

# Statement of Purpose

Rupadarshi Ray

December 1, 2025

Over the course of my training at IISER Mohali, I have explored diverse areas of mathematics, including differential geometry, dynamical systems, symplectic geometry, complex geometry, representation theory, Lie groups, ergodic theory and mathematical physics. I have had the opportunity to credit courses on knot theory, Riemannian geometry, elliptic curves, Fourier analysis in my major years at IISER Mohali. Still, I believe the course I benefited from the most was the one on curves and surfaces in third semester, which led to my interest in the theory of manifolds and algebraic topology quite early.

Engaging in above mentioned subjects led to me inculcate a diverse set of theoretical and problem solving skills, which firstly lets me engage in different conversations quickly. And in turn conversing with my peers lets me learn and relearn new and old ideas and skills. I have had fun giving talks for the mathematics club, courses, graduate seminar etc.

## 1 History

### 1.1 First and second years of BSMS

When I came to the IISER Mohali campus after completing the first semester of my BSMS online, I had some idea what groups and vector spaces were. The first semester course on mathematics called “symmetry” ended on the definition of symmetry of a subset of  $\mathbb{R}^n$  as the group of rotations, reflections and translation that keep it stable. However, I had heard the term *symplectic manifold*, eluding to the subject of geometric mechanics, or more generally topological dynamics on manifolds. Of course the definition of *manifolds*, *Riemannian* or *symplectic manifolds* did not make any sense at the time. It was at this time I opened a book on geometric mechanics in the library at IISER Mohali saying *torque free motion of a rigid body is the geodesic flow (of a left invariant metric) on  $SO(3)$* .

The second semester course was on analysis on  $\mathbb{R}$ , and throughout the course I found myself relearning all the intuitive notions about *functions* I had after learning calculus. The subjects of linear algebra and real analysis seemed to collide in the definition of *total derivative* of a multivariable function. I was also trying to learn about *differential forms* around this time, having heard it was a “proper” way to integrate on higher dimensional spaces.

Next, we had a course on Gauss’s geometry of curves and surfaces. We defined curvature of curves and surfaces in  $\mathbb{R}^3$  and I loved learning and teach differential geometry to my peers! I truly understood the notion of *total derivative* as a *map between tangent spaces that sends velocities to velocities* clearly in the setup of smooth map between surfaces.

After a semester on surfaces, I found the definition of manifolds and smooth maps between them quite simple! Me and a friend were looking at courses in mathematics we can audit for fun. Given we knew the definition of topological spaces and continuous maps between them, I suggested we look at what happens in the course on *algebraic topology*. After a few classes of looking at homology and homotopy groups and some “ugly” topological spaces, I started to really appreciate the definition of manifolds.

That summer, I luckily became a part of a reading group where we proved the de Rham isomorphism using sheaf/Cech cohomology theory from Griffiths and Harris, Bott and Tu, respectively. The worlds of *differential forms* and *algebraic topology* collided in this wonderful isomorphism. Even though I was young in doing mathematics, with practice and some blackboxes I had gained some skill in chasing a diagram of vector spaces and a background on the theory of smooth manifolds.

## 1.2 Third and fourth years of BSMS

After this, I began to understand the structure of vector fields and their flows, which are fundamental to the theory of Lie groups, the representation theory of Lie groups, and symplectic geometry. I could now understand what *geodesic flow on  $SO(3)$*  meant that I heard about more than a year ago! This is the point where I had my first course on complex analysis. Thus, I could appreciate the topological, homological and Lie theoretic aspects of the first course on complex analysis. Among other things, I discovered the motivation behind sheaf cohomology by the Mitag-Leffler problem on Riemann surfaces (from Griffiths and Harris, parts of which I already read for sheaf theoretic de Rham isomorphism, but somehow missed this section!).

The course on complex analysis ended with the construction of Etale space of the sheaf of holomorphic functions on  $\mathbb{C}$  to define in Riemann's terms "global domain of holomorphic functions". After a back and forth with this definition for a year, I started to understand what the topology on the Etale space is *doing*.

Latter in the summer, I spent some time studying representations of finite and Lie groups from Fulton and Harris' book.

## 2 My background and interests in differential geometry

After my encounter with differential forms in algebraic topology and gaining a background on the theory of smooth manifolds, the setup of mechanics using symplectic manifolds started to make sense to me. I started reading parts of Ana Cannas da Silva's text on symplectic geometry. The first de Rham cohomology popped up as a topological obstruction to Noether's theorem in the Hamiltonian side.

To get a broader feel of dynamics on smooth manifolds, I looked at some topics in hyperbolic dynamics on manifolds. Almost accidentally, I discovered Kathryn Mann's lectures on Anosov flows on manifolds, and was fascinated how people actually tried to classify such flows on compact manifolds. Surprisingly, I might encounter some objects called "Anosov" in the second half of my thesis.

Recently, I looked at the first few sections of the text by Helmut Hofer and Eduard Zehnder on *Symplectic Invariants and Hamiltonian Dynamics*. It fascinated me how a hypersurface  $S \subset \mathbb{R}^{2n}$  carries a prescribed collection of orbits without any choice of a Hamiltonian vector field!

My interests in mathematical physics such as quantum field theory, led to the field of gauge theory which is where I discovered the tangled theories of integrable systems, gauge theories, topology and physical mathematics. I started reading Hitchin's book on the complex algebraic geometry of Lax pair equations, and I heard about the conjectures and works by Hitchin and Donaldson on gauge theories that have connections to integrable theories as well.

I have been fascinated with complex manifolds, such as the theory of vector bundles on complex manifolds, hodge theory, etc., however unfortunately I have not got a chance to study them. Mirror symmetry and Calabi-Yau manifolds remain another distant goal for me.

## 3 Experiments with talks

When I was in third year, I was interested in Hamiltonian flows and topological properties of such flows. I wanted to give a talk proving that orbits in irrational translations on the 2-torus are dense in the torus. This was an exercise in the textbook by Perko. Boiling down the argument to density of orbits in irrational circle rotations was easy. However, after that I tried to prove the orbit of 1 in the circle under a irrational rotation is a subgroup which is not finite, hence it must be dense. This turned out not so well for the audience who had little experience with compactness. I recently discovered very simple proofs of the proposition about irrational circle rotations just using the fact that it is an isometry of the circle!

Slowly reading Ana Cannas da Silva's text on symplectic geometry over the course of a year, I gave a talk on the introduction to symplectic manifolds for a graduate seminar in IISER Mohali. I wanted to motivate the very definition of a symplectic manifold as the "minimum" structure on a smooth manifold needed to define Hamiltonian vector fields of smooth functions. Therefore, I first started with Hamiltonian flows on the plane: where Hamiltonian vector fields are rotation of the gradient vector field

$$J\text{grad}(H)$$

and sketched the proof of one-to-one correspondence of smooth functions and area preserving flows on the plane. This was met with appreciation from the audience! Then I moved onto the 1-sphere: because I find the

symplectic structure of the sphere in terms of the Riemannian metric and the complex structure (using cross product) a good motivation for declaring: a symplectic structure is all you need to replace the complex structure and the Riemannian metric and still have Hamiltonian flows (of  $H \in \mathcal{C}^\infty(H)$ ) which are “volume” preserving and also preserves the function  $H$ .

I repeated the same talk for a seminar on the orbit method in representation theory of nilpotent Lie groups. While preparing, I discovered the symplectic structure on co-adjoint orbits of Lie groups, and the agreement of the standard symplectic structure of the sphere and the one coming from the fact that it is a co-adjoint orbit of  $SO(3)$ !

Another talk I gave in fourth year was for the course on Arithmetic of elliptic curves. I had to prepare something related to the topics discussed in class, and we had discussed the Weierstrass function parametrizing the elliptic curve. I was interested in looking at how integrating a holomorphic 1-form gives inverse of this parameterization by the complex 2-torus. The problem I faced was: where should I begin? Should I assume the results proved in class? If yes, then the proof became an one liner (present in the textbook Arithmetic of elliptic curves). Therefore, I choose to redo the entire theory, starting from the proof that elliptic curves are homeomorphic to 2-tori, so that I can get their fundamental group to have two generators. I did this by explicitly constructing charts for the elliptic curve  $y^2 = x(x-1)(x-\lambda)$  by using the function

$$x \mapsto (x, \pm \sqrt{x(x-1)(x-\lambda)})$$

This helped in another aspect: showcasing how the integral of the holomorphic 1-form

$$\frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

can be made very explicit! After that it was simple exercise to show the subset  $\Lambda$  generated by integral of this 1-form on the fundamental group is a rank 2 lattice, and that the integral gives a well-defined holomorphic bijection onto  $\mathbb{C}/\Lambda$ .

This was the genus = 1 case of the theorems by Abel and Jacobi. In the summer, I wanted to prove the general genus case next. Me and my friends started a seminar series, with introductory talks by Prof. Kapil Hari Paranjape, on compact Riemann surfaces. We presented sketch of proofs of uniformization, Riemann-Roch and Abel-Jacobi theorems.

Next, I presented a series of talks on ergodic theorems, and then on symmetric spaces of non-compact type, their visual boundaries, Tits boundaries. Initially, it felt difficult to read topics in a week or so and present them. But now, after doing that for a couple of months, I have become much better in doing that!

## 4 My path to the boundary of symmetric spaces

As I had a little background in algebraic topology and had credited courses on Riemannian geometry, I could recover the *pre-moduli* classification of Riemann surfaces assuming the theorem of uniformization of simply connected Riemann surfaces in terms of quotients of these simply connected Riemann surfaces by discrete subgroups of their automorphism groups. This led me to discover the Teichmüller and moduli spaces of smooth surfaces. This geometric and algebraic aspects of the same object I find recurring in my thesis work.

These notions led me to discover the fascinating world of geometric group theory from Clara Loh’s book and the theory of locally symmetric spaces and arithmetic subgroups of semi-simple Lie groups from Dave Witte Morris’s book and more broadly the text on discrete subgroups of Lie groups by Raghunathan. I still did not know how ergodic theory may appear in this geometric and group theoretic world that I had discovered.

### A workshop on the rigidity of discrete groups

Later in the summer, I attended a workshop on the *rigidity of discrete groups* held in IISER Mohali, organized by Dr. Parab Sardar and Dr. Krishnendu Gangopadhyay. I was fascinated by the theorems on rigidity of hyperbolic 3-manifolds and some complex manifolds, having seen the theory of *deformations* in dimension 2.

### A seminar on the boundary of symmetric spaces

Eventually, I decided to do my masters thesis on topics related to the strong rigidity of locally symmetric spaces: boundary of (globally) symmetric spaces and application of ergodic theory in this area. I became a part

of a seminar on the same, organized by my thesis advisor Dr. Arghya Mondal, which has been a new experience for me!

To work around some pre-requisites, we started with the CAT(0) symmetric space structure on the set of all positive definite real matrices  $P(n, \mathbb{R})$ . I presented the embedding of symmetric spaces of non-compact type inside  $P(n, \mathbb{R})$  as complete and totally geodesic submanifolds and thus reducing our work to such submanifolds. Then we constructed the visual and Tits boundary of such symmetric spaces.

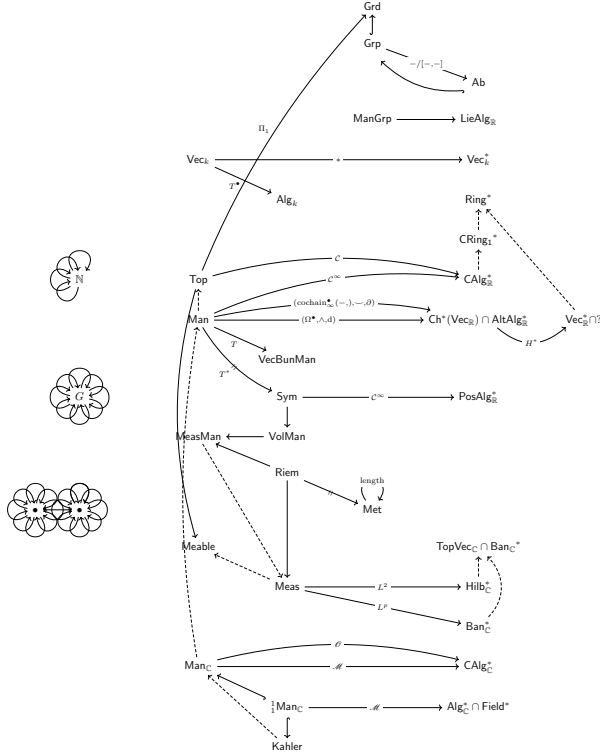
On another track, I started looking at Patterson-Sullivan theory for discrete subgroups of isometry group of rank 1 symmetric spaces of non-compact type. We wish to understand the theory of limit sets and the conformal densities given by such discrete subgroups in hopes that the applications of the theory may be generalized to higher rank as well.

I hope to have another fun semester reading, and writing for my masters thesis.

## 5 View on mathematics

My view on mathematics mainly stems from my background in geometry, topology and dynamics.

I view the various structures studied in mathematics as a quiver: categories of structures and functors between them (thus, living in the category of categories  $\mathbf{Cat}$ ). A group action is thus just a functor from the group (viewed as a category) to the category of sets  $\mathbf{Set}$ . We may consider such a functor going into  $\mathbf{Top}$ , and then compose with various functor coming out of  $\mathbf{Top}$  such as  $\pi_1$  and get *invariants*. Many such functors arise from the correspondence between geometry of spaces and algebra of functions on them.



On the other side I see the various explicit objects of study such as  $\mathbb{R}^n, \mathbb{C}^n, \mathbb{R}H^n, \mathbb{C}P^n$ , and functions on them, various other explicit Riemannian manifolds, groups, flows, group actions, etc.

Working in mathematics for me is a tug of war of information between these two worlds. We start with some explicit object and extract the correct structure out of it, which in turn gives us more explicit objects when we try to classify them. This back and forth, is what I picture when I write proofs classifying objects with some structure. For instance, it still fascinates me how proving existence of a counterexample of a statement implies we will never be able to write a (correct) proof of the statement.

My thesis work has led me to a place in between of these two worlds. Cartan had already classified (globally) symmetric spaces of non-compact type, and its geometry is also well-understood. Their isometry groups, as in semi-simple Lie groups of non-compact factors are also well-studied. Thus, quotients of such spaces, locally

symmetric spaces, are more or less explicit - but they have large moduli spaces in the case of dimension 2, and fascinating properties like rigidity, arithmeticity in other cases.

## 6 Future prospects

I wish to study topics related to (but not restricted to) the ones I already have already explored in my masters

- discrete subgroups of Lie groups and rigidity theory,
- arithmetic lattices and arithmetic manifolds,
- phenomena in non-positive curvature broadly, and
- rigidity and deformations of complex manifolds.

However, I am interested in geometry, topology and dynamics in general. I wish to join a graduate school in mathematics to continue my study of phenomena in these fields and

- foster broader connections with the mathematical community,
- teach and discuss mathematics with beginners,
- study the topics that left me fascinated

and eventually contribute to mathematics meaningfully.