CS 513: Numerical Linear Algebra

Homework 2

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Question 1

Part A

You need to compute the roots of the characteristic polynomial of this matrix which is $(\lambda-1)(\lambda-2)+2$ which gives you the roots $\frac{3\pm\sqrt{7}i}{2}$. Hence, the spectral radius is $\rho(A)=\sqrt{(3/2)^2+(\sqrt{7}/2)^2}=2$.

Part B

$||A||_1$ is the maximum one norm over all the columns of A; hence, $||A||_1=4$. $||A||_\infty$ is the maximum one norm of the rows of A, hence $||A||_\infty=3$. For the two norm we'll do something special: observe that matrix A has orthogonal (not orthonormal!) columns; hence, we know that $A'A$ is going to be diagonal. Since $||A||_2$ is going to be the square root of the largest eigenvalue of $A'A$ and we know that $A'A$ is diagonal, we just need to find the largest diagonal element of $A'A$. This turns out to be 8 corresponding to the inner product of column 2 of A with itself. Taking the square root we find that $||A||_2=2.8284$.

Part C

First note that

$$A^{-1}=\begin{bmatrix}0.5 & -0.5\\ 0.25 & 0.25\end{bmatrix}. \tag{1}$$

Next, using the methods above we find that $||A^{-1}||_1=0.75$, $||A^{-1}||_\infty=1$ and $||A^{-1}||_2=\sqrt{2}/2$. Since the condition number of a matrix is the norm of the matrix times the norm of its inverse we find that $cond_1(A)=3, cond_\infty(A)=3, cond_2(A)=2$.

Question 2

We will assume A is nonsingular, since if A is singular, every vector is an eigenvector and we couldn't make any meaningful conjecture about the relation between the eigenpairs. Our conjecture will be the following: (λ_i,v_i) *or* $(-\lambda_i,v_i)$ *is an eigenpair of A if and only if* (λ_i^2,v_i) *is an eigenpair of* $A'A$. For the proof, let us first assume $(\pm\lambda_i,v_i)$ is an eigenpair of A. Hence we know $A=U'DU$ where D is a matrix with all eigenvalues on the diagonal and U is an ortognormal matrix. $A'A=U'DUU'DU=U'D^2U$ and hence a spectral decomposition with eigenvalues being the diagonal of D^2 or λ_i^2.

The reverse argument goes as follows: let u_i be eigenvectors of A corresponding to eigenvalues δ_i and let v be some eigenvector of A^2 corresponding to eigenvalue λ^2. Since A is nonsingular, it has a complete basis for the space and $v = \sum_i a(i)u_i$. Now

$$A^2 v = \lambda^2 v \Rightarrow \tag{2}$$

$$A^2(\sum_i a(i)u_i) = \lambda^2 \sum_i a(i)u_i \Rightarrow \tag{3}$$

$$\sum_i a(i)\delta_i u_i = \lambda^2 \sum_i a(i)u_i \tag{4}$$

$$\sum_i a(i)(\delta_i - \lambda^2)u_i = 0. \tag{5}$$

Since the u_i are linearly independent, $a(i)(\delta_i - \lambda^2) = 0$ for all i. Now since $v \neq 0$ at least one $a(i)$ must be different from 0. Hence $Av = A(\sum_{i:a(i)\neq 0} a(i)u_i) = \pm|\lambda|(\sum_{i:a(i)\neq 0} a(i)u_i) = \pm|\lambda|v$ since we are only summing up over $a(i) \neq 0$ combined with our statement above.

This property doesn't hold for the following nonsymmetric matrix:

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}. \tag{6}$$

The singular values of a symmetric matrix are the absolute value of its eigenvalues.

Question 3

Let (λ, v) be an eigenpair for A. Now we know that $Av = \lambda v \rightarrow A^{-1}Av = A^{-1}\lambda v$. Rearranging gives us $1/\lambda v = A^{-1}v$. The only if part follows by symmetry.

Recall that $||A^{-1}||_2$ is equal to the largest singular value of $A^{-T}A^{-1} = (AA^T)^{-}1$. You can check that the spectrum of AA^T is is the same as that of A^TA for square matrices. Since this is a square symmetric matrix and using our claim above, we know that its largest singular value/eigenvalue is equal to one over the smallest singular value/eigenvalue of A^TA. Hence, the largest singular value of A^{-1} is equal to one over the smallest singular value of A.

Since the condition number of A is equal to $||A||_2||A^{-1}||_2 = \sqrt{\sigma(A^TA)\sigma(A^{-T}A^{-1})}$ where σ denotes the largest singular value of its argument. From our previous claim, it follows that this is equal to the square root of the largest singular value of A^TA over the smallest singular value of A^TA.

If A is square and symmetric we can apply the result in question 2.e and it follows that the condition number is equal to the largest eigenvalue in absolute value over the smallest eigenvalue in absolute value.

Question 4

First we prove that A is symmetric. This follows from the fact that $A = C'C$ and hence $A' = (C'C)' = C'C = A$. Next we show that for any vector $x \neq 0$: $x'Ax > 0$. This must be true since $x'Ax = x'C'Cx = ||Cx||^2 \geq 0$. Moreover it can never be zero since if $Cx = 0$ and $x \neq 0$, then $Ax = C'Cx = 0$ and A wouldn't be invertible.

The LDU factorization for the given matrix is:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3/2 & 1 & 1 \end{bmatrix}, \tag{7}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}, \tag{8}$$

$$U = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

Note that L and U are each others transpose. Now by taking the square root of D (meaning the matrix that has the square root of D's values on its diagonal) and multiplying one part with L and one part with U we find the two matrices that are transpose of each other one of which is upper triangular. This is the Cholesky factorization of A. It follows that A must be positive definite. Note that things can go wrong if D had negative entries on its diagonal.

One important note here: be careful when you talk about eigenvalues and the diagonal elements of D in the LDU factorization. These are not the same! The decomposition $A = U'DU$ when U is an upper triangular matrix is not the same as the spectral decomposition of A which would be $A = Q'EQ$ where Q is an orthonormal matrix.

Question 5

The idea here is that since matrix A is singular, you can think of it as a transformation that takes some direction in 3D space and projects it into the zero vector. Let us call this direction v. If there is a vector x that is the solution to the least squares problem $Ax - b$ (meaning $||Ax - b||$ is the smalles among any possible x) then one can see that $A(x + v) - b = Ax + Av - b = Ax - b$. Hence, adding a multiple of v to the solution x doesn't change the value of the least squares solution $||Ax - b||$.

Finding three solutions can be done as follows: first we try to solve the normal equations $A'Ax = A'b$. This can be done by computing the LU factorization of $A'A$ and solving for $y = L\ (A'b)$. Then, we want to solve $x = U\ y$ but this is an underdetermined system that has an infinite number of solutions. You return 3 possible solutions from this set.