36. There are 6 gentlemen and 4 ladies to dine at a round table. In how many ways can they be seated so that no two ladies are together?
37. A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if (i) there are no restrictions? (ii) there must be equal number of men and women? (iii) there must be an even number of women? (iv) there must be more women than men? (v) there must be at least 8 men?
38. 7 women and 9 men are on the faculty in the mathematics department of a college. (i) How many ways are there to select a committee of 5 members of the department if at least one woman must be on the committee? (ii) How many ways are there to select a committee of 5 members of the department if at least one woman and at least one man must be on the committee?
39. How many licence plates consisting of 3 English letters followed by 3 digits contain no letter or digit twice?
40 How many strings of 6 distinct letters from the English alphabet contain (i) the letter A? (ii) the letters A and B? (iii) the letters A and B in consecutive positions with A preceding B? (iv) the letters A and B where A is somewhere to the left of B in the string?
41. A student has to answer 10 out of 13 questions in an exam. How many choices has he (i) if there is no restriction? (ii) if he must answer the first two questions? (iii) if he must answer the first or second question but not both? (iv) if he must answer exactly three out of the first 5 questions? (v) if he must answer at least 3 of the first 5 questions?
42. In how many ways can we distribute 8 identical white balls into 4 distinct containers so that (i) no container is left empty? (ii) the fourth container has an odd number of balls in it?
43. Find the number of unordered samples of size 5 (repetition allowed) from the set of letters (A, B, C, D, E, F), if (i) there is no restriction, (ii) the letter A occurs exactly twice, (iii) the letter A occurs at least twice.
44. Find the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 21$, where $x_1 \geq 8$ and x_2, x_3, x_4 are non-negative.
45. There are 10 questions on a discrete mathematics test. How many ways are there to assign marks to the problems, if the maximum of the test paper is 100 and each question is worth at least 5 marks?
46. How many integers between 1 and 10,00,000 have the sum of the digits equal to 15?
47. Show that among $(n + 1)$ arbitrarily chosen integers, there must exist two whose difference is divisible by n.

    [*Hints:* n of $(n + 1)$ integers, when divided by n will leave any of the remainders $0, 1, 2, \ldots, (n - 1)$ and $(n + 1)^{\text{th}}$ integer also will leave one of the remainders $0, 1, 2, \ldots, (n - 1)$.]
48. If there are 5 points inside a square of side length 2, prove that two of the points are within a distance of $\sqrt{2}$ of each other.

49. Of any 5 points chosen within an equilateral triangle whose sides are of length 1, show that two are within a distance of $\frac{1}{2}$ of each other.
50. Of any 26 points within a rectangle measuring 20 cm by 15 cm, show that at least two are within 5 cm of each other.

    [*Hint:* Divide the rectangle into subrectangles of dimension 4×3 cm.]
51. Prove that, in any list of 10 natural numbers $a_1, a_2, \ldots a_{10}$, there is a string of consecutive items of the list whose sum is divisible by 10.
52. How many integers between 1 and 300 (both inclusive) are divisible by (i) at least one of 3, 5, 7? (ii) 3 and by 5, but not by 7? (iii) 5 but by neither 3 nor 7?
53. How many prime numbers are less that 200? Use the principle of inclusion-exclusion.

    [*Hint* To check if a natural number n is prime, we have to check whether the prime numbers less than or equal to $\sqrt{n}$ are divisors of n.]
54. How many solutions does the equation $x_1 + x_2 + x_3 = 13$ have, where x_1, x_2, x_3 are non-negative integers less than 6? Use the principle of inclusion-exclusion.
55. A total of 1232 students have taken a course in Tamil, 879 have taken a course in English and 114 have taken a course in Hindi. Further, 103 have taken courses in both Tamil and English, 23 have taken courses in both Tamil and Hindi and 14 have taken courses in both English and Hindi. If 2092 students have taken at least one of Tamil, English and Hindi, how many students have taken a course in all the three languages?
56. How many derangements of {1, 2, 3, 4, 5, 6} (i) begin with the integers 1, 2 and 3 in some order? (ii) end with the integers 1, 2 and 3 in some order?
57. In how many ways can a teacher distribute 10 distinct books to his 10 students (one book to each student) and then collect and redistribute the books so that each student has the opportunity to peruse two different books?
58. There are 7 letters to be delivered to 7 houses in a block, one addressed to each house. If the letters are delivered completely at random, at the rate of one letter to each house, in how many ways can this be done if
    (i) no letter arrives at the right house?
    (ii) at least one letter arrives at the right house?
    (iii) all letters arrive at the right house?
59. Twenty people check their hats at a theatre. In how many ways can their hats be returned, so that
    (i) no one receives his or her own hat?
    (ii) at least one person receives his or her own hat?
    (iii) exactly one person receives his or her own hat?
60. A child inserts letters randomly into envelopes. What is the probability that in a group of 10 letters

(i) no letter is put into the correct envelope?
(ii) exactly one letter is put into the correct envelope?
(iii) exactly 8 letters are put into the correct envelopes?
(iv) exactly 9 letters are put into the correct envelopes?
(v) all letters are put into the correct envelope?

MATHEMATICAL INDUCTION

One of the most basic methods of proof is mathematical induction, which is a method to establish the truth of a statement about all the natural numbers. It will often help us to prove a general mathematical statement involving positive integers when certain instances of that statement suggest a general pattern.

Statement of the Principle of Mathematical Induction

Let $S(n)$ denote a mathematical statement (or a set of such statements) that involves one or more occurrences of the variable n, which represents a positive integer (a) If S (1) is true and (b) If, whenever $S(k)$ is true for some particular, but arbitrarily chosen $k \in Z^+$, $S(k + 1)$ is also true, then $S(n)$ is true for all $n \in Z^+$.

Note (1) The condition (a) is known as the *basis step* and the condition (b) is known as the *inductive step*.

(2) In condition (a), the choice of 1 is not mandatory, viz., $S(n)$ may be true for some first element $n_0 \in Z$, so that the induction process has a starting place.

Strong Form of the Principle

Given a mathematical statement $S(n)$ that involves one or more occurrences of the positive integer n and if

(a) $S(1)$ is true and
(b) whenever $S(1)$, $S(2)$, … $S(k)$ are true, $S(k + 1)$ is also true, then $S(n)$ is true for all $n \in Z^+$.

Well-ordering Principle

As an application of the principle of mathematical induction, we shall now establish the *well-ordering principle* which states that every non-empty set of non-negative integers has a smallest element.

A set containing just one element has a smallest member, namely the element itself. Hence, the well-ordering principle is true for sets of size 1.

Now let us assume that the principle is true for sets of size k, viz., any set of k non-negative integers has a smallest member.

Let us not consider a set S of $(k + 1)$ numbers from which one element 'a' is removed. The remaining k numbers have a smallest element, say b. [by the induction hypothesis]. The smaller of a and b is the smallest element of S.

Hence, by the principle of mathematical induction, it follows that any finite set of non-negative integers has a smallest element.

RECURRENCE RELATIONS

Definition

An equation that expresses a_n viz., the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence, namely a_0, a_1, …, a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a non-negative integer is called a *recurrence relation* for $\{a_n\}$ or a *difference equation*.

If the terms of a sequence satisfy a recurrence relation, then the sequence is called a *solution* of the recurrence relation.

For example, let us consider the geometric progression 4, 12, 36 108, …, the common ratio of which is 3. If $\{a_n\}$ represents this infinite sequence, we see that $\frac{a_{n+1}}{a_n} = 3$ viz., $a_{n+1} = 3a_n$, $n \geq 0$ is the recurrence relation corresponding to the geometric sequence $\{a_n\}$. However, the above recurrence relation does not represent a unique geometric sequence. The sequence 5, 15, 45, 135, … also satisfies the above recurrence relation. In order that the recurrence relation $a_{n+1} = 3a_n$, $n \geq 0$ may represent a unique sequence, we should know one of the terms of the sequence, say, $a_0 = 4$. If $a_0 = 4$, then the recurrence relation represents the sequence 4, 12, 36, 108, … The value $a_0 = 4$ is called *the initial condition*. If $a_0 = 4$, then from the recurrence relation, we get $a_1 = 3(4)$, $a_2 = 3^2(4)$ and so on. In general when $n \geq 0$, $a_n = 4.3^n$. This is called the *general solution* of the recurrence relation.

As another example, we consider the famous Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots,$$

which can be represented by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n, \text{ where } n \geq 0 \text{ and } F_0 = 0, F_1 = 1$$

Definitions

A recurrence relation of the form

$c_o\, a_n + c_1\, a_{n-1} + c_2\, a_{n-2} + \ldots + c_k\, a_{n-k} = f(n)$ is called a *linear recurrence relation of degree k with constant coefficients*, where c_0, c_1, … c_k are real numbers and $c_k \neq 0$. The recurrence relation is called *linear*, because each a_r is raised to the power 1 and there are no products such as $a_r \cdot a_s$. Since a_n is expressed in terms of the pervious k terms of the sequence, *the degree or order* of the recurrence relation is said to be k. In other words the degree is the difference between the greatest and least subscripts of the members of the sequence occurring in the recurrence relation.

If $f(n) = 0$, the recurrence relation is said to be *homogeneous*; otherwise it is said to be *non-homogeneous*.

Note The recurrence relations given in the above examples are linear homogeneous recurrence relations with constant coefficients and of degrees 1 and 2 respectively.

Solving Recurrence Relations

Systematic procedures have been developed for solving linear recurrence relations with constant coefficients. Let us first consider the solution of a homogeneous relation of order 2, viz., the recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0,\ n \geq 2 \tag{1}$$

Let $a_n = r^n$ $(r \neq 0)$ be a solution of (1).

Then $$c_0 r^n + c_1 r^{n-1} + c_2 r^{n-2} = 0$$

i.e., $$c_0 r^2 + c_1 r + c_2 = 0, \text{ since } r \neq 0 \tag{2}$$

(2) is a quadratic equation in r, which is called *the characteristic equation*, whose roots r_1 and r_2 are called *the characteristic roots* of the recurrence relation.

Depending on the nature of the roots r_1 and r_2, we get 3 different forms of the solution of the recurrence relation. We state them as follows without proof:

Case (i) r_1 and r_2 are real and distinct.

The solution of the recurrence relation is $a_n = k_1 r_1^n + k_2 r_2^n$, where k_1 and k_2, are arbitrary constants determined by initial conditions.

Case (ii) r_1 and r_2 are real and equal.

The solution is $a_n = (k_1 + k_2 n) r^n$, where $r_1 = r_2 = r$.

Case (iii) r_1 and r_2 are complex conjugate.

Let the modulus-amplitude form of $r_1 = r(\cos\theta + i \sin\theta)$

Then $r_2 = r(\cos\theta - i \sin\theta)$

The solution in this case is, $a_n = r^n(k_1 \cos n\theta + k_2 \sin n\theta)$

Theorem

The solution of a linear non-homogeneous recurrence relation with constant coefficients, viz., a recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_{n-k} a_{n-k} = f(n) \tag{1}$$

where $f(n) \not\equiv 0$ is of the form $a_n = a_n^{(h)} + a_n^{(p)}$, where $a_n^{(h)}$ is the solution of the associated homogeneous recurrence relation, namely,

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \tag{2}$$

and $a_n^{(p)}$ is a particular solution of (1).

Proof

Since $a_n = a_n^{(p)}$ is a particular solution of (1),

we have $$c_0 a_n^{(p)} + c_1 a_{n-1}^{(p)} + \cdots + c_k a_{n-k}^{(p)} = f(n) \tag{3}$$

Let $a_n = b_n$ be a second solution of (1).

Then $$c_0 b_n + c_1 b_{n-1} + \cdots + c_k b_{n-k} = f(n) \tag{4}$$

(4)–(3) gives

$$c_0\{b_n - a_n^{(p)}\} + c_1\{b_{n-1} - a_{n-1}^{(p)}\} + \cdots + c_{n-k}\{b_{n-k} - a_{n-k}^{(p)}\} = 0 \tag{5}$$

Step (5) means that $b_n - a_n^{(p)}$ is a solution of recurrence relation (2), viz., $a_n^{(h)}$

$\therefore \qquad b_n = a_n^{(h)} + a_n^{(p)}$ for all n.

i.e., the general solution of relation (1) is of the form $a_n = a_n^{(h)} + a_n^{(p)}$.

PARTICULAR SOLUTIONS

There is no general procedure for finding the particular solution of a recurrence relation. However for certain functions $f(n)$ such as polynomials in n and powers of constants, the forms of particular solutions are known and they are exactly found out by the method of undetermined coefficients.

The following table gives certain forms of $f(n)$ and the forms of the corresponding particular solution, on the assumption that $f(n)$ is not a solution of the associated homogeneous relation:

Form of f(n)	*Form of $a_n^{(p)}$ to be assumed*
c, a constant	A, a constant
n	$A_0 n + A_1$
n^2	$A_0 n^2 + A_1 n + A_2$
n^t, $t \in Z^+$	$A_0 n^t + A_1 n^{t-1} + \ldots + A_n$
r^n, $r \in R$	Ar^n
$n^t r^n$	$r^n(A_0 n^t + A_1 n^{t-1} + \ldots + A_n)$
$\sin \alpha n$	$A \sin \alpha n + B \cos \alpha n$
$\cos \alpha n$	$A \sin \alpha n + B \cos \alpha n$
$r^n \sin \alpha n$	$r^n(A \sin \alpha n + B \cos \alpha n)$
$r^n \cos \alpha n$	$r^n(A \sin \alpha n + B \cos \alpha n)$

When $f(n)$ is a linear combination of the terms in the first column, then $a_n^{(p)}$ is assumed as a linear combination of the corresponding terms in the second column of the table. When $f(n) = r^n$ or $(A + Bn)r^n$ where r is a non-repeated characteristic root of the recurrence relation, then $a_n^{(p)}$ is assumed as $An\ r^n$ or $cn(A + Bn)r^n$ as the case may be. When $f(n) = r^n$, where r is a twice repeated characteristic root, then $a_n^{(p)}$ is assumed as $An^2\ r^n$ and so on.

Note For a different treatment of difference equation (recurrence relations) using the finite difference operators such as Δ and E, the students are advised to refer to the chapter on 'Difference Equations' in the author's book "Numerical Methods with Programs in *C*".

SOLUTION OF RECURRENCE RELATIONS BY USING GENERATING FUNCTIONS

Definition

The generating function of a sequence $a_0, a_1, a_2, \ldots$ is the expression

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots \infty = \sum_{n=0}^{\infty} a_n x^n$$

For example,

(i) the generating function for the sequence 1, 1, 1, 1, … is given by

$$G(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(ii) the generating function for the sequence 1, 2, 3, 4, … is given by

$$G(x) = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + \ldots = \frac{1}{(1-x)^2}$$

(iii) the generating function for the sequence 1, a, a^2, a^3, … is given by

$$G(x) = 1 + ax + a^2x^2 \cdots \infty = \frac{1}{1-ax}, \text{ for } |ax| < 1.$$

To solve a recurrence relation (both homogeneous and non-homogeneous) with given initial conditions, we shall multiply the relation by an appropriate power of x and sum up suitably so as to get an explicit formula for the associated generating function. The solution of the recurrence relation a_n is then obtained as the coefficient of x^n in the expansion of the generating function. The procedure is explained clearly in the worked examples that follow.

WORKED EXAMPLES 2(B)

Example 2.1 Prove, by mathematical induction, that

$$1^1 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1).$$

Let $S(n)$: $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

When $n = 1$,

$$S(1): 1^2 = \frac{1}{3} \cdot 1 \cdot 1 \cdot 3$$

So $S(1)$ is true, viz., the basic step is valid.

Let $S(n)$ be true for $n = k$

i.e., $1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$

Now $1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2k+1)^2$

$$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2, \text{ using the truth of } S(k)$$

$$= \frac{1}{3}(2k+1)\{k(2k-1) + 3(2k+1)\}$$

$$= \frac{1}{3}(2k+1)(2k^2 + 5k + 3)$$

$$= \frac{1}{3}(2k+1)(2k+3)(k+1) \text{ or } \frac{1}{3}(k+1)(2k+1)(2k+3)$$

i.e., $S(k+1)$ is valid.

Thus the inductive step is also true.
Hence, $S(n)$ is true for all $n \in Z^+$.

Example 2.2 Prove, by mathematical induction, that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3).$$

Let S_n: $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$.

Now S_1: $1 \cdot 2 \cdot 3 = \frac{1}{4} \cdot 1 \cdot 2 \cdot 3 \cdot 4$

Thus, the basic step S_1 is true.
Let S_k be true

i.e., $$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) = \frac{1}{4}k(k+1)(k+2)(k+3) \qquad (1)$$

Now $[1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2)] + (k+1)(k+2)(k+3)$

$$= \frac{1}{4}k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3), \text{ by (1)}$$

$$= \frac{1}{4}(k+1)(k+2)(k+3)\{k+4\}$$

Thus S_{k+1} is true, if S_k is true.
i.e., the inductive step is true.
Hence, S_n is true for all $n \in Z^+$.

Example 2.3 Prove, by mathematical induction, that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Let $$S_n: \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Then S_1: $\frac{1}{1.2} = \frac{1}{1+1}$ which is true.

i.e., the basic step S_1 is true.
Let S_k be true.

i.e., $$\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1} \qquad (1)$$

Now $$\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}, \text{ by (1)}$$

$$= \frac{1}{k+1}\left\{\frac{k(k+2)+1}{k+2}\right\}$$

$$= \frac{1}{k+1}\left\{\frac{(k+1)^2}{k+2}\right\} = \frac{k+1}{k+2} \qquad (2)$$

(2) means that S_{k+1} is also true.
i.e., the inductive step is true.
Hence, S_n is true for all $n \in Z^+$.

Example 2.4 Use mathematical induction to show that

$n! \geq 2^{n-1}$, for $n = 1, 2, 3, \ldots$

Let S_n: $n! \geq 2^{n-1}$

$\therefore$ S_1: $1! \geq 2^0$, which is true.

i.e., the basic step is true

Let S_k be true

i.e., $k! \geq 2^{k-1}$ (1)

Now
$$\begin{aligned}(k+1)! &= (k+1).\ k!\\ &\geq (k+1).\ 2^{k-1}, \text{ by (1)}\\ &\geq 2\ .\ 2^{k-1}, \text{ since } k+1 \geq 2\\ &= 2^k \qquad (2)\end{aligned}$$

Step (2) means that S_{k+1} is also true.
i.e., the inductive step is true.
Hence, S_n is true for $n = 1, 2, 3, \ldots$.

Example 2.5 Use mathematical induction to show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}, \text{ for } n \geq 2$$

Let S_n: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

$\therefore$ S_2: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$, since L.S = 1.707 and R.S = 1.414

i.e., the basic step is true for $n = 2$.
Let S_k be true.

i.e., $$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k} \qquad (1)$$

Now $$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}, \text{ by (1)}$$

Now $$\sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}} > \frac{\sqrt{k \cdot k}+1}{\sqrt{k+1}}$$

i.e., $$> \frac{k+1}{\sqrt{k+1}}$$

i.e., $$> \sqrt{k+1}$$

$$\therefore \quad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} \tag{2}$$

Step (2) means that S_{k+1} is also true.
Hence, S_n is true for $n = 2, 3, 4, \ldots$.

Example 2.6 Use mathematical induction to show that

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \le \frac{1}{\sqrt{n+1}}, \text{ for } n = 1, 2, 3, \ldots$$

Let $$S_n\text{: } \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \le \frac{1}{\sqrt{n+1}}$$

$$\therefore \quad S_1\text{: } \frac{1}{2} \le \frac{1}{2}, \text{ which is true.}$$

i.e., the basic step is true.
Let S_k be true.

i.e., $$\frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)} \le \frac{1}{\sqrt{k+1}} \tag{1}$$

Now $$\frac{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k) \cdot (2k+2)} \le \frac{1}{\sqrt{k+1}} \cdot \frac{2k+1}{2k+2}, \text{ by (1)} \tag{2}$$

Now $$\frac{2k+1}{2k+2} \le \frac{\sqrt{k+1}}{\sqrt{k+2}},$$

if $$\frac{(2k+1)^2}{(2k+2)^2} \le \frac{k+1}{k+2}$$

i.e., if $$\frac{4k^2+4k+1}{4k^2+8k+4} \le \frac{k+1}{k+2}$$

i.e., if $4k^3 + 12k^2 + 9k + 2 \le 4k^3 + 12k^2 + 12k + 4$
i.e., if $9k + 2 \le 12k + 4$
i.e., if $3k + 2 \ge 0$, which is true.
Using this in step (2), we get

$$\frac{1 \cdot 3 \cdot 5 \cdots (2k-1)(2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k)(2k+2)} \le \frac{1}{\sqrt{k+1}} \cdot \frac{\sqrt{k+1}}{\sqrt{k+2}}$$

i.e., $$\le \frac{1}{\sqrt{k+2}} \tag{3}$$

Step (3) means that S_{k+1} also true.
i.e., the induction step is true.
Hence, S_n is true for $n = 1, 2, 3, \ldots$

Example 2.7 Use mathematical induction to prove that $H_{2^n} \geq 1 + \frac{n}{2}$, where $H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$.

Let S_n: $H_{2^n} \geq 1 + \frac{n}{2}$

$\therefore$ S_1: $H_2 = 1 + \frac{1}{2} \geq 1 + \frac{1}{2}$, which is true.

i.e., the basic step is true.
Let S_k be true.

i.e., $$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} \geq 1 + \frac{k}{2} \quad (1)$$

Now $$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k}\right) + \left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}\right)$$

$$\geq \left(1 + \frac{k}{2}\right) + \left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k}\right)$$

$$\geq \left(1 + \frac{k}{2}\right) + 2k \cdot \frac{1}{2^{k+1}}$$ ($\because$ each of the 2^k terms in the second group $\geq \frac{1}{2^{k+1}}$, the last term)

i.e., $$\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2}$$

i.e., $$\geq 1 + \left(\frac{k+1}{2}\right) \quad (2)$$

Step (2) means that S_{k+1} is true.
i.e., the inductive step is true.
$\therefore$ S_n is true for $n \in Z^+$.

Example 2.8 Use mathematical induction to prove that $n^3 + 2n$ is divisible by 3, for $n \geq 1$.

Let S_n: $(n^3 + 2n)$ is divisible by 3.
$\therefore$ S_1: $(1^3 + 2)$ is divisible by 3, which is true.
i.e., the basic step is true.
Let S_k be true.
i.e., $k^3 + 2k$ is divisible by 3 (1)

Now $(k + 1)^3 + 2(k + 1)$

$$= (k^3 + 2k) + (3k^2 + 3k + 3)$$

$(k^2 + 2k)$ is divisible by 3, by (1)

Also $3k^3 + 3k + 3 = 3(k^2 + k + 1)$ is divisible by 3.

$\therefore$ The sum, namely, $(k + 1)^3 + 2(k + 1)$ is divisible by 3 (2)

i.e., S_{k+1} is also true

i.e., the inductive step is true.

$\therefore$ S_n is true for $n \geq 1$.

Example 2.9 Use mathematical induction to prove that

$n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9, for $n \geq 1$.

Let S_n: $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9.

$\therefore$ S_1: $1^3 + 2^3 + 3^3 = 36$ is divisible by 9, which is true.

i.e., the basic step is true.

Let S_k be true.

i.e., $k^3 + (k + 1)^3 + (k + 2)^3$ is divisible by 9 (1)

Now $(k + 1)^3 + (k + 2)^3 + (k + 3)^3$

$$= [k^3 + (k + 1)^3 + (k + 2)^3] + [9k^2 + 27k + 27]$$

$$= [k^3 + (k + 1)^3 + (k + 2)^3] + 9(k^2 + 3k + 3)$$

The first expression is divisible by 9 [by (1)] and the second expression is a multiple of 9.

$\therefore$ Their sum is divisible by 9

i.e., S_{k+1} is true.

i.e., the inductive step is true.

$\therefore$ S_n is true for $n \geq 1$.

Example 2.10 Use mathematical induction to prove that $(3^n + 7^n - 2)$ is divisible by 8, for $n \geq 1$.

Let S_n: $(3^n + 7^n - 2)$ is divisible by 8

$\therefore$ S_1: $(3 + 7 - 2)$ is divisible by 8, which is true.

i.e., the basic step is true.

Let S_k be true.

i.e., $(3^k + 7^k - 2)$ is divisible by 8 (1)

Now $3^{k+1} + 7^{k+1} - 2 = 3(3^k) + 7(7^k) - 2$

$$= 3\{3^k + 7^k - 2\} + 4(7^k + 1) \quad (2)$$

$3(3^k + 7^k - 2)$ is divisible by 8, by step (1)

$7^k + 1$ is an even number, for $k \geq 1$

$\therefore$ $4(7^k + 1)$ is divisible by 8

$\therefore$ R.S. of (2) is divisible by 8

i.e., $3^{k+1} + 7^{k+1} - 2$ is divisible by 8

i.e., S_{k+1} is also true.

i.e., the inductive step is true

$\therefore$ S_n is true for $n \geq 1$.

Example 2.11 Solve the recurrence relation $a_n - 2a_{n-1} = 3^n$; $a_1 = 5$

The characteristic equation of the recurrence relation is $r - 2 = 0 \quad \therefore r = 2$.

$$\therefore \qquad a_n^{(h)} = c \cdot 2^n$$

Since the R.S. of the relation is 3^n, let a particular solution of the relation be $a_n = A \cdot 3^n$. Using this in the relation, we get

$$A \cdot 3^n - 2 \cdot A \cdot 3^{n-1} = 3^n$$

i.e., $\quad 3A - 2A = 3$ or $A = 3$

$$\therefore \qquad a_n^{(p)} = 3^{n+1}$$

$\therefore$ General solution is $\quad a_n = a_n^{(h)} + a_n^{(p)} = c \cdot 2^n + 3^{n+1}$

Using the condition $\quad a_1 = 5$, we get $2c + 9 = 5$

$$\therefore \qquad c = -2$$

Hence, the required solution is $\quad a_n = 3^{n+1} - 2^{n+1}$.

Example 2.12 Solve the recurrence relation

$$a_n = 2a_{n-1} + 2^n; \; a_0 = 2$$

The characteristic equation of the R.R. is $r - 2 = 0 \therefore r = 2$

$$\therefore \qquad a_n^{(h)} = c \cdot 2^n$$

Since the R.S. of the R.R. is 2^n and 2 is the characteristic root of the R.R., let

$a_n = An \cdot 2^n$ be a particular solution of the R.R.

Using this in the R.R., we get

$$An\ 2^n - 2(n-1)2^{n-1} = 2^n$$

i.e., $\quad An - (n - 1) = 1 \therefore A = 1$

$$\therefore \qquad a_n^{(p)} = n2^n$$

$\therefore$ General solution of the R.R. is

$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= c \cdot 2^n + n \cdot 2^n$$

Given: $\quad a_0 = 2 \quad \therefore c = 2$

Hence, the required solution is $\quad a^n = (n + 2) \cdot 2^n$.

Example 2.13 *n* circular disks with different diameters and with holes in their centres can be stacked on any of the three pegs mounted on a board. To start with, the pegs are stacked on peg 1 with no disk resting upon a smaller one. The objective is to transfer the disks one at a time so that we end up with the original stack on peg 2. Each of the three pegs may be used as temporary location for any disk, but at no time a larger disk should lie on a smaller one on any peg. What is the minimum number of moves required to do this for *n* disks?

Note This problem is popularly known as the *Tower of Hanoi problem*.

Let H_n denote the number of moves required to solve the Tower of Hanoi problem with *n* disks. Let us form a recurrence relation for H_n and then solve it.

To start with, the n disks are on peg 1 in the decreasing order from bottom to top. We can transfer the top $(n - 1)$ disks to peg 3, as per the rules specified, in H_{n-1} moves (by the meaning assigned to H_n). We keep the largest disk fixed in peg 1 during these moves. Then we use one move to transfer the largest disk to peg 2. We can transfer the $(n - 1)$ disks now on peg 3 to peg 2 using H_{n-1} additional moves, placing them on top of the largest disk which remains fixed in peg 2 during the second set of H_{n-1} moves.

Since the problem cannot be solved using fewer moves, we get

$H_n = 2H_{n-1} + 1$, which is the required R.R. Obviously $H_1 = 1$, since one disk can be transferred from peg 1 to peg 2 in one move.

The characteristic equation of the R.R. is $r - 2 = 0 \quad \therefore r = 2$

$$\therefore \quad H_n^{(h)} = c \cdot 2^n.$$

Since the R.S. of the R.R. $H_n - 2H_{n-1} = 1$ is 1, let

$H_n = A$ be a particular solution of the R.R. Using this in the R.R., we have

$$A = 2A + 1$$

i.e., $\quad A = -1 \quad$ or $\quad H_n^{(p)} = -1$

$\therefore$ The general solution of the R.R. is

$$H_n = c \cdot 2^n - 1$$

Using the initial condition $H_1 = 1$, we get $2c - 1 = 1 \quad \therefore c = 1$

$\therefore$ The required solution of the Tower of Hanoi problem is $H_n = 2^n - 1$.

Example 2.14 Solve the recurrence relation $a_{n+1} - a_n = 3n^2 - n$; $n \geq 0$, $a_0 = 3$.

The characteristic equation of the R.R. is

$$r - 1 = 0 \quad \text{i.e.,} \quad r = 1$$

$$\therefore \quad a_n^{(h)} = c \cdot 1^n = c$$

Since the R.S. of the R.R. is $3n^2 - n \equiv (3n^2 - n) \cdot 1^n$, let the particular solution of the R.R. be assumed as $a_n = (A_0n^2 + A_1n + A_2)\,n$, since 1 is a characteristic root of the R.R. Using this in the R.R., we have

$$\{A_0(n + 1)^3 + A_1(n + 1)^2 + A_2(n + 1)\} - \{A_0n^3 + A_1n^2 + A_2n\} = 3n^2 - n$$

i.e., $\quad A_0(3n^2 + 3n + 1) + A_1(2n + 1) + A_2 = 3n^2 - n$

Comparing like terms, we have

$$A_0 = 1, \quad 3A_0 + 2A_1 = -1 \quad \text{and} \quad A_0 + A_1 + A_2 = 0.$$

Solving these equations, we get

$$A_0 = 1, \quad A_1 = -2 \quad \text{and} \quad A_2 = 1$$

$$\therefore \quad a_n^{(p)} = n^3 - 2n^2 + n$$
$$= n(n - 1)^2$$

$\therefore$ The general solutions of the R.R. is

$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= c + n(n - 1)^2$$

Given that $a_0 = 3$. $\therefore c = 3$

$\therefore$ The required solution of the R.R. is

$$a_n = 3 + n(n-1)^2.$$

Example 2.15 Find a formula for the general term F_n of the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, ….

The recurrence relation corresponding to the Fibonacci sequence $\{F_n\}$; $n \geq 0$ is $F_{n+2} = F_{n+1} + F_n$; $n \geq 0$ with the initial conditions $F_0 = 0$, $F_1 = 1$.

The characteristic equation of the R.R. is

$$r^2 - r - 1 = 0.$$

Solving it, we have $r = \dfrac{1 \pm \sqrt{5}}{2}$.

Since the R.S. of $F_{n+2} - F_{n+1} - F_n = 0$ is zero, the solution of the R.R. is

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

$F_0 = 0$ gives $\quad c_1 + c_2 = 0 \quad (1)$

$F_1 = 1$ gives $\quad c_1\left(\dfrac{1+\sqrt{5}}{2}\right) + c_2\left(\dfrac{1-\sqrt{5}}{2}\right) = 1 \quad (2)$

Using (1) in (2), we get $c_1 - c_2 = \dfrac{2}{\sqrt{5}} \quad (3)$

Using (1) in (3), we have $c_1 = \dfrac{1}{\sqrt{5}}$ and $c_2 = -\dfrac{1}{\sqrt{5}}$.

$\therefore$ The general term F_n of the Fibonacci sequence is given by

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n;\ n \geq 0.$$

Example 2.16 A particle is moving in the horizontal direction. The distance it travels in each second is equal to two times the distance it travelled in the previous second. If a_r denotes the position of the particle in the r^{th} second, determine a_r, given that $a_0 = 3$ and $a_3 = 10$.

Let a_r, a_{r+1}, a_{r+2} be the positions of the particle in the r^{th}, $(r+1)^{\text{st}}$ and $(r+2)^{\text{nd}}$ seconds.

Then $\quad a_{r+2} - a_{r+1} = 2(a_{r+1} - a_r)$

i.e., $\quad a_{r+2} - 3a_{r+1} + 2a_r = 0 \quad (1)$

The characteristic equation of the R.R. (1) is $m^2 - 3m + 2 = 0$

i.e., $\quad (m-1)(m-2) = 0$ or $m = 1, 2$

Since the R.S. of (1) is zero, the solution of the R.R. is

$$a_r = c_1 \cdot 1^r + c_2 \cdot 2^r$$

i.e., $a_r = c_1 + c_2 \cdot 2^r$ (2)

Using $a_0 = 3$, we have $c_1 + c_2 = 3$ (3)

Using $a_3 = 10$, we have $c_1 + 8c_2 = 10$ (4)

Solving (3) and (4), we get $c_1 = 2$; $c_2 = 1$.

$\therefore$ The required solutions is

$$a = 2^r + 2.$$

Example 2.17 Solve the recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n),\ n \geq 0,$$

given that $a_0 = 1$ and $a_1 = 4$.

The characteristic equation of the R.R. is

$$r^2 - 6r + 9 = 0 \quad \text{or} \quad (r - 3)^2 = 0$$

$\therefore$ $r = 3, 3$

$\therefore$ $a_n^{(h)} = (c_1 + c_2 n)3^n$

Noting that 3 is a double root of the characteristic equation, we assume the particular solution of the R.R. as

$$a_n = A_0 \cdot 2^n + A_1 n^2 \cdot 3^n$$

Using this in the R.R., we have

$$A_0 \cdot 2^{n+2} + A_1(n+2)^2 \cdot 3^{n+2} - 6\{A_0 \cdot 2^{n+1} + A_1 \cdot (n+1)^2 \cdot 3^{n+1}\} + 9\{A_0 \cdot 2^n + A_1 n^2 \cdot 3^n\} = 3(2^n) + 7(3^n)$$

i.e., $A_0 2^n (4 - 12 + 9) + A_1 \cdot 3^n \{9(n+2)^2 - 18(n+1)^2 + 9n^2\} = 3 \cdot (2^n) + 7 \cdot (3^n)$

i.e., $A_0 \cdot 2^n + A_1 \cdot 3^n \times 18 = 3 \cdot (2^n) + 7 \cdot (3^n)$

Comparing like terms, we get

$$A_0 = 1 \text{ and } A_1 = \frac{7}{18}$$

$\therefore$ $a_n^{(p)} = 2^n + \dfrac{7}{18} n^2 \cdot 3^n$

Hence, the general solution of the R.R. is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

i.e., $a_n = (c_1 + c_2 \cdot n) \cdot 3^n + 2^n + \dfrac{7}{18} n^2 \cdot 3^n$

Given $a_0 = 1$ $\therefore$ $c_1 + 1 = 1$

i.e., $c_1 = 0$

Given $a_1 = 4$ $\therefore$ $3c_2 + 2 + \dfrac{7}{6} = 4$ i.e., $c_2 = \dfrac{5}{18}$

$\therefore$ The required solution is

$$a_n = \frac{5}{18}\, n \cdot 3^n + 2^n + \frac{7}{18}\, n^2 \cdot 3^n.$$

Example 2.18 Solve the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n.$$

The given R.R. is $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)2^n$.

The characteristic equation of the R.R. is

$$r^2 - 4r + 4 = 0$$

i.e., $(r-2)^2 = 0$, i.e., $r = 2, 2$.

$$\therefore \quad a_n^{(h)} = (c_1 + c_2 n) \cdot 2^n$$

Since the R.S. of the R.R. is $(n+1)2^n$, where 2 is a double root of the characteristic equation, we assume the particular solution of the R.R. as

$$a_n = n^2(A_0 + A_1 n) \cdot 2^n$$

Using this in the R.R., we have

$$n^2(A_0 + A_1 n) \cdot 2^n - 4(n-1)^2 \{A_0 + A_1(n-1)\}2^{n-1} + 4(n-2)^2 \{A_0 + A_1(n-2)\}\, 2^{n-2} = (n+1)2^n$$

i.e., $$4n^2(A_0 + A_1 n) - 8\,(n-1)^2 \{A_0 + A_1(n-1)\} + 4(n-2)^2 \{A_0 + A_1(n-2)\} = 4(n+1)$$

Equating coefficients of n on both sides,

$$A_1 = \frac{1}{6}$$

Equating constant terms on both sides,

$$2A_0 - 6A_1 = 1$$

i.e., $A_0 = 1$

$$\therefore \quad a_n^{(p)} = \left(n^2 + \frac{n^3}{6}\right)2^n$$

Hence, the general solution of the R.R. is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

i.e., $$a_n = \left(c_1 + c_2 n + n^2 + \frac{n^3}{6}\right)2^n.$$

Example 2.19 Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2});\ n \geq 2 \text{ and } a_0 = 1,\ a_1 = 2.$$

The given recurrence relation is

$$a_n - 2a_{n-1} + 2a_{n-2} = 0$$

The characteristic equation of the R.R. is

$$r^2 - 2r + 2 = 0$$

Solving, we have $r = 1 \pm i$

The modulus-amplitude form of

$$1 \pm i = \sqrt{2}\left(\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4}\right)$$

Hence, the general solution of the R.R. is

$$a_n = \left(\sqrt{2}\right)^n \left\{c_1\cos\frac{n\pi}{4} \pm c_2\sin\frac{n\pi}{4}\right\} \tag{1}$$

Using the condition $a_0 = 1$ in (1), we get $c_1 = 1$
Using $a_1 = 2$ in (1), we get

$$\sqrt{2}\left\{\frac{1}{\sqrt{2}} + c_2 \cdot \frac{1}{\sqrt{2}}\right\} = 2$$

i.e., $c_2 = 1$

$\therefore$ The required solution is

$$a_n = \left(\sqrt{2}\right)^n \left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right).$$

Example 2.20 Form a recurrence relation satisfied by $a_n = \sum\limits_{k=1}^{n} k^2$ and find the value of $\sum\limits_{k=1}^{n} k^2$, by solving it

$$a_n = \sum_{k=1}^{n} k^2 \quad \text{and} \quad a_{n-1} = \sum_{k=1}^{n-1} k^2$$

Hence, $a_n - a_{n-1} = n^2$. Clearly $a_1 = 1$

The characteristic equation of the R.R. is

$$r - 1 = 0 \quad \text{or} \quad r = 1$$

$$\therefore \quad a_n^{(h)} = c \cdot 1^n = c$$

Since the R.S. of the R.R. is $n^2 = n^2 \cdot 1^n$, let the particular solution be assumed as $a_n = (A_0 n^2 + A_1 n + A_2)n$.

Using this in the R.R., we have

$$(A_0 n^2 + A_1 n + A_2)n - \{A_0(n-1)^2 + A_1(n-1) + A_2\}(n-1) = n^2$$

Equating like terms and solving, we get

$$A_0 = \frac{1}{3},\ A_1 = \frac{1}{2} \text{ and } A_2 = \frac{1}{6}$$

Hence,

$$a_n^{(p)} = \frac{n}{6}(2n^2 + 3n + 1)$$

$$= \frac{n}{6}(n+1)(2n+1)$$

Hence, the general solution of the R.R. is

$$a_n = c + \frac{n}{6}(n + 1)\ (2n + 1)$$

Using $a_1 = 1$, we get $c = 0$

$$\therefore \qquad a_n = \sum n^2 = \frac{1}{6}n(n + 1)\ (2n + 1).$$

Example 2.21 Use the method of generating function to solve the recurrence relation

$$a_n = 3a_{n-1} + 1;\ n \geq 1, \quad \text{given that} \quad a_0 = 1.$$

Let the generating function of $\{a_n\}$ be $G(x) = \sum\limits_{n=0}^{\infty} a_n x^n$.

The given R.R. is $\quad a_n = 3a_{n-1} + 1$ (1)

$$\therefore \qquad \sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} x^n\ ,$$

on multiplying both sides of (1) by x^n and summing up.

i.e., $$G(x) - a_0 = 3x\ G(x) + \frac{x}{1-x}$$

i.e., $$(1 - 3x)\ G(x) = 1 + \frac{x}{1-x} \quad (\because\ a_0 = 1)$$

$$\therefore \qquad G(x) = \frac{1}{(1-x)(1-3x)} = \frac{-\frac{1}{2}}{1-x} + \frac{\frac{3}{2}}{1-3x}$$

i.e., $$G(x) = -\frac{1}{2}(1 - x)^{-1} + \frac{3}{2}(1 - 3x)^{-1}$$

i.e., $$\sum_{n=0}^{\infty} a_n x^n = -\frac{1}{2}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} 3^n x^n$$

$$\therefore \qquad a_n = \text{coefficient of } x^n \text{ in } G(x)$$

$$= \frac{1}{2}(3^{n+1} - 1)$$

Example 2.22 Use the method of generating function to solve the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2} + 4^n\ ;\ n \geq 2, \text{ given that } a_0 = 2 \text{ and } a_1 = 8.$$

Let the generating function of $\{a_n\}$ be $G(x) = \sum\limits_{n=0}^{\infty} a_n x^n$.

Multiplying both sides of the given R.R. by x^n and summing up, we have

$$\sum_{n=2}^{\infty} a_n x^n = 4 \sum_{n=2}^{\infty} a_{n-1} x^n - 4 \sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=2}^{\infty} 4^n x^n$$

i.e, $\{G(x) - a_0 - a_1 x\} = 4x\{G(x) - a_0\} - 4x^2\, G(x) + \dfrac{1}{1-4x} - 1 - 4x.$

i.e., $(1 - 4x + 4x^2)\, G(x) = \dfrac{1}{1-4x} - 1 - 4x + 2 \quad (\because a_0 = 2 \text{ and } a_1 = 8)$

$\therefore \qquad G(x) = \dfrac{1+(1-4x)^2}{(1-2x)^2 \cdot (1-4x)}$

$= \dfrac{4}{1-4x} - \dfrac{2}{(1-2x)^2}$, on splitting into partial fractions

i.e., $G(x) = \sum_{n=0}^{\infty} a_n x^n = 4[1 + 4x + (4x)^2 + \ldots + (4x)^n + \cdots \infty]$

$- 2[1 + 2 \cdot (2x) + 3 \cdot (2x)^2 + \cdots + (n + 1)(2x)^n + \cdots \infty]$

$\therefore \qquad a_n = 4^{n+1} - (n + 2)2^{n+1}.$

Example 2.23 Use the method of generating function to solve the recurrence relation

$$a_{n+1} - 8a_n + 16a_{n-1} = 4^n;\ n \geq 1;\ a_0 = 1,\ a_1 = 8.$$

Let the generating functions of $\{a_n\}$ be

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Multiplying both sides of the given R.R. by x^n and summing up, we have

$$\sum_{n=1}^{\infty} a_{n+1} x^n - 8 \sum_{n=1}^{\infty} a_n x^n + 16 \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} (4x)^n$$

i.e., $\dfrac{1}{x}\{G(x) - a_0 - a_1 x\} - 8\{G(x) - a_0\} + 16x\, G(x) = \dfrac{1}{1-4x} - 1$

i.e., $(1 - 8x + 16x^2)\, G(x) - a_0 - a_1 x + 8a_0 x = \dfrac{4x^2}{1-4x}$

i.e., $G(x) = \dfrac{a_0 + (a_1 - 8a_0)x}{(1-4x)^2} + \dfrac{4x^2}{(1-4x)^3}$

$= \dfrac{1}{(1-4x)^2} + \dfrac{4x^2}{(1-4x)^3}$, on using the values of a_0 and a_1.

$= (1 - 4x + 4x^2)(1 - 4x)^{-3}$

i.e., $$\sum_{n=0}^{\infty} a_n x^n = (1 - 4x + 4x^2) \cdot \frac{1}{2}\{1 \cdot 2 + 2 \cdot 3\,(4x) + 3 \cdot 4(4x)^2 + \cdots + (n+1)\,(n+2)\,(4x)^n \cdots\}$$

$\therefore$ $$a_n = \frac{1}{2}[(n+1)\,(n+2)\,4^n - n(n+1)4^n + (n-1)n\,4^{n-1}]$$

$$= \frac{1}{2}4^{n-1}\{4(n^2+3n+2) - 4(n^2+n) + (n^2-n)\}$$

$$= \frac{1}{2}(n^2+7n+8) \cdot 4^{n-1}.$$

Example 2.24 Use the method of generating function to solve the recurrence relation $a_{n+2} - 4a_n = 9n^2$; $n \geq 0$.

Let the generating function of $\{a_n\}$ be

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Multiplying both sides of the given R.R. by x^n and summing up, we have

$$\sum_{n=0}^{\infty} a_{n+2} x^n - 4\sum_{n=0}^{\infty} a_n x^n = 9\sum_{n=0}^{\infty} n^2 x^n$$

i.e., $$\frac{1}{x^2}\{G(x) - a_0 - a_1 x\} - 4G(x)$$

$$= 9\sum_{n=0}^{\infty}\{n(n+1) - n\}x^n$$

$$= 9[1 \cdot 2x + 2 \cdot 3x^2 + \cdots] - 9[x + 2x^2 + 3x^3 + \cdots]$$

$$= 9x \times 2(1-x)^{-3} - 9x(1-x)^{-2}$$

i.e., $$\left(\frac{1}{x^2} - 4\right)G(x) = \frac{a_0}{x^2} + \frac{a_1}{x} + \frac{18x}{(1-x)^3} - \frac{9x}{(1-x)^2}$$

$\therefore$ $$G(x) = \frac{a_0 + a_1 x}{1 - 4x^2} + \frac{18x^3}{(1-x)^3(1-4x^2)} - \frac{9x^3}{(1-x)^2(1-4x^2)}$$

$$= \frac{a_0 + a_1 x}{(1-2x)(1+2x)} + \frac{9x^3 + 9x^4}{(1-x)^3(1-2x)(1+2x)}$$

$$= \frac{A}{1-2x} + \frac{B}{1+2x} - \frac{\frac{17}{3}}{1-x} + \frac{5}{(1-x)^2} - \frac{6}{(1-x)^3} - \frac{\frac{1}{12}}{1+2x} + \frac{\frac{27}{4}}{1-2x}$$

(On splitting into partial fractions)

$$= c_1(1-2x)^{-1} + c_2(1+2x)^{-1} - \frac{17}{3}(1-x)^{-1} + 5(1-x)^{-2} - 6(1-x)^{-3},$$

where $c_1 = A + \dfrac{27}{4}$ and $c_2 = B - \dfrac{1}{12}$

i.e.,
$$\sum_{n=0}^{\infty} a_n x^n = c_1 \sum_{n=0}^{\infty} 2^n x^n + c_2 \sum_{n=0}^{\infty} (-1)^n 2^n x^n - \frac{17}{3}\sum_{n=0}^{\infty} x^n + 5\sum_{n=0}^{\infty} (n+1)\, x^n - 3\sum_{n=0}^{\infty} (n+1)\,(n+2)x^n$$

Equating coefficients of x^n, we get the general solution of the given R.R. as

$$a_n = c_1 \cdot 2^n + c_2 \cdot (-1)^n\, 2^n - \frac{17}{3} + 5(n+1) - 3(n+1)\,(n+2)$$

i.e.,
$$a_n = c_1 \cdot 2^n + c_2 \cdot (-1)^n \cdot 2^n - 3\left(n^2 + \frac{4}{3}n + \frac{20}{9}\right).$$

Example 2.25 Use the method of generating function to solve the recurrence relation

$$a_n = 4a_{n-1} + 3n \cdot 2^n;\ n \geq 1, \text{ given that } a_0 = 4.$$

Let the generating function of $\{a_n\}$ be

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Multiplying both sides of the given R.R. by x^n and summing up, we have

$$\sum_{n=1}^{\infty} a_n x^n - 4\sum_{n=1}^{\infty} a_{n-1}\, x^n = 3\sum_{n=1}^{\infty} n(2x)^n$$

i.e.,
$$\{G(x) - a_0\} - 4x\; G(x) = 6x \cdot \sum_{n=1}^{\infty} n(2x)^{n-1}$$

i.e., $(1-4x)\; G(x) = \dfrac{6x}{(1-2x)^2} + 4 \quad [\because a_0 = 4]$

$\therefore$
$$G(x) = \frac{6x}{(1-4x)(1-2x)^2}$$

$$= \frac{10}{1-4x} - \frac{3}{1-2x} - \frac{3}{(1-2x)^2}, \text{ on splitting into partial fractions}$$

i.e.,
$$\sum_{n=0}^{\infty} a_n x^n = 10\sum_{n=0}^{\infty} (4x)^n - 3\sum_{n=0}^{\infty} (2x)^n - 3\sum_{n=0}^{\infty} (n+1)\,(2x)^n$$

Equating coefficients of x^n, we get

$$a_n = 10 \times 4^n - 3 \times 2^n - 3(n+1) \times 2^n$$
$$= 10 \times 4^n - (3n+6) \times 2^n$$

EXERCISE 2(B)

Part A: (Short answer questions)

1. What is mathematical induction? In what way is it useful?
2. State the principle of mathematical induction.
3. What are basic and inductive steps in mathematical induction?
4. State the strong form of the principle of mathematical induction.
5. What is well-ordering principle. Establish it using mathematical induction.
6. Use mathematical induction to show that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.
7. Use mathematical induction to show that $1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n + 1)$.
8. Use mathematical induction to prove that $n < 2^n$, for all positive integers n.
9. Find a formula for the sum of the first n even positive integers and prove it by induction.
10. Define a recurrence relation. What do you mean by its solution?
11. Define a linear recurrence relation. What is meant by the degree of such a relation?
12. When is a recurrence relation said to be homogeneous? Non-homogeneous?
13. Define the characteristic equation and characteristic polynomial of a recurrence relation.
14. What do you mean by particular solution of a recurrence relation?
15. Define generating function of a sequence and give an example.
16. How will you use the notion of generating function to solve a recurrence relation?

Part B

Prove, by mathematical induction, the following results:

17. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.
18. $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1} n^2 = \dfrac{(-1)^{n-1} \cdot n(n+1)}{2}$.
19. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{1}{4} n^2(n + 1)^2$.
20. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \dfrac{n(n+1)(n+2)}{3}$.
21. $1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \cdots + (2n - 1) \cdot 2n = \dfrac{1}{3} n(n + 1)(4n - 1)$.
22. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$.

23. $\frac{1}{2\cdot 4}+\frac{1\cdot 3}{2\cdot 4\cdot 6}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}+\cdots+\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n+2)}$

$$= \frac{1}{2}-\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2\cdot 4\cdot 6\cdots(2n+2)}.$$

24. $\sum_{r=1}^{n}\frac{r^2}{(2r-1)(2r+1)}=\frac{n(n+1)}{2(2n+1)}$.

Prove, by mathematical induction, the following inequalities, when $n \in Z^+$.

25. $n < 2^n$, for $n \geq 1$.
26. $n^2 < 2^n$, for $n > 4$.
27. $2^n < n^3$, for $n \geq 10$.
28. $2^n < n!$ for $n > 3$.
29. $2^n \geq (2n + 1)$, for $n \geq 3$.
30. $\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\geq\frac{1}{2n}$, for $n \geq 1$.

Prove, by mathematical indication, the following results, when $n \in Z^+$.

31. $n^3 - n$ is divisable by 6.
32. $n^5 - n$ is divisible by 5.
33. $5^n - 1$ is divisable by 4.
34. $8^n - 3^n$ is divisible by 5.
35. $5^{2n} - 2^{5n}$ is divisible by 7.
36. $10^{n+1} + 10^n + 1$ is divisible by 3.
37. $6 \times 7^n - 2 \times 3^n$ is divisible by 4.

Solve the following recurrence relations:

38. $a_{n+1} - 2a_n = 5;\ n \geq 0;\ a_0 = 1$.
39. $a_n - 2a_{n-1} = n + 5;\ n \geq 1;\ a_0 = 4$.
40. $a_{n+1} - a_n = 2n + 3;\ n \geq 0;\ a_0 = 1$.
41. $a_n - 2a_{n-1} = 2n^2;\ n \geq 1;\ a_1 = 4$.
42. $a_n - 3a_{n-1} = 2^n;\ n \geq 1;\ a_0 = 1$.
43. $a_n = 2a_{n-1} + 3 \cdot 2^n;\ n \geq 1;\ a_0 = 5$.
44. $a_n - a_{n-1} = 3(b_n - a_{n-1})$, where

$$b_n = \begin{cases} 1000\cdot(3/2)^n, & \text{for } 0 \leq n \leq 10 \\ 1000\cdot(3/2)^{10}, & \text{for } n \geq 10 \end{cases} \quad \text{given that } a_0 = 0.$$

45. $a_{n+1} = 2a_n + 3a_{n-1};\ n \geq 1$; given $a_0 = 0,\ a_1 = 8$.
46. $9a_n = 6a_{n-1} - a_{n-2};\ n \geq 2$, given $a_0 = 3,\ a_1 = -1$.
47. $a_{n+2} - a_{n+1} - 2a_n = 4;\ n \geq 0$, given $a_0 = -1,\ a_1 = 3$.
48. $a_{n+2} + 4a_{n+1} + 4a_n = 7;\ n \geq 0$; given $a_0 = 1,\ a_1 = 2$.
49. $a_{n+2} + 3a_{n+1} + 2a_n = 3^n;\ n \geq 0$; given $a_0 = 0,\ a_1 = 1$.
50. $a_{n+2} - 3a_{n+1} + 2a_n = 2^n;\ n \geq 0$; given $a_0 = 3,\ a_1 = 6$.
51. $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$.

52. $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$; $n \geq 2$, given $a_0 = 1$; and $a_1 = 4$.
53. $a_{n+2} - 4a_{n+1} + 3a_n = 2^n \cdot n^2$; $n \geq 0$; given $a_0 = a_1 = 0$.
54. $a_{n+2} - 7a_{n+1} - 8a_n = n(n-1)2^n$.

Use the method of generating functions to solve the following recurrence relations:

55. $a_n + 3a_{n-1} - 4a_{n-2} = 0$; $n \geq 2$, given $a_0 = 3$, $a_1 = -2$.
56. $a_{n+2} - 5a_{n+1} + 6a_n = 36$; $n \geq 0$; given $a_0 = a_1 = 0$.
57. $a_{n+2} - a_n = 2^n$; $n \geq 0$; given $a_0 = 0$; $a_1 = 1$.
58. $a_{n+2} - 6a_{n+1} + 9y_n = 3^n$; $n \geq 0$; given $a_0 = 2$ and $a_1 = 9$.
59. $a_{n+1} + 4a_n + 4a_{n-1} = n - 1$; $n \geq 1$, given $a_0 = 0$ and $a_1 = 1$.
60. $a_{n+2} + a_n = n \cdot 2^n$; $n \geq 0$.

ANSWERS

Exercise 2(A)

3. (i) 8!	(ii) 7!	(iii) 7!	(iv) 6!
4. 24	5. 60	6. 90	7. 720, 240
9. 252	10. 45,04,501	13. 9	16. 22; 17
17. 220	19. 1854	20. 3186	
21. (i) 1,81,440	(ii) 1,05,840	(iii) 30,240	(iv) 5040
(v) 35,280	(vi) 70,560	(vii) 75,600	
22. 12; 12; 8; 4; 16; 8		23. 240; 96; 708	
24. (i) 5040	(ii) 144	(iii) 288	(iv) 720
25. (i) 2^{10}	(ii) 3^{10}		
26. (i) 220	(ii) 299	(iii) 4017	(iv) 924
27. (i) 1024	(ii) 45	(iii) 176	(iv) 252
28. (i) 120	(ii) 968	(iii) 386	(iv) 512
29. (i) 5040	(ii) 720	(iii) 120	(iv) 120
(v) 24	(vi) 0		
30. (i) 60	(ii) 48	(iii) 78	(iv) 78
31. (i) 120	(ii) 360	(iii) 360	
32. (i) 34650	(ii) 28350		
33. (i) 24	(ii) 24		
34. (i) 720	(ii) 240		
35. (i) 2,86,000	(ii) 1,49,760		
36. 43,200			
37. (i) 1,25,970	(ii) 44,100	(iii) 63,900	(iv) 40,935
(iv) 10,695			
38. (i) 4242	(ii) 4221		
39. (i) 1,12,32,000			
40. (i) $C(25, 5) \times 6!$		(ii) $C(24, 4) \times 6!$	
(iii) $C(24, 4) \times 5!$		(iv) $15 \times C(24, 4) \times 4!$	
41. (i) 286	(ii) 165	(iii) 110	(iv) 80
(v) 276			

42. (i) 35 (ii) 70
43. (i) 252 (ii) 35 (iii) 56
44. 560
45. $C(59, 9)$
46. $C(20, 15) - 6 \times C(10, 5)$
52. (i) 162 (ii) 18 (iii) 34
53. 46
54. 6
55. 7
56. (i) 4 (ii) 36
57. $10! \times D_{10}$
58. (i) D_7 (ii) $7! - D_7$ (iii) 1
59. (i) D_{20} (ii) $20! - D_{20}$ (iii) $20 \times D_{19}$
60. (i) $D_{10}/10!$ (ii) $10 \times D_9/10!$ (iii) $C(10, 2)/10!$ (iv) 0 (v) $1/10!$

Exercise 2(B)

38. $a_n = 6(2^n) - 5$
39. $a_n = 11(2^n) - (n + 7)$
40. $a_n = (n + 1)^2$
41. $a_n = 13(2^n) - 2(n^2 + 4n + 6)$
42. $a_n = 2(3^n - 2^n)$
43. $a_n = (3n + 5)2^n$
44. $a_n = \frac{9000}{7}\left\{\left(\frac{3}{2}\right)^n - (-2)^n\right\}$, for $0 \le n \le 10$

$= 1000\left(\frac{3}{2}\right)^{10} \{1 - (-2)^{10}\}$, for $n > 10$.

45. $a_n = 2(3^n) - 2(-1)^n$
46. $a_n = (1 - 2n)/\ 3^{n-1}$
47. $a_n = 2^{n+1} + (-1)^{n+1} - 2$
48. $a_n = \left(\frac{2}{9} - \frac{5n}{6}\right)(-2)^n + \frac{7}{9}$
49. $a_n = \frac{3}{4}(-1)^n - \frac{4}{5}(-2)^n + \frac{1}{20}(3)^n$.
50. $a_n = 1 + 2^{n+1} + n \cdot 2^{n-1}$
51. $a_n = A \cdot 2^n + B \cdot 3^n - n \cdot 2^{n+1} + \frac{3}{4}(2n + 7)$
52. $a_n = \frac{1}{8} + \frac{39}{8}(3^n) - 2^{n+2} - \frac{1}{4}n^2 - \frac{5}{2}n$.
53. $a_n = 3 + 5(3^n) - (n^2 + 8) \cdot 2^n$.
54. $a_n = A \cdot 8^n + B \cdot (-1)^n - \frac{1}{54}(3n^2 - 5n + 2) \cdot 2^n$
55. $a_n = 2 + (-4)^n$
56. $a_n = 18[3^n - 2^{n+1} + 1]$
57. $a_n = \frac{1}{3}[2^n - (-1)^n]$
58. $a_n = \frac{1}{18}(n^2 + 17n + 36) \cdot 3^n$
59. $a_n = \frac{2}{27}(-2)^n - \frac{5}{9}n(-2)^n - \frac{2}{27} + \frac{1}{9}n$.
60. $a_n = A \cos\frac{n\pi}{2} + B \sin\frac{n\pi}{2} + \frac{(5n-8)}{25} \cdot 2^n$.