CS 726 Assignment 5

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With

$$p^C = -\frac{g^T g}{g^T B g} g,$$
$$p^B = -B^{-1} g,$$
$$\gamma = \frac{(g^T g)^2}{(g^T B g)(g^T B^{-1} g)}$$

and $\bar{\gamma} \in (\gamma, 1]$, we can devise the following function to characterize p in the three 'legs':

$$\tilde{p}(\tau) = \begin{cases} \tau p^C & \tau \in (0, 1] \\ p^C + (\tau - 1)(\tilde{\gamma} p^B - p^C) & \tau \in (1, 2] \\ \tilde{\gamma} p^B + (\tau - 2)(p^B - \tilde{\gamma} p^B) & \tau \in (2, 3] \end{cases}$$

As p moves along the path, τ increases from 0 to 3.

It's trivial to show that $\|\tilde{p}\|$ increases with τ in the first and third segments of the path, because $\tilde{p}$ is simply moving from the origin to p^C and from $\tilde{\gamma} p^B$ to p^B, respectively. In the second leg, if we define $\alpha = \tau - 1$ for convenience, we have

$$\begin{aligned} \frac{\partial \|\tilde{p}\|^2}{\partial \alpha} &= 2\tilde{p}^T \frac{\partial \tilde{p}}{\partial \alpha} \\ &= 2[P^C + \alpha(\tilde{\gamma} p^B - p^C)]^T (\tilde{\gamma} p^B - p^C) \\ &= 2(p^C)^T (\tilde{\gamma} p^B - p^C) + 2\alpha \left\| \tilde{\gamma} p^B - p^C \right\|^2, \end{aligned}$$

in which $2\alpha\left\|\tilde{\gamma}p^B - p^C\right\|^2 \geq 0$ and

$$\begin{aligned}
(p^C)^T(\tilde{\gamma}p^B - p^C) &= -\frac{g^Tg}{g^TBg}g^T\Big(-\tilde{\gamma}B^{-1}g + \frac{g^Tg}{g^TBg}g\Big) \\
&= \frac{(g^Tg)(g^TB^{-1}g)}{g^TBg}\tilde{\gamma} - \frac{(g^Tg)^3}{(g^TBg)^2} \\
&> \frac{(g^Tg)(g^TB^{-1}g)}{g^TBg}\cdot\frac{(g^Tg)^2}{(g^TBg)(g^TB^{-1}g)} - \frac{(g^Tg)^3}{(g^TBg)^2} \\
&= \frac{(g^Tg)^3}{(g^TBg)^2} - \frac{(g^Tg)^3}{(g^TBg)^2} \\
&= 0 \\
\Rightarrow &\frac{\partial\|\tilde{p}\|^2}{\partial\alpha} > 0,
\end{aligned}$$

in the second step of which we used $\tilde{\gamma} > \gamma = \frac{(g^Tg)^2}{(g^TBg)(g^TB^{-1}g)}$ and the fact that both B and B^{-1} are positive definite. Thus, as α increases from 0 to 1, p moves along the second segment and $\|p\|$ keeps growing.