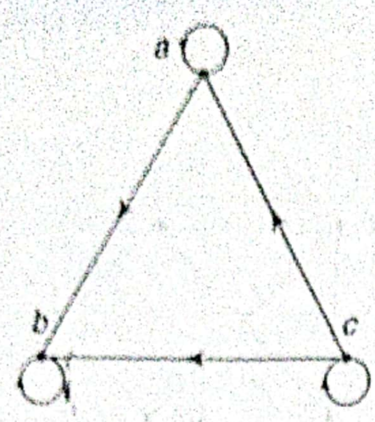


Fig. 2.20

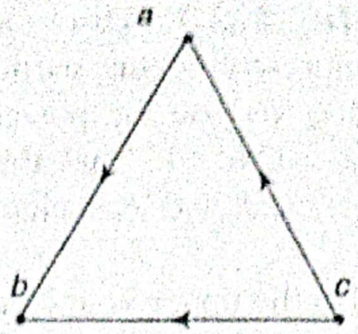


Fig. 2.21

Example 2.19 Draw the digraph representing the partial ordering $\{(a, b)\} \mid a$ divides $b\}$ on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Reduce it to the Hasse diagram representing the given partial ordering.

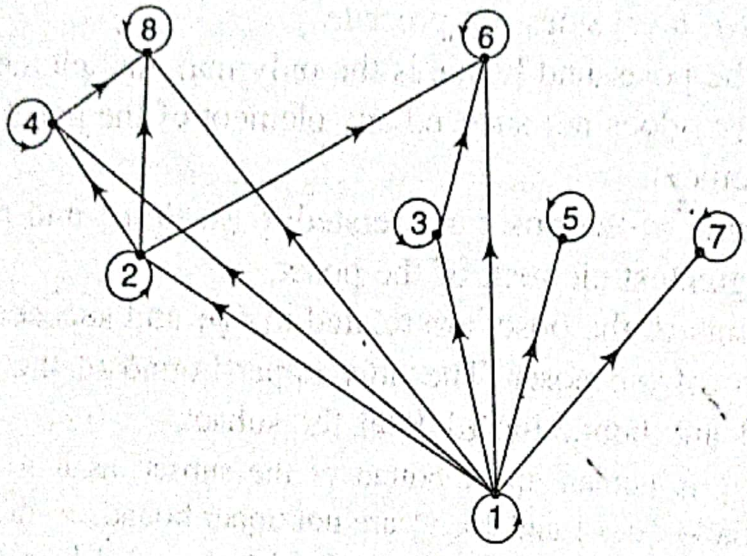
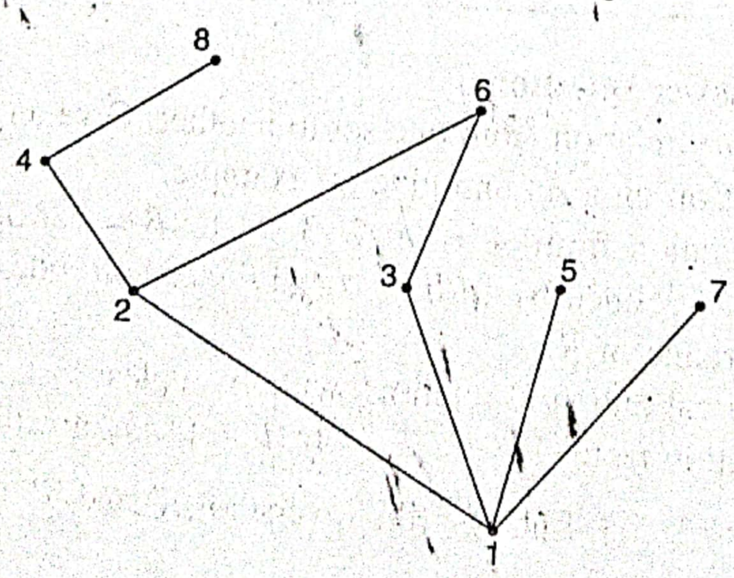


Fig. 2.22

Deleting all the loops at the vertices, deleting all the edges occurring due to transitivity, arranging all the edges to point upward and deleting all arrows, we get the corresponding Hasse diagram as given in Fig. 2.23.



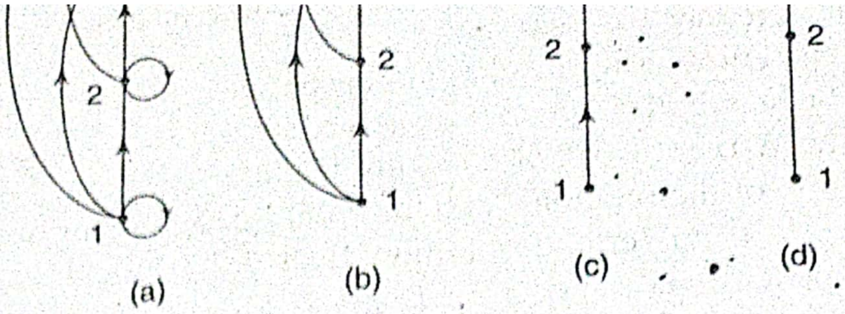


Fig. 2.15

TERMINOLOGY RELATED TO POSETS

We have already defined *poset* as a set S together with a partial order relation R. In a poset the notation $a \leq b$ (or equivalently $a \prec b$) denotes that $(a, b) \in R$. $a \leq b$ is read as "a precedes b" or "b succeeds a".

$a < b$

Definitions

When $\{P, \leq\}$ is a poset, an element $a \in P$ is called a *maximal member* of P, if there is no element $b \in P$ such that $a < b$ (viz., a strictly precedes b).

$b < a$

Similarly, an element $a \in P$ is called a *minimal member* of P, if there is no element $b \in P$ such that $b < a$.

If there exists an element $a \in P$ such that $b \leq a$ for all $b \in P$, then a is called the *greatest member* of the poset $\{P, \leq\}$.

Similarly if there exists an element $a \in P$ such that $a \leq b$ for all $b \in P$, then a is called the *least member* of the poset $\{P, \leq\}$.

Note

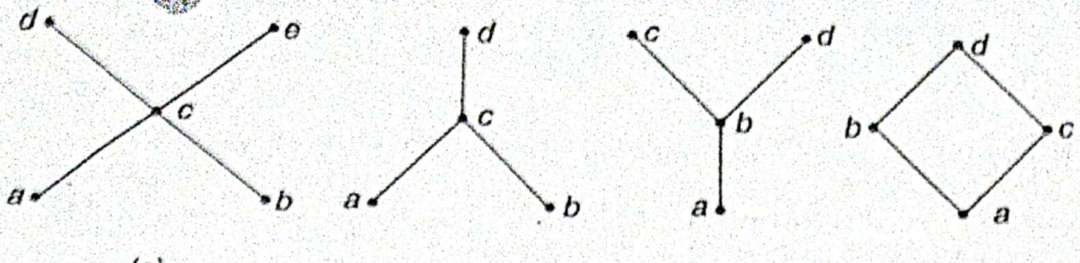
1. The maximal, minimal, the greatest and least members of a poset can be easily identified using the Hasse diagram of the poset. They are the top and bottom elements in the diagram.
2. A poset can have more than one maximal member and more than one minimal member, whereas the greatest and least members, when they exist, are unique.

For example, let us consider the Hasse diagrams of four posets given in Fig. 2.16.

For the poset with Hasse diagram 2.16(a), a and b are minimal elements and d and e are maximal elements, but the poset has neither the greatest nor the least element.

For the poset with Hasse diagram (b), a and b are minimal elements and d is the greatest element (also the only maximal element). There is no least element.

For the poset with Hasse diagram (c), a is the least element (also the only minimal element) and c & d are maximal element. There is no



(a) (b) (c) (d)

Fig. 2.16

For the poset with Hasse diagram (d), a is the least element and d is the greatest element.

Definitions

When A is a subset of a poset $\{P, \leq\}$ and if u is an element of P such that $a \leq u$ for all elements $a \in A$, then u is called an *upper bound* of A. Similarly if l is an element of P such that $l \leq a$ for all elements $a \in A$, then l is called a *lower bound* of A.

Note The upper and lower bounds of a subset of a poset are not necessarily unique.

The element x is called the *least upper bound* (LUB) or *supremum* of the subset A of a poset $\{P, \leq\}$, if x is an upper bound that is less than every other upper bound of A.

Similarly the element y is called the *greatest lower bound* (GLB) or *infimum* of the subset A of a poset $\{P, \leq\}$, if y is a lower bound that is greater than every other lower bound of A.

Note The LUB and GLB of a subset of a poset, if they exist, are unique.

For example, let us consider the poset with the Hasse diagram given in Fig. 2.17.

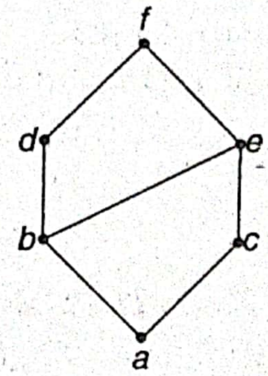


Fig. 2.17

The upper bounds of the subset $\{a, b, c\}$ are e and f. [*Note:* d is not an upper bound, since c is not related to d] and LUB of $\{a, b, c\}$ is e.

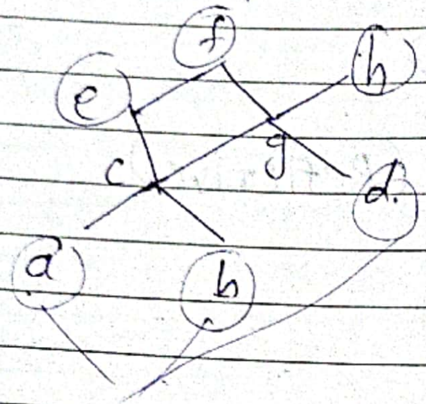
The lower bounds of the subset $\{d, e\}$ are a and b and GLB of $\{d, e\}$ is b.

Note c is not a lower bound, since c is not related to d.

⇒ Maximal Element:-

If in a POSET, an element is not related to any others element is call. Maximal element.

eg: Relationship → bottom to top



← Maximal element
↓
It does not relate with any one or relate with it

Minimal element
↓
No element is relate with it / these elements.

⇒ Minimal Element:-

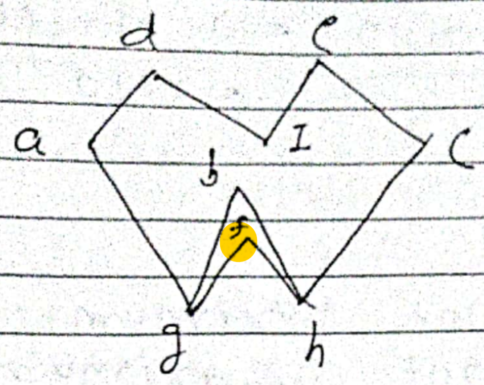
If in a POSET, no element is related to an element.

eg: as above

aRc but c$\not R$a c
bRc but c$\not R$b
dRg but g$\not R$b d. b

Maximal & Minimal both

eg.



Maximal:
f, b, d, e.

Minimal:
g, h, I

⇒ Maximum element:

If it is Maximal and every elements are related to it.

eg. I'm the tallest girl in the class.
No other student in the class taller than me

(1) optimum (Maximum)

tallest → (No one is like her, & Every one is less than her)

(2) Optimal & Unique ⇒ (Maximum)

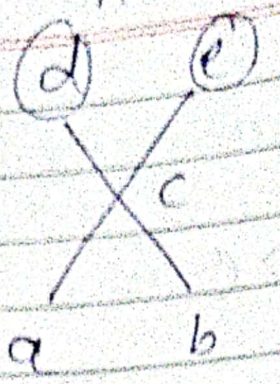
taller → (No one is bigger than her, but it is like her)

More than one solution

Maximal / optimal

Maximal alone ⟹ Maximum

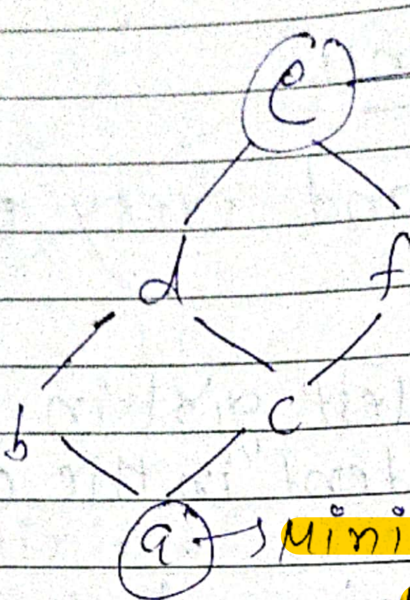
eg.



⇒ No Maximum
⇒ No Minimum

$e \not R d$ ⇒ not Maximum d.
$d \not R e$ ⇒ not maximum e

eg.



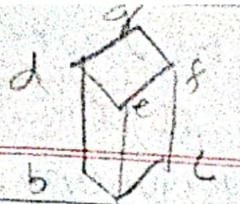
e → Maximal & Maximum.
a → Minimal & Minimum

eg. • a → Maximal & Maxim
Minimal & Minimu

eg. •a •b •c ⇒ No one relat
to each other

⇒ **Minimum elements**

If it is Minimal and it is related to every element in POSET.

(3) 

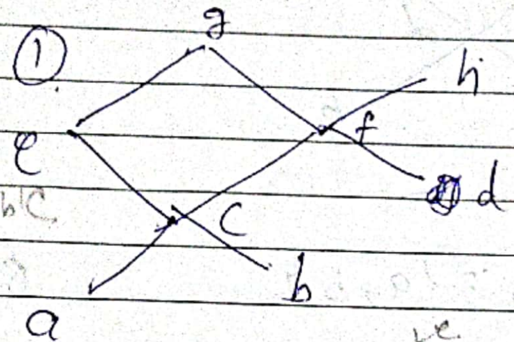
$B = \{d, g\}$
$LB = \{d, a, b, e\}$
$UB = \{g\}$

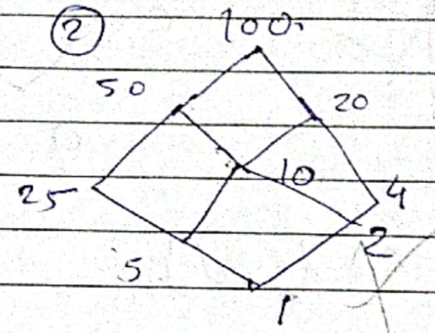
$B = \{e, f\}$
$LB =$
$UB = \{f, g\}$

⇒ Upper Bound & Lower Bound:

1) <u>Upper bound:</u> (subset ke sare element jisse relate kre)

Let B be a subset of a set A. An element $x \in A$ is in upper bound of B if $(y, x) \in$ Poset, $\forall y \in B$.

eg. (1) 

(2) 

directionally not possible

(1) $B = \{e, c\}$ — e & c relate, c relate, e → poset (self loop)

c R h
e R h
c ∉ f

$\rightarrow U.B = \{g, e\}$, $LB = \{a, b, c\}$ → e R c, lower me relate

(2) $B = \{c, f, d\}$

$U.B = \{g, h, f\}$, $LB = \phi$

(1) $B = \{5, 10\}$
$U.B. = \{100, 50, 20, 10\}$
$LB = \{5, 2, 1, 10\}$

(2) $B = \{5, 10, 2, 4\}$
$LB = \{1\}$ 5 R 2, 2 R 5, 4 ∉
$U.B. = \{100, 20\}$

2) <u>lower bound:</u> (jitne element ho jo element ke relation ko relate kre)

Let B be a subset of a set A. An element $x \in A$ is in lower bound of B if $(x, y) \in$ Poset, $\forall y \in B$

eg. above eg (1),

$L.B. = \{a, b, c\} \longrightarrow B = \{e, c\}$
$L.B. = \phi \implies B = \{c, f, d\}$

eg. (2) above eg (2)

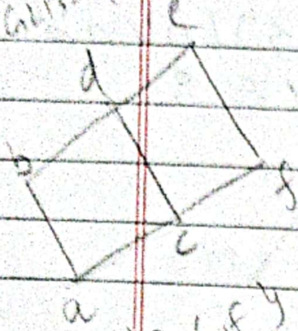
$L.B = \{1, 5\} \implies B = \{5, 10\}$
$L.B. = \{1\} \longrightarrow B = \{5, 2, 1, 4\}$

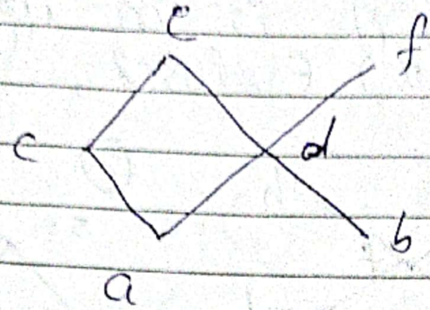
→ Greatest (Max) element in lower bound

⇒ Greatest Lower Bound (GLB) or Infimum (Meet) (∧) / disjunction

⇒ Least Upper Bound (LUB) (05) Supremum (Join) (∨) / conjunction

→ least (Minimum) element in upper bound.



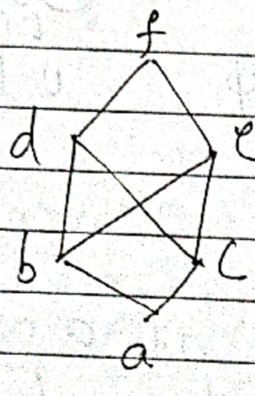


B = {c, d}
U.B = {e}
LUB = e
LB = {a}
GLB = a

B = {a, b}
UB = {e, f, d} → least value = d
LUB = {d}
LB = ϕ
GLB = ϕ

B = {e, f}
UB = {ϕ}
LUB = ϕ
LB = {a, d, b}
GLB = d

eg.



bRc, dRe so we can't find greatest value between b & c

B = {b, c}
LB = a
GLB = a
UB = {d, e, f}
LUB = ϕ
no internal relation

B = {d, e}
LB = {c, b, a} b & c can't decide
GLB = ϕ → only 1 milega
UB = f
LUB = f

Example 2.20 Draw the Hasse diagram representing the partial ordering $\{(A, B) | (A \subseteq B)\}$ on the power set $P(S)$, where $S = \{a, b, c\}$. Find the maximal, minimal, greatest and least elements of the poset.

Find also the upper bounds and LUB of the subset $(\{a\}, \{b\}, \{c\})$ and the lower bounds and GLB of the subset $(\{a, b\}, \{a, c\}, \{b, c\})$.

Here $P(S) = (\{\phi\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.

By using the usual procedure (as in the previous example), the Hasse diagram is shown, as shown in Fig. 2.24.

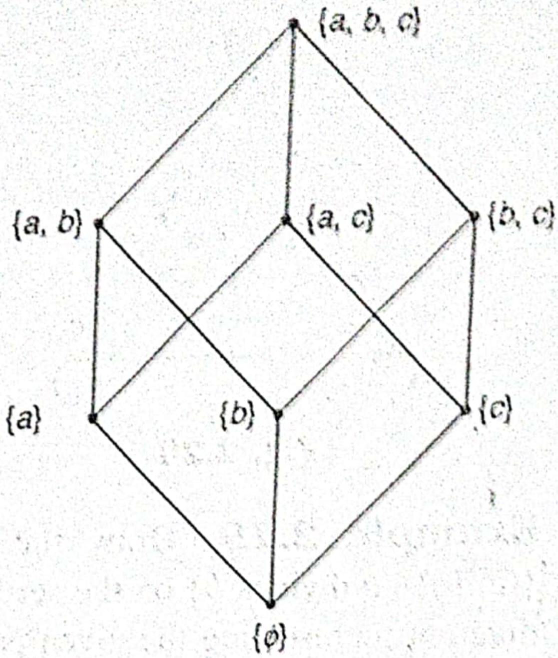


Fig. 2.24

The element $\{a, b, c\}$ does not precede any element of the poset and hence is the only maximal element of the poset.

The element $\{\phi\}$ does not succeed any element of the poset and hence is the only minimal element.

All the elements of the poset are related to $\{a, b, c\}$ and precede it. Hence $\{a, b, c\}$ is the greatest element of the poset.

All the elements of the poset are related to $\{\phi\}$ and succeed it. Hence $\{\phi\}$ is the least element of the poset. The only upper bound of the subset $(\{a\}, \{b\}, \{c\})$ is $\{a, b, c\}$ and hence the LUB of the subset.

Note $\{a, b\}$ is not an upper bound of the subset, as it is not related to $\{c\}$. Similarly $\{a, c\}$ and $\{b, c\}$ are not upper bounds of the given subset.

The only lower bound of the subset $(\{a, b\}, \{a, c\}, \{b, c\})$ is $\{\phi\}$ and hence GLB of the given subset.

Note $\{a\}, \{b\}, \{c\}$ are not lower bounds of the given subset.