CS 513 Assignment 3

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1

Please see attached `MATLAB` print out.

2

Judging from the graph, the minimum is achieved at $x = 0$, and the maximum is achieved somewhere in $(-0.7, 0.5)$ and again at $(0.5, 0.7)$. It is easy to verify that $x = 0$ is indeed the global minimum of $f(x)$, because $f(0) = 0$ and $f(x) > 0$ when $x \neq 0$. To locate the maximum more accurately, we called the `solve(y1(x)==0, x)` in `MATLAB`, and the three roots are -0.5778, 0, and 0.5778. The second root corresponds to the minima, and plugging in ± 0.5778 into `y0` we got the approximate maximum of $f(x)$ on $[-10, 10]$: $f(\pm 0.5778) = 0.1228$.

3

3.1

If we evaluate the condition number of A with l_2-norm, then

$$c_2(A) = \sqrt{\frac{\max\{\sigma(A^T A)\}}{\min\{\sigma(A^T A)\}}},$$

because $\sqrt{\max\{\sigma(A^T A)\}}$ and $\sqrt{\min\{\sigma(A^T A)\}}$ gives the largest and smallest sigular values of A, respectively, and their ratio is the condition number of A.

3.2

If A is symmetric, we have proved in the previous assignment that its sigular values are just the absolute value of its eigenvalues. Thus

$$c_2(A) = \frac{\max\{|\sigma(A)|\}}{\min\{|\sigma(A)|\}},$$

in which $|\sigma(A)|$ denotes the set of the absolute values of A.

3.3

There is no matrix in display $\mathfrak{N}$, so we are going to check only on the matrix in $\mathfrak{NN}$, namely

$$\begin{aligned} A &= \begin{bmatrix} -8 & 144 \\ 144 & -92 \end{bmatrix}, \\ A^T A &= \begin{bmatrix} 20800 & -14400 \\ -14400 & 29200 \end{bmatrix} \\ \Rightarrow \sigma(A) &= \{-200, 100\},\ \sigma(A^T A) = \{10000, 40000\}. \end{aligned}$$

The singular values of A are $\sqrt{10000} = 100$ and $\sqrt{40000} = 200$, and the condition number of A is

$$c_2(A) = \frac{200}{100} = 2,$$

which exactly matches our prediction from preceding discussions.

4

Please see attached `MATLAB` code and output.

We first tested our model against a polynomial, to see whether it can reproduce the correct coefficients. We then used our code to fit the function in Problem 2, as well as $\tan\frac{x}{3}$ and $|x-1|$ on the interval $[0, 3]$ with parameters $m = 20$, 30, 40 and $k = 4$, 5, 6. The following are our observations:

1. In all cases, when we fix k and increase m, the size of residue would increase; this is because with $k+1$ parameters we can only cancel $k+1$ entries in $Q^T b$, leaving $m-k-1$ entries not cancellable. As we increase m while fixing k, there'll be more nonzero entries in the residue, leading to a larger norm of $Ax - b$.

2. The explanation above is further supported by the fact that while we fix m and increase k, the norm of residue decreases.

3. It might be more rigorous to use a seperate test dataset to evaluate our model; using the norm of residue as the sole metric risks overfitting over the training set.

5

5.1

Given that $A = C^T C$, $C \in \mathbb{R}^{m \times m}$, for any $x \in \mathbb{R}^m$ we have

$$x^T A x = x^T C^T C x = (Cx)^T (Cx) \geq 0,$$

thus A is positive definite.

5.2

Given

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix},$$

we can LU-factorize A as the following:

$$\begin{aligned}
L_1 &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix}, \; L_1 A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 2 & \frac{7}{2} \end{bmatrix}; \\
L_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \; U = L_2 L_1 A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}; \\
L &= L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{2} & 1 & 1 \end{bmatrix}, \\
A &= LU.
\end{aligned}$$

If we further factorize U into the product of a diagonal matrix D and a unit upper triangular matrix $\tilde{U}$, we have:

$$
\begin{aligned}
D &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}, \; \tilde{U} = \begin{bmatrix} 1 & 1 & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \\
A &= LD\tilde{U}.
\end{aligned}
$$

5.3

We notice that $\tilde{U} = L^T$, so if we write D as the square of a diagonal matrix $\tilde{D}$, then $C = \tilde{D}\tilde{U}$. There are actually $2^3 = 8$ possible choices of $\tilde{D}$, since each of its three diagonal entries can carry either $+$ or $-$ sign. The following is one of the viable choices:

$$
\begin{aligned}
\tilde{D} &= \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{bmatrix}, \\
C = \tilde{D}\tilde{U} &= \begin{bmatrix} \sqrt{2} & \sqrt{2} & \frac{3\sqrt{2}}{2} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{bmatrix}, \\
A &= C^T C.
\end{aligned}
$$

And this C is also the choice of `MATLAB`.

6

With

$$
A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},
$$

we want to minimize $\|Ax - b\|_2$. First, we can do QR-decomposition to A:

$$
A = QR = \begin{bmatrix} \frac{1}{\sqrt{66}} & \frac{3}{\sqrt{11}} & -\frac{1}{\sqrt{6}} \\ 2\sqrt{\frac{2}{33}} & \frac{1}{\sqrt{11}} & \sqrt{\frac{2}{3}} \\ \frac{7}{\sqrt{66}} & -\frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{66} & 28\sqrt{\frac{2}{33}} + \sqrt{\frac{22}{3}} & 4\sqrt{\frac{6}{11}} + \sqrt{66} \\ 0 & \frac{3}{\sqrt{11}} & \frac{6}{\sqrt{11}} \\ 0 & 0 & 0 \end{bmatrix}.
$$

Since $\left\|Q^T x\right\|_2 = \|x\|_2$, minimizing $\left\|Q^T(Ax-b)\right\|_2 = \left\|Rx - Q^T b\right\|_2$ would be equivalent with minimizing $\|Ax-b\|$:

$$Rx - Q^T b = \begin{bmatrix} \sqrt{66} & 28\sqrt{\frac{2}{33}} + \sqrt{\frac{22}{3}} & 4\sqrt{\frac{6}{11}} + \sqrt{66} \\ 0 & \frac{3}{\sqrt{11}} & \frac{6}{\sqrt{11}} \\ 0 & 0 & 0 \end{bmatrix} x - \begin{bmatrix} 4\sqrt{\frac{2}{33}} \\ \frac{2}{\sqrt{11}} \\ -\sqrt{\frac{2}{3}} \end{bmatrix}.$$

We notice that the third row of R is empty, which means that there's nothing we can do about the third entry in $Q^T b$, namely $-\sqrt{\frac{2}{3}} \approx -0.816$, the absolute value of which will be the residue of our least squares problem. As for the first two entries in $Q^T b$, we can carefully choose our x so that they are cancelled out in the subtraction. Since we're now solving a linear system with 3 variables and 2 equations, we have the freedom to set one of the variables to whatever value we want. For example, if we set the ith entry in the ith solution to be zero, we have the following three least sqaures solutions:

$$x_1 = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix},\ x_2 = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix},\ x_3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix}.$$

It's easy to verify that they both lead to $\|Ax-b\|_2 = \left\|Rx - Q^T b\right\|_2 = \sqrt{\frac{2}{3}}$, which we also verified with `MATLAB`.

Though it might seems that the solution to our least squares problem is not unique in this case, all three solutions correspond to the same point in the range of A. In fact, because the difference between any two solutions of our problem belongs to the kernel of A, namely $A(x_i - x_k) = 0$, we can generate infinite new solutions just by adding a multiple of the difference between two existing solutions to any solution; however, when projected onto the range of A, they still correspond to the same point, and this is what *unique* means in this context.