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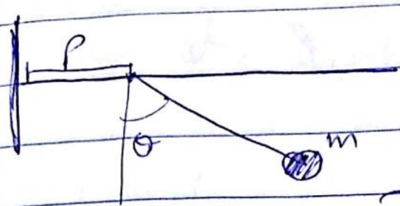
KDC homework 4



$$r = \begin{bmatrix} p + l\sin\theta \\ -l\cos\theta \end{bmatrix}$$

$$\dot{r} = \begin{bmatrix} \dot{p} + l\cos\theta\,\dot{\theta} \\ l\sin\theta\,\dot{\theta} \end{bmatrix}$$

$$T = \frac{1}{2} m \|\dot{r}\|^2$$

$$= \frac{1}{2}\left[m\left[\dot{p}^2 + l^2\cos^2\theta\,\dot{\theta}^2 + 2\dot{p}\,l\cos\theta\,\dot{\theta} + \cancel{2}\,l^2\sin^2\theta\,\dot{\theta}^2\right]\right]$$

$$T = \frac{1}{2} m\left[\dot{p}^2 + l^2\dot{\theta}^2 + 2\dot{p}\,l\cos\theta\,\dot{\theta}\right]$$

$$V = -mgl\cos\theta$$

$$L = T - V$$

$$L = \frac{1}{2}\left[m\left(\dot{p}^2 + l^2\dot{\theta}^2 + 2\dot{p}\,l\cos\theta\,\dot{\theta}\right] + mgl\cos\theta\right]$$

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$$\frac{\partial L}{\partial \theta} = \frac{1}{2} m [0 + 0 - 2\dot{p} l \sin\theta] - mgl \sin\theta$$

$$= m(-\dot{p} l \dot{\theta} \sin\theta) - mgl \sin\theta$$

$$= -m\dot{p} l \dot{\theta} \sin\theta - mgl \sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} [0 + l^2 2\dot{\theta} + 2\dot{p} l \cos\theta]$$

$$= ml^2\dot{\theta} + m\dot{p} l \cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} - m\dot{p} l \sin\theta \dot{\theta} + ml \cos\theta \ddot{p}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = ml^2\ddot{\theta} - m\dot{p} l \sin\theta \dot{\theta} + ml\cos\theta \ddot{p} + m\dot{p} l \dot{\theta} \sin\theta + mgl \sin\theta$$

$$\Rightarrow \boxed{ml^2\ddot{\theta} + ml\cos\theta \ddot{p} + mgl\sin\theta} \quad - (1)$$

$$\frac{\partial L}{\partial p} = \frac{1}{2} m (0 + 0 + 0) + 0 = 0$$

$$\frac{\partial L}{\partial \dot{p}} = \frac{1}{2} m (2\dot{p} + 0 + 2l\cos\theta \dot{\theta}) + 0$$

$$= m\dot{p} + ml \cos\theta \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right) = m\ddot{p} + ml\cos\theta \ddot{\theta} - ml \sin\theta \dot{\theta}^2$$

$$\Rightarrow \boxed{m\ddot{p} + ml\cos\theta \ddot{\theta} - ml\sin\theta \dot{\theta}^2 = 0} \quad (2)$$

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dividing eq (1) by ml & eqn (2) by m

$$\Rightarrow \begin{bmatrix} l & \cos\theta \\ l\cos\theta & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{p} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -l\sin\theta\dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{p} \end{bmatrix} + \begin{bmatrix} g\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Q2(a) Ellipsoid

$I_{xx} = \int_v \rho (x^2 + y^2)\, dv \qquad \rho = \frac{M}{\frac{4}{3}\pi abc}$

Substitute $X = \frac{x}{a}$, $Y = \frac{y}{b}$, $Z = \frac{z}{c}$

$I_{xx} = \rho \int_v [(aX)^2 + (bY)^2] \left| \frac{\partial(x.y.z)}{\partial(X,Y,Z)} \right| dX\, dY dZ$

In spherical coordinates

$x = r \sin\theta \cos\phi \qquad \phi = [0, \pi]$

$y = r \sin\theta \sin\phi \qquad \theta = [0, 2\pi)$

$z = r \cos\theta \qquad r = [0, 1]$

$I_{zz} = \rho abc \int_0^1 \int_0^{2\pi} \int_0^{\pi} (a^2 r^2 \cos^2\theta \sin^2\phi + b^2 r^2 \sin^2\theta \sin^2\phi) r^2 \sin\phi \, d\phi d\theta dr$

$= \rho abc \int_0^1 \int_0^{2\pi} \int_0^{\pi} (a^2 \cos^2\theta + b^2 \sin^2\theta) r^4 \sin^3\phi \, d\phi\, d\theta\, dr$

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1st integral →

$$\int_0^{\pi} \sin^2\phi \, d\phi = \left[\frac{-1}{3}\sin^2\phi \cos\phi\right]_0^{\pi} + \frac{2}{3}\int_0^{\pi} \sin\phi \, d\phi$$

$$= \left[\frac{-2}{3}\cos\phi\right]_0^{\pi} = \cancel{4}/3$$

2) $$\frac{4}{3}\rho abc \int_0^1 \int_0^{2\pi} \left(a^2\cos^2\theta + b^2\sin^2\theta\right) r^4 \, d\theta \, dr$$

$$2\frac{4}{3}abc \int_0^1 \left(a^2+b^2\right)\pi\, r^4 \, dr$$

$$= \frac{4\pi}{3}\frac{M}{\frac{4}{3}\pi abc}\, abc\left(a^2+b^2\right)\frac{1}{5}$$

$$= \frac{M}{5}\left(a^2+b^2\right)$$

similarly $I_{yy} = \frac{M}{5}\left(a^2+c^2\right)$

$I_{xx} = \frac{M}{5}\left(b^2+c^2\right)$

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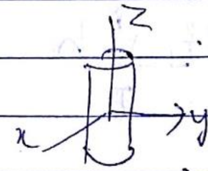
$$I_{ellipse} = \begin{bmatrix} \frac{M}{5}(b^2+c^2) & 0 & 0 \\ 0 & \frac{M}{5}(a^2+c^2) & 0 \\ 0 & 0 & \frac{M}{5}(a^2+b^2) \end{bmatrix}$$

Q2(ii) Cylinder



$\rho = \frac{M}{\pi r^2 h}$

$$I_{xx} = \rho \iiint (y^2+z^2)\,dx\,dy\,dz$$

$$\rho = \frac{M}{\pi r^2 h}$$

In cylindrical coordinates

$x' = r\cos\theta \quad y' = r\sin\theta \quad z' = z$

$$dV = r\,dr\,d\theta\,dz$$

$$I_{xx} = \rho \int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^{r} (r^2\sin^2\theta + z^2)\,r\,dr\,d\theta\,dz$$

$$= \rho \int_{-h/2}^{h/2}\int_0^{2\pi} \left(\frac{r^4}{4}\sin^2\theta + \frac{r^2}{2}z^2\right)d\theta\,dz$$

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$$= \rho \int_{-h/2}^{h/2} \left[\frac{\pi r^4}{4} + \pi r^2 z^2 \right] dz$$

$$= \rho \left[\frac{\pi r^4 h}{4} + \pi \frac{r^2 h^3}{12} \right]$$

$$= \frac{M}{\pi r^2 h} \left[\frac{\pi r^4 h}{4} + \frac{\pi r^2 h^3}{12} \right]$$

$$I_{xx} = \frac{M}{12} \left[h^2 + 3r^2 \right]$$

From symmetry $I_{xx} = I_{yy}$

$$I_{zz} = \rho \iiint (x^2 + y^2)\, dx\, dy\, dz$$

$$= \rho \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^r r^2\, r\, dr\, d\theta\, dz$$

$$= \rho \int_{-h/2}^{h/2} \frac{\pi r^4}{2}\, dz$$

$$= \frac{M}{\pi r^2 h} \cdot \frac{\pi r^4 h}{2}$$

$$= \frac{M r^2}{2}$$

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$$I = \begin{bmatrix} \frac{M}{12}\left(h^2+3r^2\right) & 0 & 0 \\ 0 & \frac{M}{12}\left(h^2+3r^2\right) & 0 \\ 0 & 0 & \frac{Mr^2}{2} \end{bmatrix}$$

Q3 –

3.1

The velocity of the COM is $Vcom = \begin{bmatrix} 0.01 \\ 0.03 \\ 0.02 \end{bmatrix}$

3.2 The inertia Tensor of the asteroid is –

$$I = \begin{bmatrix} 0.448 & -0.0004 & -0.001 \\ -0.0004 & 1.1234 & -0.0495 \\ -0.001 & -0.0495 & 1.000 \end{bmatrix}$$

3.3 The Inertia tensor of the principal axes is (found from the eigen values of the previous matrix–

$$I = \begin{bmatrix} 0.448 & 0 & 0 \\ 0 & 0.9826 & 0 \\ 0 & 0 & 1.1408 \end{bmatrix}$$

Since all 3 principal moments of inertia are different, it is probably an ellipse.

3.4 The beacon landed on the following coordinates, wrt COM (or body)–

$$Xcom = \begin{bmatrix} 0 \\ -0.0840 \\ -0.288 \end{bmatrix}$$

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The distance from the COM is the norm of the vector = 0.3 units

3.5 The average angular momentum of the asteroid is

$$H = \begin{bmatrix} 0.0209 \\ 0.2304 \\ 0.0581 \end{bmatrix}$$

4. <project proposal sent separately via email>