1. Chapter 1: Introduction

Overview of number partitioning problem.

Application: randomized control trials.

Other applications.

- Circuit design, etc.

Importance as a basic NP-complete problem.

Two questions of interest:

1. What is optimal solution.
2. How to find optimal solution.

1.1. Physical Interpretations

1.2. Statistical-to-Computational Gap

2. Number Packing Problem

Let N be the dimensionality, and $\Sigma_N := \{\pm 1\}$ be the binary cube. Suppose we're given $g \sim \mathcal{N}(0, I_N)$. We want to find $x \in \Sigma_N$ such that we minimize $|\langle x, g\rangle|$.

Definition 2.1. Let $\delta > 0$. The *solution set* $S(\delta; g)$ is the set of all $x \in \Sigma_N$ that are δ-*optimal*, satisfying

$$\frac{1}{\sqrt{N}} |\langle g, x\rangle| \leq 2^{-\delta N}. \tag{2.1}$$

(2.1) Hi

2.1. Existing Results

1. $X_i, 1 \leq i \leq n$ i.i.d. uniform from $\{1, 2, ..., M := 2^m\}$, with $\kappa := \frac{m}{n}$, then phase transition going from $\kappa < 1$ to $\kappa > 1$.
2. Average case, X_i i.i.d. standard Normal.
3. Karmarkar [KKLO86] - NPP value is $\Theta\left(\sqrt{N} 2^{-N}\right)$ whp as $N \to \infty$ (doesn't need Normality).
4. Best polynomial-time algorithm: Karmarkar-Karp [KK82] - Discrepancy $O\left(N^{-\alpha \log N}\right) = 2^{-\Theta(\log^2 N)}$ whp as $N \to \infty$
5. PDM (paired differencing) heuristic - fails for i.i.d. uniform inputs with objective $\Theta\left(n^{-1}\right)$ (Lueker).
6. LDM (largest differencing) heuristic - works for i.i.d. Uniforms, with $n^{-\Theta(\log n)}$ (Yakir, with constant $\alpha = \frac{1}{2 \ln 2}$ calculated non-rigorously by Boettcher and Mertens).
7. Krieger - $O\left(n^{-2}\right)$ for balanced partition.
8. Hoberg [HHRY17] - computational hardness for worst-case discrepancy, as poly-time oracle that can get discrepancy to within $O\left(2^{\sqrt{n}}\right)$ would be oracle for Minkowski problem.
9. Gamarnik-Kizildag: Information-theoretic guarantee $E_n = n$, best computational guarantee $E_n = \Theta\left(\log^2 n\right)$.

10. Existence of m-OGP for $m = O(1)$ and $E_n = \Theta(n)$.
11. Absence for $\omega(1) \leq E_n = o(n)$
12. Existence for $\omega\left(\sqrt{n \log_2 n}\right) \leq E_n \leq o(n)$ for $m = \omega_{n(1)}$ (with changing η, β)
    1. While OGP not ruled out for $E_n \leq \omega\left(\sqrt{n \log_2 n}\right)$, argued that it is tight.
13. For $\varepsilon \in \left(0, \frac{1}{5}\right)$, no stable algorithm can solve $\omega\left(n \log^{-\frac{1}{5}+\varepsilon} n\right) \leq E_n \leq o(n)$
14. Possible to strengthen to $E_n = \Theta(n)$ (as $2^{-\Theta(n)} \leq 2^{-o(n)}$)

3. Glossary and conventions

Conventions:

1. log means log in base 2, exp is exp base 2 - natural log/exponent is $\ln/e^x$.
2.

Glossary:

1. "instance"/"disorder" - g, instance of the NPP problem
2. "discrepancy" - for a given g, value of $\min_{x \in \Sigma_N} |\langle g, x \rangle|$
3. "energy" - negative exponent of discrepancy, i.e. if discrepancy is 2^{-E}, then energy is E. Lower energy indicates "worse" discrepancy.
4. "near-ground state"/"approximate solution"

4. Literature Review

[AC08]

- S2C gap for random constraint satisfaction

[AR06]

-

[Add+17]

-

[Ali+05]

-

[AFG96]

-

[Asp+20]

-

[APZ19]

-

[BPW18]

-

[Ban10]

-

[Bar+16]

- [BFM04]
- [BGT13]
- [BR13]
- [BM08]
- [BCP01]
- [BB19]
- [BBH19]
- [CV13]
- [Che+19]
- [CGJ78]
- [CL91]
- [CE15]
- [COY19]
- [DM15]
- [DKS17]
- [Fel+16]
- [FF98]
- [GK21]

- [Gam+22]
- [GK21]
- [GJW22]
- [GZ19]
- [GZ19]
- [GS13]
- [GJ19]
- [Gam21]
- [GJS21]
- [GS17]
- [GZ19]
- [GJ79]
- [GW98]
- [GW00]
- [Har+23]
- [HTF09]
- [HLS14]
- [Hob+16]

- [Hop+17]
- [Hop18]
- [HSS15]
- [HS25]
- [Jer92]
- [Joh+89]
- [Joh+91]
- [Kar+86]
- [KK83]
- [Kea98]
- [Kız23]
- [Koj10]
- [Kor95]
- [Kor98]
- [Kor09]
- [Kot+17]
- [KKS14]
- [KAK19]

- [KWB19]
- [LW07]
- [LRR17]
- [LM12]
- [Lue87]
- [MPW15]
- [MH78]
- [Mer03]
- [Mer01]
- [MMZ05]
- [Mic+03]
- [O'D21]
- [RSS19]
- [RV17]
- [Rot16]
- [SBD21]
- [SFD96]
- [Tsa92]

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[TMR20]

•

[Wai19]

•

[Wei20]

•

[Wen+23]

•

[Yak96]

• Showed LDM achieves $n^{\log(n)}$ performance despite being a simple heuristic, for uniform instance.

[ZK16]

•

4.1. Our Results

5. Low-Degree Algorithms

For our purposes, an *algorithm* is a function which takes as input a problem instance $g \sim \mathcal{N}(0, I_N)$ and outputs some $x \in \Sigma_N$. This definition can be extended to functions giving outputs on $\mathbf{R}^N$ (and rounding to a vertex on the hypercube Σ_N), or to taking as additional input some randomness ω, allowing for randomized algorithms. However, most of our analysis will focus on the deterministic case.

To further restrict the category of algorithms considered, we specifically restrict to *low-degree algorithms*. Compared to analytically-defined classes of algorithms (e.g. Lipschitz), these algorithms have a regular algebraic structure that we can exploit to precisely control their stability properties. In particular, our goal is to show *strong low-degree hardness*, in the sense of [HS25, Def. 3].

Definition 5.1 (Strong Low-Degree Hardness)**.** A random search problem, namely a N-indexed sequence of input vectors $y_N \in \mathbf{R}^{d_N}$ and random subsets $S_N = S_{N(y_N)} \subseteq \Sigma_N$, exhibits *strong low-degree hardness up to degree $D \leq o(D_N)$* if, for all sequences of degree $o(D_N)$ algorithms $(\mathcal{A}_N)$ with $\mathbf{E}\|\mathcal{A}(y_N)\|^2 \leq O(N)$, we have

$$\mathbf{P}(\mathcal{A}(y_N) \in S_N) \leq o(1).$$

In addition, degree D polynomials are a heuristic proxy for the class of $e^{\tilde{O}(D)}$-time algorithms [Hop18, Kot+17]. Thus, strong low-degree hardness up to $o(N)$ can be thought of as evidence of requiring exponential (i.e. $e^{\Omega(N)}$) time to find globally optimal solutions.

For the case of NPP, we consider two distinct notions of degree. One is traditional polynomial degree, which has an intuitive interpretation, but the other, which we term Efron-Stein degree, is a more flexible notion which can be applied to a much broader class of algorithms. As we will see in

Section 8, these classes of algorithms exhibit quantitatively different behavior, in line with existing heuristics for the "brittleness" of NPP.

5.1. Efron-Stein Degree and L^2 Stability

First, we consider a very general class of putative algorithms, where the notion of "degree" corresponds to how complex the interactions between the input variables can get. Given this notion, deriving stability bounds becomes a straightforward piece of functional analysis. To start, recall the notion of L^2 functions:

Definition 5.2. Let π be a probability distribution on $\mathbf{R}$. The L^2 space $L^2\left(\mathbf{R}^N, \pi^{\otimes N}\right)$ is the space of functions $f : \mathbf{R}^N \to \mathbf{R}$ with finite L^2 norm.

$$\mathbf{E}\left[f^2\right] := \int_{x=(x_1,\dots,x_n)\in\mathbf{R}^N} f(x)^2 \, \mathrm{d}\pi^{\otimes N}(x) < \infty.$$

Alternatively, this is the space of L^2 functions of N i.i.d. random variables x_i, distributed as π.

Given any function $f \in L^2\left(\mathbf{R}^N, \pi^{\otimes N}\right)$, we can consider how it depends on various subsets of the N input coordinates. In principle, everything we want to know about f should be reflected in how it acts on all possible such subsets. To formalize this intuition, we define the following coordinate projection:

Definition 5.3. Let $f \in L^2\left(\mathbf{R}^N, \pi^{\otimes N}\right)$ and $J \subseteq [N]$, with $\overline{J} = [N] \setminus J$. The *projection of f onto J* is the function $f^{\subseteq J} : \mathbf{R}^N \to \mathbf{R}$ given by

$$f^{\subseteq J}(x) = \mathbf{E}[f(x_1, \dots, x_n) \mid x_i, i \in J].$$

This is f with the $\overline{J}$ coordinates re-randomized, so $f^{\subseteq J}$ only depends on x_J.

Intuitively $f^{\subseteq J}$ is the part of f which only depends on the coordinates in J. However, depending on how f accounts for higher-order interactions, it might be the case that $f^{\subseteq J}$ is fully described by some $f^{\subseteq J'}$, for $J' \subsetneq J$. What we really want is to decompose f as

$$f = \sum_{S\subseteq[N]} f^{=S} \tag{5.1}$$

where each $f^{=S}$ only depends on the coordinates in S, but not any smaller subset. That is, if $T \nsubseteq S$ and g depends only on the coordinates in T, then $\left\langle f^{=S}, g\right\rangle = 0$.

This decomposition, often called the *Efron-Stein decomposition*, does indeed exist, and exhibits the following combinatorial construction. Our presentation largely follows [O'D21, § 8.3] (who refers to this as the *orthogonal decomposition*).

The motivating fact is that we should expect that for any $J \subseteq [N]$, we should have

$$f^{\subseteq J} = \sum_{S\subseteq J} f^{=S}. \tag{5.2}$$

Intuitively, $f^{\subseteq J}$ captures everything about f depending on the coordinates in J, and each $f^{\subseteq S}$ captures precisely the interactions within each subset S of J. The construction of $f^{=S}$ proceeds by inverting this formula.

First, we consider the case $J = \varnothing$. It is clear that $f^{=\varnothing} = f^{\subseteq\varnothing}$, which, by Definition 5.3 is the constant function $\mathbf{E}[f]$. Next, if $J = \{j\}$ is a singleton, (5.2) gives

$$f^{\subseteq\{j\}} = f^{=\varnothing} + f^{=\{j\}},$$

and as $f^{\subseteq\{j\}}(x) = \mathbf{E}\left[f \mid x_j\right]$, we get

$$f^{=\{j\}} = \mathbf{E}\left[f \mid x_j\right] - \mathbf{E}[f].$$

This function only depends on x_j; all other coordinates are averaged over, thus measuring how the expectation of f changes given x_j.

Continuing on to sets of two coordinates, some brief manipulation gives, for $J = \{i, j\}$,

$$\begin{aligned} f^{\subseteq\{i,j\}} &= f^{=\varnothing} + f^{=\{i\}} + f^{=\{j\}} + f^{=\{i,j\}} \\ &= f^{\subseteq\varnothing} + \left(f^{\subseteq\{i\}} - f^{\subseteq\varnothing}\right) + \left(f^{\subseteq\{j\}} - f^{\subseteq\varnothing}\right) + f^{=\{i,j\}} \\ \therefore\ f^{=\{i,j\}} &= f^{\subseteq\{i,j\}} - f^{\subseteq\{i\}} - f^{\subseteq\{j\}} + f^{\subseteq\varnothing}. \end{aligned}$$

We can imagine that this accounts for the two-way interaction of i and j, namely $f^{\subseteq\{i,j\}} = \mathbf{E}\left[f \mid x_i, x_j\right]$, while "correcting" for the one-way effects of x_i and x_j individually. Inductively, we can continue on and define all the $f^{=J}$ via inclusion-exclusion, as

$$f^{=J} := \sum_{S\subseteq J} (-1)^{|J|-|S|} f^{\subseteq S} = \sum_{S\subseteq J} (-1)^{|J|-|S|} \mathbf{E}[f \mid x_S].$$

This construction, along with some direct calculations, leads to the following theorem on Efron-Stein decompositions:

Theorem 5.4 ([O'D21, Thm 8.35])**.** *Let $f \in L^2\left(\mathbf{R}^N, \pi^{\otimes N}\right)$. Then f has a unique decomposition as*

$$f = \sum_{S\subseteq[N]} f^{=S}$$

where the functions $f^{=S} \in L^2\left(\mathbf{R}^N, \pi^{\otimes N}\right)$ satisfy

1. *$f^{=S}$ depends only on the coordinates in S;*
2. *if $T \subsetneq S$ and $g \in L^2\left(\mathbf{R}^N, \pi^{\otimes N}\right)$ only depends on coordinates in T, then $\left\langle f^{=S}, g\right\rangle = 0$.*

In addition, this decomposition has the following properties:

3. *Condition 2. holds whenever $S \not\subseteq T$.*
4. *The decomposition is orthogonal: $\left\langle f^{=S}, f^{=T}\right\rangle = 0$ for $S \neq T$.*
5. *$\sum_{S\subseteq T} f^{=S} = f^{\subseteq T}$.*
6. *For each $S \subseteq [N]$, $f \mapsto f^{=S}$ is a linear operator.*

In summary, this desired decomposition of any $L^2(\mathbf{R}^N, \pi^{\otimes N})$ function into it's different interaction levels not only uniquely exists, but is an orthogonal decomposition, enabling us to apply tools from elementary Fourier analysis.

We can finally define the Efron-Stein notion of "degree":

Definition 5.5. The *Efron-Stein degree* of a function $f \in L^2(\mathbf{R}^N, \pi^{\otimes N})$ is

$$\deg_{\mathrm{ES}}(f) = \max_{S \subseteq [N] \text{ s.t. } f^{=S} \neq 0} |S|.$$

If $f = (f_1, ..., f_M) : \mathbf{R}^N \to \mathbf{R}^M$ is a multivariate function, then the Efron-Stein degree of f is the maximum degree of the f_i.

Intuitively, the Efron-Stein degree is the maximum size of multivariate interaction that f accounts for. Of course, this degree is also bounded by N, very much unlike polynomial degree. Note as a special case that any multivariate polynomial of degree D has Efron-Stein degree at most D.

As we are interested in how these function behaves under small changes in its input, we are led to consider the following "noise operator," which lets us measures the effect of small changes in the input on the Efron-Stein decomposition. First, we need the following notion of distance between problem instances:

Definition 5.6. For $p \in [0, 1]$, and $x \in \mathbf{R}^N$, we say $y \in \mathbf{R}^N$ is *p-resampled from* x if y is chosen as follows: for each $i \in [N]$, independently,

$$y_i = \begin{cases} x_i & \text{with probability } p \\ \text{drawn from } \pi & \text{with probability } 1 - p \end{cases}.$$

We say (x, y) is a *p-resampled pair*.

Note that being p-resampled and being p-correlated are rather different - for one, there is a nonzero probability that, for π a continuous probability distribution, $x = y$ when they are p-resampled, even though this a.s. never occurs.

Definition 5.7. For $p \in [0, 1]$, the *noise operator* is the linear operator T_p on $L^2(\mathbf{R}^N, \pi^{\otimes N})$, defined by, for y p-resampled from x

$$T_p f(x) = \mathbf{E}_{y\ p\text{-resampled from } x}[f(y)]$$

In particular, $\langle f, T_p f \rangle = \mathbf{E}_{(x,y)\ p\text{-resampled}}[f(x) \cdot f(y)]$.

As claimed, we can compute how this operator changes the Efron-Stein decomposition:

Lemma 5.8. *Let* $p \in [0, 1]$ *and* $f \in L^2(\mathbf{R}^N, \pi^{\otimes N})$ *have Efron-Stein decomposition* $f = \sum_{S \subseteq [N]} f^{=S}$. *Then*

$$T_p f(x) = \sum_{S \subseteq [N]} p^{|S|} f^{=S}.$$

Proof: Let J denote a p-random subset of $[N]$, i.e. with J formed by including each $i \in [N]$ independently with probability p. By definition, $T_p f(x) = \mathbf{E}_J\left[f^{\subseteq J}(x)\right]$ (i.e. pick a random subset of coordinates to fix, and re-randomize the rest). We know by Theorem 5.4 that $f^{\subseteq J} = \sum_{S \subseteq J} f^{=S}$, so

$$T_p f(x) = \mathbf{E}_J\left[\sum_{S \subseteq J} f^{=S}\right] = \sum_{S \subseteq [N]} \mathbf{E}_J[I(S \subseteq J)] \cdot f^{=S} = \sum_{S \subseteq [N]} p^{|S|} f^{=S},$$

since for a fixed $S \subseteq [N]$, the probability that $S \subseteq J$ is $p^{|S|}$. □

Putting these facts together, we can derive the following stability bound on functions of bounded Efron-Stein degree.

Theorem 5.9. *Let $p \in [0,1]$ and let $f = (f_1, ..., f_M) : \mathbf{R}^N \to \mathbf{R}^M$ be a multivariate function with Efron-Stein degree D and each $f_i \in L^2(\mathbf{R}^N, \pi^{\otimes N})$. Suppose that (x, y) are a p-resampled pair under $\pi^{\otimes N}$, and $\mathbf{E}\|f(x)\|^2 = 1$. Then*

$$\mathbf{E}\|f(x) - f(y)\|^2 \leq 2\left(1 - p^D\right) \leq 2(1-p)D. \tag{5.3}$$

Proof: Observe that

$$\begin{aligned} \mathbf{E}\|f(x) - f(y)\|^2 &= \mathbf{E}\|f(x)\|^2 + \mathbf{E}\|f(y)\|^2 - 2\mathbf{E}\langle f(x), f(y)\rangle \\ &= 2 - 2\left(\sum_i \mathbf{E}[f_i(x) f_i(y)]\right) \\ &= 2 - 2\left(\sum_i \langle f_i, T_p f_i\rangle\right). \end{aligned} \tag{5.4}$$

Here, we have for each $i \in [N]$ that

$$\langle f_i, T_p f_i\rangle = \left\langle \sum_{S \subseteq [N]} f_i^{=S}, \sum_{S \subseteq [N]} p^{|S|} f_i^{=S} \right\rangle = \sum_{S \subseteq [N]} p^{|S|} \left\| f_i^{=S} \right\|^2,$$

by Lemma 5.8 and orthogonality. Now, as each f_i has Efron-Stein degree at most D, the sum above can be taken only over $S \subseteq [N]$ with $0 \leq |S| \leq D$, giving the bound

$$p^D \mathbf{E}\left[f_i(x)^2\right] \leq \langle f_i, T_p f_i\rangle = \mathbf{E}\left[f_i(x) \cdot T_p f_i(x)\right] \leq \mathbf{E}\left[f_i(x)^2\right].$$

Summing up over i, and using that $\mathbf{E}\|f(x)\|^2 = 1$, gives

$$p^D \leq \sum_i \langle f_i, T_p f_i\rangle = \mathbf{E}\langle f(x), f(y)\rangle \leq 1.$$

Finally, we can substitute into (5.4) to get

$$\mathbf{E}\|f(x) - f(y)\|^2 \leq 2 - 2p^D = 2\left(1 - p^D\right) \leq^1 2(1-p)D,$$

[1] This follows from the identity $\left(1 - p^D\right) = (1-p)\left(1 + p + p^2 + ... p^{D-1}\right)$; the bound is tight for $p \approx 1$.

as desired. □

5.2. Hermite Polynomials

Alternatively, we can consider the much more restrictive (but more concrete) class of honest polynomials. When considered as functions of independent Normal variables, such functions admit a simple description in terms of *Hermite polynomials*, which enables us to prove similar bounds as Theorem 5.9. This theory is much more classical, so we encourage the interested reader to see [O'D21, § 11] for details.

To start, we consider the following space of L^2 functions:

Definition 5.10. Let γ_N be the N-dimensional standard Normal measure on $\mathbf{R}^N$. Then the N*-dimensional Gaussian space* is the space $L^2(\mathbf{R}^N, \gamma^N)$ of L^2 functions of N i.i.d. standard Normal random variables.

Note that under the usual L^2 inner product, $\langle f, g\rangle = \mathbf{E}[f \cdot g]$, this is a separable Hilbert space.

It is a well-known fact that the monomials $1, z, z^2, \ldots$ form a complete basis for $L^2(\mathbf{R}, \gamma)$ [O'D21, Thm 11.22]. However, these are far from an orthonormal "Fourier" basis; for instance, we know $\mathbf{E}\left[z^2\right] = 1$ for $z \sim \mathcal{N}(0,1)$. By the Gram-Schmidt process, these monomials can be converted into the *(normalized) Hermite polynomials* h_j for $j \geq 0$, given as

$$h_0(z) = 1, \qquad h_1(z) = z, \qquad h_2(z) = \frac{z^2 - 1}{\sqrt{2}}, \qquad h_3(z) = \frac{z^3 - 3z}{\sqrt{6}}, \qquad \ldots \tag{5.5}$$

Note here that each h_j is a degree j polynomial.

It is then straightforward to show the following:

Theorem 5.11 ([O'D21, Prop 11.30])**.** *The Hermite polynomials* $\left(h_j\right)_{j\geq 0}$ *form a complete orthonormal basis for* $L^2(\mathbf{R}, \gamma)$.

To extend this to $L^2(\mathbf{R}^N, \gamma^N)$, we can take products. For a multi-index $\alpha \in \mathbb{N}^N$, we define the multivariate Hermite polynomial $h_\alpha : \mathbf{R}^N \to \mathbf{R}$ as

$$h_\alpha(z) := \prod_{j=1}^{N} h_{\alpha_j}(z_j).$$

The degree of h_α is clearly $|\alpha| = \sum_j \alpha_j$.

Theorem 5.12. *The Hermite polynomials* $(h_\alpha)_{\alpha\in\mathbb{N}^N}$ *form a complete orthonormal basis for* $L^2(\mathbf{R}^N, \gamma^N)$. *In particular, every* $f \in L^2(\mathbf{R}^N, \gamma^N)$ *has a unique expansion in* L^2 *norm as*

$$f(z) = \sum_{\alpha\in\mathbb{N}^N} \hat{f}(\alpha) h_\alpha(z).$$

As a consequence of the uniqueness of the expansion in , we see that polynomials are their own Hermite expansion. Namely, let $H^{\leq k} \subseteq L^2(\mathbf{R}^N, \gamma^N)$ be the subset of multivariate polynomials of degree at most k. Then, any $f \in H^{\leq k}$ can be Hermite expanded as

$$f(z) = \sum_{\alpha \in \mathbb{N}^N} \hat{f}(\alpha) h_\alpha(z) = \sum_{|\alpha| \leq k} \hat{f}(\alpha) h_\alpha(z).$$

Thus, $H^{\leq k}$ is the closed linear span of the set $\{h_\alpha : |\alpha| \leq k\}$.

When working with honest polynomials, the traditional notion of correlation is a much more natural measure of "distance" between inputs:

Definition 5.13. Let (x, y) be N-dimensional standard Normal vectors. We say (x, y) are *p-correlated* if (x_i, y_i) are p-correlated for each $i \in [N]$, and these pairs are mutually independent.

In a similar way to the Efron-Stein case, we can consider the resulting "noise operator," as a way of measuring a the effect on a function of a small change in the input.

Definition 5.14. For $p \in [0, 1]$, the *Gaussian noise operator* T_p is the linear operator on $L^2(\mathbf{R}^N, \gamma^N)$, given by

$$T_p f(x) = \mathbf{E}_{y\ p\text{-correlated to } x}[f(y)] = \mathbf{E}_{y \sim \mathcal{N}(0, I_N)}\left[f\left(px + \sqrt{1 - p^2} y\right)\right]$$

This operator admits a more classical description in terms of the Ornstein-Uhlenbeck semigroup, but we will not need those connections for what follows. As it happens, a straightforward computation with the Normal moment generating function gives the following:

Lemma 5.15 ([O'D21, Prop 11.37])**.** *Let $p \in [0, 1]$ and $f \in L^2(\mathbf{R}^N, \gamma^N)$. Then $T_p f$ has Hermite expansion*

$$T_p f = \sum_{\alpha \in \mathbb{N}^N} p^{|\alpha|} \hat{f}(\alpha) h_\alpha$$

and in particular,

$$\langle f, T_p f \rangle = \sum_{\alpha \in \mathbb{N}^N} p^{|\alpha|} \hat{f}(\alpha)^2.$$

With this in hand, we can prove a similar stability bound to Theorem 5.9.

Theorem 5.16. *Let $p \in [0, 1]$ and let $f = (f_1, ..., f_M) : \mathbf{R}^N \to \mathbf{R}^M$ be a multivariate polynomial with degree D. Suppose that (x, y) are a p-correlated pair of standard Normal vectors, and $\mathbf{E}\|f(x)\|^2 = 1$. Then*

$$\mathbf{E}\|f(x) - f(y)\|^2 \leq 2\left(1 - p^D\right) \leq 2(1 - p)D. \tag{5.6}$$

Proof: The proof is almost identical to that of Theorem 5.9 (see also [GJW22, Lem. 3.4]). The main modification is to realize that for each f_i, having degree at most D implies that $\hat{f}_i(\alpha) = 0$ for $|\alpha| > D$. Thus, as $p^D \leq p^s \leq 1$ for all $s \leq D$, we can apply Lemma 5.15 to get

$$p^D \mathbf{E}\left[f_i(x)^2\right] \leq \langle f_i, T_p f_i \rangle = \sum_{\alpha \in \mathbb{N}^N : |\alpha| \leq D} p^{|\alpha|} \hat{f}_i(\alpha)^2 \leq \mathbf{E}\left[f_i(x)^2\right].$$

From there, the proof proceeds as before. □

As a comparision to the case for functions with Efron-Stein degree D, notice that Theorem 5.16 gives a much looser bound. For instance, the function $f(x) = x_1^2 x_2^4$ has Efron-Stein degree 2, but polynomial degree 6. In exchange, being able to use p-correlation as a "metric" on the input domain will turn out to offer significant benefits in the arguments which follow, justifying equal consideration of both classes of functions.

5.3. Algorithms

Def. Randomized algorithm

Def. degree of algorithm is degree as multivariate function.

Discussion of how low-degree algs are approximate for class of Lipschitz algorithms?

Need for rounding function to land on Σ_N

Construction of randomized rounding function.

Constr. rounded algorithm.

Lemma. stability of rounding

Thrm. Stability of randomized algorithms (part 1 of Prop 1.9)

Show that Markov gives a useful bound on

Lemma 5.17. *Let $f : \mathbf{R}^N \to \mathbf{R}^N$, $p \in [0,1]$, and X, Y marginally N-dimensional standard Normal vectors. Suppose that $\mathbf{E}\|f(X)\|_2^2 = 1$ and either of the following cases hold:*

I. (X, Y) *are a p-resampled pair, and f is a degree-D function.*

II. (X, Y) *are p-correlated, and f is a degree-D polynomial.*

Then

$$\mathbf{E}\|f(X) - f(Y)\|_2^2 \leq 2\left(1 - p^D\right).$$

Definition 5.18. A *randomized algorithm* is a measurable function $\mathcal{A}^\circ : (g, \omega) \mapsto \boldsymbol{x} \in \mathbf{R}^N$, where $\omega \in \Omega_N$ is an independent random variable in some Polish space. Such an $\mathcal{A}^\circ$ is *deterministic* if it does not depend on ω.

Example. Let $\boldsymbol{U} = (U_1, ..., U_N)$ be i.i.d. Unif($[-1, 1]$). Then, we define the random coordinate-wise function

$$\mathrm{round}_{\boldsymbol{U}}(\boldsymbol{x}) = \left(\mathrm{round}_{U_1}(x_1), ..., \mathrm{round}_{U_N}(x_N)\right),$$

where we define

$$\mathrm{round}_U(x) = \begin{cases} 1 & x \geq U \\ -1 & x < U \end{cases}.$$

Example. Given a real-valued algorithm $\mathcal{A}^\circ$, we can convert it into a Σ_N-valued algorithm $\mathcal{A}$ via

$$\mathcal{A}(g, \omega, \boldsymbol{U}) := \mathtt{round}_{\boldsymbol{U}}\left(\mathcal{A}^\circ(g, \omega)\right).$$

Definition 5.19. Algorithm $\mathcal{A}$ is $(\varepsilon, \eta, p_{\text{unstable}})$-stable if, for g, g' $(1-\varepsilon)$-correlated/resampled, we have

$$\mathbf{P}\Big(\|\mathcal{A}(g) - \mathcal{A}(g')\| \leq \eta\sqrt{N}\Big) \geq 1 - p_{\text{unstable}}.$$

By the will of God (i.e. writeup pending), we have the following:

Lemma 5.20. *Algorithm $\mathcal{A}$ with degree $\leq D$ and norm $\mathbf{E}\|\mathcal{A}(g)\|^2 \leq CN$ has*

$$\mathbf{E}\|\mathcal{A}(g) - \mathcal{A}(g')\|^2 \leq 2CN\varepsilon D,$$

and (because of randomized rounding)

$$\mathbf{E}\|\mathcal{A}(g) - \mathcal{A}(g')\|^4 \leq 16CN^2\varepsilon D.$$

Thus,

$$\mathbf{P}\Big(\|\mathcal{A}(g) - \mathcal{A}(g')\| \geq \eta\sqrt{N}\Big) \leq \frac{16CN^2\varepsilon D}{\eta^4 N^2} \asymp \frac{\varepsilon D}{\eta^4}.$$

As a consequence, a degree D algorithm $\mathcal{A}$ has $p_{\text{unstable}} = o_{N(1)}$ for $\eta^4 \gg \varepsilon D$.

6. Summary of Parameters

Parameter	Meaning	Desired Direction	Intuition
N	Dimension	Large	Showing hardness *asymptotically*, want "bad behavior" to pop up in low dimensions.
δ	Solution tightness; want to find x such that $\lvert\langle g, x\rangle\rvert \leq 2^{-\delta N}$	Small	Smaller δ implies weaker solutions, e.g. $\delta = 0$ is just finding solutions ≤ 1. Want to show even weak solutions are hard to find.
E	Solution tightness; "energy level"; want to find x such that $\lvert\langle g, x\rangle\rvert \leq 2^{-E}$	Small	Smaller E implies weaker solutions, and can consider full range of $1 \ll E \ll N$. Know that $E > \left(\log^2 N\right)$ by Karmarkar-Karp
D	Algorithm degree (in either Efron-Stein sense or usual polynomial sense.)	Large	Higher degree means more complexity. Want to show even complex algorithms fail.
ε	Complement of correlation/ resample probability; (g,g') are $(1-\varepsilon)$-correlated.	Small	ε is "distance" between g, g'. Want to show that small changes in disorder lead to "breaking" of landscape.
η	Algorithm instability; $\mathcal{A}$ is stable if $\lVert\mathcal{A}(g) - \mathcal{A}(g')\rVert \leq \eta\sqrt{N}$, for (g, g') $(1-\varepsilon)$-correlated.	Large	Large η indicates a more unstable algorithm; want to show that even weakly stable algorithms fail.

Table 1: Explanation of Parameters

7. Conditional Landscape Obstruction

Explain what the obstruction is.

We start with a bound on the geometry of the binary hypercube.

Lemma 7.1. *Suppose that $K \leq N/2$, and let $h(x) = -x\log(x) - (1-x)\log(x)$ be the binary entropy function. Then, for $p := K/N$,*

$$\sum_{k \leq K} \binom{N}{k} \leq \exp(Nh(p)) \leq \exp\left(2Np\log\left(\frac{1}{p}\right)\right).$$

Proof: Consider a $\operatorname{Bin}(N, p)$ random variable S. Summing its PMF from 0 to K, we have

$$1 \geq \mathbf{P}(S \leq K) = \sum_{k \leq K} \binom{N}{k} p^k (1-p)^{N-k} \geq \sum_{k \leq K} \binom{N}{k} p^K (1-p)^{N-K}.$$

Here, the last inequality follows from the fact that $p \leq (1-p)$, and we multiply each term by $\left(\frac{p}{1-p}\right)^{K-k} < 1$. Now rearrange to get

$$\begin{aligned}\sum_{k \leq K} \binom{N}{k} &\leq p^{-K}(1-p)^{-(N-K)} \\ &= \exp(-K \log(p) - (N-K)\log(1-p)) \\ &= \exp\left(N \cdot \left(-\frac{K}{N}\log(p) - \left(\frac{N-K}{N}\right)\log(1-p)\right)\right) \\ &= \exp(N \cdot (-p\log(p) - (1-p)\log(1-p))) = \exp(Nh(p)).\end{aligned}$$

The final equality then follows from the bound $h(p) \leq 2p\log(1/p)$ for $p \leq 1/2$. □

Lemma 7.2 (Hypercube Neighborhood Size)**.** *Fix $x \in \Sigma_N$, and let $\eta \leq \frac{1}{2}$. Then the number of x' within distance $2\sqrt{\eta N}$ of x is*

$$\left|\left\{x' \in \Sigma_N \mid \|x - x'\| \leq 2\eta\sqrt{N}\right\}\right| \leq \exp(2\eta\log(1/\eta)N)$$

Proof: Let k be the number of coordinates which differ between x and x' (i.e. the Hamming distance). We have $\|x-x'\|^2 = 4k$, so $\|x - x'\| \leq 2\sqrt{\eta N}$ iff $k \leq N\eta$. Moreover, for $\eta \leq \frac{1}{2}$, $k \leq \frac{N}{2}$. Thus, by Lemma 7.1, we get

$$\sum_{k \leq N\eta} \binom{N}{k} \leq \exp(Nh(\eta)) \leq \exp(2\eta\log(1/\eta)N).$$ □

Next, we can consider what this probability is in the case of correlated Normals.

Lemma 7.3. *Suppose (g, g') are $(1-\varepsilon)$-correlated Normal vectors, and let $x \in \Sigma_N$. Then*

$$\mathbf{P}\left(|\langle g', x\rangle| \leq 2^{-E} \mid g\right) \leq \exp\left(-E - \frac{1}{2}\log(\varepsilon) + O(\log N)\right).$$

Proof: Let $\tilde{g}$ be an independent Normal vector to g, and observe that g' can be represented as $g' = pg + \sqrt{1-p^2}\tilde{g}$, for $p = 1 - \varepsilon$. Thus, $\langle g', x\rangle = p\langle g, x\rangle + \sqrt{1-p^2}\langle \tilde{g}, x\rangle$. Observe $\langle g, x\rangle$ is constant given g, and $\langle \tilde{g}, x\rangle$ is a Normal r.v. with mean 0 and variance N, so $\langle g', x\rangle \sim \mathcal{N}\left(p\langle g, x\rangle, \left(1-p^2\right)N\right)$. This random variable is nondegenerate for $\left(1-p^2\right)N > 0$, with density bounded above as

$$\begin{aligned}\varphi_g(z) = \frac{1}{\sqrt{2\pi(1-p^2)N}} e^{-\frac{(z - p\langle g, x\rangle)^2}{2(1-p^2)N}} &\leq \frac{1}{\sqrt{2\pi(1-p^2)N}} \\ \leq \frac{1}{\sqrt{2\pi\varepsilon N}} &= \exp\left(-\frac{1}{2}\log(\varepsilon) + O(\log N)\right)\end{aligned}$$

Integrating this bound over the interval $|z| \leq 2^{-E}$, we conclude that

$$\mathbf{P}\big(|\langle g', x\rangle| \leq 2^{-E} \mid g\big) = \int_{|z| \leq -2^{-E}} \varphi_{g,|S|}(z)\, \mathrm{d}z \leq \exp\!\left(-E - \frac{1}{2}\log(\varepsilon) + O(\log N)\right). \qquad \square$$

Note for instance that here ε can be exponentially small in E (e.g. $\varepsilon = \exp(-E/10)$), which for the case $E = \Theta(N)$ implies ε can be exponentially small in N.

First, we consider the probability of a solution being optimal for a resampled instance.

Lemma 7.4. *Suppose* (g, g') *are* $(1 - \varepsilon)$*-resampled Normal vectors, and let* $x \in \Sigma_N$*. Then,*

$$\mathbf{P}\big(|\langle g', x\rangle| \leq 2^{-E} \mid g, g \neq g'\big) \leq 2^{-E+O(1)}.$$

Proof: Let $S = \{i \in [N] : g_i \neq g_i'\}$ be the set of indices where g and g' differ. We can express

$$\langle g', x'\rangle = \sum_{i \in [N]} g_i' x_i = \sum_{i \notin S} g_i x_i + \sum_{i \in S} g_{i'} x_i \sim \mathcal{N}\!\left(\sum_{i \notin S} g_i x_i, |S|\right).$$

Let $\mu \coloneqq \sum_{i \notin S} g_i x_i$. The conditional distribution of interest can now be expressed as $\mathbf{P}\big(|\langle g', x'\rangle| \leq 2^{-E} \mid g, |S| \geq 1\big)$. Given $|S| \geq 1$, the quantity $\langle g', x'\rangle$ is a nondegenerate random variable, with density bounded above as

$$\varphi_{g,|S|}(z) = \frac{1}{\sqrt{2\pi|S|}} e^{-\frac{(z-\mu)^2}{2|S|}} \leq \frac{1}{\sqrt{2\pi|S|}} \leq \frac{1}{\sqrt{2\pi}}.$$

Hence, the quantity of interest can be bounded as

$$\mathbf{P}\big(|\langle g', x\rangle| \leq 2^{-E} \mid g, g \neq g'\big) \leq \int_{|z| \leq -2^{-E}} \varphi_{g,|S|}(z)\, \mathrm{d}z \leq \sqrt{\frac{2}{\pi}} 2^{-E} = 2^{-E+O(1)}. \qquad \square$$

Note that in the resampled case, we can compute the probability that $g = g'$ as

$$\mathbf{P}(g = g') = \prod_{i=1}^{N} \mathbf{P}(g_i = g_{i'}) = (1 - \varepsilon)^N,$$

which for $\varepsilon \ll 1$ is approximately $1 - N\varepsilon$. Thus, for $\varepsilon \gg \omega\!\left(\frac{1}{N}\right)$, we have

$$\mathbf{P}\big(|\langle g', x\rangle| \leq 2^{-E} \mid g\big) \leq 2^{-E+O(1)}$$

8. Proof of Strong Low-Degree Hardness

Broad setup

Throughout this section, we let $E = \delta N$ for some $\delta > 0$, and aim to rule out the existence of low-degree algorithms achieving these energy levels. This corresponds to the statistically optimal regime, as per [Kar+86].

For now, let $\mathcal{A}$ denote a Σ_N-valued deterministic algorithm. We discuss the extension to random, $\mathbf{R}^N$-valued algorithms later on in (section ???).

Setup Let g, g' be $(1-\varepsilon)$-correlated instances. We define the following events:

$$\begin{aligned} S_{\text{diff}} &= \{g \neq g'\} \\ S_{\text{solve}} &= \{\mathcal{A}(g) \in S(E; g), \mathcal{A}(g') \in S(E; g')\} \\ S_{\text{stable}} &= \left\{\|\mathcal{A}(g) - \mathcal{A}(g')\| \leq 2\sqrt{\eta N}\right\} \\ S_{\text{cond}} &= \left\{\begin{matrix} \nexists\ x' \in S(E; g') \text{ such that} \\ \|x - x'\| \leq 2\sqrt{\eta N} \end{matrix}\right\} \end{aligned}$$

In this case, set $\varepsilon = 2^{-\delta N} = o(1)$

Lemma 8.1. *There exists an $\eta > 0$ of constant order such that*

$$\mathbf{P}(S_{\text{cond}}) \geq 1 - e^{-cN}$$

for an arbitrary constant c.

$D = o(2^N)$.

$D\varepsilon = \frac{D}{2^N} * 2^{(1-\delta)N}$

Lemma 8.2. *For any $\omega(\log^2 N) \leq E \leq \Theta(N)$, there exist choices of ε, η (depending on N, E) such that $\mathbf{P}(S_{\text{ogp}}) = o(1)$.*

Proof: Observe that

$$\mathbf{P}(S_{\text{ogp}}) = \mathbf{E}[\mathbf{P}(S_{\text{ogp}} \mid g)]. \tag{8.1}$$

Conditional on g, we can compute $\mathbf{P}(S_{\text{ogp}} \mid g) = \mathbf{P}\left(\exists x' \in S(E; g'), \|x - x'\| \leq 2\eta\sqrt{N}\right)$ by setting $x = \mathcal{A}(g)$ (so x only depends on g), and union bounding Lemma 7.4 over the x' within $2\eta\sqrt{N}$ of x, as per Lemma 7.2:

$$\mathbf{P}(S_{\text{ogp}} \mid g) \leq \exp_2(-E + N\eta^2 \log_2(1/\eta^2) + O(1)).$$

We want to choose η such that

$$-E + N\eta^2 \log_2(1/\eta^2) = -\Omega(N)$$

$$\frac{E}{N} > \eta^2 \log(1/\eta^2)$$

Using the fact that $\sqrt{2x} \geq -x \log_2 x$, it suffices to pick η^2 with

$$\frac{E}{N} > 2\eta,$$

so $\eta^2 := \frac{E^2}{2N^2}$ is a valid choice.

By the choice of $\eta^2 \ll (h^{-1})(\delta) \asymp 1$, this bound gives $\mathbf{P}(S_{\text{ogp}}|g) \leq \exp_2(-O(N)) = o(1)$. Integrating over g gives the overall bound. $\square$

When $CD\varepsilon N^2 = \omega_{N(1)}$ (i.e. goes to infinity),

$$\begin{aligned} \mathbf{P}(S_{\text{stable}}) &\leq \frac{16CD\varepsilon N^2}{16\eta^4 N^2} \\ &= \frac{CD\varepsilon}{\eta^4} = \frac{4CD\varepsilon N^4}{E^4} \end{aligned}$$

$D\varepsilon \to 0$ same as $D = o\left(\frac{1}{\varepsilon}\right) = o(N)$.

Lemma 8.3. $\mathbf{P}(S_{\text{solve}}, S_{\text{stable}}) \leq \mathbf{P}(S_{\text{ogp}}) = o(1)$.

Proof: The first inequality follows from definition, with $x = \mathcal{A}(g)$ and $x' = \mathcal{A}(g')$. For the second, observe that

$$\mathbf{P}(S_{\text{ogp}}) = \mathbf{E}[\mathbf{P}(S_{\text{ogp}} \mid g)].$$

Now, let $M = \delta N$, we can compute $\mathbf{P}(S_{\text{ogp}} \mid g) = \mathbf{P}\left(\exists x' \in S(\delta; g'), \|x - x'\| \leq \eta\sqrt{N}\right)$ by setting $x = \mathcal{A}(g)$ (so x only depends on g), and union bounding Lemma 7.4 over the x' within $\eta\sqrt{N}$ of x, as per Lemma 7.2:

$$\mathbf{P}(S_{\text{ogp}} \mid g) \leq \exp_2\left(-\delta N + Nh\left(\frac{\eta^2}{4}\right) + O(\log_2 N)\right).$$

By the choice of $\eta^2 \ll (h^{-1})(\delta) \asymp 1$, this bound gives $\mathbf{P}(S_{\text{ogp}}|g) \leq \exp_2(-O(N)) = o(1)$. Integrating over g gives the overall bound. $\square$

However, by the choice of parameters above, we also have

$$\begin{aligned} \mathbf{P}(S_{\text{solve}}, S_{\text{stable}}) &\geq \mathbf{P}(S_{\text{solve}}) + \mathbf{P}(S_{\text{stable}}) - 1 \\ &\geq p_{\text{solve}}^4 + p_{\text{unstable}} \geq \Omega(1) - o(1) = \Omega(1), \end{aligned} \tag{8.2}$$

which is a contradiction.

9. Proof of Strong Low-Degree Hardness

In this section, we prove strong low-degree hardness for both low-degree polynomial algorithms and algorithms with low Efron-Stein degree.

For now, we consider Σ_N-valued deterministic algorithms. We discuss the extension to random, $\mathbf{R}^N$-valued algorithms later on in (section ???). As outlined in Section 4.1,

The key argument is as follows. Fix some energy levels E, depending on N. Suppose we have a Σ_N-valued, deterministic algorithm $\mathcal{A}$ given by a degree D polynomial (resp. an Efron-Stein degree D function), and we have two instances $g, g' \sim \mathcal{N}(0, I_N)$ which are $(1-\varepsilon)$-correlated (resp. $(1-\varepsilon)$ -resampled), for $\varepsilon > 0$. Say $\mathcal{A}(g) = x \in \Sigma_N$ is a solution with energy at least E, i.e. it "solves" this NPP instance. For ε close to 0, $\mathcal{A}(g') = x'$ will be close to x, by low-degree stability. However, by adjusting parameters carefully, we can make it so that with high probability (exponential in E), there are no solutions to g' close to x. By application of a correlation bound on the probability of solving any fixed instance, we can conclude that with high probability, $\mathcal{A}$ can't find solutions to NPP with energy E.

Intuitively, our argument can be thought of as a "conditional" version of the overlap gap property. Traditionally, the overlap gap property is a global obstruction: one shows that with high probability, one cannot find a tuple of good solutions to a family of correlated instances which are all roughly the same distance apart. Here, however, we show a local obstruction - we condition on being able to solve a single instance, and show that after a small change to the instance, we cannot guarantee any solutions will exist close to the first one. This is an instance of the "brittleness," so to speak, that makes NPP so frustrating to solve; even small changes in the instance break the landscape geometry, so that even if solutions exist, there's no way to know where they'll end up.

We start with some setup which will apply, with minor modifications depending on the nature of the algorithm in consideration, to all of the energy regimes in discussion. After proving some preliminary estimates, we establish the existence of our conditional landscape obstruction, which is of independent interest. Finally, we conclude by establishing low-degree hardness in both the linear and sublinear energy regimes.

9.1. Proof Outline and Preliminary Estimates

First, consider the case of $\mathcal{A}$ being a polynomial algorithm with degree D, where D will depend on N. Let g, g' be $(1-\varepsilon)$-correlated. We define the following events:

$$\begin{aligned} S_{\text{solve}} &= \{\mathcal{A}(g) \in S(E; g), \mathcal{A}(g') \in S(E; g')\} \\ S_{\text{stable}} &= \left\{\|\mathcal{A}(g) - \mathcal{A}(g')\| \leq 2\sqrt{\eta N}\right\} \\ S_{\text{cond}} &= \left\{\begin{matrix} \nexists\, x' \in S(E; g') \text{ such that} \\ \|x - x'\| \leq 2\sqrt{\eta N} \end{matrix}\right\} \end{aligned} \tag{9.1}$$

Next, let $\mathcal{A}$ be given by a function with Efron-Stein degree D. We now want g, g' to be $(1-\varepsilon)$-resampled. We define the following events.

$$\begin{aligned} S_{\text{diff}} &= \{g \neq g'\} \\ S_{\text{solve}} &= \{\mathcal{A}(g) \in S(E; g), \mathcal{A}(g') \in S(E; g')\} \\ S_{\text{stable}} &= \left\{\|\mathcal{A}(g) - \mathcal{A}(g')\| \leq 2\sqrt{\eta N}\right\} \\ S_{\text{cond}} &= \left\{\begin{matrix} \nexists\, x' \in S(E; g') \text{ such that} \\ \|x - x'\| \leq 2\sqrt{\eta N} \end{matrix}\right\} \end{aligned} \tag{9.2}$$

Note that these are the same events as (9.1), along with an

9.2. Conditional Landscape Obstruction

Our goal is to show that

$$p_{\mathrm{cond}} = \mathbf{P}\left(\begin{Bmatrix} \exists\ x' \in S(E; g') \text{ such that} \\ \|x - x'\| \leq 2\sqrt{\eta N} \end{Bmatrix}\right) \leq o(1).$$

10. Randomized Rounding Things

Claim: no two adjacent points on Σ_N (or pairs within $k = O(1)$ distance) which are both good solutions to the same problem. The reason is that this would require a subset of k signed coordinates $\pm g_{\{i_1\}}, ..., \pm g_{\{i_k\}}$ to have small sum, and there are only 2^k `binom`$\{N\}\{k\}l = O\left(N^k\right)$ possibilities, each of which is centered Gaussian with variance at least 1, so the smallest is typically of order $\Omega\left(N^{\{-k\}}\right)$.

Thus, rounding would destroy the solution.

- Say we're in the case where rounding changes the solution. (i.e. rounding moves x to point that is not the closest point x_*, whp.)
- Let $p_1, ..., p_N$ be the probabilities of disagreeing with x_* on each coordinate.
  - We know that $\sum p_i$ diverges (this is equivalent to the statement that rounding will changes the solution whp).
  - Reason: for each coord, chance of staying at that coordinate is $e^{-\Theta(p_i)}$.
- For each i, flip coin with heads prob $2p_i$, and keep all the heads.
  - By Borel-Cantelli type argument, typical number of heads will be $\omega(1)$.
- For every coin with a head, change coord with prob 50%, if tails, keep coord.
  - Randomized rounding in artificially difficult way. (I.e. this multistage procedure accomplishes the same thing as randomized rounding.)
- Now, randomized rounding is done by choosing a random set of $\omega(1)$ coordinates, and making those iid Uniform in $\{-1, 1\}$.
- Pick a large constant (e.g. 100), and only randomize the first 100 heads, and condition on the others (i.e. choose the others arbitrarily). Note that since $100l = \omega(1)$, there are at least 100 heads whp.
- Now rounded point is random point in 100 dimensional subcube, but at most one of them is a good solution by the claim at the top of the page.
- Combining, the probability for rounding to give a good solution is at most $o(1) + 2^{\{-100\}}$. Since 100 is arbitrary, this is $o(1)$ by sending parameters to 0 and/or infinity in the right order.

[O'D21]

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