Language Problem!

eg: consider the following statements:

i) If my checkbook is on my office table, then I paid my phone bill.

ii) I was looking at the phone bill for payment at breakfast or I was looking at the phone bill for payment in my office.

iii) If I was looking at the phone bill at breakfast, then the checkbook is on breakfast table.

iv) I did not pay my phone bill.

v) If I was looking at the phone bill in my office, then the check-book is on my office table.

Ques: where was my checkbook?

→ let

p: my checkbook is on my office table.
q: I paid my phone bill.
r: I was looking at the phone Bill for payment at breakfast.
s: I was looking at the phone bill for payment in my office.
t: the checkbook is on the breakfast table.
s: I was looking at the phone bill in my office.

Hence, in a symbolic notation, the given argument takes the form

$p \to q$

$r \vee s$

$r \to t$

$\sim q$

$s \to p$.

we now consider the following argument.

B_1 : $s \rightarrow p$ hypothesis
B_2 : $p \rightarrow q$ hypothesis
B_3 : $s \rightarrow q$ from B_1, B_2 & by hypothetical syllogism
B_4 : $\sim q$ hypothesis
B_5 : $\sim s$ from B_3, B_4 & by modus tollens
B_6 : $r \vee s$ hypothesis
B_7 : r from B_5, B_6 and by disjunctive syllogism
B_8 : $r \rightarrow t$
B_9 : t ~~fro~~ from B_7, B_8 & by modus ponens

<u>conclusion</u>: The checkbook was on the breakfast table.

⇒ <u>Proof Techniques</u>:-

Any theorem that a problem is a statement that can be shown to be true.

<u>eg</u>. If x is an integer, and x is odd, then x^2 is odd

or equivalently,

For all integers, x, if x is odd, then x^2 is odd.

This statement can be shown to be true.

~~Now~~.

As facts.

eg. 6 is an even integer (or)

$x^2+1=0$ has no sol^{n}s in real No.'s.

As implication

eg. for eg, for all integers x, if x is even, then $x+1$ is odd.

As biimplication.

eg. for all integers x, x is even if and only if x is divisible by 2.

⇒ Here, A proof may consist of previously known facts, proved results (or) previous statements of the proof.

⇒ There are several known techniques for constructing a proof.

1) Direct Proof.
2) Indirect Proof
3) Proof by contradiction
4) Proving Biimplication
5) Proving Equivalent statement.
6) Errors in the proof.

→ 1) Direct Proof:-

It includes proof of those theorems that can be expressed in the form:

$\forall x(P(x) \to Q(x))$, D is the domain of discourse

Premises — Conclusion

eg. For all Integers x, if x is odd, then x^2 is odd.

→ let us verify that the theorem is true for certain values of x.

If $x = 3 \Rightarrow x^2 = 9$ so x^2 is odd

$x = -5 \Rightarrow x^2 = 25$ " " " "

It means the theorem is true.

→ but we must prove that the theorem is true for an arbitrary value of the domain of discourse.

let $P(x)$ = " x is an odd integer"

$Q(x)$ = " x^2 is an odd integer "

Then symbolically,

$\forall x (P(x) \rightarrow Q(x))$, the domain of discourse is the set of all integers $\mathbb{Z}$

here we will start the proof by assuming 'a' is a particular but arbitrarily chosen element of $\mathbb{Z}$.

for 'a', we assume that $P(a)$ is true, then we show that $Q(a)$ is true.

सीधा Proof

$\Rightarrow$ let 'a' be an integer, such that 'a' is odd.

so we can write, ~~a = 2m + n f~~

$a = 2n+1$ for some integer 'n'

$\Rightarrow a^2 = (2n+1)^2$

$= 4n^2 + 4n + 1$

$= 2(2n^2 + 2n) + 1$

let $m = 2n^2 + 2n$ because n = integer

$\Rightarrow m\ (= 2n^2 + 2n)$ is also integer

$\therefore$ we can write,

$a^2 = 2m+1$, for some integer m

$\Rightarrow a^2 = $ odd

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Proof: let 'a' be an odd integer then

a is an odd integer.

$\Rightarrow a = 2n+1$, for some integer n

$\Rightarrow a^2 = (2n+1)^2$

$\Rightarrow a^2 = 4n^2 + 4n + 1$

$\Rightarrow a^2 = 2(2n^2 + 2n) + 1$

$\Rightarrow a^2 = 2m + 1$, $m = 2n^2 + 2n$ is an integer

$\Rightarrow a^2$ is an odd integer.

$\Rightarrow$ for all integer x, if x is odd, then x^2 is odd.

2) Indirect proof:

→ Consider the implication $p \to q$. This implication is equivalent to the implication $\sim q \to \sim p$. This means that in order to show that $p \to q$ is true, we can also show that the implication $\sim q \to \sim p$ is true.

→ to show $\sim q \to \sim p$ is true, we have to show $\sim q$ is true. & prove that $\sim p$ is true.

→ This type of proof is called Indirect proof.

Ex Prove that n is an integer, n^2+3 is odd then n is even

→ let $P(n) = n^2+3$ is an odd integer
$Q(n) = n$ is an even integer

$\forall n\ (P(n) \to Q(n))$ the domain of discourse is the set Z of all integers

Main proof → assume that n is a particular but arbitrarily chosen element of Z.

for this n, will show that $P(n) \to Q(n)$ is true.

→ it is logically equivalent to $\sim q(n) \to \sim P(n)$.
so we have to show that it is true.

→ suppose $\sim Q(n)$ is true
we have to show $\sim P(n)$ is true.

→ Because $\sim Q(n)$ is true, 'n' is not even
⇒ n is odd
So $n = 2k+1$ for some integer 'k' Thus

$$n^2+3 = (2k+1)^2+3 \quad , \text{put } n = 2k+1$$
$$= 4k^2+4k+1+3$$
$$= 4k^2+4k+4$$
$$= 2(2k^2+2k+2)$$

let us write $t = 2k^2+2k+2$
$\Rightarrow n^2+3 = 2t$, for some int t
→ or it's an

⇒ n^2+3 is an even integer
⇒ n^2+3 is not an odd ".

⇒ $\sim P(n)$ is true.

⇒ $\sim Q(n) \to \sim P(n)$ is true.
⇒ $P(n) \to Q(n)$

∴ $\forall n (P(n) \to Q(n))$, in the domain $\mathbb{Z}$ of all integers

→ by the rule universal generalization
if n^2+3 is odd
⇒ n is even

3) Proof by Contradiction:

→ In this, we assume that the conclusion is not true and then arrive at a contradiction.

eg: Show that $\sqrt{2}$ is an irration numbers.

→ let us assume that $\sqrt{2}$ is not an Irration numbers.

$\Rightarrow \sqrt{2}$ is rational numbers,

$$\Rightarrow \sqrt{2} = \frac{a}{b}, \Rightarrow a \,\&\, b = \text{integers}, \; b \neq 0.$$

here we assume that $\frac{a}{b}$ = lowest term

$\Rightarrow$ a & b have no common factor other than one

$$\therefore \sqrt{2} = \frac{a}{b}$$

$$\Rightarrow (\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$$

$$\Rightarrow 2 = \frac{a^2}{b^2}$$

$$\Rightarrow a^2 = 2b^2$$

$\Rightarrow a^2$ is an integer

$\Rightarrow a$ is an integer (by 1st eg.)

we can write $a = 2n$, for some integer 'n'

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$\Rightarrow a^2 = 4n^2$

Now substitute the value of a^2 into $a^2 = 2b^2$ to obtain

implies $2b^2 = a^2 = 4n^2$

$\Rightarrow b^2 = 2n^2$

$\Rightarrow b^2$ = even

$\Rightarrow b$ = even

$\Rightarrow$ a & b both are even.

$\Rightarrow$ both have 2 as a common factor

This contradicts our assumption that a & b have no common factors other than 1.

→ We now arrived at a contradiction

$\Rightarrow \sqrt{2}$ is an irrational number.

4) ⟹ <u>Proving</u> <u>Biimplications</u>: $\forall x\,(P(x) \leftrightarrow Q(x))$ in the domain ;

ex: consider the following theorem

An integer x is even, if and only if $x+1$ is odd.

→ $P(x)$: x is even

$Q(x)$: $x+1$ is odd.

So for all integers x, $P(x)$ iff $Q(x)$.

or

$\forall x\,(P(x) \leftrightarrow Q(x))$; in the domain Z.

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Proof: Assume that 'x' is a particular but arbitrary integer such that x is even and show that $x+1$ is odd.

→ Then assume that $x+1$ is odd and prove that x is even.

→ 1st assume that x is even

$\Rightarrow x = 2n$, for some integer n.

$\Rightarrow x+1 = 2n+1$

$\Rightarrow x+1$ is an odd integer.

→ Let us now suppose that $x+1$ is odd

$\Rightarrow x+1 = 2m+1$ for some integer m

$\Rightarrow x = 2m$

$\Rightarrow x$ is an even integer

⇒ we conclude that an integer x is even iff $x+1$ is odd.

5) Proving Equivalent statement:

→ consider the following statements:

let x be an integer.

Q1. p : x is divisible by 6.

q : x is divisible by 2 & 3.

r : x is an even number & x is divisible by 3.

we can prove that, $p \leftrightarrow q$ } In words

$p \leftrightarrow r$ } p, q & r are

→ $q \leftrightarrow r$ } equivalent statement

We have cycle $p \to q \to r \to p$

let

i) p: x is divisible by 6.
ii) q: x is divisible by 2 & 3.
iii) s: x is an even number & x is divisible by 3.

Proof: To prove that these statements are equivalent we show that,

(i) → (ii), (ii) → (iii) & (iii) → (i)

or

(i) → (ii) → (iii) → (i)

1) (i) → (ii)

Suppose x is divisible by 6
Then $x = 6n$, for some integer n.
Then $x = 6n = 2 \cdot 3n = 3 \cdot 2n$
it follows x is divisible by 2 & 3.

2) (ii) → (iii)

Suppose x is divisible by 2 & 3.
because x is divisible by 2
→ x is an even integer
Hence, x is an even integer.
and x is divisible by 3.

3) (iii) → (i)

suppose that x is an even no. & x is divisible
because x is an even integer.
$x = 2n$, for some integer n.
→ 3 is divides $2n$
thus $2n = 3t$ for some integer t

Now,

$$n = 3n - 2n$$

This implies that

$$n = 3n - 3t$$

that is,

$$n = 3(n - t)$$

let $n - t = s \Rightarrow n = 3s$, s is some integer

It follows that,

$$x = 2n = 2 \cdot 3s = 6s$$, for some integer s.

$\Rightarrow$ x is divisible by 6.

5) Errors (Fallacies) in the proofs:

- As we know, proof may consist of previously known facts, proves results or previous statements of the proof.
- However, if we are not careful, errors can occurs in the proofs.

eg.

$$
\begin{aligned}
1 &= \sqrt{1} \\
&= \sqrt{(-1)\cdot(-1)} \\
&= \sqrt{(-1)} \cdot \sqrt{(-1)} \\
&= (\sqrt{-1})^2 \\
&= -1
\end{aligned}
$$

but we know that $1 = -1$, is not true. so here, there is some error. & error is in the equality $\sqrt{(-1)\cdot(-1)} = \sqrt{-1} \cdot \sqrt{-1}$.

→ here we are using $\sqrt{ab} = \sqrt{a}\sqrt{b}$ for all real no. a & b however it is true when a & b are non-negative real numbers.