Homework Assignment # 1: Probability Review

Due: Monday, January 28, 2019, 11:59 p.m.
Total Points: 100

This assignment is designed to help you better understand concepts that were presented during class. You may work in small groups (2 or 3 individuals) on the homework assignments, but it is expected that you submit your own work. Therefore, **each student is responsible for submitting their own homework**, whether they work in groups or individually. The submitted homework must include the name of other individuals you worked with (if applicable), as well as all resources that were used to solve the problem (e.g. web sites, books, research papers, other people, etc.). Academic honesty is taken seriously; for detailed information see Indiana University Code of Student Rights, Responsibilities, and Conduct.

Your assignment must be submitted as a single pdf document on Canvas. The questions and your answers must be typed; for example, in Latex, Microsoft Word, Lyx, etc. Images may be scanned and inserted into the document if it is too complicated to draw them properly. Be sure to show all work.

All code (if applicable) should be turned in when you submit your assignment. Use Matlab, Python, R, Java or C. Be sure to include an complete explanation on how to run/execute your code.

All assignments must be submitted on time to receive credit. No late work will be accepted, unless you have a prior arrangement with the instructor.

Question 1. [10 POINTS]

There are three containers, where each container holds computer processors. Container 1 has 2000 processors, but 5% of them are defective. Container 2 has 500 processors, but 15% are defective. Similarly, container 3 has 100 defective computer processors out of 600 total. Three computer processors are picked from a randomly selected container.

(a) [5 POINTS] Find the probability that the three processors are defective.

(b) [5 POINTS] Assuming that the three processors are defective, find the probability that they came from container 2.

Question 2. [10 POINTS]

Suppose you have three coins. Coin A has a probability of heads of 0.75, Coin B has a probability of heads of 0.5, and Coin C has a probability of heads of 0.25.

(a) [5 POINTS] Suppose you flip all three coins at once, and let X be the number of heads you see (which will be between 0 and 3). What is the expected value of X, $E[X]$?

(b) [5 POINTS] Suppose instead you put all three coins in your pocket, select one at random, and then flip that coin 5 times. You notice that 3 of the 5 flips result in heads while the other 2 are tails. What is the probability that you chose Coin C?

Question 3. [20 POINTS]

X is said to be a Poisson discrete random variable with parameter λ if the PMF is:

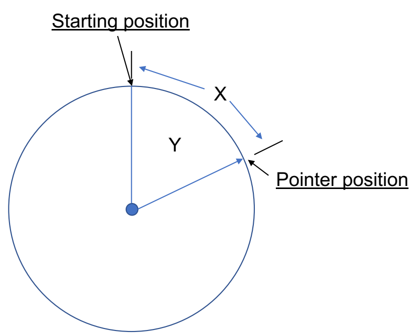
$$P_X(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}, & \text{for } x = 0, 1, 2, ... \\ 0, & \text{otherwise} \end{cases}$$

(a) [10 points] Verify that the PMF is valid.

(b) [10 points] Compute the expected value of $\frac{1}{1+X}$, (i.e. $E\left[\frac{1}{1+X}\right]$).

Question 4. [10 POINTS]

X is a random variable that denotes the position of a pointer along a wheel (e.g. arc length). The wheel has a circumference of 1. Let Y denote a random variable that specifies the area within the arc from the starting position to the position of the pointer.



(a) [2 points] What is the relationship between X and Y?

(b) [3 points] What is $F_Y(y)$?

(c) [2 points] What is $f_Y(y)$?

(d) [3 points] What is $E[Y]$?

Question 5. [20 POINTS]

A biased four-sided die is rolled and the shown face is a random variable N described by the following PMF:

$$P_N(n) = \begin{cases} n/10, & \text{if } n = 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

Given the random variable N, a biased coin is flipped and the random variable X is 1 or zero according to whether the coin shows heads or tails. The conditional PMF is

$$P_{X|N}(x|n) = \left(\frac{n+1}{2n}\right)^x \left(1 - \frac{n+1}{2n}\right)^{1-x}$$

where $x \in \{0, 1\}$. In interpreting $P(x|n)$, we take $0^0 = 1$.

(a) [5 points] Find the expectation $E[N]$ and variance $V[N]$ of N.

(b) [10 points] Find the conditional PMF for N given X, $P(N|X)$.

(c) [5 points] Find the conditional expectation $E[N|X=1]$, i.e. the expectation with respect to the conditional PMF $P(N=n|X=1)$

Question 6. [20 POINTS]

X and Y are continuous random variables, with the following joint PDF

$$f_{XY}(x,y) = \begin{cases} \frac{3}{16}(4-2x-y), & \text{if } x>0, y>0, 2x+y<4 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

(a) [10 POINTS] What is the conditional PDF of Y given, $X=x$ (i.e. $f_{Y|X}(Y|X)$)?

(b) [5 POINTS] What is the probability of Y being greater than or equal to 2, given that X equals $\frac{1}{2}$ (i.e. $P(Y \geq 2|X=\frac{1}{2})$)?

(c) [5 POINTS] Compute the conditional expectation, $E[Y|X]$

Question 7. [10 POINTS]

This question involves some simple simulations, to better visualize random variables and get some intuition for sampling, which is a central theme in machine learning. Use the attached code called `simulate.py`. This code is a simple script for sampling and plotting with python; play with some of the parameters to see what it is doing. Calling `simulate.py` runs with default parameters; `simulate.py 1 100` simulates 100 samples from a 1d Gaussian.

(a) [5 POINTS] Run the code for 10, 100 and 1000 samples with dim=1 and $\sigma = 1.0$. Next run the code for 10, 100 and 1000 samples with dim=1 and $\sigma = 10.0$. What do you notice about the sample mean?

(b) [5 POINTS] The current covariance for dim=3 is

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What does that mean about the multivariate Gaussian (i.e., about X, Y and Z)?

Change the covariance to

$$\Sigma = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

What happens?