Chapter 2

Combinatorics

INTRODUCTION

Combinatorics is an important part of discrete mathematics that solves counting problems without actually enumerating all possible cases. More specifically, combinatorics deals with counting the number of ways of arranging or choosing objects from a finite set according to certain specified rules. In other words, combinatorics is concerned with problems of permutations and combinations, which the students have studied in some detail in lower classes.

As combinatorics has wide applications in Computer Science, especially in such areas as coding theory, analysis of algorithms and probability theory, we shall briefly first review the notions of permutations and combinations and then deal with other related concepts.

PERMUTATIONS AND COMBINATIONS

Definitions

An ordered arrangement of r elements of a set containing n distinct elements is called an *r-permutation of n elements* and is denoted by $P(n, r)$ or ${}^nP_r$, where $r \le n$. An unordered selection of r elements of a set containing n distinct elements is called an *r-Combination of n elements* and is denoted by $C(n, r)$ or ${}^nC_r$ or $\binom{n}{r}$.

Note A permutation of objects involves ordering whereas a combination does not take ordering into account.

Values of $P(n, r)$ and $C(n, r)$

The first element of the permutation can be selected from a set having n elements in n ways. Having selected the first element for the first position of the permutation, the second element can be selected in $(n - 1)$ ways, as there are $(n - 1)$ elements left in the set.

Similarly, there are $(n - 2)$ ways of selecting the third element and so on. Finally there are $n - (r - 1) = n - r + 1$ ways of selecting the r^{th} element. Consequently, by the product rule (stated as follows), there are

$$n(n - 1)\,(n - 2)\,\ldots\,(n - r + 1)$$

ways of ordered arrangement of r elements of the given set.

Thus, $$P(n, r) = n(n - 1)\,(n - 2)\,\ldots\,(n - r + 1)$$
$$= \frac{n!}{(n-r)!}$$

In particular, $$P(n, n) = n!$$

Product Rule

If an activity can be performed in r successive steps and step 1 can be done in n_1 ways, step 2 can be done in n_2 ways, …, step r can be done in n_r ways, then the activity can be done in $(n_1 \cdot n_2 \ldots n_r)$ ways.

The r-permutations of the set can be obtained by first forming the $C(n, r)$ r-combinations of the set and then arranging (ordering) the elements in each r-combination, which can be done in $P(r, r)$ ways. Thus

$$P(n, r) = C(n, r) \cdot P(r, r)$$

$\therefore$ $$C(n, r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!}$$
$$= \frac{n!}{r!(n-r)!}$$

In particular, $$C(n, n) = 1.$$

Note Since the number of ways of selecting out r elements from a set of n elements is the same as the number of ways of leaving $(n - r)$ elements in the set, it follows that

$$C(n, r) = C(n, n - r)$$

This is obvious otherwise, as

$$C(n, n - r) = \frac{n!}{(n-r)!\{n-(n-r)\}!}$$
$$= \frac{n!}{(n-r)!\,r!} = C(n, r)$$

PASCAL'S IDENTITY

If n and r are positive integers, where $n \geq r$, then $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$.

Proof

Let S be a set containing $(n + 1)$ elements, one of which is 'a'. Let $S' \equiv S - \{a\}$.

The number of subsets of S containing r elements is $\binom{n+1}{r}$.

Now a subset of S with r elements either contains 'a' together with $(r - 1)$ elements of S' or contains r elements of S' which do not include 'a'.

The number of subsets of $(r - 1)$ elements of $S' = \binom{n}{r-1}$.

$\therefore$ The number of subsets of r elements of S that contain 'a' $= \binom{n}{r-1}$.

Also the number of subsets of r elements of S that do not contain 'a' = that of $S' = \binom{n}{r}$. Consequently, $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

Note This result can also be proved by using the values of $\binom{n}{r-1}, \binom{n}{r}$ and $\binom{n+1}{r}$.

Corollary

$$C(n + 1, r + 1) = \sum_{i=r}^{n} C(i, r)$$

Proof

Changing n to i and r to $r + 1$ in Pascal's identity, we get

$$C(i, r) + C(i, r + 1) = C(i + 1, r + 1)$$

i.e.,
$$C(i, r) = C(i + 1, r + 1) - C(i, r + 1) \qquad (1)$$

Putting $i = r, r + 1, \ldots, n$ in (1) and adding, we get

$$\sum_{i=r}^{n} C(i, r) = C(n + 1, r + 1) - C(r, r + 1)$$
$$= C(n + 1, r + 1) \quad [\because C(r, r + 1) = 0]$$

VANDERMONDE'S IDENTITY

If m, n, r are non-negative integers where $r \leq m$ or n, then

$$C(m + n, r) = \sum_{i=0}^{r} C(m, r - i) \cdot C(n, i)$$

Proof

Let m and n be the number of elements in sets 1 and 2 respectively.

Then the total number of ways of selecting r elements from the union of sets 1 and 2

$$= C(m + n, r)$$

The r elements can also be selected by selecting i elements from set 2 and $(n - i)$ elements from set 1, where $i = 0, 1, 2, \ldots, r$. This selection can be done in $C(m\ r - i)$. $C(n, i)$ ways, by the product rule.

The $(r + 1)$ selections corresponding to $i = 0, 1, 2, \ldots, r$ are disjoint. Hence, by the sum rule (stated as follows), we get

$$C(m + n, r) = \sum_{i=0}^{r} C(m, r - i) \cdot C(n, i) \quad \text{or} \quad \sum_{i=0}^{r} C(m, i) \cdot C(n, r - i)$$

Sum rule

If r activities can be performed in $n_1, n_2, \ldots, n_r$ ways and if they are disjoint, viz., cannot be performed simultaneously, then any one of the r activities can be performed in $(n_1 + n_2 + \ldots + n_r)$ ways.

PERMUTATIONS WITH REPETITION

Theorem

When repetition of n elements contained in a set is permitted in r-permutations, then the number of r-permutations is n^r.

Proof

The number of r-permutations of n elements can be considered as the same as the number of ways in which the n elements can be placed in r positions.

The first position can be occupied in n ways, as any one of the n elements can be used

Similarly, the second position can also be occupied in n ways, as any one of the n elements can be used, since repetition of elements is allowed.

Hence, the first two positions can be occupied in $n \times n = n^2$ ways, by the product rule. Proceeding like this, we see that the 'r' positions can be occupied by 'n' elements (with repetition) in n^r ways.

i.e., the number of r-permutations of n elements with repetition $= n^r$.

Theorem

The number of different permutations of n objects which include n_1 identical objects of type I, n_2 identical objects of type II, … and n_k identical objects of type k is equal to $\dfrac{n!}{n_1!\, n_2! \cdots n_k!}$, where $n_1 + n_2 + \cdots + n_k = n$.

Proof

The number of n-permutations of n objects is equal to the number of ways in which the n objects can be placed in n positions.

n_1 positions to be occupied by n_1 objects of the I type can be selected from n positions in $C(n, n_1)$ ways.

n_2 positions to be occupied by the n_2 objects of the II type can be selected from the remaining $(n - n_1)$ positions in $C(n - n_1, n_2)$ ways and so on. Finally n_k positions to be occupied by the n_k objects of type k can be selected from the remaining $(n - n_1 - n_2 - \cdots - n_{k-1})$ positions in $C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$ ways.

Hence, the required number of different permutations

$$= C(n, n_1) \times C(n - n_1, n_2) \times \ldots\ C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$$

(by the product rule)

$$= \frac{n!}{n_1!(n-n_1)!} \times \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \times \cdots \times \frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k!\, 0!}$$

$(\because\ n_1 + n_2 + \cdots + n_k = n)$

$$= \frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

Example

Let us consider the 3-permutations of the 3 letters A, B_1, B_2, the number of which is 3! They are: AB_1B_2, AB_2B_1. B_1AB_2, B_1B_2A, B_2AB_1 and B_2B_1A. If we replace B_1 and B_2 by B, the above permutations become

$$ABB, ABB, BAB, BBA, BAB \text{ and } BBA.$$

These permutations are not different. The different 3-permutations of the 3 letters A, B, B are ABB, BAB and BBA. Thus the number of different 3-permutations of 3 letters, of which 2 are identical of one type and 1 is of another type is equal to

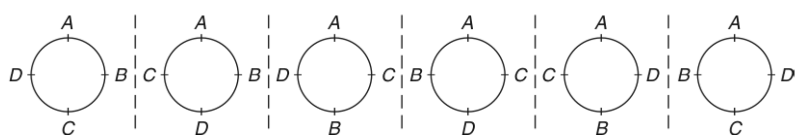
$$3 = \frac{3!}{2!1!}$$

This example illustrates the above theorem.

CIRCULAR PERMUTATION

The permutations discussed so far can be termed as linear permutations, as the objects were assumed to be arranged in a line. If the objects are arranged in a circle (or any closed curve), we get circular permutation and the number of circular permutations will be different from the number of linear permutations as seen from the following example:

We can arrange 4 elements A, B, C, D in a circle as follows: We fix one of the elements, say A, at the top point of the circle. The other 3 elements B, C, D are permuted in all possible ways, resulting in 6 = 3! different circular permutations are as follows:



Note Circular arrangements are considered the same when one can be obtained from the other by rotation, viz., The relative positions (and not the actual positions) of the objects alone count for different circular permutations.

From the example given above, we see that the number of different circular arrangements of 4 elements = (4 – 1)! = 6.

Similarly, the number of different circular arrangements of n objects = $(n - 1)!$ If no distinction is made between clockwise and counterclockwise circular arrangements [For example, if the circular arrangements in the first and the last figures are assumed as the same], then the number of different circular arrangements = $\frac{1}{2}(n - 1)!$

PIGEONHOLE PRINCIPLE

Though this principle stated as follows is deceptively simple, it is sometimes useful in counting methods. The deception often lies in recognising the problems where this principle can be applied.

Statement

If n pigeons are accommodated in m pigeon-holes and $n > m$ then at least one pigeonhole will contain two or more pigeons. Equivalently, if n objects are put in m boxes and $n > m$, then at least one box will contain two or more objects.

Proof

Let the n pigeons be labelled $P_1, P_2, \ldots, P_n$ and the m pigeonholes be labelled $H_1, H_2, \ldots, H_m$. If $P_1, P_2, \ldots, P_m$ are assigned to $H_1, H_2, \ldots, H_m$ respectively, we are left with the $(n - m)$ pigeons $P_{m+1}, P_{m+2}, \ldots, P_n$. If these left over pigeons are assigned to the m pigeonholes again in any random manner, at least one pigeonhole will contain two or more pigeons.

GENERALISATION OF THE PIGEONHOLE PRINCIPLE

If n pigeons are accommodated in m pigeonholes and $n > m$, then one of the pigeonholes must contain at least $\left\lfloor \frac{(n-1)}{m} \right\rfloor + 1$ pigeons, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, which is a real number.

Proof

If possible, let each pigeonhole contain at the most $\left\lfloor \frac{(n-1)}{m} \right\rfloor$ pigeons.

Then the maximum number of pigeons in all the pigeonholes

$$= m\left\lfloor \frac{(n-1)}{m} \right\rfloor \le m \cdot \frac{(n-1)}{m} \qquad \left\{\because \left\lfloor \frac{(n-1)}{m} \right\rfloor \le \frac{(n-1)}{m}\right\}$$

i.e., the maximum number of pigeons in all the pigeonholes $\le (n - 1)$

This is against the assumption that there are n pigeons.

Hence, one of the pigeonholes must contain at least $\left\lfloor \frac{(n-1)}{m} \right\rfloor + 1$ pigeons.

PRINCIPLE OF INCLUSION-EXCLUSION

Statement

If A and B are finite subsets of a finite universal set U, then
$|A \cup B| = |A| + |B| - |A \cap B|$, where $|A|$ denotes the cardinality of (the number of elements in) the set A.

This principle can be extended to a finite number of finite sets $A_1, A_2, \ldots, A_n$ as follows:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots$$
$$+ (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|,$$

where the first sum is over all i, the second sum is over all pairs i, j with $i < j$, the third sum is over all triples i, j, k with $i < j < k$ and so on.

Proof

Let $A \backslash B = \{a_1, a_2, \ldots, a_r\}$
$B \backslash A = \{b_1, b_2, \ldots, b_s\}$
$A \cap B = \{x_1, x_2, \ldots, x_t\}$,

where $A \backslash B$ is the set of those elements A which are not in B.

Then $A = \{a_1, a_2, \ldots, a_r, x_1, x_2, \ldots, x_t\}$

and $B = \{b_1, b_2, \ldots, b_s, x_1, x_2, \ldots, x_t\}$

Hence, $A \cup B = \{a_1, a_2, \ldots, a_r, x_1, x_2, \ldots, x_t, b_1, b_2, \ldots, b_s\}$

Now $|A| + |B| - |A \cap B| = (r + t) + (s + t) - t$

$$= r + s + t = |A \cup B| \quad (1)$$

Let us now extend the result to 3 finite sets A, B, C.

$$\begin{aligned}|A \cup B \cup C| &= |A \cup (B \cup C)| \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B| + |C| - |B \cap C| - \{(A \cap B) \cup (A \cap C)\} \text{ by (1)} \\ &= |A| + |B| + |C| - |B \cap C| - \{|A \cap B| + |A \cap C| \\ &\qquad - |(A \cap B) \cap (A \cap C)|\}, \text{ by (1)} \\ &= |A| + |B| + |C| - |A \cap B| - (B \cap C) - (C \cap A) \\ &\qquad + |A \cap B \cap C|\end{aligned}$$

Generalising, we get the required result.

WORKED EXAMPLES 2(A)

Example 2.1

(a) Assuming that repetitions are not permitted, how many four-digit numbers can be formed form the six digits 1, 2, 3, 5, 7, 8?
(b) How many of these numbers are less than 4000?
(c) How many of the numbers in part (a) are even?
(d) How many of the numbers in part (a) are odd?
(e) How many of the numbers is part (a) are multiples of 5?
(f) How many of the numbers in part (a) contain both the digits 3 and 5?

(a) The 4-digit number can be considered to be formed by filling up 4 blank spaces with the available 6 digits. Hence, the number of 4-digits numbers

$$\begin{aligned}&= \text{the number of 4-permutations of 6 numbers} \\ &= P(6, 4) = 6 \times 5 \times 4 \times 3 = 360\end{aligned}$$

(b) If a 4-digit number is to be less than 4000, the first digit must be 1, 2, or 3. Hence the first space can be filled up in 3 ways. Corresponding to any one of these 3 ways, the remaining 3 spaces can be filled up with the remaining 5 digits in $P(5, 3)$ ways. Hence, the required number = $3 \times P(5, 3)$

$$= 3 \times 5 \times 4 \times 3 = 180.$$

(c) If the 4-digit number is to be even, the last digit must be 2 or 8. Hence, the last space can be filled up in 2 ways. Corresponding to any one of these 2 ways, the remaining 3 spaces can be filled up with the remaining 5 digits in $P(5, 3)$ ways. Hence the required number of even numbers

$$= 2 \times P(5, 3) = 120.$$

(d) Similarly the required number of odd numbers $= 4 \times P(5, 3) = 240$.

(e) If the 4-digit number is to be a multiple of 5, the last digit must be 5. Hence, the last space can be filled up in only one way. The remaining 3 spaces can be filled up in $P(5, 3)$ ways.
Hence, the required number $= 1 \times P(5, 3) = 60$.

(f) The digits 3 and 5 can occupy any 2 of the 4 places in $P(4, 2) = 12$ ways.
The remaining 2 places can be filled up with the remaining 4 digits in $P(4, 2) = 12$ ways. Hence, the required number $= 12 \times 12 = 144$.

Example 2.2

(a) In how many ways can 6 boys and 4 girls sit in a row?

(b) In how many ways can they sit in a row if the boys are to sit together and the girls are to sit together?

(c) In how many ways can they sit in a row if the girls are to sit together?

(d) In how many ways can they sit in a row if *just* the girls are to sit together?

(a) 6 boys and 4 girls (totally 10 persons) can sit in a row (viz., can be arranged in 10 places) in $P(10, 10) = 10!$ ways.

(b) Let us assume that the boys are combined as one unit and the girls are combined as another unit. These 2 units can be arranged in $2! = 2$ ways.
Corresponding to any one of these 2 ways, the boys can be arranged in a row in $6!$ ways and the girls in $4!$ ways.
$\therefore$ Required number of ways $= 2 \times 6! \times 4! = 34{,}560$.

(c) The girls are considered as one unit (object) and there are 7 objects consisting of one object of 4 girls and 6 objects of 6 boys.
These 7 objects can be arranged in a row in $7!$ ways.
Corresponding to any one of these ways, the 4 girls (considered as one object) can be arranged among themselves in $4!$ ways. Hence, the required number of ways $= 7!\,4! = 1{,}20{,}960$.

(d) No. of ways in which girls only sit together
= (No. of ways in which girls sit together)
– (No of ways in which boys sit together and girls sit together)
$= 1{,}20{,}960 - 34{,}560 = 86{,}400$.

Example 2.3 How many different paths in the *xy*-plane are there from (1, 3) to (5, 6), if a path proceeds one step at a time by going either one step to the right (R) or one step upward (U)?

To reach the point (5, 6) from (1, 3), one has to traverse $5 - 1 = 4$ steps to the right and $6 - 3 = 3$ steps to the up.

Hence, the total number of 7 steps consists of 4 R's and 3 U's.

To traverse the paths, one can take R's and U's in any order.

Hence, the required number of different paths is equal to the number of permutations of 7 steps, of which 4 are of the same type (namely R) and 3 are of the same type (namely U).

$\therefore$ Required number of paths $= \dfrac{7!}{4!\,3!} = 35.$

Example 2.4 How many positive integers n can be formed using the digits 3, 4, 4, 5, 5, 6, 7, if n has to exceed 50,00,000?

In order that n may be greater than 50,00,000, the first place must be occupied by 5, 6 or 7.

When 5 occupies the first place, the remaining 6 places are to be occupied by the digits 3, 4, 4, 5, 6, 7.

The number of such numbers

$$= \frac{6!}{2!} \quad (\because \text{ the digit 4 occurs twice})$$
$$= 360.$$

When 6 (or 7) occupies the first place, the remaining 6 places are to be occupied by the digits 3, 4, 4, 5, 5, 7 (or 3, 4, 4, 5, 5, 6).

The number of such numbers

$$= \frac{6!}{2!\,2!} \quad [\because \text{ 4 and 5 each occurs twice}]$$
$$= 180$$

$\therefore$ No. of numbers exceeding 50,00,000 = 360 + 180 + 180 = 720.

Example 2.5 How many bit strings of length 10 contain (a) exactly four 1's, (b) atmost four 1's, (c) at least four 1's (d) an equal number of 0's and 1's?

(a) A bit string of length 10 can be considered to have 10 positions. These 10 positions should be filled with four 1's and six 0's.

$\therefore$ No. of required bit strings $= \dfrac{10!}{4!\,6!} = 210.$

(b) The 10 positions should be filled up with no 1 and ten 0's or one 1 and nine 0's or two 1's and eight 0's or three 1's and seven 0's or four 1's and six 0's.

$\therefore$ Required no. of bit strings

$$= \frac{10!}{0!\,10!} + \frac{10!}{1!\,9!} + \frac{10!}{2!\,8!} + \frac{10!}{3!\,7!} + \frac{10!}{4!\,6!} = 386.$$

(c) The ten positions are to be filled up with four 1's and six 0's or five 1's and five 0's etc. or ten 1's and no 0's.

$\therefore$ Required no. of bit strings

$$= \frac{10!}{4!\,6!} + \frac{10!}{5!\,5!} + \frac{10!}{6!\,4!} + \frac{10!}{7!\,3!} + \frac{10!}{8!\,2!} + \frac{10!}{9!\,1!} + \frac{10!}{10!\,0!} = 848.$$

(d) The ten positions are to be filled up with five 1's and five 0's.

$\therefore$ Required no. of bit strings

$$= \frac{10!}{5!\,5!} = 252.$$

Example 2.6 How many permutations of the letters A B C D E F G contain (a) the string BCD, (b) the string $CFGA$, (c) the strings BA and GF, (d) the strings ABC and DE, (e) the strings ABC and CDE, (f) the strings CBA and BED?

(a) Treating BCD as one object, we have the following 5 objets:

$$A, (BCD), E, F, G.$$

These 5 objects can be permuted in

$$P(5, 5) = 5! = 120 \text{ ways}$$

Note B, C, D should not be permuted in the string BCD.

(b) Treating $CFGA$ as one object, we have the following 4 objects: B, D, E, $(CFGA)$.

The no. of ways of permuting these 4 objects = 4! = 24.

(c) The objects (BA), C, D, E and (GF) can be permuted in 5! = 120 ways.

(d) The objects (ABC), (DE), F, G can be permuted in 4! = 24 ways.

(e) Even though (ABC) and (CDE) are two strings, they contain the common letter C. If we include the strings $(ABCDE)$ in the permutations, it includes both the strings (ABC) and (CDE). Moreover we cannot use the letter C twice.

Hence, we have to permute the 3 objects $(ABCDE)$, F and G. This can be done in 3! = 6 ways.

(f) To include the 2 strings (CBA) and (BED) in the permutations, we require the letter B twice, which is not allowed. Hence, the required no. of permutations = 0.

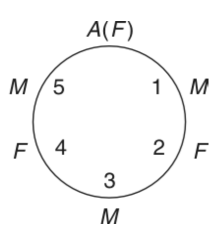
Example 2.7 If 6 people A, B, C, D, E, F are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation?

If A, B, C are females and the others are males, in how many arrangements do the sexes alternate?

The no. of different circular arrangements of n objects is $(n - 1)!$

$\therefore$ The required no. of circular arrangements = 5! = 120.

Since rotation does not alter the circular arrangement, we can assume that A occupies the top position as shown in the figure.



Of the remaining places, positions 1, 3, 5 must be occupied by the 3 males. This can be achieved in $P(3, 3) = 3! = 6$ ways.

The remaining two places 2 and 4 should be occupied by the remaining two females. This can be achieved in $P(2, 2) = 2$ ways.

$\therefore$ Total no. of required circular arrangements = $6 \times 2 = 12$.

Example 2.8 From a club consisting of 6 men and 7 women, in how many ways can we select a committee of

(a) 3 men and 4 women?
(b) 4 persons which has at least one woman?
(c) 4 persons that has at most one man?
(d) 4 persons that has persons of both sexes?
(e) 4 persons so that two specific members are not included?

(a) 3 men can be selected from 6 men in $C(6, 3)$ ways.
4 women can be selected from 7 women in $C(7, 4)$ ways.
$\therefore$ The committee of 3 men and 4 women can be selected in $C(6, 3) \times C(7, 4)$ ways. (by the product rule)

i.e., in $\dfrac{6!}{3!3!} \times \dfrac{7!}{4!3!} = 700$ ways.

(b) For the committee to have at least one woman, we have to select 3 men and 1 woman or 2 men and 2 women or 1 man and 3 women or no man and 4 women.
This selection can be done in

$$C(6, 3) \cdot C(7, 1) + C(6, 2) \cdot C(7, 2) + C(6, 1) \cdot C(7, 3) + C(6, 0) \cdot C(7, 4)$$
$$= 20 \times 7 + 15 \times 21 + 6 \times 35 + 1 \times 35$$
$$= 140 + 315 + 210 + 35 = 700 \text{ ways.}$$

(c) For the committee to have at most one man, we have to select no man and 4 women or 1 man and 3 women.
This selection can be done in

$$C(6, 0) \cdot C(7, 4) + C(6, 1) \cdot C(7, 3) = 1 \times 35 + 6 \times 35 = 245 \text{ ways.}$$

(d) For the committee to have persons of both sexes, the selection must include 1 man and 3 women or 2 men and 2 women or 3 men and 1 woman.
This selection can be done in

$$C(6, 1) \times C(7, 3) + C(6, 2) \times C(7, 2) + C(6, 3) \times C(7, 1)$$
$$= 6 \times 35 + 15 \times 21 + 20 \times 7$$
$$= 210 + 315 + 140 = 665 \text{ ways.}$$

(e) First let us find the number of selections that contain the two specific members. After removing these two members, 2 members can be selected from the remaining 11 members in $C(11, 2)$ ways. In each of these selections, if we include those 2 specific members removed, we get $C(11, 2)$ selections containing the 2 members.
The no. of selections not including these 2 members

$$= C(13, 4) - C(11, 2)$$
$$= 715 - 55 = 660.$$

Example 2.9 In how many ways can 20 students out of a class of 30 be selected for an extra-curricular activity, if

(a) Rama refuses to be selected?
(b) Raja insists on being selected?

(c) Gopal and Govind insist on being selected?
(d) either Gopal or Govind or both get selected?
(e) just one of Gopal and Govind gets selected?
(f) Rama and Raja refuse to be selected together?

(a) We first exclude Rama and then select 20 students from the remaining 29 students.
$\therefore$ Number of ways = $C(29, 20)$ = 1, 00, 15, 005.

(b) We separate Raja from the class, select 19 students from 29 and then include Raja in the selections.
$\therefore$ Number of ways = $C(29, 19)$ = 2, 00, 30, 010.

(c) We separate Gopal and Govind, select 18 students from 28 and then include both of them in the selections.
$\therefore$ Number of ways = $C(28, 18)$ = 1, 31, 23, 110

(d) Number of selections which include Gopal = $C(29, 19)$
Number of selections which include Govind = $C(29, 19)$
Number of selections which include both = $C(28, 18)$
$\therefore$ By the principle of inclusion – exclusion, the required number of selections
= $C(29, 19) + C(29, 19) - C(28, 18)$
= 2, 69, 36, 910.

(e) Number of selections including either Gopal or Govind
= (Number of selections including either Gopal or Govind or both) – (Number of selections including both)
= $[C(29, 19) + C(29, 19) - C(28, 18)] - C(28, 18)$
= 2, 69, 36, 910 – 1, 31, 23, 110 = 1, 38, 13, 800.

(f) Number of ways of selecting 20 excluding Rama and Raja together = (Total number of selections) – (Number of selections including both Rama and Raja)
= $C(30, 20) - C(28, 18)$ [as in part (c)]
= 3, 00, 45, 015 – 1, 31, 23, 110 = 1, 69, 21, 905.

Example 2.10 In how many ways can 2 letters be selected from the set $\{a, b, c, d\}$ when repetition of the letters is allowed, if (i) the order of the letters matters (ii) the order does not matter?

(i) When the order of the selected letters matters, the number of possible selections = $4^2 = 16$, which are listed below:

aa, ab, ac, ad
ba, bb, bc, bd
ca, cb, cc, cd
da, db, dc, dd

In general, the number of r-permutations of n objects, if repetition of the objects is allowed, is equal to n^r, since there are n ways to select an object from the set for each of the r-positions.

(ii) When the order of the selected letter does not matter, the number of possible selections $C(4 + 2 - 1, 2) = C(5, 2) = 10$, which are listed below:

$$\begin{array}{r} aa, ab, ac, ad \\ bb, bc, bd \\ cc, cd \\ dd \end{array}$$

In general, the number of r-combinations of n kinds of objects, if repetitions of the objects is allowed = $C(n + r - 1, r)$.

[The reader may try to prove this result.]

Example 2.11 There are 3 piles of identical red, blue and green balls, where each pile contains at least 10 balls. In how many ways can 10 balls be selected:

(a) if there is no restriction?
(b) if at least one red ball must be selected?
(c) if at least one red ball, at least 2 blue balls and at least 3 green balls must be selected?
(d) if exactly one red ball must be selected?
(e) if exactly one red ball and at least one blue ball must be selected?
(f) if at most one red ball is selected?
(g) if twice as many red balls as green balls must be selected?

(a) There are $n = 3$ kinds of balls and we have to select $r = 10$ balls, when repetitions are allowed.

$\therefore$ No. of ways of selecting $= C(n + r -1, r) = C(12, 10) = 66$.

(b) We take one red ball and keep it aside. Then we have to select 9 balls from the 3 kinds of balls and include the first red ball in the selections.

$\therefore$ No of ways of selecting = $C(11, 9) = 55$.

(c) We take away 1 red, 2 blue and 3 green balls and keep them aside. Then we selcet 4 balls from the 3 kinds of balls and include the 6 already chosen bolls in each selection.

$\therefore$ No. of ways of selecting = $C(3 + 4 - 1, 4) = 15$.

(d) We select 9 balls from the piles containing blue and green balls and include 1 red ball in each selection.

$\therefore$ No. of ways of selecting = $C(2 + 9 - 1, 9) = 10$.

(e) We take away one red ball and one blue ball and keep them aside. Then we select 8 balls from the blue and green piles and include the already reserved red and blue balls to each selection.

$\therefore$ No. of ways of selecting = $C(2 + 8 -1, 8) = 9$.

(f) The selections must contain no red ball or 1 red ball.

$\therefore$ No. of ways of selecting $= C(2 + 10 - 1, 10) + C(2 + 9 - 1, 9)$
$= 11 + 10 = 21$

(g) The selections must contain 0 red and 0 green balls or 2 red and 1 green balls or 4 red and 2 green balls or 6 red and 3 green balls.

$\therefore$ No. of ways of selecting $= C(1 + 10 - 1, 10) + C(1 + 7 - 1, 7)$
$+ C(1 + 4 - 1, 4) + C(1 + 1 - 1, 1)$
$= 1 + 1 = 1 + 1 = 4.$

Example 2.12 5 balls are to be placed in 3 boxes. Each can hold all the 5 balls. In how many diifferent ways can we place the balls so that no box is left empty, if

(a) balls and boxes are different?
(b) balls are identical and boxes are different?
(c) balls are different and boxes are identical?
(d) balls as well as boxes are identical?

(a) 5 balls can be distributed such that the first, second and third boxes contain 1, 1 and 3 balls respectively.

$\therefore$ No. of ways of distributing in this manner

$$= \frac{5!}{1!\,1!\,3!} = 20.$$

Similarly the boxes I, II, III may contain 1, 3 and 1 balls respectively or 3, 1 and 1 balls respectively. ($\because$ the boxes are different). No. of ways of distributing in each of these manners = 20.

Again the boxes I, II, III may contain 1, 2, 2 balls respectively or 2, 1, 2 balls respectively or 2, 2, 1 balls respectively. No. of ways of distributing in each of these manners $= \dfrac{5!}{1!\,2!\,2!} = 30$.

$\therefore$ Total no. of required ways

$$= 20 + 20 + 20 + 30 + 30 + 30 = 150$$

(b) Total no. of ways of distributing r identical balls in n different boxes is the same as the no. of r-combinations of n items, repetitions allowed.

It is $= C(n + r - 1, r) = C(3 + 2 - 1, 2) = 6$ since 3 balls must be first put, one in each of 3 boxes and the remaining 2 balls must be distributed in 3 boxes.

(c) When the boxes are identical, the distributions of 1, 1, 3 balls, 1, 3, 1 balls and 3, 1, 1 balls considered in (a) will be treated as identical distributions. Thus there are 20 ways of distributing 1 ball in each of any two boxes and 3 balls in the third box.

Similarly, there are 30 ways of distributing 2 balls in each of any 2 boxes and 1 ball in the third box.

$\therefore$ No. of required ways = 20 + 30 = 50.

(d) By an argument similar to that given in (c), we get from the answer in (b) that the required no. of ways $= \dfrac{6}{3} = 2$.

Example 2.13 Determine the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 32$, where

(a) $x_i \geq 0,\ 1 \leq i \leq 4$;
(b) $x_i > 0,\ 1 \leq i \leq 4$;
(c) $x_1, x_2 \geq 5$ and $x_3, x_4 \geq 7$;
(d) $x_1, x_2, x_3 > 0$ and $0 < x_4 \leq 25$.

(a) One solution of the equation is $x_1 = 15$, $x_2 = 10$, $x_3 = 7$ and $x_4 = 0$. Another solution is $x_1 = 7$, $x_2 = 15$, $x_3 = 0$ and $x_4 = 10$. These two solutions are considered different, even though the same 4 integers 15, 10, 7, 0 are used. The first solution can be interpreted as follows:

We have 32 identical chocolates and are distributing them among 4 distinct children. We have given 15, 10, 7 and 0 chocolates to the first, second, third and fourth child respectively.

Thus, each non-negative solution of the equation corresponds to a selection of 32 identical items from 4 distinct sets, repetitions allowed.

Hence, the no. of solutions $= C(4 + 32 - 1, 32)$
$= C(35, 32) = 6545$

(b) Now $x_i > 0;\ 1 \le i \le 4$
i.e., $x_i \ge 1;\ 1 \le i \le 4$
Let us put $u_i = x_i - 1$, so that $u_i \ge 0;\ 1 \le i \le 4$
Then the given equation becomes

$$u_1 + u_2 + u_3 + u_4 = 28,$$

for which the no. of non-negative integer solutions is required.

The required number $= C(4 + 28 - 1, 28)$
$= C(31, 28) = 4495.$

(c) Putting $x_1 - 5 = u_1$, $x_2 - 5 = u_2$, $x_3 - 7 = u_3$ and $x_4 - 7 = u_4$, the equation becomes $u_1 + u_2 + u_3 + u_4 = 8$, where $u_1, u_2, u_3, u_4 \ge 0$.

The required no. of solutions $= C(4 + 8 - 1, 8)$
$= C(11, 8) = 165.$

No. of solutions such that $x_1, x_2, x_3 > 0$ and $0 < x_4 \le 25$ = (No. of solutions such that $x_i > 0;\ i = 1, 2, 3, 4$) – (No. of solutions such that $x_i > 0;\ i = 1, 2, 3$ and $x_4 > 25$) $= a - b$, say.
From part (b); $a = C(31, 28) = 4495$
To find b, we put $u_1 = x_1 - 1$, $u_2 = x_2 - 1$, $u_3 = x_3 - 1$ and $u_4 = x_4 - 26$.
The equation becomes $u_1 + u_2 + u_3 + u_4 = 3$.
We have to get the solution satisfying $u_i \ge 0;\ i = 1, 2, 3, 4$.

No. of solutions $= b = C(4 + 3 - 1, 3)$
$= C(6, 3) = 20.$

$\therefore$ Required no. of solutions $= 4495 - 20$
$= 4475.$

Example 2.14 Find the number of non-negative integer solutions of the inequality $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10$?

We convert the inequality into an equality by introducing an auxiliary variable $x_7 > 0$.

Thus, we get $x_1 + x_2 + \cdots + x_6 + x_7 = 10$, where

$$x_i \ge 0,\ i = 1, 2, \ldots, 6 \text{ and } x_7 > 0 \text{ or } x_7 \ge 1.$$

Putting $x_i = y_i$, $i = 1, 2, \ldots 6$ and $x_7 - 1 = y_7$, the equation becomes

$$y_1 + y_2 + \cdots + y_7 = 10 - 1 = 9, \text{ where } y_i \ge 0, \text{ for } 1 \le i \le 7$$

The number of required solutions

$$= C(7 + 9 - 1, 9) = C(15, 9) = 5005.$$

Example 2.15 How many positive integers less than 10,00,000 have the sum of their digits equal to 19?

Any positive integer less than 10,00,000 will have a maximum of 6 digits. If we denote them by x_i; $1 \le i \le 6$, the problem reduces to one of finding the number of solutions of the equation

$$x_1 + x_2 + \cdots + x_6 = 19, \text{ where } 0 \le x_i \le 9 \quad (1)$$

There are $C(6 + 19 - 1, 19) = C(24, 5)$ solutions if $x_i \ge 0$.

We note that one of the six x_i's can be ≥ 10, but not more than one, as the sum of the x_i's= 19.

Let $x_1 \ge 10$ and let $u_1 = x_1 - 10$, $u_i = x_i$, $2 \le i \le 6$

Then the equation becomes

$$u_1 + u_2 + \cdots + u_6 = 9, \text{ where } u_i \ge 0$$

There are $C(6 + 9 - 1, 9) = C(14, 5)$ solutions for this equations.

The digit which is ≥ 10 can be chosen in 6 ways (viz., it may be $x_1, x_2, \ldots,$ or x_6).

Hence, the number of solutions of the equation $x_1 + x_2 + \cdots + x_6 = 19$, where any one $x_i \ge 10$ is $6 \times C(14, 5)$.

Hence, the required number of solutions of (1)

$$= C(24, 5) - 6 \times C(14, 5)$$
$$= 42{,}504 - 6 \times 2002 = 30{,}492.$$

Example 2.16 A man hiked for 10 hours and covered a total distance of 45 km. It is known that he hiked 6 km in the first hour and only 3 km in the last hour. Show that he must have hiked at least 9 km within a certain period of 2 consecutive hours.

Since, the man hiked 6 + 3 = 9 km in the first and last hours, he must have hiked 45 – 9 = 36 km during the period from second to ninth hours.

If we combine the second and third hours together, the fourth and fifth hours together, etc. and the eighth and ninth hours together, we have 4 time periods.

Let us now treat 4 time periods as pigeonholes and 36 km as 36 pigeons.

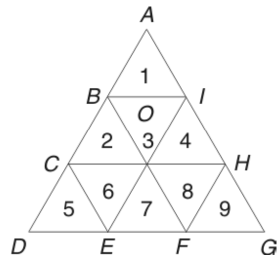
Using the generalised pigeonhole principal,

the least no. of pigeons accommodated in one pigeonhole

$$= \left\lfloor \frac{36-1}{4} \right\rfloor + 1$$
$$= \lfloor 8.75 \rfloor + 1 = 9$$

viz., the man must have hiked at least 9 km in one time period of 2 consecutive hours.

Example 2.17 If we select 10 points in the interior of an equilateral triangle of side 1, show that there must be at least two points whose distance apart is less than $\frac{1}{3}$.



Let ADG be the given equilateral triangle. The pairs of points B, C; E, F and H, I are the points of trisection of the sides AD, DG and GA respectively. We have divided the triangle ADG into 9 equilateral triangles each of side $\frac{1}{3}$.

The 9 sub-triangles may be regarded as 9 pigeonholes and 10 interior points may be regarded as 10 pigeons.

Then by the pigeonhole principle, at least one sub triangle must contain 2 interior points.

The distance between any two interior points of any sub triangle cannot exceed the length of the side, namely, $\frac{1}{3}$.

Hence the result.

Example 2.18

(i) If n pigeonholes are occupied by $(kn + 1)$ pigeons, where k is a positive integer, prove that at least one pigeonhole is occupied by $(k + 1)$ or more pigeons.

(ii) Hence, find the minimum number m of integers to be selected from $S = \{1, 2, \ldots 9\}$ so that (a) the sum of two of the m integers is even; (b) the difference of two of the m integers is 5. But there are $(kn + 1)$ pigeons. This results in a contradiction. Hence the result.

(i) If at least one pigeonhole is not occupied by $(k + 1)$ or more pigeons, each pigeonhole contains at most k pigeons. Hence, the total number of pigeons occupying the n pigeonholes is at most kn.
But there are $(kn + 1)$ pigeons. This results in a contradiction. Hence, the result

(ii) (a) Sum of 2 even integers or of 2 odd integers is even.
Let us divide the set S into 2 subsets $\{1, 3, 5, 7, 9\}$ and $\{2, 4, 6, 8\}$, which may be treated as pigeonholes. Thus $n = 2$.
At least 2 numbers must be chosen either from the first subset or from the second.
i.e., at least one pigeonhole must contain 2 pigeons
i.e., $k + 1 = 2$ or $k = 1$
$\therefore$ The minimum no. of pigeons required or the minimum number of integers to be selected is equal to

$$kn + 1 = 3.$$

(b) Let us divide the set S into the 5 subsets $\{1, 6\}$, $\{2, 7\}$, $\{3, 8\}$, $\{4, 9\}$, $\{5\}$, which may be treated as pigeonholes. Thus $n = 5$.
If $m = 6$, then 2 of integers of S will belong to one of the subsets and their difference is 5.

Example 2.19 If $(n + 1)$ integers not exceeding $2n$ are selected, show that there must be an integer that divides one of the other integers. Deduce that if 151 integers are selected from $\{1, 2, 3, \ldots, 300\}$ then the selection must include two integers x, y either of which divides the other.

Let the $(n + 1)$ integers be $a_1, a_2, \ldots, a_{n+1}$. Each of these numbers can be expressed as an odd multiple of a power of 2.
i.e., $a_i = 2^{ki} \times m_i$, where k_i is a non-negative integer and m_i is odd ($i = 1, 2, \ldots, n + 1$)
[For example, let $n = 5$ so that $2n = 10$. Let us consider $n + 1 = 6$ nos. that are less than or equal to 10, viz., 7, 5, 4, 6, 3, 10. Clearly $7 = 2^\circ \cdot 7$; $5 = 2^\circ \cdot 5$; $4 = 2^2 \cdot 1$; $6 = 2^1 \cdot 3$; $3 = 2^\circ \cdot 3$ and $10 = 2^1 \cdot 5$].

The integers $m_1, m_2, \ldots, m_{n+1}$ are odd positive integers less than $2n$ (pigeons).

But there are only n odd positive integers less than $2n$ (pigeonholes).

Hence, by the pigeonhole principle, 2 of the integers must be equal. Let them be $m_i = m_j$.

$$\therefore \qquad a_i = 2^{ki} m_i \text{ and } a_j = 2^{kj} m_j$$

$$\therefore \qquad \frac{a_i}{a_j} = \frac{2^{ki}}{2^{kj}} \qquad (\because m_i = m_j)$$

If $k_i < k_j$, then 2^{ki} divides 2^{kj} and hence a_i divides a_j.

If $k_i > k_j$, then a_j divides a_i.

Putting $n = 150$ (and hence, $2n = 300$ and $n + 1 = 151$) the deduction follows.

Example 2.20 If m is an odd positive integer, prove that there exists a positive integer n such that m divides $(2^n - 1)$.

Let us consider the $(m + 1)$ positive integers $2^1 - 1, 2^2 - 1, 2^3 - 1, \ldots, 2^m - 1$ and $2^{m+1} - 1$.

When these are divided by m, two of the numbers will give the same remainder, by the pigeonhole principle [$(m + 1)$ numbers are $(m + 1)$ pigeons and the m remainders, namely, $0, 1, 2, \ldots, (m - 1)$ are the pigeonholes].

Let the two numbers be $2^r - 1$ and $2^s - 1$ which give the same remainder r', upon division by m.
viz., let $2^r - 1 = q_1 m + r'$ and $2^s - 1 = q_2 m + r'$

$$\therefore \qquad 2^r - 2^s = (q_1 - q_2)m$$

$$\text{But} \qquad 2^r - 2^s = 2^s(2^{r-s} - 1)$$

$$\therefore \qquad (q_1 - q_2)m = 2^s(2^{r-s} - 1)$$

But m is odd and hence cannot be a factor of 2^s.

$$\therefore \qquad m \text{ divides } 2^{r-s} - 1.$$

Taking $n = r - s$, we get the required results.

Example 2.21 Prove that in any group of six people, at least three must be mutual friends or at least three must be mutual strangers.

Let A be one of the six people. Let the remaining 5 people be accommodated in 2 rooms labeled "A's friends" and "strangers to A".

Treating 5 people as 5 pigeons and 2 rooms as pigeonholes, by the generalised pigeonhole principle, one of the rooms must contain $\left\lfloor \frac{5-1}{2} \right\rfloor + 1 = 3$ people.

Let the room labeled "A's friends" contain 3 people. If any two of these 3 people are friends, then together with A, we have a set of 3 mutual friends. If no two of these 3 people are friends, then these 3 people are mutual strangers. In either case, we get the required conclusion.

If the room labeled "strangers to A" contain 3 people, we get the required conclusion by similar argument.

Example 2.22 During a four-weak vacation, a school student will attend at least one computer class each day, but he won't attend more than 40 classes in all during the vacation. Prove that, no matter how he distributes his classes during the four weeks, there is a consecutive span of days during which he will attend exactly 15 classes.

Let the student attend a_1 classes on day 1, a_2 classes on day 2 and so on a_{28} classes on day 28.

Then $b_i = a_1 + a_2 + \cdots + a_i$ will be the total no. of classes he will attend from day 1 to day i, both inclusive ($i = 1, 2, \ldots, 28$).

Clearly $\quad 1 \leq b_1 < b_2 < \ldots < b_{28} \leq 40$

and $\quad b_1 + 15 < b_2 + 15 < \ldots < b_{28} + 15 \leq 55$

Now there are 56 distinct numbers (pigeons) $b_1, b_2, \ldots, b_{28}$ and $b_1 + 15, b_2 + 15, \ldots, b_{28} + 15$.

These can take only 55 different values (1 through 55) (pigeonholes).

Hence, by the pigeonhole principle, at least two of the 56 numbers are equal.

Since $b_j > b_i$ if $j > i$, the only way for two numbers to be equal is $b_j = b_i + 15$, for some i and j where $j > i$.

$\therefore \quad b_j - b_i = 15$

i.e., $\quad a_{i+1} + a_{i+2} + \cdots + a_j = 15$

i.e., from the start of day $(i + 1)$ to the end of day j, the student will attend exactly 15 classes.

Example 2.23 If S is a set of 5 positive integers, the maximum of which is at most 9, prove that the sums of the elements in all the nonempty subsets of S cannot all be distinct.

Let the subsets of S be such that $1 \leq n_A \leq 3$ (i.e., A consists of only one or two or three elements of S).

The number of such subsets $C(5, 1) + C(5, 2) + C(5, 3)$

$$= 5 + 10 + 10 \quad (\because \text{ there are 5 elements in } S)$$
$$= 25$$

Let s_A be the sum of the elements of A.

Then $1 \leq s_A \leq 7 + 8 + 9$ ($\because$ the maximum of any element of $S = 9$)

i.e., $\quad 1 \leq s_A \leq 24$

Treating the 24 values of s_A as pigeonholes and 25 subsets A as pigeons, we get, by the pigeonhole principle, that there are 2 subsets A of S whose elements give the same sum.

Example 2.24 Find the number of integers between 1 and 250 both inclusive that are not divisible by any of the integers 2, 3, 5 and 7.

Let A, B, C, D be the sets of integers that lie between 1 and 250 and that are divisible by 2, 3, 5, and 7 respectively.
The elements of A are 2, 4, 6, …, 250

$$\therefore \qquad |A| = 125, \text{ which is the same as } \left\lfloor \frac{250}{2} \right\rfloor$$

Similarly, $$|B| = \left\lfloor \frac{250}{3} \right\rfloor = 83;\ |C| = \left\lfloor \frac{250}{5} \right\rfloor = 50,\ |D| = \left\lfloor \frac{250}{7} \right\rfloor = 35.$$

The set of integers between 1 and 250 which are divisible by 2 and 3, viz., $A \cap B$ is the same as that which is divisible by 6, since 2 and 3 are relatively prime numbers.

$$\therefore \qquad |A \cap B| = \left\lfloor \frac{250}{6} \right\rfloor = 41$$

Similarly, $$|A \cap C| = \left\lfloor \frac{250}{10} \right\rfloor = 25;\ |A \cap D| = \left\lfloor \frac{250}{14} \right\rfloor = 17$$

$$|B \cap C| = \left\lfloor \frac{250}{15} \right\rfloor = 16;\ |B \cap D| = \left\lfloor \frac{250}{21} \right\rfloor = 11;$$

$$|C \cap D| = \left\lfloor \frac{250}{35} \right\rfloor = 7;\ |A \cap B \cap C| = \left\lfloor \frac{250}{30} \right\rfloor = 8;$$

$$|A \cap B \cap D| = \left\lfloor \frac{250}{42} \right\rfloor = 5;\ |A \cap C \cap D| = \left\lfloor \frac{250}{70} \right\rfloor = 3;$$

$$|B \cap C \cap D| = \left\lfloor \frac{250}{105} \right\rfloor = 2;\ |A \cap B \cap C \cap D| = \left\lfloor \frac{250}{210} \right\rfloor = 1$$

By the Principle of Inclusion-Exclusion, the number of integers between 1 and 250 that are divisible by at least one of 2, 3, 5 and 7 is given by

$$\begin{aligned} |A \cup B \cup C \cup D| &= \{|A| + |B| + |C| + |D|\} - \{|A \cap B| + \cdots \\ &\quad + |C \cap D|\} + \{|A \cap B \cap C| + \cdots \\ &\quad + |B \cap C \cap D|\} - \{|A \cap B \cap C \cap D|\} \\ &= (125 + 83 + 50 + 35) - (41 + 25 + 17 \\ &\quad + 16 + 11 + 7) + (8 + 5 + 3 + 2) - 1 \\ &= 293 - 117 + 18 - 1 = 193 \end{aligned}$$

$\therefore$ Number of integers between 1 and 250 that are not divisible by any of the integers 2, 3, 5 and 7

$$\begin{aligned} &= \text{Total no. of integers} - |A \cup B \cup C \cup D| \\ &= 250 - 193 = 57. \end{aligned}$$

Example 2.25 How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2, x_3 are non-negative such that $x_1 \le 3$, $x_2 \le 4$ and $x_3 \le 6$? Use the principal of inclusion-exclusion.

Let the total no. of solutions with no restrictions be N.

Let P_1, P_2, P_3 denote respectively the properties $x_1 > 3$, $x_2 > 4$ and $x_3 > 6$.

Then the required no. of solutions is given by

$$N - \{|P_1| + |P_2| + |P_3| - |P_1 \cap P_2| - |P_2 \cap P_3| - |P_3 \cap P_1| + |P_1 \cap P_2 \cap P_3|\} \quad (1)$$

Now $N = C(3 + 11 - 1, 11) = 78$ (Refer to Example 2.13)

$|P_1|$ = no. of solutions subject to P_1 (viz. $x_1 \geq 4$ or $x_1 = 4, 5, 6, \ldots, 11$) $= C(3 + 7 - 1, 7) = C(9, 7) = 36$ $(\because x_2 \leq 7$ and $x_3 \leq 7)$

Similarly, $|P_2| = C(3 + 6 - 1, 6) = C(8, 6) = 28$

$|P_3| = C(3 + 4 - 1, 4) = C(6, 4) = 15$

$|P_1 \cap P_2|$ = no. of solutions subject to $x_1 \geq 4$ and $x_2 \geq 5$
$= C(3 + 2 - 1, 2) = C(4, 2) = 6$ $[\because x_3 \leq 2]$

Similarly, $|P_2 \cap P_3| = 0$ $(\because x_1 \leq -1)$ and $|P_3 \cap P_1| = C(3 + 0 - 1, 0) = 1$

$|P_1 \cap P_2 \cap P_3|$ = no. of solutions subject to $x_1 \geq 4$, $x_2 \geq 5$ and $x_3 \geq 7$
$= 0$

$\therefore$ Required number of solutions

$$= 78 - \{(36 + 28 + 15) - (6 + 0 + 1) + 0\}$$
$$= 6.$$

Example 2.26 There are 250 students in an engineering college. Of these 188 have taken a course in Fortran, 100 have taken a course in *C* and 35 have taken a course in Java. Further 88 have taken courses in both Fortran and *C*. 23 have taken courses in both *C* and Java and 29 have taken courses in both Fortran and Java. If 19 of these students have taken all the three courses, how many of these 250 students have not taken a course in any of these three programming languages?

Let F, C and J denote the students who have taken the languages Fortran, C and Java respectively.

Then $|F| = 188$; $|C| = 100$; $|J| = 35$

$|F \cap C| = 88$; $|C \cap J| = 23$; $|F \cap J| = 29$ and $|F \cap C \cap J| = 19$.

Then the number of students who have taken at least one of the three languages is given by

$$|F \cup C \cup J| = |F| + |C| + |J| - |F \cap C| - |C \cap J| - |F \cap J| + |F \cap C \cap J|$$
$$= (188 + 100 + 35) - (88 + 23 + 29) + 19$$
$$= 323 - 140 + 19 = 202.$$

No. of students who have not taken a course in any of these languages

$$= 250 - 202 = 48.$$

Example 2.27 A_1, A_2, A_3 and A_4 are subsets of a set U containing 75 elements with the following properties. Each subset contains 28 elements; the intersection of any two of the subsets contains 12 elements; the intersection of any three of the subsets contains 5 elements; the intersection of all four subsets contains 1 element.

(a) How many elements belong to none of the four subsets?

(b) How many elements belong to exactly one of the four subsets?
(c) How many elements belong to exactly two of the four subsets?

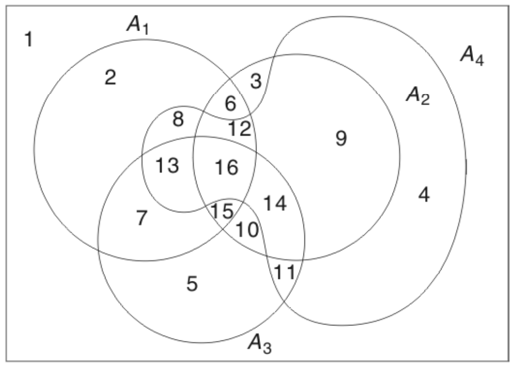


Fig. 2.1

(a) No. of elements that belong to at least one of the four subsets

$= |A_1 \cup A_2 \cup A_3 \cup A_4|$

$= [\{|A_1| + |A_2| + |A_3| + |A_4|\} - \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4|$
$+ |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|\} + \{|A_1 \cap A_2 \cap A_3|$
$+ |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|\}$
$- |A_1 \cap A_2 \cap A_3 \cap A_4|]$

$= [4 \times 28 - 6 \times 12 + 4 \times 5 - 1] = 59$

$\therefore$ No. of elements that belong to none of the four subset = 75 – 59 = 16.

(b) With reference to the Venn diagram given above Fig. 2.1, $n(A_1$ alone)

$= n[(2)]$
$= n(A_1) - [n(6) + n(7) + n(8) + n(12) + n(13) + n(15) + n(16)]$
$= n(A_1) - [\{n(6) + n(12) + n(15) + n(16)\} + \{n(7) + n(13) + n(15)$
$+ n(16)\} + \{n(8) + n(12) + n(13) + n(16)\} - n(12) - n(13) - n(15)$
$- 2n(16)]$
$= n(A_1) - [n(A_1 \cap A_2) + n(A_1 \cap A_3) + n(A_1 \cap A_4)] + [n(A_1 \cap A_2 \cap A_4)$
$+ n(A_1 \cap A_3 \cap A_4) + n(A_1 \cap A_2 \cap A_3)] - 2n[(A_1 \cap A_2 \cap A_3 \cap A_4)]$
$= 28 - 3 \times 12 + 3 \times 5 + 2 \times 1$
$= 9$

Similarly $n(A_2$ alone) $= n(A_3$ alone) $= n(A_4$ alone) $= 9$

$\therefore$ No. of elements that belong to exactly one of the subsets = 36.

(c) With reference to the Venn diagram of Fig. 2.1 given above,

$n(A_1$ and $A_2)$ only) $= n(6)$
$= n(A_1 \cap A_2) - \{n(15) + n(16)\} - \{n(12) + n(16)\} + n(16)$
$= n(A_1 \cap A_2) - n(A_1 \cap A_2 \cap A_3) - n(A_1 \cap A_2 \cap A_4)$
$+ n(A_1 \cap A_2 \cap A_3 \cap A_4)$
$= 12 - 5 - 5 + 1 = 3$

Similarly $n(A_1$ and A_3 only) $= n(A_1$ and A_4 only)
$= n(A_2$ and A_3 only) $= n(A_2$ and A_4 only) $= n(A_3$ and A_4 only) $= 3$

$\therefore$ No. of elements that belong to exactly two of the subsets = 18.

Example 2.28 Show that the number of derangements of a set of n elements is given by

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Note A *derangement* is a permutation of objects in which no object occupies its original position. For example, the derangements of 1 2 3 are 2 3 1 and 3 1 2. viz., D_3 = 2. 2 1 4 5 3 is a derangement of 1 2 3 4 5, but 2 1 5 4 3 is not a derangement of 1 2 3 4 5, since 4 occupies its original position.

Proof

Let a permutation have the property A_r, if it contains the r^{th} element in the r^{th} position.

Then D_n = the no. of the permutations having none of the properties $A_r(r = 1, 2, \ldots, n)$

$$= |A'_1 \cap A'_2 \cap \cdots \cap A'_n|$$

$$= N - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots$$

$$+ (-1)^n |A_1 \cap A_2 \ldots A_n| \qquad (1)$$

by the principle of inclusion-exclusion, where N is no. of permutations of n elements and so equals $n!$

Now $|A_i| = (n-1)!$, since $|A_i|$ is the number of permutations in which the i^{th} position is occupied by the i^{th} element, but each of the remaining positions can be filled arbitrarily.

Similarly, $|A_i \cap A_j| = (n-2)!$, $|A_i \cap A_j \cap A_k| = (n-3)!$ and so on.

Since there are $C(n, 1)$ ways of choosing one element from n, we get

$$\sum_i |A_i| = C(n, 1) \cdot (n-1)!$$

Similarly, $\sum_{i<j} |A_i \cap A_j| = C(n, 2) \cdot (n-2)!$,

$$\sum_{i<j<k} |A_i \cap A_j \cap A_k| = C(n, 3) \cdot (n-3)! \text{ and so on.}$$

Using these values in (1), we have

$$D_n = n! - C(n, 1) \cdot (n-1)! + C(n, 2) \cdot (n-2)! - \ldots + (-1)^n \cdot C(n, n) \cdot (n-n)! \qquad (2)$$

i.e., $$D_n = n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! - \cdots + (-1)^n \frac{n!}{n!\,0!}0!$$

$$= n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right\}$$

Example 2.29 Five gentlemen A, B, C, D and E attend a party, where before joining the party, they leave their overcoats in a cloak room. After the party, the overcoats get mixed up and are returned to the gentlemen in a random manner. Using the principle of inclusion-exclusive, find the probability that none receives his own overcoat.

$$\text{Required probability} = \frac{\text{No. of permutations in which none gets his overcoat}}{\text{No. of all possible permutations of the coats}}$$

$$= \frac{D_5}{5!} = \frac{5!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\}}{5!}$$

$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{11}{30}.$$

Example 2.30 In how many ways can the integers 1 through 9 be permuted such that

(a) no odd integer will be in its natural position?
(b) no even integer will be in its natural position?

(a) there are 5 odd integers between 1 and 9 inclusive.
Proceeding as in example (2.28) and from step (2) of that example,

$$\begin{aligned}\text{The required no. of ways} &= 9! - [C(5, 1) \cdot 8! - C(5, 2) \cdot 7! \\ &\quad + C(5, 3) \cdot 6! - C(5, 4) \cdot 5! + C(5, 5) \cdot 4!] \\ &= 2,05,056.\end{aligned}$$

(b) There are 4 even integers between 1 and 9.

$$\begin{aligned}\therefore \quad \text{The required no. of ways} &= 9! - [C(4, 1) \cdot 8! - C(4, 2) \cdot 7! \\ &\quad + C(4, 3) \cdot 6! - C(4, 4) \cdot 5!] \\ &= 2,29,080.\end{aligned}$$

EXERCISE 2(A)

Part A: (Short answer questions)

1. Define r-permutation and r-combination of n elements and express their values in terms of factorials.
2. Establish Pascal's identity in the theory of combinations.
3. How many permutations are there for the 8 letters *a, b, c, d, e, f, g, h*? How many of them (i) start with *a*, (ii) end with *h*, (iii) start with *a* and end with *h*?
4. In how many ways can the symbols *a, b, c, d, e, e, e, e, e* be arranged so that no *e* is adjacent to another *e*?
5. What is the number of arrangements of all the six letters in the word PEPPER?
6. How many distinct four-digit integers can one make form the digits 1, 3, 3, 7, 7 and 8?
7. In how many ways can 7 people be arranged about a circular table? If 2 of them insist on sitting next to each other, how many arrangements are possible?

8. What are the number of r-permutations and r-combinations of n objects if the repetition of objects is allowed?
9. How many different outcomes are possible when 5 dice are rolled?
10. A book publisher has 3000 copies of a Discrete Mathematics book. How many ways are there to store these books in their 3 warehouses if the copies of the book are identical?
11. State pigeonhole principle and its generalisation.
12. Show that in any group of eight people, at least two have birthdays which fall on the same day of the weak in any given year.
13. In a group of 100 people, several will have birth days in the same month. At least how many must have birth days in the same month?
14. If 20 processors are interconnected and every processor is connected to at least one other, show that at least two processors are directly connected to the same number of processors.
15. State the principle of inclusion-exclusion as applied to two finite subsets. Extend it for three finite subsets.
16. Among 30 Computer Science students, 15 know JAVA, 12 know C++ and 5 know both. How many students know (i) at least one of the two languages (ii) exactly one of the languages.
17. How many positive integers not exceeding 1000 are divisible by 7 or 11?
18. What is a derangement? Given an example.
19. Seven books are arranged in alphabetical order by author's name. In how many ways can a little boy rearrange these books so that no book is its original position?
20. How many permutations of 1, 2, 3, 4, 5, 6, 7 are not derangements?

Part B

21. (i) In how many numbers with 7 distinct digits do only the digits 1 – 9 appear?
    (ii) How many of the numbers in (i) contain a 3 and a 6?
    (iii) In how many of the numbers in (i), do 3 and 6 occur consecutively in any order?
    (iv) How many of the numbers in (i) contain neither a 3 nor a 6?
    (v) How many of the numbers in (i) contain a 3 not a 6?
    (vi) In how many of the numbers in (i) do exactly one of the numbers 3, 6 appear?
    (vii) In how many of the numbers in (i) do neither of the consecutive pairs 36 and 63 appear?
22. In how many ways can two couples Mrs. and Mr. A and Mrs. and Mr. B form a line so that (i) the A's are beside each other? (ii) the A's are not beside each other? (iii) each couple is together? (iv) the A's are beside each other but the B's are not? (v) at least one couple is together: (vi) exactly one couple is together?
23. Three couples, A's, B's and C's are going to form a line (i) In how many such lines will Mr. and Mrs. B be next to each other? (ii) In how many

such lines will Mr. and Mrs. B be next to each other and Mr. and Mrs. C be next to each other? (iii) In how many such lines will at least one couple be next to each other?

24. A Computer Science professor has 7 different programming books on a shelf, 3 of them deal with C++ and the other 4 with Java. In how many ways can the professor arrange these books on the shelf (i) if there are no restrictions? (ii) if the languages should alternate? (iii) if all the C++ books must be next to each other and all the Java books must be next to each other? (iv) if all the C++ books must be next to each other?
25. (i) In how many possible ways could a student answer a 10-question true or false test? (ii) In how many ways can the student answer the test in (i) if it is possible to leave a question unanswered in order to avoid an extra penalty for a wrong answer?
26. How many bit strings of length 12 contain (i) exactly three 1s? (ii) at most three 1s? (iii) at least three 1s? (iv) an equal number of 0s and 1s?
27. A coin is flipped 10 times where each flip comes up either head or tail. How many possible outcomes (i) are there in total? (ii) contain exactly 2 heads? (iii) contain at most 3 tails? (iv) contain the same number of heads and tails?
28. How many bit strings of length 10 have (i) exactly three 0s? (ii) at least three 1s? (iii) more 0s than 1s? (iv) an odd number of 0s?
29. How many permutations of the letters *ABCDEFGH* contain (i) the string *ED*? (ii) the string *CDE*? (iii) the strings *BA* and *FGH*? (iv) the strings *AB*, *DE* and *GH*? (v) the strings *CAB* and *BED*? (vi) the strings *BCA* and *ABF*?
30. Determine how many strings can be formed by arranging the letters *ABCDE* such that (i) *A* appears before *D*, (ii) *A* and *D* are side by side, (iii) neither the pattern *AB* nor the pattern *CD* appears, (iv) neither the pattern *AB* nor the pattern *BE* appears.
31. In how many ways can the letters *A, B, C, D, E, F* be arranged so that (i) *B* is always to the immediate left of the letter *E* (ii) *B* is always to the left of the letter *E* (iii) *B* is never to the left of the letter *E*?
32. In how different ways can the letters in the word MISSISSIPPI be arranged (i) if there is no restriction? (ii) if the two *P*s must be separated?
33. In how many ways can the letters *A, A, A, A, A, B, C, D, E* be permuted such that (i) no two *A*s are adjacent? (ii) if no two of the letters *B, C, D, E* are adjacent?
34. A computer password consists of a letter of the English alphabet followed by 3 or 4 digits. Find the number of passwords (i) that can be formed and (ii) in which no digit repeats.
35. (i) In how many ways can 7 people be arranged about a circular table? (ii) If two of the people insist on sitting next to each other, how many arrangements are possible?