CEE432/CEE532/MAE541
Developing Software for Engineering Applications

Lecture 18: Planar Frame Analysis

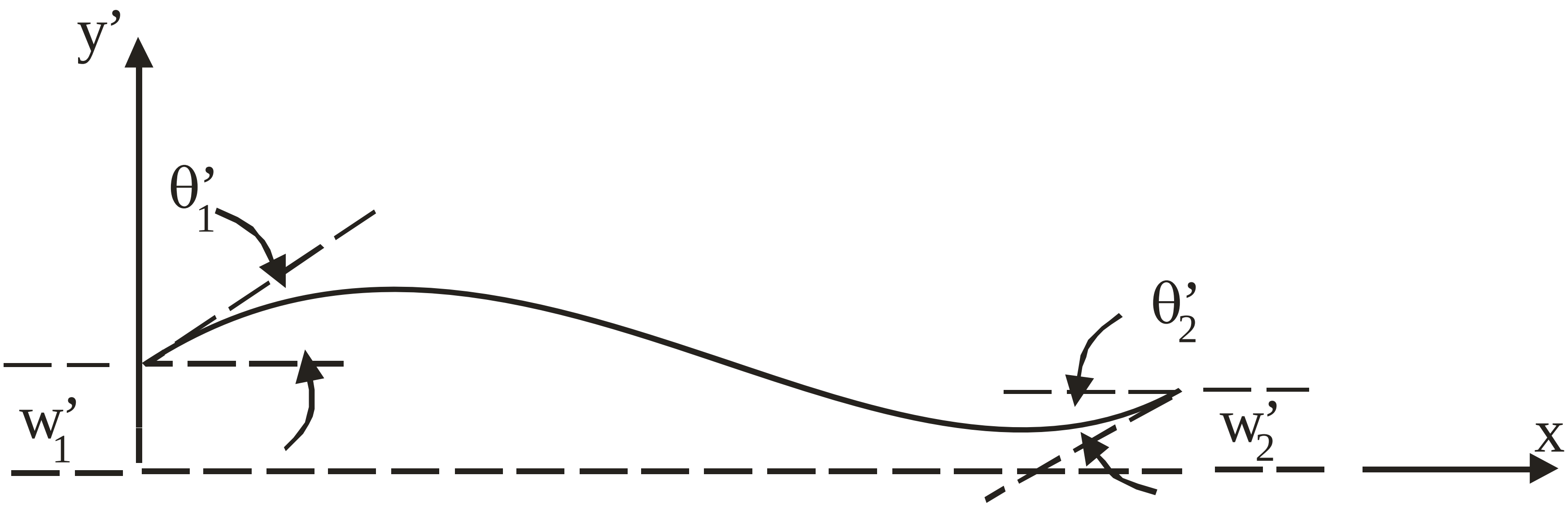
Planar Frame Analysis

- Members are slender and prismatic (essentially one-dimensional).
- Joints can be rigid, frictionless pins or in-between (typical connection).
- Loads can be applied to members or at joints.

Step 2: Element Equations

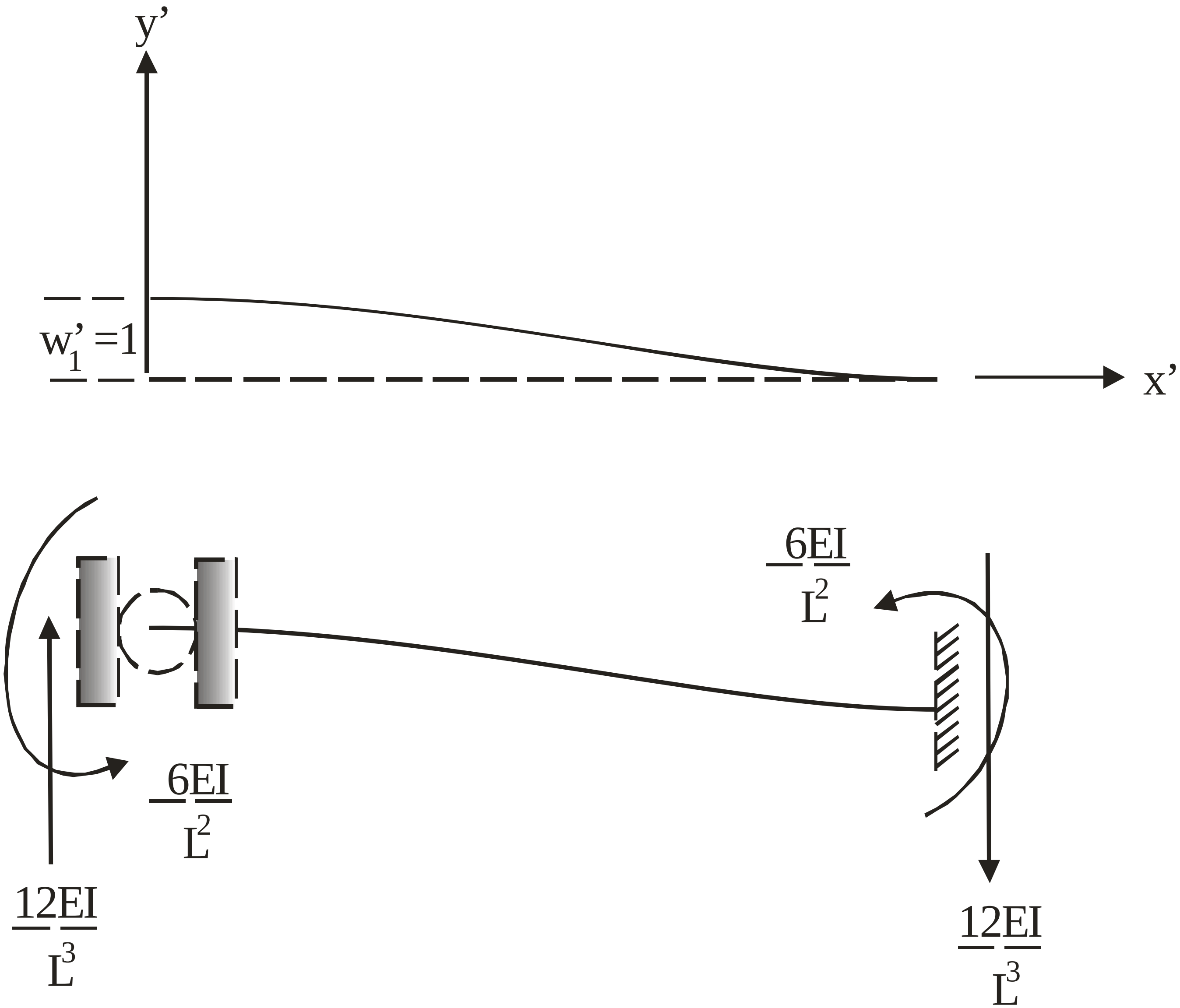
$$\mathbf{k}'_{4\times4}\mathbf{d}'_{4\times1} = \mathbf{f}'_{4\times1}$$

Typical Element

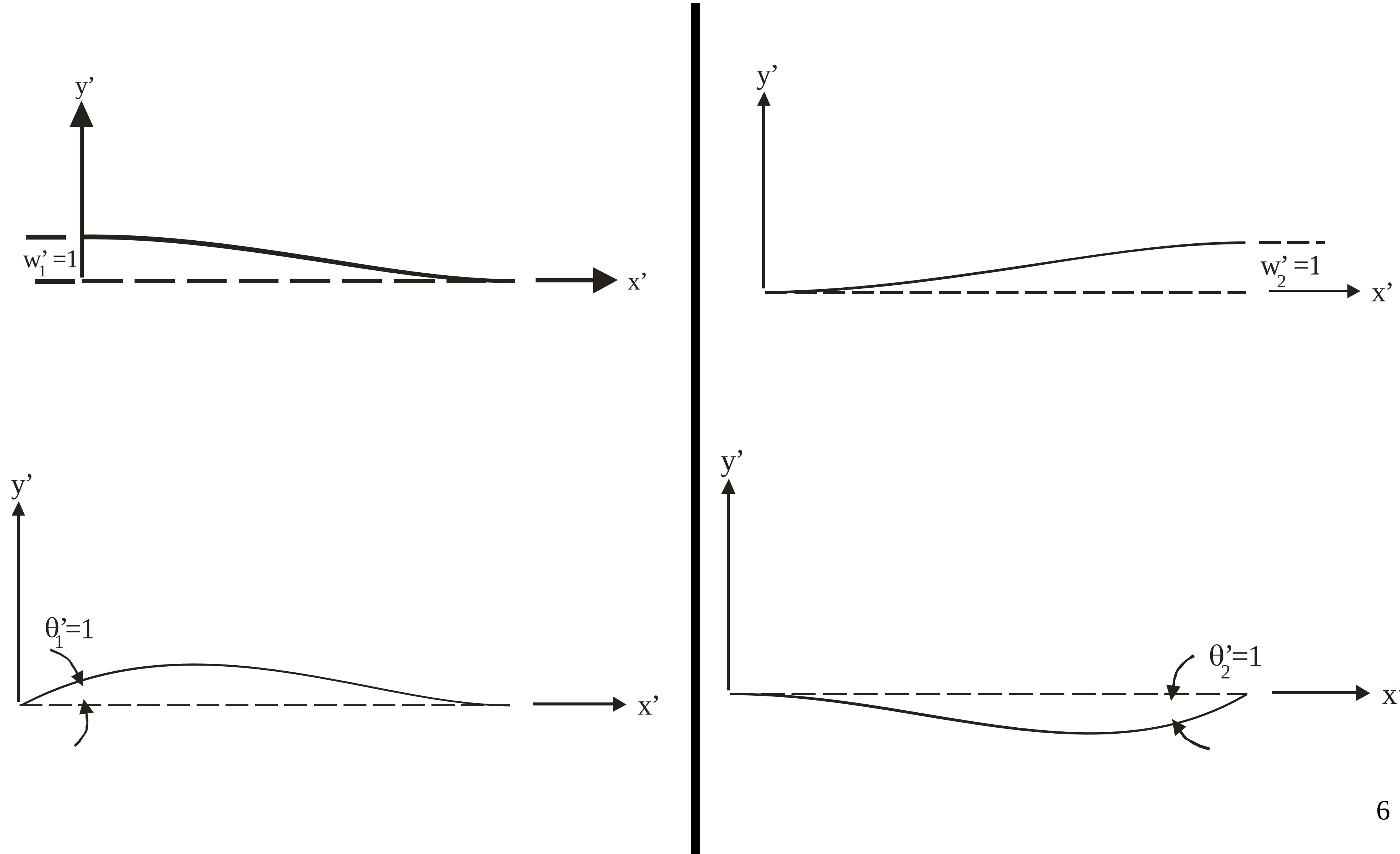


- Note local coordinate system
- 4 degrees-of-freedom

Direct Stiffness Method



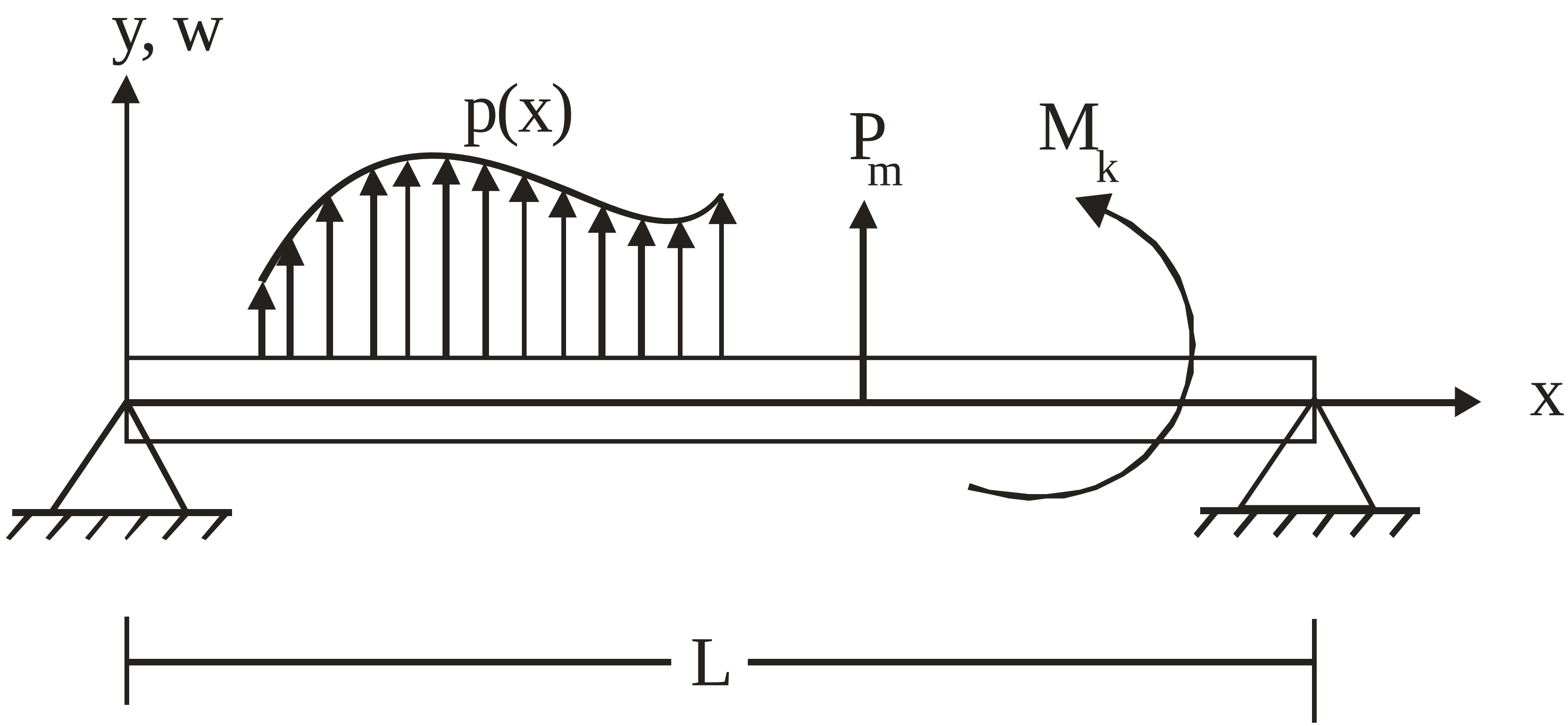
Stiffness Coefficients



Element Stiffness Matrix (Bending)

$$\mathbf{k}'_{4\times 4} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Min. Potential Energy



DE: $$\frac{d^2 w(x)}{dx^2} = \frac{M_z}{EI_z}$$

Min. Potential Energy

Strain Energy

$$\sigma_x = -\frac{M_z y}{I_z}$$

$$\sigma_x = E\varepsilon_x$$

$$U = \int_V U_0\, dV = \int_0^L \int_A \frac{1}{2}\varepsilon\sigma\, dA\, dx = \frac{1}{2}\int_0^L EI\left(\frac{d^2 w}{dx^2}\right)^2 dx$$

Total Potential Energy

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2 w}{dx^2}\right)^2 dx - \int_0^L pw\, dx - \sum_m P_m w_m - \sum_k M_k \frac{dw}{dx}$$

Min. Potential Energy

Assumed displacement

$$w(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

End Conditions

$$w(x=0) = w_1 \qquad w(x=L) = w_2$$

$$\frac{dw}{dx}(x=0) = \theta_1 \qquad \frac{dw}{dx}(x=L) = \theta_2$$

Min. Potential Energy

Assumed displacement (note interpolation idea)

$$w(x) = \phi_1 w_1 + \phi_2 \theta_1 + \phi_3 w_2 + \phi_4 \theta_2$$

Shape Functions

$$\phi_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \qquad \phi_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$$

$$\phi_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2} \qquad \phi_4 = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

Beam Element (Bending)

$$w(x) = \phi_1 w_1 + \phi_2 \theta_1 + \phi_3 w_2 + \phi_4 \theta_2$$

Differentiating Twice

$$\frac{d^2 w}{dx^2} = \left[-\frac{6}{L^2} + \frac{12x}{L^3} \;\middle|\; -\frac{4}{L} + \frac{6x}{L^2} \;\middle|\; \frac{6}{L^2} - \frac{12x}{L^3} \;\middle|\; -\frac{2}{L} + \frac{6x}{L^2} \right]_{1\times 4} \mathbf{d}_{4\times 1}$$

$$\frac{d^2 w}{dx^2} = \mathbf{B}_{1\times 4} \mathbf{d}_{4\times 1}$$

Beam Stiffness (Bending)

Strain Energy

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx$$

$$U = \frac{1}{2}\mathbf{d}^{\mathbf{T}}\left[\int_0^L \mathbf{B}^T EI\,\mathbf{B}\,dx\right]\mathbf{d}$$

$$U = \frac{1}{2}\mathbf{d}^{\mathbf{T}}_{1\times 4}\mathbf{k}_{4\times 4}\mathbf{d}_{4\times 1}$$

Beam Stiffness Matrix

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Same as Direct Stiffness Approach

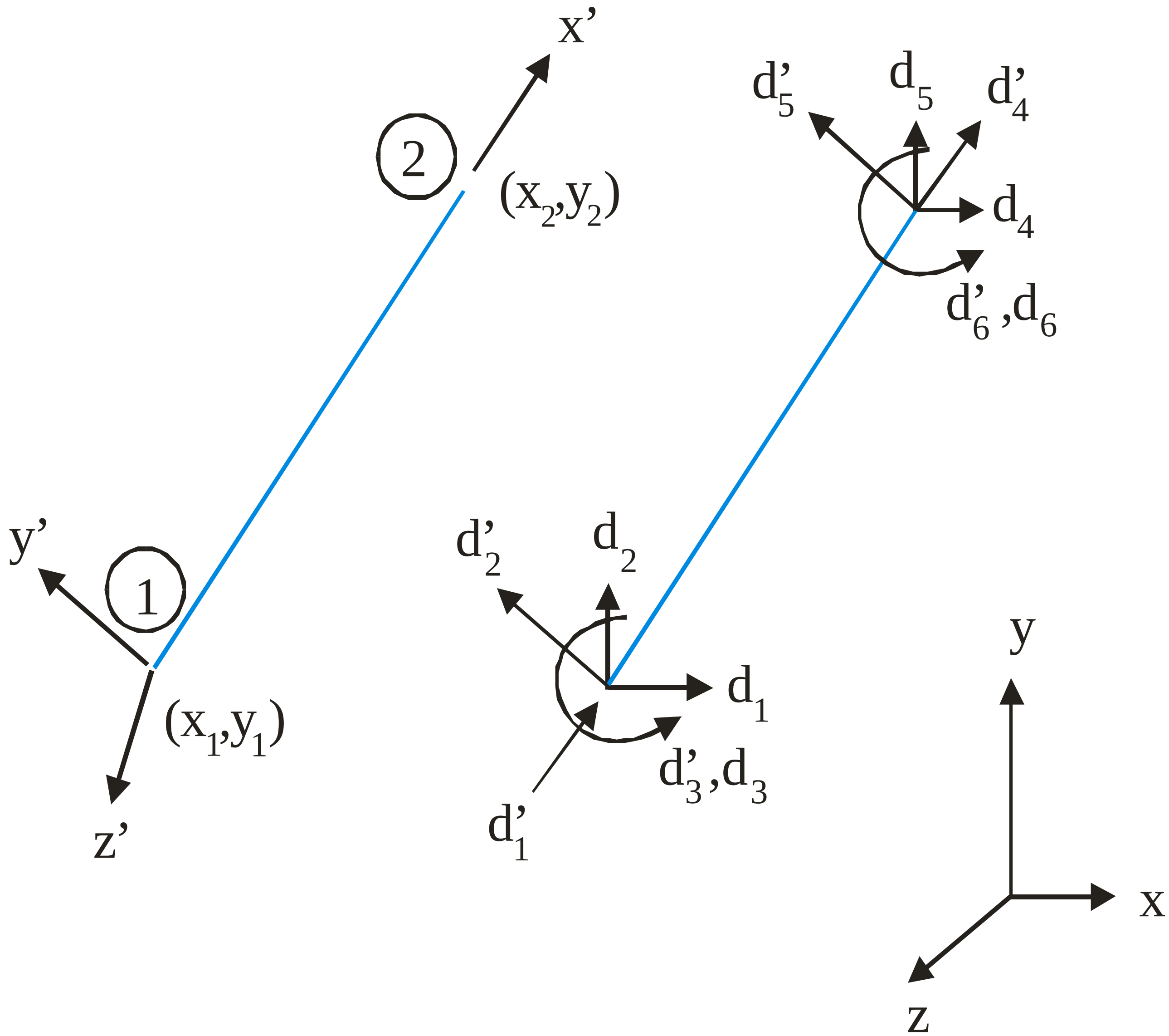
Step 2: Element Equations

$$\mathbf{k}'_{6\times 6}\,\mathbf{d}'_{6\times 1} = \mathbf{f}'_{6\times 1}$$

Element Stiffness Matrix (Axial + Bending)

$$\begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u_1' \\ w_1' \\ \theta_1' \\ u_2' \\ w_2' \\ \theta_2' \end{Bmatrix} = \begin{Bmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \\ f_5' \\ f_6' \end{Bmatrix}$$

Local-To-Global Transformation



Local-to-Global Transformation

Local axes

$$x'-axis:\left(l_{x'},m_{x'},n_{x'}\right)$$

$$y'-axis:\left(l_{y'},m_{y'},n_{y'}\right)$$

$$z'-axis:\left(0,0,1\right)$$

y' = z' x x'

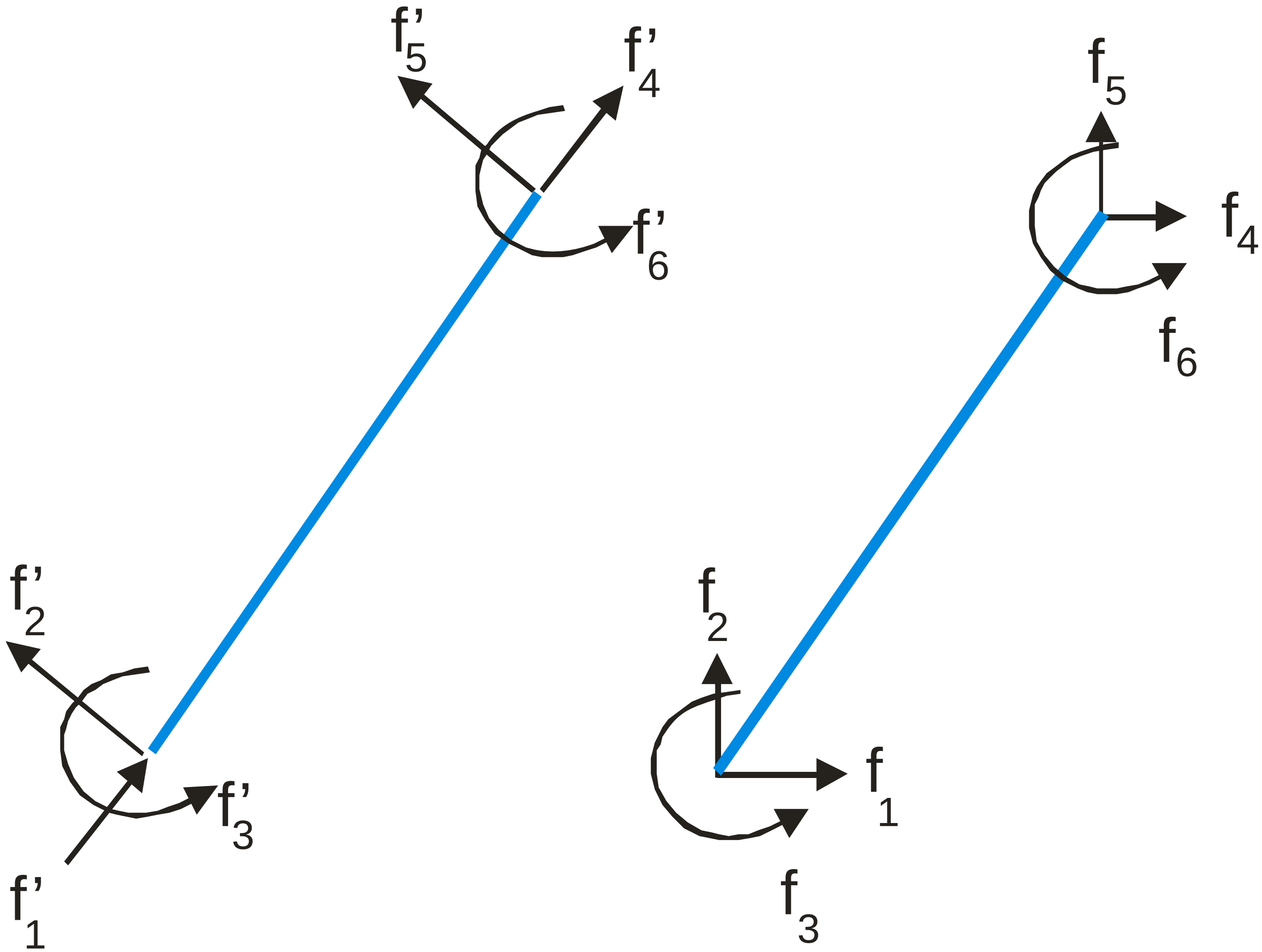
$$\left(0i+0j+1k\right)\times\left(l_{x'}i+m_{x'}j+0k\right)=$$

$$\left(-m_{x'}i+l_{x'}j+0k\right)$$

Local-to-Global Transformation

$$\begin{Bmatrix} d_1' \\ d_2' \\ d_3' \\ d_4' \\ d_5' \\ d_6' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} \Rightarrow \mathbf{d}'_{6\times 1} = \mathbf{T}_{6\times 6}\mathbf{d}_{6\times 1}$$

Global-to-Local Transformation



Global-to-Local Transformation

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix} = \begin{bmatrix} l & -m & 0 & 0 & 0 & 0 \\ m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & -m & 0 \\ 0 & 0 & 0 & m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} f_1^{'} \\ f_2^{'} \\ f_3^{'} \\ f_4^{'} \\ f_5^{'} \\ f_6^{'} \end{Bmatrix} \Rightarrow \mathbf{f}_{6\times1} = \mathbf{T}_{6\times6}^{\mathbf{T}} \mathbf{f}_{6\times1}^{'}$$

Step 2: Element Equations

Local axes

$$\mathbf{k}'_{6\times6}\mathbf{d}'_{6\times1} = \mathbf{f}'_{6\times1}$$

$$\mathbf{d}'_{6\times1} = \mathbf{T}_{6\times6}\mathbf{d}_{6\times1}$$

$$\mathbf{f}_{6\times1} = \mathbf{T}^{\mathbf{T}}_{6\times6}\mathbf{f}'_{6\times1}$$

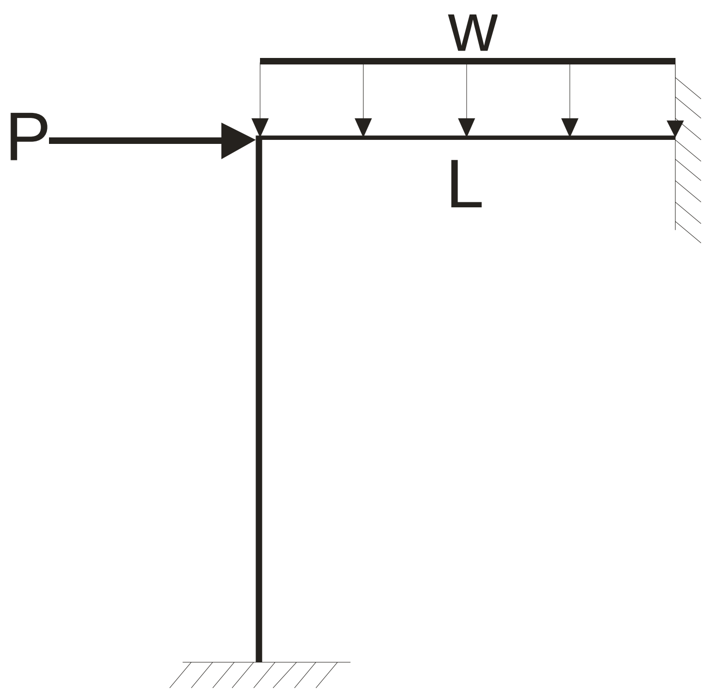
Global axes

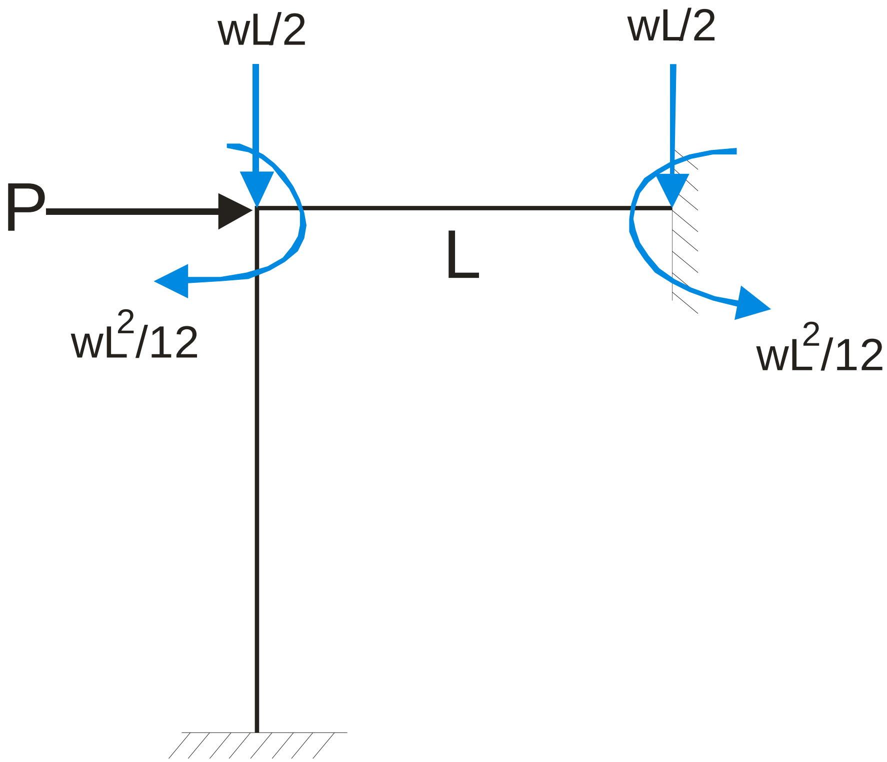
$$\mathbf{k}'_{6\times6}\mathbf{T}_{6\times6}\mathbf{d}_{6\times1} = \mathbf{f}'_{6\times1}$$

$$\mathbf{T}^{\mathbf{T}}_{6\times6}\mathbf{k}'_{6\times6}\mathbf{T}_{6\times6}\mathbf{d}_{6\times1} = \mathbf{T}^{\mathbf{T}}_{6\times6}\mathbf{f}'_{6\times1}$$

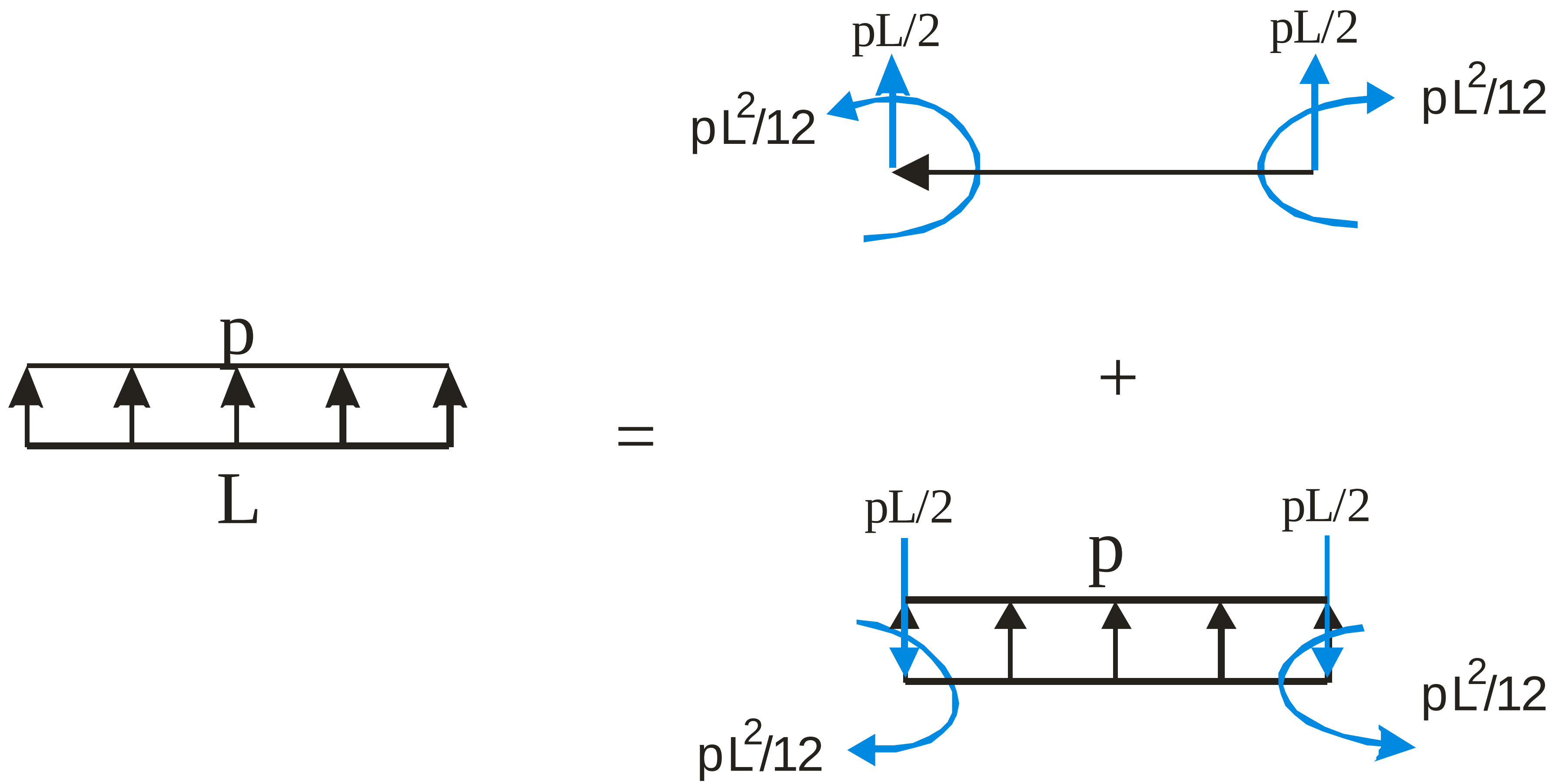
$$\mathbf{k}_{6\times6}\mathbf{d}_{6\times1} = \mathbf{f}_{6\times1}$$

Element Loads





Element Loads



Element Loads

Uniformly Distributed Loading, p(x) = p

$$q_i = \int_0^L p(x)\phi_i(x)dx = p\int_0^L \phi_i(x)dx \quad i = 1, 2, 3, 4$$

Substituting and integrating

$$\mathbf{q}'_{6\times1} = \left[0, \frac{pL}{2}, \frac{pL^2}{12}, 0, \frac{pL}{2}, -\frac{pL^2}{12}\right]^{\mathbf{T}}$$

Algorithm: Element Loads

- Form $\mathbf{q}'_{6\times1}$
- Transform $\mathbf{q}_{6\times1} = \mathbf{T}^{\mathbf{T}}_{6\times6}\mathbf{q}'_{6\times1}$
- Update $\mathbf{F} = \mathbf{F} + \mathbf{q}$
- Impose BC and solve **KD = F**.
- Compute element nodal forces

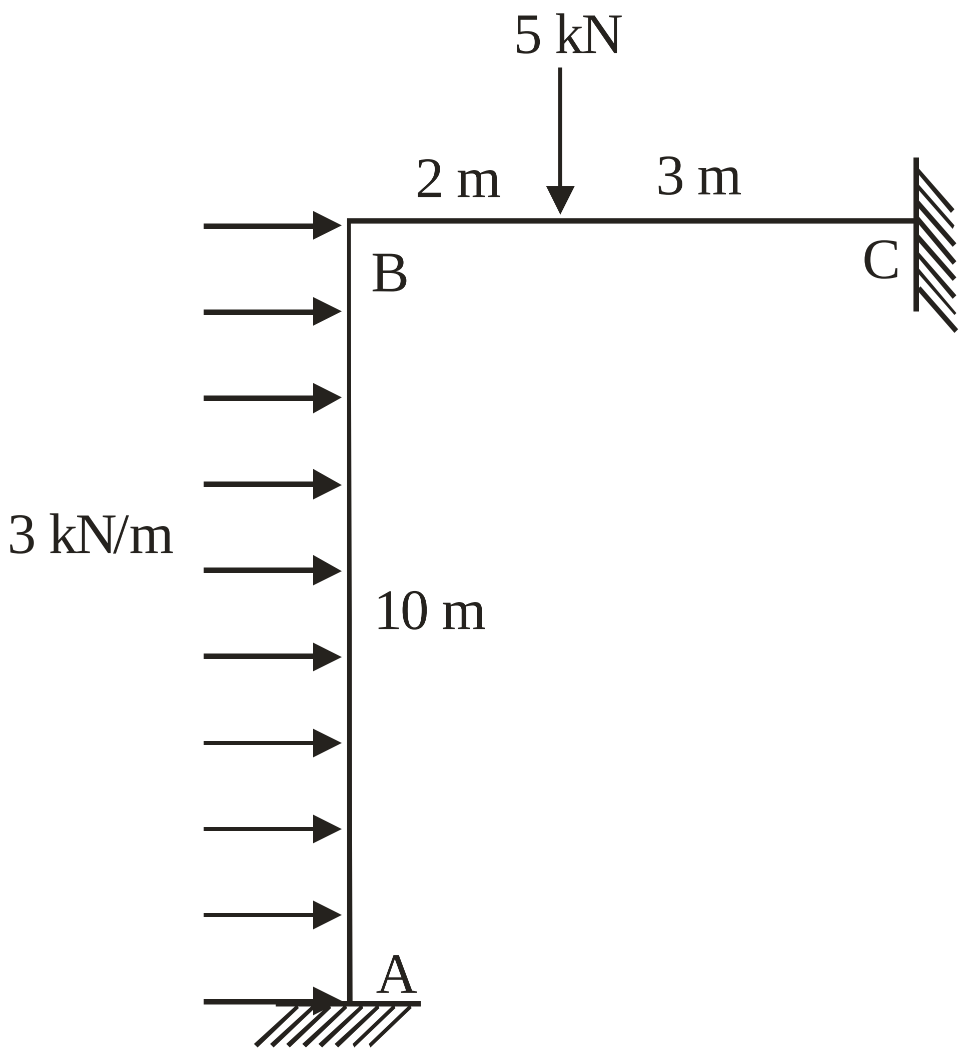
$$\mathbf{f}'_{6\times1} = \mathbf{k}'_{6\times6}\mathbf{T}_{6\times6}\mathbf{d}_{6\times1} - \mathbf{q}'_{6\times1}$$

Element Equations

$$\begin{Bmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \\ f_5' \\ f_6' \end{Bmatrix} = \begin{Bmatrix} a\left[l(d_1-d_4)+m(d_2-d_5)\right] \\ b\left[l\left(d_2-d_5\right)-m(d_1-d_4)\right]+c\left(d_3+d_6\right) \\ c\left[l\left(d_2-d_5\right)-m\left(d_1-d_4\right)\right]+d\left(2d_3+d_6\right) \\ -a\left[l(d_1-d_4)+m(d_2-d_5)\right] \\ -b\left[l\left(d_2-d_5\right)-m(d_1-d_4)\right]-c\left(d_3+d_6\right) \\ c\left[l\left(d_2-d_5\right)-m\left(d_1-d_4\right)\right]+d\left(d_3+2d_6\right) \end{Bmatrix} - \begin{Bmatrix} q_1' \\ q_2' \\ q_3' \\ q_4' \\ q_5' \\ q_6' \end{Bmatrix}$$

$$a=\frac{AE}{L} \quad b=\frac{12EI}{L^3} \quad c=\frac{6EI}{L^2} \quad d=\frac{2EI}{L}$$

Example

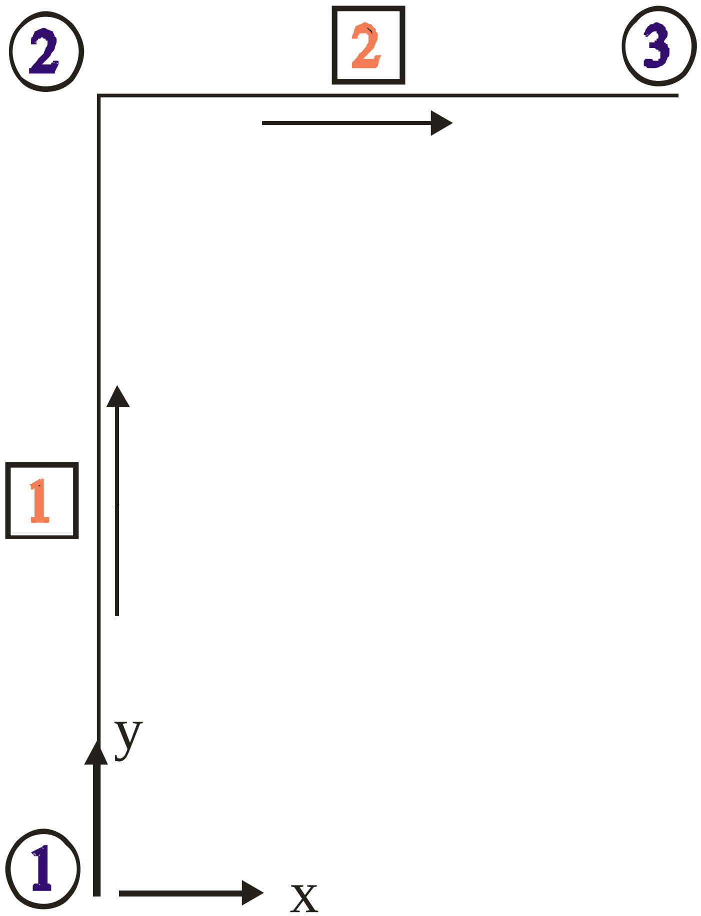


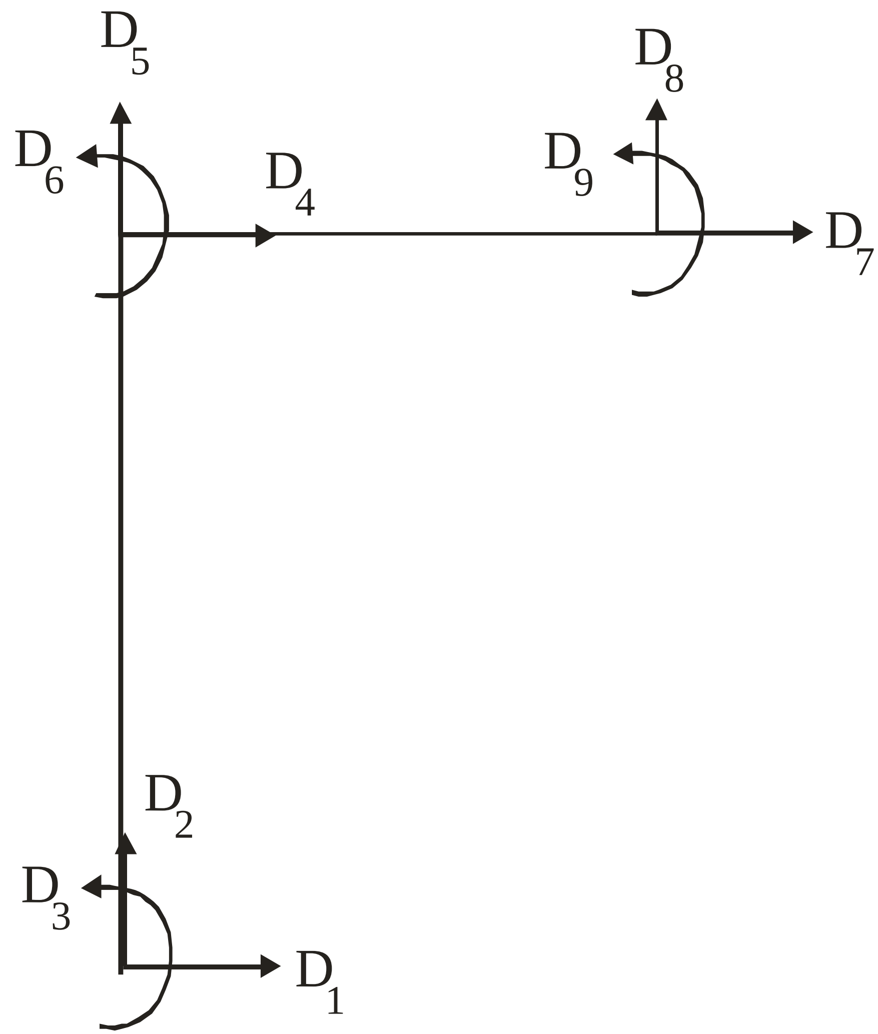
$E = 200$ GPa

$A = 0.01\ m^2$

$I = 0.0001\ m^4$

Example





Example

Element Stiffness Calculations

Element	(*l*, *m*)	(*a*, *b*, *c*, *d*)
1	(0, 1)	(2e8, 2.4e5, 1.2e6, 4e6)
2	(1, 0)	(4e8, 1.92e6, 2.8e6, 8e6)

Example

Element 1: Element Load Calculations

Transform to global coordinate system

$$\mathbf{q}'_{6\times1} = \left\{0, \frac{pL}{2}, \frac{pL^2}{12}, 0, \frac{pL}{2}, -\frac{pL^2}{12}\right\}$$

$$\mathbf{q}'_{6\times1} = \{0, 15000, -25000, 0, 15000, 25000\}$$

Element 2: Element Load Calculations

$$\mathbf{q}'_{6\times1} = \left\{0, \frac{Pb^2(L+2a)}{L^3}, \frac{Pab^2}{L^2}, 0, \frac{Pa^2(L+2b)}{L^3}, -\frac{Pa^2b}{L^2}\right\}$$

$$\mathbf{q}'_{6\times1} = \{0, -3240, -3600, 0, -1760, 2400\}$$

Example

Element 1: Element Equations (global coord. system)

$$10^5 \begin{bmatrix} 24 & 0 & -12 & -2.4 & 0 & -12 \\ 0 & 2000 & 0 & 0 & -2000 & 0 \\ -12 & 0 & 80 & 12 & 0 & 40 \\ -2.4 & 0 & 12 & 2.4 & 0 & 12 \\ 0 & -2000 & 0 & 0 & 2000 & 0 \\ -12 & 0 & 40 & 12 & 0 & 80 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 15000 \\ 0 \\ -25000 \\ 15000 \\ 0 \\ 25000 \end{Bmatrix}$$

Example

Element 2: Element Equations (global coord. system)

$$10^5 \begin{bmatrix} 4000 & 0 & 0 & -4000 & 0 & 0 \\ 0 & 19.2 & 48 & 0 & -19.2 & 48 \\ 0 & 48 & 160 & 0 & -48 & 80 \\ -4000 & 0 & 0 & 4000 & 0 & 0 \\ 0 & -19.2 & -48 & 0 & 19.2 & -48 \\ 0 & 48 & 80 & 0 & -48 & 160 \end{bmatrix} \begin{Bmatrix} D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -3240 \\ -3600 \\ 0 \\ -1760 \\ 2400 \end{Bmatrix}$$

Example

Assembly and Imposition of EBC

$$10^5 \begin{bmatrix} 4002.4 & 0 & 12 \\ 0 & 2019.2 & 48 \\ 12 & 48 & 240 \end{bmatrix} \begin{Bmatrix} D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 15000 \\ -3240 \\ 21400 \end{Bmatrix}$$

Solution

$$D_4 = 3.48(10^{-5})\,m$$

$$D_5 = -3.74(10^{-5})\,m$$

$$D_6 = 8.97(10^{-4})\,rad$$

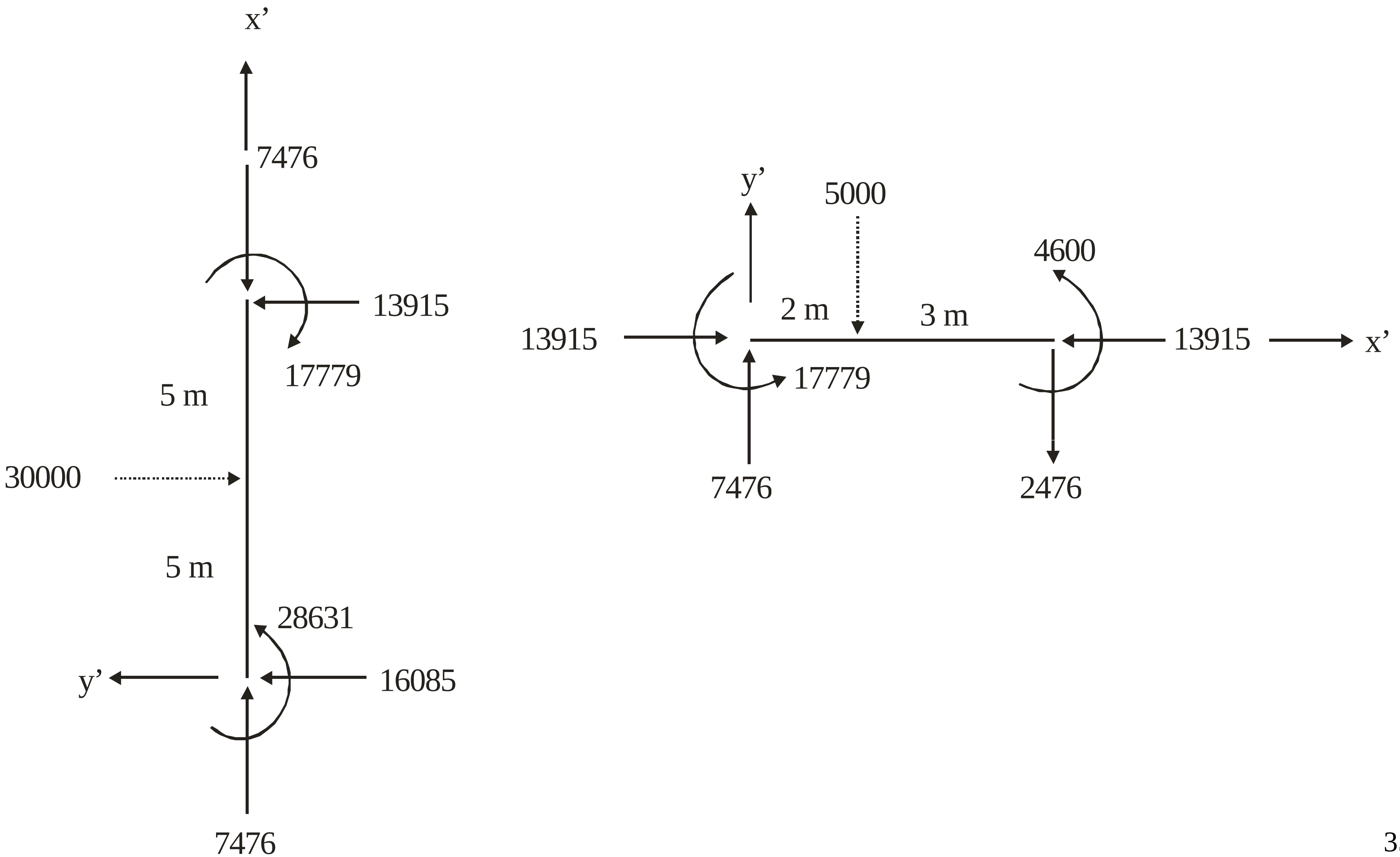
Example

Element Nodal Forces

$$\mathbf{d}'_{6\times1} = \mathbf{T}_{6\times6}\,\mathbf{d}_{6\times1}$$

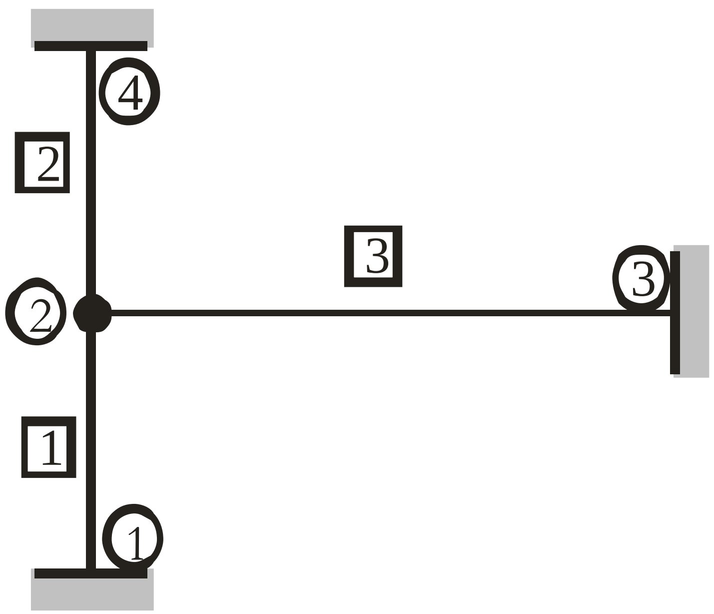
$$\mathbf{f}'_{6\times1} = \mathbf{k}'_{6\times6}\,\mathbf{d}'_{6\times1} - \sum_{i}\left(\mathbf{q}'_{6\times1}\right)_{i}$$

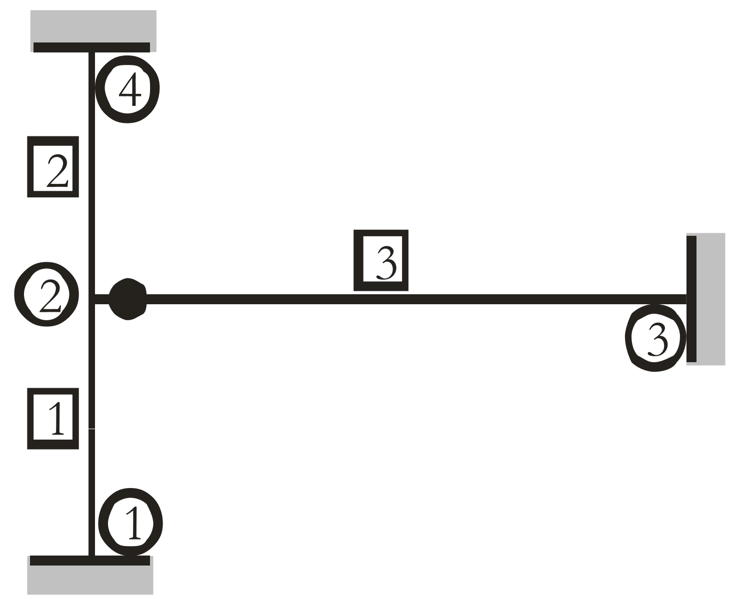
Example



x'
7476
13915
17779
5 m
30000
5 m
28631
y'
16085
7476
y'
5000
4600
2 m
3 m
13915
13915
x'
17779
7476
2476

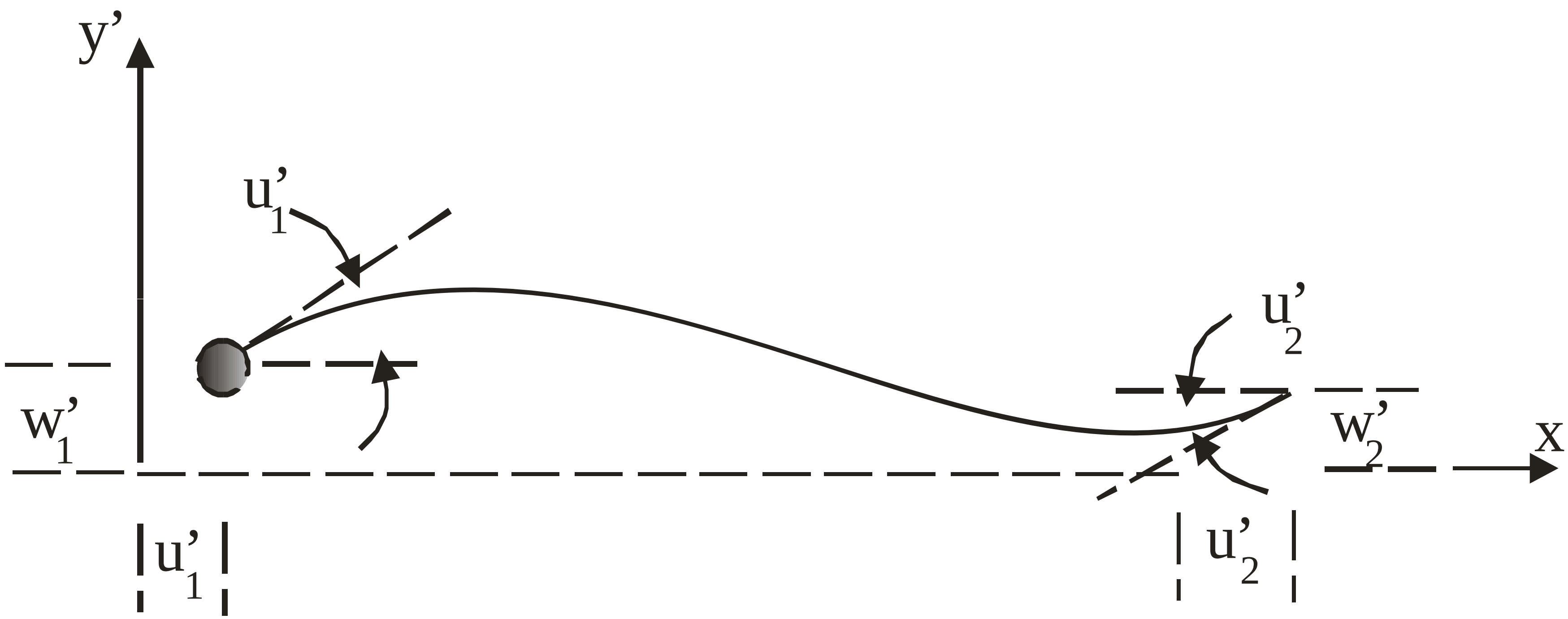
Internal Hinge





Internal Hinge

Hinge at Start Node



Internal Hinge: Start Node

Expand Element Equations

$$\mathbf{f}'_{6\times1} = \mathbf{k}'_{6\times6}\mathbf{d}'_{6\times1} - \mathbf{q}'_{6\times1}$$

$$\frac{EI}{L^3}\left(6Lw_1' + 4L^2\theta_1' - 6Lw_2' + 2L^2\theta_2'\right) - q_3' = f_3'$$

Since

$$f_3' = 0$$

$$\theta_1' = \frac{3}{2L}\left(-w_1' + w_2'\right) - \frac{1}{2}\theta_2' + \frac{L}{4EI}q_3'$$

Internal Hinge: Start Node

$$
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{3EI}{L^3} & 0 & 0 & -\frac{3EI}{L^3} & \frac{3EI}{L^2} \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{3EI}{L^3} & 0 & 0 & \frac{3EI}{L^3} & -\frac{3EI}{L^2} \\
0 & \frac{3EI}{L^2} & 0 & 0 & -\frac{3EI}{L^2} & \frac{3EI}{L}
\end{bmatrix}
\begin{Bmatrix} u_1' \\ w_1' \\ \theta_1' \\ u_2' \\ w_2' \\ \theta_2' \end{Bmatrix}
-
\begin{Bmatrix} q_1' \\ q_2' - \frac{3q_3'}{2L} \\ 0 \\ q_4' \\ q_5' + \frac{3q_3'}{2L} \\ q_6' - \frac{q_3'}{2} \end{Bmatrix}
=
\begin{Bmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \\ f_5' \\ f_6' \end{Bmatrix}
$$

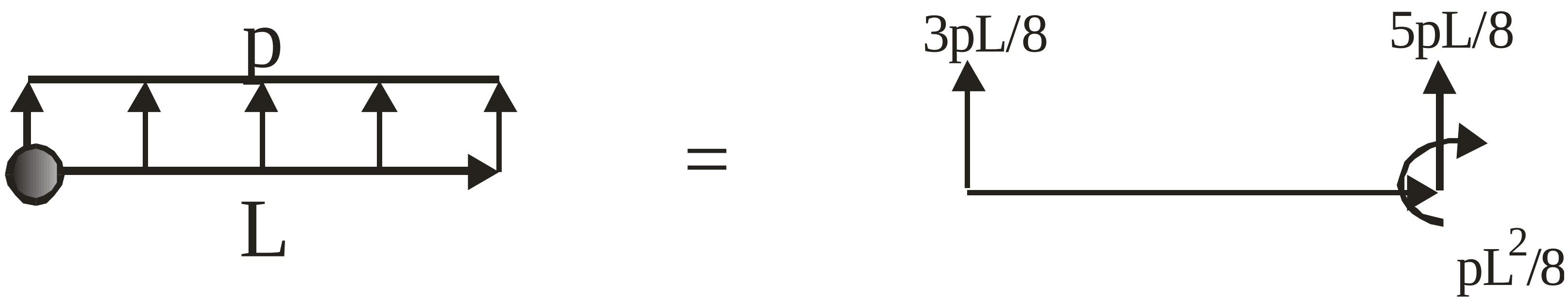
Internal Hinge: End Node

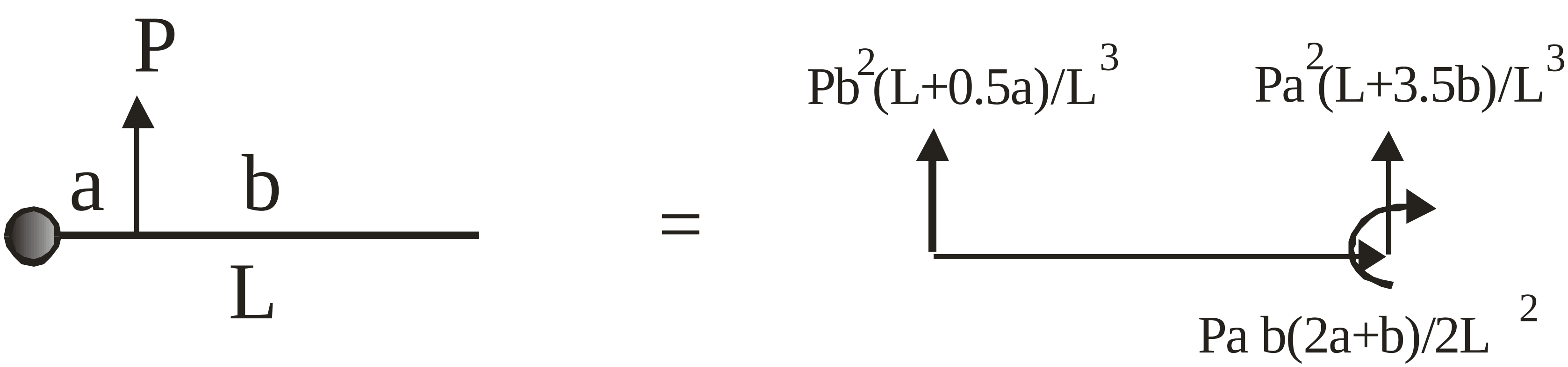
$$
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{3EI}{L^3} & \frac{3EI}{L^2} & 0 & -\frac{3EI}{L^3} & 0 \\
0 & \frac{3EI}{L^2} & \frac{3EI}{L} & 0 & -\frac{3EI}{L^2} & 0 \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{3EI}{L^3} & -\frac{3EI}{L^2} & 0 & \frac{3EI}{L^3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix} u_1^{'} \\ w_1^{'} \\ \theta_1^{'} \\ u_2^{'} \\ w_2^{'} \\ \theta_2^{'} \end{Bmatrix}
-
\begin{Bmatrix} q_1^{'} \\ q_2^{'} - \frac{3q_6^{'}}{2L} \\ q_3^{'} - \frac{q_6^{'}}{2} \\ q_4^{'} \\ q_5^{'} + \frac{3q_6^{'}}{2L} \\ 0 \end{Bmatrix}
=
\begin{Bmatrix} f_1^{'} \\ f_2^{'} \\ f_3^{'} \\ f_4^{'} \\ f_5^{'} \\ f_6^{'} \end{Bmatrix}
$$

Internal Hinge: Both Nodes

$$\begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1' \\ w_1' \\ \theta_1' \\ u_2' \\ w_2' \\ \theta_2' \end{Bmatrix} - \begin{Bmatrix} q_1' \\ q_2' - \frac{1}{L}\left(q_3' + q_6'\right) \\ 0 \\ q_4' \\ q_5' + \frac{1}{L}\left(q_3' + q_6'\right) \\ 0 \end{Bmatrix} = \begin{Bmatrix} f_1' \\ f_2' \\ f_3' \\ f_4' \\ f_5' \\ f_6' \end{Bmatrix}$$

Internal Hinge: Element Loads





Algorithm: Internal Hinge

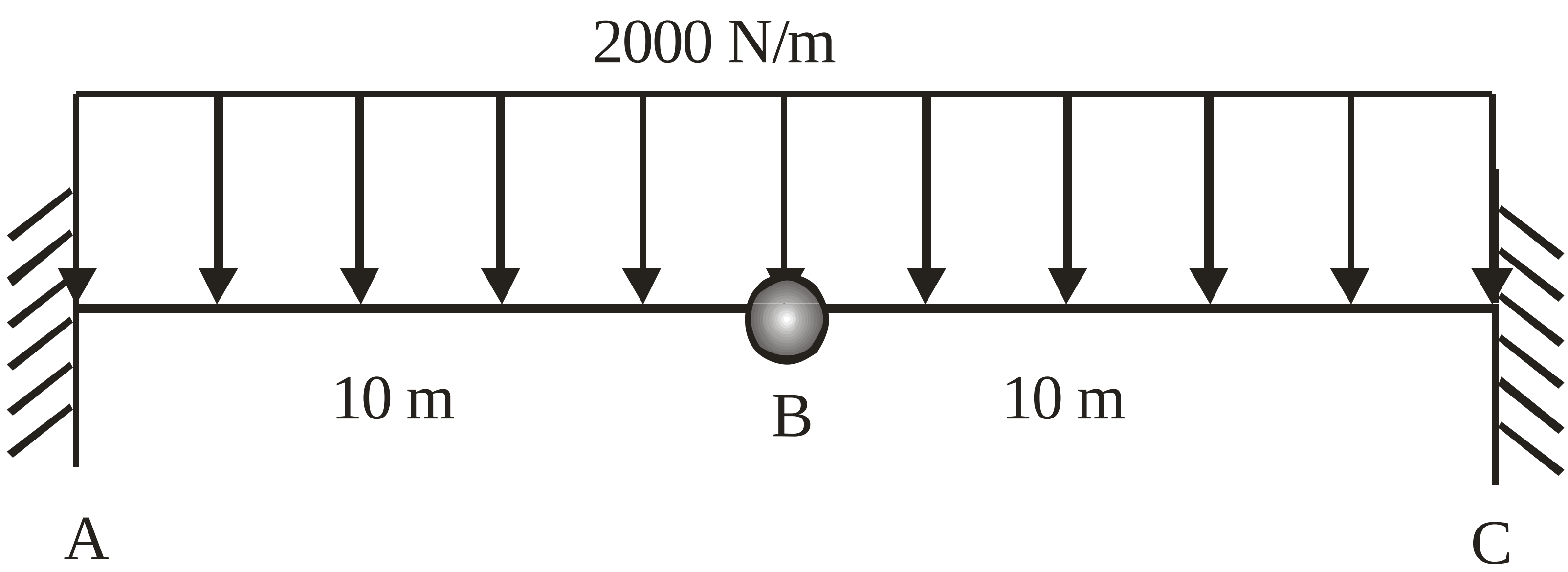
- Form $\mathbf{q}'_{6\times1}$
- Transform $\mathbf{q}_{6\times1} = \mathbf{T}^{\mathbf{T}}_{6\times6}\mathbf{q}'_{6\times1}$
- Update $\mathbf{F} = \mathbf{F} + \mathbf{q}$
- Impose BC (zero out rotation) and solve
  $\mathbf{KD} = \mathbf{F}$

Algorithm: Internal Hinge

- Loop thro' all elements
- For element with hinge $\mathbf{d}'_{6\times1} = \mathbf{T}_{6\times6}\mathbf{d}_{6\times1}$
- Recover hinge rotation
- Compute element nodal forces

$$\mathbf{f}'_{6\times1} = \mathbf{k}'_{6\times6}\mathbf{d}'_{6\times1} - \mathbf{q}'_{6\times1}$$

Example

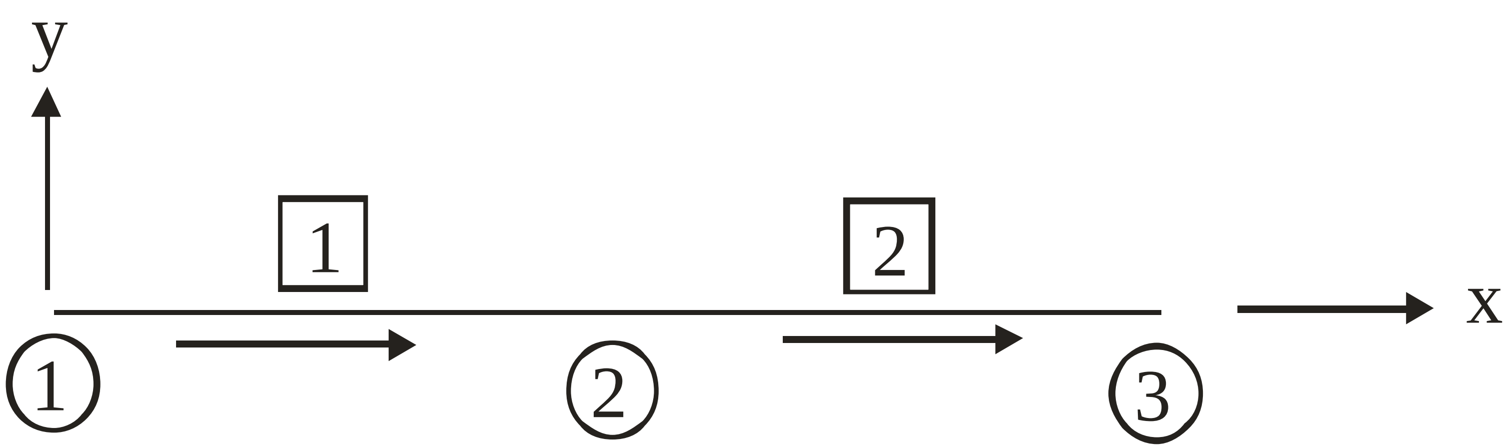


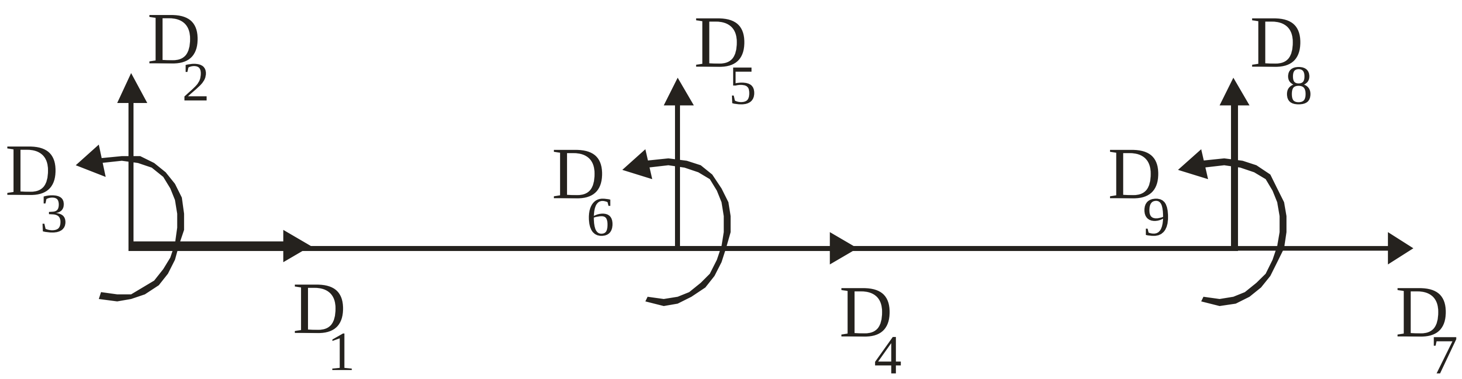
E = 200 GPa

$I = 10^{-4}$ m^4

$A = 1.0$ m^2

Example





Example

Element 1: Hinge at End Node

$$10^4 \begin{bmatrix} 2(10^6) & 0 & 0 & -2(10^6) & 0 & 0 \\ 0 & 6 & 60 & 0 & -6 & 0 \\ 0 & 60 & 600 & 0 & -60 & 0 \\ -2(10^6) & 0 & 0 & 2(10^6) & 0 & 0 \\ 0 & -6 & -60 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_{61} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -12500 \\ -25000 \\ 0 \\ -7500 \\ 0 \end{Bmatrix}$$

Example

Element 2: Hinge at Start Node

$$10^4 \begin{bmatrix} 2(10^6) & 0 & 0 & -2(10^6) & 0 & 0 \\ 0 & 6 & 0 & 0 & -6 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2(10^6) & 0 & 0 & 2(10^6) & 0 & 0 \\ 0 & -6 & 0 & 0 & 6 & -60 \\ 0 & 60 & 0 & 0 & -60 & 600 \end{bmatrix} \begin{Bmatrix} D_4 \\ D_5 \\ D_{62} \\ D_7 \\ D_8 \\ D_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -7500 \\ 0 \\ 0 \\ -12500 \\ 25000 \end{Bmatrix}$$

Example

System Equations after imposing BCs

$$10^4 \begin{bmatrix} 4(10^6) & 0 \\ 0 & 12 \end{bmatrix} \begin{Bmatrix} D_4 \\ D_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -15000 \end{Bmatrix}$$

Solution to KD = F

$$D_4 = 0 \quad D_5 = -0.125m$$

Example

Element 1: Hinge Rotation at End Node

$$\theta_2^{'} = -0.01667\, rad$$

Element 2: Hinge Rotation at Start Node

$$\theta_1^{'} = 0.01667\, rad$$

For each element compute and augment with appropriate hinge rotations

$$\mathbf{d}^{'}_{6\times1} = \mathbf{T}_{6\times6}\mathbf{d}_{6\times1}$$

Example

Finally, for each element compute element nodal forces

$$\mathbf{f}^{'}_{6\times1} = \mathbf{k}^{'}_{6\times6}\mathbf{d}^{'}_{6\times1} - \mathbf{q}^{'}_{6\times1}$$

Element 1

$$\mathbf{f}^{'}_{6\times1} = \left\{0, 20000\,N, 100000\,N-m, 0, 0, 0, 0\right\}$$

Element 2

$$\mathbf{f}^{'}_{6\times1} = \left\{0,0,0,0,20000\,N, -100000\,N-m\right\}$$

Example