Lecture 19

Wednesday, February 26, 2020 1:08 PM

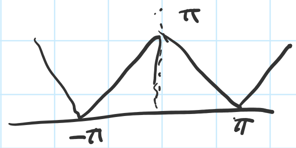
$$f(t+T) = f(t) \Rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_0 \cos(n\omega t) + b_n \sin(n\omega t)\right)$$

$$\omega = \frac{2\pi}{T}$$

Fourier Series of f

$$\left.\begin{aligned} a_n &= \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\omega t)\,dt \quad n = 0, 1, 2, 3, \dots \\ b_n &= \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\omega t)\,dt \quad n = 1, 2, 3, \dots \end{aligned}\right\} \text{Fourier coeff.}$$

Example: Triangle function



$$f(t) = \pi - |t| \qquad T = 2\pi$$

$$\omega = \frac{2\pi}{T} = 1$$

$$b_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\pi \sin nt}_{0}\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-t)\sin nt\,dt$$

$$- \frac{1}{\pi}\int_{0}^{\pi} (t)\sin nt\,dt$$

now $\displaystyle\int_{-\pi}^{0} -t\sin nt\,dt = -\frac{1}{n^2}\left[\sin nt - nt\cos nt\right]_{-\pi}^{0} = \begin{matrix} + & n \text{ even} \\ - & n \text{ odd} \end{matrix}\ \frac{\pi}{n}$

$$\int_{0}^{\pi} t\sin nt\,dt = \frac{1}{n^2}\left[\sin nt - nt\cos nt\right]_{0}^{\pi} = \begin{matrix} - & \text{even} \\ + & \text{odd} \end{matrix}\ \frac{\pi}{n}$$

$$\therefore b_n = 0 \mp \frac{1}{\pi}\frac{\pi}{n} \pm \frac{1}{\pi}\frac{\pi}{n} = 0$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = 2\,\frac{1}{\pi}\int_{0}^{\pi} (\pi - t)\cos nt\,dt$$

$$= \begin{cases} \pi & \text{if } n = 0 \\ 0 & n \neq 0,\ n \text{ is even} \\ \frac{4}{\pi n^2} & n \neq 0,\ n \text{ is odd} \end{cases}$$

$$\therefore f(t) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2}\cos((2k-1)t)$$

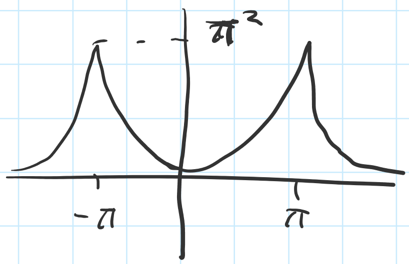
$$\therefore f(t) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2} \cos((2k-1)t)$$

$$f(0) = \pi = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{4(2k-1)^2} \cdot 1$$

$$\text{or } \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example: $f(x) = x^2$

$-\pi \le x \le \pi$



$T = 2\pi$

$w = 1$

$\Rightarrow$ From symmetry (even fcn) $\Rightarrow b_n = 0$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} F(t) \cos(nwt) = \frac{2}{2\pi} \int_{-\pi}^{\pi} t^2 \cos nt \, dt$$

$$n = 0 : \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt = \left[\frac{t^3}{3\pi}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$n \ne 0 : \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 \cos nt \, dt$$

$$\Rightarrow u^k \cos u \, du = u^k \sin u - k \int u^{k-1} \sin u \, du$$

$$\int u \sin u \, du = \sin u - u \cos u + C$$

$$a_n = \frac{1}{\pi n^3} \Big[(nt)^2 \underbrace{\sin nt}_{0} \Big]_{-\pi}^{\pi} = \frac{2}{\pi n^2} \int (nt) \sin(nt) \, d(nt)$$

$$= -\frac{2}{\pi n^3} \Big(\underbrace{\sin(nt)}_{0} - nt \cos(nt) \Big]_{-\pi}^{\pi}$$

$$= \begin{cases} -\frac{2}{\pi n^3} \left[-n\pi(-1) + n(-\pi)(-1) \right] & n \text{ odd} \\ -\frac{2}{\pi n^3} \left[-n\pi(1) + n(-\pi)(1) \right] & n \text{ even} \end{cases}$$

$$= \begin{cases} -\frac{4}{n^2} & n \text{ odd} \end{cases}$$

$$= \begin{cases} -\frac{4}{n^2} & n \text{ odd} \\ +\frac{4}{n^2} & n \text{ even} \end{cases}$$

$$\therefore f = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nt$$

All Continuous Functions are Integrable

1. General and Less-General Definition of the Integral

$$\int_a^b f(x)\,dx \equiv \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \quad (1)$$

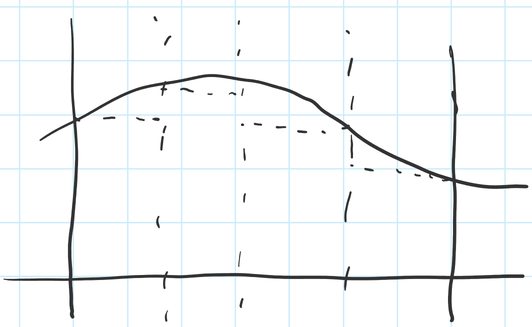
regular partition: $\Delta x = \frac{(b-a)}{n}$

right-hand end point: $x_i^* = x_i$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{(b-a)}{n} \sum_{i=1}^{n} f(x_i) \quad (2)$$

2. Proof that Continuity Guarantees Integrability based on a Restricted Def'n of the Integral

let: $$\int_a^b f(x)\,dx \equiv \lim_{n\to\infty} \frac{(b-a)}{2^n} \sum_{i=1}^{2^n} f_i^{min} \quad (3)$$



lower sum $= L_{2^n} = \frac{b-a}{2^n} \sum_{i=1}^{2^n} f_i^{min}$ (4)

$$\therefore \int_a^b f(x)\,dx = \lim_{n\to\infty} L_{2^n} \quad (5)$$

Monotonic Sequence Theorem: Every bounded monotonic sequence is convergent.

convergent.

$\therefore$ show a) L_{2^n} is bounded (above)

b) L_{2^n} is monotonic increasing

upper sum $= U_{2^n} = \frac{b-a}{2^n} \sum_{i=1}^{2^n} f_i^{max}$ (6)

Proof of (a): Define M = maximum value of $f(x)$ on $[a,b]$

m = minimum value of $f(x)$ on $[a,b]$

EVT

$\therefore f_i^{min} \geq m$ for all i

$f_i^{max} \leq M$ for all i

Lecture 28

Friday, February 28, 2020 1:12 PM

$$\int_a^b f(x)\,dx \equiv \lim_{n\to\infty} \frac{(b-a)}{2^n}\sum_{i=1}^{2^n} f_i^{min} \quad (3)$$

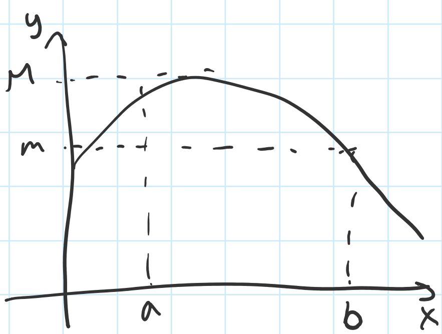
$$L_{2^n} \equiv \frac{(b-a)}{2^n}\sum_{i=1}^{2^n} f_i^{min} \quad (4) \qquad U_{2^n} = \frac{(b-a)}{2^n}\sum_{i=1}^{2^n} f_i^{max} \quad (6)$$

$$m \le f_i^{min} \le f_i^{max} \le M \quad \text{for all } i$$

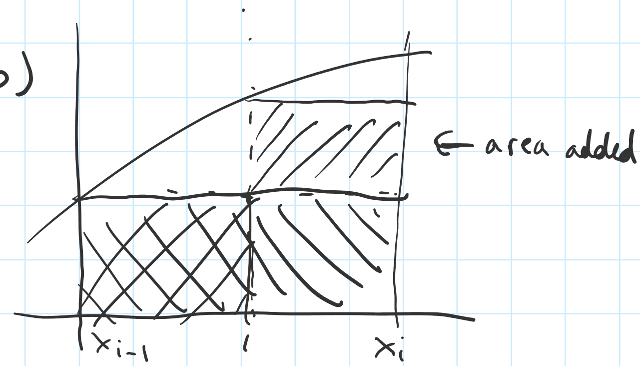
note: $\sum_{i=1}^{2^n} m = m \cdot 2^n$

$$m(b-a) = \frac{(b-a)}{2^n}\sum_{i=1}^{2^n} m \le \frac{(b-a)}{2^n}\sum_{i=1}^{2^n} f_i^{min} \le \frac{(b-a)}{2^n}\sum_{i=1}^{2^n} f_i^{max} \le \frac{(b-a)}{2^n}\sum_{i=1}^{2^n} M = M(b-a)$$

or $m(b-a) \le L_{2^n} \le U_{2^n} \le M(b-a)$



Proof of (b)



$\Rightarrow$ show $L_{2^{n+1}} \ge L_{2^n}$

$\Rightarrow$ $f_{i\,left}^{min} \ge f_i^{min} \qquad f_{i\,right}^{min} \ge f_i^{min}$

$$\Rightarrow \frac{1}{2}\left(f_{i\,left}^{min} + f_{i\,right}^{min}\right) \ge f_i^{min}$$

$$\left(\frac{b-a}{2^n}\right)\sum_{i=1}^{2^n}\left[\frac{1}{2}\left(f_{i\,left}^{min} + f_{i\,right}^{min}\right)\right] \ge \frac{(b-a)}{2^n}\sum_{i=1}^{2^n} f_i^{min}$$

$$\frac{b-a}{2\cdot 2^n} = \frac{b-a}{2^{n+1}} \Rightarrow LHS = \left(\frac{b-a}{2^{n+1}}\right)\sum_{j=1}^{2^{n+1}} f_j^{min} = L_{2^{n+1}}$$

$$\therefore L_{2^{n+1}} \ge L_{2^n}$$

$\therefore L_{2^{n+1}} \geq L_{2^n}$

$$f(x) = \begin{cases} +1 & x \text{ rational} \\ -1 & x \text{ irrational} \end{cases} \quad (7) \quad \Rightarrow f_i^{min} = -1$$

$$\therefore \int_a^b f(x)\,dx = -1(b-a)$$

from (3)

$$\lim_{n\to\infty} L_{2^n} = -1(b-a), \quad \lim_{n\to\infty} U_{2^n} = 1\cdot(b-a)$$

def'n: span of $f(x)$ in $\Delta x_i = f_i^{max} - f_i^{min}$

3. Proof that Continuity Guarantees Integrability Based on a more General Def'n of the Integral

$$R_{2^n} = \frac{(b-a)}{2^n}\sum_{i=1}^{2^n} f(x_i^*) \Rightarrow \int_a^b f(x)\,dx = \lim_{n\to\infty} R_{2^n} \quad (9)$$

Pinching Thm: 1) $L_{2^n} \leq R_{2^n} \leq U_{2^n}$ (10)

2) $\lim_{n\to\infty} L_{2^n} = \lim_{n\to\infty} U_{2^n}$ (11)

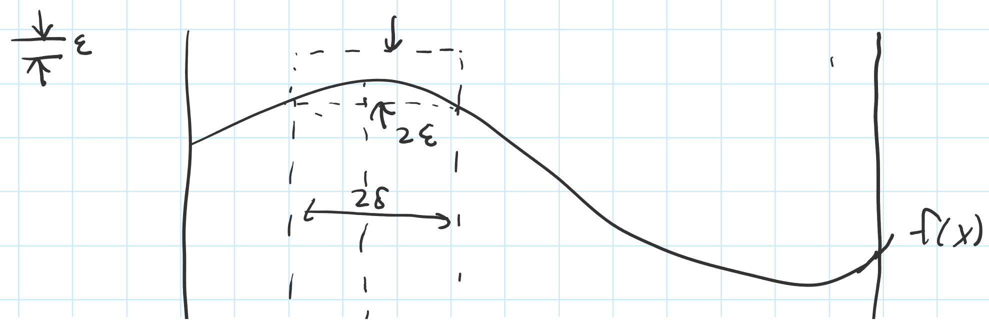
$$\Rightarrow f_i^{min} \leq f(x_i^*) \leq f_i^{max} \text{ for all } i$$

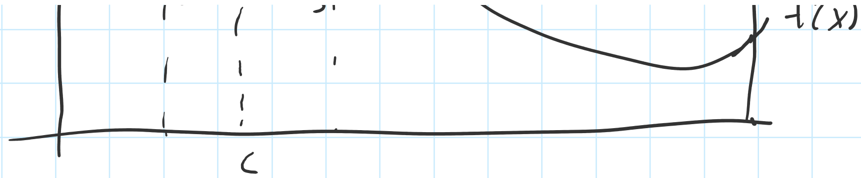
$$\therefore L_{2^n} \leq R_{2^n} \leq U_{2^n}$$

The Concept of Uniform Continuity

$f(x)$ is continuous: there exists $\delta > 0$ s.t. for all $|x - x_0| < \delta$
$|f(x) - f(x_0)| < \varepsilon$ at each point $x_0 \in [a, b]$

Def'n: A function $f(x)$ is uniformly continuous on $[a, b]$ if for each ε that is imposed, a single value $\delta > 0$ can be found s.t. for all $|x_1 - x_2| < \delta$ the values of $f(x)$ satisfy $|f(x_1) - f(x_2)| < \varepsilon$ for all $x_1, x_2 \in [a, b]$





<u>Def'n</u>: The <u>span</u> of $f(x)$ on any interval equals $M-m$, where M/m are the max/min values of $f(x)$ on that interval: span $\equiv M-m$

The small span Theorem: Given $f(x)$ continuous on $[a,b]$: for every $\varepsilon > 0$ that might be imposed, it is possible to find some partition of $[a,b]$, s.t. the span of every subinterval is $< \varepsilon$.

$\Rightarrow$ prove $\lim_{n\to\infty} \left(U_{2^n} - L_{2^n} \right) = 0 \quad (12)$

show $\left| (U_{2^n} - L_{2^n}) - 0 \right| < \varepsilon$ for $n > N$

or $\left| U_{2^n} - L_{2^n} \right| < \varepsilon$

small span theorem: $\left| f_i^{max} - f_i^{min} \right| < \varepsilon^*$ for every subinterval i, $n \geq N^*$

choose $\varepsilon^* = \dfrac{\varepsilon}{b-a}$

$\therefore$ for $n > N^*$ $\quad \left| f_i^{max} - f_i^{min} \right| < \dfrac{\varepsilon}{b-a}$

$$\Rightarrow U_{2^n} - L_{2^n} = \frac{(b-a)}{2^n} \sum_{i=1}^{2^n} \left| f_i^{max} - f_i^{min} \right| \leq \frac{(b-a)}{2^n} \sum_{i=1}^{2^n} \frac{\varepsilon}{b-a}$$

$$= \frac{(b-a)}{2^n} \cdot \frac{\varepsilon}{b-a} \cdot 2^n = \varepsilon$$

$$\Rightarrow U_{2^n} - L_{2^n} < \varepsilon$$

$\therefore \lim_{n\to\infty} U_{2^n} = \lim_{n\to\infty} L_{2^n} \qquad \therefore \lim_{n\to\infty} R_{2^n}$ exists.

Lecture 21

Monday, March 2, 2020 4:15 PM

4. Proof that Continuity Guarantees Integrability Based on the Def'n of the Integral

$$R_p = \sum_{i=1}^{n} f(x_i^*)\Delta x_i \qquad (16)$$

prove: $\lim_{\|P\|\to 0} R_p = I$

we have $I = \lim_{n\to\infty} L_{2^n} = \lim_{n\to\infty} R_{2^n} = \lim_{n\to\infty} U_{2^n}$

Given $\varepsilon > 0$, find $\delta > 0$ s.t. $\|P\| < \delta$, $|I - R_p| < \varepsilon$

$$\left.\begin{aligned} L_p &\equiv \sum_{i=1}^{n} f_i^{min}\Delta x_i \\ U_p &\equiv \sum_{i=1}^{n} f_i^{max}\Delta x_i \end{aligned}\right\} L_p \le R_p \le U_p$$

<u>Def'n</u>: Partition P' is called a refinement partition P and is obtained by adding more points of subdivision to P.

<u>Lemma A</u>: Given P' is a refinement of P. Then $L_p \le L_{p'} \le U_{p'} \le U_p$

Proof of $L_p \le L_{p'}$

$\downarrow$

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$\boxed{12}$ Vectors and the Geometry of Spaces

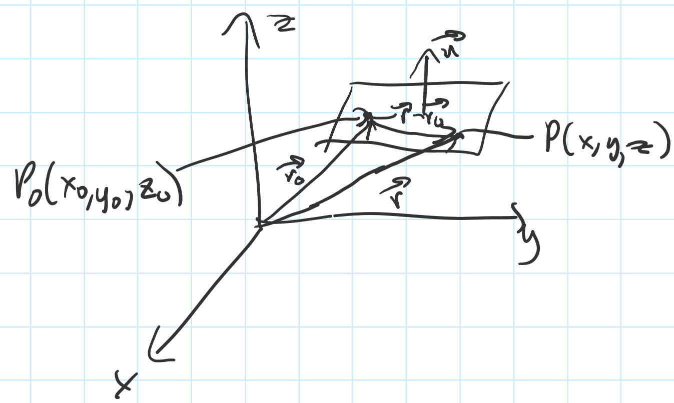
<u>12.5 Equations of Lines and Planes</u>

Vector: $\vec{a} = (1, 1, 1) = 1\hat{i} + 1\hat{j} + 1\hat{k}$

$\vec{r}$ = radius vector (starts at $(0, 0, 0)$)

Planes: $ax + by + cz = d$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = d$



$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

given $\vec{n} = (n_1, n_2, n_3)$

$\vec{r}_0 = (x_0, y_0, z_0)$

$\vec{r} = (x, y, z)$

$\therefore n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$