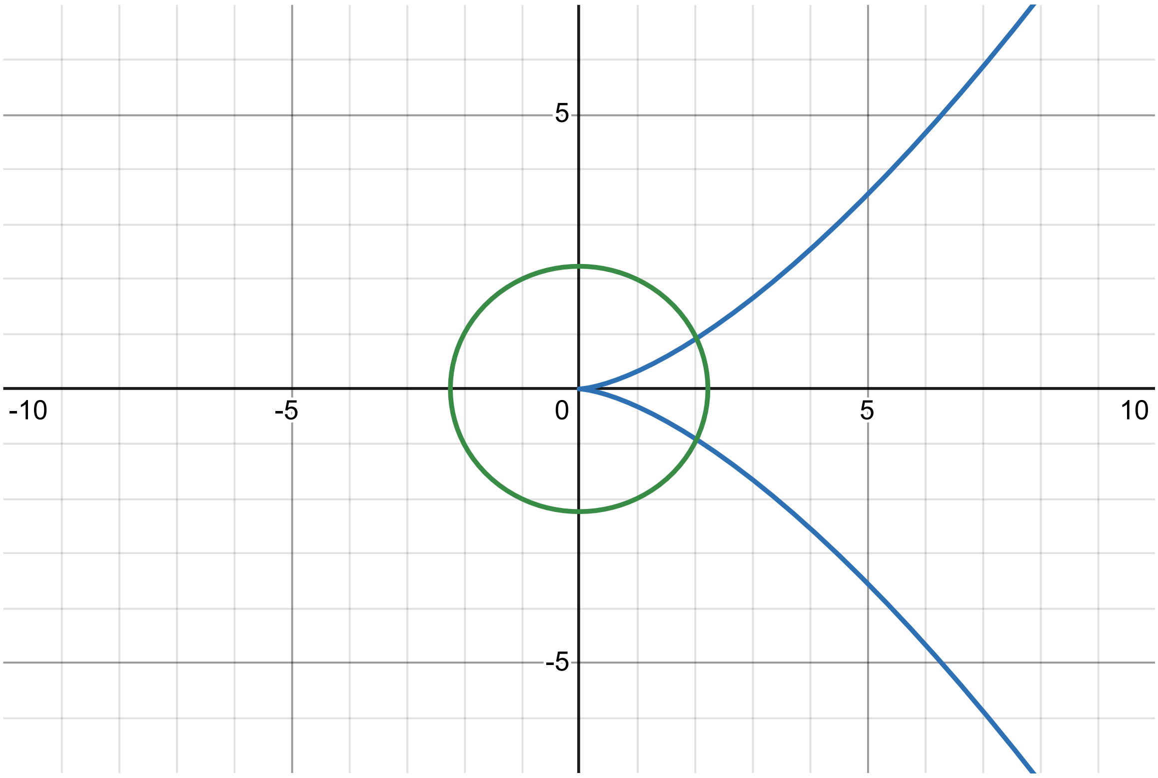
Spring and summer '25:

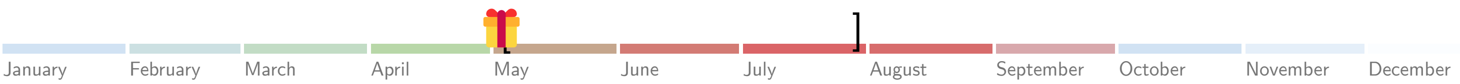
Geometry of flows and analysis on manifolds



Typeset by Rupadarshi Ray
while throwing ideas with my peers Manan Jain, Pahul Arora and Naman Narang
at IISER Mohali

Last updated: 6:04 PM - May 06, 2025

timeline



'25.

The spring semester in IISER Mohali runs from January to early May:

1 days left in semester of 122 days

99.79% complete

Then we have summer:

Summer holidarys will start from Wed 7 May, '25 and end on Mon 4 Aug, '25.

1 days to go for summer

FYI: There are 90 days of holidays.

And then the *thesis year begins* (for 3/4 of us above, anyways):

MS thesis period will start from Mon 4 Aug, '25 and end on Sat 4 Apr, '26 tentatively.

90 days to go

FYI: It is a 244 days of MS thesis year.

contents

Every note in this list is ***incomplete***, most probably ***horribly wrong***.

- **symplectic geometry I**
  - 2502 GSG symplectic geometry and Hamiltonian flows
  - Corinna Ulcigrai - Chaotic Properties of Area Preserving Flows
- **analysis I**
  - analysis
  - Holomorphic mapping onto disk, Riemann mapping theorem
- **smooth manifods II**
  - Jet Nestruev - Smooth Manifolds and Observables
  - *Global calculus*
  - *Differential forms*
  - *Vector bundles*
  - *Differential topology*
  - *Differential cohomology*
- **complex geometry I: complex 1-manifolds**
  - Universal cover of punctured plane
  - Simon Donaldson - Riemann surfaces
  - $\mathbb{C}$-manifolds of dimension 1, AKA Riemann surfaces
- **algebraic geometry I**
  - Joe Harris - Algebraic Geometry: a first course
  - varieties
  - schemes
- **geometry, groups and dynamics II: representations**

  > Representation theory of (finite, Lie) groups and Lie algebras *along with* classical algebraic (projective) geometry, symplectic geometry that arises.

  - Fulton Harris-Reps
  - Lie algebras
  - Ad^* orbits are symplectic manifolds
- **smooth manifolds III: Lie groups/ algebraic geometry II: algebraic**

  > Going hand in hand with *representations* above.

  - Reading *Bump*?

- IMSC-Linear algebraic groups

- **geometry, groups and dynamics III: with measures**
  - smooth dynamics
  - McMullen - Ergodic theory, geometry and dynamics
  - ICTP18 Dynamics
  - Luis Pesin - Smooth ergodic theory
- **geometric analysis I: Laplacian, heat**
  - Ancient solutions to geometric flows
  - Geometric analysis (Julie Rowlett)
  - Seminars on Inverse Problems, Julie Rowlett, June 7, 2022
  - Masoud Khalkhali - Spectral geometry
  - questions on spacetimes
- **analysis II**
  - Larry Guth - MIT Differential Analysis 2
  - 2503 CMI - GMT
- **geometry, groups and dynamics IV**
  - asking Riemannian symmetric questions
  - TIFR-Geometry, groups and dynamics (2017)
  - Kathryn Mann - Dynamics of Group Actions
  - Geometry, Arithmetic, and Dynamics of Discrete Groups | Fields Academy Shared Graduate Course - YouTube
  - Programa de Doutorado: Lie Groups, Representation Theory and Symmetric Spaces - YouTube
  - 10ICTS Geometry Topology and Dynamics in Negative Curvature
  - IHEs 23 Homogeneous Dynamics and Geometry in Higher-Rank Lie Groups
  - 【ETHZ】Symmetric Spaces 对称空间_哔哩哔哩_bilibili
  - Uri Bader - A course in linear groups and ergodic theory
- **geometric analysis II: Ricci flows**
  - Following Chow Lu Ni - Hamilton's Ricci flow
  - Gerhard Huisken - Ricci flow
  - Einstein metrics and Ricci solitons
  - *Recent developments in Ricci flows* https://arxiv.org/pdf/2102.12615
- **symplectic geometry II**
  - IMPA19 School of Symplectic Topology

- Denis Auroux - Fukaya categories and mirror symmetry

- **analysis III: harmonic analysis/geometric analysis III**
  - A Pirkovskii - Harmonic Analysis and Banach Algebras
  - A talk I attended in Mohali: Utsav Dewan - Boundary exceptional sets for radial limits of positive superharmonicfunctions on Harmonic manifolds
  - The three-dimensional Kakeya conjecture, after Wang and Zahl | What's new
  - Harmonic analysis on homogeneous spaces
  - Sigurdur Helgason - Groups and geometric analysis
- **complex geometry II**
  - Pierre Albin - Complex Algebraic Geometry
  - IMPA Hodge theory
  - John Milnor - Singular Points of Complex Hypersurfaces
  - Lecture 1 : Singular Levi-flat hypersurfaces by Jiri Lebl
- **algebraic geometry III**
  - 2503 GSG Rational curves and a conjecture of Drinfeld
  - March 17-19, 2025 - Geometric Aspects of Algebraic Varieties, IISER Mohali
  - Kabeer MR - Metric techniques for triangulated categories
  - GSG talk on On Character Variety of Anosov Representations by Tathagata Nayak
  - *Intersection theory*
    - - Intersection Theory (1 of 5) - YouTube Intersection theory in algebraic geometry - lccs (columbia.edu)
- **complex geometry III/geometric analysis IV**
  - **Complex Geometry (Hans-Joachim Hein, hansjoachim.hein@univ-nantes.fr)**

    https://www.bilibili.com/video/BV1Qv41177Hp/?spm_id_from=333.337.search-card.all.click

    - Prerequisites: Basics of manifolds, tensor fields, differential forms, etc. Warner, Foundations of Differentiable Manifolds and Lie Groups, Chapters 1, 2, 4, 6, contains all we need and much more.
    - Basic complex analysis as in Stein & Shakarchi, Complex Analysis, Chapters 1, 2, 3, 8.
    - Reading:
      - Huybrechts, Complex Geometry, is an excellent basic textbook with exercises.

- Lecture notes by Joel Fine: http://homepages.ulb.ac.be/~joelfine/papers.html#survey.
- Complex Monge-Ampere: http://gamma.im.uj.edu.pl/~blocki/publ/ln/tln.pdf.
- For the end of Week 2: http://arxiv.org/pdf/0803.0985.pdf, Section 5.

- Week 1: Introduction to Complex Geometry
  - Holomorphic Functions and Complex Calculus
  - Complex Manifolds
  - Holomorphic Line Bundles
  - Pseudoconvexity and Pseudoconcavity
  - The Kodaira Embedding Theorem
- Week 2: Topics in Kahler-Einstein Manifolds
  - Kahler Manifolds
  - Ricci Curvature and the Complex Monge-Ampere Equation
  - Examples of Ricci-flat Spaces
  - Basic Estimates for the Complex Monge-Ampere Equation
  - The Mukai-Umemura Manifold

- GIT → moduli of vector bundles, stability of vector bundles → Einstein-Kahler metrices

- **integrable systems**
  - Nicolai Reshetikhin - Integrable systems and representation theory: geometry, algebra and analysis
  - *Integrable PDEs*
    - KdV equation
  - *Algebraic integrable systems*
    - BIMSA - Integrable Systems and Algebraic Geometry - Beijing Summer Workshop in Mathematics and Mathematical Physics (June 24 – July 5, 2024)
- **complex geometry IV/algebraic geometry IV/smooth manifolds IV: differential equations**
  - Loray - Painlevé equations and isomonodromic deformations
  - Henryk - The Monodromy Group
  - GADEPs focused conference: Abelian and iterated integrals and Hilbert 16th problem (2022)

    - NCTS Short Course on Riemann Hilbert Method in Integrable Systems
    - algebraic study of ODEs
    - Peter J. Olver - Applications of Lie Groups to Differential Equations -Springer (2000).pdf
- **geometry, groups and dynamics V/geometric analysis V/smooth manifolds V: geometric topology**
    - 240925 GSG orderability in 3-manifold groups
    - 3-manifolds
    - 2501 GSG liftable mapping class groups
    - J. Aramayona - MCG and infinite MCG (Part 2)
    - LECTURE 3

      The proof, Part I: A volume preserving conjugacy

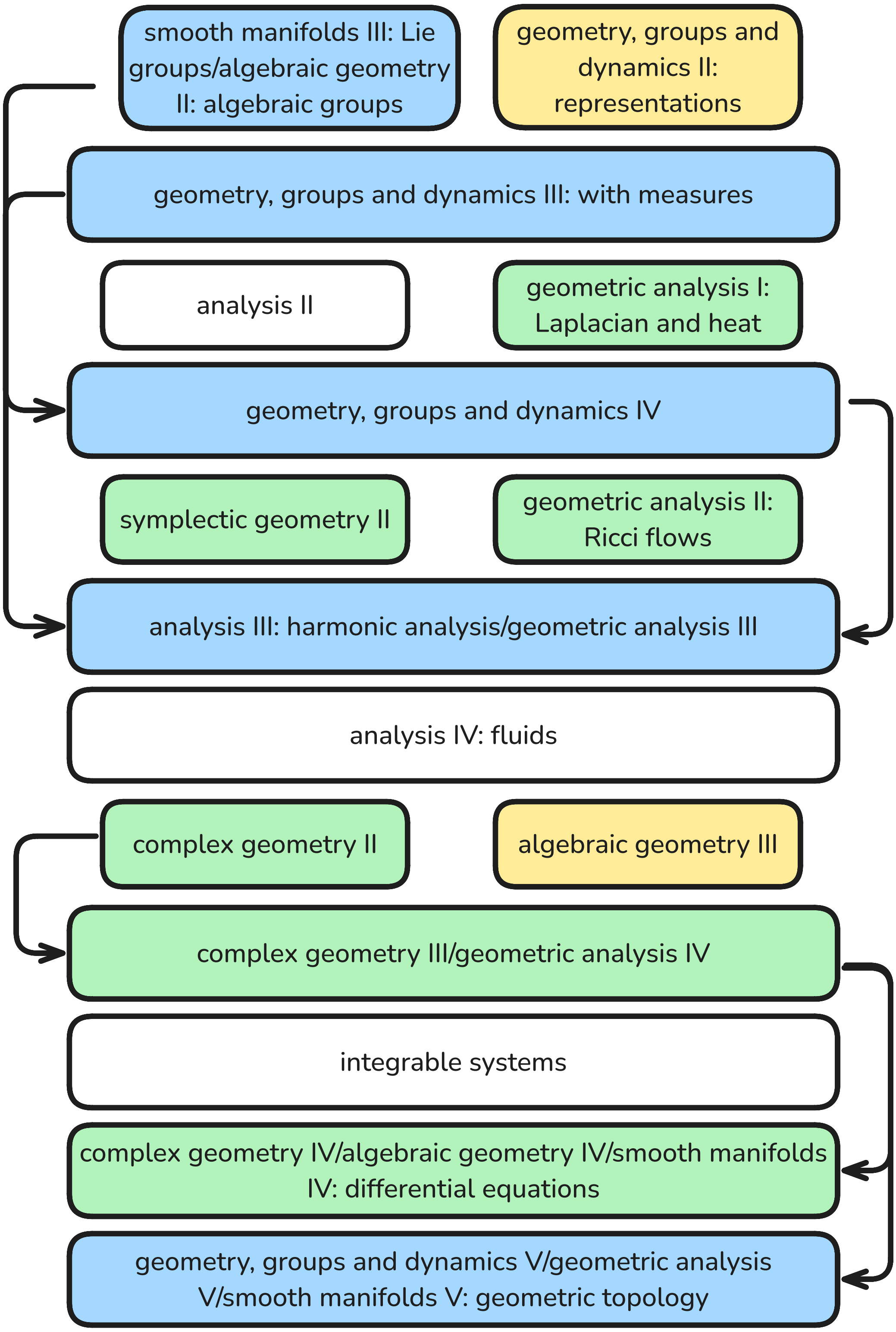
      We begin the proof of Otal's theorem. We start with:

      Proposition 3.1. *The space of negatively curved metrics on a closed surface S is path connected.*

      Idea of Proof. Hamilton's Ricci flow on surfaces preserves negative curvature and converges to a metric of constant negative curvature [20]. Given two metrics on a surface, flow both by Ricci flow to conformal metrics, and if necessary scale the metrics to have constant negative curvature -1. Then both flowed metrics belong to Teichmüller space. Since Teichmüller space is connected, the two metrics can then be connected by a path through metrics of constant negative curvature. $\diamond$

      [1]
    - Foliations
    - Young Mathematicians' Symposium 2025, IISER Mohali
    - Summer School on Rigidity of Discrete Groups, June 30 – July 4, 2025 at IISER Mohali

Thus we had a good thing going:



But we continue... now with some *fancyy* stuff

- **(co)homology $\leftrightarrow$ Lagrangian field theories $\leftrightarrow$ integrable systems**
  - paper reading Andrew Neitzke - BPS
  - Sergei Gukov - Homological algebra of knots and BPS states
  - Symplectic Duality and Geometric Langlands Duality - TIB AV-Portal
  - Following A Guest - From Quantum Cohomology to Integrable Systems
  - *Donaldson*
- **mirror symmetry**

- Calabi-Yau manifolds

Let's return back to some *lore*...

- **number theory**
- **algebraic topology**
    - homotopy extension property of cone

- endnotes
- philosophy of mathematics, 16 march '25 edition
- jerry will be back

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1. math.uchicago.edu/~wilkinso/papers/PCMI-Wilkinson.pdf?utm_source=chatgpt.com#page=29.24 ↩

manifolds

The only division rings $(F, +, \times)$ which are also smooth $\mathbb{R}$-manifolds (compatible) possible are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. [1]

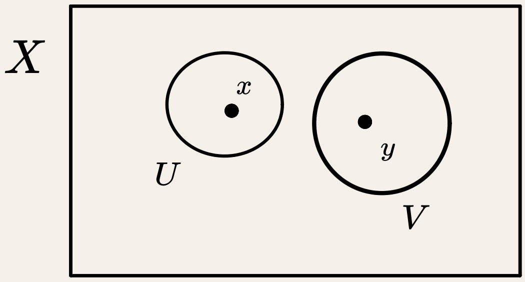
- $\mathbb{R}$-manifolds
  - **Definition. Topological manifold**

    A topological space is a real topological n-manifold or **$(\mathbb{R}^n, \mathcal{C}^0)$-manifold** for a $n \in \mathbb{Z}^+$ (or topological n-manifold) if

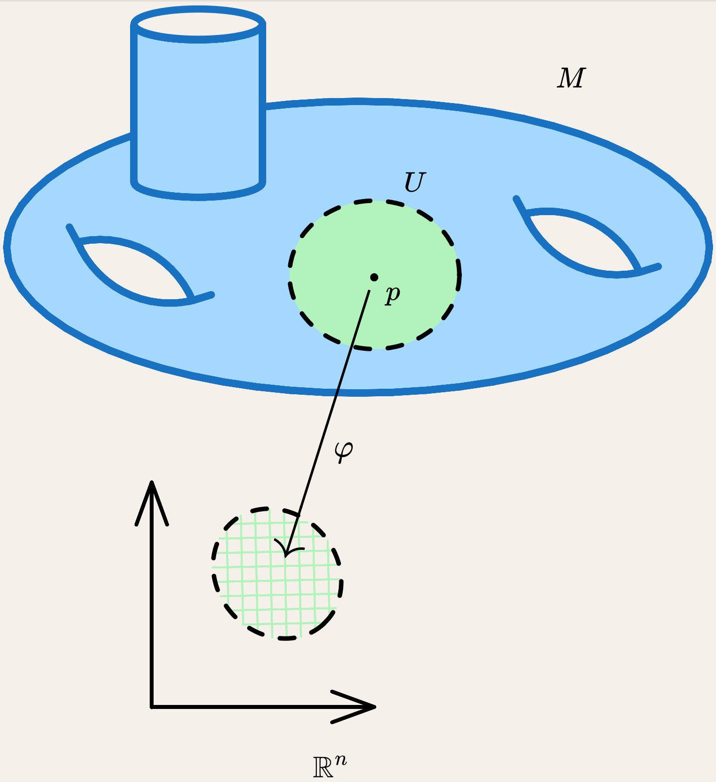
    - M is Hausdorff topological space

      Definition. T_2 topological space or Hausdorff space

      A topological space is said to be **T_2 or Hausdorff** if for any pair $x, y \in X$, $\exists U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

    - M is second countable
    - *(locally homeomorphic to $(\mathbb{R}^n, \mathcal{T}_{\text{std}})$)* for any $p \in M$, there exists



- an open subset $U \subseteq M$ with $p \in U$
- open subset $V \subseteq \mathbb{R}^n$
- and a homeomorphism $\varphi : U \to V$

- PL manifold
- $\mathcal{C}^k$-manifold
- smooth $\mathbb{R}$-manifold
  - **Definition. Homogeneous spaces**

    A **transitive** smooth Lie group action on a smooth $\mathbb{R}$-manifold is called a homogeneous G-space.
  - **Definition. Spin structure on a Riemannian vector bundle**

    Let

    $$E \to M$$

    be a oriented Riemannian $\mathbb{R}^n$-vector bundle over M and let $P_{SO}(E)$ be its bundle of oriented orthonormal frames.

    - For $n \geq 3$, we have the universal covering homomorphism

      $$\mathrm{Spin}_n : Spin(n) \to SO(n)$$

      with kernel $\{I, -I\} \cong \mathbb{Z}_2$. A **spin structure** on E is a principal

$Spin(n)$-bundle $P_{Spin}(E)$ together with a 2-sheeted covering

$$\zeta : P_{Spin}(E) \to P_{SO}(E)$$

such that $\zeta(pg) = \zeta(p)\mathrm{Spin}_n(g)$ for all $p \in P_{Spin}(E)$ and $g \in Spin(n)$

- For $n = 2$

- Definition. **A symplectic manifold (M, ω) is a smooth manifold M with a closed non-degenerate 2-form ω.**

- Definition. **Toric manifolds** (M, ω, μ)

  A symplectic manifold (M, ω) is a toric manifold if it is equipped with the (effective) Hamiltonian Lie group action of a torus $U(1)^{\dim M/2}$ with the moment map

  $$\mu : M \to \mathbb{R}^{\dim M/2}$$

  The image of the moment map $\mu(M) \subseteq \mathbb{R}^{\dim M/2}$ is called the Newton polytope of (M, ω, μ).

- Definition. **Lie group**

  A group object in Man.

- Definition. **Riemannian metric on a smooth manifold**

  A **Riemannian metric** g on a smooth manifold M is a section of $\boldsymbol{Sym}^2 T^*M$ which is positive definite.

- Definition. **Poisson manifold**

  A **Poisson manifold** (M, B) is a smooth manifold M with a lie bracket

  $$B : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

  on the $\mathbb{R}$-vector space $\mathcal{C}^\infty(M)$ which satisfies the Leibniz identity

$$f, g, h \in C^\infty(M) \implies B\{f, gh\} = B\{f, g\}h + gB\{f, h\}$$

- **Definition. Multisymplectic manifold**

  A **multisymplectic manifold** (M, ω) of degree $k \geq 1$, or a **k-plecitc manifold**, is a smooth $\mathbb{R}$-manifold with a closed differential $k+1$-form ω which is non-degnerate, that is

  $$\begin{aligned} \mathrm{Vec}(M) &\to \Omega^k(M) \\ X &\mapsto \iota_X \omega \end{aligned}$$

  is injective.

- sett.Man.R connection *Manifold with a connection*
- sett.Man.R J *Almost complex manifold*
- sett.Man.R O Sp closed

- $\mathbb{C}$-manifold complex manifold
  - sett.Man.C group
  - sett.Man.C Kahler
- manifolds based on topological vector spaces over $\mathbb{R}, \mathbb{C}$
  - Banach manifold
  - Frechet manifold
- $\mathbb{H}$-manifold Quaternionic manifold - Wikipedia [2]
- $\mathbb{O}$-manifold

- *Locally ringed spaces* are far too general?

- analytic k-manifolds where k is a field with a absolute value which is metric complete. [3] apart from $k = \mathbb{R}, \mathbb{C}$
  - $\mathbb{Q}_p$
  - $\mathbb{Q}$-manifold?

- Supermanifold - Wikipedia

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1. Are there only two smooth manifolds with field structure: real numbers and complex numbers? - MathOverflow
2. dg.differential geometry - Is there a reasonable definition of an octonionic manifold? - MathOverflow
3. 3.dvi (ucla.edu)

Counting compact topological manifolds upto homeomorphism

[1]

without boundary

Suppose there are not countably many of them, then there are an uncountable number of them for some dimension say n. Let

$$\{M_\alpha \mid \alpha \in A\}$$

be such a collection.

For each α we find a collection of imbeddings for $1 \le j \le k_\alpha$

$$h_{\alpha j} : B(2) \subset \mathbb{R}^n \to M_\alpha$$

such that

$$\{h_{\alpha j}(B(1)) | 1 \le j \le k_\alpha\} \text{ covers } M_\alpha$$

We may assume

- $k_\alpha = k$ for all α by choosing a uncountable subcollection
- $h_{\alpha j} \mid_{B(1)}$ can be extended to an imbedding of $B(k+1) \to M_\alpha$
- $M_\alpha \subset \mathbb{R}^{2n+1}$

$$\begin{aligned} \epsilon_{\alpha j m} &:= d(h_{\alpha j}(B(m)), M_\alpha - h_{\alpha j}(B(m+1))) \\ \epsilon_\alpha &:= \min_{j,m}\{\epsilon_{\alpha j m}\} \end{aligned}$$

We again assume $\exists \epsilon > 0$ such that $\epsilon_\alpha > \epsilon$ for all $\alpha \in A$ by choosing a subcollection.

Each M_α determines an imbedding

$$\begin{aligned} g_\alpha : B(k+1) &\to (\mathbb{R}^{2n+1})^k \\ g_\alpha(x) &:= (h_{\alpha,1}(x), \ldots, h_{\alpha,k}(x)) \end{aligned}$$

The set of all such imbeddings $\{g_\alpha \mid \alpha \in A\}$ under the uniform metric

$$d_u(g_\alpha, g_\beta) := \max_{x \in B(k+1)} d(g_\alpha(x), g_\beta(x))$$

is a separable metric.

Hence, g_{α_0} is a limit point of a sequence of distinct imbeddings $g_{x_1}, \ldots$.

> **Proposition:** $M_{\alpha_0} \cong M_{\alpha_i}$ for i sufficiently large. Furthermore, this homeomorphism can be taken close to $\iota : M_{\alpha_0} \hookrightarrow \mathbb{R}^n$ arbitarily d-close.

Definition. **Let**

$$V_j(m) := h_{\alpha_0 j}(B(m))$$

for $1 \leq j \leq k, 1 \leq m \leq k+1$ **and**

$$V_j'(m) := h_{\alpha_i, j}(B(m)) \subset M'$$

for $M' = M_{\alpha_i}$ **for large enough** i**.**

Definition. **Now let**

$$U_j(m) := \bigcup_{p=1}^{j} V_p(m)$$

Note $U_k(1) = M$ and $U_k'(1) = M$.

Definition.

$$f_j : V_j(k+1) \to V_j'(k+1)$$
$$h_{\alpha_i, j} \circ h_{\alpha_0, j}^{-1}$$

Suppose we construct an imbedding

$$g_j : U_j(m) \to M'$$

ϵ-close to identity by choosing M' far enough in the sequence.

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1. Counting topological manifolds - ScienceDirect ↩

Mikhail Gromov - What is a Manifold

https://www.ihes.fr/~gromov/wp-content/uploads/2018/08/manifolds-Poincare.pdf

#talk

What is a Manifold? - Mikhail Gromov - YouTube

1 Ideas and Definitions.

We are fascinated by knots and links. Where does this feeling of beauty and mystery come from? To get a glimpse at the answer let us move by 25 million years in time.

25×10^6 is, roughly, what separates us from orangutans: 12 million years to our common ancestor on the phylogenetic tree and then 12 million years back by another branch of the tree to the present day orangutans.

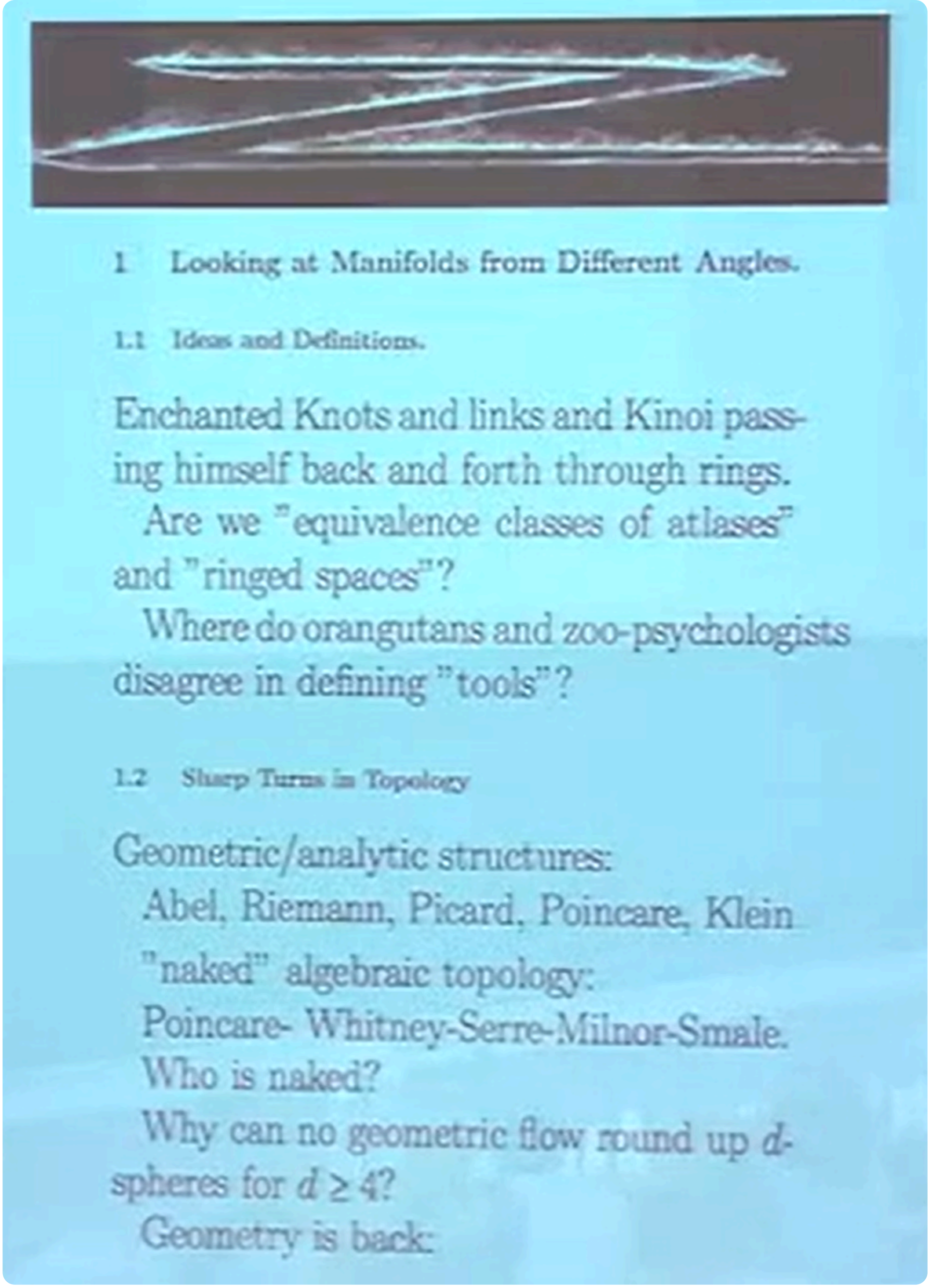
But are there topologists among orangutans?

Yes, there definitely are: many orangutans are good at "proving" the triviality of elaborate knots, e.g. they fast master the art of untying boats from their mooring when they fancy taking rides downstream in a river, much to the annoyance of people making these knots with a different purpose in mind.

A more amazing observation was made by a zoo-psychologist Anne Russon in mid 90's at Wanariset Orangutan Reintroduction Project (see p. 114 in [68]).

"... Kinoi [a juvenile male orangutan], when he was in a possession of a hose, invested every second in making giant hoops, carefully inserting one end of his hose into the other and jamming it in tight. Once he'd made his hoop, he passed various parts of himself back and forth through it – an arm, his head, his feet, his whole torso – as if completely fascinated with idea of going through the hole."

Playing with hoops and knots, where there is no visible goal or any practical gain – be it an ape or a $3D$-topologist – appears fully "non-intelligent" to a practically minded observer. But we, geometers, feel thrilled at seeing an animal whose space perception is so similar to ours.



- Poincare's "naked" algebraic topology and little bit of geometry will tell you everything in the low dim

2 Homotopies and Obstructions.

For more than half a century, starting from Poincaré, topologists have been laboriously stripping their beloved science of its geometric garments.

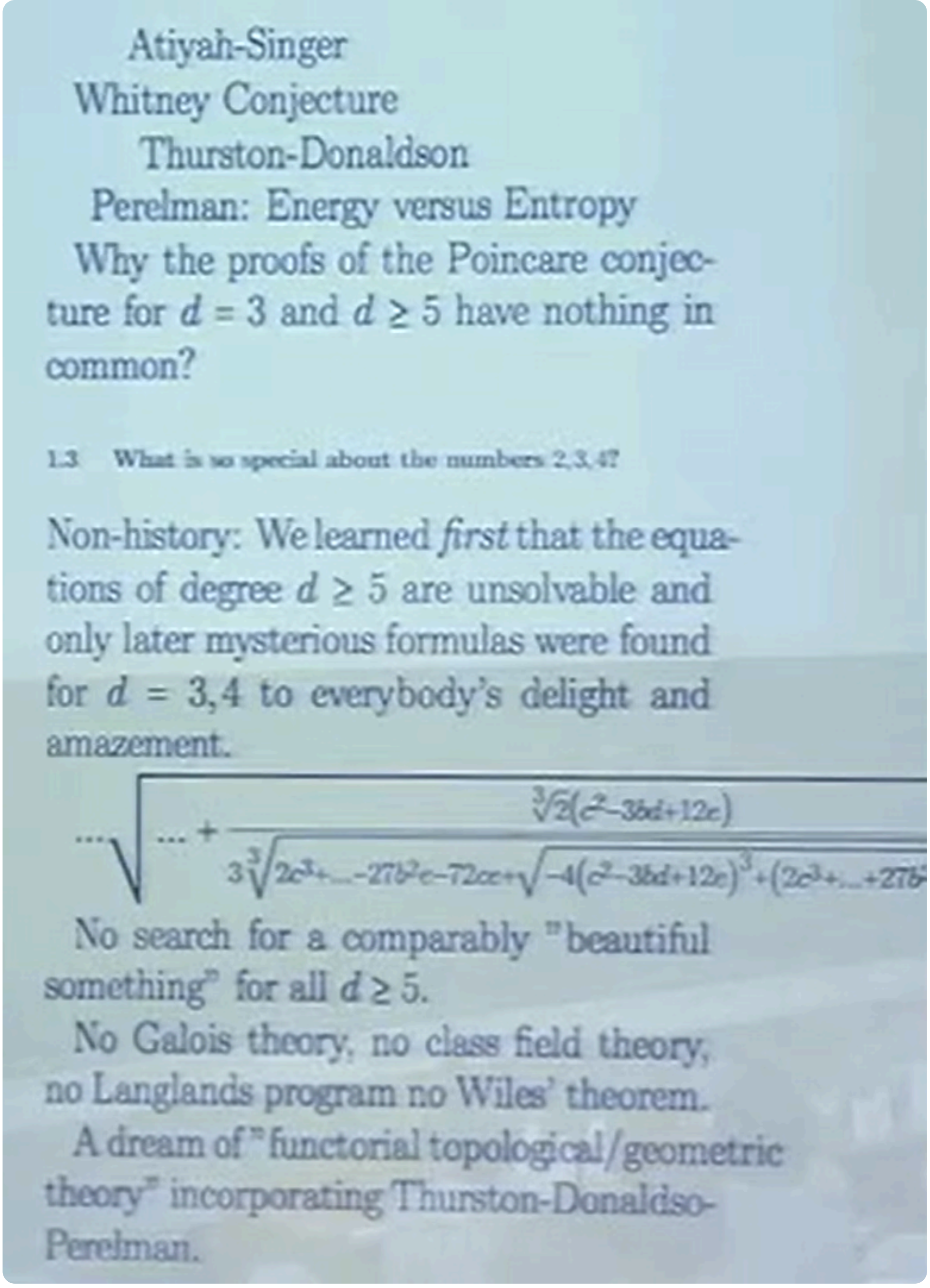
"Naked topology", reinforced by homological algebra, reached its to-day breathtakingly high plateau with the following

N	n	**homotopy classes of maps $S^{n+N} \to S^N$**
any	0	$\mathbb{Z}$; index
even	$2N-1$	contains a subgroup of finite index isomorphic to $\mathbb{Z}$

(Serre) There are at most finitely many homotopy classes of maps $S^{n+N} \to S^N$ except for two cases

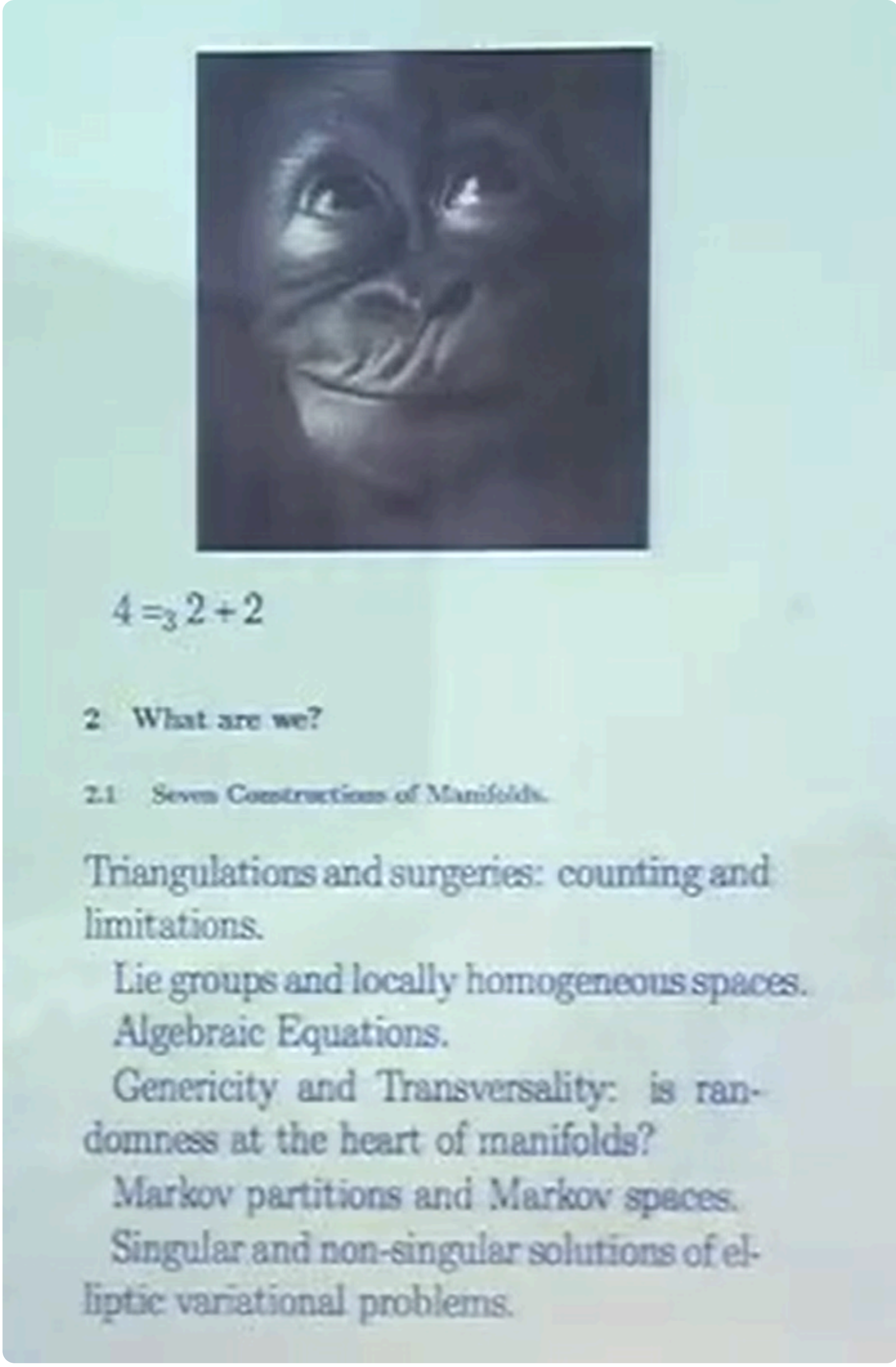
N	n	**homotopy classes of maps $S^{n+N} \to S^N$**
any	0	**$\mathbb{Z}$; index**
even	$2N-1$	**contains a subgroup of finite index isomorphic to $\mathbb{Z}$**

- Whitney 1940 - geometry of embedded non-orientable surface in $\mathbb{R}^4$
  - Analysis to topology
  - Donaldson:



Hyperbolic 3-manifolds "comes" from any 3-manifold

- Never before non-linear PDEs were saying something about topology
- Thurston, Donaldson, Hamilton (https://en.wikipedia.org/wiki/Richard_S._Hamilton?)



> **Quote**
>
> Proof of Donaldson theorem is $4 = 2 + 2$.

Riemannian connection

- *Torsion-free*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

- *metric-compactible*

$$X(g(Y,Z)) = g(\nabla_X, Z) + g(Y, \nabla_X Z)$$

$$\begin{aligned} \nabla : \mathrm{Vec}(M) &\to \mathrm{Vec}(M) \otimes \Omega^1(M) =?\boldsymbol{\Gamma}(T^*M \otimes TM) \\ X &\mapsto \nabla_{(-)} X \end{aligned}$$

$$[X, Y] \mapsto \nabla[X, Y] =?$$

expression

We compute

$$X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y))$$

and obtain

$$g(\nabla_X Y, Z) = \begin{array}{r} X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ -g(Y,[X,Z]) - g(Z,[Y,X]) + g(X,[Z,Y]) \end{array}$$

the covector field of $\nabla_X Y$ is

$$X(g(Y,-)) + Y(g(-,X)) - \mathrm{d}(g(X,Y)) - g(Y,[X,-]) - g(-,[Y,X]) + g(X,[-,Y])$$

in a chart

On a chart (U, x) we have

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_m \Gamma_{ij}^m \frac{\partial}{\partial x_m}$$

because $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ we have

Parallel transport on a curve in a semi-Riemannian manifold

Definition. Parallel vector or tensor fields along curves

Let (M, ∇) be a smooth manifold with a connection on its tangent bundle. A smooth vector or tensor field

$$V \in \mathbf{\Gamma T}^{r,s} T_\gamma M$$

along a smooth curve

$$\gamma : I \to M$$

is said to be **parallel along** γ (with respect to ∇) if

$$\nabla_{\dot{\gamma}} V \equiv 0$$

Locally, along a curve

$$x(\gamma((t)) = (\gamma_1(t), \ldots, \gamma_n(t))$$

and vector field

$$V(t) = \sum_{i=1}^{n} V_i(t) \frac{\partial}{\partial x_i}$$

is *parallel* $\iff$

$$(\nabla_{\dot{\gamma}} V)_k = \dot{V}_k + \sum_{i,j} \Gamma^k_{ij} V_j \dot{\gamma}_i = 0$$

for all components $1 \leq k \leq n$.

Curvature of a Riemannian manifold

Definition. Riemann curvature tensor of a Riemannian manifold

The assignment

$$(X, Y, Z) \mapsto ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$$

produces a (1,3)-tensor on M.
The **Riemann curvature tensor** is the (0,4)-tensor

$$R(X, Y, Z, W) := g(([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z, W)$$

The Riemann curvature (0,4)-tensor is actually

$$R \in \mathbf{\Gamma} \boldsymbol{Sym}^2 \boldsymbol{\Lambda}^2 TM$$

and it satisfies the two Branchi identities

$$R(W, X, Y, Z) + R(X, Y, W, Z) + R(Y, W, X, Z) = 0$$

Definition. Ricci and scalar tensors

$$\mathrm{Ric}(X, Y) := \mathrm{tr}(Z \mapsto)$$

		R^l_{ijk}
Riemann $\boldsymbol{Sym}^2 \boldsymbol{\Lambda}^2 TM$-tensor		$R_{ijkl} = g_{lm} R^m_{ijk}$
Ricci		$R_{ij} = R^k_{kij} = g^{km} R$
scalar	$\mathrm{sc} := \mathrm{tr}(\mathrm{Ric})$	$\kappa = R^i_i = g^{ij} R_{ij}$
total scalar curvature	$\int_M \mathrm{sc}_g \lambda_g$	

sectional: a section of $\boldsymbol{Gr}^2\boldsymbol{\Lambda}^2TM$ or a bilinear form on $\boldsymbol{\Lambda}^2TM$	$\kappa(v,w) := \dfrac{R(v,w,w,v)}{\lvert v\wedge w\rvert^2}$	
section of $T^{\mathsf{u}}M$	$\kappa(v,v)$	

List of Riemannian manifolds

Riemannian manifold	sectional curvature	Ricci	scalar
$\mathbb{R}^n$ with standard metric	0		
S^n of radius R	R^{-2}		
H^n of "radius" R	$-R^{-2}$		
$S^n \times \mathbb{R}^m$			

GSG #talk on *symplectic geometry and Hamiltonian flows*

Title: **When does a vector field kill area?**

Date: 22-Feb-25

Abstract: A Hamiltonian vector field of a given smooth function H on the plane $\mathbb{R}^2$ is the 90-degree rotation of its gradient vector field $\mathrm{grad}(H)$. With this simple definition, we can picture the flow of such a vector field: it must be along the "curves" where H is constant. Thus, we say the Hamiltonian vector field of H "preserves H". We also observe such flows preserve area on the plane, which begs the question: Are all area-preserving flows Hamiltonian? We try to answer these questions on the plane and its naive generalizations to the 2-sphere and complex vector spaces. And then, we try to generalize Hamiltonian vector fields to smooth manifolds properly: we understand that the minimal structure needed to define a Hamiltonian vector field is precisely the choice of a symplectic form. In this general setting, we shall discuss many introductory topics (Darboux theorem, Moser's theorem, Noether's theorem) and advanced questions/open problems in the field (Arnold conjecture), motivated through examples.

https://ggl.link/hamiltonian

$\mathbb{C} = \mathbb{R}^2$	S^2	$\mathbb{C}^n = \mathbb{R}^{2n}$	(M, ω) ***symplectic manifolds***
$J = [i]$	J is the complex structure on the Riemann sphere	$J = \begin{bmatrix} i & & \\ & i & \\ & & \ddots \end{bmatrix}$	not available in general!
$X_H := J\mathrm{grad}(H)$	$X_H := J\mathrm{grad}(H)$	$X_H := J\mathrm{grad}(H)$	$X_H := -\omega^{-1}\mathrm{d}H$ *notice the "$-$" sign!*
$X_H H = 0$	$X_H H = 0$	$X_H H = 0$	$X_H H = 0$
area preserving $\mathcal{L}_{X_H}\mathrm{d}x \wedge \mathrm{d}y = 0$	*area preserving* $\mathcal{L}_{X_H}\lambda_{\text{area}} = 0$	volume preserving, but preserves more...	*"k-dimensional area preserving"* $\mathcal{L}_{X_H}\omega^k = 0$

$\mathbb{C} = \mathbb{R}^2$	S^2	$\mathbb{C}^n = \mathbb{R}^{2n}$	(M, ω) ***symplectic manifolds***
Converse? True for $\mathbb{R}^2$ but not true for general *open subsets!*	Converse? true for S^2!	Converse for volume preservation is false for $n > 2$, even for linear flows!	Converse is false for $\dim M > 2$! However, it is true that vector fields that preserve the *symplectic 2-form* are Hamiltonian on a simply connected domain.
Linear maps that are *generated* by Hamiltonian vector fields are just the area preserving ones $$Sp_{\mathbb{R}}(2) = SL_{\mathbb{R}}(2)$$	What could be "linear" flows? well isometries $$SO_{\mathbb{R}}(3)$$ preserve area, so we have one nice Lie group action on the 2-sphere.	$$SL_{\mathbb{R}}(2n) \supset Sp_{\mathbb{R}}(2n) \supset U(n)$$	symplectic G-actions, Hamiltonian G-actions

on the plane

We want a vector field that *preserves* a given smooth

$$H : \mathbb{R}^2 \to \mathbb{R}$$

Then

$$\operatorname{grad}(H)$$

does the opposite: its flow goes out of $H^{-1}(c)$ *curves*.

So we should work with the 90 degrees rotation of the gradient, so

$$J\operatorname{grad}(H)$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is the clockwise 90 degrees rotation matrix.

The flow of this vector field ϕ^t such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi^t(p) = J\mathrm{grad}(H)_{\phi^t(p)}$$

which quickly implies

$$\frac{\mathrm{d}}{\mathrm{d}t}H(\phi^t(p)) = \mathrm{d}H(\frac{\mathrm{d}}{\mathrm{d}t}\phi^t(p)) = \langle \mathrm{grad}(H), J\mathrm{grad}(H)\rangle = 0$$

by chain rule!

Hamiltonian vector fields on plane preserve area

$$X_H = \begin{bmatrix} -\frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial x} \end{bmatrix}$$

$$\mathrm{div}(X_H) = -\frac{\partial}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial}{\partial y}\frac{\partial H}{\partial x} = 0$$

And divergence free vector fields preserve area (volume in $\mathbb{R}^n$)

space.R.n.Vec.volume

Volume change by flows on $\mathbb{R}^n$

Consider a **smooth** flow φ^t on some open subset of $\mathbb{R}^n$, this means

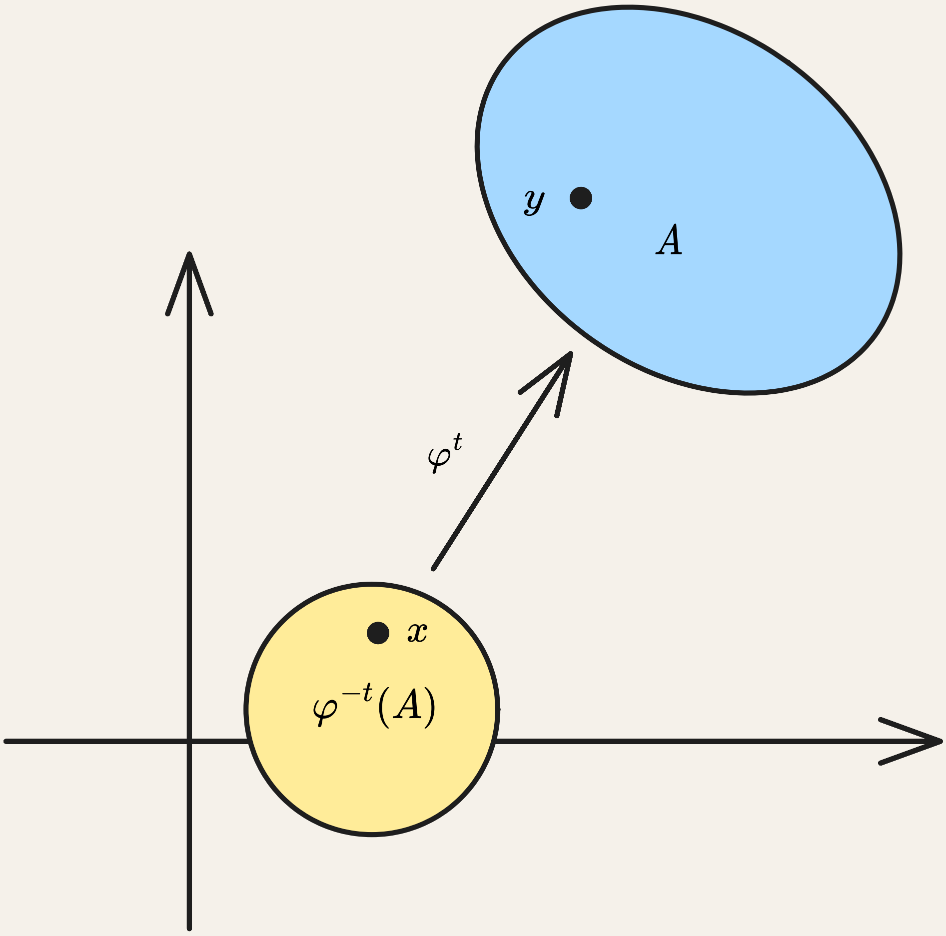
$$\varphi^t : U_t \subseteq U \to \mathbb{R}^n$$

is an injective local diffeomorphism. The *only* problem is surjectivity!

Then choose $A \subseteq \mathbb{R}^n$ so that it's preimage is not empty, then

$$\varphi^t : \varphi^{-t}(A) \to A$$

is a diffeomorphism.



We wish to compare volume of A with volume of $\varphi^{-t}(A)$. Thus we calculate volume

- By *change of variables* we get

$$\begin{aligned}\mu(A) &= \int_{y\in A} \mathrm{d}y_1 \wedge \dots \mathrm{d}y_n \\ &= \int_{x\in\varphi^{-t}(A)} \mathrm{d}\varphi_1^t(x) \wedge \cdots \wedge \mathrm{d}\varphi_n^t(x) \\ &= \int_{x\in\varphi^{-t}(A)} (\det \mathfrak{D}\varphi^t)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\end{aligned}$$

- So

$$\mu(\varphi^{-t}(A)) - \mu(A) = \int_{x\in A} (1 - \det \mathfrak{D}\varphi^t)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

- Now

$$\varphi^0 = \mathsf{Id} \implies \det \mathfrak{D}\varphi^0 = 1$$

  which means

$$\lim_{t\to 0} \frac{1}{t}(1 - \det \mathfrak{D}\varphi^t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det \mathfrak{D}\varphi^t$$

- Thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mu(\varphi^{-t}(A))=\int_{x\in A}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\det\mathfrak{D}\varphi^t\right)\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$$

So rate of change of volume boils down to $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\det\mathfrak{D}\varphi^t$.

computing derivative of $\det\mathfrak{D}\varphi^t$ using wedge products

Bug

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\det(\mathfrak{D}\varphi^t)&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\varphi^t}{\partial x_1}\wedge\cdots\wedge\frac{\partial\varphi^t}{\partial x_n}\right)\\&=\frac{\partial}{\partial x_1}\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\varphi^t}_{X(\varphi^t(x))}\wedge\frac{\partial\varphi^t}{\partial x_2}\wedge\cdots\wedge\frac{\partial\varphi^t}{\partial x_n}+\dots\\&=\left(\mathfrak{D}X\left(\frac{\partial}{\partial x_1}\varphi^t\right)\right)\wedge\\&=\underbrace{\frac{\partial X_1}{\partial x_1}\det\mathfrak{D}\varphi^t+\dots}_{\det(\mathfrak{D}\varphi^t\ )\mathrm{div}X}\end{aligned}$$

computing derivative of $\det\mathfrak{D}\varphi^t$ using Jacobi's formula

From

(Jacobi's formula)

$$\frac{\mathrm{d}}{\mathrm{d}t}\det A(t)=(\det A(t))\mathrm{tr}\left(A(t)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}A(t)\right)$$

and

$$\begin{aligned}A'(t)&=B(t)A(t)\\\implies(\det A)'(t)&=\det(A(t))\mathrm{tr}(A(t)^{-1}B(t)A(t))\\&=\det(A(t))\mathrm{tr}(B(t))\end{aligned}$$

along with

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(t,x) &= X_{\varphi(t,x)} \\ \implies \frac{\partial}{\partial t}\mathfrak{D}\varphi(t,x) &= \mathfrak{D}X_{\varphi(t,x)} \\ &= \mathfrak{D}_{\varphi(t,x)}X \circ \mathfrak{D}\varphi(t,x) \end{aligned}$$

we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{D}\varphi^t = \mathfrak{D}\frac{\mathrm{d}}{\mathrm{d}t}\varphi^t = \mathfrak{D}X_{\varphi^t} = \mathfrak{D}X \circ \mathfrak{D}\varphi^t \\ \implies \operatorname{tr}(\mathfrak{D}\varphi^{-t} \circ \mathfrak{D}X \circ \mathfrak{D}\varphi^t) = \operatorname{tr}(\mathfrak{D}X) \end{aligned}$$

So we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\det \mathfrak{D}\varphi^t_{(t,x)} &= \operatorname{tr}(\mathfrak{D}X)_{(\varphi^t(x))}\det \mathfrak{D}\varphi^t_{(t,x)} \\ &= \operatorname{div}(X)_{(\varphi^t(x))}\det \mathfrak{D}\varphi^t_{(t,x)} \end{aligned}$$

- At $t=0$, $\varphi^0 = \mathrm{Id}$ thus we have

$$\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\det \mathfrak{D}\varphi^t\right)(x) = \operatorname{div}(X)_x$$

- In general

$$\det \mathfrak{D}\varphi^t_{(t,x)} = \exp\left(\int_{[0,t]}\operatorname{div}(X)_{\varphi^t(x)}\mathrm{d}t\right)$$

[1]

[2]

computing derivative of $\det \mathfrak{D}\varphi^t$ using Liouville's formula

[3]

rate of change of volume is determined by integral of divergence

Hence, we have

For flow

$$\exp(tX) =: \varphi^t$$

of a smooth vector field X on a open subset of $\mathbb{R}^n$, the rate of change of volume of $\phi^{-t}(A), A \subseteq \mathbb{R}^n$ at $t=0$ is the volume integral of divergence of X, on A

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mu(\varphi^{-t}(A)) = \int_{x\in A} \operatorname{div}(X)_x \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

Thus

- $\operatorname{div}(X) \equiv 0 \iff$ the flow of X is *volume preserving*

using Reynolds transport theorem - Wikipedia

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1. https://math.stackexchange.com/a/3456775/1290493 ↩
2. https://mathoverflow.net/a/327773 ↩
3. https://math.stackexchange.com/a/312623/1290493 ↩

are all area preserving flows Hamiltonian vector fields?

Well,

- $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is *Hamiltonian* for some smooth H
  - $\iff$

$$\exists H \in \mathcal{C}^\infty(\mathbb{R}^2) : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -\frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial x} \end{bmatrix}$$

  - $\iff$

$$\exists H \in \mathcal{C}^\infty(\mathbb{R}^2) : \mathrm{d}H = X_2 \mathrm{d}x - X_1 \mathrm{d}y$$

- So we may use facts about 1-forms

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \leftrightarrow \alpha = X_2 \mathrm{d}x - X_1 \mathrm{d}y$$

  - We know **on a simply connected domain** α is closed $(\mathrm{d}\alpha = 0) \iff \alpha$ is exact $\exists H \in \mathcal{C}^\infty(\mathbb{R}^2) : \alpha = \mathrm{d}H$.
  - Now

$$\mathrm{d}\alpha = \underbrace{\left(-\frac{\partial X_1}{\partial x} - \frac{\partial X_2}{\partial y}\right)}_{-\mathrm{div}(X)} \mathrm{d}x \wedge \mathrm{d}y$$

  - Hence, on a **simply connected domain** flow of X is area preserving $\iff X$ is divergence free $\iff \exists H \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that

$$X = J\mathrm{grad}(H)$$

  - **Notice a minus sign pops out in $\mathrm{d}\alpha$ above.**

a area preserving flow that is *not* globally Hamiltonian on the punctured plane

The 1-form

Definition. **Angle form on $\mathbb{R}^2 \setminus \{0\}$**

The angle form on $\mathbb{R}^2 \setminus \{0\}$ is defined by

$$\frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2 + y^2}$$

using the coordinates (x, y).

is closed but not exact.

So equating this form with $-\alpha$ above we get the vector field

$$X_\theta := \frac{1}{x^2 + y^2}\begin{bmatrix} x \\ y \end{bmatrix}$$

which has zero divergence but does not have a globally defined Hamiltonian.

It is however locally Hamiltonian for the angle function

$$X_\theta = J\mathrm{grad}(\theta)$$

defined on any simply connected subset of $\mathbb{R}^2 \setminus \{0\}$.

Hamiltonian but in different coordinates

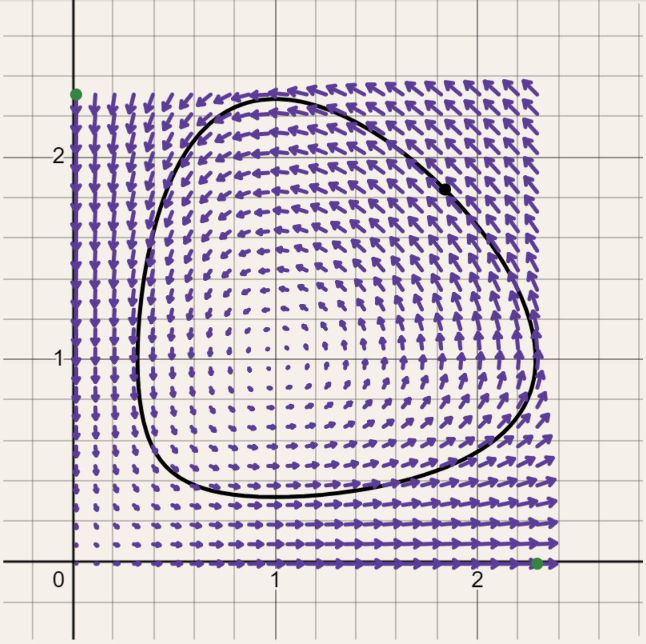
space.f.field.usual Lotka-Volterra

Lotka-Volterra vector field on $\mathbb{R}^2$

Definition. **Lotka-Volterra vector field on $\mathbb{R}^2$**

$$\begin{bmatrix} \lambda_1 x - \eta_1 xy \\ -\lambda_2 y + \eta_2 xy \end{bmatrix}$$

which looks like



on $\mathbb{R}^2_{>0}$
[1]

for $\lambda_i, \eta_i = 1$

- $$\operatorname{div}\begin{bmatrix} x - xy \\ -y + xy \end{bmatrix} = 1 - y - 1 + x = x - y$$

Intuition

The flow is not area preserving of course, but it has a "*hidden area preserving*" because the flow is periodic on $\mathbb{R}^2_{>0}$.

- However, it preserves a *different* area form on $\mathbb{R}^2_{>0}$
$$\mathfrak{L}_X\left(\frac{\mathrm{d}x \wedge \mathrm{d}y}{xy}\right) = \operatorname{div}\left(\frac{1}{xy}X\right) = 0$$
- Consider the smooth function for $(x, y) \in \mathbb{R}^2_{>0}$
$$x - \log x + y - \log y$$
  - The vector field preserves this function
$$\begin{aligned}\partial_{\begin{bmatrix} x-xy \\ -y+xy \end{bmatrix}}(x - \log x + y - \log y) &= (x - xy)\left(1 - \frac{1}{x}\right) + (-y + xy)\left(1 - \frac{1}{y}\right) \\ &= (x - xy - y + xy) - 1 + y + 1 - x \\ &= 0\end{aligned}$$

- However,

$$\operatorname{grad}(x-\log x+y-\log y)=\begin{bmatrix}1-\frac{1}{x}\\1-\frac{1}{y}\end{bmatrix}$$

$$\implies J\operatorname{grad}(x-\log x+y-\log y)=\begin{bmatrix}-1+\frac{1}{y}\\1-\frac{1}{x}\end{bmatrix}=\frac{1}{xy}\begin{bmatrix}x-xy\\-y+xy\end{bmatrix}$$

- This shows the vector field is Hamiltonian vector field with the symplectic form $\frac{1}{xy}\mathrm{d}x\wedge\mathrm{d}y$ on $\mathbb{R}^2_{>0}$.

https://www.desmos.com/calculator/l0gvaeds5s

Lotka-Volter flow

in logarithmic coordinates

- In coordinates

$$\begin{aligned}\mathbb{R}^2_{>0}&\to\mathbb{R}^2_{>0}\\\begin{pmatrix}x\\y\end{pmatrix}&\mapsto\begin{pmatrix}\log x\\\log y\end{pmatrix}\end{aligned}$$

we have

Now it is a Hamiltonian system.

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1. https://www.desmos.com/calculator/l0gvaeds5s ↩

Todo

- Lotka-Volterra's vector field
- diagonal action on $\mathbb{C}^n$

sidenote: vector fields that are (locally) Hamiltonian as well as (locally) gradient

object	equivalent to Cauchy-Riemann operator	Cauchy-Riemann operator/equivalent gives 0 which makes the object a	if Cauchy-operator/ gives 0 th primitive
a complex function $f=u+iv$	the Cauchy-Riemann operator $$\frac{\partial}{\partial\bar{z}}f=\frac{1}{2}\Big((u_x-v_y)+i(u_y+v_x)\Big)$$	Holomorphic function	$\exists g$ $\frac{\partial}{\partial}$
a (complex) differential $(1,0)$-form $f(z)\mathrm{d}z$	the exterior derivative $$\mathrm{d}(f(z)\mathrm{d}z)=-\frac{\partial f}{\partial\bar{z}}\mathrm{d}z\wedge\mathrm{d}\bar{z}$$	closed differential 1-form/Holomorphic 1-form	$\exists g\mathrm{d}z$ $\mathrm{d}g$
(real) differential 1-form $\omega=u\mathrm{d}x-v\mathrm{d}y$ and its Hodge dual $\star\omega=v\mathrm{d}x+u\mathrm{d}y$	$$\begin{aligned}\mathrm{d}\omega&=(u_y+v_x)\mathrm{d}x\wedge\mathrm{d}y\\ \mathrm{d}\star\omega&=(v_y-u_x)\mathrm{d}x\wedge\mathrm{d}y\end{aligned}$$	closed, coclosed (real) differential 1-form	$\exists\alpha,\beta$ (real $\mathrm{d}\alpha=$
(real) vector field $\bar{f}=\begin{bmatrix}u\\-v\end{bmatrix}$	$$\begin{aligned}\mathrm{curl}\bar{f}&=-v_x-u_y\\ \mathrm{div}\bar{f}&=u_x-v_y\end{aligned}$$	incompressible (area preserving/symplectic) and a solenoid (curl free) vector field	$\exists H,\phi$ such $J\mathrm{grad}(H)$

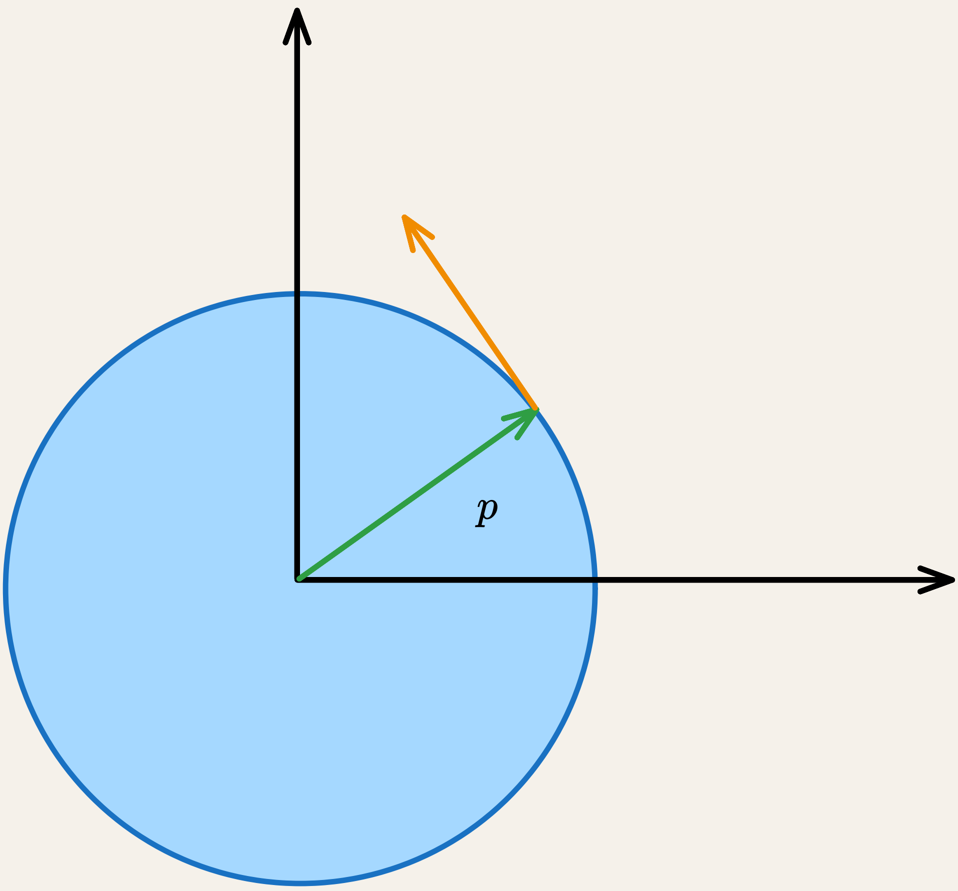
on the 2-sphere

Almost complex structure on S^2 induced from spherical metric and its area form

Definition. Almost complex structure on S^2 using cross product on $\mathbb{R}^3$.

The involution

$$\begin{aligned} J : TS^2 &\to TS^2 \\ X_p &\mapsto p \times X_p \end{aligned}$$



where we've identified $T_pS^2 \leq \mathbb{R}^3$.

Because area spanned by the parallelogram is

$$\|X\| \, \|Y\| \sin(\theta_{X,Y})$$

and *oriented area* is

- supposed to be

$$\begin{aligned} \langle X, Y \rangle &= \|X\| \, \|Y\| \cos(\theta_{X,Y}) \\ &= \|X\| \, \|Y\| \sin\left(\theta_{X,Y} + \frac{\pi}{2}\right) \\ &= \langle JX, Y \rangle \end{aligned}$$

- $$\langle JX, Y\rangle = \langle p \times X, Y\rangle = \langle p, X \times Y\rangle$$

  which, again, *should* be the oriented area of $X, Y \in T_pS^2$
- In local coordinates

$$\langle JX, Y\rangle = \left\langle J\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\rangle \mathrm{d}x \wedge \mathrm{d}y = \left\|\frac{\partial}{\partial x} \times \frac{\partial}{\partial y}\right\| \mathrm{d}x \wedge \mathrm{d}y$$

so we conclude

$$g_{\mathsf{S}}(JX, Y) = \lambda_{\mathsf{S}}(X, Y)$$

So same as before we define

Definition. Hamiltonian vector field of a smooth function on the 2-sphere $H \in \mathcal{C}^\infty(S^2)$ is the vector field

$$X_H = J\mathrm{grad}(H)$$

- It preserves H because of chain rule, again.
- It preserves area

on complex vector spaces

diagonal action on $\mathbb{C}^2$

space.f.oscillators.harmonic.2

diagonal action on $\mathbb{C}^2$

AKA the dynamics of 2 harmonic oscillators

On the symplectic 4-manifold

$$\left(\mathbb{R}^4, \sum_{i=1}^{2} \mathrm{d}p_i \wedge \mathrm{d}q_i\right) = \left(\mathbb{C}^2, \frac{1}{2i}\sum_{i=1}^{2} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i\right)$$

we have the following Hamiltonian vector field and flow

equations	vector field
$$\begin{aligned}\dot p_1 &= -a_1 q_1\\ \dot q_1 &= a_1 p_1\\ \dot p_2 &= -a_2 q_2\\ \dot q_2 &= a_2 p_2\end{aligned}$$ or in more revealing, matrix form $$\begin{bmatrix}\dot p_1\\ \dot q_1\\ \dot p_2\\ \dot q_2\end{bmatrix} = \begin{bmatrix} & -a_1 & & \\ a_1 & & & \\ & & & -a_2\\ & & a_2 & \end{bmatrix}\begin{bmatrix}p_1\\ q_1\\ p_2\\ q_2\end{bmatrix}$$ where the matrix is a block matrix with a scaled $\pi/2$-rotation matrix	$$\omega_1 \underbrace{\left(-q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1}\right)}_{X_1} + \omega_2 \underbrace{\left(-q_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial q_2}\right)}_{X_2}$$
$$\begin{aligned}\dot z_1 &= i\omega_1 z_1\\ \dot z_2 &= i\omega_2 z_2\end{aligned}$$ or $$\begin{bmatrix}\dot z_1\\ \dot z_2\end{bmatrix} = \begin{bmatrix} i\omega_1 & \\ & i\omega_2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}$$	$$\begin{aligned}&\omega_1 \underbrace{i z_1 \frac{\partial}{\partial z_1}}_{X_1} + \omega_2 \underbrace{i z_2 \frac{\partial}{\partial z_2}}_{X_2}\\ &= \omega_1 X_1 + \omega_2 X_2\end{aligned}$$

We may rephrase in terms of the diagonal action on $\mathbb{C}^2$ by $\mathbb{T}^2$

action	generating vector field	moment map
diagonal $\mathbb{T}^2$-action on $\mathbb{C}^2$ *with both frequencies = 1* given by $$\begin{aligned}\mathbb{T}^2 &\curvearrowright \mathbb{C}^2\\ \begin{bmatrix}z_1\\ z_2\end{bmatrix} &\overset{t_1,t_2}{\longmapsto} \begin{bmatrix}\exp(it_1)z_1\\ \exp(it_2)z_2\end{bmatrix}\end{aligned}$$	the generators of the $\mathbb{T}^2$-action are the vector fields $$X_1, X_2$$ with $$[X_1, X_2] = 0$$	the $$H_1, H_2$$ above with $$\{H_1, H_2\} = 0$$ give with moment map $$\begin{aligned}(H_1, H_2) : \mathbb{C}^2 &\to \mathbb{R}^2\\ (z, z_2) &\mapsto \frac{1}{2}(z_1\bar z_1, z_2 \bar z_2)\end{aligned}$$ of the Hamiltonian $\mathbb{T}^2$-action

action	generating vector field	moment map
from the homomorphism $$\begin{aligned} S^1 &\to \mathbb{T}^2 \curvearrowright \mathbb{C}^2 \\ t &\mapsto l_1 t + l_2 t \end{aligned}$$ for $l_1, l_2 \in \mathbb{Z}$ we get a S^1-action on $\mathbb{C}^2$ with *integer frequencies*	the generator is $$l_1 X_1 + l_2 X_2$$	
$$\mathbb{R} \to \mathbb{T}^2$$	$$\omega_1 X_2 + \omega_2 X_2$$	
$$\mathbb{T}^2 \to \mathbb{T}^2$$		

restricting to the invariant 3-sphere

We restrict to the 3-sphere

$$S^3_h = \{(p_1, q_1, p_2, q_2) \in \mathbb{R}^4 | p_1^2 + q_1^2 + p_2^2 + q_2^2 = 2h\} \cong_{\mathsf{Man}} S^3$$

which is invariant under the flow of $\omega_1 X_1 + \omega_2 X_2$ and has orbits of $H_1 = h$. On this sphere, we have the invariant tori

$$T^2_{h_1,h_2} := \{(p_1, q_1, p_2, q_2) \in \mathbb{R}^4 | p_1^2 + q_1^2 = 2h_1, p_2^2 + q_2^2 = 2h_2\}$$

with $\omega_1 h_1 + \omega_2 h_2 = h$.

The projection

$$\begin{aligned} S^3 \setminus \{(0,0,0,1)\} &\to \mathbb{R}^3 \\ (x, y, z, w) &\mapsto (X, Y, Z) := \frac{1}{1-w}(x, y, z) \end{aligned}$$

maps

- the orbit

  $$(p_1, q_1, 0, 0) \mapsto (p_1, q_1, 0)$$

  where $p_1^2 + q_1^2 = 2h_1$, the image is thus a circle in the x-y plane in $\mathbb{R}^3$
- the orbit

  $$(0, 0, p_2, q_2) \mapsto \left(0, 0, \frac{p_2}{1-q_2}\right)$$

whose image is the z-axis

- $$T^2_{h_1,h_2} \mapsto \left\{(X,Y,Z)\middle| X^2+Y^2 = \frac{2h_1}{(1-w)^2}, Z^2 = \frac{2h_2 - w^2}{(1-w)^2}, w \neq 1\right\}$$

  which should be a torus in $\mathbb{R}^3$

-

restricting to the invariant tori: a torus translation

The subset

$$\begin{aligned} T^2_{a,b} &:= \{(p_1,q_1,p_2,q_2) \in \mathbb{R}^4 | p_1^2+q_1^2 = 2a, p_2^2+q_2^2 = 2b\} && \subseteq (\mathbb{R}^2 \setminus \{0\})^2 \\ S^3_h &:= \{(p_1,q_1,p_2,q_2) \in \mathbb{R}^4 | p_1^2+q_1^2+p_2^2+q_2^2 = 2h\} && \subseteq \mathbb{R}^4 \setminus \{0\} \end{aligned}$$

which is preserved by the flow of $\omega_1 X_1 + \omega_2 X_2$.

Thus $(\mathbb{R}^2 \setminus \{0\})^2$ has a torus fibration

coordinates	X_1	$\omega_1 X_1 + \omega_2 X_2$
On $\mathbb{R}^4$ we have (p_1, q_1, p_2, q_2)	$-q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1}$	$\omega_1 \underbrace{\left(- q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1} \right)}_{X_1} + \omega_2 \underbrace{\left(- q_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial q_2} \right)}_{X_2}$
On $(\mathbb{R}^2 \setminus \{0\})^2$ we have the coordinates $r_1^2 = p_1^2 + q_1^2$ θ_1 $r_2 = p_2^2 + q_2^2$ θ_2	$\frac{\partial}{\partial \theta_1}$	$\omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2}$ which is a torus translation $T^2_{a,b}$

Flow of a general vector field

$$\omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2}$$

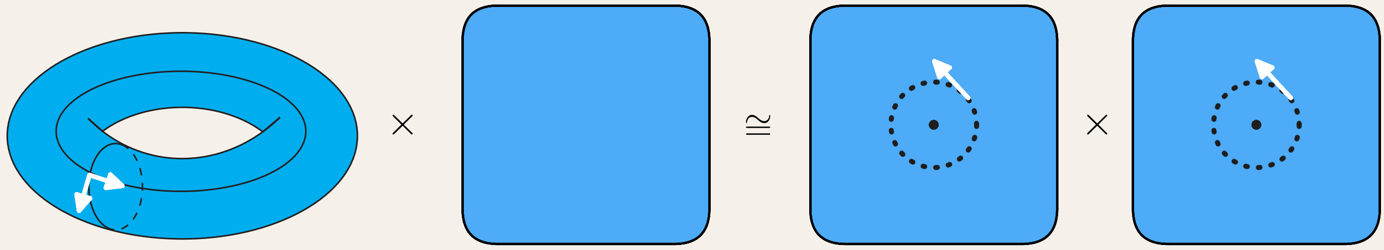
is given by

$$\exp\left(t \left(\omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2} \right) \right) \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} + t \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

whose flows have properties

if	flow is	which gives a	that is coming from
$\omega_1/\omega_2 \in \mathbb{Q}$	periodic	S^1-action on T^2	composition $S^1 \hookrightarrow T^2 \curvearrowright T^2$
$\omega_1/\omega_2 \in \mathbb{R} \setminus \mathbb{Q}$	not periodic, minimal (orbits are dense)	$\mathbb{R}$-action on T^2	composition $\mathbb{R} \hookrightarrow T^2 \curvearrowright T^2$

decomposition into torus translation $\times$ trivial

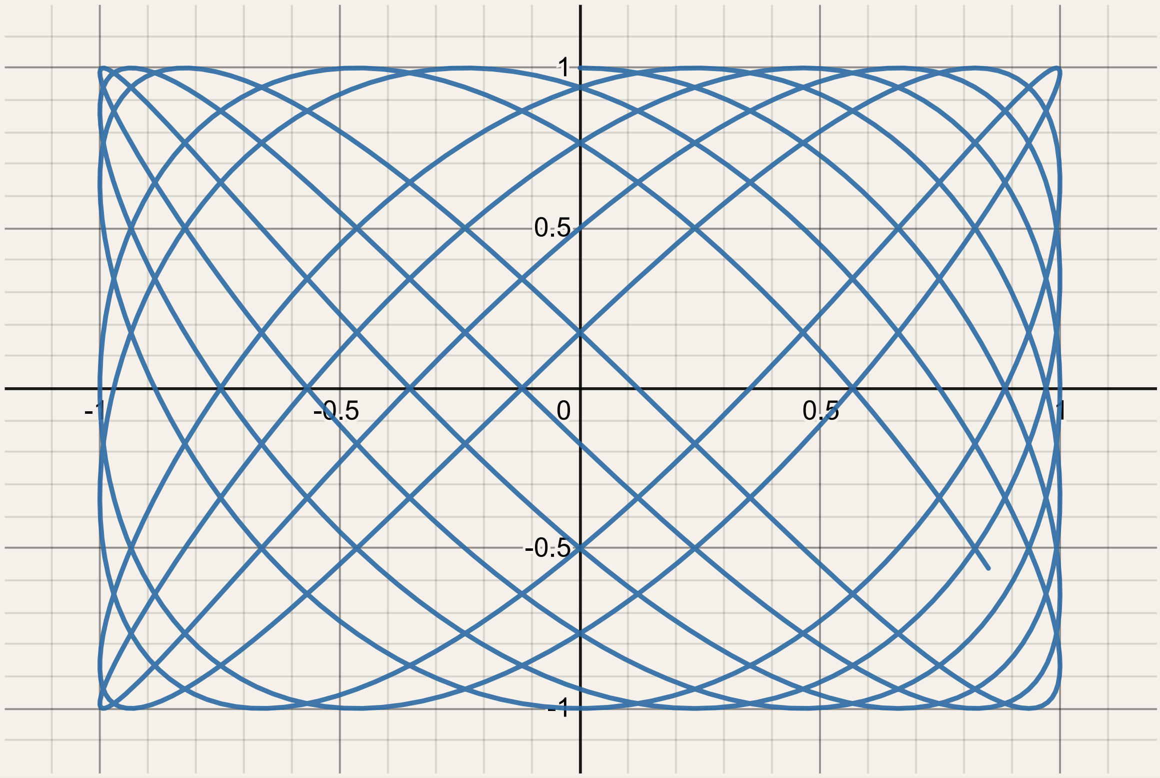


$$\begin{array}{ccccc}
\frac{\partial}{\partial \theta_1} & 0 & \longmapsto & -q_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial q_1} & 0 \\
\frac{\partial}{\partial \theta_2} & 0 & \longmapsto & 0 & -q_2\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial q_2}
\end{array}$$

$$\begin{aligned}
T^2 \curvearrowright T^2 \times \mathbb{R}^2 &\cong_{T^2} (\mathbb{R}^2 \setminus \{0\})^2 \\
v_1 \mapsto \frac{\partial}{\partial \theta_1} \times 0 \mapsto X_1 & \\
v_2 \mapsto \frac{\partial}{\partial \theta_2} \times 0 \mapsto X_2 &
\end{aligned}$$

projecting onto (q_1, q_2)

$$\{(\sin 45t, \cos 65t) | t \in [0,1]\}$$



Projecting $(0, q_1, 0, q_2)$ we have

$$(q_1, q_2) \overset{t}{\mapsto} (\sin(\omega_1 t) q_1, \sin(\omega_2 t) q_2)$$

[1]

rational oscillator when $\omega_1, \omega_2 \in \mathbb{Z}$

A $S^1 \cong U(1)$-action AKA representation on $\mathbb{R}^4 = \mathbb{C}^2$

irrational oscillator when $\omega_1/\omega_2 \in \mathbb{R} \setminus \mathbb{Q}$

as a Lax pair

as a Lax pair

We define

$$L := \begin{bmatrix} p & q \\ q & -p \end{bmatrix}$$
$$M := \frac{\omega}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then

$$\begin{aligned} [M, L] &= \frac{\omega}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p & q \\ q & -p \end{bmatrix} - \begin{bmatrix} p & q \\ q & -p \end{bmatrix} \frac{\omega}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{\omega}{2} \begin{bmatrix} -q & p \\ p & q \end{bmatrix} - \frac{\omega}{2} \begin{bmatrix} q & -p \\ -p & -q \end{bmatrix} \\ &= \omega \begin{bmatrix} -q & p \\ p & q \end{bmatrix} \end{aligned}$$

Hence, the oscillator is equivalent to

$$\dot{L} = [M, L]$$

thus giving us conserved quantities

$$\operatorname{tr}(L^2) = \operatorname{tr}\left(\begin{bmatrix} p^2 + q^2 & ? \\ ? & 0 \end{bmatrix}\right) = p^2 + q^2$$

$$L := \begin{bmatrix} p_1 & q_1 \\ q_1 & -p_1 \end{bmatrix} + \begin{bmatrix} p_2 & q_2 \\ q_2 & -p_2 \end{bmatrix} z$$
$$M := \frac{\omega}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$[M, A_1 + A_2 z] = [M, A_1] + [M, A_2] z$$

then

$$\dot{L} = [M, L]$$

should give the equation for isotropic harmonic oscillator.

Now

$$\begin{aligned} yI_2 - A_1 - A_2 z &= \begin{bmatrix} y - p_1 - zp_2 & -q_1 - zq_2 \\ -q_1 - zq_2 & y + p_1 + zp_2 \end{bmatrix} \\ \det(yI_2 - A_1 - A_2 z) &= y^2 - (p_1 + zp_2)^2 - (q_1 + zq_2)^2 \\ &= y^2 - (p_1^2 + q_1^2) - (p_2^2 + q_2^2)z^2 - 2z(p_1 p_2 + q_1 q_2) \end{aligned}$$

which is a conic section over which

$$(y, z) \mapsto \ker(yI_2 - A(z))$$

defines a line bundle.

[2]

---

1. https://www.desmos.com/calculator/irnyqunfsm ↩
2. https://www.desmos.com/calculator/2n2puejwvk ↩

on a symplectic manifold

We want a smooth tensor

$$\omega : T^*M \to TM$$

that is, smoothly varying (on $p \in M$) linear maps

$$\omega_p : T_p^*M \to T_p^*M$$

with which we want to define

$$X_H := -\omega^{-1}\mathrm{d}H \iff \mathrm{d}H(-) = -\omega(X_H, -)$$

as the Hamiltonian vector field of H such that

- **flow of X_H must preserve H**

$$0 = X_H H = \mathrm{d}H(X_H) = -\omega(X_H, X_H)$$

  $\iff$ ω is **alternating**, so a 2-form
- and **flow of X_H must preserve ω**

$$0 = \mathcal{L}_{X_H}\omega = \iota_{X_H}\mathrm{d}\omega + \cancelto{0}{\mathrm{d}\underbrace{\iota_{X_H}\omega}_{-\mathrm{d}H}}$$

  which is true when $\mathrm{d}\omega = 0$

Thus

> **Definition.** **A symplectic manifold (M, ω) is a** smooth manifold **M with a closed non-degenerate 2-form ω.**

properties of the symplectic form

> ω^n defines a volume form

Darboux local coordinates on symplectic manifolds

The Darboux Theorem

Our next theorem is one of the most fundamental results in symplectic geometry. It is a nonlinear analogue of the canonical form for a symplectic tensor given in Proposition 22.7. It illustrates the most dramatic difference between symplectic structures and Riemannian metrics: unlike the Riemannian case, there is no local obstruction to a symplectic structure being locally equivalent to the standard flat model.

Theorem 22.13 (Darboux). *Let (M,ω) be a $2n$-dimensional symplectic manifold. For any $p \in M$, there are smooth coordinates $\left(x^1,\dots,x^n,y^1,\dots,y^n\right)$ centered at p in which ω has the coordinate representation*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{22.5}$$

We will prove the theorem below. Any coordinates satisfying (22.5) theorem are called ***Darboux coordinates***, ***symplectic coordinates***, or ***canonical coordinates***.

Proof of the Darboux theorem. Let ω_0 denote the given symplectic form on M, and let $p_0 \in M$ be arbitrary. The theorem will be proved if we can find a smooth coordinate chart (U_0,φ) centered at p_0 such that $\varphi^*\omega_1 = \omega_0$, where $\omega_1 = \sum_{i=1}^n dx^i \wedge dy^i$ is the standard symplectic form on $\mathbb{R}^{2n}$. Since this is a local question, by choosing smooth coordinates $\left(x^1,\dots,x^n,y^1,\dots,y^n\right)$ in a neighborhood of p_0, we may replace M with an open ball $U \subseteq \mathbb{R}^{2n}$. Proposition 22.7 shows that we can arrange by a linear change of coordinates that $\omega_0|_{p_0} = \omega_1|_{p_0}$.

Let $\eta = \omega_1 - \omega_0$. Because η is closed, the Poincaré lemma (Theorem 17.14) shows that we can find a smooth 1-form α on U such that $d\alpha = -\eta$. By subtracting a constant-coefficient (and thus closed) 1-form from α, we may assume without loss of generality that $\alpha_{p_0} = 0$.

For each $t \in \mathbb{R}$, define a closed 2-form ω_t on U by

$$\omega_t = \omega_0 + t\eta = (1-t)\omega_0 + t\omega_1.$$

Let J be a bounded open interval containing $[0,1]$. Because $\omega_t|_{p_0} = \omega_0|_{p_0}$ is nondegenerate for all t, a simple compactness argument shows that there is some neighborhood $U_1 \subseteq U$ of p_0 such that ω_t is nondegenerate on U_1 for all $t \in \bar{J}$. Because of this nondegeneracy, the smooth bundle homomorphism $\widehat{\omega}_t \colon TU_1 \to T^*U_1$ defined by $\widehat{\omega}_t(X) = X \lrcorner\, \omega_t$ is an isomorphism for each $t \in \bar{J}$.

Define a smooth time-dependent vector field $V \colon J \times U_1 \to TU_1$ by $V_t = \widehat{\omega}_t^{-1}\alpha$, or equivalently

$$V_t \lrcorner\, \omega_t = \alpha.$$

Our assumption that $\alpha_{p_0} = 0$ implies that $V_t|_{p_0} = 0$ for each t. If $\theta \colon \mathcal{E} \to U_1$ denotes the time-dependent flow of V, it follows that $\theta(t, 0, p_0) = p_0$ for all $t \in J$, so $J \times \{0\} \times \{p_0\} \subseteq \mathcal{E}$. Because $\mathcal{E}$ is open in $J \times J \times M$ and $[0, 1]$ is compact, there is some neighborhood U_0 of p_0 such that $[0, 1] \times \{0\} \times U_0 \subseteq \mathcal{E}$.

For each $t_1 \in [0, 1]$, it follows from Proposition 22.15 that

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_1} (\theta_{t,0}^*\omega_t) &= \theta_{t_1,0}^*\left(\mathcal{L}_{V_{t_1}}\omega_{t_1} + \left.\frac{d}{dt}\right|_{t=t_1}\omega_t\right) \\
&= \theta_{t_1,0}^*\big(V_{t_1} \lrcorner\, d\omega_{t_1} + d(V_{t_1} \lrcorner\, \omega_{t_1}) + \eta\big) \\
&= \theta_{t_1,0}^*(0 + d\alpha + \eta) = 0.
\end{aligned}$$

Therefore, $\theta_{t,0}^*\omega_t = \theta_{0,0}^*\omega_0 = \omega_0$ for all t. In particular, $\theta_{1,0}^*\omega_1 = \omega_0$. It follows from Theorem 9.48(c) that $\theta_{1,0}$ is a diffeomorphism onto its image, so it is a coordinate map. Because $\theta_{1,0}(p_0) = p_0 = 0$, these coordinate are centered at p_0. $\square$

Moser's theorem on volumes

Theorem 2. (Moser) Assume M is a compact, connected and oriented manifold of dimension m without boundary. If α and β are two volume forms such that their total volumes agree, i.e.,

$$\int_M \alpha = \int_M \beta, \tag{1.40}$$

then there is a diffeomorphism u of M satisfying $u^*\beta = \alpha$.

Consequently the total volume is the only invariant of volume-preserving diffeomorphisms.

Proof. We proceed as in the proof of Darboux's theorem and deform the volume form α into β defining

$$\alpha_t = (1-t)\alpha + t\beta, \quad 0 \le t \le 1.$$

These forms α_t are volume forms since locally α and β are represented by $\alpha = a(x)dx_1 \wedge \ldots \wedge dx_m$ and $\beta = b(x)dx_1 \wedge \ldots \wedge dx_m$ with nonvanishing smooth functions a and b, which, by assumption (1.40), must have the same sign. We shall construct a family φ^t of diffeomorphisms satisfying

$$\left(\varphi^t\right)^* \alpha_t = \alpha, \quad \varphi^0 = id \tag{1.41}$$

for $0 \le t \le 1$, so that the diffeomorphism $u = \varphi^1$ will solve our problem. Since M is compact, connected and oriented we conclude from $\int\limits_M (\beta - \alpha) = 0$ that

$$\beta - \alpha = d\gamma \tag{1.42}$$

for some $(m-1)$-form γ on M. This is a special case of the de Rham theorem. Since α_t is a volume form we find a unique time-dependent vector field X_t on M solving the linear equation

$$i_{X_t}\alpha_t = -\gamma$$

for $0 \le t \le 1$. Denote by φ^t the flow of this vector field X_t satisfying $\varphi^0 = id$. Since M is compact it exists for all t. Since $d\alpha_t = 0$ for volume forms we find, again using Cartan's formula,

$$\begin{aligned}
\tfrac{d}{dt}\left(\varphi^t\right)^* \alpha_t &= \left(\varphi^t\right)^* \left(L_{X_t}\alpha_t + \tfrac{d}{dt}\alpha_t\right) \\
&= \left(\varphi^t\right)^* \left(d(i_{X_t}\alpha_t) + \beta - \alpha\right),
\end{aligned}$$

which vanishes since $d(i_{X_t}\alpha_t) + \beta - \alpha = d(i_{X_t}\alpha_t + \gamma) = 0$ by our choice of the vector field X_t. Therefore (1.41) holds and the proof is finished. ■

references

- Jack Lee's book on smooth manifolds
- http://staff.ustc.edu.cn/~wangzuoq/Courses/15S-Symp/SympGeom.html
- Ana Canas da Silver, Lectures on Symplectic Geometry, Lecture Notes in Mathematics 1764, Springer.

Corinna Ulcigrai - Chaotic Properties of Area Preserving Flows

Flows on surfaces are one of the fundamental examples of dynamical systems, studied since **Poincaré**; area preserving flows arise from many physical and mathematical examples, such as the **Novikov model of electrons in a metal, unfolding of billiards in polygons, pseudo-periodic topology**.
In this course we will focus on smooth area-preserving or locally Hamiltonian flows and their ergodic properties. The course will be self-contained, so we will define basic ergodic theory notions as needed and no prior background in the area will be assumed.
The course aim is to explain some of the many developments happened in the last decade. These include the full classification of generic mixing properties (mixing, weak mixing, absence of mixing) motivated by a conjecture by Arnold, up to very recent rigidity and disjointness results, which are based on a breakthrough adaptation of ideas originated from Marina Ratner's work on unipotent flows to the context of flows with singularities.
We will in particular highlight the role played by shearing as a key geometric mechanism which explains many of the chaotic properties in this setup. A key tool is provided by Diophantine conditions, which, in the context of higher genus surfaces, are imposed through a multi-dimensional continued fraction algorithm (Rauzy-Veech induction): we will explain how and why they appear and how they allow to prove quantitative shearing estimates needed to investigate chaotic properties.

- https://www.zora.uzh.ch/id/eprint/194106/1/Ulcigrai2020_Article_SlowChaosInSurfaceFlows.pdf
- https://homepage.mi-ras.ru/~snovikov/74.pdf

Lecture 2

Definition. **Let $\Sigma \subseteq S$ be an arc transversal to the flow Φ^t on Σ. For $x \in \Sigma$ we let the minimum return time $r(x)$ be the minimum $t > 0$ such that $\Phi^t(x) \in \Sigma$, if defined, then**

$$\begin{aligned} T : \Sigma &\to \Sigma \\ x &\mapsto \Phi^{r(x)}(x) \end{aligned}$$

2502 geometry and analysis of flows and on manifolds.analysis

2502 geometry and analysis of flows and on manifolds.integration by parts and pieces

integration by parts and pieces

vector calculus in $\mathbb{R}^3$

space.R.3.Vec.derivatives

Derivatives of vector fields in $(\mathbb{R}^3, \mathrm{DOT})$

$$\mathcal{C}^\infty(U) \overset{\text{grad}}{\to} \underset{\text{curl}\circ\text{grad}=0}{\mathrm{Vec}^\infty(U)} \overset{\text{curl}}{\to} \underset{\text{div}\circ\text{curl}=0}{\mathrm{Vec}^\infty(U)} \overset{\text{div}}{\to} \mathcal{C}^\infty(U)$$

- and we have directional derivative of scalar fields

$$\partial_X : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$$
$$\partial_X\phi = \langle X, \mathrm{grad}(\phi)\rangle = \mathrm{d}\phi(X)$$

- and vector fields

$$\partial_X : \mathrm{Vec}^\infty(U) \to \mathrm{Vec}^\infty(U)$$

- and the Laplacian of scalar fields

$$\Delta : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$$

- and vector fields

$$\Delta : \mathrm{Vec}^\infty(U) \to \mathrm{Vec}^\infty(U)$$

Note, curl, directional derivatives and two Laplacians are $\mathbb{R}$-vector space *endomorphisms*, thus we may talk about invariant subspaces, etc.

- The Leibnitz-type product rules which all comes from

$$\mathrm{d}^{k+1}\omega \wedge \eta = (\mathrm{d}^k\omega) \wedge \eta + (-1)^k \omega \wedge \mathrm{d}^l\eta$$

giving us

$$
\begin{aligned}
\partial_X(fg) &= f\partial_X g + g\partial_X f \\
\mathrm{grad}(fg) &= f\,\mathrm{grad}(g) + g\,\mathrm{grad}(f) \\
\mathrm{curl}(fX) &= f\mathrm{curl}(A) - A \times \mathrm{grad}(f) \\
\mathrm{div}(fX) &= f\mathrm{div}(A) + A \cdot \mathrm{grad}(f)
\end{aligned}
$$

- On

  $$(\mathrm{Vec}^\infty(U), +, \times, \langle , \rangle)$$

  where

  $$\langle , \rangle : \mathrm{Vec}^\infty(U) \times \mathrm{Vec}^\infty(U) \to \mathcal{C}^\infty(U)$$

  we have more "products", so more "product" rules

  $$
  \begin{aligned}
  \mathrm{grad}\,\langle X, Y\rangle &= X \times \mathrm{curl}(Y) + Y \times \mathrm{curl}(X) + \partial_X Y + \partial_Y X \\
  \mathrm{curl}(X \times Y) &= -\partial_X Y + \partial_Y X + X\mathrm{div}(Y) - Y\mathrm{div}(X) \\
  \mathrm{div}(X \times Y) &= \langle Y, \mathrm{curl}(X)\rangle - \langle X, \mathrm{curl}(Y)\rangle
  \end{aligned}
  $$

- And this one connects double derivatives

  $$\mathrm{curl} \circ \mathrm{curl} = \mathrm{grad} \circ \mathrm{div} - \Delta$$

$X = X^i \partial_{x_i}$	$\alpha = X^i \mathrm{d}x_i$
$\mathrm{curl}(X^i \partial_{x_i})$	$\mathrm{d}(X^i \mathrm{d}x_i) = \dfrac{\partial X^i}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_i$
	$\iota_X \mathrm{d}(X^i \mathrm{d}x_i) = \dfrac{\partial X^i}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_i(X, -)$
	$\alpha \wedge \beta$

calculus in $\mathbb{R}^n$

object	In Cartesian coordinates
Volume form λ	$\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$

integrating *gradient vector field* ⊗ *volume form* in $\mathbb{R}^n$

- Let $B \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^1$ boundary. Let $u \in \mathcal{C}^1(\text{open nbd of } \bar{\Omega})$ then we have

$$^{n-1}\alpha = u\widehat{\mathrm{d}x_i}$$

where $\widehat{\mathrm{d}x_i}$ is $\lambda = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ with $\mathrm{d}x_i$ removed.

  - The **exterior derivative** of this $n-1$ form is the n-form

$$\mathrm{d}\alpha = \frac{\partial u}{\partial x_i}\mathrm{d}x_i \wedge \widehat{\mathrm{d}x_i} = (-1)^{i+1}\frac{\partial u}{\partial x_i}\lambda$$

  where

$$\begin{aligned}\mathrm{d}x_i \wedge \widehat{\mathrm{d}x_i} &= \mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \dots \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n \\ &= (-1)^{i+1}\underbrace{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}_{\lambda}\end{aligned}$$

  - Then by *Stroke's theorem*

$$\begin{aligned}\int_B \mathrm{d}\alpha &= (-1)^{i+1}\int_{\partial B}\alpha \\ \implies \int_B \frac{\partial u}{\partial x_i}\lambda &= (-1)^{i+1}\int_{\partial B} u\widehat{\mathrm{d}x_i}\end{aligned}$$

Example

For $n = 2$ we have

$$\int_{\partial B} -u\mathrm{d}x = \int_B \frac{\partial u}{\partial y}\mathrm{d}x \wedge \mathrm{d}y$$

- Let us parameterize ∂B by a $\mathcal{C}^1$ function

$$\begin{aligned}U(\text{open}) \subseteq \mathbb{R}^{n-1} &\to B \\ y &\mapsto (g(y), y)\end{aligned}$$

where $g \in \mathcal{C}^1(U, \mathbb{R})$.

- Then

$$\int_{\partial B} u\widehat{\mathrm{d}x_i} = \int_U u(g(y),y)\mathrm{d}g \wedge \widehat{\mathrm{d}y_{i-1}}$$
$$= \int_U u(g(y),y)\frac{\partial g}{\partial y_{i-1}}\mathrm{d}y_{i-1} \wedge \widehat{\mathrm{d}y_{i-1}}$$
$$= (-1)^i \int_U u(g(y),y)\frac{\partial g}{\partial y_{i-1}}\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1}$$

  for $i > 1$ and

$$= \int_U u(g(y),y)\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1}$$

  for $i = 1$.
- Thus

$$\int_B \frac{\partial u}{\partial x_i}\lambda = -\int_U u(g(y),y)\frac{\partial g}{\partial y_{i-1}}\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1}$$

  for $i < n$ and

$$= \int_U u(g(y),y)\mathrm{d}y_1 \wedge \ldots . \wedge \mathrm{d}y_{n-1}$$

  for $i = 1$.
- Tensoring with $\hat{e}_i$ and summing for $1 \le i \le n$ we have

$$\int_B \operatorname{grad}(u) \otimes \lambda = \int_U u(1, -\operatorname{grad}(g)) \otimes (\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-1})$$

  where $X \otimes \lambda$ is a vector field $\otimes$ n-form, which we understand is integrated component-wise.
- We identify

$$(1, -\operatorname{grad}(g)) \otimes (\mathrm{d}y_1 \wedge \ldots . \wedge \mathrm{d}y_{n-1}) =: \hat{N}_{\partial B} \otimes \lambda_{\partial B}$$

  then we have

$$\int_B \operatorname{grad}(u) \otimes \lambda_B = \int_{\partial B} u\, \hat{N}_{\partial B} \otimes \lambda_{\partial B}$$

integration by parts in $\mathbb{R}^n$

calculus on a Riemannian manifold

divergence theorem

Let (M, g) be a **compact** manifold with boundary and $X \in \mathrm{Vec}(M)$. We know

$$\mathrm{d}(\iota_X \lambda_g) = \mathcal{L}_X \lambda_g = (\mathrm{div}_g(X))\lambda_g$$

and

$$\lambda_{\hat{g}} = \iota_{\hat{N}} \lambda_g$$

where $\hat{N}$ is the outward unit normal to ∂M and $\hat{g}$ is the induced metric on ∂M.

Then

$$\begin{aligned} X &= g(X, \hat{N})\hat{N} + X^{\perp} \\ \implies \iota_X \lambda_g &= \lambda_g(g(X, \hat{N})\hat{N} + X^{\perp}, -) \\ &= g(X, \hat{N})\lambda_{\hat{g}} + 0 \end{aligned}$$

$$\iota_X \lambda_g = g(X, \hat{N})\lambda_{\hat{g}}$$

Then for M **oriented**, by *Strokes theorem* we have

$$\int_M \mathrm{div}_g(X)\lambda_g = \int_{\partial M} \langle X, N \rangle_g \lambda_{\hat{g}}$$

When M is not orientable we have

$$\int_M \mathrm{div}(X)\mu_g = \int_{\partial M} g(X, N)\ \mu_{\hat{g}}$$

integration by parts

Because

$$\mathrm{div}(uX) = u\mathrm{div}(X) + Xu$$

we have

$$\int_M Xu\ \lambda_g = \int_{\partial M}$$

MTH441.LEx2

$$L^2(D^2)$$

For every L^2 function on the unit disk

$$u : D^2 \to \mathbb{C}$$

we can expand

$$D^2 \setminus \{0\} \cong (0,1) \times \frac{\mathbb{R}}{\mathbb{Z}} \to \mathbb{C}$$
$$u(r,x) = \sum_{n \geq 0} a_n(r) e^{2\pi i n x}$$

The Green's function for Laplacian on unit disk

$$G(z,w) = \frac{1}{2\pi} \log \left| \frac{z-w}{w(z-\bar{w})} \right|$$

The Laplacian is unbounded. But we have the compact inverse *integral* operator which solves the Poisson equation?

The Fourier-Bessel functions

$$\{J_n(\lambda_{n,m} r)\cos(n\theta), J_n(\lambda_{n,m} r)\sin(n\theta)\}$$

form an orthogonal basis for $L^2(D^2)$. Moreover they are eigenfunctions for the Laplacian on the standard metric on D^2.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import jn_zeros

# Parameters
R = 1
m_max = 5  # Maximum m value to consider
n_max = 5  # Maximum n value to consider

# Compute eigenvalues
eigenvalues = []
for m in range(m_max + 1):
    zeros = jn_zeros(m, n_max)
    eigenvalues.extend(zeros**2 / R**2)

# Sort eigenvalues
eigenvalues = np.sort(eigenvalues)
print(eigenvalues)
```

```
# Plot the spectrum on R
plt.figure(figsize=(9, 5))
plt.scatter(eigenvalues, np.zeros_like(eigenvalues), marker='|',
color='b', s=100)
plt.xlabel("Eigenvalues of the Laplacian on the unit disk")

plt.title("Spectrum of the Laplacian on the Unit Disk (Dirichlet
BCs)")
plt.grid(True, axis='x', linestyle='--', alpha=0.5)
plt.show()
```

```
from scipy import special
import numpy as np
def drumhead_height(n, k, distance, angle, t):
   kth_zero = special.jn_zeros(n, k)[-1]
   return np.cos(t) * np.cos(n*angle) * special.jn(n,
distance*kth_zero)
theta = np.r_[0:2*np.pi:50j]
radius = np.r_[0:1:50j]
x = np.array([r * np.cos(theta) for r in radius])
y = np.array([r * np.sin(theta) for r in radius])
z = np.array([drumhead_height(1, 1, r, theta, 0.5) for r in radius])

import matplotlib.pyplot as plt
fig = plt.figure()
ax = fig.add_axes(rect=(0, 0.05, 0.95, 0.95), projection='3d')
ax.plot_surface(x, y, z, rstride=1, cstride=1, cmap='RdBu_r',
vmin=-0.5, vmax=0.5)
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_xticks(np.arange(-1, 1.1, 0.5))
ax.set_yticks(np.arange(-1, 1.1, 0.5))
ax.set_zlabel('Z')
plt.show()
```

$$L^2(\Omega)$$

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$.

Theorem 9.8 (Unique solution of the elliptic boundary value problem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let the operator L in (9.20) be uniformly elliptic, with coefficients $a^{ij} \in \mathbf{L}^\infty(\Omega)$. Then, for every $f \in \mathbf{L}^2(\Omega)$, the boundary value problem (9.19) has a unique weak solution $u \in H_0^1(\Omega)$. The corresponding solution operator, which we denote as $L^{-1} : f \mapsto u$, is a compact linear operator from $\mathbf{L}^2(\Omega)$ into $H_0^1(\Omega)$.*

Theorem 9.9 (Representation of solutions as a series of eigenfunctions). *Assume $a^{ij} = a^{ji} \in \mathbf{L}^\infty(\Omega)$. Then, in the setting of Theorem 9.8, the linear operator $L^{-1} : \mathbf{L}^2(\Omega) \mapsto \mathbf{L}^2(\Omega)$ is compact, one-to-one, and self-adjoint.*

The space $\mathbf{L}^2(\Omega)$ admits a countable orthonormal basis $\{\phi_k\,;\ k \geq 1\}$ consisting of eigenfunctions of L^{-1}, and one has the representation

$$L^{-1} f \;=\; \sum_{k=1}^{\infty} \lambda_k (f, \phi_k)_{\mathbf{L}^2}\, \phi_k\,. \tag{9.26}$$

The corresponding eigenvalues λ_k satisfy

$$\lim_{k\to\infty} \lambda_k = 0\,, \qquad \lambda_k > 0 \text{ for all } k \geq 1. \tag{9.27}$$

[1]

$$L^2(M, g)$$

[2]

[3]

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1. https://math.stackexchange.com/a/1707821/1290493 ↩
2. https://math.stackexchange.com/q/1754700/1290493 ↩
3. https://www.math.mcgill.ca/toth/spectral%20geometry.pdf ↩

IMPA (2019) Functional analysis

Doctorate program: Functional Analysis (2019) - YouTube

Sobolev spaces and generalized derivatives on $\mathbb{R}^N$

Let

$$G \subset \mathbb{R}^N$$

be an open, bounded set.

Consider the *test* functions

$$\mathcal{C}_{\mathsf{c}}^{\infty}(G)$$

which are smooth functions on G with **compact support**, whose value on ∂G is defined to be 0.

We have by integration by parts that

$$\int_G (\partial_i u) v = -\int_G u(\partial_j v)$$

for $u, v \in \mathcal{C}_{\mathsf{c}}^{\infty}(G)$ because the boundary values of u, v is 0.

Then

Definition. Generalized derivatives of $L^2(G)$

Let $G \subset \mathbb{R}^N$ be an open, bounded set. If for $u, w \in L^2(G)$ we have

$$f \in \mathcal{C}_{\mathsf{c}}^{\infty}(G) \implies \int_G wf = -\int_G u(\partial_j f)$$

Then we say the **generalized derivative** of u in the direction $\hat{x}_j$ is w that is

$$\partial_j u = w$$

on G.

We observe

- If there are *two* $w_1, w_2 \in L^2(G)$ such that

$$f \in \mathcal{C}_c^\infty(G) \int_G w_1 f = -\int_G u(\partial_j f) = \int_G w_2 f$$

then

$$\int_G (w_1 - w_2) f = 0$$

for all compactly supported smooth functions f on G.

  - As $\mathcal{C}_c^\infty(G)$ is dense in $L^2(G)$ we have a sequence $f_n \in \mathcal{C}_c^\infty(G)$

    $$f_n \xrightarrow{n\to\infty, L^2(G)} f$$

    we have

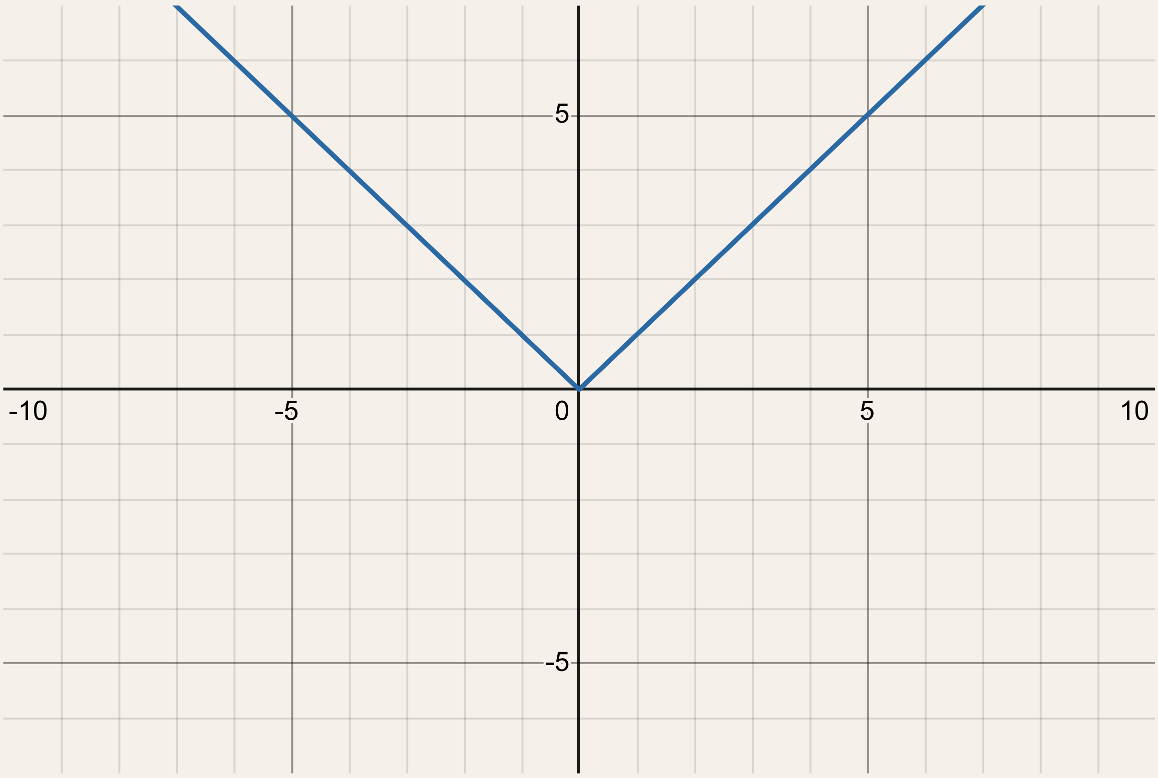
    $$\int_G (w_1 - w_2) f_n = 0 \implies \int_G (w_1 - w_2) f = 0$$

    Thus as f is arbitrary, $w_1 = w_2$ almost everywhere.
  - Thus, generalized derivatives are unique almost everywhere, that is, unique in $L^2(G)$.

Example

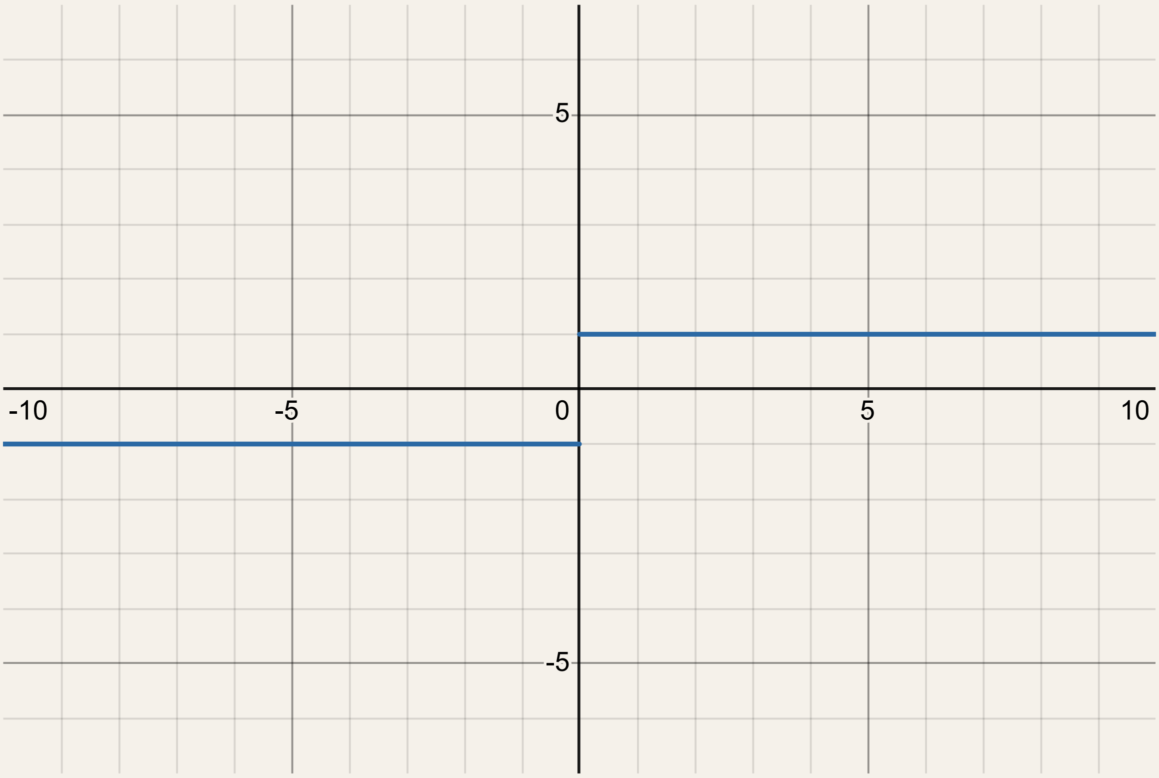
For $G = (-1, 1)$ the function

$|x|$



The *generalized derivative* of this function is

$$\begin{cases} -1 & x < 0 \\ \text{any } c \in \mathbb{R} & x = 0 \\ 1 & x > 0 \end{cases}$$



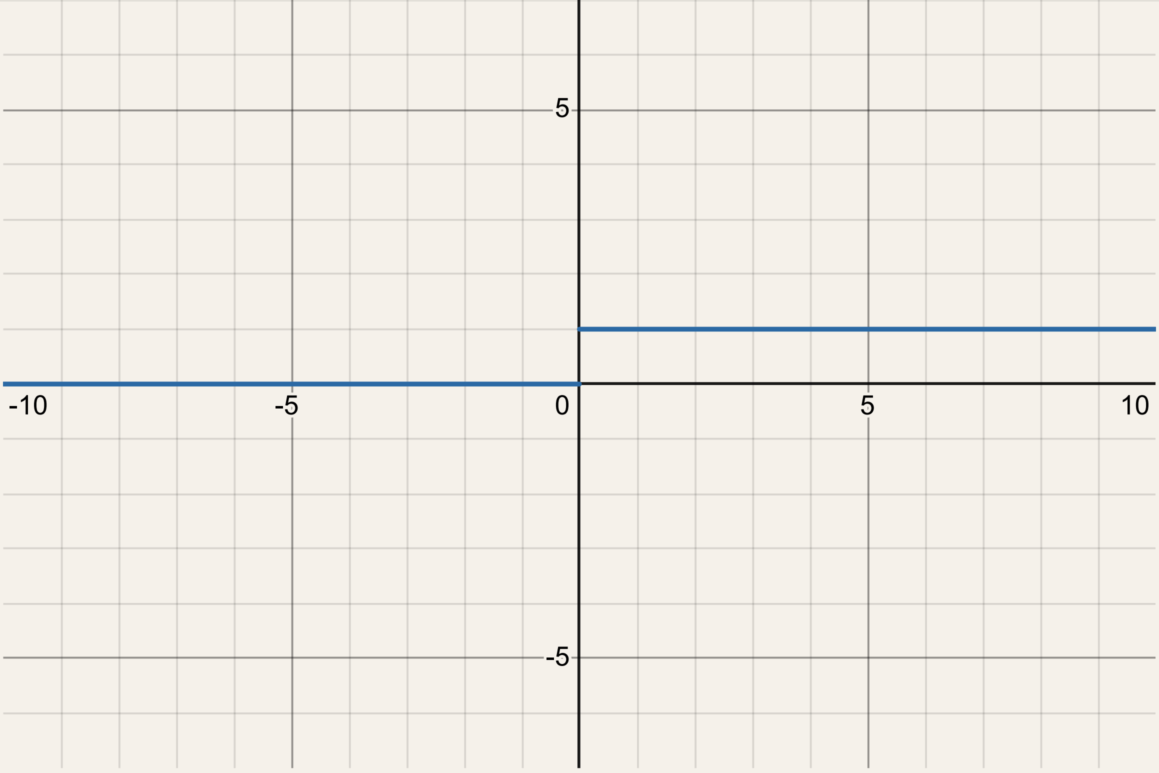
as

$$-\int_{(-1,1)} |x| f(x) = -\int_{(-1,0)} (-1) f(x) - \int_{(0,1)} (1) f(x)$$

Non-existence of generalized derivative in $L^2(G)$

For $G = (-1,1)$, the *generalized derivative* of

$$\chi_{(0,1)}$$



w must satisfy

$$-\int_{(-1,1)} wf = \int_{(-1,1)} \chi_{(0,1)} f' = \int_{(0,1)} f' = -f(0)$$

for all $f \in \mathcal{C}_{\mathrm{c}}^{\infty}(G)$.
Then for every such f with support outside $(-\epsilon, \epsilon)$ we have

$$\int_G wf = 0$$

thus $w = 0$ on $(-\epsilon, \epsilon)$ for all $\epsilon > 0$. Then w is only non-zero at $0 \in (-1, 1)$, but this means

$$\int_{(-1,1)} wf = \int_{(-1,1)\setminus\{0\}} \cancel{w}^{0} f = 0$$

Hence, $\chi_{(0,1)}$ does not have a generalized derivative in $L^2(G)$.

derivative of distributions

Intuition

What we must do is define the derivative of $\chi_{(0,1)}$ as the *linear functional*

$$\frac{\mathrm{d}}{\mathrm{d}x}\chi_{(0,1)} = \begin{aligned} \delta_0 : \mathcal{C}_\mathrm{c}^\infty(-1,1) &\to \mathbb{R} \\ f &\mapsto f(0) \end{aligned}$$

Example

The derivative of the Dirac delta

$$\begin{aligned} \delta_0 : \mathcal{C}_\mathrm{c}^\infty(-1,1) &\to \mathbb{R} \\ f &\mapsto f(0) \end{aligned}$$

is the distribution

$$\begin{aligned} \mathcal{C}_\mathrm{c}^\infty(-1,1) &\to \mathbb{R} \\ f &\mapsto -f'(0) \end{aligned}$$

denoted by δ_0'. Hence, we have the *generalized derivatives*

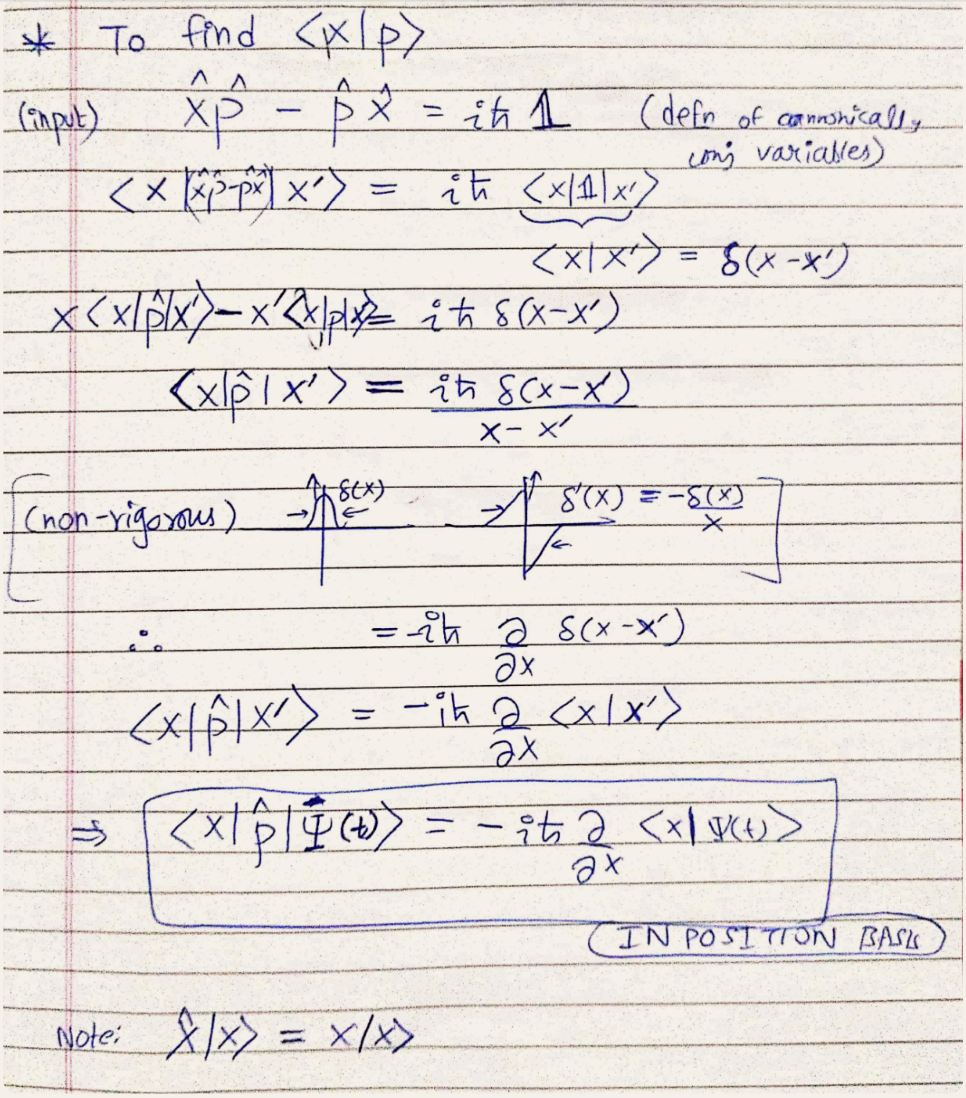
$$\cdots \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} \frac{x^k}{k!}\chi_{(0,1)} \cdots \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} \frac{x^2}{2}\chi_{(0,1)} \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} x\chi_{(0,1)}(x) \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} \chi_{(0,1)} \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} \delta_0' \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} \delta_0'' \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} \cdots \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} \delta_0^{(k)}$$

derived by integrating "functions" starting from the right, where

$$\begin{aligned} \delta_0^{(k)} : \mathcal{C}_\mathrm{c}^\infty(-1,1) &\to \mathbb{R} \\ f &\mapsto (-1)^k f^{(k)}(0) \end{aligned}$$

Intuition

A intuition behind the derivative of Delta:



* To find $\langle x|p\rangle$

(input) $\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar \mathbb{1}$ (defn of canonically conj variables)

$$\langle x|\hat{x}\hat{p}-\hat{p}\hat{x}|x'\rangle = i\hbar \underbrace{\langle x|\mathbb{1}|x'\rangle}$$

$$\langle x|x'\rangle = \delta(x-x')$$

$$x\langle x|\hat{p}|x'\rangle - x'\langle x|\hat{p}|x'\rangle = i\hbar\,\delta(x-x')$$

$$\langle x|\hat{p}|x'\rangle = \frac{i\hbar\,\delta(x-x')}{x-x'}$$

(non-rigorous) $\delta(x)$ $\quad \delta'(x) = -\frac{\delta(x)}{x}$

$$\therefore \quad = -i\hbar \frac{\partial}{\partial x}\delta(x-x')$$

$$\langle x|\hat{p}|x'\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|x'\rangle$$

$$\Rightarrow \boxed{\langle x|\hat{p}|\Psi(t)\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\Psi(t)\rangle}$$

IN POSITION BASIS

Note: $\hat{x}|x\rangle = x|x\rangle$

from the Lecture 7 on *Quantum Physics* by V. Balakrishnan (physicist) - Wikipedia

W_2^1

Definition. W_2^1

Let $G \subset \mathbb{R}^N$ be an open, bounded set. We have the space of all $L^2(G)$ functions whose generalized derivatives exist in all directions:

$$W_2^1(G) := \left\{ u \in L^2(G) \mid_{f \in \mathcal{C}_c^\infty(G)\,:\, \int_G (\partial_j f)u = -\int_G w_j f}^{\forall 1 \le j \le N \exists w_j \in L^2(G)\,:} \right\}$$

with an inner product

$$\begin{aligned}\langle u, v\rangle_{1,2} &:= \langle u, v\rangle_{L^2} + \sum_{j=1}^N \langle \partial_j u, \partial_j v\rangle_{L^2} \\ &=: \langle u, v\rangle_{L^2} + \langle \mathrm{grad}(u), \mathrm{grad}(v)\rangle_{L^2}\end{aligned}$$

Proposition:

- Convergence in $\langle , \rangle_{1,2}$ implies convergence in L^2
- Thus

$$(W_2^1, \langle\ ,\ \rangle_{1,2})$$

is a Hilbert space.

Definition.

$$\overset{\circ}{W}{}_2^1(G) := \overline{(\mathcal{C}_c^\infty(G), \langle , \rangle_{1,2})}$$

Intuition

We have the *technically wrong* idea

$$\overset{\circ}{W}{}_2^1(G) = \{u \in W_2^1(G) \mid u\Big|_{\partial G} = 0\}$$

A more precise statement, in dim 1, is the following:

Lemma: (For $\dim G = 1$) Let $(a, b) \subset \mathbb{R}$ be a finite interval. For every $u \in \overset{\circ}{W}{}_2^1(a, b)$,

$$\exists v \in \mathcal{C}^0[a, b] : u = v \text{ ae }, v(a) = 0 = v(b)$$

- Fix $x \in (a, b)$. Consider $f \in \mathcal{C}_c^\infty(a, b)$ then

$$\begin{aligned}|f(x)|^2 = |f(x) - f(a)|^2 = \left|\int_{[a,x]} f'\right| &< (x - a) \int_{[a,x]} |f'|^2 \\ &\leq (b - a)\|f'\|_2\end{aligned}$$

- Then

$$f \in \mathcal{C}_c^\infty(a, b) \implies \|f\|_\infty \leq \sqrt{(b - a)}\underbrace{\|f'\|_2}_{\|f\|_{1,2}}$$

Let u be in

Definition.

$$\overset{\circ}{W}{}_2^1(G) := \overline{(\mathcal{C}_c^\infty(G), \langle , \rangle_{1,2})}$$

then

$$f_n \xrightarrow{n\to\infty,\langle,\rangle_{1,2}} u$$

where $f_n \in \mathcal{C}_c^\infty(a,b)$.

- Then

$$\|f_n - f_m\|_\infty \leq \sqrt{b-a}\|f_n - f_m\|_{1,2}$$

  thus f_n is Cauchy in $\|\ \|_\infty$. So it converges in $\mathcal{C}^0[a,b]$ to a continuous function

$$f_n \xrightarrow{n\to\infty,\|\ \|_\infty} f$$

- By

  Proposition:

  - Convergence in $\langle,\rangle_{1,2}$ implies convergence in L^2
  - Thus

$$(W_2^1, \langle\ ,\ \rangle_{1,2})$$

    is a Hilbert space.

  we know

$$f_n \xrightarrow{n\to\infty} u \in L^2(a,b)$$

- As convergence in $\|\ \|_\infty$ implies convergence in L^2 we know

$$u = f \text{ ae}$$

  where $f \in \mathcal{C}^0[a,b]$ and

$$f(0) = \lim_{n\to\infty} f_n(0) = 0 = \lim_{n\to\infty} f_n(1) = f(1)$$

Let $G \subset \mathbb{R}^N$ be an open, bounded set such that ∂G is a smooth submanifold of $\mathbb{R}^N$ (or locally smooth?), and $u \in \overset{\circ}{W}{}_2^1(G)$. Then

$$\int_{\partial G} u^2 \leq c\|u\|_{1,2}^2$$

(Proof for G being a rectangle in $\mathbb{R}^2$)

Evans PDEs

source: Lawrence C. Evans - Partial Differential Equations-AMS(2010).pdf

Chapter 2

Heat equation

Definition. **Heat equation on $U \times (0,T)$**

$$u_t - \nabla u = 0 \text{ on } U \times (0,T)$$

Definition. **Fundamental solution to the heat equation**

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\} &\to \mathbb{R} \\ (x,t) &\mapsto G(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \end{cases} \end{aligned}$$

For

$$g \in \mathcal{C}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

and

Definition. **Fundamental solution to the heat equation**

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\} &\to \mathbb{R} \\ (x,t) &\mapsto G(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \end{cases} \end{aligned}$$

the convolution

$$u(x,t) := \int_{y \in \mathbb{R}^n} G(x-y,t) g(y)$$

is smooth $u \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ and solves

Definition. **Heat equation on $U \times (0,T)$**

$$u_t - \nabla u = 0 \text{ on } U \times (0, T)$$

for $T = \infty$ and for all $a \in \mathbb{R}^n$

$$\lim_{(x,t)\in\mathbb{R}\times(0,\infty)\to(a,0)} = g(a)$$

Definition. **Heat ball**

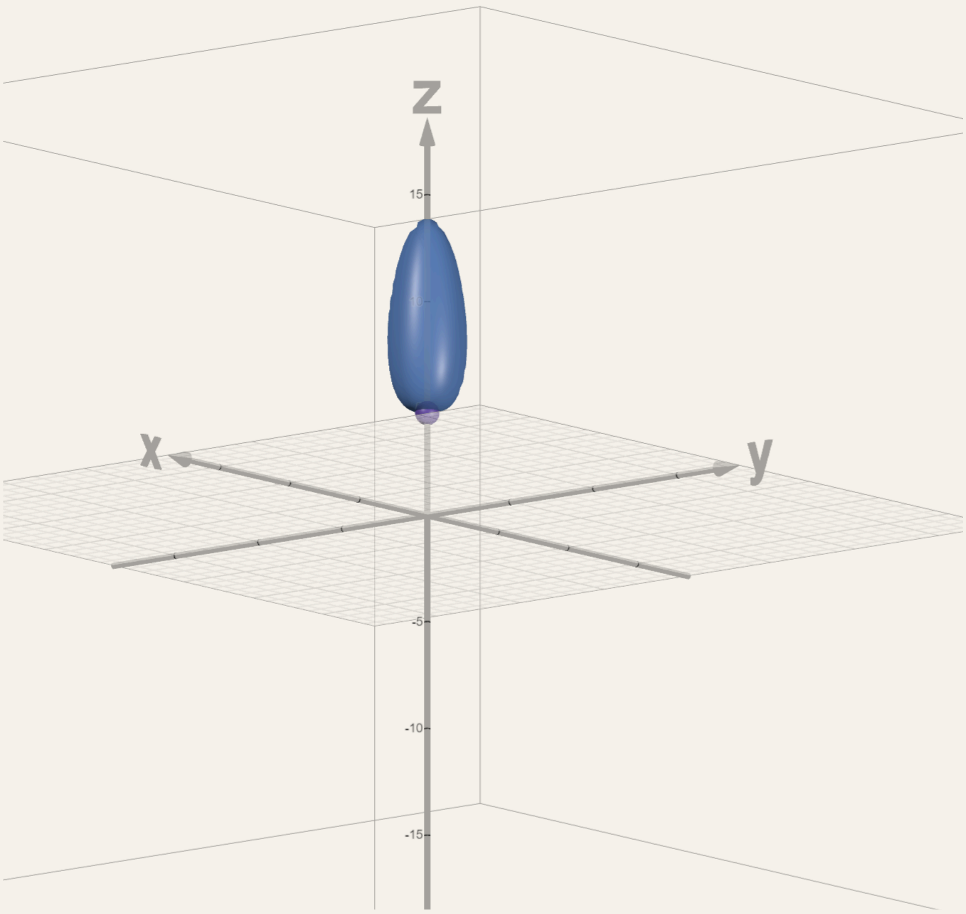
For

Definition. **Fundamental solution to the heat equation**

$$\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\} \to \mathbb{R}$$
$$(x,t) \mapsto G(x,t) := \left\{ \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \right.$$

we define the heat ball *starting* at $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ of *size* $r > 0$ as

$$E(x,t,r) := \left\{ (a,s) \in \mathbb{R}^n \times \mathbb{R} \mid s \leq t, G(a-y, t-s) \geq \frac{1}{r^n} \right\}$$



[1]

Definition. Parabolic cylinder and boundary of bounded open sets in $U \subseteq \mathbb{R}^n$

Assume $U \subseteq \mathbb{R}^n$ is **open, bounded**, and fix $T > 0$.

- the **parabolic cylinder** is

$$U_T := U \times (0, T]$$

- the **parabolic boundary** is

$$\Gamma_T := \overline{U_T} \setminus U_T = \underbrace{U \times \{0\}}_{\text{bottom}} \cup \underbrace{\partial U \times [0, T]}_{\text{sides}}$$

Hence

$$\partial U = \Gamma_T \cup \underbrace{U \times \{T\}}_{\text{top}}$$

Definition. Definition

(viii) *Functions of x and t.* It is occasionally useful to introduce spaces of functions with differing smoothness in the x- and t-variables, although there is no standard notation for such spaces. We will for this book write

$$C_1^2(U_T) = \{u : U_T \to \mathbb{R} \mid u, D_x u, D_x^2 u, u_t \in C(U_T)\}.$$

In particular, if $u \in C_1^2(U_T)$, then u, $D_x u$, etc. are continuous up to the top $U \times \{t = T\}$.

(Mean value property for heat equation on bounded open $U \subseteq \mathbb{R}^n$) Let

$$u \in C_1^2(U_T)$$

solves the heat equation. Then

$$u(x,t) = \frac{1}{4r^n} \int_{E(x,t,r)} u(a,s) \frac{|a-y|^2}{(t-s)^2}$$

whenever $E(x,t,r) \subset U_T$.

$$
\begin{aligned}
\phi(r) &:= \frac{1}{r^n}\int_{E(0,0,r)} u(a,s)\frac{|a|^2}{s^2}\mathrm{d}y\wedge\mathrm{d}s \\
&= \int_{E(0,0,1)} u(ra,r^2 s)\frac{|a|^2}{s^2}\mathrm{d}y\wedge\mathrm{d}s
\end{aligned}
$$

- Then

$$
\phi'(r) = \int_{E(0,0,1)}
$$

Chapter 3

- **First-order PDE:**

$$
\begin{aligned}
F(\mathrm{d}_x u, u(x), x) &= 0 \text{ on } x\in U \\
u &= g \text{ on } \Gamma \subseteq \partial U
\end{aligned}
$$

- **complete integral:** $I(x;a)$ is a complete integral for F on $U\times A$ if

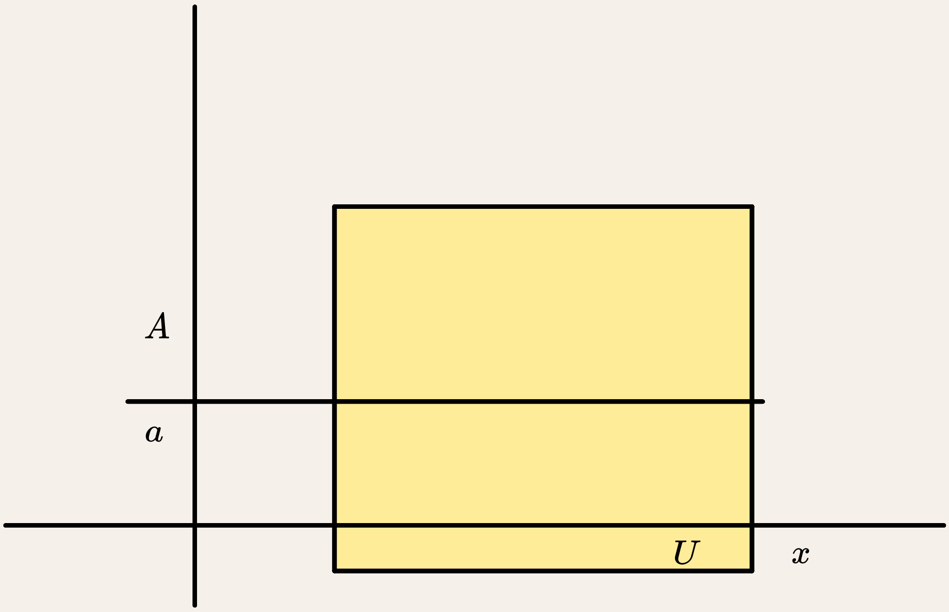
$$
u(x) = I(x,a)
$$

  solves the PDE

  - **First-order PDE:**

  $$
  \begin{aligned}
  F(\mathrm{d}_x u, u(x), x) &= 0 \text{ on } x\in U \\
  u &= g \text{ on } \Gamma \subseteq \partial U
  \end{aligned}
  $$

  which means



$$F(\mathrm{d}_x I\Big|_{U\times\{a\}}, I\Big|_{U\times\{a\}}(x), x) = 0$$

for each $a \in A$ and

$$\operatorname{rank}\left(\mathfrak{D}\begin{bmatrix} I \\ \partial_{x_1,0} I \\ \partial_{x_2,0} I \\ \dots \\ \partial_{x_n,0} I \end{bmatrix}_{(x,-)}\right)^{\mathsf{T}} = n$$

for each $x \in U$.

PDE	complete integral
$\mathrm{d}u(x) + f(\mathrm{d}u) = u$	$I(x;a) = a \cdot x + f(a)$ $a \in \mathbb{R}^n$
$\lVert \mathrm{d}u \rVert = 1$	$a \cdot x + b$ $a \in \mathbb{R}^n, \lVert a \rVert = 1, b \in \mathbb{R}$
$u_{x_{n+1}} + H(\mathrm{d}u) = 0$	$a \cdot x - tH(a) + b$

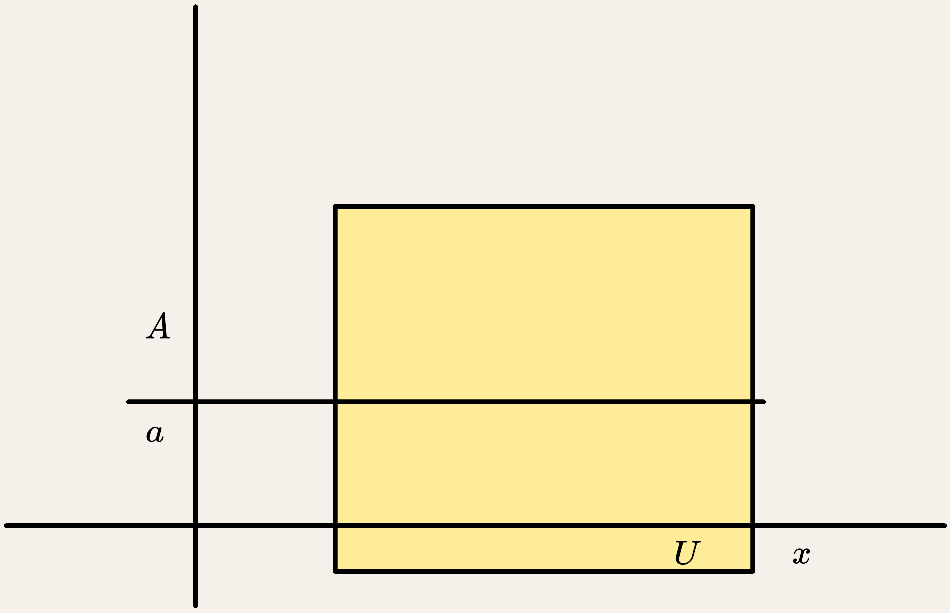
- Let
  - **complete integral:** $I(x;a)$ is a complete integral for F on $U \times A$ if

    $$u(x) = I(x,a)$$

    solves the PDE
    - **First-order PDE:**

      $$F(\mathrm{d}_x u, u(x), x) = 0 \text{ on } x \in U$$
      $$u = g \text{ on } \Gamma \subseteq \partial U$$

    which means



$$F\left(\mathrm{d}_x I\Big|_{U\times\{a\}}, I\Big|_{U\times\{a\}}(x), x\right) = 0$$

for each $a \in A$ and

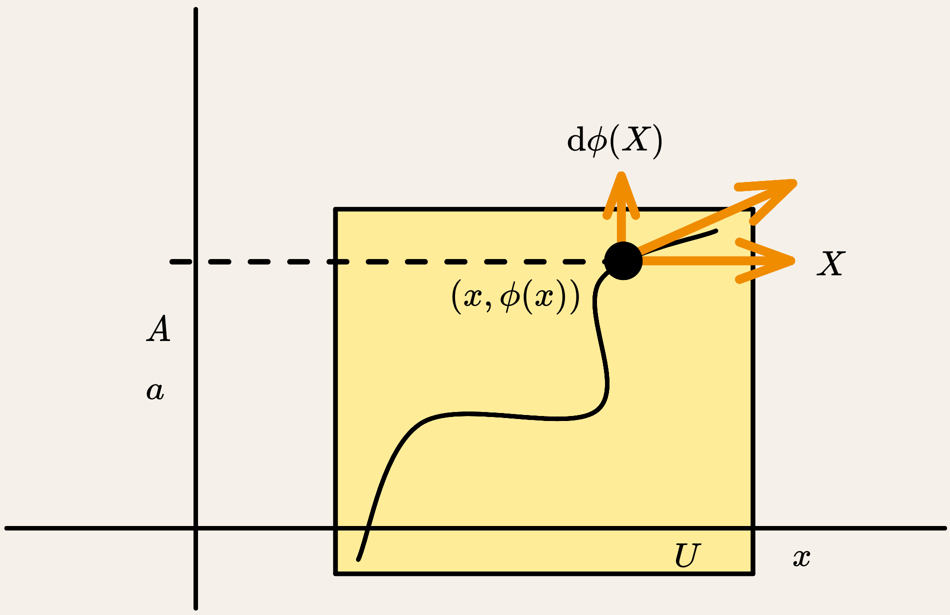
$$\operatorname{rank}\left(\mathfrak{D}\begin{bmatrix} I \\ \partial_{x_1,0} I \\ \partial_{x_2,0} I \\ \dots \\ \partial_{x_n,0} I \end{bmatrix}_{(x,-)}\right)^{\mathsf{T}} = n$$

for each $x \in U$.

- Let $\phi : U \to A$ is a $\mathcal{C}^1$ function such that

$$\mathrm{d}_{(x,\phi(x))} I \begin{bmatrix} 0 \\ \mathrm{d}\phi(X) \end{bmatrix} = 0$$

- Then the function $v(x) := I(x, \phi(x))$ is called **singular integrals** and it satisfies



$$\mathrm{d}_x v(X) = \mathrm{d}_{(x,\phi(x))} I \begin{bmatrix} X \\ 0 \end{bmatrix} + \cancelto{0}{\mathrm{d}_{(x,\phi(x))} I \begin{bmatrix} 0 \\ \mathrm{d}\phi(X) \end{bmatrix}}$$

which means

$$\mathrm{d}_x v = \mathrm{d}_x I\Big|_{U\times\{a\}}$$

hence because $I(-,a)$ is a solution of the PDE with F, v also satisfies

$$F(\mathrm{d}_x v, v(x), x) = F(\mathrm{d}_x I\Big|_{U\times\{a\}}, I\Big|_{U\times\{a\}}(x), x) = 0$$

- **general integrals** (rather than "complete integrals") are of the form

$$J(x;a') = I(x;a',h(a'))$$

PDE	complete integral, general integrals	singular integrals,
$u^2(1+\|Du\|^2) = 1$	$I(x,a) = \pm\sqrt{1-\|x-a\|^2}$ for $\|x-a\| < 1$	$\mathrm{d}_{(x,\phi(x))} I^\mathsf{T} = \mp\frac{x-a}{\sqrt{1-\|x-a\|^2}}$ which is zero iff $a = \phi(x) = x$, but then $I(x,\phi(x)) = \pm 1$
$u_{x_{n+1}} + H(\mathrm{d}u) = 0$ for $H(p) = \|p^2\|$	for $b = 0$ $I(x,t;a) := a\cdot x - tH(a)$ $= a\cdot x - t\|a^2\|$	$\mathrm{d}_{(x,\phi(x))} I^\mathsf{T} = x - 2ta$ which is zero iff $a = \phi(x,t) = \frac{x}{2t}$ $v(x) = \left(\frac{x}{2t}\right)\cdot x - t\left\|\frac{x}{2t}\right\|^2 = \frac{1}{4}\frac{\|x\|}{4t}$ must solve the Hamilton-Jacobi equations

- **characteristic ODE:** Say u solves the PDE $F(\mathrm{d}u, u, x) = 0$
  - and then define

$$z(s) := u(x(s))$$
$$p(s) := \mathrm{d}_{x(s)} u$$

then

$$
\begin{aligned}
z'(s) &= \mathrm{d}_{x(s)}u(x'(s)) \\
p'(s) &= \mathfrak{D}^2_{x(s)}u(-, x'(s))
\end{aligned}
$$

- Because

  $$F(\mathrm{d}u, u, x) = 0$$

  for all $x \in U$ we have

  $$\left(\mathfrak{D}F(\mathfrak{D}^2u(-, X), 0, 0) + \mathfrak{D}F(0, \mathrm{d}u(X), 0) + \mathfrak{D}F(0, 0, X)\right)\Big|_{(\mathrm{d}u, u, x)} = 0$$

  for all $X \in V$.

- $$F : V^* \times \mathbb{R} \times U \to \mathbb{R}$$

  so the derivative in the first slot

  $$\mathfrak{D}F(-, 0, 0) : V^* \to \mathbb{R}$$

  is naturally corresponds to an element of V by $V \cong V^{**}$.

  - We set

    $$\mathfrak{D}F(\underbrace{\varphi}_{\in V^*}, 0, 0) = \varphi(x')$$

- By

  - and then define

    $$
    \begin{aligned}
    z(s) &:= u(x(s)) \\
    p(s) &:= \mathrm{d}_{x(s)}u
    \end{aligned}
    $$

    then

    $$
    \begin{aligned}
    z'(s) &= \mathrm{d}_{x(s)}u(x'(s)) \\
    p'(s) &= \mathfrak{D}^2_{x(s)}u(-, x'(s))
    \end{aligned}
    $$

  $$\mathfrak{D}F(\mathfrak{D}^2u(-, -), 0, 0) = \dot{p} = \mathfrak{D}^2u(-, x')$$

  we get

  $$\left(\cancelto{\dot{p}(s)}{\mathfrak{D}F(\mathfrak{D}^2u(-, -), 0, 0)} + \mathfrak{D}F(0, \cancelto{p(s)}{\mathrm{d}u(X)}, 0) + \mathfrak{D}F(0, 0, -)\right)\Big|_{(s)} = 0$$

  which means

$$\dot{p}_{\square} = -\mathfrak{D}F(0, p(s)_{\square}, 0) - \mathfrak{D}F(0, 0, \square)$$

- $$z'(s) = \mathrm{d}_{x(s)}u(x'(s)) = p(s)(x'(s)) = \mathfrak{D}F(p(s), 0, 0)$$

by

- We set

$$\mathfrak{D}F(\underbrace{\varphi}_{\in V^*}, 0, 0) = \varphi(x')$$

Definition. Characteristic equations for PDEs

The following equations are called **characteristic equations**

$$\begin{aligned}\mathfrak{D}F(\varphi, 0, 0) &= \varphi(x'),\ \forall \varphi \in V^* \\ \dot{p}_{\square} &= -\mathfrak{D}F(0, p(s)_{\square}, 0) - \mathfrak{D}F(0, 0, \square) \\ z'(s) &= \mathfrak{D}F(p(s), 0, 0)\end{aligned}$$

for the PDE

$$\begin{aligned}F : V^* \times \mathbb{R} \times U &\to \mathbb{R} \\ F(\mathrm{d}u, u, x) &= 0\end{aligned}$$

which forms an ODE in $V^* \times \mathbb{R} \times U$ and solves for

$$u(x(s)) = z(s)$$

which is obtained from some given boundary conditions.

$$\begin{aligned}\dot{x} &= \mathrm{grad}_{(p)}(F) \\ \dot{p} &= -\mathrm{grad}_{(x)}(F) - \mathrm{grad}_{(z)}(F) \\ \dot{z} &= \mathfrak{D}F(p(s), 0, 0)\end{aligned}$$

PDE	characteristic equations
linear $\mathrm{d}u(B) + cu = 0$	$\dot{x} = B(x)$, $\dot{z} = -c(x)z$
quasi-linear $\mathrm{d}u(B_{x,u(x)}) + c(x, u(x)) = 0$	$\dot{x} = B(x, z)$, $\dot{z} = -c(x, z)$

PDE	characteristic equations
conservation laws $$G(\mathrm{d}u, u_t, u, x, t) = u_t + \underbrace{\mathrm{div}(F_u)}_{\mathrm{d}u(F')}$$ where $$F : \mathbb{R} \to U$$	
Hamilton-Jacobi equations $$G(\mathrm{d}u, u_t, u, x, t) = u_t + H(\mathrm{d}u, x)$$	$$\dot{x} = \mathrm{grad}_{(p)}(H)$$ $$\dot{p} = -\mathrm{grad}_{(x)}(H)$$ $$\dot{z} = \mathrm{d}H(p, 0, 0) - H$$ first two are *Hamilton's equations*

1. https://www.desmos.com/3d/lwzefumnnn ↩

#book Walter Rudin - Real and complex analysis AKA *Papa Rudin*

sett.measure.cpx

$\mathbb{C}$-measures

Definition. $\mathbb{C}$**-measures on a measure space**

Let $\mathfrak{M}$ be a σ-algebra on a set X. A **$\mathbb{C}$-measure** on $\mathfrak{M}$ is a function

$$\mu : \mathfrak{M} \to \mathbb{C}$$

such that

$$\{E_i\} \text{ partitions } E \in \mathfrak{M} \implies \mu(E) = \sum_{i \geq 1} \mu(E_i)$$

- The **convergence of $\sum_{i\geq 1} \mu(E_i)$** is now a requirement for every partition of E, unlike for positive measures where the series could converge or diverge to ∞.
- Because $E = \bigcup_{i\geq 1} E_i$ does not change under a rearrangement of (E_i) the series $\sum_{i\geq 1} \mu(E_i)$ must actually **converge absolutely**.

dominating a complex measure with positive measures

Any positive measure $(X, \mathfrak{M}, \lambda)$ that dominates a complex measure $(X, \mathfrak{M}, \mu)$, meaning

$$E \in \mathfrak{M} \implies |\mu(E)| \leq \lambda(E)$$

must have

$$\{E_i\} \text{ partitions } E \in \mathfrak{M} \implies \lambda(E) = \sum_{i \geq 1} \lambda(E_i) \geq \sum_{i \geq 1} |\mu(E_i)|$$

So $\lambda(E)$ is at least equal to the supremum of all possible sums on the right that is

Definition. **The total variation of a complex measure $(X, \mathfrak{M}, \mu)$ is**

$$\begin{aligned} |\mu| : \mathfrak{M} &\to [0, \infty) \\ E &\mapsto |\mu|(E) := \sup_{\{E_i\} \text{ paritions } E} \sum_{i \geq 1} |\mu(E_i)| \end{aligned}$$

So for any such λ we have

$$\lambda(E) \geq |\mu|(E)$$

$$|\mu|(E) \geq |\mu(E)|$$

But actually $|\mu|$ itself is a measure!

Definition. **The total variation of a complex measure $(X, \mathfrak{M}, \mu)$ is**

$$\begin{aligned} |\mu| : \mathfrak{M} &\to [0, \infty) \\ E &\mapsto |\mu|(E) := \sup_{\{E_i\} \text{ paritions } E} \sum_{i \geq 1} |\mu(E_i)| \end{aligned}$$

is a positive measure on $(X, \mathfrak{M})$ and

$$|\mu|(X) < \infty$$

Proof Let $\{E_i\}$ be a partition of $E \in \mathfrak{M}$. Let t_i be real numbers such that $t_i < |\mu|(E_i)$. Then each E_i has a partition $\{A_{ij}\}$ such that

$$\sum_j |\mu(A_{ij})| > t_i \qquad (i = 1, 2, 3, \ldots). \tag{1}$$

Since $\{A_{ij}\}$ $(i, j = 1, 2, 3, \ldots)$ is a partition of E, it follows that

$$\sum_i t_i \le \sum_{i,j} |\mu(A_{ij})| \le |\mu|(E). \tag{2}$$

Taking the supremum of the left side of (2), over all admissible choices of $\{t_i\}$, we see that

$$\sum_i |\mu|(E_i) \le |\mu|(E). \tag{3}$$

To prove the opposite inequality, let $\{A_j\}$ be any partition of E. Then for any fixed j, $\{A_j \cap E_i\}$ is a partition of A_j, and for any fixed i, $\{A_j \cap E_i\}$ is a partition of E_i. Hence

$$\begin{aligned} \sum_j |\mu(A_j)| &= \sum_j \left| \sum_i \mu(A_j \cap E_i) \right| \\ &\le \sum_j \sum_i |\mu(A_j \cap E_i)| \\ &= \sum_i \sum_j |\mu(A_j \cap E_i)| \le \sum_i |\mu|(E_i). \end{aligned} \tag{4}$$

So our problem of dominating μ with a positive measure does have a solution.

space of all complex measures

Definition. Normed space of all complex measures on $(X, \mathfrak{M})$

Let $(X, \mathfrak{M})$ be a measurable space then space of all complex measures on $\mathfrak{M}$

$$\mathrm{Meas}(X, \mathfrak{M}, \mathbb{C})$$

is a $\mathbb{C}$-vector space with norm

$$\|\mu\| := |\mu|(X)$$

decomposition of signed measures

For a $\mathbb{R}$-measure μ we may decompose

$$\mu^+ := \frac{1}{2}(|\mu| + \mu)$$
$$\mu^- := \frac{1}{2}(|\mu| - \mu)$$

which are positive measure called **positive** and **negative variations** of μ respectively.

Absolute continuity

decomposition of complex measures

(Lebesgue decomposition of complex measure relative to a positive measure and Radon-Nikodym description of λ_a) Let $(X, \mathfrak{M}, \mu)$ be a positive σ-finite measure space and λ be a complex measure on $\mathfrak{M}$. Then there is a unique pair of complex measures λ_a, λ_s on $\mathfrak{M}$ such that

$$\lambda = \lambda_a + \lambda_s$$
$$\lambda_a \ll \mu$$
$$\lambda_s \perp \mu$$

called the **Lebesgue decomposition** of λ relative to μ. If λ is positive and finite, then so are λ_a, λ_s. Moreover there is a unique $h \in L^1(X, \mathfrak{M}, \mu)$ such that

$$E \in \mathfrak{M} \implies \lambda_a(E) = \int_E h \, \mathrm{d}\mu$$

consequences of the decomposition

(Polar decomposition of $\mathbb{C}$-measures) Let $(X, \mathfrak{M}, \mu)$ be a $\mathbb{C}$-measure space. Then there is a measurable function

$$h : X \to U(1) \subset \mathbb{C}$$

such that

$$\mathrm{d}\mu = h \mathrm{d}|\mu|$$

6.12 Theorem *Let μ be a complex measure on a σ-algebra $\mathfrak{M}$ in X. Then there is a measurable function h such that $|h(x)| = 1$ for all $x \in X$ and such that*

$$d\mu = h \, d|\mu|. \tag{1}$$

Riesz representation

sett.Top.t.2 and loc cpt

Locally compact Hausdorff spaces

Definition. **Locally compact Hausdorff spaces**

A Hausdorff space is locally compact if every point has a neighbourhood whose closure is compact.

2.4 Theorem *Suppose K is compact and F is closed, in a topological space X. If $F \subset K$, then F is compact.*

PROOF If $\{V_\alpha\}$ is an open cover of F and $W = F^c$, then $W \cup \bigcup_\alpha V_\alpha$ covers X; hence there is a finite collection $\{V_{\alpha_i}\}$ such that

$$K \subset W \cup V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

Then $F \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. ////

Corollary *If $A \subset B$ and if B has compact closure, so does A.*

2.5 Theorem *Suppose X is a Hausdorff space, $K \subset X$, K is compact, and $p \in K^c$. Then there are open sets U and W such that $p \in U$, $K \subset W$, and $U \cap W = \varnothing$.*

PROOF If $q \in K$, the Hausdorff separation axiom implies the existence of disjoint open sets U_q and V_q, such that $p \in U_q$ and $q \in V_q$. Since K is compact, there are points $q_1, \ldots, q_n \in K$ such that

$$K \subset V_{q_1} \cup \cdots \cup V_{q_n}.$$

Our requirements are then satisfied by the sets

$$U = U_{q_1} \cap \cdots \cap U_{q_n} \quad \text{and} \quad W = V_{q_1} \cup \cdots \cup V_{q_n}.$$ ////

Corollaries

(*a*) *Compact subsets of Hausdorff spaces are closed.*
(*b*) *If F is closed and K is compact in a Hausdorff space, then $F \cap K$ is compact.*

Corollary (*b*) follows from (*a*) and Theorem 2.4.

positive measures and positive functions on $\mathcal{C}_\mathrm{c}$

A finite positive Borel measure μ on X gives a functional

$$\Lambda : \mathcal{C}_\mathrm{c}(X) \to k$$
$$\Lambda(f) := \int_{(X,\mu)} f$$

Thus functional is *positive* in the sense that if $f(X) \subset (0,1)$ then $(\Lambda f)(X) \subset (0,1)$.

[1]

The converse of this statement is true!

(Riesz representation of positive functionals on $\mathcal{C}_\mathrm{c}$) Let X be locally compact Hausdorff space and Λ be a positive linear functional on $\mathcal{C}_\mathrm{c}(X)$. Then there exists a σ-algebra in X which contains all Borel sets in X and there exists a **unique positive Radon measure** μ in this σ-algebra

such that

$$\Lambda(f) = \int_{(X,\mu)} f$$

for every $f \in \mathcal{C}_\mathsf{c}(X)$ **and for every compact** $K \subseteq X$

$$\mu(K) < \infty$$

Walter Rudin - Real and complex analysis-McGraw-Hill (1987), p.40

1. math.uchicago.edu/~may/REU2023/REUPapers/Espejo.pdf#page=9.37 ↩

M E Taylor - PDEs I

chapter 3

solving of constant-coefficient PDE on locally on $\mathbb{R}^n$ and globally on T^n

Let

$$P \in \mathbb{C}[X]$$

then we wish to solve for $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ or $\mathscr{D}(\mathbb{R}^n)$

$$P(D)u = f \text{ on } B_R \subset \mathbb{R}^n$$

when $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ or $\mathscr{D}(\mathbb{R}^n)$ for every $R > 0$.

- $$P(D)u = f \iff P(D+\alpha)e^{-i\langle\alpha,x\rangle} = e^{-i\langle\alpha,x\rangle} f$$
- $$P(D)u = f \text{ on } B_R \iff P(D)w = h \text{ on } [0,1]^n$$
  so we may work with the problem on T^n
- $$P(\widehat{D+\alpha}) = P(k+\alpha)$$
  so inversion of $P(D+\alpha)$ boils down to a lower bound of the $P(k+\alpha)$ operator

chapter 4: Sobolev spaces

Definition. Sobolev spaces in $\mathbb{R}^n$

For $\alpha \in \mathbb{R}$

$$H^\alpha(\mathbb{R}^n) := \{u \in \mathcal{S}^*(\mathbb{R}^n) \mid \langle\xi\rangle^\alpha \hat{u} \in L^2(\mathbb{R}^n)\}$$

Proposition:

$$\alpha > \frac{n}{2} \implies H^\alpha(\mathbb{R}^n) \subset \mathcal{C}_\mathrm{b}(\mathbb{R}^n)$$

Fourier inversion and Cauchy-Swartz inequality.

Corollary:

$$\alpha > \frac{n}{2} + k \implies H^{\alpha}(\mathbb{R}^n) \subset \mathcal{C}^k(\mathbb{R}^n)$$

chapter 5: linear elliptic equations

Let

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

be a differential operator with principal symbol

$$P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

is invertible for $\xi \in \mathbb{R}^n \setminus \{0\}$.

Theorem 11.1. *If $P(x, D)$ is elliptic of order m and $u \in \mathcal{D}'(M)$, $P(x, D)u = f \in H^s(M)$, then $u \in H^{s+m}_{loc}(M)$, and, for each $U \subset\subset V \subset\subset M$, $\sigma < s+m$, there is an estimate*

$$\|u\|_{H^{s+m}(U)} \leq C\|P(x, D)u\|_{H^s(V)} + C\|u\|_{H^\sigma(V)}. \tag{11.3}$$

Holomorphic mapping onto disk

Riemann mapping onto disk

(Riemann mapping theorem) Let $\Omega \subset \mathbb{C}$ be a simply connected open, proper subset of $\mathbb{C}$. Then there exists a biholomorphism from Ω onto the unit disk

$$f : \Omega \to D$$

Moreover, given $a \in \Omega$ there is a unique biholomorphism

$$\begin{gathered} f : \Omega \to D \\ f(a) = 0, f'(a) > 0 \end{gathered}$$

(uniqueness) For a give $a \in \Omega$, let there are two biholomorphisms

$$f, g : \Omega \to D$$

then

$$f \circ g^{-1} : D \to D$$

is a automorphism of the disk such that

$$\begin{aligned} f \circ g^{-1}(0) &= f(a) = 0 \\ (f \circ g^{-1})'(0) &= f'(g^{-1}(0)) \frac{1}{g'(g^{-1}(0))} < 1 \end{aligned}$$

- By

  (Swartz inequality) Any holomorphic map

  $$f : D \to D, f(0) = 0$$

  satisfies both

  $$\begin{gathered} |f'(0)| \leq 1 \\ |f(z)| \leq |z| \end{gathered}$$

  Moreover, equality in *either* of these inequalities $\iff$ its a rotation $f(z) = cz, |c| = 1$.

  we conclude

  $$f \circ g^{-1}(w) = cw, |c| = 1$$

- This means

  $$f(z) = cg(z)$$

  but

  $$0 < f'(a) = cg'(a), g'(a) > 0 \implies c = 1$$

  this $f = g$.

[1]

constructing biholomorphisms into the unit disk

- Assume $0 \notin \Omega$. Then there exists a branch of square root $\sqrt{}$ on Ω. Let the image be $\sqrt{\Omega}$.
  - The square root on Ω is injective.
  - If $w \in \sqrt{\Omega}$ then $-w \notin \sqrt{\Omega}$, as otherwise there are $z_1, z_2 \in \Omega$ such that

    $$\sqrt{z_1} = w = -\sqrt{z_2} \implies z_1 = z_2 \text{ or } w = -w$$

    which implies $0 \in \Omega$ which is a contradiction.
  - As $\sqrt{\Omega}$ is open

    $$\sqrt{\Omega} \cap B_\delta(w_0) = \emptyset$$

  - So

    $$\begin{aligned} \left|\sqrt{z} - w_0\right| &> \delta \\ \frac{\delta}{\left|\sqrt{z} - w_0\right|} &< 1 \end{aligned}$$

  - Thus

    $$\begin{aligned} f : \Omega &\to D \\ z &\mapsto \frac{\delta}{\sqrt{z} - w_0} \end{aligned}$$

    is injective into the unit disk D.

by maximizing modulus

Proposition: The unique biholomorphism in

(Riemann mapping theorem) Let $\Omega \subset \mathbb{C}$ be a simply connected open, proper subset of $\mathbb{C}$. Then there exists a biholomorphism from Ω onto the unit disk

$$f : \Omega \to D$$

Moreover, given $a \in \Omega$ there is a unique biholomorphism

$$\begin{gathered} f : \Omega \to D \\ f(a) = 0, f'(a) > 0 \end{gathered}$$

is the one attains the **maximum** of

$$\begin{aligned} \{h : \Omega \to D \mid h \text{ is holomorphic, injective } \} &\to \mathbb{R} \\ h &\mapsto |h'(a)| \end{aligned}$$

(that is this maximum is also surjective).

- *(surjectivity)* For such an Ω, let

  $$f : \Omega \to D$$

  be an *injective* holomorphic map such that

  $$f'(0) = \sup_{g \in \mathcal{F}} |g'(0)|$$

  - Assume $\alpha \in D \setminus f(\Omega)$. Let $R_\alpha : D \to D$ be an holomorphic automorphism such that $R_\alpha(\alpha) = 0$, then

    $$R_\alpha(f(\Omega))$$

    avoids the origin $0 \in D$.
  - As Ω is simply connected

Montel's normal families of holomorphic functions and a converging sequence in them

Definition. **Montel's normal family of holomorphic functions**

A family of holomorphic functions on a open $\Omega \subseteq \mathbb{C}$

$$\mathcal{F} = \{f_\alpha \mid \alpha \in \mathcal{A}\} \subset \mathscr{O}(\Omega)$$

is a normal family if for each compact $K \subset \Omega$, the family is *uniformly bounded on K*

$$\sup_{z\in K}\sup_{\alpha\in\mathcal{A}}|f_\alpha(z)| < \infty$$

(Montel's normal family theorem) For any normal family of holomorphic functions

Definition. Montel's normal family of holomorphic functions

A family of holomorphic functions on a open $\Omega \subseteq \mathbb{C}$

$$\mathcal{F} = \{f_\alpha \mid \alpha \in \mathcal{A}\} \subset \mathscr{O}(\Omega)$$

is a normal family if for each compact $K \subset \Omega$, the family is *uniformly bounded on K*

$$\sup_{z\in K}\sup_{\alpha\in\mathcal{A}}|f_\alpha(z)| < \infty$$

there exists a sequence f_n in $\mathcal{F}$ that converges uniformly on compact subsets of Ω.

Let f be holomorphic in a disk around $z \in \mathbb{C}$ then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{S^1_r}\frac{f(w)}{(w-z)^{n+1}}\mathrm{d}w$$

- So $\{f_\alpha^{(k)} \mid \alpha \in \mathcal{A}\}$ is uniformly bounded on compact subsets of Ω for all $k \geq 0$.
-

using a normal family

- We assume $\Omega \subset D$ and $0 \in \Omega$. Consider the family

$$\mathcal{F} := \{f : \Omega \to D \text{ holomorphic, injective} \mid f(0) = 0, f'(0) > 0\}$$

  - In particular, $f'(0) \in \mathbb{R}$. As $\mathrm{Id}_\Omega \in \mathcal{F}$ it is not empty.

- As

$$|f(z)| < 1 \implies \sup_{z \in D} \sup_{f \in \mathcal{F}} |f(z)| < 1$$

so $\mathcal{F}$ is a normal family

Definition. Montel's normal family of holomorphic functions

A family of holomorphic functions on a open $\Omega \subseteq \mathbb{C}$

$$\mathcal{F} = \{f_\alpha \mid \alpha \in \mathcal{A}\} \subset \mathscr{O}(\Omega)$$

is a **normal family** if for each compact $K \subset \Omega$, the family is *uniformly bounded on K*

$$\sup_{z \in K} \sup_{\alpha \in \mathcal{A}} |f_\alpha(z)| < \infty$$

- Consider the sequence

$$f_m \in \mathcal{F}$$

such that

$$f'_m(0) \xrightarrow{m \to \infty} \sup_{h \in \mathcal{F}} f'(0)$$

This sequence is also a normal family

  - By

(Montel's normal family theorem) For any normal family of holomorphic functions

Definition. Montel's normal family of holomorphic functions

A family of holomorphic functions on a open $\Omega \subseteq \mathbb{C}$

$$\mathcal{F} = \{f_\alpha \mid \alpha \in \mathcal{A}\} \subset \mathscr{O}(\Omega)$$

is a normal family if for each compact $K \subset \Omega$, the family is *uniformly bounded on K*

$$\sup_{z \in K} \sup_{\alpha \in \mathcal{A}} |f_\alpha(z)| < \infty$$

there exists a sequence f_n in $\mathcal{F}$ that converges uniformly on compact subsets of Ω.

there is a subsequence f_n which converges

$$f_n \xrightarrow{n\to\infty} f_\infty \in \mathscr{O}(\Omega)$$

uniformly on compact subsets of Ω, such that

$$f'_\infty(0) = \sup_{f\in\mathcal{F}} f'(0)$$

- Then $f_\infty(0) = 0, f'_\infty(0) > 0$, hence the map is not constant.
- As $f_\infty(\Omega) \subset \bar{D}$ by

holomorphic functions to closed sets must map to interior

Let

$$f : \Omega \to \bar{U}$$

is a *non-constant* holomorphic function from a open $\Omega \subseteq \mathbb{C}$ to a closed set $\bar{U}$ with open interior U.

Then by

(Open mapping theorem for open subsets in $\mathbb{C}$) Let f be *non-constant* holomorphic function on a connected open set U then $f(U)$ is open (and connected).

non-constant holomorphic function locally looks like w^k on a change of coordinates

Let we have a holomorphic function f on some open subset of $\mathbb{C}$ containing z_0.

> **Proposition:** If $f(z_0) \neq 0$ and *any* positive integer k, we compose f with the function $z^{1/k}$ to obtain a function h such that
>
> $$h^k = f \text{ on } V$$
>
> for a smaller open V containing z_0.

Proposition: If $f : U \to \mathbb{C}$ is a *non-constant* holomorphic function on a open subset $U \subseteq \mathbb{C}$,

- then around any $z_0 \in U$ there is a holomorphic $g(z)$

$$\begin{aligned} f(z) &= f(0) + (z - z_0)^k g(z) \text{ around } z_0 \\ g(z_0) &\neq 0 \end{aligned}$$

- Moreover there is a smaller neighborhood around z_0 and a holomorphic $h(z)$ such that

$$\begin{aligned} f(z) &= f(0) + ((z - z_0)h(z))^k \text{ around } z_0 \\ h(z_0) &\neq 0 \end{aligned}$$

  with further smaller neighborhood around z_0 so that $(z - z_0)h(z)$ is a biholomorphism to its (open) image.

- Now, we write

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n \text{ on } z \in B_R(z_0)$$

- If f is **not constant**, there is some smallest $k \geq 1$ (**multiplicity of the zero**) such that $a_k \neq 0$, meaning

$$\begin{aligned} f(z) &= a_0 + (z - z_0)^k g(z) \text{ on } z \in B_R(z_0) \\ g(z) &= \sum_{n \geq 0} a_{n+k} (z - z_0)^n \\ g(z_0) &\neq 0 \end{aligned}$$

- Then by

  > **Proposition:** If $f(z_0) \neq 0$ and *any* positive integer k, we compose f with the function $z^{1/k}$ to obtain a function h such that
  >
  > $$h^k = f \text{ on } V$$
  >
  > for a smaller open V containing z_0.

  we find a smaller open V containing z_0 and a holomorphic h such that

$$\begin{aligned} f(z) &= a_0 + (z - z_0)^k (h(z))^k \text{ on } z \in V \\ h(z_0) &\neq 0 \end{aligned}$$

- Thus we have

$$f(z) = a_0 + (\underbrace{(z - z_0)h(z)}_{h_1(z)})^k \text{ on } z \in V$$

with $h(z_0) \neq 0$, so

$$h_1(z) = (z - z_0)h(z)$$
$$h_1'(z) = h(z) + (z - z_0)h'(z)$$
$$h_1'(z_0) = h(z_0) \neq 0$$

- As h_1 has a non-vanishing derivative at z_0, so there must be *another* smaller open W containing z_0 where $h_1(W)$ is open and

$$h_1 : W \to h_1(W)$$

is a homeomorphism.
- We can choose W so that $h_1(W)$ is a disk centered at $h_1(z_0) = 0$.

Proposition:

$$f : W \to f(W) = a_0 + D$$
$$f(z) = a_0 + (h_1(z))^k$$

is an **open** map.

- Given open set $A \subseteq W$ its image $h_1(A)$ under the homeomorphism h_1 is open set in a disk containing 0.
- Image of $h_1(A)$ under $z \mapsto z^k$ is open and translation $w \mapsto a_0 + w$ takes open sets to open sets.
- Thus $f(A)$ is open.

we must have

$$f(\Omega) \subseteq U$$

Even, by (the statement of) maximum modulus we may conclude the same for function

$$f : \Omega \to B_R(0)$$

going to a ball centered at 0: if there were a point in the image which is in the boundary of the ball

$$p \in U, f(p) \in f(\Omega) \cap S^1_R$$

then p would be a maxima for $|f|$.

This has implications:

- Let

  $$f_n : \Omega \to U$$

  be a sequence of functions converging to f uniformly on compact subsets of Ω. By

  > **Let $f_n : U \to \mathbb{C}$ be holomorphic on open $U \subseteq \mathbb{C}$ and**
  >
  > $$f_n \xrightarrow{\text{uniformly as } n \to \infty} f \text{ on } K$$
  >
  > ***for every $K \subset U$ compact* then f is holomorphic in U.**

  the limit f is holomorphic. Then as $f_n(z) \in U$ the limit must be in $\bar{U}$, but because image must be open, so we conclude the limit is a holomorphic

  $$f : \Omega \to U$$

  > - So we have
  >
  > $$z \in \Omega \implies \lim_{n \to \infty} f_n(z) \in U$$
  >
  >   even though $f_n(z) \in U$ and U is open.

$f(\Omega) \subseteq D$, thus we have a holomorphic

$$f : \Omega \to D$$

-

by solving a Dirichlet problem

[2]

- Let $\Omega \subset D$ and $0 \in \Omega$. We solve the Dirichilet problem for

  $$\log|\zeta| \text{ for } \zeta \in \partial\Omega$$

and obtain a harmonic

$$u : \Omega \to \mathbb{R}$$

such that

$$u(z) \xrightarrow{z \to \zeta \in \partial\Omega} \log|\zeta|$$

- Since Ω is simply connected there is a harmonic conjugate v to u, making

$$f := z \exp(-(u + iv)) : \Omega \to \mathbb{C}$$

  holomorphic on Ω such that

$$|f(z)| \xrightarrow{z \to \zeta \in \partial\Omega} 1$$

- Hence, $f(\Omega) \subset \bar{D}$ and by open mapping or maximum modulus $f(\Omega) \subset D$. Thus

$$f : \Omega \to D$$

- Applying the argument principle to

---

1. https://www.math.stonybrook.edu/~bishop/classes/math626.F08/rmt.pdf ↩
2. complex analysis - Solution verification —— Proof of the Riemann mapping theorem using Perron's solution of the Dirichlet problem. - Mathematics Stack Exchange ↩

Jet Nestruev - Smooth Manifolds and Observables

- Let A be a commutative, associative $\mathbb{R}$-algebra with 1_A.
- Let $|A| := \operatorname{HomAlg}_{\mathbb{R}}(A, \mathbb{R})$ be all unital $\mathbb{R}$-algebra homomorphisms.
  - Every element

    $$x : A \to \mathbb{R}$$

    defines a kernel which is either zero or maximal $\mathfrak{m}_x$
- $$\tilde{A} := \{\mathrm{ev}_f : |A| \to \mathbb{R} \text{ for some } f \in A\}$$

  is a commutative $\mathbb{R}$-algebra
  - These evaluation maps

    $$\mathrm{ev}_f : |A| \to \mathbb{R}$$
- There is a natural surjective homomorphism

  $$\begin{aligned} A &\to \tilde{A} \\ f &\mapsto \mathrm{ev}_f \end{aligned}$$

  but it is not injective in general.

Definition. **An $\mathbb{R}$-algebra A is called geometric if**

$$\bigcap_{x \in |A|} \ker x = 0$$

And

$$f \in \ker(A \to \tilde{A}) \iff \mathrm{ev}_f = 0 \iff \forall x \in |A|,\ x(f) = 0 \iff f \in \bigcap_{x \in |A|} \ker x$$

thus

Any geometric $\mathbb{R}$-algebra A is canonically isomorphic to the $\mathbb{R}$-algebra of functions on the dual space

$$|A|$$

We give the weak-* topology on $|A|$ such that every

$$f \equiv \mathrm{ev}_f : M \to \mathbb{R}$$

in A is continuous.

> **Proposition:** This makes $|A|$ a Hausdorff space.

Thus we understand

$$A \hookrightarrow \mathcal{C}(|A|, \mathbb{R})$$

Let A be a geometric $\mathbb{R}$-algebra and $S \subset |A|$. Then the restriction

$$A(S) := \{f : S \to \mathbb{R} \mid \forall x \in S \exists \bar{f} \in A, U_x \subset S : f|_U = \bar{f}|_U\}$$

and we have the restriction

$$\mathrm{res}_S : A \to A(S)$$

Definition. **A geometric $\mathbb{R}$-algebra is complete if the restriction**

$$\mathrm{res}_{|A|} : A \to A(|A|)$$

is *surjective*.

on $\mathcal{C}(X)$

- $\mathrm{maxSpec}\,\mathcal{C}(X)$ for non-cpt (nice enough) X are points of βX (Stone-Cech compactification) https://mathoverflow.net/a/3876
- but $\beta\mathbb{R} \setminus \mathbb{R}$ (new points in compactified R) is horribly large https://math.stackexchange.com/a/1790467 and hard to describe (AoC blinds us); its supposed to be larger than one/two points because $\sin(x)$ is continuous on $\beta\mathbb{R}$

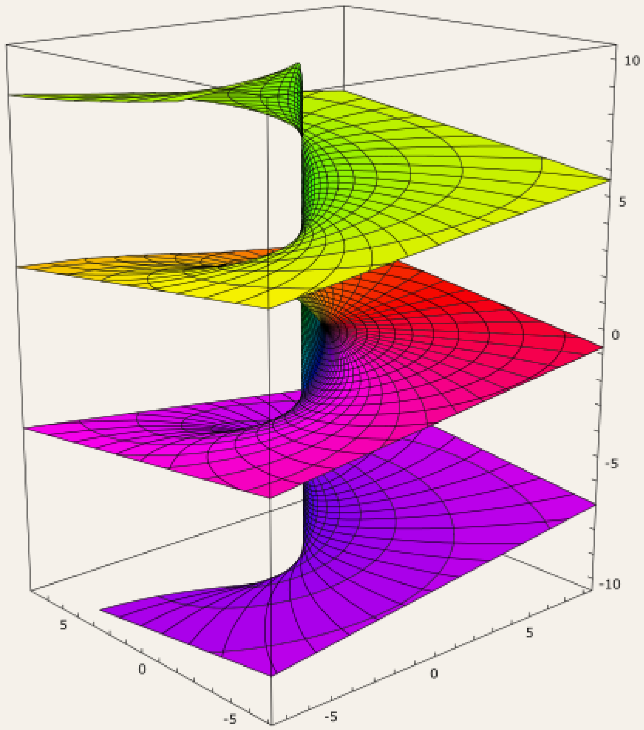
$\mathbb{Z}$-sheeted cover of punctured plane $U(\mathbb{R}^2 \setminus \{(0,0)\})$

#wiki/Man/R/2

We construct the connected, simply connected, 2 dimensional smooth $\mathbb{R}$-manifold (diffeomorphic to $\mathbb{R}^2$) that is the universal cover of $\mathbb{R}^2 \setminus \{(0,0)\}$.

Definition.

$$\mathbb{Z} \curvearrowright (U(\mathbb{C}^\times), +) \xrightarrow{\pi} (\mathbb{C}^\times, \times)$$



Consider

$$\mathbb{C} \setminus \{0\}$$

and

$$U(\mathbb{C} \setminus \{0\}) := \frac{\mathbb{C} \setminus (-\infty, 0] \times \mathbb{Z} \coprod (-\infty, 0) \times (-1, 1) \times \mathbb{Z}}{(z, k)}$$

- by taking

$$\coprod_{\mathbb{Z}} \mathbb{R}^2 \setminus \{(0, x) | x \leq 0\}$$

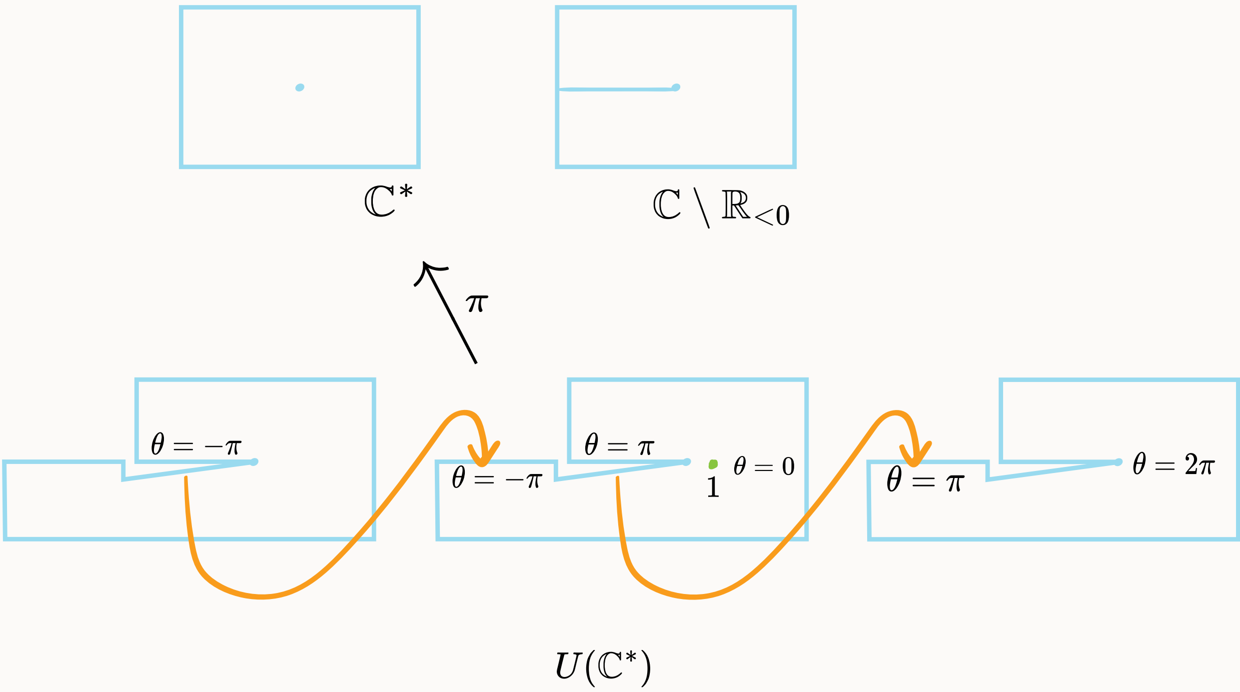
and gluing the edge of the n-th piece with the edge of $n+1$-th piece.

- The elements look like

$$(x, y, \theta)$$

  where θ is defined by adding $2\pi n$ to $\tan^{-1}(y/x)$.
- This forms simply connected (smooth) cover of $\mathbb{R}^2 \setminus \{0\}$ with covering map

$$\begin{aligned}\pi : U(\mathbb{C}^\times) &\to \mathbb{C}^\times \\ (x, y, \theta) &\mapsto (x, y)\end{aligned}$$



  $\mathbb{C}$, where elements look like (z, θ) whose complex structure is given by the covering map π.

- We lift the group structure

$$\begin{aligned}&(U(\mathbb{C}^\times), +) \\ &(z_1, \theta_1) + (z_2, \theta_2) = (z_1 z_2, \theta_1 + \theta_2)\end{aligned}$$

  from $(\mathbb{C}^\times, \times)$ such that π is a group homomorphism by choosing $(1, 0)$ as the identity.
  - Already a choice has been made for the identity on the cover, now we let the two identity to be base points, we have a canonical lift

$$\begin{aligned}\mathbb{C}^\times \setminus \mathbb{R}_{<0} &\hookrightarrow U(\mathbb{C}^\times) \\ z = x + iy &\mapsto \tilde{z} := (z, \tan^{-1}(y/x))\end{aligned}$$

    such that $\pi^{-1}(z) = \tilde{z} + (1, 2\pi i\mathbb{Z})$
  - We can now write

$$(z, \theta) = \tilde{z} + (1, 2\pi n)$$

- With the covering group action by $\mathrm{Deck}(U(\mathbb{C}^\times) \to \mathbb{C}^\times) \cong \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$ is generated by σ given by

$$z \overset{\sigma}{\mapsto} z + (1, 2\pi i)$$

$U(\mathbb{C}^\times)$ is a Riemann surface biholomorphic to $\mathbb{C}$ which is the complex Lie group covering of $\mathbb{C}^\times$. The map

$$\theta : U(\mathbb{C}^\times) \to \mathbb{R}$$

is smooth and surjective.

embedding inside $\mathbb{R}^3$

$$\begin{aligned} U(\mathbb{R}^2 \setminus \{0\}) &\hookrightarrow \mathbb{R}^3 \\ (x, y, \theta) &\mapsto (x, y, \theta) \end{aligned}$$

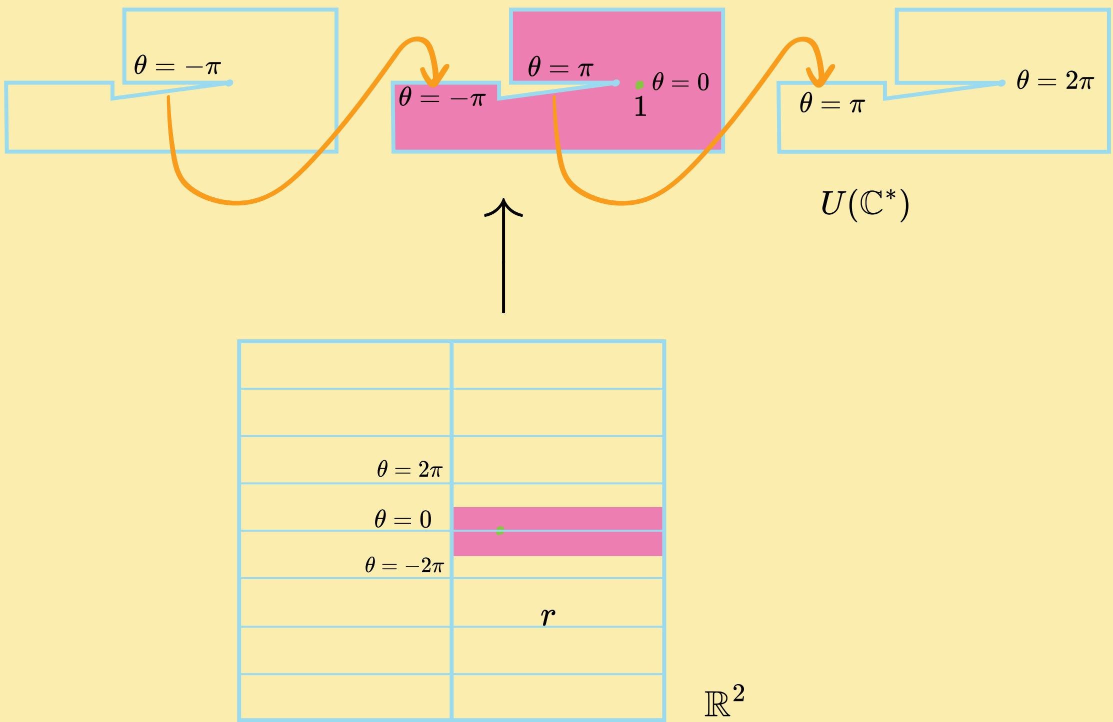
The image is a subset of

$$\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}$$

polar coordinates actually cover the universal cover!

The "polar coordinates on $\mathbb{R}^2 \setminus \{0\}$" actually covers the infinite helicoid

$$\begin{aligned} \mathbb{R}_{>0} \times \mathbb{R} &\to U(\mathbb{C}^\times) \\ (r, \theta) &\mapsto (r\cos\theta, r\sin\theta, \theta) \end{aligned}$$



is a diffeomorphism.

which in complex coordinates this is

$$
\begin{aligned}
\mathbb{C}_{\mathrm{Re}>0} &\to U(\mathbb{C}^\times) \\
(r,\theta) &\mapsto (r\exp(i\theta),\theta)
\end{aligned}
$$

but it is of course **not** a bi-holomorphism?

A better way to think about this is: we think of

$$
\begin{aligned}
\mathbb{C}_{\mathrm{Re}>0} &\to \mathbb{C}^\times \\
(r,\theta) &\mapsto r\exp(i\theta)
\end{aligned}
$$

as the covering map π itself, where we think of the right half plane *as* the universal cover, (but still this is not holomorphic?).

$$
\begin{tikzcd}
\begin{array}{c}(0,\infty)\times\mathbb{R}\\ (|z|,\theta)\end{array} \arrow[rr, bend left] \arrow[dr] & & \begin{array}{c}U(\mathbb{C}\setminus\{0\})\\ (z,\theta)\end{array} \arrow[ll, bend left] \arrow[dl] \\
& \begin{array}{c}(0,\infty)\times S^1 \cong \mathbb{C}\setminus\{0\}\\ |z|e^{i\theta}=z\end{array} &
\end{tikzcd}
$$

We need

$$\exp : \mathbb{C} \to \mathbb{C}^\times$$

as the holomorphic map, which is the covering map.

the complex logarithm

complex logarithm on the universal cover of punctured plane

The $U(\mathbb{C}^\times)$ is the Riemann surface of $\log(z)$

$$
\begin{aligned}
\log : U(\mathbb{C}^\times) &\to \mathbb{C} \\
(z,\theta) &\mapsto \log_{\mathbb{R}}(|z|) + i\theta
\end{aligned}
$$

$$
\begin{tikzcd}
\begin{matrix} U(\mathbb{C}\setminus\{0\}) \\ (z,\theta) \end{matrix} \arrow[rr, bend left] \arrow[dr] & & \begin{matrix} \mathbb{C} \\ ? \end{matrix} \arrow[ll, bend left] \arrow[dl, "\exp"] \\
& \begin{matrix} \mathbb{C}\setminus\{0\} \\ z=\exp(w) \end{matrix} &
\end{tikzcd}
$$

$$z = \exp(f(z,\theta))$$

the pullback of inexact closed forms on punchered plane

$$H^1(\Omega^\bullet(\mathbb{R}^2\setminus\{0\})) = \mathbb{R}\left[\frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2}\right]$$

$$\pi^*\frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2} = \mathrm{d}\theta$$

Simon Donaldson #book on Riemann surfaces

Chapter 7

> **Proposition:** Let
>
> $$X := V(p) \subseteq \mathbb{C}P^2$$
>
> be a smooth complex curve of degree d for the homogeneous polynomial p with non-vanishing (on X) partial derivatives. Then the genus of X is
>
> $$g_X = \frac{1}{2}(d-1)(d-2)$$

So this means

degree of $p(X) \in \mathbb{C}[X]$		$g_X(d)$	Riemann surface $V(p)$
1		0	sphere
2	conic	0	sphere
3	cubic	1	torus
nope!		2	
4		3	

💡 As

$$p(x,y) = 0$$

on X we have

$$p_x \mathrm{d}x + p_y \mathrm{d}y = 0$$
$$\theta := \frac{\mathrm{d}x}{p_y} = -\frac{\mathrm{d}y}{p_x}$$

as p_x, p_y do not vanish on X.

Chapter 11

algebraic degree and degree of meromorphic maps

Let Σ be a compact Riemann surface. It admits a non-trivial holomorphic

$$f : \Sigma \to \mathbb{C}P^1$$

Then

$$\begin{aligned} f_* : \mathbb{C}[\mathbb{C}P^1] \cong \mathbb{C}(z) &\hookrightarrow \mathbb{C}(\Sigma) \\ p(z) &\mapsto p(f(-)) \end{aligned}$$

is an embedding of fields.

$$[\mathbb{C}(\Sigma), \mathbb{C}(z)] = \deg f$$

Let (by translation if necessary) $0 \in \mathbb{C}P^1$ is a regular value of f so $f^{-1}(0)$ is a set of $d := \deg f$ distinct points

$$p_1, \ldots, p_d \in \Sigma$$

- By *existence* for each i there is a meromorphic

  $$g_i : \Sigma \to \mathbb{C}P^1$$

  that has a simple pole at p_i but is holomorphic around p_j for each $j \neq i$.
- Suppose

  $$\sum_j \lambda_j g_j = 0$$

  where $\lambda_j \in \mathbb{C}(z) \hookrightarrow \mathbb{C}(\Sigma)$. By multiplying a suitable z^k we can assume λ_j are holomorphic around $z = 0$ and do not all vanish at $z = 0$.
- Consider

  $$\sum_j \lambda_j(f(p_i)) g_j(p_i) = 0$$

  The only non-holomorphic term is $\lambda_i g_i$ so

  $$\lambda_j(\underbrace{f(p_i)}_{0}) = 0$$

  which is a contradiction.
- Thus

  $$[\mathbb{C}(\Sigma), \mathbb{C}(z)] \geq \deg f$$

- By *primitive element theorem* it suffices to show any $g \in \mathbb{C}(\Sigma)$ satisfies

$$P(f, g) = \lambda_0(f) + g\lambda_1(f) + g^2\lambda_2(f) + \cdots + g^d\lambda_d(f)$$

  - Let z be a regular value of f so

  $$f^{-1}(z) = \{p_1, \ldots, p_d\} \subset \Sigma$$

  such that

  $$w_i := g(p_i) \in \mathbb{C} \subset \mathbb{C}P^1$$

  - If

  $$a_1 := \sum_i w_i$$

  and so on are elementary symmetric functions we have

  $$\sum_{r=0}^{d} (-1)^r a_r g^r = 0$$

holomorphic maps and field extensions

Corollary of

> **Let Σ be a compact Riemann surface. It admits a non-trivial holomorphic**
>
> $$f : \Sigma \to \mathbb{C}P^1$$
>
> **Then**
>
> $$\begin{aligned} f_* : \mathbb{C}[\mathbb{C}P^1] \cong \mathbb{C}(z) &\hookrightarrow \mathbb{C}(\Sigma) \\ p(z) &\mapsto p(f(-)) \end{aligned}$$
>
> **is an embedding of fields.**
>
> $$[\mathbb{C}(\Sigma), \mathbb{C}(z)] = \deg f$$

If

$$f : \Sigma_1 \to \Sigma_2$$

is a non-constant holomorphic map between compact connected Riemann surfaces then

$$[\mathbb{C}(\Sigma_1), \mathbb{C}(\Sigma_2)] = \deg f$$

Consider

$$f_0 : \Sigma_2 \to \mathbb{C}P^1$$

then

$$\mathbb{C}(z) \hookrightarrow \mathbb{C}(\Sigma_2) \hookrightarrow \mathbb{C}(\Sigma_1)$$

which gives

$$[\mathbb{C}(\Sigma_1), \mathbb{C}(z)] = [\mathbb{C}(\Sigma_1), \mathbb{C}(\Sigma_2)][\mathbb{C}(\Sigma_2), \mathbb{C}(z)]$$

fields of transcendence degree $= 1$

- We know

  $$\mathbb{C}(z) \hookrightarrow \mathbb{C}(\Sigma)$$

  Thus the transcendence degree is at least 1.
- Suppose

  $$\mathbb{C}(z_1, z_2) \hookrightarrow \mathbb{C}(\Sigma)$$

  then

  $$z_1 \mapsto f : \Sigma \to \mathbb{C}P^1$$

  (non-constant). By

  Let Σ be a compact Riemann surface. It admits a non-trivial holomorphic

  $$f : \Sigma \to \mathbb{C}P^1$$

  Then

  $$\begin{aligned} f_* : \mathbb{C}[\mathbb{C}P^1] \cong \mathbb{C}(z) &\hookrightarrow \mathbb{C}(\Sigma) \\ p(z) &\mapsto p(f(-)) \end{aligned}$$

  is an embedding of fields.

  $$[\mathbb{C}(\Sigma), \mathbb{C}(z)] = \deg f$$

we know

$$[\mathbb{C}(\Sigma), \mathbb{C}(z_1)] < \infty \implies [\mathbb{C}(z_1, z_2), \mathbb{C}(z_1)] < \infty$$

which is a contradiction.

Let K is any field extension of $\mathbb{C}$ and of transcendence degree 1 then for every

$$\iota : \mathbb{C}(z) \hookrightarrow K$$

there is a compact connected Riemann surface Σ with $K = \mathbb{C}(\Sigma)$ and

$$f : \Sigma \to \mathbb{C}P^1$$

such that $f_* = \iota$.

valuations

$$\begin{array}{r} \text{cpt conn Riemann surfaces} \\ \text{non-constant holomorphic maps} \end{array} \leftrightarrow \begin{array}{l} \text{fields of transcendence degree 1} \\ \text{field inclusions} \end{array}$$

complex 1-manifolds AKA *Riemann surfaces*

Definition. **Connected complex 1-manifolds are called Riemann surfaces.**

We may quickly look at the classification:

Every *non-compact simply connected* Riemann surface is biholomprhic to the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$.

Every *compact simply connected* Riemann surface is biholomorphic to the Riemann sphere $\mathbb{C}P^1$

homeomorphism class	hol classes	moduli space	universal covering	Aut
plane $\mathbb{R}^2$	$\mathbb{C}$	$\{\bullet, \bullet\}$	$\mathbb{C}$	$\mathbb{C}$
	disk/upper half plane $D \cong H$		$D \cong H$	$PSL(2, \mathbb{R})$
cylinder $\cong \mathbb{R}^2 \setminus \{0\}$ $\cong D \setminus \{(0,0)\}$	$\mathbb{C}$	$\{\bullet, \bullet\} \cup (0,1)$	$\mathbb{C}$	$\mathbb{C}$
	$D^* \cong H \setminus \{i\}$		$D \cong H$ by exponential map	$SO(2, \mathbb{R})$
	$D_r, r \in (0,1)$		$D \cong H$	??
sphere S^2	$\mathbb{C}P^1$	$\mathcal{M}_0 \cong \{\bullet\}$	$\mathbb{C}P^1$	$PGL(2, \mathbb{C})$
disk with $n > 1$ points removed		$\mathcal{M}_{0,1}$	$D \cong H$	?
torus T^2	a rank 2 lattice $\Lambda \subseteq \mathbb{C}$ upto homothety	$\mathcal{M}_1 \cong \dfrac{H}{SL(2, \mathbb{Z})} \cong \mathbb{C}$	$\mathbb{C}$ quotient by a rank 2 lattice	

homeomorphism class	hol classes	moduli space	universal covering	Aut
higher genus surfaces $T_g^2, g > 1$		$\mathcal{M}_g \cong \mathbb{C}^{3g-3}$	$D \cong H$	finite
compact genus $g > 0$ surface with n points removed		$\mathcal{M}_{g,n}$	$D \cong H$ $D \cong H$	
compact genus g surface with infinite points removed			$D \cong H$	
infinite genus				
...				

We have

- sett.Man.C 2.Hol.loc open
- sett.Man.C 2.T C complex tangent space, differential forms, Stroke's theorem
- sett.Man.C 2.Hol holomorphic functions, forms, sooo many theorems
- sett.Man.C 2.Mer field of meromorphic functions
- vector bundle
  - line bundles
-

If X is a compact connected Riemann surface, then

$$\mathcal{O}(X) \cong \mathbb{C}$$

Take any holomorphic $f \in \mathcal{O}(X)$ then

- $|f|$ is a continuous function from compact to $\mathbb{R}$, so there exists a maximum, that is, there exists $x_0 \in X$ such that

$$|f(x_0)| = \sup\{|f(x)| : x \in X\}$$

- $f^{-1}(x_0)$ is a closed subset of X
- take $x \in f^{-1}(x_0)$ and a chart nbd $U\alpha$ of x and move $f\mid_{U\alpha}$ onto a function between subsets of $\mathbb{C}$, then because modulus of non-constant holomorphic functions cannot attain its maximum in the interior of its domain, it must be a constant on U_α, which

is a open nbd of x. So every element of $f^{-1}(x_0)$ has a open nbd inside it. So $f^{-1}(x_0)$ is **open**

Because X is connected, $f^{-1}(x_0) = X$. Thus f is constant on X.

sett.Man.C 2.T C.forms These are smooth complex differential forms on a Riemann surface X

$$\begin{array}{ccccc}
\Omega^0(X,\mathbb{C}) & \to & \Omega^{1,0}(X,\mathbb{C}) \oplus \Omega^{0,1}(X,\mathbb{C}) & \to & \Omega^2(U,\mathbb{C}) \\
\phi & \overset{\mathrm{d}}{\mapsto} & \mathrm{d}\phi = \frac{\partial \phi}{\partial z}\mathrm{d}z + \frac{\partial \phi}{\partial \bar{z}}\mathrm{d}\bar{z} & & \\
 & & f\mathrm{d}z + g\mathrm{d}\bar{z} & \overset{\mathrm{d}}{\mapsto} & \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right)\mathrm{d}z \wedge \mathrm{d}\bar{z} \\
 & & & & h\mathrm{d}z \wedge \mathrm{d}\bar{z}
\end{array}$$

These are $\mathcal{C}^\infty(M,\mathbb{C})$-differential forms on a Riemann surface and the exterior derivative.

The Hodge diamond of Riemann surfaces

$$\begin{array}{ccc}
 & h^{0,0} & \\
h^{1,0} & & h^{0,1} \\
 & h^{1,1} &
\end{array}$$

Line bundles

On a Riemann surface M

$$\{\text{hol line bundles on } M\}_{\cong} \cong H^1(M, \mathscr{O}^*)$$

Let M be a Riemann surface. Then

$$p > 2 \implies H^p(M, \mathscr{O}(L)) = 0 = H^p(M, \mathbb{C}) = H^p(M, \mathbb{Z})$$

for a line bundle L.

(Serre duality) If L is a line bundle on a compact Riemann surface M then

$$H^1(M, L) \cong H^0(M, K_M \otimes L^*)^*$$

$$H^0(M, LL_p^{-1}) \cong H^0(M, L)$$

Definition. Degree of a line bundle on a Riemann surface

From the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathscr{O} \xrightarrow{\exp(2\pi i -)} \mathscr{O}^* \to 1$$

which on a **compact** Riemann surface gives a long exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to H^1(M, \mathbb{Z}) \to H^1(M, \mathscr{O}) \to H^1(M, \mathscr{O}^*) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathscr{O}) \to \dots$$

where we have

- $\mathbb{C} \to \mathbb{C}^*$ is surjective which means $H^1(M, \mathbb{Z}) \to H^1(M, \mathscr{O})$ is injective
- $H^2(M, \mathscr{O}) = 0$
- $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$

so the equation becomes

$$0 \to \underbrace{\frac{H^1(M, \mathscr{O})}{H^1(M, \mathbb{Z})}}_{\cong \mathbb{C}^g/\mathbb{Z}^{2g} \cong (S^1)^{2g}} \to \underbrace{H^1(M, \mathscr{O}^*)}_{\text{space of line bundles on } M} \xrightarrow{\delta} \underbrace{H^2(M, \mathbb{Z})}_{\cong \mathbb{Z}} \to 0$$

where $\delta([L])$ is called the **degree or the first Chern class** of the line bundle L

Line bundles	transition functions live in $$H^1(M, \mathscr{O}^*)$$	sections	s
$L \to M$ with $U_\alpha \subseteq M$	$g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{C}^*$ (say)	s (say)	
$L_1 \otimes L_2 := L_1 L_2$	$g_{\alpha\beta}^{(L_1)} g_{\alpha\beta}^{(L_2)}$	$s_1 s_2$?	
the bundle $L \otimes L^*$ is trivial	?	holomorphic line bundle hom	

Line bundles	transition functions live in $$H^1(M, \mathscr{O}^*)$$	sections	sp
$L^{-1} := L^*$ the dual bundle is a "multiplicative inverse"	$(g_{\alpha\beta})^{-1}$		
the canonical bundle $K_M := T^{(1,0)}M$			h
L_p for any $p \in M$	On U_p with coordinate z such that $z(p) = 0$ then $$g_{01} := z : U_p \cap \{p\}^\complement \to \mathbb{C}^*$$	Has a **canonical section** s_p: take the two functions z on U_p and 1 on $\{p\}^\complement$ which becomes a section since $$z = z1 = g_{01}1$$ This section vanishes only at p	
$>$ L has a section that vanishes in $(p_i)_{i=1}^N$ with multiplicity m_i then it is $$\bigotimes_i (L_{p_i})^{\otimes m_i}$$			
pullback bundle $$f^*L := \{(x, q) \in L \times M : \pi(x) = f(q)\}$$	$$\gamma_{\alpha\beta} \circ f$$	$s : \tilde{M} \to L$ such that $\pi \circ s = f$	
On $\mathbb{C}P^1$ we define the bundle $$\mathcal{O}(n)$$	on usual patches U_0, U_1 we have the transition function $$g_{01} := z^n : U_0 \cap U_1 \to \mathbb{C}^*$$	a section of this line bundle is given by holomorphic $s_0, s_1 : \mathbb{C} \to \mathbb{C}$ such that $$s_0(z) = z^n s_1\left(\frac{1}{z}\right)$$ expanding, we get $s_0 = \sum_{i=0}^n a_i z^i$	H p a

If $\deg L < 0$ then L only has trivial sections.

If M is a compact Riemann surface of genus g then

$$\dim H^1(M, K_M) = 2g$$

Jacobian

Definition. Degree of a line bundle on a Riemann surface

From the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathscr{O} \xrightarrow{\exp(2\pi i -)} \mathscr{O}^* \to 1$$

which on a **compact** Riemann surface gives a long exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to H^1(M, \mathbb{Z}) \to H^1(M, \mathscr{O}) \to H^1(M, \mathscr{O}^*) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathscr{O}) \to \dots$$

where we have

- $\mathbb{C} \to \mathbb{C}^*$ is surjective which means $H^1(M, \mathbb{Z}) \to H^1(M, \mathscr{O})$ is injective
- $H^2(M, \mathscr{O}) = 0$
- $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$

so the equation becomes

$$0 \to \underbrace{\frac{H^1(M, \mathscr{O})}{H^1(M, \mathbb{Z})}}_{\cong \mathbb{C}^g/\mathbb{Z}^{2g} \cong (S^1)^{2g}} \to \underbrace{H^1(M, \mathscr{O}^*)}_{\text{space of line bundles on } M} \xrightarrow{\delta} \underbrace{H^2(M, \mathbb{Z})}_{\cong \mathbb{Z}} \to 0$$

where $\delta([L])$ is called the **degree or the first Chern class** of the line bundle L

$$\mathrm{Jac}(M) := \frac{H^1(M, \mathscr{O})}{H^1(M, \mathbb{Z})}$$
$$\mathrm{Jac}^d(M) := \{\text{line bundles of degree } d\}_{\cong} \cong \mathrm{Jac}(M)$$

There is a holomorphic map

$$\begin{aligned} M^{g-1} &\to J^{g-1} \\ (p_1, \ldots, p_{g-1}) &\mapsto L_{p_1} \ldots L_{p_{g-1}} \end{aligned}$$

whose image is called the **theta divisor**.

Vector bundles

	transition functions	rank	degree
E	$g_{\alpha\beta}$		$\deg(E) := \deg(\det(E))$
$E_1 \oplus E_2$		$\mathrm{rk}(E_1) + \mathrm{rk}(E_2)$	
$E_1 \otimes E_2$			
$\det E := \Lambda^{\mathrm{rk}(E)} E$	$\det(g_{\alpha\beta})$	1	

(Riemann-Roch) If $E \to M$ is a holomorhic vector bundle on a compact Riemann surface of genus g then

$$\dim H^0(M, \mathscr{O}(E)) - \dim H^1(M, \mathscr{O}(E)) = \deg(E) + (1-g)\mathrm{rk} E$$

- If E is the trivial line bundle then $\mathscr{O}(E) = \mathscr{O}$ and

  $$\dim H^0(M, \mathscr{O}(E)) - \dim H^1(M, \mathscr{O}(E)) = 1 - g$$

  so the formula holds.
- Let L be a line bundle on M and we consider the short exact sequence

  $$0 \to \mathscr{O}(L) \xrightarrow{s_p} \mathscr{O}(LL_p) \to \mathscr{O}_p(LL_p) \to 0$$

  which gives the long exact sequence

  $$0 \to H^0(M, \mathscr{O}(L)) \to H^0(M, LL_p) \to \mathbb{C} \to H^1(M, \mathscr{O}(L)) \to H^1(M, \mathscr{O}(LL_p)) \to 0$$

  - By exactness the alternating sum of the dimensions must be zero therefore

---

(Birkoff-Grothendieck) If E is a rank m holomorphic vector bundle over $\mathbb{C}P^1$ then

$$E \cong \bigoplus_{i=1}^{m} \mathscr{O}(a_i)$$

for some $a_i \in \mathbb{Z}$.

- Let E be a holomorphic vector bundle over $\mathbb{C}P^1$ then E is trivial $\iff$

  $$\deg E = 0 \text{ and } H^0(\mathbb{C}P^1, E(-1)) = 0$$

$$\deg f := \deg(f^* L_p)$$

Let $f : N \to M$ be a holomorphic map between compact Riemann surfaces and L is a line bundle on N. Then

$$H^0(M, f_* \mathscr{O}(L)) \cong H^0(N, \mathscr{O}(L))$$

Any rank $= \deg f$ holomorphic vector bundle $E \to M$ has

$$f_* \mathscr{O}(L) = \mathscr{O}(E)$$

If $V \to M$ is a holomorphic vector bundle then

$$f_* \mathscr{O}(L \otimes f^* V) \cong \mathscr{O}(E \otimes V)$$

> Each point $p \in M$ has a nbd U for which
>
> $$f_* \mathscr{O}(L)(U) \cong \bigoplus_{i=1}^{m} \mathscr{O}(U)$$

> With L, E as above
>
> $$\deg E = \deg L + (1 - g(N)) - (1 - g(M)) \deg f$$

$$f : N \to \mathbb{C}P^1$$

then

$$\deg E = \deg L + (1 - g(N)) - \deg f$$

On $\mathbb{C}P^1$

For

$$f : \underbrace{M}_{\text{genus } g} \to \mathbb{C}P^1$$

for a generic line bundle $L(-1) \in J^{g-1}$ will give

$$f_* \mathscr{O}(L) \to \mathbb{C}P^1$$

will be the trivial bundle E over $\mathbb{C}P^1$.

We now take a section

$$w \in H^0(M, f^* \mathcal{O}(n))$$

Let $U \subseteq \mathbb{C}P^1$ then multiplication by w defines a linear map

$$w(-) : H^0(f^{-1}(U), L) \to H^0(f^{-1}(U), L(n))$$

By definition of E this is a homomorphism

$$W : H^0(U, E) \to H^0(U, E(n))$$

Since W is globally defined and E is trivial

$$W : H^0(\mathbb{C}P^1, E) \to H^0(\mathbb{C}P^1, E(n)) \cong \mathbb{C}^n \otimes H^0(\mathbb{C}P^1, \mathcal{O}(n))$$

which is a $m \times m$ matrix-valued holomorphic section of $\mathcal{O}(n)$ which is a polynomial degree $\leq n$ with matrix coefficients

$$A(z) = A_0 + A_1 z + \cdots + A_n z^n$$

Dedekind-Weber theory

22.1. Reconstruction of a Riemann surface from its function field. Let X be a connected Riemann surface. For any nonzero meromorphic function f on X and any point $p \in X$, we have

$$f(z) = \sum_{n \geq \mathrm{ord}_p f} a_n z^n$$

for some local holomoprhic coordinate z of X around p, and some integer N with $a_N \neq 0$. The integer N is called the order of f of p, denoted by $\mathrm{ord}_p f$. The function ord_p is a discrete valuation of $\mathscr{M}(X)$, defined as follows.

Definition 22.1. Let K be a field. A *discrete valuation* of K is a surjective map

$$v : K^\times \twoheadrightarrow \mathbf{Z}$$

with $v(0) := \infty$, such that for every $f, g \in K$, we have

- $v(f+g) \geq \min\{\, v(f), v(g)\,\}$;
- $v(fg) = v(f) + v(g)$.

We have the following fundamental result in the theory of Riemann surfaces.

Theorem 22.2 (Dedekind, Weber). *Let X be a connected compact Riemann surface. The map*

$$\begin{aligned} X &\to \{\, \textit{Discrete valuations on } \mathscr{M}(X)\,\} \\ p &\mapsto \mathrm{ord}_p \end{aligned}$$

is bijective.

Remark 22.3. The above result also holds for non-compact Riemann surfaces, which was proven by Hironaka, under the pseudonym "Iss'sa".

Exercise 22.4. Let X be the Riemann sphere. Show that $\mathrm{ord}_\infty f = -\deg f$ for every $f \in \mathscr{M}(X) = \mathbf{C}(z)$.

Finally, note that $\mathrm{ord}_p(f)$ only depends on the local behavior of f at p. For instance, if f is holomorphic at p, then $\mathrm{ord}_p(f)$ only depends on $f \in \mathscr{O}_{X,p}$ regarded as a germ of holomorphic function at p.

[1]

[2]

Counting holomorphic maps between compact Riemann surfaces

https://en.wikipedia.org/wiki/De_Franchis_theorem : There are only finitely many non-constant holomorphic mappings between two fixed compact Riemann surfaces of genus greater than 1.

bounds on that number: https://www.cambridge.org/core/services/aop-cambridge-core/content/view/4F55ECA45BC877D46534577A07CA2C56/S0013091507000223a.pdf

${}_gX \to {}_hX$	$g = 0$	1	2	3	4	5	6
$h = 0$							

${}_gX \to {}_hX$	$g=0$	**1**	**2**	**3**	**4**	**5**	**6**
1							
2							
3							
4							
5							
6							

[3]

Classification of branched covers of $\mathbb{C}P^1$

[4]

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1. http://homepage.ntu.edu.tw/~hsuehyunglin/Modern_Algebra_II.pdf ↩
2. https://www.math.columbia.edu/~calebji/teaching/2%20-%20Dedekind.pdf ↩
3. Hurwitz Schemes and Irreducibility of Moduli of Algebraic Curves ↩
4. https://arxiv.org/pdf/1509.07820 ↩

Joe Harris - Algebraic Geometry: a first course

Let $k = \bar{k}$ be a field.

twisted cubic

varieties

affine verities

affine n-space	$$\boldsymbol{A}^n(\bar{k}) := \bar{k}^n$$
k-points of $\boldsymbol{A}^n$	$$\boldsymbol{A}^n(k) := \boldsymbol{A}^n(\bar{k}) \cap k^n = k^n$$
	$$\begin{gathered}\mathrm{Gal}(\bar{k}/k) \curvearrowright \boldsymbol{A}^n(\bar{k}) \\ (x_1, x_2, \ldots, x_n) \overset{\sigma}{\mapsto} (x_1^\sigma, x_2^\sigma, \ldots, x_n^\sigma)\end{gathered}$$
	then $\boldsymbol{A}^n(k)$ is the set fixed pointwise by the Galois group
affine algebraic set	for an ideal $I \subseteq \bar{k}[X_1, \ldots, X_n]$ $$V_{\bar{k}}(I) := \{p \in \boldsymbol{A}^n(\bar{k}) \mid f \in I \implies f(p) = 0\}$$
k-points of $V_{\bar{k}}(I)$	$$V_k(I) := V_{\bar{k}}(I) \cap \boldsymbol{A}^n(k)$$
geometric ideal in k	$$\begin{aligned} I(V_k(I)) &:= \{f \in k[X_1, \ldots, X_n] \mid p \in V_k(I) \implies f(p) = 0\} \\ &= I(V_k(I)) \cap k[X_1, \ldots, X_n]\end{aligned}$$
$V_k(I)$ is an **affine variety**	if $I(V_{\bar{k}}(I))$ is a **prime ideal** in $\bar{k}[X_1, \ldots, X_n]$
affine coordinate ring of an affine variety $V = V_k(I)$	the *integral domain* $$k[V] := \frac{k[X_1, \ldots, X_n]}{I(V)}$$
function field of an affine variety V	field of fraction of affine coordinate ring $$k(V)$$
	$$\bar{k}[V] := \frac{\bar{k}[X_1, \ldots, X_n]}{I(V_{\bar{k}}(I))}$$

the evaluation map on affine coordinate ring	$$\mathrm{ev}_p : \bar{k}[V] \to \bar{k}$$ which is surjective
dimension of variety	*transcendence degree* of $\bar{k}(V)$ over $\bar{k}$
maximal ideal of coordinate functions vanishing at a point	$$\mathfrak{m}_p(V) := \{f \in \bar{k}[V] \mid f(p) = 0\} = \ker \mathrm{ev}_p$$ which is a maximal ideal
cotangent space at $p \in V$	the $\bar{k}$-vector space $$T_p^* V := \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2}$$
non-singular or smooth point $p \in V$	$$\mathrm{rank}\begin{bmatrix} \frac{\partial f}{\partial X_1} & \dots & \frac{\partial f}{\partial X_n} \end{bmatrix}_p = n - \dim(V)$$ $\iff$ $$\dim \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2} = \dim V$$
local ring of V at p of "germs of functions at p"	$$\bar{k}[V]_p := (\bar{k}[V])_{\mathfrak{m}_p}$$ whose elements are said to be *regular at p*

projective varieties

projective n-space	$$\bar{k}P^n := P(k^{n+1})$$
minimal field for $p \in \bar{k}P^n$	
homogeneous polynomial	
projective algebraic set	$$V(I)$$
projective variety	if $$I(V)$$ is a prime ideal in $\bar{k}[X]$

dimension	Choose $\boldsymbol{A}^n \subset P^n$ such that $V \cap \boldsymbol{A}^n \neq \emptyset$, then $$\dim V := \dim V \cap \boldsymbol{A}^n$$
non-singular/smooth point $p \in V$	if $p \in V \cap \boldsymbol{A}^n$ is non-singular at p
local ring of V at p	$$\bar{k}[V]_p$$ is the local ring of $V \cap \boldsymbol{A}^n$ at p
rational $$f : V_1 \to V_2$$ between projective varieties $V_1, V_2 \subset P^n$	$$f = (f_0, \ldots, f_n) \in \bar{k}(V_1)^n$$ such that for every point $p \in V_1$ where all $f_0, \ldots, f_1$ are defined $$(f_0(p), \ldots f_n(p)) \in V_2$$
rational is *regular* at $p \in V_1$	if there is a $g \in \bar{k}(V_1)$ such that gf_i are well defined at p and $(gf_i)(p) \neq 0$ for some p
morphism of projective varieties	a *rational* map $f : V_1 \to V_2$ which is *regular* at every point

curves

Definition. **A projective curve is a projective variety of dimension one.**

Proposition: Let C be a curve and $p \in C$ be a smooth point. Then

$$\bar{k}[C]_p$$

is a DVR.

Definition. **Order of functions on curves**

Let C be a curve and $p \in C$ be a smooth point. The **normalized valuation** on $\bar{k}[C]_p$ is

$$\begin{aligned}\operatorname{ord}_p : \bar{k}[C]_p &\to \mathbb{N} \cup \{\infty\} \\ \operatorname{ord}_p(f) &:= \sup\{d \in \mathbb{Z} \mid f \in \mathfrak{m}_p^d\}\end{aligned}$$

and we extend this map to

$$\mathrm{ord}_p : \bar{k}(C) \to \mathbb{Z} \cup \{\infty\}$$

by $\mathrm{ord}_p(f/g) := \mathrm{ord}_p(f) - \mathrm{ord}_p(g)$. The **order** of $f \in \bar{k}(C)$ at p is $\mathrm{ord}_p(f)$.

A **uniformizer** for C at p is any $t \in \bar{k}(C)$ with $\mathrm{ord}_p(t) = 1$, that is, a generator for ideal $\mathfrak{m}_p$.

Proposition:

- If $\mathrm{ord}_p(f) > 0$ then f has a zero at p.
- If $\mathrm{ord}_p(f) \geq 0$ then f is *regular* (or *defined*) at p, and we may evaluate $f(p)$.
- If $\mathrm{ord}_p(f) < 0$ then f as a pole at p and $f(p) = \infty$.

Proposition: Let C be a smooth curve and $f \in \bar{k}(C)$ with $f \neq 0$. Then

- there are only finitely many points at which f has a pole or zero
- if f has no poles then $f \in \bar{k}$

Proposition: Let C/k be a curve and $t \in k(C)$ be a uniformizer at some non-singular point $p \in C/k$. Then

$$k(C) \geq k(t)$$

is a finite separable extension.

intersection multiplicities	Let $p \in \boldsymbol{A}^2(k)$ and two curves F, G then $$\mu_p(F,G) := \dim \frac{k(\boldsymbol{A}^2)}{\langle F, G\rangle}$$

schemes

commutative ring with 1_R	R
affine scheme	**Definition. Spectrum of a commutative ring** The set of all prime ideals of a ring is called **spectrum** of a commutative ring written $$\mathrm{Spec}R$$ The set of all maximal ideals are written $\mathrm{maxSpec}R$. The subsets of the form $$Z(I) := \{\mathfrak{p} \supset I \text{ is a prime ideal of } R\}$$ are called **Zariski-closed subsets** of $\mathrm{Spec}R$ which define a topology on the spectrum of R.
	$$\mathscr{O}_\mathfrak{p} := \text{field of fractions of } \frac{R}{\mathfrak{p}}$$
evaluating $f \in R$ at $\mathfrak{p}$	$$f(\mathfrak{p}) \in \mathscr{O}_\mathfrak{p}$$ that is image under $$R \to \frac{R}{\mathfrak{p}} \to \mathscr{O}_\mathfrak{p}$$
zero locus of $S \subset R$	$$Z(S) = \{\mathfrak{p} \in \mathrm{Spec}R \mid f \in S \implies f(\mathfrak{p}) = 0\}$$

vector bundles on schemes

Let X be a scheme.	
locally free sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules of **rank** r	if there is a open cover $\{U_i\}$ of X such that $$\mathscr{F}_{U_i} \cong \mathscr{O}_{U_i}^{\oplus r}$$
vector bundle of rank r on X over a field k	a k-scheme E and a *morphism* $$\pi : F \to X$$ ...
differentials	$$\Omega_{R/S}$$

Fulton Harris-Reps

chapter 2

exercise 2.33

exercise 2.37

> if $G \to GL(V)$ is injective then any irrep of G is contained in some tensor power $V^{\otimes n}$

Use character theory.

Consider

$$a_n := \langle \chi_W, \chi_V^n \rangle$$

where both W, V are fixed. Then $f(z) := \sum_{n \geq 1} a_n z^n$ then

$$\begin{aligned} f(z) &= \frac{1}{|G|} \sum_{n \geq 1} \sum_{g \in G} \overline{\chi_W(g)} (\chi_V(g))^n z^n \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \sum_{n \geq 1} (\chi_V(g))^n z^n \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \left(\frac{\chi_V(g) z}{1 - \chi_V(g) z} \right) \end{aligned}$$

chapter 12

Representations of $\mathfrak{sl}(3, \mathbb{C})$. Part 1.

> As we will see, a number of the basic constructions need to be modified, or at least rethought. There are, however, two pieces of good news that should be borne in mind.
> Fulton Harris Representation theory, page 175

- Let $\mathfrak{h} \leq \mathfrak{sl}(3, \mathbb{C})$ of diagonal matrices. So

  $$\mathfrak{h} = \left\{ \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & -a_1 - a_2 \end{bmatrix} : a_1, a_2 \in \mathbb{C} \right\}$$

  which is of dim $= 2$.
-

- Every finite-dim rep of $\mathfrak{sl}(3, \mathbb{C})$ decomposes into

$$\bigoplus_{\alpha} V_{\alpha}$$

where every $H \in \mathfrak{h}$ is (simul) diagonalizable on V_α.

- $$[D, M]_{ij} = (D_i - D_j)M_{ij}$$

- E_{ij} has (i, j)-entry $= 1$ and rest 0. Then $[D, M]$ is a multiple of M for all D if and only if $M = cE_{ij}$.

$$[D, E_{ij}] = (D_i - D_j)E_{ij}$$

- We want

$$\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$$

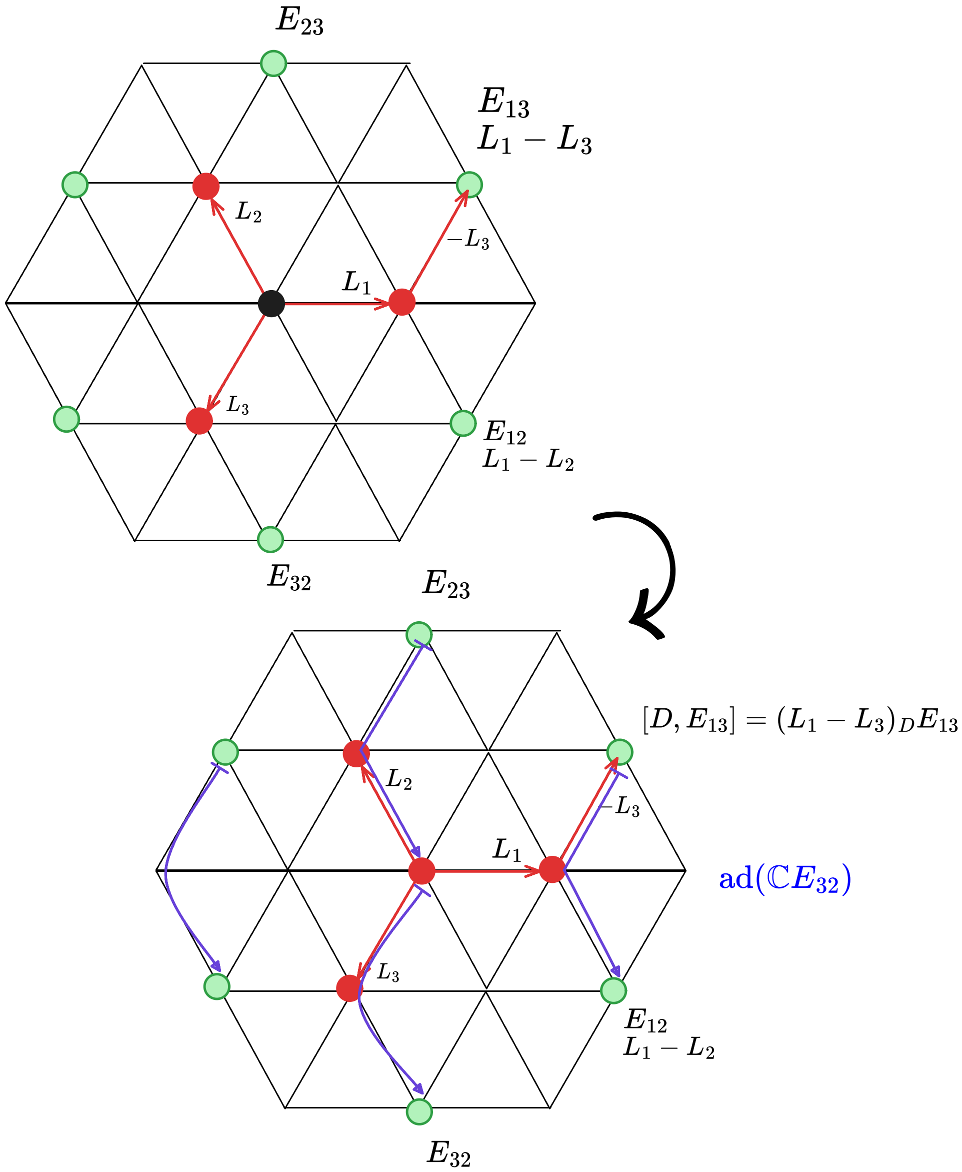
such that

$$[\underbrace{H}_{\in \mathfrak{h}}, Y] = \mathrm{ad}_H(Y) = \underbrace{\alpha}_{\in \mathfrak{h}^*}(H)Y$$

- But by previous calculations, the new subspaces needs to be $\mathbb{C}\{E_{ij} | i \neq j\}$ which is of dim = 9-3= 6.
- This gives the decomposition we wanted because 6+2 is the dimension of $2, \mathbb{C}$.
- $$\mathfrak{h}^* \cong \frac{\mathbb{C}\{L_1, L_2, L_3\}}{\{L_1 + L_2 + L_3 = 0\}}$$

where

$$L_i() = a_i$$


E_{23}
E_{13}
$L_1 - L_3$
L_2
$-L_3$
L_1
L_3
E_{12}
$L_1 - L_2$
E_{32}
E_{23}
$[D, E_{13}] = (L_1 - L_3)_D E_{13}$
L_2
$-L_3$
L_1
$\mathrm{ad}(\mathbb{C}E_{32})$
L_3
E_{12}
$L_1 - L_2$
E_{32}

$\mathfrak{sl}(2, \mathbb{C})$

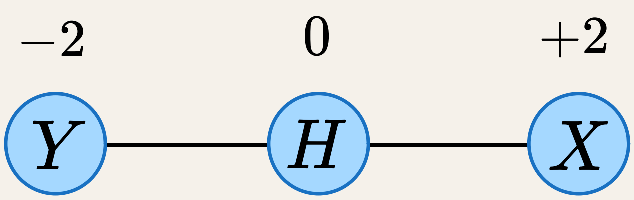
The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is the set of all **2 x 2 complex matrices with trace = 0**. This space is spanned by

$$H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{(traceless diagonal)}$$
$$X := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (= E_{12})$$
$$Y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (= E_{21})$$

with

$$[H, X] = 2X$$
$$[H, Y] = -2Y$$
$$[X, Y] = H$$

whose structure is understood by the diagram



Thus we have

$$\mathfrak{sl}_2\mathbb{C} \cong_{\mathsf{Alg}} \mathbb{C}\left\langle H, X, Y \,\middle|\, \begin{matrix} [H, X] = 2X \\ [H, Y] = -2Y \\ [X, Y] = H \end{matrix} \right\rangle$$

For each $n \in \mathbb{N}$ there exists an unique $n+1$ dimensional $\mathrm{Vec}_{\mathbb{C}}$ irreducible representation

$$\mathfrak{sl}(2, \mathbb{C}) \to \mathrm{EndVec}_{\mathbb{C}}(Sym^n\mathbb{C})$$

which is the n-th symmetric power of the standard representation on $\mathbb{C}^2$.

space.R.SL C.2.rep.irr FinVecC

$\mathsf{FinVec}_{\mathbb{C}}$ irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ and $SL_{\mathbb{C}}(2)$

classification of $\mathsf{FinVec}_{\mathbb{C}}$ irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$

We want all representations

$$\underbrace{\mathfrak{sl}(2,\mathbb{C})}_{\cong_{\mathsf{Alg}}\mathbb{C}\langle H,X,Y|[H,X]=2X,[H,Y]=-2Y,[X,Y]=H\rangle} \to \mathbf{End}\mathsf{FinVec}_{\mathbb{C}}(V)$$

up to isomorphism.

- Start with an arbitrary finite dim irrep V. How does H, X, Y act on V?
  - The adjoint action by H

    $$H \overset{\text{ad}}{\curvearrowright} \mathfrak{sl}(2,\mathbb{C})$$

    is diagonal. By *preservation of Jordan decomposition* under the representation

    $$\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$$

    the action $H \curvearrowright V$ must also be diagonal: hence we have a decomposition

    $$V = \bigoplus V_\alpha$$

    where V_α is the subspace such that

    $$v \in V_\alpha \implies H(v) = \alpha v$$

  - If $v \in V_\alpha$,

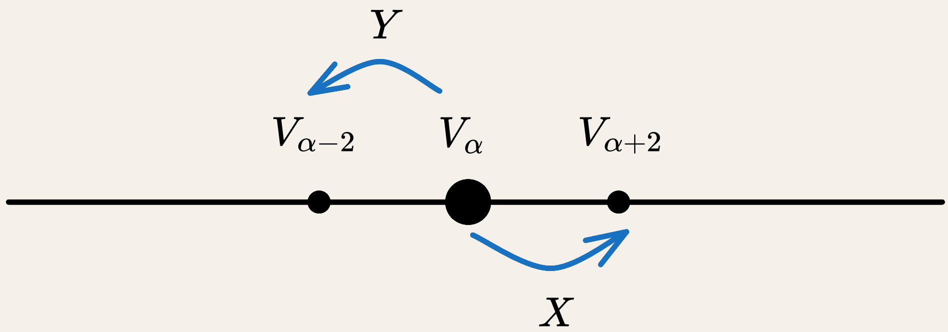
    $$\begin{aligned} H(X(v)) &= X(H(v)) + [H,X](v) \\ &= X(\alpha v) + 2X(v) \\ &= (\alpha+2)X(v) \end{aligned}$$

    Hence, $X : V_\alpha \to V_{\alpha+2}$.

- If $v \in V_\alpha$,

$$\begin{aligned} H(Y(v)) &= Y(H(v)) + [H, Y](v) \\ &= Y(\alpha v) - 2Y(v) \\ &= (\alpha - 2)Y(v) \end{aligned}$$

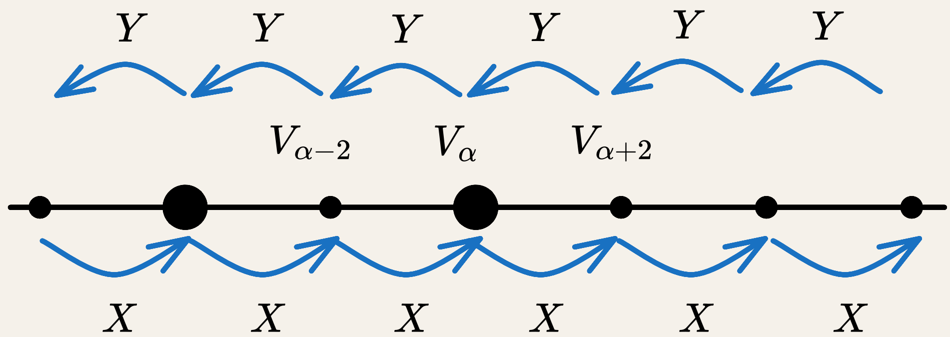
  Hence, $Y : V_\alpha \to V_{\alpha-2}$.
- So,



- But

$$\sum_{n \in \mathbb{Z}} V_{\alpha_0 + 2n}$$

is invariant under X, Y, H

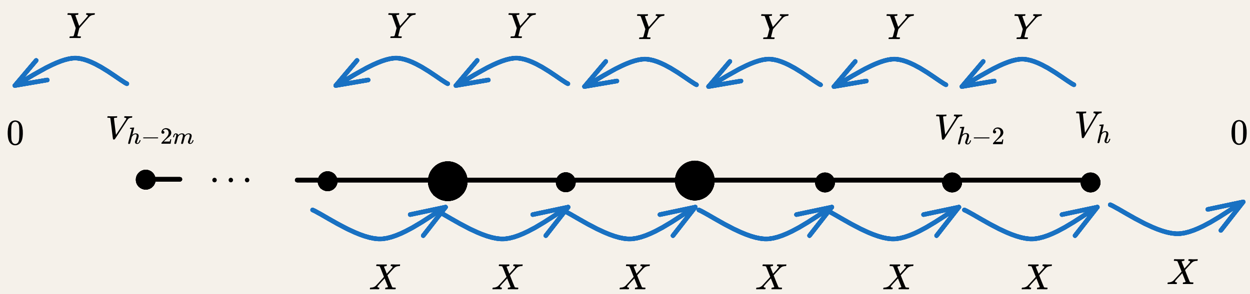


implies this subspace is invariant under the $\mathfrak{sl}_2\mathbb{C}$ action. So, irreducibility implies $V = \bigoplus V_{\alpha_0+2n}$.

- Because V is finite dim, V_α that appear must satisfy

$$\alpha \in \{h, h-2 \ldots, h-2m\}$$

  for some $h \in \mathbb{C}$ ("highest" eigenvalue for H) and $m \in \mathbb{N}$ (the "size" of the representation) where the rest of the spaces are 0.
- Thus we have



- Pick $0 \neq v \in V_n$ and then we get the subspace

$$W := \mathbb{C}\{v, Y(v), \ldots, Y^m(v)\} \leq V$$

  - Y preserves W by definition.
  - We know $Y^k(v) \in V_{h-2k}$, so H acts by

$$H(Y^k(v)) = (h-2k)Y^k(v)$$

  - So H preserves W.
  - Now $X(v) = 0$ because v belongs to the space with last V_α.
    - **Base case:**

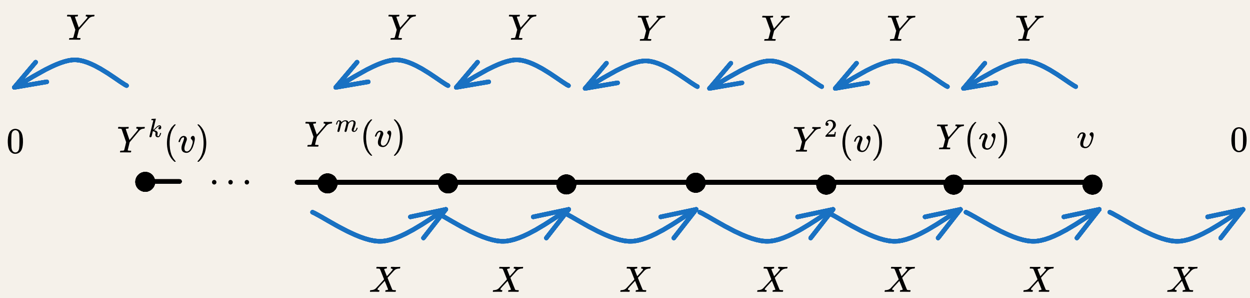
$$\begin{aligned} X(Y(v)) &= [X, Y](v) + Y(X(v)) \\ &= H(v) + Y(0) \\ &= hv \end{aligned}$$

    - **Induction hypothesis:** We assume

$$X(Y^k(v)) = k(h-k+1)Y^{k-1}(v)$$

    - **Induction step:** We obtain

$$\begin{aligned} XY^{k+1}(v) = XY(Y^k(v)) &= [X, Y](Y^k v) + YXY^k v \\ &= HY^k(v) + Y(k(h-k+1)Y^{k-1}(v)) \\ &= (h-2m)Y^k(v) + k(h-k+1)Y^k(v) \\ &= h + k(h-k-1)Y^k(v) \\ &= (k+1)(h-k)Y^k(v) \end{aligned}$$

    - Hence, $X(Y^k(v)) = k(h-k+1)Y^{k-1}(v)$ is true for all $k \geq 1$.
  - So, X preserves W.
  - 

- Hence, *by irreducibility*, we must have $V = W$. Hence, $\{v, Y(v), \ldots, Y^k(v)\}$ is a basis of V.
  - $V_{n-2m} = \mathbb{C}Y^m(v)$ (one dimensional)
  - Because $Y^{m+1}(v) = 0$ we have

$$0 = XY^{m+1}(v) = (m+1)(h-m)Y^k(v)$$

but $Y^k(v) \neq 0$ by definition, so

$$h - m = 0 \implies h = m$$

and thus this implies the highest eigenvalue $h \in \mathbb{N}$ is a natural number and equals to the size of the representation.

- Hence, for any such irreducible representation, we have a $n \in \mathbb{N}$ such that the eigenvalues of H are

$$\{\underbrace{n}_{h}, n-2, \ldots, -n+2, -n\}$$

and dimension of the representation is $n+1$. *This does now show existence, yet.*

- **Existence:** For each $n \in \mathbb{N}$ there exists a finite dim $\mathbf{Vec}_{\mathbb{C}}$ irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$:
  - $n = 0$ is the trivial representation
  - $n = 1$ the standard representation

$$\mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}\mathbf{Vec}_{\mathbb{C}}(\underbrace{\mathbb{C}^2}_{\mathbb{C}\{x,y\}})$$

where $Hx = x, Hy = -y$

  - and for any $n \in \mathbb{N}$ we have the representation

$$\mathfrak{sl}(2, \mathbb{C}) \to \mathrm{End}\mathbf{Vec}_{\mathbb{C}}(\underbrace{Sym^n\mathbb{C}^2}_{\mathbb{C}\langle x^n, x^{n-1}y, \ldots, y^n\rangle})$$

such that $H(x^{n-k}y^k) = (n-2k)x^{n-k}y^k$

decomposing tensor product of irreps

geometric side

The automorphism group of $\mathbb{C}P^n$ - either as a algebraic variety or as a complex manifold - is the group

$$PGL_{\mathbb{C}}(n+1)$$

Correspondences

For any vector space W of dim $n+1$,

$$\boldsymbol{Sym}^k W^* \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{\boldsymbol{P}(W)}_{= \text{space of lines in } W}\}$$

Dually,

$$\begin{aligned}\boldsymbol{Sym}^k W \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{\boldsymbol{P}(W^*)}\} \\ = \text{space of lines in } W^* \\ = \text{space of hyperplanes in } W\end{aligned}$$

where the last correspondence is

$$\begin{aligned}\text{lines in } W^* &\leftrightarrow \text{hyperplanes in } W \\ [\phi] &\leftrightarrow \ker \phi\end{aligned}$$

Now because zero sets of polynomials are *hypersurfaces*(?)

$$P(\boldsymbol{Sym}^k W) \leftrightarrow \{\text{hypersurfaces of deg } k \text{ in } P(W^*)\}$$

so to derive results about $\boldsymbol{Sym}^k W$ we must work with $P(W^*)$.

Definition. Veronese embedding

For any vector space V we have the *Veronese embedding*

$$\begin{aligned}P(V^*) &\hookrightarrow P(\boldsymbol{Sym}^k V^*) \\ [v] &\mapsto [v^n]\end{aligned}$$

For $\dim(V) = 2$, $P(V^*) \cong \mathbb{C}P^1$, $\boldsymbol{Sym}^n V^*$ is of dim $n+1$, so it is $P(\boldsymbol{Sym}^n V^*) \cong \mathbb{C}P^n$ naturally and we have

$$\begin{aligned}\iota_n : \mathbb{C}P^1 &\hookrightarrow \mathbb{C}P^n \\ x\hat{x} + y\hat{y} &\mapsto (x\hat{x} + y\hat{y})^n \\ [x, y] &\mapsto [x^n, x^{n-1}y, x^{n-2}y^2, \ldots, xy^{n-1}, y^n]\end{aligned}$$

whose image is called *rational normal curve of degree n*.

Now we have the representation

$$\begin{aligned}SL_{\mathbb{C}}(2) &\curvearrowright \boldsymbol{Sym}^n V^* \cong \mathbb{C}^{n+1} \\ \hat{x}^k \hat{y}^{n-k} &\overset{S}{\mapsto} (S\hat{x})^k (S\hat{y})^{n-k}\end{aligned}$$

which quotients down to

$$\begin{aligned} SL_{\mathbb{C}}(2) &\curvearrowright \boldsymbol{PSym}^n V^* \cong \mathbb{C}P^n \\ \hat{x}^k \hat{y}^{n-k} &\overset{S}{\mapsto} (S\hat{x})^k (S\hat{y})^{n-k} \end{aligned}$$

the exact action on $\mathbb{C}P^n$ that preserves the *rational normal curve*.

weight character of a $\mathfrak{sl}(2,\mathbb{C})$ representation

Definition. **Weight character of a $\mathfrak{sl}(2,\mathbb{C})$ representation**

Because any $\mathsf{FinVec}_{\mathbb{C}}$ representation of $\mathfrak{sl}(2,\mathbb{C})$ is reducible, we may decompose it into its irreducible pieces

$$V \cong \bigoplus_{a\in\mathbb{N}} \boldsymbol{Sym}^a(\mathbb{C}^2)^{\otimes d_a}$$

and further, the irreps into its weight subspaces, that is, the eigenvalue of H which is brought all together as

$$V = \sum_{n\in\mathbb{Z}} \ker(H - n\mathsf{Id})$$

We define the **weight character** of this representation as the $\mathbb{Z}[t,t^{-1}]$ polynomial

$$\eta(V) := \sum_{n\in\mathbb{Z}} \dim\ker(H - n\mathsf{Id}_V)t^n$$

$$\eta(V \oplus W) = \eta(V) + \eta(W)$$

$$V \cong_{\mathfrak{sl}(2,\mathbb{C})} W \iff \eta(V) = \eta(W)$$

- It is clear that $V \cong W \implies \eta(V) = \eta(W)$.
- Let $\eta(V) = \eta(W)$ then in particular their dimensions are equal because they have same weight space decomposition. Say their dimension is n.
  - The case $n = 0$ is vacuously true.
  - Consider $n > 0$ and let $\lambda \in \mathbb{Z}$ is the highest weight of both V and W. Then we may decompose

$$
\begin{aligned}
V &= V' + V_\lambda \\
W &= W' + W_\lambda
\end{aligned}
$$

where V_λ, W_λ are irr representations with highest weight λ. Now

$$\eta(V) = \eta(V') + \eta(V_\lambda) = \eta(W) = \eta(W') + \eta(W_\lambda)$$

so we have $\eta(W') = \eta(V')$.

- The *induction hypothesis* on n yields that V' and W' are isomorphic as representations.
- Thus V and W are isomorphic as representations.

We know the weight character of the irreducible reps:

$$
\begin{aligned}
\eta(\boldsymbol{Sym}^n(\mathbb{C}^2)) &= t^{-n} + t^{-n+2} + \cdots + t^{n-2} + t^n \\
&= \frac{t^{n+2} - t^{-n}}{t^2 - 1}
\end{aligned}
$$

But the main use of the ring of polynomials is here:

$$\eta(V \otimes W) = \eta(V)\eta(W)$$

$$
\begin{aligned}
H(v \otimes w) &= Hv \otimes w + v \otimes Hw \\
&= (n+m) v \otimes w
\end{aligned}
$$

Hence,

$$\ker_{V\otimes W}(H - k\mathsf{Id}) = \bigoplus_{n+m=k} \ker_V(H - n\mathsf{Id}) \otimes \ker_W(H - m\mathsf{Id})$$

Thus

$$
\begin{aligned}
\eta(V \otimes W) &= \sum_{k\in\mathbb{Z}} \dim \ker_{V\otimes W}(H - k\mathsf{Id}) t^k \\
&= \sum_{n\in\mathbb{Z}} \sum_{n+m=k} \Big(\dim \ker_V(H - n\mathsf{Id})\Big)\Big(\dim \ker_V(H - m\mathsf{Id})\Big) t^k \\
&= \left(\sum_{n\in\mathbb{Z}} \dim \ker_V(H - n\mathsf{Id}) t^n\right)\left(\sum_{n\in\mathbb{Z}} \dim \ker_V(H - m\mathsf{Id}) t^m\right) \\
&= \eta(V)\eta(W)
\end{aligned}
$$

which Laurent polynomials are achieved by representations?

$$\sum_{n\in\mathbb{N}} a_n(t^{-n}+t^{-n+2}+\cdots+t^{n-2}+t^n) = \sum_{i\in\mathbb{N}}\sum_{j} a_j(t^{-i}+t^i)$$

decompositions of tensor powers of irreps into sums of irreps

- $2\otimes 3 = 5\oplus 3\oplus 1$

The tensor product

$\boldsymbol{Sym}^a(\mathbb{C}^2)\otimes\boldsymbol{Sym}^b(\mathbb{C}^2)$	$-a$	$-a+2$	$\dots$	$a-2$	a
$-b$	$-a-b$	$-a-b+2$		$a-b-2$	$a-b$
$-b+2$	$-a-b+2$	$-a-b+4$		$a-b$	$a-b+2$
$\dots$			$\dots$		
$b-2$	$b-a-2$	$b-a$		$a+b-4$	$a+b-2$
b	$b-a$	$b-a+2$		$a+b-2$	$a+b$

gives the weight decomposition from top-left to bottom right

	$-a-b$	$-a-b+2$	
	1	2	

$$\boldsymbol{Sym}^a(\mathbb{C}^2)\otimes\boldsymbol{Sym}^b(\mathbb{C}^2)\cong_{SL_{\mathbb{C}}(2)}\bigoplus_{i=0}^{b}\boldsymbol{Sym}^{a+b-2i}(\mathbb{C}^2)$$

From

$$\eta(\boldsymbol{Sym}^n(\mathbb{C}^2)) = t^{-n}+t^{-n+2}+\cdots+t^{n-2}+t^n = \frac{t^{n+2}-t^{-n}}{t^2-1}$$

the weight character of the right side is

$$\begin{aligned}\sum_{i=0}^{b}\frac{t^{a+b-2i+2}-t^{-(a+b-2i)}}{t^2-1} &= \frac{\sum_{i=0}^{b}t^{a+b-2i+2}-\sum_{i=0}^{b}t^{-(a+b-2i)}}{t^2-1}\\ &= \frac{t^{a+2}\sum_{i=0}^{b}t^{b-2i}-t^{-a}\sum_{i=0}^{b}t^{-(b-2i)}}{t^2-1}\\ &= \frac{t^{a+2}-t^{-a}}{t^2-1}(t^n+t^{n-2}+\cdots+t^{-n})\\ &= \eta(\boldsymbol{Sym}^a(\mathbb{C}^2))\eta(\boldsymbol{Sym}^b(\mathbb{C}^2))\end{aligned}$$

So by

$$V \cong_{\mathfrak{sl}(2,\mathbb{C})} W \iff \eta(V)=\eta(W)$$

- It is clear that $V \cong W \implies \eta(V)=\eta(W)$.
- Let $\eta(V)=\eta(W)$ then in particular their dimensions are equal because they have same weight space decomposition. Say their dimension is n.
  - The case $n=0$ is vacuously true.
  - Consider $n>0$ and let $\lambda\in\mathbb{Z}$ is the highest weight of both V and W. Then we may decompose

    $$\begin{aligned}V&=V'+V_\lambda\\ W&=W'+W_\lambda\end{aligned}$$

    where V_λ, W_λ are irr representations with highest weight λ. Now

    $$\eta(V)=\eta(V')+\eta(V_\lambda)=\eta(W)=\eta(W')+\eta(W_\lambda)$$

    so we have $\eta(W')=\eta(V')$.
  - The *induction hypothesis* on n yields that V' and W' are isomorphic as representations.
  - Thus V and W are isomorphic as representations.

and

$$\eta(V\otimes W)=\eta(V)\eta(W)$$

$$\begin{aligned}H(v\otimes w)&=Hv\otimes w+v\otimes Hw\\ &=(n+m)v\otimes w\end{aligned}$$

Hence,

$$\ker_{V\otimes W}(H-k\mathsf{Id}) = \bigoplus_{n+m=k} \ker_V(H-n\mathsf{Id})\otimes \ker_W(H-m\mathsf{Id})$$

Thus

$$\begin{aligned}\eta(V\otimes W) &= \sum_{k\in\mathbb{Z}} \dim\ker_{V\otimes W}(H-k\mathsf{Id})t^k \\ &= \sum_{n\in\mathbb{Z}}\sum_{n+m=k}\Big(\dim\ker_V(H-n\mathsf{Id})\Big)\Big(\dim\ker_V(H-m\mathsf{Id})\Big)t^k \\ &= \left(\sum_{n\in\mathbb{Z}}\dim\ker_V(H-n\mathsf{Id})t^n\right)\left(\sum_{n\in\mathbb{Z}}\dim\ker_V(H-m\mathsf{Id})t^m\right) \\ &= \eta(V)\eta(W)\end{aligned}$$

the statement follows.

decompositions of symmetric powers of irreps into sums of irreps

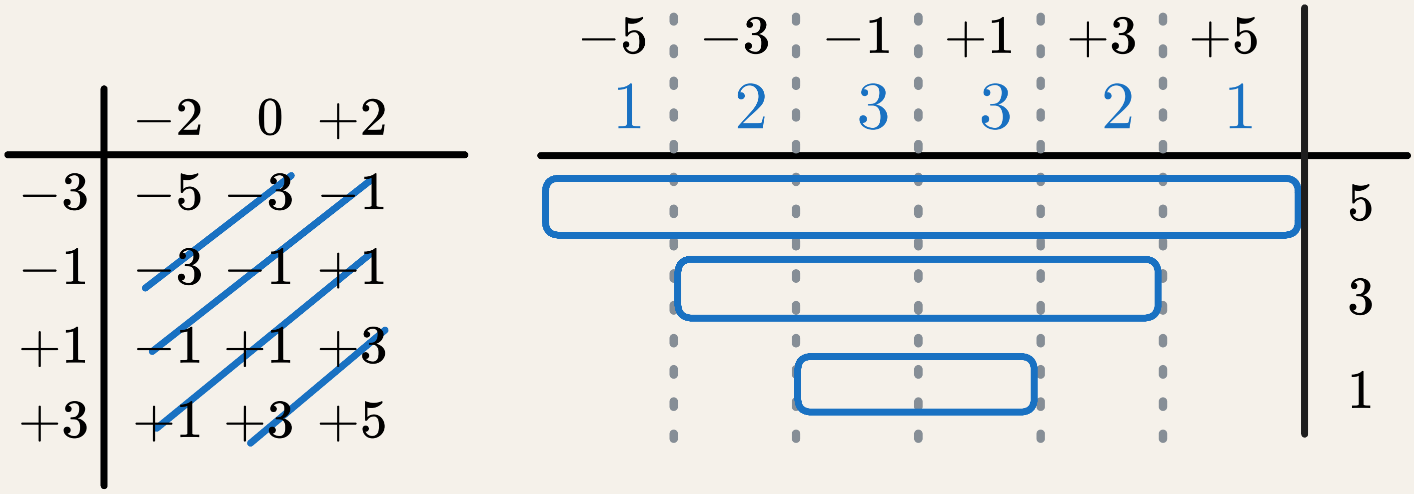
- $\boldsymbol{Sym}^2\boldsymbol{Sym}^2(\mathbb{C}^2) \cong_{SL_{\mathbb{C}}(2)} \boldsymbol{Sym}^0(\mathbb{C}^2)\oplus\boldsymbol{Sym}^4(\mathbb{C}^2)$
-

space.R.SL C.2.rep.irr FinVecC.2 x 3 =5 + 3 + 1

$$\mathbf{2}\otimes\mathbf{3}\cong_{\mathfrak{sl}(2,\mathbb{C})}\mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}$$

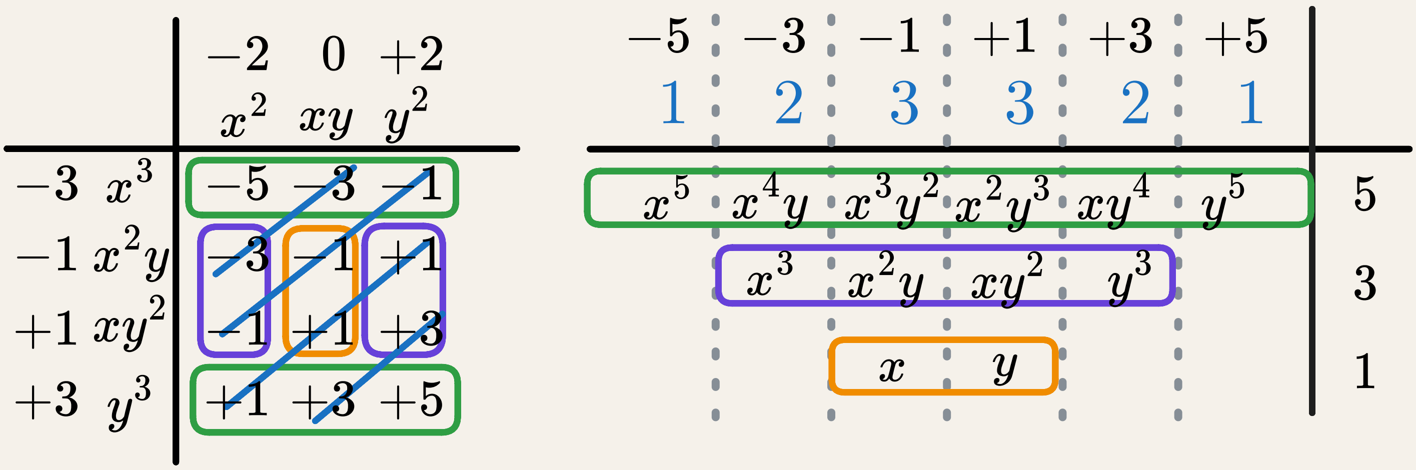
The tensor product

$$\boldsymbol{Sym}^2(\mathbb{C}^2)\otimes\boldsymbol{Sym}^3(\mathbb{C}^2)$$



decomposes as

$$\boldsymbol{Sym}^5(\mathbb{C}^2) \oplus \boldsymbol{Sym}^3(\mathbb{C}^2) \oplus \boldsymbol{Sym}^1(\mathbb{C}^2)$$



where we choose the projection

$$\begin{array}{ccccccc} \boldsymbol{Sym}^2(\mathbb{C}^2) \otimes \boldsymbol{Sym}^3(\mathbb{C}^2) & \to & \boldsymbol{Sym}^5(\mathbb{C}^2) & \oplus & \boldsymbol{Sym}^3(\mathbb{C}^2) & \oplus & \boldsymbol{Sym}^1(\mathbb{C}^2) \\ |x^2, x^3\rangle & \mapsto & |x^5\rangle & & & & \end{array}$$

- the weight character

$$\begin{aligned} &(t^{-2}+1+t^2)(t^{-3}+t^{-1}+t+t^3) \\ &= t^{-5}+2t^{-3}+3t^{-1}+3t+2t^3+t^5 \end{aligned}$$

- the quotient projection

$$\boldsymbol{Sym}^2(\mathbb{C}^2) \otimes \boldsymbol{Sym}^3(\mathbb{C}^2) \to \boldsymbol{Sym}^5(\mathbb{C}^2)$$

  is multiplication of polynomials but carefully

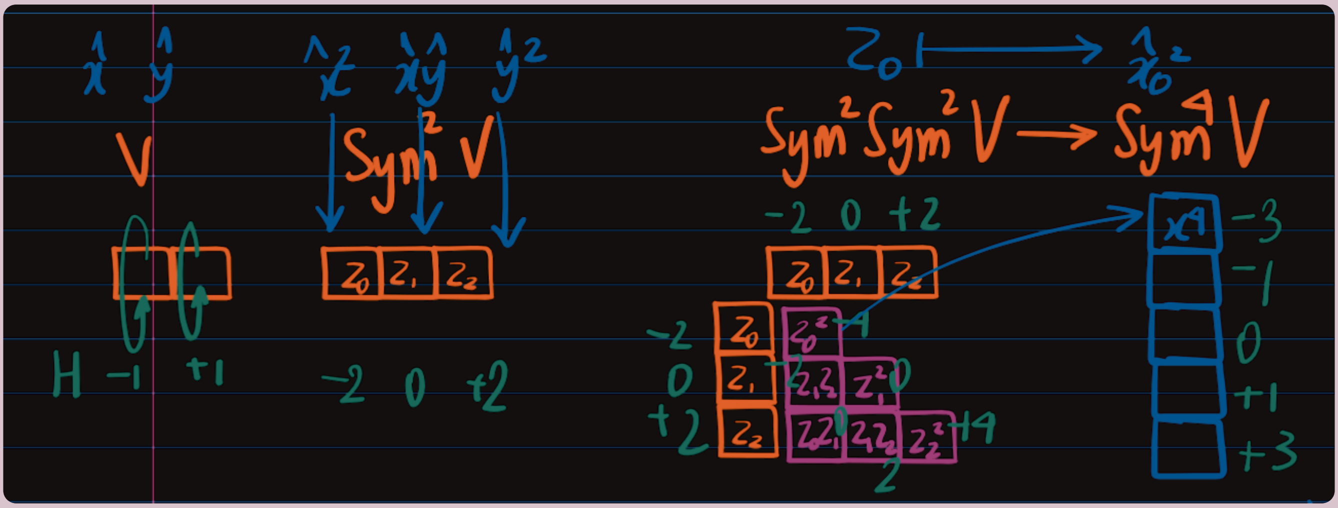
-

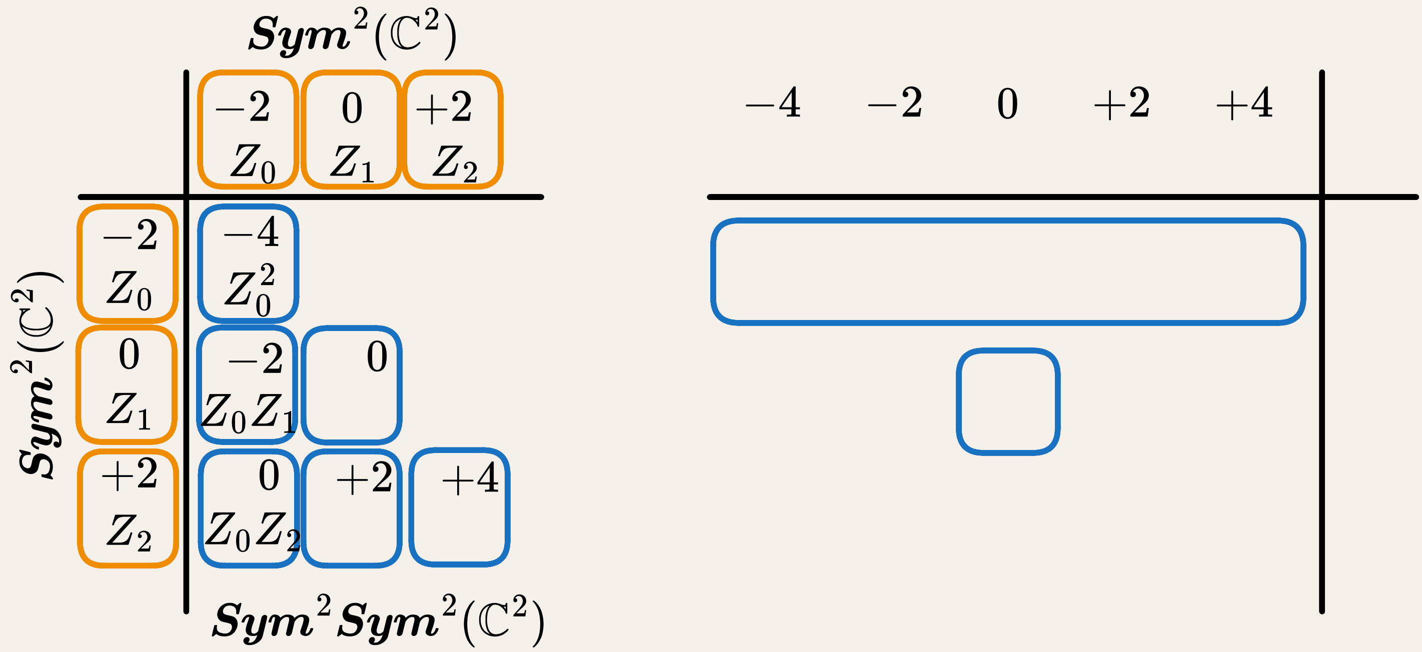
space.R.SL C.2.rep.irr FinVecC.Sym2 of 2 = 0 + 4

$$\boldsymbol{Sym}^2(2) \cong_{\mathfrak{sl}(2,\mathbb{C})} 0 \oplus 4$$

$\boldsymbol{Sym}^2\boldsymbol{Sym}^2(\mathbb{C}^2) \cong_{\mathfrak{sl}(2,\mathbb{C})} \boldsymbol{Sym}^0(\mathbb{C}^2) \oplus \boldsymbol{Sym}^4(\mathbb{C}^2)$

using weights





decomposition as homogeneous polynomials

Correspondences

For any vector space W of dim $n+1$,

$$\boldsymbol{Sym}^k W^* \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{\boldsymbol{P}(W)}\}$$
$$= \text{space of lines in } W$$

Dually,

$$\boldsymbol{Sym}^k W \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{\boldsymbol{P}(W^*)}\}$$
$$= \text{space of lines in } W^*$$
$$= \text{space of hyperplanes in } W$$

where the last correspondence is

$$\text{lines in } W^* \leftrightarrow \text{hyperplanes in } W$$
$$[\phi] \leftrightarrow \ker \phi$$

Now because zero sets of polynomials are *hypersurfaces*(?)

$$P(\boldsymbol{Sym}^k W) \leftrightarrow \{\text{hypersurfaces of deg } k \text{ in } P(W^*)\}$$

so to derive results about $\boldsymbol{Sym}^k W$ we must work with $P(W^*)$.

We understand

- $$\boldsymbol{Sym}^2 \boldsymbol{Sym}^2 V = \{\text{homg deg 2 polynomials on } \boldsymbol{P}(\boldsymbol{Sym}^2 V)^*\}$$
- for $V = \mathbb{C}^2 = \mathbb{C}\{x, y\}$, we have
$$\boldsymbol{Sym}^2 \mathbb{C}\{x, y\} \cong \mathbb{C}\{x^2, xy, y^2\} \cong \mathbb{C}\{Z_0, Z_1, Z_2\}$$
$$x^2 \mapsto Z_0$$
$$xy \mapsto Z_1$$
$$y^2 \mapsto Z_2$$
- which gives
$$\boldsymbol{Sym}^2 \boldsymbol{Sym}^2 \mathbb{C}^2 = \underbrace{\mathbb{C}\begin{Bmatrix} Z_0^2, Z_1^2, Z_2^2, \\ Z_0 Z_1, Z_1 Z_2, Z_2 Z_0 \end{Bmatrix}}_{\text{homg polynomials on } \boldsymbol{P}\mathbb{C}\{Z_0, Z_1, Z_2\} \cong \mathbb{C}P^2}$$
- thus using

- for $V = \mathbb{C}^2 = \mathbb{C}\{x, y\}$, we have

$$\boldsymbol{Sym}^2\mathbb{C}\{x,y\} \cong \mathbb{C}\{x^2, xy, y^2\} \cong \mathbb{C}\{Z_0, Z_1, Z_2\}$$
$$x^2 \mapsto Z_0$$
$$xy \mapsto Z_1$$
$$y^2 \mapsto Z_2$$

we have the evaluation map

$$\underbrace{\boldsymbol{Sym}^2\boldsymbol{Sym}^2\mathbb{C}^2}_{} \to \underbrace{\boldsymbol{Sym}^4\mathbb{C}^2}_{}$$

$$\mathbb{C}\left\{\begin{matrix} Z_0^2, Z_1^2, Z_2^2, \\ Z_0Z_1, Z_1Z_2, Z_2Z_0 \end{matrix}\right\} \to \mathbb{C}\left\{\begin{matrix} x^4 \\ x^3y, x^2y^2, xy^3 \\ y^4 \end{matrix}\right\}$$
$$Z_0^2 \mapsto x^4$$
$$Z_1^2 \mapsto {\color{blue}x^2y^2}$$
$$Z_2^2 \mapsto y^4$$
$$Z_0Z_1 \mapsto x^3y$$
$$Z_1Z_2 \mapsto xy^3$$
$$Z_2Z_0 \mapsto {\color{blue}x^2y^2}$$

- this map surjective because the chosen basis is contained in the image
- and it is easily seen

$$Z_2Z_0 - Z_1^2 \mapsto 0$$

  and this polynomial spans the kernel because the map is surjective and it goes from a dim 6 to a dim 5 space
- So we get an **exact sequence**

$$0 \to \mathbb{C}\{Z_2Z_0 - Z_1^2\} \hookrightarrow \boldsymbol{Sym}^2\boldsymbol{Sym}^2\mathbb{C}^2 \to \boldsymbol{Sym}^4\mathbb{C}^2 \to 0$$

- But what about the split $\boldsymbol{Sym}^4\mathbb{C}^2 \hookrightarrow \boldsymbol{Sym}^2\boldsymbol{Sym}^2\mathbb{C}^2$? We must define something like

$$\boldsymbol{Sym}^4\mathbb{C}^2 \hookrightarrow \boldsymbol{Sym}^2\boldsymbol{Sym}^2\mathbb{C}^2$$
$$x^4 \mapsto Z_0^2$$
$$x^3y \mapsto Z_0Z_1$$
$$x^2y^2 \mapsto \frac{1}{2}(Z_1^2 + Z_0Z_2)$$
$$xy^3 \mapsto Z_1Z_2$$
$$y^4 \mapsto Z_2^2$$

$\mathbb{C}P^2$ geometry behind the decomposition

For $n = 2$ we have the embedding

$$[x, y] \mapsto [x^2, xy, y^2]$$

whose image satisfies the equation

$$F(Z_0, Z_1, Z_2) := Z_0 Z_2 - (Z_1)^2 = 0$$

in $\mathbb{C}P^2$ with coordinates

$$[Z_0, Z_1, Z_2] := [x^2, xy, y^2]$$

whose zero set is called the *plane conic* (rational normal curve of degree 2).

The action

$$\begin{aligned} SL_{\mathbb{C}}(2) \quad &\curvearrowright P\boldsymbol{Sym}^2 V^* \cong \mathbb{C}P^2 \\ \hat{x}^k \hat{y}^{n-k} \quad &\overset{S}{\mapsto} (S\hat{x})^k (S\hat{y})^{n-k} \end{aligned}$$

is exactly the group of motions of $\mathbb{C}P^2$ that carry the plane conic c_2 onto itself

$$\begin{tikzcd}
P(V) \arrow[rrr, hook, "\dot{i}_2"] \arrow[ddd, "SL_{\mathbb{C}}(2)"'] & & & P(Sym^2 V) \arrow[ddd] \\
& {[v]} \arrow[r, mapsto] \arrow[d, mapsto, "s"'] & {[v^2]} \arrow[d, mapsto, "s"] & \\
& {[Sv]} \arrow[r, mapsto] & {[(Sv)^2]} & \\
P(V) \arrow[rrr, hook, "\dot{i}_2"'] & & & P(Sym^2 V)
\end{tikzcd}$$

Thus the action

$$SL_{\mathbb{C}}(2) \curvearrowright \boldsymbol{Sym}^2(\boldsymbol{Sym}^2(V)) \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } P\underbrace{(W^*)}_{\boldsymbol{Sym}^2(V)}\}$$

must preserve $\mathbb{C}\{F\}$ spanned the polynomial for c_2.

The split

$$\boldsymbol{Sym}^4(V) \to$$

Hence

$$0 \to \underbrace{\mathbb{C}\{F\}}_{\boldsymbol{Sym}^0(V)} \hookrightarrow \boldsymbol{Sym}^2(\boldsymbol{Sym}^2(V)) \to \boldsymbol{Sym}^4(V) \to 0$$

is exact as $SL_{\mathbb{C}}(2)$-modules, that splits giving us an isomorphism

$$\boldsymbol{Sym}^2(\boldsymbol{Sym}^2(V)) \cong_{SL_{\mathbb{C}}(2)} \boldsymbol{Sym}^4(V) \oplus \boldsymbol{Sym}^0(V)$$

Ad^* orbits are symplectic manifolds

Definition. Kirillov form on a coadjoint orbit of a matrix group

Let

$$\Omega \subset \mathfrak{g}^*$$

be a coadjoint orbit of a matrix group G. Let $F \in \Omega$.

- By definition Ω is a transitive G-space. Thus the surjective

  $$\begin{aligned} G &\to \Omega \\ g &\mapsto gFg^{-1} \end{aligned}$$

  descends to an isomorphism

  $$\frac{G}{\mathrm{stab}_G(F)} \cong \Omega$$

- The derivative of this isomorphism gives

  $$\frac{\mathfrak{g}}{\mathrm{Lie}(\mathrm{stab}_G(F))} \cong T_F\Omega$$

- We construct a 2-form

  $$\omega_F : T_F\Omega \times T_F\Omega \to \mathbb{R}$$

  by

  $$\begin{aligned} \omega_F : \frac{\mathfrak{g}}{\mathrm{Lie}(\mathrm{stab}_G(F))} \times \frac{\mathfrak{g}}{\mathrm{Lie}(\mathrm{stab}_G(F))} &\to \mathbb{R} \\ (X, Y) &\mapsto \langle F, [X, Y] \rangle \end{aligned}$$

[1]

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1. studenttheses.uu.nl/bitstream/handle/20.500.12932/7207/Maes%2CJMA2011.pdf?sequence=1&isAllowed=y#page=40.00 ↩

IMSC-Linear algebraic groups

> ...the first 4 lectures were missed, I suggest reading from the start of chapter 2, up to Proposition 2.2.5 of Springer's textbook 'Linear algebraic groups'. This is where he starts from (after an overview).

smooth dynamics

for real matrices

- For a real matrix acting on $\mathbb{R}^n$ we think of it has a matrix acting on $\mathbb{C}^n$

  $$\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$$

  use the decomposition

  - The vector space $\mathbb{C}^n$ decomposes into subspaces

    $$\mathbb{C}^n = \sum_{\lambda} \ker(A-\lambda)^{d_\lambda}$$

    where for each eigenvalue the generalized eigenspace

    $$\ker(A-\lambda)^{d_\lambda}$$

    (for large enough $d_\lambda \leq n$) is invariant under A.

  and restrict our attention to the $\mathbb{R}$-subspace $\mathbb{R}^n$.

If A is an invertible nxn $\mathbb{R}$-matrix then the linear autonomous system $\dot{y} = Ay$ has a critical point at origin in $\mathbb{R}^n$ which is

strictly stable if real parts of eigenvalues of A are negative,

stable if A has at least one pair of pure imaginary eigenvalues of multiplicity 1 and

unstable otherwise.

$$\mathbb{R}^n = \sum_{\lambda\in\mathbb{R}} \ker(A-\lambda)^{d_\lambda} + \sum_{\lambda\notin\mathbb{R}} \Re\ker(A-\lambda)^{d_\lambda} + \Re\ker(A-\bar{\lambda})^{d_\lambda}$$

$$\mathbb{R}^n = \bigoplus_{\lambda_j\in\mathbb{R}} \ker(A-\lambda_j I)^{d_j} \bigoplus_{\lambda_j\notin\mathbb{R}} \ker(A^2-|\lambda_j|^2 I)^{d_j}$$

Then the solution

$$\exp(tA)(y_0)$$

sett.Man.R.iterations

Iterations of a smooth map on a smooth manifold

Definition. Hyperbolic fixed points of a smooth manifold diffeomorphism

A *fixed point* p of a diffeomorphism $f : M \to M$ of a smooth manifold M is said to be **hyperbolic** if all (including complex) eigenvalues of $\mathrm{d}_p f : T_pM \to T_pM$ have absolute value different from 1.
For such a fixed point p, the tangent space splits into $\mathrm{d}_p f$-invariant subspaces called **stable** and **unstable** tangent subspaces

$$T_pM = E_p^{\mathsf{u}} \oplus E_p^{\mathsf{s}}$$

where $E_p^{\mathsf{u}}, E_p^{\mathsf{s}}$ are generalized subspaces with (complex) eigenvalues with real part more and less than 1 respectively.
Moreover, for such a fixed point p and its neighborhood U we have the **local stable and unstable subsets**

$$W^{\mathsf{s}}(p,U) = \{x \in U | \lim_{n\to\infty} f^n(x) = p\}$$
$$W^{\mathsf{u}}(p,U) = \{x \in U | \lim_{n\to\infty} f^{-n}(x) = p\}$$

The global stable and unstable subsets are $W^{\mathsf{s}}(p,M)$ and $W^{\mathsf{u}}(p,M)$.

(Discrete Hartman-Grobman theorem) If p is a *hyperbolic fixed point* of a diffeomorphism $f : M \to M$ on a smooth manfold M then f near p is topologically conjugate to $\mathrm{d}_p f : T_pM \to T_pM$ near $0 \in T_pM$.

of 0

This allows us to classify hyperbolic fixed points locally by classifying the same only for $\mathbb{R}$-linear maps.

Let $f : M \to M$ is a $\mathcal{C}^k$-diffeomorphism of a $\mathcal{C}^k$-manifold M and p is a hyperbolic fixed point of f. Then there exists a neignbourhood U of p such that the local *stable* and *unstable* subsets $W^{\mathsf{s}}(p,U)$ and $W^{\mathsf{u}}(p,U)$ are $\mathcal{C}^k$-embedded submanifolds of M with

$$T_pW^{\mathsf{s}}(p,U) = E_p^{\mathsf{s}},\ T_pW^{\mathsf{u}}(p,U) = E_p^{\mathsf{u}}$$

The global stable and unstable sets are manifolds, but *immersed* not embedded submanifolds in general.

space.R.n.Vec.fixed

Fixed points

Definition. **Fixed point**

For a flow on $\mathbb{R}^n$ $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}$ is a **fixed point** if $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.

[1]

	sign of real part of eigenvalues		stability
	all 0		

	sign of real part of eigenvalues		stability
hyperbolic	all > 0	source	unstable
hyperbolic	all < 0	sink	asymptotically stable
hyperbolic	mixed but $\neq 0$	saddle	unstable
	mixed with 0		

for $n = 2$

	sign of real part of eigenvalues		stability	complex parts are all 0, so $\lambda_1, \lambda_2 \in \mathbb{R}$	complex parts are $\neq 0$, so $\lambda \pm \mu$	
	both 0				stable **center**	
hyperbolic	both > 0	source	unstable		unstable **spiral**	
hyperbolic	both < 0	sink	asymptotically stable		stable **spiral**	
hyperbolic	mixed but $\neq 0$	saddle	unstable		(not possible)	
	mixed with 0				(not possible)	

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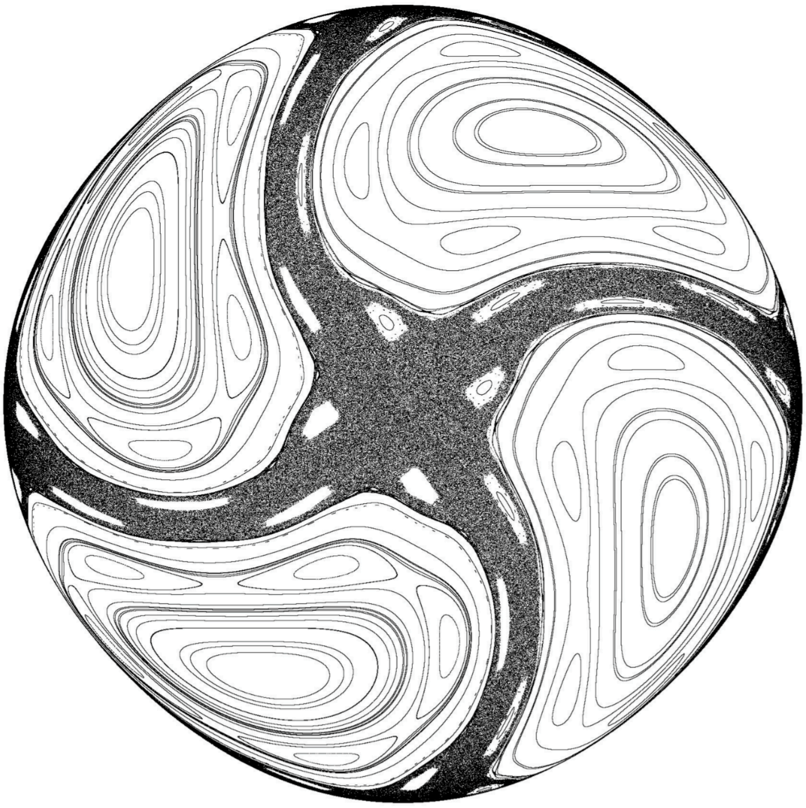
1. VBalki.Non-Linear Dynamics.L02 > Critical points of a dynamical system ↩

Hyperbolic dynamics

McMullen - Ergodic theory, geometry and dynamics



Abstract

Math 275 - Tu Th 12-1:15 - Virtual Reality
Harvard University - Fall 2020

- **Instructor:** Curtis T McMullen
- **Description:** A survey of fundamental results and current research. Topics may include:
  - Dynamics on the circle and the torus
  - Lie groups and ergodic theory
  - Hyperbolic surfaces and SL2(R)
  - Kazhdan's property T and SL3(R)
  - Amenability and expanding graphs
  - Martingales and Furstenberg's theorem
  - Hyperbolic 3-manifolds and Mostow rigidity
  - Ratner's theorem

- Conjectures of Oppenheim and Littlewood
- Planes in hyperbolic 3-manifolds
- Dynamics on moduli space Mg

- **Main Course Notes**

- C. McMullen, Notes on Ergodic Theory
  These notes will evolve during the course; be sure to use the current version, linked above.
- *See also:* C. McMullen, Notes on Teichmüller Theory and Complex Dynamics

- **Suggested Texts**

- M. B. Bekka and M. Mayer, *Ergodic theory and topological dynamics of group actions on homogeneous spaces,* Cambridge University Press, 2000.
- Bekka, de la Harpe and Valette, *Kazhdan's Property (T),* 2007.
- Benedetti and Petronio, *Lectures on Hyperbolic Geometry,* Springer-Verlag, 1992.
- E. Ghys, *Dynamique des flots unipotents sur les espaces homogènes,* Sem. Bourbaki 1991/92; Asterisque 206.
- M. Gromov, *Volume and bounded cohomology*
- R. Mañé, *Ergodic Theory and Differentiable Dynamics*
- D. Witte Morris, *Ratner's Theorems on Unipotent Flows,* Chicago Lectures in Math. Series, 2005.
- J. Ratcliffe, *Foundations of Hyperbolic Manifolds,* 2nd Edition. Springer, 2006.
- W. P. Thurston, *Three-Dimensional Geometry and Topology,* Princeton University Press, 1997.

- people.math.harvard.edu/~ctm/home/text/class/harvard/275/20/html/index.html
  - notes https://people.math.harvard.edu/~ctm/home/text/class/harvard/275/20/html/home/course/course.pdf

powers of 2 that starts with 1 in decimal

Here log has base 10.

We want to know when we have

$$10^n \leq 2^k < 2(10^n)$$

taking log this becomes

$$n \leq k \log_{10} 2 < (\log_{10} 2) + n$$

modulo $\mathbb{Z}$ it boils down to how often

$$\begin{aligned} & 0 \leq k \log_{10} 2 \bmod \mathbb{Z} \leq \log_{10} 2 \\ \iff & k \log_{10} 2 \bmod \mathbb{Z} \in [0, \log_{10} 2] \subset [0,1] \end{aligned}$$

ICTP18 Dynamics

[Summer School in Dynamics (Introductory and Advanced) | (smr 3226 -smr 3253)) - YouTube]

Gauss map

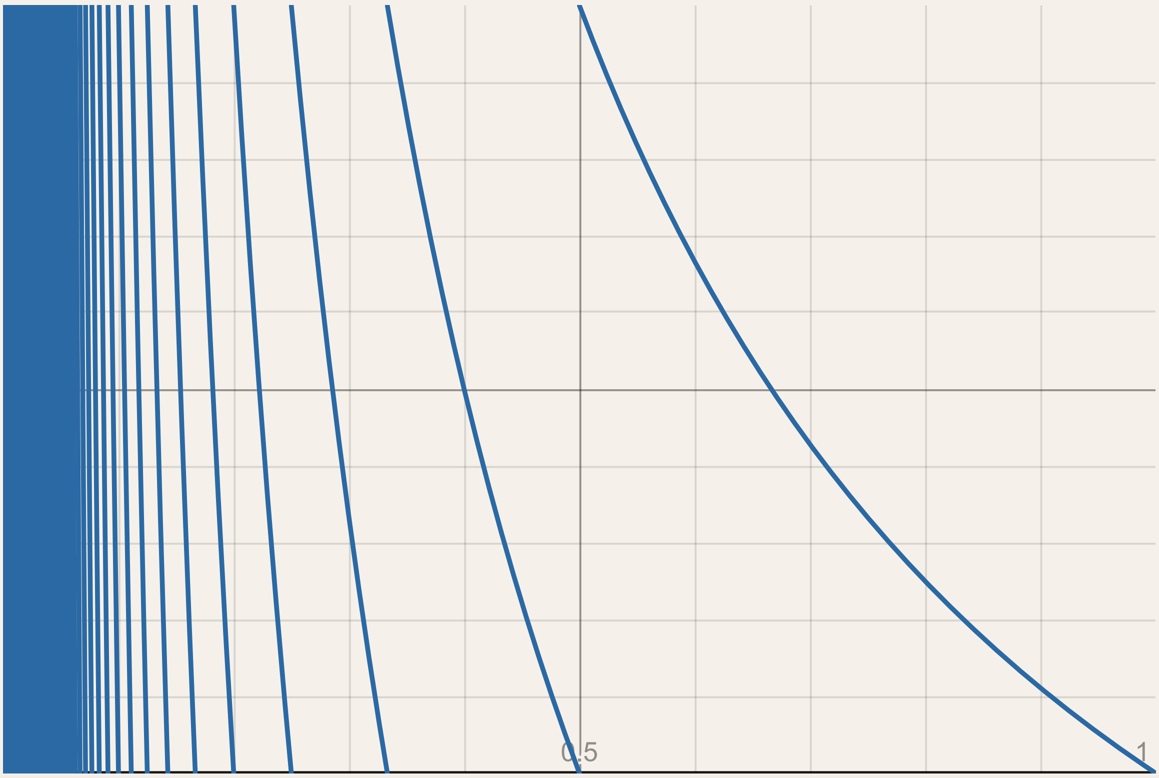
[1]

[2]

Definition. Gauss map

The Gauss map

$$\begin{aligned} G : [0,1] &\to [0,1] \\ x &\mapsto \begin{cases} 0 & x = 0 \\ \left\{ \frac{1}{x} \right\} = \frac{1}{x} \bmod 1 & x \neq 0 \end{cases} \end{aligned}$$



- As

$$\left[\frac{1}{x} \right] = n \iff n \leq \frac{1}{x} < n+1 \iff \frac{1}{n+1} < x \leq \frac{1}{n}$$

- so

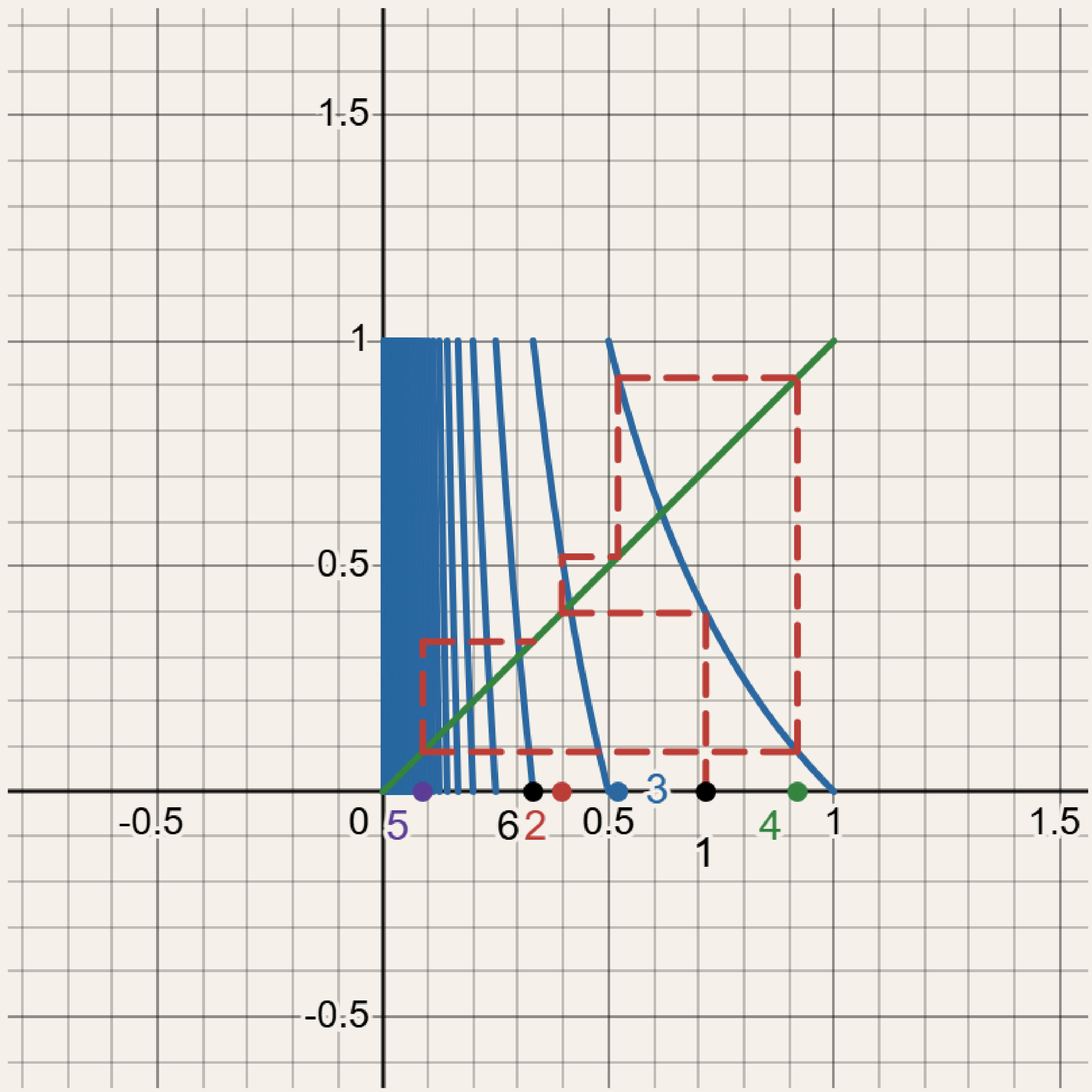
$$\frac{1}{n+1} < x < \frac{1}{n} \iff x \in \underbrace{\left(\frac{1}{n+1}, \frac{1}{n}\right]}_{P_n} \implies G(x) = \frac{1}{x} - n$$

- Thus G restricted to an interval

$$\begin{aligned} G : P_n := \left(\frac{1}{n+1}, \frac{1}{n}\right] &\to [0,1) \\ x &\mapsto \frac{1}{x} - n \end{aligned}$$

a *branch* of G, is monotonically decreasing, $\mathbb{R}$-analytic in the interior, and surjective onto $[0,1)$.

https://www.desmos.com/calculator/6gg4totiql



Definition. **The itineary of orbit of $x \in [0,1]$ under Gauss map G is a sequence of integers**

$$(a_0, a_1, \ldots)$$

such that

$$G^i(x) \in P_{a_i}$$

- From
  - As

    $$\left[\frac{1}{x}\right] = n \iff n \le \frac{1}{x} < n+1 \iff \frac{1}{n+1} < x \le \frac{1}{n}$$

    - so

      $$\frac{1}{n+1} < x < \frac{1}{n} \iff x \in \underbrace{\left(\frac{1}{n+1}, \frac{1}{n}\right]}_{P_n} \implies G(x) = \frac{1}{x} - n$$

  we know

  $$x \in P_n \iff \left[\frac{1}{x}\right] = n$$

- In particular

  $$G^0(x) = x \in P_{a_0} \implies a_0 = \left[\frac{1}{x}\right]$$

  - **Base case:**

    $$G(x) = \frac{1}{x} - \left[\frac{1}{x}\right] = \frac{1}{x} - a_0 \iff x = \frac{1}{a_0 + G(x)}$$

  - **Induction hypothesis:** We shall assume

    $$a_n = \left[\frac{1}{G^n(x)}\right]$$

and

$$x = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \dots \cfrac{1}{a_n + G^{n+1}(x)}}}$$

[3]

- **Induction step:** We know

$$G^{n+1}(x) \in P_{a_n+1} \implies a_{n+1} = \left[\frac{1}{G^{n+1}(x)}\right]$$

Then

$$G^{n+2}(x) = \frac{1}{G^{n+1}(x)} - \underbrace{\left[\frac{1}{G^{n+1}(x)}\right]}_{a_{n+1}} \iff G^{n+1}(x) = \frac{1}{a_{n+1} + G^{n+2}(x)}$$

This gives us

$$x = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \dots \cfrac{1}{a_n + \cfrac{1}{a_{n+1} + G^{n+2}(x)}}}}$$

Thus

Definition. **The itineary of orbit of $x \in [0,1]$ under Gauss map G is a sequence of integers**

$$(a_0, a_1, \dots)$$

such that

$$G^i(x) \in P_{a_i}$$

Then

$$x = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \dots \cfrac{1}{a_n + \cfrac{1}{a_{n+1} + G^{n+2}(x)}}}}$$

---

1. Rotations of the circle and renormalization 2 ↩
2. people.maths.bris.ac.uk/~ip13935/dyn/C1L2.pdf ↩
3. **Definition. Continued fraction notation**

   $$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}} \equiv b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \dots$$

   ↩

Luis, Pesin - Smooth ergodic theory

[Luis Barreira, Yakov Pesin - Introduction to Smooth Ergodic Theory (2013, American Mathematical Society).pdf]

#talk Ancient solutions to geometric flows

Ancient solutions to geometric flows - Panagiota Daskalopoulos - YouTube

Heat equation

Definition. Heat equation on a Riemannian manifold

A function

$$u : I \times M \to \mathbb{R}$$

solves the **heat equation** if

$$\frac{\partial u}{\partial t} = \Delta_g u$$

on a Riemannian manifold (M, g)

- $t \geq 0$ usually, so $I = [0, \infty)$
- Model for *diffusion* processes.
- **Scaling**
  - If u solves

    Definition. Heat equation on a Riemannian manifold

    A function

    $$u : I \times M \to \mathbb{R}$$

    solves the **heat equation** if

    $$\frac{\partial u}{\partial t} = \Delta_g u$$

    on a Riemannian manifold (M, g)

    on $\mathbb{R}^n$ then

    $$(x, t) \mapsto \beta u(\alpha x, \alpha^2 t)$$

    also solves it.
- Finite speed of propagation

- Fundamental solution on $\mathbb{R}^n$

$$\frac{1}{\sqrt{(4\pi t)^n}} e^{-|x|^2/4t}$$

[1]

Definition. **An ancient solution to a *parabolic equation* is a solution defined for all $t \in (-\infty, T]$ where $T \in \mathbb{R} \cup \infty$.**

Keywords

"Blowup analysis of non-linear parabolic gives ancient solutions", "singularities"

Outline for lectures on *ancient solutions*

1. Heat equation
2. semi-linear heat equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \; p > 1$$

3. Curve shortening floe & mean curvature flow
4. 2 dimensional Ricci flow (much simpler than 3 dimensional counterpart)

examples of ancient solutions

Example

$$u(x, t) = \phi(x)$$

where ϕ is harmonic which means

$$\Delta u = 0$$

Example

$$u(x, t) = e^{x_1 + t}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, t \in \mathbb{R}$

Then

$$u_t = e^{x_1+t} = u_{xx} = \Delta u$$

Example

$$u(t,x) = \frac{1}{\sqrt{(4\pi(T-t))^n}} e^{-|x|^2/4(T-t)}$$

for $t < T$

uniqueness theorems for ancient solutions for Heat equation

(Yau, 1975) Let M be a complete non-compact Riemannian manifold of non-negative Ricci curvature ($\mathrm{Ricci}(M) \geq 0$). If $u \geq 0$ is a harmonic function on M, that is $\Delta_g u = 0$ on M, then u is constant.

whose proof is based on the *estimate*

(Cheng-Yau, 1975) Let M be a complete non-compact Riemannian manifold of non-negative Ricci curvature ($\mathrm{Ricci}(M) \geq 0$). If $u \geq 0$ is a harmonic function on M, that is $\Delta_g u = 0$ on M, then

$$\frac{\|\mathrm{grad}(u)\|(x)}{u(x)} \leq \frac{C_n}{R}$$

for all $x \in B_R(x_0)$ and $x_0 \in M$ where $C_n \in \mathbb{R}$ only depends on $\dim M$.

- This estimate reminds us of the *gradient estimate* for harmonic functions

$$\sup_{B_R(x_0)} |\mathfrak{D}u| \leq \frac{C_n}{R} \sup_{B_R(x_0)} |u|$$

Question

Does the analogue of Yau's result hold for ancient solution of the heat equation: a $u \geq 0$ such that

$$u_t = \Delta_g u$$

on complete non-compact Riemannian manifold $\times(-\infty, T)$?

The answer is **noooo!** We have seen the counterexample precisely:

Example

$$u(x,t) = e^{x_1+t}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, t \in \mathbb{R}$
Then

$$u_t = e^{x_1+t} = u_{xx} = \Delta u$$

But this is *not* really an analogue because t has no meaning in the parabolic(?) setting

(Souplet, Zang, 2007?) Let M be a complete non-compact Riemannian manifold of non-negative Ricci curvature ($\mathrm{Ricci}(M) \geq 0$) and let

$$u : M \times (-\infty, T) \to \mathbb{R}$$

be a solution of heat equation

$$u_t = \Delta_g u$$

then
1. If $u \geq 0$ and

$$u(x,t) = e^{o(d(x,x_0)+\sqrt{|t|})} \text{ as } d(x,x_0) \to \infty$$

then u is a constant on M.
2. If

$$u(x,t) = o(d(x,x_0) + \sqrt{|t|}) \text{ as } |x| \to \infty$$

then $u \equiv 0$.

Let M be a complete non-compact Riemannian manifold of non-negative Ricci curvature ($\mathrm{Ricci}(M) \geq 0$) and let

$$u : M \times (-\infty, T) \to \mathbb{R}$$

be a solution of heat equation

$$u_t = \Delta_g u$$

then

$$\frac{\|\mathrm{grad}(u)\|}{u} \leq c_n \left(\frac{1}{R} + \frac{1}{\sqrt{T}}\right)\left(1 + \ln \frac{L}{u}\right)$$

which holds on

$$Q_{R,T}(x_0) := B_R(x_0) \times [t_0 - T, t_0]$$

and

$$u \leq L \text{ on } Q_{2R,2T}(x_0)$$

(For $\mathbb{R}^n$, we use the *maximal principle*!)

proof of Souplet-Zang from the heat estimate on $\mathbb{R}^n$

- Assume $L = 1$ that is $u \leq 1$ on $Q_{2R,2T}$ where $T = 4R^2$
- $$\ln(u+1) \leq o(|x| + \sqrt{t})$$

references

- In UIMP 2015 Summer Course *"Liouville theorems for parabolic equations with emphasis on geometric flows"* ***Slides*** **1** **2** **3**
- 0.pdf
- rnoti-p467.pdf

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1. PDE and Boundary-Value Problems Winter Term 2014/2015 ↩

Masoud Khalkhali - Spectral geometry

Spectral Geometry

Weyl's asymptotic law

Let $\Omega \subset \mathbb{R}^n$ is open with piecewise smooth boundary and with **Laplacian** $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ let

$$\Delta\varphi = \lambda\varphi$$
$$\varphi\Big|_{\partial\Omega} = 0$$

be the Helmholtz equation with Dirichlet boundary condition.

- the **spectrum**

  $$\begin{aligned}\operatorname{spec}(\Omega) &:= \sigma_{\mathrm{p}}(\Delta : L^2(\Omega) \to L^2(\Omega)) \\ &= \{\lambda \mid \Delta\varphi = \lambda\varphi\}\end{aligned}$$

  is known to be **discrete** with

  $$0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$$

- the **eigenfunctions** of Δ form an **orthonormal basis** of $L^2(\Omega)$ (when normalized)

(Weyl's asymptotic law, 1911) Let $\Omega \subset \mathbb{R}^n$ is open with piecewise smooth boundary. Then element λ_k of spectrum of Ω grow like

$$\lambda_k \sim \frac{(2\pi)^2}{(\omega_n \operatorname{vol}(\Omega))^{2/n}} k^{2/n}$$

as $k \to \infty$ where $\omega_n = \operatorname{vol}(D^n)$ is volume of the unit ball $D^n \subset \mathbb{R}^n$.

So the spectrum determines the volume!

Definition. **Isospectral open sets of $\mathbb{R}^n$**

Two open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are called **isospectral** if

$$\operatorname{spec}(\Omega_1) = \operatorname{spec}(\Omega_2)$$

and the multiplicity of each eigenvalue are also equal, that is for each

$\lambda \in \operatorname{spec}(\Omega_1) = \operatorname{spec}(\Omega_2)$ we have

$$\dim \ker(\Delta\Big|_{L^2(\Omega_1)} - \lambda \mathsf{Id}) = \dim \ker(\Delta\Big|_{L^2(\Omega_2)} - \lambda \mathsf{Id})$$

Isometric open sets are isospectral!

questions on spacetimes

Birkhoff's theorem: any non-flat vacuum spacetime is locally isometric to Schwarzschild spacetime

https://annegretburtscher.wordpress.com/wp-content/uploads/2019/11/willemvanoosterhout_bscthesis_2019.pdf

Larry Guth - MIT Differential Analysis 2

"An analyst is a person who knows how to use Hölder's inequality."

I heard that J. Bourgain said this.

Loomis-Whitney volume inequality

- Consider

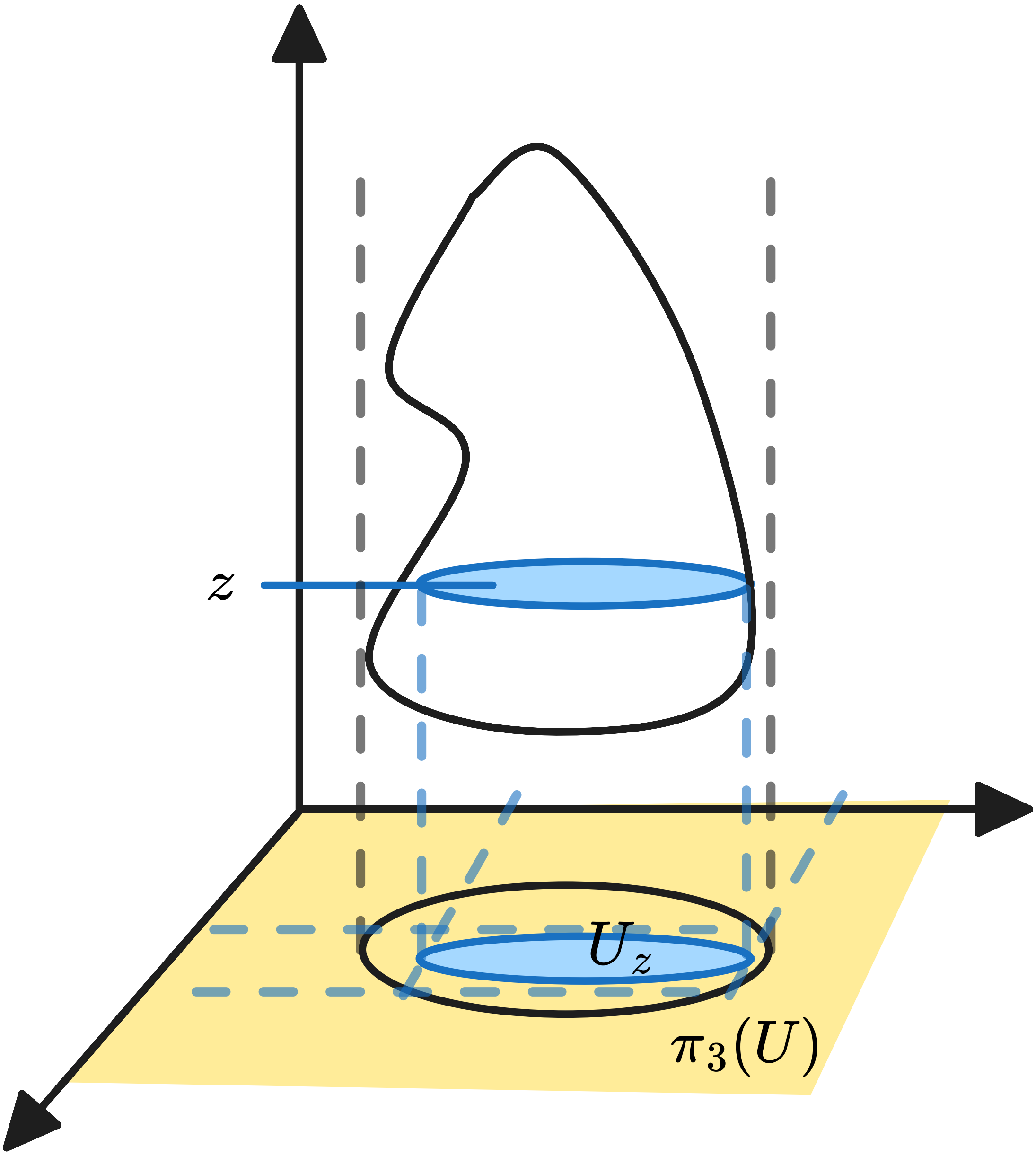
$$\mathbb{R} \times D^2$$

- Consider

$$[0, \epsilon] \times \left[0, \frac{1}{\epsilon}\right] \times \left[0, \frac{1}{\epsilon}\right]$$

(Loomis-Whitney) If $U \subseteq \mathbb{R}^3$ is open then

$$\mu_{\mathbb{R}^3}(U) \leq \prod_{j \in \{1,2,3\}} \sqrt{\mu_{\mathbb{R}^2}(U)}$$



$$\mu_{\mathbb{R}^3}(U) = \int_{z \in \mathbb{R}} \mu_{\mathbb{R}^2}(U_z)$$

- We have

$$\mu_{\mathbb{R}^2}(U_z) \leq \mu_{\mathbb{R}^3}(\pi_3(U))$$

- Also

$$\mu_{\mathbb{R}^2}(U_z) \leq \mu_{\mathbb{R}}(\pi_1(U_z))\mu_{\mathbb{R}}(\pi_2(U_z))$$

where

$$\mu(\pi_1(U)) = \int_{z \in \mathbb{R}} \mu(\pi_1(U_z))$$

Philosophical question: If we have two bounds for something, what should we do?

1. Pick one bound
2. Use both $f(x) \leq \boxed{1}^{\alpha}\boxed{2}^{1-\alpha}$
3. Wait and try to figure out for each x which bound is better for $f(x)$.

Choosing (2) in this case we have

- $$\begin{aligned}\mu_{\mathbb{R}^3}(U) &= \int_{z\in\mathbb{R}} \mu_{\mathbb{R}^2}(U_z) \\ &\leq \int_{z\in\mathbb{R}} \mu_{\mathbb{R}^3}(\pi_3(U))^{\alpha}(\mu_{\mathbb{R}}(\pi_1(U_z))\mu_{\mathbb{R}}(\pi_2(U_z)))^{1-\alpha}\end{aligned}$$
  - How should we choose α?

CMI March '25 - Mini course on Geometric measure theory

Abstract

There will be a series of seven lectures on Geometric measure theory at CMI from 10 March to 21 March 2025.

The first five lectures will be given by Dr. Aswin Govindan Sheri on 10, 11, 13, 14, and 17 March 2025. These lectures will be in hybrid mode.

The last two lectures will be given by Prof. Malabika Pramanik and will be online only.

Date, Time and Venue:

- March 10, 13, 14, and 17, 2025 at 3:30 PM in Seminar Hall
- March 11, 2025 at 5:00 PM in Seminar Hall

Zoom Meeting details (Dr. Aswin Govindan Sheri)
https://cmi-ac-in.zoom.us/j/84329164349

Meeting ID: 843 2916 4349
Passcode: 165489

- March 20 and 21, 2025 at 9:00 PM (online talks by Prof. Malabika Pramanik)

Zoom Meeting details (Prof. Malabika Pramanik)
https://cmi-ac-in.zoom.us/j/89698106713

Meeting ID: 896 9810 6713
Passcode: 880632

First Five lectures by Dr Sheri would cover the following topics:

Hausdorff measure and dimension, Examples of various types of constructions, Fourier transform and it's basic properties, Frostman's lemma, energy integrals, it's reformulation using Fourier inversion, Distance set conjecture

The mini-course on Geometric Measure Theory will conclude with two lectures on 20, 21 March 2025 by Malabika Pramanik. These lectures aim to showcase the applications of the foundational concepts introduced by Dr. Sheri, while also providing the audience with a glimpse into active areas of research in the field.

The first lecture will explore problems related to pattern recognition, demonstrating how geometric measure theory offers valuable tools for identifying and analyzing patterns in various mathematical and real-world contexts. The second lecture will focus on variants of distance problems, highlighting their significance and complexity in geometric and analytic settings.

These sessions promise to offer both depth and accessibility, intended for participants with a range of mathematical backgrounds.

Prerequisites: Basic course on measure theory

Lecture series abstract: https://drive.google.com/file/d/1Wf-ZSvyd8ZqIDT_2KTBFWuh-bGSuCF9C/view?usp=sharing

AN INTRODUCTION TO GEOMETRIC MEASURE THEORY

ASWIN GOVINDAN SHERI

1. Lecture series outline

The purpose of the lecture series is to introduce the students to some of the basic concepts in geometric measure theory. Towards the end of the series, we will also touch upon the topic of distance sets, an active research area in the field. Topics to be covered are given as follows:

- Hausdorff measure and dimension [2, §4.1 - §4.9]
- constructions of various sets; Cantor sets in $\mathbb{R}$ and $\mathbb{R}^n$, and other self-similar sets [2, §4.10 - §4.15], [3, §8]
- energy integrals and Frostman's lemma [2, §8], [3, §2.5]
- Fourier transform, reformulation of the energy integral [2, §12]
- distance sets and Falconer's distance set conjecture [2, §12], a brief sketch of the partial result for the case $\dim A > \frac{n+1}{2}$, $A \subset \mathbb{R}^n$ [3, Theorem 4.6]

Reference textbooks: *Geometry of Sets and Measures in Euclidean Spaces* [2], *Fourier analysis and Hausdorff dimension* [3], *The geometry of fractal sets* [1]

References

[1] K. J. Falconer. *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.

[2] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[3] P. Mattila. *Fourier analysis and Hausdorff dimension*, volume 150 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2015.

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Department of Mathematics, Indian Institute of Science Education and Research Berhampur

#people Dr. Aswin Govindan Sheri

I am a post doctoral fellow at the department of Mathematics, IISER Berhampur. Previously, I taught at National Institute of Technology, Calicut as an ad-hoc faculty. I graduated with a Ph.D.
in mathematics from the University of Edinburgh under Dr. Jonathan Hickman.

- https://sites.google.com/view/agovindansheri/
- MS thesis: Kakeya Sets in Harmonic Analysis
- PhD: On certain geometric maximal functions in Harmonic analysis

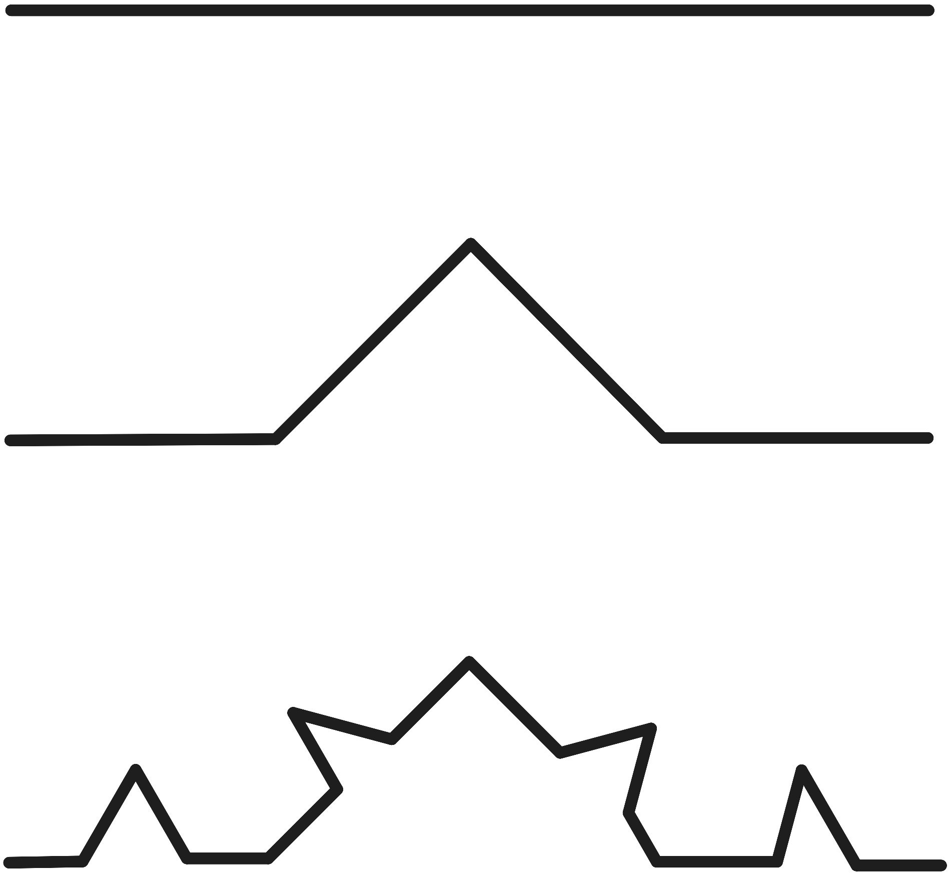
- Lecture 1 by Dr Aswin Govindan Sheri
- Lecture 2 by Dr Aswin Govindan Sheri
- Lecture 3 by Dr Aswin Govindan Sheri
- Lecture 4 by Dr Aswin Govindan Sheri

Middle third Cantor set

$$E := \bigcap_{k\geq 1} E_k$$

- self-similar
- fine structure, details at every scale
- defined recursively
- "complicated" local geometry
- $\mu(E) = 0$ "thin set"

Von Koch curve



$$E_k$$

us a union of 4^k line segments of length

$$\frac{1}{3^k}$$

$$E := \lim_{k \to \infty} E_k$$

- self-simmilar
- recursive definition

Hausdorff measure through Carathéodory construction

Definition. generalized Hausdorff measure

Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^d$ along with

$$\mathscr{C} : \mathcal{F} \to [0, \infty]$$

such that

- for each $\delta > 0$ there exists $E_i \in \mathcal{F}, i \geq 1$ such that

$$d(E_i) \leq \delta, \mathbb{R}^d = \bigcup_{i \geq 1} E_i$$

- for each $\delta > 0$ there exists $E \in \mathcal{F}$ such that

$$d(E) \leq \delta, \mathscr{C}(E) \leq \delta$$

where $d(E)$ is diameter of E. Then for $A \subset \mathbb{R}^d, \delta > 0$ we define

$$\psi_\delta^{\mathscr{C}}(A) := \inf \left\{ \sum_{i \geq 1} \mathscr{C}(E_i)^d \middle| A \subset \bigcup_{i \geq 1} E_i, E_i \in \mathcal{F}, d(E_i) < \delta \right\}$$

The **generalized Hausdorff measure** based on $\mathscr{C}$ is

$$\psi^{\mathscr{C}}(A) := \sup_{\delta > 0} \psi_\delta^{\mathscr{C}}(A)$$

Proposition:

- ψ_δ is monotonic, countably sub-additive but not additive
- $$U := B(0, \delta)$$

  then

  $$\psi_\delta(U) = \psi_\delta(\bar{U}) = \psi_\delta(\partial U) = \delta^s$$

- $\psi_\delta(\emptyset) = 0$

The measure ψ is a Borel measure and if $\mathcal{F}$ consists of Borel sets then H is Borel regular.

Use

(Caratheodory criterion) If μ is an outer measure on $\mathbb{R}^d$ then μ is Borel measurable iff

$$\mu(A \sqcup B) = \mu(A) + \mu(B)$$

when $A, B \subseteq \mathbb{R}^d$ and $d(A, B) > 0$.

- *(verifying ceriterion)*

examples

$\mathcal{F}$	$\mathscr{C}$	ψ
$$A \in 2^{\mathbb{R}^d}$$	$$d(A)^s$$ $s \in [0, \infty]$	s-Hausdorff measure $$H^s$$
	1	H^0 is counting measure
$$\left\{ R = \prod_i [a_i, b_i] \mid a_i < b_i \right\}$$ in $\mathbb{R}^d$	$$\mathrm{vol}(R)$$	Lebesgue measure $$\mu_{\mathbb{R}^d}$$ and we can see $$\mu_{\mathbb{R}^d} = cH^d$$ where $$c = \mu_{\mathbb{R}^d}(B_1(0))$$
all balls in $\mathbb{R}^d$	$$d(E)^t$$ for $t \in [0, \infty)$	$$S^t$$ where it is clear that $$H^t(A) \leq S^t(A)$$

Exercise

$$S^t(A) \leq 2^t H^t(A)$$

then

$$\mathcal{H}^s(\varphi(A)) \le C^s \mathcal{H}^s(A).$$

4. Theorem: If

(a) $\mathcal{F} = \{E \subseteq \mathbb{R}^d : E \text{ is open}\}$ or

(b) $\mathcal{F} = \{E \subseteq \mathbb{R}^d : E \text{ is closed}\}$ or

(c) $\mathcal{F} = \{K \subseteq \mathbb{R}^d ; K \text{ is convex},$

then $\Phi([\mathrm{diam}(\cdot)]^s, \mathcal{F}) = \mathcal{H}^s$

asking symmetric questions

Is every smooth manifold homogeneous?

- Given any two points in a connected manifold, there exists a diffeomorphism taking one to the other. [1]
- However, not all smooth manifolds are homogeneous G-spaces for (finite dim) Lie groups G, there are necessary conditions [2]

(Mostow, 2005) M is a compact homogeneous G-space $\implies \chi(M) \geq 0$ [3]

Is a quotient of a Lie-homogeneous again Lie-homogeneous?

No. By

(Mostow, 2005) M is a compact homogeneous G-space $\implies \chi(M) \geq 0$ [1]

---

1. A Structure Theorem for Homogeneous Spaces | Geometriae Dedicata ↩

higher genus surfaces are not Lie-homogeneous, but their universal cover is $\mathbb{R}^2$ which is Lie-homogeneous.

The fact that they are not Riemannian homogeneous is easily proven by Riemann surface techniques. [4]

Is every Lie-homogeneous smooth manifold Riemannian homogeneous?

No. [5]

> **Quote**
>
> For example, consider $G = PGL(2, \mathbb{C})$ acting on the Riemann sphere S^2 via *Möbius transformations*. Suppose there is a G-invariant distance on S^2. Let $H \subseteq G$ denote the isotropy group at $p := (1, 0, 0)$. That is, for $h \in G$, $h \in H$ iff $h * p = p$. For any $\epsilon > 0$, because G preserves distance, H will preserve the set $\{q \in S^2 : d(p, q) = \epsilon\}$. For small ϵ, this set should be a circle. However, since G acts 3-transitively on S^2, it follows that H acts transitively on $S^2 \setminus \{p\}$, so H cannot preserve any circle. [6]

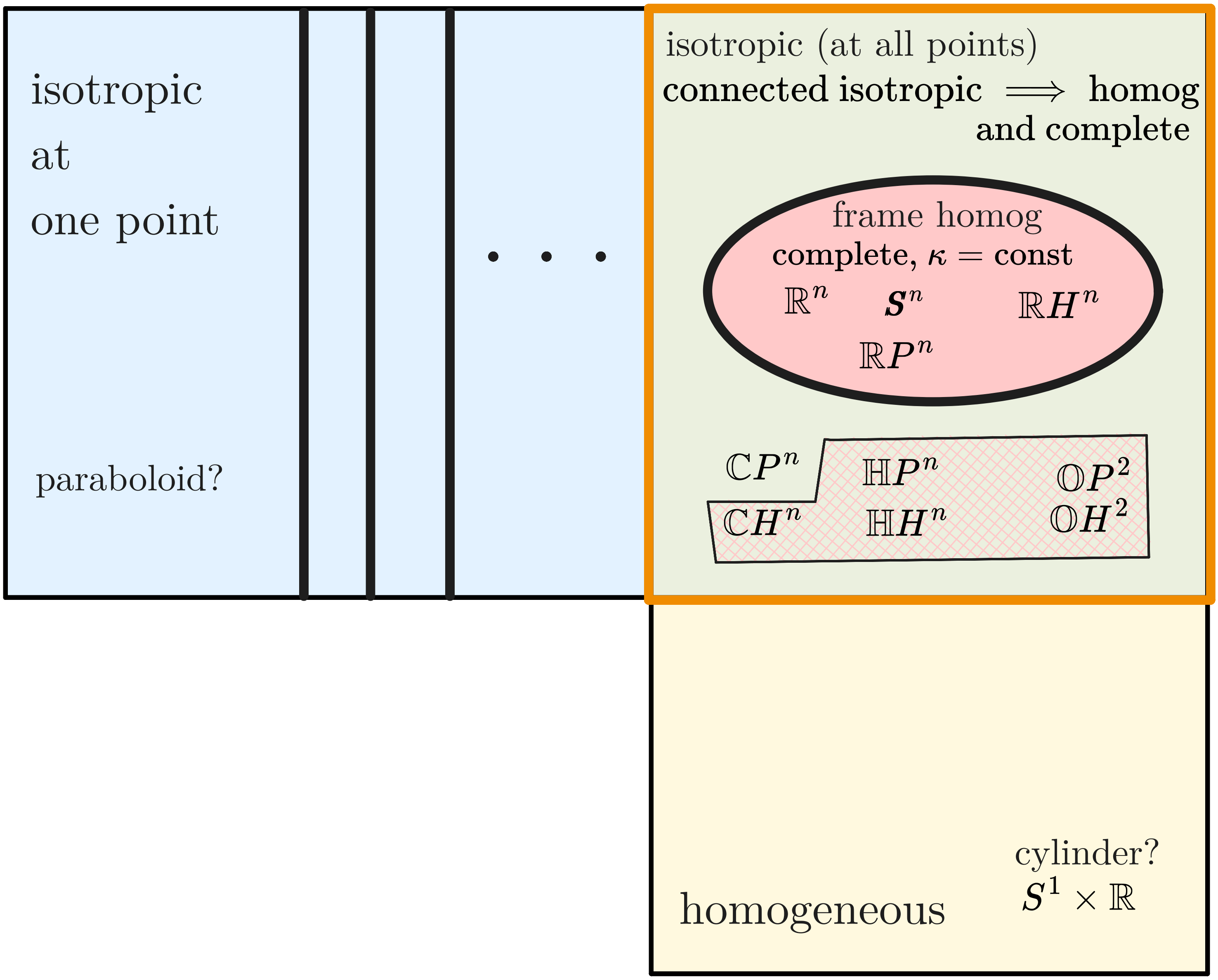
> **Quote**
>
> If a **compact** Lie group G acts transitively on a smooth manifold, then by averaging one can produce a Riemann metric on that manifold which is G-invariant. [7] [6-1]

Classification of connected isotropic Riemannian manifolds

[8]

[9]

connected Riemannian manifolds



8.12.2 Theorem. *Let M be a riemannian symmetric space. Then the following conditions are equivalent.*

(i) *M is two point homogeneous.*
(ii) *Either M is a euclidean space or M is irreducible and of rank 1.*
(iii) *M is isometric to one of the spaces*

(a) $\mathbf{R}^n = \mathbf{E}(n)/\mathbf{O}(n)$, *euclidean space,*
(b) $\mathbf{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, *sphere,*
(c) $\mathbf{P}^n(\mathbf{R}) = \mathbf{SO}(n+1)/\mathbf{O}(n)$, *real projective space,*
(d) $\mathbf{P}^n(\mathbf{C}) = \mathbf{SU}(n+1)/\mathbf{U}(n)$, *complex projective space,*

SEC. 8.12 TWO POINT HOMOGENEOUS SPACES 295

(e) $\mathbf{P}^n(\mathbf{Q}) = \mathbf{Sp}(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$, *quaternionic projective space,*

(f) $\mathbf{P}^2(\mathbf{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$, *Cayley projective plane,*

(b′) = (c′) $\mathbf{H}^n(\mathbf{R}) = \mathbf{SO}^1(n+1)/\mathbf{SO}(n)$, *real hyperbolic space,*

(d′) $\mathbf{H}^n(\mathbf{C}) = \mathbf{SU}^1(n+1)/\mathbf{U}(n)$, *complex hyperbolic space,*

(e′) $\mathbf{H}^n(\mathbf{Q}) = \mathbf{Sp}^1(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$, *quaternionic hyperbolic space,*

(f′) $\mathbf{H}^2(\mathbf{Cay}) = \mathbf{F}_4^*/\mathbf{Spin}(9)$, *Cayley hyperbolic plane.*

Here ′ denotes noncompact dual.

Classifying homogeneous Riemannian manifolds

Much harder.

	Homogeneous G-space M (that is $G \curvearrowright M$ transitively)	$$M \cong_G \frac{G}{H}$$ where H is a closed subgroup
compact G ?	G-invariant metrics on M	H-invariant inner products on $\mathfrak{g}/\mathfrak{h}$?

Isometric embedding of Riemannian 2-manifolds into $(\mathbb{R}^n, \mathrm{DOT})$

[10]

---

1. differential geometry - Proving that given any two points in a connected manifold, there exists a diffeomorphism taking one to the other - Mathematics Stack Exchange ↩
2. dg.differential geometry - Example of a manifold which is not a homogeneous space of any Lie group - MathOverflow ↩
3. A Structure Theorem for Homogeneous Spaces | Geometriae Dedicata ↩
4. https://mathoverflow.net/a/104106 ↩
5. https://mathoverflow.net/questions/346364/is-every-homogeneous-space-riemannian-homogeneous ↩
6. https://math.stackexchange.com/a/3876991/1290493 ↩ ↩

7. dg.differential geometry - How do you see that higher genus surfaces are not homogeneous? - MathOverflow ↩
8. https://math.stackexchange.com/a/4920625 ↩
9. https://math.stackexchange.com/a/4921317 ↩
10. dg.differential geometry - Construct embedding given metric - MathOverflow ↩

TIFR-Geometry, groups and dynamics (2017)

- Geometry, Groups and Dynamics (GGD) - 2017 - YouTube
- Geometry, Groups and Dynamics (GGD) - 2017 | ICTS

Course Outline

The program focuses on geometry, dynamical systems and group actions. Topics are chosen to cover the modern aspects of these areas in which research has been very active in recent times. The program will have two parts, a preschool in the first week and an advanced school during the remaining two weeks. The pre-school will be devoted to covering the necessary background material for the advanced school during the remaining two weeks. The courses for the first week of pre-school include:

1. Discrete subgroups of Lie groups, by Pralay Chatterjee (IMSc, Chennai)
2. Introduction to hyperbolic geometry, by Subhojoy Gupta (IISc, Bangalore)
3. Introduction to Geometric Group Theory, by Pranab Sardar (IISER, Mohali)
4. Crash course on Riemannian geometry, by Harish Seshadri (IISc, Bangalore)
5. Teichmüller Theory, old and new, by Athanase Papadopoulos (Univ. Strasbourg, France)

More details of the courses will be given soon on this webpage.

The advanced school will consist of courses aimed at graduate students and young researchers who are either working or want to work in the broad theme of the program. The courses will be supplemented by a few research level surveys and lectures by some eminent experts in the field. These supplementary lectures are meant to connect the topics of the advanced school to cutting edge research to open new horizons of research for the participants of the school.

The following courses and mini courses for the two-week advanced school have been planned.

1. Riemann Surfaces and the Absolute Galois Group, by Norbert A'Campo (Univ. Basel, Switzerland)
2. Hierarchically hyperbolic space, by Jason Behrstock (CUNY, USA)
3. Projection Complexes, by Mladen Bestvina (Univ. Utah, USA)
4. Geodesic flow of the WeilPetersson metric, by Keith Burns (Northwestern Univ., USA)
5. Growth in groups, by Rostislav Grigorchuk (TAMU, USA)
6. Higher Teichmüller Theory, by Francois Labourie (Univ. Nice, France) and Misha Kapovich (Univ. Davis, USA)

7. Boundary theory of hyperbolic groups, by Mahan Mj. (TIFR, India)
8. Geometrically infinite Kleinian groups, by Ken'ichi Ohshika (Osaka Univ., Japan)
9. Teichmüller Theory, old and new, by Athanase Papadopoulos (Univ. Strasbourg, France)
10. Deformation of complex hyperbolic lattices, by Pierre Will (Univ. Grenoble, France)

There will be some expository lectures as well. The program is followed by ICTS discussion meeting : Surface Group Representations and Geometric Structures.

Talks

Week - 1: Basics on Geometry and Groups

1. **Norbert A'Campo**

   Riemann surfaces: algebra, analysis, geometry

   Riemann started the study of the possibility of finding meromorphic functions with prescribed truncated Laurent expansions at its zero's and poles on compact Riemann surfaces. This study was seminal and has forced discoveries in many, perhaps all, branches of mathematics. Here only some key words: topology, manifold, dierential forms, cohomology, hyperbolic geometry, Gauss-Bonnet theorem, harmonic analysis, fundamental group, Teichmueller space, Chow's Theorem, ... The lectures will have as goal the so-called Riemann Existence Theorem, Uniformisation Theorem for compact connected Riemann surfaces of genus $g > 1$, the Universal Curve,.....

2. **Pralay Chatterjee**

   Symmetric spaces

   Starting with the basics of symmetric spaces and examples, we will explore Chapter 2 of the book "Geometry of nonpositively curved manifolds" by P. Eberlein which deals with the structure of symmetric spaces of non-compact type.

3. **Subhojoy Gupta**

   Hyperbolic surfaces and their Teichmüller spaces

   In the first talk, we shall introduce the Teichmüller space T of a compact oriented surface S as the deformation space of hyperbolic structures on S. In the second talk, we shall discuss the relation with surface group representations to PSL(2,R) and in the case S is closed, identify T with a component comprising discrete and faithful representations. In the final talk, we shall introduce the mapping class group of S and discuss its action on Teichmüller space T.

I will only assume some basic familiarity with hyperbolic plane and its isometries.

4. **Mahan Mj**

   Introduction to hyperbolic groups

   Hyperbolic metric spaces were discovered by Gromov in the 80's to give a unified treatment of manifolds of negative curvature and discrete groups satisfying certain combinatorial conditions. We shall describe various equivalent notions of hyperbolicity. We shall also describe the Gromov boundary of such spaces. (Reference: Bridson-Haefliger--Spaces of non-positive curvature).

5. **Ken'ichi Ohshika**

   Boundaries of quasi-Fuchsian spaces and continuous/discontinuous phenomena

   In this series talks, I shall first explain what kind of Kleinian groups appear on boundaries of quasi-Fuchsian spaces. Relying on examples given in the first part, I shall next show that as quasi-Fuchsian groups converge to a Kleinian group on the boundary, there are things which vary continuously and other things which vary discontinuously. In the last part I shall illustrate that this difference can be understood using geometric limits.

6. **Athanase Papadopoulos**

   Lecture 1: Introduction to spherical geometry I

   Lecture 2: Introduction to spherical geometry II

   I will prove some basic theorems of spherical geometry : In a triangle, the angle sum is greater than two right angles ; the segment joining the midpoints of the two legs is greater than half of the basis, and I will give two formulas for the area (one in terms of angles, and one in terms of side lengths). I will discuss the relation with hyperbolic geometry, and more generally with Funk and Hilbert geometries.

7. **Pranab Sardar**

   Geometry of the symmetric space SL(n,R)/SO(n,R)

I will discuss the geometry of SL(n,R)/SO(n,R) following the book Metric Spaces of Non-positive Curvature by Bridson-Haefliger (Part II, Chapter 10). Starting from the definition of the metric we will prove that it is a CAT(0) space; we will describe the flats, and Weyl chambers etc. Time permitting we will touch on the Tits boundary of this space.

8. **Harish Seshadri**

   Crash course in Riemanian geometry

   I will discuss some basic techniques and results in Riemannian geometry relevant to the study of negatively curved manifolds. These will include Jacobi fields, the Cartan-Hadamard theorem and the Cartan-Ambrose-Hicks theorem.

   The text " Riemannian Geometry" By Gallot, Hulin and Lafontaine will be the reference.

Week - 2: Concentration Period: Geometry and Groups

1. **Jason Behrstock**

   Hierarchy Hyperbolic Spaces

   In the first lecture we will introduce the mapping class group (MCG) of a surface and discuss the curve complex of a surface. This leads to several very powerful tools developed by Masur, Minsky, Behrstock, and others which we will survey. In the second lecture we will discuss fundamental groups of compact cube complexes, focussing in particular on the interesting special case of right angled Artin groups (RAAGs). After introducing these groups, we will begin to discuss recent tools developed by Behrstock-Hagen-Sisto which are analogues of the machinery discussed for the mapping class group in the first lecture; we will give details of the tools for RAAGs and describe the general framework of a hierarchically hyperbolic space (HHS). In the third lecture we will outline some of the techniques for working with HHS and give a sketch of the proof of some applications, including the quasi-flats theorems, which proves a number of outstanding conjectures.

   References:

   1. This is a survey which is a good introduction to the topics which will be covered in this mini-course: "What is a hierarchically hyperbolic space?", by Alessandro Sisto, https://arxiv.org/abs/1707.00053.
   2. This paper won't be discussed explicitly, but it contains and applies a number of the topics discussed in the first lecture, so is a good reference to be aware of. "Geometry

and rigidity of mapping class groups", by Jason Behrstock, Bruce Kleiner, Yair Minsky, and Lee Mosher, https://arxiv.org/abs/0801.2006

3. The second lecture will be cover several of the topics of this paper in the special case of RAAGs: Hierarchically hyperbolic spaces I: curve complexes for cubical groups, by Jason Behrstock, Mark F. Hagen, and Alessandro Sisto, https://arxiv.org/abs/1412.2171
4. The main results of this paper will be discussed in lecture 3: "Quasiflats in hierarchically hyperbolic spaces", by Jason Behrstock, Mark Hagen, and Alessandro Sisto, https://arxiv.org/abs/1704.04271

2. **Mladen Bestvina**

Constructing group actions on quasi-trees and applications

Starting from relatively simple axioms one can construct a quasi-tree in a natural way. These axioms turn out to be satisfied in a variety of situations that arise in geometric group theory, most notably in the setting of Masur-Minsky subsurface projections, or in the presence of rank 1 group elements. The original construction is presented in arXiv:1006.1939, and it is joint work with Ken Bromberg and Koji Fujiwara. There is a simplification of the construction which I will present in the minicourse, and it is a joint work with Bromberg, Fujiwara, and Alessandro Sisto.

3. **Rostislav Grigorchuk**

Growth of finitely generated groups and related topics

I will explain what is the growth of a group (semigroup, algebra,...), the relation between growth of a group and of Riemannian manifold. Describe the main results about growth of groups from Shwartz and Milnor's contributions (50th-60th of 20th century) till our days. The main focus will be given to groups of intermediate growth (between polynomial and exponential). Starting from my original construction of such groups (that answered the question of Milnor and solved some other problems) I will finish by recent constructions by Bartholdi-Erschler and by Nekrashevych. Also such topics as actions on rooted trees, self-similar groups, automaton groups, amenable groups, and iterated monodromy groups will appear occasionally.

4. **Michael Kapovich**

Discrete subgroups of higher rank Lie groups

I will talk about recent advances in the theory of discrete subgroups of higher rank Lie groups, which exhibit some "rank one" behavior and related aspects of coarse geometry of higher rank symmetric spaces. This is based mostly on my work with Bernhard Leeb and Joan Porti.

5. **Francois Labourie**

   Mini Course 1: Dynamics of Anosov representations

   In this series of lectures I will give the definition of Anosov subgroups of linear groups emphasizing their dynamical nature. In particular, I will explain that they are associated to currents and that they have natural geodesic flows with dynamical properties. My talk intend to be elementary, starting from Anosov representations of the group Z.

   Mini course 2: Introduction to Higgs bundles

   In this series of lecture, I will first start with very elementary classical result on line bundles on Riemann surfaces. Following this historical point of view, I will move to the definition of Higgs bundles. My goal is to motivate and explain the definition and state the major theorems of the field.

6. **Ken'ichi Ohshika**

   Boundaries of quasi-Fuchsian spaces and continuous/discontinuous phenomena

   In this series talks, I shall first explain what kind of Kleinian groups appear on boundaries of quasi-Fuchsian spaces. Relying on examples given in the first part, I shall next show that as quasi-Fuchsian groups converge to a Kleinian group on the boundary, there are things which vary continuously and other things which vary discontinuously. In the last part I shall illustrate that this difference can be understood using geometric limits.

7. **Michah Sageev**

   CAT(0) cube complexes and group theory

   CAT(0) cube complexes are a particular class of CAT(0) spaces that carry a combinatorial structure which gives them the look and feel of “generalized trees". We will discuss the st ructure of CAT(0) cube complexes, what this structure tells us about the groups that act on them, and how to get build actions of groups on such complexes.

Nov 18: Special Program: Dynamics and Its Interaction with Number Theory

1. **Anish Ghosh**

   Dynamical systems on homogenous spaces and number theory

   I will survey some of S. G. Dani's many influential contributions to the above subject.

2. **Athanase Papadopoulos**

   On some theorems on spherical geometry from Menelaus' Spherics (November 18 Special talk)

   The « Spherics » by Menelaus of Alexandria (1st-2nd c. A.D.) is the most important book ever written on spherical geometry. It is a profound work. It contains 91 propositions, and some of them are very difficult to prove. An edition, from Arabic texts (the Greek original does not survive), is being published now by De Gruyter, in their series Scientia Graeco-Arabica, No. 21.

   Link

   This publication contains in particular the first English translation of Menelaus' treatise. In this talk, I will explain some of the major theorems on spherical geometry contained in this work.

Week-3: Concentration Period: Geometry and Dynamics

1. **Jason Behrstock**

   Hierarchy Hyperbolic Spaces

   In the first lecture we will introduce the mapping class group (MCG) of a surface and discuss the curve complex of a surface. This leads to several very powerful tools developed by Masur, Minsky, Behrstock, and others which we will survey. In the second lecture we will discuss fundamental groups of compact cube complexes, focussing in particular on the interesting special case of right angled Artin groups (RAAGs). After introducing these groups, we will begin to discuss recent tools developed by Behrstock-Hagen-Sisto which are analogues of the machinery discussed for the mapping class group in the first lecture; we will give details of the tools for RAAGs and describe the general framework of a hierarchically hyperbolic space (HHS). In the third lecture we will outline some of the techniques for working with HHS and give a sketch of the proof of some applications, including the quasi-flats theorems, which proves a number of outstanding conjectures.

References:

1. This is a survey which is a good introduction to the topics which will be covered in this mini-course: "What is a hierarchically hyperbolic space?", by Alessandro Sisto, https://arxiv.org/abs/1707.00053.
2. This paper won't be discussed explicitly, but it contains and applies a number of the topics discussed in the first lecture, so is a good reference to be aware of. "Geometry and rigidity of mapping class groups", by Jason Behrstock, Bruce Kleiner, Yair Minsky, and Lee Mosher, https://arxiv.org/abs/0801.2006
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4. The main results of this paper will be discussed in lecture 3: "Quasiflats in hierarchically hyperbolic spaces", by Jason Behrstock, Mark Hagen, and Alessandro Sisto, https://arxiv.org/abs/1704.04271

2. **Keith Burns**

Ergodicity of the Weil-Petersson geodesic flow

The Weil-Petersson metric is a Riemannian metric on the Teichmueller space of a Riemann surface. It is invariant under the action of the mapping class group and descends to a Riemannian metric with volume on the moduli space.

The Weil-Peterson metric has negative sectional curvatures but is incomplete. The curvature and its derivatives blow up as one approaches the boundary of Teichmueller space. The effect of negative curvature on the behaviour of the geodesic flow is well understood. In particular Hopf and Anosov showed that the geodesic flow of a compact manifold with negative curvatures is ergodic. Their results extend to the Weil-Petersson geodesic flow, but the incompleteness of the metric creates considerable additional difficulties.

The lectures will attempt to explain the arguments of Hopf and Anosov and to indicate the additional ideas needed to apply them to the Weil-Petersson geodesic flow.

3. **Shrikrishna Dani**

Hyperbolic geometry, the modular group and Diophantine approximation

Study of dynamics of the geodesic flow associated with the modular surface, consisting of the Poincare plane viewed modulo the action of the modular group SL(2, Z) acting as isometries, has been applied to study the distribution of values of quadratic forms at points on the Euclidean plane with integer coordinates. In these talks we shall discuss the framework and some results in this respect. Relation with some other questions in the area of Diophantine approximation will also be discussed.

4. **Anish Ghosh**

   Dynamical systems on homogenous spaces and number theory

   I will survey some of S. G. Dani's many influential contributions to the above subject.

5. **Francois Labourie**

   Mini course 2: Introduction to Higgs bundles

   In this series of lecture, I will first start with very elementary classical result on line bundles on Riemann surfaces. Following this historical point of view, I will move to the definition of Higgs bundles. My goal is to motivate and explain the definition and state the major theorems of the field.

6. **Athanase Papadopoulos**

   Lecture 3 and 4 : The arc metric on Teichmüller space

   I will give an overview of the arc metric on the Teichmüller space of surfaces with boundary, including a study of its geodesics, its horofunction boundary, and the fact that the arc metric converges in an appropriate way to the Thurston metric of surfaces without boundary.

   Lecture 5 : Transitional geometry

   I will give an introduction to transitional geometry, that is, the subject of continuous passages between geometries.

7. **Pierre Will**

   Discrete groups in complex hyperbolic geometry

   I will discuss the geometry of complex hyperbolic space, and describe examples of dicrete subgroups of its isometry group. I will mainly focus on the complex hyperbolics analogues of quasi Fuchsian groups. My goal is to end up describing recent progresses concerning spherical CR structures on 3-manifolds, which appear on the boundary at infinity of quotients of the complex hyperbolic plane by discrete subgroups.

8. **Michael Wolf**

   Harmonic Maps between surfaces and Teichmuller theory, I and II

   We describe some of the basics of classical Teichmuller theory from the point of view of harmonic maps. The goal is to give an introduction to the Hitchin component of the PSL(2,R) character variety that one would undertake after one has identified a convenient setting and phrasing for the Hitchin equations, which in this case take the form of the 'Bochner equation' for harmonic maps of surfaces. We describe elements of the local deformation theory as well as the asymptotics. We will not assume any prior knowledge of harmonic maps.

Mahan Mj - Introduction to Hyperbolic groups

Example of hyperbolic groups (differential geometry): fundamental groups of closed negatively curved manifolds

For example let

$$\Gamma \curvearrowright H^n$$

act by isometries, properly discontinuously and cocompactly then

$$\frac{H^n}{\Gamma}$$

is a closed manifold.
Then

$$\pi_1\left(\frac{H^n}{\Gamma}\right)$$

is a *hyperbolic group*.

Example of hyperbolic groups (combinatorial): combinatorial group theory

Small cancellation groups.

The above two very *different* examples were unified by Gromov (1982-87) to give the definition of *hyperbolic groups*.

Here, we study

Definition. Geodesic metric spaces

Metric spaces

$$(X, d)$$

with a **geodesic** between every pair of points, that is $\forall p, q \in X$ there exists an isometric embedding

$$\begin{aligned}[0, d(p,q)] &\to X \\ 0 &\mapsto p \\ d(p,q) &\mapsto q\end{aligned}$$

are called **geodesic metric spaces**.

Example of geodesic metric spaces (differential geometric): complete Riemannian manifolds

Hopf-Rinow theorem.

Example of geodesic metric spaces (discrete): connected graphs with d defined by edge length 1

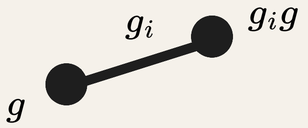
Any such graph, by connectedness has a *geodesic* between every pair of points.
For example, the **Cayley graph of a finitely generated group**

$$G = \langle \underbrace{g_1, \dots, g_n}_{=:S} \mid r_i, i \in J \rangle$$

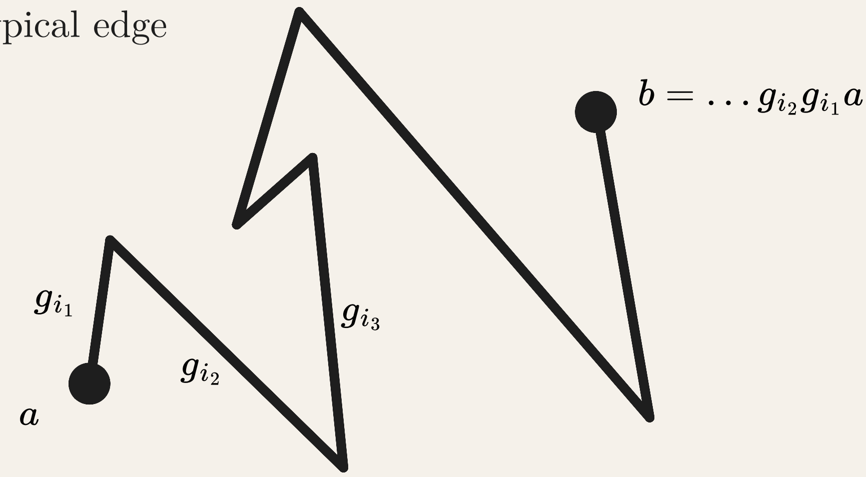
Then the Caley graph

$$\Gamma(G, S) := \begin{cases} V := \{g \mid g \in G\} = G \\ E := \{(g,h) \mid g^{-1}h \in S\} \end{cases}$$

has edges between two points if they are related by a left multiplication by generators in S



A typical edge



A path between two points

Thus a path between $a, b \in G$ gives a word g_α in S that expresses b as

$$b = g_\alpha b$$

So length of this path is the number of edges. Thus distance $d(a, b)$ is the minimum length of word in S required to write $b = g_\beta a$.

length spaces

Definition. Length of a path in a metric space

Let (X, d) be a metric space and

$$\gamma : I \subseteq \mathbb{R} \to X$$

be a curve where I is a non-empty interval ($\implies$ connected) set. Then **length of γ** is

$$\mathfrak{L}(\gamma) := \sup \left\{ \sum_{i \in I} d(\gamma(t_{i-1}), \gamma(t_i)) \middle| \text{all sequences } t_0 \leq t_1 \leq \cdots \leq t_k \text{ in } I \right\}$$

We say γ is **rectifiable** if $\mathfrak{L}(\gamma) < \infty$.

Definition. Length metric of a metric space

Let (X, d) be a metric space. The **length metric** associated to d is the function

$$d_{\mathsf{L}} : X \times X \to [0, \infty]$$
$$(x, y) \mapsto \inf\{\mathfrak{L}(\gamma) | \gamma : [0, 1] \to X \text{ rectifiable path} \gamma(0) = x, \gamma(1) = y\}$$

The metric space (X, d) is a **length space** if $d = d_{\mathsf{L}}$.

Proposition:

- The function d_{L} is a metric on X $\iff$ it is finite $\iff$ every pair of points in X can be joined by a rectifiable curve
  - In particular, a length space has this property.
- $d_{\mathsf{L}} \geq d$
- $(d_{\mathsf{L}})_{\mathsf{L}} = d_{\mathsf{L}}$

Example

For a connected Riemannian manifold (X, g) the induced metric space (X, d) is a **length space**.

Proposition: Every geodesic space is a length space.

As

$$d \leq d_{\mathsf{L}}$$

and

$$d_{\mathsf{L}}(x, y) \leq \mathfrak{L}(\gamma_{x,y}) = d(x, y)$$

We conclude

$$d = d_{\mathsf{L}}$$

(Hopf-Rinow, Cohn Vossen 1935) Let X be a complete, locally compact length space. Then X is proper (every closed and bounded subset of X is compact) and a geodesic space.

quasi-isometries

Intuition

We need a notion of morphism that is *robust* enough so that $\Gamma(G, S)$ is independent of the choice of the generating set S

$$\Gamma(G, S_1) \xrightarrow{\mathrm{Id}_G} \Gamma(G, S_2)$$

should be a morphism in the category of complete metric spaces we're working with. This is precisely **quasi-isometries**.

Definition. Quasi-isometric embedding

A **K, ϵ-quasi-isometric embedding** between complete metric spaces is a map

$$f : (X, d) \to (Y, \rho)$$

if $\forall x_1, x_2 \in X$ we have

$$\frac{1}{K} d(x_1, x_2) - \epsilon \leq \rho(f(x_1), f(x_2)) \leq K d(x_1, x_2) + \epsilon$$

Definition. Quasi-isometies

A **(K, ϵ, C)-quasi-isometry** of complete metric spaces

$$f : (X, d) \to (Y, \rho)$$

is a *(K, ϵ)-quasi-isometric embedding* and a C-coarsely surjective map, that is, the ball of radius C around the image is entirety of Y C

$$Y = B_C(f(X))$$

Warning

Quasi-isometries *may* be discontinuous!

Slogan

- Calculus, differential geometry $\leftrightarrow$ good eyesight
- quasi-differential geometry $\leftrightarrow$ bad eyesight, coarse graining

Proposition: A (K, ϵ, C)-quasi-isometry f is equivalent to an (K, ϵ)-quasi-isometric embedding f and a (K', ϵ') quasi-isometric embedding such that

$$f \circ g, g \circ f$$

are C'-close to **Id** for some $C' > 0$, that is

$$d(g \circ f(x), x), d(f \circ g(y), y) \leq C'$$

Proposition: Let S_1, S_2 are finite symmetric ($g \in S_1 \iff g^{-1} \in S_1$) generating set of a group G. Then

$$\mathsf{Id}_G : \Gamma(G, S_1) \to \Gamma(G, S_2)$$

(on the vertex set) is a **quasi-isometry**.

Take

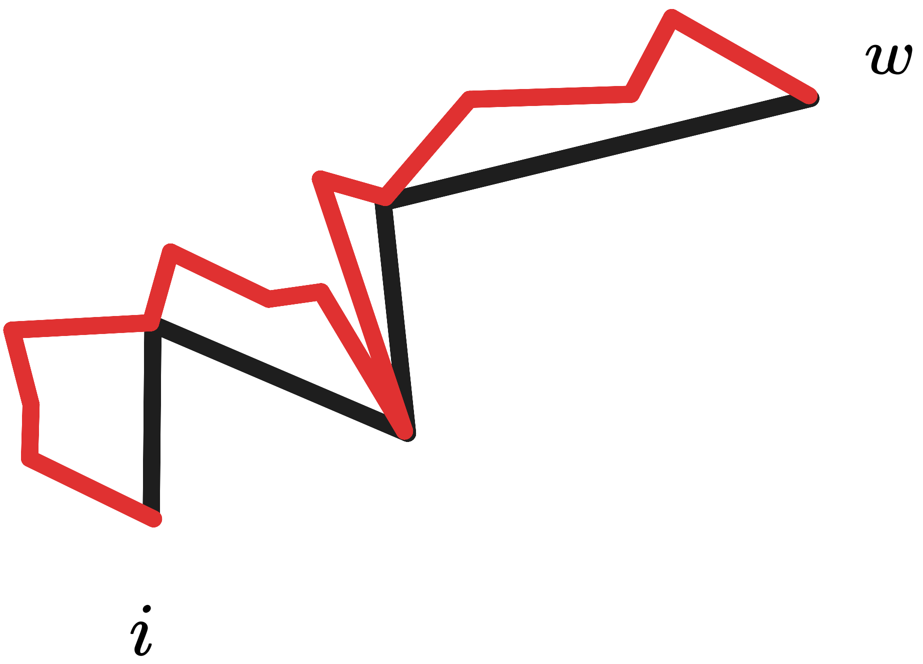
$$S_1 = \{g_1, \ldots, g_n\}$$
$$S_2 = \{h_1, \ldots, h_m\}$$

- Then each h_i is a word in $\{g_1, \ldots, g_n\}$ of some length l_i. Now

$$L := \max\{l_i\}$$

- This means

$$d_1(i, w) \leq L d_2(i, w)$$



- Doing the opposite we have

$$d_2(i, w) \leq L' d_1(i, w)$$

- Hence we obtain a bi-Lipschitz map $\mathbf{Id}_G$.

Definition. **A geometric group action is one that acts free, properly discontinuous, cocompact and by isometries.**

(Milnor-Svarc lemma for complete Riemannian manifolds) Let M be a complete Riemannian manifold and Γ be a finitely generated group acting geometrically on M. Then

$$(M, g) \cong_{\text{quasi}} \Gamma$$

Kathryn Mann - Dynamics of Group Actions

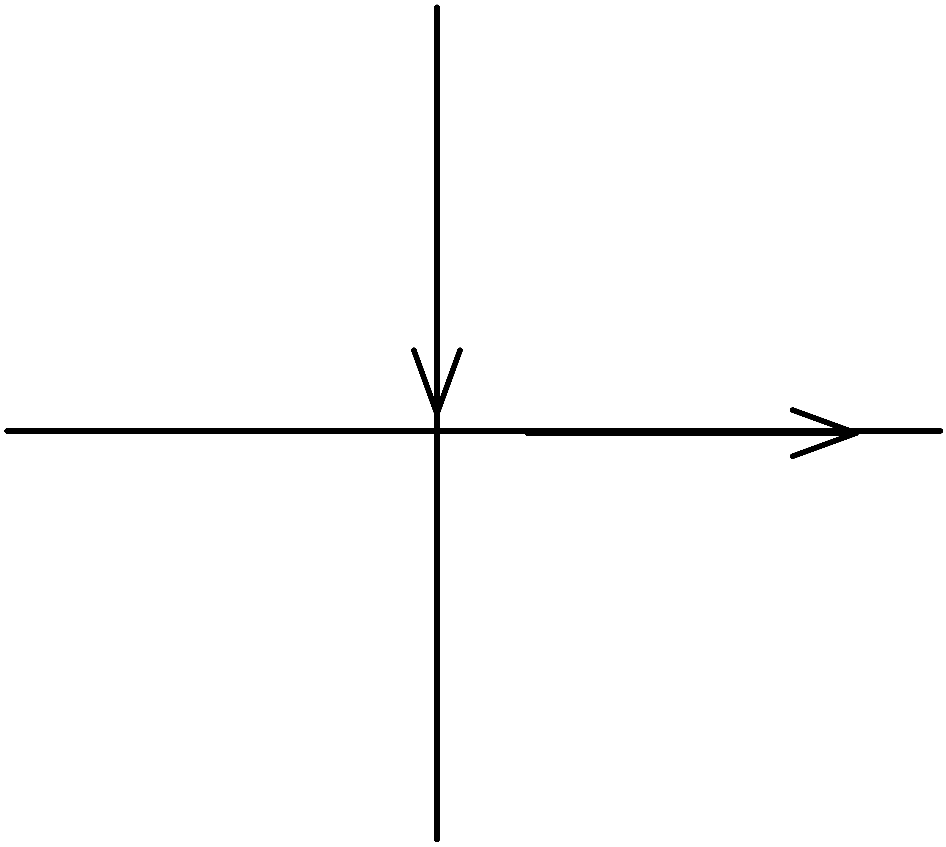
The broad theme of this minicourse is the relationship between the dynamics of a group acting on a space, the algebraic structure of the group, and some geometric structure responsible for the action; and how one of these can often determine the other two. A paradigm example of this is the "convergence group theorem" of Tukia, Gabai, and Casson--Jungries, which says that a convergence group (a purely dynamical condition) acting on the circle is a Fuchsian group, i.e isomorphic to the fundamental group of a hyperbolic surface or orbifold; and the action on the circle is the action on the boundary of the surface or orbifold's universal cover.

In a similar spirit, Thurston defined an "extended convergence group" for groups acting on the line, and showed using ideas of Barbot and Fenley that these are always fundamental groups of 3-manifolds, and the action on the line comes from an Anosov flow on the 3-manifold. However, not all 3-manifolds with Anosov flows give rise to extended convergence groups; instead giving rise to a more general class called "Anosov-like actions on bi-foliated planes"

In the course, we'll study Anosov-like actions from the axioms, and see how some simple dynamical assumptions can lead to a rich structure theory. This is a perspective that Barthelmé, Bonatti, Fenley, Frankel and I (in various combinations and in still ongoing work) have recently successfully used towards the study and classification of Anosov flows on 3-manifolds, building on work of Barbot, Fenley, and others.

- geodesic flow on round n-sphere
    - trajectories periodic
    - global instability
- geoedsic flows of n-manifolds with negative curvature
    - trajectories are "chaotic"
    - global stability (Anosov '60)
- "looks" like iterations of

$$\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}, \lambda > 1$$



Anosov flows

Definition. A flow on M is **Anosov** if $\exists$ continous splitting

$$TM = \underbrace{X}_{\text{flow field}} \oplus \underbrace{E^{\mathrm{ss}}}_{\text{contracted}} \oplus \underbrace{E^{\mathrm{uu}}}_{\text{expanded}}$$

where contracted means

$$\exists C, K > 0 : v \in E^{\mathrm{ss}}, t > 0 \implies \left\|\phi_*^t(v)\right\| < Ce^{-|t|K} \,||\, v \,||$$

Definition. A flow on M is **Anosov** if $\exists$ continous splitting

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where contracted means

$$\exists C, K > 0 : v \in E^{\mathrm{ss}}, t > 0 \implies \left\|\phi_*^t(v)\right\| < Ce^{-|t|K} \,||\, v \,||$$

$X \oplus E^{\mathrm{ss}}$ is tangent to some foliation $\mathcal{F}^{\mathrm{s}}$ invariant under ϕ^t and similarly for $X \oplus E^{\mathrm{uu}}$ is tangent to $\mathcal{F}^{\mathrm{u}}$

Definition. Two flows ϕ^t, ψ^t on manifolds M_1, M_2 are **topologically orbit equivalent** if there is a homeomorphism $h : M_1 \to M_2$ that takes orbits of ϕ to orbit of ψ.

Two Anosov flows that are **$\mathcal{C}^1$-stable**, that is, if the two vector fields that generate the Anosov flows are $\mathcal{C}^1$-close, then the flows are ***topologically orbit equivalent***.

examples: suspensions and geodesic flows

<table>
<tr><th>manifold</th><th>flow</th><th>stable and unstable</th></tr>
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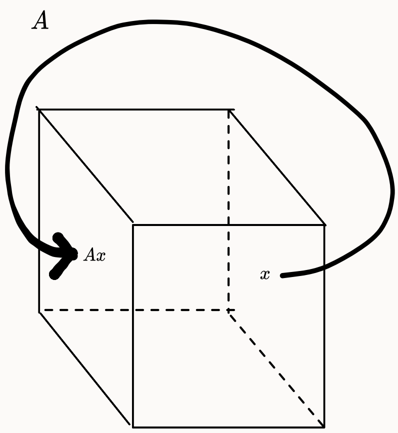
$$M := \frac{\frac{\mathbb{R}^2}{\mathbb{Z}^2} \times [0,1]}{(x,1) \sim (Ax, 0)}$$



with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

where the flow is

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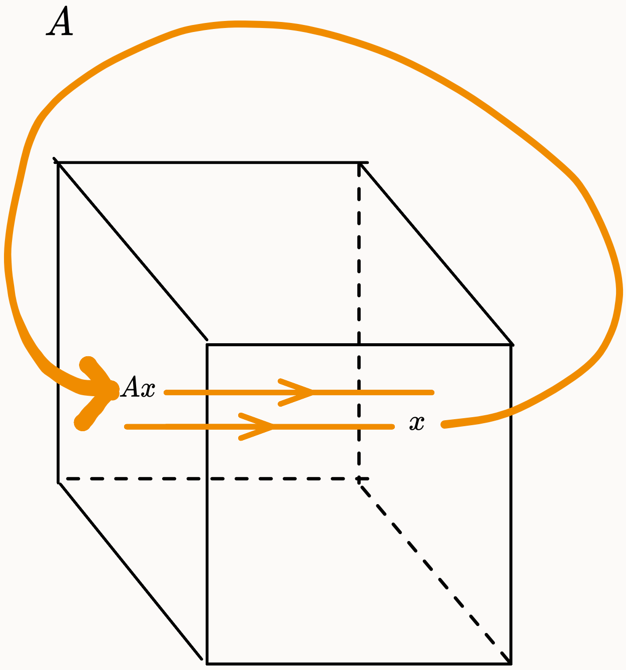

We may write the flow on

$$M = \frac{\frac{\mathbb{R}^2}{\mathbb{Z}^2} \times \mathbb{R}}{(x,s) \sim (Ax, s-1)}$$

as

$$\phi^t(x,s) := (x, s+t)$$

[3]M has a SOL (metric) structure and for this metric Anosov conditions hold with $C = 1$

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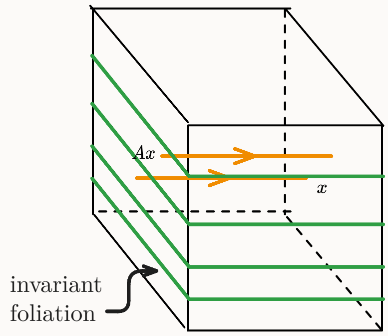
here E^{ss} and E^{uu} are eigendirections of A with foliations



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unit tangent bundle

$$T^{\mathsf{u}}(N)$$

of a hyperbolic surface N

</td>
<td></td>
<td>horocylce foliations</td>
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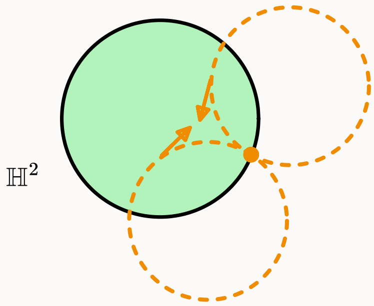
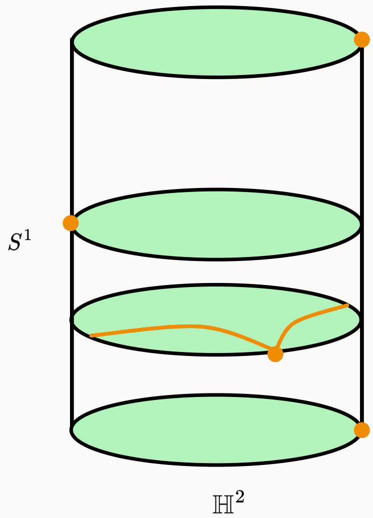
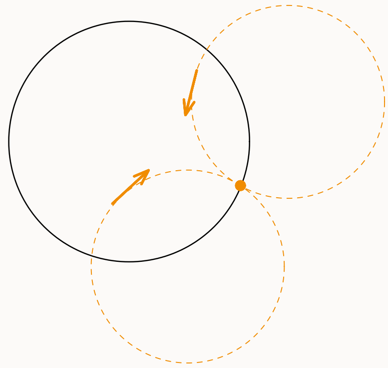
manifold	flow	stable and unstable
$T^u\mathbb{H}^2 = \mathbb{H}^2 \times S^1$  		the foliation is given by 

$$\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & -\frac{i}{2} \end{bmatrix}$$

the classification problem

> **Proposition:** stability $\implies$ upto orbit equivalence there are only countably many Anosov flows on a given compact smooth manifold

Classification problem of Anosov flows (Smale, '66)

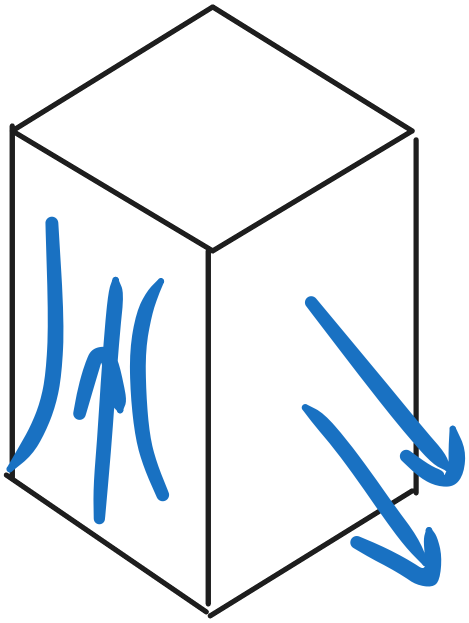
We hope to classify all Anosov flows

> **Proposition:** stability $\implies$ upto orbit equivalence there are only countably many Anosov flows on a given compact smooth manifold

on all (countably many) compact smooth manifolds by discrete algebraic invariants.

- For $\dim > 3$, still very few examples known.

- For $\dim = 3$, there are many examples beyond suspensions and geodesic flows.
  - '70s-80s: New examples on 3-manifolds via "*surgery*"
  - 80s Goodman: $(1, n)$ Dehn surgery procedure: cut and glue along annulus transverse to the flow
  - Fried: Dehn surgeries along periodic orbits
  - Bonatti Langevin: "building-block construction" define a flow on $[0,1]^3$ and



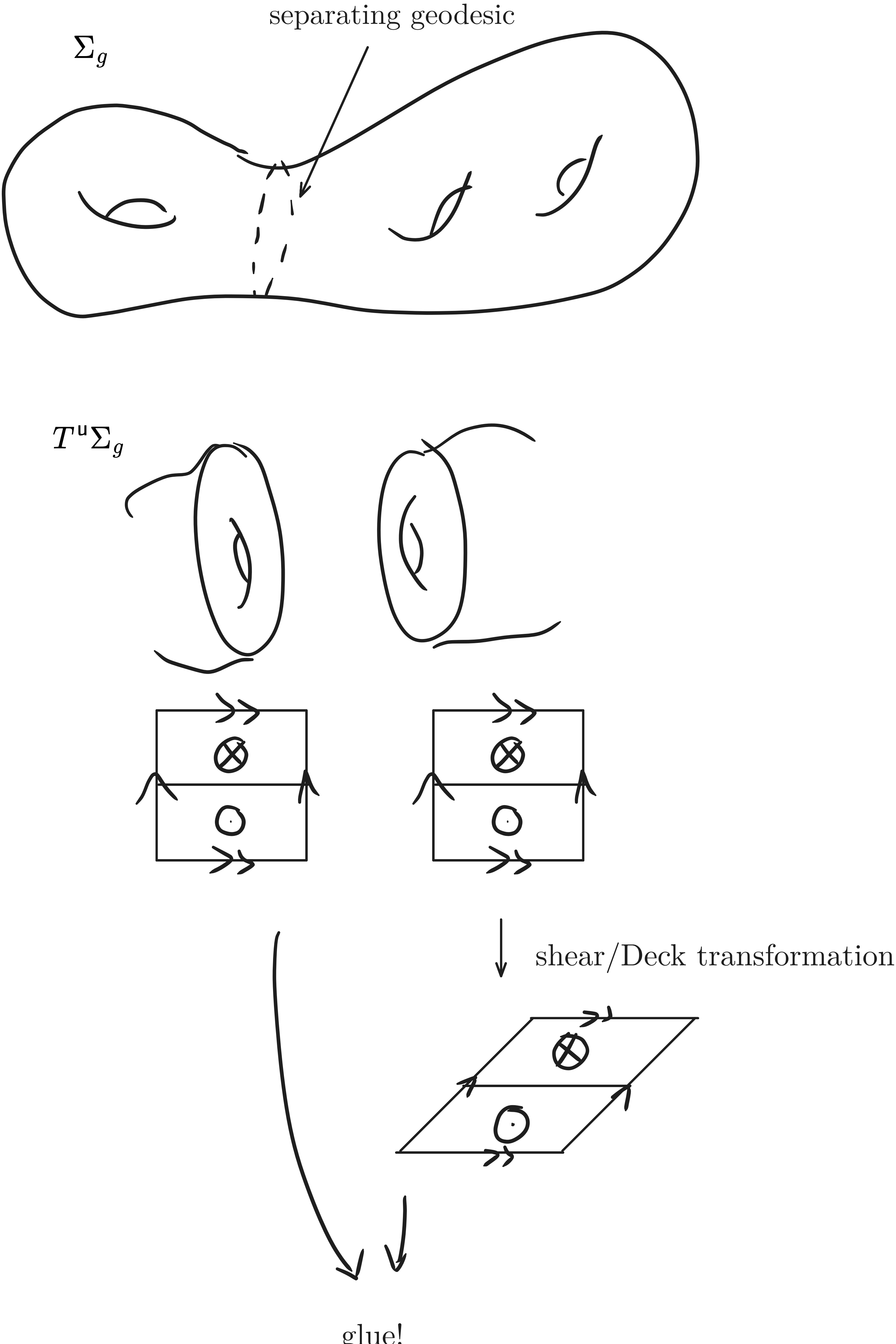
  - More recently, Bengnin Bonatti-Yi: Graph manifolds: $\forall n$ there exists a compact Graph manifold with $\geq n$ distinct (orbit eqv) classes of Anosov flows
  - Clay-Pinky: different approach
  - Boaden-Mann: $\forall n$ there exists a hyperbolic ${}^3M$ with $\geq n$ Anosov flows

Classification problem

- Which 3 manifolds admit a Anosov flow?
  - obstructions are known: necessary conditions for ${}^3M$ to have Anosov flows:
    - aspherical (have universal cover is contractible),
    - fundamental group has exponential growth
    - $\pi_1({}^3M)$ is left orderable
    - ...
- How to classify all examples on a given manifold?
  -

'80: *Anosov Flows on New Three Manifolds*, Michael Handel and William P. Thurston

[1]


separating geodesic
Σ_g
$T^{\mathsf{u}}\Sigma_g$
shear/Deck transformation
glue!

Here the manifold obtained is new, and the new flow is Anosov.

orbit space

Definition. Orbit space of a Anosov flow

Given flow ϕ^t on ${}^3M$ lift to $\tilde{\phi}^t$ on $\tilde{M}$ universal cover. Then **orbit space of ϕ^t** is

$$\mathcal{O}_\phi := \frac{\tilde{M}}{x \sim \phi^t(x)}$$

(Barbot, Fenley) If ϕ^t is Anosov on M then the universal cover $\tilde{M} \cong \mathbb{R}^3$ and

$$\mathcal{O}_\phi \cong \mathbb{R}^2$$

Moreover $\pi_1({}^3M) \curvearrowright \tilde{M}$ (preserves $\tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u$) descends to an action on $\mathcal{O}_\phi$ by homeomorphisms (which preserves induced foliations).

(Barbot) Let ϕ^t, ψ^t are Anosov flows on ${}^3M$. They are orbit equivalent via a map $\Phi \iff \exists$ a homeomorphism

$$h : \mathcal{O}_\phi \to \mathcal{O}_\psi$$

that takes stable/unstable foliations to foliations such that

$$\forall \gamma \in \pi_1(M), h \circ \gamma = \Phi(\gamma) \circ h$$

references

https://drive.google.com/file/d/1JMfXIAs6i6f8YXAERFWHz-HVhd0mrvPv/view

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1. EUDML | Anosov Flows on New Three Manifolds. ↩

ICTS '10 - Geometry Topology and Dynamics in Negative Curvature

https://www.icts.res.in/sites/default/files/1280718981TitleAbstract.pdf

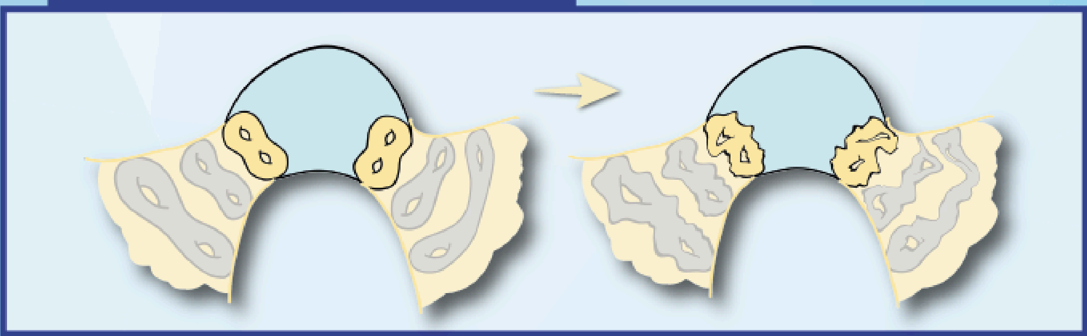
Martin Bridson

space	$K(\mathbb{Z}^n, 1)$	$K(\Sigma_g, 1)$	$K(F_n, 1)$
group	$SL(n, \mathbb{Z})$	$\mathrm{MCG}_g \cong \mathrm{Out}(\Sigma_g)$	$\mathrm{Out}(F_n)$
acts by	diffeomorphisms	diffeomorphisms	homotopy equivalence
classifying space?	$\mathbb{R}^? \cong \dfrac{SL(n,\mathbb{R})}{SO(n,\mathbb{R})}$	Teichmuller space $\mathscr{T}_g$ $\cong \mathbb{R}^{6g-g}$	outer space, a finite dimensional contractible space
which is a	complete CAT(0) manifold	incomplete CAT(0)	no non-positive curved metric
the group is generated by		Dehn twists	homotopy equivalence
		pseudo-Anosov	
group automorphisms			
quotients	$PSL(n, \mathbb{Z})$ or finite quotients like $SL(n, \mathbb{Z}_m)$	$1 \to \mathrm{Tor}_g \to \mathrm{MCG}_g \to Sp(2g, \mathbb{Z})$	Take characteristic $H < F_n$ we get $\mathrm{Out}(F_n) \to \mathrm{Out}\left(\dfrac{F_n}{H}\right.$ for example subgroup in lower central series?

IHEs '23 Homogeneous Dynamics and Geometry in Higher-Rank Lie Groups

– WORKSHOP –

HOMOGENEOUS DYNAMICS AND GEOMETRY IN HIGHER-RANK LIE GROUPS



June 19 > 23 2023

IHES
Marilyn and James Simons Conference Center
35 route de Chartres
91440 Bures-sur-Yvette

Organizers
Martin Bridgeman, *Boston College*
Richard Canary, *Univ. of Michigan*
Fanny Kassel, *CNRS & IHES*
Hee Oh, *Yale Univ.*
Maria Beatrice Pozzetti, *Univ. Heidelberg*
Jean-François Quint, *CNRS & Univ. de Bordeaux*

Information:
WWW.IHES.FR

Minicourses

> **Jean-François Quint**, *CNRS & Univ. de Bordeaux*
"Dynamics on Higher-rank Lie Groups"

> **Tengren Zhang**, *National Univ. of Singapore*
"Anosov Representations"

> **Andrés Sambarino**, *CNRS & IMJ-PRG*
& **Minju Lee**, *Univ. of Chicago*
"Recent Developments"

Speakers

> **Pierre-Louis Blayac**, *Univ. of Michigan*
> **Leon Carvajales**, *Univ. de la Republica*
> **Nguyen-Thi Dang**, *Univ. Paris-Saclay*
> **Sam Edwards**, *Durham Univ.*
> **François Labourie**, *Univ. Côte d'Azur*
> **Sara Maloni**, *Univ. of Virginia*
> **Giuseppe Martone**, *Yale Univ.*
> **Wenyu Pan**, *Univ. of Toronto*
> **Rafael Potrie**, *Univ. de la Republica*
> **Pratyush Sarkar**, *UC San Diego*
> **Andrew Zimmer**, *Univ. of Wisconsin*

FONDATION MATHÉMATIQUE JACQUES HADAMARD

STRUCTURES CLUSTER OF EXCELLENCE

L'IHES, membre fondateur de université PARIS-SACLAY

Homogeneous Dynamics and Geometry in Higher-Rank Lie Groups

The goal of the workshop is to explore links between homogeneous dynamics and recent developments on infinite-covolume discrete subgroups of Lie groups, including images of Anosov representations and generalizations. There will be three minicourses and a dozen research talks.

This workshop is organized by **Martin Bridgeman** (Boston College), **Richard Canary** (University of Michigan), **Fanny Kassel** (CNRS & IHES), **Hee Oh** (Yale University), **Maria Beatrice Pozzetti** (Universität Heidelberg), and **Jean-François Quint** (CNRS & Université de Bordeaux).

Minicourses:

- **Jean-François Quint** (CNRS & Université de Bordeaux) - *"Dynamics on Higher-rank Lie Groups"* [3h]
- **Tengren Zhang** (National University of Singapore) - *"Anosov Representations"* [3h]
- **Andrés Sambarino** (CNRS & IMJ-PRG) & **Minju Lee** (University of Chicago) - *"Recent Developments"* [2+2h]

Research Talks:

- **Pierre-Louis Blayac** (University of Michigan)
- **León Carvajales** (Universidad de la Republica, Montevideo)
- **Nguyen-Thi Dang** (Université Paris-Saclay)
- **Sam Edwards** (Durham University)
- **François Labourie** (Université Côte d'Azur)
- **Sara Maloni** (University of Virginia)
- **Giuseppe Martone** (Yale University)
- **Wenyu Pan** (University of Toronto)
- **Rafael Potrie** (Universidad de la Republica, Montevideo)
- **Pratyush Sarkar** (UC San Diego)
- **Andrew Zimmer** (University of Wisconsin, Madison)

https://www.youtube.com/playlist?list=PLx5f8IelFRgENNBbT_A7c15Cbk1oaqEsF

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Uri Bader - A course in linear groups and ergodic theory

Consider

$$\underbrace{SL(2,\mathbb{R})}_{=:G} \curvearrowright \underbrace{\mathbb{R}^2}_{=:X}$$

There are two orbits

$$\{0\}$$

and

$$\mathbb{R}^2 \setminus \{0\}$$

(disjoint of course).

By orbit-stabilizer theorem

$$\mathbb{R}^2 \setminus \{0\} \cong \frac{G}{\mathcal{U}}$$

where stabilizer is the subgroup of strictly upper triangular matrices

$$\mathcal{U} := \mathrm{stab}_G \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\} = \exp \left\{ \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \right\} \cong (\mathbb{R}, +)$$

Question

Find all G-invariant maps

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2$$

that is

$$\forall g \in G, x \in \mathbb{R}^2, \phi(gx) = g\phi(x)$$

Because G-orbits will go to G-orbits under such a map we conclude

$$\phi(0) = 0$$

But if we choose where one point of a orbit will go, then we know where the other elements will go

$$\underbrace{g\phi(x)}_{\text{known}} = \underbrace{\phi(gx)}_{\text{unkown}}$$

But to know

$$\phi\begin{pmatrix}1\\0\end{pmatrix}$$

we observe

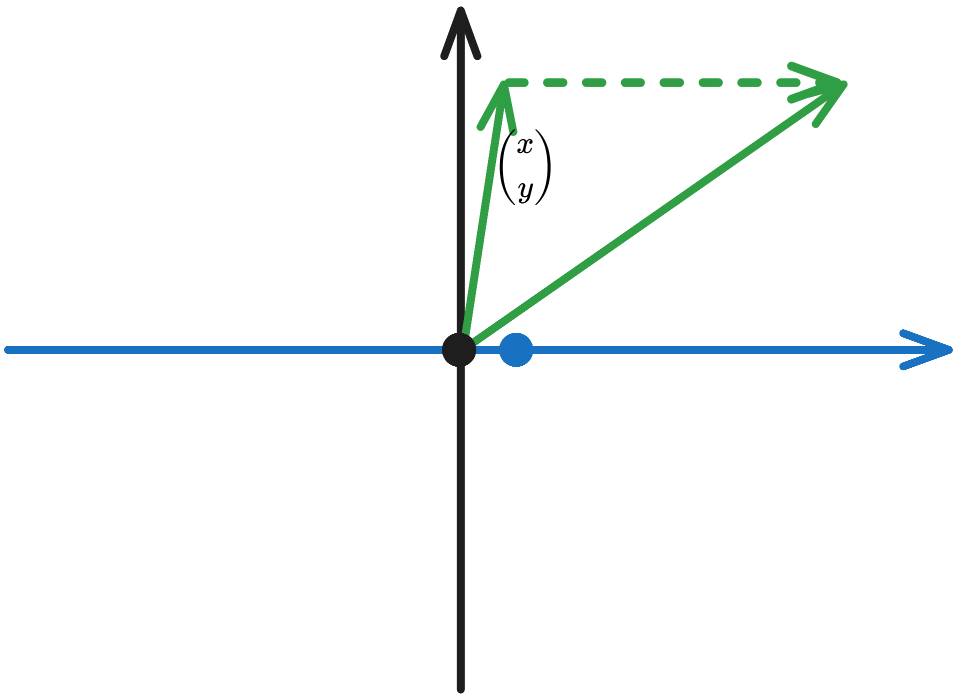
$$g\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \implies \phi\begin{pmatrix}1\\0\end{pmatrix} = \phi(g\begin{pmatrix}1\\0\end{pmatrix}) = g\phi\begin{pmatrix}1\\0\end{pmatrix}$$

means

$$\mathcal{U} \subset \mathrm{stab}_G(\phi\begin{pmatrix}1\\0\end{pmatrix})$$

So we must look at *the orbit structure of $\mathcal{U} \curvearrowright \mathbb{R}^2$* to know what point does it *stabilizes*:

$$\begin{bmatrix}1 & \alpha\\0 & 1\end{bmatrix} \in \mathcal{U} \implies \begin{bmatrix}1 & \alpha\\0 & 1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix} + \alpha\begin{pmatrix}y\\0\end{pmatrix}$$



So

$$\mathcal{U} \subset \mathrm{stab}_G(\phi\begin{pmatrix}1\\0\end{pmatrix}) \iff \phi\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\alpha\\0\end{pmatrix}$$

Then

$$\phi(y) = \phi(g\begin{pmatrix}1\\0\end{pmatrix}) = g\phi\begin{pmatrix}1\\0\end{pmatrix} = g\begin{pmatrix}\alpha\\0\end{pmatrix} = \alpha g\begin{pmatrix}1\\0\end{pmatrix} = \alpha y$$

Thus, every G-invariant map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a scaling

$$\phi(x) = \alpha x$$

by some $\alpha \in \mathbb{R}$, which is invertible if $\alpha \in \mathbb{R}^\times$.

Exercise

$$\mathrm{Aut}G\mathsf{Set}\left(\frac{G}{H}\right) \cong \frac{N_G(H)}{H}$$

For our case $H = \mathcal{U}$

$$N_G(\mathcal{U}) = \left\{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}\right\} \cong \mathbb{R}^\times \times \mathbb{R}$$

and

$$\frac{N_G(\mathcal{U})}{\mathcal{U}} \cong \mathbb{R}^\times$$

Chow Lu Ni - Hamilton's Ricci flow

chapter 2

standard shrinking sphere

Proposition: Let $M = S^n$ and g_{S^n} be the standard metric on S^n. Then for $r_0 > 0$

$$g_0 := r_0^2 g_{S^n}$$
$$g(t) := (r_0^2 - 2(n-1)t) g_{S^n}$$

is the solution (called **standard shrinking n-sphere** of initial radius r_0) to *Ricci flow* with $g(0) = g_0$ for maximal time interval

$$(-\infty, r_0^2/2(n-1))$$

Gerhard Huisken - Ricci flow

https://www.mfo.de/about-the-institute/staff/prof-dr-gerhard-huisken/lectures/introduction-to-ricci-flow

Einstein metrics and Ricci solitons

[1]

Definition. (M, g) **is an Einstein manifold if**

$$\mathrm{Ric}g = \lambda g$$

where $\lambda \in \mathcal{C}^\infty(M)$

- Taking trace both sides we have

$$\mathrm{RicS} = \lambda \dim M \implies \lambda = \frac{\mathrm{RicS}}{\dim M}$$

Definition. Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}g(t)$$

curvature in 2 dim

The Riemann curvature tensor

$$R_{ijkl}$$

is determined by one component

$$R_{1212}$$

so if a tensor is equal to the Riemann tensor if it has the same symmetries and

---

1. https://www.youtube.com/watch?v=ZUCq0oMb-FE ↩

IMPA19 School of Symplectic Topology

School of Symplectic Topology (2019)

Floer homologies

Let

$$(M, \omega)$$

be a symplectic manifold and

$$H : \mathbb{R} \times M \to \mathbb{R}$$

be smooth.

Then

$$X_H := -\omega^{-1} \mathrm{d}H$$

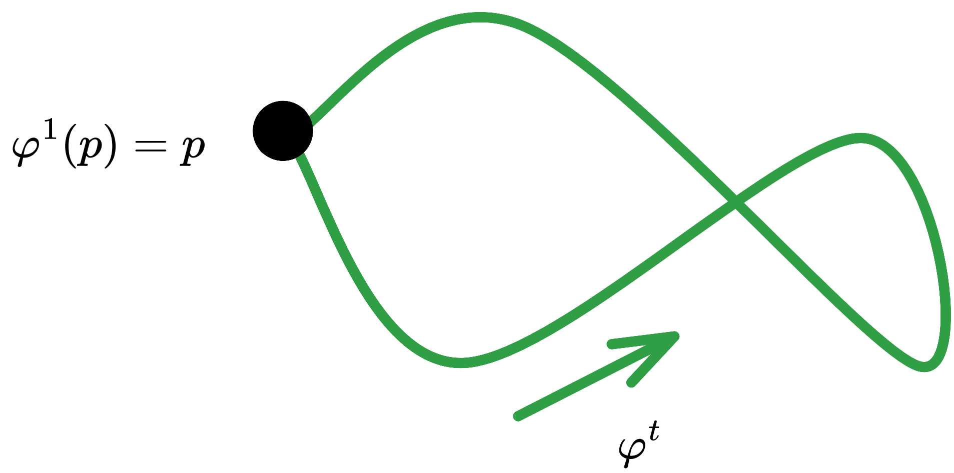
is its Hamiltonian vector field and

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_H^t = X_{\varphi_H^t}$$

is its flow.

Then φ^t is a symplectomorphism

$$(\varphi_H^t)^* \omega = \omega$$



Then 1-periodic orbits are fixed points of φ^1.

We can reparameterize to get a new

$$H : \frac{\mathbb{R}}{\mathbb{Z}} \times M \to \mathbb{R}$$

which has same 1-periodic orbits.

Arnold conjecture

The number of fixed points of the Hamiltonian diffeomorphism is bounded from below

$$\left|\mathrm{fix}(\varphi_H^1)\right| \geq \sum_k \dim H_{\mathsf{M}k}(M, \mathbb{Z}_2)$$

when φ_H^1 is non-degenerate.

Morse homology

Definition. Morse functions

Let M be a compact smooth manifold and

$$f : M \to \mathbb{R}$$

and $p \in \mathrm{crit}(f)$. Then we have a symmetric bilinear form

$$H_p : T_pM \times T_pM \to \mathbb{R}$$

The critical point p is **non-degenerate** if H_p is a non-degenerate bilinear form.
The function f is a **Morse function** if all its critical points are non-degenerate.

(Morse lemma) For a Morse function $f \in \mathcal{C}^\infty(M)$ and a critical point p, there exists a neighborhood of p and a chart on U on which f looks like

$$f(p) - x_1^2 - x_2^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$$

Definition. Morse index of a critical point

Morse index of a critical point of a Morse function is the i appearing in

(Morse lemma) For a Morse function $f \in \mathcal{C}^\infty(M)$ and a critical point p, there exists a neighborhood of p and a chart on U on which f looks like

$$f(p) - x_1^2 - x_2^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$$

Definition. Smale condition

A pseudo-gradient field satisfies the Smale condition if for all $p, q \in \mathrm{crit}(f)$ the unstable and stable manifold of p, q respectively intersect transversally

$$W^{\mathsf{u}}(p) \pitchfork W^{\mathsf{s}}(q)$$

Their intersection is a manifold in this case.

$$\dim W^{\mathsf{u}}(p) \pitchfork W^{\mathsf{s}}(q) = \mathrm{ind}(p) - \mathrm{ind}(q)$$

Definition. The space of trajectories connecting p to q is

$$\frac{W^{\mathsf{u}}(p) \pitchfork W^{\mathsf{s}}(q)}{x \sim \varphi^s(x)} =: L(p,q)$$

whose dimension is $\mathrm{ind}(p) - \mathrm{ind}(q) - 1$.

Definition. Morse homology

The chain complex with vector spaces

$$\mathrm{C}_k := \mathbb{Z}_2\{p \in \mathrm{crit}(f) \mid \mathrm{ind}(p) = k\}$$

with boundary operator

$$\begin{aligned} \partial_k : \mathrm{C}_k &\to \mathrm{C}_{k-1} \\ p &\mapsto \partial_k(p) := \sum_{q \in \mathrm{crit}_{k-1}(f)} (|L(p,q)| \bmod 2) q \end{aligned}$$

where $|L(p,q)| \bmod 2$ counts the number of trajectories mod 2.

$$\sum_{q \in \mathrm{cir}_k} (|L(p,q)| \bmod 2)(|L(q,r)| \bmod 2) = 0$$

Denis Auroux - Fukaya categories and mirror symmetry

【Fukaya范畴与镜像对称】【Columbia】Fukaya categories and mirror symmetry by Denis Auroux_哔哩哔哩_bilibili

http://web.archive.org/web/20191224163140/http://math.columbia.edu/~topology/L1_Luis.pdf

A Pirkovskii - Harmonic Analysis and Banach Algebras

Harmonic Analysis and Banach Algebras. A. Pirkovskii. Fall 2024.

Abstract

Harmonic analysis on locally compact abelian groups is a natural generalization of the classical Fourier analysis usually studied by undergraduate students in mathematics (that is, of the theory of trigonometric Fourier series and of the Fourier transform on the real line). The most elegant approach to harmonic analysis on abelian groups is based on the theory of commutative Banach algebras, which was initiated by Gelfand in the early 1940s and was further developed by Raikov, Naimark, Shilov and many other brilliant mathematicians. This approach, in particular, yields a relatively simple analytic proof of the Pontryagin duality based on the Plancherel theorem. In this course, we discuss the basics of Banach algebra theory and apply it to constructing the harmonic analysis on a locally compact abelian group. If time permits, some nonabelian groups will also be considered.

the circle

The Fourier series of $f \in L^2(S^1)$ states

$$f = \sum_{k \in \mathbb{Z}} \hat{f}(k) \chi_k$$

where

$$\begin{aligned} \chi_k : S^1 &\to S^1 \\ z &\mapsto z^k \end{aligned}$$

$$\chi_k(ab) = \chi_k(a)\chi_k(b)$$

so

$$\chi_k \in \mathrm{Hom}\textsf{TopGrp}(S^1, S^1)$$

Definition. **If G is an Abelian topological group, we say the Abelian group of all homomorphisms $G \to S^1$**

$$\hat{G} := \mathrm{Hom}\textsf{TopGrp}(G, S^1)$$

that is, the group of all unitary characters of G, is the dual group of G.

Thus we may write

$$f = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi$$

as we shall show that all unitary characters of S^1 are of the form χ_k for some $k \in \mathbb{Z}$.

overview

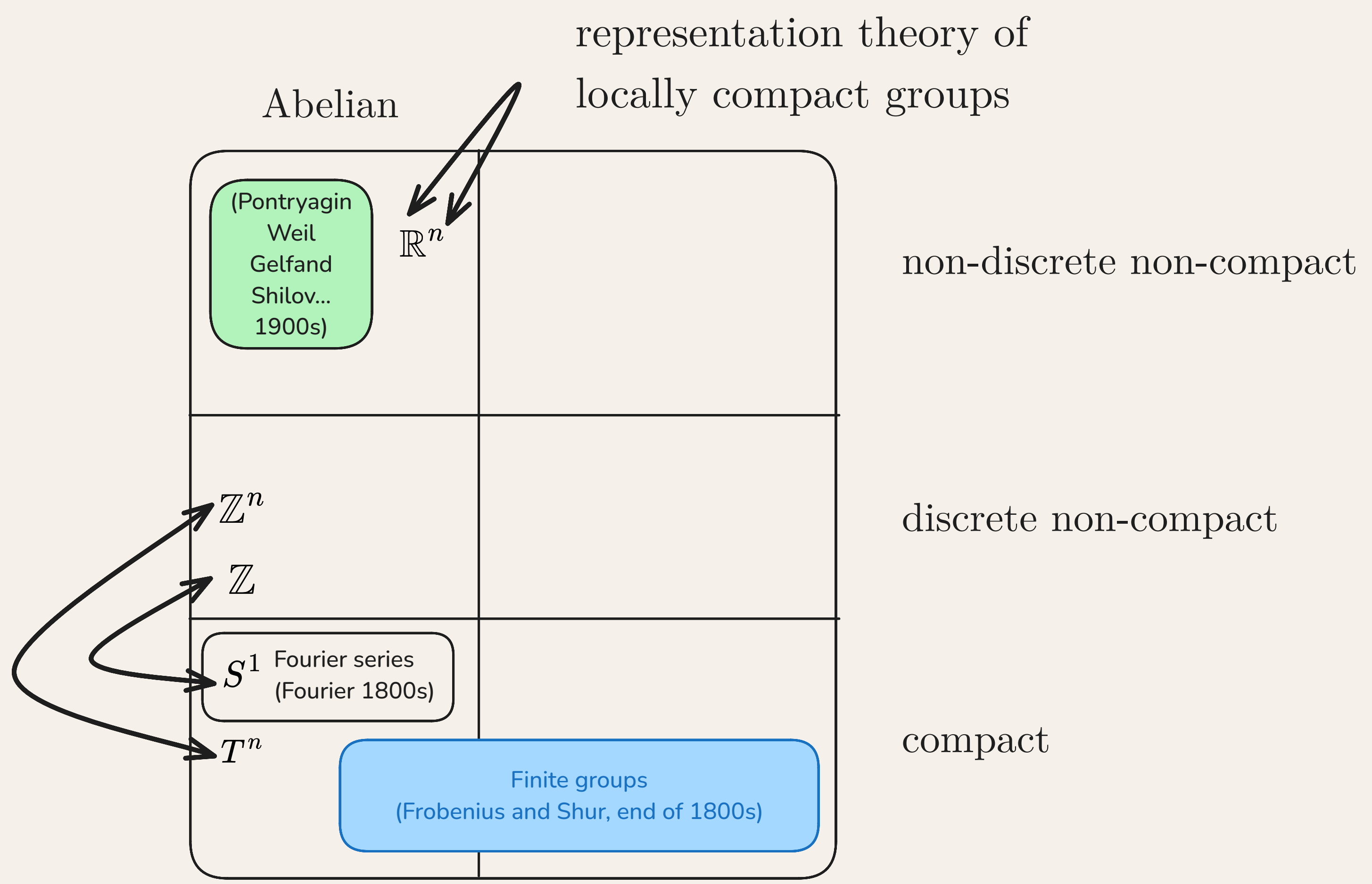
A representation

$$G \to GL(V)$$

makes

$$G \curvearrowright L^2(V)$$

Harmonic analysis and representation theory are closely related



finite Abelian groups

For finite Abelian group G

$$\hat{G} = \mathrm{HomGrp}(G, S^1)$$

are the set of unitary characters of

If $G = \langle x \rangle$ is a finite cylic group then

$$\begin{aligned} \hat{G} &\to \mu_n := \{z \in \mathbb{C} \mid z^n = 1\} \\ \chi &\mapsto \chi(x) \end{aligned}$$

is an isomorphism.

So $G \cong \hat{G}$ is G is cylic but not canonically.

Proposition: Let G, G_1, G_2 be finite Abelian groups

- $$\widehat{G_1 \times G_2} = \hat{G}_1 \times \hat{G}_2$$
- $x \in G, x \neq i_G$ implies
$$\exists \chi \in \hat{G} : \chi(x) \neq 1_{S^1}$$
- ("Pontryagin duality" for finite Abelian groups)
$$\begin{aligned} G &\cong \hat{\hat{G}} \\ x &\mapsto \mathrm{ev}_x \end{aligned}$$
- $$\chi \in \hat{G}, \chi \neq 1 \implies \sum_{x \in G} \chi(x) = 0$$

Definition. Fourier transform for a finite Abelian group

The **Fourier transform** for a finite Abelian group is

$$\begin{aligned} \hat{}: \mathbb{C}^G &\to \mathbb{C}^{\hat{G}} \\ f &\mapsto \hat{f} \end{aligned}$$

where

$$\hat{f}(\chi) := \sum_{x \in G} f(x)\chi(x)$$

Proposition:

- $$\hat{\delta}_x = \mathrm{ev}_x$$

Utsav Dewan #talk Harmonic manifolds

> **Quote**
>
> **Speaker**: Utsav Dewan (ISI Kolkata, Stat Math Unit)
>
> **Time**: Friday, 31/1/25, 5PM
>
> **Place**: AB2-2A
>
> **Title**: Boundary exceptional sets for radial limits of positive superharmonic functions on Harmonic manifolds
> **Abstract**: By classical Fatou (resp. Littlewood) type theorems in various setups, it is well-known that positive harmonic (resp. superharmonic) functions have non-tangential (resp. radial) limits at almost every point on the boundary. In this talk, in the setting of non-positively curved Harmonic manifolds of purely exponential volume growth, we will see some size estimates of the exceptional sets of points on the boundary at infinity, where a suitable function blows up faster than a prescribed growth rate, along radial geodesic rays, in terms of its Hausdorff dimension or Hausdorff outer measure.

https://en.wikipedia.org/wiki/Fatou%27s_theorem

[2309.05661] Boundary exceptional sets for radial limits of superharmonic functions on non-positively curved Harmonic manifolds of purely exponential volume growth

Colloquium: The Fourier transform on harmonic manifolds by Kingshook Biswas.

Harmonic analysis on homogeneous spaces

#reference

- Analysis on Homogeneous Spaces Class Notes Spring 1994
- 88-871 (Dynamics of Group Actions)
- Dynamics on homogeneous spaces

Sigurdur Helgason - Groups and geometric analysis

§1. Harmonic Analysis on Homogeneous Spaces

1. General Problems

Let X be a locally compact Hausdorff space acted on transitively by a locally compact group G. We assume G leaves invariant a positive measure μ on X. Let T_X denote the (unitary) representation of G on the Hilbert space $L^2(X)$ defined by $(T_X(g)f)(x) = f(g^{-1} \cdot x)$ for $g \in G$,

Chapter 1: Integral geometry and Radon transforms

Invariant measures on coset spaces

Definition. Invariant forms

Let M be a smooth manifold and

$$\Phi : M \to M$$

be a diffeomorphism. A differential form $\omega \in \Omega(M)$ is **Φ-invariant** if

$$\Phi^* \omega = \omega$$

Definition. Left and right invariant forms on a Lie group

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A differential form $\omega \in \Omega(G)$ is **left-invariant** if

$$x \in G \implies L_x^* \omega = \omega$$

and **right-invariant** if

$$x \in G \implies R_x^* \omega = \omega$$

Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ (left-invariant vector fields). Then we have n many 1-forms

$$\omega_1, \ldots, \omega_n$$

such that

$$\omega_i(X_j) = \delta_{ij}$$

These 1-forms are left-invariant and thus

$$\lambda := \omega_1 \wedge \cdots \wedge \omega_n$$

is also a left-invariant n-form on G and it is the unique left-invariant n-form on G up to a constant factor.

Radon transform on $\mathbb{R}^n$

Let

$$\boldsymbol{P}(\mathbb{R}^{n*}) \cong \mathbb{R}\boldsymbol{P}^{n-1}$$

be the compact smooth $n-1$-manifold of all hyperplanes in $\mathbb{R}^n$ passing through origin.

Each hyperplane ξ can be written

$$\xi = \{x \in \mathbb{R}^n \mid \langle x, \hat{n} \rangle = \alpha\}$$

This produces a *surjective* map onto the space of all hyperplanes $\boldsymbol{Pl}^n$ in $\mathbb{R}^n$

$$\begin{aligned} S^{n-1} \times \mathbb{R} &\to \boldsymbol{Pl}^n \\ (\hat{n}, \alpha) &\mapsto \xi = \{x \in \mathbb{R}^n \mid \langle x, \hat{n} \rangle = \alpha\} \end{aligned}$$

that is a smooth double covering(?) as

$$(\hat{n}, \alpha), (-\hat{n}, -\alpha)$$

produce the same plane. Thus we identify

$$\mathcal{C}^\bullet(\boldsymbol{Pl}^n) \leftrightarrow \{f \in \mathcal{C}^\bullet(S^{n-1} \times \mathbb{R}) \mid f(\hat{n}, \alpha) = f(-\hat{n}, -\alpha)\}$$

Definition. **Radon transform and its dual**

Let

$$f : \mathbb{R}^n \to \mathbb{C}$$

be integrable on each hyperplane in $\mathbb{R}^n$. Let $\boldsymbol{P}^n$ denote the compact smooth n-manifold of all hyperplanes in $\mathbb{R}^n$. The **Radon transform** of f is

$$\begin{aligned} J(f) : \boldsymbol{P}^n &\to \mathbb{C} \\ \xi &\mapsto \hat{J}(f)(\xi) := \int_{(\xi, m_{\mathbb{R}^2})} f \end{aligned}$$

and the **dual Radon transform** of $\phi \in \mathcal{C}(\boldsymbol{P}^n)$

$$\begin{aligned} \mathbb{R}^n &\to \mathbb{C} \\ x &\mapsto \check{J}(\phi)(x) := \int_{x \in \xi} \phi \end{aligned}$$

Pierre Albin - Complex Algebraic Geometry

Math 514, Complex Algebraic Geometry (Fall 2020)

- Home
  - Lecture Notes And Videos
  - Piazza
  - Moodle
  - Links

Text

Griffiths & Harris, *Principles of Algebraic Geometry*
Ballmann, *Lectures on Kähler Manifolds*
Voisin, *Hodge Theory and Complex Algebraic Geometry, I*

Description

This course is an introduction to the geometry of Kähler manifolds and the Hodge structure of cohomology.

Kähler manifolds are at the intersection of complex analytic geometry, Riemannian geometry, and symplectic geometry. Moreover, every smooth projective variety is a Kähler manifold. All of this structure is reflected in a rich theory of geometric and topological invariants. In this course we will develop techniques from sheaf theory and linear elliptic theory to study the cohomology of Kähler manifolds.

In the setting of complex projective varities one can use geometric and analytic methods to address questions in algebraic geometry. The results and examples then serve as guides in more algebraic approaches.

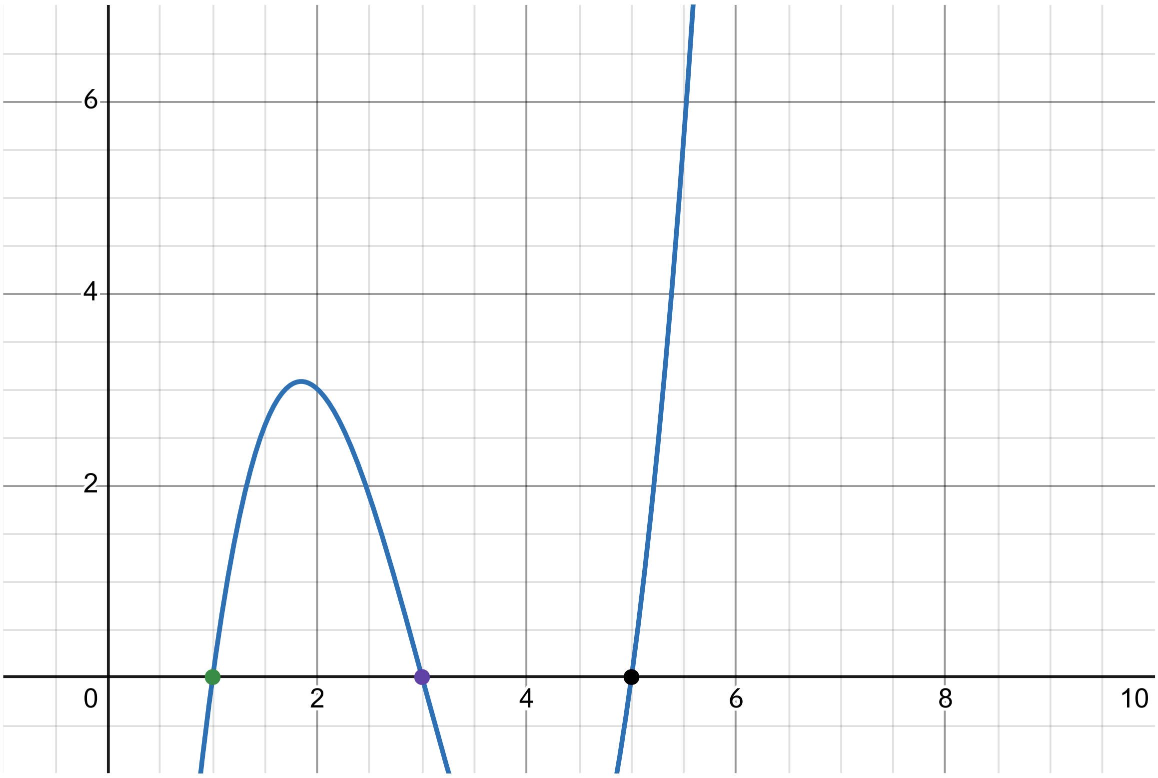
There is a lot of current research taking place in the setting of Kähler manifolds. For instance, very recently a couple of teams of researchers have managed to relate a notion of stability to the existence of Käher-Einstein metrics on Fano spaces. For another example, one of the Clay millenium problems is to prove the Hodge conjecture: every Hodge class of a non-singular projective variety over $\mathbb{C}$ is a rational linear combination of cohomology classes of algebraic cycles. This course should be useful for students interested in research in geometry, be it differential or algebraic, and topology.

elliptic integrals

Let

$$p(X) = a_3X^3 + a_2X^2 + a_1X + a_0 \in \mathbb{R}[X]$$

be a polynomial of degree 3 with three distinct real roots $\alpha_1 < \alpha_2 < \alpha_3$.



Then if

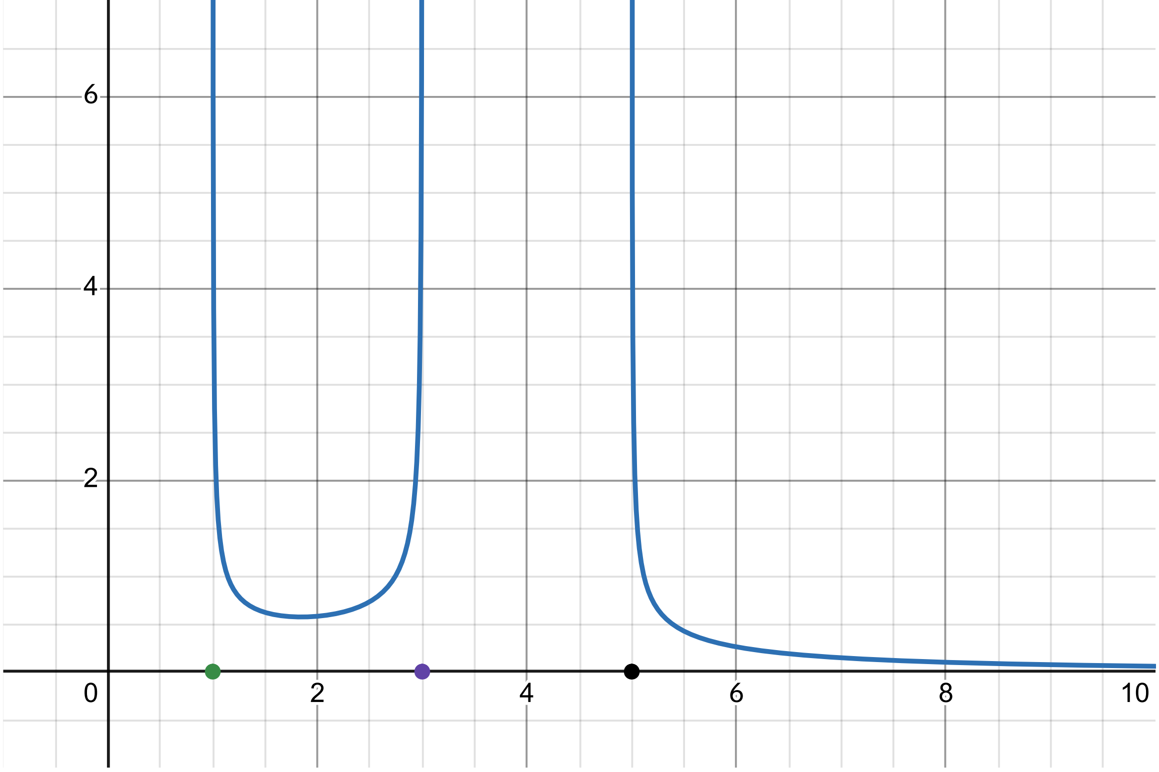
$$[a, b] \subset (-\infty, \alpha_1) \vee (\alpha_1, \alpha_2) \vee (\alpha_2, \alpha_3) \vee (\alpha_3, \infty)$$

such that

$$p([a, b]) \subset (0, \infty)$$

we may compute

$$\int_{[a,b]} \frac{1}{\sqrt{p(x)}} \mathrm{d}x$$



Rather, on the elliptic curve

$$E_\alpha := \{(x, y) \in \mathbb{C}^2 \mid y^2 = p(x)\}$$

the integral of the 1-form

$$\int_\delta \frac{\mathrm{d}x}{y} = c_0 \int \frac{1}{\sqrt{p(x)}} \mathrm{d}x$$

for $\sigma \in H_1(E, \mathbb{Z})$

John Milnor - Singular Points of Complex Hypersurfaces

Chapter 1

Definition. Complex hypersurfaces

Let

$$f \in \mathbb{C}[z_1, z_2, \dots, z_{n+1}]$$

be a non-constant polynomial then

$$V(f)$$

us called **complex hypersurface**, an algebraic set consisting of the zeros of f.
A point $p \in V$ is a **regular point** of f if some $\frac{\partial f}{\partial z_i}$ does not vanish at p, making V a smooth $2n$-manifold near p.

Let $p \in V(f)$ and we consider

$$K := V(f) \cap \partial B_\epsilon(p)$$

where $\partial B_\epsilon \subset \mathbb{C}^{n+1}$ is the $2n+1$-sphere of radius $\epsilon > 0$ centered at p.

If p is a regular point, and $\epsilon > 0$ small enough, the intersection

$$K := V(f) \cap \partial B_\epsilon(p)$$

is a smooth $2n-1$-manifold, embedded in

$$K \hookrightarrow \partial B_\epsilon(p) \cong S^{2n+1}$$

in an *unknotted manner*.

However,

Proposition: Consider the polynomial

$$f(z_1, z_2) := z_1^p + z_2^q$$

that has a critical point at $p = (0, 0)$. Assuming p, q are relatively prime integers and ≥ 2

we can conclude

$$^1K := V(f) \cap \underbrace{\partial B_\epsilon(p)}_{S^3}$$

for small enough $\epsilon > 0$ is the torus knot of (p, q) type

$$^1K \hookrightarrow S^3$$

$$K = \{z_1^p + z_2^q = 0\} \cap \{|z_1|^2 + |z_2|^2 = \epsilon\}$$

- $$\left(ae^{iq\theta}, be^{ip\theta}\right) \in K$$

  implies

  $$a^p e^{iqp\theta} + b^q e^{ipq\theta} = 0, a^2 + b^2 = \epsilon$$

  that is

  $$b^2 = \epsilon - a^2 = (-a^p)^{2/q}$$

GSG on Rational curves and a conjecture of Drinfeld

Abstract

Everyone is invited to the next talk of the Graduate Students' Group seminar, whose details are as follows:

Speaker: Dr. Sarbeswar Pal

Topic: Rational curves and an application of it to a conjecture of Drinfeld.

Abstract: In this lecture I will introduce the basic notions of rational curves in a smooth projective variety over complex numbers. We will see some interesting properties of the variety of rational curves and see how this helps us to prove a conjecture of Drinfeld.

Date: 15 March, 2025 (Tomorrow)
Time: 6:00 PM - 7:00 PM
Venue: AB2 2A or 2B

Question

Let $X \subset \mathbb{C}P^M$ be a smooth projective variety. Does there exist a rational curve

$$\mathbb{C}P^1 \to X$$

In general, no.

> TX is not **nef**.

$$c_1(TX) =$$

We know

$$\mathrm{Mor}_d(P^1, X)$$

is a quasi-projective variety.

Suppose

$$f : P^1 \to X$$

is a rational curve and

$$TX \to X$$

is the projection from the tangent bundle.

Now the pullback

$$f^*TX \to P^1$$

is a vector bundle on P^1.

Grothendieck. Any vector bundle on P^1 is a direct sum of line bundle

$$\bigoplus_i \mathcal{O}(a_i)$$

for $a_i \in \mathbb{Z}$

As

$$\dim X = n \implies TX \text{ has rank } n \implies f^*TX \text{ has rank } n$$

Then

$$f^*TX = \bigoplus_{i=1}^{n} \mathcal{O}(a_i)$$

for

$$a_1 \geq \cdots \geq a_n$$

Definition. **f is free if $a_n \geq 0$ and is non-free otherwise**

X is Fano if...

$$K_X := \Lambda^n TX$$

(canonical bundle)

Let L is a globally generated line bundle?

$$\Phi_L : X \to P(H^0(L)^*)$$

Then ...

Definition. **X is Fano if K_X ample.**

Fact: $\exists$ an open set $X^{\text{free}} \subset X$ such that any rational curve which intersects X^{free} is **free**.

Question

is X^{free} empty? is it proper?

We don't know, in general.

Definition. $X^{\text{nf}} := X \setminus X^{\text{free}}$

Question

X be a Fano projective manifold of (Picard rank 1) such that TX is not nef. Then X^{nf} is nonempty.

1. What can we say about $\dim X^{\text{nf}}$?

Guess: X^{nf} is pure of co-dim 1.

Example

- P^n is Fano of Picard rank 1, and tangent bundle is...
- $$X = \mathrm{Gr}(r, \mathbb{C}^n)$$

Examples of non-nef

Let X be a general hypersurface in P^n of degree $d \leq n$.

- Intersection of two quadratics
- X is curve

  $$M(r, n)$$

  be moduli of stable vec buns. It is smooth, Fano of Picard rank 1

$E \in M(r,n)$ is called **wobbly** if

$$\exists \phi : E \to E \otimes K_C$$

such that $\phi^m = 0 : E \to E \otimes K_C$.

Drinfeld's conjecture

Locus of wobbly bundles is pure of codim 1

A vector bundle is wobbly if $\exists$ a non-free rational curve passing through E.

Conjecture

Let X be Fano, Picard rank 1, different from P^n. Then any non-constant endomorphism

$$X \to X$$

is isomorphism.

All proved examples (Grassmanians, homogeneous spaces?) have TX nef.

Guess

Let X Fano, Picard rank 1, $X \neq P^n$, TX is not nef

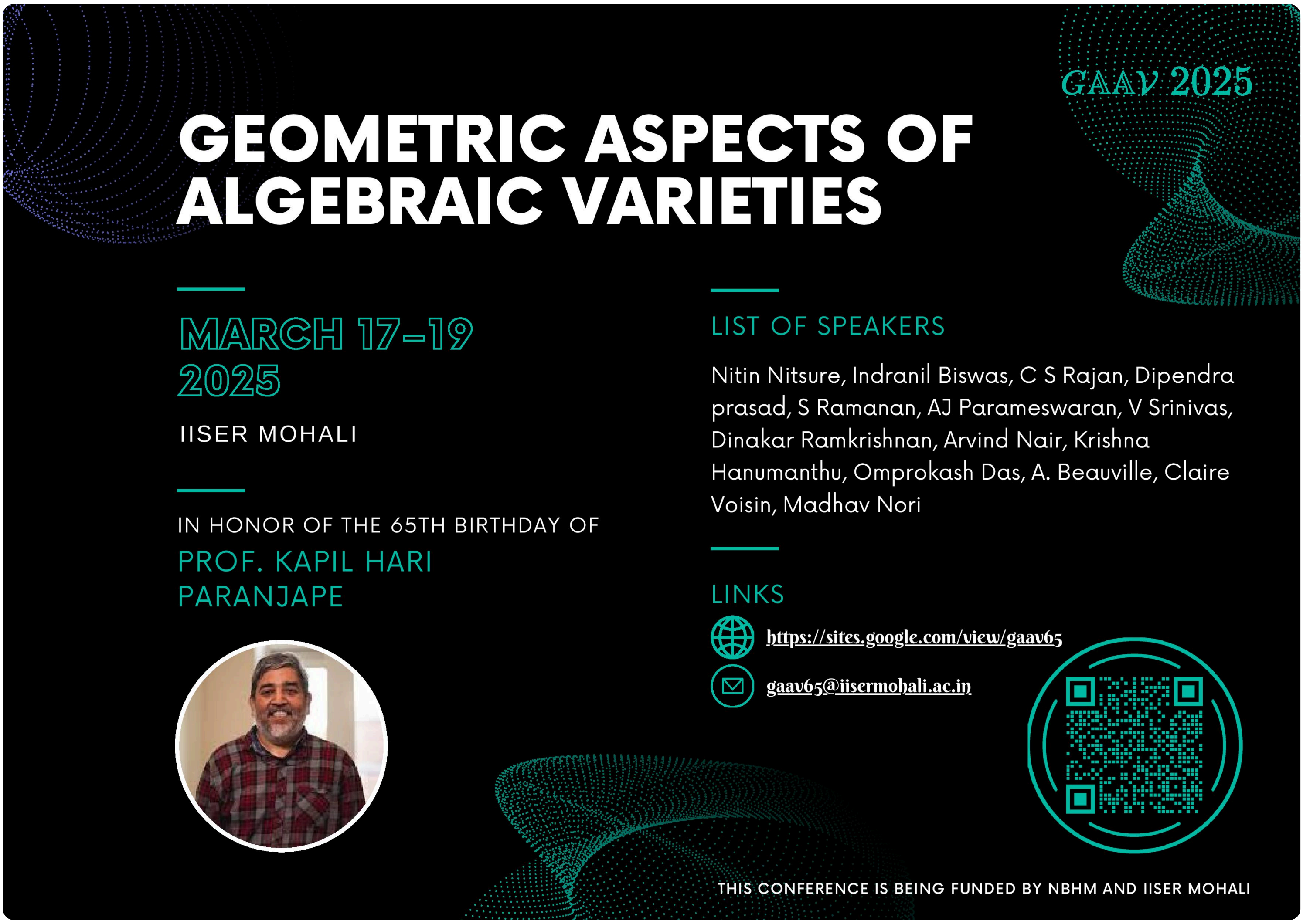
$$f : X \to X$$

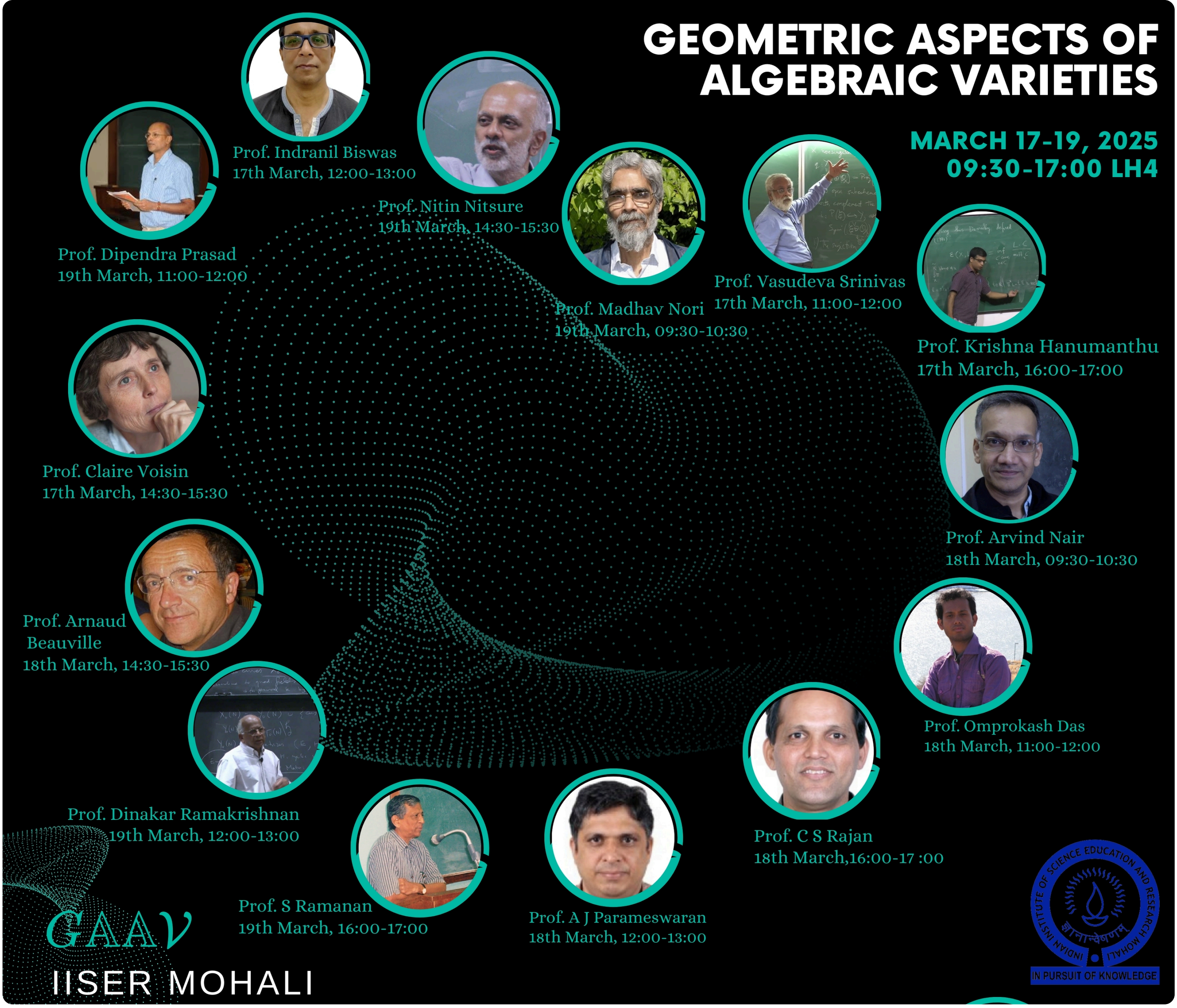
then

$$f^{-1}(X^{\mathrm{nf}}) = X^{\mathrm{nf}}$$

This proves the conjecture.

March 17-19, 2025 - Geometric Aspects of Algebraic Varieties, IISER Mohali





An event in honor of Professor Kapil Paranjape's 65th birthday.

Geometric Aspects of Algebraic Varieties
(Supported by NBHM and IISER Mohali)
IN PURSUIT OF KNOWLEDGE
MARCH 17–19 2025
IISER MOHALI
NO SMOKING
Conference celebrating the 65'th Birthday of Prof. Kapil Hari Paranjape
LIST OF SPEAKERS
Indranil Biswas, Arvind Nair
Nitin Nitsure, C S Rajan
Dinakar Ramkrishnan
A. J. Parameswaran
Dipendra Prasad
Omprokash Das
Arnaud Beauville*
Vasudevan Srinivas
Krishna Hanumanthu
Sundararaman Ramanan
Madhav Nori*, Claire Voisin*
GAAV
2025
VENUE
LH4, Lecture Hall Complex, IISER Mohali
*Online talk

[Geometric Aspects of Algebraic Varieties : Day 1 - YouTube]

17th March		Talks
9:30-10:30	Registration	
10:30-11:00	Tea	
11:00-12:00	Vasudevan Srinivas, University of Buffalo—SUNY (USA)	*Title: Some finiteness results for the étale fundamental group in positive characteristics* In this talk will discuss some results on étale fundamental groups of varieties over an algebraically closed field of characteristic $p > 0$, based on joint work with Helne Esnault and other coauthors. One result, along with Mark Schusterman, is that the tame fundamental group is finitely presented for such a variety which is the complement of an SNC divisor in a smooth projective variety. A second, along with Jakob Stix, is to give an obstruction for a smooth projective variety to admit a lifting to characteristic 0, in terms of the structure of its étale fundamental group as a profinite group. We will finally touch on some open questions.
12:00-13:00 ✅	Prof. Indranil Biswas, Shiv Nadar University Delhi	*Title: A canonical connection on vector bundles on Riemann surfaces and Quillen connection on the theta bundle* We investigate the differential geometric aspects of some canonical torsors on the moduli space of vector bundles on a compact Riemann surface. (Joint work with Jacques Hurtubise.)
13:00-14:30	Lunch	
14:30-15:30	Prof. Claire Voisin, Jussieu, Paris	*Title: Universally defined cycles* I prove that universally defined cycles on smooth varieties of dimension d are uniquely given by polynomials in Chern classes. I prove a similar result for universally defined cycles on products of smooth varieties of dimension $d_1, \ldots, d_n$. I will also discuss a related statement, which is still open, concerning universally defined cycles on powers of smooth varieties of a given dimension, and explain the motivation for this work.
15:30-16:00	Tea	

17th March		Talks
16:00-17:00	Prof. Krishna Hanumanthu, CMI	*Title: Some recent work on Seshadri constants* Let X be a projective variety, and let L be an ample line bundle on X. For a point x in X, the Seshadri constant of L at x is defined as the infimum, taken over all curves C passing through x, of the ratios L.Cm, where L.C denotes the intersection product of L and C, and m is the multiplicity of C at x. This was introduced by J.-P. Demailly in 1990, inspired by Seshadri's ampleness criterion. These constants contain interesting and surprising information about the properties of the line bundle as well as the geometry of the variety. This notion has been extended in many directions, such as multi-point Seshadri constants and Seshadri constants for vector bundles of arbitrary rank. Seshadri constants are also related to certain well-known conjectures about linear system of plane curves. We will discuss some recent developments in this active area.

18th March		Talks
9:30-10:30	Prof. Arvind Nair	*D-modules on flag varieties and extensions in some categories of representations* I will show how to make some computations of groups in categories of representations of real reductive groups using D-modules/constructible sheaves of (enhanced) flag varieties, via the Beilinson-Bernstein localization theory. This leads to some curious numerical facts that I will try to motivate from arithmetic geometry.
10:30-11:00	Tea	
11:00-12:00	Prof. Omprokash Das	
12:00-13:00	Prof. A.J. Parameswaran	
13:00-14:30	Lunch	
14:30-15:30	Prof. Arnaud Beauville	
15:30-16:00	Tea	

18th March		Talks
16:00-17:00	C. S. Rajan, Ashoka University	*Title: Non-lacunarity of coefficients of linear combinations of L-functions of l-adic Galois representations* We extend results of Serre on non-lacunarity of prime and integral coefficients of L-functions attached to l-adic Galois representations to linear combinations of such L-functions. We apply them to Fourier coefficients of L-functions attached to modular forms and L-functions of induced representations. This is joint work with Rishabh Agnihotri and Mihir Sheth.

19th March		Talks
9:30-10:30	Prof. Madhav Nori	
10:30-11:00	Tea	
11:00-12:00	Prof. Dipendra Prasad	
12:00-13:00	Prof. Dinakar Ramakrishnan	
13:00-14:30	Lunch	
14:30-15:30	Prof. Nitin Nitsure	
15:30-16:00	Tea	
16:00-17:00	Prof. S. Ramanan	*Title: Quadric Geometry and Hyper-elliptic curves* The relationship between Quadric Geometry and hyper-elliptic curves has been much studied and goes back about 50 years - Indeed in 6 dimensions and genus 2, even 150 years! Recently I reworked the general case, particularly in order to make things work over a number field. I will talk about the modified approach and also refer to some recent results and open questions.

Prof. Indranil Biswas, Shiv Nadar University Delhi

Quote

During the early 90s, KHP used to talk about Hodge theory during lunch time.
- Prof. Indranil Biswas

Jacques Hurtubise (mathematician) - Wikipedia

[2102.00624] A canonical connection on bundles on Riemann surfaces and Quillen connection on the theta bundle

Let X with genus ≥ 2 be a compact RS and

$$E \to X$$

be degree 0 vector bundle.

E is **stable** if degree of every sub-bundle $F \leq E$ is non-zero

M moduli space of stable vec bun of rank r degree 0

NS: A stable E admits a unique unitary flat connection.

$$M = \frac{\mathrm{Hom}(\pi_1(X, u_0), U(r))}{U(r)}$$

$$T_E M = H$$

[1210.1643] Rank one connections on abelian varieties, II

Kabeer Manali Rahul - Metric techniques for triangulated categories

> **Quote**
>
> **Speaker**: Kabeer M. R. (Australian National University)
> **Time**: Thursday, 20/3/25, 5PM.
> **Room**: AB2-2A
> **Title** : Introduction to metric techniques for (triangulated) categories
> **Abstract** : Neeman has recently introduced certain techniques for triangulated categories which are analogous to notions related to a metric space. These techniques have been used to prove many interesting results in algebraic geometry, including three conjectures. In this talk, I will try to give some motivation, and an overview of these techniques. If time permits, I will also talk about some new representability theorems which have been proven using them.

> #people **Kabeer Manali Rahul**
>
> - MS Thesis: DSpace@IISERMohali: Riemann Surfaces
>   - 210.212.36.82:8080/jspui/bitstream/123456789/1538/3/MS Thesis MS15152.pdf
> - Kabeer Manali Rahul | kabeermr

- arxiv.org/pdf/1901.01453
- Triangulated category - Wikipedia

examples of triangulated categories

Let A be an Abelian category (eg $\mathsf{Mod}(R)$) and $\mathsf{Ch}(A)$ be category of chain complexes

- K(A)
- D(A) derived category

Given a Noetherian R

- $\mathsf{K}^{\mathsf{b}}(\text{proj-}R)$
- $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}(R))$

Given a Noeth scheme X

- $\mathsf{Perf}(X)$
- $\mathsf{D}^{\mathsf{b}}(\mathsf{coh}(X))$

> $(1) \cong (2) \iff X$ is regular $\iff R$ is ?regular?

keywords

- Morita theory

(Rieckerd) Let R, S be Noeth rings then

GSG on Anosov representations

Abstract

Everyone is invited to the next talk of the Graduate Students' Group seminar, whose details are as follows:

Speaker: Tathagata Nayak

Title:- On Character Variety of Anosov Representations

Abstract: Let Γ be the fundamental group of an k-punctured, k ≥ 0, closed connected orientable surface of genus g ≥ 2. In this talk, it will be shown that the character variety of the (Q+, Q-)-Anosov irreducible representations, resp. the character variety of the (P +, P −)-Anosov Zariski dense representations of Γ into SL(n, C), n ≥ 2, is a complex manifold of complex dimension (2g + k − 2)(n^2 − 1). For Γ = π1(Σg), these character varieties are holomorphic symplectic manifolds. This talk is based on a joint work with Prof. Krishnendu Gongopadhyay.

Date: 25 Jan 2025 (Saturday)

Time: 4 30 PM - 5 30 PM

Venue: AB2 2A/2B

We look forward to your presence. Feel free to ask questions to the speaker between and after the talk.

https://arxiv.org/pdf/1108.0733
https://arxiv.org/pdf/2409.07316

GIT → moduli of vector bundles, stability of vector bundles → Einstein-Kahler metrices

> Literature recommendations in between Algebraic and Differential Geometry : r/math (reddit.com)
>
> On the more analytic side, people are interested in using algebraic+differential techniques to solve differential equations (canonical metrics on varieties, vector bundles) but usually the algebraic input is *fairly* limited (you sort of know it ahead of time, and then all the true difficulty is in analysis). The standard books here are Gabor's *Introduction to Extremal Kahler metrics* and Kobayashi *Differential geometry of complex vector bundles* to get started.

- Donaldson's recent work centers on a problem in complex differential geometry concerning a conjectural relationship between algebro-geometric "stability" conditions for smooth projective varieties and the existence of "extremal" Kähler metrics, typically those with constant scalar curvature (see for example cscK metric). [1]
  - Donaldson obtained results in the toric case of the problem (see for example Donaldson (2001)).
  - He then solved the Kähler–Einstein case of the problem in 2012, in collaboration with Chen and Sun. This latest spectacular achievement involved a number of difficult and technical papers.
  - The first of these was the paper of Donaldson & Sun (2014) on Gromov–Hausdorff limits.
  - The summary of the existence proof for Kähler–Einstein metrics appears in Chen, Donaldson & Sun (2014). Full details of the proofs appear in Chen, Donaldson, and Sun (2015a, 2015b, 2015c).

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1. https://en.wikipedia.org/wiki/Simon_Donaldson#Research ↩

Conjecture on Fano manifolds and Veblen Prize

See also: K-stability and K-stability of Fano varieties

In 2019, Donaldson was awarded the Oswald Veblen Prize in Geometry, together with Xiuxiong Chen and Song Sun, for proving a long-standing conjecture on Fano manifolds, which states "that a Fano manifold admits a Kähler–Einstein metric if and only if it is K-stable". It had been one of the most actively investigated topics in geometry since its proposal in the 1980s by Shing-Tung Yau after he proved the Calabi conjecture. It was later generalized by Gang Tian and Donaldson. The solution by Chen, Donaldson and Sun was published in the *Journal of the American Mathematical Society* in 2015 as a three-article series, "Kähler–Einstein metrics on Fano manifolds, I, II and III".[12]

#reference

- Lectures on GEOMETRIC INVARIANT Theory & MODULI 1 | RADU LAZA
- [1111.3032] GIT and moduli with a twist
- https://matematiflo.github.io/SoSe_2023/Vector_bundles_on_curves.pdf
- THE LANDAU LECTURES 2014 - Prof. Gang Tian: Lecture 1
- Stability of algebraic varieties and Kahler geometry - Simon K. Donaldson
- Differential geometry of holomorphic vector bundles on a curve - Florent Schaffhauser
- LECTURES ON HERMITIAN-EINSTEIN METRICS FOR STABLE BUNDLES AND K̈AHLER-EINSTEIN METRICS by Yum-Tong Siu
- Simon Donaldson: Kaehler-Einstein metrics and algebraic geometry I
- Simon Donaldson: Kaehler Einstein metrics and algebraic geometry II
- Simon Donaldson - The Ding functional, Berndtsson convexity and moment maps
- Simon Donaldson - Kahler Geometry
- Kahler-Einstein Metrics, Extremal Metrics and Stability
- Informal Talk on Kahler-Einstein Geometry, Pt. 1
- Informal Talk on Kahler-Einstein Geometry, Pt. 2
- Informal Talk on Kahler-Einstein Geometry, Pt. 3
- Informal Talk on Kahler-Einstein Geometry, Pt. 4
- Song Sun: Kahler-Einstein metrics #ICBS2024

    - In this talk the speaker is represented to explain the paper X. Chen, S. Donaldson and S. Sun, "Kahler-Einstein metrics on Fano manifolds, I, II, III, JAMS 2015" which proves the equivalence between the existence of Kahler-Einstein metrics on a Fano manifold and K-stability. The key ingredient involves bridging differential geometry and algebraic geometry of singularity formations of Kahler manifolds. Time permitting we will also discuss some recent progress along this general direction by researchers in both the differential and algebro-geometric community.
- S. Donaldson, Imperial College
- Notes on GIT and symplectic reduction for bundles and varieties - R.P. Thomas

Nicolai Reshetikhin - Integrable systems and reppresentation theory: geometry, algebra and analysis

Abstract

This talk is an overview of some of the research directions that I am currently interested in. We will start with basic notions. The notion of an integrable system is clearly defined in the Hamiltonian mechanics, it extends naturally to quantum systems and to models of statistical mechanics where the transfer matrix is considered as a quantum evolution. After this introduction, I will focus on some examples that clearly illustrate the relation between integrable systems and representation theory. If time permit, I will also mention another research direction which is very important for quantum field theory but is not a part of the "world of integrable systems". It is the quantization of gauge theories. Professor Nicolai Reshetikhin was born in Leningrad, former Soviet Union, now St. Petersburg, Russia. In 1982, he graduated from Leningrad State University with a bachelor's degree and a master's degree. In 1984, he graduated from the Steklov Institute of Mathematics and obtained a PhD degree. He has taught at well-known universities such as Harvard University, the University of California, Berkeley and the University of Amsterdam. He was invited twice to give a talk at the ICM International Conference of Mathematicians, one of which was a plenary talk. Professor Reshetikhin's main research interests include quantum topology, quantum groups and their representations, classical and quantum integrable systems, and integrable models statistical mechanics. He is one of the founders of quantum group theory, one of the authors of Reshetikhin-Turaev invariant, has important results in the theory of quantum integrable system, in Poisson and symplectic geometry, in the theory of quantum Kac Moody algebra. In 2010, he was elected as a Foreign member of the Royal Danish Academy. In 2021, he became a Fellow of the American Mathematical Society.

A Hamiltonian system in $SU(n)$

We have (almost) coordinates on $SU(n)$

$$g_{ij}$$

Cotangent bundle to $SU(n)$

$$T^*SU(n) \cong \mathfrak{su}(n, \mathbb{C})^* \times SU(n)$$

with symplectic structure

$$\omega = \sum_{ij} \mathrm{d}\Pi_{ij} \wedge \mathrm{d}g_{ij}$$

where $\sum_{i=1}^{n} \Pi_{ii} = 0$.

Let $X \in \mathfrak{su}(n, \mathbb{C})^*$

$$H_k := \mathrm{tr}(X^k)$$

Proposition:

$$\{H_k, H_l\} = 0$$

The Hamiltonian vector fields can be solved explicitly

$$\begin{aligned} X(t) &= X \\ g(t) &= \exp\left(\sum_{k=2}^{n} t_k X^{k-1}\right) g \end{aligned}$$

projection method

Consider the conjugation action

$$\begin{aligned} SU(n) &\curvearrowright T^*SU(n) \\ (X, h) &\overset{h}{\mapsto} (hXh^{-1}, hgh^{-1}) \end{aligned}$$

This action is Hamiltonian with moment map

$$\begin{aligned} \mu : T^*SU(n) &\to \mathfrak{su}(n)^* \\ (X, h) &\mapsto X - h^{-1}Xh \end{aligned}$$

Consider a $SU(n)$-orbit

$$\mathcal{O}_X = \{gXg^{-1} \mid g \in SU(n)\} \subset \mathfrak{su}(n)^*$$

Proposition:

$$\mu^{-1}(\mathcal{O}_X) \subset T^*SU(n)$$

is invariant under the flow

$$\mathcal{S}(\mathcal{O}) = \frac{\mu^{-1}(\mathcal{O})}{SU(n)}$$

is a *stratified* symplectic space, the Hamiltonian reduced space

The KdV equation

[1]

KdV	$$u_t - 6uu_x + u_{xxx} = 0$$
mKdV	$$v_t - 6v^2 v_x v_{xxx} = 0$$
Schrodinger eigenfunction equation with potential V	$$\psi_{xx} + (\lambda - V)\psi = 0$$

Proposition: If v solves mKdV then

$$u := v^2 + v_x \text{ at } (x, t)$$

solves the KdV equation.

$$\begin{aligned} u &= v^2 + v_x \\ u_t &= 2vv_t + v_{tx} \\ u_x &= 2vv_x + v_{xx} \\ u_{xx} &= 2(v_x)^2 + 2vv_{xx} + v_{xxx} \\ u_{xxx} &= 4v_x v_{xx} + 2v_x v_{xx} + 2vv_{xxx} + v_{xxxx} \end{aligned}$$

- $$u_t - 6uu_x + u_{xxx} = 2vv_t + v_{tx} - 6(v^2 + v_x)(2vv_x + v_{xx}) +$$

Let us fix $t_0 \in \mathbb{R}$ and get

$$\psi(x, t_0)$$

such that

$$v = \frac{\psi_x}{\psi} \text{ at } (x, t_0)$$

in

$$u = v^2 + v_x \text{ at } (x, t_0)$$

produces

$$
\begin{aligned}
u &= \left(\frac{\psi_x}{\psi}\right)^2 + \frac{\psi_{xx}}{\psi} - \frac{(\psi_x)^2}{\psi^2} \\
u &= \frac{\psi_{xx}}{\psi} \\
\psi_{xx} &= u\psi
\end{aligned}
$$

that is

$$
\psi_{xx}(x, t_0) - u(x, t_0)\psi(x, t_0) = 0 \text{ for all } t_0 \in \mathbb{R}
$$

which is a linear equation.

Say we know

$$
u(x, 0)
$$

Then

$$
u_2(x, t) := u(x + 6\lambda t, t) + \lambda
$$

also solves the KdV equation we obtain

$$
\psi_{xx} + (\lambda - u)\psi = 0
$$

direct scattering: Schrodinger eigenfunction equation with potential $u(x, t)$

Let $u(x, 0)$ have compact support. Then eigenfunctions that satisfy

$$
-\psi_{xx} + u(x, 0)\psi = \lambda\psi
$$

have

- **bound states/discrete eigenvalues** $\lambda = -\kappa_n^2$ for $n \in \{1, \ldots, N\}$

$$
\psi_n(x, 0) = N_n(0) \exp(-\kappa_n x)
$$

- **continuum of positive eigenvalues** $\lambda = k^2$

$$
\psi_k(x, 0) \sim \begin{cases} \underbrace{T(k, 0)}_{\text{transmission}} \exp(-ikx) & \text{as } x \to -\infty \\ \exp(-ikx) + \underbrace{R(k, 0)}_{\text{reflection}} \exp(ikx) & \text{as } x \to \infty \end{cases}
$$

The scattering data at $t = 0$ is

$$\Big((\kappa_1, N_1), \ldots, (\kappa_N, N_N), R(k, 0), T(k, 0)\Big)$$

evolution of scattering data

If

$$L : \mathcal{C}^\infty(M \times I) \to \mathcal{C}^\infty(M \times I)$$

is thought of as

$$L(t) : \mathcal{C}^\infty(M \times I) \to \mathcal{C}^\infty(M)$$

then

$$\begin{aligned}
\frac{\partial}{\partial t} L\psi &= \frac{\mathrm{d}L}{\mathrm{d}t}\psi + L\frac{\partial \psi}{\partial t} \\
L &:= -6\partial_x^2 - u(x,t) \\
M &:= -4\partial_x^3 - u\partial_x - \frac{u_x}{2}
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial t} L\psi = \frac{\partial}{\partial t}(-6\partial_x^2\psi - u\psi) &= -u_t + L\frac{\partial \psi}{\partial t} \\
\implies \frac{\mathrm{d}L}{\mathrm{d}t} &= -u_t
\end{aligned}$$

$$\begin{aligned}
[L, M] &= LM\psi - ML\psi \\
&= (-6\partial_x^2 - u)\left(-4\partial_x^3\psi - u\partial_x\psi - \frac{u_x}{2}\psi\right) \\
&\quad - \left(-4\partial_x^3 - u\partial_x - \frac{u_x}{2}\right)(-6\partial_x^2\psi - u\psi) \\
&= (-6\partial_x^2)\left(\cancelto{0}{-4\partial_x^3\psi} - u\partial_x\psi - \frac{u_x}{2}\psi\right) \\
&\quad + (-u)\left(-4\partial_x^3\psi - u\partial_x\psi - \cancelto{0}{\frac{u_x}{2}\psi}\right) \\
&\quad - \left(-4\partial_x^3\right)(\cancelto{0}{-6\partial_x^2\psi} - u\psi) \\
&\quad - (-u\partial_x)(-6\partial_x^2\psi - u\psi) \\
&\quad - \left(-\frac{u_x}{2}\right)(-6\partial_x^2\psi - \cancelto{0}{u\psi}) \\
&= -uu_x - u_{xxx}
\end{aligned}$$

Let M, L be self adjoint operators and $\lambda \in \mathbb{R}$. If

$$u_t - \mathcal{N}(u) = 0$$

can be expressed as

$$L_t + [L, M] = 0$$

and if

$$L\psi = \lambda\psi$$

then $\lambda_t = 0$ and $\psi_t = M\psi$ for $t > 0$.

$$L\psi(x,t) = \lambda(t)\psi(x,t)$$

differentiated with respect to t

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t$$

- $$\begin{aligned} \lambda_t\psi &= (L-\lambda)\psi_t + [M, L]\psi \\ &= (L-\lambda)\psi_t + \lambda M\psi - LM\psi \\ &= (L-\lambda)(\psi_t - M\psi) \end{aligned}$$
- Taking inner product

$$\begin{aligned} \langle\psi,\psi\rangle\lambda_t &= \langle\psi, (L-\lambda)(\psi_t - M\psi)\rangle \\ &= \langle(L-\lambda)\psi, \psi_t - M\psi\rangle \\ &= 0 \end{aligned}$$

-

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1. ethz.ch/content/dam/ethz/special-interest/phys/theoretical-physics/itp-dam/documents/gaberdiel/proseminar_fs2018/11_Schalch.pdf ↩

BIMSA - Integrable Systems and Algebraic Geometry - Beijing Summer Workshop in Mathematics and Mathematical Physics (June 24 – July 5, 2024)

#talk/school/online

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Introduction

This series of summer workshops is organized by the Beijing Institute of Mathematical Sciences and Applications (BIMSA). It aims at introducing young researchers to some of the active research areas in Mathematics and Mathematical Physics via a series of short lecture courses taught by some of the world's best mathematicians, combined with research talks given by world-renowned experts in the field. The theme for the inagural workshop in Summer 2024 is *Integrable Systems and Algebraic Geometry.*

Lecturers

- Pavel Etingof (Massachusetts Institute of Technology, USA)
- Samuel Grushevsky (Simons Center for Geometry and Physics, Stony Brook University, USA)
- Nikita Nekrasov (Simons Center for Geometry and Physics, Stony Brook University, USA)
- Andrei Okounkov (Columbia University, USA)
- Pavel Etingof (Massachusetts Institute of Technology, USA)
  Title: The Hitchin System and its Quantization
  (Lecture 1: PDF and Youtube Video, Lecture 2: PDF and Youtube Video, Lecture 3: PDF and Youtube Video, Lecture 4: PDF and Youtube Video, Lecture 5: PDF (preliminary) and Youtube Video)
  Abstract: Let G be a simple complex Lie group. I will review the classical Hitchin integrable system on the cotangent bundle to the moduli space $BunG(X)$ of principal G-bundles on a smooth complex projective curve X (possibly with punctures), as well as its quantization by Beilinson and Drinfeld using the loop group LG. I will explain how this system unifies many important integrable systems, such as Toda, Calogero-Moser, and Gaudin systems. Then I'll discuss opers (for the dual group Gv), which parametrize the (algebraic) spectrum of the quantum Hitchin system. Finally, I will discuss the analytic problem of defining and computing the spectrum of the quantum Hitchin system on the Hilbert space $L2(BunG(X))$, and will show that (modulo some conjectures, known in genus 0 and 1) this spectrum is discrete and parametrized by opers with real monodromy. Moreover, we will see that the quantum Hitchin system commutes with certain mutually commuting compact integral operators HL,v called Hecke operators (depending on a point $x\in X$ and a representation V of Gv), whose eigenvalues on the quantum Hitchin eigenfunction ψL corresponding to a real oper L are real analytic solutions $\beta(x,x)$ of certain differential equations $D\beta=0$, $D\beta=0$ associated to L and V. This constitutes the analytic Langlands correspondence, developed in my papers with E. Frenkel and Kazhdan following previous work by Braverman-Kazhdan, Kontsevich, Langlands, Nekrasov, Teschner, and others. I will review the analytic Langlands correspondence and explain how it is connected with arithmetic and geometric Langlands correspondence.
  - Samuel Grushevsky (Simons Center for Geometry and Physics, Stony Brook University, USA)
    Title: Integrable systems approach to the Schottky problem and related questions

(Lecture 1: Youtube Video, Lecture 2: Youtube Video, Lecture 3: Youtube Video, Lecture 4: Youtube Video, Lecture 5: Youtube Video)

Abstract: We will review the integrable systems approach to the classical Schottky problem of characterizing Jacobians of Riemann surfaces among all principally polarized complex abelian varieties. Starting with the Krichever's construction of the spectral curve from a pair of commuting differential operators, we will proceed to show that theta functions of Jacobians satisfy the KP hierarchy, and will review Novikov's conjecture (proven by Shiota) solving the Schottky problem by the KP equation. We will finally discuss some of the motivation for Krichever's proof of Welters' trisecant conjecture, and related characterizations for Prym varieties.

- Nikita Nekrasov (Simons Center for Geometry and Physics, Stony Brook University, USA)
  Title: Integrable many-body systems and gauge theories
  (Lecture 1: Youtube Video, Lecture 2: Youtube Video, Lecture 3: Youtube Video, Lecture 4: Youtube Video, Lecture 5: Youtube Video).
  Abstract: Elliptic Calogero-Moser and Toda systems, Gaudin and other spin chains are algebraic integrable systems which have intimate connections to gauge theories in two, three, and four dimensions. I will explain two such connections: first, classical, through Hamiltonian reduction and second, quantum, through dualities of supersymmetric gauge theories.

- Andrei Okounkov (Columbia University, USA)
  Title: From elliptic genera to elliptic quantum groups
  (Lecture 1: Youtube Video, Lecture 2: Youtube Video, Lecture 3: Youtube Video, Lecture 4: Youtube Video, Lecture 5: Youtube Video).
  Abstract: This course will be an example-based introduction to elliptic cohomology, Krichever elliptic genera, rigidity, and related topics. We will work our way towards the geometric construction of elliptic quantum groups.
  Problem Sessions: Problems for the course
  You can present your solutions either in person during the discussion session or by sending a pdf file to the course discussion groups in WeChat and in Telegram:
  - Telegram: https://t.me/+0w-1UHGT9bk0Njcx

Invited Speakers (* = online talk)

- Mikhail Bershtein (The University of Edinburgh, UK)
- Alexander Bobenko* (Technische Universität Berlin, Germany)
- Alexei Borodin (Massachusetts Institute of Technology, USA)
- Thomas Bothner* (University of Bristol, UK)

- Alexander Braverman (University of Toronto, Canada)
- Ivan Cherednik (University of North Carolina, USA)
- Anton Dzhamay (BIMSA, China and The University of Northern Colorado, USA)
- Sergei Lando (Higher School of Economics and Krichever Center for Advanced Studies, Russia)
- Henry Liu (Kavli IPMU, Japan)
- Andrei Marshakov (Krichever Center for Advanced Studies, Russia)
- Grigori Olshanski* (Krichever Center for Advanced Studies and HSE, Russia)
- Senya Shlosman (Krichever Center for Advanced Studies, Russia and BIMSA, China)
- Stanislav Smirnov (Geneva University, Switzerland and Krichever Center for Advanced Studies and St Petersburg State University, Russia)
- Alexander Veselov (Loughborough University, UK)
- Paul Wiegmann (University of Chicago, USA and BIMSA, China)
- Anton Zabrodin (Krichever Center for Advanced Studies, Russia)
- Da-jun Zhang (Shanghai University, China)
- Youjin Zhang (Tsinghua University, China)

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- Nicolai Reshetikhin (YMSC, Tsinghua University and BIMSA, China)

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Conference Poster

The poster can be downloaded here.

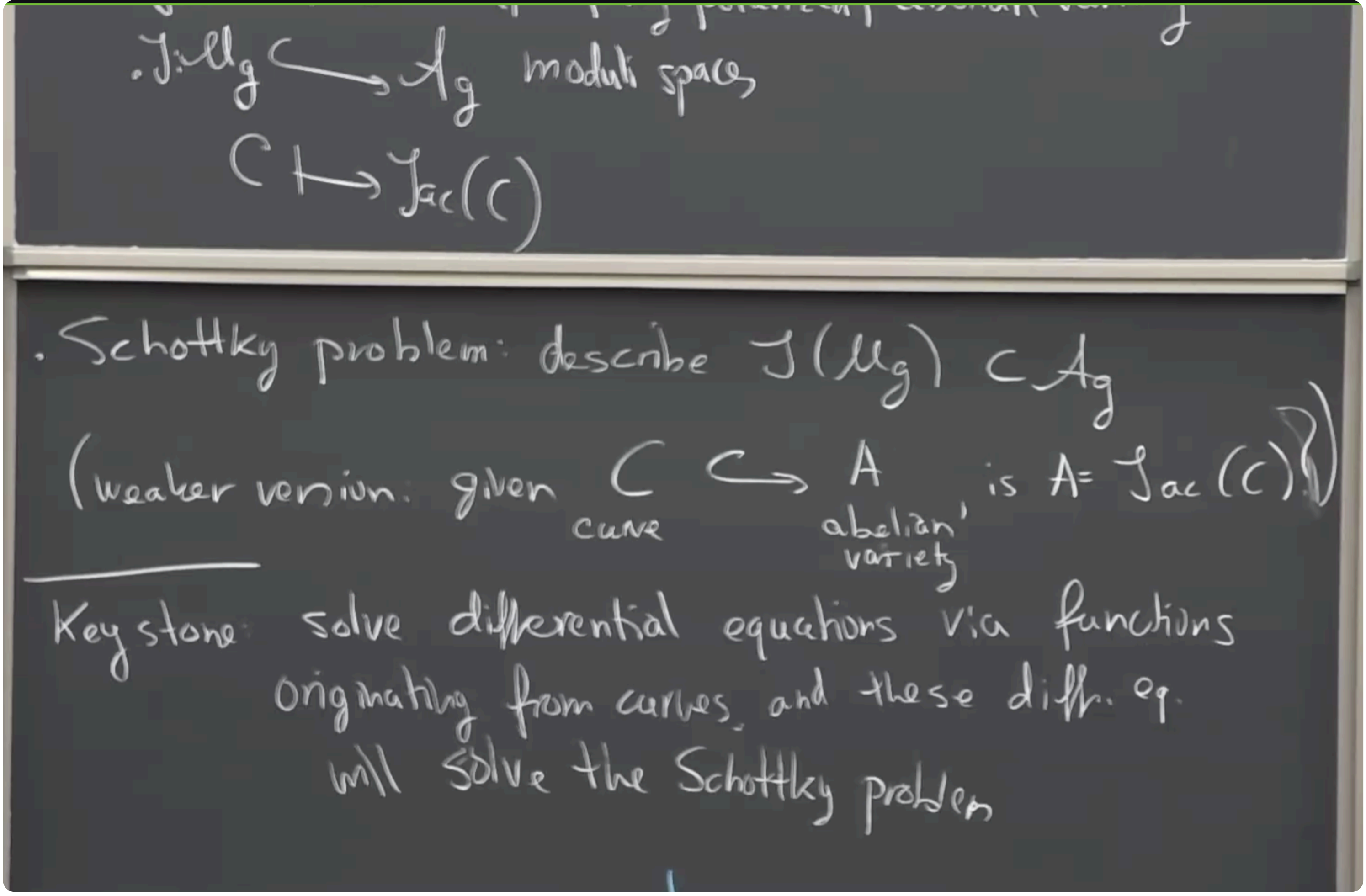
Integrable systems approach to the Schottky problem and related questions

Samuel Grushevsky (Simons Center for Geometry and Physics, Stony Brook University, USA)

Summary

We will review the integrable systems approach to the classical Schottky problem of characterizing Jacobians of Riemann surfaces among all principally polarized complex abelian varieties.

- Starting with the Krichever's construction of the spectral curve from a pair of commuting differential operators,
- we will proceed to show that theta functions of Jacobians satisfy the KP hierarchy,
- and will review Novikov's conjecture (proven by Shiota) solving the Schottky problem by the KP equation.
- We will finally discuss some of the motivation for Krichever's proof of Welters' trisecant conjecture,
- and related characterizations for Prym varieties.



commuting diff operators

A general diff operator in one variable $x \in \mathbb{C}$

$$L = \sum_i u \frac{\mathrm{d}^i}{\mathrm{d}x^i}$$

want eigenfunctions of L

$$L|\psi\rangle = (\mathrm{const})|\psi\rangle$$

> Show that it is enough to consider
>
> $$L = \frac{\mathrm{d}^n}{\mathrm{d}x^n} + \sum_i u_i \frac{\mathrm{d}^i}{\mathrm{d}x^i}$$

Consider

$$L = \frac{\mathrm{d}^n}{\mathrm{d}x^n} + \sum_i u_i \frac{\mathrm{d}^i}{\mathrm{d}x^i}$$

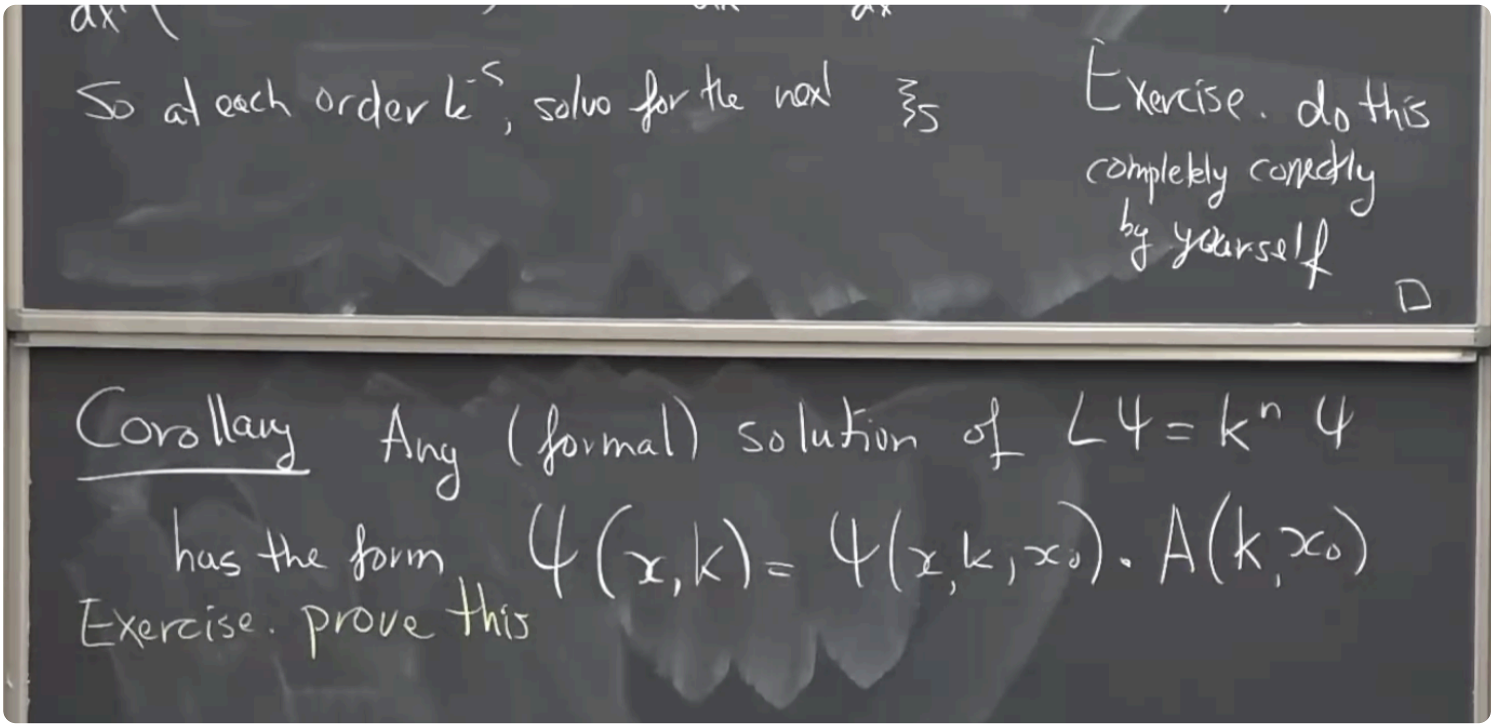
then for all $x_0 \in \mathbb{C}$, $\exists!$ formal solution $|\psi\rangle$ of

$$L|\psi\rangle = k^n|\psi\rangle$$

of the form

$$\psi(x) = \sum_{s=0}^{\infty} z_s(x) k^{-s} e^{k(x-x_0)}$$

such that $z_0(x_0) = 1$ and for all s $z_s(x_0) = 1$



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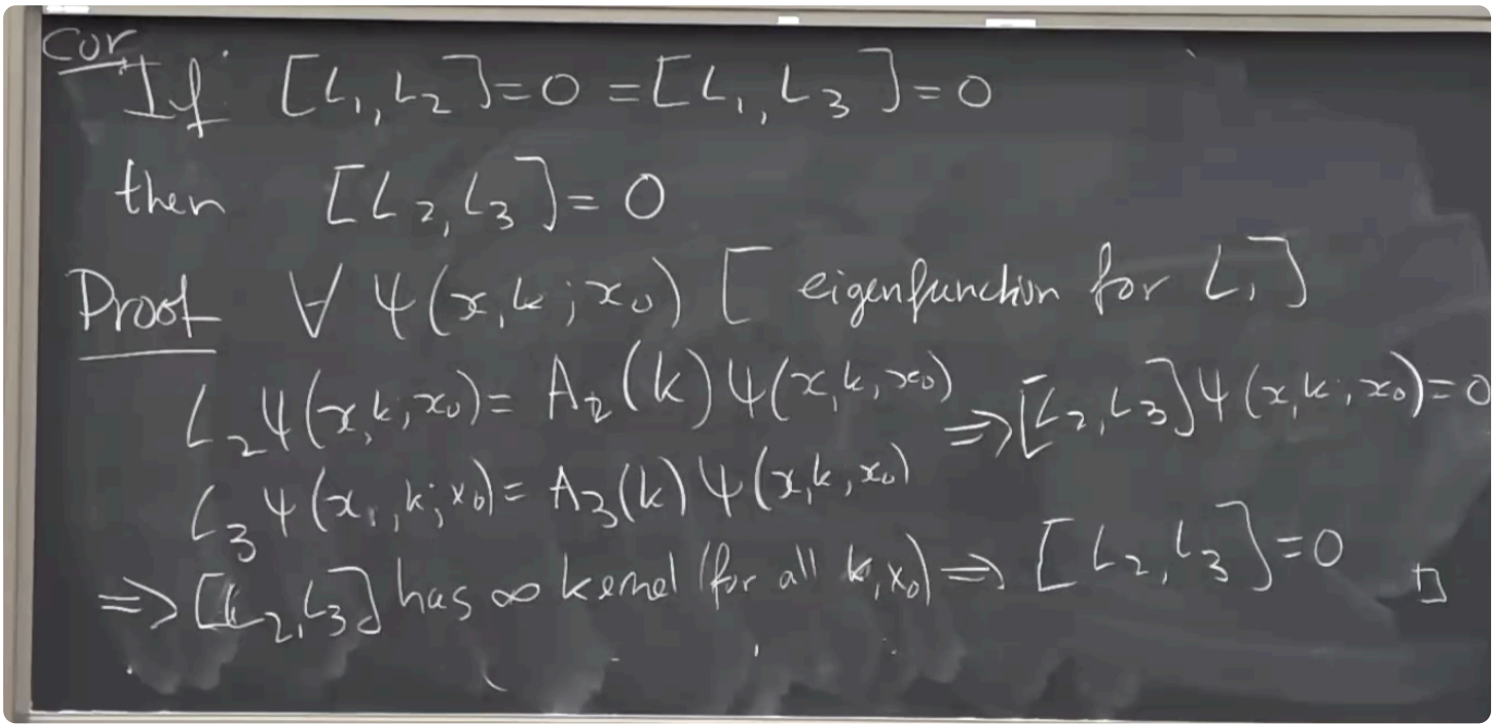


($\forall E$ finite-dim)

Then $L_2 : \mathcal{Y}(E) \longrightarrow \mathcal{Y}(E)$. Let $Q_E(\alpha)$ be the characteristic polynomial of

Claim: $Q_E(\alpha)$ depends polynomially on E

So $Q(\alpha, \beta) \in \mathbb{C}[\alpha, \beta]$

But then $Q(L_1, L_2)\Big|_{\mathcal{Y}(E)} = 0 \quad \forall E$

$\Rightarrow Q(L_1, L_2) \equiv 0$

Centralizer of L_1 is fin. gen

Growth of polynomial ring in two variables

Exercise:

1) Show that if $[L_1, L_2] = 0$ $\Rightarrow L_2 = \frac{d^m}{dx^m} + \ldots$

2) dim L_2. $[L_1, L_2] = 0$ $\leq m+1$

Loray - Painlevé equations and isomonodromic deformations

Summary

Abstract - In these lectures, we use the material of V. Heu and H. Reis' lectures to introduce and study Painlevé equations from the isomonodromic point of view. The main objects are rank 2 systems of linear differential equations on the Riemann sphere, or more generally, rank 2 connections. We will mainly focus on the case they have 4 simple poles, corresponding to the Painlevé VI equation, while other Painlevé equations correspond to confluence of these poles.

First, we settle the Riemann-Hilbert correspondance which establish, roughly speaking, a one-to-one correspondance between connections and their monodromy data, once the poles are fixed. This correspondance is analytic, but not algebraic, very transcendental. Then isomonodromic deformations arise when we deform poles and connection without deforming the monodromy representation. Although the deformation is also transcendental in general, the coefficients satisfy a non linear polynomial differential equation, namely the Painlevé VI equation. By constructing an universal isomonodromic deformation, we explain how Malgrange proved the Painlevé property for isomonodromic deformation equations: solutions admit analytic continuation (with poles) outside of a fixed singular set. At the end, we can describe the Okamoto space of initial conditions for Painlevé VI equation, as well as its non linear monodromy. This can be used to prove the irreductibility of Painlevé VI equation, i.e. the absence of special first integrals, and therefore the transcendance of the general solution.

[1]

[2]

space.f.Painlevé equations

Painlevé equations

Definition. **Painlevé equations**

		Ha de
I (Painlevé)	$$\frac{d^2y}{dt^2} = 6y^2 + t$$	
II (Painlevé)	$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$	wit we $H($
III (Painlevé)	$$\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$	
IV (Gambier)	$$\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \tfrac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$	
V (Gambier)	$$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$	
VI (R. Fuchs)	$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left\{\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right\}$$	

(Poincare, Fuchs) The differential equations of the form

$$P(x, y, y') = 0$$

for $P \in \mathbb{C}[X, Y, Z]$ whose local solutions

$$y(x)$$

are transcendental with *good* analytic continuation are Riccati equation

$$y' = a(x)y^2 + b(x)y + c(x)$$

and Weirstrass equation

$$(y')^2 = y^3 + ay + b$$

Here, "good" analytic continuation means:

Definition. Painleve property

A differential equation

$$P(x, y, y') = 0$$

is said to have the **Painleve property** if exists a finite $Z \subset \mathbb{C}P^1$ such that any local solution

$$y(x)$$

that is defined at the neighborhood of $x_0 \in \mathbb{C}P^1 \setminus Z$ can be meromorphically continuated along any path

$$\gamma : [0,1] \to \mathbb{C}P^1 \setminus Z$$

(sketch of proof)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{P(x,y)}{Q(x,y)}$$

where $P, Q \in \mathbb{C}[X, Y]$

- This equation have foliations whose leaves are graphs of local solutions
- This is the foliation associated to

  $$Q\mathrm{d}y - P\mathrm{d}x = 0$$

  or equivalently to the vector field

  $$Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y}$$

-

1. https://hal.science/medihal-02273534v1 ↩
2. https://av.tib.eu/series/1587 ↩

NCTS Short Course on Riemann Hilbert Method in Integrable Systems

NCTS Short Course

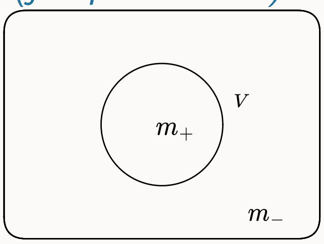
Riemann-Hilbert Method in Integrable Systems

Prof. Peter Miller (U. Michigan)

There will be 5 lectures. Each lecture takes about 2 hours, with outline as below.

Lecture I: Riemann-Hilbert Problems for Orthogonal Polynomials and Painlevé Equations

Abstract: This lecture will introduce the notion of Riemann-Hilbert problems in the theory of orthogonal polynomials and in the theory of Painlevé equations. We begin with orthogonal polynomials and develop the Fokas-Its'-Kitaev representation of them via a matrix Riemann-Hilbert problem. Then we discuss the representation of solutions of Painlevé equations via Riemann-Hilbert problems associated to the inverse monodromy problem for some linear differential equations. We develop isomonodromic Schlesinger transformations as a tool for generating families of solutions of Painlevé equations, focusing on the example of the rational solutions of Painlevé-II.

	RHP(A)	RHP(B)
given	smooth $$V : S^1 \to \mathbb{C}$$	smooth $$V : S^1 \to \mathbb{C}$$
find	functions m_+, m_- analytic inside and outside S^1 respectively such that $m_-(\infty) = 0$ and $$m_+(e^{i\theta}) = m_-(e^{i\theta}) + V(e^{i\theta})$$ *(jump condition)* 	functions m_+, m_- analytic inside anc outside S^1 respectively such that $m_-(\infty) = 0$ and $$m_+(e^{i\theta}) = m_-(e^{i\theta})V(e^{i\theta})$$ *(multiplicative jump condition)*

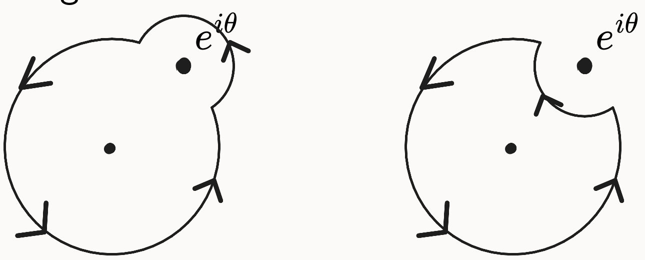
	RHP(A)	RHP(B)
solution 1	Conditions imply $$m_+(z) = \sum_{n\geq 0} a_n z^n$$ convergent inside S^1 and $$m_-(z) = \sum_{n\leq -1} b_n z^n$$ with $$V(e^{i\theta}) = \sum_{n\in\mathbb{Z}} v_n e^{in\theta},\ \theta \in (-\pi, \pi]$$ where the *jump condition* takes the form $$\sum_{n\geq 0} a_n z^n = \sum_{n\leq -1} b_n z^n + \sum_{n\in\mathbb{Z}} v_n e^{in\theta}$$ giving us $$\begin{aligned} n \in \mathbb{N} &\implies a_n = v_n \\ n < 0 &\implies b_n = -v_n \end{aligned}$$	**Take logs!** reduces the problem to RHP(A) $$m_\pm(z) = \exp\left(\frac{1}{2\pi i}\int_{S^1} \frac{\log(V(w))\mathrm{d}w}{w-z}\right.$$ is smooth iff index of V ("winding number of V) is zero around S^1.
Uniqueness	Let $m_\pm, \tilde{m}_\pm$ be two solutions. Then $$\Delta m_\pm := \tilde{m}_\pm - m_\pm$$ by jump condition $$\Delta m_+ = \Delta m_- \text{ on } S^1$$ Then Δm is entire, and its bounded, so by Louville it has to be 0.	

<table>
<tr><th></th><th>RHP(A)</th><th>RHP(B)</th></tr>
<tr><td>solution 2</td><td>

$$m(z) := \underbrace{\frac{1}{2\pi i}\int_{w\in S^1}\frac{V(w)\mathrm{d}w}{w-z}}_{\mathsf{C}_{S^1}[V](z)}$$

which goes to 0 as $z \to \infty$

Integrate on these contours



then

$$m_-(e^{i\theta}) - m_-(e^{i\theta}) = \frac{1}{2\pi i}\int_{S^1_{e^{i\theta}}}\frac{V(w)}{w-e^{i\theta}}$$

(Plemelj) which

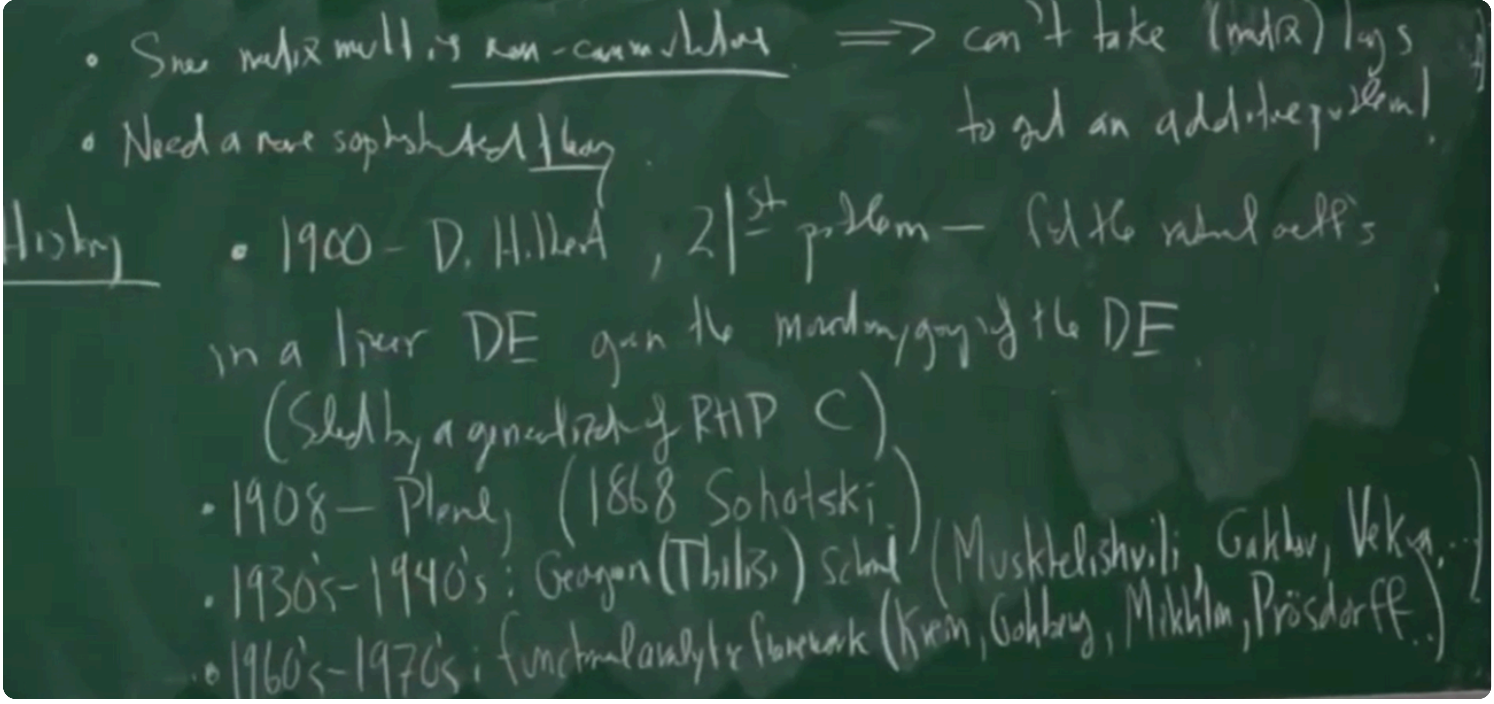
$$= V(e^{i\theta})$$

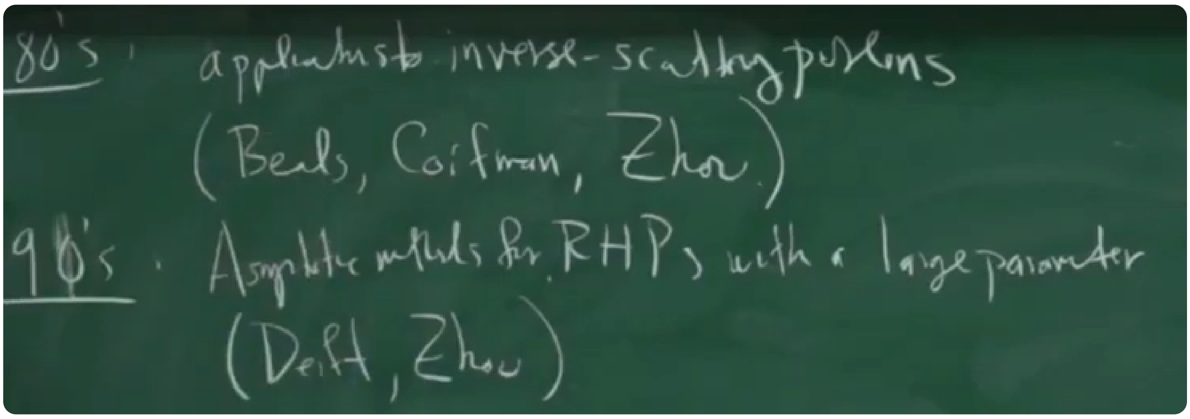
by residue theorem.

</td><td></td></tr>
</table>

History

[1]





- 1849
    - RHP(B) for $V(e^{i\theta}) = 1$ by B. Riemann in his thesis (where he cooked complex analysis and Riemann surfaces)
- 1900
    - D. Hilbert, 21st problem: Proof of the existence of linear differential equations having a prescriped monodromic group
- 1908
    - Plemelj solved the problem by a generalization of RHP(C)
    - 1868 solution?
- 1930s - 40s
    - Physics inspired problems
    - Georgian (Thilisi) school by Musktelisvili, Gakhov, Vek
- 1960s-1970s
    - functional analytical framework by Kren, G
- 1980s
    - application to inverse scattering and integrable systems

- generally for regular holonomic D-modules by Masaki Kashiwara (1980, 1984) and Zoghman Mebkhout (1980, 1984) independently

- 1990s
    - asymtyotic methods with large parameters

RHP type C

- Given: $V : S^1 \to GL()$

orthogonal polynomials

Orthogonal Polynomials via Riemann-Hilbert Problems

Orthogonal polynomials.

Let $w : \mathbb{R} \to \mathbb{R}$, $w(x) \geq 0$, $w(x) \not\equiv 0$, be a *weight* with finite moments:

$$\int_{\mathbb{R}} w(x)|x|^n \, dx < \infty, \quad \forall n = 0, 1, 2, 3, \ldots .$$

The associated system of *orthogonal polynomials* is a sequence $\{p_n(x)\}_{n=0}^{\infty}$ of polynomials $p_n(x) = \gamma_n x^n + \cdots$ with $\gamma_n > 0$ and such that

$$\langle p_j, p_k \rangle_w = \delta_{jk}, \quad \langle f, g \rangle_w := \int_{\mathbb{R}} f(x)g(x)w(x)\, dx.$$

The existence/uniqueness of the orthogonal polynomials follows from their sequential construction via the *Gram-Schmidt process*: For $n = 0, 1, 2, \ldots$,

$$\pi_n(x) := x^n - \sum_{k=0}^{n-1} \langle p_k, x^n \rangle_w p_k(x), \quad \gamma_n := \frac{1}{\sqrt{\langle \pi_n, \pi_n \rangle_w}}, \text{ and } p_n(x) := \gamma_n \pi_n(x).$$

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1. arxiv.org/pdf/2003.14374 ↩

algebraic study of ODEs

Let $P(X) \in \mathbb{C}[X]$ be a monic polynomial. We consider the ODE: looking for functions f such that

$$P(\mathfrak{D})f = 0$$

where $\mathfrak{D}$ is the derivative operator.

using primary decomposition for complex case

Now over the field $\mathbb{C}$, if we factorize the polynomial

$$P(X) = (X-\alpha_1)^{r_1} \dots (X-\alpha_k)^{r_k}$$

where r_k is the multiplicity of the root α_k then by using

(General primary decomposition) Let T be a endomorphism on a vector space V and

$$p = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$$

be prime factorization of a monic polynomial p such that $p(T) = 0$. Then

$$V = \sum_{i=1}^{k} \ker p_i^{r_i}(T)$$

projection on the kernel of prime factors

- Let T be a endomorphism on a vector space V and

$$p = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$$

  be prime factorization of a monic polynomial p such that $p(T) = 0$.
- For each i, let

$$f_i := \frac{p}{p_i^{r_i}} = \prod_{i \neq j} p_j^{r^j}$$

  - Since p_i are distinct prime polynomials, f_i are **relatively prime**. Thus there are polynomials g_i such that

$$\sum_{i=1}^{k} f_i g_i = 1$$

- Then

$$i \neq j \implies f_i f_j = p \underbrace{\frac{p}{p_i^{r_i} p_j^{r_j}}}_{\text{a polynomial}} \implies p \mid f_i f_j$$

- Let

$$\begin{aligned} h_i &:= f_i g_i \\ E_i &:= h_i(T) \end{aligned}$$

then we have

$$h_1 + h_2 + \dots + h_k = 1$$

- In terms of E_i we have

$$\begin{aligned} E_1 + E_2 + \cdots + E_k &= \mathrm{Id} \\ i \neq j \implies E_i E_j &= (\underbrace{f_i f_j}_{\text{divisible by } p} g_i g_j)(T) = 0 \end{aligned}$$

This E_i projects onto the direct sum decomposition

$$V = \sum_{i=1}^{k} \operatorname{im} E_i$$

- **Proposition:**

$$\ker p_i^{r_i}(T) = \operatorname{im} E_i$$

- $$\begin{aligned} v \in \operatorname{im} E_i \implies v &= E_i v \\ p_i^{r_i}(T) v &= p_i^{r_i}(T) E_i v \\ &= (p_i^{r_i} f_i g_i)(T)(\alpha) \\ &= (g_i p)(T)(\alpha) \\ &= 0 \\ \implies v &\in \ker p_i^{r_i}(T) \end{aligned}$$

- $$v \in \ker p_i^{r_i}(T) \implies p_i^{r_i}(T)(v) = 0$$

but as $f_j = p/p_j^{r_j}$ we have

$$i \neq j \implies p_i^{r_i} \mid f_j g_j$$

so $i \neq j \implies E_j(v) = 0$ so v must be in E_i

- Thus

$$V = \sum_{i=1}^{k} \ker p_i^{r_i}(T)$$

on the space of all solutions $V = \ker(P(D) : \mathcal{C}^n(\mathbb{R}, \mathbb{C}) \to \mathcal{C}^{n-1}(\mathbb{R}, \mathbb{C}))$ we have the operator

$$D : V \to V$$

which satisfies the polynomial $P(D) = 0$, so we get

$$V = \ker(D - \alpha_1)^{r_1} + \cdots + \ker(D - \alpha_k)^{r_k}$$

So we must only solve the differential equation

$$(D - \alpha)^r f = 0$$

- **Base case:**

$$f' - \alpha f = e^{\alpha x}(e^{-\alpha x} f)'$$

- **Induction hypothesis:** We assume

$$(D - \alpha)^r f = e^{\alpha x} D^r (e^{-\alpha x} f)$$

  holds for some r

- **Inductive step:**

$$\begin{aligned}
(D - \alpha)^{r+1} f &= (D - \alpha) e^{\alpha x} D^r (e^{-\alpha x} f) \\
&= \alpha e^{\alpha x} D^r (e^{-\alpha x} f) + e^{\alpha x} D^{r+1} (e^{-\alpha x} f) - \alpha e^{\alpha x} D^r (e^{-\alpha x} f) \\
&= e^{\alpha x} D^{r+1} (e^{-\alpha x} f)
\end{aligned}$$

  - So the proposition holds for $r + 1$.

- Hence

$$(D - \alpha)^r f = 0 \iff D^r (e^{-\alpha x} f) = 0$$

which is true if and only if

$$e^{-\alpha x} f(x) = b_0 + b_1 x + \cdots + b_{r-1} x^{r-1}$$

- which means

$$\ker(D - \alpha)^r = \mathbb{C}\{e^{\alpha x}, x e^{\alpha x}, \ldots, x^{r-1} e^{\alpha x}\}$$

Thus the functions

$$x^m e^{\alpha_i x}$$

for $0 \leq m \leq r_i - 1$ for all $1 \leq i \leq k$ form a basis for V, the space of all solutions of the differential equation.

In particular

$$\dim V = \deg p$$

GSG on orderability in 3-manifold groups

Abstract

Speaker: Debattam Das

Title: Orderibility in 3-manifolds groups

Abstract: A group is said to be left orderable if there is a left invariant order of the elements, presents in the group. In this talk, We will discuss the orderability on $ 3 $-manifold groups and also the relation between orderbility and generalised $ n $- torsion elements. In the ending note, we will see some recent results on that.

Date: 21 Sep 2024 (Saturday)

Time: 3 30 PM - 5 00 PM

Venue: AB2 2B

We look forward to your presence. Feel free to ask questions to the speaker between and after the talk.

A left order on a group, bi order on a group Linearly ordered group - Wikipedia

$$t \in G \implies g < h \implies tg < th$$

(Conrad, 1959) A group G is left-orderable if and only if it acts effectively on a linearly ordered set X by order-preserving bijections.

#paper [math/0211110v2] Orderable 3-manifold groups personal.math.ubc.ca/~rolfsen/papers/ord/ord.final.pdf - Steven Boyer, Dale Rolfsen, Bert Wiest

$$G = G_1 * G_2 * \cdots * G_n$$

G is left orderable $\iff$ G_i are left orderable

(Prime decomposition theorem) M be cpt, conn, orientable 3-man then

$$M \cong M_1 \# M_2 \ldots \# M_n$$

> M_i prime manifold...

- prime manifold
- irreducible manifold
- reducible prime manifolds are $S^2 \times S^1$, ...

> ...and?
>
> $$\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_n)$$

- Prime manifold - Wikipedia
- > For a prime manifold M possibly with bd with $H^1(M)$ is infinite then $\pi_1(M)$ is L.O.
- Seifert fiber space - Wikipedia
- classify
- $\pi_1(M)$
- horizontal foliation on SFS is 2-foliation which is transverse to a ?
- > M be cpt, connected SFS, orientable then M admits transverse foliations $\implies$ $\pi_1(M)$ is L.O.
- generalized torsion elements and Teragaento conjecure
- fund grp of hyperbolic 3-manifolds

  $$\pi_1(M^3_{p,q,m}) = \langle t, a, b \mid t^{-1}at = aba^{m-1}, t^{-1}bt = a^{-1}, t^{-p} = (aba^{-1}b^{-1})^q \rangle$$

  it not L.O. by thw conjecture? where the torsion element is

  $$t^{p^2}$$

- the gen 2-tosion element in (M, ϕ)
- Hyperbolic group - Wikipedia
- ams.org/journals/jams/2003-16-03/S0894-0347-03-00426-0/S0894-0347-03-00426-0.pdf
- Hyperbolic group - Wikipedia

Boyer (2005) - Orderable 3-manifold groups

#paper [math/0211110v2] Orderable 3-manifold groups personal.math.ubc.ca/~rolfsen/papers/ord/ord.final.pdf - Steven Boyer, Dale Rolfsen, Bert Wiest

Torsion elements in 3-manifold groups (2024)

#paper [2406.03754] Classification of generalized torsion elements of order two in 3-manifold groups - Keisuke Himeno, Kimihiko Motegi, Masakazu Teragaito (2024)

#paper [2403.01541v3] Reversible and other generalised torsion elements in Seifert-fibered spaces - Anushree Das, Debattam Das (2024)

3-manifolds

3-Manifolds.

For 3-manifold theory there are several books:

- W Thurston. *Three-Dimensional Geometry and Topology.* Princeton University Press, 1997. [$55]

A geometric introduction by the master. Also useful for the geometry of surfaces.

- A Hatcher. *Basic Topology of 3-Manifolds.* Unpublished notes available online at http://www.math.cornell.edu/˜hatcher

The more classical topological aspects of 3-manifold theory.

- J Hempel. *3-Manifolds.* Annals of Math Studies 86. Princeton University Press, 1976. [$30]
- P Scott. *Geometries of 3-manifolds.* Bull. London Math. Soc. 15: 401-487, 1983.

A clear presentation of seven of Thurston's eight possible geometric structures on 3-manifolds, all but hyperbolic geometry, the most subtle case by far.

- N Saveliev. *Invariants for Homology 3-Spheres.* Springer, 2002. [$99]
- M Kapovich. *Hyperbolic Manifolds and Discrete Groups.* Birkh¨auser, 2001. [$77] Knot Theory.

https://people.dm.unipi.it/martelli/Geometric_topology.pdf

sett.Man.R.group action manifolds

Locally $G \curvearrowright X$-manifolds

Definition. Locally $G \curvearrowright X$-manifolds

Let G is a group action on a smooth manifold by diffeomorphisms

$$G \curvearrowright X$$

Then a manifold M is said to be **locally $G \curvearrowright X$** if

- **locally diffeomorphic to open sets of X:** there is a open cover $\{U_\alpha\}$ of M and a family of diffeomorphisms

$$\{\phi_\alpha : U_\alpha \subseteq M \to V_\alpha \subseteq X\}$$

onto open sets $V_\alpha \subseteq X$

- and when $U_\alpha \cap U_\beta \neq \emptyset$ there is a $g \in G$ such that

$$g\Big|_{V_\alpha \cap V_\beta} = \phi_\alpha \circ \phi_\beta^{-1}$$

that is, the **transition map is given by a restriction of an element of G**.

[1]

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1. (G,X)-manifold - Wikipedia ↩

Model geometry

Definition. Model geometry

[1]

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1. cmi.ac.in/~vijayr/nachiketa_thurston.pdf#page=10.43 ↩

GSG on liftable mapping class groups

Quote

Dear all,

Everyone is invited to the next talk of the Graduate Students' Group seminar, whose details are as follows:

Speaker: Pankaj Kapdi

Title:- Generating the liftable mapping class groups of regular $\mathbb{Z}n$-covers of the closed and oriented surface Sg of genus $g>0$.

Abstract: Consider a regular cover $p:S\to Sg$ of the closed and oriented surface Sg of genus $g>0$ with the Deck group $\mathbb{Z}n$. The subgroup of the mapping class group $Mod(Sg)$ consisting of mapping classes represented by homeomorphisms that lift under the cover p is called the liftable mapping class group, denoted by $LModp(Sg)$, associated with the cover p. It is known that $LModp(Sg)$ is finitely generated. The question is, can one derive a finite generating set of $LModp(Sg)$. Consider the symplectic representation $\Psi:Mod(Sg)\to Sp(2g;\mathbb{Z})$ of $Mod(Sg)$. The kernel of Ψ is known as the Torelli group denoted by $\mathcal{J}(Sg)$. For $g\neq 2$, it is known that $\mathcal{J}(Sg)$ is finitely generated. Hence, for $g\neq 2$, one can obtain a finite generating set for $LModp(Sg)$ by combining a finite generating set of $\mathcal{J}(Sg)$ and lifting a finite generating set of $\Psi(LModp(Sg))$. *This approach does not work for* g=2 as $\mathcal{J}(Sg)$ is not finitely generated. In this talk, I will discuss a method to obtain a finite generating set for $LModp(S2)$.

Date: 11 Jan 2025 (Saturday)

Time: 11 00 AM - 12 00 PM

Venue: AB2 2A/2B

We look forward to your presence. Feel free to ask questions to the speaker between and after the talk.

Regards,

Ananya Gaur

Member, Organizing Committee

Graduate Students Group

PS: Please visit https://sites.google.com/view/iiserm-gsg/home for the details of the past talks. For queries and suggestions, you may contact :

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Let

$$p : \tilde{X} \to X$$

be a **regular** covering.

Lift of a homeomorphism

$$f : X \to X$$

is such that

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & \tilde{X} \\ \Big\downarrow p & & \\ X & & \end{array}$$

Lifting criteria:

$$f : Y \to X$$

lifts to

$$\tilde{f} : Y \to \tilde{X}$$

$\iff f_* \pi_1(Y) \leq p_* \pi_1(\tilde{X})$

If $\mathrm{Deck}(p)$ is Abelian then $\pi_1(X) \to \mathrm{Deck}(p)$ factors through H_1

$$\frac{\mathrm{Homeo}^+(S_g)}{\text{isotopy}} =: \mathrm{Mod}(S_g)$$

and the liftable mapping classes

$$\mathrm{LMod}_p(S_g)$$

$\mathrm{Mod}(S_g)$ is finitely presented

$$\mathrm{Mod}(S_g) = \langle \text{Dehn twists along } a_i, b_i, c_j \mid 1 \leq i \leq g, 1 \leq j \leq g-1 \rangle$$

Lot of the Dehn twists *lift*.

> For $g = 2$ we may **conjecture**
>
> $$\mathrm{LMod}(S_2) = \langle T_a, T_b^3, T_c, T_d, T_e \rangle$$

symplectic rep of Mod

$$\mathrm{Mod}(S_g) \to \mathrm{Aut}(H_1(S_g, \mathbb{Z}))$$

where the automorphisms are $2g \times 2g$ integer matrices, but they actually go inside the **symplectic group**

$$\mathrm{Mod}(S_g) \to Sp(2g, \mathbb{Z})$$

its kernel is called the **Torelli group** $I(S_g)$

> $I(S_g) \leq \mathrm{LMod}(S_g)$

lifting criteria

> **Symplectic criteria:**
>
> $$f_\# = (a_{ij})$$
>
> lifts $\iff$ $n \mid a_{2j}$ for $j \neq 2$ and $\gcd(a_{22}, n) = 1$

By symplectic crietria,

$$\mathrm{Mod}(S_g)[n] \leq \mathrm{LMod}_p(S_g)$$

index

$$[\mathrm{Mod}(S_g) : \mathrm{LMod}(S_g)] = [\mathrm{Sp}(2g, \mathbb{Z}_n) : \psi_n] < \infty$$

generating sets

- can't use symplectic rep?
- $$1 \to I(S_g) \to \mathrm{LMod}(S_g) \to \psi(\mathrm{LMod}_p(S_g)) \to 1$$

We know some generating sets of $I(S_g)$

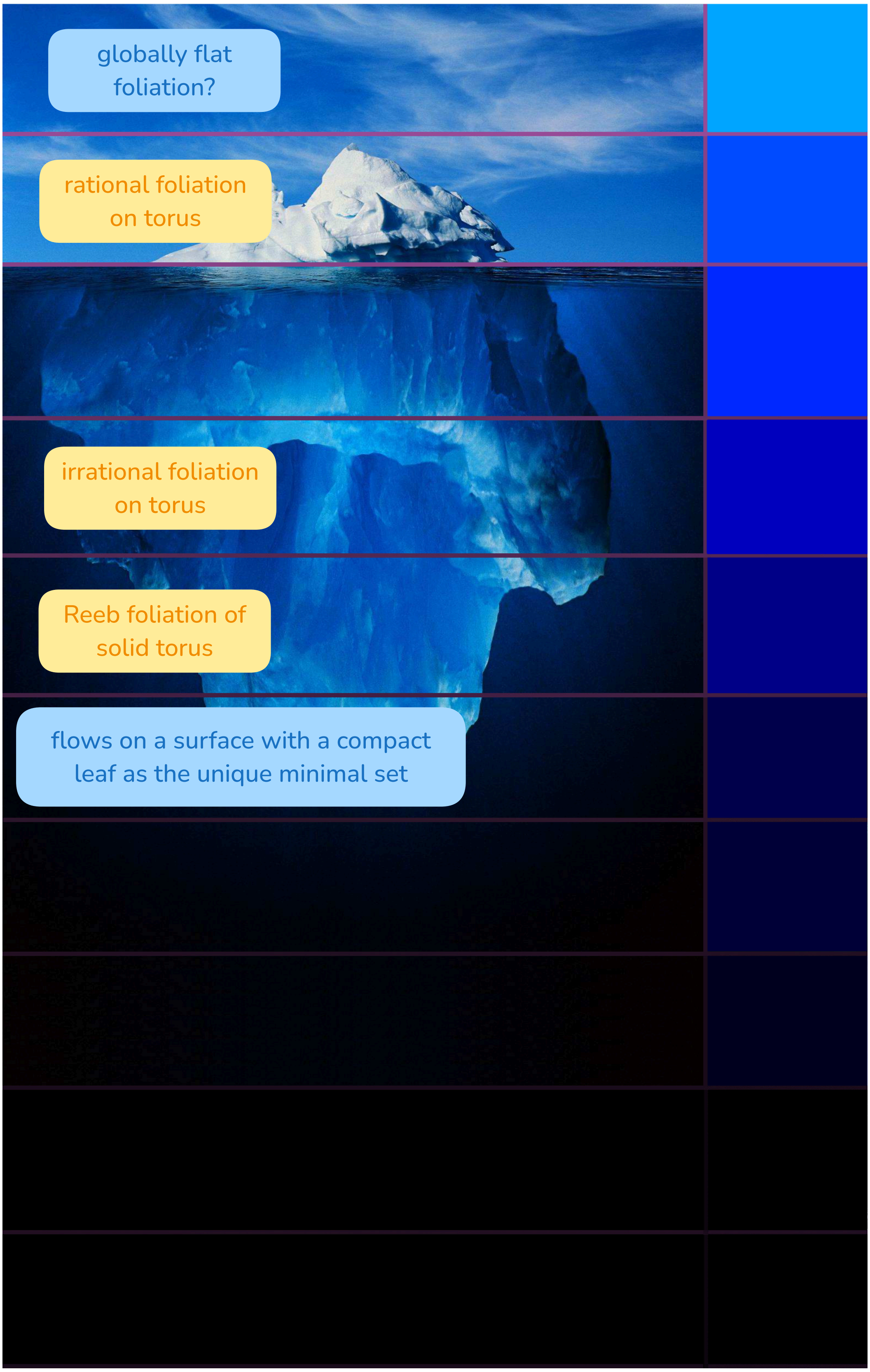
motivation

> 1971, Birman-Hilden

Foliations

#reference

- https://homepages.math.uic.edu/~hurder/papers/70manuscript.pdf
- https://webspace.science.uu.nl/~ban00101/foliations2006/foliations.pdf
- https://mathoverflow.net/questions/230426/what-is-a-foliation-and-why-should-i-care
- https://www.math.unl.edu/~mbrittenham2/papers/folnotes/lectures_1_to_10.pdf
- https://www.youtube.com/watch?v=epEaXLS7ajM
- Adolfo Guillot: Complete holomorphic vector fields and their singular points - lecture 3



leaves are parallel
and contractible,
hence the foliation
has no germinal holonomy

Every orbit limits to the circle,
which is the forward
(and backward)
limit set for all leaves.

tangent distributions and involutivity

$$F(t, x(t), x'(t)) = 0 \iff \begin{cases} \gamma(t) = (t, x(t), p(t)) \\ \gamma \subset F^{-1}(0) \\ (\mathrm{d}x - p\mathrm{d}t)(\gamma') = 0 \end{cases}$$

Definition. Tangent distributions

A (topological/smooth) **tangent k-distribution on a smooth manifold M** is a rank-k (topological/smooth) subbundle of TM.

Given a smooth tangent distribution $D \subseteq TM$, a nonempty immsersed submanifold $N \subseteq M$ is an **integral manifold of D** if

$$T_pN = D_p$$

at each $p \in N$.

foliations

Definition. Foliation on a smooth manifold

Let M be a smooth n-manifold and $\mathcal{F}$ be a collection of k-submanifolds of M. A smooth chart onto a rectangle

$$\varphi : U \subset M \twoheadrightarrow \prod_{i=1}^{n}[a_i, b_i]$$

is a **flat chart for $\mathcal{F}$** if each submanifold in $\mathcal{F}$ intersects U either in the empty set or in a countable union of k-dimensional slices of the form

$$\{x^{k+i} = c_i \mid i \geq 1\}$$

A **k-foliation on smooth manifold M** is a collection $\mathcal{F}$ of disjoint, connected, nonempty immersed sub-k-manifolds of M (called **leaves**) whose union is M and such that in a neighborhood of each $p \in M$ there exists a flat chart for $\mathcal{F}$.

involutive tangent distributions $\leftrightarrow$ foliations

Young Mathematicians' Symposium 2025, IISER Mohali

- 09/05/2025 - 11/05/2025
- Venue: 5A, 5B (5th floor) Academic Block II

The Young Mathematicians' Symposium is an annual event that celebrates the vibrant research culture within our department. It serves as a platform for our PhD scholars and postdoctoral fellows to showcase their work, engage with peers, and spark new ideas through collaboration.

Like every year, the symposium will feature a series of plenary research talks by distinguished invited speakers. Alongside these, we proudly present contributed talks by our own students and postdoctoral fellows, highlighting the diverse and cutting-edge work happening within our community.

YoMathS' 25

9 May 2025	Event/speaker	Title and abstract
9:00-9:30	Inauguration + registration	
9:30-10:30	Prof. Mahuya Datta, ISI Kolkata	Title: Foliations with geometric structures Abstract: A foliation F on a manifold M gives a decomposition of M into immersed submanifolds L, called leaves. The simplest example of k-dimensional foliation on $\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^{n-k}$ is given by the leaves R x {y}, y in R^(n-k). If M is of dimension n, then a foliation on M is locally modelled on one of the above type of foliations on R^n. In the first half of the talk, we will discuss the main questions and results in foliation theory after briefly introducing the basic concepts. In the second half of the talk we will discuss foliations with additional geometric structures on the leaves, focussing mainly on symplectic foliations.
10:30-11:00	Tea/Coffee break	

9 May 2025	Event/speaker	Title and abstract
11:00-11:30	Pabitra Barman	Title: Dominating Surface-Group Representations in PSL_n(C) and PU(2, 1). Abstract: Let S be a connected, oriented, punctured surface of negative Euler characteristic. In this talk, we present two comparison results for representations of π_1(S) into higher-rank Lie groups. First, we show that a generic representation ρ : π_1(S) → PSLn(C) is dominated by a 'positive' representation ρ_0 : π_1(S) → PSLn(R) in both the Hilbert length spectrum and the translation length spectrum in the symmetric space X_n = PSLn(C)/PSU(n), while preserving the lengths of the peripheral curves. This is a joint work with Subhojoy Gupta. Second, we extend this perspective to complex hyperbolic geometry: we show that a T-bent representation ρ : π_1(S) → PU(2, 1) is dominated by a discrete and faithful representation ρ_0 : π_1(S) → PO(2, 1) in the Bergman translation length spectrum, again preserving the lengths of peripheral curves. This is a joint work with Krishnendu Gongopadhyay. These results offer new insights into the role of positivity in higher-rank Teichm üller theory and complex hyperbolic geometry.
11:30-12:00	Lokenath Kundu	Title: The Dehn Function of Palindromic Automorphism Group of Free Group Abstract: The palindromic automorphism group ΠA(F_n) of a free group of rank n is a subgroup of Aut(F_n). The Dehn function of the group Aut(F_n) and Out(F_n) is exponential for n>=3 and is known from the work of Bridson and Vogtmann. In this talk, we present a joint work with Krishnendu Gongopadhyay in which we determine the Dehn function of the palindromic automorphism group of Aut(F_n) for n>=3.

9 May 2025	Event/speaker	Title and abstract
12:00-12:30	Rahul Mondal	Title: Kulkani Limit Sets for Cyclic Quaternionic Projective Groups Abstract: In this talk, I will present results on the dynamics of cyclic subgroups of PSL(n + 1, H) acting on the quaternionic projective space P^n_H, with particular focus on their Kulkarni limit sets. The elements of such groups have been classified as elliptic, loxodromic, or parabolic based on their spectral properties and Jordan forms. Building on this classification, we describe the structure of the corresponding Kulkarni limit sets and the associated domains of discontinuity. This is a joint work with Krishnendu Gongopadhyay and Sandipan Dutta.
12:30-2:00	Lunch break	
2:00-3:00	Prof. Anand Sawant, ITFR Mumbai	Title: R-equivalence in algebraic groups from the perspective of A^1-homotopy theory. Abstract: The group of R-equivalence classes in an algebraic group is an important classical invariant in the study of their near-rationality properties. I will describe its relationship with the sheaves of naive and genuine A^1-connected components and discuss how some of the classical questions involving R-equivalence can be approached using modern techniques. The talk will be based on some old, recent and ongoing joint works with Chetan Balwe and Amit Hogadi.

9 May 2025	Event/speaker	Title and abstract
3:00-3:30	Shruti Rastogi	Title: Motivic Intersection Complex of a Threefold Abstract: The triangulated category of motivic sheaves DM(X) over a scheme X, with rational coefficients, admits a realization functor to the bounded derived category of mixed constructible sheaves D^b_m(X, Q_l). The latter is equipped with a perverse and a weight structure, as well as the S. Morel's t-structure. In this talk, we explore an analogue of Morel's t-structure within a suitable subcategory of DM(X). A key object in the theory of perverse sheaves is the intersection complex IC_X, which appears as a simple object in the heart of the perverse t-structure and can also be characterized using Morel's weight truncation functors. Motivated by this perspective, we define its motivic analogue EM_X for a three-dimensional variety X, using the motivic version of Morel's truncation. We show that EM_X defines a pure motive, and more precisely a Chow motive offering a parallel to the classical intersection complex in the motivic setting.
3:30-4:00	Tea/Coffee break	

9 May 2025	**Event/speaker**	**Title and abstract**
4:00-4:30	Nidhi Gupta	Title: A^1-Connected Components of Affine Quadrics Abstract: Let X be a variety over a field k. Denote by π^A1_0(X) the sheaf of A^1-connected components of X, and by S(X) the sheaf of naive A^1-connected components of X. A result of Balwe, Rani, and Sawant shows that for any field extension F/k, the canonical map π^A1_0(X)(F) → lim_n S^n(X)(F) is an isomorphism. In this talk, we show that for any field F/k and any smooth affine quadratic hypersurface X, this isomorphism stabilizes at n=2. That is, π^A1_0(X)(F) = S^2(X)(F). As an application, we combine this result with Morel's characterization of A^1-connected spaces in terms of the triviality of field-valued sections of π^A1_0 to give a complete characterization of A^1-connected affine quadrics. This talk is based on joint work with Dr. Chetan Balwe.

9 May 2025	Event/speaker	Title and abstract
4:30-5:00	Sohan Ghosh	Title: Euler Characteristics of weight 1 modular forms Abstract: Let E be an elliptic curve over Q. The Selmer group is a fundamental cohomological tool capturing deep arithmetic information about elliptic curves. Fix an odd prime p, and let Zp denote the ring of p-adic integers. For each n ≥ 1, let ζ*p^n be a primitive p^n-th root of unity in \bar{Q }. The cyclotomic Z_p-extension, denoted by Q_cyc, is the unique extension of Q in Q(\cup*{n≥1} ζ_p^n) whose Galois group Γ := Gal(Qcyc/Q) ~= Z_p. When E has good ordinary reduction at p, the Euler characteristic of the p∞-Selmer group over Q_cyc is conjecturally related to the special value L_E(1)/Ω_E, under the Birch and Swinnerton-Dyer conjecture and some suitable assumptions. Here, L_E(s) denotes the Hasse–Weil (complex) L-function of E over Q, and Ω_E is the smallest positive real period of E. In this talk, we raise questions about how similar results might extend to modular forms of weight 1.

9 May 2025	Event/speaker	Title and abstract
5:00-5:30	Chitrarekha Sahu	Title: Picard-Vessiot Extensions with Unipotent Differential Galois Groups Abstract: A differential field is a field F equipped with a derivation $D : F \to F$, an additive map satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$ for all $a, b \in F$. Analogous to the classical Galois theory for polynomial equations, differential Galois theory addresses a similar theory for linear differential equations. Informally, a Picard-Vessiot extension E of a differential field F is a differential field extension of F generated by the solutions of a linear homogeneous differential equation with coefficients in F. The group of all differential automorphisms of E that fix F and commute with the derivation D is called the differential Galois group of the extension. For a differential field F having an algebraically closed field of constants, we analyze the structure of Picard-Vessiot extensions of F whose differential Galois groups are unipotent linear algebraic groups, and as an application of these results we will give a generalization of Liouville's theorem [Ros68]. This talk is based on the joint work with Dr. Matthias Seiss and Dr. Varadharaj R. Srinivasan [SSS25].

10 May 2025	Event/speaker	Title and abstract
9:30-10:30	Prof. Manjunath Krishnapur, IISc	Title: On lemniscates of polynomials Abstract: The (filled) t-lemniscate of a complex polynomial is the set $\{z : \|p(z)\| \leq 1\}$. On many occasions Erdos raised fascinating questions on the extremal properties of lemniscates, most famously in an influential paper with Herzog and Piranian in 1958. Many of these questions are still unanswered. One of them is the area question: What is the minimum possible area of the 1-lemniscate of a degree n polynomial whose zeros are inside the closed unit disk? The best available upper bound was 1/(log log n)^{0.5+\epsilon} due to Wagner and a lower bound of 1/n^4 due to Pommerenke. In joint work with Erik Lundberg and Koushik Ramachandran, we have improved these bounds to 1/log log n and 1/log n respectively, but the gap still remains. The talk will survey some of the questions in the area and outline a proof of this and other related results. The lecture is meant to be accessible to anyone with a basic knowledge of real and complex analysis.
10:30-11:00	Tea/Coffee break	
11:00-11:30	Anusha Bhattacharya	Title: Graph Discretization of Riemannian Manifolds for Spectral Approximation and Learning Applications Abstract: This talk discusses the approximation of eigenvalues and eigenfunctions of the Laplace–Beltrami operator on compact Riemannian manifolds without boundary via weighted graph discretization. Building on the work of Burago et al, I present a joint work with Dr. Soma Maity which generalizes their results to a broader class of manifolds satisfying lower bounds on the Ricci curvature and injectivity radius and upper bounds on diameter and sectional curvature. The construction yields uniform spectral convergence in this class. I will conclude by illustrating connections to spectral graph neural networks, where such discretizations underpin the use of graph Laplacian to approximate manifold-based learning architectures.

10 May 2025	Event/speaker	Title and abstract
11:30-12:00	Yogesh	Title: Preferential Attachment Based on Reinforcement Abstract: We consider a preferential attachment random graph model with self-reinforcement. Each time a new vertex is added, it attaches to an existing vertex with a probability proportional to the sum of the degrees of that vertex across all previous time steps. The resulting growing graph forms a random tree whose vertex degrees increase at a polynomial rate over time. We compute the growth exponent of this model and demonstrate that it is strictly larger than the growth exponent in models without self reinforcement. Our analysis provides insights into how self-reinforcement influences network evolution and accelerates degree growth. This is joint work with Prof. Frank den Hollander.
12:00-12:30	Partha Kumbhakar	Title: Liouville's Theorem for Integration in Terms of Abelian Integrals and Abelian Elements. Abstract: We advance Liouville's theorem on integration in finite terms ([1]) to include abelian integrals and abelian elements. This is a joint work with Sudip Pandit and Varadharaj Ravi Srinivasan.
12:30-2:00	Lunch break	
2:00-3:00	Prof. Tanmay Deshpande, TIFR Mumbai	Title: Character sheaves on tori over local fields Abstract: I will begin with an introduction to the theory of character sheaves and Grothendieck's sheaf-function correspondence in the simplest case, namely that of connected commutative algebraic groups (e.g. the additive group, the multiplicative group, abelian varieties) defined over finite fields and also in the case of pro-algebraic groups. I will then show that character sheaves on tori defined over local fields are naturally parametrized by inertial local Langlands parameters and will relate this result to the local Langlands correspondence for tori due to Langlands. The talk is based on joint work with Saniya Wagh.
3:00-3:15	Tea/Coffee break	

10 May 2025	Event/speaker	Title and abstract
3:15-3:45	Diksha Mukhija	Title: Studying descent of Quadratic forms in characteristic 2 Abstract: Most courses on quadratic forms over fields start with the hypothesis of characteristic being different than 2. In this talk, we will work with quadratic forms over characteristic 2 fields. After reviewing some basic properties, we'll address the question of Descent. The descent problem seeks conditions under which a K-form (quadratic or bilinear) is defined over F for a field extension K/F. We'll answer it for some small dimensional cases when K is the function field of a quadric and address the difficulties we encounter while answering this question in general.
3:45-4:15	Harish Kishnani	Title: Products of Cycle-Classes of Symmetric Groups Abstract: Abstract. Let G be a finite group and C_1, C_2, · · · , C_k be non-trivial conjugacy classes of G. The product of these classes defined by C_1C_2 · · · C_k := {x_1x_2 · · · x_k|x_i ∈ C_i, 1 ≤ i ≤ k} is invariant under conjugation and thus a union of conjugacy classes of G. In particular, for an integer k ≥ 2, the k-th power of a conjugacy class C of G, denoted by C_k, is simply the set CC · · · C (k-times). The main question is, for a given collection of non-trivial conjugacy classes of a finite group G, how large is the set C_1 · · · C_k? In particular, when does this set cover the entire group G? It is a well-established theme in finite group theory which is very active at the current moment. The works of Bertram, Herzog, Lev, Kaplan, Guralnick, Malle, etc. are good sources of motivation for this problem. In this talk, I will first discuss some already known results in this direction. Then, a joint work with Dr. Rijubrata Kundu and Dr. Sumit C. Mishra will be discussed. In the symmetric group, we take the power of a conjugacy class of cycles of a fixed length and determine conditions under which it will cover the alternating group. We will provide a complete answer to both the conjectures of Herzog, Lev and Kaplan [Herzog, Marcel; Kaplan, Gil; Lev, Arieh, Covering the alternating groups by products of cycle classes. J. Combin. Theory Ser. A 115 (2008), no. 7, 1235–1245].

10 May 2025	Event/speaker	Title and abstract
4:15-4:45	Manujith K Michel	Title: Splitting iterative derivations on central simple algebras Abstract: The splitting of a derivation d on a k−central simple algebra A by a Picard- Vessiot extension of (k, d|k) is well studied in the case that the constants of (k, d|k) is an algebraically closed field of characteristic zero. In fact in this case the Picard-Vessiot extension K for the differential module A is the smallest no new constant extension splitting d. One can also characterize the structure of the differential algebra A using the structure of the differential Galois group of K. Similar results are obtained in positive characteristic if we use iterative derivations and Picard-Vessiot theory of ID−modules. More precisely, we show that if d is an iterative derivation on a k−central simple algebra A such that the constants of (k, d|k) is algebraically closed then the Picard-Vessiot extension K for the ID−module A is the smallest no new constant extension splitting d. We also characterise the structure of the ID−algbera A when the differential Galois group of K is linearly reductive or solvable.
4:45-5:30	Tea break and Poster presentation	

11 May 2025	Event/speaker	Title and abstract
9:30-10:30	Prof. Sayani Bera, IACS Kolkata	Title: Attracting Basins and Automorphisms of C^n, n ≥ 2 Abstract: Unlike the complex plane, the automorphisms, i.e., the bijective holomorphic maps of C^n, n≥2 forms a complicated group (under composition). The goal of this talk will be to explore (iterative) dynamics of an automorphism of C^2, near an attracting fixed point. Additionally, we will generalise this model for non-autonomous (random) dynamical systems of automorphisms, which leads to the solution of a long-standing open problem called Bedford's Conjecture. Also, if time permits, we will briefly emphasize on the relevant ideas that extend the above result to C^n, n≥3. This is a joint work with Kaushal Verma.
10:30-11:00	Tea/Coffee break	
11:00-11:30	Aditya Tiwari	Title: Dirichlet Domains in Complex Hyperbolic Bidisks Abstract: The geometry of bidisks -- the product of two hyperbolic planes -- has been widely studied. In this talk, we explore a complex analogue: the product of two complex hyperbolic planes. Focusing on cyclic group actions, we investigate the structure of Dirichlet domains in this space. We describe the behaviour of equidistant surfaces, introduce the notions of visibility and invisibility in this context, and demonstrate that for loxodromic actions, the Dirichlet domain remarkably has exactly two faces. Along the way, we highlight the interplay between the product metric, holomorphic sectional curvature, and the boundary behaviour of the level sets. This exploration opens new avenues for understanding discrete group actions in higher-rank complex hyperbolic geometry. This is a joint work with Prof. Krishnendu Gongopadhyay and Dr. Lokenath Kundu.

11 May 2025	Event/speaker	Title and abstract
11:30-12:00	Tathagata Nayak	Title: On Character Variety of Anosov Representations Abstract: Let Γ be the fundamental group of a k-punctured, k ≥ 0, closed connected orientable surface of genus g ≥ 2. In this talk, it will be shown that the character variety of the (Q+, Q−)-Anosov irreducible representations, resp. the character variety of the (P+, P−)-Anosov Zariski dense representations of Γ into SL(n, C), n ≥ 2, is a complex manifold of complex dimension (2g + k − 2)(n^2 − 1). For Γ = π_1(Σ_g), these character varieties are holomorphic symplectic manifolds. This talk is based on a joint work with Prof. Krishnendu Gongopadhyay.
12:00-12:30	Rifat Siddique	Title: Character space and Gelfand type representation of locally C^∗-algebra Abstract: In this talk, we present a suitable notion of character space of commutative unital locally C^∗-algebra using the notion of inductive limit of topological spaces and study its properties. We prove that every commutative unital locally C^∗-algebra is isomorphic (as a topological ∗-algebra) to the space of continuous functions on its character space. As a consequence, we define unital locally C^∗-algebra generated by a locally bounded normal operator T and its character space is homeomorphic to the local spectrum of T. Also, the functional calculus and spectral mapping theorem are proved in this framework. The talk is based on a joint work with Dr. Santhosh Kumar Pamula.
12:30-2:00	Lunch break	
2:00-3:00	Prof. Somnath Basu	Title: On manifolds homeomorphic to spheres. Abstract: We shall discuss Reeb's Theorem and basic differential topology of Morse functions. This was used by Milnor to prove the existence of exotic spheres in 7 dimensions. We shall propose a generalization of Reeb's Theorem and discuss a proof of it. This is joint work with Sachchidanand Prasad.
3:00-3:30	Pragya Belwal	

11 May 2025	Event/speaker	Title and abstract
3:30-4:00	Tea/Coffee break	
4:00-4:30	Neeraj Kumar Dhanwani	Title: Generating the Liftable Mapping Class Groups Abstract: For a closed orientable surface S_g, the mapping class group Mod(S_g) is defined as isotopy classes of orientation-preserving homeomorphisms of S_g. A mapping class is called liftable if it has a representative homeomorphism that lifts under a regular cover. In this talk, we discuss a generating set for a liftable mapping class group under k-sheeted regular free covers p:S_{k(g-1)+1}--> S_g.
4:30-5:00	Deepanshi Sarraf	
5:00-5:30	Pravin Kumar	Title: Orderability of big mapping class groups. Abstract: An orderable group is a group that is totally ordered, with the ordering being invariant under left-multiplications. In this talk, we will see that a generalized ideal arc system for surfaces determines a left ordering on the mapping class group of a connected, oriented infinite-type surface with a non-empty boundary. We will also explore the conditions under which two generalized ideal arc systems induce the same left ordering. Finally, we will conclude the talk by comparing the usual topology of mapping class groups with the order topology. This is a joint work with Prof. Mahender Singh and Dr. Apeksha Sanghi.
5:30-6:00	Note of thanks and closing.	

Summer School on Rigidity of Discrete Groups, June 30 – July 4, 2025 at IISER Mohali

(Supported by IISER Mohali and The Indian Mathematics Consortium)

IISERM-Rigidity.docx - Google Docs

Organisers:

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In the proposed meeting, we shall focus on some aspects of the broad topic Rigidity in Geometry and Topology. The geometric form of Mostow's celebrated rigidity theorem states that every isomorphism between fundamental groups of closed hyperbolic manifolds of dimension at least 3, is induced by an isometry. This result was followed by a similar theorem of Prasad for nonuniform lattices. However, in a spectacular breakthrough, G. A. Margulis obtained a far reaching generalization of these theorems for irreducible lattices in higher rank groups. This result is now known as the Margulis' Superrigidity Theorem. The ideas behind proofs of these theorems have become very important and fruitful in modern mathematics. In the proposed workshop, we aim to provide a gentle introduction to these results along with similar rigidity theorem in geometry and topology through mini courses and survey talks.

***Speakers*:** Marc Bourdon (U. Lille), Pralay Chatterjee (IMSc), Soma Maity (IISER Mohali), Arghya Mondal (IISER Mohali), C. S. Rajan (Ashoka), Pranab Sardar (IISER Mohali), Riddhi Shah (JNU), T. N. Venkataramana (ICTS Bengaluru).

Expected Participants: Faculty members, PhD students and postdoctoral fellows working in related areas. The topics to be covered in this school require some prior knowledge of geometric group theory, smooth manifolds and Lie groups.

Due to limited funding, we shall be able to provide local hospitality to only a limited number of participants who will be selected based on the proximity of their research interests with the theme of the summer school. Participants are requested to arrange their own travel funding.

Andrew Neitzke - Lectures on BPS states and spectral networks

https://gauss.math.yale.edu/~an592/exports/bps-lectures.pdf

Given a Riemannian manifold

$$(M, g)$$

we have

(classical)	(quantum)
On $$T^*M$$ the Hamiltonian flow of $$H = \frac{\|p\|_g^2}{2} + V$$	$$L^2(M)$$ the flow of the operator $$H := \frac{1}{2}\Delta_g + V$$

Definition. $\mathcal{N} = 2, \dim = 1$ **SUSY Lie superalgebra**

We define a Lie superalgebra ($\mathbb{Z}_2$-graded Lie algebra)

$$\mathcal{A} := \mathcal{A}^0 \oplus \mathcal{A}^1$$

as

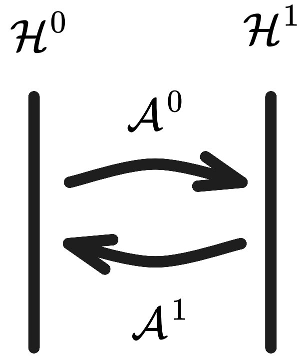
$$\mathcal{A}^0 = \mathbb{C}H$$
$$\mathcal{A}^1 = \mathbb{C}Q \oplus \mathbb{C}\bar{Q}$$

with the brackets

$$[Q, \bar{Q}] = 2H$$
$$[Q, H] = 0 = [\bar{Q}, H] = 0$$

A $\mathbb{Z}_2$-graded representation of $\mathcal{A}$ is a representation of $\mathcal{A}$ on a $\mathbb{Z}_2$-graded vector space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ where

$$\mathcal{A}^i : \mathcal{H}^j \to \mathcal{H}^{j+i}$$



$T_1[M]$

Definition. $T_1[M]$

Consider the Laplace operator on differential forms

$$\begin{aligned}\Delta : \Omega^\star_{L^2}(M) \to \Omega^\star_{L^2}(M) \\ \Delta = [\mathrm{d}, \mathrm{d}^\star]\end{aligned}$$

We have the following representation

$$\begin{aligned}\mathcal{H}^0 &= \bigoplus_{k \geq 0} \Omega^{2k}_{L^2}(M) \\ \mathcal{H}^1 &= \bigoplus_{k \geq 0} \Omega^{2k+1}_{L^2}(M)\end{aligned}$$

and

$$\begin{aligned}Q &\mapsto \mathrm{d} \\ \bar{Q} &\mapsto \mathrm{d}^\star \\ H &\mapsto \frac{1}{2}\Delta\end{aligned}$$

- This representation is *unitary*.
- $$\begin{aligned}2\langle \psi, H\psi \rangle &= \left\langle \psi, Q\bar{Q}\psi \right\rangle + \left\langle \psi, \bar{Q}Q\psi \right\rangle \\ &= \|Q\psi\|^2 + \left\|\bar{Q}\psi\right\|^2\end{aligned}$$
    - Thus eigenvalues of H are non-negative.
    - Moreover,

$$H\psi = 0 \iff Q\psi = 0 \text{ and } \bar{Q}\psi = 0$$

- In particular each

$$\psi \in \ker H = \ker Q \cap \ker \bar{Q}$$

generate

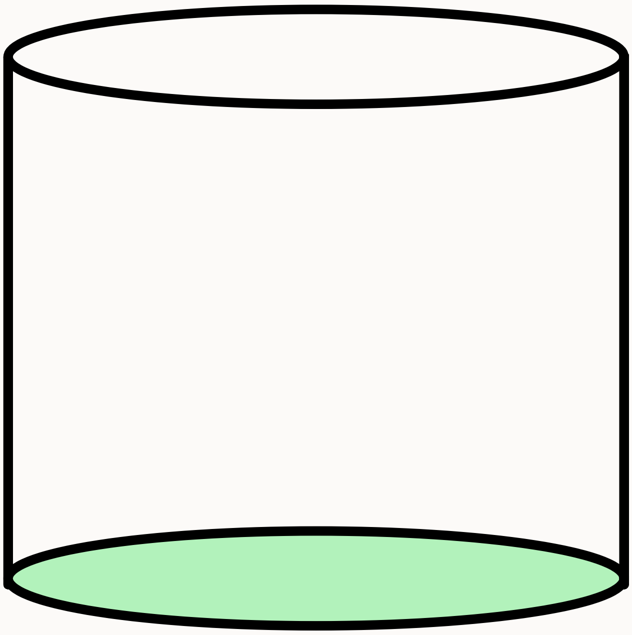
$$\mathbb{C}\{\psi\}$$

a 1-dim (trivial) representation of $\mathcal{A}$.

more examples

$\mathcal{N}$	dim	**given**		**(co)homology**
2	1	(M, g) is Riemannian	(QM) $T_1[M, g]$	de Rham
(2,2)	2	M is Kahler	(QFT) $T_2[M]$	Lagrangian Floer homology
(2,0)	6	$\mathfrak{g}$ be a ADE type Lie algebra, L be a link in $\mathbb{R}^3$	(CFT) $T_6[\mathfrak{g}, L]$	Kovanov homology

Sergei Gukov - Homological algebra of knots and BPS states

(co)homology	physics	index
	QM?	
de Rham $\cong$ simplicial	SUSY QM	Euler characteristic χ
quantum de Rham	2d QFT: A-model on $\mathbb{R} \times S^1$ 	Hirzebruch polynomial
Dolveault $H^{p,q}_{\bar{\partial}}(X)$	B-model	genus?
Kovanov homology $\mathrm{Kh}^{p,r}$	???	Jones polynomial $J(q)$
symplectic Floer homology $\mathrm{HF}_{\mathsf{sym}}(L_1, L_2)$	A-model on $\mathbb{R} \times I$ with boundary conditions L_1, L_2 (choice of Lagrangian submanifolds)	$\mid L_1 \cap L_2 \mid$
instanton Floer homology $\mathrm{HF}_{\mathsf{sym}}(^3M)$	4d gauge theory	Casson invariant

A Guest - From Quantum Cohomology to Integrable Systems

Let

$$\langle\ ,\ \rangle : H^i(M;\mathbb{Z}) \times H_i(M;\mathbb{Z}) \to \mathbb{Z}$$

denote the natural pairing, and also the extended pairing

$$\langle\ ,\ \rangle : H^*(M;\mathbb{Z}) \times H_*(M;\mathbb{Z}) \to \mathbb{Z}$$

where $\langle a, B\rangle = 0$ when $|a| \neq |B|$. In de Rham notation, $\langle a, B\rangle = \int_B a$. Since there is no torsion, these pairings are non-degenerate.

The *intersection pairing* is defined by

$$(\ ,\ ) : H^*(M;\mathbb{Z}) \times H^*(M;\mathbb{Z}) \to \mathbb{Z}, \quad (a, b) = \langle ab, M\rangle = \int_M a \wedge b.$$

We have $\langle ab, M\rangle = \langle a, B\rangle = \langle b, A\rangle$. It follows that the intersection pairing $(\ ,\ )$ is a non-degenerate symmetric bilinear form.

3-point Gromov-Witten invariants

$$\langle A|B|C\rangle_0 = \langle abc, M\rangle = \int_M a \wedge b \wedge c =$$

Definition. Definition

Let M be a $\mathbb{C}$-manifold and $p \in \mathbb{C}P^1$. Then

$$\mathscr{O}_D((\mathbb{C}P^1, p), (M, A))$$

Calabi-Yau manifolds

Definition. Calabi-Yau manifolds

A **Calabi-Yau manifold** is a compact Kahler manifold

$$(M, J, g)$$

such that either one of the following equivalent conditions occur

- $\mathrm{Ric}(g) = 0$
- $c_1(M) = 0$
- $\mathrm{Hol}(g) \subseteq SU(n)$
- the canonical bundle K_M is trivial
- there is a global, non-vanishing $(n, 0)$-form

$\dim_{\mathbb{C}}$	name
1	Calabi-Yau elliptic curve
2	K3 surface
3	Calabi-Yau 3-fold

Hodge diamond

Proposition: The Hodge diamond of a Calabi-Yau 3-fold looks like

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

number theory

> **Question**
>
> Is *algebraic number theory* study of *algebraic numbers* or subfield of number theory that uses *algebraic* tools?

algebraic numbers and integers

> **Definition. Algebraic numbers and integers**
>
> A **number field** is a finite extension
>
> $$K \geq \mathbb{Q}$$
>
> whose elements are **algebraic numbers** in K and an **algebraic integer** in K is a root $\alpha \in p^{-1}(0) \subset K$ of a monic polynomial $p \in \mathbb{Z}[X] \subset \mathbb{Q}[X] \subset K[X]$.
> The set of algebraic integers in K is $\mathcal{O}_K$ called **ring of integers** of K.

[1]

> **Proposition:** The minimal polynomial m_α of an algebraic number α has integer coefficients iff it is an algebraic number, that is,
>
> $$m_\alpha(X) \in \mathbb{Z}[X] \iff \alpha \in \mathcal{O}_K$$

$\implies$ is easy. So we assume α is an algebraic integer, so there is a monic polynomial $f \in \mathbb{Z}[X]$ such that

$$f(\alpha) = 0$$

and we know

$$\frac{f}{m_\alpha}(X) = g(X) \in \mathbb{Q}[X]$$

- Assume $g(X) \notin \mathbb{Z}[X]$ then there is a coefficient which in reduced form has a denominator $\neq 1 \implies$ there is a prime p which divides it.
- Let $u > 0$ be the largest integer such that $p^u g$ does not have any denominators divisible by p. And say $v > 0$ be the smallest integer such that $p^v h$ has no

denominator divisible by p. We have

$$p^u g p^v h = p^{u+h} f$$

- Going modulo p we have

$$(p^u g \bmod p)(p^v h \bmod p) = 0 \in \mathbb{Z}_p[X]$$

where the left side is a product of non-zero polynomials $p^u g \bmod p$ and $p^v h \bmod p$ (non-zero as they do not have any denominators divisible by p), but it's product is zero, which contradicts $\mathbb{Z}_p$ being a field.

Corollary:

$$\mathscr{O}_{\mathbb{Q}} = \mathbb{Z}$$

The minimal polynomial of $\frac{a}{b}$ is $X - \frac{a}{b}$ which has integer coefficients $\iff \frac{a}{b} \in \mathbb{Q}$.

Let K be a number field and $\alpha \in K$. Then α is an algebraic integer $\iff$ the Abelian group

$$\mathbb{Z}[\alpha]$$

is finitely generated.

Corollary: Let K be a number field then $\mathscr{O}_K$ is a ring and

$$\mathbb{Q}\mathscr{O}_K = K$$

Example

$$\mathbb{Z}\left[\frac{1}{2}\right]$$

is **not** finitely generated as $\frac{1}{2}$ is not an algebraic integer in $\mathbb{Q}$.

norms and traces

Definition. **Definition**

Let $L \geq K$ be a finite extension of number field K. Then

$$K \to \text{EndVec}_K(L)$$

as scalars. We call

$$N_{L/K}(\alpha) := \det(\alpha) \in K$$

as the norm of α and

$$\operatorname{tr}_{L/K}(\alpha) := \operatorname{tr}(\alpha)$$

as the trace.

- Norm is multiplicative and trace is additive as $\det, \operatorname{tr}$ are like that resp.
- For $a \in K$ we have

  $$N_{L/K}(a\alpha) = a^{\deg L/K} N_{L/K}(\alpha)$$

  and

  $$\operatorname{tr}_{L/K}(a\alpha) = \operatorname{tr} a_{L/K}(\alpha)$$

Corollary: Let K be a number field.

- $\alpha \in \mathscr{O}_K \implies$ norm and trace of α belongs to $\mathbb{Z}$
- norm of α is $\pm 1 \iff \alpha$ is a unit in $\mathscr{O}_K$

Proposition: Let K be a number field. Then

$$\mathscr{O}_K$$

is a free Abelian group of rank $[K, \mathbb{Q}]$.

ideal numbers

Definition. **Let $I \subseteq R$ be a non-zero ideal, then the norm of I is**

$$N(I) := \left|\frac{R}{I}\right|$$

Proposition: Let I be a non-zero ideal of $\mathscr{O}_K$ then $N(I)$ is finite and

$$N(\alpha\mathscr{O}_K) = \left|N_{K/\mathbb{Q}}(\alpha)\right|$$

The set of all non-zero fractional ideals

$$I_K$$

of a number field K forms a group under multiplication.

discriminant

Definition. **Let K be a number field of degree n with r_1 number of real embeddings and r_2 number of complex embeddings. Then**

$$\operatorname{sig}(K) := (r_1, r_2)$$

where $n = r_1 + 2r_2$.

Given n embeddings

$$\sigma_k : K \to \mathbb{C}$$

we have the product

$$\begin{aligned} \sigma : K &\to \mathbb{C}^n \\ x &\mapsto (\sigma_1(x), \ldots, \sigma_n(x)) \end{aligned}$$

And given a $\mathbb{Z}$-basis $\{\alpha_1, \ldots, \alpha_n\}$ of $\mathscr{O}_K$ we have $n \times n$ matrix

$$[\sigma]_{\{\alpha_1, \ldots, \alpha_n\}} = (\sigma_i(\alpha_j))_{i,j}$$

The determinant

$$\det(\sigma_i(\alpha_j))$$

computes "volume" of $K/\mathscr{O}_K$, which serves as a is a measure of *density* of $\mathscr{O}_K$ in K.

To get a real number we rather consider

$$\det([\sigma]^2_{\{\sigma_1, \ldots, \sigma_n\}}) = \det \operatorname{tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j) \in \mathbb{Z}$$

Definition. **Definition**

Let $\alpha_1, \ldots, \alpha_n \in K$. Then

$$\operatorname{disc}(\alpha_1, \ldots, \alpha_n) := \det \operatorname{tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j) \in \mathbb{Z}$$

In particular, if $\alpha_1, \ldots, \alpha_n$ is a $\mathbb{Z}$-basis of $\mathscr{O}_K$ we write

$$\Delta_K :=$$

Proposition: The symmetric bilinear form

$$\begin{aligned} K \times K &\to \mathbb{Q} \\ (x, y) &\mapsto \operatorname{tr}_{K/\mathbb{Q}}(xy) \end{aligned}$$

is non-degenerate.

Hence, $\Delta_K \neq 0$.

prime decomposition and ramification

Proposition: Let $\mathfrak{p} \subset \mathcal{O}_K$ be prime ideal. Then

- $N(\mathfrak{p}) \in \mathfrak{p} \cap \mathbb{Z}$
- $\mathfrak{p} \cap \mathbb{Z}$ is a **prime ideal** of $\mathbb{Z}$
- Hence,

$$\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$$

  for some prime $p \in \mathbb{Z}$.

Definition. Prime ideals above primes

When

Proposition: Let $\mathfrak{p} \subset \mathcal{O}_K$ be prime ideal. Then

- $N(\mathfrak{p}) \in \mathfrak{p} \cap \mathbb{Z}$
- $\mathfrak{p} \cap \mathbb{Z}$ is a **prime ideal** of $\mathbb{Z}$
- Hence,

$$\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$$

  for some prime $p \in \mathbb{Z}$.

we say **$\mathfrak{p}$ is above p**

$$\begin{array}{c} \mathfrak{p} \subset \mathcal{O}_K \subset K \\ \Big| \\ p\mathbb{Z} \subset \mathbb{Z} \subset \mathbb{Q} \end{array}$$

We have the morphisms

$$\mathbb{Z} \xhookrightarrow{\iota} \mathcal{O}_K \to \mathcal{O}_K/\mathfrak{p}$$

whose kernel is

$$\{a \in \mathbb{Z} \mid a \in \mathfrak{p}\} = \mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$$

so by quotienting

$$\mathbb{Z}_p \hookrightarrow \mathscr{O}_K/\mathfrak{p}$$

which is a finite field extension of $\mathbb{Z}_p$.

Definition. **The inertial degree of $\mathfrak{p} \subset \mathscr{O}_K$ *above* $p \in \mathbb{Z}$ is**

$$f(\mathfrak{p}) := \dim_{\mathbb{Z}_p}(\mathscr{O}_K/\mathfrak{p})$$

$$N(\mathfrak{p}) = |\mathscr{O}_K/\mathfrak{p}| = p^{f(\mathfrak{p})}$$

Definition. **Ramification index of $\mathscr{O}_K$**

Let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime ideal *above* $p \in \mathbb{Z}$. We call **ramification index** of $\mathfrak{p}$, $e(\mathfrak{p})$, the power of $\mathfrak{p}$ that divides $p\mathscr{O}_K$.
Hence,

$$p\mathscr{O}_K = \mathfrak{p}_1^{e(\mathfrak{p}_1)} \dots \mathfrak{p}_g^{e(\mathfrak{p}_g)}$$

We say p is **ramified** if $e(\mathfrak{p}_i) > 1$ for some i, otherwise p is non-ramified (that is, when $p\mathscr{O}_K = \mathfrak{p}_1 \dots \mathfrak{p}_g$).

Proposition: In

Definition. **Ramification index of $\mathscr{O}_K$**

Let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime ideal *above* $p \in \mathbb{Z}$. We call **ramification index** of $\mathfrak{p}$, $e(\mathfrak{p})$, the power of $\mathfrak{p}$ that divides $p\mathscr{O}_K$.
Hence,

$$p\mathscr{O}_K = \mathfrak{p}_1^{e(\mathfrak{p}_1)} \dots \mathfrak{p}_g^{e(\mathfrak{p}_g)}$$

We say p is **ramified** if $e(\mathfrak{p}_i) > 1$ for some i, otherwise p is non-ramified (that is, when $p\mathscr{O}_K = \mathfrak{p}_1 \dots \mathfrak{p}_g$).

we have

$$[K:\mathbb{Q}] = \sum_{i=1}^{g} e(\mathfrak{p}_i) f(\mathfrak{p}_i)$$

As

$$p\mathcal{O}_K = \mathfrak{p}_1^{e(\mathfrak{p}_1)} \dots \mathfrak{p}_g^{e(\mathfrak{p}_g)}$$
$$p^{[K:\mathbb{Q}]} = N(p\mathcal{O}_K) = N(\mathfrak{p}_1^{e(\mathfrak{p}_1)} \dots \mathfrak{p}_g^{e(\mathfrak{p}_g)})$$

and since norm is multiplicative and

$$N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}| = p^{f(\mathfrak{p})}$$

we have

$$N(\mathfrak{p}_1^{e(\mathfrak{p}_1)} \dots) = \prod_i p^{f(\mathfrak{p}_i)e(\mathfrak{p}_i)}$$

Proposition: Let K be a number field, $\exists \theta$ such that

$$\mathcal{O}_K = \mathbb{Z}[\theta]$$

and $p \in \mathbb{Z}$ prime. Let

$$m_\theta(X) \bmod p = \prod_i \phi_i(X)^{e_i} \text{ in } \mathbb{Z}_p[X]$$

with ϕ_i coprime and irreducible. Then setting

$$\mathfrak{p}_i := (p, \tilde{\phi}_i(\theta)) = p\mathcal{O}_K + \tilde{\phi}_i(\theta)\mathcal{O}_K$$

where $\tilde{\phi}_i$ is any lift of ϕ_i to $\mathbb{Z}[X]$, gives us the factorization of $p\mathcal{O}_K = \mathcal{O}_K$

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}^{e_g}$$

Definition. Inert, totally ramified primes

	e	f	g	
inert	1	n	1	
totally ramified	n	1	n	

Let K is a number field. Then p is ramified $\iff p \mid \Delta_K$.

[2]

Corollary of

> **Let K is a number field. Then p is ramified $\iff p \mid \Delta_K$.**

There are only a finite number of ramified primes for a number field K.

[3]

glossary

K	algebraic integers	elements	norm	trace	number of embeddings into $\mathbb{C}$	Δ_K
$\mathbb{Q}$	$\mathbb{Z}$					
$\mathbb{Q}(\sqrt{2})$	$\mathbb{Z}[\sqrt{2}]$?	$$a+b\sqrt{2}$$	a^2-2b^2	$2a$	2; $$\sqrt{2}\mapsto\sqrt{2},-\sqrt{2}$$	
$\mathbb{Q}(i)$ that is $\mathbb{Q}(\sqrt{-1})$	$\mathbb{Z}$?					
$\mathbb{Q}(\sqrt{5})$	$$\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$$				2; $$\sqrt{5}\mapsto\sqrt{5},-\sqrt{5}$$	5
$\mathbb{Q}(\sqrt{m})$	?					

number fields vs elliptic curves

number fields $$K$$	elliptic curves $$E := V(y^2 - f(x))$$ where $f(X)$ is a smooth cubic
ring of integers $$\mathscr{O}_K$$ whose elements are roots of monic polynomials	rational points on the elliptic curves $$E(\mathbb{Q}) := E \cap \mathbb{Q}P^2$$

number fields	elliptic curves
$$K$$	$$E := V(y^2 - f(x))$$ where $f(X)$ is a smooth cubic
Proposition: Let K be a number field. Then $$\mathscr{O}_K$$ is a free Abelian group of rank $[K, \mathbb{Q}]$.	**(Mordell-Weil theorem)** F be a number field (finite extension of $\mathbb{Q}$) and $E(F)$ be an elliptic curve curve ($y^2 = f(x)$) with a F-rational point $O \in E(F)$ then $$(E(F), +, O)$$ is a finitely generated Ab group. $$\implies E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^{\text{rank}(E/\mathbb{Q})}$$
$\mathscr{O}_K^\times$ is a finitely generated Abelian group with rank $r_1 + r_2 - 1$, $$\implies \mathscr{O}_K^\times = \underbrace{\mu(K)}_{\text{torsion}} \oplus \mathbb{Z}^{r_1+r_2-1}$$	
Class group $$\text{Cl}(K)$$	$$Ш(E/\mathbb{Q})$$ *Shafarevich-Tate group*

number fields	elliptic curves
K	$E := V(y^2 - f(x))$ where $f(X)$ is a smooth cubic
Dedekind zeta function $$\zeta_K(s) := \sum_{I \subseteq \mathcal{O}_K} \frac{1}{N_{K/\mathbb{Q}}(I)^s} = \prod_{\mathfrak{p} \subseteq \mathcal{O}_K} \frac{1}{1 - N_{K/\mathbb{Q}}(\mathfrak{p})^s}$$	• Hasway-Weil L-function $L(E, s)$ is a meromorphic function on $\mathbb{C}$ where $$L(E,s) = \sum_{n \geq 1} \frac{a_n}{n^s} = \prod_{\text{primes } p} \left(1 - \frac{a_p}{p^s} + \epsilon(p) p^{1-2s}\right)^{-1}$$ where $$\epsilon(p) = \begin{cases} 0 \\ 1 \end{cases}$$ • a_n is recovered via a_p for p primes • $$a_p = \begin{cases} p + 1 - \lvert\{E(\mathbb{F}_p)\}\rvert & \text{good red} \\ \pm 1 & \text{mult red} \\ 0 & \text{not good red} \end{cases}$$

number fields	elliptic curves
K	$$E := V(y^2 - f(x))$$ where $f(X)$ is a smooth cubic
(Analytic class number formula ~1839) $$\lim_{s\to 0}\frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{R_K h_K}{\lvert\mu(K)\rvert}$$	⚠ **Birch and Swinnerton-Dyer conjecture** Let $E(F)$ where F be a number field. Then analytic rank(E) = order at $s = 1$ of $L(E, s)$ = algebraic rank$(E) = r$. (implicit in this conjecture is $L(E, s)$ is holomorphic at $s = 1$) Finer version using other refined invariants of E

ramification in number theory and algebraic geometry

> **Proposition:** Let D be any Dedekind domain with fraction field K, L/K a finite separable field extension and R the integral closure of D in L. Then R is again a Dedekind domain.

So

Definition. Ramified primes in Dedekind domains

From

> **Proposition:** Let D be any Dedekind domain with fraction field K, L/K a finite separable field extension and R the integral closure of D in L. Then R is again a Dedekind domain.

given any prime ideal $\mathfrak{p} \subset D$ of a Dedekind domain D, we have a primary decomposition

$$\mathfrak{p}R = \beta_1^{e_1} \dots \beta_r^{e_r}$$

where e_k is called the **ramification index** of the prime β_k. A prime $\mathfrak{p}$ is called **ramified** if some ramification index $e_k > 1$.

We say L/K is *unramified* if all prime ideals of R are *unramified* in L.

Definition. Ramification points of morphisms of curves

Let X, Y be two curves (complete, non-singular, 1 dimensional integral scheme over $k = \bar{k}$). Let

$$f : X \to Y$$

be a finite morphism, $P \in X$ and $Q = f(P)$. Then f is **unramified** at P if

- the induced map on stalks

  $$\mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$$

  gives

  $$\mathfrak{m}_Q \mathcal{O}_{X,P} = \mathfrak{m}_P$$

- and the extension of residue fields

  $$k(Q) \to k(P)$$

  is separable

and otherwise it is **ramified** at P.

The map f is *unramified* if it is unramified at every point, and *ramified* if it is ramified at some point.

From

> **Proposition: Let D be any Dedekind domain with fraction field K, L/K a finite separable field extension and R the integral closure of D in L. Then R is again a Dedekind domain.**

let

$$\iota : D \hookrightarrow R$$

induce the morphism

$$\iota^* : \mathrm{Spec}(R) \to \mathrm{Spec}(D)$$

Then a point $\mathfrak{p}$ is a ramification point of the map $\iff$ the prime $\mathfrak{p}$ is a ramified prime.

Let $\mathfrak{p} \subset R$ be prime, that is $\mathfrak{p} \in \mathrm{Spec}(R)$. Then

$$\mathfrak{p} \mapsto \mathfrak{q} \iff f(\mathfrak{p}) = \mathfrak{p} \cap D = \mathfrak{q}$$

[4]

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2. https://math.stackexchange.com/a/700951 ↩
3. https://feog.github.io/ANT10.pdf ↩
4. https://www.wstein.org/edu/2010/581b/projects/matthew_ward/581bfinalproject.pdf ↩

#problem homotopy extension property of cone

Problem

Show that the cone over X

$$X \hookrightarrow \mathrm{Cone}(X)$$

has the *homotopy extension property.*

Quote

Not all inclusions $A \subset X$ are created equal.
[1]

Definition. Homotopy extension property

Let X be a topological space and let $A \subset X$. We say (X, A) has the **homotopy extension property** if for a homotopy $f_\bullet : A \times I \to Y$ and a map $g_0 : X \to Y$ such that $f_0 = g_0|_A$ there exists a homotopy

$$g_\bullet : X \times I \to Y$$

that starts from $g_0 : X \to Y$ and *extends* f, that is,

$$f_t = g_t|_A, \forall t \in I$$

The property amounts to lifting $f_\bullet, g_0$ in

$$\begin{tikzcd} X \arrow[r, "g_0"] & Y \\ A \arrow[u, hook] \arrow[r, "f_\bullet"'] & Y^I \arrow[u, "\pi_0"'] \end{tikzcd}$$

to a $f_\bullet$ such that

$$
\begin{tikzcd}
X \arrow[r, "g_0"] \arrow[dr, "g_\bullet" description] & Y \\
A \arrow[u, hook] \arrow[r, "f_\bullet"'] \arrow[r, phantom, shift left=3, "\Uparrow"] & Y^I \arrow[u, "\pi_0"']
\end{tikzcd}
$$

Now a cone

$$\mathrm{Cone}(X) := \frac{X \times I}{X \times \{1\}}$$

$$X \cong X \times \{0\} \hookrightarrow \mathrm{Cone}(X)$$

gives

[2]

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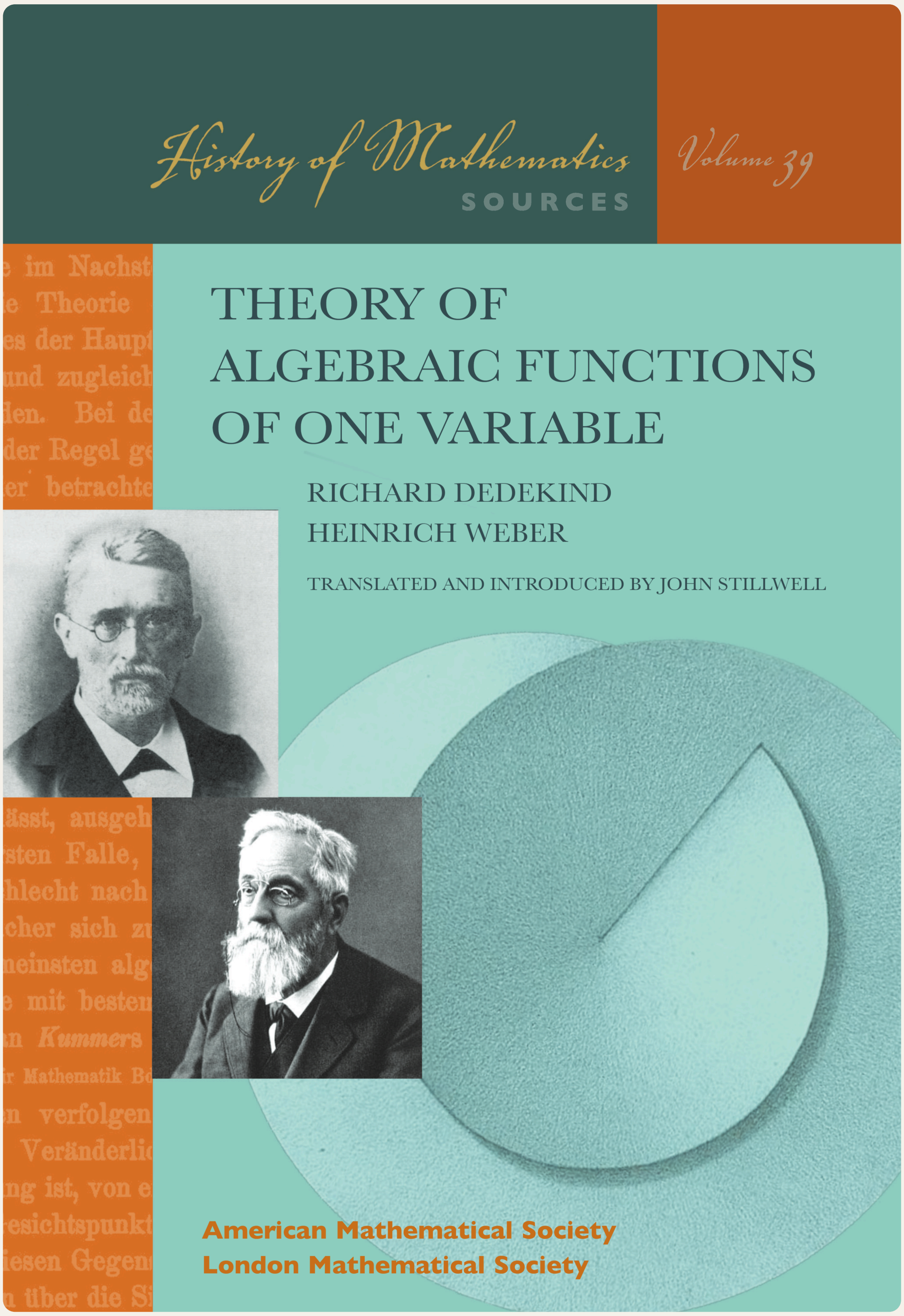
1. math.jhu.edu/~jmb/note/hep.pdf ↩
2. algebraic topology - Homotopy Extension Property of mapping cone - Mathematics Stack Exchange ↩

logbook

- 14 April

#logbook/spring25

- Was studying branched covering properties of $z^3 - 3z$
  - Cubic function fields with prescribed ramification
  - Mating quadratic maps with the modular group II
- Adding more topics in complex 1-manifolds, AKA Riemann surfaces
- Looking into Dedekind-Weber theory
  - https://mathoverflow.net/a/176887
  - The History of Algebraic Geometry
  - the classic paper



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- 1 April

  #logbook/spring25

  - started looking at Jet Nestruev - Smooth Manifolds and Observables
  - will create a note on Farey sequence - Wikipedia and hopefully its relation with the Gauss map later

references

smooth manifolds

#reference smooth manifolds, *continued*

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- algebraic topology / differential topology
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  - Vinicius Ramos-Differential Topology
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  - ### *Differential Topology*

    A saying in MathOverflow

    Run don't walk your way to *Milnor's Topology from the Differentiable Viewpoint*

    - Differential topology - difftop.pdf (ethz.ch)
      - follows Milnor and completes exposition
    - Differential Topology by Guillemin and Pollack
- geometric topology
- differential forms
  - Bott Tu - Differential forms in algebraic topology
  - - Isu Vaisman – Differential Forms & Cohomology
  - Morita differential forms

PDEs, geometric analysis, harmonic analysis

- ### PDEs
  - Doctorate course: Partial Differential Equations and Applications (IMPA, 2022) with notes in Portuguese
- ### Ricci flow
  - https://www.mfo.de/about-the-institute/staff/prof-dr-gerhard-huisken/lectures/introduction-to-ricci-flow

- https://www.dropbox.com/scl/fo/le0chdpiegjxj2i0yss6c/ADkJqSxPmNu3NGu9FOm_ayI/recordings?rlkey=rdva7s9rqdnrjcqmvsp33xih4&subfolder_nav_tracking=1&st=4rl2afuu&dl=0

REF spectral geometry

#reference spectral geometry

- Olivier Lablée. - Spectral Theory in Riemannian Geometry (2015, European Mathematical Society).pdf

  Moreover, spectral geometry is an inter-disciplinary field of mathematics; it involves

  - analysis of ODE and PDE
  - dynamical systems: classical and quantum completely integrable systems, quantum chaos, geodesic flows and Anosov flows on (negatively curved) manifolds (for example, Ruelle resonances are related to the spectrum of the Laplacian, see the recent articles [Fa-Ts1], [Fa-Ts2]).
  - geometry and topology (the main purpose of these notes is to explain this)
  - geometric flow: the parabolic behaviour of scalar curvature

    $$\frac{\partial R_{g(t)}}{\partial t} = \Delta R_{g(t)} + 2\left|\mathrm{Ric}_{g(t)}\right|^2;$$

    the Ricci flow (see also Sections 7.8.3 and 7.8.4)

    $$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)}$$

  - probability: the Brownian motion on a Riemannian manifold (M, g) is defined to be a diffusion on M (generator operator is given by $\frac{1}{2}\Delta_g$).
  - etc. ...

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- https://www.michaellevitin.net/Book/TSG230529.pdf
- https://math.mit.edu/~dyatlov/files/2015/restalk.pdf
- Semyon Dyatlov's Homepage

- **fourier/harmonic analysis, geometric measure theory**
  - Abstract harmonic analysis
  - HIM Lectures: Trimester Program "Harmonic Analysis and Partial Differential Equations"

- Programa de Doutorado: Topics in Analysis: Sphere packings, Fourier analysis and beyond (2018)
- Lectures on Harmonic Analysis - YouTube
- IPS on Geometric Measure Theory, Harmonic Analysis, Partial Diff. Equations - MAA MathFest 2023
- restriction problem
  - Hong Wang: The restriction problem and the polynomial method, Lecture I
  - The Polynomial Method and the Restriction Problem - Larry Guth
  - Larry Guth (MIT) - 1/3 Introduction to decoupling [MSRI 2017]
- L01 Harmonic analysis and affine Hecke algebras by Eric Opdam
- Eugenia Malinnikova: Frequency function methods in PDE I

geometry, groups and dynamics

REF dynamics.Anosov flows Birkhoff sections 3-man

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solitons

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- bose einstein condensate - Link between integrability and soliton solutions - Physics Stack Exchange

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geometry

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  - Andreas Gathmann - Class Notes: Algebraic Geometry (rptu.de)
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- Joe Harris (mathematician) - Wikipedia, Griffiths
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    - E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris - Geometry of Algebraic Curves_ Volume I (1985).pdf
      - [1] **The reader is assumed to have a working knowledge of basic algebraic geometry such as is given, for example, in the first chapter of Hartshorne's book Algebraic Geometry.**
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[2]

commutative algebra

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Richard E Borcherds: Commutative algebra and algebraic geometry

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- sett.field.poly n.permutation action

Algebraic geometry

- Algebraic geometry I: Varieties
- Algebraic geometry II: Schemes
- Algebraic geometry: extra topics

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2. https://math.stackexchange.com/a/24447/1290493 ↩

#reference complex analysis and complex geometry

complex analysis

> Quote
>
> Advanced Complex Analysis - Harvard
>
> Complex analysis is a nexus for many mathematical fields, including:
>
> 1. Algebra (theory of fields and equations);
> 2. Algebraic geometry and complex manifolds;
> 3. Geometry (Platonic solids; flat tori; hyperbolic manifolds of dimen sions two and three);
> 4. Lie groups, discrete subgroups and homogeneous spaces (e.g. H/ SL2(Z);
> 5. Dynamics (iterated rational maps);
> 6. Number theory and automorphic forms (elliptic functions, zeta functions);
> 7. Theory of Riemann surfaces (Teichmuller theory, curves and their Jacobians);
> 8. Several complex variables and complex manifolds;
> 9. Real analysis and PDE (harmonic functions, elliptic equations and distributions).

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T. E. Venkata Balaji - An Introduction to Families, Deformations and Moduli

Riemann surfaces

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- (Princeton Mathematical Series_ 65) Lars Valerian Ahlfors_ Leo Sario - Riemann Surfaces_ (PMS-26)-Princeton University Press (2015).pdf

- Renzo Cavalieri, Eric Miles - Riemann Surfaces and Algebraic Curves_ A First Course in Hurwitz Theory-Cambridge University Press (2016).pdf
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  - Eberhard Freitag - Complex Analysis 2 Riemann Surfaces, Several Complex Variables, Abelian Functions, Higher Modular Functions (2011).pdf
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      - Phillip Griffiths, Joseph Harris, - Principles of algebraic geometry-Wiley (2011).pdf
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        - [1] **The reader is assumed to have a working knowledge of basic algebraic geometry such as is given, for example, in the first chapter of Hartshorne's book Algebraic Geometry.**
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- https://adebray.github.io/lecture_notes/m392c_RSnotes.pdf
- Lecture notes by McMullen
- Riemann.pdf (berkeley.edu)
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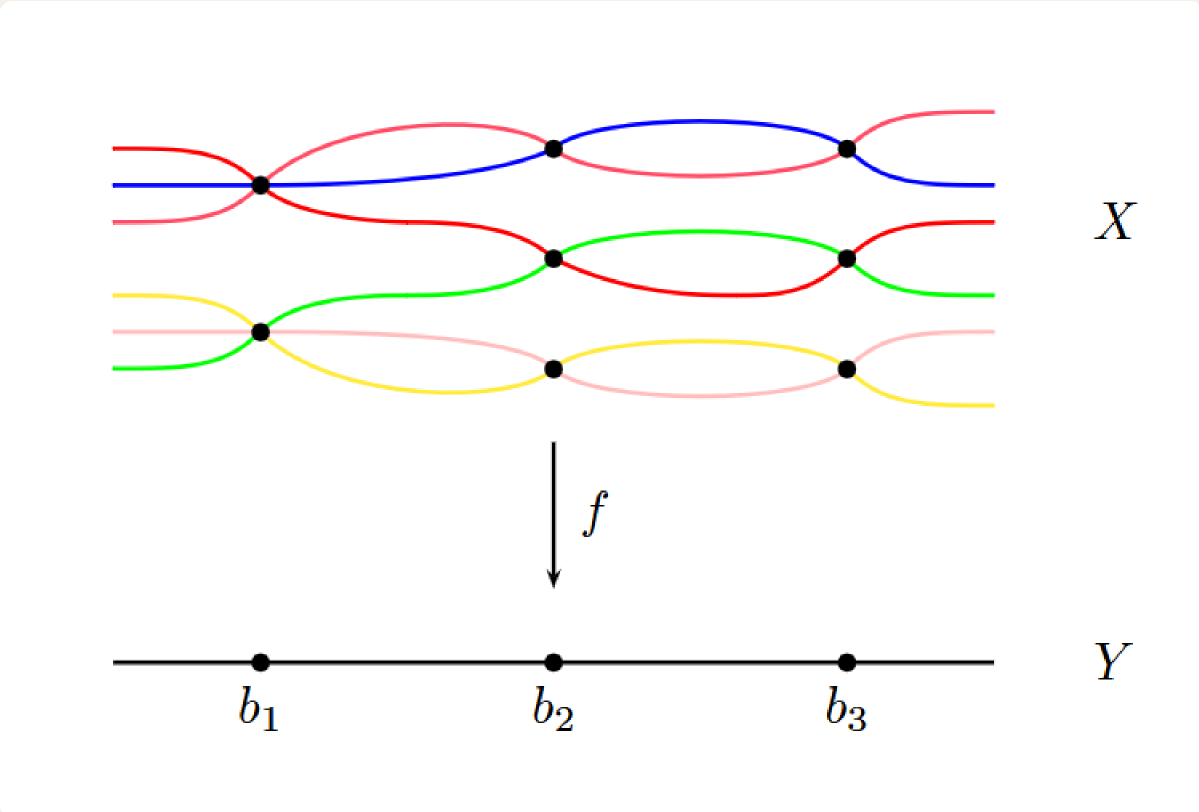
- https://mathoverflow.net/questions/313254/references-for-riemann-surfaces



- https://www.icts.res.in/sites/default/files/Weaver-LectureNotes-Groups&Surfaces.pdf

- The following theorem is sometimes called the Riemann existence theorem. It constructs a Riemann surface from a finite covering of a Riemann surface (usually the Riemann sphere) with a number of points deleted. In this version it can be viewed as a purely topological property of the existence of extensions of coverings of punctured surfaces.

  Theorem 4.5. *If X is a Riemann surface and $S \subset X$ is a closed discrete subset, then any finite covering $\phi' : Y' \to X' = X \setminus S$ (which we suppose connected) can be extended to a proper holomorphic map $\phi : Y \to X$, where Y is a Riemann surface containing Y' such that $Y \setminus Y'$ is a closed discrete subset.*

  Proof. At a point $s \in S$ there exists a neighborhood U_s with $U_s \cap S = \{s\}$ and a coordinate chart $\phi_s : U_s \to D$ where D is the unit disc centered at the origin. As ϕ' is a finite covering, there exists a finite number of components $\phi'^{-1}(U_s \setminus \{s\})$. In fact, $\phi' : \phi'^{-1}(U_s \setminus \{s\}) \to U_s \setminus \{s\}$ is a covering. Let V' be one of the components. As $\phi'_{|V'}$ is a finite covering of the unit punctured disc, there exists a map $\psi' : V' \to D \setminus \{0\}$ so that $\phi_s \circ \phi \circ \psi'^{-1} : D \setminus \{0\} \to D \setminus \{0\}$ and such that $\phi_s \circ \phi \circ \psi'^{-1}(z) = z^k$ and therefore we can add the point 0 to $D \setminus \{0\}$ and obtain a holomorphic map from D to D. Let V be the set obtained by adding an abstract point to V' so that $\psi : V \to D$ is a homeomorphism and defines a holomorphic chart. $\phi_{|V}$ becomes a branched holomorphic covering. Repeating the procedure for each component above every $U_s \setminus \{s\}$ for $s \in S$ we obtain the Riemann surface Y. □

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differential equations in complex domain

- with monodromy ordinary differential equations - Diff Eq Textbook with Monodromy? - Mathematics Stack Exchange
  - E. Hille, Ordinary differential equations in the complex domain. It has a section 5.7 (with exercises in the end) on monodromy. Einar Hille - Ordinary Differential Equations in the complex domain-Dover Publications (1997).pdf
  - For more challenging reading: Y. Ilyashenko, S. Yakovenko, Lectures on Analytic Differential Equations. Chapter III deals with monodromy.
  - Or you can try D.Anosov, A.Bolibruch, The Riemann-Hilbert Problem. The book works out many examples but **is not** a textbook.
  - Lastly: H.Zoladek, The Monodromy Group. Henryk Żołądek - The Monodromy Group-Birkhäuser Basel (2006).pdf
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complex geometry

Why are the two kinds of algebraic geometry (complex manifolds/schemes) considered the same subject? : r/math (reddit.com)
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1. E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris - Geometry of Algebraic Curves_ Volume I (1985).pdf > page=13&selection=86,0,109,1 ↩

Lecture Notes by Zbigniew Błocki

Literature recommendations in between Algebraic and Differential Geometry : r/math (reddit.com)

Read Griffiths & Harris *Principles of Algebraic Geometry* chapters 0,1,2. Especially chapter 2 is beautiful.

For complex geometry, read Huybrechts or Wells' *Differential Analysis on Complex Manifolds* (which is more analytically focused).

You won't be able to escape proper analysis for the **Hodge theorem on Kahler manifolds**: its simply unavoidable. Modern Hodge theory in algebraic geometry is a very advanced subject. You can study it without actually learning the proof of the Hodge theorem proper, but if you're going to do so you better be doing a PhD in algebraic geometry, in which case your supervisor will instruct you on where to learn. For actual proofs of the Hodge theorem, I learned it from Warner *Foundations...* but that's pretty dated. The "true" proof uses pseudodifferential operators and is mostly of interest to people looking to go into geometric analysis, and there are other cool proofs (such as in Roe's *Elliptic operators, topology, and asymptotic methods*) but none of them will give great algebraic insights. If you're not interested in spending 2 months learning elliptic operator theory in some depth then you're better off just reading what you see in Huybrechts or another book on Kahler manifolds (I love Ballmann *Lectures on Kahler Manifolds*) so you know all the consequences of the Hodge theorem.

Once you have a good foundation in complex geometry, it's much easier to see which direction you want to go in.

- On the more algebraic side people are doing stability conditions, enumerative geometry, toric mirror symmetry, etc. which involve plenty of complex-geometric thinking but are mainly algebraic.
- A fun thing to start reading might be *Lectures on Bridgeland Stability* by Macri to see if you like it.
- On the more analytic side, people are interested in using algebraic+differential techniques to solve differential equations (canonical metrics on varieties, vector bundles) [1] but usually the algebraic input is *fairly* limited (you sort of know it ahead of time, and then all the true difficulty is in analysis). The standard books here are Gabor's *Introduction to Extremal Kahler metrics* and Kobayashi *Differential geometry of complex vector bundles* to get started.

Keep in mind that to appreciate these modern threads of the theory there are a lot of things you need to "know" to really "get it," but not to get going. You'll want to eventually have an appreciation for:

- geometric invariant theory,
- symplectic reduction,
- construction of moduli spaces in both algebraic and differential geometry,
- the origins of enumerative invariants via integration over moduli spaces,
- some standard examples of moduli (Riemann surfaces,
- vector bundles over Riemann surfaces, starting with the Jacobian (see Chapter 2 of Griffiths&Harris) etc.

Due to its intersectional nature, complex geometry has a lot more of this background information to learn than most other subjects, but you don't have to learn it all at once. For example note Gabor's book doesn't get to the thing he's really interested in (K-stability) until chapter 6, because of all the background knowledge he has to go through.

Another very new area in the modern theory is non-Archimedean geometry. This is being used by pure arithmetic geometers by Scholze and others, but its also growing in popularity in traditional complex geometry due to the works of people like Boucksom et al (in the study of canonical Kahler metrics) and Yang Li (in the study of the SYZ conjecture). It seems to be one of the only new tools to really understand Calabi-Yau manifolds, which is the next unknown in this branch of complex geometry, so it's likely to be a hot topic for a little while. It has a very "pure algebraic geometry" flavour, as you're basically doing Hartshorne-style stuff over non-Archimedean fields. Somehow it produces fairly concrete results though.

If all of that seems overwhelming, then the correct thing to do is go read several of Donaldson's surveys of the subject, which will be both interesting and inspiring.

Textbook recs for intro to complex algebraic geometry : r/math (reddit.com)

- Griffiths & Harris *Principles of Algebraic Geometry* (There are better CG intros than Chapter 0, but chapters 1 and 2 are essential reading, and the later chapters are great for building examples to help you understand the theory) Phillip Griffiths, Joseph Harris, - Principles of algebraic geometry-Wiley (2011).pdf

- Voisin *Hodge theory and complex algebraic geometry I and II* (second volume is very hard, special focus on Hodge theory/Hodge structures and deformation theory)
- Huybrechts *Complex geometry* (good first book in CG)
- Eisenbud & Harris *3264 and all that* (good second book, a good introduction to intersection theory which is a super important tool in CAG)
- Demailly *Complex analytic and differential geometry* (good if you want to do geometric analysis on complex manifolds)
- Wells *Differential analysis of complex manifolds* (good first book)
- Sze`kelyhidi *An introduction to extremal Kahler metrics* (excellent book, specifically about K-stability, includes a complete proof of the Calabi conjecture)
- Vakil *Rising sea* (this is more pure algebraic geometry)
- Huybrechts *Lectures on K3 surfaces* (good book, again good for seeing techniques and examples)
- Clay Mirror symmetry book from 2000 (bad book, but has a completely comprehensive state of play of mirror symmetry from 20 years ago; later chapters 39-40 are good reading even if you don't understand anything in detail) clay-Mirror-Symmetry.pdf
- Huybrechts & Lehn *Moduli spaces of sheaves* (good introduction to sheaf-theoretic things, lots of important ideas that you should be familiar with here)

There have been some big advancements in complex algebraic geometry over the last 25 years in different directions, and no one source covers all of it. Some big recent topics are: Classification of Fano threefolds, K-stability, Bridgeland stability conditions, derived complex algebraic geometry, birational geometry, Gromov--Witten theory, and mirror symmetry. There have been some results in pure Hodge theory but I don't know too much about this (have a look at some survey's, perhaps of Voisin, but she tends to write in French as much as English).

There is generally no good single reference for understanding mirror symmetry, which is a broad subject, and you are better off looking up various surveys and lecture notes. The book *Calabi-Yau manifolds and related geometries: lectures at a summer school in Nordfjordeid, Norway* is a possible starting point after you've studied complex manifolds a bit more deeply, and this considers the SYZ conjecture. You can also come at the topic from the pure algebraic geometry perspective through derived categories, which I know less about, or you can look into Gromov--Witten theory and toric mirror symmetry.

The subject requires a lot of moving in and out of various areas and reading the key papers, rather than reading a textbook or two and gaining a comprehensive understanding of what is going on. You can have a look at some survey's and lecture notes of people like Donaldson, Dominic Joyce, Richard Thomas from the last 20 years. If you have a specific area you have in mind I may be able to suggest some key papers.

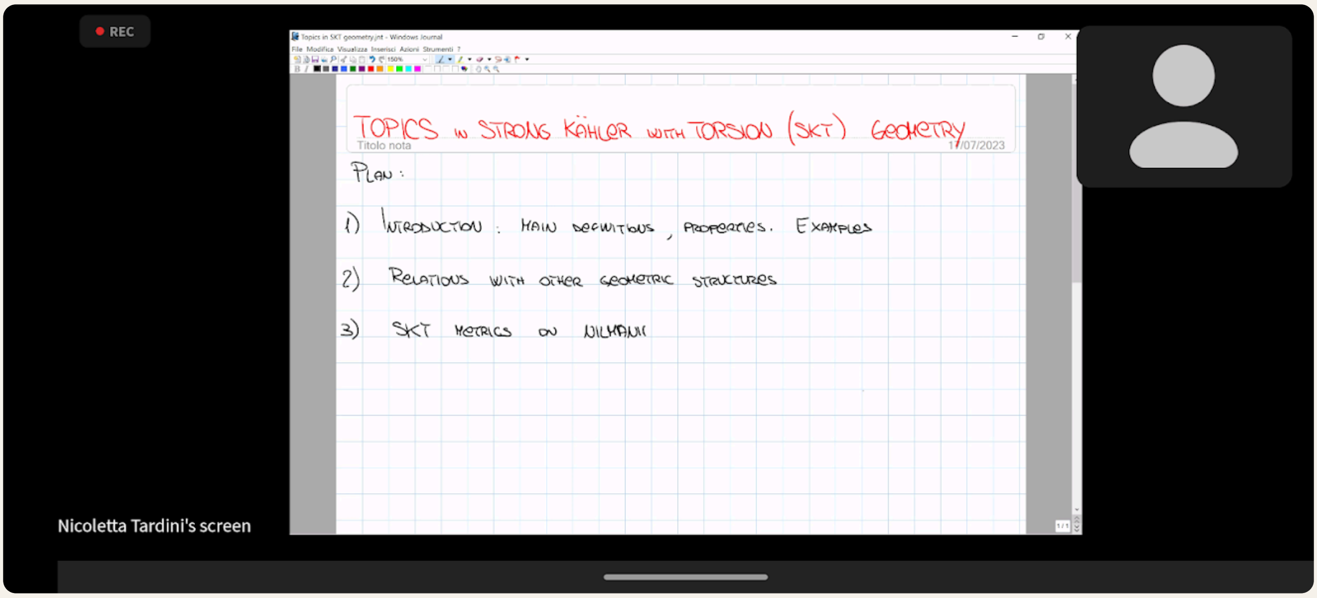
If you want to get into the singular theory that involves a lot of study by example. I don't know good reference texts for this but you will want to look at works by birational geometers who have learned many techniques about how to deal with singular varieties. Key tools are blow ups, Q-divisors, intersection theory, normalisation, and cohomology calculations. You generally learn how to do these things by reading the references above and by reading papers where people study many examples. If you want to see such things in action look up papers by Ivan Cheltsov to get a feel for it.

- Intersection Theory (1 of 5) - YouTube Intersection theory in algebraic geometry - lccs (columbia.edu)

Kahler manifolds

- http://people.maths.ox.ac.uk/~joyce/Nairobi2019/BallmannKahlerManifolds.pdf
- https://hal.science/file/index/docid/1136/filename/kg.pdf

SKT geometry

- Nicoletta Tardini - Lectures on strong Kähler with torsion (SKT) metrics - YouTube
  - 

- Hodge theory of SKT manifolds - ScienceDirect

SKT Geometry | PIRSA

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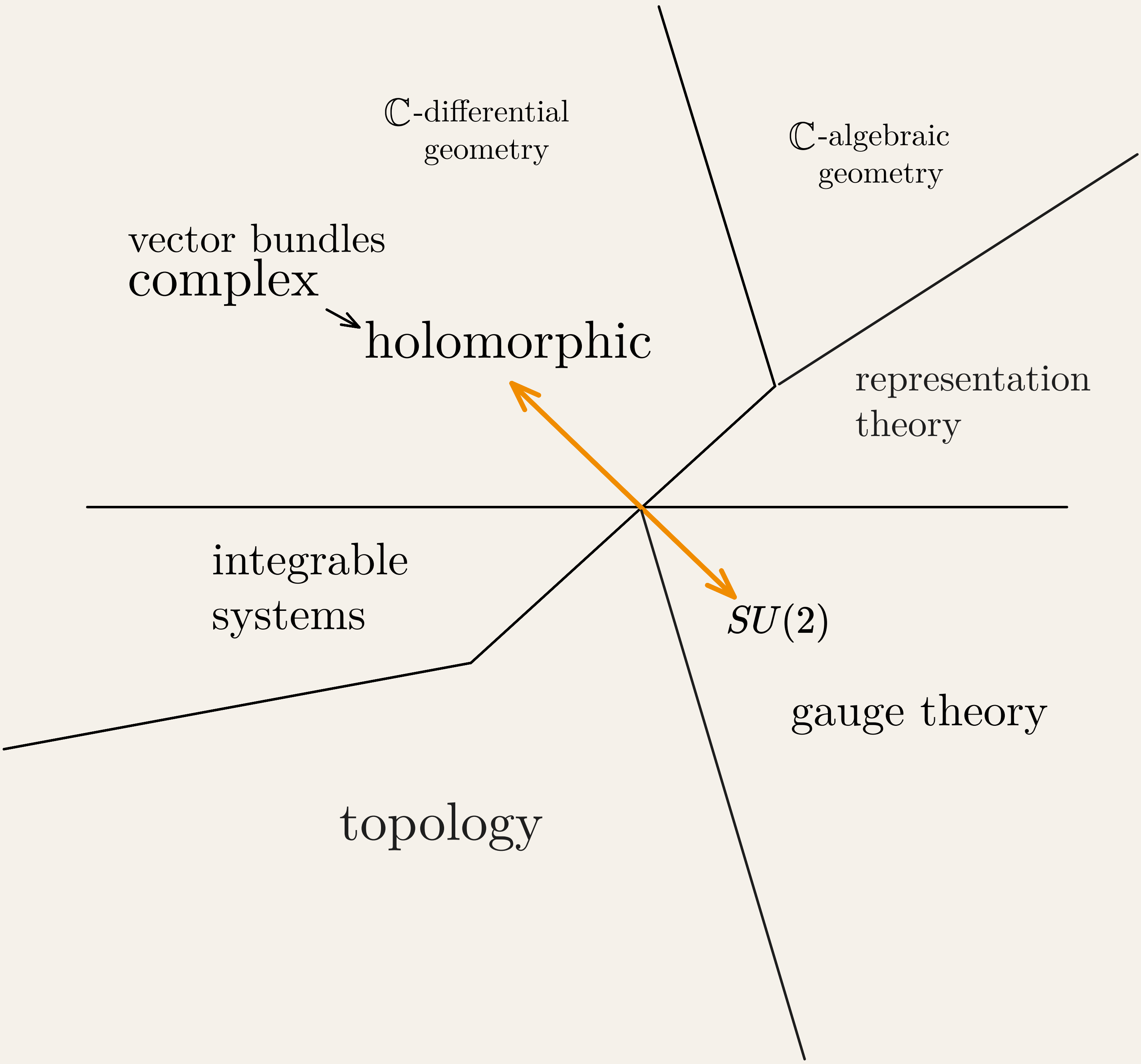
1. REF cpx vec bun x gauge x geometry ↩

#reference **complex vector bundles, gauge, geometry**

Literature recommendations in between Algebraic and Differential Geometry : r/math (reddit.com)

On the more analytic side, people are interested in using algebraic+differential techniques to solve differential equations (canonical metrics on varieties, vector bundles) but usually the algebraic input is *fairly* limited (you sort of know it ahead of time, and then all the true difficulty is in analysis). The standard books here are Gabor's *Introduction to Extremal Kahler metrics* and Kobayashi *Differential geometry of complex vector bundles* to get started.



- $\mathbb{C}$-algebraic geometry $\leftrightarrow$ $\mathbb{C}$-differential geometry
- Vector bundles $\leftrightarrow$ gauge theories
  - Kobayashi–Hitchin
- "integrable geometry"
  - How to get into integrable systems? : r/math (reddit.com)

    I've only had a tangential interaction with integrable systems. Specifically, I study what *you might call integrable geometry*. The "niceness" that makes a system integrable (spectral deformation, Backlund transformations, etc.) corresponds to nice transformations/deformations of a certain family of submanifolds (e.g. the sine-Gordon equation and K-surfaces). For me this really boils down to a there being family of flat connections somewhere so I haven't really ever had to get into the PDE side of things.

    From this perspective, the most useful thing I ever read on integrable systems was the book **Integrable Systems by Hitchin, Segal and Ward**. If nothing else, read Hitchin's introduction.

books

- Griffiths and Harris-Alg geom
- John M. Lee - Introduction to Complex Manifolds-American Mathematical Society (2024).pdf

relating everything

- https://impa.br/wp-content/uploads/2017/04/25CBM_02.pdf
- https://walpu.ski/Teaching/WS2122/DifferentialGeometry3/Topics.pdf
- walpu.ski/Teaching/WS2324/DifferentialGeometry3/GaugeTheoryS.pdf

Integrable systems and complex geometry

- https://arxiv.org/pdf/0706.1579

Hitchin and Higgs

Kobayashi–Hitchin

Kobayashi–Hitchin correspondence - Wikipedia

> In differential geometry, algebraic geometry, and gauge theory, the **Kobayashi–Hitchin correspondence** (or Donaldson–Uhlenbeck–Yau theorem) relates stable vector bundles over a complex manifold to Einstein–Hermitian vector bundles. The correspondence is named after Shoshichi Kobayashi and Nigel Hitchin, who independently conjectured in the 1980s that the moduli spaces of stable vector bundles and Einstein–Hermitian vector bundles over a complex manifold were essentially the same. This was proven by Simon Donaldson for projective algebraic surfaces and later for projective algebraic manifolds, by Karen Uhlenbeck and Shing-Tung Yau for compact Kähler manifolds, and independently by Buchdahl for non-Kahler compact surfaces, and by Jun Li and Yau for arbitrary compact complex manifolds.The theorem can be considered a vast generalisation of the Narasimhan–Seshadri theorem concerned with the case of compact Riemann surfaces, and has been influential in the development of differential geometry, algebraic geometry, and gauge theory since the 1980s. In particular the Hitchin–Kobayashi correspondence inspired conjectures leading to the nonabelian Hodge correspondence for Higgs bundles, as well as the Yau–Tian–Donaldson conjecture about the existence of Kähler–Einstein metrics on Fano varieties, and the Thomas–Yau conjecture about existence of special Lagrangians inside isotopy classes of Lagrangian submanifolds of a Calabi–Yau manifold.

Simon Donaldson

- https://www.ma.ic.ac.uk/~skdona/EMP.PDF
-
  - Donaldson's recent work centers on a problem in complex differential geometry concerning a conjectural relationship between algebro-geometric "stability" conditions for smooth projective varieties and the existence of "extremal" Kähler metrics, typically those with constant scalar curvature (see for example cscK metric). [1]
    - Donaldson obtained results in the toric case of the problem (see for example Donaldson (2001)).
    - He then solved the Kähler–Einstein case of the problem in 2012, in collaboration with Chen and Sun. This latest spectacular achievement involved a number of difficult and technical papers.
    - The first of these was the paper of Donaldson & Sun (2014) on Gromov–Hausdorff limits.

- The summary of the existence proof for Kähler–Einstein metrics appears in Chen, Donaldson & Sun (2014). Full details of the proofs appear in Chen, Donaldson, and Sun (2015a, 2015b, 2015c).

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1. https://en.wikipedia.org/wiki/Simon_Donaldson#Research ↩

- Simon Donaldson - Einstein-Kahler, algebraic geometry

Siberg-Witten

> **Quote**
>
> Maybe Seiberg-Witten theory. In the 80s, topologists studied manifolds using the Donaldson equations. This was grueling work, but they learned wondrous things. In 1994, Witten suggested the Seiberg-Witten equations, which were much easier to study. /
> 1:26 PM · Aug 21, 2024
> Practically overnight, all the major results of Donaldson theory were reproven with 1/10th the length, plus many more exciting things. The old tools were obsolete.
>
> Nowadays, the donaldson equations are making a comeback, doing some things the Seiberg-Witten eqs can't. Not a permanent usurping. But for those in the room where it happened, topology turned on its head.
>
> Here is an email from that time. It's electric.
> https://web.ma.utexas.edu/users/benzvi/math/S-W

REF singularity theory

#reference singularity theory

- V I Arnold.Singularity theory volume I
- Henryk Żołądek - The Monodromy Group-Birkhäuser Basel (2006).pdf

- https://www.maths.ed.ac.uk/~v1ranick/papers/greuel.pdf [1]
- John Milnor - Singular Points of Complex Hypersurfaces (AM-61), Volume 61-Princeton University Press (2016).pdf
- PCompanionMathematics.pdf > page=391
- https://math.berkeley.edu/~giventh/papers/arn.pdf
- http://www2.egr.uh.edu/~ychen11/PUBLICATIONS/BookChapter.pdf
- https://arxiv.org/pdf/2305.19842.pdf
- https://www.birs.ca/workshops/2012/12w5067/report12w5067.pdf
- https://www.sissa.it/fa/download/publications/remizov.pdf
- A singular mathematical promenade

previous references

references

https://chessapig.github.io/files/presentations/A_B_Models.pdf

- Arnold, Dynamical systems VI: Singularity theory, V. I. Arnold, V. V. Goryunov, O. V. Lyashko, V. A. Vasil'ev - Singularity Theory I-Springer-Verlag Berlin Heidelberg (1998).pdf
  - Math: Phenomenological account of singularity theory (few proofs). Reasonably quick to read, Arnold is good at writing.
  - https://www.springer.com/gp/book/9783540637110
- V.I. Arnold, A.N. Varchenko & S.M. Gusein-Zade, Singularities of Differentiable Maps Volume II: Monodromy and Asymptotic Integrals, 2012
  - Math: More detailed account of above book. Has section on asymptotic oscillating integrals.
  - https://www.springer.com/gp/book/9781461284086
- Greene and Vafa, Calabi-Yau Manifolds and Renormalization Group flows, 1989
  - Physics: **Describes CY mflds as RG fixed points of LG models**
  - REF CYMan x RG flows
- Vafa and Warner, Catastrophes and the classification of Conformal Theories, 1988
  - Physics: *Uses singularity theory to classify CFTs, derives ADE classification*
- Vafa, Topological Mirrors and Quantum Rings, 1991
  - *Physics for Math readers: Survey style summary of mirror symmetry*
- Griffiths, Periods of Integrals on Algebraic Manifolds, 1969
  - Math: *Survey of variations of hodge structures and period maps*
- Cecotti, Girardello, and Pasquinucci, Singularity Theory and N=2 Supersymmetry, 1990
  - Physics: *Describes dictionary between singularity theory and LG model physics*
- Witten, Mirror Manifolds and Topological Field Theory, 1991
  - Physics: *Notes on mirror symmetry in context of topological field theory*
- Barannikov, Quantum Periods 1: Semi-infinite variations of Hodge structures, 2000

- Math: *Introduced VSHSs, On moduli space of noncommunative cmplx deformations*

descriptions

NSF Award Search: Award # 1104329 - International Conference on Singularity Theory and Applications

Singularity theory is a meeting place of many disparate areas of mathematics, where different types of ideas, techniques and results merge together. The modern theory of singularities dates back to the 1960s, with the pioneering work of Thom, Hironaka, Brieskorn, Zariski, and many other renowned mathematicians. More recently, *Singularity theory promoted vigorous interchanges among mathematical fields such as algebraic and geometric topology, algebraic geometry, number theory, and more applied fields such as the study of configurations in robot motion planning*.

In the last century, considerable effort has been directed towards studying manifolds - spaces that locally look uniform, at each point and in each direction. This effort has been immensely successful; a substantial part of our insight has been gained through the study of various invariants (e.g., characteristic numbers and classes), the surgery program, etc. In recent decades, *topologists have studied "singular" spaces with increasing interest, due to their numerous occurrences and applications within pure mathematics (algebraic geometry, number theory) and outside pure mathematics (mathematical physics, robot motion planning)*. In contrast to a manifold, a singular space may locally look different from point to point. The study of topological properties of singular spaces developed into the field of Singularity theory. The proposed conference will focus around recent developments in this fast advancing field of research.

Singularity theory - Isaac Newton Institute

The main goal in most problems of singularity theory is to understand the dependence of some objects of analysis and geometry, or physics, or from some other science on parameters. For generic points in the parameter space their exact values influence only the quantitative aspects of the phenomena, their qualitative, topological features remaining stable under small changes of parameter values.

However, for certain exceptional values of the parameters these qualitative features may suddenly change under a small variation of the parameter. This change is called a perestroika, bifurcation or catastrophe in different branches of the sciences. A typical example is that of Morse surgery, describing the perestroika of the level variety of a function as the function crosses through a critical value. *(This has an important complex counterpart - the Picard-Lefschetz theory concerning the branching of integrals.)* Other familiar examples include caustics and outlines or profiles of surfaces obtained from viewing or projecting from a point, or in a given direction.

In spite of its fundamental character, and the central position it now occupies in mathematics, singularity theory is a surprisingly young subject. So, for example, one can consider the singularities arising from the orthogonal projections a generic surface in 3-space, a problem of surely classical interest. *Their classification was completed as recently as 1979. In one sense singularity theory can be viewed as the modern equivalent of the differential calculus, and this explains its central position and wide applicability.* In its current form the subject started with the fundamental discoveries of Whitney (1955), Thom (1958), Mather (1970), Brieskorn (1971). Substantial results and exciting new developments within the subject have continued to flow in the intervening years, while the theory has embodied more and more applications.

Newton Institute: Singularity theory and hyperbolicity, 2024

singularity theory and Bayesian statistics

Picard–Lefschetz theory, monodromy

Picard–Lefschetz theory - Wikipedia
Monodromy - Wikipedia

- analytic continuation and covering spaces
  Monodromy - Wikipedia

Henryk Żołądek - The Monodromy Group-Birkhäuser Basel (2006), p.7

> After publication of "Analysis situs" by H. Poincaré, investigation of the topology of algebraic varieties began.

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1. BIMSA - Introduction to Singularity Theory ↩

representation theory

REF representation theory

#reference representation theory

- Fulton Harris Representation theory.pdf
- Peter Woit - Quantum Theory, Groups and Representations_ An Introduction(2017).pdf
- Modern Trends in Algebra and Representation Theory (cambridge.org)

https://arxiv.org/pdf/0901.0827v5.pdf

```
frame: PDF
style: height: 900px;
urlSuffix: arxiv.org/pdf/0901.0827v5.pdf
```

lecture videos

- Programa de Doutorado: Lie Groups, Representation Theory and Symmetric Spaces - YouTube
- Representation Theory: A Combinatorial Viewpoint - YouTube
- M S Raghunathan.cpt Lie and reps
- CMI Lie algebras - YouTube Anupam Singh - AIS on Lie Algebras (4-23 July 2011) CMI-IMSc, Chennai (google.com)
- Introduction to Lie Algebras Doutorado IMPA Verao 2011 Reimundo Heluani 03/01/2011 aula 01 parte1 (youtube.com)
- IMPA Meeting: Quantum Groups and 3-Manifold Invariants - YouTube

usual and beyond

- https://math.mit.edu/~etingof/reprbook.pdf
- https://ocw.mit.edu/courses/18-755-lie-groups-and-lie-algebras-ii-spring-2024/mit18_755_s24_lec_full.pdf
- Lecture notes on Cherednik Algebras | Double Affine Hecke Algebras in Representation Theory, Combinatorics, Geometry, and Mathematical Physics | Mathematics | MIT OpenCourseWare

- https://www.mat.uniroma2.it/~damiani/RepresentationTheory_Etingof.pdf
- *moduli of quiver reps and VBs*

geometric representation theory

geometric representation theory in nLab (ncatlab.org)

Representation theory is the study of the basic symmetries of mathematics and physics. Symmetry groups come in many different flavors: finite groups, Lie groups, p-adic groups, loop groups, adelic groups,.. A striking feature of representation theory is the persistence of fundamental structures and unifying themes throughout this great diversity of settings. One such theme is the Langlands philosophy, a vast nonabelian generalization of the Fourier transform of classical harmonic analysis, which serves as a visionary roadmap for the subject and places it at the heart of number theory.

The fundamental aims of geometric representation theory are to uncover the deeper geometric and categorical structures underlying the familiar objects of representation theory and harmonic analysis, and to apply the resulting insights to the resolution of classical problems. *A groundbreaking example of its success is Beilinson-Bernstein's generalization of the Borel-Weil-Bott theorem, giving a uniform construction of all representations of Lie groups via the geometric study of differential equations on flag varieties.*

The geometric study of representations often reveals deeper layers of structure in the form of categorification. Categorification typically replaces numbers (such as character values) by vector spaces (typically cohomology groups), and vector spaces (such as representation rings) by categories (typically of sheaves). It is a primary explanation for miraculous integrality and positivity properties in algebraic combinatorics. A recent triumph of geometric methods is Ngô's proof of the Fundamental Lemma, a key technical ingredient in the Langlands program. The proof relies on the cohomological interpretation of orbital integrals, which makes available the deep topological tools of algebraic geometry (such as Hodge theory and the Weil conjectures).

- Geometric Representation Theory and Gauge Theory: Cetraro, Italy 2018 | SpringerLink STUDYING gauge theory
- Raphaël Rouquier - Geometric representation theory as representation-theoretic geometry (youtube.com)

- QGpublic.pdf (ed.ac.uk)
- Lecture 1 | Geometric representation theory (youtube.com)
- Representation theory as gauge theory - Clay Mathematics Institute
- 2021 IHES Summer School - Enumerative Geometry, Physics and Representation Theory
- A.Okounkov's lectures, math.columbia.edu/~okounkov/33lectures.pdf
- Math 569: Flag Varieties

- Elliot Kienzle Understanding hamiltonian G-spaces through quantization | Elliot Kienzle (chessapig.github.io)
- HIM Lectures: Trimester Program "Symplectic Geometry and Representation Theory" - YouTube

https://escholarship.org/content/qt07t5s31b/qt07t5s31b_noSplash_b004dfc53a64ff1ad5536bb1e3849d3.pdf

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- Representation Theory and Complex Geometry | SpringerLink
- Hodge Theory, Complex Geometry, and Representation Theory (ams.org)

integrable systems

REF representation theory and integrable systems

- Representation Theory, Mathematical Physics, and Integrable Systems: In Honor of Nicolai Reshetikhin | SpringerLink
- Integrable Systems, Spectral Curves and Representation Theory A. Lesfari
- Leonid Rybnikov Representation Theory, Integrable Systems, Quantization, Kashiwara Crystals
- Ivan Motorin Cluster Algebras, Resolution of Singularities, Representation Theory, Integrable Systems
- Representation Theory and Integrable Systems (iupui.edu)
- Representation theory, gauge theory, and integrable systems (3-8 February 2019): Scientific Programme · Kavli IPMU Indico System (Indico)

MSRI summer school 2014: Complex geometry and geometric analysis on complex manifolds

https://www.slmath.org/ckeditor_assets/attachments/106/bernstein_hein_naber_syllabus.pdf

Riemann Surfaces (Jacob Bernstein, bernstein@math.jhu.edu)

https://www.bilibili.com/video/BV1fW41197nr/?spm_id_from=333.337.search-card.all.click

- **Prerequisites:**
  - Knowledge of basic complex analysis—at the level of Ahlfors, Complex Analysis, Chapters 1-5—will be assumed. Some basic familiarity with (abstract) surface theory and differential forms will be helpful. However, I will review this material as needed.
- Reading: The main text will be
  - Donaldson, Riemann Surfaces; get at http://www2.imperial.ac.uk/~skdona/RSPREF.PDF.
- Other useful references:
  - Farkas and Kra, Riemann Surfaces; a classical text on the subject.
  - Miranda, Algebraic Curves and Riemann Surfaces; a more algebraic perspective.
- Week 1: Introduction to Riemann Surfaces
  - Surfaces and Topology
  - Riemann Surfaces and Holomorphic Maps
  - Maps between Riemann Surfaces
  - Calculus on Riemann Surfaces
  - De Rham Cohomology
- Week 2: Geometric Analysis on Riemann Surfaces
  - Elliptic Functions and Integrals
  - Meromorphic Functions
  - Inverting the Laplacian
  - The Uniformization Theorem
  - Riemann Surfaces and Minimal Surfaces

Geometric Analysis (Aaron Naber, anaber@math.northwestern.edu)

https://www.bilibili.com/video/BV1fW41197nr/?spm_id_from=333.337.search-card.all.click

- **Prerequisites:** Basics of manifolds, tensors, and differential forms. Basics of pde theory, for instance Evans's book Partial Differential Equations, in particular those chapters on second order elliptic and parabolic equations. Familiarity with exponential maps, injectivity radius, and geodesics would be helpful, for instance chapter one of Jost's book Riemannian Geometry and Geometric Analysis is more than sufficient.
- **Reading:** The main source will be Petersen's book on Riemannian Geometry. We will also rely on Jost's Riemannian Geometry and Geometric Analysis, and on the book by Cheeger Degeneration of Riemannian Metrics Under Ricci Curvature Bounds. More advanced topics will use relevant papers in the field.
- Week 1: Introduction to Geometric Analysis
  - Review of Manifolds and Smooth Structure
  - Introduction to Curvature and Geodesic Coordinates
  - Laplacians and Harmonic Coordinates
  - Heat Kernels and Geometry
  - Sectional Curvature and Finite Diffeomorphism Theorems
- Week 2: Topics in Regularity Theory
  - Ricci Curvature, Volume Monotonicity and Rigidity Theorems
  - Ricci Curvature and Almost Rigidity Theorems
  - Lower Ricci Curvature and Stratification Theorems
  - Bounded Ricci Curvature and ε-regularity Theorems
  - Outline of Regularity Theory for Einstein Manifolds

Complex Geometry (Hans-Joachim Hein, hansjoachim.hein@univ-nantes.fr)

https://www.bilibili.com/video/BV1Qv41177Hp/?spm_id_from=333.337.search-card.all.click

- Prerequisites: Basics of manifolds, tensor fields, differential forms, etc. Warner, Foundations of Differentiable Manifolds and Lie Groups, Chapters 1, 2, 4, 6, contains all we need and much more.
- Basic complex analysis as in Stein & Shakarchi, Complex Analysis, Chapters 1, 2, 3, 8.
- Reading:
  - Huybrechts, Complex Geometry, is an excellent basic textbook with exercises.

    - Lecture notes by Joel Fine: http://homepages.ulb.ac.be/~joelfine/papers.html#survey.
    - Complex Monge-Ampere: http://gamma.im.uj.edu.pl/~blocki/publ/ln/tln.pdf.
    - For the end of Week 2: http://arxiv.org/pdf/0803.0985.pdf, Section 5.
- Week 1: Introduction to Complex Geometry
    - Holomorphic Functions and Complex Calculus
    - Complex Manifolds
    - Holomorphic Line Bundles
    - Pseudoconvexity and Pseudoconcavity
    - The Kodaira Embedding Theorem
- Week 2: Topics in Kahler-Einstein Manifolds
    - Kahler Manifolds
    - Ricci Curvature and the Complex Monge-Ampere Equation
    - Examples of Ricci-flat Spaces
    - Basic Estimates for the Complex Monge-Ampere Equation
    - The Mukai-Umemura Manifold

18.966 Geometry of Manifolds II

Abstract

Differential forms, introduction to Lie groups, the DeRham theorem, Riemannian manifolds, curvature, the Hodge theory. 18.966 is a continuation of 18.965 and focuses more deeply on various aspects of the geometry of manifolds. Contents vary from year to year, and can range from Riemannian geometry (curvature, holonomy) to symplectic geometry, complex geometry and Hodge-Kahler theory, or smooth manifold topology. Prior exposure to calculus on manifolds, as in 18.952, recommended.

- 1992
    - `Guillemin, V., Course 18.966 - Geometry of Manifolds, M.I.T., Spring of 1992` from Lectures on Symplectic Geometry
- 1998
    - `T. Mrowka, 18.966 Lecture Notes, Spring 1998` from Notes on J-Holomorphic Maps
- 2004 (Alan Edelman)
- 2005 (Tomasz Mrowka)

- 2006 (Victor Guillemin)
- 2007 (Denis Auroux)
  - Geometry of Manifolds | Mathematics | MIT OpenCourseWare
- 2008 (Denis Auroux)
- 2009 (Tomasz Mrowka)
- 2010 (Tomasz Mrowka)
- 2011 (Peter Ozsváth)
- 2012 (Colding)
- 2013
  - 18.966 -- Spring 2013 (Guth)
    - The theme of the class is the connection between analysis on the one hand and the topology of manifolds on the other hand. There are three main topics.
    - The first is transversality, covering Sard's theorem and its applications in topology. The applications include degrees of maps, linking numbers, and the Hopf invariant.
    - The second topic is vector bundles, connections, and characteristic classes. We will study the Euler class, and Chern and Pontryagin classes. One of the main results we will study is the Gauss-Bonnet-Chern theorem.
    - The third topic is Morse theory, connecting the critical points of a function to the topology of the manifold. We will begin with Morse theory on finite dimensional manifolds, and then study Morse theory on the space of loops on a manifold, building up to a proof of the Bott periodicity theorem on the homotopy groups of the unitary group.
    - **Texts:** In the first unit, we will use the book Topology from the Differentiable Viewpoint, by John Milnor. In the third unit, we will use the book Morse Theory, also by John Milnor. For the second unit, we will use in-class lectures, and I may post some references on the webpage.
- 2014
  - 18.966 -- Spring 2014 (Murphy)
    - Contact geometry is a topic at the intersection of many fields, closely related to classical dynamics, complex analysis, and geometric topology. It is best thought of as the odd-dimensional cousin to symplectic geometry. The beginning of the class will cover the basic theory, and afterwards we will specialize to the 3-dimensional case, where a number of additional tools strengthen our knowledge of that dimension (particularly Giroux flexibility).
    - Contact isotopies. Gray stability, Darboux, and Legendrian neighborhood theorems. Connections between contact and symplectic geometry. Geometry of

hypersurfaces.
    - Tight contact structures and Bennequin's inequality. Applications of Bishop disk fillings. Giroux flexibility. Eliashberg's proof of Cerf's theorem. Connect sum decompositions of tight contact manifolds. Classification of overtwisted contact structures. Legendrian knots.
    - As time allows confoliation theory and/or topics from high dimensional contact geometry will be covered.
    - **Text Book:** No official text, but Geiges' book is an excellent resource that covers most of the topics discussed. There will also be class notes.
  - https://math.mit.edu/classes/18.966/2014SP/notes.pdf
    - found the link in dg.differential geometry - Characterization of contact vector fields - MathOverflow
- 2015 (Colding)
- 2016 (Paul Seidel)
- 2017 (Minicozzi)
- 2018
  - 18.966 -- Spring 2018 (Melrose)
- 2019
  - 18.966 -- Spring 2019 (Collins)
  - Math 18.966: Geometry of Manifolds II
  - 18.966_2019Spring_Comparison Geometry and Ricci Spaces_Collins.pdf - Google Drive by Ao Sun's homepage - Notes
    - > **This is the course note of 18.966 at MIT taught by Professor Tristan Collins in 2019 Spring. The note was taken together with Feng Gui. This note contains basic materials of comparison geometry and Ricci Spaces.**
- 2020 (Minicozzi)
  - 18-966-full.pdf - Google Drive by Sinho Chewi
- 2021 (Colding)
  - https://math.mit.edu/~tangkai/note/HE.pdf by Tang-Kai Lee
  - 【MIT数学课程】 流形上的几何：18.966 Geometry of Manifolds II 2021 Spring_哔哩哔哩_bilibili
- 2022
  - 18.966 -- Spring 2022 (Colding)
  - https://math.mit.edu/classes/18.966/2022SP/

- Form and shape can be described by differential equations. Many of these equations originate in various branches of science and engineering. They are fundamental and, in a sense, canonical. The fact that they make sense geometrically means that they are relevant everywhere and have fundamental properties that appear over and over again in many settings. Understanding them requires simultaneous insight into analysis and geometry and the interplay between these.
- In this class we will discuss a number of different ideas and estimates that have a wide range of applications to many fields including geometry, analysis, probability and applied mathematics. Common for them all is that they originate in geometry.
- Topics will include (but not restricted to):
  - Continues and discrete Laplacian.
  - Drift Laplacian and weighted inequalities.
  - Gradient estimates.
  - Harnack inequalities.
  - Sharp gradient estimate and monotonicity.
  - L2 eigenfunctions.
  - Ornstein-Uhlenbeck operator and Hermite polynomials.
  - Li-Yau differential Harnack inequality.
  - Hamilton's matrix maximum principle.
  - Perelman's monotonicity.

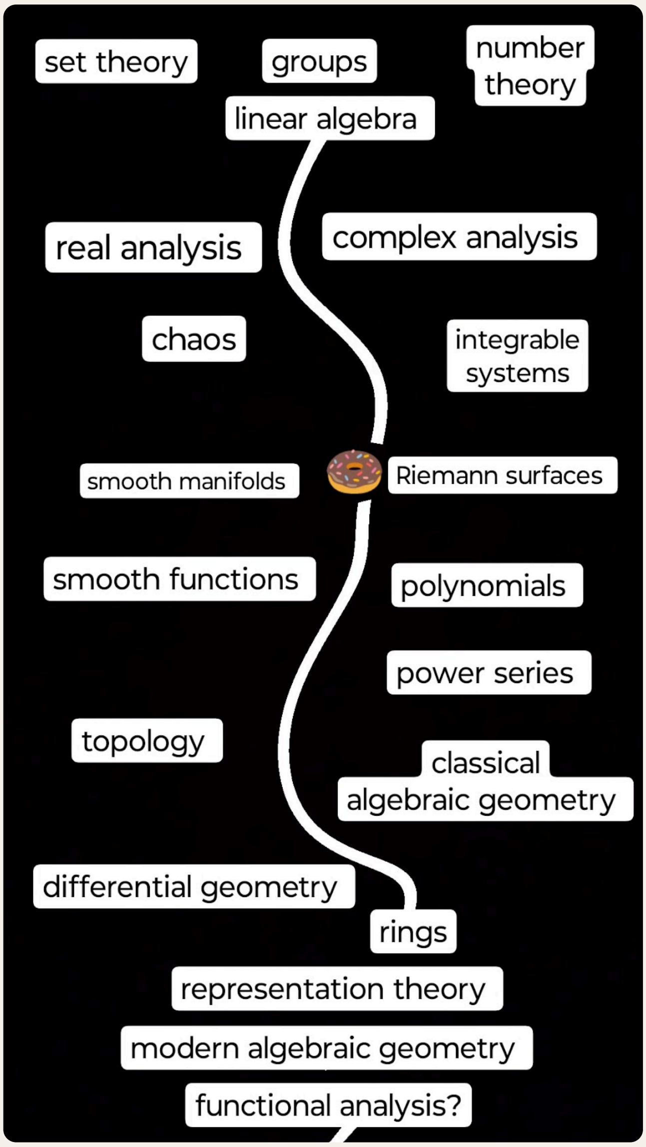
- 2023
  - 18.966 -- Spring 2023 (Colding)
  - Non-Blinding | MIT Admissions The Meta Home Paige
- 2024
  - 18.966 -- Spring 2024 (Minicozzi)
- Not offered in Spring 2025.

more MIT courses

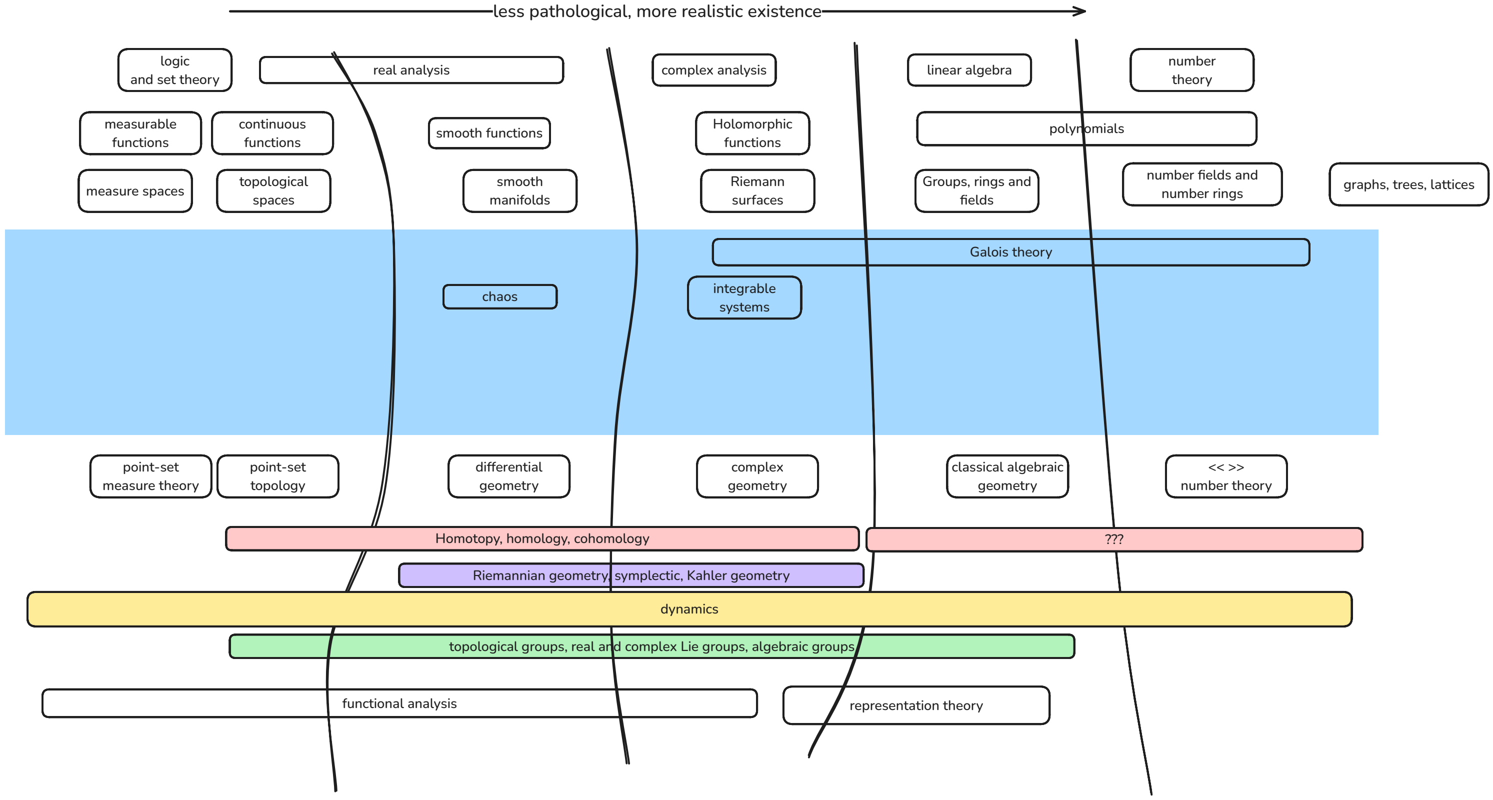
- 18.969 - Topics in Geometry - Spring 2009
- holdenlee.github.io/coursework/math/18.965/main.pdf
- mathweb.ucsd.edu/~ebelmont/965-notes.pdf

philosophy of mathematics, 16 march '25 edition

Exactly one year ago, I made the following distinction



I want to improve on this art:


less pathological, more realistic existence
logic and set theory
real analysis
complex analysis
linear algebra
number theory
measurable functions
continuous functions
smooth functions
Holomorphic functions
polynomials
measure spaces
topological spaces
smooth manifolds
Riemann surfaces
Groups, rings and fields
number fields and number rings
graphs, trees, lattices
Galois theory
chaos
integrable systems
point-set measure theory
point-set topology
differential geometry
complex geometry
classical algebraic geometry
<< >> number theory
Homotopy, homology, cohomology
???
Riemannian geometry, symplectic, Kahler geometry
dynamics
topological groups, real and complex Lie groups, algebraic groups
functional analysis
representation theory


YOU HAVE SCROLLED ALL THE WAY TO BIKINI BOTTOM
BIKINI
BOTTOM
imgflip.com


Jerry will be back soon with more disturbing facts