We note that some logicians and many texts use the word "implies" in the same sense as we ... "logically implies", and so they distinguish between "implies" and "if... then". These two distinct con... cepts are, of course, intimately related as seen in the above theorem.

4.11 PROPOSITIONAL FUNCTIONS, QUANTIFIERS

Let A be a given set. A *propositional function* (or: an *open sentence* or *condition*) defined on A is an expression

$$p(x)$$

which has the property that $p(a)$ is true or false for each $a \in A$. That is, $p(x)$ becomes a statement (with a truth value) whenever any element $a \in A$ is substituted for the variable x. The set A is called the *domain* of $p(x)$, and the set T_p of all elements of A for which $p(a)$ is true is called the *truth set* of $p(x)$. In other words,

$$T_p = \{x: x \in A, p(x) \text{ is true}\} \quad \text{or} \quad T_p = \{x: p(x)\}$$

Frequently, when A is some set of numbers, the condition $p(x)$ has the form of an equation or inequality involving the variable x.

Example 4.9

Find the truth set of each propositional function $p(x)$ defined on the set $\mathbf{N}$ of positive integers.

(a) Let $p(x)$ be "$x + 2 > 7$". Its truth set is

$$\{x: x \in \mathbf{N}, x + 2 > 7\} = \{6, 7, 8, \ldots\}$$

consisting of all integers greater than 5.

(b) Let p(x) be "$x + 5 < 3$". Its truth set is

$$\{x: x \in \mathbf{N}, x + 5 < 3\} = \varnothing$$

the empty set. In other words, $p(x)$ is not true for any positive integer in $\mathbf{N}$.

(c) Let $p(x)$ be "$x + 5 > 1$". Its truth set is

$$\{x: x \in \mathbf{N}, x + 5 > 1\} = \mathbf{N}$$

Thus $p(x)$ is true for every element in $\mathbf{N}$.

Remark: The above example shows that if $p(x)$ is a propositional function defined on a set A then $p(x)$ could be true for all $x \in A$, for some $x \in A$, or for no $x \in A$. The next two subsections discuss quantifiers related to such propositional functions.

Universal Quantifier

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\forall x \in A)p(x) \quad \text{or} \quad \forall x\, p(x)$$

which reads "For every x in A, $p(x)$ is a true statement" or, simply, "For all x, $p(x)$". The symbol

$$\forall$$

which reads "for all" or "for every" is called the *universal quantifier*. The statement (*4.1*) is equivalent to the statement

$$T_p = \{x: x \in A, p(x)\} = A \tag{4.2}$$

that is, that the truth set of $p(x)$ is the entire set A.

The expression $p(x)$ by itself is an open sentence or condition and therefore has no truth value. However, $\forall x\, p(x)$, that is $p(x)$ preceded by the quantifier $\forall$, does have a truth value which follows from the equivalence of (*4.1*) and (*4.2*). Specifically:

Q_1: If $\{x: x \in A, p(x)\} = A$ then $\forall x\, p(x)$ is true; otherwise, $\forall x\, p(x)$ is false.

Example 4.10

(a) The proposition $(\forall n \in \mathbf{N})(n + 4 > 3)$ is true since

$$\{n: n + 4 > 3\} = \{1, 2, 3, \ldots\} = \mathbf{N}$$

(b) The proposition $(\forall n \in \mathbf{N})(n + 2 > 8)$ is false since

$$\{n: n + 2 > 8\} = \{7, 8, \ldots\} \neq \mathbf{N}$$

(c) The symbol $\forall$ can be used to define the intersection of an indexed collection $\{A_i: i \in I\}$ of sets A_i as follows:

$$\cap(A_i: i \in I) = \{x: \forall_i \in I, x \in A_i\}$$

Existential Quantifier

Let $p(x)$ be a propositional function defined on a set A, Consider the expression

$$(\exists x \in A)p(x) \quad \text{or} \quad \exists x,\ p(x) \tag{4.3}$$

which reads "There exists an x in A such that $p(x)$ is a true statement" or, simply, "For some x, $p(x)$". The symbol

$$\exists$$

which reads "there exists" or "for some" or "for at least one" is called the *existential quantifier*. Statement (*4.3*) is equivalent to the statement

$$T_p = \{x: x \in A, p(x)\} \neq \varnothing \tag{4.4}$$

i.e., that the truth set of $p(x)$ is not empty. Accordingly, $\exists x\, p(x)$, that is, $p(x)$ preceded by the quantifier $\exists$, does have a truth value. Specifically:

Q_2: If $\{x: p(x)\} \neq \varnothing$ then $\exists x\, p(x)$ is true; otherwise, $\exists x\, p(x)$ is false.

Example 4.11

(a) The proposition $(\exists n \in \mathbf{N})\ (n + 4 < 7)$ is true since $\{n: n + 4 < 7\} = \{1, 2\} \neq \varnothing$.

(b) The proposition $(\exists n \in \mathbf{N})\ (n + 6 < 4)$ is false since $(n: n + 6 < 4\} = \varnothing$.

(c) The symbol $\exists$ can be used to define the union of an indexed collection $\{A_i: i \in I\}$ of sets A_i as follows:

$$\cup(A_i: i \in I) = \{x: \exists i \in I, x \in A_i\}$$

Notation

Let $A = \{2, 3, 5\}$ and let $p(x)$ be the sentence "x is a prime number" or, simply "x is prime". Then

"Two is prime and three is prime and five is prime" (*)

can be denoted by

$$p(2) \wedge p(3) \wedge p(5) \quad \text{or} \quad \wedge(a \in A, p(a))$$

which is equivalent to the statement

"Every number in A is prime" or $\forall a \in A, p(a)$ (**)

Similarly, the proposition

"Two is prime or three is prime or five is prime."

can be denoted by

$$p(2) \wedge p(3) \wedge p(5) \quad \text{or} \quad \vee(a \in A, p(a))$$

which is equivalent to the statement

"At least one number in A is prime" or $\exists a \in A, p(a)$

In other words

$$\wedge(a \in A, p(a)) \equiv \forall a \in A, p(a) \quad \text{and} \quad \vee(a \in A, p(a)) \equiv \exists a \in A, p(a)$$

Thus the symbols $\wedge$ and $\vee$ are sometimes used instead of $\forall$ and $\exists$.

Remark: If A were an infinite set, then a statement of the form (*) cannot be made since the sentence would not end; but a statement of the form (**) can always be made, even when A is infinite.

4.12 NEGATION OF QUANTIFIED STATEMENTS

Consider the statement: "All math majors are male". Its negation reads:

"It is not the case that all math majors are male" or, equivalently,

"There exists at least one math major who is a female (not male)"

Symbolically, using M to denoted the set of math majors, the above can be written as

$$\neg\,(\forall x \in M)(x \text{ is male}) \equiv (\exists x \in M)(x \text{ is not male})$$

or, when $p(x)$ denotes "x is male",

$$\neg\,(\forall x \in M)\; p(x) \equiv (\exists x \in M)\; \neg\, p(x) \quad \text{or} \quad \neg\forall x p(x) \equiv \exists x \neg p(x)$$

The above is true for any proposition $p(x)$. That is:

Theorem 4.5 (DeMorgan): $\neg\,(\forall x \in A)\; p(x) \equiv (\exists x \in A)\; \neg\, p(x)$.

In other words, the following two statements are equivalent:

1. It is not true that, for all $a \in A$, $p(a)$ is true.
2. There exists an $a \in A$ such that $p(a)$ is false.

There is an analogous theorem for the negation of a proposition which contains the existential quantifier.

Theorem 4.6 (DeMorgan): $\neg\,(\exists x \in A)\; p(x) \equiv (\forall x \in A)\; \neg\, p(x)$.

That is, the following two statements are equivalent:

1. It is not true that for some $a \in A$, $p(a)$ is true.
2. For all $a \in A$, $p(a)$ is false.

Example 4.12

(a) The following statements are negatives for each other:

"For all positive integers n we have $n + 2 > 8$"

"There exists a positive integer n such that $n + 2 \not> 8$"

(b) The following statements are also negatives of each other:

"There exists a (living) person who is 150 years old"

"Every living person is not 150 years old"

Remark: The expression $\neg\, p(x)$ has the obvious meaning; that is:

"The statement $\neg\, p(a)$ is true when $p(a)$ is false, and vice versa"

Previously, $\neg$ was used as an operation on statements; here $\neg$ is used as an operation on propositional functions. Similarly, $p(x) \wedge q(x)$, read "$p(x)$ and $q(x)$", is defined by:

"The statement $p(a) \wedge q(a)$ is true when $p(a)$ and $q(a)$ are true"

Similarly, $p(x) \vee q(x)$, read "$p(x)$ or $q(x)$", is defined by:

"The statement $p(a) \vee q(a)$ is true when $p(a)$ or $q(a)$ is true"

Thus in terms of truth sets:

(i) $\neg\, p(x)$ is the complement of $p(x)$.
(ii) $p(x) \wedge q(x)$ is the intersection of $p(x)$ and $q(x)$.
(iii) $p(x) \vee q(x)$ is the union of $p(x)$ $q(x)$.

One can also show that the laws for propositions also hold for propositional functions. For example, we have DeMorgan's laws:

$$\neg(p(x) \wedge q(x)) \equiv \neg p(x) \vee \neg q(x) \quad \text{and} \quad \neg(p(x) \vee q(x)) \equiv \neg p(x) \wedge \neg q(x)$$

Counterexample

Theorem 4.6 tells us that to show that a statement $\forall x, p(x)$ is false, it is equivalent to show that $\exists x\ \neg p(x)$ is true or, in other words, that there is an element x_0 with the property that $p(x_0)$ is false. Such an element x_0 is called a *counterexample* to the statement $\forall x, p(x)$.

Example 4.13

(a) Consider the statement $\forall x \in \mathbf{R}, |x| \neq 0$. The statement is false since 0 is a counterexample, that is, $|0| \neq 0$ is not true.

(b) Consider the statement $\forall x \in \mathbf{R}, x^2 \geq x$. The statement is not true since, for example, $\frac{1}{2}$ is a counterexample. Specifically, $\left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$ is not true, that is $\left(\frac{1}{2}\right)^2 < \frac{1}{2}$.

(c) Consider the statement $\forall x \in \mathbf{N}, x^2 \geq x$. This statement is true where $\mathbf{N}$ is the set of positive integers. In other words, there does not exist a positive integer n for which $n^2 < n$.

Propositional Functions with More than One Variable

A propositional function (of n variables) defined over a product set $A = A_1 \times \ldots \times A_n$ is an expression

$$p(x_1, x_2, \ldots, x_n)$$

which has the property that $p(a_1, a_2, \ldots, a_n)$ is true or false for any n-tuple $(a_1, \ldots a_n)$ in A. For example,

$$x + 2y + 3z < 18$$

is a propositional function on $\mathbf{N}^3 = \mathbf{N} \times \mathbf{N} \times \mathbf{N}$. Such a propositional function has no truth value. However, we do have the following:

Basic Principle: A propositional function preceded by a quantifier for each variable, for example

$$\forall x\, \exists y, p(x, y) \quad \text{or} \quad \exists x\, \forall y\, \exists z, p(x, y, z)$$

denotes a statement and has a truth value.

Example 4.14

Let $B = \{1, 2, 3, \ldots, 9\}$ and let $p(x, y)$ denote "$x + y = 10$". Then $p(x, y)$ is a propositional function on $A = B^2 = B \times B$.

(a) The following is a statement since there is a quantifier for each variable:

$\forall x\, \exists y, p(x, y)$, that is, "For every x, there exists a y such that $x + y = 10$"

This statement is true. For example, if $x = 1$, let $y = 9$; if $x = 2$, let $y = 8$, and so on.

(b) The following is also a statement:

$\exists y\ \forall x,\ p(x, y)$, that is, "There exists a y such that, for every x, we have $x + y = 10$"

No such y exists; hence this statement is false.

Note that the only difference between (a) and (b) is the order of the quantifiers. Thus a different ordering of the quantifiers may yield a different statement. We note that, when translating such quantified statements into English, the expression "such that" frequently follows "there exists".

Negating Quantified Statements with More than One Variable

Quantified statements with more than one variable may be negated by successively applying Theorems 4.5 and 4.6. Thus each $\forall$ is changed to $\exists$ and each $\exists$ is changed to $\forall$ as the negation symbol $\neg$ passes through the statement from left to right. For example,

$$\neg\,[\forall x \exists y \exists z,\ p(x, y, z)] \equiv \exists x \neg\,[\exists y \exists z,\ p(x, y, z)] \equiv \exists x \forall y[\neg \exists z,\ p(x, y, z)$$
$$\equiv \exists x\ \forall y\ \forall z,\ \neg\, p(x, y, z)$$

Naturally, we do not put in all the steps when negating such quantified statements.

Example 4.15

(a) Consider the quantified statement:

"Every student has at least one course where the lecturer is a teaching assistant"

Its negation is the statement:

"There is a student such that in every course the lecturer is not a teaching assistant"

(b) The formal definition that L is the limit of a sequence $a_1, a_2, \ldots$ follows:

$$\forall \in > 0,\ \exists\, n_0 \in \mathbf{N},\ \forall n > n_0,\ |a_n - L| < \in$$

Thus L is not the limit of the sequence $a_1, a_2, \ldots$ when:

$$\exists \in > 0,\ \forall n_0 \in \mathbf{N},\ \exists n > n_0,\ |a_n - L| \geq \in$$