AN INTRODUCTION TO STRUCTURAL DYNAMICS, NON-LINEAR DYNAMICS, WIND TUNNEL AND A STUDY ON THE LORENZ SYSTEM

A PROJECT REPORT

submitted by

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ABSTRACT

A large part of all the practical dynamical processes that we observe in nature around us follows the principles of non-linear dynamics. The application of the non-linear dynamics is vast and as such they are found in the study of turbulent fluid flow, aero elastic fluttering and, climate and weather models. Yet to have a very strong mathematical as well as an intuitive mind into the complex physics associated with the non-linear dynamics, one must have a firm grasp on the relatively easier topic of linear dynamics.

The response of linear systems (in the form of Single Degree Of Freedom systems (SDOFs)) subjected to damped vibration with varying values of damping coefficients and initial conditions is studied and is found to be in good agreement both by numerical and analytical methods. A brief ride is taken into the basics of non-linear dynamics by studying the various possibilities of bifurcations in one dimension. In the later part, a bit more complicated but deeply interesting non-linear system called the Lorenz system is explored, which begins with a parametric study of the Lorenz system followed by a study on the Hopf bifurcation and next amplitude map. Studies have also been conducted to figure out basins of attraction and phase space trajectories for different initial conditions. A thorough mathematical insight has also been undertaken into the concepts of Lyapunov exponents, Poincaré maps, time series analysis and recurrence plots.

Finally, we conclude our work by a study on the experimental setups of wind tunnel and vibration testing machines and presenting the obtained results from vibration tests.

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CHAPTER 1
INTRODUCTION TO STRUCTURAL DYNAMICS

1.1 Introduction

Structures may be subjected to both static and dynamic loads. Dead loads are the static forces which remain constant for an extended time. Whereas dynamic loads are the live loads which are usually unstable or moving loads. The dynamic loads may involve the considerations like impact load, momentum, vibration etc.

The branch of Structural Engineering which deals with the behaviour of a physical structure subjected to a force is generally called Structural Analysis. The force may be a static or a dynamic load. Generally the forces which we are concerned about are due to people, furniture, wind, snow etc. or the excitations due to the seismic loads. All these loads are dynamic loads since at a certain time these loads did not have any existence. Even the self-weight of the structure is considered as a dynamic load. The static and dynamic analysis of the structures can be distinguished by the fact that if the applied action has enough acceleration in comparison to the natural frequency of the structure, then it is said to be the dynamic analysis of the structure. In this project we concentrated only on the Dynamic behaviour of the structures.

Structural Dynamics can therefore be defined as the type of structural analysis which is concerned with the behaviour of the structure under the action of dynamic loading, i.e. actions having high accelerations. The examples include wind, waves, earthquakes, traffic, blasts etc.

Structural Dynamics is an important topic in Civil Engineering, more specifically in Earthquake Engineering. We need to know the dynamic behaviour of the structure due to vibration under the action of a seismic load and by analysing that we must take necessary precautions so that the structure does not fail. Broadly speaking, Structural Dynamics is not only a topic of Civil Engineering or Mechanical Engineering, even it has applications in Aerospace Engineering and also in the other branches of Engineering. So, there are many scopes in the field of structural dynamics and there are many research topics in this field. Therefore, we may call Structural Dynamics as the Science of the present day and surely it will open many new dimensions in Science and Engineering in future.

1.2 Topics Covered

Vibration of Single Degree Of Freedom (SDOF) Systems

- SDOF System-Definition
- Classification of vibration
- Mathematical Background
- Brief Discussion on Harmonic Motion
- Damping
- Damped Free Vibration of SDOF System
- Damped Forced Oscillation
- Simple Pendulum-Response with viscous damping

1.3 SDOF System-Definition

To know the definition of SDOF System, at first we need to know the definition of the Degree of freedom of a system.

The minimum number(s) of independent coordinates which is/are required to determine the positions of all parts of a system at any instant of time is defined as the Degree of Freedom of the System.

A Single Degree of Freedom (SDOF) System is defined as the system in which only one coordinate is sufficient to determine its position at any instant of time. Few examples are given in the figures below.

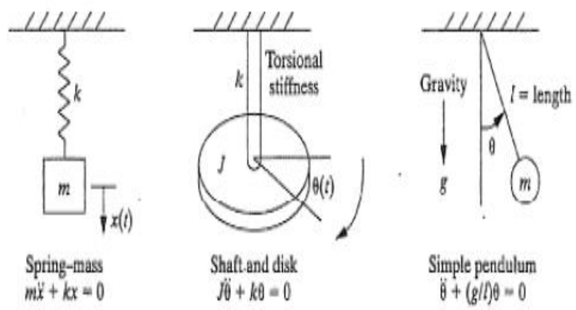

Figure 1.1 Examples of a few SDOF Systems (Source→sdof.pdf by Dr. Pelin Gundes Bakir)

1.4 Classification of Vibration

- Free Vibration → If a system vibrates on its own after an initial disturbance, then the vibration is defined as a free vibration. In this case, no external force acts on the system. One example is the oscillation of a simple pendulum.
- Forced Vibration → If a system is subjected to an external force which is usually a repeating type of force, then the resulting vibration is known as the forced vibration.

(If the frequency of the external force coincides with one of the frequencies of the system, then resonance occurs)

- Undamped vibration →If no energy is lost or dissipated due to friction or any other resistance during oscillation, then the vibration is known as undamped vibration.
- Damped vibration →If any energy is lost or dissipated due to friction or any other resistance during oscillation, then the vibration is known as damped vibration.

1.5 Mathematical Background

The knowledge of Linear Differential Equation with constant coefficients is required for this project.

1.6 Brief Discussion on Harmonic Motion

If the motion is repeated after equal intervals of time, then it is termed as a periodic motion. The simplest type of periodic motion is Harmonic motion. The definitions related to harmonic motion like amplitude, frequency, time period etc. are not discussed here due to the lack of scope in this report.

1.7 Damping

The response of a spring-mass model predicts the indefinite oscillation of the system. But actually we observe that most of the freely oscillating systems eventually die out and reduce to zero motion. The phenomenon associated with this observation is known as damping.

1.8 Damped Free Vibration of SDOF System

- **Problem Statement:**

We consider a Spring Mass Damper System to observe its unforced response. The governing equation for this system is

$$m\ddot{x}+c\dot{x}+kx=0$$

Here, m = 5 kg; k = 1000 N/m; x (0) = 5 cms; $\dot{x}$ (0) =v (0) =0;

Damping factor (ξ) = {0, 0.1 0.25, 0.5, 0.75, 1.0}.

$$\xi = \frac{c}{c_c} = \frac{c}{2m\omega_n}$$

Where C_c is the critical damping coefficient.

- **Solution and Plotting:**

We solved the equation numerically by MATLAB coding with the help of ode45 and also we solved this equation analytically and then used the obtained expression for MATLAB coding.

Finally we plotted the response of the system (x vs. t and v vs. t) for at least six cycles and compared the graphs of numerical and analytical solutions for different damping factor values. The different plots for analytical and numerical solution are shown in the same graph below.

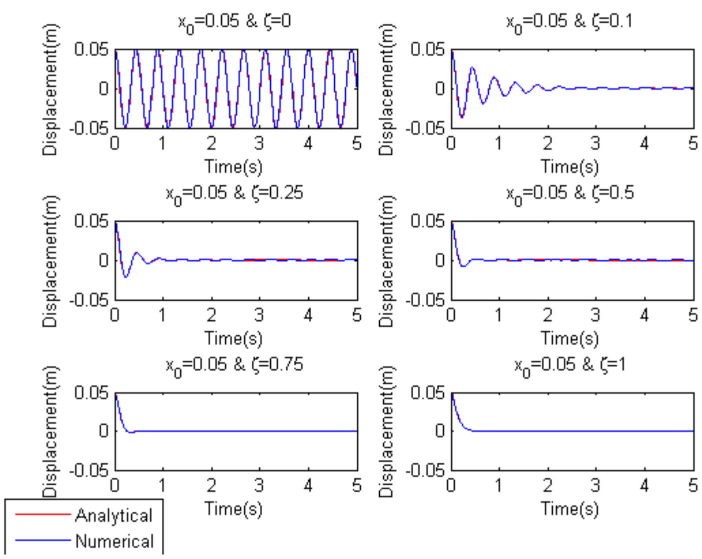

Figure 1.2 Displacement vs. Time Plot for Damped Free Vibration of SDOF System

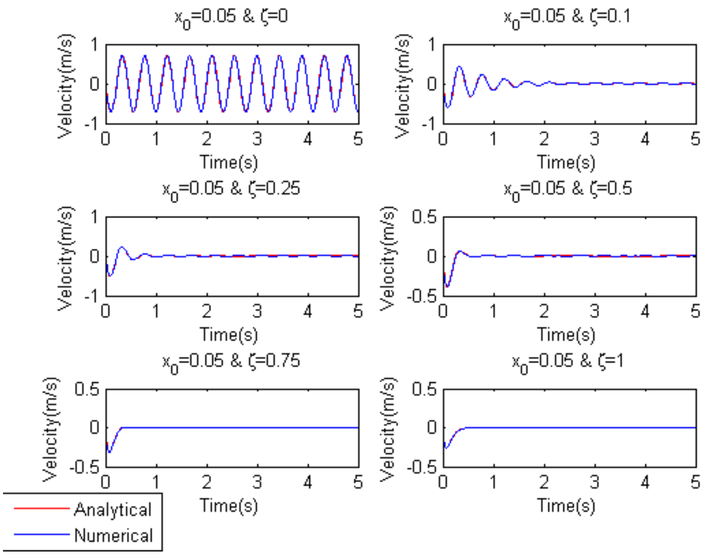

Figure 1.3 Velocity vs. Time Plot for Damped Free Vibration of SDOF System

- **<u>Inference</u>:**

The graphs obtained from the numerical solution by ode45 are exactly superimposing on the graphs obtained from the coding with the help of analytical solution. Therefore, the numerical and analytical solutions are in accordance with each other.

1.9 Damped Forced Oscillation

- **<u>Problem Statement</u>:**

We consider a Spring Mass Damper System to observe its forced response now. The governing equation for this system is $m\ddot{x}+c\dot{x}+kx=f\sin\omega t$

Here, m = 5 kg; k = 1000 N/m; x (0) = 5 cms; $\dot{x}$ (0) =v (0) =0; f=50 N, ω=4ωn

Damping factor (ξ) = {0, 0.1 0.25, 0.5, 0.75, 1.0}.

- **<u>Plotting</u>:**

As in the previous case, we compared the two graphs obtained numerically and analytically. The graphs are given below.

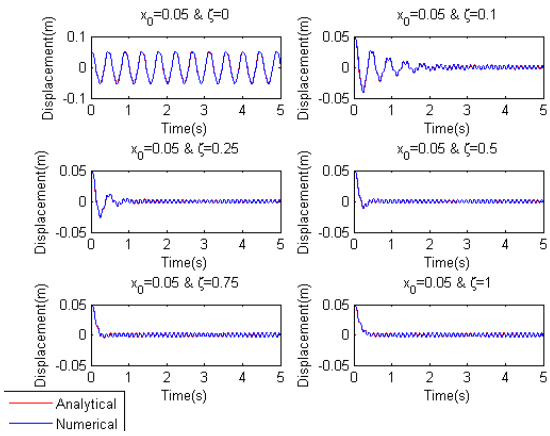

Figure 1.4 Displacement vs. Time Plot for Damped Forced Oscillation

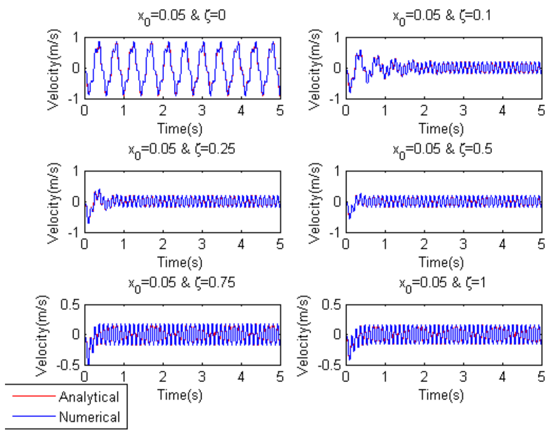

Figure 1.5 Velocity vs. Time Plot for Damped Forced Oscillation

- **Inference:**

Here the numerical and analytical solutions are exactly in accordance with each other.

1.10 Simple Pendulum-Response with viscous damping

- **Problem Statement:**

To compute the linear response of a simple pendulum the equation of motion which is considered (assuming θ to be very small and sin θ= θ) is

$$ml^2\ddot{\theta} + c\dot{\theta} + mgl\theta = 0$$

Here Damping Factor(ξ) = 0.1,θ (0) = {15°, 30°, 45°, 60°, 90°}, θ'(0)=0,M = 10 g ; l = 5 cms.

- **Plotting:**

We plotted the linear response of the system (θ vs. t and θ' vs. t) and compared the graphs of numerical and analytical solutions for different initial angular displacement values. The different plots for analytical and numerical solution are shown in the same graph below.

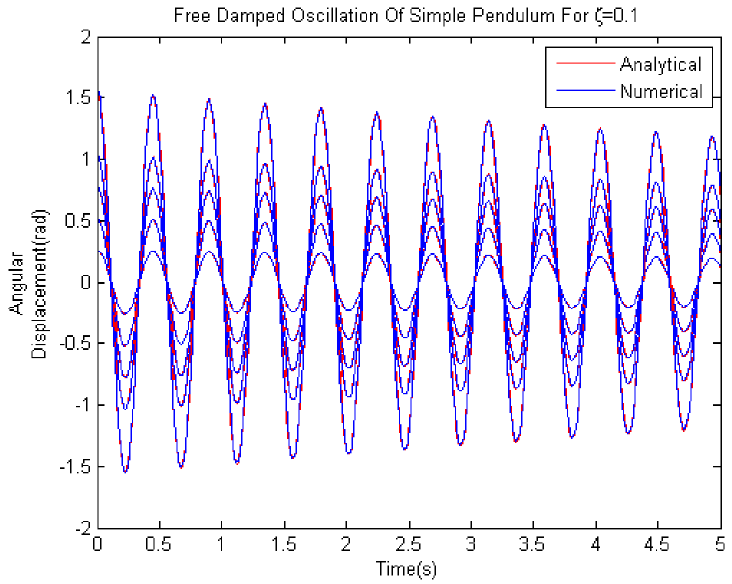

Figure 1.6 Angular Displacement vs. Time for Simple Pendulum

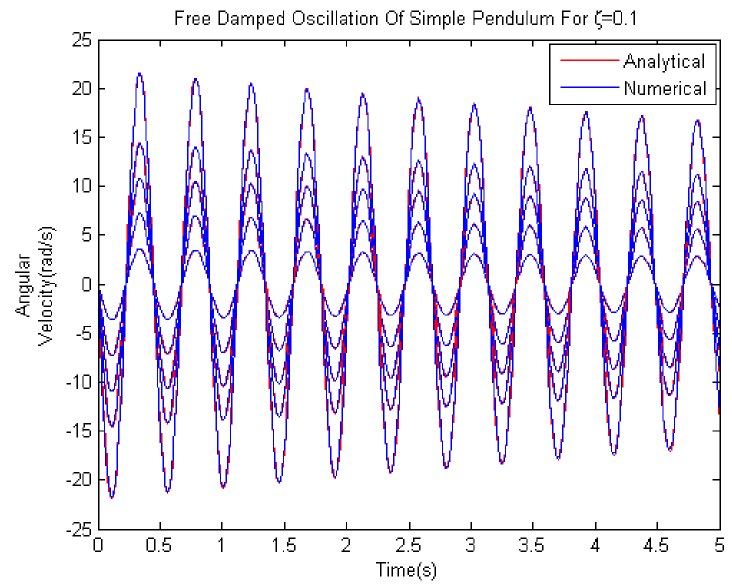

Figure 1.7 Angular Velocity vs. Time for Simple Pendulum

- **<u>Inference</u>:**

The graphs obtained from the numerical solution by ode45 are exactly superimposing on the graphs obtained from the coding with the help of analytical solution. Therefore, the numerical and analytical solutions are in accordance with each other.

1.11 Applications of Structural Dynamics

- To simulate space vehicles and airplanes, Aerospace engineers must know the structural dynamics of these vehicles.
- To control the vibration of different machines, Mechanical engineers must know the dynamics of these machineries.
- In Civil Engineering, Structural Dynamics has extensive applications. For designing purpose, it is absolutely essential to know the dynamics of the structure.
- In order to withstand severe dynamic loading from earthquakes, hurricanes, strong wind etc., Civil Engineers must know the dynamics of the structure so that they can take the necessary precautions during construction or can retrofit the structure even after a slight amount of damage.

CHAPTER 2

INTRODUCTION TO NON-LINEAR DYNAMICS

2.1 Concept of a Non-Linear Dynamical System

A non-linear dynamical system does not satisfy the principle of superposition, i.e. the output of a non-linear dynamical system is not directly proportional to the input. Typically, the behaviour of a non-linear system can be described by a non-linear system of equations.

Non-linear problems are given so much importance since actually most of the systems that we encounter in nature are non-linear. Often differential equations are helpful to solve problems related to non-linear systems. A differential equation is linear if it is linear in terms of unknown function and its derivatives, even if it is non-linear in terms of other variables appearing in it.

In non-linear system generally there are multiple equilibrium points. Also, in this system the state can go to infinity in finite time.

2.2 General Idea of Non-Linear Dynamics

Non-linear dynamics is the study of non-linear systems which are governed by equations in which a small change in one variable can induce a large amount of change in the other variable. Unlike a linear system, a non-linear system is more dependent on the initial conditions of the system. An example of a non-linear dynamical system is pendulum.

Non-linear dynamical system problems are very tough to analyse since the principle of superposition does not work in non-linear systems. Thus it is impossible to break the non-linear problem into smaller parts and that is why its difficulty level increases.

2.3 Chaos Theory

Chaos Theory is the study of Non-linear dynamics in which the events, though seems to be random, actually they are predictable from simple deterministic equations. Chaos theory is one of the most marvellous fields of modern Mathematics. More about Chaos Theory will be discussed later in the next chapter.

2.4 Topics Covered

The topics that were covered in this project are given below in a bulleted form.

Topic 1: One Dimensional Flow (Flows on the line)

- Vector Field
- Fixed Points-Stable and Unstable
- Phase Point
- Trajectory
- Phase Portrait
- Brief Discussion on Linear Stability Analysis
- Potentials

Topic 2: Bifurcation Theory

- Bifurcation and Bifurcation Points
- Saddle Node Bifurcation
- Blue-Sky Bifurcation
- Transcritical Bifurcation
- Pitchfork Bifurcation
    i. Supercritical Pitchfork Bifurcation
    ii. Subcritical Pitchfork Bifurcation
- Prevention of blow up
- Imperfect Bifurcation

2.5 One Dimensional Flow (Flows on the line)

A general dynamical system is given by the equations

$$\dot{x}_1=f_1(x_1,\ldots\ldots,x_n)$$

$$.$$

$$.$$

$$\dot{x}_n=f_n(x_1,\ldots\ldots, x_n)$$

The solution of this system can be visualised as trajectories flowing through an n dimensional phase space with coordinates $(x_1,\ldots\ldots,x_n)$

When n=1, the system of equations reduces to a single equation $\dot{x}=f(x)$

Here x (t) is a real-valued function of time t, and f(x) is a smooth real-valued function of x. Such equations are called one-dimensional or first-order systems or 1D Flow equation.

2.5.1 Vector Field

Let us consider the one dimensional flow equation $\dot{x}=f(x)$

Here x is the position of the imaginary particle moving along the real line, $\dot{x}$ is the velocity of the particle and t is the time.

This differential equation represents a vector field on the line, i.e. it gives the velocity vector $\dot{x}$ at each x. $\dot{x}$ vs. x graph is plotted to obtain the vector field.

2.5.2 Fixed Point-Stable and Unstable

Let us imagine a fluid is flowing steadily along the x-axis with a varying velocity according to the equation $\dot{x}=f(x)$

The flow is to the right when $\dot{x}>0$ and is to the left when $\dot{x}<0$. At the point(s) where $\dot{x}=0$, there is no flow and that point is called a fixed point.

If the flow from both the sides of a fixed point is towards the fixed point, then the fixed point is termed as a stable fixed point or attractor. If the flow diverges from a fixed point in both the directions, then the point is termed as Unstable fixed point or repeller. If the flow is towards a fixed point from one side and is away from that point in another side, then the fixed point is termed as Half-stable.

2.5.3 Phase Point

To find the solution of the equation $\dot{x}=f(x)$ starting from an initial condition x_0, we place an imaginary particle at x_0 and watch how it is carried along the flow. The imaginary particle is called the phase point. It is basically any point along the flow path.

2.5.4 Trajectory

After solving the differential equation $\dot{x}=f(x)$ starting from an initial condition x_0, we will obtain a solution x (t) based on the given initial condition. Basically the phase point moves along the x-axis according to this function x (t). This function is called the trajectory based at x_0.

2.5.5 Phase Portrait

A picture which shows all the qualitatively different trajectories of the system is called a phase portrait.

2.5.6 Brief Discussion on Linear Stability Analysis

To have a quantitative measure of stability of fixed points, we rely on the linear stability analysis.

Let x* be a fixed point and let n (t) = x (t) – x* be a small perturbation away from x*. Now after differentiating we obtain $n' = \dot{x}$

Now after using the Taylor's series expansion, we obtain $n' = nf'(x^*)$

This is a linear equation in n and is called linearization about x*.

Therefore the perturbation grows exponentially if $f'(x^*)>0$ and decays if $f'(x^*)<0$. If $f'(x^*)=0$, then the higher order terms in the Taylor's series expansion is not negligible and we need to go for non-linear analysis then. So, the slope is negative at the stable fixed point and is positive at the unstable fixed point.

2.5.7 Potentials

The potential V(x) is defined by $f(x) = -\frac{dV}{dx}$

Applying chain rule, we obtain

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}$$

Again for a first order system,

$$\frac{dx}{dt} = -\frac{dV}{dx}$$

Therefore, $\frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0$

Thus, V (t) decreases along trajectories, so the particle always moves toward lower potential. When dv/dx=0, V remains constant in that case. dv/dx implies that dx/dt is zero which corresponds to a fixed point.

The local maxima of V (t) corresponds to an unstable fixed point and the local minima of V (t) corresponds to a stable fixed point.

2.6 Bifurcation Theory

Bifurcation theory is the mathematical study of changes in the qualitative structure of a given family, such as the solutions of a family of differential equations.

2.6.1 Bifurcation and Bifurcation points

'Bifurcation' means the division of something into two branches or parts.

Fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are termed as Bifurcation.

Bifurcation points are defined as the parameter values at which bifurcation occurs.

Example → Suppose a small weight is placed on top of a beam. In this condition the beam is in stable state. But when a large amount of weight is placed on top of the beam, the beam no longer remains stable but it buckles. Thus the stability of the beam changes and weight is the control parameter in this case. So, this is a typical type of bifurcation where the stability changes.

2.6.2 Saddle Node Bifurcation

- Fixed points are created or destroyed.
- As the parameter is varied, two fixed points move toward each other, collide, and mutually annihilate
- Model equation: $\dot{x}=r+x^2$
- $\dot{x}$ vs. x plots:

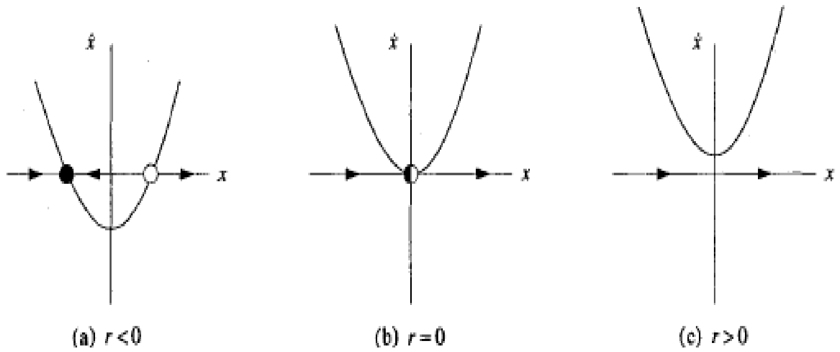

Figure 2.1 Phase Space Plot for Saddle Node Bifurcation (Source→ Strogatz)

- Bifurcation diagram

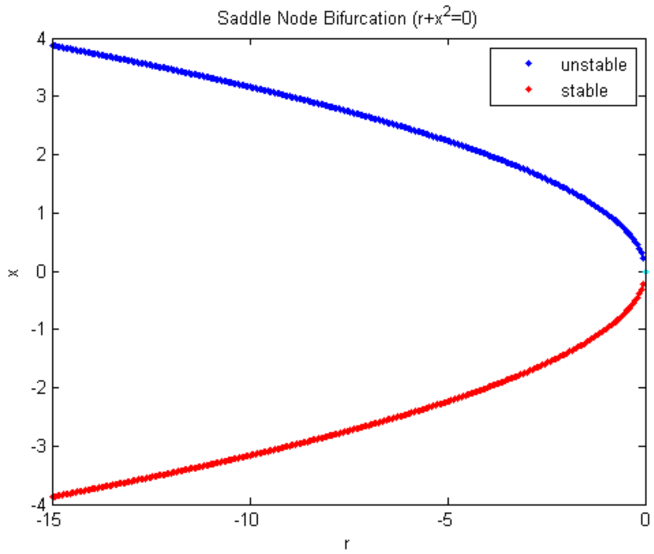

Figure 2.2

- Trajectory Plots

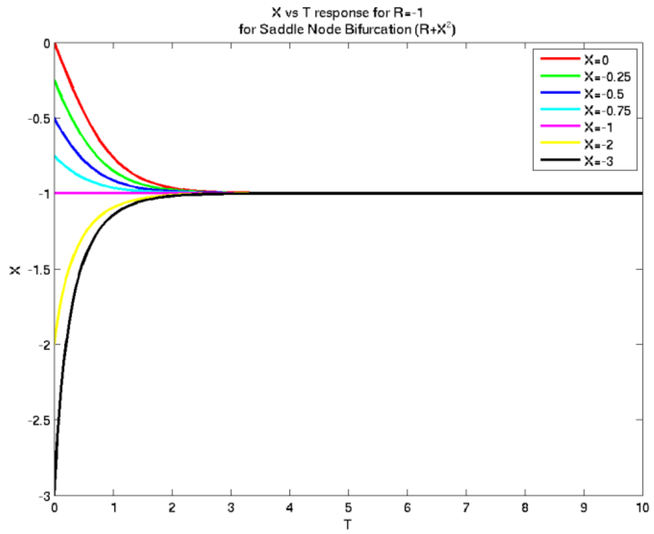

Figure 2.3

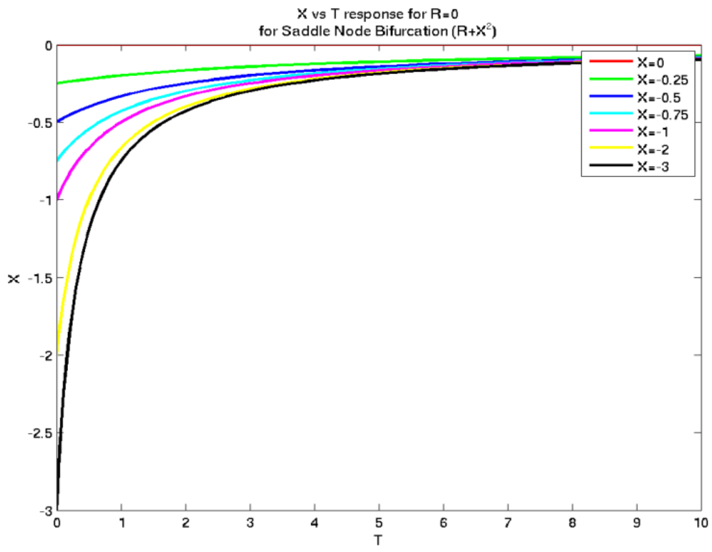

Figure 2.4

2.6.3 Blue-Sky Bifurcation

- It may also be called a Saddle Node Bifurcation
- Model equation: $\dot{x}=r-x^2$
- Bifurcation diagram

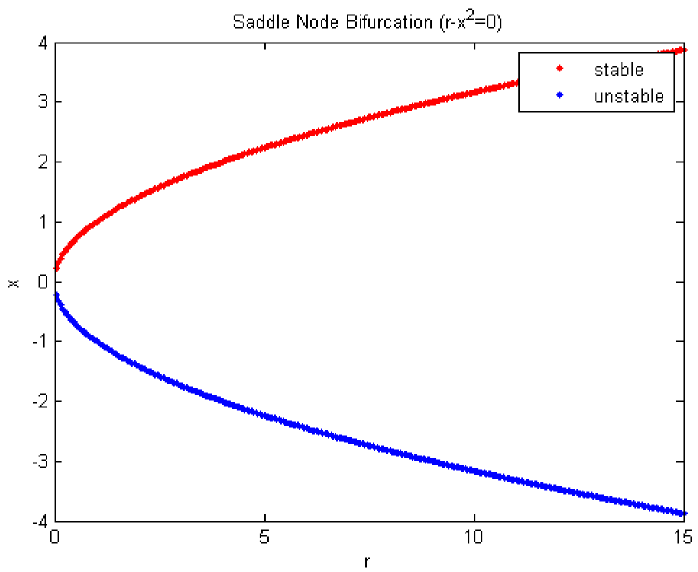

Figure 2.5 Bifurcation Diagram for Blue-Sky Bifurcation

2.6.4 Transcritical Bifurcation

- Fixed Point changes its stability as the parameter value is varied.
- Model equation: $\dot{x}=rx-x^2$ or $\dot{x}=rx+x^2$

- Bifurcation Diagrams

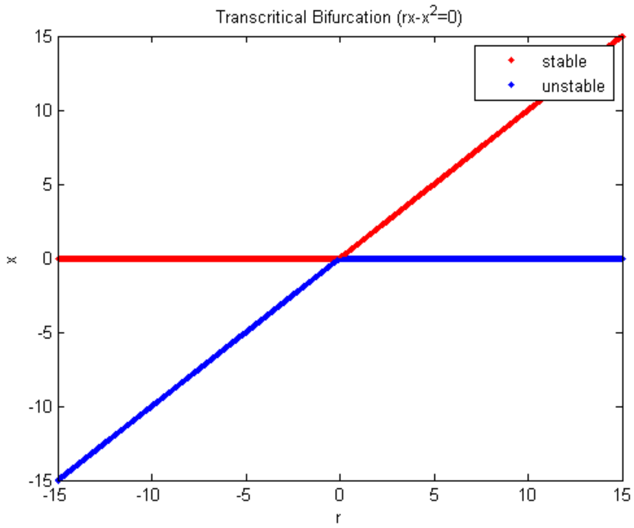

Figure 2.6

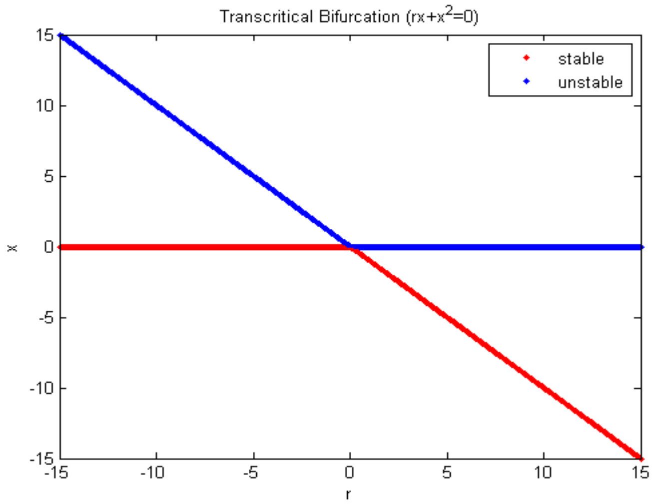

Figure 2.7

- Trajectory Plots

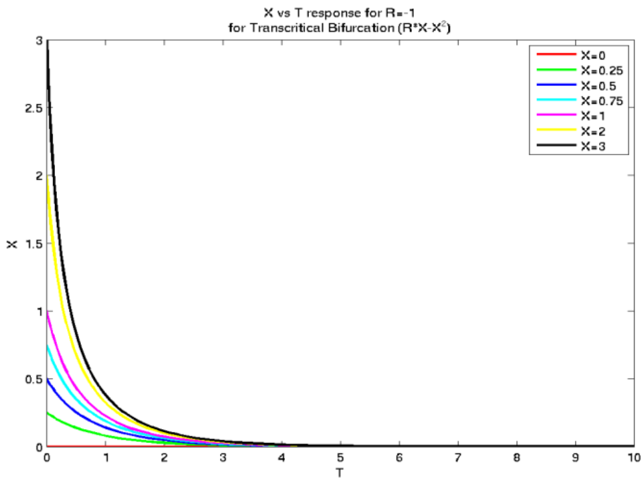

Figure 2.8

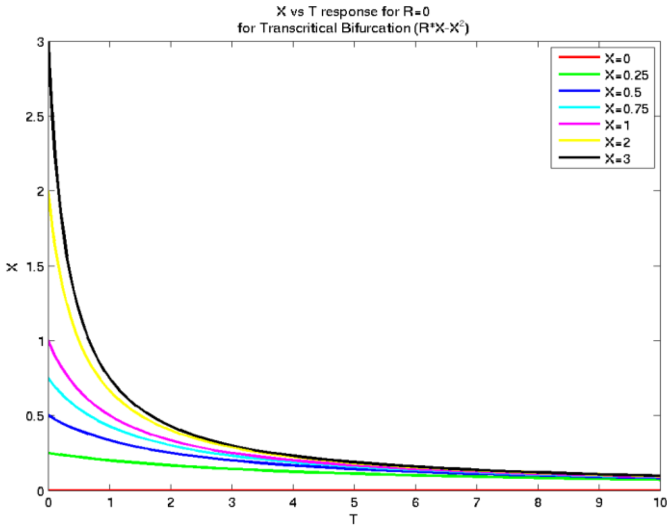

Figure 2.9

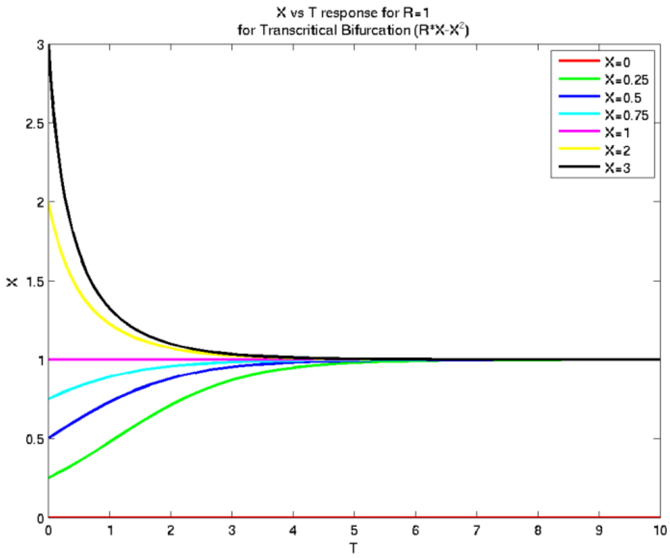

Figure 2.10

2.6.5 Pitchfork Bifurcation

- This bifurcation is common in physical problems that have a symmetry
- Fixed points tend to appear and disappear in symmetrical pairs
- The shape of the bifurcation curve looks like a pitchfork

2.6.5.1 Supercritical Pitchfork Bifurcation

- Model equation: $\dot{x}=rx-x^3$

- Bifurcation Diagram

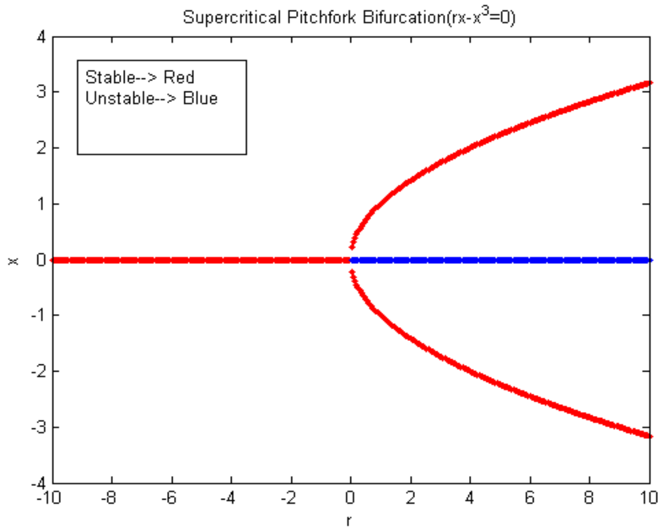

Figure 2.11

- Trajectory Plots

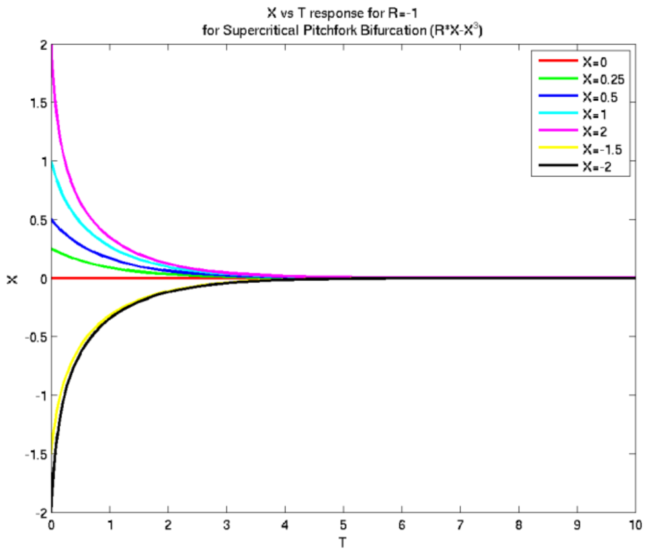

Figure 2.12

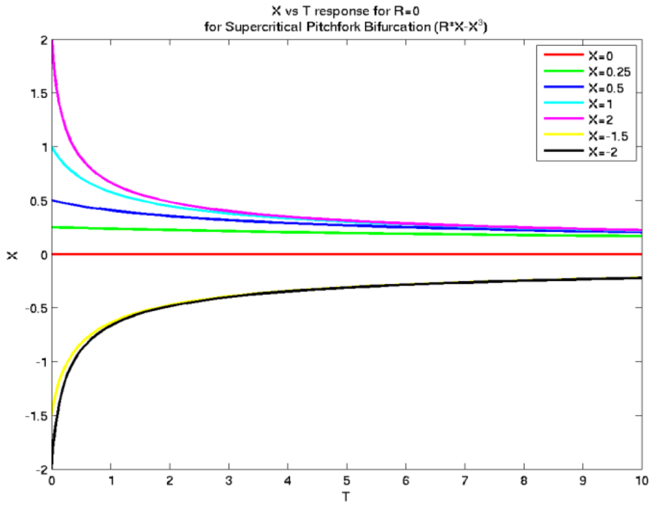

Figure 2.13

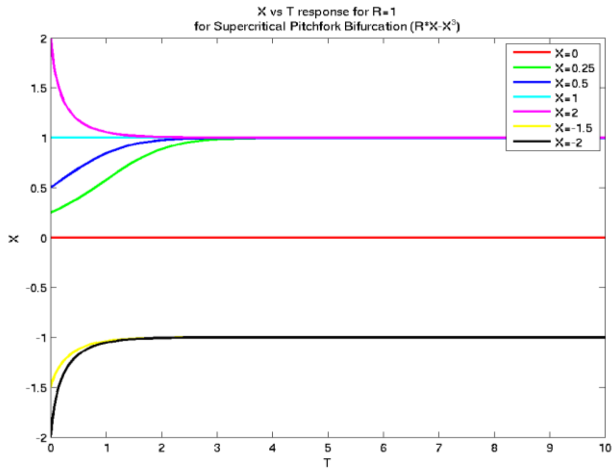

Figure 2.14

2.6.5.2 Subcritical Pitchfork Bifurcation

- Model equation: $\dot{x}=rx+x^3$
- Trajectories can be driven out to infinity in this case
- Bifurcation Diagram

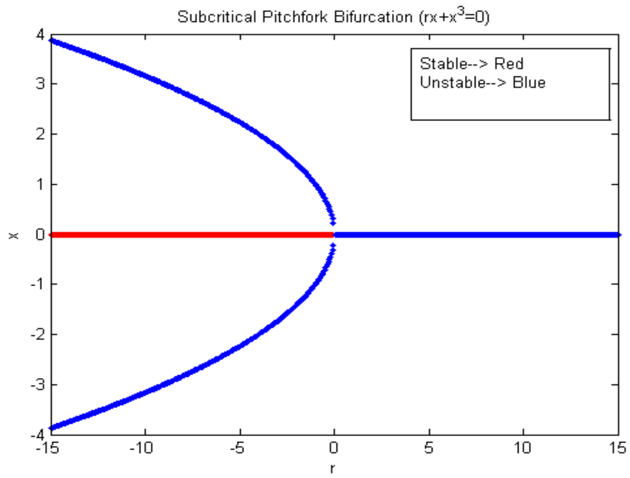

Figure 2.15

2.6.6 Prevention of Blow up

- Trajectories are not driven out to infinity in this case.
- Symmetry of the system remains intact.
- Model equation: $\dot{x}=rx+x^3-x^5$

- Bifurcation Diagram

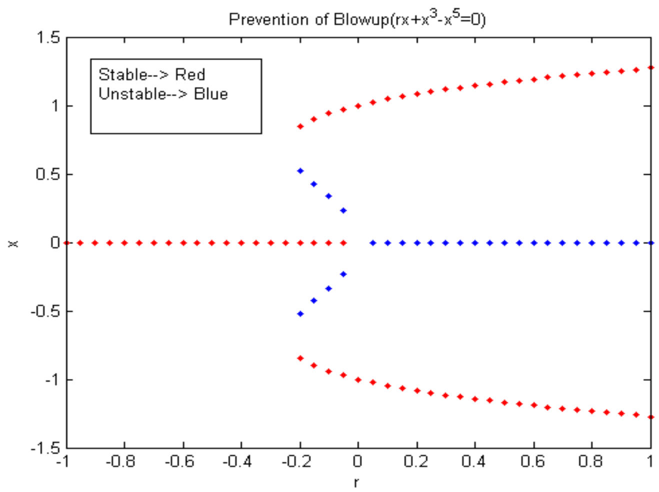

Figure 2.16

- Trajectory Plots

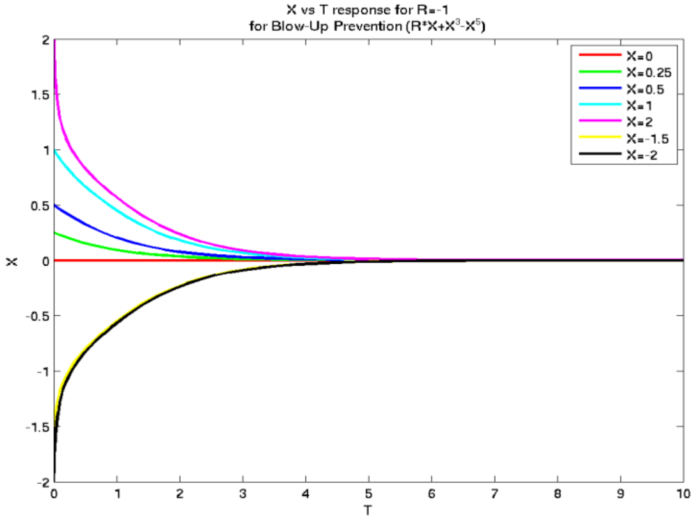

Figure 2.17

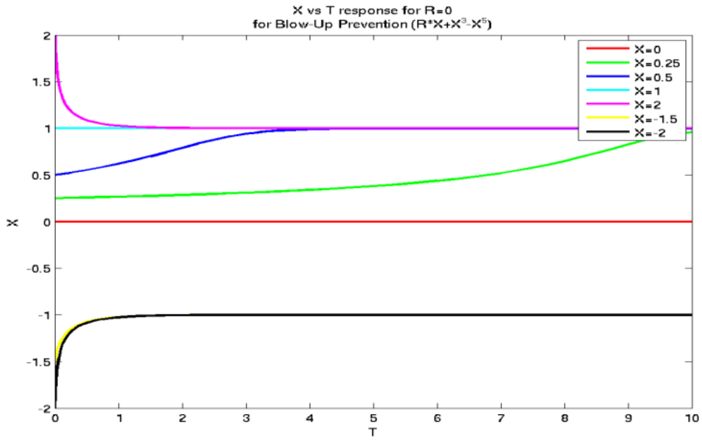

Figure 2.18

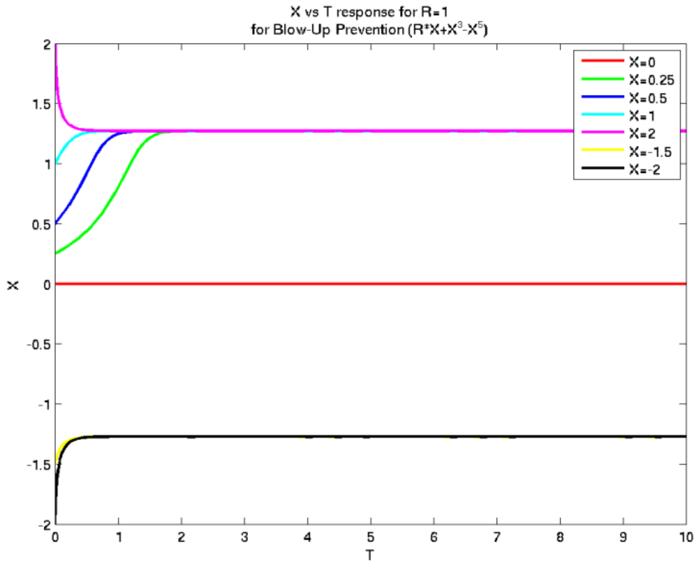

Figure 2.19

2.6.7 Imperfect Bifurcation

- In many real-world circumstances, the symmetry is only approximate
- We introduce an imperfection parameter h to break the symmetry
- Model equation: $\dot{x}=h+rx-x^3$
- Bifurcation diagram

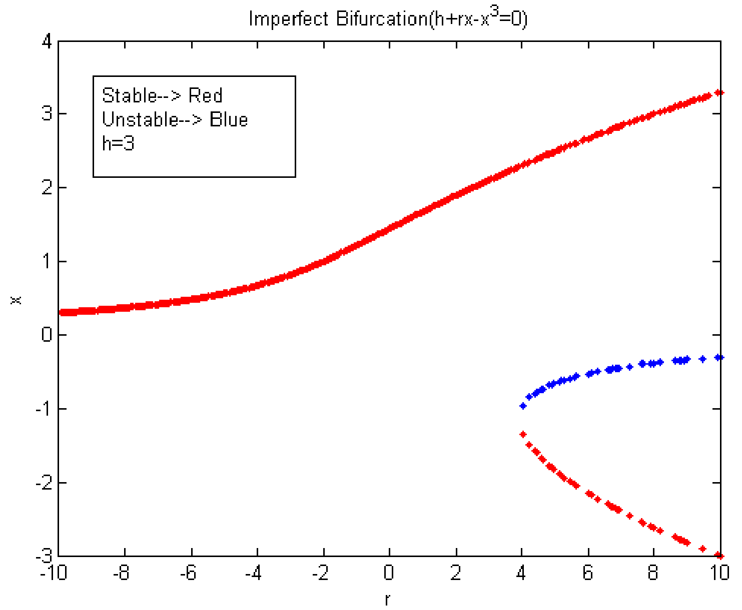

Figure 2.20

- x vs. h plot

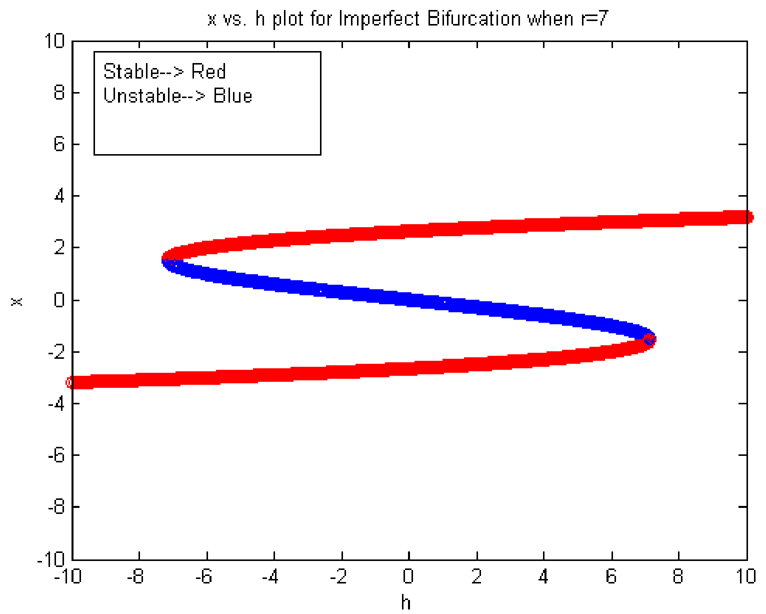

Figure 2.21

- h vs. r plot

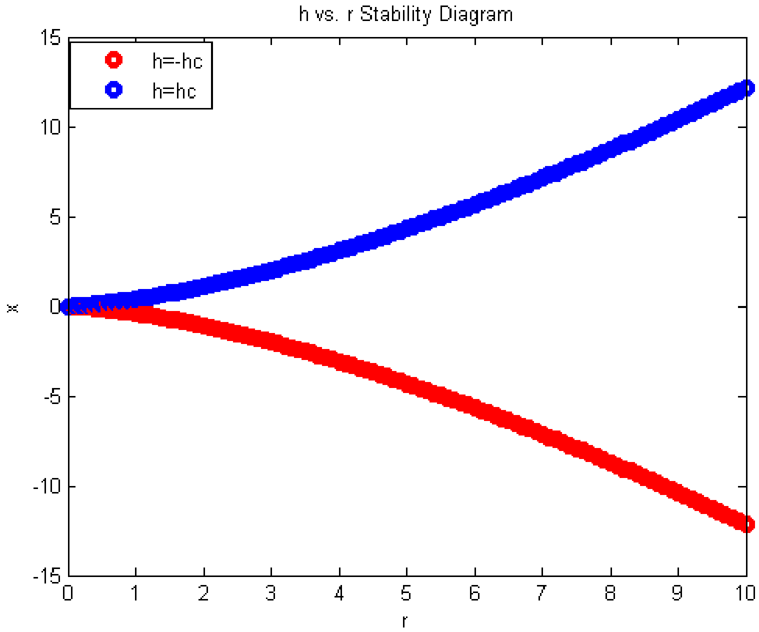

Figure 2.22

2.6.8 Applications of Bifurcation

- By studying different types of bifurcation, we got to know about the parameter values for which the fixed point is either stable or unstable.
- Stability of the limit cycle can also be known from the bifurcation theory
- Bifurcation theory also gives us the idea whether a trajectory flies out to a distant attractor or not.
- Bifurcation theory also has many biological applications.

CHAPTER 3

A STUDY ON THE LORENZ SYSTEM

3.1 Introduction

In the previous chapter, we introduced a term named Chaos Theory. In this chapter we will discuss about chaos more extensively. The basic idea behind chaos is that it is a phenomenon which is dependent on the initial conditions of the system. Small changes in the initial conditions can lead to broad changes in the output data of a system. This is the fundamental aspect regarding chaos.

In this chapter we will discuss about how the famous meteorologist Ed N. Lorenz discovered chaos and thus we will study more about the Lorenz System and the Butterfly Effect. Later we will study the famous Lorenz Equations and will observe the behaviour of the system under the variation of parameters like the Rayleigh Number and Prandtl Number etc. We will have an idea about the periodic and chaotic trajectories and also about the intermittency which is the route to chaos. The calculation of the Lyapunov exponent is also an essential part of our study by which we can make a comparison between the parameter values for which the Lorenz system is chaotic (obtained both from the graph and the value of the Lyapunov exponent). We will study whether the two different approaches are in accordance with each other or not. The knowledge of Poincaré Map is also important in this chapter. We will discuss many other relevant topics which will be mentioned later in this chapter.

3.2 Lorenz Equations

The study of chaos can be done with the help of the Lorenz Equations. The Lorenz Equations are

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz$$

Here, σ, r, b>0 are parameters. σ is the Prandtl number, r is the Rayleigh number, and b has no name.

Since the terms xz and xy are non-linear terms, that is why Lorenz System is a non-linear dynamical system.

3.3 History of Lorenz Equations

Being a meteorologist, Ed Lorenz was quite fascinated by the study of the weather phenomena. In 1961, he devised a mathematical model of the atmosphere involving twelve differential equations. He was using his computer in order to continue his calculations by running a programme.

Lorenz obtained certain results by running his programme. He intended to carry his calculations further. But it was painfully slow to start the whole programme again. In this situation he started the programme in the middle of the calculation by inputting the data as calculated by the computer at that middle point. Surprisingly, Lorenz obtained different results this time around over the same range in which he made the calculations earlier. At first, he thought that there might have been a hardware problem since same input data must give the same results all the time.

Eventually Lorenz found out that the input data of the second run was slightly different from the first one since the number of decimal places that was given as input and the number of decimal places up to which the data was stored in the computer during the first run were different. So, there was a slight change in the initial condition in the second run. But there was a drastic change in the calculations compared to the first run. Thus Lorenz understood that a small change in the initial data can lead to major changes in the output data. Thus Lorenz discovered Chaos.

Later in 1963 Lorenz derived a three-dimensional system from a drastically simplified model of convection rolls in the atmosphere. The same equations also arise in models of lasers and dynamos. Lorenz intended to model some of the unpredictable behaviour of the weather system and this desire led to the discovery of the set of three dimensional ordinary differential equations which is now famously known as the Lorenz equations.

3.4 Butterfly Effect

Due to nonlinearities in weather processes, a butterfly flapping its wings in Brazil can, in theory, produce a tornado in Texas. This strong dependence of outcomes on slight changes in the initial conditions is a hallmark of the mathematical behaviour known as chaos. But actually the flapping of the wing of the butterfly does not cause the tornado directly. The flap of the wings is just a part of the initial conditions. Had the butterfly not flapped its wings, the trajectory of the system might have been very different. Thus the initial condition is only influenced by the flapping of the wings and these small changes in the initial conditions of the system may lead to a tornado in some other place. This is the general idea behind the butterfly effect. But the problem with the weather system is that due to uncertainties it is really hard to determine the initial conditions. So it is really difficult to study the weather system and for this reason a number of forecasting is done from the perturbed initial conditions.

3.5 Limit Cycles

A limit cycle is an isolated closed trajectory. Here, isolated means that the neighbouring trajectories are not closed, either they spiral toward the limit cycle or spiral away from it.

- **Stable Limit Cycle**→ If the neighbouring trajectories spiral toward the limit cycle
- **Unstable Limit Cycle**→ If the neighbouring trajectories spiral away from the limit cycle
- **Half-stable Limit Cycle**→ If a set of neighbouring trajectories spiral toward the limit cycle and another set of neighbouring trajectories spiral away from the limit cycle

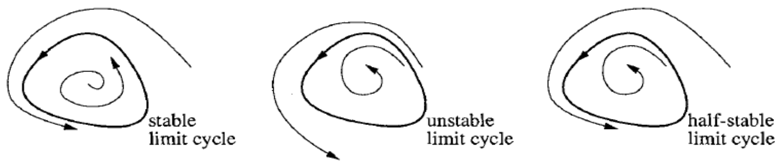

Figure 3.1 Different Types of Limit Cycle (Source→ Strogatz)

3.6 Few Important types of Bifurcations

- Hopf Bifurcation
  1. Supercritical Hopf Bifurcation
  2. Subcritical Hopf Bifurcation
- Homoclinic Bifurcation

3.6.1 Hopf Bifurcation

- In this bifurcation, the real part of the complex conjugate eigenvalues changes sign
- It occurs in phase spaces of any dimension ≥ 2

3.6.1.1 Supercritical Hopf Bifurcation

- In this case, a stable spiral transforms into an unstable spiral surrounded by a small, nearly elliptical limit cycle
- Model equations:
  $\dot{r} = \mu r - r^3$
  $\theta' = \omega + br^2$ (where θ' is $d\theta/dt$)

3.6.1.2 Subcritical Hopf Bifurcation

- In this case, the trajectories must jump to a distant attractor which may be a fixed point, another limit cycle, chaotic attractor or infinity
- This type of bifurcation is potentially dangerous in engineering applications
- Model equations:
  $\dot{r} = \mu r + r^3 - r^5$

  $\theta' = \omega + br^2$ (where θ' is $d\theta/dt$)

3.6.2 Homoclinic Bifurcation

- It is a kind of infinite-period bifurcation
- Part of a limit cycle moves closer and closer to a saddle point
- At the bifurcation by touching the saddle point, the cycle becomes a homoclinic orbit

3.7 Simple Properties of Lorenz Equations

We will now have a brief discussion on the basic properties of the Lorenz System of Equations. The properties are listed below.

- **Nonlinearity**

  The Lorenz System has only two nonlinearities, the quadratic terms xy and xz.

- **Symmetry**
  If we replace x and y by –x and –y respectively in the Lorenz Equations, then the equations remain the same. Hence if (x (t), y (t), z (t)) is a solution, then (-x (t),-y (t), z (t)) is also a solution of the system of equations.
- **Invariant Z-axis**
  In the Lorenz Equations, if x (0) = 0 and y (0) = 0, then x and y remain zero for all t. Thus the z-axis is an orbit, on which $\dot{z} = -bz$
  Hence, z (t) = z (0) e^{-bt}, for x, y = 0
  Thus the z-axis is always a part of the stable manifold for the equilibrium at the origin.
- **Volume Contraction**
  The Lorenz System is dissipative. The volumes in phase space shrink exponentially fast.
  div (f) = $-1-\sigma-b < 0$
  Volume contraction imposes strong constraints on the possible solutions of the Lorenz equations.
- **Fixed Points**
  The Lorenz System has two types of fixed points. Obviously, the origin is a fixed point for all parameter values. For $r>1$, there is a symmetric pair of fixed points. Lorenz named them as C^+ and C^-. They are given by $x^* = y^* = \pm\sqrt{(b(r-1))}$, $z^* = r-1$. As $r \to 1^+$, C+ and C- coalesce with the origin in a pitchfork bifurcation.
- **Linear Stability of the Origin**
  The origin is a stable node for $r<1$
- **Global Stability of the Origin**
  For $r<1$, every trajectory approaches the origin as $t \to \infty$. So, the origin is globally stable and hence there can be no limit cycles or chaos for $r<1$
  Now, we plot the phase space diagram of the Lorenz System for $r<1$, here $r=0.5$ by varying the initial conditions. We observe that all the trajectories approach the origin and hence the global stability of the origin for $r<1$ is proved.

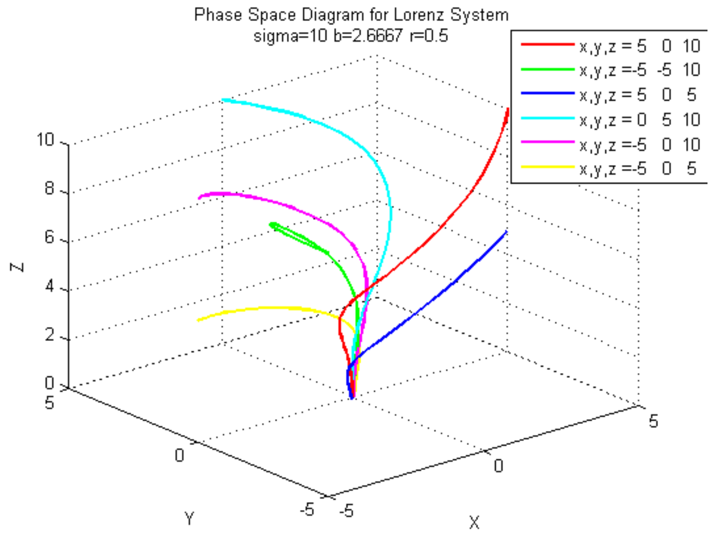

Figure 3.2

- **Stability of C^+ and C^-**

  When r>1, C^+ and C^- both exist. They are linearly stable for $1<r<r_H=\frac{\sigma(\sigma+b+3)}{\sigma-b-1}$ assuming that σ-b-1>0

  We used the subscript H since C^+ and C^- lose stability in a Subcritical Hopf Bifurcation at $r=r_H$

  So, the limit cycles are unstable and exist only for $r<r_H$

  For $r>r_H$, there are no attractors in the neighbourhood and hence the trajectories must fly away to a distant attractor. But we do not know the nature or behaviour of the attractor. That is why Lorenz named it as a Strange Attractor.

  We plot the bifurcation diagram for different Rayleigh Numbers below.

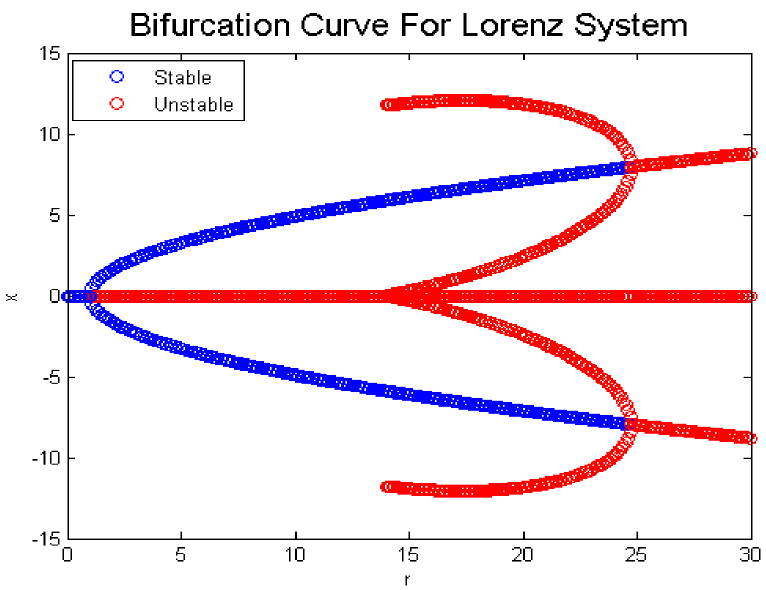

Figure 3.3

3.8 Qualitative Analysis of Lorenz Equations

3.8.1 Varying the Rayleigh Number Parameter

- **0<r<1**

In this range origin is the only fixed point. We need to figure out the nature of this fixed point.

The Jacobian of the Lorenz System is

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

At the origin the last equation has no significance since the right hand side becomes zero and thus the Jacobian becomes

$$J = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$$

Trace (τ) $=-\sigma-1 < 0$ Determinant (Δ) $=\sigma(1-r) > 0$ for $0<r<1$

Again, $\tau^2-4\Delta = (\sigma+1)^2 - 4\sigma(1-r) = (\sigma-1)^2 + 4\sigma r > 0$

<u>**Figure for the stability analysis of a system**</u>

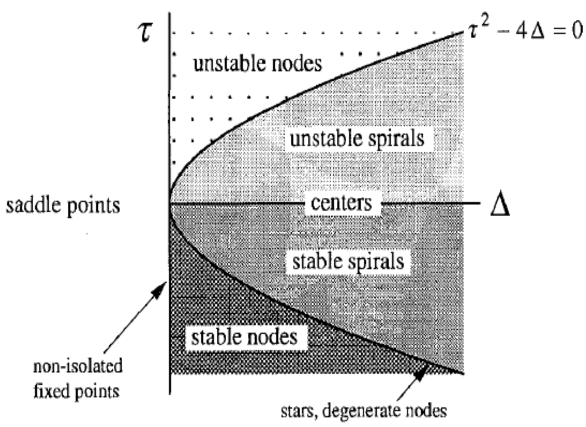

Figure 3.4 (Source→ Strogatz)

Therefore, from our knowledge of Linear Systems we can conclude that the origin is a stable fixed node for 0<r<1 which matches with our previous discussion about the stability of the origin.

Now, we plot the trajectories and phase space diagram for 0<r<1, keeping σ=10 and b=8/3 (Since Lorenz studied a particular case of chaos for parameter values σ=10, b=8/3 and r=28, we are varying r only here)

We took the initial conditions as (0, 1, 0) in this case. Since the origin is a fixed point we observe that the trajectories approach the origin.

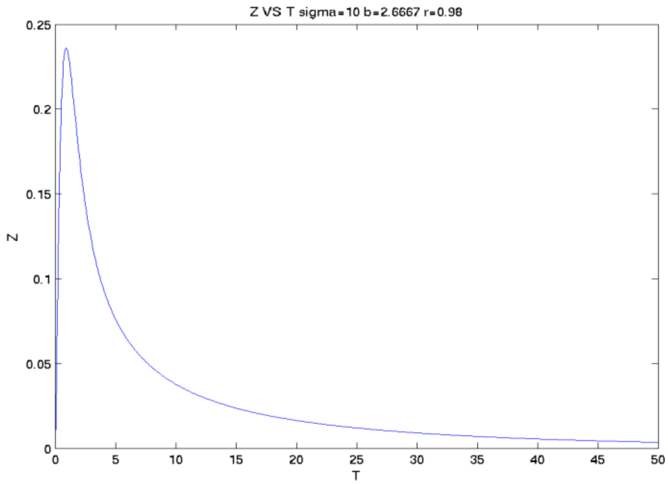

Figure 3.5

From the phase space diagrams, it is also clear that all the trajectories lead to the origin and hence it is justified that the origin is a fixed point.

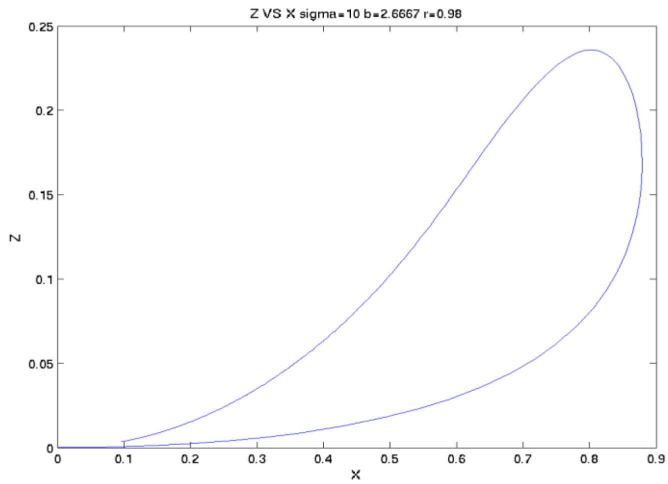

Figure 3.6

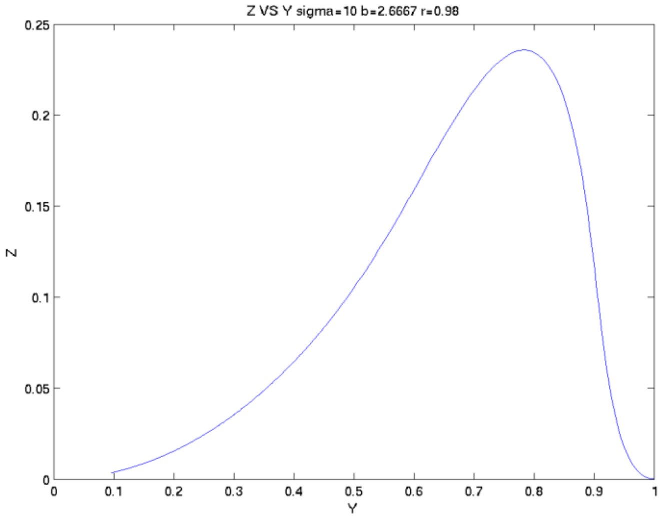

Figure 3.7

- **1<r<13.926**

For r>1, both C^+ and C^- exist and the determinant becomes negative in this case for the origin. So the origin is a saddle node in this case.

Again, the two more fixed points are given by $[\pm\sqrt{(b\,(r-1))}, \pm\sqrt{(b\,(r-1))}, r-1]$

Now by making the calculations for the trace and determinant of the Jacobian matrix by putting all the parameter values, we will see that the trajectories of C^+ and C^- are stable spirals. We will see that in the figures below. We took the value r=6

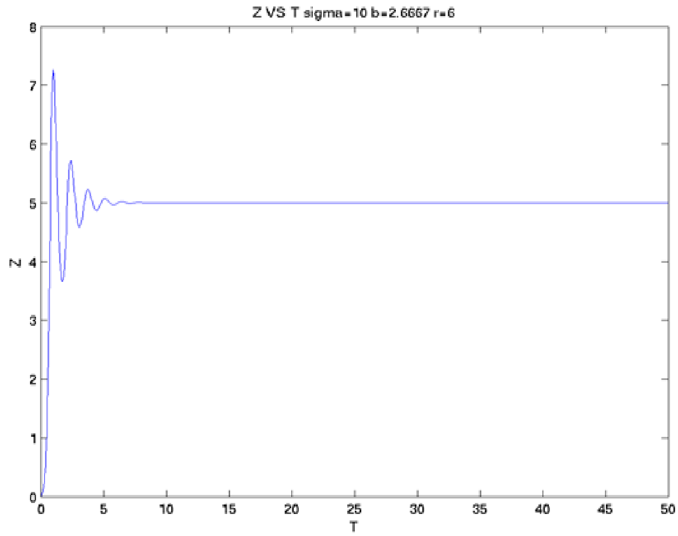

Figure 3.8

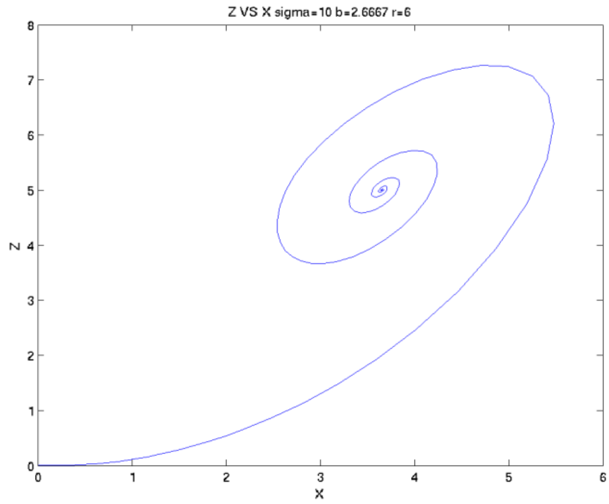

Figure 3.9

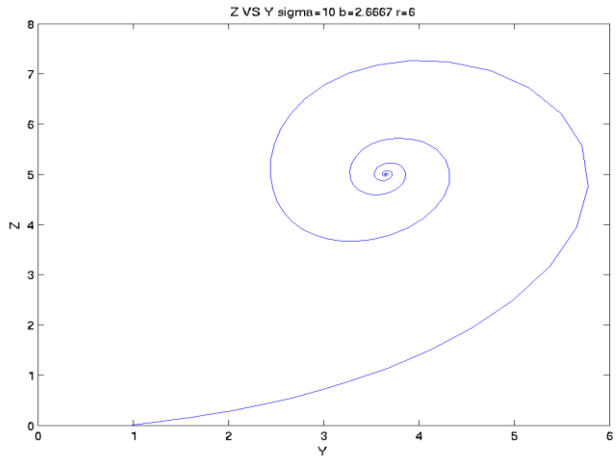

Figure 3.10

Thus we observe that the trajectories become stable after having undergone some initial oscillations. Again from the phase space diagrams it is evident that the stability of the two fixed points C^+ and C^- can be termed as a stable spiral.

- **$r>13.926$ and $r<r_H$**

For the values of r>13.926, the calculations for the Jacobian matrix becomes difficult. We may also observe the nature of the eigenvalues in order infer about the trajectory followed by the system at that particular r value. For values of r >13.926, the solution leaving the origin, loops around the nearer spiral node and then settles onto the other spiral fixed point. Homoclinic bifurcation will occur at the point r=13.926 and there will be two homoclinic orbits corresponding to both C^+ and C^-. For r>13.926, these orbits become unstable limit cycle and they enclose both C^+ and C^-

We now plot the trajectories and phase space diagram for r=15

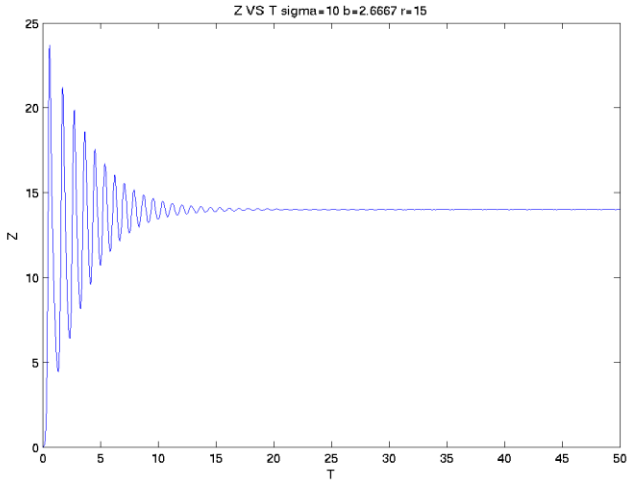

Figure 3.11

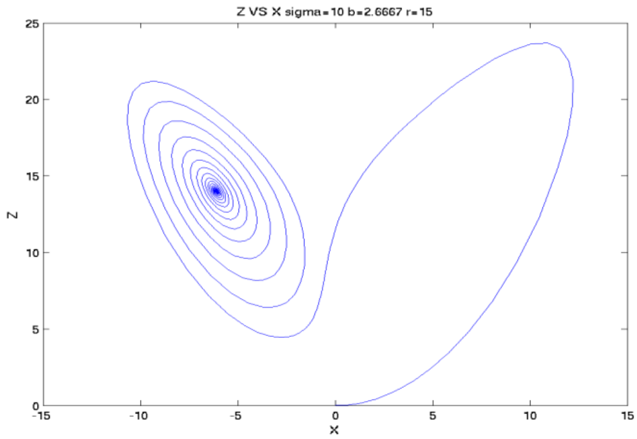

Figure 3.12

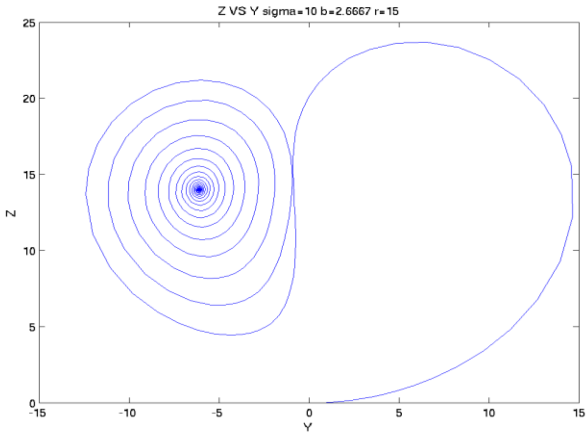

Figure 3.13

From the trajectory plot we see that the amplitude decreases with time and finally the trajectory becomes stable.

On the other hand we see from the phase space plot that the two nearby trajectories move far apart which is an example of an unstable limit cycle. Trajectories can wander in an invariant set for a while but eventually they escape and settle down to C^+ and C^- . The time spent in the set gets longer and longer as the parameter value r increases.

- **r=21**

Transient chaos occurs when r=21. The figure is not plotted here. We will observe the phenomena, chaos later.

We plot the trajectories and phase space diagrams for $r=24.5<r_H$ here in order to have some idea about the transient chaos.

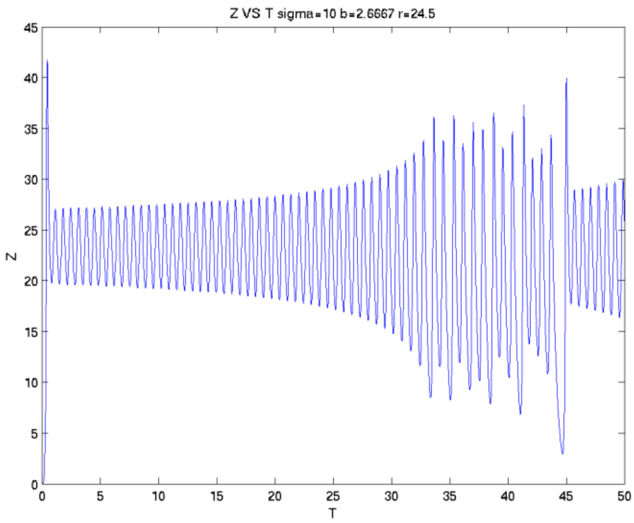

Figure 3.14

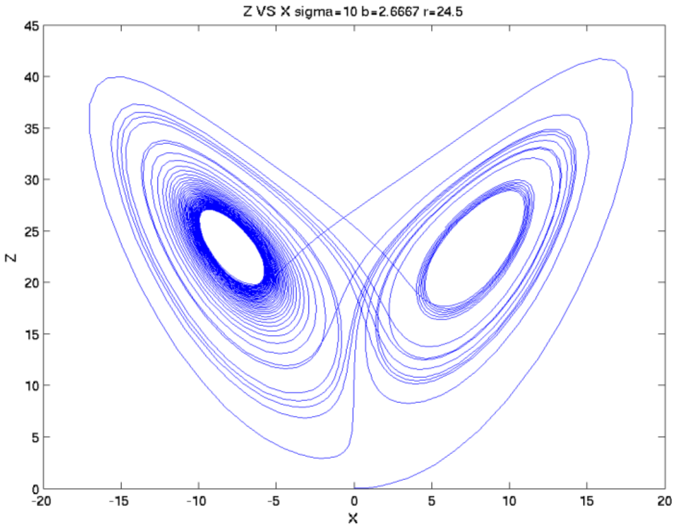

Figure 3.15

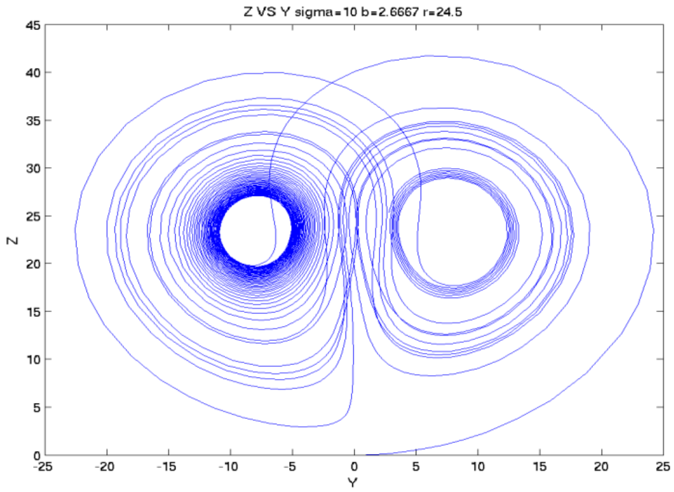

Figure 3.16

- **$r=r_H$**

  The two limit cycles involve in Subcritical Hopf Bifurcation at $r=r_H$. The unstable limit cycles actually slowly shrink and finally absorb the stable spiral nodes at $r = r_H$. The value of r_H is considered to be as 24.74 for σ=10 and b=8/3.We are not plotting the trajectories in this case. Of course, we will have some idea about this by observing the bifurcation diagram given earlier.

- **$r>r_H$**

  1. In this case there are no stable fixed points.
  2. The trajectories cannot escape to infinity. They are bounded.
  3. The attractor does not contain any fixed points.
  4. The system behaviour changes to chaotic.

We now plot the trajectory and phase space diagrams for the Lorenz attractor with parameter values σ=10, b=8/3 and r=28. We can observe the chaotic nature clearly.

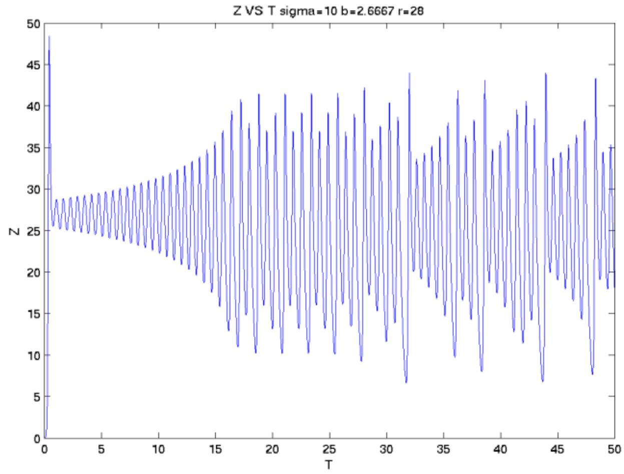

Figure 3.17

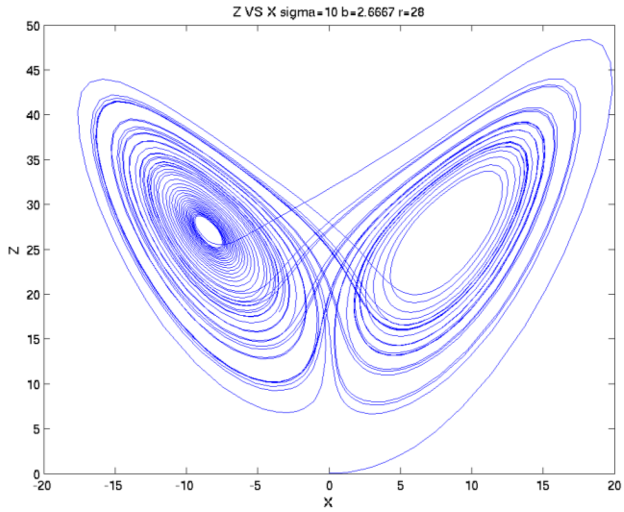

Figure 3.18

The plot Z vs. Y is termed as the **Lorenz Map.** According to Lorenz,

“The trajectory apparently leaves one spiral only after exceeding some critical distance from the centre. Moreover, the extent to which this distance is exceeded appears to determine the point at which the next spiral is entered; this in turn seems to determine the number of circuits to be executed before changing spirals again. It therefore seems that some single feature of a given circuit should predict the same feature of the following circuit.”

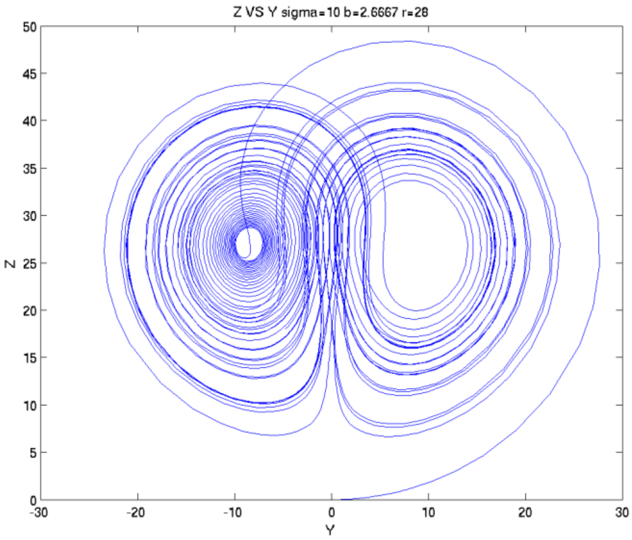

Figure 3.19

- **Study for the Intermittency**

Intermittency is defined as the route to chaos. For $r>r_H$ here, we are obtaining the chaotic behaviour of the system. Of course, at some greater value of r, the system behaviour will change from chaotic to periodic and again it will change from periodic to chaotic. Hence we need to find that critical value of r for which the system behaviour just starts to change from periodic to chaotic. This phenomenon is known as Intermittency.

To observe intermittency, we need to look for a period bursting that means if the response of the system is periodic at first, then there is some period bursting and finally with the increase in time the system returns to its original periodicity. Thus we can tell that we have found intermittency in the system.

We observed that for r=165, the trajectory is periodic and for r=166, the trajectory is chaotic. So, we infer that the intermittency takes place between the r values 165 and 166. Further we narrowed down the domain of r in order to obtain the exact value of r for intermittency. We searched the value of r up to three decimal places and finally we obtained r=165.850 for which Intermittency occurs.

We will now look at the plots in order to justify our statement as mentioned earlier.

For r=165.849 (Periodic)

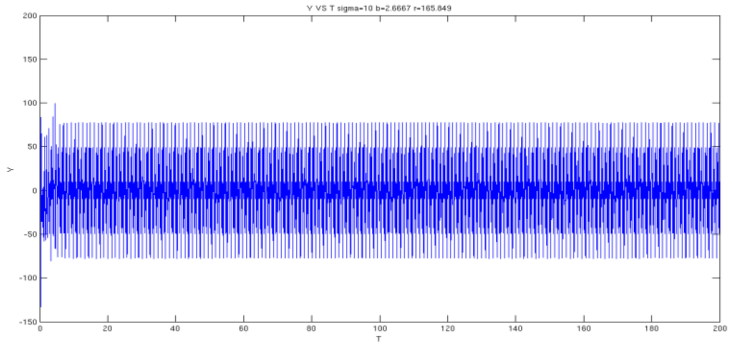

Figure 3.20

For r=165.850 (Intermittent)

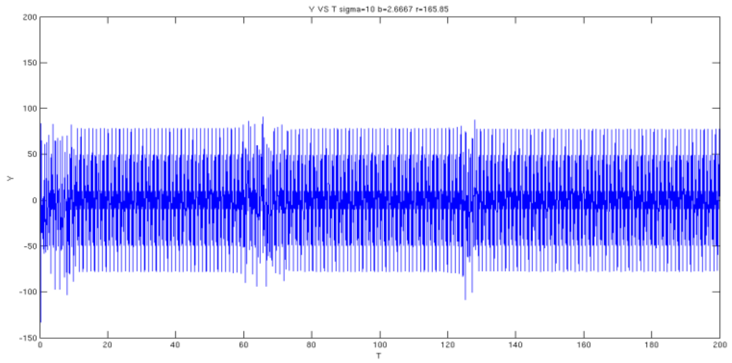

Figure 3.21

For r=166 (Chaotic)

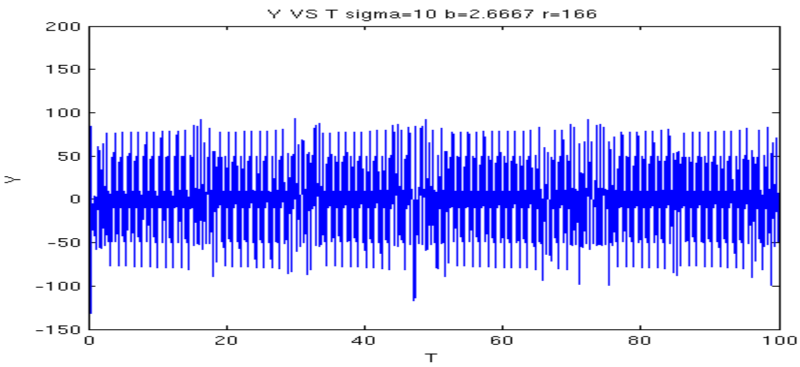

Figure 3.22

3.8.2 Varying the Prandtl Number Parameter

Now by varying the Prandtl Number, we will observe the trajectory followed by the Lorenz System. In this case we keep the values of r and b as constants, i.e. r=28 and b=8/3

- **σ=1, 5, 18, 19, 22, 100 (respectively)**

  1. Z vs. t Plots

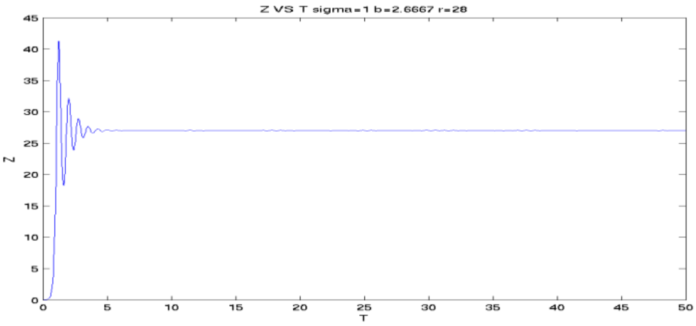

Figure 3.23

When σ=1, we observe that after an initial transience the trajectory reaches stability.

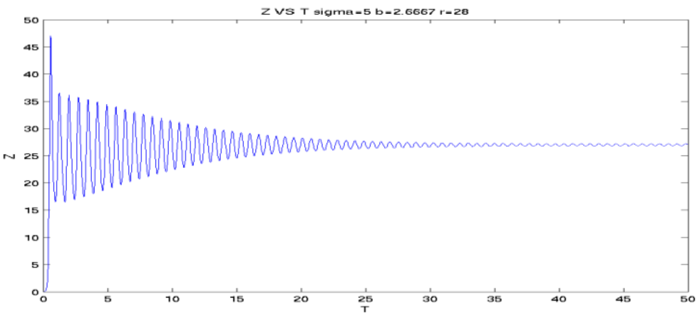

Figure 3.24

When σ=5, the amplitude decreases gradually but the trajectory does not reach stability before t=50s. Small oscillations take place for this σ value.

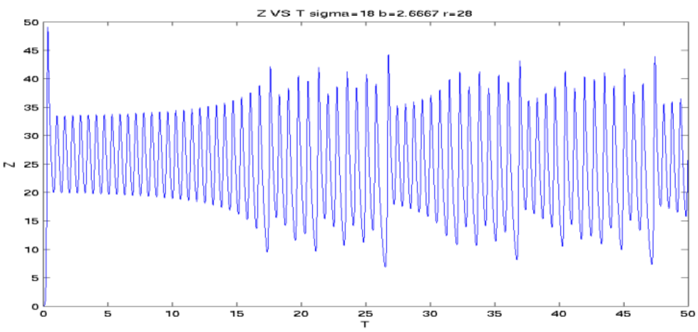

Figure 3.25

When σ=18, after an initial transience the trajectory is chaotic. Actually for σ values ranging between 10 and 18 (roughly), the system behaviour remains chaotic.

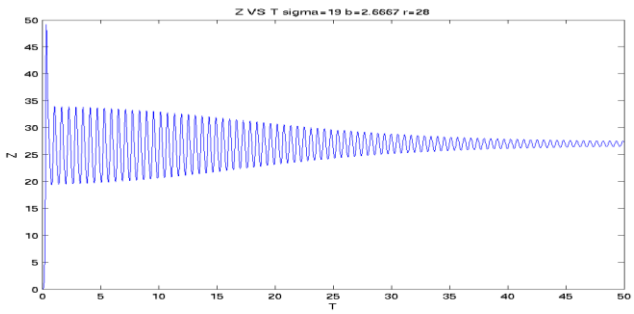

Figure 3.26

When σ=19, though the chaotic behaviour of the system to some extent remains intact, we observe the amplitude gradually decreases and at a greater value of time, very small oscillations take place.

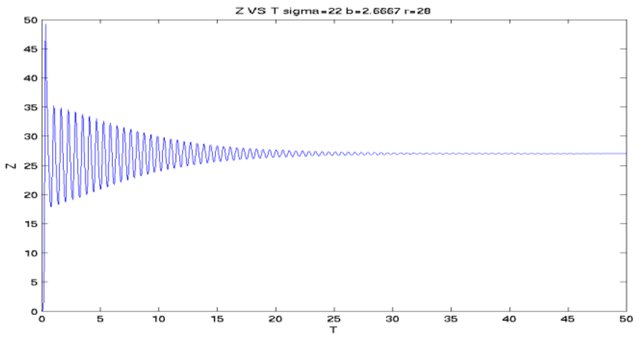

Figure 3.27

When σ=22, after an initial transience the system tends to become stable and eventually becomes stable for a t value close to 40-45 s.

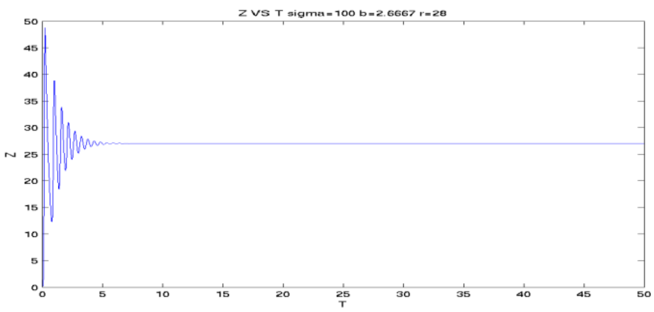

Figure 3.28

When σ value increases to 100, the stability of the system is reached earlier. So, we can infer that for σ>19, with the increase in σ value, the system reaches equilibrium earlier.

2. Z vs. x Plots

To back our observations in trajectory plots, we now plot the phase space diagram. Now we again observe the stability of the system.

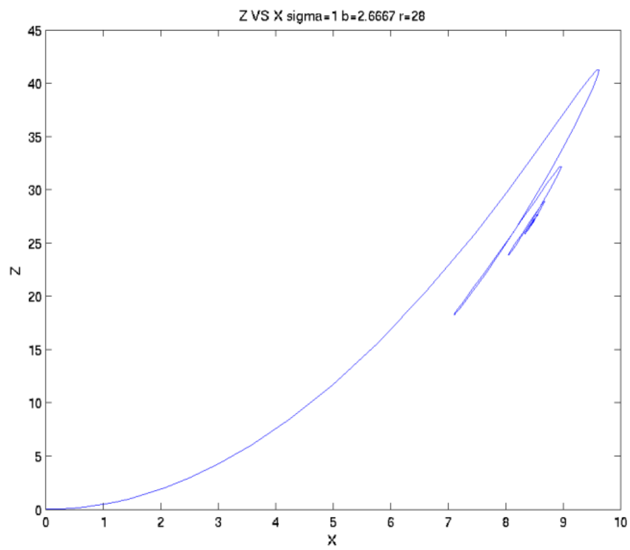

Figure 3.29

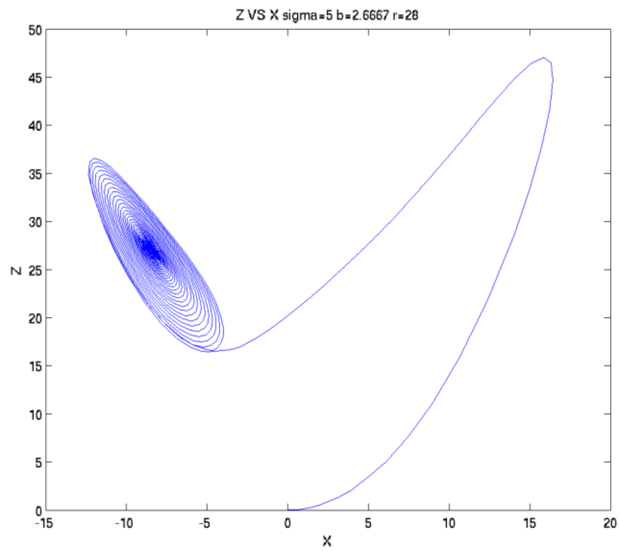

Figure 3.30

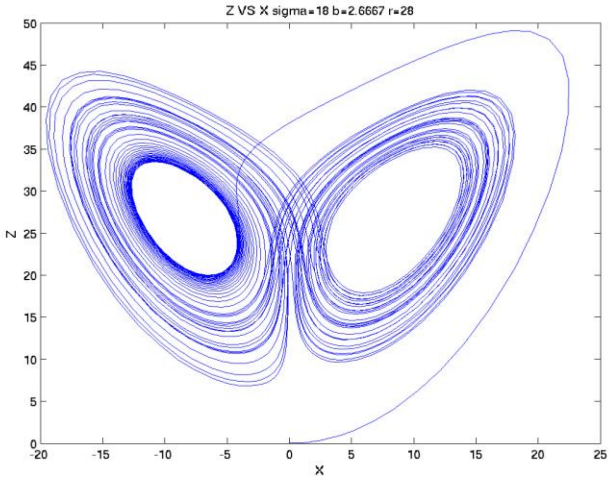

Figure 3.31

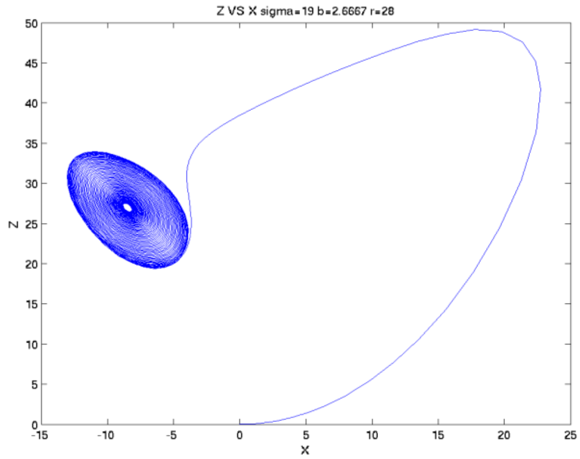

Figure 3.32

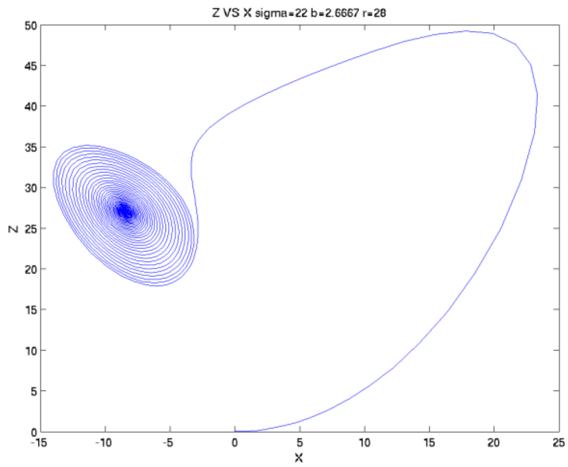

Figure 3.33

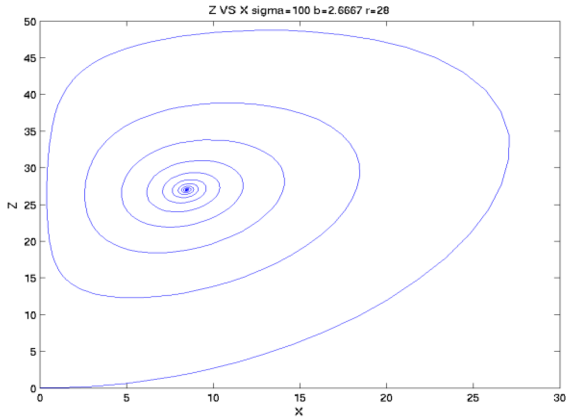

Figure 3.34

Clearly, we can justify our inference obtained from the trajectory plot by this Phase Space Plot.

3.8.3 Varying the Parameter value b

Keeping σ=10 and r=28, we vary the parameter b now. We observe the trajectory plot for different values of b and try to infer something from the plots.

- **b=0.1, 1, 8/3, 6, 10, 100 (respectively)**

  1. Z vs. t Plot

  When b=0.1, we observe that there are many discrete peaks in the trajectory, i.e. there are some values of t for which the trajectory nearly jumps to a local maxima (and also local minima is obtained).

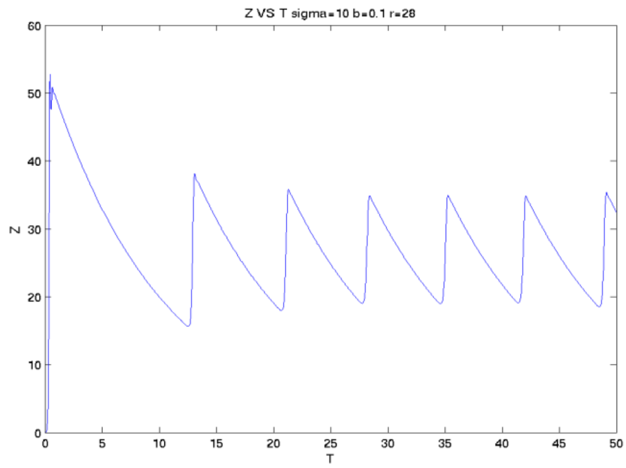

Figure 3.35

When b=1, we observe that the behaviour of the Lorenz System is chaotic.

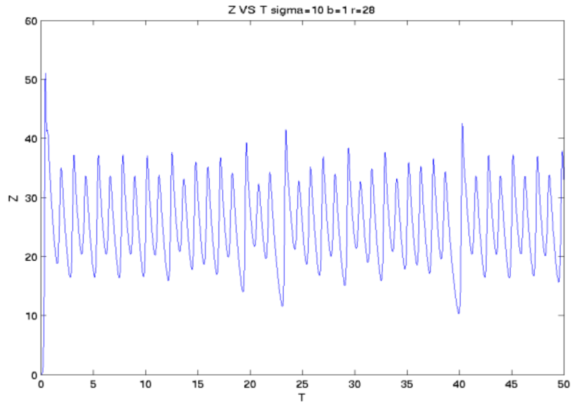

Figure 3.36

When b=8/3, we observe that the behaviour of the Lorenz System is chaotic after an initial transience.

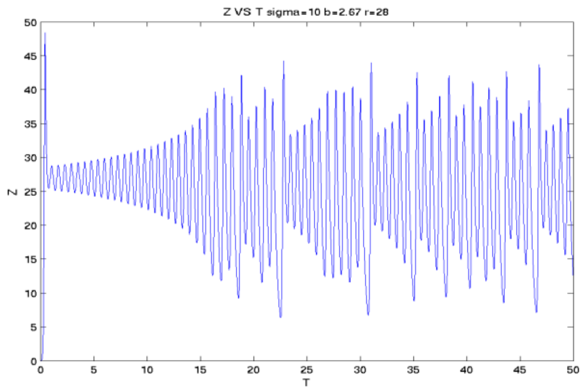

Figure 3.37

When b=6, after an initial transience, small oscillations take place in the system, i.e. both the amplitude and period are very less after a certain time span and the trajectory becomes nearly stable.

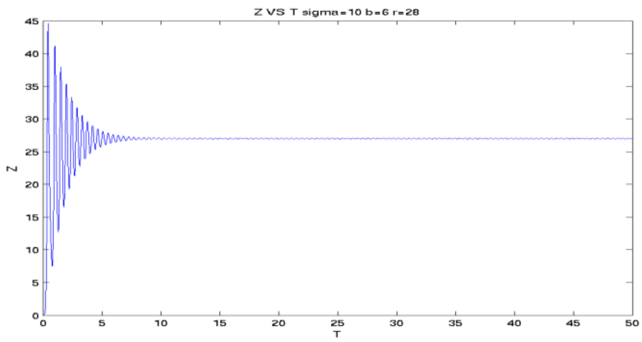

Figure 3.38

When b=10, the system reaches stability very quickly.

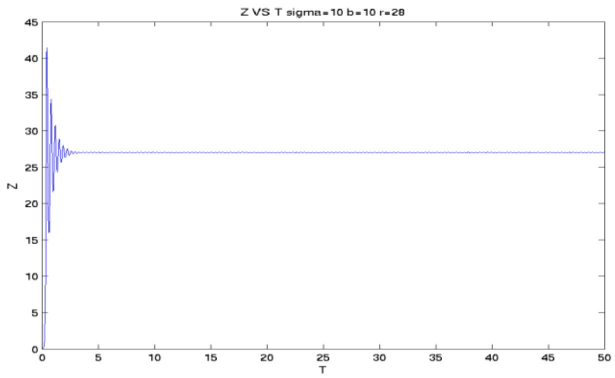

Figure 3.39

With further increase in the value of b, say when b=100, we observe that the trajectory is nearly stable even from a very low value of time.

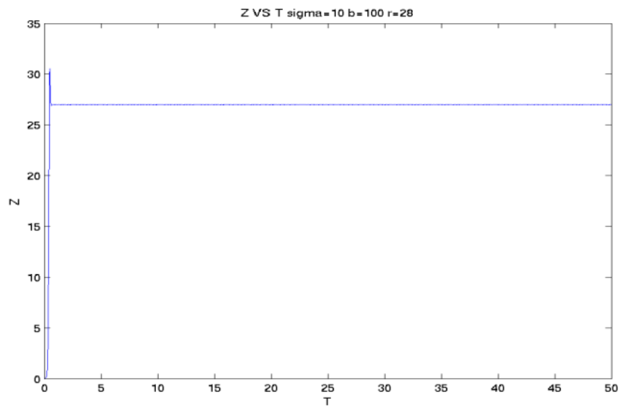

Figure 3.40

2. **Z vs. X Plot**

   Now we plot the phase space diagram for the Lorenz System with varying b values and we observe the stability of the system.

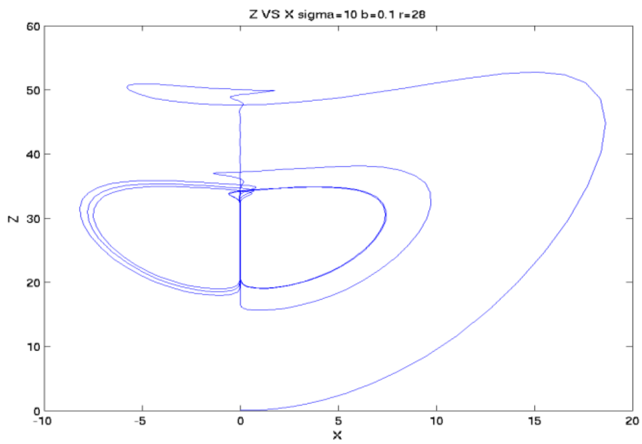

Figure 3.41

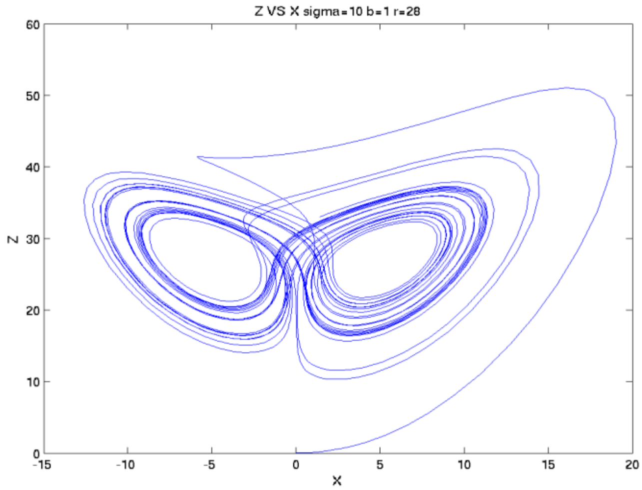

Figure 3.42

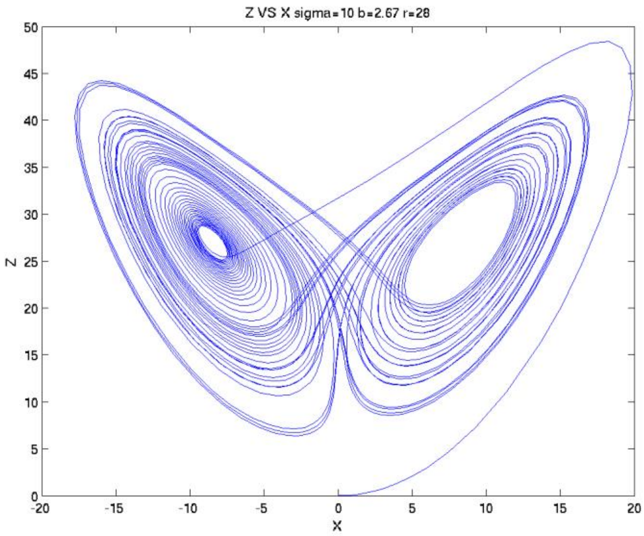

Figure 3.43

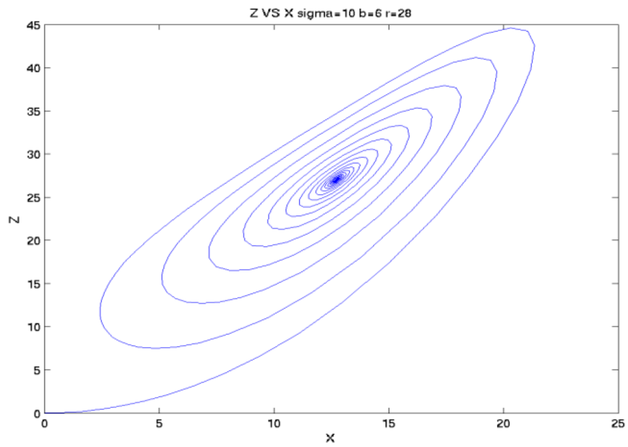

Figure 3.44

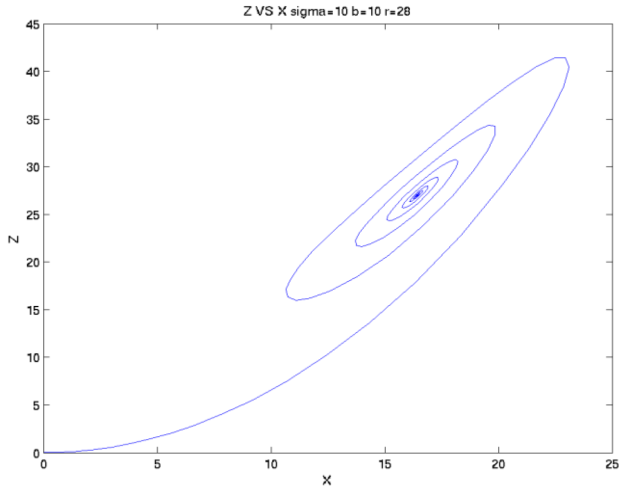

Figure 3.45

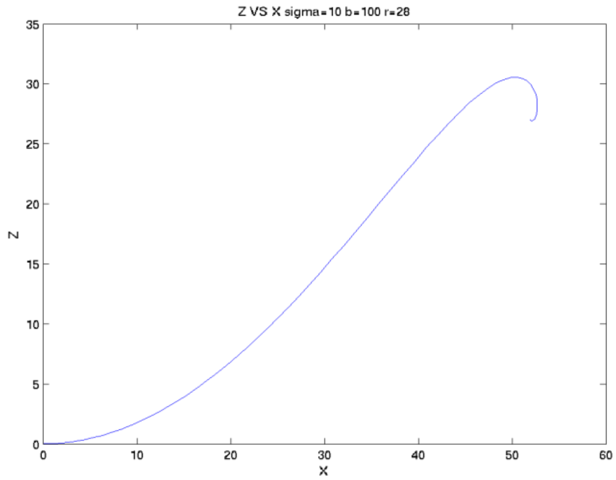

Figure 3.46

Clearly, the stability is reached quicker with the increasing value of the parameter b.

3.9 Next Amplitude Map for the Lorenz System

When we plot the Z vs. t graph for the Lorenz System for the parameter values σ=10, b=8/3 and r=28, we designate the nth local maxima of the plot as z_n and the (n+1)th local maxima as z_{n+1}

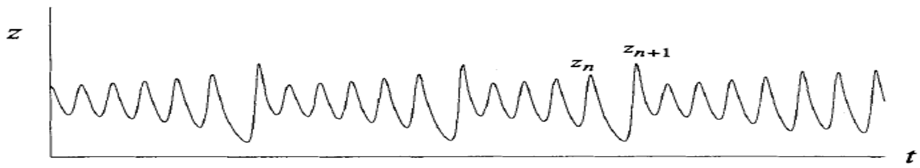

Figure 3.47 (Source→ Strogatz)

Now the plot of z_n vs. z_{n+1} is termed as the Next Amplitude Map. The next amplitude map for the Lorenz System is given below

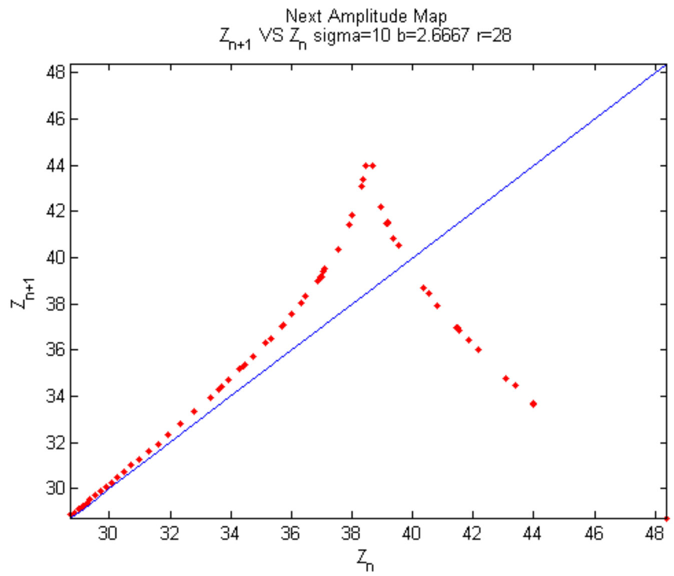

Figure 3.48

3.10 Lyapunov Exponent

Chaos is aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions. The mathematical tool to predict the chaos in a system is the Lyapunov exponent. In mathematics, Lyapunov exponent or Lyapunov characteristic exponent (LCE) of a dynamical system is defined as a quantity that characterizes the rate of separation of two infinitesimally close trajectories.

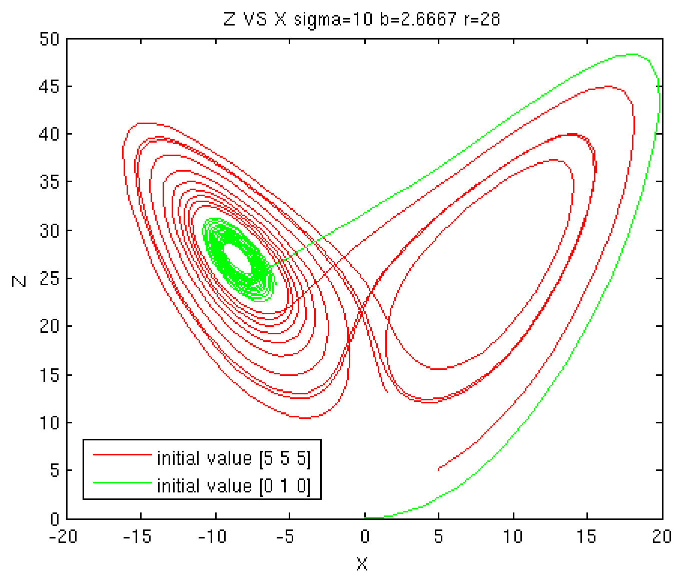

Figure 3.49 Z vs. X plot (Idea of basin of attraction also)

In case of Lorenz system, the trajectory in the phase plane for two nearby initial conditions diverge rapidly and have two completely different future after a small time interval. Theoretically, the rate of divergence of the trajectories are exponential. For an *n*-dimensional system, there are *n*-direction of divergence of two close trajectories, and as such, there are *n* Lyapunov exponents. The largest of these Lyapunov exponents is called the *maximal Lyapunov characteristic exponent (mLCE)*.

The Lyapunov exponent gives a measure of the chaos in a system.

- For an attractor to be chaotic, at least one of the Lyapunov exponents must be positive.
- For a limit cycle attractor, the largest Lyapunov exponent must be zero with the remaining values negative.
- For a fixed point attractor, all the Lyapunov exponents must be negative.

3.10.1 Gram-Schmidt and modified Gram-Schmidt (MGS) method

We define the projection operator as,

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u},$$

Where the left hand side gives the projection of vector **v** on vector **u**. Then by the Gram-Schmidt process, we have

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} \\
&\vdots & &\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

But the Gram-Schmidt process being numerically unstable, the modified Gram-Schmidt (MGS) algorithm is generally used in numerical methods. Instead of calculating the vector $\mathbf{u}_k$ as,

$$\mathbf{u}_k = \mathbf{v}_k - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_k) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_k) - \cdots - \mathrm{proj}_{\mathbf{u}_{k-1}}(\mathbf{v}_k),$$

We calculate it as,

$$\begin{aligned}
\mathbf{u}_k^{(1)} &= \mathbf{v}_k - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_k), \\
\mathbf{u}_k^{(2)} &= \mathbf{u}_k^{(1)} - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{u}_k^{(1)}), \\
&\ \ \vdots \\
\mathbf{u}_k^{(k-2)} &= \mathbf{u}_k^{(k-3)} - \mathrm{proj}_{\mathbf{u}_{k-2}}(\mathbf{u}_k^{(k-3)}), \\
\mathbf{u}_k^{(k-1)} &= \mathbf{u}_k^{(k-2)} - \mathrm{proj}_{\mathbf{u}_{k-1}}(\mathbf{u}_k^{(k-2)}).
\end{aligned}$$

Which is numerically stable.

3.10.2 QR decomposition

Consider the modified Gram-Schmidt process applied to the columns of full column rank matrix, **A**= [$\mathbf{a}_1, \ldots, \mathbf{a}_n$], then,

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{a}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{a}_2 - \mathrm{proj}_{\mathbf{e}_1}\mathbf{a}_2, & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{a}_3 - \mathrm{proj}_{\mathbf{e}_1}\mathbf{a}_3 - \mathrm{proj}_{\mathbf{e}_2}\mathbf{a}_3, & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
&\ \ \vdots & &\ \ \vdots \\
\mathbf{u}_k &= \mathbf{a}_k - \sum_{j=1}^{k-1}\mathrm{proj}_{\mathbf{e}_j}\mathbf{a}_k, & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}
\end{aligned}$$

Thus the matrix **A** can be written as, **A**=**QR**, where,

Q= [$\mathbf{e}_1, \ldots, \mathbf{e}_n$] and **R**=$\mathbf{Q}^T$ **A**.

We have,

det (**A**) =det (**Q**).det (**R**)

Since **Q** is unitary, | det (**Q**) |=1. Therefore,

$$|\det(\mathbf{A})| = |\det(\mathbf{R})| = |\prod_i r_{ii}|,$$

Where r_{ii} are the diagonal elements of the matrix **R**. Furthermore, because the determinant equals the product of the eigenvalues, we have,

$$|\prod_i r_{ii}| = |\prod_i \lambda_i|,$$

Where, λ_i are the eigenvalues of **A**.

3.10.3 Calculation of Lyapunov exponents

Numerically, the Lyapunov exponent is calculated by the modified Gram-Schmidt or MGS process. For the Lorenz system, the Jacobian is defined as,

$$\mathbf{J_j} = \begin{bmatrix} -\sigma & \sigma & \mathbf{0} \\ \boldsymbol{r - z_j} & \mathbf{-1} & \boldsymbol{-x_j} \\ \boldsymbol{y_j} & \boldsymbol{x_j} & \boldsymbol{-b} \end{bmatrix}$$

We start with $\mathbf{Q_1}=\mathbf{I_3}$, where $\mathbf{I_3}$ is a 3x3 identity matrix serving as the initial orthonormal bases. At any step of iteration, ***j,*** we have $\mathbf{A}=\mathbf{Q_j J_j}=\mathbf{Q_{j+1} R_j}$, where an ellipse (except for the first step when we start with a circle) whose axes are defined by the orthonormal bases $\mathbf{Q_j}$ is mapped into another ellipse defined by orthonormal bases $\mathbf{Q_{j+1}}$, according to the MGS algorithm. The diagonal elements of the upper triangular matrix give the three eigenvalues along the three orthonormal direction defined by $\mathbf{Q_{j+1}}$.

Therefore, the Lyapunov exponents are given by,

$$\Lambda_i = \frac{1}{T}\sum_{j=1}^{j=m} ln\,(|r_{ii}|)$$

Where, r_{ii} are the diagonal elements of $\mathbf{R_j}$, m is the total number of time steps and T is the total time over which the Lyapunov exponents are calculated. The numerically obtained Lyapunov exponents for different time intervals of numerical integration and different time steps are tabulated below.

Time of numerical integration (T)	Time step (dt)	Lyapunov exponents (Λ_i)	Sum of the Lyapunov exponents ($\sum_{i=1}^{3} \Lambda_i$)
100	0.001	0.9023, 0.0161, -14.6201	-13.7016
1000	0.001	0.9607, 0.0155, -14.6796	-13.7034

In any step the sum of the Lyapunov exponent is equal to the trace of the Jacobian of the Lorenz system which is a constant. For σ=10, b=8/3,

$$\sum_{i=1}^{3} \Lambda_i = -(\sigma+b+1) = -13.667$$

3.11 Poincaré Map

In dynamical system, a Poincaré map is the intersection of a periodic orbit of a state space of a continuous dynamical system with a lower dimensional subspace, called Poincaré section, transverse to the flow direction in the state space. In recurrent dynamics, a trajectory enters a region of state space and then re-enters with a definite return time. This gives rise to an iterative mapping and helps in representation of the flow in a space of lesser dimension. This iterative technique of mapping is called a Poincaré map.

- **Definition**

Let us consider an *n*-dimensional space vector, $\mathbf{x}$=f($\mathbf{x}$). Let *S* be any Poincaré section of dimension (*n-1*) of the Poincaré map *P*. Then *S* should be transverse to the direction of the flow, that is, all the trajectories starting on *S* should flow through it and not parallel to it.
The Poincaré map *P* is a map from *S* to itself. If $\mathbf{x}_k \in S$ be k^{th} intersection on the Poincaré map then,

$$\mathbf{x}_{k+1}=P(\mathbf{x}_k)$$

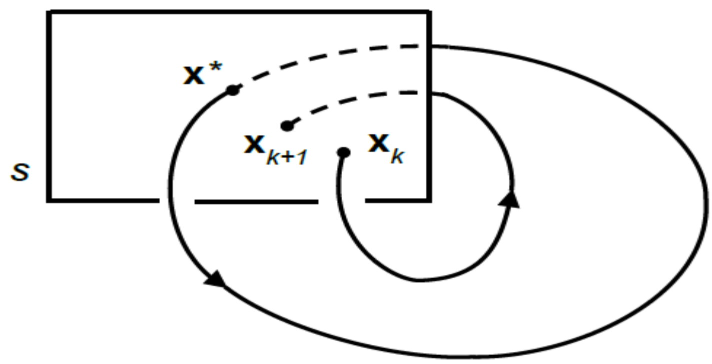

Figure 3.50 Poincaré Map (Source→ Strogatz)

If $\mathbf{x^*}$ be a fixed point on the section *S,* then *P*($\mathbf{x^*}$) =$\mathbf{x^*}$. That is, a trajectory starting from the $\mathbf{x^*}$ returns to $\mathbf{x^*}$ after a time period of T.
The Poincaré map converts problems about closed orbits (which are difficult) into problems about fixed points of a mapping (relatively easier in general). The choice of the Poincaré section S is altogether arbitrary. It is rarely possible

to define a single section that cuts across all trajectories of interest. With a sufficiently clever choice of a Poincaré section or a set of sections, any orbit of interest intersects a section.

3.12 Recurrence Plot

- **Definition**

  Recurrence Plot is defined as an advanced technique of non-linear data analysis. It is basically a graph of a square matrix, in which the matrix elements correspond to those times at which a state of a dynamical system recurs. So basically Recurrence Plot refers to all the times when the phase space trajectory of the dynamical system visits roughly the same area in the phase space.

- **Figure**

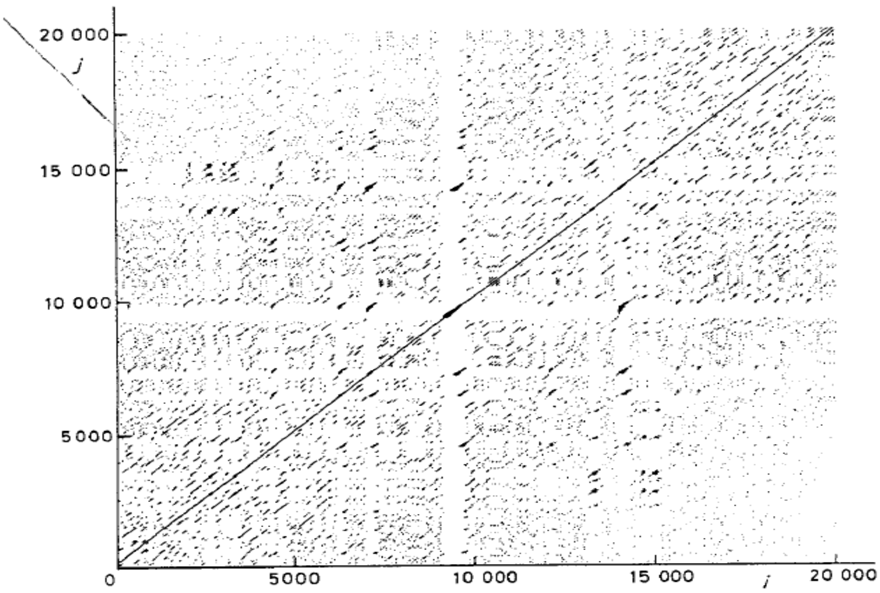

Figure 3.51 (Source→ Eckmann)

The Lorenz system with 10% drift for a total time of 200. The embedding dimension is 3, using the three coordinates as 3 channels of data. A total of $20\,000 \times 3$ data points is used.

- **Description**

  The recurrence plot is basically a graph in which time is plotted both along the x-axis and y-axis. There is a main diagonal of the recurrence plot which is inclined at an angle of 45° with both the axes since at a particular time the trajectory visits only a particular point. So, the coordinates (j, j) lies on the recurrence plot.

For a periodic system, the other points in the recurrence plot will lie on a straight line and the straight line will be parallel to the main diagonal.

On the other hand, for a chaotic system the points on the recurrence plot are scattered and there is no pattern that we can follow.

- **Types**
  1. **Cross recurrence plot**→It shows all those times at which a state in one dynamical system occurs simultaneously in a second dynamical system. So, cross recurrence plot reveals all the times when the phase space trajectory of the first system visits roughly the same area in the phase space where the phase space trajectory of the second system is.
  2. **Joint recurrence plot**→It shows all those times at which a recurrence in one dynamical system occurs simultaneously with a recurrence in a second dynamical system.

- **Recurrence Quantification Analysis**

  Recurrence Quantification Analysis is a method of nonlinear data analysis for the investigation of the dynamical systems. It quantifies the number and duration of recurrences of a dynamical system presented by its phase space trajectory.

- **Applications of the Recurrence Plot**
  1. Recurrence plots are an effective way to visualize the geometry of a dynamical system's behaviour
  2. Recurrence Plot brings out distance correlation of a time series
  3. Recurrence plots make it instantly apparent whether a system is periodic or chaotic
  4. Recurrence plots may be a better way to do a quick comparison of two dynamical systems.

3.13 Introduction to Time Series Analysis

- The analysis of time series is based on the assumption that successive values in the data file represent consecutive measurements taken at equally spaced time intervals

- The two main goals of the Time Series Analysis are
  1. Identifying the nature of the phenomenon represented by the sequence of observations
  2. Forecasting, which is basically the method to predict the future values of the time series variable
- Other objectives of the Time Series Analysis are
  1. Data compression
  2. Relationship with other variables
  3. Signal Processing
- Example of a time series:
  Measurement of the value of retail sales each month of the year would comprise a time series. This is because sales revenue is well defined, and consistently measured at equally spaced intervals.
- An observed time series can be decomposed into three components
  1. The trend
  2. The Seasonal
  3. The irregular

Further discussions on Time Series Analysis are not done in this project due to shortage of time. But surely, by having an elementary idea about the Recurrence Plot and Time Series Analysis, we can work in these topics in future.

3.14 Phase Space Reconstruction-Basic Idea

- From the embedding theorem, we know that it is possible to reconstruct the phase space from the time series
- A frequently used method for the phase space reconstruction is the method of time delay
- The purpose of the phase space reconstruction is to construct the system state by using its history and viewing it in a higher dimensional space

3.15 Applications of the Lorenz System and Chaos

- Lorenz System is one of the most simplest systems in order to study Chaos
- Chaotic behaviour exists in many natural systems like weather and climate. This behaviour can be studied through the Lorenz System.

Therefore, Lorenz System has an extensive application in the field of Weather System

- Secret messages can be sent using chaos of the Lorenz System
- Private communications can also be done using synchronized chaos
- Chaotic Waterwheel can be modelled using the Lorenz System of Equations

CHAPTER 4

INTRODUCTION TO WIND TUNNEL AND VIBRATION EXPERIMENTS

Figure 4.1 (Source→ Internet)

A Wind Tunnel in Aerospace Engineering Department, IIT Madras

4.1 What is a Wind Tunnel?

A wind tunnel is a machine that can simulate the flow of air past any solid object. In a wind tunnel, air is made to flow around the test body commonly called wind tunnel model, which is provided with suitable sensors to measure pressure distribution, aerodynamic forces and other aerodynamic related properties. It is a popular tool for scientists, engineers and researchers where a scaled model of a real life object can be studied in varying aerodynamic conditions by varying the flow parameters in a wind tunnel.

4.2 Working Principle

In a wind tunnel air is made to flow with the help of a powerful fan system controlled by a motor or a turbofan. Air is blown or sucked through the duct of the wind tunnel depending on the direction of rotation of the fan.

The model is preferably kept at the centre of the tunnel to avoid boundary turbulence and the duct is equipped with a viewing port. For very large wind tunnels, blowing of air through one large fan is not practical. In this case, air is made to flow with the help of multiple fans connected in series. The air flow entering the tunnel section from the fan system is highly turbulent. The turbulence is not a big problem when wind is sucked from downstream of the tunnel, in which case the wind turbulence can be neglected. Nevertheless, the problem is solved by using horizontal and vertical air vanes to smoothen the wind turbulence before the wind acts on the test object. Further to overcome the effect of turbulence, the cross section of the tunnel is generally made circular instead of square or other shapes, because the air flow will be more constricted in the corners of a square rather than in a circular section. The inside surface of the tunnel is made as smooth as possible. All these factors ensure that the air flow through the wind tunnel is relatively laminar and turbulence-free. There are correction factors to relate wind tunnel results to open-air results. The following operations are generally done in the wind tunnel:

- **Force Measurement**

Aerodynamic forces on the model can be determined by beam balances, connected to test model by beams, springs and cables.

- **Pressure Measurement**

Pressure distribution on the model can be measured by 'Pressure-Sensitive Paints (PSP)', substances which show reduced fluorescence at points of high local pressure. Pressure on model can also be determined if it includes 'pressure belts', in which multiple ultra-miniaturized pressure sensor modules are integrated into a flexible strip. The strip is attached to the aerodynamic surface with tape which sends signals depicting the pressure distribution along its surface.

- **Wind Velocity Measurement**

The wind velocity is commonly measured in the wind tunnel with the help of a static pitot tube. A more sophisticated way of measuring the wind velocity is by high sensitivity anemometers.

- **Displacement Measurement**

The displacement of the deformed structure of the test model by the action of the flowing wind is generally measured by the use of lasers of sensitivity of the order of one micron.

- **Data Acquisition**

Data acquisition is the process of sampling real world analog data recorded by the various sensors into digital form to be fed into a computer for further processing. This is normally done by devices called Data Acquisition Systems (DASs or DAQs). The components of data acquisition systems include:

- Sensors that convert physical parameters to electrical signals.
- Signal conditioning circuitry to convert sensor signals into a form that can be converted to digital values.

- Analog to digital converters, which convert conditioned sensor signals to digital values.
- **Aerodynamic Parameter Scaling**

Certain aerodynamic properties of the test model are made similar to the actual real life object. These are usually:

- Geometric Similarity: The dimensions of the model are proportionately scaled to the real object.
- Mach number: The ratio of the airspeed to the speed of sound should be identical for the scaled model and the actual object.
- Reynolds number: The ratio of the inertial force to the gravity force should be kept constant. This property is difficult to achieve and has led to the development of cryogenic and pressurized wind tunnel in which the viscosity of the working can be greatly regulated for scaled models.
- Froude number, etc.

4.3 Flow Visualisation

Air is transparent and as such it is difficult to observe air movement directly. Instead, multiple methods of both quantitative and qualitative flow visualization methods have been developed for testing in a wind tunnel.

- **Qualitative Methods:**
  - Smoke
  - Tuft: Tufts are applied to a model and remain attached during testing. Tufts can be used to gauge air flow patterns and flow separation.
  - Evaporating Suspensions: Evaporating suspensions are simply a mixture of some sort of fine powder, talc, or clay mixed into a liquid with a low latent heat of evaporation. When the wind is turned on the liquid quickly evaporates leaving behind the clay in a pattern characteristic of the air flow.
  - Oil: When oil is applied to the model surface it can clearly show the transition from laminar to turbulent flow as well as flow separation.
  - Fog: Fog (usually from water particles) is created with an ultrasonic piezoelectric nebulizer. The fog is transported inside the wind tunnel (preferably of the closed circuit & closed test section type). An electrically heated grid is inserted before the test section which evaporates the water particles at its vicinity thus forming fog sheets. The fog sheets function as streamlines over the test model when illuminated by a light sheet.

4.4 Classification

- **Subsonic (Low Speed) Wind Tunnel**

  Low speed wind tunnels are used for low wind speed (~134m/s) and low Mach number (M=0.4). They may be of open flow type (also called Eiffel type) and closed flow type (also called Prandtl type).

- **Transonic Wind Tunnel**

  High subsonic wind tunnels ($0.4 < M < 0.75$) and transonic wind tunnels ($0.75 < M < 1.2$) are designed on the same principles as the low speed wind tunnels. The Mach number is approximately 1 with combined subsonic and supersonic flow regions. Testing at transonic speeds presents additional problems, mainly due to the reflection of the shock waves from the walls of the test section. Therefore, perforated or slotted walls are required to reduce shock reflection from the walls.

4.5 Applications

Wind tunnel is used to study any kind of aero elastic and aerodynamic behaviour of a structure. Here are some of the common applications of it.

1. *In the study of fluttering in structures like aircraft wings and windmill blades:* Structures exposed to aerodynamic forces like wings, aerofoils, chimneys and bridges are designed carefully within known parameters to avoid fluttering. In complex structures where both the aerodynamics and the mechanical properties of the structure are not fully understood, flutter can be discounted only through detailed testing which are generally undertaken in a wind tunnel.
2. *In the study of harvesting of wind energy by vibrating structures using piezoelectric substances:* An energy harvester comprising a cantilever attached to piezoelectric patches and a proof mass is developed for wind energy harvesting, from a vortex induced vibration of the cantilever, by the electromechanical coupling effect of piezoelectric materials. The vibration of the cantilever is induced by the air pressure owing to a vortex shedding phenomenon that occurs on the leeward side of the cantilever. To describe the energy harvesting process, a theoretical model considering the vortex induced vibration on the piezoelectric coupled cantilever energy harvester is developed, to calculate the charge and the voltage from the harvester. The influences of the length and location of the piezoelectric patches on the generated electric power are investigated. Results from numerous works show that the total generated electric power can be maximum when the resonant frequency of the cantilever harvester is close to the vortex shedding

frequency. This type of research work with the help of wind tunnel facilitates an effective and compact wind energy harvesting device.

3. *For checking the safety and design efficiency of cars, aircrafts and spacecraft:* The design efficiency of racing cars and aircrafts needs to be modelled against aerodynamics forces satisfying certain safety requirements. Such models are validated by wind tunnel experiments.
4. *For the validation of results obtained from using commercial CFD packages:* The results obtained from various CFD softwares need to be verified by wind tunnel experiments for their accuracy before implementing them in real life.

4.6 Experiments on vibration

- **Apparatus Used**

Function Generator: A function generator is an electronic device embedded with software that is used to generate different types of electrical signals over a wide range of frequencies. The most common types of signals generated are the sine, square, triangular and saw-toothed waveform. These waveforms can be either repetitive or single-shot.

Figure 4.2 Function Generator (Source→ Internet)

Power Amplifier: An amplifier is an electronic device which increases the power of a signal. It does so by taking power from a supply source and controlling the output signal by matching it with the input signal shape but with larger amplitude.

Shaker: It is used to feed the signal obtained after magnifying from the amplifier to the structural element under test.

<u>Accelerometer:</u> These are electronic sensors used for measuring the acceleration of the structural element when it is subjected to the generated signal.

<u>Oscilloscope:</u> An oscilloscope is an electronic instrument which is used to capture the response of one or more signals, generally plotting them as a function of time in two dimensions.

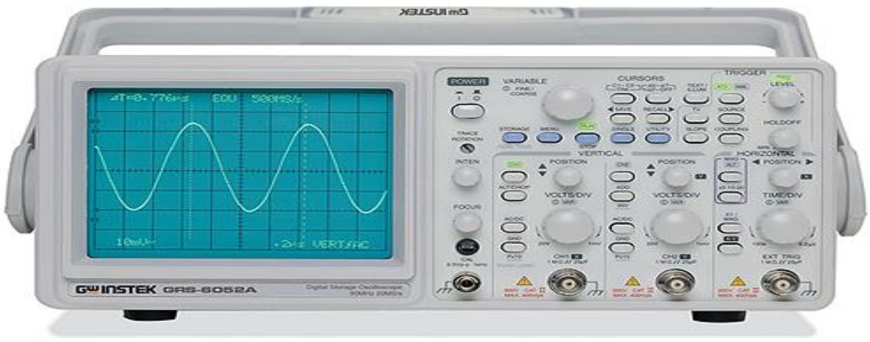

Figure 4.3 Oscilloscope (Source→ Internet)

- **Procedure of Experiment**

Firstly, the signal is generated by the function generator which is then amplified in strength by the power amplifier. The amplified signal is fed to the shaker which is kept in contact with a structural member (which in our case is a cantilever). The signal causes the member to vibrate. The accelerometer sensors attached to the member receives the response of the structure which after being amplified is fed into an oscilloscope for visualisation of the response and for further processing. The response signal obtained from the oscilloscope is not free from noise. The sources of the noise are, the inherent frequency of the AC supply to the amplifier and the loose attachment of the shaker and the accelerometers on the structural element. In our experimental work, the data set or the time response is being fed into a MATLAB code and plotted as shown below.

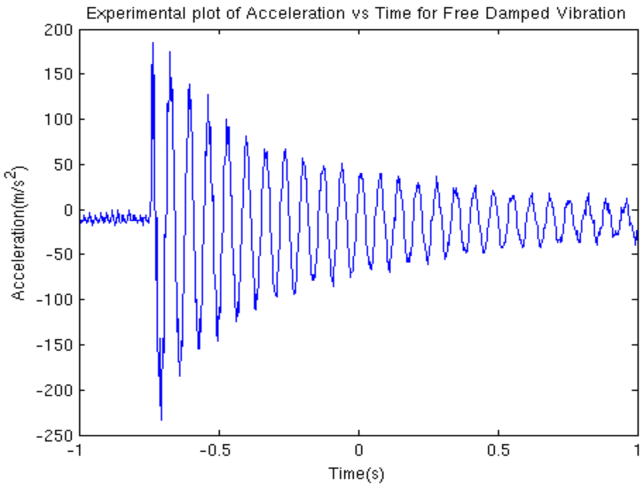

Figure 4.4

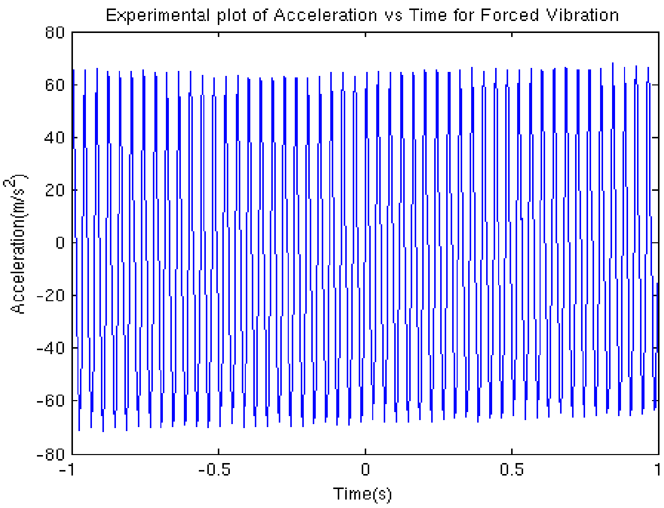

Figure 4.5

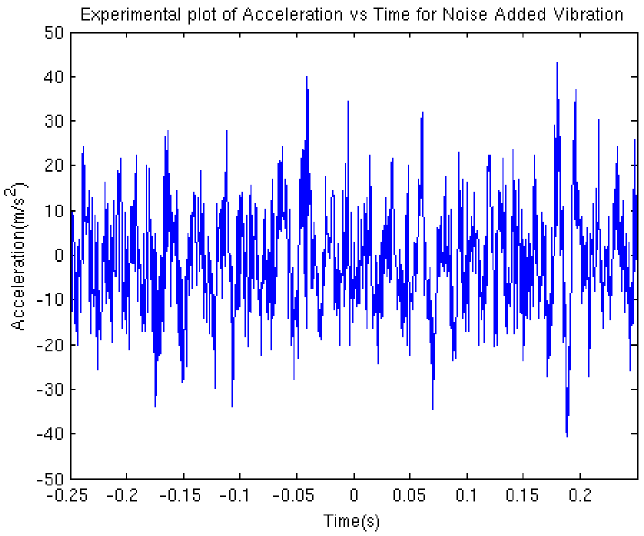

Figure 4.6

CHAPTER 5

CONCLUSION

In this project, at first we studied about the basics of Structural Dynamics. We came to know about the characteristics of SDOF System and later we plotted the response of free damped vibration, forced damped vibration and also the vibratory motion of a Simple Pendulum in MATLAB and studied the behaviour of the responses.

We were then introduced to the concepts of Non-Linear Dynamics which is quite an interesting domain of studying Higher Physics. We learnt about One Dimensional Flow and Bifurcations. We studied different types of Bifurcations and plotted the Bifurcation Curves in MATLAB. We also plotted the trajectories for different types of bifurcations.

Our main project work involved the study of the Lorenz System. At first, we observed the behaviour of the trajectories of the Lorenz System by varying the three parameters. We also plotted the bifurcation diagram and the Next Amplitude Map. We observed the global stability of the origin for Rayleigh Number (r) less than 1. We also tried to find out the basin of attraction by varying the initial conditions of the system. We learnt about the Lyapunov Exponent and by finding this for a particular value of the parameter r, we tried to infer whether the system is chaotic or periodic. We also found the intermittency in the Lorenz System.

Our study also involved the knowledge of Poincaré Maps. We studied briefly about the Recurrence Plot and Time Series Analysis. Finally we attended a Wind Tunnel Demonstration and later we did some vibration experiments which concluded our Project.

This project was undoubtedly a superb experience for us where we had a little exposure to a Non-linear Dynamical system. Surely, in future we can work in this field of Non-linear dynamics again and then the experience gathered by us from this project will be a great help. Learning many new topics was the key in this project and obviously it was a great learning curve for us. Though we did not work in Recurrence Plot and Time Series Analysis more extensively due to shortage of time, both of us are eager to know these things and work in these topics in future.

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