# Homework Assignment 2

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**Note:** Homework was discussed with Varun Miranda and Bhushan Malgaonkar

**Ans 1.**

$$f(x/\alpha) = \alpha^x (1-\alpha)^{1-x}$$

Given $\alpha$ is selected from a uniform distribution(0,1)

$$\text{Therefore } f(\alpha) = \frac{1}{1-0} = 1$$

$$argmax_{\alpha}(f(\alpha)f(x/\alpha)) = 1.\prod_{i=1}^{n} \alpha^x (1-\alpha)^{1-x}$$

Taking log

$$= \sum_{i=1}^{n} (x\log(\alpha) + (1-x)\log(1-\alpha))$$

Taking derivative w.r.t $\alpha$

$$0 = \frac{1}{\alpha}\sum_{i=1}^{n} x - \frac{1}{1-\alpha}\sum_{i=1}^{n}(1-x)$$

$$\frac{1}{\alpha}\sum_{i=1}^{n} x = \frac{1}{1-\alpha}\sum_{i=1}^{n}(1-x)$$

$$\sum_{i=1}^{n} x - \alpha\sum_{i=1}^{n} x = N\alpha - \alpha\sum_{i=1}^{n} x$$

$$\hat{\alpha}_{MAP} = \frac{1}{N}\sum_{i=1}^{n} x = \bar{x}$$

**Ans 2.**

$$argmax_{\theta}\prod_{i=0}^{n}\theta e^{-\theta x}$$
$$\text{Taking log}$$
$$\sum_{i=0}^{n}\log\theta-\theta x$$
$$\text{Taking derivative w.r.t.}\theta$$
$$\sum_{i=0}^{n}(\frac{1}{\theta}-x_i)=0$$
$$\frac{n}{\theta}=\sum_{i=0}^{n}x_i$$
$$\hat{\theta}_{MLE}=\frac{1}{\bar{x}}$$

**Ans 3.** *a).*

$$\text{For H0,}$$
$$f_X(x)=f_N(\frac{x+s}{a})$$
$$\text{For H1,}$$
$$f_X(x)=f_N(\frac{x-s}{b})$$
$$f_N(\frac{x+s}{a})\underset{H1}{\overset{H0}{\gtrless}}f_N(\frac{x-s}{b})$$
$$\text{Since } a=b$$
$$f_N(\frac{x+s}{a})\underset{H1}{\overset{H0}{\gtrless}}f_N(\frac{x-s}{a})$$
$$\frac{1}{\pi(1+(\frac{x+s}{a})^2)}\underset{H1}{\overset{H0}{\gtrless}}\frac{1}{\pi(1+(\frac{x-s}{a})^2)}$$
$$1+(\frac{x-s}{a})^2\underset{H1}{\overset{H0}{\gtrless}}1+(\frac{x+s}{a})^2$$
$$x^2-2xs+s^2\underset{H1}{\overset{H0}{\gtrless}}x^2+2xs+s^2$$
$$0\underset{H1}{\overset{H0}{\gtrless}}4xs$$
$$\text{Since s>0, s is positive}$$
$$0\underset{H1}{\overset{H0}{\gtrless}}x$$

$\square$

*b).*

$$r(\hat{y} = \int_{H0} f(x|y = H1).P(y = H1)dx + \int_{H1} f(x|y = H0).P(y = H0)dx$$

Since priors are equal $P(y = H1) = P(y = H0) = \frac{1}{2}$

$$r(\hat{y} = \frac{1}{2}\int_{H0} f(x|y = H1)dx + \int_{H1} f(x|y = H0)dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} f_N(\frac{x+s}{a}) + \frac{1}{2}\int_{0}^{\infty} f_N(\frac{x-s}{b})$$

$$= \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\pi(1+(\frac{x+s}{a})^2)} + \frac{1}{2}\int_{0}^{\infty} \frac{1}{\pi(1+(\frac{x-s}{b})^2)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} \frac{1}{1+(\frac{x+s}{a})^2} + \frac{1}{2\pi}\int_{0}^{\infty} \frac{1}{1+(\frac{x-s}{b})^2}$$

put $t = \dfrac{x+s}{a}$ and $u = \dfrac{x-s}{b}$

When $x = -\infty$, $t = -\infty$. When $x = 0$, $t = \dfrac{s}{a}$

When $x = 0$, $u = \dfrac{-s}{b}$. When $x = \infty$, $u = \infty$

$$r(\hat{y} = \frac{1}{2\pi}\int_{-\infty}^{s/a} \frac{1}{1+t^2}dt + \frac{1}{2\pi}\int_{-s/b}^{\infty} \frac{1}{1+u^2}du$$

$$= \frac{1}{2\pi}(\tan^{-1}(\frac{s}{a}) + \frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1}(\frac{-s}{b}))$$

$$= \frac{1}{2} + \frac{1}{2\pi}(\tan^{-1}(\frac{s}{a}) - \tan^{-1}(\frac{-s}{b}))$$

$\square$

**Ans 4.** *a).*

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X|Y}(x|y).f_Y(y)}{f_X(x)} \\
f_Y(y) &= \alpha e^{-y\alpha} \\
f_X(x) &= \int_0^\infty f_{X|Y}(x|y).f_Y(y)dy \\
&= \int_0^\infty ye^{-yx}.\alpha e^{-y\alpha}dy \\
&= \alpha \int_0^\infty ye^{-y(x+\alpha)}dy \\
&\text{Using Integration by parts and LIATE Rule} \\
&= \alpha(y\int e^{-y(x+\alpha)})dy - \int(\frac{d(y)}{dy}\int e^{-y(x+\alpha)})dy)dy) \\
&= \alpha[\frac{-ye^{-y(x+\alpha)}}{x+a} - \frac{e^{-y(x+\alpha)}}{(x+\alpha)^2}]_0^\infty \\
&= \alpha[\frac{-e^{-y(x+\alpha)}}{x+a}(y+\frac{1}{x+\alpha})]_0^\infty \\
&= \alpha[\frac{e^{-y(x+\alpha)}}{x+a}(y+\frac{1}{x+\alpha})]_\infty^0 \\
&= \alpha[\frac{1}{x+\alpha}(0+\frac{1}{x+\alpha})-0] \\
&= \frac{\alpha}{(x+\alpha)^2} \\
f_{Y|X}(y|x) &= \frac{ye^{-yx}\alpha e^{-y\alpha}}{\frac{\alpha}{(x+\alpha)^2}} \\
&= (x+\alpha)^2.y.e^{-y(x+\alpha)}
\end{aligned}
$$

□

*b).*

□

*c).*

□

**Ans 5.** *a).*

$$\begin{aligned} P(S=1|V=1) &= P(S=1) \text{ (Since S is independent of V)} \\ P(S=1) &= P(S=1|G=0).P(G=0) + P(S=1|G=1).P(G=1) \\ &= (1-\gamma)\alpha + (1-\beta)(1-\alpha) \end{aligned}$$

□

*b).* P(S=1—V=0) = P(S=1) (Since S is independent of V)
= P(S=1—V=1)
Explanation: S is independent of V. Hence value of V doesn't affect S

□

**Ans 6.** *a).*

$$\begin{aligned}
P(X_2 = salmon) &= P(X_1 = Winter).P(X_2 = salmon|X_1 = Winter)+\\
&P(X_1 = Autumn).P(X_2 = salmon|X_1 = Autumn)\\
&= 0.5 * 0.9 + 0.5 * 0.8\\
&= 0.85\\
P(X_2 = seabass) &= P(X_1 = Winter).P(X_2 = seabass|X_1 = Winter)+\\
&P(X_1 = Autumn).P(X_2 = seabass|X_1 = Autumn)\\
&= 0.5 * 0.1 + 0.5 * 0.2\\
&= 0.15
\end{aligned}$$

□

*b).*

$$\begin{aligned}
P(Winter) &= P(X_1 = Winter) * P(X_2 = salmon|X_1 = Winter) * P(X_3 = Dark|X_2 = salmon)\\
&* P(X_4 = Wide|X_2 = salmon) + P(X_1 = Winter) * P(X_2 = seabass|X_1 = Winter)\\
&* P(X_3 = Dark|X_2 = seabass) * P(X_4 = Wide|X_2 = seasbass)\\
&= 0.25(0.9 * 0.34 * 0.4 + 0.1 * 0.1 * 0.95)\\
&= 0.0323\\
P(Spring) &= P(X_1 = Spring) * P(X_2 = salmon|X_1 = Spring) * P(X_3 = Dark|X_2 = salmon)\\
&* P(X_4 = Wide|X_2 = salmon) + P(X_1 = Spring) * P(X_2 = seabass|X_1 = Spring)\\
&* P(X_3 = Dark|X_2 = seabass) * P(X_4 = Wide|X_2 = seasbass)\\
&= 0.25(0.3 * 0.34 * 0.4 + 0.7 * 0.1 * 0.95)\\
&= 0.0268\\
P(Summer) &= P(X_1 = Summer) * P(X_2 = salmon|X_1 = Summer) * P(X_3 = Dark|X_2 = salmon)\\
&* P(X_4 = Wide|X_2 = salmon) + P(X_1 = Summer) * P(X_2 = seabass|X_1 = Summer)\\
&* P(X_3 = Dark|X_2 = seabass) * P(X_4 = Wide|X_2 = seasbass)\\
&= 0.25(0.4 * 0.34 * 0.4 + 0.6 * 0.1 * 0.95)\\
&= 0.02785\\
P(Autumn) &= P(X_1 = Autumn) * P(X_2 = salmon|X_1 = Autumn) * P(X_3 = Dark|X_2 = salmon)\\
&* P(X_4 = Wide|X_2 = salmon) + P(X_1 = Autumn) * P(X_2 = seabass|X_1 = Autumn)\\
&* P(X_3 = Dark|X_2 = seabass) * P(X_4 = Wide|X_2 = seasbass)\\
&= 0.25(0.8 * 0.34 * 0.4 + 0.2 * 0.1 * 0.95)\\
&= 0.03195\\
&\text{It is most likely to be winter}
\end{aligned}$$

□

**Ans 7.**

**Ans 8.** *b).*

$\square$