Lecture Notes For Important Topics

In this chapter, we will discuss the word-based problems on the number system, Arithmetic progression, Geometry progression, Remainder theorem and Unit digits. All these chapters are very important for all aptitude exams.

Word Based Problem On Number System

We can solve these problems with the help of:

1. Linear equations.
2. Identify the variable in the equation.
3. Think numerically and logically.

Problem 1:

5 year ago Anjali was 5 times as old as her son. 5 years hence her age will be 8 less than three times the corresponding age of her son. Find the present age of Anjali.

a. 30 years
b. 35 years
c. 40 years
d. 45 years

Solution:

	5 Years Ago	**Present**	**5 Years Later**
Anjali	5x	5x+5	5x+10
Son	x	x+5	x+10

Now it is given that after 5 years her age will be 8 less than three times her Son's age.

Hence, $5x+10+8 = 3(x+10)$

$3x+30 = 5x+18$

$2x = 12$

$x = 6.$

Therefore, Anjali's present age = $5 \times 6+5 = 35$ years.

2nd method:

We can check from the options.

Let's take the option (a) 30 years.

So, Anjali's present age = 30 years

5 year ago her age = 30 - 5 = 25 years and her son's age 5 years ago =(1/5) × 25 = 5years

Present age of son = 5+5 = 10.

5 years later age of her son = 10+5 = 15 and Anjali's age = 35

5 years later her age will 3 times of her son's age = 15 × 5 = 45 years.

Anjali,s age 8 years less than her son's age after 5 years, that means Son's age 8 year more than 35 years. So, 35+8 = 43 years

Hence this option is wrong.

The same way we can check for option(b).

	5 Years Ago	**Present**	**5 Years Later**
Anjali	30	35	40
Son	6	11	16

5 year age Son's age = (1/5) × 30 = 6.

After 5 years Anjali's age is 3 times of her son's age i.e. 16 × 3 = 48.

And after 5 years Anjali's age is 8 years less than 3times of her son's age i.e. 16 × 3 - 8 = 40 years.

Problem 2:

Girish's youth lasted one-sixth of his life. He grew a beard after one twelfth more. After one seventh more of his life, he married. 5 year later, he and his wife had a son. The son lived exactly one half as long as his father and Girish died four years after his son. How many years did Girish live?

a. 76 years
b. 80 years
c. 84 years
d. 88 years

Solution:

Lets the Girish final age = x

Final age = youth lasted age + grew beard age + marriage + 5 + son's age + 4

$x = x/6 + x/12 + x/7 + 5 + x/2 + 4$

X = 84 years.

2nd method:
Girish youth lasted = 14/6 of his life.
So, His life should be divisible by 6 because age always be an integer. Only option (c) 84 will be divisible by 6.
Hence, the age of Girish = 84 years.

Problem 3:
5 children who are born at an interval of 4 years and sum of their ages is 60. What is the age of the oldest child?

Solution:
Children are born at a 4-year interval. It means it is an A.P. of +4.
Sum of their ages = 60.
Average of the ages = 60/5 = 12
There are only 5 children, so an average of 5 will be the middle term.

1st children	2nd children	3rd children	4th children	5th children
4	8	**12**	16	20

Hence, the age of the oldest children = 20 years.

2nd method:
Let the age of 1st children = a
Hence, the age of 2nd children = a+4
age of 3rd children = a+8
age of 4th children = a+12
age of 5th children = a+16

Sum of their ages = 60
a+a+4+a+8+a+12+a+16 = 60
5a + 40 = 60
a = 4.
Hence, the age of oldest children = 4+16 = 20 years.

Arithmetic Progressions (AP)

Any series that has the property of the same number getting added to get the next number every time is called Arithmetic Progression (AP).

***For example*:**

1. 5,8,11,14,17,20
   Here 3 is added to get the next number.
2. 3,10,17,24,31
   Here 7 is added to get the next number.

This number getting added is referred to as the ***Common Difference*** of an A.P. and it is denoted by **'d'.** OR ***common difference*** is the number which is the difference between two consecutive terms of an A.P.

Important notations:

a. ***'a'*** is defined as the first term of an A.P.
b. ***'n'*** is the total number of terms in an A.P.
c. **'** a_n **'** denotes the nth term of an A.P.

Formulae:

1. n^{th} term of an A.P, a_n = a + (n-1)d
2. Sum of an A.P is $S_n = \frac{n}{2}$ (2a+(n-1)d)
3. $S_n = \frac{n}{2}$ (a+l) here, 'l' is the last term of an A.P.
4. Number of terms 'n' = $\frac{D}{d}$ + 1, here, D is the difference of 1st and last term of an A.P and 'd' is the common difference.

Problem 1:
A series 3,10,17,...........up to 20 terms. Find the 20th term of this sequence.

Solution:
Here, a = 3 , n = 20 and d = 10-3 = 7
Last term, a_{20} = a + (n-1)d
a_{20} = 3 + (20-1)7
a_{20} = 3+133 = 136.

Problem 2:
A series 3,10,17,.........,381. Find the number of terms.

Solution:
a = 3, d = 7 and a_n = 381.
a_n = a + (n-1)d
381 = 3+(n-1)*7
Hence, n = 55.

Or we can calculate by n = $\frac{D}{d}$ + 1, D = 381-3 = 378 and d = 7.

n = $\frac{378}{7}$ + 1 = 54+1 = 55
Hence, the total number of terms is 55.

Problem 3:
A sequence 3,7,11,15,19,23. Find the sum of this sequence.

Solution:
Addition of 1st and last term = 3+23 = 26.
Addition of 2nd and 2nd last term = 7+19 = 26
Addition of 3rd and 3rd last term = 11+15 = 26
Avg of each pair = 26/2 = 13
And 13 is an average of this A.P. and there are 6 terms in this A.P
Hence, sum of the A.P = 6 × 13 = 78.

In an A.P. when n is even: Then the average of A.P. comes from two middle terms.
eg 3,5,7,9,11,13
7 and 9 are two middle terms. So, Average = (7+9)/2 = 8.

In an A.P. when n is odd: Then the middle term, itself is an average.
eg. 2,5,8,11,14
8 is the middle term. So, average = 8.

NOTE: Sum of an A.P. = n × average.

Problem 4:

How many numbers between 100 and 200 leave a remainder 3, when divided by 7 and what are some of the numbers?

Solution:

This forman A.P of common difference = 7

101,108,115,..........,192,199

D = 199 - 101 = 98 and d = 7

Number of terms = $\frac{D}{d} + 1$

$= \frac{98}{7} + 1 = 14+1 = 15.$

Average = (101+199)/2 = 150

And, sum of these number = 15 × 150 = 2250

Problem 5:

Two A.P's are 3,10,17,24,.........up to 200 terms and 2,10,18,26,............,up to 200 terms. How many common terms exist between these A.P's?

Solution:

In 1st AP, $a_{200} = 3 + (200\text{-}1)7 = 1396$

In 2nd AP, $a_{200} = 2 + (200\text{-}1)8 = 1594$

1st common term is 10

Common terms between the AP will themselve form an AP and the common difference of this AP is LCM (d1, d2).

d1 = 7 and d2 = 8

d = LCM(7,8) = 56.

So, AP formed by common terms is;

10,66,122,178,.................

This series is 56n+10 and we have to limit this series to 1396(because last term would be less than 1396)

Let take n = 20, 56 × 20+10 = 1130 and the next terms are 1130,1186,1242,1298,1354.

Now, number of terms = $\frac{D}{d} + 1$, D = 1354-10 = 1344 and d = 56

$= \frac{1344}{56} + 1 = 25.$

Hence, the total number of common terms = 25.

Geometric Progressions (GP)

Geometric progression is a sequence in which any number after the first number is obtained by multiplying the proceeding number by a constant value, then the sequence is called geometric progression(GP).
And that constant value is called the common ratio, which is denoted by ‘r’.

***For example*:** The sequence 4,12,36,108,324……. Is a GP with common ratio 3.

Important notations:

a. ***‘a’*** is defined as the first term of a G.P.
b. ***‘n’*** is the total number of terms in a G.P.
c. ‘a_n’ denotes the nth term of a G.P.

Formulae:

1. n^{th} term of a G.P, $a_n = \mathrm{a} \times r^{(n-1)}$
2. Sum of ‘n’ terms
   a. $S_n = \frac{a(r^n-1)}{(r-1)}$, when r > 1.
   b. $S_n = \frac{a(1-r^n)}{(1-r)}$, when 0 < r < 1.
3. Sum of infinite terms: $S_\infty = \frac{a}{(1-r)}$, when 0 < r < 1.
4. The geometric mean between two quantities: GM = $\sqrt{a \times b}$

NOTE: Common ratio can be negative. eg 10,-20,40,-80,160.., here r = -2

Problem 1:
Find the sum of the series: 5,10,20,40,...........up to 18 terms.

Solution:
Here, a = 5, r = 2 and n = 18.
Now, sum of this G.P is
$S_{18} = \frac{5\times(2^{18}-1)}{(2-1)}$

Problem 2:

A ball is dropped from a height of 400 ft and bounce back half of its height and drops again and keeps bouncing and coming back to half of its height until it comes to rest. What is the total distance covered by the ball before it comes to rest?

Solution:

In this case, we find two GP,

1. G.P when the ball dropped: 400+200+100+................

   Here, a = 400 , r = 1/2. Then the sum is, $S_\infty = \frac{a}{(r-1)}$

   $S_\infty = \frac{400}{(1-1/2)} = 800.$
2. G.P when the ball bounced back: 200,100,50,................

   Here, a = 200 , r = 1/2. Then the sum is, $S_\infty = \frac{a}{(r-1)}$

   $S_\infty = \frac{200}{(1-1/2)} = 400.$

Hence, total distance covered by the ball = 800+400 = 1200 ft.

Eliminate the formula ($S_\infty = \frac{a}{(r-1)}$):

We have,

1st term	**Common Ratio**	S_∞
a	1/2	2a
a	1/3	3a/2
a	2/5	5a/3

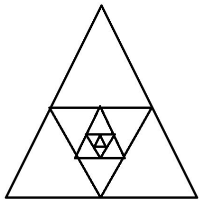
So, we can conclude that denominator common ratio becomes a multiplier for 'a' and difference between denominator and numerator of the common ratio becomes the denominator of the sum. So, without using the formula for infinite terms sum, we can directly calculate the sum of infinite terms.

Problem 3;

Midpoints of a triangle join to form another triangle, whose midpoints again join to form another triangle and process are repeating infinite times to form infinite triangles by continuously joining midpoints. P is the perimeter of the 1st triangle and A is the area of the 1st triangle. What is the sum of all perimeters and all areas?

Solution:

By joining midpoints of a triangle we will get the following figure.



By the midpoints theorem, the perimeter of the next triangle becomes ½ of the first triangle. So, we have G.P

P, P/2, P/4, P/8, P/16,.................

Here, a = P and r = 1/2

So, S_∞ = 2P

In case of area, all the triangles are similar to each other, So, using the similarity concept, if the perimeter becomes half then lengths also becomes half and area will become 1/4th. So, we have G.P

A, A/4, A/16, …………...

Here, a = A and r = 1/4

So, S_∞ = 4P/3

Hence, the sum of all perimeters = 2P and the sum of all areas = 4A/3.

Remainder Theorem

The logic of remainder theorem:

Consider a question, 17×23

We want to find the remainder of this expression when divided by 7.

$\frac{17\times23}{7}$ = 391/7 = 6/7.

Hence, the remainder of this expression is 6.

This is the school time approach and this is time-consuming. To solve this question we have to use the Remainder theorem.

We can write this as:

$17 \times 23 = (14+3) \times (21+2)$

Which can be expanded:

$(14+3) \times (21+2) = 14 \times 21 + 14 \times 2 + 3 \times 21 + 3 \times 2$

When you divide this expression by 7, you will realise that remainder depends on last terms.

Thus, $\frac{14\times21 + 14\times2 + 3\times21 + 3\times2}{7} = \frac{6}{7}$

Hence, the remainder of this expression is 6.

Remainder theorem transformation:

The remainder theorem transformation denote it by the sign $\xrightarrow{R}$

If we use the remainder theorem transformation, then we can take the remainder of individual number 17 and 23 (the above question data) when divided by 7. When 17 divided by 7 gives remainder 3 and when 23 divided by 7, gives remainder 2 and then by multiplying these remainder you will get the remainder of the original expression.

$$\frac{17\times23}{7} \xrightarrow{R} \frac{3\times2}{7} \xrightarrow{R} \frac{6}{7}$$

Hence, the remainder is 6.

Problem 1:

Find the remainder of 1421 × 1423 × 1425, when divided by 12.

Solution:

The individual remainder are,

		Remainder
1.	$\frac{1421}{12}$	5
2.	$\frac{1423}{12}$	7
3.	$\frac{1425}{12}$	9

$$\frac{1421\times1423\times1425}{7} \xrightarrow{R} \frac{5\times7\times9}{7} = \frac{35\times9}{7} \xrightarrow{R} \qquad \xrightarrow{R} \frac{11\times9}{7} \quad \frac{3}{7}$$

Hence, the remainder of this expression is 3.

Using Negative Remainder:

Consider the following question:

Find the remainder when: 53 × 54 divided by 55.

$$\frac{53\times54}{55} \xrightarrow{R} \frac{-2\times-1}{55} = \frac{2}{55}$$

Hence, the remainder is 2.

Some you might find a question which does not allow simple calculation and that will involve long calculations. Hence, the principle is that you should use negative remainders wherever you can.

When Answer Comes Negative:

Find the remainder when: $52 \times 53 \times 54$ divided by 55.

$$\frac{52\times53\times54}{55} \xrightarrow{R} \frac{-3\times-2\times-1}{55} = \frac{-6}{55}$$

But we know that remainder can't be negative i.e. -6. So, the remainder of this expression will be 55-(-6) = 49.

Use Of Cutting In Remainder Theorem problem:

Find the remainder when: $42 \times 31 \times 17$ divided by 12.

When we go with remainder theorem,

$\frac{42\times31\times17}{12} \xrightarrow{R} \frac{6\times7\times5}{12} = \frac{210}{12}$, hence remainder is 6.

Instead of doing this, if we write this expression as:

$$\frac{42\times31\times17}{12} = \frac{7\times31\times17}{2} \xrightarrow{R} \frac{1\times1\times1}{2} = \frac{1}{2}$$

Hence, the remainder, in this case, is 1. But we can see the answer is not the same.

We have transformed $\frac{42}{12}$ into $\frac{7}{2}$ by dividing the numerator and the denominator by 6. The result is that the original remainder 6 is also divided by 6 giving us 1 as the remainder. Thus to get the original remainder 1 is multiplied by 6.

Dealing With Large Power:

a. +1 remainder rule:

For example $\frac{17}{8} \xrightarrow{R} +1$

$$\frac{17\times17}{8} \xrightarrow{R} \frac{1\times1}{8} \xrightarrow{R} +1$$

$$\frac{17\times17\times17}{8} \xrightarrow{R} \frac{1\times1\times1}{8} \xrightarrow{R} +1$$

Thus, $\frac{17^{5555}}{8} \xrightarrow{R} \frac{1^{5555}}{8} \xrightarrow{R} +1$

So, we can say that there is no use of power.

You will not always get the situation of remainder 1.

For example $\frac{10^{800}}{7}$ in this situation +1 rule has to be used in an oblique way to get the point where you can use +1 remainder rule.

Step 1: $\frac{10^{800}}{7} \xrightarrow{R} \frac{10\times10\times10\times10.......}{7} \xrightarrow{R} \frac{3\times3\times3\times3.......}{7}$

$\frac{10^{800}}{7} \xrightarrow{R} \frac{3^{800}}{7}$

Step 2: Try to find out the power of 3 when divided by 7 gives the remainder 1.

		Remainder
1.	$\frac{3^1}{7}$	3
2.	$\frac{3^2}{7}$	2
3.	$\frac{3^3}{7}$	6
4.	$\frac{3^4}{7}$	4
5.	$\frac{3^5}{7}$	5
6.	$\frac{3^6}{7}$	1

Thus, $\frac{3^{6^{133}}\times3^2}{7}$ [because 800/6, gives quotient 133 and remainder 2]

$\frac{3^{6^{133}}\times3^2}{7} \xrightarrow{R} \frac{1^{133}\times3^2}{7} \xrightarrow{R} \frac{3^2}{7} \xrightarrow{R} +2$

Hence, the remainder for this expression is +2.

b. +1 remainder rule:

1. $\frac{16^1}{17} \xrightarrow{R} -1$
2. $\frac{16^2}{17} \xrightarrow{R} \frac{-1\times-1}{8} \xrightarrow{R} +1$
3. $\frac{16^3}{17} \xrightarrow{R} \frac{-1\times-1\times-1}{8} \xrightarrow{R} -1$
4. $\frac{16^4}{17} \xrightarrow{R} \frac{-1\times-1\times-1\times-1}{8} \xrightarrow{R} +1$

This situation where you have $\frac{a^{power}}{a}$ The remainder in such a situation depends upon the value of power.

a. **If power is odd:** the remainder would be -1, then the original remainder would be a+1-1 = a
b. **If power is even:** the remainder would be +1.

Find Power Which Leaves Remainder 1:

$\frac{A^{P-1}}{P}$ always gives remainder 1, if P is a prime number and A should not be multiple of P.

Problem 1:

$\frac{3^P}{7}$ what power of 3 gives remainder 1?

Solution:

$\frac{3^P}{7}$ compare with standard form $\frac{A^{P-1}}{P}$

Thus, P = 7 - 1 = 6.

Hence, 3^6 gives a remainder 1 when divided by 7.

Unit Digits

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3 kinds of cyclicity of unit digits we need to understand.

1. Cyclicity of 1 value in unit digit:

a. Number ending in 1 and raised to any power, the unit digit is always 1.
e.g: $(91)^{12}$ the unit digit will be 1.

b. Number ending in 5,6 and 0 and raised to any power, the unit digit remains the same.
e.g: $(25)^{12}$, the unit digit will be 5.
$(26)^{12}$, the unit digit will be 6.
$(20)^{12}$, the unit digit will be 0.

2. Cyclicity of 2 value in unit digit:

a. **Number ending in 4**

$54 \rightarrow$ unit digit is 4

$54 \times 54 \rightarrow$ unit digit is 6

$54 \times 54 \times 54 \rightarrow$ unit digit is 4

$(4)^{odd} \rightarrow$ unit digit remains the same.
$(4)^{even} \rightarrow$ unit digit is 6.

b. Number ending in 9

$(9)^{odd} \rightarrow$ unit digit is 1.
$(9)^{even} \rightarrow$ unit digit remains the same.

3. Cyclicity of 4 values in unit digit:

a. Number ending in 2

Number	**Unit digit**
2^1	2
2^2	4
2^3	8
2^4	6
2^5	2
2^6	4 ……….. And so on

We conclude that 2^{4n+1} gives unit digit 2, 2^{4n+2} gives unit digit 4, 2^{4n+3} gives unit digit 6, 2^{4n+4} gives unit digit 8.
Hence, 2^{power} and power $\div$ 4,then check for 4n+1,4n+2,4n+3 and 4n+1.

b. Number ending in 3

Number	**Unit digit**
3^1	3
3^2	9
3^3	7
3^4	1
3^5	3
3^6	9 ……….. And so on

We conclude that 3^{4n+1} gives unit digit 3, 3^{4n+2} gives unit digit 9, 3^{4n+3} gives unit digit 7, 3^{4n+4} gives unit digit 1.
Hence, 3^{power} and power $\div$ 4,then check for 4n+1,4n+2,4n+3 and 4n+1.

c. **Number ending in 7**

Number	**Unit digit**
7^1	7
7^2	9
7^3	3
7^4	1
7^5	7
7^6	9 ……….. And so on

We conclude that 7^{4n+1} gives unit digit 7, 7^{4n+2} gives unit digit 9, 7^{4n+3} gives unit digit 3, 7^{4n+4} gives unit digit 1.
Hence, 7^{power} and power $\div$ 4,then check for 4n+1,4n+2,4n+3 and 4n+1.

d. **Number ending in 8**

Number	**Unit digit**
8^1	8
8^2	4
8^3	2
8^4	6
8^5	8
8^6	4 ……….. And so on

We conclude that 8^{4n+1} gives unit digit 8, 8^{4n+2} gives unit digit 4, 8^{4n+3} gives unit digit 2, 8^{4n+4} gives unit digit 6.
Hence, 8^{power} and power $\div$ 4,then check for 4n+1,4n+2,4n+3 and 4n+1.

Summary: Unit digit

	If the value of the power is			
Number ending in	**4n + 1**	**4n + 2**	**4n + 3**	**4n**
1	1	1	1	1
2	2	4	8	6
3	3	9	7	1
4	4	6	4	6
5	5	5	5	5
6	6	6	6	6
7	7	9	3	1

8	8	4	2	6
9	9	1	9	1

Some Question For Practice

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1. Find the value of the expression:
   1 – 6 + 2 – 7 + 3 – 8 +……… to 100 terms
   Ans: -250.
2. Find a_{10} and S_{10} for the following series:
   1, 8, 15,22,29…
   Ans : 64,325.
3. Find the sum to 200 terms of the series
   1 + 4 + 6 + 5 + 11 + 6 +…
   Ans : 30200.
4. How many terms are there in GP 5, 20, 80, 320,… 20480?
   Ans: 7.
5. If the fifth term of a GP is 81 and the first term is 16, what will be the 4th term of the GP?
   Ans: 54.
6. Find the remainder when 73 × 75 × 78 × 57 × 197 is divided by 34.
   Ans: 22.
7. Find the remainder when 4177 is divided by 7
   Ans: 6.
8. Find the remainder when 73 + 75 + 78 + 57 + 197 is divided by 34.
   Ans: 4.
9. Find the Units digit in 67 × 37 × 43 × 91 × 42 × 33 × 42.
   Ans: 4.
10. Find the Units digit in 67 × 35 × 45 × 91 × 42 × 33 × 81.
    Ans : 0.