CS726, Spring 2018

Homework 1 (posted 1/28/18; due Friday 2/9/18.)

Please submit your answers in the order listed below.

Exercise numbers refer to the course text, *Numerical Optimization* (second edition, 2006).

Some of these questions require knowledge of Appendix A of *Numerical Optimization*, and the other background material posted on Canvas.

1. A polyhedron is the intersection of a finite number of linear inequalities. Say which of the following sets are polyhedra. If they are polyhedral, express them in the form $\{x \mid Ax \geq b, Fx = g\}$.
   (a) $S = \{x \in \mathbb{R}^n \mid x \geq 0,\ \sum_{i=1}^n x_i = 1,\ \sum_{i=1}^n a_i x_i = b,\ \sum_{i=1}^n a_i^2 x_i = c\}$, where $a_1, a_2, \ldots, a_n, b, c \in \mathbb{R}$ are given constants.
   (b) $S = \{x \in \mathbb{R}^n \mid x \geq 0,\ x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1\}$.
   (c) $S = \{x \in \mathbb{R}^n \mid x \geq 0,\ x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_1 = 1\}$.

2. Exercise 2.6: Prove that all isolated minima are strict. (Hint: One way to do this is to prove that "not strict" $\Rightarrow$ "not isolated".)

3. (a) Give an example of a matrix that is *not* positive definite despite having all positive entries.
   (b) If A is a positive definite matrix, must its diagonal elements all be positive? Explain.

4. Exercise 2.1 from the text.

5. For each value of the scalar β, find the set of all stationary points (that is, the points x for which $\nabla f(x) = 0$) of the following function:
$$f(x) = x_1^2 + x_2^2 + \beta x_1 x_2 + x_1 + 2x_2.$$
   Which of these points is a global minimizer?

6. Let f be a twice continuously differentiable function on $\mathbb{R}^n$. Prove that f is convex if and only if $\nabla^2 f(x) \succeq 0$ for all x. Hint: The following characterization of convexity of a smooth convex function on $\mathbb{R}^n$ may be helpful: $f(y) \geq f(x) + \nabla f(x)^T (y - x)$ for all x and y.

7. For the following functions of two variables, answer these questions and justify your answers fully using optimality conditions. (A saddle point is a point that is neither a local minimum nor a local maximum.)

   (a) Show that $f(x_1, x_2) := (x_1^2 - 4)^2 + x_2^2$ has two global minima and one saddle point.

   (b) Find all local minima of $f(x_1, x_2) = \frac{1}{2}x_1^2 + x_1 \cos x_2$.

   (c) Show that $f(x_1, x_2) := (x_2 - x_1^2)^2 - x_1^2$ has only one stationary point, which is a saddle point.