Reasoning about bounded arithmetic within Lean 4

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- **Aim:** formalize theorems in the weakest adequate system.

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- define a sorting function
- prove standard graph theorems.

The goals of this presentation

1. Why formalize arithmetic?

These theories correspond nicely to complexity classes.

We want to formalize theorems of the form $I\Delta_0 \vdash \phi(x, y)$ to explore computational content of the proofs.

2. Demonstrate that formalizing it is possible.

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Technicality: require the $=$ symbol be the *actual* equality on underlying objects. Will skip equality axioms later.

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- if A is a formula and x is a variable, then $\forall x A$, $\exists x A$ are formulas

The deduction system: what is "$\vdash$" in "$I\Delta_0 \vdash \phi(x, y)$"?

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Syntactic sugar: $A \to B := \neg A \vee B$.

The axioms: what is $I\Delta_0$? 1-BASIC axioms

Table 1: 1-BASIC axioms

Axiom	Statement
B1.	$x + 1 \neq 0$
B2.	$x + 1 = y + 1 \implies x = y$
B3.	$x + 0 = x$
B4.	$x + (y + 1) = (x + y) + 1$
B5.	$x \cdot 0 = 0$
B6.	$x \cdot (y + 1) = (x \cdot y) + x$
B7.	$(x \leq y \wedge y \leq x) \implies x = y$
B8.	$x \leq x + y$
C.	$0 + 1 = 1$

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Definition (Induction Scheme).
If Φ is a set of formulas, then Φ-IND axioms are the formulas

$$\left(\varphi(0)\ \land\ \forall x\,(\varphi(x) \to \varphi(x+1))\right)\ \to\ \forall z\,\varphi(z),$$

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By carefully controlling Φ, we obtain **interesting** theories.

Complexity of formulas

Definition (Bounded Quantifiers).

$$\exists x \leq t\, A \;:=\; \exists x\, (x \leq t \wedge A)$$
$$\forall x \leq t\, A \;:=\; \forall x\, (x \leq t \to A)$$

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A formula is Δ_0 (**bounded**) if every quantifier in it is bounded.
A formula is Σ_1 if it is of the form $\exists x_1, \ldots, \exists x_k \phi$ and ϕ is bounded.

Case 2: $\Phi = \Delta_0$

1-BASIC axioms together with induction for bounded formulas only give us a well-studied system called $I\Delta_0$.

The following formulas (and their universal closures) are theorems of $I\Delta_0$ [@Cook_Nguyen_2010]:

- $x + y = y + x$ *(commutativity of* $+$*)*

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- $x \leq x$

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- $\forall x \, \forall y \, (0 < x \to \exists q \, \exists r \, (r < x \land y = x \cdot q + r))$ *(division theorem)*

Defining new functions in $I\Delta_0$

We say that a function $f(\vec{x})$ is *provably total* in $I\Delta_0$ if there is a formula $\phi(\vec{x}, y)$ in Σ_1 (i.e. of the form $\exists \ldots \exists \psi$ for ψ bounded) such that:

$$I\Delta_0 \vdash \forall x \exists ! y \phi(\vec{x}, y)$$

and that

$$y = f(\vec{x}) \iff \phi(\vec{x}, y)$$

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NOTE: the computational content of $I\Delta_0$ is well-studied.
NOTE: $I\Delta_0$ doesn't align well with practical computer science.

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VL	FLOGSPACE	$\cdots$
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  - `num` (representing unary numbers)

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V^1	FPTIME	$\cdots$

If time allows, we will get back to details of the definition.

Theories corresponding to complexity classes

The idea is similar.

- We still operate in first-order, classical logic.
- Instead of one sort, we have two:
  - `num` (representing unary numbers)
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Theory	Characterizes	Examples
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  - `str` (representing binary strings)
- Instead of induction we have finite set comprehension (finite sets $\equiv$ binary strings)

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How would you even formalize this field?

Recall: Axiom schema of induction

Definition (Induction Scheme).
If Φ is a set of formulas, then Φ-IND axioms are the formulas

$$\left(\varphi(0) \;\wedge\; \forall x\,(\varphi(x) \to \varphi(x+1))\right) \;\to\; \forall z\,\varphi(z),$$

where $\varphi \in \Phi$. $\varphi(x)$ may have free variables other than x.

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The theory having axioms **B1**-**B8**, together with induction for arbitrary formulas from our vocabulary, *is* the **Peano** arithmetic (a very strong system).

By carefully controling Φ, we obtain **interesting** theories.

Approaches: obvious and wrong

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Sometimes we can "switch-off" some axioms of proof assistants, but this doesn't go far enough. E.g. we can (and shouldn't) "switch-on" "-impredicative-set" in Rocq.

Introducing new axioms to meta-mathematical tools

- Introducing new axioms to Rocq and Lean is not considered a good practice
- We have tools designed specifically to define new theories from scratch
- Metamath, Isabelle (not Isabelle/HOL)
- In Isabelle/Pure, we can easily define our BASIC axioms (without induction). Example:
- Metamath seems to be a strictly less useful tool than Isabelle for our problem

Isabelle/Pure

```
(* ===== Axioms of 2-BASIC ===== *)
axiomatization where
  (* Basic arithmetic axioms (B1-B8) *)
  (* B1. x + 1 != 0 *)
  B1: "~((x + 1) = 0)" and
  (* B2. x + 1 = y + 1 implies x = y *)
  B2: "((x + 1) = (y +1)) --> (x = y)" and
```

Isabelle/Pure: working proof

```
(* Exercise 5.1.a: not x < 0 *)
lemma exercise_5_1_a: "~ (x < 0)"
proof
  assume "x < 0"
  then have "x <= 0" and "x ~= 0" by simp_all
  from B9 have "0 <= x" .
  then have "((x <= 0) \<and> (0 <= x)) --> (x = 0)"
    using B7 by blast
  then have "x = 0" using `x <= 0` `0 <= x` by blast
  with `x ~= 0` show False by contradiction
qed
```

How to express Δ_0 formulas?

- When trying to define our Δ_0-induction axiom scheme, we hit a wall: every expression of the form $\forall x, \exists z, \phi(x, z)$ is just a (Isabelle-equivalent of) `Prop`.
- The same is true for Rocq and Lean. If we "shallowly" define a formula in these systems, we get an object of type `Prop` and can do strictly nothing with it.
- We need to define from scratch what it means to be a Formula.
- We have no advantage of using Isabelle for this as compared to Rocq and Lean.

Defining Formulas: Hilbert

```
inductive Formula where
| var : Name -> Formula
| imp : Formula -> Formula -> Formula
deriving Repr, DecidableEq

notation:60 a " ==> " b => Formula.imp a b

def A1 (phi psi : Formula)
  := phi ==> psi ==> phi
def A2 (phi psi ksi : Formula)
  := (phi ==> psi ==> ksi) ==> (phi ==> psi) ==> phi ==> ksi
```

Need for macros / metaprogramming

Defining formulas as

```
Formula.imp (Formula.leq a b) (Formula.binOp add c d)
```

is not going to work.

A realistic formula we will want to embed:

```
x ≠ 0 -> (∃i y, (y <= x ∧ (y + 1) = x))
```

Syntactical overlay over Rocq or Lean

- A good idea is to try enabling user to enter a special “proof mode”, in which we as programmers have full control over what is allowed and what is not.
- We will define what it means to be a Formula and (perhaps) what it means to be Derivable.
- As working with objects defined this way will be a nightmare, we need to apply metaprogramming.
- Rocq syntactical metaprogramming is spread across multiple tools: `Notation` command, MetaRocq, OCaml plugins (full power).
- For tactics, we have Ltac, Ltac2, Mtac2, and also OCaml plugins.
- Lean 4 metaprogramming is done in Lean 4.

Lean 4 metaprogramming world

- Parser reads source text and produces objects of type `Syntax`.
- We can define custom macros `Syntax -> Syntax`.
- Elaborator reads `Syntax` and produces a single typed `Expr` object.
- We can define custom elaborator rules `Syntax -> Expr`.
- The internal state of all these components is exposed through monads
- A function `f : a -> MetaM b` can introduce a new variable.
- A function `f : a -> TacticM b` can change the goal of a theorem being proved.

Defining Derivability: Hilbert

```
inductive Derivable : (List Formula) -> Formula -> Prop where
| assumption {Γ} {φ} :
  (φ ∈ Γ)
  -> Derivable Γ φ
| axK {Γ} {phi psi} :
  Derivable Γ $ K phi psi
| axS {Γ} {phi psi ksi} :
  Derivable Γ $ S phi psi ksi
| mult_mp {Γ1 Γ2} {phi psi} :
  Derivable Γ2 (phi ==> psi)
  -> Derivable Γ1 phi
  -> Derivable (Γ1 ++ Γ2) psi
```

Lean 4: Macros for Hilbert-style proof mode

```
-- Syntax category for Hilbert proof steps
declare_syntax_cat hilbert_tactic
syntax "have" ident ":" logic_expr "by" "assumption"
  : hilbert_tactic
syntax "have" ident ":=" "axK" logic_expr "," logic_expr
  : hilbert_tactic
syntax "have" ident ":=" "axS" ...
syntax "have" ident ":=" "mult_mp" ident ident : hilbert_tactic
syntax "exact " ident : hilbert_tactic

syntax "begin_hilbert " (hilbert_tactic)* : tactic
```

Lean 4: Demo of Hilbert-style proof mode

After we define our custom syntax `[Logic| ..]`, transforming syntax of logical formula into our previously defined `Formula` object, we can already get a Hilbert-style proof mode. If the proof begins with “by begin_hilbert”, there is no way to cheat.

```
example : Derivable [] [Logic| φ -> φ] := by
  begin_hilbert
    have a := axS φ, φ -> φ, φ
    have b := axK φ, φ -> φ
    have c := mult_mp a b
    have d := axK φ, φ
    have e := mult_mp c d
    exact e
```

How far will we go this way?

- To make our Hilbert-proof mode usable, we would need to implement a few tactics, etc.

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- To make our Hilbert-proof mode usable, we would need to implement a few tactics, etc.
- But: in reality we don't prove Hilbert-style.
- To scale this approach to a proper Gentzen-style deduction system is orders of magnitude more work

Coq Library for First-Order Logic

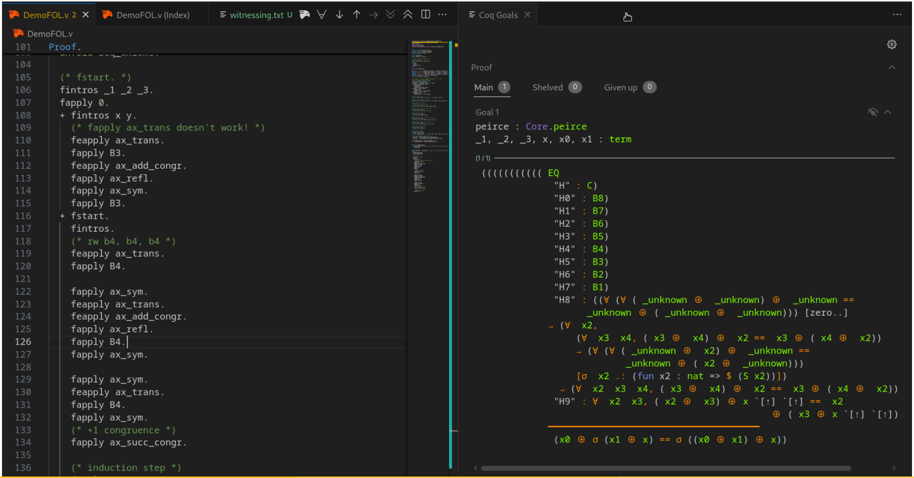
Luckily, someone has already done it.

```
Lemma add_assoc :
  BEq_axioms' ⊢ << ∀' x y z,
    (((x ⊕ y) ⊕ z) == (x ⊕ (y ⊕ z))).
Proof.
  unfold BEq_axioms'.
  unfold Beq_axioms.

  (* fstart. *)
  fintros _1 _2 _3.
  fapply 0.
  + fintros x y.
    (* fapply ax_trans doesn't work! *)
    feapply ax_trans.
```

More control = more responsibility

This really is a deduction system written from scratch. You have all the control over the axioms, but also there will be no more features than you implement on your own.



Circle back

If we can do proofs so easily without axiom schemes, we really would like to not lose this functionality.

```
class NatModel (num : Type)
  extends Zero num, One num, Add num, LE num where
  B3: ∀ x : num, x + 0 = x
  B8: ∀ x y : num, x <= x + y

theorem le_refl (M : Type) (h : NatModel M)
  : ∀ a : M, a <= a :=
by
  intro x
  conv => right; rw [<- h.B3 x]
  apply h.B8
```

Mathlib ModelTheory

ModelTheory library is a byproduct of formalization of the independence of the continuum hypothesis (the Flypitch project). Contains:

- definition of a first-order language (e.g. Peano: $0, 1, +, \cdot, \leqslant$)
- definition of logical terms, formulas in a language; substitution, variable relabeling
- definition of a formula being realized in a model

For our purposes:

```
Model.Realizes("forall x, x + 1 != 0") : Prop
Model.Realizes("forall x, x + 1 != 0") <-> forall x:Model, x + 1 != 0
```

Adding induction scheme!

Here:

- `phi.IsOpen` traverses `Formula` recursively and ensures there are no quantifiers
- `peano.Formula` - the formula has to be in the language of Peano
- `IsEnum` just gives an enumeration of type a
- `mkInductionSentence phi` turns phi into `phi(0) -> .. -> all x, phi(x)`

```
class IOPENModel (num : Type*) extends BASICModel num where
  open_induction {a : Type} [IsEnum a]
    (phi : peano.Formula (Vars1 ⊕ a)) :
    phi.IsOpen -> (mkInductionSentence phi).Realize num
```

Interoperability with Mathlib

We can easily enable ourselves to use standard Lean symbols such as `0`, `=`, `<=` in our formulas. Typeclasses are used for this purpose:

```
inductive PeanoFunc : Nat -> Type*
  | zero : PeanoFunc 0
  ...

instance : Zero (peano.Term a) where
  zero := Constant PeanoFunc.zero
```

How does it look like?

```
theorem add_assoc
  : ∀ x y z : M, (x + y) + z = x + (y + z) :=
by
  let phi : peano.Formula (Vars3 .z .x .y) :=
    ((x + y) + z) =' (x + (y + z))
  have ind := iopen.open_induction $ display3 .z phi
  unfold phi at ind
  simp_complexity at ind; simp_induction at ind
  rw [forall_swap_231]
  apply ind ?base ?step; clear ind
  · intro x y
    rw [B3 (x + y)]; rw [B3 y]
  · intro z hInd x y
    rw [B4]; rw [B4]; rw [B4]
    rw [<- (B2 (x + y + z) (x + (y + z)))]
    rw [hInd]
```

What can be proved?

Example III.1.9. The following formulas (and their universal closures) are theorems of $I\Delta_0$:

- **D1.** $x \neq 0 \rightarrow \exists y \leq x\,(x = y + 1)$ *(Predecessor).*
- **D2.** $\exists z\,(x + z = y \ \vee \ y + z = x)$.
- **D3.** $x \leq y \leftrightarrow \exists z\,(x + z = y)$.
- **D4.** $(x \leq y \wedge y \leq z) \rightarrow x \leq z$ *(Transitivity).*
- **D5.** $x \leq y \ \vee \ y \leq x$ *(Total order).*
- **D6.** $x \leq y \leftrightarrow x + z \leq y + z$.
- **D7.** $x \leq y \rightarrow x \cdot z \leq y \cdot z$.
- **D8.** $x \leq y + 1 \leftrightarrow (x \leq y \ \vee \ x = y + 1)$ *(Discreteness 1).*
- **D9.** $x < y \leftrightarrow x + 1 \leq y$ *(Discreteness 2).*
- **D10.** $x \cdot z = y \cdot z \wedge z \neq 0 \rightarrow x = y$ *(Cancellation law for $\cdot$).*

Interoperability with Mathlib

Our design (using `class`) was not an accident, it fosters applying Mathlib theorems to our natural numbers. So, we can automate proving *inside* of $I\Delta_0$ by using standard Lean proof-search tactics such as `apply?`.

```
variable {M : Type u} [iopen : IOPENModel M]
instance : IsRightCancelAdd M where ...
instance : PartialOrder M where ...
instance : IsOrderedRing M where ...
instance : CommSemiring M where ...
```

Proving these instances requires first proving D1-D10 manually.

Crucial: having these instances means that we will *not* have to prove every theorem about natural numbers from scratch!

Extending to Cook and Nguyen's two-sorted logics

This is possible to extend this approach to Cook and Nguyen's V^i arithmetics, which have a very well studied computational content and characterize complexity classes.

This is the main interest of my master's thesis.

Thanks!

https://github.com/ruplet/formalization-of-bounded-arithmetic

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Bibliography

- Jiatu Li's introduction from 1st July 2025: https://eccc.weizmann.ac.il/report/2025/086/