

**FINAL JEE(Advanced) EXAMINATION - 2020****(Held On Sunday 27<sup>th</sup> SEPTEMBER, 2020)****PAPER-1****TEST PAPER WITH SOLUTIONS****PART-1 : PHYSICS****SECTION-1 : (Maximum Marks : 18)**

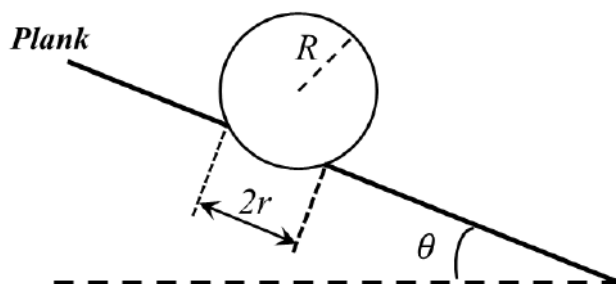
- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

*Full Marks* : +3 If **ONLY** the correct option is chosen;

*Zero Marks* : 0 If none of the options is chosen (i.e. the question is unanswered);

*Negative Marks* : -1 In all other cases.

1. A football of radius  $R$  is kept on a hole of radius  $r$  ( $r < R$ ) made on a plank kept horizontally. One end of the plank is now lifted so that it gets tilted making an angle  $\theta$  from the horizontal as shown in the figure below. The maximum value of  $\theta$  so that the football does not start rolling down the plank satisfies (figure is schematic and not drawn to scale) -



(A)  $\sin \theta = \frac{r}{R}$

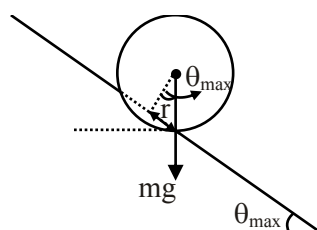
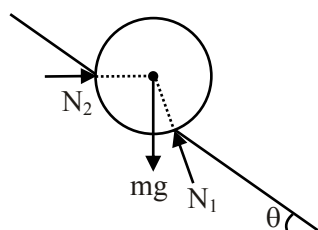
(B)  $\tan \theta = \frac{r}{R}$

(C)  $\sin \theta = \frac{r}{2R}$

(D)  $\cos \theta = \frac{r}{2R}$

**Ans. (A)**

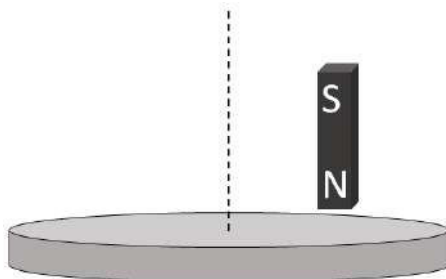
**Sol.**



For  $\theta_{\max}$ , the football is about to roll, then  $N_2 = 0$  and all the forces ( $Mg$  and  $N_1$ ) must pass through contact point

$$\therefore \cos(90^\circ - \theta_{\max}) = \frac{r}{R} \Rightarrow \sin \theta_{\max} = \frac{r}{R}$$

2. A light disc made of aluminium (a nonmagnetic material) is kept horizontally and is free to rotate about its axis as shown in the figure. A strong magnet is held vertically at a point above the disc away from its axis. On revolving the magnet about the axis of the disc, the disc will (figure is schematic and not drawn to scale)-



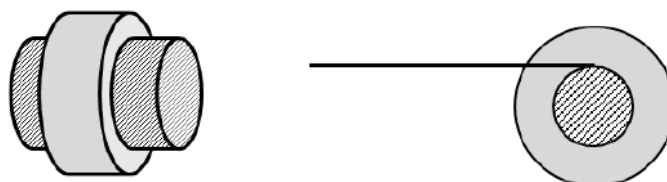
- (A) rotate in the direction opposite to the direction of magnet's motion  
 (B) rotate in the same direction as the direction of magnet's motion  
 (C) not rotate and its temperature will remain unchanged  
 (D) not rotate but its temperature will slowly rise

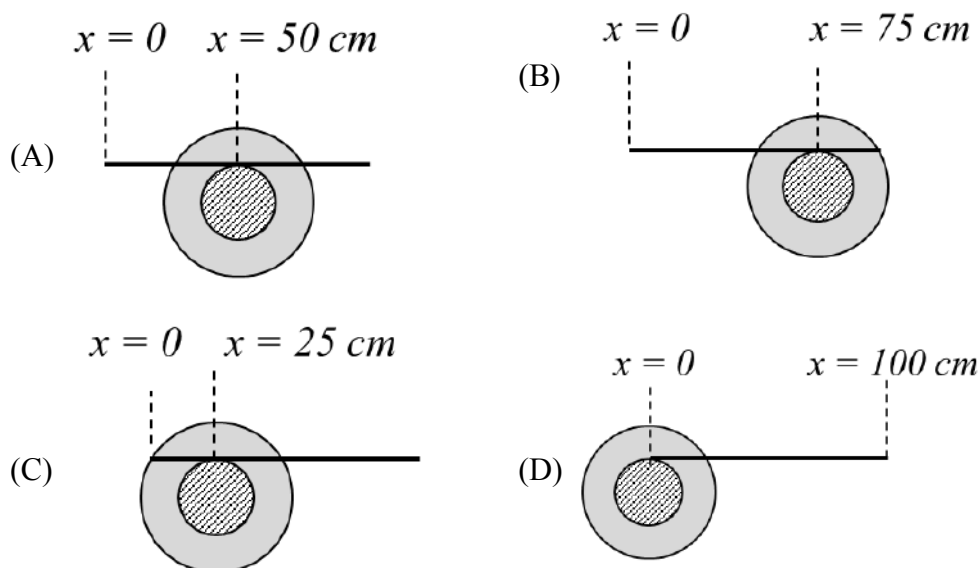
**Ans. (B)**

**Sol.** When the magnet is moved, it creates a state where the plate moves through the magnetic flux, due to which an electromotive force is generated in the plate and eddy currents are induced. These currents are such that it opposes the relative motion  $\Rightarrow$  disc will rotate in the direction of rotation of magnet.

Note : This apparatus is called Arago's disk.

3. A small roller of diameter 20 cm has an axle of diameter 10 cm (see figure below on the left). It is on a horizontal floor and a meter scale is positioned horizontally on its axle with one edge of the scale on top of the axle (see figure on the right). The scale is now pushed slowly on the axle so that it moves without slipping on the axle, and the roller starts rolling without slipping. After the roller has moved 50 cm, the position of the scale will look like (figures are schematic and not drawn to scale)-





**Ans. (B)**

**Sol.** For no slipping at the ground,

$$V_{\text{centre}} = \omega R \quad (R \text{ is radius of roller})$$

$$\therefore \text{Velocity of scale} = (V_{\text{centre}} + \omega r) \quad [r \text{ is radius of axle}]$$

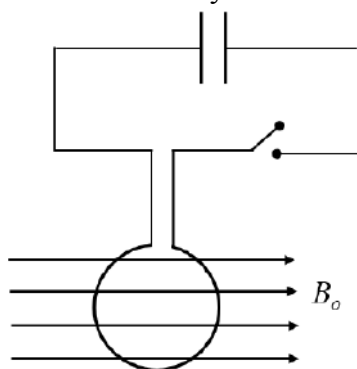
$$\text{Given, } V_{\text{centre}} \cdot t = 50 \text{ cm}$$

$$\therefore \text{Distance moved by scale} = (V_{\text{centre}} + \omega r)t$$

$$= \left( V_{\text{centre}} + \frac{V_{\text{centre}} r}{R} \right) t = \frac{3 V_{\text{centre}}}{2} \cdot t = 75 \text{ cm}$$

Therefore relative displacement (with respect to centre of roller) is  $(75 - 50) \text{ cm} = 25 \text{ cm}$ .

4. A circular coil of radius  $R$  and  $N$  turns has negligible resistance. As shown in the schematic figure, its two ends are connected to two wires and it is hanging by those wires with its plane being vertical. The wires are connected to a capacitor with charge  $Q$  through a switch. The coil is in a horizontal uniform magnetic field  $B_0$  parallel to the plane of the coil. When the switch is closed, the capacitor gets discharged through the coil in a very short time. By the time the capacitor is discharged fully, magnitude of the angular momentum gained by the coil will be (assume that the discharge time is so short that the coil has hardly rotated during this time)-



(A)  $\frac{\pi}{2} N Q B_0 R^2$

(B)  $\pi N Q B_0 R^2$

(C)  $2\pi N Q B_0 R^2$

(D)  $4\pi N Q B_0 R^2$

**Ans. (B)**

**Sol.** Torque experienced by circular loop =  $\vec{M} \times \vec{B}$

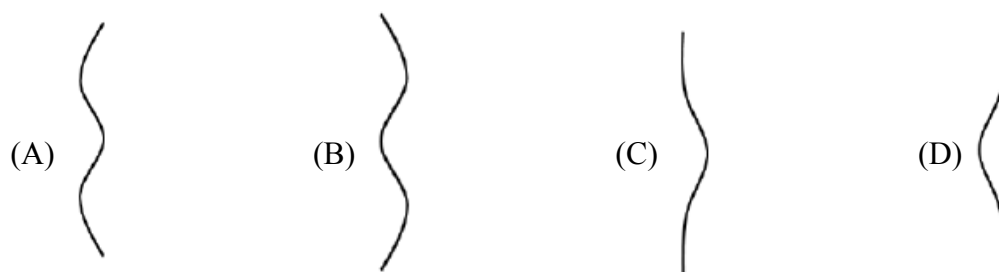
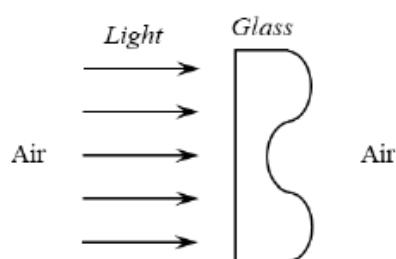
where  $\vec{M}$  is magnetic moment

$\vec{B}$  is magnetic field

$$\therefore \tau = i\pi R^2 N B_0 \text{ [at the instant shown } \theta = \pi/2]$$

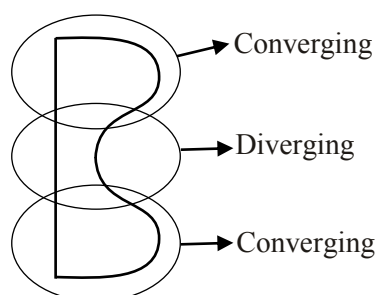
$$\therefore \tau dt = d\vec{L} = i\pi R^2 N B_0 dt = Q\pi R^2 N B_0 [idt = Q]$$

5. A parallel beam of light strikes a piece of transparent glass having cross section as shown in the figure below. Correct shape of the emergent wavefront will be (figures are schematic and not drawn to scale)-

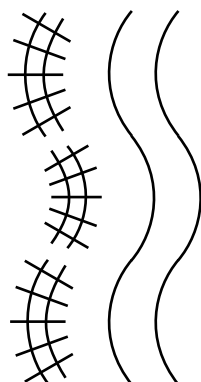


**Ans. (A)**

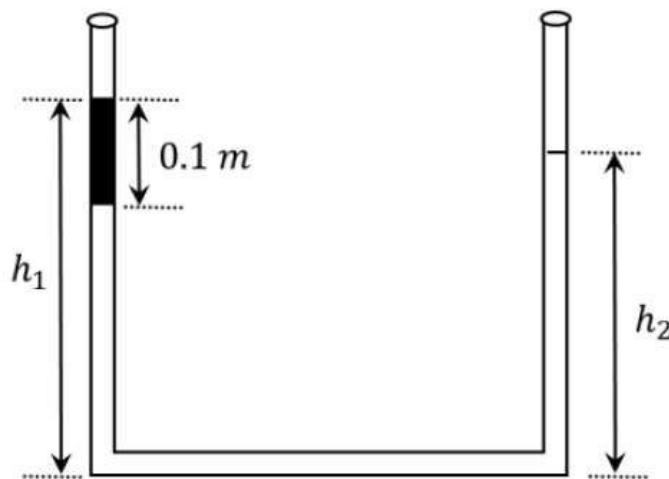
**Sol.**



(A)



6. An open-ended U-tube of uniform cross-sectional area contains water (density  $10^3 \text{ kg m}^{-3}$ ). Initially the water level stands at 0.29 m from the bottom in each arm. Kerosene oil (a water-immiscible liquid) of density  $800 \text{ kg m}^{-3}$  is added to the left arm until its length is 0.1 m, as shown in the schematic figure below. The ratio  $\left(\frac{h_1}{h_2}\right)$  of the heights of the liquid in the two arms is-



(A)  $\frac{15}{14}$

(B)  $\frac{35}{33}$

(C)  $\frac{7}{6}$

(D)  $\frac{5}{4}$

Ans. (B)

Sol.  $h_1 + h_2 = 0.29 \times 2 + 0.1$

$$h_1 + h_2 = 0.68 \quad \dots\dots(1)$$

$$\Rightarrow P_0 + \rho_k g(0.1) + \rho_w g(h_1 - 0.1) \quad [\rho_k = \text{density of kerosene} \ \& \ \rho_w = \text{density of water}]$$

$$-\rho_w g h_2 = P_0$$

$$\Rightarrow \rho_k g(0.1) + \rho_w g h_1 - \rho_w g \times (0.1)$$

$$= \rho_w g h_2$$

$$\Rightarrow 800 \times 10 \times 0.1 + 1000 \times 10 \times h_1$$

$$- 1000 \times 10 \times 0.1 = 1000 \times 10 \times h_2$$

$$\Rightarrow 10000 (h_1 - h_2) = 200$$

$$\Rightarrow h_1 - h_2 = 0.02 \quad \dots\dots(2)$$

$$\Rightarrow \boxed{h_1 = 0.35}$$

$$\Rightarrow \boxed{h_2 = 0.33}$$

$$\text{So, } \boxed{\frac{h_1}{h_2} = \frac{35}{33}}$$

**SECTION-2 : (Maximum Marks : 24)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :
 

<i>Full Marks</i>	:	+4	If only (all) the correct option(s) is(are) chosen;
<i>Partial Marks</i>	:	+3	If all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	:	0	If none of the options is chose (i.e. the question is unanswered);
<i>Negative Marks</i>	:	-2	In all other cases

7. A particle of mass  $m$  moves in circular orbits with potential energy  $V(r)=Fr$ , where  $F$  is a positive constant and  $r$  is its distance from the origin. Its energies are calculated using the Bohr model. If the radius of the particle's orbit is denoted by  $R$  and its speed and energy are denoted by  $v$  and  $E$ , respectively, then for the  $n^{\text{th}}$  orbit (here  $h$  is the Planck's constant)-

(A)  $R \propto n^{1/3}$  and  $v \propto n^{2/3}$

(B)  $R \propto n^{2/3}$  and  $v \propto n^{1/3}$

(C)  $E = \frac{3}{2} \left( \frac{n^2 h^2 F^2}{4\pi^2 m} \right)^{1/3}$

(D)  $E = 2 \left( \frac{n^2 h^2 F^2}{4\pi^2 m} \right)^{1/3}$

**Ans. (B,C)**

**Sol.**  $U = Fr$

[Using  $U$  = Potential energy and  $v$  = velocity, to avoid confusion between their symbols]

$$\Rightarrow \text{Force} = \frac{-dU}{dr} = -F$$

$$\Rightarrow \text{Magnitude of force} = \text{Constant} = F$$

$$\Rightarrow F = \frac{mv^2}{R} \quad \dots\dots(1)$$

$$\Rightarrow mvR = \frac{nh}{2\pi} \quad \dots\dots(2)$$

$$\Rightarrow F = \frac{m}{R} \times \frac{n^2 h^2}{4\pi^2} \times \frac{1}{m^2 R^2}$$

$$\Rightarrow R = \left( \frac{n^2 h^2}{4\pi^2 m F} \right)^{1/3} \quad \dots\dots(3)$$

$$\Rightarrow v = \frac{nh}{2\pi m R}$$

$$\Rightarrow v = \frac{nh}{2\pi m} \left( \frac{4\pi^2 mF}{n^2 h^2} \right)^{1/3}$$

$$\Rightarrow v = \frac{n^{1/3} h^{1/3} F^{1/3}}{2^{1/3} \pi^{1/3} m^{2/3}} \dots\dots(4)$$

(B) is correct

$$\Rightarrow E = \frac{1}{2}mv^2 + U$$

$$= \frac{1}{2}mv^2 + FR$$

$$\Rightarrow E = \frac{1}{2}m \left( \frac{n^{2/3} h^{2/3} F^{2/3}}{2^{2/3} \pi^{2/3} m^{4/3}} \right) + F \times \left( \frac{n^2 h^2}{4\pi^2 mF} \right)^{1/3}$$

$$\Rightarrow E = \left( \frac{n^2 h^2 F^2}{4\pi^2 m} \right)^{1/3} \left[ \frac{1}{2} + 1 \right]$$

$$= \frac{3}{2} \left( \frac{n^2 h^2 F^2}{4\pi^2 m} \right)^{1/3}$$

8. The filament of a light bulb has surface area  $64 \text{ mm}^2$ . The filament can be considered as a black body at temperature  $2500 \text{ K}$  emitting radiation like a point source when viewed from far. At night the light bulb is observed from a distance of  $100 \text{ m}$ . Assume the pupil of the eyes of the observer to be circular with radius  $3 \text{ mm}$ . Then

(Take Stefan-Boltzmann constant  $= 5.67 \times 10^{-8} \text{ Wm}^{-2} \text{ K}^{-4}$ , Wien's displacement constant  $= 2.90 \times 10^{-3} \text{ m-K}$ , Planck's constant  $= 6.63 \times 10^{-34} \text{ Js}$ , speed of light in vacuum  $= 3.00 \times 10^8 \text{ ms}^{-1}$ )-

- (A) power radiated by the filament is in the range  $642 \text{ W}$  to  $645 \text{ W}$   
 (B) radiated power entering into one eye of the observer is in the range  $3.15 \times 10^{-8} \text{ W}$  to  $3.25 \times 10^{-8} \text{ W}$   
 (C) the wavelength corresponding to the maximum intensity of light is  $1160 \text{ nm}$   
 (D) taking the average wavelength of emitted radiation to be  $1740 \text{ nm}$ , the total number of photons entering per second into one eye of the observer is in the range  $2.75 \times 10^{11}$  to  $2.85 \times 10^{11}$

Ans. (B,C,D)

**Sol.**  $A = 64 \text{ mm}^2$ ,  $T = 2500 \text{ K}$  ( $A$  = surface area of filament,  $T$  = temperature of filament,  $d$  is distance of bulb from observer,  $R_e$  = radius of pupil of eye)

Point source  $d = 100 \text{ m}$

$$R_e = 3 \text{ mm}$$

$$\begin{aligned} \text{(A)} \quad P &= \sigma A e T^4 \\ &= 5.67 \times 10^{-8} \times 64 \times 10^{-6} \times 1 \times (2500)^4 \quad (e = 1 \text{ black body}) \\ &= 141.75 \text{ W} \end{aligned}$$

Option (A) is wrong

**(B)** Power reaching to the eye

$$\begin{aligned} &= \frac{P}{4\pi d^2} \times (\pi R_e^2) \\ &= \frac{141.75}{4\pi \times (100)^2} \times \pi \times (3 \times 10^{-3})^2 \\ &= 3.189375 \times 10^{-8} \text{ W} \end{aligned}$$

Option (B) is correct

**(C)**  $\lambda_m T = b$

$$\begin{aligned} \lambda_m \times 2500 &= 2.9 \times 10^{-3} \\ \Rightarrow \lambda_m &= 1.16 \times 10^{-6} \\ &= 1160 \text{ nm} \end{aligned}$$

Option (C) is correct

**(D)** Power received by one eye of observer =  $\left(\frac{hc}{\lambda}\right) \times \dot{N}$

$\dot{N}$  = Number of photons entering into eye per second

$$\begin{aligned} \Rightarrow 3.189375 \times 10^{-8} &= \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{1740 \times 10^{-9}} \times \dot{N} \end{aligned}$$

$$\Rightarrow \dot{N} = 2.79 \times 10^{11}$$

Option (D) is correct

**9.** Sometimes it is convenient to construct a system of units so that all quantities can be expressed in terms of only one physical quantity. In one such system, dimensions of different quantities are given in terms of a quantity  $X$  as follows: [position] =  $[X^\alpha]$ ; [speed] =  $[X^\beta]$ ; [acceleration] =  $[X^p]$ ; [linear momentum] =  $[X^q]$ ; [force] =  $[X^r]$ . Then -

$$\text{(A)} \quad \alpha + p = 2\beta$$

$$\text{(B)} \quad p + q - r = \beta$$

$$\text{(C)} \quad p - q + r = \alpha$$

$$\text{(D)} \quad p + q + r = \beta$$

**Ans. (A,B)**



**Sol.** Given  $L = x^\alpha$  .....(1)

$LT^{-1} = x^\beta$  .....(2)

$LT^{-2} = x^p$  .....(3)

$MLT^{-1} = x^q$  .....(4)

$MLT^{-2} = x^r$  .....(5)

$$\frac{(1)}{(2)} \Rightarrow T = x^{\alpha-\beta}$$

From (3)

$$\frac{x^\alpha}{x^{2(\alpha-\beta)}} = x^p$$

$$\Rightarrow \alpha + p = 2\beta \quad (A)$$

From (4)

$$M = x^{q-\beta}$$

$$\text{From (5)} \Rightarrow x^q = x^r x^{\alpha-\beta}$$

$$\Rightarrow \alpha + r - q = \beta \quad \text{.....(6)}$$

Replacing value ' $\alpha$ ' in equation (6) from (A)

$$2\beta - p + r - q = \beta$$

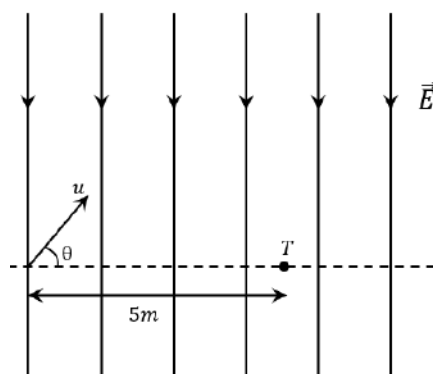
$$\Rightarrow p + q - r = \beta \quad (B)$$

Replacing value of ' $\beta$ ' in equation (6) from (A)

$$2\alpha + 2r - 2q = \alpha + p$$

$$\alpha = p + 2q - 2r$$

10. A uniform electric field,  $\vec{E} = -400\sqrt{3}\hat{y} \text{ NC}^{-1}$  is applied in a region. A charged particle of mass  $m$  carrying positive charge  $q$  is projected in this region with an initial speed of  $2\sqrt{10} \times 10^6 \text{ ms}^{-1}$ . This particle is aimed to hit a target  $T$ , which is 5 m away from its entry point into the field as shown schematically in the figure. Take  $\frac{q}{m} = 10^{10} \text{ Ckg}^{-1}$ . Then-



- (A) the particle will hit T if projected at an angle  $45^\circ$  from the horizontal
- (B) the particle will hit T if projected either at an angle  $30^\circ$  or  $60^\circ$  from the horizontal
- (C) time taken by the particle to hit T could be  $\sqrt{\frac{5}{6}} \mu\text{s}$  as well as  $\sqrt{\frac{5}{2}} \mu\text{s}$
- (D) time taken by the particle to hit T is  $\sqrt{\frac{5}{3}} \mu\text{s}$

**Ans. (B,C)**

**Sol.**  $a_y = -400\sqrt{3} \times 10^{10} [qE_y = ma_y]$

$$R = 5 = \frac{40 \times 10^{12} \sin 2\theta}{400\sqrt{3} \times 10^{10}} \left[ R(\text{range}) = \frac{u^2 \sin 2\theta}{a_y} \right]$$

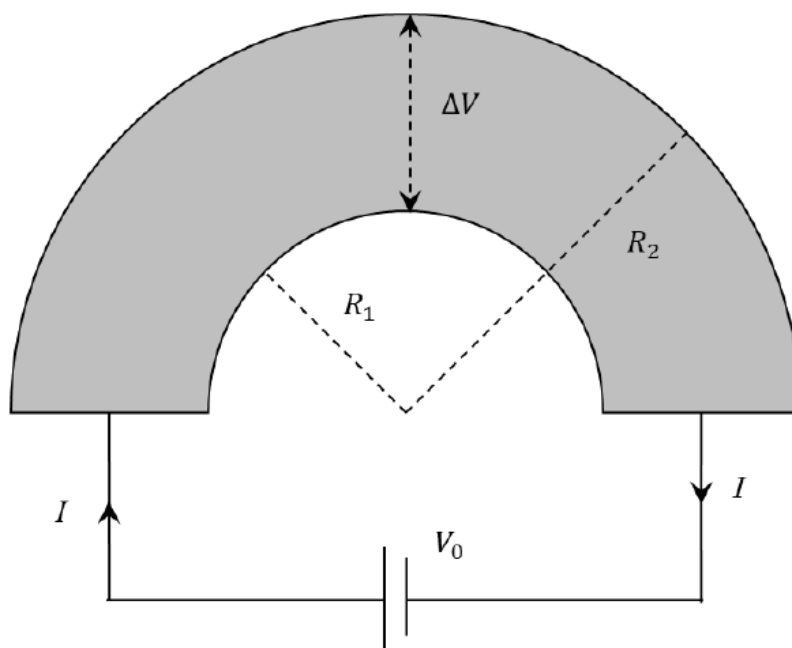
$$\sin 2\theta = \frac{\sqrt{3}}{2}$$

$$2\theta = 60^\circ, 120^\circ \Rightarrow \theta = 30^\circ, 60^\circ$$

$$\text{Time of flight } T_1 = \frac{2 \times 2\sqrt{10} \times 10^6 \times \frac{1}{2}}{400\sqrt{3} \times 10^{10}} = \sqrt{\frac{5}{6}} \mu\text{s} \quad (\text{for } \theta = 30^\circ)$$

$$\text{Time of flight } T_2 = \frac{2 \times 2\sqrt{10} \times 10^6 \times \frac{\sqrt{3}}{2}}{400\sqrt{3} \times 10^{10}} = \sqrt{\frac{5}{2}} \mu\text{s} \quad (\text{for } \theta = 60^\circ)$$

11. Shown in the figure is a semicircular metallic strip that has thickness  $t$  and resistivity  $\rho$ . Its inner radius is  $R_1$  and outer radius is  $R_2$ . If a voltage  $V_0$  is applied between its two ends, a current  $I$  flows in it. In addition, it is observed that a transverse voltage  $\Delta V$  develops between its inner and outer surfaces due to purely kinetic effects of moving electrons (ignore any role of the magnetic field due to the current). Then (figure is schematic and not drawn to scale)-



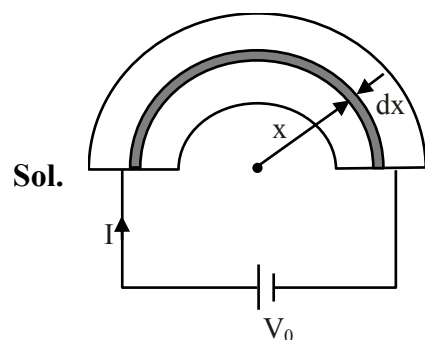
(A)  $I = \frac{V_0 t}{\pi \rho} \ln\left(\frac{R_2}{R_1}\right)$

(B) the outer surface is at a higher voltage than the inner surface

(C) the outer surface is at a lower voltage than the inner surface

(D)  $\Delta V \propto I^2$

Ans. (A,C,D)



All the elements are in parallel

$$\therefore \int \frac{1}{dr} = \int_{R_1}^{R_2} \frac{t dx}{\rho \pi x}$$

$$\frac{1}{r} = \frac{t}{\pi \rho} \ln\left(\frac{R_2}{R_1}\right)$$

$$\text{Resistance} = \frac{\pi \rho}{t \ln\left(\frac{R_2}{R_1}\right)}$$

$$i = \frac{V_0 t \ln\left(\frac{R_2}{R_1}\right)}{\pi \rho} \quad (\text{A})$$

$(-e\vec{E})$  will be inward direction in order to provide centripetal acceleration. Therefore electric field will be radially outward

$$V_{\text{outer}} < V_{\text{inner}} \quad (\text{C})$$

$$\frac{mV_d^2}{r} = q\vec{E}$$

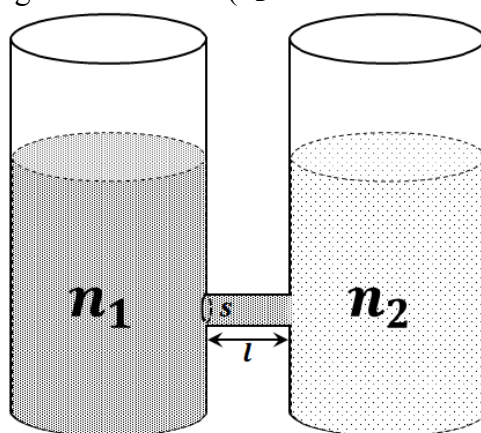
$$E = \frac{mV_d^2}{qr} \quad (I = neAV_d \Rightarrow V_d \propto i)$$

$$\Delta V = \int \vec{E} \cdot d\vec{r}$$

$$\Delta V \propto V_d^2$$

$$\Delta V \propto I^2$$

12. As shown schematically in the figure, two vessels contain water solutions (at temperature  $T$ ) of potassium permanganate ( $\text{KMnO}_4$ ) of different concentrations  $n_1$  and  $n_2$  ( $n_1 > n_2$ ) molecules per unit volume with  $\Delta n = (n_1 - n_2) \ll n_1$ . When they are connected by a tube of small length  $\ell$  and cross-sectional area  $S$ ,  $\text{KMnO}_4$  starts to diffuse from the left to the right vessel through the tube. Consider the collection of molecules to behave as dilute ideal gases and the difference in their partial pressure in the two vessels causing the diffusion. The speed  $v$  of the molecules is limited by the viscous force  $-\beta v$  on each molecule, where  $\beta$  is a constant. Neglecting all terms of the order  $(\Delta n)^2$ , which of the following is/are correct? ( $k_B$  is the Boltzmann constant)-



(A) the force causing the molecules to move across the tube is  $\Delta n k_B T S$

(B) force balance implies  $n_1 \beta v \ell = \Delta n k_B T$

(C) total number of molecules going across the tube per sec is  $\left(\frac{\Delta n}{\ell}\right) \left(\frac{k_B T}{\beta}\right) S$

(D) rate of molecules getting transferred through the tube does not change with time

Ans. (A,B,C)

**Sol.**  $n_1 \gg (n_1 - n_2) = \Delta n$

$$p_1 = \frac{n_1 RT}{N_A} \quad p_2 = \frac{n_2 RT}{N_A}$$

$$F = (n_1 - n_2)k_B TS = \Delta n k_B TS \quad (A)$$

$$V = \frac{\Delta n k_B TS}{\beta}$$

Force balance  $\Rightarrow$  Pressure  $\times$  Area = Total number of molecules  $\times \beta v$

$$\Delta n k_B TS = \ell n_1 S \beta v$$

$$\Rightarrow n_1 \beta v \ell = \Delta n k_B T \quad (B)$$

$$\text{Total number of molecules/sec} = \frac{(n_1 v dt) S}{dt}$$

$$= n_1 v S = \frac{\Delta n k_B T v S}{\beta v \ell}$$

$$= \left( \frac{\Delta n}{\ell} \right) \left( \frac{k_B T}{\beta} \right) S \quad (C)$$

As  $\Delta n$  will decrease with time therefore rate of molecules getting transfer decreases with time.

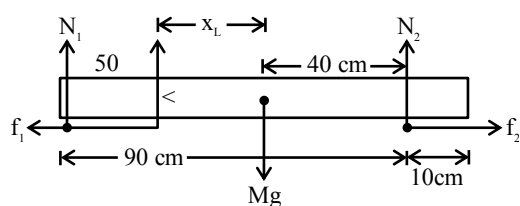
### SECTION-3 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme :  
*Full Marks* : +4 If ONLY the correct numerical value is entered;  
*Zero Marks* : 0 In all other cases.

- 13.** Put a uniform meter scale horizontally on your extended index fingers with the left one at 0.00 cm and the right one at 90.00 cm. When you attempt to move both the fingers slowly towards the center, initially only the left finger slips with respect to the scale and the right finger does not. After some distance, the left finger stops and the right one starts slipping. Then the right finger stops at a distance  $x_R$  from the center (50.00 cm) of the scale and the left one starts slipping again. This happens because of the difference in the frictional forces on the two fingers. If the coefficients of static and dynamic friction between the fingers and the scale are 0.40 and 0.32, respectively, the value of  $x_R$  (in cm) is \_\_\_\_\_.

**Ans. (25.60)**

Sol.



Initially

$$\begin{array}{l|l}
 N_1 + N_2 = Mg & N_1 = \frac{4Mg}{9} \\
 (\tau_n = 0)_{\text{about centre}} N_1(50) = N_2(40) & N_2 = \frac{5Mg}{9} \\
 \hline
 5N_1 = 4N_2 & \\
 \hline
 f_{1K} = \mu_K N_1 & f_{1L} = \mu_S N_1 \\
 f_{1K} = 0.32 N_1 & f_{1L} = 0.4 N_1 \\
 \hline
 f_{2K} = 0.32 N_2 & f_{2L} = 0.4 N_2
 \end{array}$$

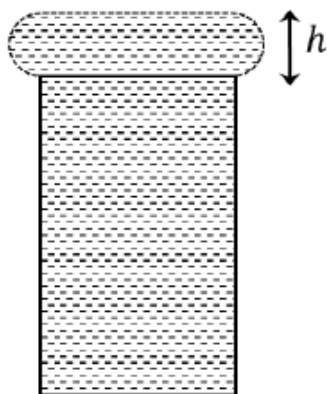
Suppose  $x_L$  = distance of left finger from centre when right finger starts moving

$$\begin{aligned}
 (\tau_n = 0)_{\text{about centre}} &\Rightarrow N_1 x_L = N_2(40) \\
 f_{K1} = f_{L2} &\Rightarrow 0.32 N_1 = 0.40 N_2 \\
 4N_1 &= 5N_2 \\
 N_1 x_L &= \frac{4N_1}{5}(40) \\
 x_L &= 32
 \end{aligned}$$

Now  $x_R$  = distance when right finger stops and left finger starts moving

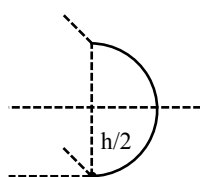
$$\begin{aligned}
 (\tau_n = 0)_{\text{about centre}} &\Rightarrow N_1 x_L = N_2(x_R) \\
 f_{L1} = f_{K2} &\Rightarrow 0.4 N_1 = 0.32 N_2 \\
 5N_1 &= 4N_2 \\
 \frac{4N_2}{5}(32) &= N_2 x_R \\
 x_R &= \frac{128}{5} = 25.6 \text{ cm}
 \end{aligned}$$

14. When water is filled carefully in a glass, one can fill it to a height  $h$  above the rim of the glass due to the surface tension of water. To calculate  $h$  just before water starts flowing, model the shape of the water above the rim as a disc of thickness  $h$  having semicircular edges, as shown schematically in the figure. When the pressure of water at the bottom of this disc exceeds what can be withstood due to the surface tension, the water surface breaks near the rim and water starts flowing from there. If the density of water, its surface tension and the acceleration due to gravity are  $10^3 \text{ kg m}^{-3}$ ,  $0.07 \text{ Nm}^{-1}$  and  $10 \text{ ms}^{-2}$ , respectively, the value of  $h$  (in mm) is \_\_\_\_\_.



Ans. (3.74)

Sol.



$$r = h/2$$

Pressure at the bottom of disc = pressure due to surface tension

$$\rho gh = T \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$R_1 \gg R_2$$

$$\text{So } \frac{1}{R_1} \ll \frac{1}{R_2} \text{ and } R_2 = h/2$$

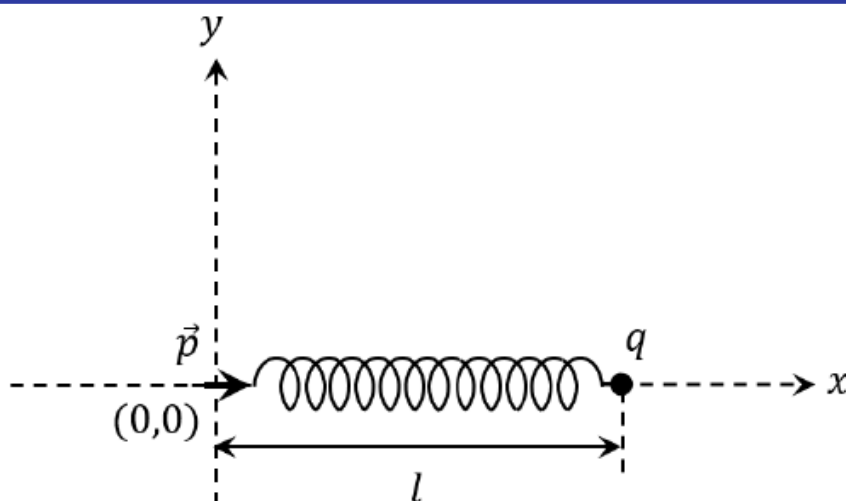
$$\therefore \rho gh = T \left[ \frac{1}{R_1} + \frac{1}{R_2} \right] = T \left[ 0 + \frac{1}{h/2} \right]$$

$$h^2 = \frac{2T}{\rho g}$$

$$h = \sqrt{\frac{2T}{\rho g}} = \sqrt{\frac{2 \times 0.07}{10^3 \times 10}} = \sqrt{\frac{14 \times 100}{10^4 \times 100}}$$

$$h = \sqrt{14} \text{ mm} = 3.741$$

15. One end of a spring of negligible unstretched length and spring constant  $k$  is fixed at the origin  $(0,0)$ . A point particle of mass  $m$  carrying a positive charge  $q$  is attached at its other end. The entire system is kept on a smooth horizontal surface. When a point dipole  $\vec{p}$  pointing towards the charge  $q$  is fixed at the origin, the spring gets stretched to a length  $\ell$  and attains a new equilibrium position (see figure below). If the point mass is now displaced slightly by  $\Delta \ell \ll \ell$  from its equilibrium position and released, it is found to oscillate at frequency  $\frac{1}{\delta} \sqrt{\frac{k}{m}}$ . The value of  $\delta$  is \_\_\_\_\_.

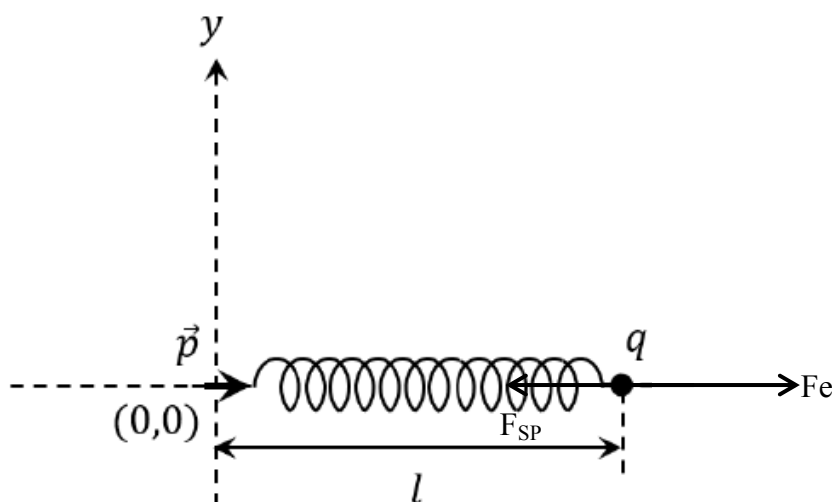


**Ans. (3.14)**

**Sol.**  $\Delta \ell \rightarrow x$

At  $\ell$  :  $F_e = F_{SP}$

$$k\ell = \frac{2kpq}{\ell^3}$$



$$F_{\text{net}} = F_{\text{sp}} - F_e = k(\ell + x) - \frac{q(2kp)}{(\ell + x)^3}$$

$$= k(x + \ell) - \frac{q(2kp)}{\ell^3(1 + x/\ell)^3}$$

$$= kx + k\ell - q\left(\frac{2kp}{\ell^3}\right)\left(1 - \frac{3x}{\ell}\right)$$

$$= kx + k\ell - q\left(\frac{2kp}{\ell^3}\right) + \frac{2kpq}{\ell^3} \cdot \frac{3x}{\ell}$$



$$F_N = kx + k\ell \left( \frac{3x}{\ell} \right) = 4kx$$

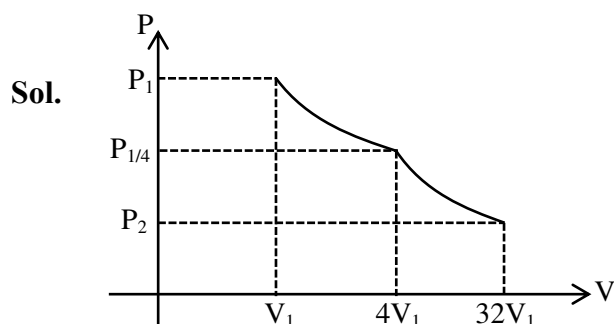
$$k_{eq} = 4k \quad T = 2\pi\sqrt{\frac{m}{4k}} = \pi\sqrt{\frac{m}{k}}$$

$$f = \frac{1}{\pi}\sqrt{\frac{k}{m}}$$

$$\text{So } \delta = \pi = 3.14$$

16. Consider one mole of helium gas enclosed in a container at initial pressure  $P_1$  and volume  $V_1$ . It expands isothermally to volume  $4V_1$ . After this, the gas expands adiabatically and its volume becomes  $32V_1$ . The work done by the gas during isothermal and adiabatic expansion processes are  $W_{iso}$  and  $W_{adia}$ , respectively. If the ratio  $\frac{W_{iso}}{W_{adia}} = f \ln 2$ , then  $f$  is \_\_\_\_\_.

Ans. (1.77 to 1.78)



$$\frac{P_1}{4} (4V_1)^{5/3} = P_2 (32V_1)^{5/3}$$

$$P_2 = \frac{P_1}{4} \left( \frac{1}{8} \right)^{5/3} = \frac{P_1}{128}$$

$$W_{adi} = \frac{P_1 V_1 - P_2 V_2}{\gamma - 1} = \frac{P_1 V_1 - \frac{P_1}{128} (32V_1)}{\frac{5}{3} - 1}$$

$$= \frac{P_1 V_1 (3/4)}{2/3} = \frac{9}{8} P_1 V_1$$

$$W_{iso} = P_1 V_1 \ln \left( \frac{4V_1}{V_1} \right) = 2P_1 V_1 \ln 2$$

$$\frac{W_{\text{iso}}}{W_{\text{adio}}} = \frac{2P_1 V_1 \ln 2}{\frac{9}{8} P_1 V_1} = \frac{16}{9} \ln 2 = f \ln 2$$

$$f = \frac{16}{9} = 1.7778 \approx 1.78$$

17. A stationary tuning fork is in resonance with an air column in a pipe. If the tuning fork is moved with a speed of  $2 \text{ ms}^{-1}$  in front of the open end of the pipe and parallel to it, the length of the pipe should be changed for the resonance to occur with the moving tuning fork. If the speed of sound in air is  $320 \text{ ms}^{-1}$ , the smallest value of the percentage change required in the length of the pipe is \_\_\_\_\_.

**Ans. (0.62 to 0.63)**

**Sol.**  $f \propto \frac{1}{\ell_1} \Rightarrow f = \frac{k}{\ell_1} \dots (1)$

( $\ell_1 \Rightarrow$  initial length of pipe)

$$\left( \frac{V}{V - V_T} \right) f = \frac{k}{\ell_2} \quad \{V_T \text{ Speed of tuning fork, } \ell_2 \rightarrow \text{new length of pipe}\} \dots (2)$$

$$(1) \div (2)$$

$$\frac{V - V_T}{V} = \frac{\ell_2}{\ell_1}$$

$$\frac{\ell_2}{\ell_1} - 1 = \frac{V - V_T}{V} - 1$$

$$\frac{\ell_2 - \ell_1}{\ell_1} = \frac{-V_T}{V}$$

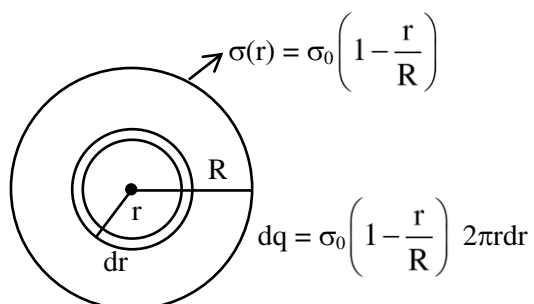
$$\frac{\ell_2 - \ell_1}{\ell_1} \times 100 = \frac{-2}{320} \times 100 = -0.625$$

Therefore smallest value of percentage change required in the length of pipe is 0.625

18. A circular disc of radius  $R$  carries surface charge density  $\sigma(r) = \sigma_0 \left( 1 - \frac{r}{R} \right)$ , where  $\sigma_0$  is a constant and  $r$  is the distance from the center of the disc. Electric flux through a large spherical surface that encloses the charged disc completely is  $\phi_0$ . Electric flux through another spherical surface of radius  $\frac{R}{4}$  and concentric with the disc is  $\phi$ . Then the ratio  $\frac{\phi_0}{\phi}$  is \_\_\_\_\_.

**Ans. (6.40)**

Sol.



$$\phi_0 = \frac{\int dq}{\epsilon_0} = \frac{\int_0^R \sigma_0 \left(1 - \frac{r}{R}\right) 2\pi r dr}{\epsilon_0}$$

$$\phi = \frac{\int dq}{\epsilon_0} = \frac{\int_0^{R/4} \sigma_0 \left(1 - \frac{r}{R}\right) 2\pi r dr}{\epsilon_0}$$

$$\therefore \frac{\phi_0}{\phi} = \frac{\sigma_0 2\pi \int_0^R \left(r - \frac{r^2}{R}\right) dr}{\sigma_0 2\pi \int_0^{R/4} \left(r - \frac{r^2}{R}\right) dr}$$

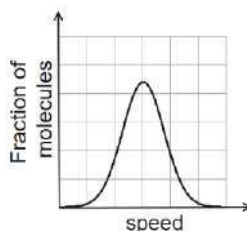
$$= \frac{\frac{R^2}{2} - \frac{R^2}{3}}{\frac{R^2}{32} - \frac{R^2}{3 \times 64}} = \frac{32}{5}$$

$$= 6.40$$

**FINAL JEE(Advanced) EXAMINATION - 2020****(Held On Sunday 27<sup>th</sup> SEPTEMBER, 2020)****PAPER-1****TEST PAPER WITH SOLUTION****PART-2 : CHEMISTRY****SECTION-1 : (Maximum Marks : 18)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :  
*Full Marks* : +3 If **ONLY** the correct option is chosen;  
*Zero Marks* : 0 If none of the options is chosen (i.e. the question is unanswered);  
*Negative Marks* : -1 In all other cases.

1. If the distribution of molecular speeds of a gas is as per the figure shown below, then the ratio of the most probable, the average and the roots mean square speeds, respectively, is



- (A) 1 : 1 : 1  
 (B) 1 : 1 : 1.224  
 (C) 1 : 1.128 : 1.224  
 (D) 1 : 1.128 : 1

**Ans. (B)**

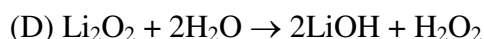
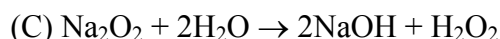
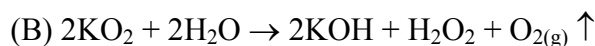
**Sol.** Graph represents symmetrical distribution of speed and hence, the most probable and the average speed should be same. But the root mean square speed must be greater than the average speed.

2. Which of the following liberates  $O_2$  upon hydrolysis?

- (A)  $Pb_3O_4$  (B)  $KO_2$  (C)  $Na_2O_2$  (D)  $Li_2O_2$

**Ans. (B)**

**Sol.** (A)  $Pb_3O_4$  is insoluble in water or do not react with water.



3. A colorless aqueous solution contains nitrates of two metals, **X** and **Y**. When it was added to an aqueous solution of  $NaCl$ , a white precipitate was formed. This precipitate was found to be partly soluble in hot water to give a residue **P** and a solution **Q**. The residue **P** was soluble in aq.  $NH_3$  and also in excess sodium thiosulfate. The hot solution **Q** gave a yellow precipitate with  $KI$ . The metals **X** and **Y**, respectively, are

- (A)  $Ag$  and  $Pb$  (B)  $Ag$  and  $Cd$  (C)  $Cd$  and  $Pb$  (D)  $Cd$  and  $Zn$

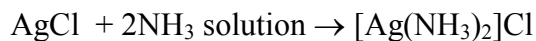
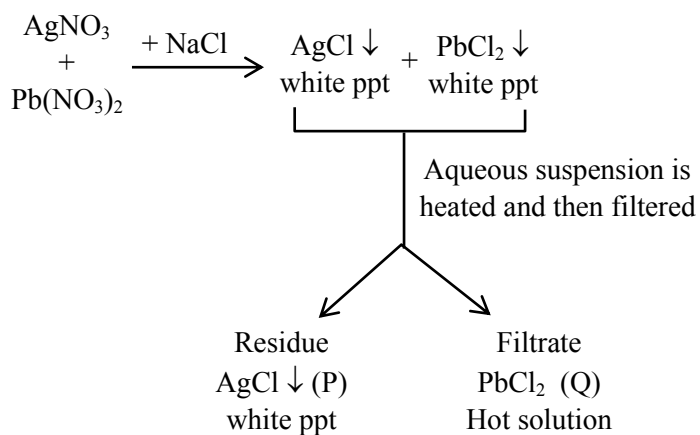
Ans. (A)

Sol. X : Ag

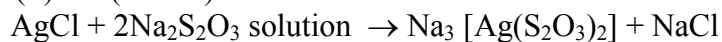
P : AgCl

Y : Pb

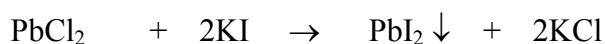
Q : PbCl<sub>2</sub>



(P) (excess) clear solution



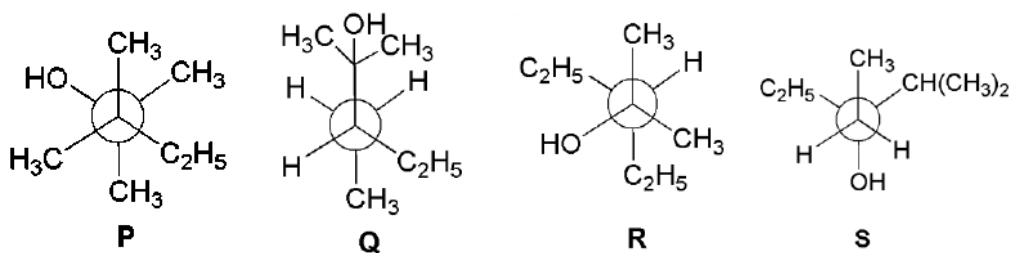
(P) (excess) clear solution



Hot solution (yellow ppt)

(Q)

4. Newman projections **P**, **Q**, **R** and **S** are shown below :



Which one of the following options represents identical molecules ?

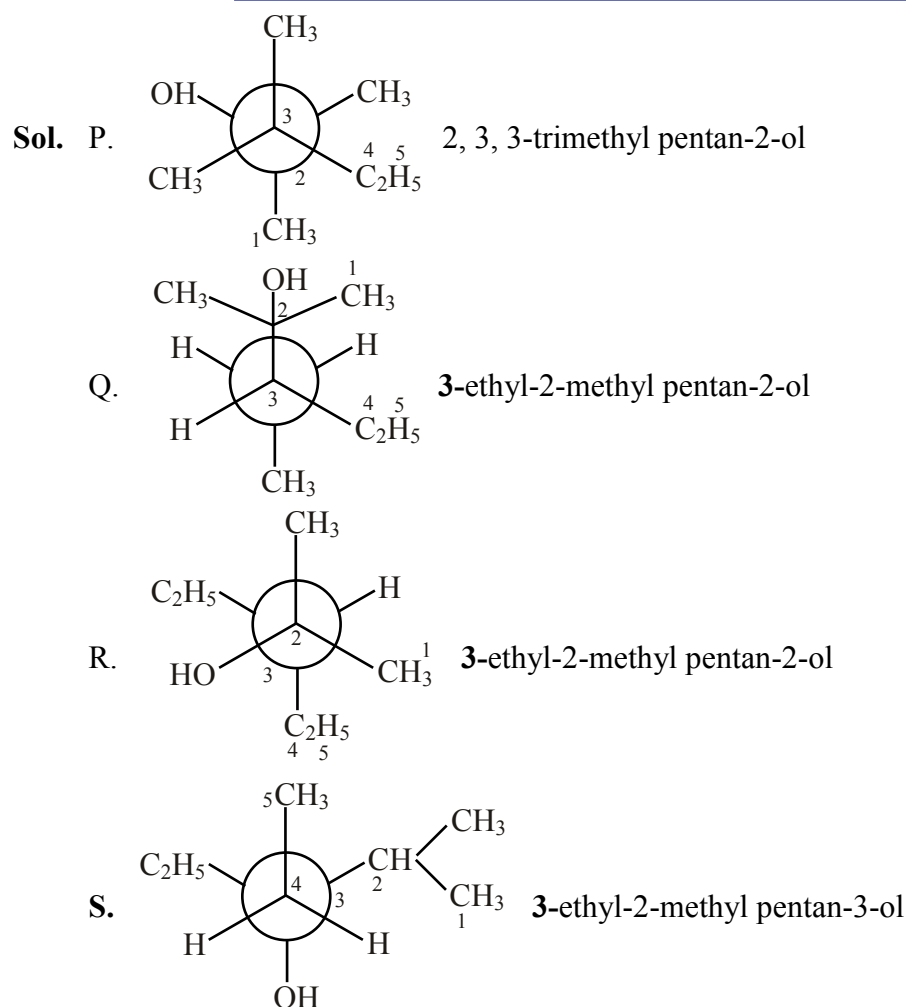
(A) **P** and **Q**

(B) **Q** and **S**

(C) **Q** and **R**

(D) **R** and **S**

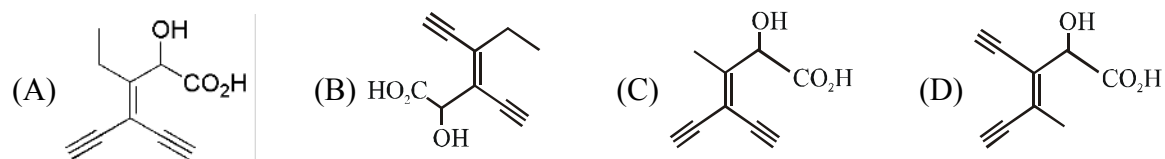
Ans. (C)



Q and R is same.

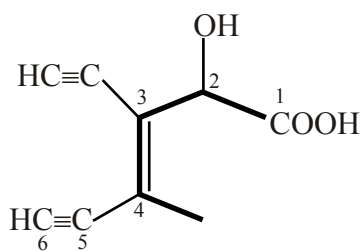
5. Which one of the following structures has the IUPAC name

3-ethynyl-2-hydroxy-4-methylhex-3-en-5-ynoic acid ?



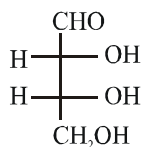
**Ans. (D)**

Sol.



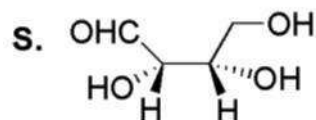
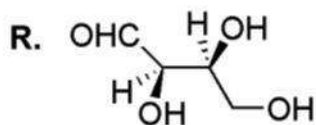
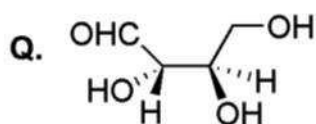
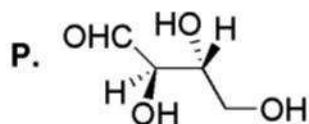
3-ethynyl-2-hydroxy-4-methyl-hex-3-en-5-ynoic acid.

6. The Fischer projection of D-erythrose is shown below.



D-Erythrose

D-Erythrose and its isomers are listed as **P**, **Q**, **R**, and **S** in **Column-I**. Choose the correct relationship of **P**, **Q**, **R**, and **S** with D-erythrose from **Column II**.

**Column-I****Column-II**

1. Diastereomer
2. Identical
3. Enantiomer

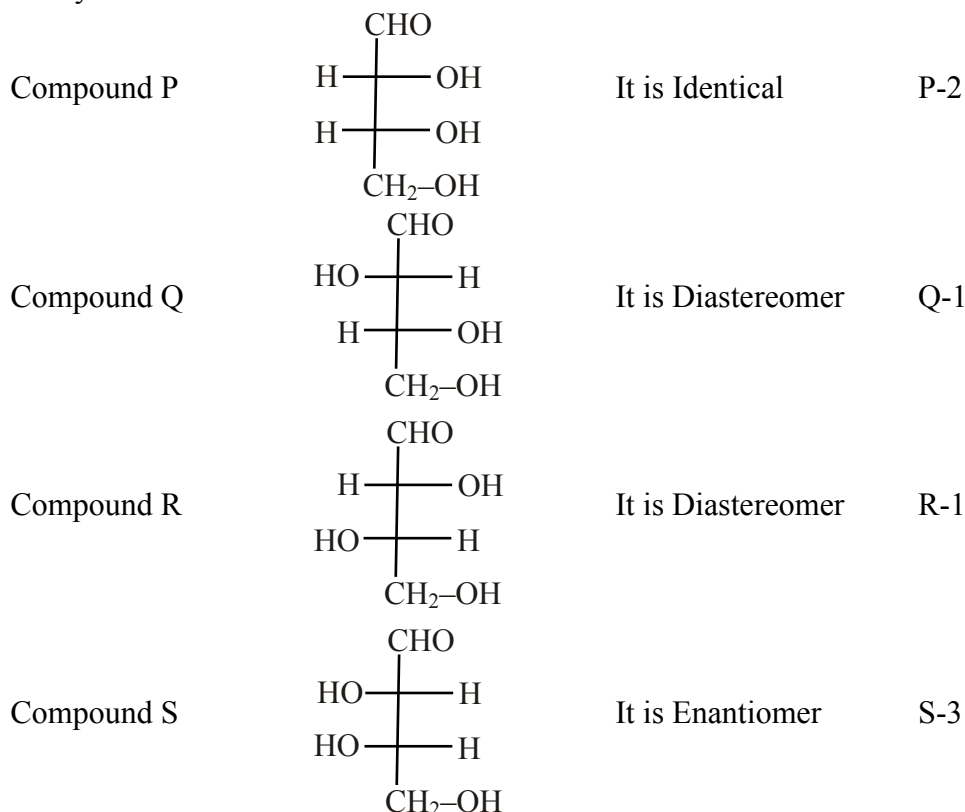
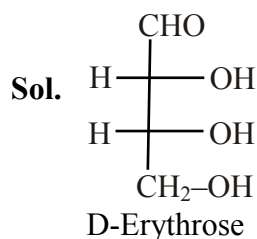
(A) P → 2, Q → 3, R → 2, S → 2

(B) P → 3, Q → 1, R → 1, S → 2

(C) P → 2, Q → 1, R → 1, S → 3

(D) P → 2, Q → 3, R → 3, S → 1

Ans. (C)



Ans. P-2, Q-1, R-1, S-3

**SECTION-2 : (Maximum Marks : 24)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :
 

<i>Full Marks</i>	: +4	If only (all) the correct option(s) is(are) chosen;
<i>Partial Marks</i>	: +3	If all the four options are correct but <b>ONLY</b> three options are chosen;
<i>Partial Marks</i>	: +2	If three or more options are correct but <b>ONLY</b> two options are chosen, both of which are correct;
<i>Partial Marks</i>	: +1	If two or more options are correct but <b>ONLY</b> one option is chosen and it is a correct option;
<i>Zero Marks</i>	: 0	If none of the options is chose (i.e. the question is unanswered);
<i>Negative Marks</i>	: -2	In all other cases



7. In thermodynamics the P-V work done is given by

$$w = -\int dV P_{\text{ext}}.$$

For a system undergoing a particular process, the work done is ,

$$w = -\int dV \left( \frac{RT}{V-b} - \frac{a}{V^2} \right).$$

This equation is applicable to a

- (A) System that satisfies the van der Waals equation of state.
- (B) Process that is reversible and isothermal.
- (C) Process that is reversible and adiabatic.
- (D) Process that is irreversible and at constant isothermal.

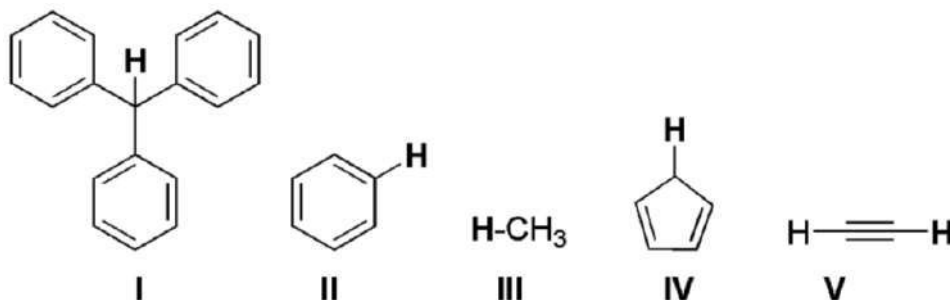
**Ans. (A, B, C)**

**Sol.** For 1 mole Vander Waal's gas

$$P = \frac{RT}{V-b} - \frac{a}{V^2}$$

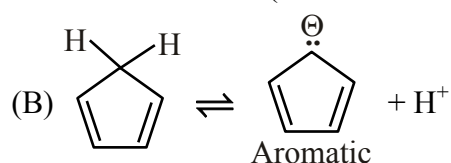
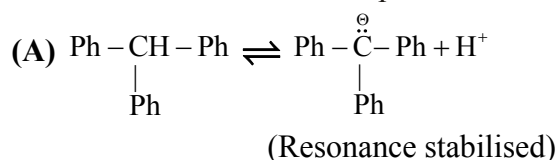
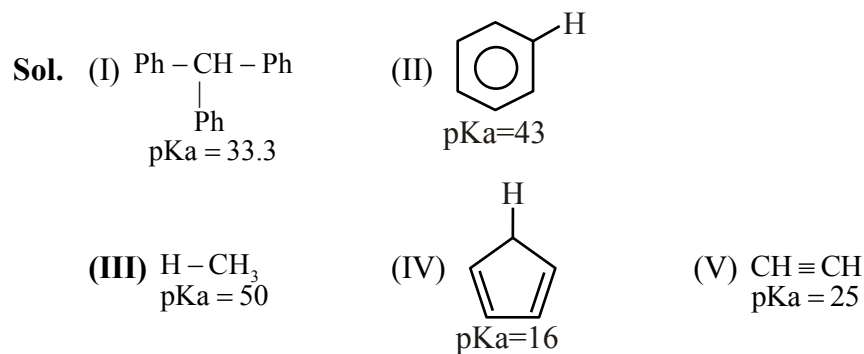
If  $P_{\text{ext}} = P$ , means process is reversible. For Vanderwaal gas, expression is correct for all reversible process.

8. With respect to the compounds **I-V**, choose the correct statement(s).



- (A) The acidity of compound **I** is due to delocalization in the conjugate base.
- (B) The conjugate base of compound **IV** is aromatic.
- (C) Compound **II** becomes more acidic, when it has a  $-\text{NO}_2$  substituent.
- (D) The acidity of compounds follows the order **I** > **IV** > **V** > **II** > **III**.

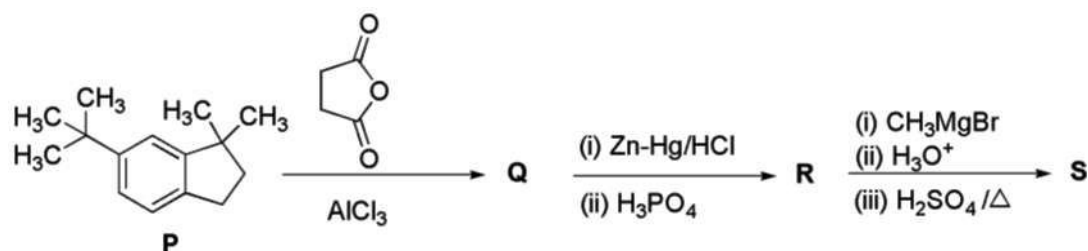
**Ans. (A, B, C)**



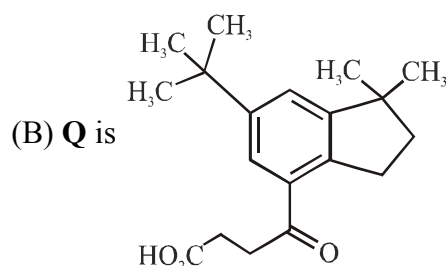
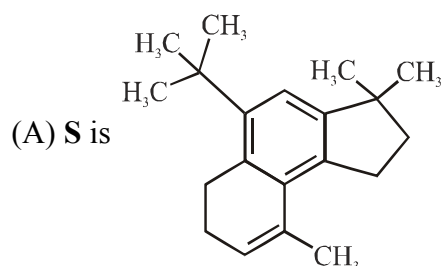
(C)  $-\text{NO}_2$  is  $-I$  group (electron withdrawing group). It increases acid strength.

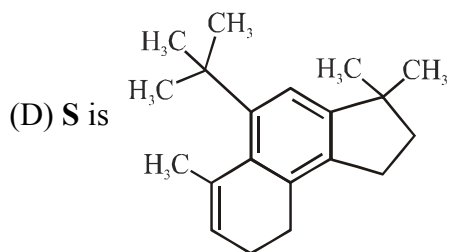
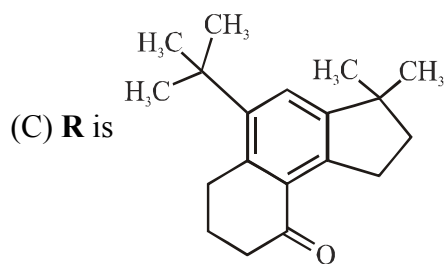
(D) Acid strength order :  $\text{IV} > \text{V} > \text{I} > \text{II} > \text{III}$

9. In the reaction scheme shown below **Q**, **R** and **S** are the major products.



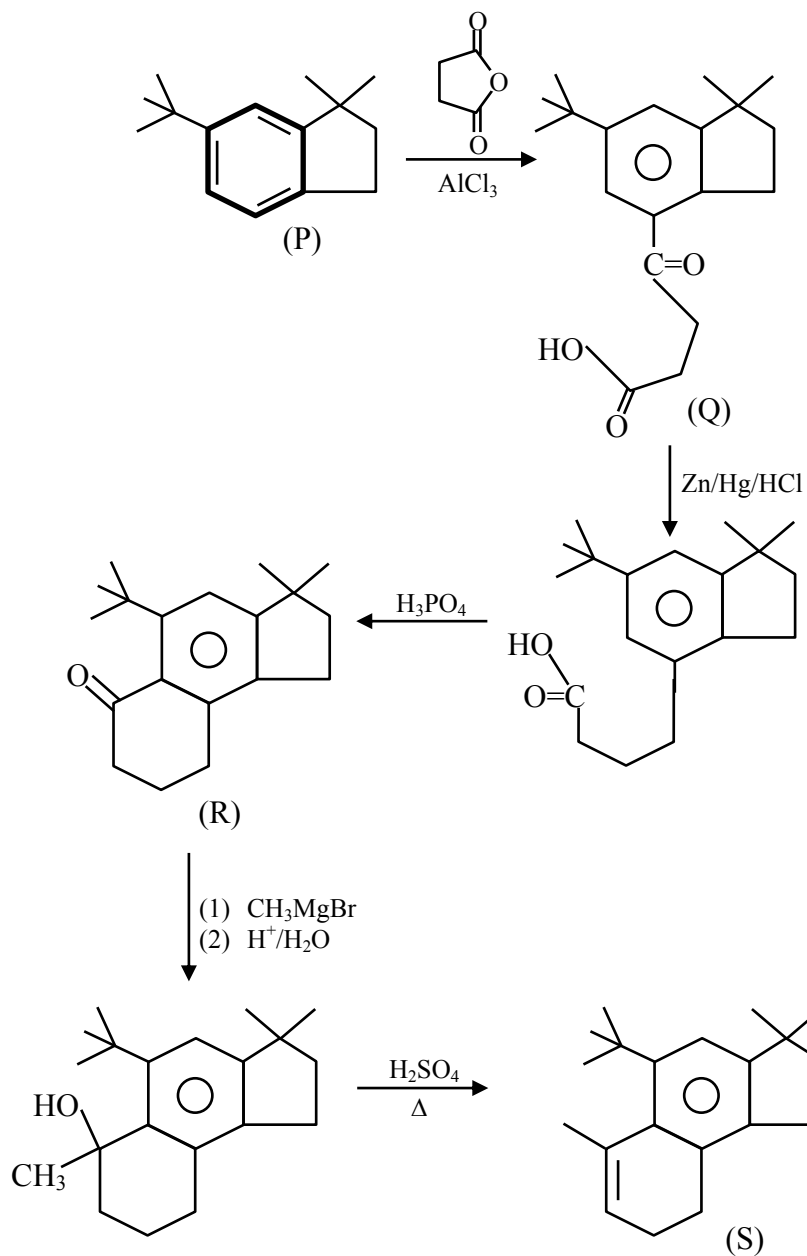
The correct structure of





Ans. (B,D)

Sol.

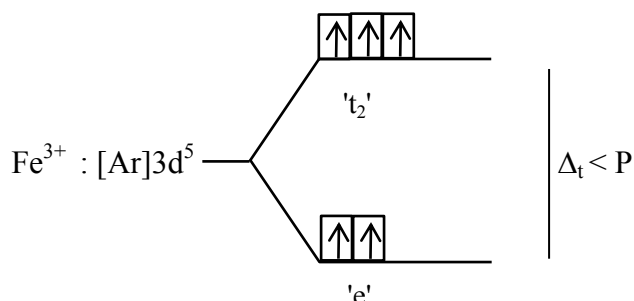


10. Choose the correct statement(s) among the following :

- (A)  $[\text{FeCl}_4]^-$  has tetrahedral geometry.
- (B)  $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$  has 2 geometrical isomers.
- (C)  $[\text{FeCl}_4]^-$  has higher spin-only magnetic moment than  $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$ .
- (D) The cobalt ion in  $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$  has  $\text{sp}^3\text{d}^2$  hybridization.

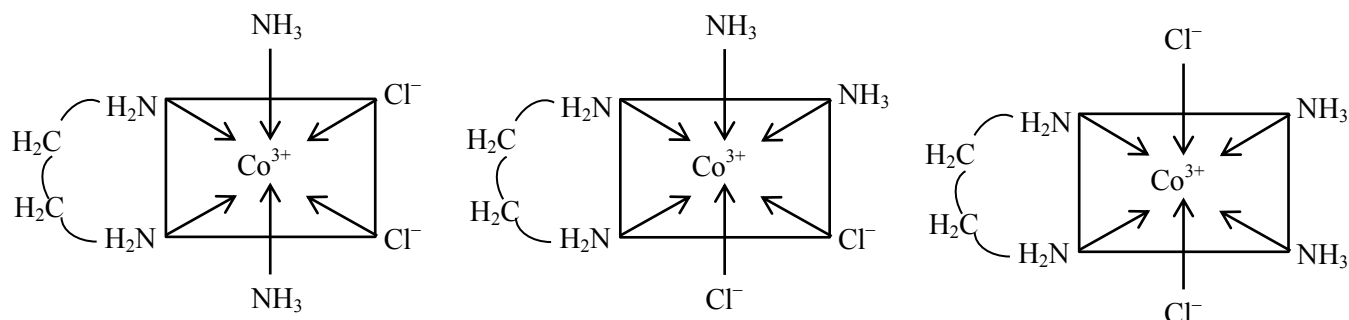
Ans. (A,C)

Sol. (A)  $[\text{FeCl}_4]^-$

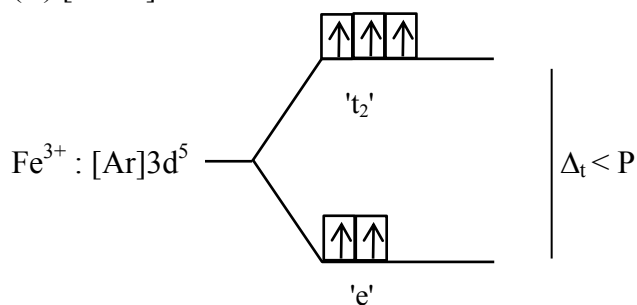


$[\text{FeCl}_4]^-$  is  $\text{sp}^3$  hybridised and has tetrahedral geometry with 5 unpaired electrons.

(B)  $[\text{Co}(\text{en})(\text{NH}_3)_2\text{Cl}_2]^+$  has three geometrical isomers.

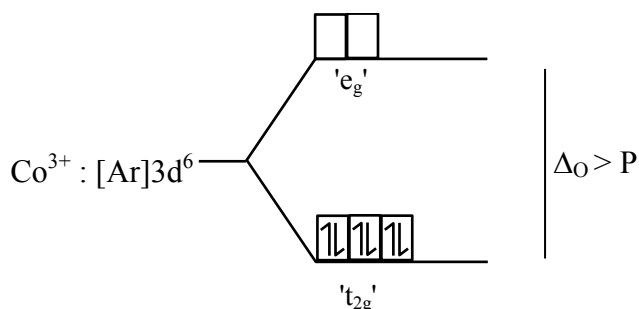


(C)  $[\text{FeCl}_4]^-$



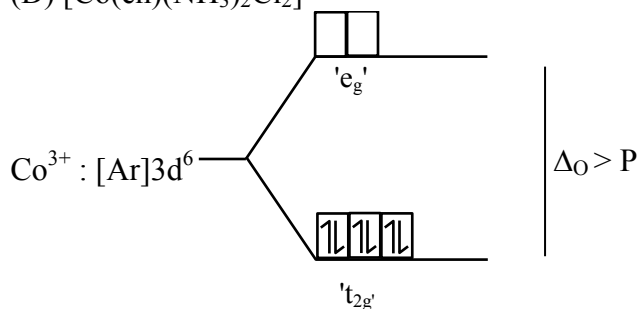
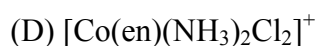
Number of unpaired electrons ( $n$ ) = 5

$$\begin{aligned}\text{Spin only magnetic moment} &= \sqrt{n(n+2)} \text{ B.M.} \\ &= 5.92 \text{ B.M.}\end{aligned}$$



Number of unpaired electrons ( $n$ ) = 0

$$\begin{aligned}\text{Spin only magnetic moment} &= \sqrt{n(n+2)} \text{ B.M.} \\ &= 0\end{aligned}$$



$[\text{Co(en)}(\text{NH}_3)_2\text{Cl}_2]^+$  is  $d^2sp^3$  hybridised and has octahedral geometry with 0 unpaired electron.

11. With respect to hypochlorite, chlorate and perchlorate ions, choose the correct statement(s).

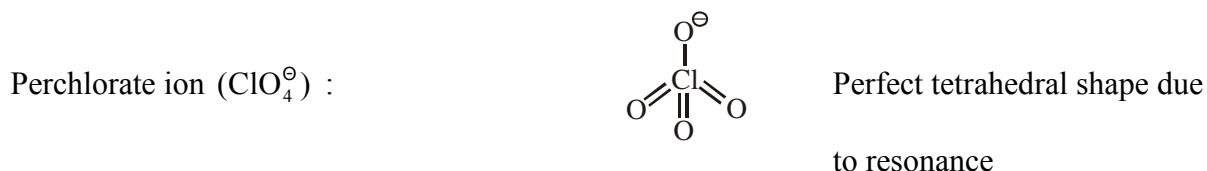
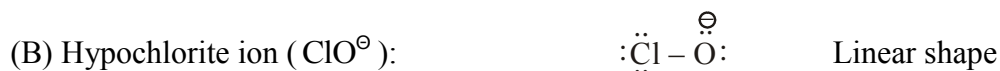
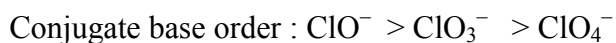
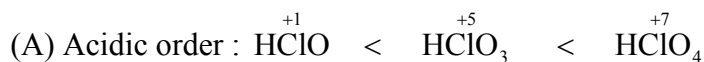
- (A) The hypochlorite ion is the strongest conjugate base.
- (B) The molecular shape of only chlorate ion is influenced by the lone pair of electrons of Cl.
- (C) The hypochlorite and chlorate ions disproportionate to give rise to identical set of ions.
- (D) The hypochlorite ion oxidizes the sulfite ion.

Ans. (A,B,D)

Sol. Hypochlorite ion :  $\text{ClO}^\ominus$

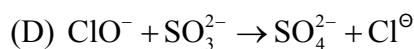
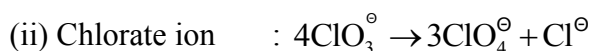
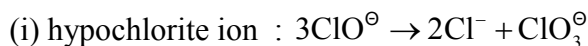
Chlorate ion :  $\text{ClO}_3^\ominus$

Per chlorate ion :  $\text{ClO}_4^\ominus$

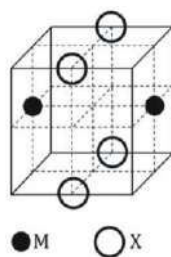


In chlorate ion bond angle changes due to presence of lone pair on chlorine atom. While hypochlorite ion is linear and perchlorate ion is tetrahedral and there is no effect of lone pair on hypochlorite ion.

(C) Disproportionation reaction of



12. The cubic unit cell structure of a compound containing cation M and anion X is shown below. When compared to the anion, the cation has smaller ionic radius. Choose the correct statement(s).



- (A) The empirical formula of the compound is MX.  
 (B) The cation M and anion X have different coordination geometries.  
 (C) The ratio of M-X bond length to the cubic unit cell edge length is 0.866.  
 (D) The ratio of the ionic radii of cation M to anion X is 0.414.

Ans. (A,C)

Sol. (A)  $Z_M = 2 \times \frac{1}{2} = 1$

$$Z_X = 4 \times \frac{1}{4} = 1$$

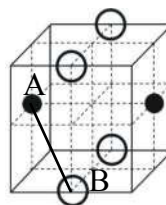
$\therefore$  Empirical formula is MX

(B) Coordinate numbers of both M and X is 8.

(C) Bond length of M – X bond

$$= AB = \sqrt{3} \cdot \frac{a}{2} = 0.866a$$

(D)  $r_M : r_X = (\sqrt{3} - 1) : 1 = 0.732 : 1.000$



### SECTION-3 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme :

*Full Marks* : +4 If ONLY the correct numerical value is entered;

*Zero Marks* : 0 In all other cases.

13. 5.00 mL of 0.10 M oxalic acid solution taken in a conical flask is titrated against NaOH from a burette using phenolphthalein indicator. The volume of NaOH required for the appearance of permanent faint pink color is tabulated below for five experiments. What is the concentration, in molarity, of the NaOH solution ?

Exp. No.	Vol. of NaOH (mL)
1	12.5
2	10.5
3	9.0
4	9.0
5	9.0

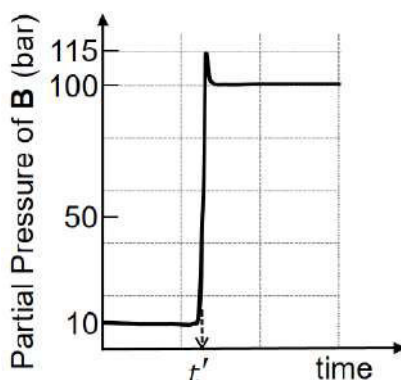
Ans. (0.11)

**Sol.** No. of eq. of oxalic acid = No. of eq. of NaOH

$$\text{or } \frac{5.00 \times 0.10}{1000} \times 2 = \frac{9.0 \times M}{1000} \times 1$$

$$\therefore \text{Molarity of NaOH solution} = \frac{1}{9} = 0.11\text{M}$$

- 14.** Consider the reaction  $\text{A} \rightleftharpoons \text{B}$  at 1000 K. At time  $t'$ , the temperature of the system was increased to 2000 K and the system was allowed to reach equilibrium. Throughout this experiment the partial pressure of A was maintained at 1 bar. Given below is the plot of the partial pressure of B with time. What is the ratio of the standard Gibbs energy of the reaction at 1000 K to that at 2000 K?



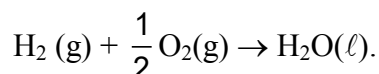
**Ans. (0.25)**

**Sol.**  $K_{\text{eq}} = \frac{[\text{B}]}{[\text{A}]}$

$$K_{1000} = \frac{10}{1} = 10 \text{ and } K_{2000} = \frac{100}{1} = 100$$

$$\text{Now, } \frac{\Delta G_{1000}^0}{\Delta G_{2000}^0} = \frac{(-RT \ln K_{\text{eq}})_{1000}}{(-RT \ln K_{\text{eq}})_{2000}} = \frac{1000 \times \ln 10}{2000 \times \ln 100} = 0.25$$

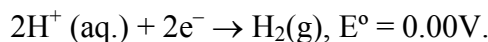
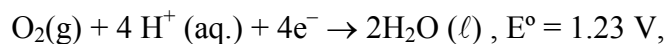
- 15.** Consider a 70% efficient hydrogen-oxygen fuel cell working under standard conditions at 1 bar and 298 K. Its cell reaction is



The work derived from the cell on the consumption of  $1.0 \times 10^{-3}$  mol of  $\text{H}_2(\text{g})$  is used to compress 1.00 mol of a monoatomic ideal gas in a thermally insulated container. What is the change in the temperature (in K) of the ideal gas?



The standard reduction potentials for the two half-cells are given below.



Use  $F = 96500 \text{ C mol}^{-1}$ ,  $R = 8.314 \text{ J mol}^{-1} \text{ K}^{-1}$

**Ans. (13.32)**

**Sol.**  $E_{\text{cell}}^0 = 1.23 - 0.00 = 1.23 \text{ V}$

$$\Delta G_{\text{cell}}^0 = -nF E_{\text{cell}}^0 = -2 \times 96500 \times 1.23 \text{ J}$$

$\therefore$  Work derived from this fuel cell

$$= \frac{70}{100} \times (-\Delta G_{\text{cell}}^0) \times 10^{-3} = x \text{ J}$$

Since insulated vessel, hence  $q = 0$

From equation, for monoatomic gas,

$$w = \Delta U \quad \Rightarrow \quad x = nC_{V,m}\Delta T \left\{ C_{V,m} = \frac{3R}{2} \right\}$$

$$\text{or, } \frac{70}{100} \times (2 \times 96500 \times 1.23) \times 10^{-3} = 1 \times \frac{3}{2} \times 8.314 \times \Delta T$$

$$\therefore \Delta T = 13.32$$

- 16.** Aluminium reacts with sulfuric acid to form aluminium sulfate and hydrogen. What is the volume of hydrogen gas in liters (L) produced at 300 K and 1.0 atm pressure, when 5.4 g of aluminium and 50.0 mL of 5.0 M sulfuric acid are combined for the reaction ?

(Use molar mass of aluminium as  $27.0 \text{ g mol}^{-1}$ ,  $R = 0.082 \text{ atm L mol}^{-1} \text{ K}^{-1}$ )

**Ans. (6.15)**

**Sol.**  $2\text{Al} + 3\text{H}_2\text{SO}_4 \longrightarrow \text{Al}_2(\text{SO}_4)_3 + 3\text{H}_2$

$$\text{Moles of Al taken} = \frac{5.4}{27} = 0.2$$

$$\text{moles of H}_2\text{SO}_4 \text{ taken} = \frac{50 \times 5.0}{1000} = 0.25$$

$$\text{As } \frac{0.2}{2} > \frac{0.25}{3}, \text{ H}_2\text{SO}_4 \text{ is limiting reagent}$$

Now, moles of  $\text{H}_2$  formed =  $\frac{3}{3} \times 0.25 = 0.25$

$$\therefore \text{Volume of } \text{H}_2 \text{ gas formed} = \frac{nRT}{P}$$

$$= \frac{0.25 \times 0.082 \times 300}{1} = 6.15 \text{ L}$$

17.  ${}^{238}_{92}\text{U}$  is known to undergo radioactive decay to form  ${}^{206}_{82}\text{Pb}$  by emitting alpha and beta particles. A rock initially contained  $68 \times 10^{-6} \text{ g}$  of  ${}^{238}_{92}\text{U}$ . If the number of alpha particles that it would emit during its radioactive decay of  ${}^{238}_{92}\text{U}$  to  ${}^{206}_{82}\text{Pb}$  in three half-lives is  $Z \times 10^{18}$ , then what is the value of  $Z$ ?

Ans. (1.20)

Sol. Initial moles of  $\text{U}^{238} = \frac{68 \times 10^{-6}}{238} = x$

Moles of  $\text{U}^{238}$  decayed in three half-lives =  $\frac{7}{8}x$

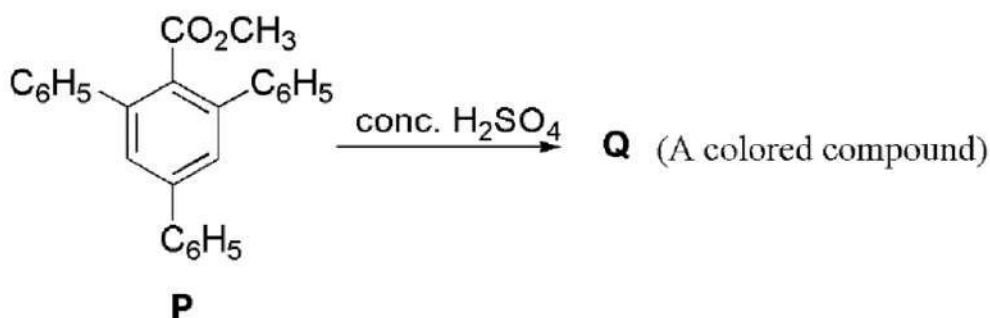
In decay from  $\text{U}^{238}$  to  $\text{Pb}^{206}$ , each  $\text{U}^{238}$  atom decays and produces 8  $\alpha$ -particles and hence, total number of  $\alpha$ -particles emitted out

$$= \left( \frac{7}{8}x \right) \times 8 \times N_A$$

$$= 7 \times \frac{68 \times 10^{-6}}{238} \times 6.022 \times 10^{23}$$

$$= 1.204 \times 10^{18}$$

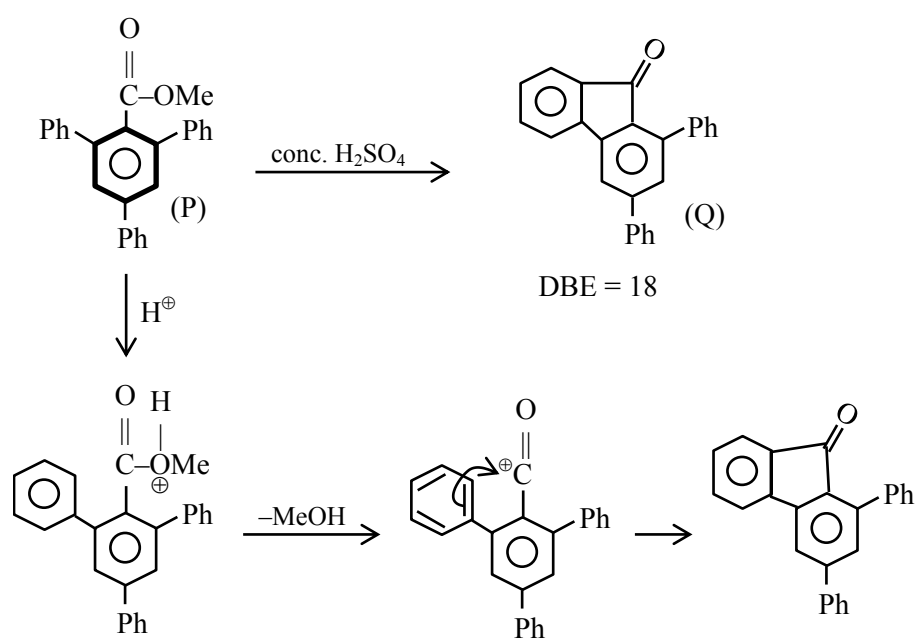
18. In the following reaction, compound **Q** is obtained from compound **P** via an ionic intermediate



What is the degree of unsaturation of **Q**?

Ans. (18.00)

Sol.



**FINAL JEE(Advanced) EXAMINATION - 2020****(Held On Sunday 27<sup>th</sup> SEPTEMBER, 2020)****PAPER-1****TEST PAPER WITH SOLUTION****PART-3 : MATHEMATICS****SECTION-1 : (Maximum Marks : 18)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

*Full Marks* : +3 If ONLY the correct option is chosen;

*Zero Marks* : 0 If none of the options is chosen (i.e. the question is unanswered);

*Negative Marks* : -1 In all other cases.

1. Suppose  $a, b$  denote the distinct real roots of the quadratic polynomial  $x^2 + 20x - 2020$  and suppose  $c, d$  denote the distinct complex roots of the quadratic polynomial  $x^2 - 20x + 2020$ . Then the value of

$$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$$

is

- (A) 0                      (B) 8000                      (C) 8080                      (D) 16000

**Ans. (D)**

**Sol.**  $x^2 + 20x - 2020 = 0$  has two roots  $a, b \in \mathbb{R}$

$x^2 - 20x + 2020 = 0$  has two roots  $c, d \in \text{complex}$

$$\begin{aligned}
 & ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d) \\
 &= a^2c - ac^2 + a^2d - ad^2 + b^2c - bc^2 + b^2d - bd^2 \\
 &= a^2(c + d) + b^2(c + d) - c^2(a + b) - d^2(a + b) \\
 &= (c + d)(a^2 + b^2) - (a + b)(c^2 + d^2) \\
 &= (c + d)((a + b)^2 - 2ab) - (a + b)((c + d)^2 - 2cd) \\
 &= 20[(20)^2 + 4040] + 20[(20)^2 - 4040] \\
 &= 20[(20)^2 + 4040 + (20)^2 - 4040] \\
 &= 20 \times 800 = 16000
 \end{aligned}$$

2. If the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = |x|(x - \sin x)$ , then which of the following statements is **TRUE** ?

- (A)  $f$  is one-one, but **NOT** onto  
 (B)  $f$  is onto, but **NOT** one-one  
 (C)  $f$  is **BOTH** one-one and onto  
 (D)  $f$  is **NEITHER** one-one **NOR** onto

**Ans. (C)**

**Sol.**  $f(x)$  is a non-periodic, continuous and odd function

$$f(x) = \begin{cases} -x^2 + x \sin x, & x < 0 \\ x^2 - x \sin x, & x \geq 0 \end{cases}$$

$$f(-\infty) = \lim_{x \rightarrow -\infty} (-x^2) \left( 1 - \frac{\sin x}{x} \right) = -\infty$$

$$f(\infty) = \lim_{x \rightarrow \infty} x^2 \left( 1 - \frac{\sin x}{x} \right) = \infty$$

$$\Rightarrow \text{Range of } f(x) = \mathbb{R}$$

$$\Rightarrow f(x) \text{ is an onto function... (1)}$$

$$f'(x) = \begin{cases} -2x + \sin x + x \cos x, & x < 0 \\ 2x - \sin x - x \cos x, & x \geq 0 \end{cases}$$

For  $(0, \infty)$

$$f'(x) = (x - \sin x) + x(1 - \cos x)$$

always +ve    always +ve  
or 0                    or 0

$$\Rightarrow f'(x) > 0$$

$$\Rightarrow f'(x) \geq 0, \forall x \in (-\infty, \infty)$$

$$\text{equality at } x = 0$$

$$\Rightarrow f(x) \text{ is one-one function .... (2)}$$

From (1) & (2),  $f(x)$  is both one-one & onto.

3. Let the functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = e^{x-1} - e^{-|x-1|} \text{ and } g(x) = \frac{1}{2}(e^{x-1} + e^{1-x}).$$

Then the area of the region in the first quadrant bounded by the curves  $y = f(x)$ ,  $y = g(x)$  and  $x = 0$  is

(A)  $(2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$

(B)  $(2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})$

(C)  $(2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})$

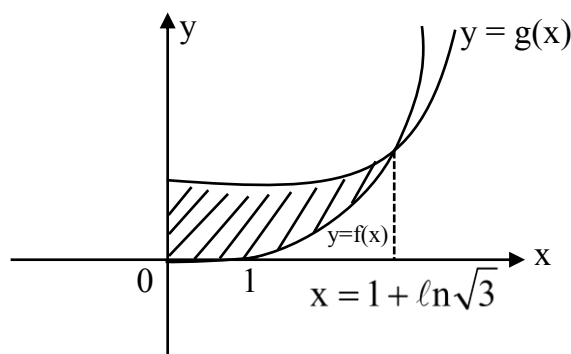
(D)  $(2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})$

**Ans. (A)**

**Sol.** Here

$$f(x) = \begin{cases} 0 & x \leq 1 \\ e^{x-1} - e^{1-x} & x \geq 1 \end{cases}$$

$$\& g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$$



$$\text{solve } f(x) \& g(x) \Rightarrow x = 1 + \ln \sqrt{3}$$

$$\text{So bounded area} = \int_0^1 \frac{1}{2}(e^{x-1} + e^{1-x}) dx + \int_1^{1+\ln \sqrt{3}} \frac{1}{2}(e^{x-1} + e^{1-x}) - (e^{x-1} - e^{1-x}) dx$$

$$= \frac{1}{2} [e^{x-1} - e^{1-x}]_0^1 + \left[ -\frac{1}{2} e^{x-1} - \frac{3}{2} e^{1-x} \right]_1^{1+\ln \sqrt{3}}$$

$$= \frac{1}{2} \left[ e - \frac{1}{e} \right] + \left[ \left( -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \right) + 2 \right] = 2 - \sqrt{3} + \frac{1}{2} \left( e - \frac{1}{e} \right)$$

Ans. (A)

4. Let  $a$ ,  $b$  and  $\lambda$  be positive real numbers. Suppose  $P$  is an end point of the latus rectum of the parabola  $y^2 = 4\lambda x$ , and suppose the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  passes through the point  $P$ . If the tangents to the parabola and the ellipse at the point  $P$  are perpendicular to each other, then the eccentricity of the ellipse is

- (A)  $\frac{1}{\sqrt{2}}$       (B)  $\frac{1}{2}$       (C)  $\frac{1}{3}$       (D)  $\frac{2}{5}$

Ans. (A)

**Sol.**  $y^2 = 4\lambda x$ ,  $P(\lambda, 2\lambda)$

Slope of the tangent to the parabola at point  $P$

$$\frac{dy}{dx} = \frac{4\lambda}{2y} = \frac{4\lambda}{2 \times 2\lambda} = 1$$

Slope of the tangent to the ellipse at  $P$

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

As tangents are perpendicular  $y' = -1$

$$\Rightarrow \frac{2\lambda}{a^2} - \frac{4\lambda}{b^2} = 0 \Rightarrow \frac{a^2}{b^2} = \frac{1}{2}$$

$$\Rightarrow e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Ans. A

5. Let  $C_1$  and  $C_2$  be two biased coins such that the probabilities of getting head in a single toss are  $\frac{2}{3}$  and  $\frac{1}{3}$ , respectively. Suppose  $\alpha$  is the number of heads that appear when  $C_1$  is tossed twice, independently, and suppose  $\beta$  is the number of heads that appear when  $C_2$  is tossed twice, independently. Then probability that the roots of the quadratic polynomial  $x^2 - \alpha x + \beta$  are real and equal, is

- (A)  $\frac{40}{81}$  (B)  $\frac{20}{81}$  (C)  $\frac{1}{2}$  (D)  $\frac{1}{4}$

Ans. (B)

Sol.  $P(H) = \frac{2}{3}$  for  $C_1$

$P(H) = \frac{1}{3}$  for  $C_2$

for  $C_1$

No. of Heads ( $\alpha$ )	0	1	2
Probability	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$

for  $C_2$

No. of Heads ( $\beta$ )	0	1	2
Probability	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

for real and equal roots

$$\alpha^2 = 4\beta$$

$$(\alpha, \beta) = (0, 0), (2, 1)$$

$$\text{So, probability} = \frac{1}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{4}{9} = \frac{20}{81}$$

6. Consider all rectangles lying in the region

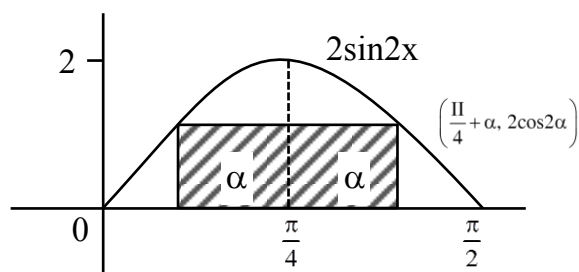
$$\left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq \frac{\pi}{2} \text{ and } 0 \leq y \leq 2\sin(2x) \right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

- (A)  $\frac{3\pi}{2}$                       (B)  $\pi$                       (C)  $\frac{\pi}{2\sqrt{3}}$                       (D)  $\frac{\pi\sqrt{3}}{2}$

Ans. (C)

Sol.



$$\text{Perimeter} = 2(2\alpha + 2\cos 2\alpha)$$

$$P = 4(\alpha + \cos 2\alpha)$$

$$\frac{dP}{d\alpha} = 4(1 - 2\sin 2\alpha) = 0$$

$$\sin 2\alpha = \frac{1}{2}$$

$$2\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\frac{d^2P}{d\alpha^2} = -4\cos 2\alpha$$

$$\text{for maximum } \alpha = \frac{\pi}{12}$$

$$\text{Area} = (2\alpha)(2\cos 2\alpha)$$

$$= \frac{\pi}{6} \times 2 \times \frac{\sqrt{3}}{2} = \frac{\pi}{2\sqrt{3}}$$



**SECTION-2 : (Maximum Marks : 24)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :
 

<i>Full Marks</i>	: +4	If only (all) the correct option(s) is(are) chosen;
<i>Partial Marks</i>	: +3	If all the four options are correct but ONLY three options are chosen;
<i>Partial Marks</i>	: +2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
<i>Partial Marks</i>	: +1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
<i>Zero Marks</i>	: 0	If none of the options is chose (i.e. the question is unanswered);
<i>Negative Marks</i>	: -2	In all other cases

7. Let the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^3 - x^2 + (x-1) \sin x$  and let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be an arbitrary function. Let  $fg: \mathbb{R} \rightarrow \mathbb{R}$  be the product function defined by  $(fg)(x) = f(x)g(x)$ . Then which of the following statements is/are TRUE ?
- (A) If  $g$  is continuous at  $x = 1$ , then  $fg$  is differentiable at  $x = 1$
- (B) If  $fg$  is differentiable at  $x = 1$ , then  $g$  is continuous at  $x = 1$
- (C) If  $g$  is differentiable at  $x = 1$ , then  $fg$  is differentiable at  $x = 1$
- (D) If  $fg$  is differentiable at  $x = 1$ , then  $g$  is differentiable at  $x = 1$

**Ans. (A,C)**

**Sol.**  $f: \mathbb{R} \rightarrow \mathbb{R}$   $f(x) = (x^2 + \sin x)(x-1)$   $f(1^+) = f(1^-) = f(1) = 0$

$$fg(x) : f(x).g(x) \quad fg : \mathbb{R} \rightarrow \mathbb{R}$$

$$\text{let } fg(x) = h(x) = f(x).g(x) \quad h: \mathbb{R} \rightarrow \mathbb{R}$$

$$\text{option (c)} \quad h'(x) = f'(x)g(x) + f(x)g'(x)$$

$$h'(1) = f'(1)g(1) + 0,$$

$$(\text{as } f(1) = 0, g'(x) \text{ exists})$$

$\Rightarrow$  if  $g(x)$  is differentiable then  $h(x)$  is also differentiable (true)

option (A) If  $g(x)$  is continuous at  $x = 1$  then  $g(1^+) = g(1^-) = g(1)$

$$h'(1^+) = \lim_{h \rightarrow 0^+} \frac{h(1+h) - h(1)}{h}$$

$$h'(1^+) = \lim_{h \rightarrow 0^+} \frac{f(1+h)g(1+h) - 0}{h} = f'(1)g(1)$$

$$h'(1^-) = \lim_{h \rightarrow 0^+} \frac{f(1-h)g(1-h) - 0}{-h} = f'(1)g(1)$$

So  $h(x) = f(x).g(x)$  is differentiable

at  $x = 1$  (True)

option (B) (D)  $h'(1^+) = \lim_{h \rightarrow 0^+} \frac{h(1+h) - h(1)}{-h}$

$$h'(1^+) = \lim_{h \rightarrow 0^+} \frac{f(1+h)g(1+h)}{h} = f'(1)g(1^+)$$

$$h'(1^-) = \lim_{h \rightarrow 0^+} \frac{f(1-h)g(1-h)}{-h} = f'(1).g(1^-)$$

$$\Rightarrow g(1^+) = g(1^-)$$

So we cannot comment on the continuity and differentiability of the function.

8. Let  $M$  be a  $3 \times 3$  invertible matrix with real entries and let  $I$  denote the  $3 \times 3$  identity matrix. If  $M^{-1} = \text{adj}(\text{adj } M)$ , then which of the following statement is/are ALWAYS TRUE ?

- (A)  $M = I$                       (B)  $\det M = 1$                       (C)  $M^2 = I$                       (D)  $(\text{adj } M)^2 = I$

**Ans. (B,C,D)**

**Sol.**  $\det(M) \neq 0$

$$M^{-1} = \text{adj}(\text{adj } M)$$

$$M^{-1} = \det(M).M$$

$$M^{-1}M = \det(M).M^2$$

$$I = \det(M).M^2 \quad \dots (i)$$

$$\det(I) = (\det(M))^5$$

$$1 = \det(M) \quad \dots (ii)$$

$$\text{From (i)} \quad I = M^2$$

$$(\text{adj } M)^2 = \text{adj}(M^2) = \text{adj } I = I$$

9. Let  $S$  be the set of all complex numbers  $z$  satisfying  $|z^2 + z + 1| = 1$ . Then which of the following statements is/are TRUE ?

(A)  $\left|z + \frac{1}{2}\right| \leq \frac{1}{2}$  for all  $z \in S$

(B)  $|z| \leq 2$  for all  $z \in S$

(C)  $\left|z + \frac{1}{2}\right| \geq \frac{1}{2}$  for all  $z \in S$

(D) The set  $S$  has exactly four elements

**Ans. (B,C)**

**Sol.**  $|z^2 + z + 1| = 1$

$$\Rightarrow \left| \left( z + \frac{1}{2} \right)^2 + \frac{3}{4} \right| = 1$$

$$\Rightarrow \left| \left( z + \frac{1}{2} \right)^2 + \frac{3}{4} \right| \leq \left| z + \frac{1}{2} \right|^2 + \frac{3}{4}$$

$$\Rightarrow 1 \leq \left| z + \frac{1}{2} \right|^2 + \frac{3}{4} \Rightarrow \left| \left( z + \frac{1}{2} \right)^2 \right| \geq \frac{1}{4}$$

$$\Rightarrow \left| z + \frac{1}{2} \right| \geq \frac{1}{2}$$

also  $|(z^2 + z) + 1| = 1 \geq ||z^2 + z| - 1|$

$$\Rightarrow |z^2 + z| - 1 \leq 1$$

$$\Rightarrow |z^2 + z| \leq 2$$

$$\Rightarrow ||z^2| - |z|| \leq |z^2 + z| \leq 2$$

$$\Rightarrow |r^2 - r| \leq 2$$

$$\Rightarrow r = |z| \leq 2; \forall z \in S$$

Also we can always find root of the equation  $z^2 + z + 1 = e^{i\theta}; \forall \theta \in \mathbb{R}$

Hence set 'S' is infinite

- 10.** Let  $x, y$  and  $z$  be positive real numbers. Suppose  $x, y$  and  $z$  are lengths of the sides of a triangle opposite to its angles  $X, Y$  and  $Z$ , respectively. If

$$\tan \frac{X}{2} + \tan \frac{Z}{2} = \frac{2y}{x+y+z},$$

then which of the following statements is/are TRUE?

(A)  $2Y = X + Z$

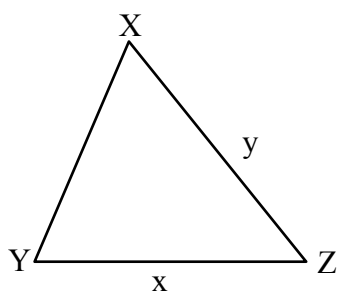
(B)  $Y = X + Z$

(C)  $\tan \frac{X}{2} = \frac{x}{y+z}$

(D)  $x^2 + z^2 - y^2 = xz$

**Ans. (B,C)**

Sol.



$$\tan \frac{x}{2} + \tan \frac{z}{2} = \frac{2y}{x+y+z}$$

$$\frac{\Delta}{S(S-x)} + \frac{\Delta}{S(S-z)} = \frac{2y}{2S}$$

$$\frac{\Delta}{S} \left( \frac{2S-(x+z)}{(S-x)(S-z)} \right) = \frac{y}{S}$$

$$\Rightarrow \frac{\Delta y}{S(S-x)(S-z)} = \frac{y}{S}$$

$$\Rightarrow \Delta^2 = (S-x)^2 (S-z)^2$$

$$\Rightarrow S(S-y) = (S-x)(S-z)$$

$$\Rightarrow (x+y+z)(x+z-y) = (y+z-x)(x+y-z)$$

$$\Rightarrow (x+z)^2 - y^2 = y^2 - (z-x)^2$$

$$\Rightarrow (x+z)^2 + (x-z)^2 = 2y^2$$

$$\Rightarrow x^2 + z^2 = y^2 \Rightarrow \angle Y = \frac{\pi}{2}$$

$$\Rightarrow \angle Y = \angle X + \angle Z$$

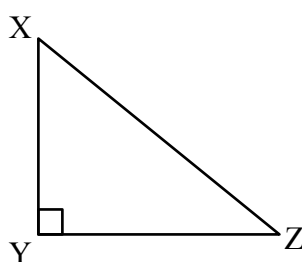
$$\tan \frac{x}{2} = \frac{\Delta}{S(S-x)}$$

$$\tan \frac{x}{2} = \frac{\frac{1}{2}xz}{\frac{(y+z)^2 - x^2}{4}}$$

$$\tan \frac{x}{2} = \frac{2xz}{y^2 + z^2 + 2yz - x^2}$$

$$\tan \frac{x}{2} = \frac{2xz}{2z^2 + 2yz} \quad (\text{using } y^2 = x^2 + z^2)$$

$$\tan \frac{x}{2} = \frac{x}{y+z}$$



11. Let  $L_1$  and  $L_2$  be the following straight line.

$$L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3} \text{ and } L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$$

Suppose the straight line

$$L : \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$$

lies in the plane containing  $L_1$  and  $L_2$ , and passes through the point of intersection of  $L_1$  and  $L_2$ . If the line  $L$  bisects the acute angle between the lines  $L_1$  and  $L_2$ , then which of the following statements is/are TRUE?

(A)  $\alpha - \gamma = 3$

(B)  $l + m = 2$

(C)  $\alpha - \gamma = 1$

(D)  $l + m = 0$

Ans. (A,B)

Sol. Point of intersection of  $L_1$  &  $L_2$  is  $(1, 0, 1)$

Line  $L$  passes through  $(1, 0, 1)$

$$\frac{1-\alpha}{l} = -\frac{1}{m} = \frac{1-\gamma}{-2} \quad \dots(1)$$

acute angle bisector of  $L_1$  &  $L_2$

$$\vec{r} = \hat{i} + \hat{k} + \lambda \left( \frac{\hat{i} - \hat{j} + 3\hat{k} - 3\hat{i} - \hat{j} + \hat{k}}{\sqrt{11}} \right)$$

$$\vec{r} = \hat{i} + \hat{k} + t(\hat{i} + \hat{j} - 2\hat{k})$$

$$\Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{-2}{-2} \Rightarrow l = m = 1$$

$$\text{From (1)} \quad \frac{1-\alpha}{1} = -1 \Rightarrow \alpha = 2$$

$$\& \frac{1-\gamma}{-2} = -1 \Rightarrow \gamma = -1$$

12. Which of the following inequalities is/are TRUE?

(A)  $\int_0^1 x \cos x dx \geq \frac{3}{8}$

(B)  $\int_0^1 x \sin x dx \geq \frac{3}{10}$

(C)  $\int_0^1 x^2 \cos x dx \geq \frac{1}{2}$

(D)  $\int_0^1 x^2 \sin x dx \geq \frac{2}{9}$

Ans. (A,B,D)

**Sol.** (A)  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x \geq 1 - \frac{x^2}{2}$$

$$\int_0^1 x \cos x \geq \int_0^1 x \left(1 - \frac{x^2}{2}\right) = \frac{1}{2} - \frac{1}{8}$$

$$\int_0^1 x \cos x \geq \frac{3}{8} \quad (\text{True})$$

(B)  $\sin x \geq x - \frac{x^3}{6}$

$$\int_0^1 x \sin x \geq \int_0^1 x \left(x - \frac{x^3}{6}\right) dx$$

$$\int_0^1 x \sin x \geq \frac{1}{3} - \frac{1}{30} \Rightarrow \int_0^1 x \sin x \, dx \geq \frac{3}{8} \quad (\text{True})$$

(D)  $\int_0^1 x^2 \sin x \, dx \geq \int_0^1 x^2 \left(x - \frac{x^3}{6}\right) dx$

$$\int_0^1 x^2 \sin x \, dx \geq \frac{1}{4} - \frac{1}{36}$$

$$\int_0^1 x^2 \sin x \, dx \geq \frac{2}{9} \quad (\text{True})$$

(C)  $\cos x < 1$

$$x^2 \cos x < x^2$$

$$\int_0^1 x^2 \cos x \, dx < \int_0^1 x^2 dx$$

$$\int_0^1 x^2 \cos x \, dx < \frac{1}{3}$$

So option 'C' is incorrect.

## SECTION-3 : (Maximum Marks : 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme :  
*Full Marks* : +4 If ONLY the correct numerical value is entered;  
*Zero Marks* : 0 In all other cases.

13. Let  $m$  be the minimum possible value of  $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$ , where  $y_1, y_2, y_3$  are real numbers for which  $y_1 + y_2 + y_3 = 9$ . Let  $M$  be the maximum possible value of  $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$ , where  $x_1, x_2, x_3$  are positive real numbers for which  $x_1 + x_2 + x_3 = 9$ . Then the value of  $\log_2(m^3) + \log_3(M^2)$  is \_\_\_\_\_.

Ans. (8.00)

Sol.  $\frac{3^{y_1} + 3^{y_2} + 3^{y_3}}{3} \geq \left[ 3^{(y_1+y_2+y_3)} \right]^{\frac{1}{3}}$

$$\Rightarrow 3^{y_1} + 3^{y_2} + 3^{y_3} \geq 3^4$$

$$\Rightarrow \log_3(3^{y_1} + 3^{y_2} + 3^{y_3}) \geq 4$$

$$\Rightarrow m = 4$$

Also,  $\frac{x_1 + x_2 + x_3}{3} \geq \sqrt[3]{x_1 x_2 x_3}$

$$\Rightarrow x_1 x_2 x_3 \leq 27$$

$$\Rightarrow \log_3 x_1 + \log_3 x_2 + \log_3 x_3 \leq 3$$

$$\Rightarrow M = 3$$

$$\begin{aligned} \text{Thus, } \log_2(m^3) + \log_3(M^2) &= 6 + 2 \\ &= 8 \end{aligned}$$

14. Let  $a_1, a_2, a_3, \dots$  be a sequence of positive integers in arithmetic progression with common difference 2. Also, let  $b_1, b_2, b_3, \dots$  be a sequence of positive integers in geometric progression with common ratio 2. If  $a_1 = b_1 = c$ , then the number of all possible values of  $c$ , for which the equality

$$2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$$

holds for some positive integer  $n$ , is \_\_\_\_\_

Ans. (1.00)

**Sol.** Given  $2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$

$$\Rightarrow 2 \times \frac{n}{2} (2c + (n-2)x_2) = c \left( \frac{2^n - 1}{2 - 1} \right)$$

$$\Rightarrow 2n^2 - 2n = c(2^n - 1 - 2n)$$

$$\Rightarrow c = \frac{2n^2 - 2n}{2^n - 1 - 2n} \in \mathbb{N}$$

$$\text{So, } 2n^2 - 2n \geq 2^n - 1 - 2n$$

$$\Rightarrow 2n^2 + 1 \geq 2^n \Rightarrow n < 7$$

$$\Rightarrow n \text{ can be } 1, 2, 3, \dots,$$

Checking  $c$  against these values of  $n$

we get  $c = 12$  (when  $n = 3$ )

Hence number of such  $c = 1$

**15.** Let  $f : [0, 2] \rightarrow \mathbb{R}$  be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right)$$

If  $\alpha, \beta \in [0, 2]$  are such that  $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$ , then the value of  $\beta - \alpha$  is \_\_\_\_\_

**Ans. (1.00)**

**Sol.** Let  $\pi x - \frac{\pi}{4} = \theta \in \left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$

$$\text{So, } \left(3 - \sin\left(\frac{\pi}{2} + 2\theta\right)\right) \sin \theta \geq \sin(\pi + 3\theta)$$

$$\Rightarrow (3 - \cos 2\theta) \sin \theta \geq -\sin 3\theta$$

$$\Rightarrow \sin \theta [3 - 4\sin^2 \theta + 3 - \cos 2\theta] \geq 0$$

$$\Rightarrow \sin \theta (6 - 2(1 - \cos 2\theta) - \cos 2\theta) \geq 0$$

$$\Rightarrow \sin \theta (4 + \cos 2\theta) \geq 0$$

$$\Rightarrow \sin \theta \geq 0$$

$$\Rightarrow \theta \in [0, \pi] \Rightarrow 0 \leq \pi x - \frac{\pi}{4} \leq \pi$$

$$\Rightarrow x \in \left[\frac{1}{4}, \frac{5}{4}\right]$$

$$\Rightarrow \beta - \alpha = 1$$

**16.** In a triangle PQR, let  $\vec{a} = \overrightarrow{QR}$ ,  $\vec{b} = \overrightarrow{RP}$  and  $\vec{c} = \overrightarrow{PQ}$ . If

$$|\vec{a}| = 3, |\vec{b}| = 4 \text{ and } \frac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}, \text{ then the value of } |\vec{a} \times \vec{b}|^2 \text{ is } \underline{\hspace{2cm}}$$

**Ans. (108.00)**



**Sol.** We have  $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

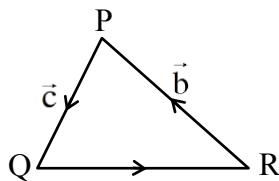
$$\Rightarrow \vec{c} = -\vec{a} - \vec{b}$$

$$\text{Now, } \frac{\vec{a} \cdot (-\vec{a} - 2\vec{b})}{(-\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})} = \frac{3}{7}$$

$$\Rightarrow \frac{9 + 2\vec{a} \cdot \vec{b}}{9 - 16} = \frac{3}{7}$$

$$\Rightarrow \vec{a} \cdot \vec{b} = -6$$

$$\Rightarrow |\vec{a} \times \vec{b}|^2 = a^2 b^2 - (\vec{a} \cdot \vec{b})^2 = 9 \times 16 - 36 = 108$$



**17.** For a polynomial  $g(x)$  with real coefficient, let  $m_g$  denote the number of distinct real roots of  $g(x)$ .

Suppose  $S$  is the set of polynomials with real coefficient defined by

$$S = \{(x^2 - 1)^2 (a_0 + a_1 x + a_2 x^2 + a_3 x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

For a polynomial  $f$ , let  $f'$  and  $f''$  denote its first and second order derivatives, respectively. Then the minimum possible value of  $(m_{f'} + m_{f''})$ , where  $f \in S$ , is \_\_\_\_\_

**Ans. (5.00)**

**Sol.**  $f(x) = (x^2 - 1)^2 h(x)$ ;  $h(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\text{Now, } f(1) = f(-1) = 0$$

$$\Rightarrow f'(\alpha) = 0, \alpha \in (-1, 1) \quad [\text{Rolle's Theorem}]$$

Also,  $f'(1) = f'(-1) = 0 \Rightarrow f'(x) = 0$  has atleast 3 root,  $-1, \alpha, 1$  with  $-1 < \alpha < 1$

$$\Rightarrow f''(x) = 0 \text{ will have at least 2 root, say } \beta, \gamma \text{ such that}$$

$$-1 < \beta < \alpha < \gamma < 1 \quad [\text{Rolle's Theorem}]$$

$$\text{So, } \min(m_{f''}) = 2$$

$$\text{and we find } (m_{f'} + m_{f''}) = 5 \text{ for } f(x) = (x^2 - 1)^2.$$

Thus, Ans. 5

**18.** Let  $e$  denote the base of the natural logarithm. The value of the real number  $a$  for which the right hand limit

$$\lim_{x \rightarrow 0^+} \frac{(1-x)^{\frac{1}{x}} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is \_\_\_\_\_.

**Ans. (1.00)**

**Sol.**  $\lim_{x \rightarrow 0^+} \frac{e^{\left(\frac{\ell n(1-x)}{x}\right)} - \frac{1}{e}}{x^a}$

$$= \lim_{x \rightarrow 0^+} \frac{1}{e} \frac{e^{\left(1 + \frac{\ell n(1-x)}{x}\right)} - 1}{x^a}$$

$$= \frac{1}{e} \lim_{x \rightarrow 0^+} \frac{1 + \frac{\ell n(1-x)}{x}}{x^a}$$

$$= \frac{1}{e} \lim_{x \rightarrow 0^+} \frac{\ell n(1-x) + x}{x^{(a+1)}}$$

$$= \frac{1}{e} \lim_{x \rightarrow 0^+} \frac{\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots\right) + x}{x^{a+1}}$$

Thus,  $a = 1$