Random future lifetime

LIFE INSURANCE PRODUCTS VALUATION IN R

Katrien Antonio, Ph.D.
Professor, KU Leuven and University of Amsterdam

The random future lifetime

- (x) denotes an individual aged x at this moment, with $x \geq 0$.
- The random variable T_x is the **future lifetime** of (x).
- Thus, **age at death** of (x) is $x + T_x$.

The life table in R

- Human Mortality Database (HMD, **www.mortality.org**).
- `life_table` contains the period life table for males in Belgium of 2013.

```
head(life_table, 10)
```

```
   age      qx     lx  dx    ex
1    0 0.00381 100000 381 77.95
2    1 0.00047  99619  47 77.24
3    2 0.00019  99572  19 76.28
4    3 0.00015  99553  15 75.30
5    4 0.00013  99538  13 74.31
6    5 0.00010  99525  10 73.32
7    6 0.00011  99514  11 72.32
8    7 0.00008  99504   8 71.33
9    8 0.00011  99496  11 70.34
10   9 0.00008  99485   8 69.34
```

Mortality rates and survival probabilities

- The one-year probability of **dying**

$$q_x = \Pr(T_x \leq 1).$$

  q_x is the **mortality rate** at age x.

- The one-year probability of **surviving**

$$p_x = \Pr(T_x > 1).$$

- Thus, $p_x = 1 - q_x$.

Mortality rates of Belgian sportsmen in R

- Eden Hazard is a Belgian footballer who plays for Chelsea and is 27 years old.

```
age <- life_table$age
qx <- life_table$qx
qx[age == 27]
```

```
0.00062
```

```
qx[27 + 1]
```

```
0.00062
```

- Eddy Merckx is a Belgian cyclist who won the Tour de France 5 times and is 72.

```
qx[age == 72]
```

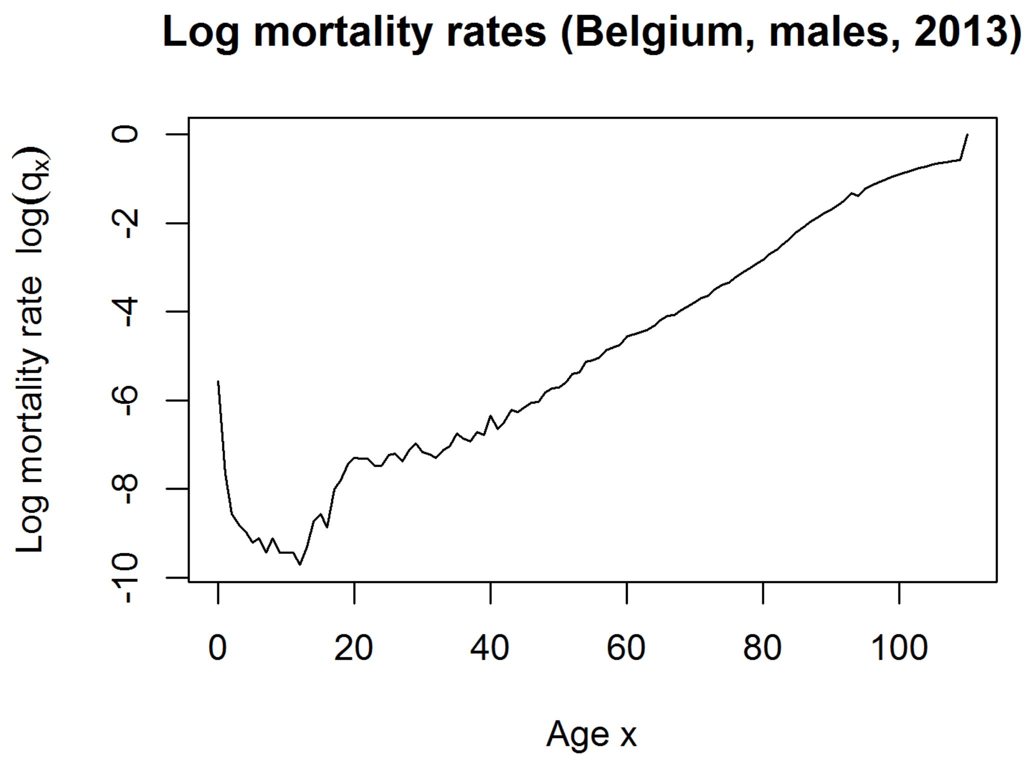
```
0.02631
```

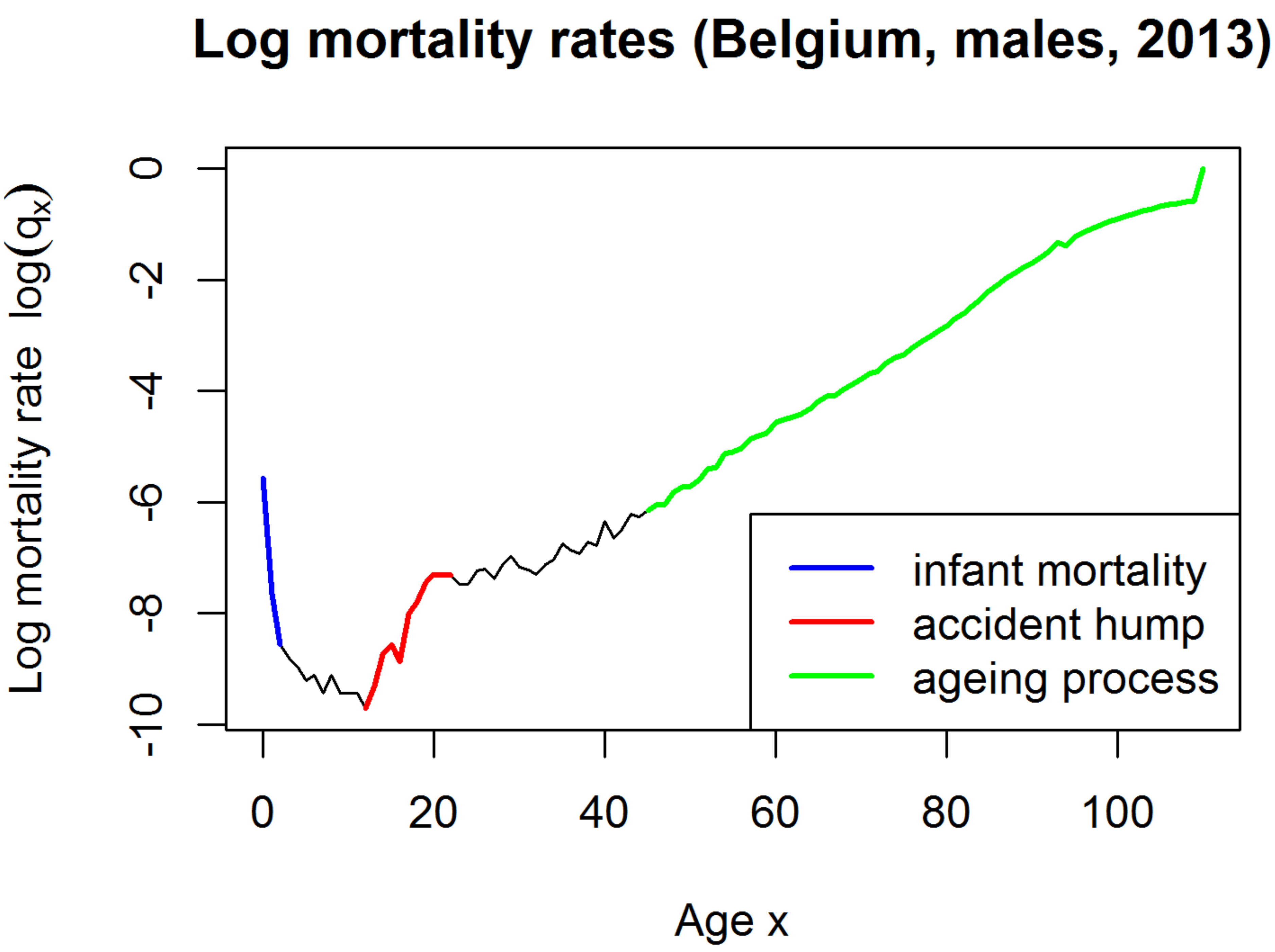
```
qx[72 + 1]
```

```
0.02631
```

Picturing Belgian mortality rates q_x in R

```
plot(age, log(qx), main = "Log mortality rates (Belgium, males, 2013)",
     xlab = "Age x", ylab = expression(paste("Log mortality rate ", log(q[x]))),
     type = "l")
```




Log mortality rates (Belgium, males, 2013)
Log mortality rate log(qx)
0
-2
-4
-6
-8
-10
0
20
40
60
80
100
Age x
infant mortality
accident hump
ageing process

The life expectancy

- The (complete) **expected future lifetime** of (x) is $E[T_x]$
- For Eden Hazard who is 27 years old:

```
ex <- life_table$ex
ex[27 + 1]
```

```
51.74
```

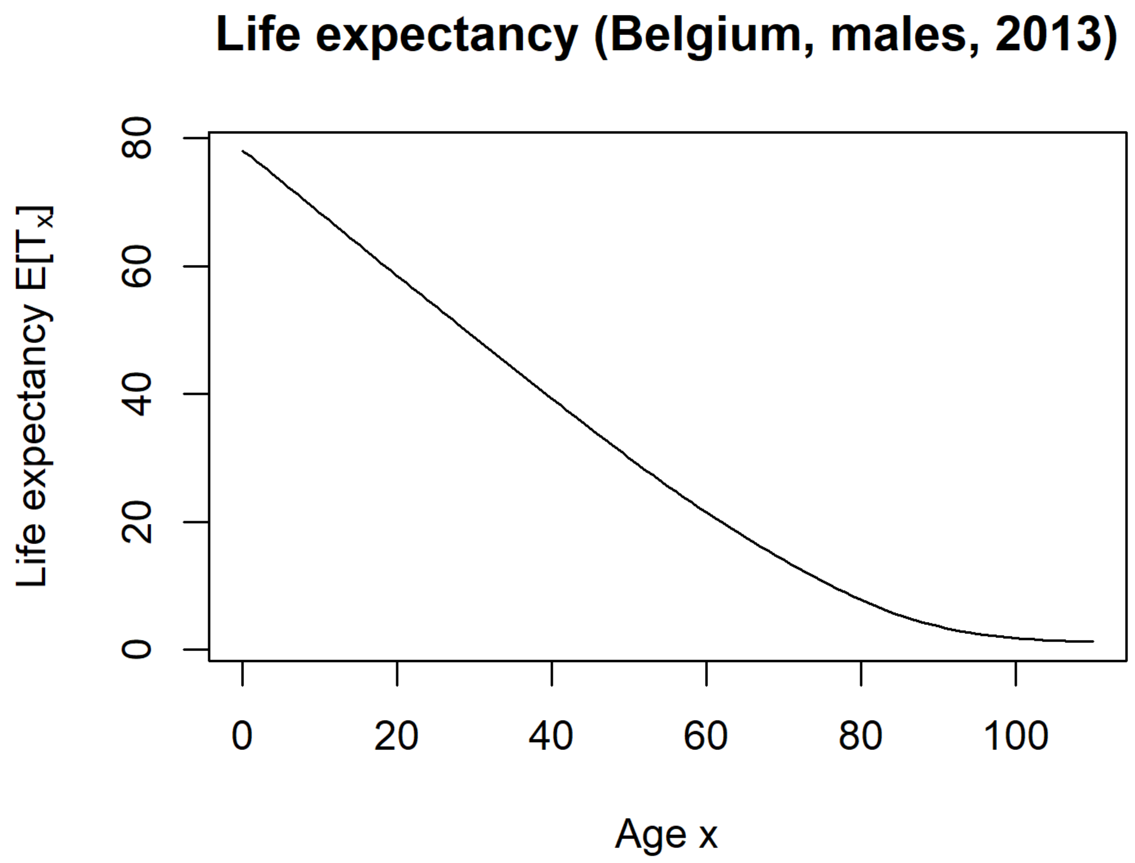
- For Eddy Merckx who is 72 years old:

```
ex[72 + 1]
```

```
12.67
```

Picturing the life expectancy in R

```
plot(age, ex, main = "Life expectancy (Belgium, males, 2013)", xlab = "Age x",
     ylab = expression(paste("Life expectancy E[", T[x], "]")), type = "l")
```



Let's practice!

LIFE INSURANCE PRODUCTS VALUATION IN R

Binomial experiments

LIFE INSURANCE PRODUCTS VALUATION IN R

Roel Verbelen, Ph.D.
Statistician, Finity Consulting

The life table in R

- `life_table` contains the period life table for males in Belgium of 2013.

```
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7    6 0.00011  99514  11 72.32
8    7 0.00008  99504   8 71.33
9    8 0.00011  99496  11 70.34
10   9 0.00008  99485   8 69.34
```

```
# Variables used in this video
qx <- life_table$qx
px <- 1 - qx
lx <- life_table$lx
dx <- life_table$dx
```

A binomial experiment: surviving one year

- Focus on ℓ_x in `life_table` .

```
lx[0 + 1]
```

```
1e+05
```

A binomial experiment: surviving one year

- The number of survivors up to age $x + 1$ follows a BIN(ℓ_x, p_x).

```
lx[72 + 1]
```

```
73977
```

```
px[72+ 1]
```

```
0.97369
```

```
rbinom(n = 1, size = lx[72 + 1], prob = px[72 + 1])
```

```
72022
```

A binomial experiment: surviving one year

- Now in a vectorized way!

```
sims <- rbinom(n = length(lx), size = lx, prob = px)
head(sims)
```

```
99637 99567 99553 99546 99525 99515
```

A binomial experiment: surviving k years

- The number of **1-year** survivors follows a BIN(ℓ_x, p_x).

  Expected value:

$$\ell_{x+1} = \ell_x \cdot p_x.$$

- The number of k-**year** survivors follows a BIN(ℓ_x, ${}_kp_x$).

  Expected value:

$$\ell_{x+k} = \ell_x \cdot \ {}_kp_x.$$

  Thus:

$${}_kp_x = \frac{\ell_{x+k}}{\ell_x}.$$

A binomial experiment: the number of deaths

- The number of deaths follows a BIN(ℓ_x, q_x).

  Expected value:

$$
\begin{aligned}
d_x &= \ell_x \cdot q_x \\
&= \ell_x \cdot (1 - p_x) \\
&= \ell_x - \ell_{x+1}.
\end{aligned}
$$

```
dx[72 + 1]
```

```
1946
```

```
lx[72 + 1] - lx[73 + 1]
```

```
1946
```

Survival probabilities in R

Compute ${}_5p_{65} = \frac{\ell_{70}}{\ell_{65}}$.

```
# Probability that (65) survives 5 more years
lx[age == 70] / lx[age == 65]
```

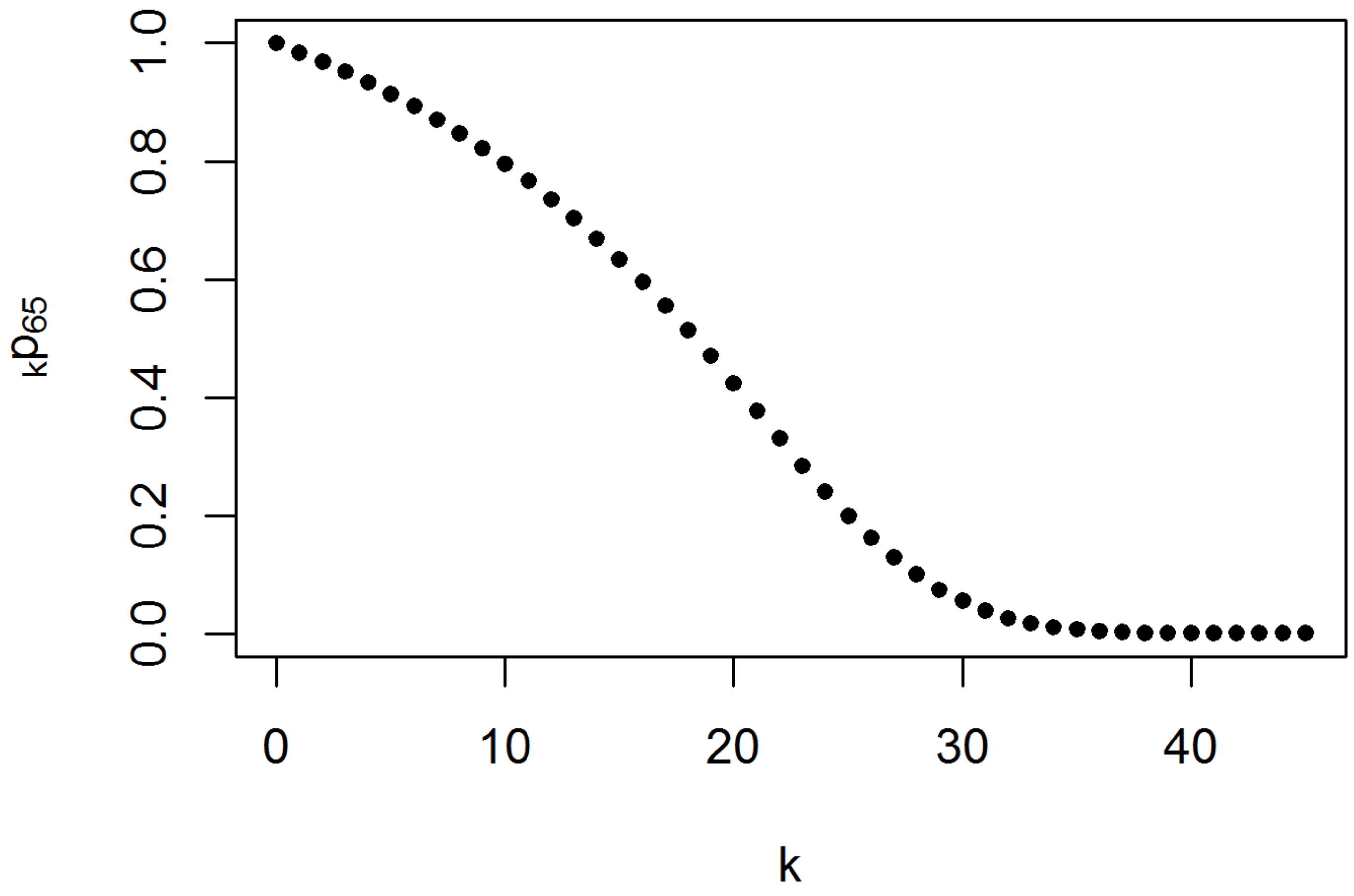
```
0.9143957
```

```
# Alternatively
lx[70 + 1] / lx[65 + 1]
```

```
0.9143957
```

Picturing survival probabilities in R

```
# probability that (65) survives to age 65 + k
k <- 0:45
plot(k,
     lx[65 + k + 1] / lx[65 + 1],
     pch = 20,
     xlab = "k",
     ylab = expression(paste(""[k], "p"[65])))
```



Let's practice!

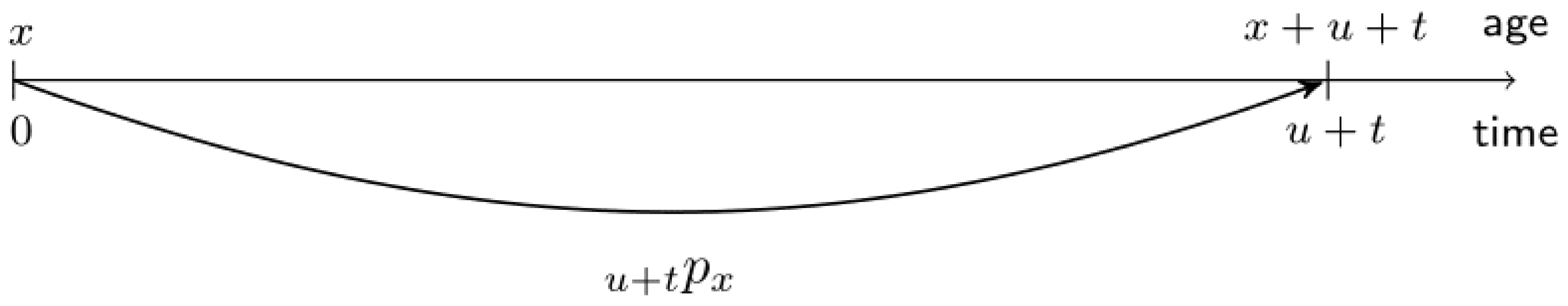
LIFE INSURANCE PRODUCTS VALUATION IN R

Calculating probabilities

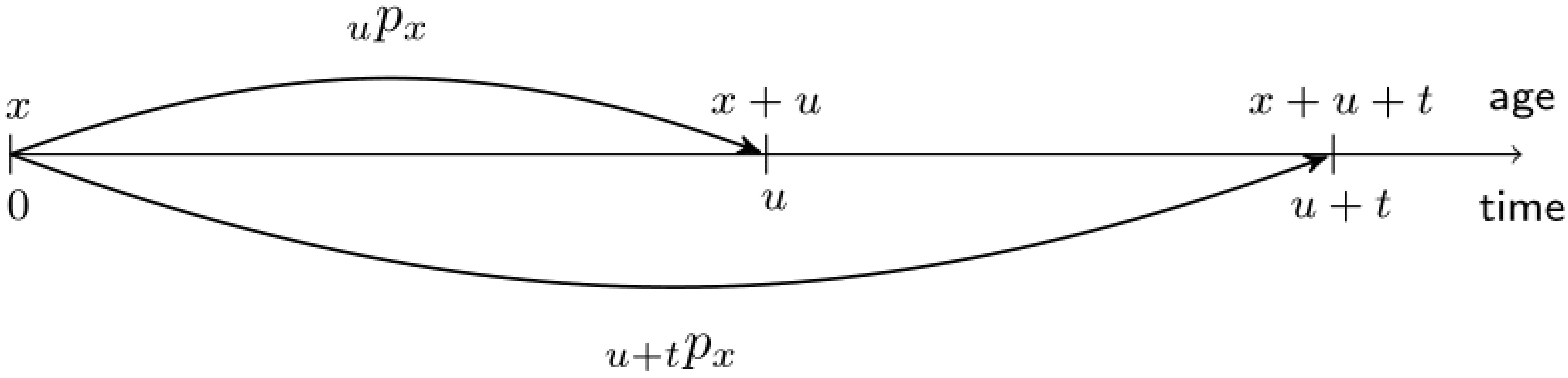
LIFE INSURANCE PRODUCTS VALUATION IN R

Katrien Antonio, Ph.D.
Professor, KU Leuven and University of Amsterdam

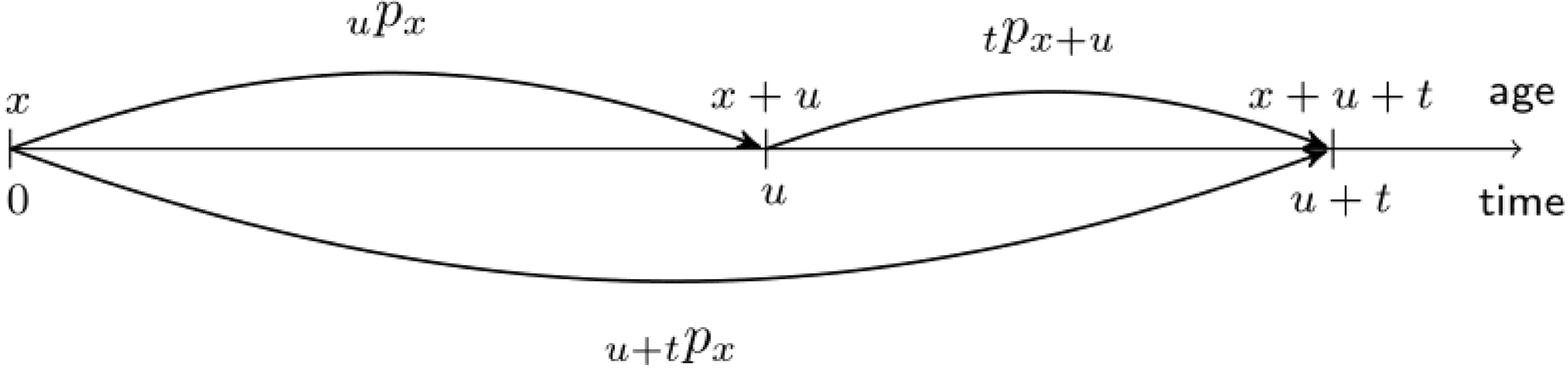
From one-year to multi-year survival probabilities



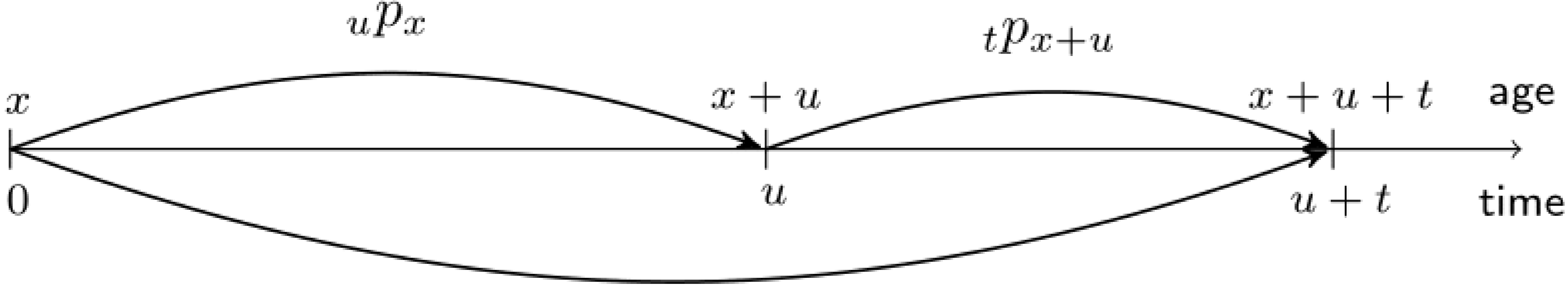
From one-year to multi-year survival probabilities



From one-year to multi-year survival probabilities



From one-year to multi-year survival probabilities



$${}_{u+t}p_x = {}_u p_x \cdot {}_t p_{x+u}$$

The multiplication rule

- Rewriting the **survival probabilities:**

$$ {}_{t+u}p_x = {}_{u}p_x \cdot {}_{t}p_{x+u}. $$

The multiplication rule

- Rewriting the **survival probabilities:**

$$_{t+u}p_x = \; _up_x \cdot \; _tp_{x+u}.$$

- With k an **integer** we obtain:

$$\begin{aligned} _kp_x &= p_x \cdot \; _{k-1}p_{x+1} \\ &= p_x \cdot p_{x+1} \cdots p_{x+k-1} \\ &= \prod_{l=0}^{k-1} p_{x+l} \end{aligned}$$

which is a **product** of **one-year** survival probabilities.

Calculating survival probabilities in R

Compute ${}_5p_{65} = p_{65} \cdot p_{66} \cdot p_{67} \cdot p_{68} \cdot p_{69}$.

```
# One-year survival probabilities
px <- 1 - life_table$qx
px[(65 + 1):(69 + 1)]
```

```
0.98491 0.98320 0.98295 0.98091 0.97935
```

```
# Probability that (65) survives 5 more years
prod(px[(65 + 1):(69 + 1)])
```

```
0.9144015
```

Compute ${}_5p_{65} = \frac{\ell_{70}}{\ell_{65}}$.

```
# Alternatively (difference due to
                    rounding)
lx[70 + 1] / lx[65 + 1]
```

```
0.9143957
```

Cumulative product of survival probabilities in R

Compute ${}_kp_{65}$ for $k = 1, 2, 3, 4, 5.$

```
# One-year survival probabilities
px[(65 + 1):(69 + 1)]
```

```
0.98491 0.98320 0.98295 0.98091 0.97935
```

```
# Multi-year survival probabilities of (65)
cumprod(px[(65 + 1):(69 + 1)])
```

```
0.9849100 0.9683635 0.9518529 0.9336820
0.9144015
```

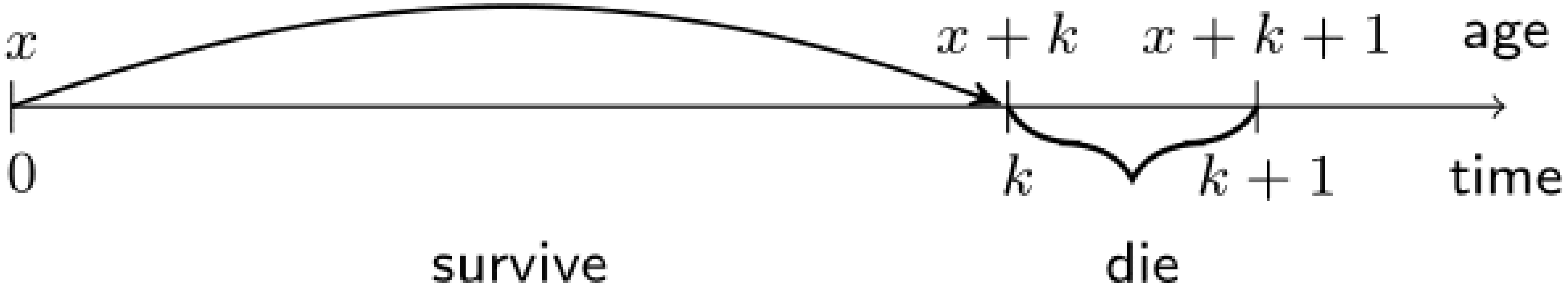
Compute ${}_kp_{65}$ for $k = 0, 1, 2, 3, 4, 5.$

```
# Multi-year survival probabilities of (65)
c(1, cumprod(px[(65 + 1):(69 + 1)]))
```

```
1.0000000 0.9849100 0.9683635 0.9518529
0.9336820 0.9144015
```

A deferred mortality probability

- Focus on a specific **deferred mortality probability:**



- (x) survives k whole years, but dies before reaching age $x+k+1$:

$$_{k|}q_x = {}_kp_x \cdot q_{x+k}.$$

A deferred mortality probability in R

Compute ${}_{5|}q_{65} = {}_{5}p_{65} \cdot q_{70}$.

```
# 5-year deferred mortality probability of (65)
prod(px[(65 + 1):(69 + 1)]) * qx[70 + 1]
```

```
0.02086664
```

Compute ${}_{5|}q_{65} = \frac{d_{70}}{\ell_{65}}$.

```
# Alternatively (difference due to rounding)
dx[70 + 1] / lx[65 + 1]
```

```
0.02086817
```

Deferred mortality probabilities in R

Compute ${}_{k|}q_{65} = {}_kp_{65} \cdot q_{65+k}$ for $k = 0, 1, 2, \ldots$

```
# Survival probabilities of (65)
kpx <- c(1, cumprod(px[(65 + 1):(length(px) - 1)]))
head(kpx)
```

```
1.0000000 0.9849100 0.9683635 0.9518529 0.9336820 0.9144015
```

```
# Deferred mortality probabilities of (65)
kqx <- kpx * qx[(65 + 1):length(qx)]
head(kpx)
```

```
0.01509000 0.01654649 0.01651060 0.01817087 0.01928053 0.02086664
```

Let's practice!

Calculating life expectancies

LIFE INSURANCE PRODUCTS VALUATION IN R

Roel Verbelen, Ph.D.
Statistician, Finity Consulting

The curtate future lifetime

- $K_x = \lfloor T_x \rfloor$, the **number of whole years lived by** (x) in the future.

$$\begin{aligned} Pr(K_x = k) &= Pr(k \leq T_x < k+1) \\ &= {}_kp_x \cdot q_{x+k} \\ &= {}_kp_x - {}_{k+1}p_x, \end{aligned}$$

- Compute $Pr(K_{65} = 5) = {}_5p_{65} \cdot q_{70} = {}_5p_{65} - {}_6p_{65}$.

```
# 5-year deferred mortality probability of (65)
prod(px[(65 + 1):(69 + 1)]) * qx[70 + 1]
```

```
0.02086664
```

```
# Alternatively
prod(px[(65 + 1):(69 + 1)]) -
  prod(px[(65 + 1):(70 + 1)])
```

```
0.02086664
```

The curtate life expectancy

- The expected value of K_x is called the **curtate life expectancy**:

$$
\begin{aligned}
E[K_x] &= \sum_{k=0}^{\infty} k \cdot Pr(K_x = k) \\
&= \sum_{k=0}^{\infty} k \cdot ({}_kp_x - {}_{k+1}p_x) \\
&= \ldots \\
&= \sum_{k=1}^{\infty} {}_kp_x .
\end{aligned}
$$

The life expectancy of a superhero

Mr. Incredible is 35 years old and lives in Belgium.

As an independent superhero he needs to take care of his **financial planning**.

What is a good estimate of his **curtate future lifetime**?

Can you help?

The life expectancy of a superhero in R

Compute $E[K_{35}] = \sum_{k=1}^{\infty} {}_kp_{35}$.

```
# one-year survival probabilities
head(px[(35 + 1):length(px)])
```

```
0.99883 0.99896 0.99902 0.99879 0.99887
0.99824
```

```
# k-year survival probabilities of (35)
kp35 <- cumprod(px[(35 + 1):length(px)])
head(kp35)
```

```
0.9988300 0.9977912 0.9968134 0.9956072
0.9944822 0.9927319
```

```
# curtate expected future lifetime of (35)
sum(kp35)
```

```
43.53192
```

The expected future lifetime

- How to step from $E[K_x]$ to $E[T_x]$?
- Thus: from the **curtate** to the **complete** life expectancy?
- Use:

$$E[T_x] \approx E[K_x] + \frac{1}{2},$$

The life expectancy of a superhero in R

Compare $E[K_{35}]$ to $E[T_{35}]$.

```
# Curtate life expectancy of (35)
sum(kp35)
```

```
43.53192
```

```
# Complete life expectancy of (35)
ex <- life_table$ex
ex[35 + 1]
```

```
44.03
```

Let's practice!

LIFE INSURANCE PRODUCTS VALUATION IN R

Dynamics

LIFE INSURANCE PRODUCTS VALUATION IN R

Katrien Antonio, Ph.D.
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Multiple dataframes with mortality rates

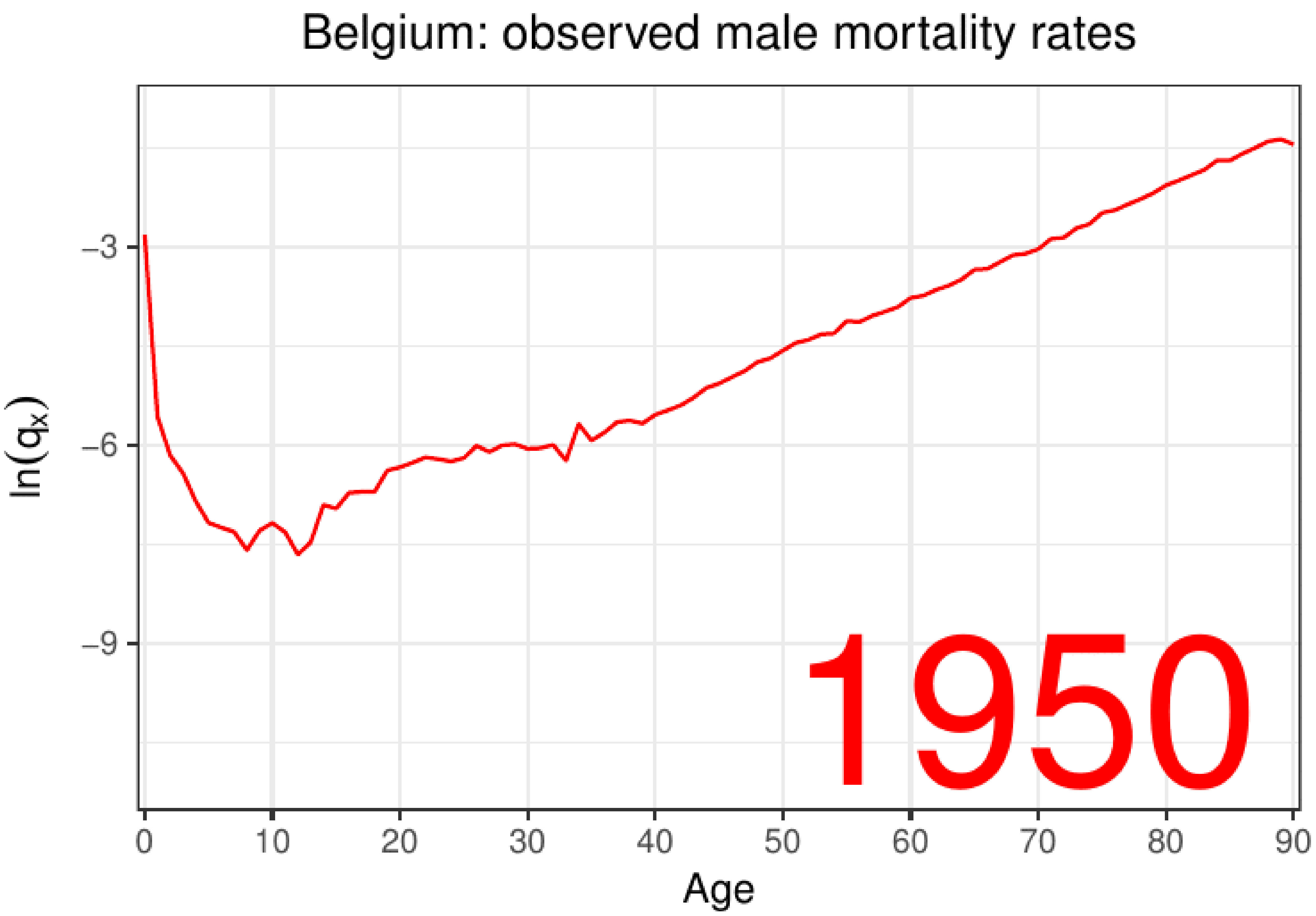
- Look at $\{q_x\}_{x\geq 0}$ or $\{p_x\}_{x\geq 0}$ for multiple periods of time:

1841	1842	$\cdots$	2015
$q_{x,1841}$	$q_{x,1842}$		$q_{x,2015}$

The evolution of mortality rates over time

$q_{x,t}$ is now the mortality rate for an x-year-old in period t.

Age	Period				
	$\ldots$	$t-1$	t	$t+1$	$\ldots$
0	$\ldots$	$q_{0,t-1}$	$q_{0,t}$	$q_{0,t+1}$	$\ldots$
1	$\ldots$	$q_{1,t-1}$	$q_{1,t}$	$q_{1,t+1}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x	$\ldots$	$q_{x,t-1}$	$q_{x,t}$	$q_{x,t+1}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$


Belgium: observed male mortality rates
−3
−6
−9
ln(qx)
0
10
20
30
40
50
60
70
80
90
Age
1950

The period approach

- Calculate survival probability ${}_kp_{x,t}$ in a **vertical** way:

$${}_kp_{x,t} = p_{x,t} \cdot p_{x+1,t} \cdot \ldots \cdot p_{x+k-1,t}$$

Age	Period				
	$\ldots$	$t-1$	t	$t+1$	$\ldots$
0	$\ldots$	$q_{0,t-1}$	$q_{0,t}$	$q_{0,t+1}$	$\ldots$
1	$\ldots$	$q_{1,t-1}$	$q_{1,t}$	$q_{1,t+1}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x	$\ldots$	$q_{x,t-1}$	$q_{x,t}$	$q_{x,t+1}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The cohort approach

- Calculate survival probability ${}_kp_{x,t}$ in a **diagonal** way:

$${}_kp_{x,t} = p_{x,t} \cdot p_{x+1,t+1} \cdot \ldots \cdot p_{x+k-1,t+k-1}$$

Age	Period				
	$\ldots$	t	$t+1$	$t+2$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x	$\cdots$	$p_{x,t}$	$\cdots$	$\cdots$	$\cdots$
$x+1$	$\cdots$	$\cdots$	$p_{x+1,t+1}$	$\cdots$	$\cdots$
$x+2$	$\cdots$	$\cdots$	$\cdots$	$p_{x+2,t+2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Life tables over time in R

```
head(life_table)
```

```
  year age      qx     lx    dx    ex
1 1841   0 0.16580 100000 16580 40.28
2 1841   1 0.07148  83420  5963 47.22
3 1841   2 0.03905  77457  3025 49.82
4 1841   3 0.02306  74432  1716 50.82
5 1841   4 0.01693  72716  1231 51.01
6 1841   5 0.01233  71485   881 50.88
```

```
tail(life_table)
```

```
      year age      qx lx dx   ex
19420 2015 105 0.49719 40 20 1.46
19421 2015 106 0.51410 20 10 1.40
19422 2015 107 0.52979 10  5 1.35
19423 2015 108 0.54425  5  3 1.31
19424 2015 109 0.55749  2  1 1.28
19425 2015 110 1.00000  1  1 1.26
```

The period survival probabilities of a famous Belgian in R

- Jacques Brel is a Belgian singer who was born in 1929 and died at age 49.
- What is the probability ${}_{49|}q_{0,1929}$ that a newborn, born in 1929, dies at age 49?
- Period or vertical way:

```
period_life_table <- subset(life_table, year == 1929)
qx <- period_life_table$qx
px <- 1 - qx
prod(px[1:(48 + 1)]) * qx[49 + 1]
```

```
0.008456378
```

The cohort survival probabilities of a famous Belgian in R

- Jacques Brel is a Belgian singer who was born in 1929 and died at age 49.
- What is the probability ${}_{49|}q_{0,1929}$ that a newborn, born in 1929, dies at age 49?
- Cohort or diagonal way:

```
cohort_life_table <- subset(life_table, year - age == 1929)
qx <- cohort_life_table$qx
px <- 1 - qx
prod(px[(0 + 1):(48 + 1)]) * qx[49 + 1]
```

```
0.006410343
```

Let's practice!

LIFE INSURANCE PRODUCTS VALUATION IN R