HW 4. C.R.V, Uniform, Normal, derived from Normal

5.6 3) $f(x) = \begin{cases} 0 & x < 0 \\ cx & 0 < x < 1.5 \\ c(3-x) & 1.5 < x < 3 \\ 0 & x > 3 \end{cases}$ $f: \mathbb{R} \to \mathbb{R}$

(a) If f is a pdf, $f(x) \geq 0$ for all x. &
Area $= 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$1 = \int_0^{1.5} cx\,dx + \int_{1.5}^{3} (3c - cx)\,dx.$$

$$1 = \int_0^{1.5} cx\,dx + \int_{1.5}^{3} 3c\,dx - \int_{1.5}^{3} cx\,dx$$

$$1 = c\left(\frac{x^2}{2}\right)_0^{1.5} + 3c(x)_{1.5}^{3} - c\left(\frac{x^2}{2}\right)_{1.5}^{3}$$

$$1 = c\left[\frac{(1.5)^2}{2} + 9 - 4.5 - \frac{(3)^2}{2} + \frac{(1.5)^2}{2}\right]$$

$$2 = c\left[(1.5)^2 + 18 - 9 - 9 + (1.5)^2\right]$$

$$2 = c\left[(1.5)^2 + (1.5)^2\right].$$

$$1 = (1.5)^2 c$$

$$c = \frac{1}{(1.5)^2} = \frac{1}{(3/2)^2} = \boxed{\frac{4}{9}}$$

(b) $EX = \int_{-\infty}^{\infty} x\, f(x)\, dx.$

$$= \frac{4}{9}\int_{0}^{1.5} x \cdot x \, dx + \frac{4}{9}\int_{1.5}^{3} x \cdot 3 \, dx - \frac{4}{9}\int_{1.5}^{3} x \cdot x \, dx.$$

$$= \frac{4}{9}\left[\left(\frac{x^3}{3}\right)_{0}^{1.5} + \left(\frac{3x^2}{2}\right)_{1.5}^{3} - \left(\frac{x^3}{3}\right)_{1.5}^{3}\right]$$

$$= \frac{4}{9}\left[\frac{(1.5)^3}{3} + \frac{3^3 - 3(1.5)^2}{2} - \frac{3^3}{3} + \frac{(1.5)^3}{3}\right]$$

$$= \frac{4}{9}\left[\frac{2\cdot(1.5)^3 + 3^4 - 3^2(1.5)^2 - 3^3\cdot 2 + 2(1.5)^3}{6}\right]$$

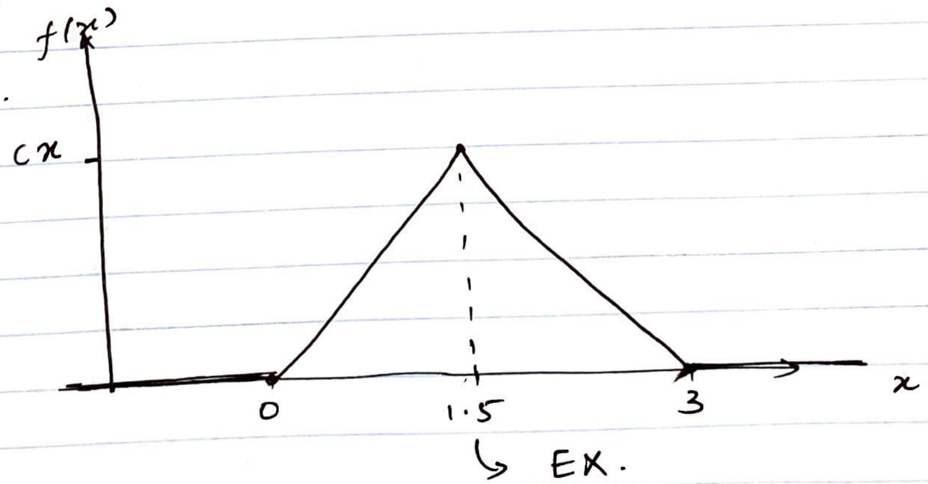
$$= \frac{4}{9}\left[\frac{3\cdot(1.5)^2 + 3^4 - 3^2(1.5)^2 - 3^2\cdot 2 + 3(1.5)^2}{6}\right]$$

$$= \frac{2\times 2}{9}\left[\frac{2(1.5)^2 + 3^3 - 3(1.5)^2 - 6}{2}\right].$$

$$= \frac{4}{9}\left[1.125 + 2.25\right].$$

$$= \boxed{1.5}.$$

Hint :-



$$E(X^2) = c\left[\int_0^{1.5} x^2 x\,dx + \cancel{3}\int_{1.5}^{3} x^2 \cdot 3\,dx \;\cancel{+}\!-\int_{1.5}^{3} x^2 \cdot x\,dx\right].$$

$$= c\left[\left(\frac{x^4}{4}\right)_0^{1.5} + 3\left(\frac{x^3}{3}\right)_{1.5}^{3} \;\cancel{+}\!-\left(\frac{x^4}{4}\right)_{1.5}^{3}\right]$$

$$= c\left[\frac{(1.5)^4}{4} + (3)^3 - (1.5)^3 - \frac{(3)^4}{4} + \frac{(1.5)^4}{4}\right]$$

$$= c\left[\frac{(1.5)^4}{2} + 23.625 - \frac{(3)^4}{4}\right].$$

$$= c\left[\;\cdots\;\right].$$ was not needed.

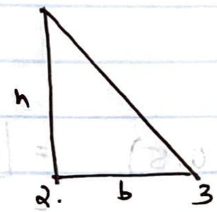
$$= c\left[1.265 + 4.64\right].$$

$$= 2.65$$

(c) $P(X > 2) = 1 - P(X \le 2)$

$= 1 - P(X < 2)$ [for continous R.V]

$=$ Area $(2, \infty)$.



$b = 1 = 3 - 2.$

$h,\ f(x = 2) = c(3-2) = c = 4/9 = 0.45$

Area $= \frac{1}{2} \times bh = \frac{1}{2} \times 0.45 \times 1 = \boxed{0.225}$.

(d) $Y \sim$ Uniform $(0, 3)$ $\mu = 0,\ \sigma^2 = 3.$

Var $X = EX^2 - (EX)^2 = 2.65 - (1.5)^2 = 0.4.$

Var $Y = \frac{(b-a)^2}{12} = \frac{(3-0)^2}{12} = \frac{9}{12} = \frac{3}{4} = 0.75.$

$\therefore$ Y has larger variance..

7) $X \sim \text{Normal}(\mu = -5, \sigma^2 = 100)$.

(a) $P(X < 0) = \text{pnorm}(0, -5, 10) = \boxed{0.69}$

$$P\left(\frac{X-\mu}{\sigma} < \frac{0-(-5)}{10}\right)$$

$$P\left(\frac{X-\mu}{\sigma} < 0.5\right) = \text{pnorm}(0.5) = \boxed{0.69}$$
$$= \Phi(0.5).$$

(b) $P(X > 5) = 1 - P(X \leq 5)$

$$= 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{5+5}{10}\right)$$

$$= 1 - P\left(\frac{X-\mu}{\sigma} \leq 1\right)$$

$$= 1 - \text{pnorm}(1) = 1 - \Phi(1)$$

$$= \boxed{0.158} = 1 - \text{pnorm}(5, -5, 10)$$

(c) $P(-3 < X < 7) = P(X > -3)$ and $P(X < 7)$

$= P(X < 7) - P(X < -3)$ · [acc to area].

$= \boxed{0.305}$

$$P\left(\frac{-3+5}{10} < \frac{X-\mu}{\sigma} < \frac{7+5}{10}\right)$$

$$= P\left(\frac{1}{5} < \frac{X-\mu}{\sigma} < \frac{12}{10}\right)$$

$$= P\left(0.2 < \frac{X-\mu}{\sigma} < 1.2\right) = \text{pnorm}(1.2) - \text{pnorm}(0.2)$$
$$= \boxed{0.305}$$

$$= \Phi(1.2) - \Phi(0.2)$$

(d) $P(|X+5|<10)$; $|X+5|<10$

$X+5<10$, $X+5>-10$

$X<5$, $X>-15$

$P(-15<X<5) = -\text{pnorm}(-15,-5,10) + \text{pnorm}(5,-5,10)$

$= \boxed{0.682}$

$P\left(\frac{-15+5}{10} < \frac{X-\mu}{\sigma} < \frac{5+5}{10}\right) = P\left(\frac{-10}{10} < \frac{X-\mu}{\sigma} < \frac{10}{10}\right)$

$\Phi(1) - \Phi(-1) = \text{pnorm}(1) - \text{pnorm}(-1)$

$= \boxed{0.682}$.

(e) $P(|X-3|>2) = P(-2 < X-3 < 2)$

$= P(1 < X < 5)$

$= P\left(\frac{1+5}{10} < X < \frac{5+5}{10}\right)$

$= -\Phi(0.6) + \Phi(1)$

$= \text{pnorm}(1) - \text{pnorm}(0.6)$

$= \boxed{0.115}$

$|X-3|>2$.

$X-3>2$ $\quad$ $X-3<-2$

$X>5$ $\quad$ $X<1$ $\Rightarrow$

$1-P(X\leq 5) + P(X<1)$

$1-P\left(\frac{X-\mu}{\sigma} \leq \frac{5+5}{10}\right) + P\left(\frac{X-\mu}{\sigma} < \frac{1+5}{10}\right)$

$1-\Phi(1) + \Phi(0.6)$

$= \boxed{0.884}$

8) $X_1 \sim \text{Normal}(1, 9)$ $\sigma_1 = 3$

$X_2 \sim \text{Normal}(3, 16)$ $\sigma_2 = 4$

$X_1 \perp X_2$

(a) $X_1 + X_2$. ~~Z~~ $\sim \text{Normal}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$\sim \text{Normal}(\underline{4, 25})$.

μ, var.

(b) $-X_2$. $\sim \text{Normal}(\mu_2, \sigma_2^2)$

$\sim \text{Normal}(\underline{-3, 16})$

$\left[\begin{array}{l} \because E(cX) = c\,EX \\ \because Var(cX) = c^2\,Var\,X \end{array} \right]$

(c) $X_1 - X_2 \sim \text{Normal}(1 + (-3), 9 + 16)$

$\sim \text{Normal}(\underline{-2, 25})$.

(d) $2X_1 \sim$ ~~$X_1 + X_1 \sim$ Normal~~

$(2)X_1 \sim \text{Normal}(2\mu_1, (2)^2\sigma_1^2)$

$\sim \text{Normal}(2 \times 1, 4 \times 9)$

$\sim \text{Normal}(\underline{2, 36})$.

(e) $2X_1 - 2X_2$ $E(2X_1 + (-2)X_2) = 2 \times 1 + (-2) \times 3$

$= 2 - 6$

$= -4$

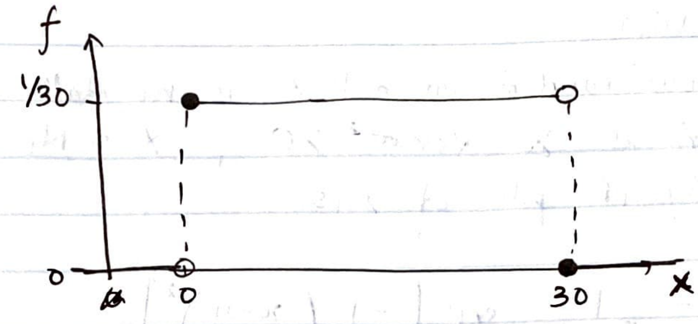
$Var(2X_1 + (-2)X_2) = (2)^2 \times 9 + (-2)^2 \times 16$

$= 36 + 64$

$= 100$

$\therefore 2X_1 - 2X_2 \sim \text{Normal}(\underline{-4, 100})$.

OTP 1)



PDF.

(a) $P(X \in [25,30)) = \text{Area}_{[25,30)} = 5 \times \frac{1}{30} = \underline{\underline{\frac{1}{6}}}$.

(b) $P(X \in [0,5)) = \text{Area}_{[0,5)} = \cancel{5\times} (5-0)\frac{1}{30} = \underline{\underline{\frac{1}{6}}}$.

yes ~~P(X~~ Prob of order arriving in first five minutes = prob of order arriving in the last five minutes.

(c) $\mu = 18, \sigma = 6$.

$$P(X > 25) = 1 - P(X \leq 25)$$
$$= 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{25-18}{6}\right)$$
$$= 1 - \Phi(7/6)$$
$$= \boxed{0.121}$$

(d) $P(X < 0) = P\left(\frac{X-\mu}{6} < \frac{0-18}{6}\right)$

$= \cancel{\Phi(-0.3)}$ $\quad \cancel{\Phi(0.3)}$ $\Phi(-3)$

$= \boxed{\cancel{0.382}}$ $\quad = \boxed{0.001}$

(e) $P(X > 30) = 1 - P(X \leq 30)$
$$= 1 - P\left(\frac{X-\mu}{6} < \frac{30-18}{6}\right)$$
$$= 1 - \Phi(2)$$
$$= \boxed{0.022}$$

Hence only 2.2% deliveries are taking more time than 30 minutes

∴ I think that business is doing well

OTP 3) • $N(\mu, \sigma^2)$ here expected value or population mean is 'μ' & Variance is 'σ^2'. and

- Hence Standard deviation is 'σ'.
- And, the pdf for Normal distribution is symmetric about 'μ'.
- around 99% of data is within '$\mu+3\sigma$'. i.e 3 standad deviations. & $N(S) \in (-\infty, \infty)$.
- In real life, it is observed that all (most of) of the data is normally distributed.
- For, Normal Distribution, most of the data is around mean, hence the pdf has a spike like structure on the mean & flat as we go far away.

• $N(0,1)$, here mean = 0, Variance = 1.

- This is a special case of Normal distribution
- Since Normally distributed data are symmetric about the mean, we can convert any Normally distributed data to standard form using '$\frac{X-\mu}{\sigma}$' = $\Phi(\_)$.

• $\chi^2(n)$. , here n = degrees of freedom.

- It is derived from Normal distribution (standard).
- where $n = [1 \text{ to } \infty)$

$$\chi^2(n) \Leftarrow Y = Z_1^2 + Z_2^2 + \dots Z_n^2.$$

where Z_x^2, $x \in [1, \infty]$ are Normal (0,1)

$\therefore Y \sim \chi^2(n)$.

$Y(S) = [0, \infty)$ $[\because Z_1^2 \geq 0]$.

- Expected value of Y is 'n'. because $E Z_x^2 = 1$.

& $Y = \Sigma Z_x^2$.

- Chi-square is the distribution of sum of squared- Normal distributions.

- $t(\nu)$, ν = degree of freedom.

- $T = \dfrac{Z}{\sqrt{Y/\nu}}$ where Z is standard Normally dist R.V. & Y is $\chi^2(\nu)$.
- $T(S) \in (-\infty, \infty)$. $[\because \alpha\ Z]$.
- The shape is similar to normal distribution i.e symmetric about the origin but has slightly higher values in tails.
- But when degrees of freedom is large enough it is (t) approximately same as standard Normal distributi

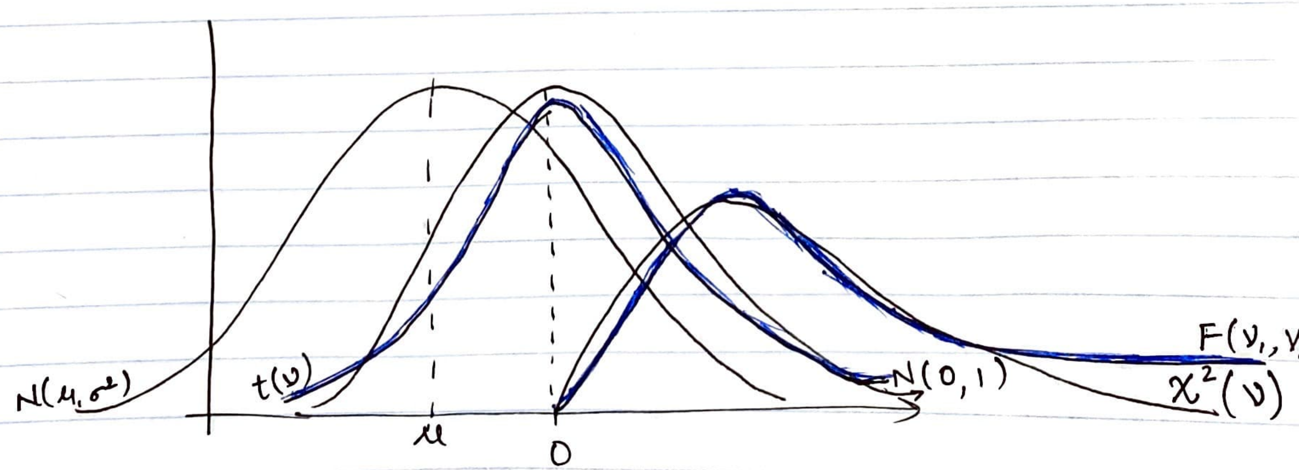
- $F(\nu_1, \nu_2)$. ν_1 = numerator degrees of freedom
  ν_2 = denominator degree of freedom

- where $F = \dfrac{Y_1/\nu_1}{Y_2/\nu_2}$, Y_1, Y_2 are $\chi^2(\nu_1), \chi^2(\nu_2)$

$F(S) = [0, \infty)$ $\because [\chi^2(S) = [0, \infty)]$

~~here, if $\nu_1 = 1$ then~~

- If T belongs to student's t distribution with degree of freedom $= \nu$. then T^2. ~~is~~ follows F distribution with $\nu_1 = 1$ and $\nu_2 = \nu$.



OTP2). As we can see from the figure that df increases , t distribution seems more like the Normal distribution.

here, red with df = 10 is more similar to Normal dist (the green line) than the blue curve which is t distributed with df = 3.

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R 4.2.1

temp.R HW4_OTP2.R

```r
#Standard Normal
x = seq(-10,10, length.out=10^6)
fxn=dnorm(x,0,1)
plot(x=x,y=fxn,type='l',col='green')

# T with df=3
fxt1=dt(x,df=3)
lines(x=x,y=fxt1,col='blue')

# T with df=10
fxt2=dt(x,df=10)
lines(x=x,y=fxt2,col='red')
```

8:29 (Top Level) R Script

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