Counting

6.1 INTRODUCTION—BASIC COUNTING PRINCIPLES

Combinatorial analysis, which includes the study of permutations, combinations, and partitions, is concerned with determining the number of logical possibilities of some event without necessarily identifying every case. There are two basic counting principles used throughout.

Sum Rule Principle

Suppose some event E can occur in m ways and a second event F can occur in n ways, and suppose both events cannot occur simultaneously. Then E or F can occur in $m + n$ ways. More generally, suppose an event E_1 can occur in n_1 ways, a second event E_2 can occur in n_2 ways, a third event E_3 can occur in n_3 ways, ..., and suppose no two of the events can occur at the same time. Then one of the events can occur in $n_1 + n_2 + n_3 + \cdots$ ways.

Example 6.1

(a) Suppose there are 8 male professors and 5 female professors teaching a calculus class. A student can choose a calculus professor in 8 + 5 = 13 ways.

(b) Suppose E is the event of choosing a prime number less than 10, and suppose F is the event of choosing an even number less than 10. Then E can occur in four ways [2, 3, 5, 7], and F can occur in 4 ways [2, 4, 6, 8]. However E or F cannot occur in 4 + 4 = 8 ways since 2 is both a prime number less than 10 and an even less than 10. In fact, E or F can occur in only 4 + 4 − 1 = 7 ways.

(c) Suppose E is the event of choosing a prime number between 10 and 20, and suppose F is the event of choosing an even number between 10 and 20. Then E can occur in 4 ways [11, 13, 17, 19], and F can occur in 4 ways [12, 14, 16, 18]. Then E or F can occur in 4 + 4 = 8 ways since now none of the even numbers is prime.

Product Rule Principle

Suppose there is an event E which can occur in m ways and, independent of this event, there is a second event F which can occur in n ways. Then combinations of E and F can occur in mn ways. More generally, suppose an event E_1 can occur in n_1 ways, and, following E_1, a second event E_2 can occur in n_2 ways, and, following E_2, a third event E_3 can occur in n_3 ways, and so on. Then all the events can occur in the order indicated in $n_1 \cdot n_2 \cdot n_3 \cdots$ ways.

Example 6.2

(a) Suppose a license plate contains two letters followed by three digits with the first digit not zero. How many different license plates can be printed?

Each letter can be printed in 26 different ways, the first digit in 9 ways and each of the other two digits in 10 ways. Hence

$$26 \cdot 26 \cdot 9 \cdot 10 \cdot 10 = 608\,400$$

different plates can be printed.

(b) In How many ways can an organization containing 26 members elect a president, treasurer, and secretary (assuming no person is elected to more than one position)?

The president can be elected in 26 different ways; following this, the treasurer can be elected in 25 different ways (since the person chosen president is not eligible to be treasurer); and, following this, the secretary can be elected in 24 different ways. Thus, by the above principle of counting, there are

$$26 \cdot 25 \cdot 24 = 15\,600$$

different ways in which the organization can elect the officers.

There is a set theoretical interpretation of the above two counting principles. Specifically, suppose $n(A)$ denotes the number of elements in a set A. Then:

1. ***Sum Rule Principle***: If A and B are disjoint sets, then
$$n(A \cup B) = n(A) + n(B)$$
2. ***Product Rule Principle***: Let $A \times B$ be the Cartesian product of sets A and B. Then
$$n(A \times B) = n(A) \cdot n(B)$$

6.2 FACTORIAL NOTATION

The product of the positive integers from 1 to n inclusive is denoted by $n!$ (read "n factorial"):

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-2)\,(n-1)n$$

In other words, $n!$ is defined by

$$1! = 1 \quad \text{and} \quad n! = n \cdot (n-1)!$$

It is also convenient to define $0! = 1$.

Example 6.3

(a) $2! = 1 \cdot 2 = 2,\quad 3! = 1 \cdot 2 \cdot 3 = 6,\quad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24,$
$5! = 5 \cdot 4! = 5 \cdot 24 = 120,\quad 6! = 6 \cdot 5! = 6 \cdot 120 = 720.$

(b) $\frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} = 8 \cdot 7 = 56,\ 12 \cdot 11 \cdot 10 = \frac{12 \cdot 11 \cdot 10 \cdot 9!}{9!} = \frac{12!}{9!},\ \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 12 \cdot 11 \cdot 10 \cdot \frac{1}{3!} = \frac{12!}{3!9!}.$

(c) $n(n-1)\cdots(n-r+1) = \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3 \cdot 2 \cdot 1}{(n-r)(n-r-1)\cdots 3 \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}$

$\frac{n(n-1)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots (r-1)r} = n(n-1)\cdots(n-r+1) \cdot \frac{1}{r!} = \frac{n!}{(n-r)!} \cdot \frac{1}{r!} = \frac{n!}{r!(n-r)!}.$

6.3 BINOMIAL COEFFICIENTS

The symbol $\binom{n}{r}$ (*read* "*nCr*"), where r and n are positive integers with $r \le n$, is defined as follows

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots (r-1)r}$$

By Example 6.3(c), we see that

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{1 \cdot 2 \cdot 3 \cdots (r-1)r} = \frac{n!}{r!(n-r)!}$$

But $n - (n - r) = r$; hence we have the following important relation:

$$\binom{n}{n-r} = \binom{n}{r} \text{ or, in other words, if } a + b = n \text{ then } \binom{n}{a} = \binom{n}{b}$$

Example 6.4

(a) $\binom{8}{2} = \frac{8 \cdot 7}{1 \cdot 2} = 28,\quad \binom{9}{4} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 126,\quad \binom{12}{5} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 792,$

$\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120,\quad \binom{13}{1} = \frac{13}{1} = 13.$

Note that $\binom{n}{r}$ has exactly r factors in both the numerator and the denominator.

(b) Compute $\binom{10}{7}$. By definition,

$$\binom{10}{7} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = 120$$

On the other hand, $10 - 7 = 3$ and so we can also compute $\binom{10}{7}$ as follows:

$$\binom{10}{7} = \binom{10}{3} = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120$$

Observe that the second method saves space and time.

Binomial Coefficients and Pascal's Triangle

The numbers $\binom{n}{r}$ are called the *binomial coefficients* since they appear as the coefficients in the expansion of $(a + b)^n$. Specifically, one can prove that

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The coefficients of the successive powers of $a + b$ can be arranged in a triangular array of numbers, called Pascal's triangle, as pictured in Fig. 6.1. The numbers in Pascal's triangle have the following intersecting properties:

(i) The first number and the last number in each now is 1.

(ii) Every other number in the array can be obtained by adding the two numbers appearing directly above it. For example $10 = 4 + 6$, $15 = 5 + 10$, $20 = 10 + 10$.

Since the numbers appearing in Pascal's triangle are binomial coefficients, property (ii) of Pascal's triangle comes from the following theorem (proved in Problem 6.7):

Theorem 6.1: $\binom{n+1}{r} = \binom{n}{n-1} + \binom{n}{r}$.

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$
$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Fig. 6.1 Pascal's triangle

6.4 PERMUTATIONS

Any arrangement of a set of n objects in a given order is called a *permutation* of the objects (taken all at a time). Any arrangement of any $r \le n$ of these objects in a given order is called an *r-permutation* or a *permutation of the n objects taken r at a time*. Consider, for example, the set of letters a, b, c, and d. Then:

(i) *bdca*, *dcba* and *acdb* are permutations of the four letters (taken all at a time);
(ii) *bad*, *adb*, *cbd* and *bca* are permutations of the four letters taken three at a time;
(iii) *ad*, *cb*, *da* and *bd* are permutations of the four letters taken two at a time.

The number of permutations of n objects taken r at a time is denoted by

$$P(n, r),\ {}_nP_r,\ P_{n,r},\ P_r^n, \text{ or } (n)_r$$

We shall use $P(n, r)$. Before we derive the general formula for $P(n, r)$ we consider a particular case.

Example 6.5

Find the number of permutations of six objects, say A, B, C, D, E, F, taken three at a time. In other words, find the number of "three-letter words" using only the given six letters without repetitions.

Let the general three-letter word be represented by the following three boxes:

$$\square \quad \square \quad \square$$

Now the first letter can be chosen in six different ways; following this, the second letter can be chosen in five different ways; and, following this, the last letter can be chosen in four different ways. Write each number in its appropriate box as follows:

$$\boxed{6} \quad \boxed{5} \quad \boxed{4}$$

Thus by the fundamental principle of counting there are $6 \times 5 \times 4 = 120$ possible three-letter words without repetitions from the six letters, or there are 120 permutations of six objects taken three at a time:

$$P(6, 3) = 120$$

Derivation of the Formula for P(n, r)

The derivation of the formula for the number of permutations of n objects taken r at a time, or the number of r-permutations of n objects, $P(n, r)$, follows the procedure in the preceding example. The first element in an r-permutation of n objects can be chosen in n different ways; following this, the second element in the permutation can be chosen in $n - 1$ ways; and, following this, the third element in the permutation can be chosen in $n - 2$ ways. Continuing in this manner, we have that the rth (last) element in the r-permutation can be chosen in $n - (r - 1) = n - r + 1$ ways. Thus, by the fundamental principle of counting, we have

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

By Example 6.3(c), we see that

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Thus we have proven the following theorem.

Theorem 6.2: $P(n, r) = \dfrac{n!}{(n-r)!}$.

In the special case in which $r = n$, we have

$$P(n, n) = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 = n!$$

Accordingly, we have the following corollary.

Corollary 6.3: There are $n!$ permutations of n objects (taken all at a time).

For example, there are $3! = 1 \cdot 2 \cdot 3 = 6$ permutations of the three letters a, b, and c. These are abc, acb, bac, bca, cab, cba.

Permutations with Repetitions

Frequently we want to know the number of permutations of a multiset, that is, a set of objects some of which are alike. We will let

$$P(n; n_1, n_2, \ldots, n_r)$$

denote the number of permutations of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike. The general formula follows:

Theorem 6.4: $P(n; n_1, n_2, \ldots, n_r) = \dfrac{n!}{n_1! n_2! \cdots n_r!}$.

We indicate the proof of the above theorem by a particular example. Suppose we want to form all possible five-letter "words" using the letters from the word "BABBY". Now there are $5! = 120$ permutations of the objects B_1, A, B_2, B_3, Y, where the three Bs are distinguished. Observe that the following six permutations

$$B_1B_2B_3AY, \quad B_2B_1B_3AY, \quad B_3B_1B_2AY, \quad B_1B_3B_2AY, \quad B_2B_3B_1AY, \quad B_3B_2B_1AY,$$

produce the same word when the subscripts are removed. The 6 comes from the fact that there are $3! = 3 \cdot 2 \cdot 1 = 6$ different ways of placing the three Bs is in the first three positions in the permutation. This is true for each set of three positions in which the Bs can appear. Accordingly there are

$$P(5; 3) = \frac{5!}{3!} = \frac{120}{6} = 20$$

different five-letter words that can be formed using the letters from the word "BABBY".

Example 6.6

(a) How many seven-letter words can be formed using the letters of the word "BENZENE"? We seek the number of permutations of seven objects of which three are alike (the three Es), and two are alike (the two Ns). By Theorem 6.4, the number of such words is

$$P(7; 3, 2) = \frac{7!}{3!2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 420$$

(b) How many different signals, each consisting of eight flags hung in a vertical line, can be formed from a set of four indistinguishable red flags, three indistinguishable white flags, and a blue flag? We seek the number of permutations of eight objects of which four are alike and three are alike. The number of such signals is

$$P(8;\ 4,\ 3) = \frac{8!}{4!3!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1\cdot 3\cdot 2\cdot 1} = 280$$

6.5 COMBINATIONS

Suppose we have a collection of n objects. A *combination* of these n objects taken r at a time is any selection of r of the objects where order does not count. In other words, an *r-combination* of a set of n objects is any subset of r elements. For example, the combinations of the letters a, b, c, d taken three at a time are

$$\{a, b, c\},\quad \{a, b, d\},\quad \{a, c, d\},\quad \{b, c, d\}\quad \text{or simply}\quad abc,\ abd,\ acd,\ bcd$$

Observe that the following combinations are equal:

$$abc,\ acb,\ bac,\ bca,\ cab, \text{ and } cba$$

That is, each denotes the same set $\{a, b, c\}$.

The number of combinations of n objects taken r at a time is denoted by $C(n, r)$. The symbols ${}_nC_r$, $C_{n,r}$ and C^n_r also appear in various texts. Before we give the general formula for $C(n, r)$, we consider a special case.

Example 6.7

Find the number of combinations of the four objects, a, b, c, d, taken three at a time.

Each combination consisting of three objects determines $3! = 6$ permutations of the objects in the combination as pictured in Fig. 6.2. Thus the number of combinations multiplied by 3! equals the number of permutations; that is,

$$C(4,\ 3) \times 3! = P(4,\ 3) \quad \text{or} \quad C(4,\ 3) = \frac{P(4,3)}{3!}$$

But $P(4, 3) = 4 \cdot 3 \cdot 2 = 24$ and $3! = 6$; hence $C(4, 3) = 4$ as noted in Fig. 6.2.

Combination	Permutations
abc	*abc, acb, bac, bca, cab, cba*
abd	*abd, adb, bad, bda, dab, dba*
acd	*acd, adc, cad, cda, dac, dca*
bcd	*bcd, bdc, cbd, cdb, dbc, dcb*

Fig. 6.2

Formula for C(n, r)

Since any combination of n objects taken r at a time determines $r!$ permutations of the objects in the combination, we can conclude that

$$P(n, r) = r!\ C(n, r)$$

Thus we obtain

Theorem 6.5: $C(n, r) = \dfrac{P(n,r)}{r!} = \dfrac{n!}{r(n-r)!}$;

Recall that the binomial coefficient $\binom{n}{r}$ was defined to be $\dfrac{n!}{r!(n-r)!}$; hence

$$C(n, r) = \binom{n}{r}$$

We shall use $C(n, r)$ and $\binom{n}{r}$ interchangeably.

Example 6.8

(a) How many committees of three can be formed from eight people?

Each committee is, essentially, a combination of the eight people taken three at a time. Thus the number of committees that can be formed is

$$C(8, 3) = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

(b) A farmer buys 3 cows, 2 pigs and 4 hens from a man who has 6 cows, 5 pigs and 8 hens. How many choices does the farmer have?

The farmer can choose the cows in $\binom{6}{3}$ ways, the pigs in $\binom{5}{2}$ ways, and the hens in $\binom{8}{4}$ ways.

Hence altogether he can choose the animals in

$$\binom{6}{3}\binom{5}{2}\binom{8}{4} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \cdot \frac{5 \cdot 4}{1 \cdot 2} \cdot \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 20 \cdot 10 \cdot 70 = 14000 \text{ ways}$$

6.6 THE PIGEONHOLE PRINCIPLE

Many results in combinatorial theory come from the following almost obvious statement.

Pigeonhole Principle: If n pigeonholes are occupied by $n + 1$ or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.

This principle can be applied to many problems where we want to show that a given situation can occur.

Example 6.9

(a) Suppose a department contains 13 professors. Then two of the professors (pigeons) were born in the same month (pigeonholes).

(b) Suppose a laundry bag contains many red, white, and blue socks. Then one need only grab four socks (pigeons) to be sure of getting a pair with the same color (pigeonholes).

(c) Find the minimum number of elements that one needs to take from the set $S = \{1, 2, 3, \ldots, 9\}$ to be sure that two of the numbers add up to 10.
Here the pigeonholes are the five sets $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 6\}$, $\{5\}$. Thus any choice of six elements (pigeons) of S will guarantee that two of the numbers add up to ten.

The Pigeonhole Principle is generalized as follows.

Generalized Pigeonhole Principle

If n pigeonholes are occupied by $kn + 1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by $k + 1$ or more pigeons.

Example 6.10

(a) Find the minimum number of students in a class to be sure that three of them are born in the same month. Here the $n = 12$ months are the pigeonholes and $k + 1 = 3$, or $k = 2$. Hence among any $kn + 1 = 25$ students (pigeons), three of them are born in the same month.

(b) Suppose a laundrybag contains many red, white, and blue socks. Find the minimum number of socks that one needs to choose in order to get two pairs (four socks) of the same color. Here there are $n = 3$ colors (pigeonholes) and $k + 1 = 4$, or $k = 3$. Thus among any $kn + 1 = 10$ socks (pigeons), four of them have the same color.

6.7 ORDERED AND UNORDERED PARTITIONS

Ordered Partitions

Suppose a bag A contains seven marbles numbered 1 through 7. We compute the number of ways we can draw, first, two marbles from the bag, then three marbles from the bag, and lastly two marbles from the bag. In other words, we want to compute the number of ordered partitions

$$[A_1, A_2, A_3]$$

of the set of seven marbles into cells A_1 containing two marbles, A_2 containing three marbles and A_3 containing two marbles. We call these ordered partitions since we distinguish between

$$[\{1, 2\}, \{3, 4, 5\}, \{6, 7\}] \quad \text{and} \quad [\{6, 7\}, \{3, 4, 5\}, \{1, 2\}]$$

each of which determines the same partition of A.

Now we begin with seven marbles in the bag, so there are $\binom{7}{2}$ ways of drawing the first two marbles, i.e. of determining the first cell A_1; following this, there are five marbles left in the bag and so there are $\binom{5}{3}$ ways of drawing the three marbles, i.e. of determining the second cell A_2; finally, there are two marbles left in the bag and so there are $\binom{2}{2}$ ways of determining the last cell A_3. Hence there are

$$\binom{7}{2}\binom{5}{3}\binom{2}{2} = \frac{7 \cdot 6}{1 \cdot 2} \cdot \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} \cdot \frac{2 \cdot 1}{1 \cdot 2} = 210$$

different ordered partitions of A into cells A_1 containing two marbles, A_2 containing three marbles, and A_3 containing two marbles.

Now observe that

$$\binom{7}{2}\binom{5}{3}\binom{2}{2} = \frac{7!}{2!5!} \cdot \frac{5!}{3!2!} \cdot \frac{2!}{2!0!} = \frac{7!}{2!3!2!}$$

since each numerator after the first is canceled by the second term in the denominator of the previous factor.

The above discussion can be shown to hold in general. Namely,

Theorem 6.6: Let A contain n elements and $n_1, n_2, \ldots, n_r$ be positive integers whose sum is n, that is, $n_1 + n_2 + \cdots + n_r = n$. Then there exist

$$\frac{n!}{n_1! n_2! n_3! \cdots n_r!}$$

different ordered partitions of A of the form $[A_1, A_2, \ldots, A_r]$ where A_1 contains n_1 elements, A_2 contains n_2 elements, ..., and A_r contains n_r elements.

We apply this theorem in the next example.

Example 6.11

Find the number m of ways that nine toys can be divided between four children if the youngest child is to receive three toys and each of the other two toys.

We wish to find the number m of ordered partitions of the nine toys into four cells containing 3, 2, 2, 2 toys respectively. By Theorem 6.6,

$$m = \frac{9!}{3!2!2!2!} = 7560$$

Unordered Partitions

Frequently, we want to partition a set A into a collection of subsets $A_1, A_2, \ldots, A_r$ where the subsets are now unordered. Just as the number of permutations with repetitions was obtained from the number of permutations by dividing by $k!$ when k objects were alike, so too we can obtain the number of unordered partitions from the number of ordered partitions by dividing by $k!$ when k of the sets have the same number of elements. This is illustrated in the next example, where we solve the problem in two ways.

Example 6.12

Find the number m of ways that 12 students can be partitioned into three teams, A_1, A_2, and A_3, so that each team contains four students.

Method 1: Let A denote one of the students. Then there are $\binom{11}{3}$ ways to choose three other students to be on the same team as A. Now let B denote a student who is not on the same team as A; then there are $\binom{7}{3}$ ways to choose three students of the remaining students to be on the same team as B. The remaining four students constitute the third team. Thus, altogether, the number of ways to partition the students is

$$m = \binom{11}{3} \cdot \binom{7}{3} = 165 \cdot 35 = 5775$$

Method 2: Observe that each partition $\{A_1, A_2, A_3\}$ of the students can be arranged in $3! = 6$ ways as an ordered partition. By Theorem 6.8, there are $\frac{12!}{4!4!4!} = 34\,650$ such ordered partitions. Thus there are $m = 34\,650/6 = 5775$ (unordered) partitions.

6.8 GENERATION OF PERMUTATIONS AND COMBINATIONS

Let a, b and c be three distinct objects. We are interested in enlisting all $3! = 6$ permutations as *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*. For 1, 2, 3 and 4 as four distinct objects, it is not easy to enlist all $4! = 24$ permutations. While enlisting the permutations, one has to be careful about writing down all the permutations with no omissions and no repetitions. We need a suitable algorithm, which, upon input of a positive integer n, generates a complete list of all $n!$ permutations for n distinct objects.

Many different algorithms have been developed to enlist $n!$ permutations for given n distinct objects. One of them is based on lexicographic ordering. Lexicographic order generalizes ordinary dictionary order. The way words are ordered in a dictionary is one way to order permutation. Given two words, to determine which one precedes other in dictionary, we compare letters in the words. Comparison results in one of the following two possibilities.

- Words are of different or of the same lengths, but letters at some position differ.
- Words are of different lengths. Each letter of shorter word is same as that of corresponding letter of longer word.

For first possibility, we locate the leftmost position p at which the letters differ. The order of letters at position P decides order of words in dictionary. For example the word 'computation' precedes the word 'computer'. For the second possibility the shorter word precedes the longer, like the word ... precedes the word 'catwalk'.

Lexicographic order generalizes ordinary dictionary order by replacing the alphabet by any ... symbols on which an order has been defined. For example, 1357264 precede 1357624 and ... 1347256. With respect to lexicographic ordering, the procedure of generating permutations, finds ... permutations which follows given permutation. Let us study the algorithm for the same.

An algorithm constructs the next permutation in lexicographic order, following a given permu... $a_1, a_2, \ldots, a_n$. Suppose that $a_{n-1} < a_n$, interchange a_{n-1} and a_n to obtain a larger permutation. No ot... permutation is both larger than the original permutation and smaller than the permutation obtained ... interchanging a_{n-1} and a_n. Algorithm permutation of n objects starts with permutation 1 2 3 4 ... n ... stops at permutation n … 4 3 2 1.

Algorithm

Given a natural number n, enlist the $n!$ permutations of given n distinct objects say 1, 2, 3, ... (ascending order).

1. Let $C = 1$, print first permutation as 1 2 3 … n; if $n = 1$, go to Step 3.
2. For $C = 2$ to $n!$, generate next permutation from given permutation $a_1, a_2, \ldots, a_n$ by following the steps given below:
   (a) Scan the digits of given permutation from right to left and note the first consecutive pair (a_{n-1} and a_n) such that $a_{n-1} < a_n$.
   Remember the position of a_{n-1}. Let $m = n - 1$
   (b) Search for the smallest digit among digits $a_{m+1}, a_{m+2}, \ldots a_n$ that is larger than a_m, call it X.
   (c) Interchange a_m and X.
   (d) Arrange all digits a_{m+1} to a_n in increasing order as $a_{m+1} < a_{m+2} < a_n$.
   (e) Print $a_1, a_2, \ldots a_n$.
3. Stop.

Consider given permutation as 1374652. Scan from right to left, we find pair (4 and 6) as 4<6, we replace 4 by smallest digit on its right which is larger than 4. Among 6, 5, and 2, digit 5 is the smallest digit which is greater than 4. So, exchange 4 and 5. Now first m digits of new permutation are 1375. Now sorting remaining digits in increasing order, we get 1375246 as next permutation.

Using this algorithm let us generate all permutations of 1, 2, 3 and 4 as:

1234	1243	1324	1342	1423	1432	2134
2143	2314	2341	2413	2431	3124	3142
3214	3241	3412	3421	4123	4132	4213
4231	4312	4321				

For example, for set as {physics, chemistry, mathematics, biology}, we generate permutations by mapping the elements with elements in set {1, 2, 3, 4}.

Combinations

As with permutations, combinations will be listed lexicographically but being careful to avoid listing both 9284 and 4829, since these represent the same combination.

Algorithm

{Select k objects out of n distinct objects, say 1, 2, 3, 4, ... n.}

1. List all n digits in ascending order.
2. Select first k digit, print them.
3. $J = 1, 2, \ldots k$ be index on k elements.
4. Assign maximum value to each jth position as $n - k + j$.
5. Start scanning the list from right to left, stop at location where there are consecutive digits such as $a_{n-1} < a_n$. let $m = n - 1$. if no such pair formed, go to Step 8.
6. Leave the digits before a_m as they were, follow a_m by $a_{m+1}, a_{m+2}, \ldots$ and until k digits in all have been listed down, such that no digit crosses the limit assigned as in Step 4.
7. Go to Step 5.
8. Stop.

Solved Problems

FACTORIAL NOTATION AND BINOMIAL COEFFICIENTS

6.1 Compute 4!, 5!, 6!, and 7!.

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24,$

$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5 \cdot (4!) = 5 \cdot (24) = 120,$

$6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 6 \cdot (5!) = 6 \cdot (120) = 720,$

$7! = 7 \cdot (6!) = 7 \cdot (720) = 5040.$

6.2 Compute (a) $\dfrac{13!}{11!}$; (b) $\dfrac{7!}{10!}$;

(a) $$\frac{13!}{11!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 12 = 156.$$

Alternatively, this could be solved as follows:

$$\frac{13!}{11!} = \frac{13 \cdot 12 \cdot 11!}{11!} = 13 \cdot 12 = 156.$$

(b) $$\frac{7!}{10!} = \frac{7!}{10 \cdot 9 \cdot 8 \cdot 7!} = \frac{1}{10 \cdot 9 \cdot 8} = \frac{1}{720}.$$

6.3 Simplify: (a) $\dfrac{n!}{(n-1)!}$; (b) $\dfrac{(n+2)!}{n!}$.

(a) $$\frac{n!}{(n-1)!} = \frac{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1} = n; \text{ alternatively, } \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n.$$

(b) $\dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = (n+2)(n+1) = n^2 + 3n + 2;$

alternatively, $\dfrac{(n+2)!}{n!} = \dfrac{(n+2)(n+1)n!}{n!} = (n+2)(n+1) = n^2 + 3n + 2.$

6.4 Compute: (a) $\dbinom{16}{3}$; (b) $\dbinom{12}{4}$.

Recall that there are as many factors in the numerator as in the denominator.

(a) $\dbinom{16}{3} = \dfrac{16\cdot 15\cdot 14}{1\cdot 2\cdot 3} = 560;$ (b) $\dbinom{12}{4} = \dfrac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4} = 495.$

6.5 Compute: (a) $\dbinom{8}{5}$; (b) $\dbinom{9}{7}$.

(a) $\dbinom{8}{5} = \dfrac{8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 56$ or, since $8 - 5 = 3$, $\dbinom{8}{5} = \dbinom{8}{3} = \dfrac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3} = 56.$

(b) Since $9 - 7 = 2$, we have $\dbinom{9}{7} = \dbinom{9}{2} = \dfrac{9\cdot 8}{1\cdot 2} = 36.$

6.6 Prove: $\dbinom{17}{6} = \dbinom{16}{5} + \dbinom{16}{6}$

Now $\dbinom{16}{5} + \dbinom{16}{6} = \dfrac{16!}{5!11!} + \dfrac{16!}{6!10!}$. Multiply the first fraction by $\dfrac{6}{6}$ and the second by $\dfrac{11}{11}$ to obtain the same denominator in both fractions; and then add:

$$\binom{16}{5} + \binom{16}{6} = \frac{6\cdot 16!}{6\cdot 5!\cdot 11!} + \frac{11\cdot 16!}{6!\cdot 11\cdot 10!} = \frac{6\cdot 16!}{6!\cdot 11!} + \frac{11\cdot 16!}{6!\cdot 11!}$$

$$= \frac{6\cdot 16! + 11\cdot 16!}{6!\cdot 11!} + \frac{(6+11)\cdot 16!}{6!\cdot 11!} = \frac{17\cdot 16!}{6!\cdot 11!} = \frac{17!}{6!\cdot 11!} = \binom{17}{6}$$

6.7 Prove Theorem 6.1: $\dbinom{n+1}{r} = \dbinom{n}{r-1} + \dbinom{n}{r}$.

(The technique in this proof is similar to that of the preceding problem.)

Now $\dbinom{n}{r-1} + \dbinom{r}{n} = \dfrac{n!}{(r-1)!\cdot(n-r+1)!} + \dfrac{n!}{r!\cdot(n-r)!}$. To obtain the same denominator in both fractions, multiply the first fraction by $\frac{r}{r}$ and the second fraction by $\frac{n-r+1}{n-r+1}$. Hence

$$\binom{n}{r-1}+\binom{n}{r} = \frac{r\cdot n!}{r\cdot(r-1)!\cdot(n-r+1)!}+\frac{(n-r+1)\cdot n!}{r!\cdot(n-r+1)\cdot(n-r)!}$$

$$= \frac{r\cdot n!}{r!(n-r+1)!}+\frac{(n-r+1)\cdot n!}{r!(n-r+1)!}$$

$$= \frac{r\cdot n!+(n-r+1)\cdot n!}{r!(n-r+1)!}+\frac{[r+(n-r+1)]\cdot n!}{r!(n-r+1)!}$$

$$= \frac{(n+1)+n!}{r!(n-r+1)!}+\frac{(n+1)!}{r!(n-r+1)!}=\binom{n+1}{r}$$

PERMUTATIONS

6.8 There are four bus lines between A and B; and three bus lines between B and C. In how many ways can a man travel (a) by bus from A to C by way of B? (b) round-trip by bus from A to C by way of B? (c) round-trip by bus from A to C by way of B, if he does not want to use a bus line more than once?

(a) There are four ways to go from A to B and three ways to go from B to C; hence there are $4\cdot 3 = 12$ ways to go from A to C by way of B.

(b) There are 12 ways to go from A to C by way of B, and 12 ways to return. Hence there are $12\cdot 12 = 144$ ways to travel round-trip.

(c) The man will travel from A to B to C to B to A. Enter these letters with connecting arrows as follows:

$$A\longrightarrow B\longrightarrow C\longrightarrow B\longrightarrow A$$

The man can travel four ways from A to B and three ways from B to C, but he can only travel two ways from C to B and three ways from B to A since he does not want to use a bus line more than once. Enter these numbers above the corresponding arrows as follows:

$$A\xrightarrow{\ 4\ } B\xrightarrow{\ 3\ } C\xrightarrow{\ 2\ } B\xrightarrow{\ 3\ } A$$

Thus there are $4\cdot 3\cdot 2\cdot 3 = 72$ ways to travel round-trip without using the same bus line more than once.

6.9 Suppose repetitions are not permitted. (a) How many three-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9? (b) How many of these numbers are less than 400? (c) How many are even?

In each case draw three boxes $\square\ \square\ \square$ to represent an arbitrary number, and then write in each box the number of digits that can be placed there.

(a) The box on the left can be filled in six ways; following this, the middle box can be filled in five ways; and, lastly, the box on the right can be filled in four ways; $\boxed{6}\ \boxed{5}\ \boxed{4}$. Thus, there are $6 \cdot 5 \cdot 4 = 120$ numbers.

(b) The box on the left can be filled in only two ways, by 2 or 3, since each number must be less than 400; the middle box can be filled in five ways; and, lastly, the box on the right can be filled in four ways: $\boxed{2}\ \boxed{5}\ \boxed{4}$. Thus, there are $2 \cdot 5 \cdot 4 = 40$ numbers.

(c) The box on the right can be filled in only two ways, by 2 or 6, since the numbers must be even; the box on the left can then be filled in five ways; and lastly, the middle box can be filled in four ways: $\boxed{5}\ \boxed{4}\ \boxed{2}$. Thus, there are $5 \cdot 4 \cdot 2 = 40$ numbers.

6.10 Find the number of ways that a party of seven persons can arrange themselves: (a) in a row of seven chairs; (b) around a circular table.

(a) The seven persons can arrange themselves in a row in $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7!$ ways.

(b) One person can sit at any place in the circular table. The other six persons can then arrange themselves in $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ ways around the table.

This is an example of a *circular permutation*. In general, n objects can be arranged in a circle in $(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = (n-1)!$ ways.

6.11 Find the number of distinct permutations that can be formed from all the letters of each word: (a) RADAR; (b) UNUSUAL.

(a) $\dfrac{5!}{2!2!} = 30$, since there are five letters of which two are R and two are A.

(b) $\dfrac{7!}{3!} = 840$, since there are seven letters of which three are U.

6.12 In how many ways can four mathematics books, three history books, three chemistry books, and two sociology books be arranged on a shelf so that all books of the same subject are together?

First the books must be arranged on the shelf in four units according to subject matter: $\boxed{}\ \boxed{}\ \boxed{}\ \boxed{}$. The box on the left can be filled by any of the four subjects; the next by any three remaining subjects, the next by any two remaining subjects, and the box on the right by the last subject: $\boxed{4}\ \boxed{3}\ \boxed{2}\ \boxed{1}$. Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 4!$ ways to arrange the books on the shelf according to subject matter.

Now, in each of the above cases, the mathematics books can be arranged in 4! ways, the history books in 3! ways, the chemistry books in 3! ways, and the sociology books in 2! ways. Thus, altogether, there are $4! \cdot 4! \cdot 3! \cdot 3! \cdot 2! = 41\,472$ arrangements.

6.13 Find n if: (a) $P(n, 2) = 72$; (b) $P(n, 4) = 42\, p(n, 2)$; (c) $2p(n, 2) + 50 = P(2n, 2)$.

(a) $P(n, 2) = n(n-1) = n^2 - n$; hence $n^2 - n = 72$ or $n^2 - n - 72 = 0$ or $(n-9)(n+8) = 0$. Since n must be positive, the only answer is $n = 9$.

(b) $P(n, 4) = n(n-1)(n-2)(n-3)$ and $P(n, 2) = n(n-1)$. Hence

$n(n-1)(n-2)(n-3) = 42n(n-1)$ or, if $n \neq 0, n \neq 1$, $(n-2)(n-3) = 42$

or $n^2 - 5n + 6 = 42$ or $n^2 - 5n - 36 = 0$ or $(n-9)(n+4) = 0$

Since n must be positive, the only answer is $n = 9$.

(c) $P(n, 2) = n(n-1) = n^2 - n$ and $P(2n, 2) = 2n(2n-1) = 4n^2 - 2n$. Hence

$2(n^2 - n) + 50 = 4n^2 - 2n$ or $2n^2 - 2n + 50 = 4n^2 - 2n$ or $50 = 2n^2$ or $n^2 = 25$

Since n must be positive, the only answer is $n = 5$.

14. In how many ways can five examinations be scheduled in a week so that no two examinations are scheduled on the same day considering Sunday as a holiday?

Excluding Sunday, five examinations are to be scheduled on six days so that there is at most one examination on a particular day.

$$\text{Total schedules} = {}^6P_5 = \frac{6!}{5!} = 6.$$

15. In a certain programming language, variable should be of length three and should be made up of two letters followed by a digit or of length two made up of a letter followed by a digit. How many possible variables? What if letters are not to be repeated?

(a) Total variables of length three $= 26 \times 25 \times 10$

Total variables of length two $= 26 \times 10$

$= 260$

Total possible variables of length three or two $= 26 \times 25 \times 10 + 260 = 4760$

(b) If repetition is allowed then,

Total variables $= 26 \times 26 \times 10 + 26 \times 10 = 7320$

16. Six boys and 6 girls are to be seated in a row, how many ways can they be seated if

(a) All boys are to be seated together and all girls are to be seated together

All possible ways girls can be seated = 6!

All possible arrangements of boys = 6!

Arrangement of boys and girls (Boys on left or girls on left) = 2

Hence, possible arrangement such that all boys are seated together and all girls seated together $= 6! \times 6! \times 2$

(b) No two girls should be seated together.

All girls can be seated in 6! ways. There are five in between positions among girls, which are to be filled in with boy (or boys) so that no two girls are together.

Total way of making 5 boys occupies those positions $= {}^6P_5 = \dfrac{6!}{1!} = 6!$

Now, remaining one boy can be made seated has in all 12 positions as choices to be seated.

Hence, total possible ways $= 6! \times 6! \times 12$

(c) Boys occupy extreme positions.
Two boys will be seated at two extreme positions in $P(6, 2)$ ways. The remaining ten people will be seated in remaining ten positions in the 10! ways.
Hence, total possible ways = ${}^6P_2 \times 10!$

6.17 How many ways can the letters in word MISSISSIPPI can be arranged? What if P's are to be separated?

Total ways to arrange letters in the word MISSISSPPI = $\dfrac{11!}{4!4!2!}$

And if P's are to be separated, total ways to arrange =
(Total ways to arrange letters in the word MISSISSPPI) – (all arrangements in which P's are kept together) = $\dfrac{11!}{4!4!2!} - \dfrac{10!}{4!4!}$

Hence total ways with no two P's together = $\dfrac{11!}{4!4!2!} - \dfrac{10!}{4!4!}$

6.18 In how many ways can letters a, b, c, d, m and n be arranged if

(a) m and n always appear together = $5! \times 2$
Here, in 5! ways the letters a, b, c, d and mn can be arranged. Here we consider m and n as a single entity. And the pair m and n can be arranged in 2 ways.

(b) Total number of ways such that m is always immediate left of $n = 5!$

6.19 If repetitions are not permitted, how many four digit numbers can be formed from digits 1, 2, 3, 7, 8 and 5.

(a) Total four digit numbers = ${}^6P_4 = \dfrac{6!}{2!} = 360$

(b) Total number less than 5000
For number to be less than 5000, first position can be occupied by either of 1, 2 or 3,

$$\text{Hence four digits number less 5000 } = 3 \times {}^5P_3 = 3 \times \frac{5!}{2!}$$
$$= 3 \times 5 \times 4 \times 3$$
$$= 180$$

(c) Total four digit even numbers formed from digits 1, 2, 3, 7, 8 and 5 are
$$= 2\times {}^5P_3 = 2 \times 5 \times 4 \times 3$$
$$= 120$$

(d) Total odd four digit numbers formed from these digits = $4 \times {}^5P_3 = 240$

(e) Total number of numbers that contain both the digits 3 and 5.
Let us compute this using inclusion principle,
Let A denote set of numbers excluding 3.
Let B denote set of numbers excluding 5.
Let $A \cap B$ denote set of numbers excluding both 3 and 5

Total four digit numbers formed from digits 1, 2, 3, 5, 7 and 8 containing both 3 and 5 = (Total four digit numbers) – (Total four digit numbers excluding 3 or 5 or both)

$$= 360 - [|A \cup B|]$$
$$= 360 - [|A| + |B| - |A \cap B|]$$
$$= 360 - [{}^5P_4 + {}^5P_4 - {}^4P_4]$$
$$= 360 - [5! + 5! - 4!]$$
$$= 360 - [240 - 24]$$
$$= 360 - 216$$
$$= 144$$

6.20 A word that reads the same when read in forward or backward is called as palindrome. How many seven-letter palindromes can be formed from English alphabets?

The total number of seven letter palindromes is to be computed. As the word is palindrome of length seven last, that is, seventh letter is same as first, sixth letter is same as second and fifth letter should be same as third. Hence only all possible ways to fill up first four positions is to be computed.

Each of the first four (or last four) has 26 choices, hence total seven-letter palindromes $= 26 \times 26 \times 26 \times 26 = 26^4$.

COMBINATIONS

6.21 In how many ways can a committee consisting of three men and two women be chosen from seven men and five women?

The three men can be chosen from the seven men in $\binom{7}{3}$ ways, and the two women can be chosen from the five women in $\binom{5}{2}$ ways. Hence the committee can be chosen in

$$\binom{7}{3}\binom{5}{2} = \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}\cdot\frac{5\cdot 4}{1\cdot 2} = 350 \text{ ways}$$

6.22 A bag contains six white marbles and five red marbles. Find the number of ways four marbles can be drawn from the bag if (a) they can be any color; (b) two must be white and two red; (c) they must all be of the same color.

(a) The four marbles (of any color) can be chosen from the eleven marbles in $\binom{11}{4} = \frac{11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4} = 330$ ways.

(b) The two white marbles can be chosen in $\binom{6}{2}$ ways, and the two red marbles can be chosen in $\binom{5}{2}$ ways. Thus there are $\binom{6}{2}\binom{5}{2} = \frac{6\cdot 5}{1\cdot 2}\cdot\frac{5\cdot 4}{1\cdot 2} = 150$ ways of drawing two white marbles and two red marbles.

(c) There are $\binom{6}{4} = 15$ ways of drawing four white marbles, and $\binom{5}{4} = 5$ ways of drawing four red marbles. Thus there are $15 + 5 = 20$ ways of drawing four marbles of the same color.

6.23 How many committees of five with a given chairperson can be selected from 12 persons?
The chairperson can be chosen in 12 ways and, following this, the other four on the committee can be chosen from the eleven remaining in $\binom{11}{4}$ ways. Thus there are $12\cdot\binom{11}{4} = 12\cdot 330 = 3960$ such committees.

6.24 Out of 12 employees, a group of four trainees is to be sent for 'Software testing and QA' training of one month.

(a) In how many ways can the four employees be selected?

Selecting 4 persons of $12 = {}^{12}C_4 = \dfrac{12!}{4!\times 8!}$

(b) What if there are two employees who refuse to go together for training?
Let A and B be the two employees who refuse to go together.
Total possible ways to select include
- (i) Both A and B do not go $= {}^{10}C_4$
- (ii) A is selected, hence B refuses $= {}^{10}C_3$
- (iii) B is selected, hence A refuses $= {}^{10}C_3$

$\therefore$ Total ways of selection with this constraint $= {}^{10}C_4 + {}^{10}C_3 + {}^{10}C_3$

(c) There are two employees who want to go together that is either they both go or both do not go for training.
Let C and D be the two employees who want to go together that is either they both go or both do not go for training $= {}^{10}C_4 + {}^{10}C_2$

(d) There are two employees who want to go together and there are two employees who refuse to go together.
Let A and B are two employees who refuse to go together and C and D are two who want to go together. Now let us consider the following cases
- (i) A and B both do not go and C and D both go $= {}^{8}C_2$.
- (ii) A and B and also C and D do not go $= {}^{8}C_4$.

(iii) C and D both and either of A or B go for training $= 2 \times {}^8C_1$

(iv) C and D both do not go for training and either of A or B go for training $= 2 \times {}^8C_3$

$\therefore$ Total ways of selecting 4 of 12 with given constraints $= {}^8C_4 + {}^8C_2 + 2 \times {}^8C_1 + 2 \times {}^8C_3$

25. There are 50 students in each of the senior and junior classes. Each class has 25 male and 25 female students. In how many ways can an eight students committee be formed so that there are four females and three juniors in the committee?

To find of number of ways of forming such a committee, we have to consider all cases.

Juniors		Seniors	
Female	Male	Female	Male
3	0	1	4
2	1	2	3
1	2	3	2
0	3	4	1

From above table, we can easily compute total possible ways of selecting eight students committee which includes 4 female and three juniors.

Total ways of selection

$$= 2 \times ({}^{25}C_3 \times {}^{25}C_1 \times {}^{25}C_4) + 2 \times ({}^{25}C_3 \times {}^{25}C_1 \times {}^{25}C_2 \times {}^{25}C_2)$$

These selections confirm that there are exactly four female and three juniors.

$\therefore$ Total ways of forming a committee of eight students from 100 students where 50 of them are from junior class and 50 from senior class where each class has 25 male and 25 female students such that committee includes 3 juniors and 4 females $= {}^{93}C_1 \times [2 \times ({}^{25}C_3 \times {}^{25}C_1 \times {}^{25}C_4) + 2 \times ({}^{25}C_2 \times {}^{25}C_1 \times {}^{25}C_2 \times {}^{25}C_3)]$

26. Show that $C(2n, 2) = 2C(n, 2) + n^2$

Let
$$\text{L.H.S.} = C(2n, 2)$$
$$= \frac{2n!}{(2n-2)!2!}$$
$$= \frac{2n \times (2n-1) \times (2n-2)!}{(2n-2)!2!}$$
$$= n(2n-1)$$
$$= 2n^2 - n$$
$$= n^2 + n^2 - n$$
$$= n^2 + n(n-1)$$

Let us multiply and divide second term by $2(n-2)!$

$$= n^2 + \frac{n(n-1)2(n-2)!}{2(n-2)!}$$

$$= n^2 + \frac{2n!}{2(n-2)!}$$
$$= n^2 + \frac{2 \times n!}{2(n-2)!}$$
$$= n^2 + 2\,C(n, 2)$$
$$= 2\,C(n, 2) + n^2 = \text{R.H.S. Hence proved.}$$

ORDERED AND UNORDERED PARTITIONS

6.27 *In how many ways can nine students be partitioned into three teams containing four, three, and two students, respectively?*

Since all the cells contain different numbers of students, the number of unordered partitions equals the number of ordered partitions, $\frac{9!}{4!3!2!} = 1260$.

6.28 *There are 12 students in a class. In how many ways can the 12 students take four different tests if three students are to take each test?*

Method 1: We seek the number of ordered partitions of the 12 students into cells containing three students each. By Theorem 6.8, there are $\frac{12!}{3!3!3!3!} = 369\,600$ such partitions.

Method 2: There are $\binom{12}{3}$ ways to choose three students to take the first test; following this, there are $\binom{9}{3}$ ways to choose three students to take the second test, and $\binom{6}{3}$ ways to choose three students to take the third test. The remaining students take the fourth test. Thus, altogether, there are $\binom{12}{3}\binom{9}{3}\binom{6}{3} = (220)\,(84)\,(20) = 369\,600$ ways for the students to take the tests.

6.29 In how many ways can 12 students be partitioned into four teams, A_1, A_2, A_3, and A_4, so that each team contains three students?

Method 1: Observe that each partition $\{A_1, A_2, A_3, A_4\}$ of the students can be arranged in $4! = 24$ ways as an ordered partition. Since (see the preceding problem) there are $\frac{12!}{3!3!3!3!} = 369\,600$ such ordered partitions, there are $369\,600/24 = 15\,400$ (unordered) partitions.

Method 2: Let A denote one of the students. Then there are $\binom{11}{2}$ ways to choose two other students to be on the same team as A. Now let B denote a student who is not on the same team as A. Then there are $\binom{8}{2}$ ways to choose two students from the remaining students to be on the same team as B. Next let C denote a student who is not on the same team as A or B. Then there are $\binom{5}{2}$ ways to choose two students to be on the same team as C. The remaining three students constitute the fourth team. Thus, altogether, there are $\binom{11}{2}\binom{8}{2}\binom{5}{2} = (55)\ (28)\ (10) = 15\ 400$ ways to partition the students.

THE PIGEONHOLE PRINCIPLE

6.30 Assume there are n distinct pairs of shoes in a closet. Show that if you choose $n + 1$ single shoes at random from the closet, you are certain to have a pair.

The n distinct pairs constitute n pigeonholes. The $n + 1$ single shoes correspond to $n + 1$ pigeons. Therefore, there must be at least one pigeonhole with two shoes and thus you will certainly have drawn at least one pair of shoes.

6.31 Assume there are three men and five women at a party. Show that if these people are lined up in a row, at least two women will be next to each other.

Consider the case where the men are placed so that no two men are next to each other and not at either end of the line. In this case, the three men generate four potential locations (pigeonholes) in which to place women (at either end of the line and two locations between men within the line). Since there are five women (pigeons), at least one slot will contain two women who must, therefore, be next to each other. If the men are allowed to be placed next to each other or at the end of the line, there are even fewer pigeonholes and, once again, at least two women will have to be placed next to each other.

6.32 Find the minimum number of students needed to guarantee that five of them belong to the same class (Freshman, Sophomore, Junior, Senior).

Here the $n = 4$ classes are the pigeonholes and $k + 1 = 5$ so $k = 4$. Thus among any $kn + 1 = 17$ students (pigeons), five of them belong to the same class.

6.33 A student must take five classes from three areas of study. Numerous classes are offered in each discipline, but the student cannot take more than two classes in any given area.

(a) Using the pigeonhole principle, show that the student will take at least two classes in one area.

(b) Using the inclusion-exclusion principle, show that the student will have to take at least one class in each area.

(a) The three areas are the pigeonholes and the student must take five classes (pigeons). Hence, the student must take at least two classes in one area.

(b) Let each of the three areas of study represent three disjoint sets, A, B, and C. Since the sets are disjoint, $m(A \cup B \cup C) = 5 = n(A) + n(B) + n(C)$. Since the student can take at most two classes in any area of study, the sum of classes in any two sets, say A and B, must be less than or equal to four. Hence, $5 - [n(A) + n(B)] = n(C) \geq 1$. Thus, the student must take at least one class in any area.

6.34 Let L be a list (not necessarily in alphabetical order) of the 26 letters in the English alphabet (which consists of 5 vowels, A, E, I, O, U, and 21 consonants). (a) Show that L has a sublist consisting of four or more consecutive consonants. (b) Assuming L begins with a vowel, say A, show that L has a sublist consisting of five or more consecutive consonants.

(a) The five letters partition L into $n = 6$ sublists (pigeonholes) of consecutive consonants. Here $k + 1 = 4$ and so $k = 3$. Hence $nk + 1 = 6(3) + 1 = 19 < 21$. Hence some sublist has at least four consecutive consonants.

(b) Since L begins with a vowel, the remainder of the vowels partition L into $n = 5$ sublists. Here $k + 1 = 5$ and so $k = 4$. Hence $kn + 1 = 21$. Thus some sublist has at least five consecutive consonants.

6.35 Find the minimum number n of integers to be selected from $S = \{1, 2, ..., 9\}$ so that: (a) the sum of two of the n integers is even; (b) the difference of two of the n integers is 5.

(a) The sum of two even integers or of two odd integers is even. Consider the subsets $\{1, 3, 5, 7, 9\}$ and $\{2, 4, 6, 8\}$ of S as pigeonholes. Hence $n = 3$.

(b) Consider the five subsets $\{1, 6\}$, $\{2, 7\}$, $\{3, 8\}$, $\{4, 9\}$, $\{5\}$ of S as pigeonholes. Then $n = 6$ will guarantee that two integers will belong to one of the subsets and their difference will be 5.

THE INCLUSION-EXCLUSION PRINCIPLE

6.36 There are 22 female students and 18 male students in a classroom. How many students are there in total?

The sets of male and female students are disjoint; hence the total $t = 22 + 18 = 40$ students.

6.37 Of 32 people who save paper or bottles (or both) for recycling, 30 save paper and 14 save bottles. Find the number m of people who (a) save both, (b) save only paper, and (c) save only bottles.

Let P and B denote the sets of people saving paper and bottles, respectively. By Theorem 6.7:

(a) $m = n(P \cap B) = n(P) + n(B) - n(P \cup B) = 30 + 14 - 32 = 12$.

(b) $m = n(P \backslash B) = n(P) - n(P \cap B) = 30 - 12 = 18$.

(c) $m = n(B \backslash P) = n(B) - n(P \cap B) = 14 - 12 = 2$.

6.38 Let A, B, C, D denote, respectively, art, biology, chemistry, and drama courses. Find the number N of students in a dormitory given the data:

12 take A,	5 take A and B,	3 take A, B, C,
20 take B,	7 take A and C,	2 take A, B, D,
20 take C,	4 take A and D,	2 take B, C, D,

8 take D, 16 take B and C, 4 take B and D, 3 take C and D, 3 take A, C, D, 2 take all four, 71 take none.

Let T be the number of students who take at least one course. By the inclusion–exclusion principle (Theorem 6.7),

$$T = s_1 - s_2 + s_3 - s_4, \text{ where}$$
$$s_1 = 12 + 20 + 20 + 8 = 60, \quad s_2 = 5 + 7 + 4 + 16 + 4 + 3 = 39$$
$$s_3 = 3 + 2 + 2 + 3 = 10, \quad s_4 = 2$$

Thus $$T = 29, \text{ and } N = 71 + T = 100.$$

6.39 Prove that if A and B are disjoint finite sets, $A \cup B$ is finite and that

$$n(A \cup B) = n(A) + n(B)$$

In counting the elements of $A \cup B$, first count those that are in A. There are $n(A)$ of these. The only other elements of $A \cup B$ are those that are in B but not in A. But since A and B are disjoint, no element of B is in A, so there are $n(B)$ elements that are in B but not in A. Therefore, $n(A \cup B) = n(A) + n(B)$.

6.40 Prove Theorem 6.7 for two sets: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

In counting the elements of $A \cup B$ we count the elements in A and count the elements in B. There are $n(A)$ in A and $n(B)$ in B. However, the elements $A \cap B$ were counted twice. Thus

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

as required. Alternatively, we have the disjoint unions

$$A \cup B = A \cup (B \backslash A) \text{ and } B = (A \cap B) \cup (B \backslash A)$$

Therefore, according to the previous problem

$$n(A \cup B) = n(A) + n(B \backslash A) \text{ and } n(B) = n(A \cap B) + n(B \backslash A)$$

Thus $$n(B \backslash A) = n(B) - n(A \cap B) \text{ and hence}$$
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

as required.

6.41 A student wants to make up a schedule for seven-day span during which he/she will study one subject each day. He/she is taking four subjects mathematics, physics, chemistry and economics. Among 4^7 schedules, how many schedules devote at least one day to each subject.

Total schedules with required constraint can be computed as = (Total schedules) – (Schedules in which one or more of the subjects are excluded).

Let us compute no. of schedules in which one or more of the subjects are excluded.

Let A, B, C and D denote sets in which maths, physics, chemistry and economics are excluded respectively.

Now $|A| = |B| = |C| = |D| = 3^7$

Also, $|A \cap B| = |A \cap C| = |A \cap D| \cdots = |C \cap D| = 2^7$

And $|A \cap B \cap C| = |A \cap B \cap D| = |B \cap C \cap D| = |A \cap C \cap D| = 1^7$

And $|A \cap B \cap C \cap D| = 0$

Using inclusion–exclusion principle,

$\therefore$ $|A \cup B \cup C \cup D|$ = Number of schedules excluding one or more subjects.

$= |A| + |B| + |C| + |D| - (|A \cap B| + |A \cap C| + |A \cap D| + \cdots + |C \cap D|) + |A \cap B \cap C| + |A \cap B \cap D| + |B \cap C \cap D| + |A \cap C \cap D| - |A \cap B \cap C \cap D|$.

$= 4(3^7) - 6(2^7) + 4(1)^7 - 0 = 4(3^7) - 6(2^7) + 4$

Hence among 4^7 schedules, schedules that devote at least one day to each subject

$$= 4^7 - [4(3^7) - 6(2^7) + 4(1)^7 - 0 = 4(3^7) - 6(2^7) + 4\ ]$$

Miscellaneous Problems

6.42 A sample of 80 car owners revealed that 24 owned station wagons and 62 owned cars which are not station wagons. Find the number k of people who owned both a station wagon and some other car.

6.43 Suppose 12 people read the *Wall Street Journal* (W) or *Business Week* (B) or both. Given three people read only the *Journal* and six read both, find the number k of people who read only *Business Week*.

6.44 Show that any set of seven distinct integers includes to integers, x and y, such that either $x + y$ or $x - y$ is divisible by 10.

6.45 Consider a tournament in which each of n players plays against every other player and each player wins at least once. Show that there are at least two players having the same number of wins.

Objective Type Problems

6.1 In a room containing 28 people, there are 18 people who speak English, 15 people who speak Hindi and 22 people who speak Kannada whereas 13 persons speak both Kannada and English. How many people speak all the three languages?

(a) 6
(b) 7
(c) 8
(d) 9

6.2 In a group of 72 students, 47 have background in electronics, 59 have background in mathematics and 42 have background in both the subjects. How many students do not have background in any of the subjects?

(a) 8
(b) 13
(c) 25
(d) 34

6.3 In a beauty contest, half the number of experts voted for Mr. A and two thirds voted for Mr. B. 10 voted for both and 6 did not vote for either. How many experts were there in all?

(a) 18 (b) 36
(c) 24 (d) None of these

6.4 The number of different permutations of the word BANANA is

(a) 720 (b) 60
(c) 120 (d) 360

6.5 The number of ways in which a team of eleven players can be selected from 22 players including 2 of them and excluding 4 of them is

(a) ${}^{16}C_{11}$ (b) ${}^{16}C_5$
(c) ${}^{16}C_9$ (d) ${}^{20}C_9$

6.6 Ramesh has 6 friends. In how many ways can he invite one or more of them at a dinner?

(a) 61 (b) 62
(c) 63 (d) 64

6.7 If ${}^nC_{12} = {}^nC_8$, then n is equal to

(a) 20 (b) 12
(c) 6 (d) 30

6.8 The sides AB, BC, CA of a triangle ABC have 3, 4 and 5 interior points respectively on them. The total number of triangles that can be constructed by using these points as vertices is

(a) 220 (b) 204
(c) 205 (d) 195

6.9 Three persons enter a railway compartment. If there are 5 seats vacant, in how many ways can they take these seats?

(a) 60 (b) 20
(c) 15 (d) 125

6.10 There are three identical red balls and four identical blue balls in a bag. Three balls are drawn. The number of different color combinations is

(a) 20 (b) 35
(c) 40 (d) 30

SUPPLEMENTARY PROBLEMS

FACTORIAL NOTATION

6.1 Simplify: (a) $\dfrac{(n+1)!}{n!}$; (b) $\dfrac{n!}{(n-2)!}$; (c) $\dfrac{(n-1)!}{(n+2)!}$; (d) $\dfrac{(n-r+1)!}{(n-r-1)!}$.

6.12 A woman has 11 close friends of whom six are also women.
(a) In how many ways can she invite three or more to a party?
(b) In how many ways can she invite three or more of them if she wants the same number of men as women (including herself)?

6.13 A student is to answer 10 out of 13 questions on an exam.
(a) How many choices has he?
(b) How many if he must answer the first two questions?
(c) How many if he must answer the first or second question but not both?
(d) How many if he must answer exactly three out of the first five questions?
(e) How many if he must answer at least three of the first five questions?

PARTITIONS

6.14 In how many ways can 10 students be divided into three teams, one containing four students and the others three?

6.15 In how many ways can 14 people be partitioned into six committees where two of the committees contain three members and the others two?

6.16 (a) Assuming that a cell can be empty, in how many ways can a set with three elements be partitioned into (i) three ordered cells, (ii) three unordered cells?
(b) In how many ways can a set with four elements be partitioned into (i) three ordered cells, (ii) three unordered cells?

ANSWERS TO OBJECTIVE TYPE PROBLEMS

6.1 (a) 6.2 (a) 6.3 (c) 6.4 (b) 6.5 (c)
6.6 (c) 6.7 (a) 6.8 (c) 6.9 (a) 6.10 (d)

ANSWERS TO SUPPLEMENTARY PROBLEMS

6.1 (a) $n + 1$; (b) $n(n - 1) = n^2 - n$; (c) $1/[n(n + 1)(n + 2)]$; (d) $(n - r)(n - r + 1)$.

6.2 (a) 10; (b) 35; (c) 91; (d) 15; (e) 1140; (f) 816.

6.3 *Hints*: (a) Expand $(1 + 1)^n$; (b) Expand $(1 - 1)^n$.

6.4 (a) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 = 468\,000$; (b) $26 \cdot 25 \cdot 9 \cdot 9 \cdot 8 = 421\,200$.

6.5 (a) 24; (b) 576; (e) 360.

6.6 360.

6.7 (a) 120; (b) 48; (c) 24; (d) 12.

6.8 $3!\,5!\,4!\,3! = 103\,680$.

6.9 (a) 120; (b) 24; (c) 24; (d) 12.

6.10 (a) 120; (b) 24; (c) 48.

6.11 (a) 462; (b) 210; (c) 252.

6.12 (a) $2^{11} - 1 - \binom{11}{2} - \binom{11}{2} = 1981$ or $\binom{11}{3} + \binom{11}{4} + \cdots + \binom{11}{11} = 1981$.

(b) $\binom{5}{5}\binom{6}{4} + \binom{5}{4}\binom{6}{3} + \binom{5}{3}\binom{6}{2} + \binom{5}{2}\binom{6}{1} = 325.$

6.13 (a) 286; (b) 165; (c) 110; (d) 80;. (e) 276.

6.14 $\frac{10!}{4!3!3!} \cdot \frac{1}{2!} = 2100$ or $\binom{10}{4}\binom{5}{2} = 2100.$

6.15 $\frac{14!}{3!3!2!2!2!2!} \cdot \frac{1}{2!4!} = 3153\ 150.$

6.16 (a) (i) $3^3 = 27$ (each element can be placed in any of the three cells).

(ii) The number of elements in the three cells can be distributed as follows:

(a) [{3}, {0}, {0}]; (b) [{2}, {1}, {0}]; (c) [{1}, {1}, {1}].

Thus, the number of partitions is $1 + 3 + 1 = 5$.

(b) (i) $3^4 = 81$.

(ii) The number of elements in the three cells can be distributed as follows:

(a) [{4}, {0}, {0}]; (b) [{3}, {1}, {0}]; (c) [{2}, {2}, {0}]; (d) [{2}, {1}, {1}]

Thus, the number of partitions is $1 + 4 + 3 + 6 = 14$

6.17 By Theorem 6.7, $k = 62 + 24 - 80 = 6$.

6.18 Note $W \cup B = (W \backslash B) \cup (W \cap B) \cup (B \backslash W)$ and the union is disjoint. Thus $12 = 3 + 6 + $ ... or $k = 3$.

6.19 Let $X = \{x_1, x_2, ..., x_7\}$ be a set of seven distinct integers and let r_i be the remainder when x_i is divided by 10. Consider the following partition of X:

$H_1 = \{x_i: r_i = 0\}$ $\quad H_2 = \{x_i: r_i = 5\}$

$H_3 = \{x_i: r_i = 1 \text{ or } 9\}$ $\quad H_4 = \{x_i: r_i = 2 \text{ or } 8\}$

$H_5 = \{x_i: r_i = 3 \text{ or } 7\}$ $\quad H_6 = \{x_i: r_i = 4 \text{ or } 6\}$

There are six pigeonholes for seven pigeons. If x and y are in H_1, or in H_2, then both $x + y$ and $x - y$ are divisible by 10. If x and y are in one of the other four subsets, then either $x - y$ or $x + y$ is divisible by 10, but not both.

6.20 The number of wins for a player is at least 1 and at most $n - 1$. These $n - 1$ numbers correspond to $n - 1$ pigeonholes which cannot accommodate n player-pigeons. Thus, at least two players will have the same number of wins.

6.2 Evaluate: (a) $\binom{5}{2}$; (b) $\binom{7}{3}$; (c) $\binom{14}{2}$; (d) $\binom{6}{4}$; (e) $\binom{20}{17}$; (f) $\binom{18}{15}$.

PERMUTATIONS

6.3 Show that: (a) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}=2^n$;

(b) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}=0$.

6.4 (a) How many automobile license plates can be made if each plate contains two differen followed by three different digits?

(b) Solve the problem if the first digit cannot be 0.

6.5 There are six roads between A and B and four roads between B and C. Find the numb ways that one can drive: (a) from A to C by way of B; (b) round-trip from A to C by way (c) round-trip from A to C by way of B without using the same road more than once.

6.6 Find the number of ways in which six people can ride a toboggan if one of a subset of three drive.

6.7 (a) Find the number of ways in which five persons can sit in a row.

(b) How many ways are there if two of the persons insist on sitting next to one another?

(c) Solve part (a) assuming they sit around a circular table.

(d) Solve part (b) assuming they sit around a circular table.

6.8 Find the number of ways in which five large books, four medium-size books, and three books can be placed on a shelf so that all books of the same size are together.

6.9 (a) Find the number of permutations that can be formed from the letters of the word ELE

(b) How many of them begin and end with E?

(c) How many of them have the three Es together?

(d) How many begin with E and end with N?

6.10 (a) In how many ways can three boys and two girls sit in a row?

(b) In how many ways can they sit in a row if the boys and girls are each to sit together?

(c) In how many ways can they sit in a row if just the girls are to sit together?

COMBINATIONS

6.11 A woman has 11 close friends.

(a) In How many ways can she invite five of them to dinner?

(b) In how many ways if two of the friends are married and will not attend separately?

(c) In how many ways if two of them are not on speaking terms and will not attend toge