Logic and Propositional Calculus

4.1 INTRODUCTION

Many proofs in mathematics and many algorithms in computer science use logical expressions such as

$$\text{“IF } p \text{ THEN } q\text{”} \qquad \text{or} \qquad \text{“IF } p_1 \text{ AND } p_2\text{, THEN } q_1 \text{ OR } q_2\text{”}$$

It is therefore necessary to know the cases in which these expressions are either TRUE or FALSE: what we refer to as the truth values of such expressions. We discuss these issues in this section.

We also investigate the truth value of quantified statements, which are statements which use the logical quantifiers "for every" and "there exists".

4.2 PROPOSITIONS AND COMPOUND PROPOSITIONS

A *proposition* (or *statement*) is a declarative sentence which is either true or false, but not both. Consider, for example, the following eight sentences:

(i) Paris is in France.
(ii) $1 + 1 = 2$.
(iii) $2 + 2 = 3$.
(iv) London is in Denmark.
(v) $9 < 6$.
(vi) $x = 2$ is solution of $x^2 = 4$.
(vii) Where are you going?
(viii) Do your homework.

All of them are propositions except (vii) and (viii). Moreover, (i), (ii), and (vi) are true, whereas (iii), (iv), and (v) are false.

Compound Propositions

Many propositions are *composite*, that is, composed of *subpropositions* and various connecti cussed subsequently. Such composite propositions are called *compound propositions*. A propos said to be *primitive* if it cannot be broken down into simpler propositions, that is, if it is not com

Example 4.1

(a) "Roses are red and violets are blue" is a compound proposition with subpropositions "Ros are red" and "Violets are blue".

(b) "John is intelligent or studies every night" is a compound proposition with subpropositio "John is intelligent" and "John studies every night".

(c) The above propositions (i) through (vi) are all primitive propositions; they cannot be brol down into simpler propositions.

4.3 BASIC LOGICAL OPERATIONS

This section discusses the three basic logical operations of conjunction, disjunction, and negation correspond, respectively, to the English words "and", "or", and "not"

Conjunction, $p \wedge q$

Any two propositions can be combined by the word "and" to form a compound proposition calle *conjunction* of the original propositions. Symbolically,

$$p \wedge q$$

read "p and q", denotes the conjunction of p and q. Since $p \wedge q$ is a proposition it has a truth value this truth value depends only on the truth values of p and q. Specifically:

Definition 4.1: If p and q are true, then $p \wedge q$ is true; otherwise $p \wedge q$ is false:

The truth value of $p \wedge q$ may be defined equivalently by the table in Fig. 4.1(a). Here, the first line is a short way of saying that if p is true and q is true, then $p \wedge q$ is true. The second line says that if p is true and q is false, then $p \wedge q$ is false. And so on. Observe that there are four lines corresponding to the four possible combinations of T and F for the two subpropositions p and q. Note that $p \wedge q$ is true only when both p and q are true.

and

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(a) "p and q"

or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(b) "p or q"

p	
T	F
F	T

(c) "

Fig. 4.1

The fundamental property of a compound proposition is that its truth value is completely determined by the values of its subpropositions together with the way in which they are connected to form the compound proposit The next section studies some of these connectives.

Example 4.2

Consider the following four statements:

(i) Paris is in France and $2 + 2 = 4$.

(ii) Paris is in France and $2 + 2 = 5$.

(iii) Paris is in England and $2 + 2 = 4$.

(iv) Paris is in England and $2 + 2 = 5$.

Only the first statement is true. Each of the other statements is false, since at least one of its substatements is false.

Disjunction, $p \vee q$

Any two propositions can be combined by the word "or" to form a compound proposition called the *disjunction* of the original propositions. Symbolically,

$$p \vee q$$

read "p or q", denotes the disjunction of p and q. The truth value of $p \vee q$ depends only on the truth values of p and q as follows.

Definition 4.2: If p and q are false, then $p \vee q$ is false; otherwise $p \vee q$ is false.

The truth value of $p \vee q$ may be defined equivalently by the table in Fig. 4.1(b). Observe that $p \vee q$ is false only in the fourth case when both p and q are false.

Example 4.3

Consider the following four statements:

(i) Paris is in France or $2 + 2 = 4$.

(ii) Paris is in France or $2 + 2 = 5$.

(iii) Paris is in England or $2 + 2 = 4$.

(iv) Paris is in England or $2 + 2 = 5$.

Only the last statement (iv) is false. Each of the other statements is true since at least one of its substatements is true.

Remark: The English word "or" is commonly used in two distinct ways. Sometimes it is used in the sense of "p or q or both", i.e., at least one of the two alternatives occurs, as above, and sometimes it is used in the sense of "p or q but not both", i.e., exactly one of the two alternatives occurs. For example, the sentence "He will go to Harvard or to Yale" uses "or" in the latter sense, called the *exclusive disjunction*. Unless otherwise stated, "or" shall be used in the former sense. The discussion points out the precision we gain from our symbolic language: $p \vee q$ is defined by its truth table and *always* means "p and/or q".

Negation, $\neg p$

Given any proposition p, another proposition, called the *negation* of p, can be formed by writing "It is not the case that ..." or "It is false that ..." before p or, if possible, by inserting in p the word "not". Symbolically,

$$\neg p$$

read "not p", denotes the negation of p. The truth value of p depends on the truth value of p as follows.

Definition 4.3: If p is true, then $\neg p$ is false; and if p is false, then $\neg p$ is true.

The truth value of $\neg p$ may be defined equivalently by the table in Fig. 4.1(c). Thus the truth value of the negation of p is always the opposite of the truth value of p.

Example 4.4

Consider the following six statements:

(a_1) Paris is in France.

(a_2) It is not the case that Paris is in France

(a_3) Paris is not in France.

(b_1) $2 + 2 = 5$.

(b_2) It is not the case that $2 + 2 = 5$.

(b_3) $2 + 2 \neq 5$.

Then (a_2) and (a_3) are each the negation of (a_1); and (b_2) and (b_3) are each the negation of (b_1). Since (a_1) is true, (a_2) and (a_3) are false; and since (b_1) is false, (b_2) and (b_3) are true.

Remark: The logical notation for the connectives "and", "or", and "not" is not completely standardized. For example, some texts use:

p & q, $p \cdot q$ or pq	for $p \wedge q$
$p + q$	for $p \vee q$
p', $\bar{p}$ or $\sim p$	for $\neg p$

4.4 PROPOSITIONS AND TRUTH TABLES

Let $P(p, q, ...)$ denote an expression constructed from logical variables $p, q, ...,$ which take on the value TRUE (T) or FALSE (F), and the logical connectives, $\wedge$, $\vee$, and $\neg$ (and others discussed subsequently). Such an expression $P(p, q, ...)$ will be called a *proposition*.

The main property of a proposition $P(p, q, ...)$ is that its truth value depends exclusively upon the truth values of its variables, that is, the truth value of a proposition is known once the truth value of each of its variables is known. A simple concise way to show this relationship is through a *truth table*. We describe a way to obtain such a truth table below.

Consider, for example, the proposition $\neg (p \wedge \neg q)$. Figure 4.2(a) indicates how the truth table of $\neg (p \wedge \neg q)$ is constructed. Observe that the first columns of the table are for the variables $p, q, ...$ and hat there are enough rows in the table to allow for all possible combinations of T and F for these ariables. (For 2 variables, as above, 4 rows are necessary; for 3 variables, 8 rows are necessary; and, in eneral, for n variables, 2^n rows are required.) There is then a column for each "elementary" stage of the onstruction of the proposition, the truth value at each step being determined from the previous stages y the definitions of the connectives $\wedge$, $\vee$, $\neg$. Finally we obtain the truth value of the proposition, which ppears in the last column.

The actual truth table of the proposition $\neg(p \wedge \neg q)$ is shown in Fig. 4.2(*b*). It consists precisely of the columns in Fig. 4.2(*a*) which appear under the variables and under the proposition; the other columns were merely used in the construction of the truth table.

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(*a*)

p	q	$\neg(p \wedge \neg q)$
T	T	T
T	F	F
F	T	T
F	F	T

(*b*)

Fig. 4.2

Remark: In order to avoid an excessive number of parentheses, we sometimes adopt an order of precedence for the logical connectives. Specifically,

$\neg$ has precedence over $\wedge$ which has precedence over $\vee$

For example, $\neg p \wedge q$ means $(\neg p) \wedge q$ and not $\neg(p \wedge q)$.

Alternative Method for Constructing a Truth Table

Another way to construct the truth table for $\neg(p \wedge \neg q)$ follows:

(a) First we construct the truth table shown in Fig. 4.3. That is, first we list all the variables and the combinations of their truth values. Then the proposition is written on the top row to the right of its variables with sufficient space so that there is a column under each variable and each connective in the proposition. Also there is a final row labeled "Step".

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T					
T	F					
F	T					
F	F					
Step						

Fig. 4.3

(b) Next, additional truth values are entered into the truth table in various steps as shown in Fig. 4.4. That is, first the truth values of the variables are entered under the variables in the proposition, and then there is a column of truth values entered under each logical operation. We also indicate the step in which each column of truth values is entered in the table.

The truth table of the proposition then consists of the original columns under the variables and the last step, that is, the last column entered into the table.

(4) Implication: ($\rightarrow$) (or) conditional

- Let p and q be 2 statements then if p then q is a statement called implication or a condition written $p \rightarrow q$.

Truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- The implication $p \rightarrow q$ is considered false, when p is true & q is false. Otherwise it is considered true.

- In the implication $p \rightarrow q$, p is called hypothesis & q is called conclusion

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Note: let p & q be statements then
- The statement,
1) $q \to p$ is called the converse of the implication $p \to q$.

2) ~~The s~~ $\sim p \to \sim q$ is called the inverse of the implication $p \to q$.

3) $\sim q \to \sim p$ is called the contrapositive of the implication $p \to q$.

(4) Biimplication ($\leftrightarrow$) (or) biconditional:

let p & q be 2 statements, then "p if and only if q" written $p \leftrightarrow q$ is called the biimplication (or) biconditional of statements p & q.

- It can ($p \leftrightarrow q$) also be read as
"p is necessary and sufficient for q" (or)
"q is " " " " p" (or)
"q if and only if p" (or)
"q when and only when p"

$p \leftrightarrow q$ is considered as true if both have same truth values and false otherwise

Truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

4.6 LOGICAL EQUIVALENCE

Two propositions $P(p, q, \ldots)$ and $Q(p, q, \ldots)$ are said to be *logically equivalent*, or simply *equivalent* or *equal*, denoted by

$$P(p, q, \ldots) \equiv Q(p, q, \ldots)$$

if they have identical truth tables. Consider, for example, the truth tables of $\neg (p \wedge q)$ and $\neg p \vee \neg q$ appearing in Fig. 4.6. Observe that both truth tables are the same, that is, both propositions are false in the first case and true in the other three cases. Accordingly, we can write

$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$

In other words, the propositions are logically equivalent.

Remark: Consider the statement

"It is not the case that roses are red and violets are blue"

This statement can be written in the form $\neg (p \vee q)$ where:

p is "roses are red" and q is "violets are blue"

However, as noted above, $\neg (p \wedge q) \equiv \neg p \vee \neg q$. Thus the statement

"Roses are not red, or violets are not Blue."

has the same meaning as the given statement.

p	q	$p \wedge q$	$\neg (p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(*a*) $\neg (p \wedge q)$

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(*b*) $\neg p \vee \neg q$

Fig. 4.6

4.7 ALGEBRA OF PROPOSITIONS

Propositions satisfy various laws which are listed in Table 4.1. (In this table, T and F are restricted to the truth values "true" and "false", respectively.) We state this result formally.

Theorem 4.2: Propositions satisfy the laws of Table 4.1.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

(a) $(p \to q)$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(b) $(p \leftrightarrow q)$

Fig. 4.7

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

$\neg p \vee q$

Fig. 4.8

In other words, the conditional statement "If p then q" is logically equivalent to the statement "Not p or q" which only involves the connectives $\vee$ and $\neg$ and thus was already a part of our language. We may regard $p \to q$ as an abbreviation for an oft-recurring statement.

4.9 ARGUMENTS

An *argument* is an assertion that a given set of propositions $P_1, P_2, \ldots, P_n$, called *premises*, yields (has a consequence) another proposition Q, called the *conclusion*. Such an argument is denoted by

$$P_1, P_2, \ldots, P_n \vdash Q$$

The notion of a "logical argument" or "valid argument" is formalized as follows:

Definition 4.4: An argument $P_1, P_2, \ldots, P_n \vdash Q$ is said to be *valid* if Q is true whenever all the premises $P_1, P_2, \ldots, P_n$ are true.

An argument which is not valid is called a *fallacy*.

Example 4.5

(a) The following argument is valid:

$$p, p \to q \vdash q \quad \textbf{\textit{(Law of Detachment)}}$$

The proof of this rule follows from the truth table in Fig. 4.9. Specifically, p and $p \to q$ are true simultaneously only in Case (row) 1, and in this case q is true.

(b) The following argument is a fallacy:

$$p \to q, q \vdash p$$

For $p \to q$ and q are both true in Case (row) 3 in the truth table in Fig. 4.9, but in this case p is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Fig. 4.9

Now the propositions $P_1, P_2, ..., P_n$ are true simultaneously if and only if the proposition $P_1 \wedge ...$ P_n is true. Thus the argument $P_1, P_2, ..., P_n \vdash Q$ is valid if and only if Q is true whenever $P_1 \wedge ...$ P_n is true or, equivalently, if the proposition $(P_1 \wedge P_2 \wedge ... \wedge P_n) \rightarrow Q$ is a tautology. We state result formally.

Theorem 4.3: The argument $P_1, P_2, ..., P_n \vdash Q$ is valid if and only if the proposition $(P_1 \wedge ... P_n) \rightarrow Q$ is a tautology.

We apply this theorem in the next example.

Example 4.6

A fundamental principle of logical reasoning states:

"If p implies q and q implies r, then p implies r."

That is, the following argument is valid:

$$p \rightarrow q,\ q \rightarrow r \vdash p \rightarrow r \quad \textit{(Law of Syllogism)}$$

This fact is verified by the truth table in Fig. 4.10 which shows that the following proposition is a tautology:

$$[(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$$

Equivalently, the argument is valid since the premises $p \rightarrow q$ and $q \rightarrow r$ are true simultaneously only in Cases (rows) 1, 5, 7 and 8, and in these cases the conclusion $p \rightarrow r$ is also true. (Observe that the truth table required $2^3 = 8$ lines since there are three variables p, q and r.)

p	q	r	$[(p$	$\rightarrow$	$q)$	$\wedge$	$(q$	$\rightarrow$	$r)]$	$\rightarrow$	$(p$	$\rightarrow$	$r)$
T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	F	F	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F	T	F	T	T	F	F
F	T	T	F	T	T	T	T	T	T	T	F	T	T
F	T	F	F	T	T	F	T	F	F	T	F	T	F
F	F	T	F	T	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	T	F	T	F	T	F	T	F
Step			1	2	1	3	1	2	1	4	1	2	1

Fig. 4.10

We now apply the above theory to arguments involving specific statements. We emphasize that the validity of an argument does not depend upon the truth values nor the content of the statements appearing in the argument, but upon the particular form of the argument. This is illustrated in the following example.

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T			T
T	F		T			F
F	T		F			T
F	F		F			F
Step			1			1

(*a*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T		F	T
T	F		T		T	F
F	T		F		F	T
F	F		F		T	F
Step			1		2	1

(*b*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T		T	F	F	T
T	F		T	T	T	F
F	T		F	F	F	T
F	F		F	F	T	F
Step			1	3	2	1

(*c*)

p	q	$\neg$	$(p$	$\wedge$	$\neg$	$q)$
T	T	T	T	F	F	T
F	T	F	T	T	T	F
F	F	T	F	F	F	T
F	F	T	F	F	T	F
Step		4	1	3	2	1

(*d*)

Fig. 4.4

.5 TAUTOLOGIES AND CONTRADICTIONS

›me propositions $P(p, q, \ldots)$ contain only T in the last column of their truth tables or, in other word ey are true for any truth values of their variables. Such propositions are called *tautologies*. Anal ously, a proposition $P(p, q, \ldots)$ is called a *contradiction* if it contains only F in the last column of i uth table or, in other words, if it is false for any ruth values of its variables. For example, the propo- ition "p or not p", that is, $p \vee \neg p$, is a tautology, and the proposition "p and not p", that is, $p \wedge \neg p$, is a contradiction. This is verified by looking at their truth tables in Fig. 4.5. (The truth tables have only two rows since each proposition has only the one variable p.)

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(*a*) $p \vee \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(*b*) $p \wedge \neg p$

Fig. 4.5

Note that the negation of a tautology is a contradiction since it is always false, and the negation of contradiction is a tautology since it is always true.

Now let $P(p, q, \ldots)$ be a tautology, and let $P_1(p, q, \ldots)$, $P_2(p, q, \ldots)$, ... be any propositions. Since $P($ $q, \ldots)$ does not depend upon the particular truth values of its variables $p, q, \ldots$, we can substitute P_1 for P_2 for q, ... in the tautology $P(p, q, \ldots)$ and still have a tautology. In other words:

Theorem 4.1 (Principle of Substitution): If $P(p, q, \ldots)$ is a tautology, then $P(P_1, P_2, \ldots)$ is a taut ogy for any propositions $P_1, P_2, \ldots$.

Table 4.1 *Laws of the algebra of propositions*

Idempotent laws	(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws	(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (p \wedge r)$
Commutative laws	(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws	(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws	(5a) $p \vee \mathrm{T} \equiv p$	(5b) $p \wedge \mathrm{T} \equiv p$
	(6a) $p \vee \mathrm{T} \equiv \mathrm{T}$	(6b) $p \wedge \mathrm{F} \equiv \mathrm{F}$
Complement laws	(7a) $p \vee \neg p \equiv \mathrm{T}$	(7b) $p \wedge \neg p \equiv \mathrm{F}$
	(8a) $\neg \mathrm{T} \equiv \mathrm{F}$	(8b) $\neg \mathrm{F} \equiv \mathrm{T}$
Involution law	(9) $\neg \neg p \equiv p$	
DeMorgan's laws	(10a) $\neg (p \vee q) \equiv \neg p \wedge \neg q$	(10b) $\neg (p \wedge q) \equiv \neg p \vee \neg q$

4.8 CONDITIONAL AND BICONDITIONAL STATEMENTS

Many statements, particularly in mathematics, are of the form "if p then q". Such statements are cal *conditional* statements and are denoted by

$$p \to q$$

The conditional $p \to q$ is frequently read "p implies q" or "p only if q".

Another common statement is of the form "p if and only if q". Such statements are called *bicon- tional* statements and are denoted by

$$p \leftrightarrow q$$

The truth values of $p \to q$ and $p \leftrightarrow q$ are defined by the tables in Fig. 4.7. Observe that:

(a) The conditional $p \to q$ is false only when the first part p is true and the second part q is fal Accordingly, when p is false, the conditional $p \to q$ is true regardless of the truth value of q

(b) The biconditional $p \leftrightarrow q$ is true whenever p and q have the same truth values and false otherwi

The truth table of the proposition $\neg p \vee q$ appears in Fig. 4.8. Observe that the truth tables $\neg p \vee q$ and $p \to q$ are identical, that is, they are both false only in the second case. Accordingl $p \to q$ is logically equivalent to $\neg p \vee q$; that is,

$$p \to q \equiv \neg p \vee q$$