4.13 NORMAL FORMS

In logic theory, it is very often needed to determine whether a given statement is a tautology or a contradiction. Let $A(p_1, p_2, \dots p_n)$ be a statement formula where $p_1, p_2, \dots p_n$ are the atomic variables. If we consider all possible assignments of truth-values of $p_1, p_2, \dots p_n$ and obtain resulting truth-values of the formula A, then we get truth table for A. Such a truth table has 2^n rows. If A has true value for all possible assignments then A is said to be tautology. If A has false value for all possible assignments, then

A is said to be contradiction. If A has the truth value, T for at least one combination of truth values of p_1, p_2, ... p_n then A is said to be *satisfiable*.

The problem of finding whether a given statement is tautology or contradiction or satisfiable in a finite number of steps, is called as decision problem. For decision problems, the construction of truth tables may not be practical always. We therefore consider alternate procedure known as reduction to normal forms. The two such forms are:

- Disjunctive Normal Form (DNF)
- Conjunctive Normal Form (CNF)

4.13.1 Disjunctive Normal Form (dnf)

A conjunction of statement variables and (or) their negations is called as a fundamental conjunction (It is also called as a min term). For example: $p, \sim p, \sim p \wedge q, p \wedge q, p \wedge \sim p \wedge q$ are fundamental conjunctions.

We know that $p \wedge \sim p$ is always false. Hence if a fundamental conjunction contains at least one pair of factors in which one is the negation of the other, it will be false. A statement form, which consists of a *disjunction of fundamental conjunctions*, is called a *disjunctive normal* form (abbreviated as *dnf*).

Few examples of *dnf* are as follows:

(i) $(p \wedge q \wedge r) \vee (p \wedge r) \vee (q \wedge r)$

(ii) $(p \wedge \sim q) \vee (p \wedge r)$

(iii) $(p \wedge q \wedge r) \vee \sim r$

(iv) $(p \wedge q) \vee \sim q$

(v) $(\sim p \wedge q) \vee (p \wedge q) \vee q$

Let us see few examples to reduce the given statement to *dnf*, by using logical equivalence and without using truth table.

Examples 4.16

(i) Obtain the dnf of the form $(p \rightarrow q) \wedge (\sim p \wedge q)$.

Let us rewrite the given proposition as:

$p \rightarrow q \equiv \sim p \vee q$ (Elimination of conditional)

Hence $(p \rightarrow q) \wedge (\sim p \wedge q)$

can be written as,

$$\equiv (\sim p \vee q) \wedge (\sim p \wedge q)$$

using the distributive law

$$\equiv (\sim p \wedge \sim p \wedge q) \vee (q \wedge \sim p \wedge q)$$

using idempotence law and commutative law

$$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q)$$

(ii) Obtain the dnf of the form $\sim (p \rightarrow (q \wedge r))$

$$\begin{aligned}
&\sim (p \rightarrow (q \wedge r)) \\
&\equiv \sim (\sim p \vee (q \wedge r)) \\
&\equiv \sim (\sim p \vee (q \wedge r)) \\
&\equiv \sim (\sim p) \wedge \sim (q \wedge r) \quad \text{(De Morgan's Laws)} \\
&\equiv p \wedge (\sim q \vee \sim r) \quad \text{(Idempotent laws and De Morgan's laws)} \\
&\equiv (p \wedge \sim q) \vee (p \wedge \sim r)
\end{aligned}$$

4.13.2 Conjunctive Normal Form (cnf)

A disjunction of statement variables and (or) their negations is called a *fundamental disjunction* or *maxterms*. For example: p, $\sim p$, $\sim p \vee q$, $p \vee q$, $p \vee \sim p \vee q$ are fundamental disjunctions. We know that, $p \vee \sim p$ is always true. Hence if a fundamental disjunctions contains at least one pair of factors in which one is the negation of the other, it will true ($p \vee \sim p \vee r$ is logically equivalent to a tautology).

A statement form, which consists of a *conjunction of fundamental disjunctions*, is called a *conjunctive normal form* (abbreviated as *cnf*).

Note that a *cnf* is a tautology if and only if every fundamental disjunction contained in it is a tautology.

Few examples are as follows:

(i) $p \wedge r$

(ii) $\sim p \wedge (p \vee r)$

(iii) $(p \vee q \vee r) \wedge (\sim p \vee r)$

Let us see few examples to learn to obtain *cnf* of a given statement without using truth tables.

Example 4.17

(i) Obtain the *cnf* of the form $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

$$\begin{aligned}
&(p \wedge q) \vee (\sim p \wedge q \wedge r) \\
&\equiv (p \vee (\sim p \wedge q \wedge r)) \wedge (q \vee (\sim p \wedge q \wedge r)) \quad \text{(Distributive law)} \\
&\equiv ((p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r)) \, L \, ((q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r)) \\
&\equiv (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge q \wedge (q \vee r))
\end{aligned}$$

(ii) Obtain the *cnf* of the form $(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$

$$\begin{aligned}
&(\sim p \rightarrow r) \wedge (p \leftrightarrow q) \\
&\equiv (\sim p \rightarrow r) \wedge ((p \rightarrow q) \wedge (q \rightarrow p)) \\
&\equiv (\sim(\sim p) \vee r) \wedge ((\sim p \vee q) \wedge (\sim q \vee p)) \quad (\text{as } p \rightarrow q \equiv \sim p \vee q) \\
&\equiv (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)
\end{aligned}$$

Truth Table Method to find dnf

Let P be a statement form containing n variables $p_1, p_2, \ldots p_n$. Its *dnf* form the truth table obtained follows. For each row in which P assumes value T, form the conjuction $p_1 \wedge p_2 \wedge \ldots \wedge \sim p_j \wedge$ where p_j is put, if there is T in the j-th position in the row and $\sim p_j$ if there is F. Such a term is ca[lled] *minterm*. The disjunction of the minterm is the *dnf* of the given form.

Example 4.18

Find the conjunctive normal form and disjunctive normal form for:

(i) $p \leftrightarrow (\bar{p} \vee \bar{q})$

(ii) $(p \vee \bar{q}) \rightarrow q$

(i) $p \leftrightarrow (p \vee q) \equiv (\bar{p} \vee (\bar{p} \vee \bar{q})) \wedge ((\bar{p} \vee \bar{q}) \vee p)$

$\equiv (\bar{p} \vee \bar{p} \vee \bar{q}) \wedge ((\bar{\bar{p}} \wedge \bar{\bar{q}}) \vee p)$

$\equiv (\bar{p} \vee \bar{q}) \wedge (p \vee p) \wedge (q \vee p)$

$\equiv (\bar{p} \vee \bar{q}) \wedge p \wedge (q \vee p)$ is the required *cnf* form

$\equiv ((\bar{p} \wedge p) \vee (\bar{q} \wedge p)) \wedge (q \vee p)$

$\equiv (F \vee (\bar{q} \wedge p)) \wedge (q \vee p)$

$\equiv (\bar{q} \wedge p) \wedge (q \vee p)$

$\equiv (\bar{q} \wedge p \wedge q) \vee (\bar{q} \wedge p \wedge p)$

$\equiv (F \wedge p) \vee (\bar{q} \wedge p)$

$\equiv (F \vee (\bar{q} \wedge p)$

$\equiv (\bar{q} \wedge p)$ is the required *dnf* form

(ii) $(p \vee \bar{q}) \rightarrow q \equiv \overline{(p \vee \bar{q})} \vee q$

$\equiv (\bar{p} \wedge \bar{\bar{q}}) \vee q$

$\equiv (\bar{p} \wedge q) \vee q$ is the required *dnf* form

$(\bar{p} \wedge q) \vee q \equiv (\bar{p} \vee q) \wedge (q \vee q)$

$\equiv (\sim p \wedge q) \wedge q$ is the required *cnf* form

Example 4.19

Find the *dnf* of $(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$

The truth table for the same is as:

Table 4.2

p	q	r	$\sim p$	$(\sim p \rightarrow r)$	$(p \leftrightarrow q)$	$(\sim p \rightarrow r) \wedge (p \leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

ere the rows of p, q, r in which has truth valueT appears in the last column.
ence the required *dnf* is $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r)$

xample 4.20

otain the conjunctive normal form and disjunctive normal form of the following

i) $p \wedge (p \rightarrow q)$

ii) $(p \vee q) \leftrightarrow (p \wedge q)$

i) $p \wedge (p \rightarrow q) \equiv p \wedge (\sim p \vee q)$

$$p \wedge (\sim p \vee q) \equiv (p \wedge \sim p) \vee (p \wedge q)$$
$$\equiv F \vee (p \wedge q)$$
$$\equiv (p \wedge q)$$

ii) $\sim(p \vee q) \leftrightarrow (p \wedge q)$

as $p \leftrightarrow q$ is equivalent to $p \rightarrow q \wedge q \rightarrow p$,

$$\equiv (\sim\sim (p \vee q) \vee (p \wedge q)) \wedge ((\sim p \wedge q) \vee \sim (p \vee q))$$
$$\equiv ((p \vee q) \vee (p \wedge q)) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q))$$
$$\equiv (p \vee q) \wedge ((\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q))$$
$$\equiv (p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q)$$
$$\equiv (p \vee q) \wedge (\sim p \vee \sim q)$$

and $(p \vee q) \wedge (\sim p \vee \sim q)$

$$\equiv ((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q)$$

$$\equiv (p \wedge \sim p) \vee (q \wedge \sim p) \vee ((p \wedge \sim q) \vee (q \wedge \sim q))$$

$$\equiv F \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee F$$

$$\equiv (q \wedge \sim p) \vee (p \wedge \sim q)$$