Groups :

Let G is non empty set, with binary operations $*$ on elements of G, then $(G, *)$ is said to be a group if,

(1) "$*$" is closed.

(2) "$*$" is ~~acco~~ associative.

(3) Identity element exist in G. i.e there exist $e \in G$, such that $e * a = a$, $a * e = a$ $\forall a \in G$.

(4) Inverse of each element exist in G i.e for a, $\exists a^{-1} \in G$ such that $a * a^{-1} = e$, $a^{-1} * a = e$.

Q Is the set of natural numbers and group under multiplication $(N, *)$

(1) $a, b \in N$ $\because a \times b = \underline{c \in N}$ $\because$ hence closed.

(2) $ab(c) = (a)(bc)$ $\rightarrow$ Hence associative.

(3) $1 \in N$ $\because 1 * a = a$, $a * 1 = a$. $\because$ Hence Indentity element exist.

(4) For $2 \in N$, $2^{-1} = \frac{1}{2}$ for multiplication

$\because 2 \times \frac{1}{2} = 1$ but $\frac{1}{2} \notin N$. Hence Inverse of each element doesn't exist.

$\rightarrow$ $\because$ It is not a Group.

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ex:- 2 $(R, *)$

(1) $a, b \in R$ ∵ $a \times b \in R$ ∵ hence closed.

(2) $a(bc) = (ab)c$ → Hence associative.

(3) $1 \in R$ ∵ $1 * a = a$, $a * 1 = a$ ∵ Hence Indentity exist

(4) Inverse of each element exist. Hence It is group.

If binary operation is commutive for group $(G, *)$ then $(G, *)$ is said to be abelian.

Finize sets are called self-inversible sets.

Reflexive $\forall a \in A, (a, a) \in R$

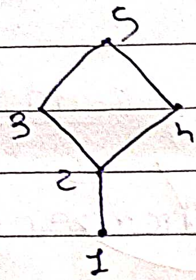
Irreflexive $\forall a \in A, (a, a) \notin R$

ex:- $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2) (3,2) (2,1)\}$
↳ neither reflexive nor irreflexive.

⇒ Sub lattice:
If $(L, \leq)$ lattice, then $S \subseteq L$ is said to be sub lattice of $(L, \leq)$ is $(S, \leq)$ is itself a lattice.

eg:



(L, ≤)

$S_1 = \{2, 3, 4\}$ ✗

$S_2 = \{1, 2, 3\}$ ✓

$S_3 = \{1, 2, 4, 3\}$ ✗

$S_4 = \{3, 4, 5\}$ ✗

$S_5 = \{2, 3, 5\}$ ✓

$S_6 = \{1, 4\}$ ✓

✓ - Sub lattice , ✗ - not s...

⇒ If Glb and lub exist then sublattice.

* **Direct product of lattice :**

⇒ $(L_1, \le_1)$, $(L_2, \le_2)$

$L_1 \times L_2 = \{ (x_i, y_j) \mid x_i \in L_1, y_j \in L_2 \}$

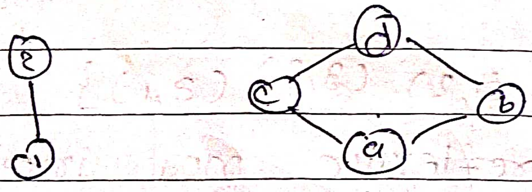
→ Glb and lub will be defined as,

$glb\{(x_1, y_1), (x_2, y_2)\} = (glb(x_1, x_2), glb(y_1, y_2))$

OR $(x_1, y_1) \wedge (x_2, y_2) = ((x_1 \wedge_1 x_2), (y_1 \wedge_2 y_2))$

$lub\ (x_1, y_1) \vee (x_2, y_2) = ((x_1 \vee_1 x_2), (y_1 \vee_2 y_2))$

ex:- find the direct product of lattices given.



L_1 L_2

$L_1 \times L_2 = \{(1,a), (1,b), (1,c), (1,d), (2,a), (2,b), (2,c), (2,d)\}$.

$\Rightarrow (1,a) \wedge (1,d) = ((1 \wedge 1), (a \wedge d)) = (1,a)$

$\Rightarrow (1,a) \vee (1,d) = ((1 \vee 1), (a \vee d)) = (1,d)$

$\Rightarrow (2,c) \wedge (1,b) = (1,a)$

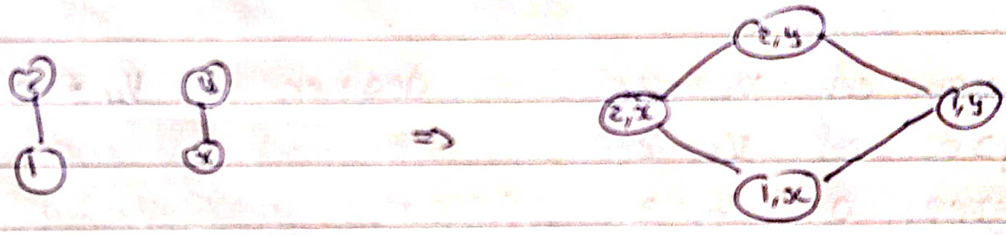
$(2,c) \vee (1,b) = (2,d)$

$\Rightarrow (1,a) \wedge (2,b) = (1,a)$

$(1,a) \vee (2,b) = (2,b)$

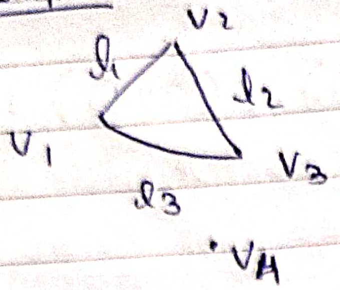
$\Rightarrow$

Ex:-



Graph theory

* **Graphs:**

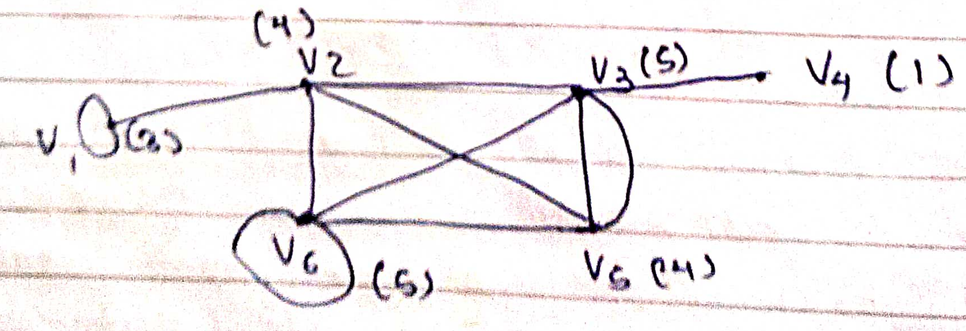


$G, (V, E)$
$V = \{ v_1, v_2, v_3, v_4 \}$
$E = \{ (v, v_2), (v_2, v_3), (v_3, v_1) \}$
$= \{ l_1, l_2, l_3 \}$

=> $G(V, E)$:-
A graph G is consist of a non-empty set, V: set of vertices/nodes/poi[nts] and set of edge is defined on V as
$E = \{ (v_i, v_j) / v_i, v_j \in V \}$

Degree of $v_1 = 2$ degree of $v_4 = 0$
Degree of $v_2 = 2$
Degree of $v_3 = 2$

-> A vertex v is said to pendant vertex if the degree of that vertex is 1.
-> A vertex is said to be isolated vertex if its degree is zero (0).
-> An edge is said to be self loop if both vertices are same (v_i, v_i)



-> What is max degree in graph?
-> What is min degree in graph?
-> Find pendant vertices and isolated one

$$2e = \sum_{odd} \text{degree } v_i + \sum_{even} \text{degree } v_i$$

No. of odd vertices are always even.

NOTE :-

Each edge will contribute 2 degree in graph total degree of G with 2 edges = 2 × e.

odd vertex : vertex with odd degree
even vertex : vertex with even degree

* Types of Graph :-

1) Simple graph :

A graph G is said to be simple graph if there is no self such loop and parallel edges in the graph.

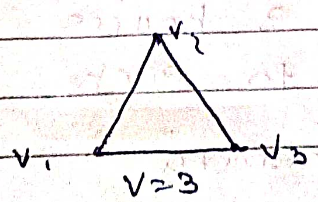
* Parellel edge :-

Edges are called parelled edges if they have same end vertices

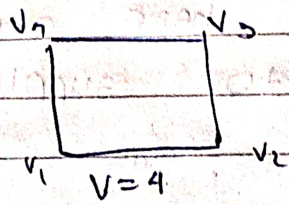
NOTE :- In a simple graph max degree possible is $n-1$.

2) Regular graph :

A simple graph G is said to be regular if all vertices have same degree.



Degree of $v_1, v_2, v_3 = 3$



Degree of $v_1, v_2, v_3, v_4 = 4$

Total degree of regular graph (n, r) → degree $= n \cdot r$

3) <u>Complete graph</u> :-

A simple graph, having edges between each pair of vertices.

∴ degree of vertex $= n(n-1)$

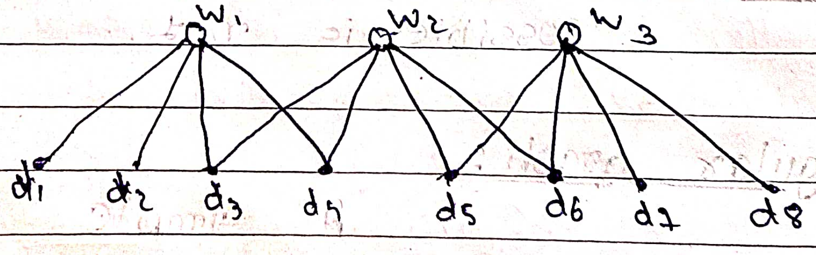
∴ no. of edges in a complete graph $= \frac{n(n-1)}{2}$

∴ Total degree of $G = n(n-1)$.

4) <u>Bipertite graph</u> :-

A graph G is said to be bipertite graph if set of vertices V can be partitioned into two partitions V_1 and V_2 such that $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 = V$ such that no vertex $V_i \in V_1$ is not adjecent to any vertex in V_1. i.e each edge in G will have one end point in V_1 and another end point in V_2

<u>ex</u>:-



* <u>Utility problem</u> :-

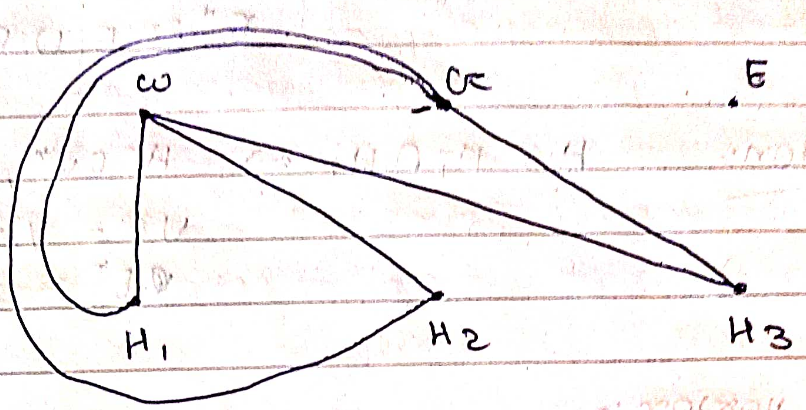
There are 3 houces and three utility to supply to each house

Water Gas Electricity

H_1 H_2 H_3

Can you supply this utilities without any crossover.

Ans:-



There is no such way to connect E to H, H_2, H_3 without crossover.

* Complete bipertite :-

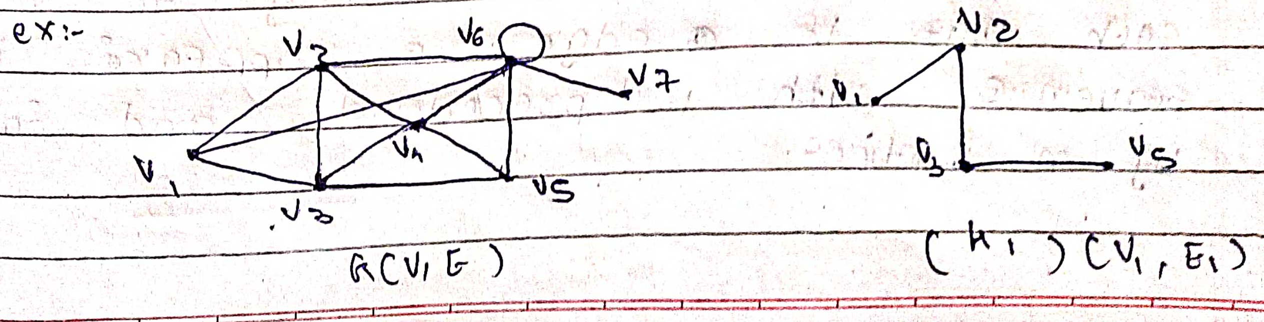
If G is a Bipertite graph it is said to be complete bipertite graph if each vertex in first partition is adjucent to all vertices in second portition.

* OPERATIONS ON GRAPH :-

→ Subgraph :-

Let G consist of (V, E) is a graph. $H(V_1, E_1)$ is said to be subgraph of G if $V_1 \subseteq V$ & $E_1 \subseteq E$, such that adjacancy relation is preserved.

ex:-



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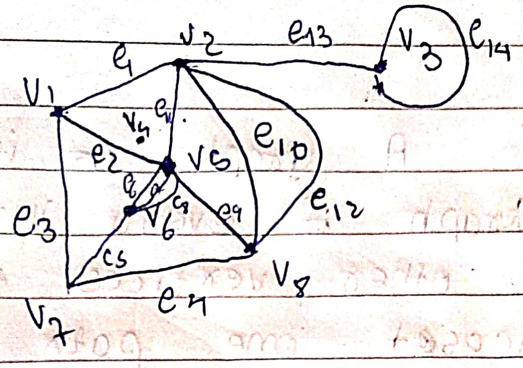
(i) Union :- $H = H_1 \cup H_2 \Rightarrow H(V^*, E^*)$
$V^* = V_1 \cup V_2$
$E^* = E_1 \cup E_2$

(ii) Intersection:- $H = H_1 \cap H_2 \Rightarrow H(V^*, E^*)$
$V^* = V_1 \cap V_2$
$E^* = E_1 \cap E_2$

* Deletion of vertex:-
In a graph G deletion of and vertex V_i results into a new graph with vertex set $V - \{V_i\}$ and all the edges incident to V_i is deleted.

* Fusion of 2 vertex V_i & V_j :-
In a graph G(V, E) results into a graph replacing V_i, V_j by single vertex V such that all edges incident with V_i, V_j will be now incident to V new vertex V.

* Graph traversing (Reachability):
A finite alternate sequence of vertices and edges starting with a vertex and ending to the vertex such that if edge e appears in the sequence with it precedend and followed by n vertices.



walk :-

ex 1 : $V_7 e_7 V_8 e_{12} V_2 e_{13} V_3 e_{14} V_3 e_{13} V_2 e_1 V_1$

=> If starting and ending vertex is same It is called closed walk.

* Construct a walk that travels through every vertex except V_4.

Ans :- $V_1 e_3 e_5 V_6 e_6 V_5 e_9 V_8 e_{10} V_2 e_{13} V_{13}$.

* **Path :** A walk when we are not repeating any edge.
  * No vertex and no edge is repeated except starting edge.

* If starting and ending vertex of path is same it is said to be closed path.

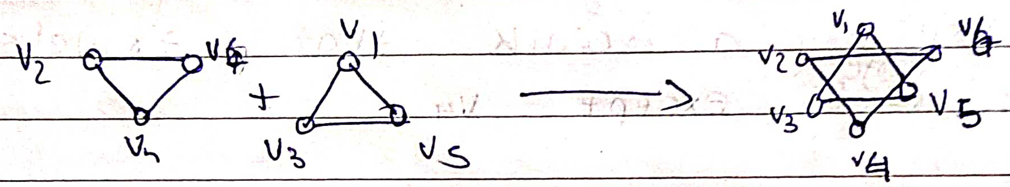
* A closed path is said to be a ~~ckt~~ circuit.

* Smallest circuit :- self loop.

* **Rechability :-** In a graph G a vertex V_i is said to be reachable from V_J if there exist one path between V_i & V_J.

• V (trivial graph)
(only one vertex)

* **Connected Graph :-** A graph G is said to be connected Graph if every vertex is reachable to all other vertices that is there exist atleast one path between each and every pair of vertices. Otherwise G is said to be disconnected graph.



(connected subgraph) → (component) (disconnected graph)

* A disconnected graph is collection of two or more then two components such that each component is a connected graph.

* Consider a simple graph G with n vertices and K components then G can have atmost $\frac{(n-K) * (n-K+1)}{2}$ edges.

⇒ Let us consider a simple graph with n vertices and K components.

so $n \geq K$.

⇒ Let ith component has n_i vertices.

∴ Hence max no. of edges in ith components can be $\frac{n_i(n_i-1)}{2}$

⇒ Therefore, maximum number of edges in

$$G = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k}(n_i)^2 - \sum_{i=1}^{k} n_i\right]$$

$$\cancel{= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right)}$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k}(n_i^2) - n\right]$$

$$= \frac{1}{2}\sum_{i=1}^{k}(n_i-1)^2 + \sum_{i=1}^{k} n_i - \frac{1}{2}\sum_{i=1}^{n} 1 - \frac{1}{2}n$$

$$= \frac{1}{2}\sum_{i=1}^{k}(n_i-1)^2 + \frac{n^2}{2} - \frac{n}{2}$$

* Null Graph :-

If there are vertices but not edges it is called Null graph.

* Connectivity :-

* Edge connectivity :- In a connected graph theory edge connectivity is the minimum no. of edges who's deletion makes the

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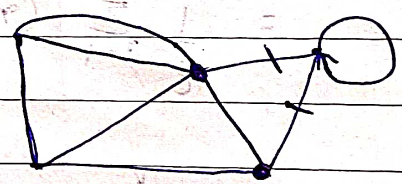
graph disconnected.

* Vertex connectivity :- Minimum number of vertices who's deletion will be result into disconnected graph.

💡 = If we are denoting edge connectivity by $\lambda(G)$ and vertex connectivity by $K(G)$.

Here, $K(G)$ can not exceed $\lambda(G)$.

ex :-

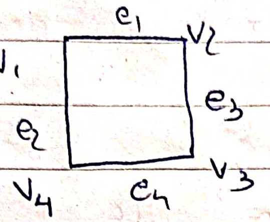


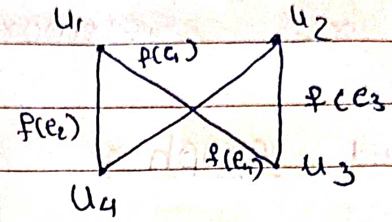
$\lambda(G) = 2$
$K(G) = 2$

* Cut set : It is a set of edges who's deletion increases the no. of components in the Graph theory such that no proper subset of this set is a cutset.

* cutpoint : A vertex in a graph (if exist) is said to be cut point if deletion of it results into disconnected graph.

* Isomorphism :



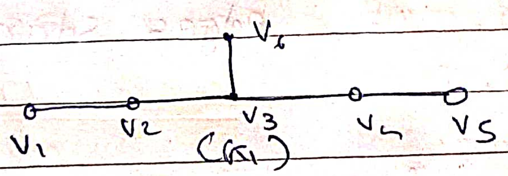


(G_1) $(\because G_1 \cong G_2)$ (G_2) , $f^{-1}: G_2 \to G_1$.

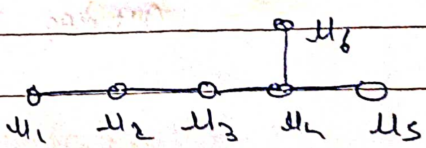
$f(V_1) = u_1$
$f(V_2) = u_3$
$f(V_4) = u_4$
$f(V_3) = u_2$

* Conditions :-

1) No. of vertices same.
2) No. of edges same
3) degree sequence must be same.
4) Self loop / parellel edges (if exist) must be same.



(G_1)



(G_2)

In G_1 and G_2 the vertices V_3 and u_4 have degree 3 hence the isomorphism mapping from $(f: G_1 \to G_2)$ V_3 to u_4

* But in graph G_1 V_3 is adjacent to only one pendant vertex V_6, In G_2 u_4 is adjacent to 2 pendant vertices u_6, u_5.

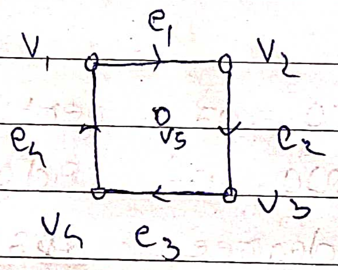
Hence, $f(v_3) = u_4$ cannot preserve adjacency relationship hence graphs are not isomorphic.

* **Directed graph :**

A graph is said to be directed graph if direction is asign to each and every ~~vertexe~~ edge to the graph and edges are represented by ordered pair of vertices as their n^{th} vertices.

$$e_k = \langle v_i , v_j \rangle = (v_i , v_j)$$

ex:-



$e_1 = (v_1, v_2) = \langle v_1, v_2 \rangle$

indeg (v_i) = no. of edges coming to v_i.

Out deg (v_i) = no. of edges starting from v_i.

⇒ $\deg(G) = \sum_{\substack{i=0 \\ \text{in deg}}}^{n} v_i + \sum_{\substack{i=0 \\ \text{out deg}}}^{n} v_i$

⇒ If indeg $(v_i) = 0$ and only outdeg $(v_i) > 0$ v_i is known as source vertex.

→ If outdeg $(v_i) = 0$ and only indeg $(v_i) > 0$ v_i is known as sink vertex.

* The path to reach from v_1 to v_2 will be $v_1 e_4 v_4 e_3 v_3 e_2 v_2 e_6$ $v_1 e_1 v_2$ we can't use second coz it's direction is different

* **Connectivity of digraph:**

A digraph is said to be ~~weakly~~ weakly connected if ~~we connect~~ it is a connected graph without directions of edges.

* **Unilaterly connected:**

A digraph is said to be unilaterly connected if for every pair of vertices (v_i, v_j) there exist path from $(v_i$ to $v_j)$ or from v_j to v_i.

* **Strongly connected:**

A digraph is said to be strongly connected if each vertex v_i is reachable from each vertex v_j in graph G.

* For a directed graph a matrix representation is taken by a matrix with rows and colums filled with entries a_{ij} such that,

$$a_{ij} = \begin{cases} 1, & \text{if } v_i, v_j \text{ are edgecent} \\ 0, & \text{otherwise.} \end{cases}$$

there is edge from v_i to v_j.

* Properties:-

(i) A graph without direction has symmetric matrix.

(ii) A digraph may or may not have symmetric matrix.

(iii) For a directed graph sum of entries in jth coloum will be indegree of v_j.

(iv) For a directed graph sum of entries in ith row will be outdegree of V_i.

(v) If A is a matrix representing a graph G then entry in the (i, j) cell of A^n represents no. of path of length n between vertices V_i and V_j.

* weighted graph :

A graph G is said to be weighted graph if every edge is G is assigned a real number $w(e_i)$. For example $w(e)$ may be cost, time, distance etc.

* Graph algorithms :

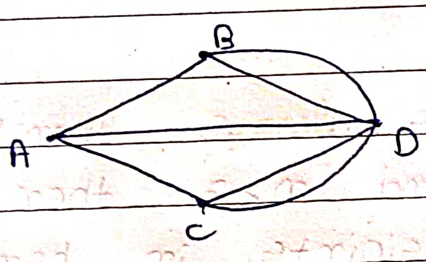
* Euler graph / Eulerion graph / Eulerian line/ walk / circuit :

A graph (connected) G is said to be Euler graph if there exist euler walk (closed) in the graph.

* Closed euler walk : (circuit)

A closed walk in a graph G is said to be eular walk if it passes through all the edges in the graph.

ex *



(Not eular graph)

* Euler line :

An open walk in a connected graph G if exist which passes through every edge is said to be euler line.

* Necessary and sufficient condition for a eular graph :

1) A connected graph G is a Eulerian graph if and only if all vertices in G are of even degree.

* ## Hemiltonion graph :

* **Hemiltonion circuit :** In A connected graph G a circuit is said to be Hemiltonion circuit if it contain all the vertices exactly one except starting and ending vertices.

=> A graph which has hamiltonian circuit is said to be Hemitonion graph

- Complete graph is always Hemiltonion.
- If n is odd and $n \geq 3$ then G will have $\frac{(n-1)}{2}$ edge disjoints, in hemitonion circuit.

* G is a complete graph with n vertices wher n is odd and $n \geq 3$,

we know that in a complete graph with n ~~vert~~ vertices ther are $\frac{n(n-1)}{2}$ edges

also a circuit with n vertices has n edges.

Therefore from $n\left(\frac{n-1}{2}\right)$ edges we can hav atmost $\frac{(n-1)}{2}$ edge disjoints.