Finite Elements for Engineers

Lecture 2: Preparing for Isoparametric Formulation - Shape Functions and Numerical Integration

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Family of 1D-C^0 Elements

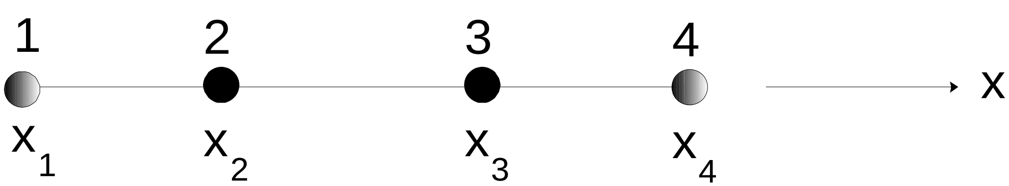
Linear

$$y(x) = a_0 + a_1 x$$

Quadratic

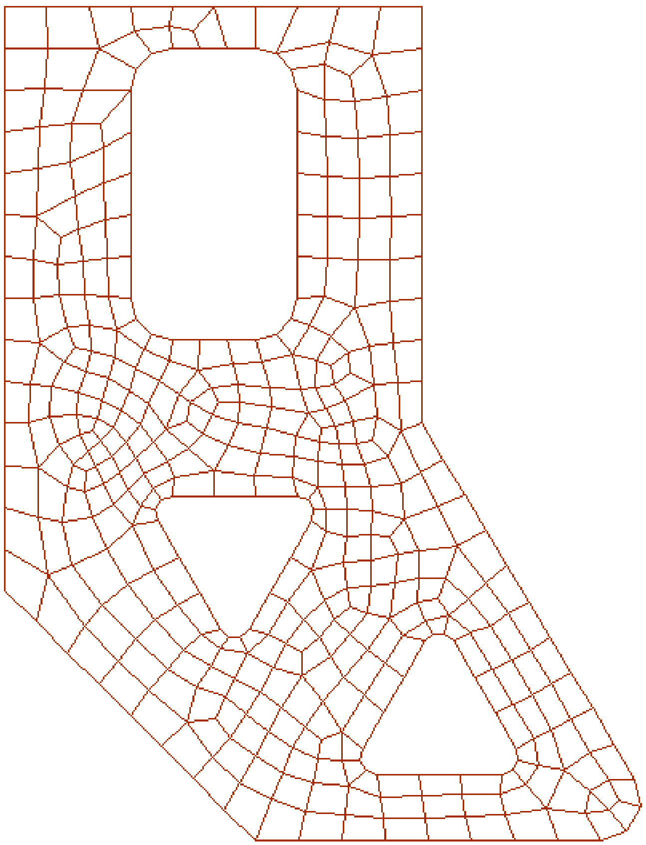
$$y(x) = a_0 + a_1 x + a_2 x^2$$

Cubic

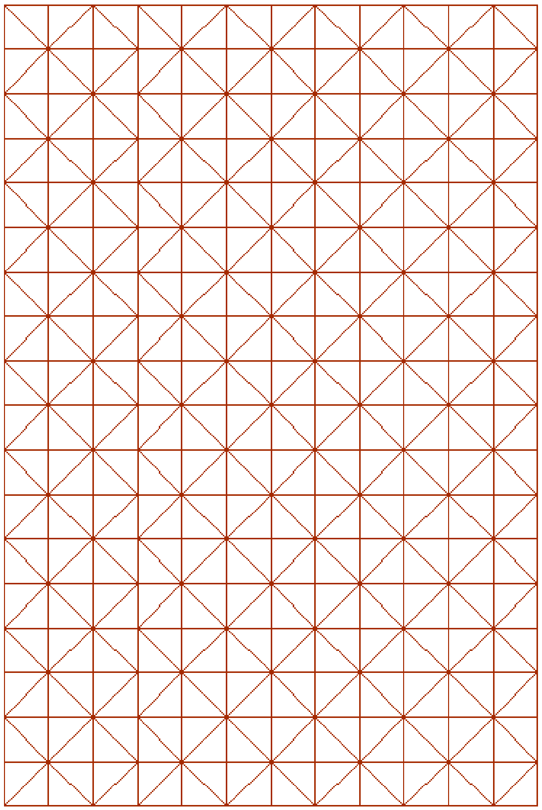


$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Meshing 2D Domains



Automatic Nonuniform Mesh



Mapped Uniform Mesh

Generating Shape Functions

Pascal Triangle (2D Elements)

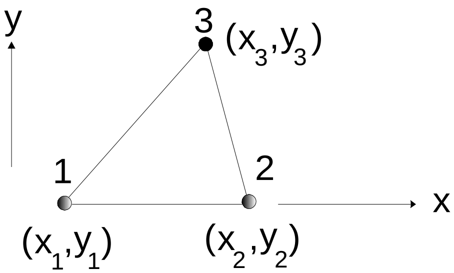
				1					Constant
			x		y				Linear
		x^2		xy		y^2			Quadratic
	x^3		x^2y		xy^2		y^3		Cubic
x^4		x^3y		x^2y^2		xy^3		y^4	Quartic

Properties

(1) Geometric Isotropy

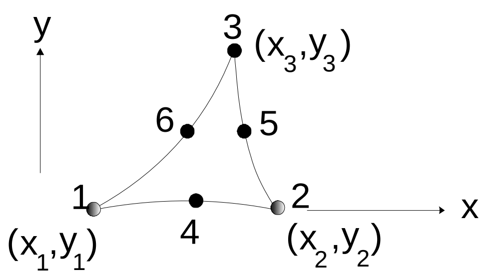
(2) Complete

Family of C^0 Triangular Elements



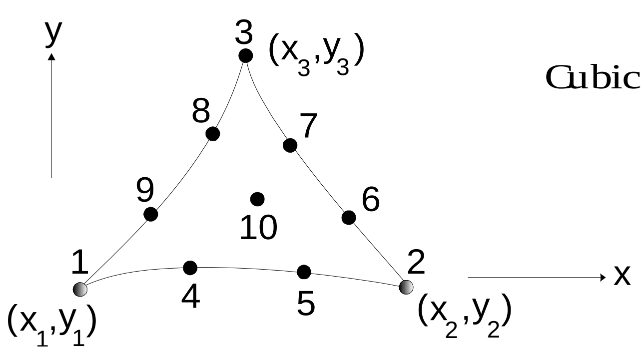
Linear

$$u(x,y) = a_0 + a_1 x + a_2 y$$



Quadratic

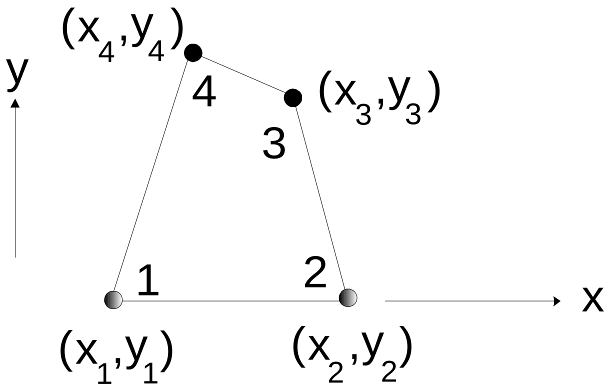
$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2$$



Cubic

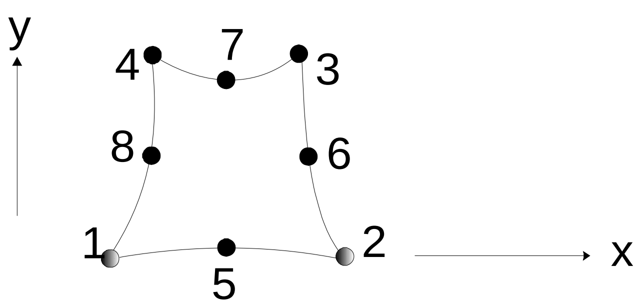
$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^3$$

Family of "Serendipity" C^0 Quadrilateral Elements



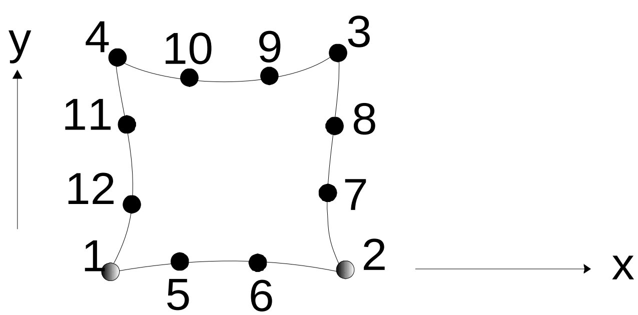
Linear

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$$



Quadratic

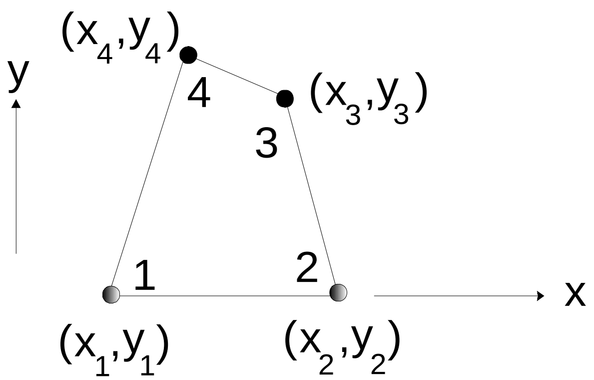
$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$$
$$+ a_4 x^2 + a_5 y^2 + a_6 x^2 y$$
$$+ a_7 xy^2$$



Cubic

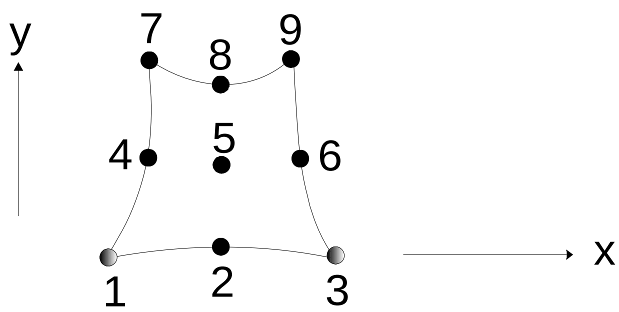
$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$$
$$+ a_4 x^2 + a_5 y^2 + a_6 x^2 y + a_7 xy^2$$
$$+ a_8 x^3 + a_9 y^3 + a_{10} x^3 y + a_{11} xy^3$$

Family of "Lagrange" C^0 Quadrilateral Elements



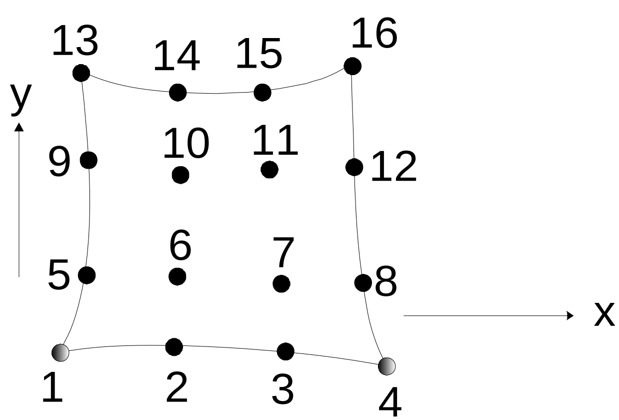
Linear

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$$



Quadratic

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + a_6 x^2 y + a_7 xy^2 + a_8 x^2 y^2$$



Cubic

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + a_6 x^2 y + a_7 xy^2 + a_8 x^3 + a_9 y^3 + a_{10} x^3 y + a_{11} xy^3 + a_{12} x^2 y^2 + a_{13} x^4 + a_{14} y^4 + a_{15} x^3 y^3$$

Beware

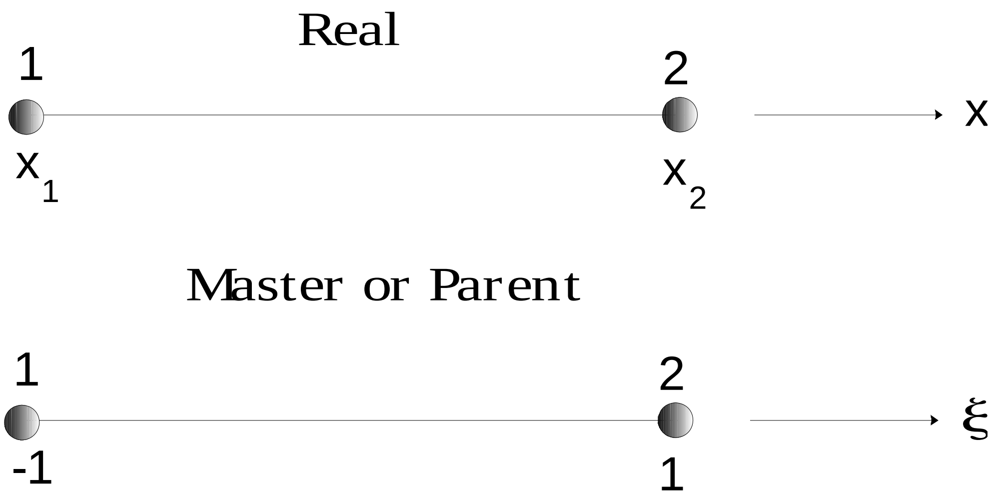
- This approach of generating and using shape functions in the cartesian coordinate system will lead to difficulties because we need to evaluate integrals in triangular and quadrilateral domains.

$$\iint_{R_{xy}} f(x, y)\, dxdy$$

Natural Coordinates

These are “normalized” coordinate systems where the coordinate values are between –1 and 1 , or between 0 and 1.

1D-C^0 Linear Element



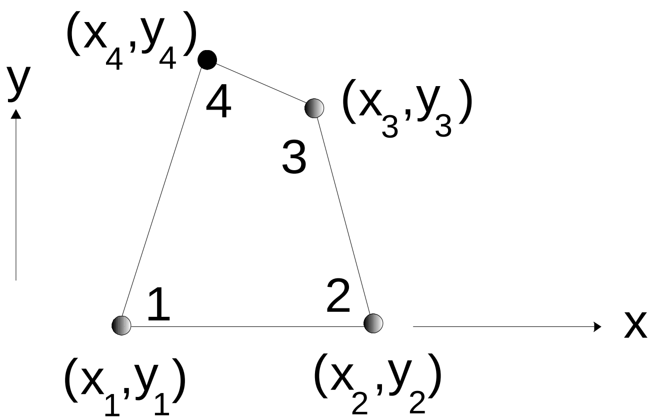
Mapping Function

$$x = x(\xi)$$

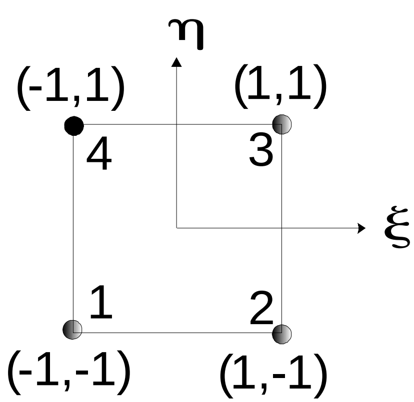
$$-1 \le \xi \le 1$$

Natural Coordinates

2D-C^0 Bilinear Quadrilateral Element



Real



Master or Parent

Mapping Functions

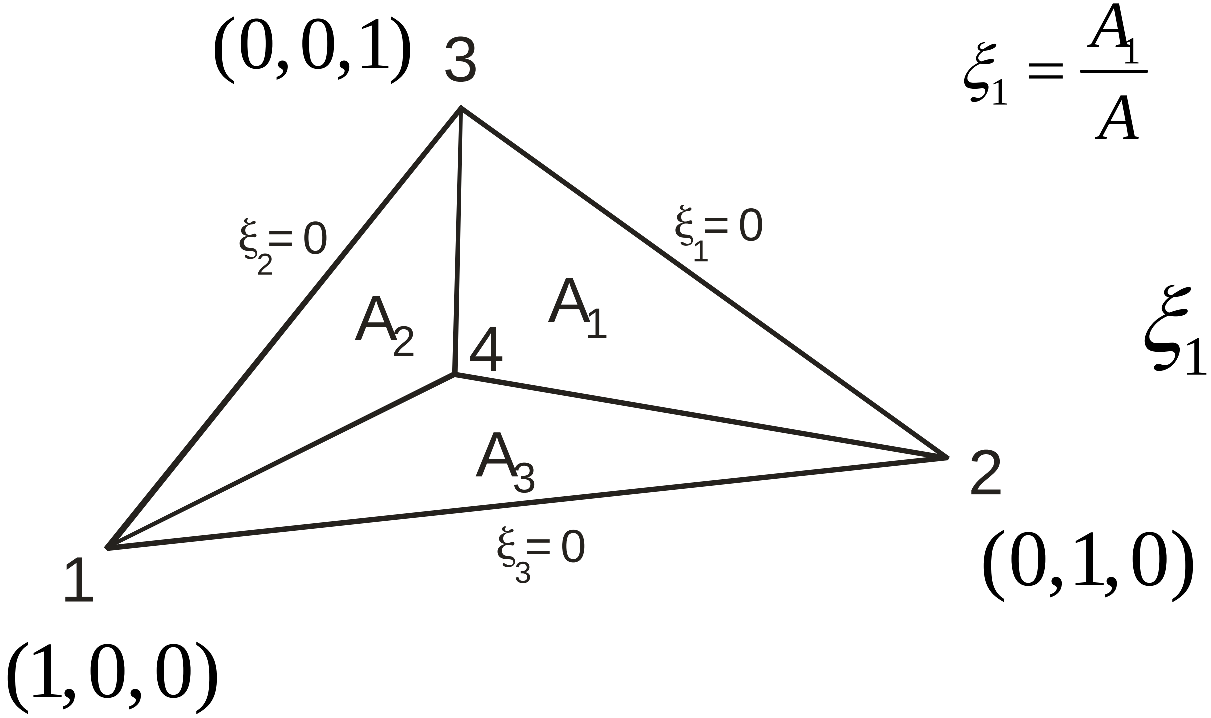
$$x = x(\xi, \eta)$$

$$y = y(\xi, \eta)$$

$$-1 \le \xi \le 1$$

$$-1 \le \eta \le 1$$

Area Coordinates

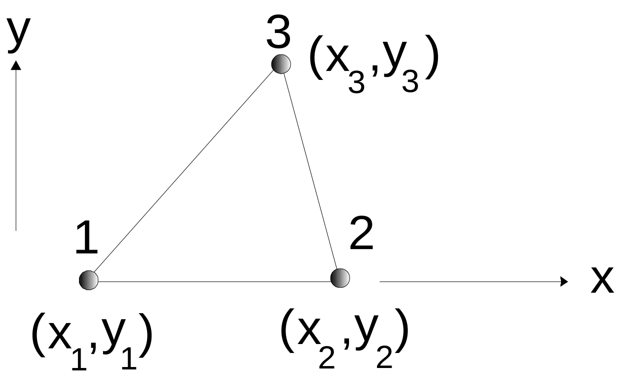


$$\xi_1 = \frac{A_1}{A} \qquad \xi_2 = \frac{A_2}{A} \qquad \xi_3 = \frac{A_3}{A}$$

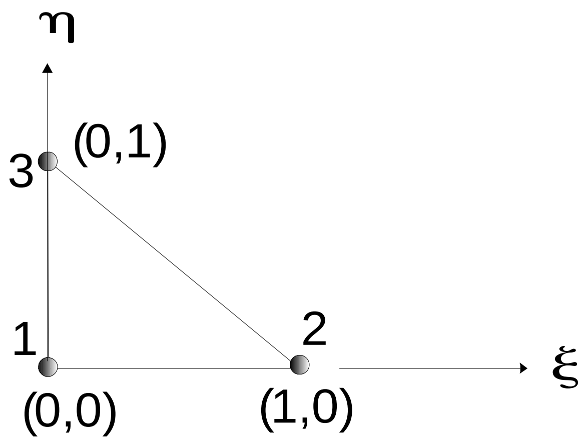
$$\xi_1 + \xi_2 + \xi_3 = 1$$

Natural Coordinates

2D-C^0 Bilinear Triangular Element



Real



Master or Parent

Area Coordinates

$$\xi + \eta + \zeta = 1$$

$$x = x(\xi, \eta) \qquad y = y(\xi, \eta)$$

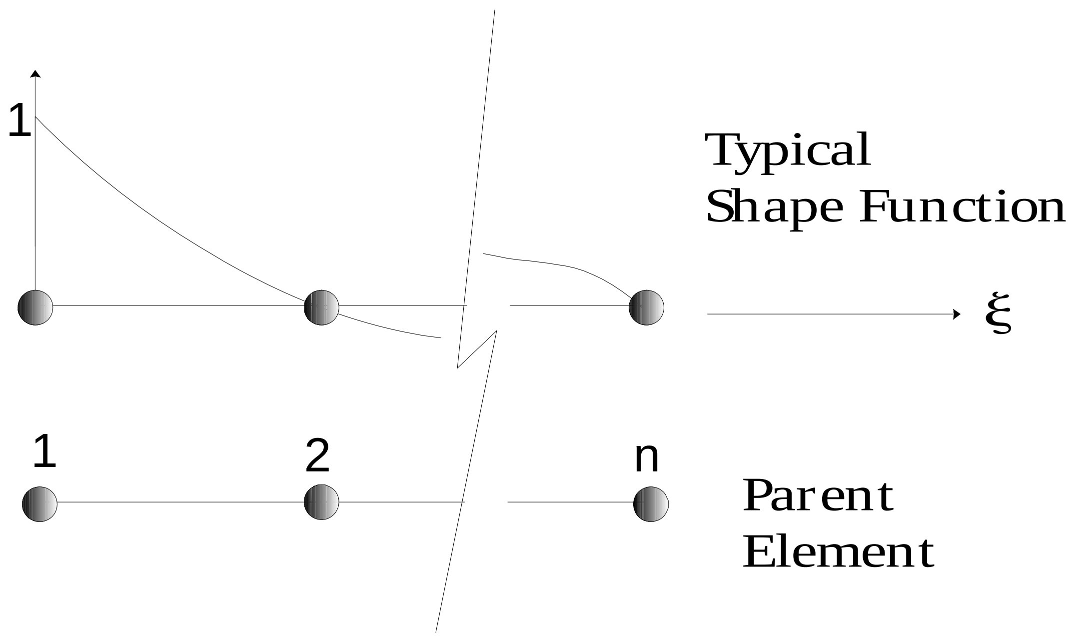
$$0 \le \xi \le 1 \qquad 0 \le \eta \le 1$$

Generating Shape Functions

Lagrange Polynomial (unit value at k and zero elsewhere)

$$l_k^n(\xi) = \frac{(\xi - \xi_0)(\xi - \xi_1)\ldots(\xi - \xi_{k-1})(\xi - \xi_{k+1})\ldots(\xi - \xi_n)}{(\xi_k - \xi_0)(\xi_k - \xi_1)\ldots(\xi_k - \xi_{k-1})(\xi_k - \xi_{k+1})\ldots(\xi_k - \xi_n)}$$

By definition $l_0^0 = 1$



$\Leftarrow l_1^n$

1D Elements

1D C^0 Linear Element

Node 1 $$\phi_1 = l_0^1 = \frac{(\xi - \xi_1)}{(\xi_0 - \xi_1)} = \frac{(\xi - 1)}{(-1-1)} = \frac{1-\xi}{2}$$

Node 2 $$\phi_2 = l_1^1 = \frac{(\xi - \xi_0)}{(\xi_1 - \xi_0)} = \frac{(\xi - (-1))}{(1-(-1))} = \frac{1+\xi}{2}$$

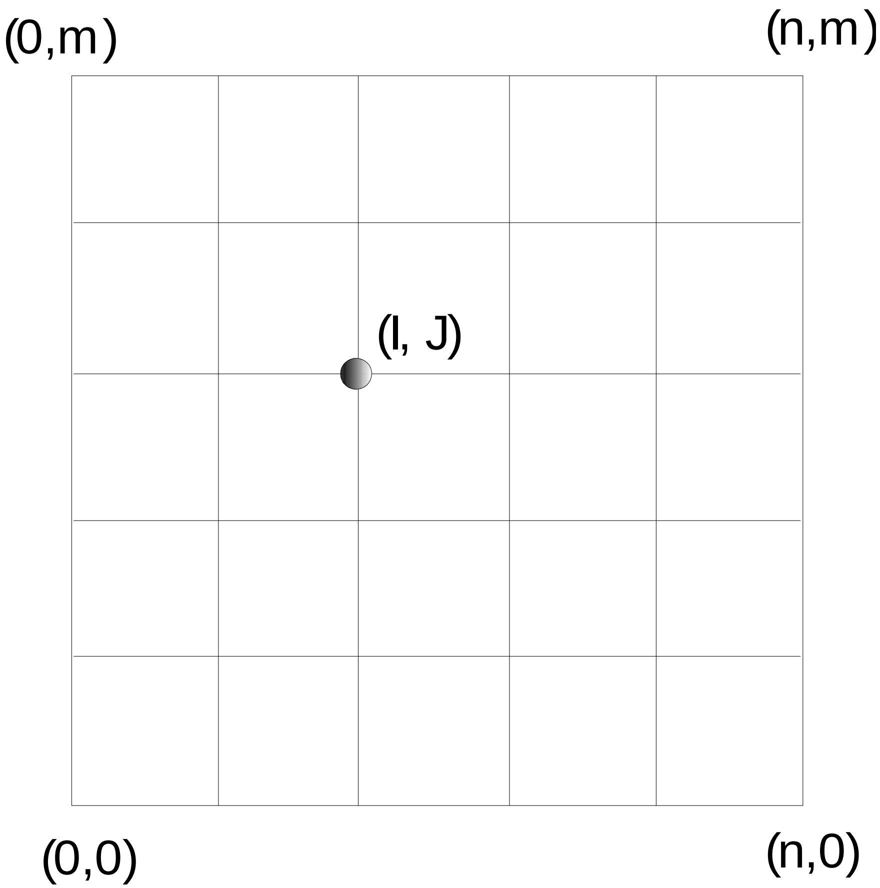
1D C^0 Quadratic Element

Node 1 $$\phi_1 = l_0^2 = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_0 - \xi_1)(\xi_0 - \xi_2)} = \frac{(\xi - 0)(\xi - 1)}{(-1-0)(-1-1)} = \frac{1}{2}\xi(\xi - 1)$$

Node 2 $$\phi_2 = l_1^2 = \frac{(\xi - \xi_0)(\xi - \xi_2)}{(\xi_1 - \xi_0)(\xi_1 - \xi_2)} = \frac{(\xi - (-1))(\xi - 1)}{(0-(-1))(0-1)} = 1 - \xi^2$$

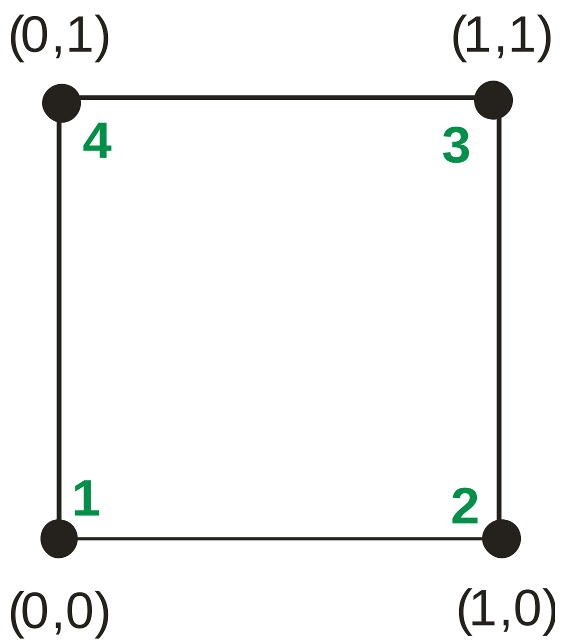
Node 3 $$\phi_3 = l_2^2 = \frac{(\xi - \xi_0)(\xi - \xi_1)}{(\xi_2 - \xi_0)(\xi_2 - \xi_1)} = \frac{(\xi - (-1))(\xi - 0)}{(1-(-1))(1-0)} = \frac{1}{2}\xi(\xi + 1)$$

2D Lagrange Elements

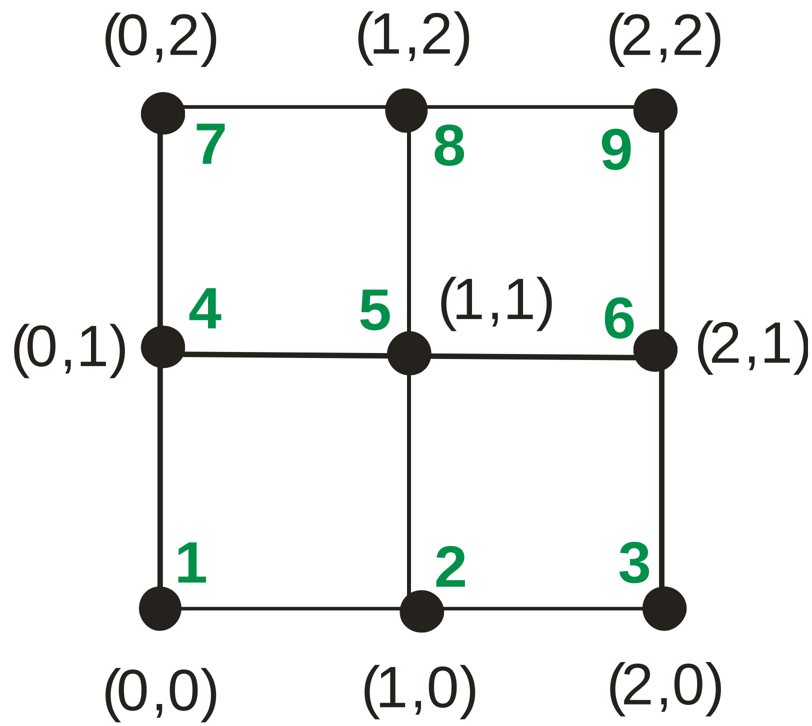


$$\phi_i \equiv \phi_{IJ} = l_I^n(\xi)\, l_J^m(\eta)$$

2D Lagrange Elements



2D-C^0 Bilinear Quadrilateral Element



2D-C^0 Quadratic Quadrilateral Element

Shape Functions

2D-C^0 Bilinear Quadrilateral Element

Node 3 $$\phi_3 = N_{11} = l_1^1(\xi) l_1^1(\eta) = \frac{(\xi - \xi_0)}{(\xi_1 - \xi_0)} \frac{(\eta - \eta_0)}{(\eta_1 - \eta_0)} = \frac{(\xi - (-1))}{(1 - (-1))} \frac{(\eta - (-1))}{(1 - (-1))}$$

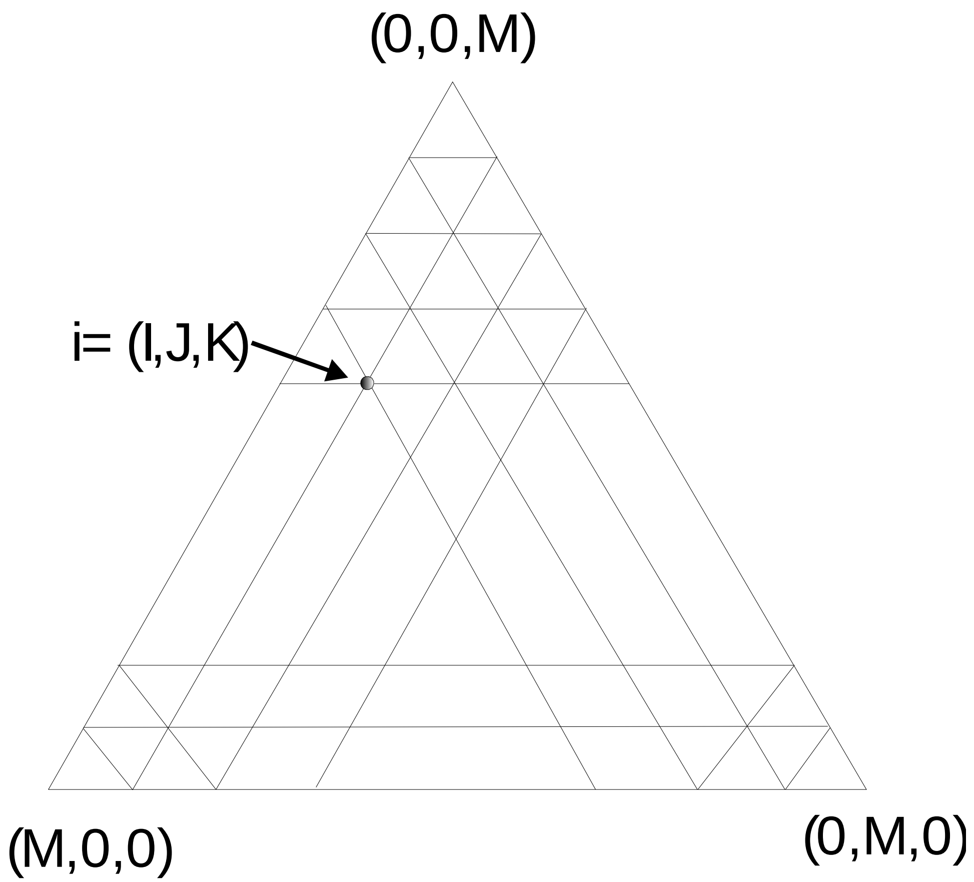
$$\phi_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

2D-C^0 Quadratic Quadrilateral Element

Node 6 $$\phi_6 = N_{21} = l_2^2(\xi) l_1^2(\eta) = \frac{(\xi - \xi_0)(\xi - \xi_1)}{(\xi_2 - \xi_0)(\xi_2 - \xi_1)} \frac{(\eta - \eta_0)(\eta - \eta_2)}{(\eta_1 - \eta_0)(\eta_1 - \eta_2)}$$

$$\phi_6 = \frac{(\xi+1)(\xi-0)}{(1+1)(1-0)} \frac{(\eta+1)(\eta-1)}{(0+1)(0-1)} = \frac{1}{2}\xi(1+\xi)(1-\eta^2)$$

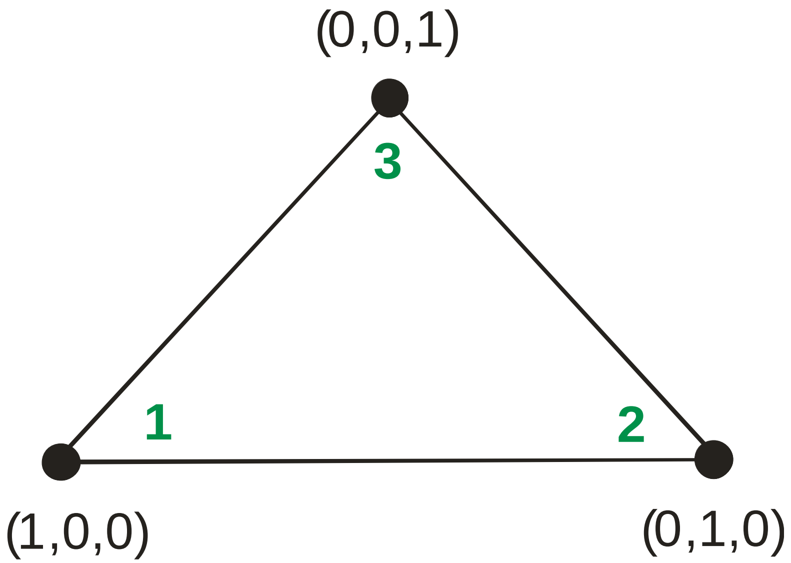
2D Triangular Elements



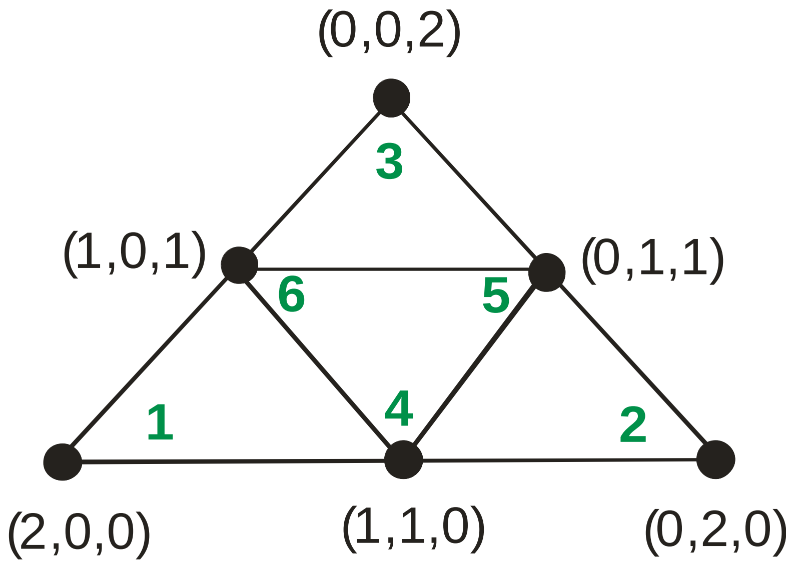
$$\phi_i = l_I^I(\xi)\, l_J^J(\eta)\, l_K^K(\zeta)$$

$$I + J + K = M$$

2D Triangular Elements



2D-C^0 Linear Triangular Element



2D-C^0 Quadratic Triangular Element

Shape Functions

2D-C^0 Linear Triangular Element

Node 3

$$\phi_3 = l_0^0(\xi)\, l_1^1(\eta)\, l_0^0(\zeta) = (1)\frac{(\eta-\eta_0)}{(\eta_1-\eta_0)}(1) = \frac{(\eta-0)}{(1-0)} = \eta$$

$$\phi_3 = \eta$$

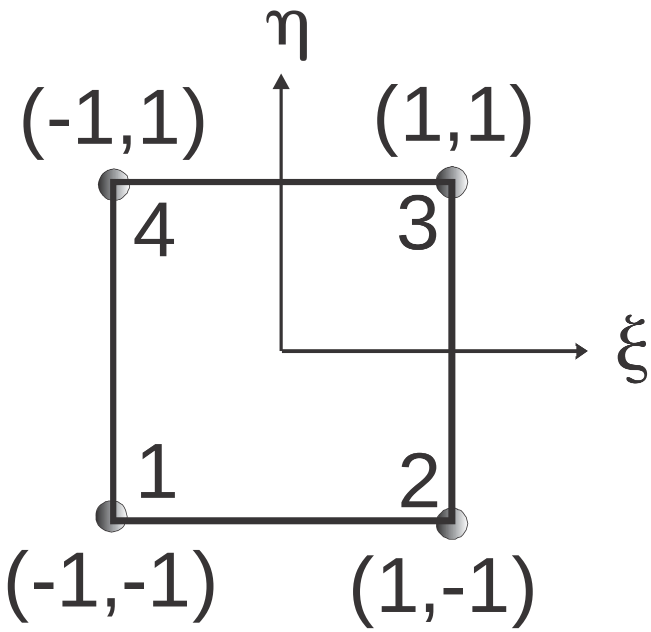
2D-C^0 Quadratic Triangular Element

Node 4

$$\phi_4 = l_1^1(\xi)\, l_0^0(\eta)\, l_1^1(\zeta) = \frac{(\xi-\xi_0)}{(\xi_1-\xi_0)}(1)\frac{(\zeta-\zeta_0)}{(\zeta_1-\zeta_0)} = \frac{(\xi-0)}{(1/2-0)}\frac{(\zeta-0)}{(1/2-0)}$$

$$\phi_4 = 4\xi\zeta = 4\xi(1-\xi-\eta)$$

2D-C^0 Bilinear Quadrilateral Element



$$u(\xi,\eta) = \sum_{i=1}^{4} \phi_i(\xi,\eta)\, u_i$$

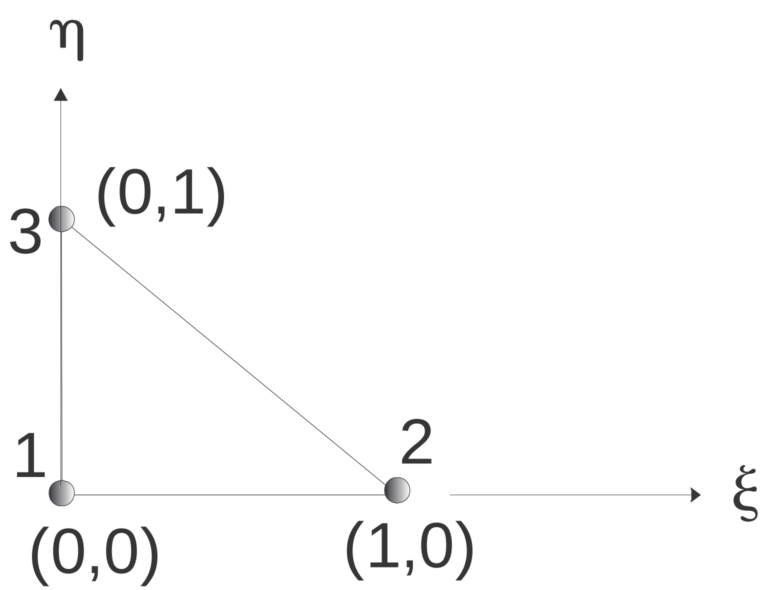
$$\phi_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$\phi_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$\phi_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$\phi_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

2D-C^0 Linear Triangular Element



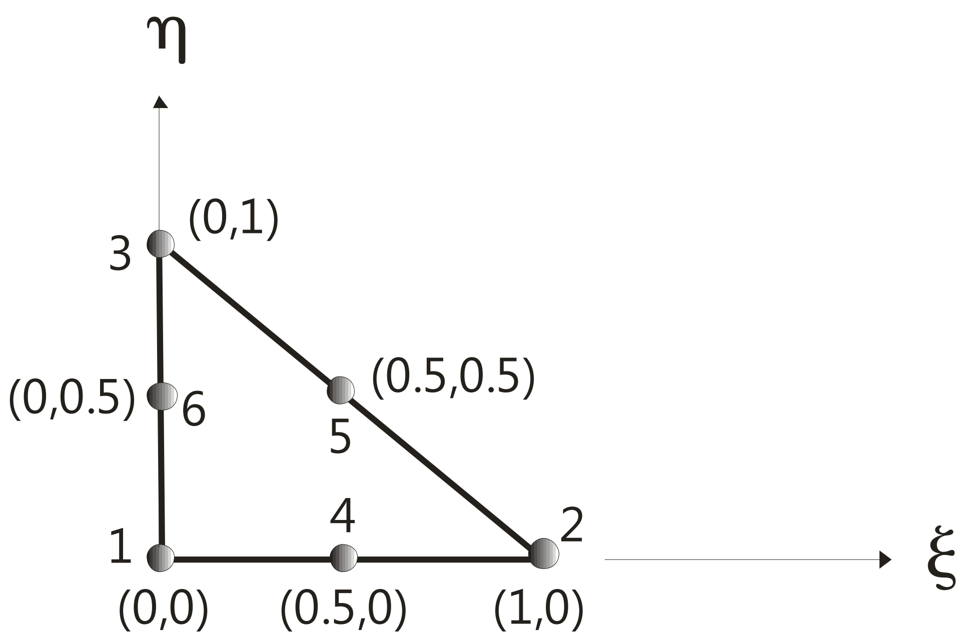
$$u(\xi,\eta)=\sum_{i=1}^{3}\phi_i(\xi,\eta)u_i$$

$$\phi_1=\zeta=1-\xi-\eta$$

$$\phi_2=\xi$$

$$\phi_3=\eta$$

2D-C^0 Quadratic Triangular Element



$$u(\xi,\eta)=\sum_{i=1}^{6}\phi_i(\xi,\eta)u_i$$

$$\phi_1=\zeta(2\zeta-1)$$

$$\phi_2=\xi(2\xi-1)$$

$$\phi_3=\eta(2\eta-1)$$

$$\phi_4=4\xi\zeta$$

$$\phi_5=4\xi\eta$$

$$\phi_6=4\eta\zeta$$

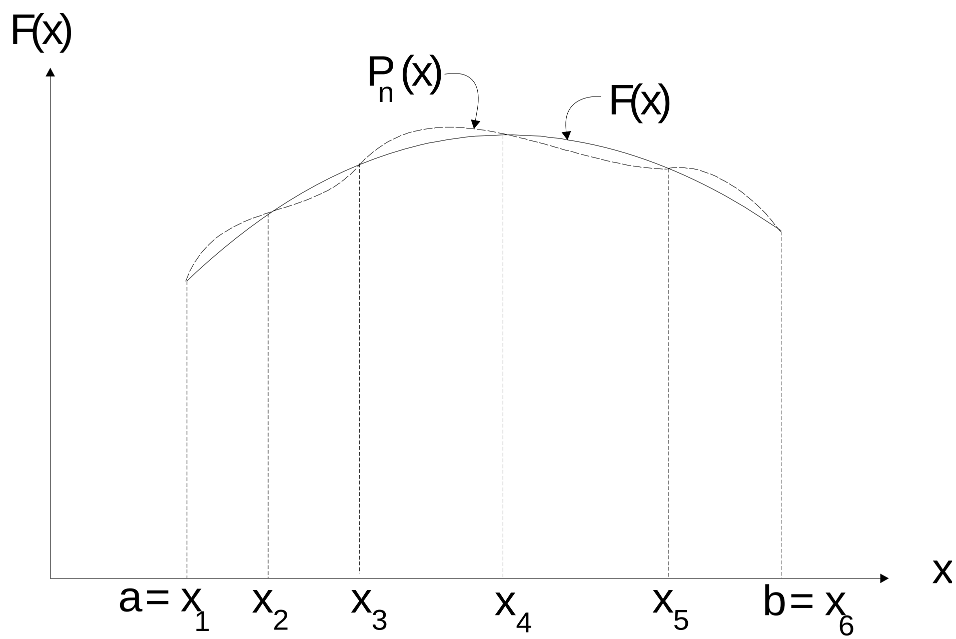
Numerical Integration

Integral $$I = \int_a^b F(x)\,dx$$

(1) *F(x)* is dependent on a jacobian

(2) *F(x)* is known at a few discrete points

$$I = \int_a^b F(x)\,dx \approx \int_a^b P_n(x)\,dx$$



Numerical Integration

Gauss-Legendre Quadrature

$$\int_a^b F(x)\,dx = \int_{-1}^{1} \widehat{F}(\xi)\,d\xi = \sum_{i=1}^{n} w_i \widehat{F}(\xi_i)$$

w_i : weights

ξ_i : locations

where $$F(x)\,dx = F(x(\xi))\frac{dx}{d\xi}d\xi = F(x(\xi))\cdot J\left(\xi\right)d\xi = \widehat{F}(\xi)$$

Note

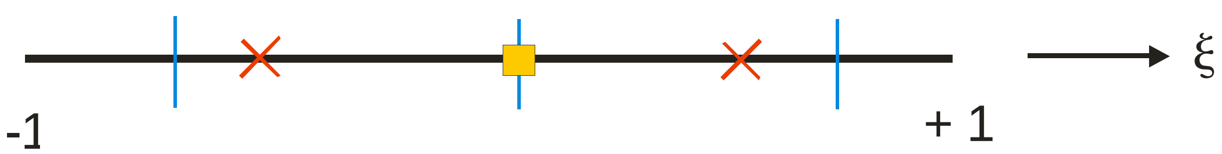
(1) G-LQ requires fewer base points than Newton Cotes.

(2) A polynomial of degree n is integrated exactly by (n+1)/2 Gauss points.

Gauss-Quadrature

Gauss points and weights

Order, *n*	Weight	Location
1	2.0	0.0
2	1.0	0.57735 02691
	1.0	-0.57735 02691
3	0.55555 55555	0.77459 66692
	0.55555 55555	-0.77459 66692
	0.88888 88888	0.0



Example 1

$$I = \int_{-1}^{1} x^4 dx = \int_{-1}^{1} \xi^4 d\xi = \sum_{i=1}^{n} w_i \xi_i^4$$

n	w	ξ	$f(\xi) = \xi^4$	$w_i \xi_i^4$	I
1	2	0	0	0	**0**
2	1	0.577350269	0.111111	0.111111	
	1	-0.577350269	0.111111	0.111111	**0.222222**
3	0.555556	0.774596669	0.36	0.2	
	0.555556	-0.774596669	0.36	0.2	
	0.888889	0	0	0	**0.4**

Example 2

$$I = \int_2^5 x^4 dx = \sum_{i=1}^{n} w_i \widehat{F}\left(\xi_i\right)$$

Preprocessing

$$x = \frac{1-\xi}{2} x_1 + \frac{1+\xi}{2} x_2 = \frac{3}{2}\xi + \frac{7}{2} \Rightarrow J = \frac{dx}{d\xi} = \frac{3}{2}$$

$$\widehat{F}(\xi) = F\left(x\left(\xi\right)\right) J = \left(\frac{3}{2}\xi + \frac{7}{2}\right)^4 \frac{3}{2}$$

Example 2 (cont'd)

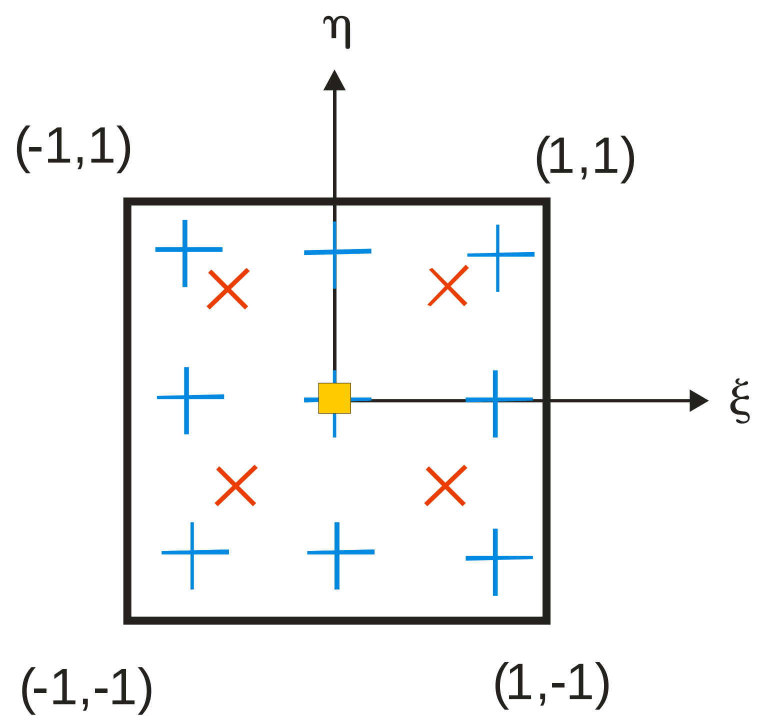
n	w	ξ	$\widehat{F}(\xi)$	$w_i \widehat{F}(\xi_i)$	I
1	2	0	225.09	450.19	**450.19**
2	1	0.577350269	545.05	545.05	
	1	-0.577350269	72.2	72.2	**617.25**
3	0.555556	0.774596669		393.609	
	0.555556	-0.774596669		24.9048	
	0.888889	0		200.083	**618.6**

2D Example

$$I = \int_c^d \int_a^b F\left(x, y\right) dx\, dy = \int_{-1}^{1} \int_{-1}^{1} F\left(x(\xi,\eta), y(\xi,\eta)\right) \left|J\right| d\xi d\eta$$

$$I = \sum_{j=1}^{n} \sum_{i=1}^{n} w_i w_j f\left(\xi_i, \eta_j\right)$$

$$\left|J\right| = \det(\mathbf{J}) = \det \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}_{2\times 2}$$



Example 3

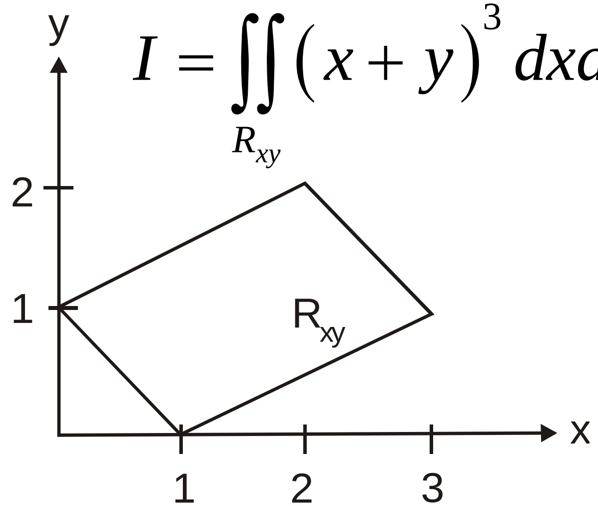
$$I = \int_{-1}^{1}\int_{-1}^{1} x^2 dx\, dy = \int_{-1}^{1}\int_{-1}^{1} \xi^2 d\xi\, d\eta = \sum_{j=1}^{n}\sum_{i=1}^{n} w_i w_j \xi_i^2$$

n=2

ξ_i	η_j	w_i	w_j	$f(\xi_i,\eta_j)$	$w_i w_j f(\xi_i,\eta_j)$
-0.5773502691	-0.5773502691	1.0	1.0	0.333333333	0.333333333
0.5773502691	-0.5773502691	1.0	1.0	0.333333333	0.333333333
0.5773502691	0.5773502691	1.0	1.0	0.333333333	0.333333333
-0.5773502691	0.5773502691	1.0	1.0	0.333333333	0.333333333
				TOTAL	1.333333333

Example 4

$$I = \iint_{R_{xy}} \left(x+y\right)^3 dxdy$$



Node	x	y
1	1	0
2	3	1
3	2	2
4	0	1

$$x = \sum_{i=1}^{4} \phi_i\left(\xi,\eta\right) x_i$$

$$y = \sum_{i=1}^{4} \phi_i\left(\xi,\eta\right) y_i$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}_{2\times 2} = \frac{1}{4}\begin{bmatrix} 4 & 2 \\ -2 & 2 \end{bmatrix}$$

$$\det(\mathbf{J}) = \frac{3}{4}$$

Example 4

$$I = \iint_{R_{xy}} (x+y)^3 \, dxdy = \int_{-1}^{1}\int_{-1}^{1} \widehat{F}(\xi_i, \eta_j) d\xi \, d\eta = \sum_{j=1}^{n}\sum_{i=1}^{n} w_i w_j \widehat{F}(\xi_i, \eta_j)$$

n=1

ξ_i	η_j	w_i	w_j	$F(\xi_i, \eta_j)$	$w_i w_j \det(\mathbf{J}) F(\xi_i, \eta_j)$
0	0	2.0	2.0	$\left(\frac{5}{2}\right)^3$	46.875

$$x + y = \left(\sum_{i=1}^{4} \phi_i x_i\right) + \left(\sum_{i=1}^{4} \phi_i y_i\right) = \left(\frac{1}{4}(1+3+2+0) + \frac{1}{4}(0+1+2+1)\right) = \frac{5}{2}$$

Example 4

n=2

ξ_i	η_j	w_i	w_j	ϕ_1	ϕ_2	ϕ_3	ϕ_4	x	y	$(x+y)^3$
-0.57735	-0.57735	1.0	1.0	0.622008	0.166667	0.044658	0.166667	1.21133	0.42265	4.362508
0.57735	-0.57735	1.0	1.0	0.166667	0.622008	0.166667	0.044658	2.36603	1	38.13748
0.57735	0.57735	1.0	1.0	0.044658	0.166667	0.622008	0.166667	1.78868	1.57735	38.13748
-0.57735	0.57735	1.0	1.0	0.166667	0.044658	0.166667	0.622008	0.63398	1	4.362508
								$I = \det(J)\sum(x+y)^3$		63.75

Summary

- Generating Shape Functions
  – Properties (geometric isotropy, complete, linearly independent)
  – Lagrange Polynomials
- Shape Functions for
  – 1D-C^0 elements
  – Quadrilateral “Serendipity” elements
  – Quadrilateral Lagrange elements
  – Triangular elements

Summary

- Natural coordinates
- Area coordinates
- Numerical Integration
  - Gauss-Legendre Quadrature
  - Mapping
  - Jacobian

C++ Program

Further Reading

- Web sites

  http://www.cs.kuleuven.ac.be/~ronald/