CS 513 Assignment 6

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May 3, 2018

1

If we use the `rand` function in `Matlab` to generate a 100×100 random matrix, which we denote with A, then the typical distribution of the absolute values of its spectrum is shown in Figure. After multiple experiments, we see that there is one eigenvalue that 'dominates' over all other eigenvalues in modulus, and it is located around 50. Apart from this outlier, the other eigenvalues are of realatively comparable sizes.

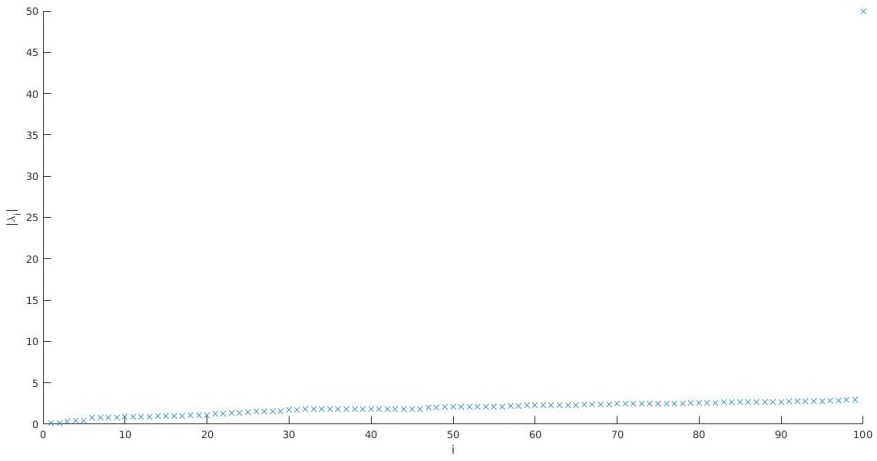

Figure 1: Distribution of $|\lambda_i|$ of a random 100×100 matrix generated by `rand` function.

Figure 2 shows the distribution of the spectrum of A and its Arnoldi approximations with $n = 1, 2, 3$ on the complex plane. We see that all of the eigenvalues of A has positive real parts, and the complex eigenvalues of A appear in $\sigma(A)$ with their complex conjugates. This is an expected property for a real matrix. In addition, the convergence of eigenvalues approximated by the Arnoldi method towards the largest eigenvalue is much faster than those to other smaller eigenvalues: even at n values as small as 2 or 3, the Arnoldi approximation of the eigenvalue is almost identical to the real value of $\lambda_{\max}$. For the other eigenvalues, however, we do not see any convergence at such small ns.

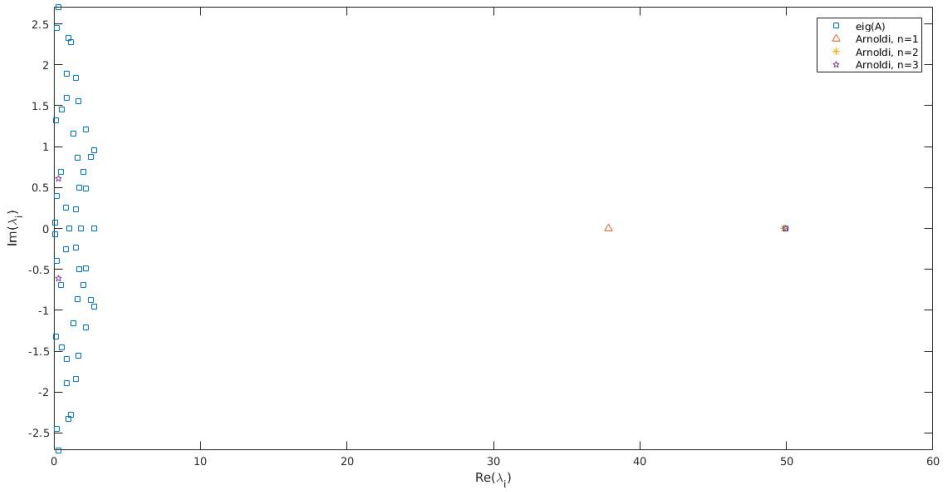

Figure 2: Distribution of $\sigma(A)$ and its Arnoldi approximation at $n = 1, 2, 3$, with A being a random 100×100 matrix generated by `rand` function.

The observation from Figure 1 can be explained by the Perron-Frobenius Theorem: when we have a real square matrix with only possitive entries, there will be one real eigenvalue that has its modulus strictly larger than all other eigenvalues. The observation from Figure 2 is explained by the known fact that the Arnoldi iterations give geometric convergence towards the 'outlier' eigenvalue in the spectrum of a matrix, and the convergence to other eigenvalues is much slower.

Table 1: Errors generated in GMRES with different ϵs and ns for a 100×100 random matrix with eigenvalues centered around 4, 5, and 10 with radius ϵ.

	n			
$-\log_{10}\epsilon$	5	10	20	90
1	-3.62	-9.09	-12.22	-12.57
4	-6.55	-13.14	-12.52	-12.50
7	-8.44	-12.37	-12.37	-12.37
10	-12.75	-13.28	-13.30	-12.62

2

From Table 1 we can see that, if ϵ is small, GMRES could converge to the correct solution faster. This matches our analysis in class, because in the limit of $\epsilon \to 0$, we would expect GMRES to find the solution in 3 steps. Actually, for $\epsilon = 10^{-10}$, we do see this type of fast convergence. Another obsevation is that, in general, as n increases, the error becomes smaller. This is the result of an expanding Krylov subspace. However, in the cases of $\epsilon = 10^{-4}$ and 10^{-10}, error increased a bit from the order of 10^{-13} to 10^{-12} from $n = 10$ to $n = 90$. This might be the result of error accumulation in numerical computations.

Table 2:

k	$\left\|(I - uu^T)A\right\|_2$	$\left\|uu^T A\right\|_2$
1	2.09	4.25×10^1
10	4.43	1.37×10^2
10^2	5.63	1.12×10^3
10^3	5.74	1.10×10^4
10^4	5.75	1.10×10^5
10^5	5.75	1.10×10^6

3

If we define the first 2 columns of U as u, then uu^T is the projection matrix to range(u), and $I - uu^T$ is the projection matrix to $\mathbb{R}^m \backslash \text{range}(u)$. In addition, we note that the two subspaces constitute a partition of $\mathbb{R}^m$. If a_i denotes the columns of A, then the results in Table 2 shows that, as k increases, the components of a_i in range(u) would increase accordingly, while the components in $\mathbb{R}^m \backslash \text{range}(u)$ remain relatively constant. This means that, as k increases, the columns of A can be more accurately approximated by linear combinations of the columns of u.