Appendix I: Local Lie Group Representations

1.1 Definition of Invariants

Let us consider the problem by analysing a system transform with specified invariance properties. We seek a system that is described by a Lagrangian functional:

$$J(x) = \int_a^b L(t, x, \dot{x}) \quad , \qquad [133]$$

where L is the Lagrangian which is integrated in order to obtain the output function $J(x)$. It should be noticed that, along with the specifiable variables t and x, the Lagrangian also specifies a function in terms of the derivative signal $\dot{x}$.

We define a transformation, T as a mapping $T : R^2 \to R^2$, which maps a point (t, x) into a point (t', x') by:

$$T : t' = \phi(t, x), \qquad x' = \psi(t, x) \qquad [134]$$

where $\phi(t, x)$ and $\psi(t, x)$ are specified transformation functions. Composition of transformations is represented by:

$$S : t' = \gamma(t, x), \qquad x' = \omega(t, x) \qquad [135]$$

then

$$ST : t'' = \gamma(t', x'), \qquad x'' = \omega(t', x') \qquad [136]$$

and by substitution from Equation 135 we arrive at the composition functional:

$$ST : t'' = \gamma(\phi(t, x), \omega(t, x)), \qquad x'' = \omega(\phi(t, x), \omega(t, x)) \quad . \qquad [137]$$

In order for a transformation to belong to the group it must have a corresponding inverse transform within some bounded region of the transform:

$$T^{-1} : t = \Phi(t', x'), \qquad x = \Psi(t', x') \quad . \qquad [138]$$

Another necessary component of a group is the identity element which is represented by the compositions $T^{-1}T$ and TT^{-1}. We represent the identity element with the notation T_0.

Auditory group transformations are dependent upon a real parameter ε, for all ε in an open interval $|\varepsilon| < \varepsilon_0$. We can define this family one parameter transformations in the following manner:

$$T_\varepsilon : t' = \phi(t, x, \varepsilon), \qquad x' = \psi(t, x, \varepsilon) \quad . \qquad [139]$$

As well as the existence of inverse and identity transforms for the functionals we also want the transforms to exhibit the *local closure property.* This property states that for ε_1 and ε_2 sufficiently small, there exists an ε_3 such that:

$$T_{\varepsilon_1} T_{\varepsilon_2} = T_{\varepsilon_3} \quad . \qquad [140]$$

A one-parameter family of transformations that satisfies the above requirements of inverse, identity and closure is called a *local Lie group,* [Logan87] [Moon95] [Gilmore74] [Bourbaki75].

1.2 Transformations of points

Consider the transform:

$$T_\varepsilon : t' = t\cos\varepsilon - x\sin\varepsilon, \qquad x' = t\sin\varepsilon + x\cos\varepsilon \qquad [141]$$

The composition of these functions has the form $T_{\varepsilon_1} T_{\varepsilon_2} = T_{\varepsilon_1 + \varepsilon_2}$, thus the parameters sum under composition. The identity transform is $T_0 = I$ and $T_\varepsilon^{-1} = T_{-\varepsilon}$. By these four properties the above transformation forms a local Lie group, the *rotation group.*

A Taylor series expansion of Equation 141 about $\varepsilon = 0$ yeilds the following form for T_ε:

$$t' = \phi(t, x, 0) + \phi_\varepsilon(t, x, 0)\varepsilon + \frac{1}{2}\phi_{\varepsilon\varepsilon}(t, x, 0)\varepsilon^2 + \dots \quad [142]$$
$$= t + \tau(t, x)\varepsilon + o(\varepsilon)$$
$$x' = \psi(t, x, 0) + \psi_\varepsilon(t, x, 0)\varepsilon + \frac{1}{2}\psi_{\varepsilon\varepsilon}(t, x, 0)\varepsilon^2 + \dots$$
$$= x + \xi(t, x)\varepsilon + o(\varepsilon)$$

where $o(\varepsilon) \to 0$ faster than $\varepsilon \to 0$. The function subscripts ϕ_ε and ψ_ε denote a partial derivative, e.g. $\phi_x = \frac{\partial\phi}{\partial x}$. This representation constitutes a global representation of the local Lie group T_ε. The quantities τ and ξ are the *generators* of T_ε and they are defined as the partial derivatives of the component functions of the transformation:

$$\tau(t, x) = \phi_\varepsilon(t, x, 0), \qquad \xi(t, x) = \psi_\varepsilon(t, x, 0) \quad . \quad [143]$$

The generators are also used to obtain an *infinitessimal representation*, or local representation, for the local Lie group obtained for small ε:

$$t' = t + \varepsilon\tau(t, x) + o(\varepsilon), \quad . \quad [144]$$
$$x' = x + \varepsilon\xi(t, x) + o(\varepsilon)$$

For linear transforms the infinitessimal representation can be used to specify the global representation. This is not true for non-linear transformations for which the generators represent a local-linear transform of the vector field of the transformation for a small region of ε.

The rotation group transform, given by Equation 141, is a linear transform. We define the properties of a linear transform in Section ??? where we considered the class of normal subgroups. We can obtain the generators for the rotation group by solving the partial derivatives for the component functions, ϕ_ε and ψ_ε, evaluating at $\varepsilon = 0$ gives:

$$\tau(t, x) = \frac{\partial}{\partial\varepsilon}\phi(t, x, \varepsilon)\Big|_{\varepsilon = 0} = -x \quad [145]$$

and

$$\xi(t, x) = \frac{\partial}{\partial\varepsilon}\psi(t, x, \varepsilon)\Big|_{\varepsilon = 0} = t \quad [146]$$

substituting into the infinitessimal representation, Equation 144, we obtain the local representation of the rotation group transform:

$$t' = t - \varepsilon x + o(\varepsilon), \qquad [147]$$

$$x' = x + \varepsilon t + o(\varepsilon)$$

the generators of which are $\tau(t, x) = x$ and $\xi(t, x) = t$ thus specifying the global representation:

$$t' = t - x\varepsilon + o(\varepsilon), \qquad [148]$$

$$x' = x + t\varepsilon + o(\varepsilon)$$

We shall see the importance of the rotation group later when we consider a general class of transforms that operates in the time-frequency plane as time shifts and time-frequency rotations. This class of transforms will be used in the following chapters to characterize particular classes of sound structure invariant.

1.3 Transformations of functions

We now consider the effects of transforming a function by a local Lie group. Let $x = h(t)$, where $h(t) \in C$, the set of complex numbers. Under the local Lie group transformation T_ε produces a mapping from $x = h(t)$ to $x' = h'(t')$. We find the form of h' by noting that T_ε maps t to $t' = \phi(t, h(t), \varepsilon)$. Now, for ε sufficiently small there exists an inverse mapping of $\phi(t, h(t), \varepsilon)$, denoted by K, such that $t = K(t', \varepsilon)$. Then under T_ε :

$$x' = \psi(t, h(t), \varepsilon) = \psi(K(t', \varepsilon), h(K(t', \varepsilon)), \varepsilon) \equiv h'(t') \quad , \qquad [149]$$

where r t' and x' are the transformations of the Lagrangian $L(t, x, \dot{x})$. In order to fully specify the behaviour of the function under transformation we must also determine how derivatives of functions are transformed under T_ε :

By the chain rule:

$$\frac{d}{dt'}h' = \psi_t \frac{dt}{dt} + \psi_x \frac{dh}{dt}\frac{dt}{dt'} = (\psi_t + \psi_x \dot{h})\frac{dt}{dt'} \qquad [150]$$

where ψ_t and ψ_x are evaluated at $(K(t', \varepsilon), h(K(t', \varepsilon)), \varepsilon)$. Using this derivative we arrive at an extended group of transforms which represent the effect of the transform T_ε upon the Lagrangian $L(t, x, \dot{x})$:

$$t' = \phi(t, x, \varepsilon), x' = \psi(t, x, \varepsilon), \dot{x}' = \frac{\psi_t(t, x, \varepsilon) + \psi_x(t, x, \varepsilon)\dot{x}}{\phi_t(t, x, \varepsilon) + \phi_x(t, x, \varepsilon)\dot{x}} \quad [151]$$

The generators for the derivative element of the extended group is derived by the global representation method, described above, and it is given by:

$$\dot{x}' = \dot{x} + (\dot{\xi} - \dot{x}\dot{\tau})\varepsilon + o(\varepsilon) \quad [152]$$

the linear generator η is thus given by:

$$\eta = \dot{\xi} - \dot{x}\dot{\tau} \quad [153]$$

which extends the generators previously given in Equation 144.