CS 513: Numerical Linear Algebra

Homework 1

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Question 1

Q 1.1 p 9

Remember from class that if we multiply a matrix to the left, we are taking linear combinations of the rows of the matrix, while if we multiply from the right, we are taking linear combinations of the columns of the matrix. This rule helps us construct appropriate matrices; we will number the matrices $X_1, \cdots X_7$ corresponding to the numbers of the question in the textbook. We then find:

1.
$$X_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

2.
$$X_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

3.
$$X_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

4.
$$X_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

5.
$$X_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{5}$$

6.

$$X_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

7.

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Where we use the following ordering: $X_5X_3X_2BX_1X_4X_6X_7$ and hence $A = X_5X_3X_2$ and $C = X_1X_4X_6X_7$.

Q 2.3 p15

Part a. Let λ be any eigenvalue of A with corresponding eigenvector x. Hence

$$\begin{aligned} Ax &= \lambda x && (8) \\ &\to x^T Ax = \lambda x^T x && (9) \\ &\to x^T A^* x = \lambda^* x^T x && (10) \\ &\to \lambda x^T x = \lambda^* x^T x \qquad \text{because } A^* = A && (11) \\ &\to \lambda = \lambda^* \to \lambda \in \mathcal{R}. && (12) \end{aligned}$$

Part b. Let λ_x, λ_y any distinct eigenvalues corresponding to eigenvectors x, y w.r.t A. We know that $Ax = \lambda_x x$ and $Ay = \lambda_y y$. Moreover, since A is hermitian, $x^T Ay = \lambda_y x^T y$ and $x^T Ay = y^T Ax = \lambda_x y^T x = \lambda_x x^T y$. Hence $(\lambda_x - \lambda_y)x^T y = 0$. Since $\lambda_x \neq \lambda_y$, it must be that $x^T y = 0$ and hence the two eigenvectors are orthogonal.

Question 2

Part A

Let λ be any eigenvalue of $Q^T Q$ with corresponding eigenvector x. Hence

$$\begin{aligned} Q^T Qx &= \lambda x && (13) \\ &\to x^T Q^T Qx = \lambda x^T x && (14) \\ &\to ||Qx||_2^2 = \lambda ||x||^2 && (15) \\ &\to \lambda = 1. && (16) \end{aligned}$$

Part B

First of all, note that from the basic rules of transposition $(Q^T Q)^T = Q^T (Q^T)^T = Q^T Q$ and hence $Q^T Q$ is symmetric. From that we know that the matrix is diagonalizable: $Q^T Q = P^{-1} DP$ where D is the matrix with all eigenvalues on the diagonal. From our previous part, we know these are all ones and hence $D = I$ which commutes with any other matrix. Hence $Q^T Q = P^{-1} D P = P^{-1} I P = I$. Now if we look at the meaning of $Q^T Q$ we find that every entry in the product corresponds to an inner product between two columns. Moreover, since the product is equal to I, all inner products are zero except if the two columns have the same index. Exactly the orthogonality property we were trying to prove.

Question 3

There are many ways to solve this problem. By looking at the algorithm, you can see that a loop is executed n times in which the most expensive operation is an $O(n^3)$ operation. Hence, we can expect an $O(n^4)$ algorithm. By executing the method on some random matrices and counting the number of operations performed, one can get a plot through which you can fit a polynomial of degree 4. Depending on how many experiments you did, you will probably find a constant around 2.