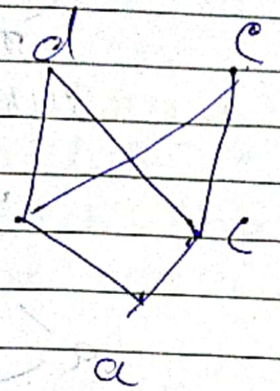
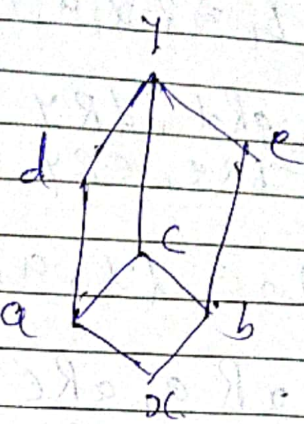
⇒ Sub-lattice:

① 

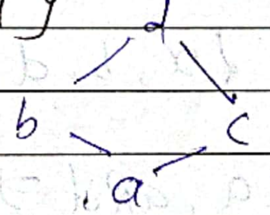
② 

⇒ Sol^n: ① Lattice ⇒ Join & Meet.

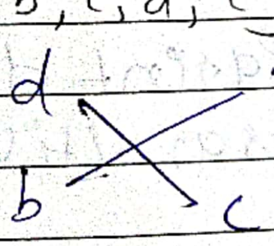
a) Is it Lattice?
1 → GLB & LUB single bottom.
No single bottom. d & e → Join not exist

Not a Lattice

b) Subset, $m = \{a, b, c, d\}$
GLB = a
LUB = d
⇒ L

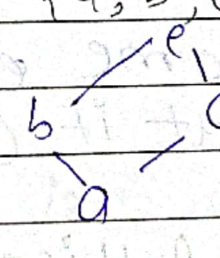


c) $\{b, c, d, e\}$



⇒ Not lattice.

d) $\{a, b, c, e\}$

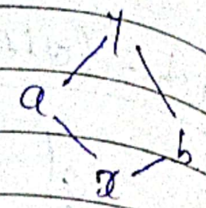


→ Latti

② Sol^n ②

LUB = y
GLB = x
} ⇒ lattice

1) $L_1 = \{x, a, y, b\}$

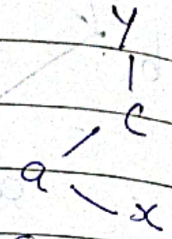


transitivity $aRd,\ dRY \Rightarrow aRY$

$bRe,\ eRY \Rightarrow bRY$

Sublattice

2) $L_2 = \{x, a, c, y\}$

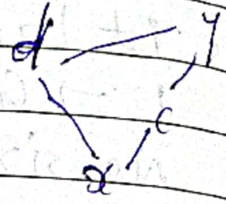


$xRa,\ aRc \Rightarrow xRc$

$aRd,\ dRY \Rightarrow aRY$

Sublattice

3) $L_3 = \{x, c, d, y\}$

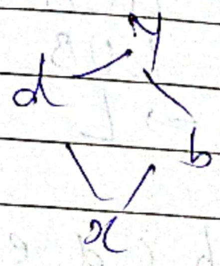


$c \not R d \Rightarrow$ but it's sublattice.

(i) $xRa,\ aRd \Rightarrow xRd$

$xRa,\ aRc \Rightarrow xRc.$

4) $L_4 = \{x, b, d, y\}$



$xRa,\ aRd \Rightarrow dRy$

$bRe,\ eRy \Rightarrow bRy$

Some edges are not in parent diagram but it can draw due to transitivity.

⇒ Sublattices: (definition)

A sublattice of a lattice L is a non-empty subset of L that is lattice with the same meet and join operations as L.

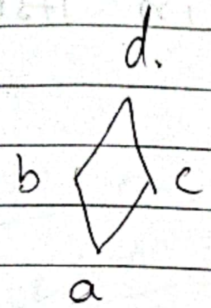
$a \vee I = I$
$a \wedge 0 = 0$

Ques: Is sublattice is lattice in itself ?

⇒ Upper Bound (Maximum - I) of a lattice.
- In a lattice 'L', if there exists an element I, such that $\forall a \in L$ (aRI), then 'I' is called upper bound of the lattice.

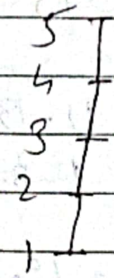
⇒ Lower Bound (Minimum - 0) of a lattice :
- In a lattice 'L' if there exists an element 0, such that $\forall a \in L$, $(0Ra)$, then '0' is called lower bound of the lattice.

Ex



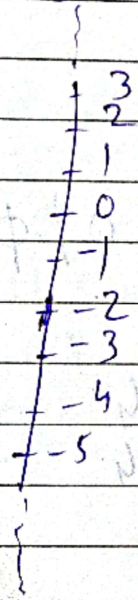
UB = d
LB = a

Ex



UB = 5
LB = 1

⇒ Is lattice is always finite ?
Ans: No, it can be infinite.



It's infinite.

It have Join & Meet but NO Lower bound & Upper bound

⇒ It is Lattice

If Lattice → finite = bounded
⇓
UB & LB exists.

⇒ we compare it with <u>set theory</u>:

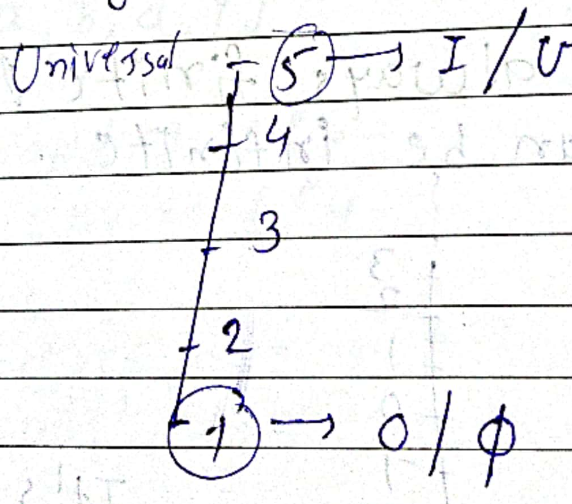
U.B. of set theory = Universal set
L.B. 〃 〃 〃 = ϕ, Null set

$A \cup U = U$, $A \cap U = A$
$A \cup \phi = A$, $A \cap \phi = \phi$

→ Lattice is also an Universe in itself.

U.B. = Universal set
L.B. = empty set.

according to set theory,



$U \longleftrightarrow I$
$\phi \longleftrightarrow 0$

2 & 4, U.B. = 4
LUB = 4

2 & 4, LB = 2
GLB = 2

Now,

Join with U.B. ⇒ U.B. $a \vee I = I$, $a \wedge I = a$
Meet with L.B. ⇒ L.B. $a \vee 0 = a$, $a \wedge 0 = 0$

all lattice are bounded (finite).
some unbounded lattice are also there

⇒ complement:

If set $= A$
comple. $= A^C$

Comple. law ⇒ $A \cup A^C = U$ → Join in lattice
$A \cap A^C = \phi$ → Meet in lattice

eg: complement of 2 = ?

Meet & Join
$a \vee a^c = I$
$a \wedge a^c = 0$

⇒ complement of an element in a lattice:

- In a bounded lattice L, for any element $a \in L$, if there exist an element $b \in L$, such that $a \vee b = I$, $a \wedge b = 0$, that b is called complement of 'a'. We can say a & b are complement of each other.

Now,

As we knows

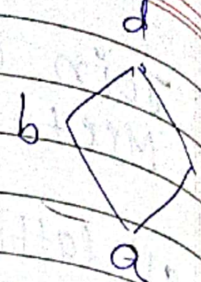
$A \cup A^c = U$ $a \vee a^c = I$

$A \cap A^c = \phi$ $a \wedge a^c = 0$

4 cases

$U^c = \phi \Rightarrow I^c = 0$

$\phi^c = U \Rightarrow 0^c = I$



for a & d.

1) $a^c = d$ & $d^c = a \Rightarrow$ check condition

$\left.\begin{array}{l} a \vee d = d \\ a \wedge d = a \end{array}\right\}$ we have to show

UB $= d$, LUB $= d$

LB $= a$, GLB $= a$

only one complement

$\Rightarrow a^c = d$ & $d^c = a$

commutative

$\boxed{a \,\&\, d \text{ are Pair}}$

for b & c

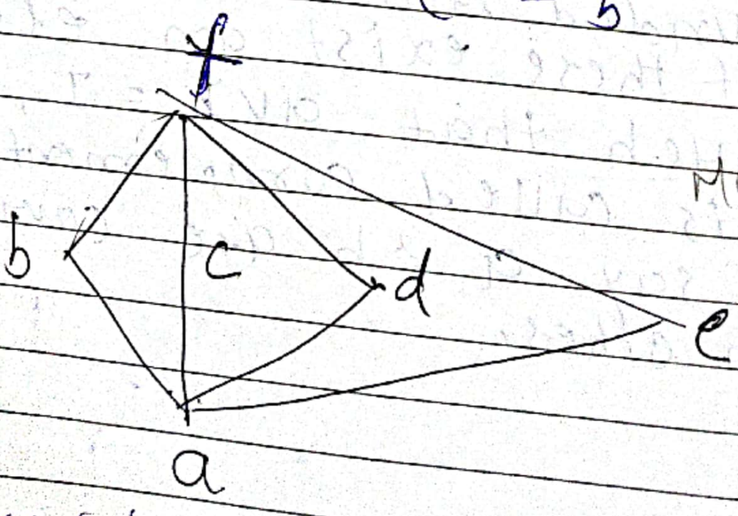
2) $b^c = c$ & $c^c = b$

$b \vee c = d \Rightarrow$ Inter. d UB

$b \wedge c = a \Rightarrow$ " a LB

$b^c = c$ & $c^c = b$

eg.



More than one complement

Now here,

in set theory, there is complement for all set
Has set ka sirf 1 complement hota he.

$a^c = f$ & $f^c = a$ $\quad a \vee f = f$

$a \wedge f = a$

for b & c $\Rightarrow$ $L \cdot B = a$ $\quad b \wedge c = a$

$U \cdot B = f$ $\quad b \vee c = f$

$b^c = c$ & $c^c = b$

for b & d $\Rightarrow$ $LB = a$ $\quad b \vee d = f$

$UB = f$ $\quad b \wedge d = a.$

$b^c = d$ & $d^c = b$

$\Rightarrow \boxed{b^c = c, d \;,\; d^c = b \;,\; c^c = b}$.

for e & b $\Rightarrow$ $LB = a$ $\quad e \vee b = f$

$UB = f$ $\quad e \wedge b = a.$

$e^c = b$ & $b^c = e$

$$\Rightarrow \boxed{\begin{aligned} a^c &= y \\ b^c &= c, d, e \\ c^c &= b, d, e \\ d^c &= b, c, e \\ e^c &= b, c, d. \end{aligned}}$$

$\to$ Join UB, $\to$ Meet LB

There exist complement for all element.

$\Rightarrow$ More than 1 comple. exist $\Rightarrow$ Distri. law fails

$\Downarrow$

$\Rightarrow$ Error Probability arise

element $\rightarrow$ more than 1 complete distributive law fails

$\rightarrow$ No complement. complementation law fails

for that only

complemented & distributive lattice

PROPERTIES OF LATTICES

Property 1

If $\{L, \leq\}$ is a lattice, then for any $a, b, c \in L$,

L_1: $a \vee a = a$ $\qquad$ $(L_1)'$: $a \wedge a = a$ $\qquad$ (Idempotency)

L_2: $a \vee b = b \vee a$ $\qquad$ $(L_2)'$: $a \wedge b = b \wedge a$ $\qquad$ (Commutativity)

L_3: $a \vee (b \vee c) = (a \vee b) \vee c$ $\qquad$ $(L_3)'$: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ $\qquad$ (Associativity)

L_4: $a \vee (a \wedge b) = a$ $\qquad$ $(L_4)'$: $a \wedge (a \vee b) = a$ $\qquad$ (Absorption)

Proof

(i) $a \vee a = \text{LUB}(a, a) = \text{LUB}(a) = a$. Hence L_1 follows.

(ii) $a \vee b = \text{LUB}(a, b) = \text{LUB}(b, a) = b \vee a$ $\{\because$ LUB (a, b) is unique.$\}$
Hence L_2 follows.

(iii) Since $(a \vee b) \vee c$ is the LUB $\{(a \vee b), c\}$, we have

$$a \vee b \leq (a \vee b) \vee c \qquad (1)$$

and $$c \leq (a \vee b) \vee c \qquad (2)$$

Since $a \vee b$ is the LUB $\{a, b\}$, we have

$a \leq a \vee b$ (3)

and $b \leq a \vee b$ (4)

From (1) and (3), $a \leq (a \vee b) \vee c$ by transitivity (5)

From (1) and (4), $b \leq (a \vee b) \vee c$ by transitivity (6)

From (2) and (6), $b \vee c \leq (a \vee b) \vee c$ by definition of join (7)

From (5) and (7), $a \vee (b \vee c) \leq (a \vee b) \vee c$ by definition of join (8)

Similarly, $a \leq a \vee (b \vee c)$ (9)

$b \leq b \vee c \leq a \vee (b \vee c)$ (10)

and $c \leq b \vee c \leq a \vee (b \vee c)$ (11)

From (9) & (10) $a \vee b \leq a \vee (b \vee c)$ — (12)

From (11) & (12) $(a \vee b) \vee c \leq a \vee (b \vee c)$ (13)

From (8) and (13), by antisymmetry of $\leq$, we get
$a \vee (b \vee c) = (a \vee b) \vee c$.
Hence L_3 follows.

(iv) Since $a \wedge b$ is the GLB $\{a, b\}$, we have

$$a \wedge b \leq a \quad (1)$$

Also $$a \leq a \quad (2)$$

From (1) and (2), $a \vee (a \wedge b) \leq a$ (3)

Also $$a \leq a \vee (a \wedge b)$$

by definition of LUB (4)

$\therefore$ From (3) and (4), by antisymmetry, we get $a \vee (a \wedge b) = a$.

Hence L_4 follows.

Now the identities $(L_1)'$ to $(L_4)'$ follow from the principle of duality.

Property 2

If $\{L, \leq\}$ is a lattice in which $\vee$ and $\wedge$ denote the operations of join and meet respectively, then for $a, b \in L$,

$$a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a.$$

In other words,

(i) $a \vee b = b$, if and only if $a \leq b$.

(ii) $a \wedge b = a$, if and only if $a \leq b$.

(iii) $a \wedge b = a$, if and only if $a \vee b = b$.

Proof

(i) Let $a \leq b$.

Now $b \leq b$ (by reflexivity).

$\therefore$ $a \vee b \leq b$ (1)

Since $a \vee b$ is the LUB (a, b),

$b \leq a \vee b$ (2)

From (1) and (2), we get $a \vee b = b$ (3)

Let $a \vee b = b$.

Since $a \vee b$ is the LUB (a, b),

$a \leq a \vee b$

i.e., $a \leq b$, by the data (4)

From (3) and (4), result (i) follows. Result (ii) can be probed in a way similar to the proof (i).

From (i) and (ii), result (iii) follows.

Note Property (2) gives a connection between the partial ordering relation $\leq$ and the two binary operations $\vee$ and $\wedge$ in a lattice $\{L, \leq\}$.

Property 3 (Isotonic Property)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$, the following properties hold good:

If $b \leq c$, then (i) $a \vee b \leq a \vee c$ and (ii) $a \wedge b \leq a \wedge c$.

Proof

Since $b \leq c$, $b \vee c = c$, by property 2(i). Also $a \vee a = a$, by idempotent property

Now $a \vee c = (a \vee a) \vee (b \vee c)$, by the above steps

$= a \vee (a \vee b) \vee c$, by associativity

$= a \vee (b \vee a) \vee c$, by commutativity

$= (a \vee b) \vee (a \vee c)$, by associativity

This is of the form $x \vee y = y$. $\therefore x \leq y$, by property 2(i).

i.e. $a \vee b \leq a \vee c$, which is the required result (i).

Similarly, result (ii) can be proved.

Property 4 (Distributive Inequalities)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$,

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

Proof

Since $a \wedge b$ is the GLB(a, b), $a \wedge b \leq a$ (1)

Also $a \wedge b \leq b \leq b \vee c$ (2)

since $b \vee c$ is the LUB of b and c.

From (1) and (2), we have $a \wedge b$ is a lower bound of $\{a, b \vee c\}$

$\therefore$ $a \wedge b \leq a \wedge (b \vee c)$ (3)

Similarly $a \wedge c \leq a$

and $a \wedge c \leq c \leq b \vee c$

$\therefore$ $a \wedge c \leq a \wedge (b \vee c)$ (4)

From (3) and (4), we get

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$$

i.e. $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$, which is result (i).

Result (ii) follows by the principle of duality.

Property 5 (Modular Inequality)

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$, $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$. (1)

Proof

Since $a \leq c$, $a \vee c = c$ (1), by property 2(i)

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ (2), by property 4(ii)

i.e. $a \vee (b \wedge c) \leq (a \vee b) \wedge c$ (3), by (1)

Now $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

$\therefore$ $a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$, by the definitions of LUB and GLB (4)

i.e. $a \leq c$

From (3) and (4), we get

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c.$$

LATTICE AS ALGEBRAIC SYSTEM

A set t[illegible]ther with certain operations (rules) for combining the elements of the [illegible]
set to form other elements of the set is usually [illegible] introduced [illegible]eory

system. Lattice L was introduced

LATTICE HOMOMORPHISM

ordered set in which for every pair of elements $a, b \in L$, LUB$(a, b) = a \vee b$ and GLB$(a, b) = a \wedge b$ exist in the set. That is, in a Lattice $\{L, \leq\}$, for every pair of elements a, b of L, the two elements $a \vee b$ and $a \wedge b$ of L are obtained by means of the operations $\vee$ and $\wedge$. Due to this, the operations $\vee$ and $\wedge$ are considered as binary operations on L. Moreover we have seen that $\vee$ and $\wedge$ satisfy certain properties such as commutativity, associativity and absorption. The formal definition of a lattice as an algebraic system is given as follows:

Definition

A lattice is an algebraic system $(L, \vee, \wedge)$ with two binary operations $\vee$ and $\wedge$ on L which satisfy the commutative, associative and absorption laws.

Note We have not explicity included the idempotent law in the definition, since the absorption law implies the idempotent law as follows:

$a \vee a = a \vee [a \wedge (a \vee a)]$, by using $a \vee a$ for $a \vee b$ in $(L_4)'$ of property 1

$= a$, by using $a \vee a$ for b in L_4 of property 1.

$a \wedge a = a$ follows by duality.

Though the above definition does not assume the existence of any partial ordering on L, it is implied by the properties of the operations $\vee$ and $\wedge$ as explained below:

Let us assume that there exists a relation R on L such that for $a, b \in L$,

$$aRb \text{ if and only if } a \vee b = b$$

For any $a \in L$, $a \vee a = a$, by idempotency

$\therefore$ aRa or R is reflexive.

Now for any $a, b, \in L$, let us assume that aRb and bRa.

$\therefore$ $a \vee b = b$ and $b \vee a = a$

Since $a \vee b = b \vee a$ by commutativity, we have $a = b$ and so R is antisymmetric.

Finally let us assume that aRb and bRc

$\therefore$ $a \vee b = b$ and $b \vee c = c$.

Now $a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = b \vee c = c$

viz. aRc and so R is transitive.

Hence R is a partial ordering.

Thus the two definitions given for a lattice are equivalent.

SUBLATTICES

Definition

A non-empty subset M of a lattice $\{L, \vee, \wedge\}$ is called a *sublattice* of L, iff M is closed under both the operations $\vee$ and $\wedge$. viz. if $a, b, \in M$, then $a \vee b$ and $a \wedge b$ also $\in M$.

From the definition, it is obvious that the sublattice itself is a lattice with respect to $\vee$ and $\wedge$.

For example if aRb whenever a divides b, where $a, b \in Z^+$ (the set of all positive integers) then $\{Z^+, R\}$ is a lattice in which $a \vee b =$ LCM (a, b) and $a \wedge b =$ GCD(a, b).

If $\{S_n, R\}$ is the lattice of divisors of any positive integer n, then $\{S_m, R\}$ is a sublattice of $\{Z^+, R\}$.