# 1 Hamiltonian of the System without Adjoint

The Hamiltonian of the system without adjoint variables is given by:

$$H = \frac{1}{2}m\boldsymbol{v}^2 - \frac{GM_1 m}{|\boldsymbol{r} - \boldsymbol{r}_1|} - \frac{GM_2 m}{|\boldsymbol{r} - \boldsymbol{r}_2|} \tag{1}$$

where $\boldsymbol{r}$ is the position of the planet, $\boldsymbol{v}$ is its velocity, $G$ is the gravitational constant, $M_1$ and $M_2$ are the masses of the two suns, and $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are their positions.

## 1.1 Hamilton's Equations

In classical mechanics, Hamilton's equations describe the evolution of a physical system. These equations relate the Hamiltonian of the system to the rates of change of the position and momentum (or velocity, in the case of unit mass). For a system described by the Hamiltonian $H$, the equations are:

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial H}{\partial \boldsymbol{p}} \tag{2}$$

$$\frac{d\boldsymbol{p}}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}} \tag{3}$$

where $\boldsymbol{p}$ is the momentum vector of the planet, which, for a particle of unit mass $m = 1$, equals the velocity vector $\boldsymbol{v}$.

## 1.2 State Equations Derived from Hamiltonian

The state equations of our system describe how the position and velocity of the planet evolve over time. Using Hamilton's equations, we derive these state equations directly from the Hamiltonian:

1. **Position Equation:** From Hamilton's equation, the rate of change of position is given by the derivative of the Hamiltonian with respect to the momentum (velocity):

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial H}{\partial \boldsymbol{v}} = \boldsymbol{v} \tag{4}$$

This indicates that the velocity of the planet is the rate of change of its position, as expected in physical systems.

2. **Velocity Equation:** Similarly, the rate of change of momentum (or velocity, for unit mass) is given by the negative gradient of the Hamiltonian with respect to position:

$$\frac{d\boldsymbol{v}}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}} = -\left(\frac{GM_1(\boldsymbol{r}-\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^3} + \frac{GM_2(\boldsymbol{r}-\boldsymbol{r}_2)}{|\boldsymbol{r}-\boldsymbol{r}_2|^3}\right) \tag{5}$$

This equation reflects the gravitational forces acting on the planet due to the two suns, pointing towards decreasing the potential energy, which is consistent with the nature of conservative forces in physics.

### 1.3 Finite Differencing

To compute the gradients using finite differencing, we approximate the derivatives as:

$$\frac{\partial H}{\partial \boldsymbol{r}} \approx \frac{H(\boldsymbol{r}+\Delta\boldsymbol{r},\boldsymbol{v}) - H(\boldsymbol{r},\boldsymbol{v})}{\Delta\boldsymbol{r}} \tag{6}$$

$$\frac{\partial H}{\partial \boldsymbol{v}} \approx \frac{H(\boldsymbol{r},\boldsymbol{v}+\Delta\boldsymbol{v}) - H(\boldsymbol{r},\boldsymbol{v})}{\Delta\boldsymbol{v}} \tag{7}$$

### 1.4 Gradient Descent for Optimization

Using gradient descent, we update the position and velocity at every $n$-th timestep:

$$\boldsymbol{r}_{n+1} = \boldsymbol{r}_n - \eta\nabla a(\boldsymbol{r}_n) \tag{8}$$

where $\eta$ is the learning rate and $\nabla a(\boldsymbol{r})$ is the gradient of the objective function with respect to $\boldsymbol{r}$.

## 2 Hamiltonian of the System with Adjoint

The Hamiltonian for a system with adjoint variables is constructed to include terms that account for the dynamics (state equations) and constraints of the system. It is given by:

$$H = a(\boldsymbol{r}) + \boldsymbol{\lambda}_{\boldsymbol{r}} \cdot \boldsymbol{v} + \boldsymbol{\lambda}_{\boldsymbol{v}} \cdot \left(-\frac{GM_1(\boldsymbol{r}-\boldsymbol{r}_1)}{|\boldsymbol{r}-\boldsymbol{r}_1|^3} - \frac{GM_2(\boldsymbol{r}-\boldsymbol{r}_2)}{|\boldsymbol{r}-\boldsymbol{r}_2|^3}\right) \tag{9}$$

where $a(\boldsymbol{r})$ is the objective function representing the deviation of the planet's position $\boldsymbol{r}$ from a desired trajectory $\boldsymbol{r}_{\text{figure8}}(t)$.

## 2.1 Derivation of Adjoint Equations

In optimal control theory, adjoint equations are derived from the Hamiltonian using Pontryagin's Maximum Principle, which provides a way to find the optimal control laws for a dynamic system. The principle states that the optimal trajectory and control must minimize the Hamiltonian at each point in time.

The adjoint equations describe the evolution of the adjoint variables ($\boldsymbol{\lambda_r}$ and $\boldsymbol{\lambda_v}$), which are Lagrange multipliers associated with the state variables of the system. These equations are obtained by taking the partial derivatives of the Hamiltonian with respect to the state variables:

1. **Adjoint Equation for $\boldsymbol{\lambda_r}$:** This is derived by differentiating the Hamiltonian with respect to the position $\boldsymbol{r}$ and taking the negative of the result:

$$\frac{d\boldsymbol{\lambda_r}}{dt} = -\frac{\partial H}{\partial \boldsymbol{r}} = -\left(\nabla a(\boldsymbol{r}) + \boldsymbol{\lambda_v} \cdot (\boldsymbol{H_1} + \boldsymbol{H_2})\right) \tag{10}$$

Here, $\nabla a(\boldsymbol{r})$ is the gradient of the objective function with respect to $\boldsymbol{r}$, and $\boldsymbol{H_1}$, $\boldsymbol{H_2}$ are the Hessians of the gravitational potentials due to the suns, representing the curvature of the potential field at $\boldsymbol{r}$.

2. **Adjoint Equation for $\boldsymbol{\lambda_v}$:** This equation is obtained by differentiating the Hamiltonian with respect to the velocity $\boldsymbol{v}$ and taking the negative of the result:

$$\frac{d\boldsymbol{\lambda_v}}{dt} = -\frac{\partial H}{\partial \boldsymbol{v}} = -\boldsymbol{\lambda_r} \tag{11}$$

This reflects that the change in the adjoint variable $\boldsymbol{\lambda_v}$ is directly influenced by the adjoint position variable $\boldsymbol{\lambda_r}$.

## 2.2 Hessians of the Gravitational Potential

The Hessians $\boldsymbol{H_1}$ and $\boldsymbol{H_2}$, which are 2x2 matrices in this 2D system, are calculated as follows, taking into account only the components involved in the motion plane:

$$\boldsymbol{H_1} = \begin{bmatrix} \frac{-GM_1(3(\boldsymbol{r}_x - \boldsymbol{r}_{1x})^2 - |\boldsymbol{r}-\boldsymbol{r}_1|^2)}{|\boldsymbol{r}-\boldsymbol{r}_1|^5} & 0 \\ 0 & \frac{-GM_1(3(\boldsymbol{r}_y - \boldsymbol{r}_{1y})^2 - |\boldsymbol{r}-\boldsymbol{r}_1|^2)}{|\boldsymbol{r}-\boldsymbol{r}_1|^5} \end{bmatrix} \tag{12}$$

$$\boldsymbol{H_2} = \begin{bmatrix} \frac{-GM_2(3(\boldsymbol{r}_x - \boldsymbol{r}_{2x})^2 - |\boldsymbol{r}-\boldsymbol{r}_2|^2)}{|\boldsymbol{r}-\boldsymbol{r}_2|^5} & 0 \\ 0 & \frac{-GM_2(3(\boldsymbol{r}_y - \boldsymbol{r}_{2y})^2 - |\boldsymbol{r}-\boldsymbol{r}_2|^2)}{|\boldsymbol{r}-\boldsymbol{r}_2|^5} \end{bmatrix} \tag{13}$$

## 2.3 Update Based on Adjoint Simulation

The update of the positional term based on the adjoint simulation is given by:

$$\frac{d\boldsymbol{\lambda_r}}{dt} = -\left(\nabla a(\boldsymbol{r}) + \boldsymbol{\lambda_v} \cdot (\boldsymbol{H_1} + \boldsymbol{H_2})\right) \tag{14}$$

To compute $\boldsymbol{\lambda_r}$ and $\boldsymbol{\lambda_v}$ at each timestep:

$$\boldsymbol{\lambda_r}^{n+1} = \boldsymbol{\lambda_r}^n - \left(\nabla a(\boldsymbol{r}^n) + \boldsymbol{\lambda_v}^n \cdot (\boldsymbol{H_1} + \boldsymbol{H_2})\right) \Delta t \tag{15}$$

$$\boldsymbol{\lambda_v}^{n+1} = \boldsymbol{\lambda_v}^n - \boldsymbol{\lambda_r}^n \Delta t \tag{16}$$

## 2.4 Adjoint Gradient Descent for Optimization

Using the adjoint variables, we compute the gradient and update the parameters:

$$\frac{\partial J}{\partial \boldsymbol{r}} = \boldsymbol{\lambda_r} \tag{17}$$

The positions are updated at every $n$-th timestep as:

$$\boldsymbol{r}_{n+1} = \boldsymbol{r}_n - \eta \boldsymbol{\lambda_r}^n \tag{18}$$