Uncertainty and Optimal Monetary Policy

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May 1, 2020

Abstract

We study optimal monetary policy in a New Keynesian model in which the policy maker faces uncertainty about the economy. Uncertainty creates stochastic deviations between the policy maker's expectations and the truth, which cause endogenous and stochastic macroeconomic fluctuations in equilibrium. Relative to the full-information benchmark, output responds less to supply shocks but more to demand shocks. Severe enough uncertainty causes equilibrium indeterminacy. Quantitatively, output and inflation are positively correlated under the optimal policy, in contrast to full-information policies that prescribe either price stability or leaning against the wind.

Keywords: optimal monetary policy, rational inattention

JEL Classification: D8, E3, E5

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Uncertainty about the current state of the economy is a chronic problem for policymakers. At best, official data represent incomplete snapshots of various aspects of the economy, and even then they may be released with a substantial lag and be revised later.

- Ben Bernanke, Economic Policy Conference (2007)

1 Introduction

When setting monetary policy, a policy maker must confront uncertainty about the state of the economy. In particular, policy must be designed and implemented with limited information about both the efficient path of the economy that is best to target, and current measures of economic activity such as inflation and the output gap. In a world of pervasive uncertainty, what is the optimal monetary policy?

In this paper, we explicitly model the uncertainty faced by policy makers using tools from information theory and rational inattention (Sims 2003). We solve for the optimal monetary policy in a textbook New Keynesian model augmented with a constraint that limits the policy maker's information about the economy. The nature of the information that the monetary authority chooses to obtain, as well as the corresponding reduction in uncertainty achieved, is determined as part of the optimal policy.[1]

Our main analytical result shows that the optimal monetary policy induces endogenous and stochastic macroeconomic fluctuations. We demonstrate that the policy maker sets the nominal interest rate equal to her subjective expectation of the efficient real interest rate, which deviates stochastically from the truth due to limited information. In equilibrium, these stochastic deviations cause endogenous and stochastic fluctuations of the macroeconomy.

We interpret these shocks as endogenous monetary policy shocks because a positive shock causes the nominal rate to rise, but output and inflation to fall. In contrast to the literature on *exogenous* monetary policy shocks, our theory shows how information frictions can generate them endogenously.

Uncertainty also affects the economy's responses to exogenous shocks. We show that the real interest rate response is muted relative to the full information benchmark. Intuitively, in the absence of complete information, the policy maker uses the long run mean of the efficient

[1]An earlier literature has studied the design of monetary policy when current inflation and the output gap are observed with exogenous noise. We discuss how our paper connects to this body of work in the literature review at the end of the introduction.

real rate as an anchor point when setting policy. As such, the dynamics of the real interest rate are attenuated towards this value in equilibrium.

This attenuation dampens the output response to supply shocks, but amplifies the output response to demand shocks.[2] In response to a supply shock, the muted real rate movement causes a weakened intertemporal substitution of consumption by households. In response to a demand shock, the muted real rate movement results in extra substitution by households relative to the efficient economy.

We also derive a feedback rule that implements the optimal monetary policy, and specifies the nominal interest rate as a function of error-ridden measurements of output and inflation. Unlike standard Taylor rules, the coefficients on output and inflation are endogenously linked to the information friction faced by the policy maker. This dependence ties the information friction to the determinacy of the optimal equilibrium.

We show that if the information friction is severe enough, then the optimal equilibrium cannot be implemented uniquely. In equilibrium, limited information attenuates the response of the nominal rate to endogenous outcomes. If this attenuation is strong enough, then there exist multiple equilibria driven by "sunspot" shocks. In such an equilibrium, a decline in the real interest rate caused by a positive sunspot shock to inflation is not offset by a large enough increase in the nominal rate. This causes an increase in aggregate demand, which equilibrates the initial shock to inflation.

We also solve our model numerically to determine the quantitative bite of the uncertainty faced by the policy maker. We calibrate the information friction so that monetary policy shocks contribute 25% of total output variance, in line with the evidence on monetary policy shocks presented by Christiano et al. (2005). The remaining parameters are set in line with the New Keynesian literature.

Quantitatively, the responses of output to exogenous shocks are hump-shaped and are muted relative to the full information benchmark. The non-monotonic pattern is a result of the persistent output gaps that occur under the optimal policy, which the policy maker tries to smooth out over time to minimize their welfare loss.

In the calibrated model, endogenous volatility causes output and inflation to co-move positively in equilibrium, with a correlation of 0.49. This finding suggests that a positive co-movement of output and inflation is not evidence of poor policy design, and contrasts with policies that prescribe zero or negative correlations when the policy maker has full informa-

[2]In line with the literature, we model supply shocks as shocks to aggregate TFP, and demand shocks as shocks to the discount rate of households.

tion.[3] Using equilibrium correlations to determine optimal policy can be misleading if we do not account for the effect of uncertainty and limited information.

Throughout our analysis, we use the concept of mutual information to measure information processing costs. As a robustness check, we show that our main insights are unchanged if we instead consider the Fisher information cost proposed by Hébert and Woodford (2020). While different information cost functions affect how the policy maker divides her attention among competing fundamental shocks, they generate the same equilibrium predictions: non-fundamental volatility, endogenous monetary policy shocks, and possible indeterminacy.

Related Literature Our results offer new insights on how uncertainty about the economy affects both the optimal path of nominal interest rates, and the optimal reduction in uncertainty through information processing.

We build on existing work by Aoki (2003), Svensson and Woodford (2003), and Boehm and House (2019), who study a policy maker who is subject to exogenously restricted information: her observations of current output and inflation are contaminated by noise. We complement their analyses by studying endogenous information acquisition by the policy maker. We show that endogenous information acquisition creates feedback effects between exogenous and endogenous sources of volatility that drive the business cycle. In particular, we show how endogenous volatility changes the response of the macroeconomy to exogenous shocks, and can create indeterminacy.

More broadly, we contribute to the literature that merges the New Keynesian framework with models of incomplete information by studying how policy makers should act when information is scarce. This contrasts to recent work by Paciello and Wiederholt (2014) and Angeletos and La'O (2019), who study optimal monetary policy when firms have limited information. Our results add a layer of nuance to their policy prescriptions since the positive co-movement between output and inflation driven by the endogenous shocks that we uncover may dominate the co-movements prescribed by optimal policy when the policy maker's information is complete.

The paper proceeds as follows. Section 2 lays out the economic environment and describes key features of the equilibrium. We introduce information frictions and set up the optimal monetary policy problem in section 3. We present our analytical results in section 4, and our quantitative findings in section 5. Section 7 concludes.

[3] The classical prescription of price stability is an example of zero correlation. See Angeletos and La'O (2019) for a recent example of a negative correlation (leaning against the wind) prescription.

2 Model

We situate our analysis in a standard New Keynesian model in order to cleanly isolate the effects of policy maker uncertainty. Households consume final goods and supply labor in each period. Monopolistically competitive firms maximize profits by setting prices subject to the demand curves that they face. A policy maker sets the nominal interest rate on risk-free bonds, that are in zero net supply.

2.1 Physical Environment

Time denoted by $t = 0, 1, 2, ...$ is discrete and infinite.

Households There is a representative household, who consumes an index of final goods and supplies labor in each period. She has preferences given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \prod_{s=1}^{t} \left(\frac{1}{1+\rho_s} \right) \left(\frac{c_t^{1-\frac{1}{\gamma}} - 1}{1-\frac{1}{\gamma}} - \frac{n_t^{1+\varphi}}{1+\varphi} \right)$$

where c_t is a consumption index over the unit continuum of final goods,

$$c_t = \left(\int_0^1 c_{j,t}^{\frac{\Phi-1}{\Phi}} dj \right)^{\frac{\Phi}{\Phi-1}}$$

and n_t is hours worked. $\rho_t > 0$ is the discount rate and is subject to aggregate shocks. Φ is the elasticity of substitution between goods, γ is the elasticity of intertemporal substitution, and φ is the inverse of the Frisch labor supply elasticity. The expectation is taken over sequences of shocks described below. Note that this expectation is the standard full-information rational expectations operator. The incomplete-information expectations operator of the policy maker is described in Section 3.

In each period, the household faces the flow budget constraint

$$\int_0^1 p_{j,t} c_{j,t} dj + B_t = P_t w_t n_t + P_t d_t - P_t T_t + (1 + \iota_{t-1}) B_{t-1}$$

where $p_{j,t}$ is the nominal price of good j, B_t is the household's bond position, ι_t is the nominal bond return, w_t is the real wage, d_t is real non-labor income, T_t is a real lump sum tax, and P_t is the aggregate nominal price level. We assume that $B_0 = 0$.

The household chooses paths of consumption of each final good, labor supply, and bond positions to maximize her utility subject to the sequence of flow budget constraints, and a no-Ponzi condition.

Standard optimization yields the familiar demand curves and price index,

$$c_{j,t} = \left(\frac{p_{j,t}}{P_t}\right)^{-\Phi} c_t$$

and

$$P_t = \left(\int_0^1 p_{j,t}^{1-\Phi} dj\right)^{\frac{1}{1-\Phi}}.$$

Firms Each good $j \in [0,1]$ is produced by a monopolistically competitive firm operating a constant returns to scale production function

$$y_{j,t} = A_t n_{j,t}$$

where A_t is aggregate total factor productivity (TFP), and is subject to aggregate shocks, and $n_{j,t}$ is the labor input of firm j in period t.

Each firm chooses its path of prices to maximize the discounted sum of profits subject to quadratic price adjustment costs (Rotemberg 1982):

$$\max_{\{p_{j,t}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_t \left(\frac{p_{j,t}}{P_t} y_{j,t} - (1-\tau_n) w_t \frac{y_{j,t}}{A_t} - \frac{\xi}{2}\left(\frac{p_{j,t}}{p_{j,t-1}} - 1\right)^2 Y_t\right)$$

subject to

$$y_{j,t} = \left(\frac{p_{j,t}}{P_t}\right)^{-\Phi} Y_t$$

$$p_{j,0} = P_0$$

where $\beta_t = \prod_{s=1}^{t}\left(\frac{1}{1+\rho_s}\right)\left(\frac{c_t}{c_0}\right)^{-\frac{1}{\gamma}}$ is the stochastic discount factor of the household, τ_n is a labor subsidy that is financed by the lump sum tax on households, and ξ is the Rotemberg price adjustment parameter.

We focus on the symmetric equilibrium, as is standard, in which $p_{j,t} = P_t$ and $y_{j,t} = Y_t$ for all $j \in [0,1]$. In this case, the aggregate dividend in period t is given by

$$d_t = Y_t\left(1 - (1-\tau_n)\frac{w_t}{A_t} - \frac{\xi}{2}\pi_t^2\right)$$

where $\pi_t = \frac{P_t - P_{t-1}}{P_{t-1}}$ is the inflation rate.

Fiscal Policy The fiscal authority finances the labor subsidy via a lump sum tax on households, and is subject to the flow budget constraint

$$P_t T_t + B_t = \tau_n \frac{P_t w_t}{A_t} \int_0^1 y_{j,t} dj + (1 + \iota_{t-1}) B_{t-1}$$

where we have already imposed the bond market clearing condition.

In order to fully offset the monopoly distortion in the equilibrium with flexible prices, we follow the New Keynesian literature and assume that $\tau_n = \frac{1}{\Phi}$.

Monetary Policy The policy maker sets a path for the nominal interest rate $\{\iota_t\}_{t=0}^{\infty}$, which is non-neutral due to the price-adjustment friction that firms face. We are ultimately interested in deriving an optimal path $\{\iota_t\}_{t=0}^{\infty}$ subject to an informational constraint that we introduce in Section 3.

Aggregate Shocks We assume that the discount rate and aggregate TFP follow AR(1) processes,

$$\rho_t = (1 - \delta_\rho) \rho + \delta_\rho \rho_{t-1} + e_t^\rho$$

and

$$\log A_t = (1 - \delta_a) \log A + \delta_a \log A_{t-1} + e_t^a$$

where $e_t^\rho \sim N\left(0, \sigma_\rho^2\right)$ and $e_t^a \sim N\left(0, \sigma_a^2\right)$. Let $\hat{\rho} = \rho_t - \rho$ denote the deviation of the discount rate from its mean level, and let $\hat{a}_t = \log A_t - \log A$ denote the log deviation of aggregate TFP from its mean level.

2.2 Equilibrium

Conditional on a path of nominal interest rates, equilibrium is defined in the usual way.

Definition 1. Given a path of nominal interest rates $\{\iota_t\}_{t=0}^{\infty}$, an equilibrium is a sequence of allocations $\left\{\{c_{j,t}\}_j, n_t, d_t, T_t, B_t\right\}_{t=0}^{\infty}$ and prices $\left\{\{p_{j,t}\}_j, P_t, w_t\right\}_{t=0}^{\infty}$ such that

1. $\left\{\{c_{j,t}\}_j, n_t\right\}_t$ solve the household problem taking prices and non-labor income as given.
2. $\{p_{j,t}\}_t$ solve the firm problem for each j.
3. Non-labor income satisfies the dividend equation in each period.

4. The fiscal budget constraint holds in each period.

5. Markets clear in each period.

2.3 Equilibrium Analysis

We now use log-linear approximations to describe key features of the equilibrium conditional on a given path of nominal rates that describe the conduct of monetary policy. The following analysis follows very closely the textbook exposition of the New Keynesian model found in Galí (2015).

Flexible Price Benchmark In the absence of price-adjustment frictions ($\xi \to 0$), monetary policy is neutral, and the equilibrium is Pareto efficient. Optimal monetary policy will therefore try to attain this allocation when prices are sticky.

In order to characterize key features of the efficient equilibrium, we adopt a log-linear approximation of the economy around its deterministic steady state in which $\rho_t = \rho$ and $A_t = A$ for all t. Let $\hat{y}_t^e = \log y_t^e - \log y$ denote the log deviation of the efficient level of output from its value in the deterministic steady state, and define r_t^e as the real interest rate in the efficient equilibrium. The formal proofs of the following lemma and all other results are contained in the appendix.

Lemma 1. *Let $\xi \to 0$. Then, the equilibrium allocation is efficient, and satisfies the log-linear equations*

$$\hat{y}_t^e = \frac{1+\varphi}{\frac{1}{\gamma}+\varphi}\hat{a}_t$$

and

$$r_t^e = \rho + \hat{\rho}_t - \frac{1+\varphi}{1+\gamma\varphi}\left(1-\delta_a\right)\hat{a}_t.$$

The equilibrium output path follows from combining the household optimality condition $c_t^{\frac{1}{\gamma}} n_t^{\varphi} = w_t$ with the production function, and the fact that firm profit maximization implies that $w_t = A_t$.[4] The path for the efficient real interest rate follows from the log-linearized consumption Euler equation,

$$\mathbb{E}_t\left[\hat{y}_{t+1}^e\right] - \hat{y}_t^e = \gamma\left(r_t^e - \rho_t\right)$$

[4]Recall that the firm subsidy undoes the distortion from monopoly power.

and ensures that the bond market clears in every period.[5]

Sticky Price Equilibrium Following the New Keynesian literature (e.g. Galí 2015), we describe the equilibrium with sticky prices ($\xi > 0$) in terms of deviations from the efficient benchmark. To this end, define $\hat{y}_t = \log y_t - \log y$ as the log deviation of equilibrium output, and denote $\tilde{y}_t = \hat{y}_t - \hat{y}_t^e$ as the output gap: the log deviation of equilibrium output from its efficient level.

Lemma 2. *Given a policy $\{\iota_t\}_{t=0}^{\infty}$, the equilibrium with sticky prices is described by*

$$\mathbb{E}_t\left[\tilde{y}_{t+1}\right] - \tilde{y}_t = \gamma\left(\iota_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^e\right)$$

and

$$\pi_t = \varphi_y \tilde{y}_t + \frac{1}{1+\rho}\mathbb{E}_t\left[\pi_{t+1}\right]$$

where $\varphi_y = \frac{\Phi-1}{\xi}\left(\frac{1}{\gamma} + \varphi\right)$.

The equations in lemma 2 are the much-studied consumption Euler equation and the New Keynesian Phillips curve. The Euler equation describes how households adjust their demand (and hence expected output growth) in response to changes in the real interest rate via intertemporal substitution. The New Keynesian Phillips curve describes how firms adjust prices (and hence inflation) in response to changes in aggregate demand or future expected inflation. We emphasize that the expectations of households and firms that enter these equations are the usual full-information rational expectations operators. Information frictions only affect the policy maker, whose expectations operator diverges from $\mathbb{E}_t$ in general. We now turn to this issue.

3 Information Frictions and the Ramsey Problem

In this section, we introduce the information frictions faced by the policy maker and then frame the optimal policy problem.

[5]Given a path for nominal rates $\{\iota_t\}$, inflation is any path satisfying

$$\mathbb{E}_t\left[\pi_{t+1}\right] = \iota_t - r_t^e$$

3.1 Information Frictions

In the absence of constraints on monetary policy, it is well known that setting $\iota_t = r_t^e$ in equilibrium ensures that $\tilde{y}_t = \pi_t = 0$ for all t, so that "divine coincidence" holds and the sticky price equilibrium coincides with the efficient benchmark (e.g. Galí 2015).

However, implementing this policy assumes that the policy maker is able to process enough information about the economy in period t to remove all uncertainty about r_t^e. We now depart from this assumption, and instead suppose that the policy maker possesses limited information processing capacity. We think of this limitation as capturing both the cognitive effort required to process data and form expectations about r_t^e and issues such as data lags and revisions that make it difficult to obtain accurate real-time data on the economy.

In order to model uncertainty reduction with limited information, we follow the literature on rational inattention (Sims 2003). We impose that the policy maker cannot remove all uncertainty about r_t^e when setting monetary policy. This restriction is consistent with the limited information that the policy maker has about the path for efficient real rates when setting policy.

Formally, let $r^{e,t} = \left(r_t^e, r_{t-1}^e, r_{t-2}^e, ...\right)$ denote the history of efficient real rates up to period t, and let $r^t = (r_t, r_{t-1}, r_{t-2}, ...)$ denote the history of realized equilibrium real rates.

These histories must respect the following information constraint,

$$I\left(r^t; r^{e,t}\right) \leq \kappa_M$$

where I measures the mutual information between the stochastic processes for efficient real rates and equilibrium real rates. Mutual information uses the concept of entropy to measure the reduction in uncertainty about a random variable X that occurs in the process of choosing another random variable Y.[6]

In words, this constraint forces the amount of information about efficient real rates contained in the process for equilibrium real rates to not exceed some upper bound κ_M, which parameterizes the information processing capacity of the policy maker. In the limit as $\kappa_M \to \infty$, the processing capacity grows infinitely large, and setting $r_t = r_t^e$ becomes feasible.

[6] For random variables Y and X with joint density f and marginals f_Y and f_X, their mutual information is defined as

$$I(Y;X) = H(X) - H(X|Y) = -\int_x f_X(x)\log_2(f_X(x))\,dx + \int_{x,y} f(y,x)\log_2\left(\frac{f(y,x)}{f_Y(y)}\right)dxdy$$

where $H(X)$ is the entropy of X and $H(X|Y)$ is the entropy of X conditional on Y.

Our formulation of information processing and uncertainty reduction allows us to model policy as both the path of nominal interest rates, and the information that the policy maker chooses to use when deciding on a path of nominal rates. We therefore develop a theory of how a monetary policy maker should optimally use information to help design policy. This contrasts with existing work that exogenously restricts the information obtained by the policy maker (e.g. Aoki 2003 and Boehm and House 2019).

3.2 Welfare Criterion

In order to determine optimal policy, we assume that the policy maker wishes to maximize the utility payoff of the household. Since the flexible price economy features an efficient allocation, it is natural to compare outcomes to that benchmark. Therefore, we follow the large literature on optimal monetary policy, with and without information frictions, and compute the second order utility loss relative to the efficient benchmark.

Lemma 3. *The second order household utility loss is proportional to*

$$\frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \left(\tilde{y}_t^2 + \frac{\xi}{\frac{1}{\gamma}+\varphi}\pi_t^2\right).$$

The utility loss is composed of two terms. The first is the discounted sum of squared output gaps, which captures utility losses from inefficient output levels. The second is the discounted sum of squared inflation rates. This term captures the utility loss from the resources used when firms adjust their prices.

3.3 The Ramsey Problem

The Ramsey problem for monetary policy is to choose a path of nominal interest rates in order to minimize the household utility loss, subject to the constraints imposed by equilibrium, and the information frictions faced by the policy maker.

The Ramsey problem is given by

$$\min_{\{\iota_t\}_{t=0}^{\infty}} \frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \left(\tilde{y}_t^2 + \frac{\xi}{\frac{1}{\gamma}+\varphi}\pi_t^2\right)$$

subject to

$$\pi_t = \varphi_y \tilde{y}_t + \frac{1}{1+\rho}\mathbb{E}_t\left[\pi_{t+1}\right]$$

$$\mathbb{E}_t\left[\tilde{y}_{t+1}\right] - \tilde{y}_t = \gamma\left(\iota_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^e\right)$$

$$I\left(r^t; r^{e,t}\right) \leq \kappa_M$$

Before characterizing the optimal monetary policy and the solution to the Ramsey problem, we first establish an important preliminary result.

Lemma 4. *Price stability, $\pi_t = 0$ for all t, is infeasible and cannot be part of a solution to the Ramsey problem.*

Lemma 4 establishes that the classical objective of monetary policy, to stabilize prices, is an infeasible goal in our environment of limited information. This is a direct consequence of the informational constraint that the policy maker faces when implementing policy. In order to attain $\pi_t = 0$ in all periods, the Phillips curve requires that $\tilde{y}_t = 0$ too, so that output attains its efficient level. However, this implies that the stochastic processes for r^t and $r^{e,t}$ are identical, so that their mutual information is infinitely large. This violates the information processing constraint that the policy maker faces, thus rendering such an outcome infeasible.

4 Optimal Monetary Policy: Analytical Solution

In order to solve the Ramsey problem in closed form, we temporarily assume that aggregate shocks are i.i.d. over time. We return to the case of persistent shocks in our numerical exercise.

Assumption 1. $\delta_\rho = \delta_a = 0$.

In order to build intuition for the solution to the Ramsey problem, we proceed in stages. We first characterize the optimal monetary policy, and then derive the equilibrium paths for macroeconomic variables. Next, we derive the optimal information allocation and show how to implement the optimal policy using a plausible feedback rule that specifies the nominal interest rate as a function of observables. Finally, we discuss determinacy of the equilibrium.

4.1 Optimal Monetary Policy

In the presence of informational frictions, the policy maker does not know r_t^e with certainty in period t. Instead, the policy maker may only form an expectation of r_t^e based on her available information. To formalize this, let $\mathbb{E}_{M,t}$ denote the expectations operator conditioned on the policy maker's information set in period t. Using this expectation, we can derive an intuitive expression for the optimal monetary policy.

Proposition 1. *Under assumption 1, the optimal monetary policy is given by*

$$\iota_t = \mathbb{E}_{M,t}\left[r_t^e\right].$$

In the presence of limited information, the policy maker sets the nominal interest rate equal to her expectation of the current efficient real rate, where the expectation is conditional on her limited information. Were the policy maker to obtain complete information, $\mathbb{E}_{M,t}$ would coincide with $\mathbb{E}_t$, and the optimal policy would be $\iota_t = r_t^e$ as in the textbook exposition.[7]

Next, we evaluate the policy authority's expectation operator so that we can further describe optimal monetary policy. The following steps offer a method to do so, and are explained in more detail in the appendix. Using the expression for the efficient real rate, we can rewrite the equation for the optimal nominal interest rate as

$$\iota_t = \rho + \mathbb{E}_{M,t}\left[\hat{\rho}_t\right] - \frac{1+\varphi}{1+\gamma\varphi}\mathbb{E}_{M,t}\left[\hat{a}_t\right].$$

Given the assumption of i.i.d. exogenous shocks, we can guess and later verify that the nominal rate, output, and inflation are also i.i.d. processes in equilibrium, so that $\iota_t = r_t$.[8] Furthermore, when all processes are i.i.d. over time, the past history of shocks is irrelevant when computing the mutual information between r^t and $r^{e,t}$ in period t. Therefore, we can write the information constraint as

$$I\left(\rho + \mathbb{E}_{M,t}\left[\hat{\rho}_t\right] - \frac{1+\varphi}{1+\gamma\varphi}\mathbb{E}_{M,t}\left[\hat{a}_t\right]; \rho + \hat{\rho}_t - \frac{1+\varphi}{1+\gamma\varphi}\hat{a}_t\right) \leq \kappa_M$$

To simplify the mutual information operator, we appeal to the fact that the TFP and discount rate shocks are independent normal random variables. Given this, and the scale invariance

[7] The certainty equivalence of proposition 1 is similar to the results in Aoki (2003) and Svensson and Woodford (2003): in the presence of risk, the optimal policy is the expectation of the optimal policy without risk.

[8] In general, the nominal rate, real rate, and inflation rate must satisfy the condition $r_t = \iota_t - \mathbb{E}_t\left[\pi_{t+1}\right]$. When inflation is i.i.d., this simplifies to $\iota_t = r_t$.

property of mutual information, we can separate the mutual information into two pieces,

$$\underbrace{I\left(\mathbb{E}_{M,t}\left[\hat{\rho}_t\right];\hat{\rho}_t\right)}_{\kappa_\rho}+\underbrace{I\left(\mathbb{E}_{M,t}\left[\hat{a}_t\right];\hat{a}_t\right)}_{\kappa_a}\leq\kappa_M$$

where κ_ρ and κ_a denote the mutual information concerning discount rate shocks and TFP shocks respectively. The information constraint then states that the information processing capacity devoted to each shock cannot exceed the total capacity available: $\kappa_\rho+\kappa_a\leq\kappa_M$.

Finally, we exploit the fact that when the underlying shocks are normal, it is optimal for the policy maker's conditional expectations to also follow normal distributions (Sims 2003). This allows us to use the result that the mutual information of two normal random variables X and Y is given by

$$I\left(X;Y\right)=-\frac{1}{2}\log_2\left(1-\rho_{X,Y}^2\right)$$

where $\rho_{X,Y}$ is the correlation of X and Y. Applying this result allows us to compute the expectations.

Proposition 2. *Let assumption 1 hold, and suppose that* $I\left(\mathbb{E}_{M,t}\left[\hat{a}_t\right];\hat{a}_t\right)=\kappa_a$ *and* $I\left(\mathbb{E}_{M,t}\left[\hat{\rho}_t\right];\hat{\rho}_t\right)=\kappa_\rho$. *Then, the expectations satisfy*

$$\mathbb{E}_{M,t}\left[\hat{a}_t\right]=\left(1-\frac{1}{2^{2\kappa_a}}\right)\hat{a}_t+\sqrt{\frac{2^{2\kappa_a}-1}{2^{4\kappa_a}}\sigma_a^2}v_t$$

$$\mathbb{E}_{M,t}\left[\hat{\rho}_t\right]=\left(1-\frac{1}{2^{2\kappa_\rho}}\right)\hat{\rho}_t+\sqrt{\frac{2^{2\kappa_\rho}-1}{2^{4\kappa_\rho}}\sigma_\rho^2}u_t$$

where v_t *and* u_t *are i.i.d. standard normal random variables.*

To understand proposition 2, let us consider the expression for $\mathbb{E}_{M,t}\left[\hat{a}_t\right]$; the interpretation for $\mathbb{E}_{M,t}\left[\hat{\rho}_t\right]$ is analogous. The expression is composed of two pieces. The first shows that the policy maker attenuates her expectation of the TFP shock towards its long run mean of zero. Intuitively, when the policy maker updates her posterior mean of the shock using her limited information, she places positive weight on her prior mean, which is the long run value when shocks are i.i.d. over time. The strength of this attention is decreasing in the information capacity devoted to the shock κ_a.

The second piece is driven by the random variable v_t, and adds pure noise to the policy maker's expectation. This noise reflects the uncertainty that the policy maker faces about the true TFP shock, which causes her expectation to stochastically deviate from the truth over time. We stress that this noise is an endogenous outcome of the policy maker's expectation formation, and is caused by the information constraint.

The variance of the expectation depends on the information processing capacity devoted to TFP shocks. When $\kappa_a = 0$, the policy maker does not track TFP shocks at all. In this case, her best guess of $\hat{a}_t$ is its long run mean value, $\mathbb{E}_{M,t}\left[\hat{a}_t\right] = 0$. As κ_a increases, the variance of the errors increases but then decreases as she processes more and more information about the shock. In the limit as $\kappa_a \to \infty$, the policy maker obtains complete information and sets $\mathbb{E}_{M,t}\left[\hat{a}_t\right] = \hat{a}_t$.

4.2 Equilibrium Dynamics

Having evaluated the expectations, we can now write the optimal path of nominal (equal to expected real) rates in explicit form,

$$\iota_t = \rho + \left(1 - \frac{1}{2^{2\kappa_\rho}}\right)\hat{\rho}_t + \sqrt{\frac{2^{2\kappa_\rho}-1}{2^{4\kappa_\rho}}\sigma_\rho^2}u_t - \frac{1+\varphi}{1+\gamma\varphi}\left(\left(1-\frac{1}{2^{2\kappa_a}}\right)\hat{a}_t + \sqrt{\frac{2^{2\kappa_a}-1}{2^{4\kappa_a}}\sigma_a^2}v_t\right).$$

Comparing this expression to the efficient path of real interest rates shows how the equilibrium real rate responses to exogenous TFP and discount rate shocks are muted relative to the efficient path. This attenuation directly follows from the attenuation of the policy maker's expectations of the true shocks towards zero.

Combining this equation with the New Keynesian Phillips curve and the Euler equation yields a three equation system, that can be solved for the full equilibrium dynamics of the economy.

Proposition 3. *Under assumption 1, the optimal paths for output and inflation are*

$$\hat{y}_t = \gamma\left(\frac{1}{2^{2\kappa_\rho}}\hat{\rho}_t - \sqrt{\frac{2^{2\kappa_\rho}-1}{2^{4\kappa_\rho}}\sigma_\rho^2}u_t\right) + \gamma\frac{1+\varphi}{1+\gamma\varphi}\left(\left(1-\frac{1}{2^{2\kappa_a}}\right)\hat{a}_t + \sqrt{\frac{2^{2\kappa_a}-1}{2^{4\kappa_a}}\sigma_a^2}v_t\right)$$

$$\pi_t = \varphi_y\gamma\left(\frac{1}{2^{2\kappa_\rho}}\hat{\rho}_t - \sqrt{\frac{2^{2\kappa_\rho}-1}{2^{4\kappa_\rho}}\sigma_\rho^2}u_t\right) + \varphi_y\gamma\frac{1+\varphi}{1+\gamma\varphi}\left(-\frac{1}{2^{2\kappa_a}}\hat{a}_t + \sqrt{\frac{2^{2\kappa_a}-1}{2^{4\kappa_a}}\sigma_a^2}v_t\right)$$

Proposition 3 shows how the macroeconomy is driven by the endogenous shocks, v_t and u_t, in addition to the exogenous TFP and discount rate shocks. Therefore, the economy is subject to endogenous and stochastic fluctuations under the optimal monetary policy. These fluctuations are caused by the information constraint, which creates stochastic deviations between the nominal interest rate and the efficient real interest rate. In equilibrium, these deviations affect the paths for output and inflation.

Recall that the efficient path of output is given by $\hat{y}_t^e = \gamma \frac{1+\varphi}{1+\gamma\varphi} \hat{a}_t$. Comparing this path to the process for equilibrium output, we see that the response of output to TFP shocks is muted, while the response to discount rate shocks is amplified relative to the efficient path.

These differences stem from muted responses of the real interest rate to exogenous shocks. The muted response of the real rate to TFP shocks translates into the output response since households do not substitute as much future consumption to the current period when the price of future consumption does not increase as much.

In contrast, the muted real rate response to discount rate shocks causes output to respond more than the efficient path. In this case, the gap between the real interest rate and discount rate causes households to substitute consumption across periods more than they would in the efficient economy.

4.3 Endogenous Monetary Policy Shocks

The endogenous shocks v_t and u_t cause a positive co-movement of output and inflation, but a negative co-movement of output and the nominal interest rate. Therefore, v_t and u_t have the characteristics of a monetary policy shock, as defined in the vast literature on monetary policy transmission (e.g. Christiano et al. 2005).

The information constraint offers a theory of the origin of monetary policy shocks. In equilibrium, there are unanticipated shocks to the nominal interest rate that stem from the limited information that the policy maker has about the efficient path of the economy she would like to target. The variances of these shocks are determined endogenously as part of the optimal policy, and reflect both the total information processing capacity of the policy maker, and her optimal division of this capacity among competing sources of exogenous variation.

4.4 Optimal Information Allocation

So far, we have described the optimal monetary policy and equilibrium in terms of primitives, and the information processing variables κ_a and κ_ρ. In equilibrium, these quantities are determined as part of the optimal policy: given the policy maker's total capacity for processing information κ_M, she must optimally decide how to divide this capacity between the two sources of exogenous variation in the efficient real interest rate. For brevity, we focus on the interior solution in which $0 < \kappa_a < \kappa_M$.

Proposition 4. *Under assumption 1, the optimal information allocation satisfies*

$$\kappa_a = \frac{1}{2}\kappa_M + \frac{1}{4}\log_2\left(\left(\frac{1+\varphi}{1+\gamma\varphi}\right)^2 \frac{\sigma_a^2}{\sigma_\rho^2}\right)$$

$$\kappa_\rho = \frac{1}{2}\kappa_M - \frac{1}{4}\log_2\left(\left(\frac{1+\varphi}{1+\gamma\varphi}\right)^2 \frac{\sigma_a^2}{\sigma_\rho^2}\right)$$

The optimal information allocation depends on the relative variances of each of the sources of exogenous variation. Intuitively, if the efficient real rate is mainly driven by TFP shocks, then $\left(\frac{1+\varphi}{1+\gamma\varphi}\right)^2 \sigma_a^2 > \sigma_\rho^2$, and it is optimal for the policy maker to devote more of her information processing capacity to this source of variation.[9]

4.5 Implementation

Having described the equilibrium in terms of primitives, we now discuss how the optimal monetary policy may be implemented in practice. We derive a feedback rule that specifies the optimal nominal interest rate as a function of observables.

Proposition 5. *Under assumption 1, the optimal monetary policy can be implemented using the rule*

$$\iota_t = \rho + 2^{2\kappa_\rho}\frac{2^{2\kappa_a}-1}{2^{2\kappa_a}}\frac{1}{\gamma\varphi_y}\pi_t^o + \frac{2^{2\kappa_\rho}-2^{2\kappa_a}}{2^{2\kappa_a}}\frac{1}{\gamma}\hat{y}_t^o$$

where

$$\pi_t^o = \pi_t - \gamma\varphi_y\frac{1+\varphi}{1+\gamma\varphi}\sqrt{\frac{1}{2^{2\kappa_a}-1}\sigma_a^2}v_t$$

$$\hat{y}_t^o = \hat{y}_t + \gamma\frac{2^{2\kappa_a}}{2^{2\kappa_\rho}-2^{2\kappa_a}}\sqrt{\left(2^{2\kappa_\rho}-1\right)\sigma_\rho^2}u_t$$

are noisy observations of output and inflation.

The proposition shows how the optimal path of nominal interest rates can be implemented using a feedback rule based on noisy observations of equilibrium output and inflation. The fact that the noise processes and the coefficients on output and inflation in the policy rule are functions of the optimal information allocation provide two practical methods by which the policy maker can deal with her limited information friction.

[9] If the relative variance of TFP shocks is large or small enough then the optimal information allocation features a corner solution, $\kappa_a = \kappa_M$ or $\kappa_a = 0$.

First, the policy maker should try to measure output more accurately than inflation when discount rate shocks are the more important driver of variation in the efficient real rate. To see this, define $m_t^y = \hat{y}_t^o - \hat{y}_t$ and $m_t^\pi = \pi_t^o - \pi_t$ as the measurement errors in output and inflation. The variance of m_t^y is decreasing in κ_ρ, while the variance of m_t^π is increasing in κ_ρ. Since κ_ρ is increasing in σ_ρ^2, the variance of m_t^y must be decreasing in σ_ρ^2.

The optimality of focusing on output measurement accuracy in the face of discount rate shocks follows from the fact that efficient output does not respond to such shocks. Therefore, it is optimal for the policy maker to track equilibrium output very closely in order to discern whether the path of the economy is close to efficient or not.

Second, the dependence of the coefficients in the policy rule on the information allocation implies that the policy maker should respond positively to output only when $\kappa_\rho > \kappa_a$, i.e. when discount rate shocks are the main driver of variation in the efficient economy. In this case, observing a positive deviation of output above steady state indicates an inefficiency, which the policy maker should respond to by raising the nominal interest rate.

In contrast, the policy maker should always respond to inflation positively. Since inflation indicates a positive output gap via the New Keynesian Phillips curve, it is optimal for the policy maker to increase the nominal rate in response.

Finally, we note that in the knife-edge case in which discount rate shocks and TFP shocks contribute equally to variation in the efficient real rate, $\kappa_a = \kappa_\rho$, and the policy maker needs only to respond to inflation in order to implement the optimal policy. In this special case, inflation depends linearly on the path of efficient real rate deviations $\hat{\rho}_t - \frac{1+\varphi}{1+\gamma\varphi}\hat{a}_t$. Therefore, linear dependence of the nominal rate on inflation is sufficient to implement the optimal path of nominal interest rates.

This result offers an interpretation of the well-known result that a Taylor rule with an infinitely large coefficient on inflation can implement the efficient equilibrium with arbitrary accuracy in the economy without information frictions. If we let $\kappa_M \to \infty$, then $\kappa_a, \kappa_\rho \to \infty$ and $\frac{\kappa_a}{\kappa_\rho} \to 1$, so that the logic above applies: the coefficient on output tends to zero, while the coefficient on inflation goes to infinity. In this sense, our economy converges to the standard model as $\kappa_M \to \infty$.

4.6 Determinacy

Combining the optimal policy rule with the Euler equation and New Keynesian Phillips curve yields a three equation system. The following result establishes the determinacy of such a system, and hence the uniqueness of the optimal policy equilibrium.

Proposition 6. *The optimal equilibrium can be implemented uniquely when*

$$\kappa_M > \log_2 \left(\frac{1}{1+\rho} \frac{1+\gamma\varphi}{1+\varphi} \frac{\sigma_\rho}{\sigma_a} + \left(\frac{\rho}{1+\rho} + \gamma\varphi_y \right) \frac{1+\varphi}{1+\gamma\varphi} \frac{\sigma_a}{\sigma_\rho} \right).$$

Similar to the textbook New Keynesian model, determinacy holds when the coefficients in the policy rule are large enough, thus ruling out self-fulfilling sunspot equilibria.

In our setting, this condition holds when the policy maker's information processing capacity is large enough. Intuitively, the larger is κ_M, the more accurately the policy maker can track changes in inflation and output. Sufficiently accurate tracking is important to ensure that the nominal rate responds enough to changes in inflation and output to rule out sunspot equilibria. For example, a strong enough response to inflation guarantees that the real interest rate increases in response to inflation, which rules out equilibria in which an increase in inflation driven by a sunspot is supported by a fall in the real interest rate and an increase in output.

5 Optimal Monetary Policy: Numerical Results

Having studied a tractable version of the Ramsey problem to build intuition for the key economic forces at play, we now relax assumption 1, and return to the case of persistent shocks. We solve the Ramsey problem numerically in order to determine the quantitative bite of the information constraint. Our numerical algorithm is described in the appendix.

5.1 Calibration

Table 1 summarizes the calibration of the model at a quarterly frequency. We set parameters governing preferences and technology in line with the vast literature that studies quantitative New Keynesian models.

The remaining parameters govern the stochastic processes for exogenous shocks, and the severity of the information friction. We set the persistences and standard deviations of TFP and discount rate shocks in line with the empirical evidence for the persistence of demand and supply shocks, and their relative contributions to total output volatility (e.g. Smets and Wouters 2007). In equilibrium, TFP shocks account for 90% of the variance in output attributable to exogenous shocks.

In order to calibrate κ_M, we appeal to the fact that the endogenous shocks generated by the information friction have similar effects to the monetary policy shocks studied in the literature. Therefore, we set $\kappa_M = 0.35$ to match the contribution of monetary policy shocks to overall output volatility, as estimated by Christiano et al. (2005). In equilibrium, these endogenous shocks account for 25% of total output variance.

Parameter	Value	Description
γ	1	Elasticity of Intertemporal Substitution
φ	1	Frisch Elasticity of Labor Supply
ρ	0.01	Discount Rate
Φ	6	Elasticity of Substitution
ξ	100	Price Adjustment Cost
δ_a	0.9	Persistence of TFP shocks
σ_a	0.015	Standard Deviation of TFP innovations
δ_ρ	0.8	Persistence of Discount Rate shocks
σ_ρ	0.015	Standard Deviation of Discount Rate innovations
κ_M	0.35	Information Processing Capacity

Table 1: Calibrated Parameter Values

5.2 Quantitative Results

Output and Inflation Responses Figure 1 plots the impulse responses of output and inflation to one standard deviation TFP, discount rate, and endogenous shocks.[10] For reference, we also plot the efficient response of output in the flexible price economy in red-dash. All responses are expressed in percentage points and are plotted for the first 25 quarters.

A number of features are worth noting. First, the output response to TFP shocks is muted while the response to discount rate shocks is significantly amplified relative to the efficient economy (recall that efficient output does not respond to discount rates at all). This attenuation and amplification follows the same logic as in the i.i.d. case: limited information causes the policy maker to use the long-run mean efficient real interest rate as an anchor point when designing policy. As such, output responds less to TFP shocks but more to discount rate shocks than in the efficient case.

[10] When we solve the Ramsey problem, we freely normalize the standard deviation of endogenous shocks to 1%.

In addition, the optimal response of output to discount rate shocks is hump-shaped. This feature follows from the persistent nature of the underlying shocks. Intuitively, limited information about the persistent shocks creates persistent output gaps, which the policy maker would like to smooth over time in order to minimize the corresponding utility loss. This smoothing motive results in a hump-shaped response of output.

In contrast to exogenous shocks, the output response to endogenous shocks is much shorter-lived, and reflects the fact that the exogenous responses converge to their efficient counterparts relatively quickly. Intuitively, when the exogenous shocks are persistent, the policy maker is able to use her limited information about current shocks to improve her knowledge of past shocks. Quantitatively, this build up of information implies that the policy maker learns about the true past shocks within 10 quarters of their impact.

The responses of inflation have similar features to the output responses, which follow from the relationship between output and inflation dictated by the New Keynesian Phillips curve.

Macroeconomic Volatility To gain further insight into the effect that limited information has on optimal policy design, Table 2 displays the unconditional variances of output and inflation in the efficient, and optimal sticky price equilibria. For ease of comparison, we have normalized all statistics by the variance of efficient output.

Relative to the flexible price economy, output variance increases by 30% under the optimal monetary policy. Furthermore, the increase in volatility is predominantly due to the endogenous monetary shocks that occur when the policy maker is subject to limited information. Exogenous shocks account for only 75% of output volatility under the optimal policy, as targeted by our calibration.

Finally, we note that, under the optimal policy, output and inflation are strongly positively correlated, with a correlation coefficient of 0.49. This positive co-movement is a result of the endogenous monetary shocks, that move output and inflation in the same direction, and account for a significant portion of overall macroeconomic volatility.

The finding contrasts with targeting rules that prescribe either zero correlation (price stability) or negative correlation (leaning against the wind) between output and inflation. In our setting, the positive correlation is purely an artifact of the limited information under which policy must be conducted. Estimating a positive co-movement between output and inflation does not imply that the underlying policy is suboptimal.

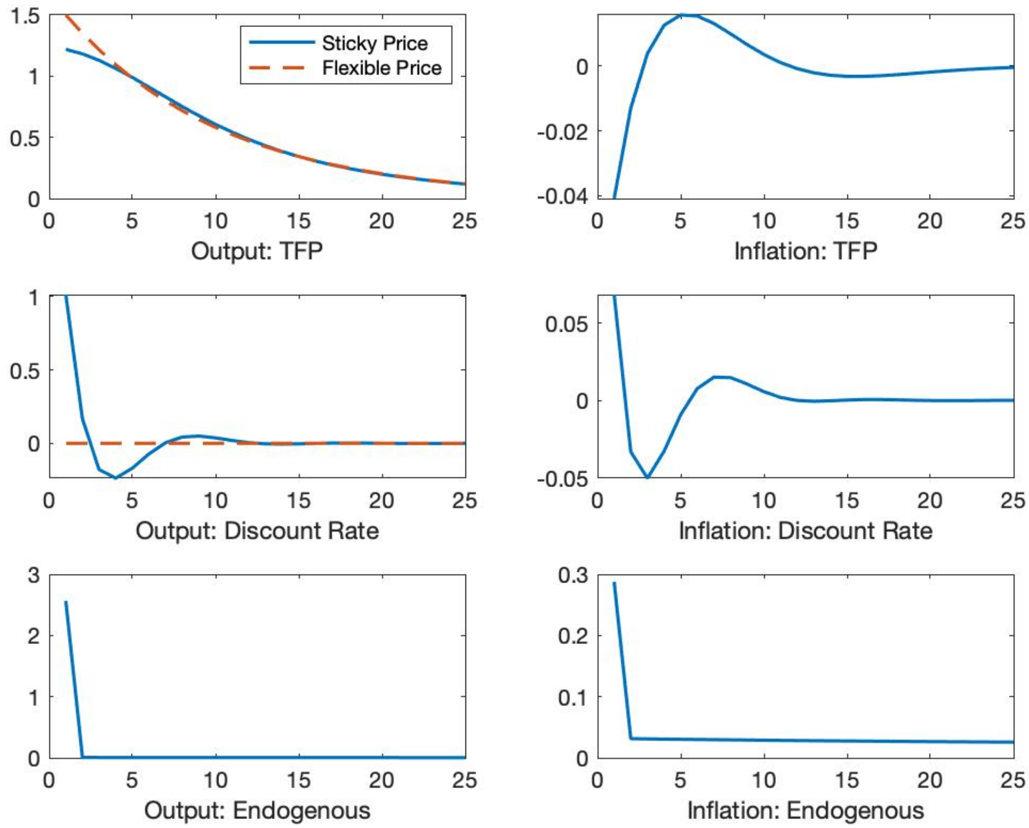

Figure 1: Impulse Response Functions of output and inflation to one standard deviation TFP, discount rate, and endogenous shocks. The standard deviation of endogenous shocks is normalized to 1%. Responses are in percentage points at a quarterly frequency.

Economy	Var[$\hat{y}$]	Var[π]	Endog. share of Var[$\hat{y}$]	Corr($\hat{y}, \pi$)
Flexible Price	1	-	0%	-
Sticky Price	1.30	0.006	24%	0.49

Table 2: Macroeconomic volatility statistics. The variances are normalized by the variance of efficient output.

Interest Rate Responses Figure 2 plots the impulse responses of real and nominal interest rates to the same shocks. For reference, the efficient real rate responses are plotted in red-dash.

Relative to the efficient economy, the real rate responses display muted and hump-shaped behavior, much like output and inflation. Similarly, the response to endogenous shocks is large, but short-lived.

The responses of the nominal interest rate display significant non-monotonicities, and are long-lived even for endogenous shocks. Intuitively, the path of nominal rates must account for how both the real interest rate and inflation will respond in equilibrium in order to attain the optimal balance of output and inflation volatility.

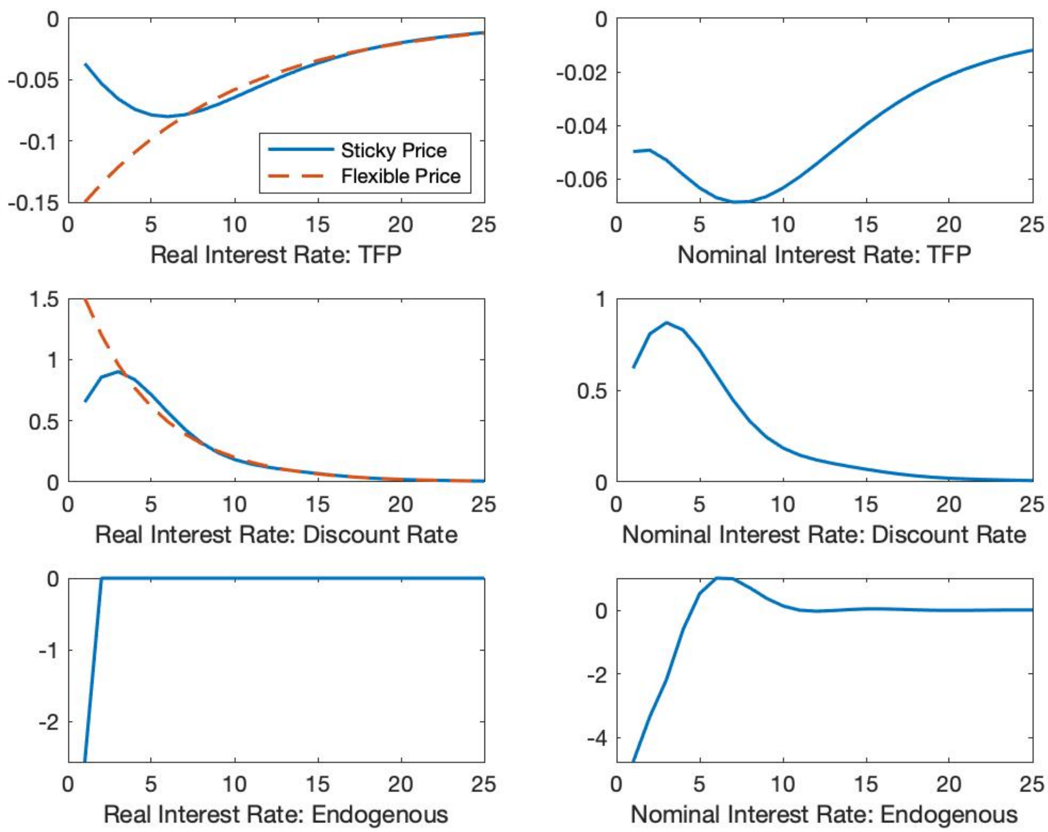

Figure 2: Impulse Response Functions of interest rates to one standard deviation TFP, discount rate, and endogenous shocks. The standard deviation of endogenous shocks is normalized to 1%. Responses are in percentage points at a quarterly frequency.

6 An Alternative Information Cost

Our benchmark analysis assumes that mutual information is a good model of the information processing frictions faced by the policy maker. In this section, we show that our main results continue to hold when we consider an alternative information cost function proposed by Hébert and Woodford (2020).

The authors propose a family of "neighborhood-based" information cost functions. In contrast to mutual information, this family of cost functions allows for the notion that certain subgroups of hidden states are easier to distinguish than others. For example, it may be less costly for the policy maker to distinguish between negative and positive TFP shocks, than to distinguish between positive shocks of different magnitude. Hébert and Woodford (2020) argue that this feature allows such cost functions to make accurate predictions about behavior in perceptual experiments, where mutual information cannot.

In our linear-quadratic setting with normal shocks, we can use the result in Hébert and Woodford (2020) that the cost function takes the form of the average Fisher information to retain tractability.

6.1 Analytical Results

Consider the case of i.i.d. aggregate shocks (impose assumption 1). Applying the same steps as before, it is straightforward to establish that proposition 1 continues to hold, and that the information constraint can be expressed as

$$\underbrace{I^F\left(\mathbb{E}_{M,t}\left[\hat{\rho}_t\right];\hat{\rho}_t\right)}_{\kappa_\rho^F}+\underbrace{I^F\left(\mathbb{E}_{M,t}\left[\hat{a}_t\right];\hat{a}_t\right)}_{\kappa_a^F}\leq\kappa_M^F$$

where I^F is the average Fisher information cost function, and κ_ρ^F and κ_a^F denote the information allocation under the Fisher cost.

We now use the result in Hébert and Woodford (2020) that when the underlying shocks are normal, it is optimal for the policy maker to form normally distributed conditional expectations.[11] Given this, we can exploit the fact that the average Fisher information cost

[11] Technically, Hébert and Woodford (2020) show that the optimal signal structure with be normal, which implies that the updated expectations will also be normal.

generated by two scalar normal random variables X and Y is given by

$$I^F(X;Y) = \int \phi(x) \left(\left(\frac{\partial \mu_{Y|X=x}}{\partial X} \right)^2 / \sigma^2_{Y|X=x} \right) dx$$

where $\mu_{Y|X=x}$ is the mean of Y conditional on $X = x$, $\sigma^2_{Y|X=x}$ is the variance of Y conditional on $X = x$, and $\phi(x)$ is the normal density function of X. Applying this result allows us to compute the expectations (the analogous result to proposition 2).

Proposition 7. *Let assumption 1 hold, and suppose that $I^F(\mathbb{E}_{M,t}[\hat{a}_t]; \hat{a}_t) = \kappa_a^F$ and $I^F(\mathbb{E}_{M,t}[\hat{\rho}_t]; \hat{\rho}_t) = \kappa_\rho^F$. Then, the expectations satisfy*

$$\mathbb{E}_{M,t}[\hat{a}_t] = \left(\frac{\kappa_a^F}{\kappa_a^F + \frac{1}{\sigma_a^2}} \right) \hat{a}_t + \frac{\sqrt{\kappa_a^F}}{\kappa_a^F + \frac{1}{\sigma_a^2}} v_t$$

$$\mathbb{E}_{M,t}[\hat{\rho}_t] = \left(\frac{\kappa_\rho^F}{\kappa_\rho^F + \frac{1}{\sigma_\rho^2}} \right) \hat{\rho}_t + \frac{\sqrt{\kappa_\rho^F}}{\kappa_\rho^F + \frac{1}{\sigma_\rho^2}} u_t$$

where v_t and u_t are i.i.d. standard normal random variables.

Comparing this result to proposition 2 demonstrates the key effect that the Fisher information cost specification has on the equilibrium. Specifically, we see that an increase in the variance of a fundamental shock increases the correlation between the policy maker's expectation of the shock and the shock itself, all else equal. This contrasts with the mutual information case, in which the correlation did not directly depend on the variance.

This difference stems from the "neighborhood" feature of the Fisher information cost. When the variance of a fundamental shock increases, it becomes less costly for the policy maker to discriminate between nearby states at every point in the state space. Therefore, for a given information allocation, the policy maker can make more accurate forecasts of the current state in each period.

Given these expectations, the construction of the equilibrium follows the same steps as before. As a result, the main qualitative features are unchanged from the mutual information cost model. The economy features stochastic fluctuations driven by endogenous monetary policy shocks. The optimal equilibrium can be implemented using a Taylor rule based on noisy observations of output and inflation, and it may not be determinate.

7 Conclusion

Monetary authorities around the world face significant uncertainty about the current state of the economy due to data lags or cognitive limitations. In light of this, what is optimal monetary policy? Within a New Keynesian model, we study the design of optimal monetary policy when the policy maker faces information constraints.

Unable to know the efficient path of the economy, the classical objective of monetary policy, price stability, is rendered infeasible. In fact, the deviations of the policy maker's beliefs from the truth result in endogenous macroeconomic volatility. The policy maker's uncertainty also affects the economy's response to exogenous shocks. For example, the real interest rate responds less to exogenous shocks than in the full-information framework, as the monetary authority uses the efficient real rate's long-run mean as an anchor point. Furthermore, the attenuation of the real interest rate results in output's response to demand shocks being amplified and output's response to supply shocks being dampened. In a quantitative exercise, we demonstrate that policy maker uncertainty leads to a positive comovement of inflation and output. Unlike existing work that suggests this correlation can be the result of poor policy, we show it can be a ramification of information constraints.

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Appendix

A Proofs

Proof of Lemma 1

Household optimality implies the following log-linear first-order conditions (FOCs)

$$\frac{1}{\gamma}\hat{c}_t + \varphi \hat{n}_t = \hat{w}_t$$

$$\mathbb{E}_t\left[\hat{c}_{t+1}\right] - \hat{c}_t = \gamma\left(r_t^e - \rho_t\right)$$

In the absence of price-adjustment frictions, profit maximization by firms implies that

$$\hat{w}_t = \hat{a}_t$$

Combining these conditions with the production function and market clearing, $\hat{y}_t = \hat{a}_t + \hat{n}_t$ and $\hat{c}_t = \hat{y}_t^e$ yields

$$\hat{y}_t^e = \frac{1+\varphi}{\frac{1}{\gamma}+\varphi}\hat{a}_t$$

$$r_t^e = \rho + \hat{\rho}_t - \left(1-\delta_a\right)\frac{1+\varphi}{1+\gamma\varphi}\hat{a}_t$$

Proof of Lemma 2

In the presence of price-adjustment frictions, profit maximization by firms yields the following FOC

$$\beta_t\left(\left(1-\Phi\right)p_{j,t}^{-\Phi}P_t^{\Phi-1}Y_t + \left(1-\tau_n\right)\frac{w_t}{A_t}\Phi p_{j,t}^{-\Phi-1}P_t^{\Phi}Y_t - \xi\left(\frac{p_{j,t}}{p_{j,t-1}}-1\right)Y_t\frac{1}{p_{j,t-1}}\right) + \mathbb{E}_t\left[\beta_{t+1}\xi\left(\frac{p_{j,t+1}}{p_{j,t}}-1\right)Y_{t+1}\frac{p_{j,t+1}}{p_{j,t}^2}\right] = 0.$$

Imposing symmetry yields

$$\pi_t\left(1+\pi_t\right) = \left(\frac{1-\Phi}{\xi}\right) + \left(1-\tau_n\right)\frac{w_t}{A_t}\frac{\Phi}{\xi} + \xi\mathbb{E}_t\left[\frac{\beta_{t+1}}{\beta_t}\pi_{t+1}\left(1+\pi_{t+1}\right)\frac{Y_{t+1}}{Y_t}\right].$$

Log-linearizing around the zero-inflation deterministic steady state yields

$$\pi_t = \frac{\Phi-1}{\xi}\left(\hat{w}_t - \hat{a}_t\right) + \frac{1}{1+\rho}\mathbb{E}_t\left[\pi_{t+1}\right]$$

Using the static household FOC and the production function implies

$$\pi_t = \frac{\Phi - 1}{\xi} \left(\frac{1}{\gamma} + \varphi \right) \tilde{y}_t + \frac{1}{1+\rho} \mathbb{E}_t \left[\pi_{t+1} \right]$$

so that $\varphi_y = \frac{\Phi-1}{\xi} \left(\frac{1}{\gamma} + \varphi \right)$

Proof of Lemma 3

Let

$$u_t = \beta_t \left(\frac{c_t^{1-\frac{1}{\gamma}}}{1-\frac{1}{\gamma}} - \frac{n_t^{1+\varphi}}{1+\varphi} \right)$$

with $\beta_t = \prod_{s=1}^t \left(\frac{1}{1+\rho+\hat{\rho}_s} \right)$ and $\beta = \frac{1}{(1+\rho)^t}$ denote the flow utility to the household in period t. A second order approximation around the deterministic steady state yields

$$u_t \approx u + \beta c^{1-\frac{1}{\gamma}} \left(\hat{c}_t \left(1 + \hat{\beta}_t \right) + \frac{1-\frac{1}{\gamma}}{2} \hat{c}_t^2 - \hat{n}_t \left(1 + \hat{\beta}_t \right) - \frac{1+\varphi}{2} \hat{n}_t^2 \right) + \beta \left(\frac{c^{1-\frac{1}{\gamma}}}{1-\frac{1}{\gamma}} - \frac{n^{1+\varphi}}{1+\varphi} \right) \left(\hat{\beta}_t + \frac{1}{2} \hat{\beta}_t^2 \right)$$

Use market clearing $c_t = Y_t \left(1 - \frac{\xi}{2} \pi_t^2 \right)$ to obtain

$$\hat{c}_t \approx \hat{y}_t - \frac{\xi}{2} \pi_t^2$$

so that

$$u_t \approx u + \beta c^{1-\frac{1}{\gamma}} \left(\left(\hat{y}_t - \frac{\xi}{2} \pi_t^2 \right) \left(1 + \hat{\beta}_t \right) + \frac{1-\frac{1}{\gamma}}{2} \hat{y}_t^2 - \hat{n}_t \left(1 + \hat{\beta}_t \right) - \frac{1+\varphi}{2} \hat{n}_t^2 \right) + \beta \left(\frac{c^{1-\frac{1}{\gamma}}}{1-\frac{1}{\gamma}} - \frac{n^{1+\varphi}}{1+\varphi} \right) \left(\hat{\beta}_t + \frac{1}{2} \hat{\beta}_t^2 \right)$$

Use the production function to obtain

$$\begin{aligned} u_t \approx u + \beta c^{1-\frac{1}{\gamma}} & \left(-\frac{\xi}{2} \pi_t^2 - \frac{1}{2} \left(\frac{1}{\gamma} + \varphi \right) \hat{y}_t^2 + (1+\varphi) \hat{y}_t \hat{a}_t + \hat{a}_t \left(1 + \hat{\beta}_t \right) - \frac{1+\varphi}{2} \hat{a}_t^2 \right) \\ & + \beta \left(\frac{c^{1-\frac{1}{\gamma}}}{1-\frac{1}{\gamma}} - \frac{n^{1+\varphi}}{1+\varphi} \right) \left(\hat{\beta}_t + \frac{1}{2} \hat{\beta}_t^2 \right) \end{aligned}$$

Now substitute $\hat{a}_t = \frac{\frac{1}{\frac{1}{\gamma}} + \varphi}{1+\varphi} \hat{y}_t^e$ to get

$$\frac{u_t - u}{c^{1-\frac{1}{\gamma}}} \approx -\beta \frac{1}{2} \left(\frac{1}{\gamma} + \varphi \right) \left(\tilde{y}_t^2 + \frac{\xi}{\frac{1}{\gamma} + \varphi} \pi_t^2 \right) + \text{t.i.p.}$$

where t.i.p. are terms independent of policy. Therefore, the per-period utility loss is proportional to

$$L_t = \frac{1}{2}\beta\left(\tilde{y}_t^2 + \frac{\xi}{\frac{1}{\gamma}+\varphi}\pi_t^2\right)$$

Proof of Proposition 1

When the exogenous shocks are i.i.d. processes, the optimal policy will only depend on the policy maker's expectations of current outcomes. As such, the equilibrium will only depend on current shocks, so that $\mathbb{E}_t\left[\tilde{y}_{t+1}\right] = \mathbb{E}_t\left[\pi_{t+1}\right] = 0$. In this case, the Ramsey problem simplifies to

$$\min_{\{\iota_t\}_{t=0}^{\infty}} \frac{1+\alpha\varphi_y^2}{2}\gamma^2\mathbb{E}_0\sum_{t=0}^{\infty}\left(\frac{1}{1+\rho}\right)^t\left(\iota_t - r_t^e\right)^2$$

subject to

$$I\left(\iota_t; r_t^e\right) = \kappa_M$$

This is a sequence of mean-squared error minimization problems. The solution is given by

$$\iota_t = \mathbb{E}_{M,t}\left[r_t^e\right]$$

where the expectation satisfies the mutual information constraint.

Proof of Proposition 2

The proof is a direct application of Proposition 3 in Mackowiak and Wiederholt (2009).

Proof of Proposition 3

Consider the system of equations

$$\iota_t = \rho + \left(1 - \frac{1}{2^{2\kappa_\rho}}\right)\hat{\rho}_t + \sqrt{\frac{2^{2\kappa_\rho}-1}{2^{4\kappa_\rho}}\sigma_\rho^2}u_t - \frac{1+\varphi}{1+\gamma\varphi}\left(\left(1-\frac{1}{2^{2\kappa_a}}\right)\hat{a}_t + \sqrt{\frac{2^{2\kappa_a}-1}{2^{4\kappa_a}}\sigma_a^2}v_t\right)$$

$$\pi_t = \varphi_y\tilde{y}_t + \frac{1}{1+\rho}\mathbb{E}_t\left[\pi_{t+1}\right]$$

$$\mathbb{E}_t\left[\tilde{y}_{t+1}\right] - \tilde{y}_t = \gamma\left(\iota_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^e\right)$$

Solving this system using a guess and verify approach yields

$$\tilde{y}_t = \gamma\left(\frac{1}{2^{2\kappa_\rho}}\hat{\rho}_t - \sqrt{\frac{2^{2\kappa_\rho}-1}{2^{4\kappa_\rho}}\sigma_\rho^2}u_t\right) + \gamma\frac{1+\varphi}{1+\gamma\varphi}\left(-\frac{1}{2^{2\kappa_a}}\hat{a}_t + \sqrt{\frac{2^{2\kappa_a}-1}{2^{4\kappa_a}}\sigma_a^2}v_t\right)$$

so that

$$\hat{y}_t = \gamma \left(\frac{1}{2^{2\kappa_\rho}} \hat{\rho}_t - \sqrt{\frac{2^{2\kappa_\rho} - 1}{2^{4\kappa_\rho}} \sigma_\rho^2} u_t \right) + \gamma \frac{1+\varphi}{1+\gamma\varphi} \left(\left(1 - \frac{1}{2^{2\kappa_a}} \right) \hat{a}_t + \sqrt{\frac{2^{2\kappa_a} - 1}{2^{4\kappa_a}} \sigma_a^2} v_t \right)$$

$$\pi_t = \varphi_y \gamma \left(\frac{1}{2^{2\kappa_\rho}} \hat{\rho}_t - \sqrt{\frac{2^{2\kappa_\rho} - 1}{2^{4\kappa_\rho}} \sigma_\rho^2} u_t \right) + \varphi_y \gamma \frac{1+\varphi}{1+\gamma\varphi} \left(- \frac{1}{2^{2\kappa_a}} \hat{a}_t + \sqrt{\frac{2^{2\kappa_a} - 1}{2^{4\kappa_a}} \sigma_a^2} v_t \right)$$

Proof of Proposition 4

Given the equilibrium dynamics, the utility loss simplifies to

$$\gamma^2 \left(\frac{1}{2^{4\kappa_\rho}} \sigma_\rho^2 + \frac{2^{2\kappa_\rho} - 1}{2^{4\kappa_\rho}} \sigma_\rho^2 \right) + \gamma^2 \left(\frac{1+\varphi}{1+\gamma\varphi} \right)^2 \left(\frac{1}{2^{4\kappa_a}} \sigma_a^2 + \frac{2^{2\kappa_a} - 1}{2^{4\kappa_a}} \sigma_a^2 \right)$$

which must be minimized subject to

$$\kappa_a + \kappa_\rho \leq \kappa_M$$

The FOCs yield

$$\kappa_a = \frac{1}{2} \kappa_M + \frac{1}{4} \log_2 \left(\frac{\left(\frac{1+\varphi}{1+\gamma\varphi} \right)^2 \sigma_a^2}{\sigma_\rho^2} \right)$$

and

$$\kappa_\rho = \frac{1}{2} \kappa_M - \frac{1}{4} \log_2 \left(\frac{\left(\frac{1+\varphi}{1+\gamma\varphi} \right)^2 \sigma_a^2}{\sigma_\rho^2} \right)$$

Proof of Proposition 5

We use the expressions for output and inflation to write

$$\frac{1+\varphi}{1+\gamma\varphi} \hat{a}_t = \frac{\hat{y}_t - \frac{1}{\varphi_y} \pi_t}{\gamma}$$

and

$$\hat{\rho}_t = 2^{2\kappa_\rho} \left(1 - \frac{1}{2^{2\kappa_a}} \right) \frac{1}{\gamma\varphi_y} \pi_t + 2^{2\kappa_\rho} \sqrt{\frac{2^{2\kappa_\rho} - 1}{2^{4\kappa_\rho}} \sigma_\rho^2} u_t + \frac{2^{2\kappa_\rho}}{2^{2\kappa_a}} \frac{1}{\gamma} \hat{y}_t - 2^{2\kappa_\rho} \frac{1+\varphi}{1+\gamma\varphi} \sqrt{\frac{2^{2\kappa_a} - 1}{2^{4\kappa_a}} \sigma_a^2} v_t$$

Substituting these into the equation for the nominal interest rate yields

$$\iota_t = \rho + 2^{2\kappa_\rho} \frac{2^{2\kappa_a} - 1}{2^{2\kappa_a}} \frac{1}{\gamma\varphi_y} \pi_t + \frac{2^{2\kappa_\rho} - 2^{2\kappa_a}}{2^{2\kappa_a}} \frac{1}{\gamma} \hat{y}_t - 2^{2\kappa_\rho} \frac{1+\varphi}{1+\gamma\varphi} \sqrt{\frac{2^{2\kappa_a} - 1}{2^{4\kappa_a}} \sigma_a^2} v_t + 2^{2\kappa_\rho} \sqrt{\frac{2^{2\kappa_\rho} - 1}{2^{4\kappa_\rho}} \sigma_\rho^2} u_t$$

Hence

$$\iota_t = \rho + 2^{2\kappa_\rho} \frac{2^{2\kappa_a} - 1}{2^{2\kappa_a}} \frac{1}{\gamma\varphi_y} \pi_t^o + \frac{2^{2\kappa_\rho} - 2^{2\kappa_a}}{2^{2\kappa_a}} \frac{1}{\gamma} \hat{y}_t^o$$

where

$$\pi_t^o = \pi_t - \gamma\varphi_y \frac{1+\varphi}{1+\gamma\varphi} \sqrt{\frac{1}{2^{2\kappa_a} - 1} \sigma_a^2} v_t$$

$$\hat{y}_t^o = \hat{y}_t + \gamma \frac{2^{2\kappa_a}}{2^{2\kappa_\rho} - 2^{2\kappa_a}} \sqrt{\left(2^{2\kappa_\rho} - 1\right) \sigma_\rho^2} u_t$$

Proof of Proposition 6

Rewrite the path of nominal rates as

$$\begin{aligned} \iota_t = \rho + 2^{2\kappa_\rho} \frac{2^{2\kappa_a} - 1}{2^{2\kappa_a}} \frac{1}{\gamma\varphi_y} \pi_t + \frac{2^{2\kappa_\rho} - 2^{2\kappa_a}}{2^{2\kappa_a}} \frac{1}{\gamma} \tilde{y}_t \\ + \frac{2^{2\kappa_\rho} - 2^{2\kappa_a}}{2^{2\kappa_a}} \frac{1+\varphi}{1+\gamma\varphi} \hat{a}_t - 2^{2\kappa_\rho} \frac{1+\varphi}{1+\gamma\varphi} \sqrt{\frac{2^{2\kappa_a} - 1}{2^{4\kappa_a}} \sigma_a^2} v_t + 2^{2\kappa_\rho} \sqrt{\frac{2^{2\kappa_\rho} - 1}{2^{4\kappa_\rho}} \sigma_\rho^2} u_t \end{aligned}$$

Combining this with

$$\pi_t = \varphi_y \tilde{y}_t + \frac{1}{1+\rho} \mathbb{E}_t \left[\pi_{t+1}\right]$$

$$\mathbb{E}_t \left[\tilde{y}_{t+1}\right] - \tilde{y}_t = \gamma \left(\iota_t - \mathbb{E}_t \left[\pi_{t+1}\right] - r_t^e\right)$$

yields a three equation system. Following the steps in Galí (2015), this system is determinate when

$$\kappa_M > \log_2 \left(\frac{1}{1+\rho} \frac{\sigma_\rho}{\left(\frac{1+\varphi}{1+\gamma\varphi}\right) \sigma_a} + \left(\frac{\rho}{1+\rho} + \gamma\varphi_y\right) \frac{\left(\frac{1+\varphi}{1+\gamma\varphi}\right) \sigma_a}{\sigma_\rho} \right)$$

B Numerical Method for Quantitative Results

We look for solutions of the form

$$\tilde{y}_t = \sum_{s=0}^{\infty} b_s^y e_{t-s}^a + \sum_{s=0}^{\infty} c_s^y e_{t-s}^v + \sum_{s=0}^{\infty} f_s^y e_{t-s}^\rho + \sum_{s=0}^{\infty} g_s^y e_{t-s}^u$$

$$\pi_t = \sum_{s=0}^{\infty} b_s^\pi e_{t-s}^a + \sum_{s=0}^{\infty} c_s^\pi + \sum_{s=0}^{\infty} f_s^\pi e_{t-s}^\rho + \sum_{s=0}^{\infty} g_s^\pi e_{t-s}^u$$

where $e_{t-s}^v, e_{t-s}^u \sim N(0,1)$ for all s. Setting $Var\left[e_{t-s}^v\right] = Var\left[e_{t-s}^u\right] = 1$ is a free normalization.

In practice, we approximate the infinite moving averages up to a terminal index $N > 0$ such that all coefficients are zero for $s \geq N$.

Substituting these guesses into the New Keynesian Phillips curve yields the iterative relationship

$$b_s^\pi = \varphi_y b_s^y + \frac{1}{1+\rho} b_{s+1}^\pi$$

$$c_s^\pi = \varphi_y c_s^y + \frac{1}{1+\rho} c_{s+1}^\pi$$

$$f_s^\pi = \varphi_y f_s^y + \frac{1}{1+\rho} f_{s+1}^\pi$$

$$g_s^\pi = \varphi_y g_s^y + \frac{1}{1+\rho} g_{s+1}^\pi$$

Substitution into the Euler equation yields the implies expression for the nominal interest rate

$$\begin{aligned}\iota_t = \rho + \sum_{s=0}^{\infty} \left(b_{s+1}^\pi + \frac{1}{\gamma}\left(b_{s+1}^y - b_s^y\right) - \frac{1+\varphi}{1+\gamma\varphi}\left(1-\delta_a\right)\delta_a^s \right) e_{t-s}^a + \sum_{s=0}^{\infty} \left(c_{s+1}^\pi + \frac{1}{\gamma}\left(c_{s+1}^y - c_s^y\right)\right) e_{t-s}^v \\ + \sum_{s=0}^{\infty} \left(f_{s+1}^\pi + \frac{1}{\gamma}\left(f_{s+1}^y - f_s^y\right) + \delta_\rho^s \right) e_{t-s}^\rho + \sum_{s=0}^{\infty} \left(g_{s+1}^\pi + \frac{1}{\gamma}\left(g_{s+1}^y - g_s^y\right)\right) e_{t-s}^u\end{aligned}$$

Given the guesses, the information constraint becomes

$$\begin{aligned} I\left(\left\{ \sum_{s=0}^{\infty} \left(\frac{1}{\gamma}\left(b_{s+1}^y - b_s^y\right) - \frac{1+\varphi}{1+\gamma\varphi}\left(1-\delta_a\right)\delta_a^s \right) e_{t-s}^a + \sum_{s=0}^{\infty} \frac{1}{\gamma}\left(c_{s+1}^y - c_s^y\right) e_{t-s}^v \right\} ; \left\{ - \sum_{s=0}^{\infty} \frac{1+\varphi}{1+\gamma\varphi}\left(1-\delta_a\right)\delta_a^s e_{t-s}^a \right\} \right) \\ + I\left(\left\{ \sum_{s=0}^{\infty} \left(\frac{1}{\gamma}\left(f_{s+1}^y - f_s^y\right) + \delta_\rho^s \right) e_{t-s}^\rho + \sum_{s=0}^{\infty} \frac{1}{\gamma}\left(g_{s+1}^y - g_s^y\right) e_{t-s}^u \right\} ; \left\{ \sum_{s=0}^{\infty} \delta_\rho^s e_{t-s}^\rho \right\} \right) \leq \kappa_M \end{aligned}$$

which can be written as

$$I\left(\left\{ \sum_{s=0}^{\infty} B_s e_{t-s}^a + \sum_{s=0}^{\infty} C_s e_{t-s}^v \right\} ; \left\{ \sum_{s=0}^{\infty} A_s e_{t-s}^a \right\} \right) + I\left(\left\{ \sum_{s=0}^{\infty} F_s e_{t-s}^\rho + \sum_{s=0}^{\infty} G_s e_{t-s}^u \right\} ; \left\{ \sum_{s=0}^{\infty} H_s e_{t-s}^\rho \right\} \right) \leq \kappa_M$$

with

$$A_s = -\frac{1+\varphi}{1+\gamma\varphi}\left(1-\delta_a\right)\delta_a^s$$

$$B_s = \frac{1}{\gamma}\left(b_{s+1}^y - b_s^y\right) - \frac{1+\varphi}{1+\gamma\varphi}\left(1-\delta_a\right)\delta_a^s$$

$$C_s = \frac{1}{\gamma}\left(c_{s+1}^y - c_s^y\right)$$

$$F_s = \frac{1}{\gamma}\left(f_{s+1}^y - f_s^y\right) + \delta_\rho^s$$

$$G_s = \frac{1}{\gamma}\left(g_{s+1}^y - g_s^y\right)$$

$$H_s = \delta_\rho^s$$

Then, the mutual informations can be written as

$$I\left(\left\{\sum_{s=0}^{\infty} B_s e_{t-s}^a + \sum_{s=0}^{\infty} C_s e_{t-s}^v\right\}; \left\{\sum_{s=0}^{\infty} A_s e_{t-s}^a\right\}\right) = -\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \log_2\left(1 - \frac{1}{1 + \left|\tilde{C}(\omega)\right|^2 / \left(\sigma_a^2 \left|\tilde{B}(\omega)\right|^2\right)}\right) d\omega$$

$$I\left(\left\{\sum_{s=0}^{\infty} F_s e_{t-s}^\rho + \sum_{s=0}^{\infty} G_s e_{t-s}^u\right\}; \left\{\sum_{s=0}^{\infty} H_s e_{t-s}^\rho\right\}\right) = -\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \log_2\left(1 - \frac{1}{1 + \left|\tilde{G}(\omega)\right|^2 / \left(\sigma_\rho^2 \left|\tilde{F}(\omega)\right|^2\right)}\right) d\omega$$

where $\tilde{X}(\omega)$ is the Fourier transform of X:

$$\tilde{B}(\omega) = \sum_{s=0}^{\infty} B_s e^{-2\pi i s\omega}$$

$$\tilde{C}(\omega) = \sum_{s=0}^{\infty} C_s e^{-2\pi i s\omega}$$

$$\tilde{F}(\omega) = \sum_{s=0}^{\infty} F_s e^{-2\pi i s\omega}$$

$$\tilde{G}(\omega) = \sum_{s=0}^{\infty} G_s e^{-2\pi i s\omega}$$

The household utility loss is proportional to

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\Bigg(&\sum_{s=0}^{\infty}(b_s^y)^2\sigma_a^2 + \sum_{s=0}^{\infty}(c_s^y)^2 + \sum_{s=0}^{\infty}(f_s^y)^2\sigma_\rho^2 + \sum_{s=0}^{\infty}(g_s^y)^2 + \\ &\alpha\left(\sum_{s=0}^{\infty}(b_s^\pi)^2\sigma_a^2 + \sum_{s=0}^{\infty}(c_s^\pi)^2 + \sum_{s=0}^{\infty}(f_s^\pi)^2\sigma_\rho^2 + \sum_{s=0}^{\infty}(g_s^\pi)^2\right)\Bigg)\end{aligned}$$

Hence we solve

$$\min_{\{b^y, c^y, f^y, g^y\}} \mathcal{L}$$

subject to

$$b_s^\pi = \varphi_y b_s^y + \frac{1}{1+\rho} b_{s+1}^\pi$$

$$c_s^\pi = \varphi_y c_s^y + \frac{1}{1+\rho} c_{s+1}^\pi$$

$$f_s^\pi = \varphi_y f_s^y + \frac{1}{1+\rho} f_{s+1}^\pi$$

$$g_s^\pi = \varphi_y g_s^y + \frac{1}{1+\rho} g_{s+1}^\pi$$

$$B_s = \frac{1}{\gamma}\left(b^y_{s+1} - b^y_s\right) - \frac{1+\varphi}{1+\gamma\varphi}\left(1-\delta_a\right)\delta_a^s$$

$$C_s = \frac{1}{\gamma}\left(c^y_{s+1} - c^y_s\right)$$

$$F_s = \frac{1}{\gamma}\left(f^y_{s+1} - f^y_s\right) + \delta_\rho^s$$

$$G_s = \frac{1}{\gamma}\left(g^y_{s+1} - g^y_s\right)$$

$$\tilde{B}\left(\omega\right) = \sum_{s=0}^{\infty} B_s e^{-2\pi i s \omega}$$

$$\tilde{C}\left(\omega\right) = \sum_{s=0}^{\infty} C_s e^{-2\pi i s \omega}$$

$$\tilde{F}\left(\omega\right) = \sum_{s=0}^{\infty} F_s e^{-2\pi i s \omega}$$

$$\tilde{G}\left(\omega\right) = \sum_{s=0}^{\infty} G_s e^{-2\pi i s \omega}$$

$$-\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \log_2\left(1 - \frac{1}{1 + \left|\tilde{C}\left(\omega\right)\right|^2 / \left(\sigma_a^2 \left|\tilde{B}\left(\omega\right)\right|^2\right)}\right) d\omega - \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \log_2\left(1 - \frac{1}{1 + \left|\tilde{G}\left(\omega\right)\right|^2 / \left(\sigma_\rho^2 \left|\tilde{F}\left(\omega\right)\right|^2\right)}\right) d\omega \leq \kappa_M$$

This minimization can be solved in MATLAB using the fmincon routine. In practice, we set $N = 200$.