\,'}{du}(\hat{u}) = 0,
\end{aligned}$$

which implies that $\hat{u} \in \hat{\mathcal{A}}_G$. Using this and $(A.2i)$, we see that $G(\hat{u}, \hat{v}) = G(\bar{u}, \bar{v})$. As $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u, v) \setminus \{(\bar{u}, \bar{v})\}$ was arbitrary, we have the desired result. ■

B. The Uniformly Morse Case

Under condition $(A.1)$ and a condition that G be uniformly Morse (see $(A.4)$ below), we will find that Assumption $(A.3')$ holds (and consequently, $(A.3)$). Hence, one may apply Theorem 3. More specifically, we assume the following.

For all $(\hat{u}, \hat{v}) \in \mathcal{A} \times \mathcal{B}$, $G_{vv}(\hat{u}, \hat{v}) \in \mathcal{L}(\mathcal{V}; \mathcal{V})$ is invertible.
$(A.4)$

1) When the full staticization is known to exist:

Theorem 8: Assume $(A.2f)$. Then, $\operatorname{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)$ exists, and $\operatorname{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} G(u,v)$.

The proof of this theorem is delayed until after the next lemmas.

Lemma 9: Assume $(A.2f)$. Then, there exists $\epsilon > 0$ and $\breve{v} \in C^1(\mathcal{A}_G \cap B_\epsilon(\bar{u}); \mathcal{B})$ such that $\breve{v}(\bar{u}) = \bar{v}$ and $G_v(u, \breve{v}(u)) = 0$ for all $u \in \mathcal{A}_G \cap B_\epsilon(\bar{u})$.

Proof: This is simply the implicit mapping theorem, cf. [10]. ∎

By Lemma 9 and the definition of $\mathcal{A}_G$, $\bar{G}^1(u) = \operatorname{stat}_{v\in\mathcal{B}} g^{1,u}(v) = G(u,\breve{v}(u))\ \forall\, u \in \mathcal{A}_G \cap B_\epsilon(\bar{u})$.

$$\bar{G}^1(u) = \operatorname*{stat}_{v\in\mathcal{B}} g^{1,u}(v) = G(u,\breve{v}(u))\ \forall\, u \in \mathcal{A}_G \cap B_\epsilon(\bar{u}). \tag{15}$$

Then, by (15), Then, by the chain rule, $(A.1)$ and Lemma 9, $\bar{G}^1(\cdot) \in C^1(\mathcal{A}_G \cap B_\epsilon(\bar{u}); \mathcal{F})$.

$$\bar{G}^1(\cdot) \in C^1(\mathcal{A}_G \cap B_\epsilon(\bar{u}); \mathcal{F}). \tag{16}$$

Also, by Lemma 9, we immediately have the following.

Lemma 10: Assume $(A.2f)$. Then, there exists $K < \infty$ and $\delta \in (0,\epsilon)$ such that $|\breve{v}(u) - \breve{v}(\bar{u})| = |\breve{v}(u) - \bar{v}| \le K|u - \bar{u}|$ for all $u \in \mathcal{A}_G \cap B_\delta(\bar{u})$.

Proof: By Lemma 9, $\frac{d\breve{v}}{du}(\cdot)$ is continuous on $\mathcal{A}_G \cap B_\epsilon(\bar{u})$. Let $K_0 \doteq \left|\frac{d\breve{v}}{du}(\bar{u})\right|$. By the continuity, there exists $\delta > 0$ such that $\left|\frac{d\breve{v}}{du}(u)\right| < K \doteq K_0 + 1$ for all $u \in \mathcal{A}_G \cap B_\delta(\bar{u})$. Hence, by the mean value theorem, we have the asserted bound. ∎

Note that Lemma 10 implies that Assumption $(A.3)$ is satisfied, and hence one may apply Theorem 3.

Lemma 11: Assume $(A.2f)$. Then, $\operatorname{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)$ exists.

Proof: *(Proof of Theorem 8)* The assertion of the existence of $\operatorname{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)$ is simply Lemma Then, noting that Lemma 10 implies that Assumption $(A.3)$ is one may apply Theorem 3 to obtain the second assertion of the theorem. ∎

2) When the iterated staticization is known to exist:

Lemma 12: Assume $(A.2i)$. Then, $\operatorname{stat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ exists.

By Lemma 12 and Theorem 8 we have the following.

Theorem 13: Assume $(A.2i)$. Then, $\operatorname{argstat}_{(u,v)\in\mathcal{A}\times\mathcal{B}} G(u,v)$ exists, and $\operatorname{stat}_{(u,v)\in\mathcal{U}\times\mathcal{V}} G(u,v) = G(\bar{u},\bar{v}) = \operatorname{stat}_{u\in\mathcal{A}_G}\bar{G}^1(u)$.

V. Applications to Astrodynamics and the Schrödinger Equation

A. TPBVPs in Astrodynamics

One may obtain fundamental solutions to TPBVPs in astrodynamics through a stationary-action based approach [8], [9], [16], [18]. We briefly recall the case of the n-body problem. In this case, the action functional with an appended terminal cost (cf. [18]) takes the form indicated in (1), where now $x = \big((x^1)^T, (x^2)^T, \dots (x^n)^T\big)^T$, where the bodies are indexed by $j \in \mathcal{N} \doteq \{1, 2, \dots n\}$, and $\xi_\cdot$, $u_\cdot$ are similarly constructed. The kinetic-energy term is $T(u_r) \doteq \frac{1}{2}\sum_{j=1}^n m_j |u_r^j|^2$, where m_j is the mass of the j^{th} body. The additive inverse of the potential at any $x \in \mathbb{R}^{3n}$ is given by

$$\begin{aligned}
-V(x) &= \sum_{(i,j)\in\mathcal{I}^\Delta} \frac{G m_i m_j}{|x^i - x^j|} \\
&= \max_{\alpha\in(0,\infty)^{M_n}} \sum_{(i,j)\in\mathcal{I}^\Delta} \left(\tfrac{3}{2}\right)^{3/2} G\, m_i m_j \left[\alpha_{i,j} - \frac{\alpha_{i,j}^3 |x^i - x^j|^2}{2}\right] \\
&\doteq \max_{\alpha\in(0,\infty)^{M_n}} \tilde{V}(x,\alpha) = \operatorname*{stat}_{\alpha\in(0,\infty)^{M_n}} \tilde{V}(x,\alpha),
\end{aligned}$$

where G is the universal gravitational constant, $\mathcal{I}^\Delta \doteq \{(i,j) \in \mathcal{N}^2 \,|\, j > i\}$. and $M_n \doteq \#\mathcal{I}^\Delta$. Letting $\mathcal{U}_{(0,t)} \doteq L_2\big((0,t);\mathbb{R}^{3n}\big)$ and $\mathcal{A}_{(0,t)} \doteq L_2\big((0,t);(0,\infty)^{M_n}\big)$, one finds that the problem becomes that of finding the stationary-action value function given by

$$W(t,x) = \operatorname*{stat}_{u\in\mathcal{B}} \operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}_{(0,t)}} J(t,x,u,\tilde{\alpha}), \quad \text{where}$$

$$J(t,x,u,\tilde{\alpha}) \doteq \left\{ \int_0^t T(u_r) - \tilde{V}(\xi_r,\tilde{\alpha}_r)\, dr + \phi(\xi_t) \right\}, \tag{17}$$

where $\mathcal{B} \subseteq \mathcal{U}_{(0,t)}$. If one is able to reorder the staticization operations, the result may be decomposed as

$$\begin{aligned}
W(t,x) &= \operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}_{(0,t)}} \tilde{W}(t,x,\tilde{\alpha}), \\
\tilde{W}(t,x,\tilde{\alpha}) &\doteq \operatorname*{stat}_{u\in\mathcal{B}} J(t,x,u,\tilde{\alpha}).
\end{aligned}$$

Suppose ϕ is a quadratic form, say $\phi(x;z) \doteq \frac{1}{2}(x - z)^T P_0 (x - z) + \gamma_0$. (More generally, one may take ϕ to be a staticization of quadratic forms.) Then,

$$\tilde{W}(t,x,\tilde{\alpha}) = \tfrac{1}{2}(x^T P_t^{\tilde{\alpha}} x + x^T Q_t^{\tilde{\alpha}} z + z^T Q_t^{\tilde{\alpha}} x + z^T R_t^{\tilde{\alpha}} z + \gamma_t^{\tilde{\alpha}}),$$

where $P_\cdot^{\tilde{\alpha}}, Q_\cdot^{\tilde{\alpha}}, R_\cdot^{\tilde{\alpha}}$ may be obtained from solution of $\tilde{\alpha}$-indexed DREs, and $\gamma_t^{\tilde{\alpha}}$ is simply an integral. The means by which this may be utilized for efficient generation of fundamental solutions is indicated in the above-noted references. *The key is, of course, the ability to invert the order of the staticization operators, which is obtained in the following theorem.* Noting that J is semi-quadratic in u and uniformly Morse in $\tilde{\alpha}$, one obtains the following.

Theorem 14: Let $t \in (0,\infty)$ and $x \in \mathbb{R}^{3n}$. Suppose $\operatorname{stat}_{u\in\mathcal{B}} \operatorname{stat}_{\tilde{\alpha}\in\mathcal{A}_{(0,t)}} J(t,x,u,\tilde{\alpha})$ exists, where J is given by (17). Then,

$$\begin{aligned}
\operatorname*{stat}_{u\in\mathcal{B}} \operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}_{(0,t)}} J(t,x,u,\tilde{\alpha}) &= \operatorname*{stat}_{(u,\tilde{\alpha})\in\mathcal{B}\times\mathcal{A}_{(0,t)}} J(t,x,u,\tilde{\alpha}) \\
&= \operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}_{(0,t)}} \operatorname*{stat}_{u\in\mathcal{B}} J(t,x,u,\tilde{\alpha}) = W(t,x).
\end{aligned}$$

Remark 15: By construction, the existence assumed in the theorem will be equivalent to existence of a solution of the TPBVP under consideration.

B. Schrödinger IVPs

We briefly discuss the application to Schrödinger IVPs. In this case, the dynamics are now stochastic and complex-valued. In this case we consider only the problem of a single particle in a central Coulomb field. The IVP is

$$0 = i\hbar\psi_t(s,y) + \tfrac{\hbar^2}{2m}\Delta\psi(s,y) - \psi(s,y)V(y), \quad (s,y) \in \mathcal{D},$$

$$\psi(0,y)=\psi_0(y), \quad y\in\mathbb{R}^n,$$

where $m\in(0,\infty)$ denotes particle mass, initial condition ψ_0 takes values in $\mathbb{C}$, V denotes the Coulomb potential function, Δ denotes the Laplacian with respect to the space (second) variable, $\mathcal{D}\doteq(0,t)\times\mathbb{R}^n$, and subscript t will denote the derivative with respect to the time variable (the first argument of ψ here) regardless of the symbol being used for time in the argument list. We consider what is sometimes referred to as the Maslov dequantization of the solution of the Schrödinger equation (cf. [11]), which as noted above, is $S:\bar{\mathcal{D}}\to\mathbb{C}$ given by $\psi(s,y)=\exp\{\frac{i}{\hbar}S(s,y)\}$. We also extend the space from $\mathbb{R}^n$ to $\mathbb{C}^n$, and reverse the time variable. The resulting transformed problem is given by [12], [14], [15]

$$\begin{aligned}
&0=S_t(s,x)+\tfrac{i\hbar}{2m}\Delta S(s,x)+H(x,S_x(s,x)),\\
&\quad (s,x)\in\mathcal{D}_{\mathbb{C}}\doteq(0,t)\times\mathbb{C}^n, && (18)\\
&S(t,x)=\phi(x), \quad x\in\mathbb{C}^n. && (19)\\
&H(x,p)\doteq-\big[\tfrac{1}{2m}|p|_c^2+V(x)\big]\\
&\qquad =\operatorname*{stat}_{u^0\in\mathbb{C}^n}\big\{(u^0)^Tp+\tfrac{m}{2}|u^0|_c^2-V(x)\big\},
\end{aligned}$$

where for $x\in\mathbb{C}^n$, $|x|_c^2\doteq\sum_{j=1}^n x_j^2$. We fix $t\in(0,\infty)$, and allow s to vary in $(0,t]$.

Under certain conditions, the solution of this dequantized form of the Schrödinger IVP has a representation in the form of the value function of staticization controlled diffusion equation [15]. In particular, we suppose the solution satisfies $|S_{xx}|\le C(1+|x|^{2q})$ for some $q\in I\!\!N$. We let $(\Omega,\mathcal{F},P)$ be a probability triple. Let $\{\mathcal{F}_s\,|\,s\in[0,t]\}$ denote a filtration on $(\Omega,\mathcal{F},P)$, and let $B_\cdot$ denote an $\mathcal{F}_\cdot$-adapted Brownian motion taking values in $\mathbb{R}^n$. For $s\in[0,t]$, let

$$\begin{aligned}\mathcal{U}_s\doteq\{u:\,[s,t]\times\Omega\to\mathbb{C}^n\,|\,&u \text{ is } \mathcal{F}_\cdot\text{-adapted, right-continuous}\\ &\text{and such that } \mathbb{E}\textstyle\int_s^t|u_r|^m\,dr<\infty\ \forall m\in I\!\!N\}.\end{aligned}$$

We will be interested in diffusion processes given by

$$\xi_r=\xi_r^{(s,x)}=x+\int_s^r u_\rho\,d\rho+\sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\int_s^r dB_\rho$$

where $x\in\mathbb{C}^n$, $s\in[0,t]$, $u\in\mathcal{U}_s$. For $s\in(0,t)$ and $\hbar\in(0,1]$, we define payoff $J(s,\cdot,\cdot):\mathbb{C}^n\times\mathcal{U}_s\to\mathbb{C}$ and stationary value, $\bar{S}:\mathcal{D}_{\mathbb{C}}\to\mathbb{C}$ by

$$\begin{aligned}&J(s,x,u)\doteq\mathbb{E}\left\{\int_s^t\tfrac{m}{2}|u_r|_c^2-V(\xi_r)\,dr+\phi(\xi_t)\right\},\\ &S(s,x)\doteq\operatorname*{stat}_{u\in\mathcal{U}_s}J(s,x,u)\ \ \forall\,(s,x)\in\mathcal{D}_{\mathbb{C}}.\end{aligned}$$

The Coulomb potential generated by a point charge in the central field (extended to complex domain) takes the form of $V(x)\doteq-\frac{\hat{\mu}}{\sqrt{|x|_c^2}}=-(\frac{3}{2})^{3/2}\hat{\mu}\,\mathrm{stat}_{\bar{\alpha}\in\mathcal{A}^R}\{\alpha-\frac{\bar{\alpha}^3|x|_c^2}{2}\}\doteq-\bar{c}\,\mathrm{stat}_{\bar{\alpha}\in\mathcal{A}^R}\{\alpha-\frac{\bar{\alpha}^3|x|_c^2}{2}\}$, where $\mathcal{A}^R\doteq\{\bar{\alpha}=r[\cos(\theta)+i\sin(\theta)]\in\mathbb{C}\,|\,r\ge0,\,\theta\in(\frac{-\pi}{2},\frac{\pi}{2}]\}$, and for $x\in\mathbb{C}^n$, $|x|_c^2\doteq\sum_{j=1}^n x_j^2$, $\hat{\mu}$ is a constant, and $\bar{c}\doteq(\frac{3}{2})^{3/2}\hat{\mu}$.

Lemma 16: Let $\mathcal{A}_{(s,t)}\doteq L^2(\Omega;L^2([s,t];\mathcal{A}^R))$. Then

$$\begin{aligned}&\operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}_{(s,t)}}\mathbb{E}\left\{\int_s^t\tfrac{m}{2}|u_r|_c^2+\bar{c}[\tilde{\alpha}_r-\frac{\tilde{\alpha}_r^3|\xi_2|_c^2}{2}]\,dr+\phi(\xi_t)\right\}\\ &=\mathbb{E}\left\{\int_s^t\tfrac{m}{2}|u_r|_c^2+\bar{c}\operatorname*{stat}_{\bar{\alpha}\in\mathcal{A}^R}[\bar{\alpha}-\frac{\bar{\alpha}^3|\xi_2|_c^2}{2}]\,dr+\phi(\xi_t)\right\}.\end{aligned}$$

The problem of solving for $S(s,x)$ then becomes that of finding the stationary-action value function given by

$$\begin{aligned}S(s,x)\doteq\operatorname*{stat}_{u\in\mathcal{U}_s}\operatorname*{stat}_{\tilde{\alpha}\in\mathcal{A}_{(s,t)}}\mathbb{E}\bigg\{\int_s^t\tfrac{m}{2}|u_r|_c^2+\bar{c}[\tilde{\alpha}_r-\frac{\tilde{\alpha}_r^3|\xi_2|_c^2}{2}]\,dr\\ +\phi(\xi_t)\bigg\}\ \ \forall\,(s,x)\in\mathcal{D}_{\mathbb{C}}.\end{aligned}$$

Again, the problem becomes that of interchanging the order of the staticization operators, where we note that the functional if semi-quadratic in u and uniformly Morse in $\tilde{\alpha}$. Once that is achieved, the functional inside the staticization over $\tilde{\alpha}$ may be solved through DREs.

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