Finite Elements for Engineers

Lecture 5: Axisymmetric Problems

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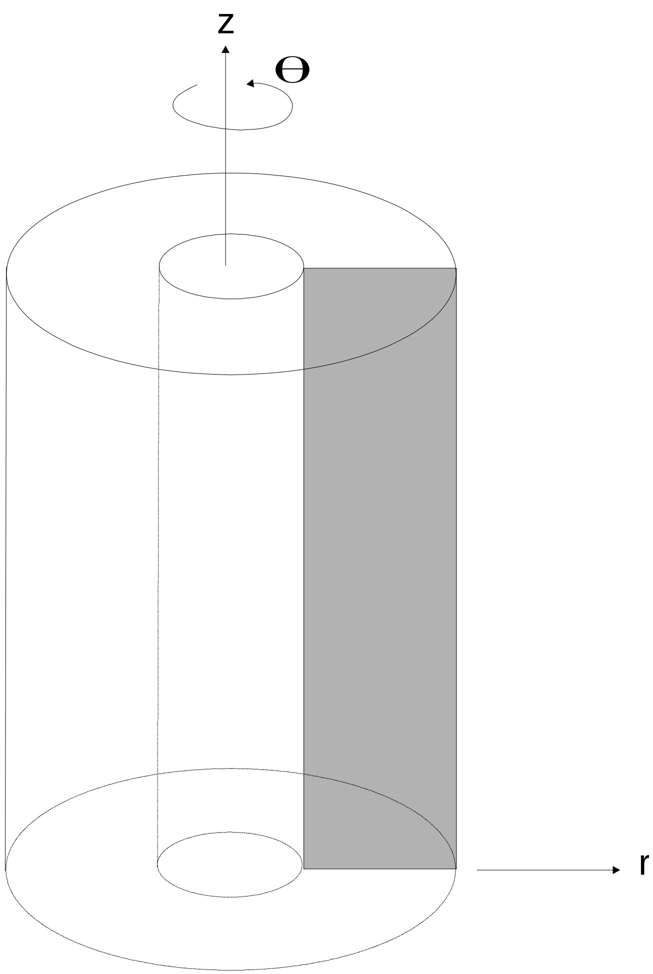
Axisymmetric Problem



- Entire FE model (geometry, properties, boundary conditions) are functions of **r** and **z**.
- Cylindrical coordinate system (r, z, θ)
- **r** is the radial direction
- **z** is the axial direction
- Sometimes called "2.5D" problems

Axisymmetric Problems

Displacement field

$$u = u(r, z)$$

$$w = w(r, z)$$

Strain-displacement relations

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \gamma_{rz} \\ \varepsilon_\theta \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial r \\ \partial w / \partial z \\ \partial u / \partial z + \partial w / \partial r \\ u / r \end{Bmatrix}$$

Axisymmetric Problems

Stress-strain relations

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \tau_{rz} \\ \sigma_\theta \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & \nu \\ & 1-\nu & 0 & \nu \\ & & \frac{1}{2}-\nu & 0 \\ SYM & & & 1-\nu \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \gamma_{rz} \\ \varepsilon_\theta \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \alpha \\ 0 \\ \alpha \end{Bmatrix} \Delta T \right)$$

Axisymmetric Problems

Total Potential Energy

$$\Pi = \frac{1}{2}\int_0^{2\pi}\int_A \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}\, r\, dA\, d\theta - \int_0^{2\pi}\int_A \mathbf{u}^{\mathrm{T}}\mathbf{f}\, r\, dA\, d\theta - \int_0^{2\pi}\int_\Gamma \mathbf{u}^{\mathrm{T}}\mathbf{T}\, r\, dl\, d\theta - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i$$

Simplifies to

$$\Pi = 2\pi\left(\frac{1}{2}\int_A \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}\, r\, dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f}\, r\, dA - \int_\Gamma \mathbf{u}^{\mathrm{T}}\mathbf{T}\, r\, dl\right) - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i$$

Linear Triangular Element

Assumed displacement field

$$u = a_1 + a_2\xi + a_3\eta$$

$$w = b_1 + b_2\xi + b_3\eta$$

Note

$$\left(x, y\right) \equiv \left(r, z\right)$$

Shape Functions

$$\phi_1 = \xi \qquad \phi_2 = \eta \qquad \phi_3 = 1 - \xi - \eta$$

$$r = \phi_1 r_1 + \phi_2 r_2 + \phi_3 r_3$$

$$z = \phi_1 z_1 + \phi_2 z_2 + \phi_3 z_3$$

$$\Rightarrow \quad \mathbf{J}_{2\times 2} = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix}$$

Linear Triangular Element

$$\det(J) = r_{13}z_{23} - r_{23}z_{13} \qquad \boldsymbol{\Gamma}_{2\times 2} = \frac{1}{\det(J)}\begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & r_{13} \end{bmatrix}$$

Strain-displacement relations

$$\mathbf{B}_{4\times 6} = \begin{bmatrix} \mathbf{B}^1_{3\times 6} \\ \hline \mathbf{B}^2_{1\times 6} \end{bmatrix} \Rightarrow \mathbf{B}^1_{3\times 6} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{11} & \Gamma_{12} \end{bmatrix}_{3\times 4} \begin{bmatrix} \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \phi_{3,\xi} & 0 \\ \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & \phi_{3,\eta} & 0 \\ 0 & \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \phi_{3,\xi} \\ 0 & \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & \phi_{3,\eta} \end{bmatrix}_{4\times 6}$$

$$\mathbf{B}^2_{1\times 6} = \begin{bmatrix} \frac{\phi_1}{r} & 0 & \frac{\phi_2}{r} & 0 & \frac{\phi_3}{r} & 0 \end{bmatrix}_{1\times 6}$$

Linear Triangular Element

$$\mathbf{B}_{4\times 6} = \begin{bmatrix} \frac{z_{23}}{\det(J)} & 0 & \frac{z_{31}}{\det(J)} & 0 & \frac{z_{12}}{\det(J)} & 0 \\ 0 & \frac{r_{32}}{\det(J)} & 0 & \frac{r_{13}}{\det(J)} & 0 & \frac{r_{21}}{\det(J)} \\ \frac{r_{32}}{\det(J)} & \frac{z_{23}}{\det(J)} & \frac{r_{13}}{\det(J)} & \frac{z_{31}}{\det(J)} & \frac{r_{21}}{\det(J)} & \frac{z_{12}}{\det(J)} \\ \frac{\phi_1}{r} & 0 & \frac{\phi_2}{r} & 0 & \frac{\phi_3}{r} & 0 \end{bmatrix}$$

Linear Triangular Element

$$\mathbf{k}_{6\times6} = 2\pi \iint_A \mathbf{B}^{\mathbf{T}}_{6\times4} \mathbf{D}_{4\times4} \mathbf{B}_{4\times6}\, r dA$$

One can integrate this exactly by expressing r in terms of the shape functions.

$$r = \sum_{i=1}^{n} \phi_i r_i$$

Or approximately as

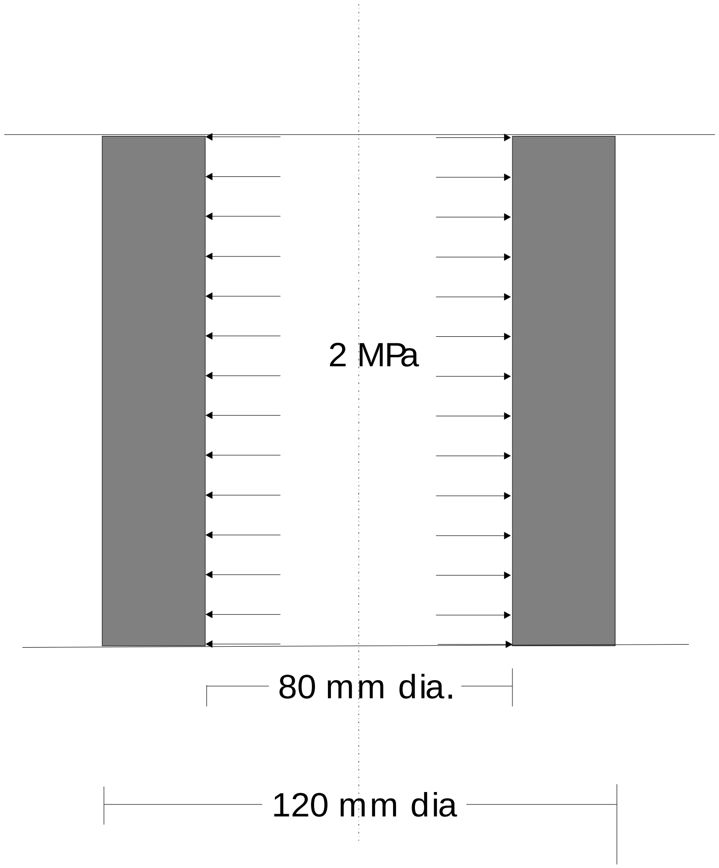
$$\bar{r} = \frac{r_1 + r_2 + r_3}{3} \quad \Rightarrow \quad \mathbf{k}_{6\times6} = 2\pi \bar{r} A \mathbf{B}^{\mathbf{T}}_{6\times4} \mathbf{D}_{4\times4} \mathbf{B}_{4\times6}$$

Computation of Strains and Stresses

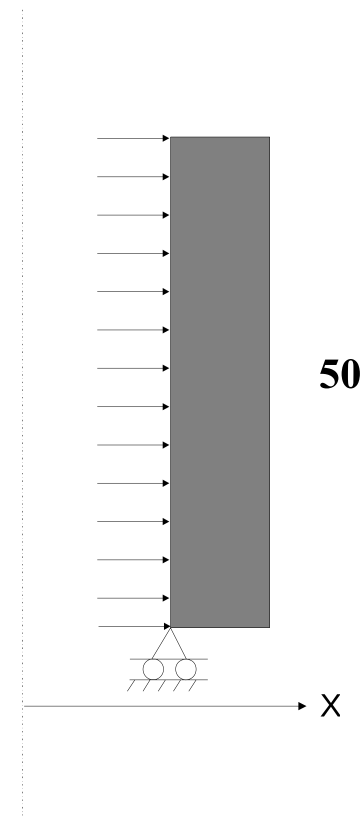
$$\boldsymbol{\varepsilon}_{4\times1} = \overline{\mathbf{B}}_{4\times6}\,\mathbf{d}_{6\times1}$$

$$\boldsymbol{\sigma}_{4\times1} = \mathbf{D}_{4\times4}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0\right)_{4\times1}$$

Example: Hollow Cylinder

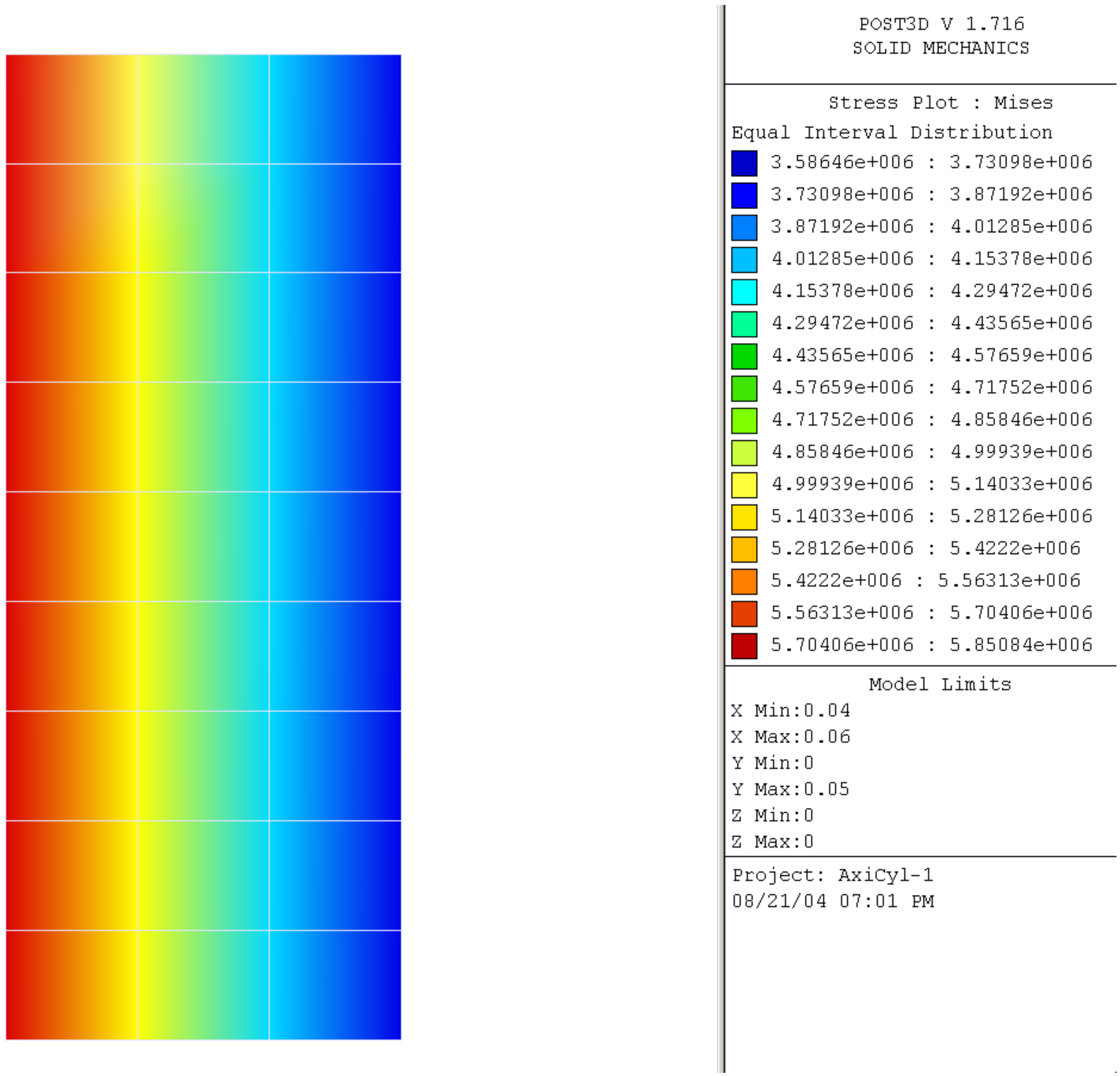


FE Model



$$E = 200\,GPa \qquad \nu = 0.3$$

FE Results


POST3D V 1.716
SOLID MECHANICS
Stress Plot : Mises
Equal Interval Distribution
3.58646e+006 : 3.73098e+006
3.73098e+006 : 3.87192e+006
3.87192e+006 : 4.01285e+006
4.01285e+006 : 4.15378e+006
4.15378e+006 : 4.29472e+006
4.29472e+006 : 4.43565e+006
4.43565e+006 : 4.57659e+006
4.57659e+006 : 4.71752e+006
4.71752e+006 : 4.85846e+006
4.85846e+006 : 4.99939e+006
4.99939e+006 : 5.14033e+006
5.14033e+006 : 5.28126e+006
5.28126e+006 : 5.4222e+006
5.4222e+006 : 5.56313e+006
5.56313e+006 : 5.70406e+006
5.70406e+006 : 5.85084e+006
Model Limits
X Min:0.04
X Max:0.06
Y Min:0
Y Max:0.05
Z Min:0
Z Max:0
Project: AxiCyl-1
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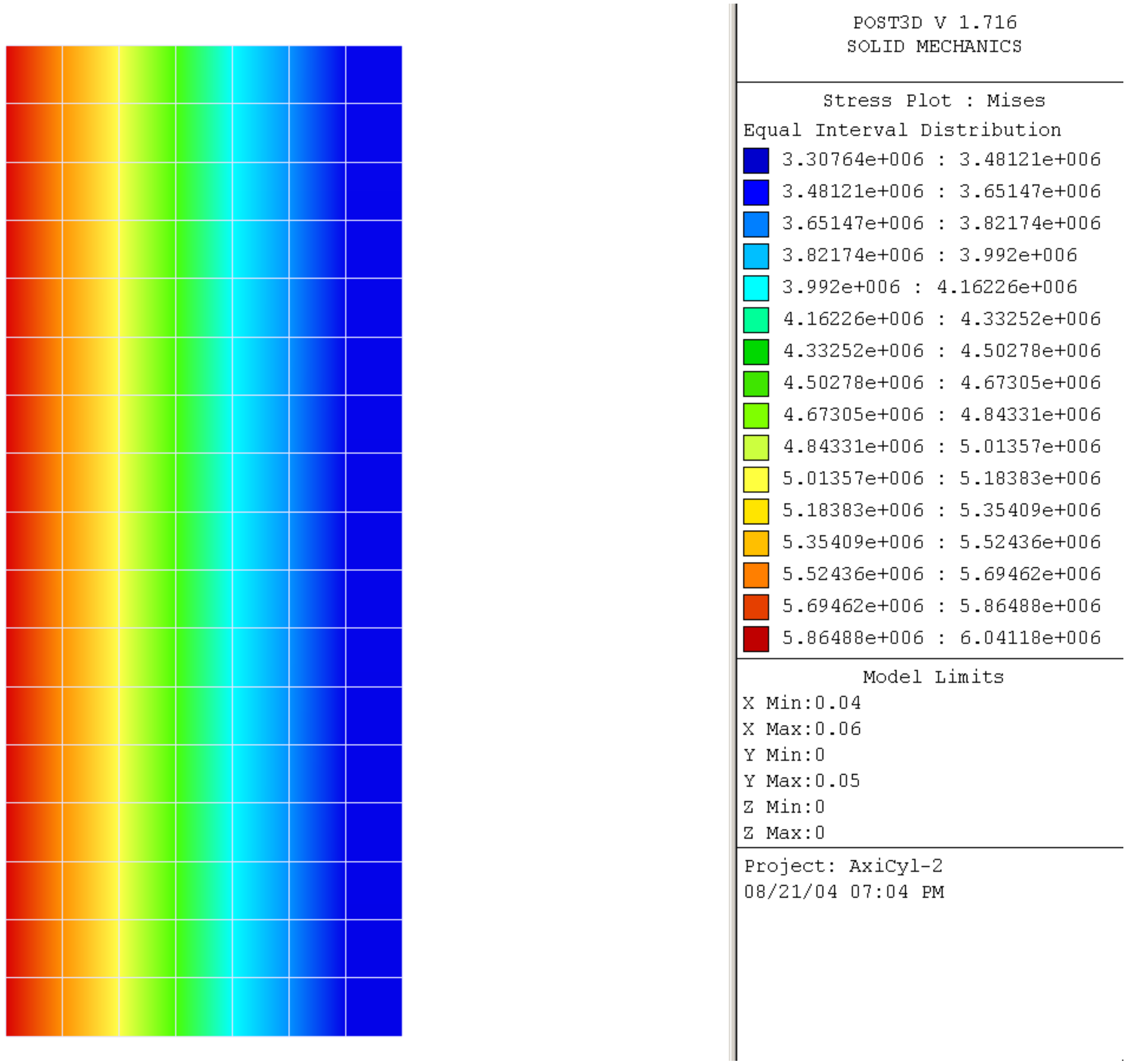
FE Results


POST3D V 1.716
SOLID MECHANICS
Stress Plot : Mises
Equal Interval Distribution
3.30764e+006 : 3.48121e+006
3.48121e+006 : 3.65147e+006
3.65147e+006 : 3.82174e+006
3.82174e+006 : 3.992e+006
3.992e+006 : 4.16226e+006
4.16226e+006 : 4.33252e+006
4.33252e+006 : 4.50278e+006
4.50278e+006 : 4.67305e+006
4.67305e+006 : 4.84331e+006
4.84331e+006 : 5.01357e+006
5.01357e+006 : 5.18383e+006
5.18383e+006 : 5.35409e+006
5.35409e+006 : 5.52436e+006
5.52436e+006 : 5.69462e+006
5.69462e+006 : 5.86488e+006
5.86488e+006 : 6.04118e+006
Model Limits
X Min:0.04
X Max:0.06
Y Min:0
Y Max:0.05
Z Min:0
Z Max:0
Project: AxiCyl-2
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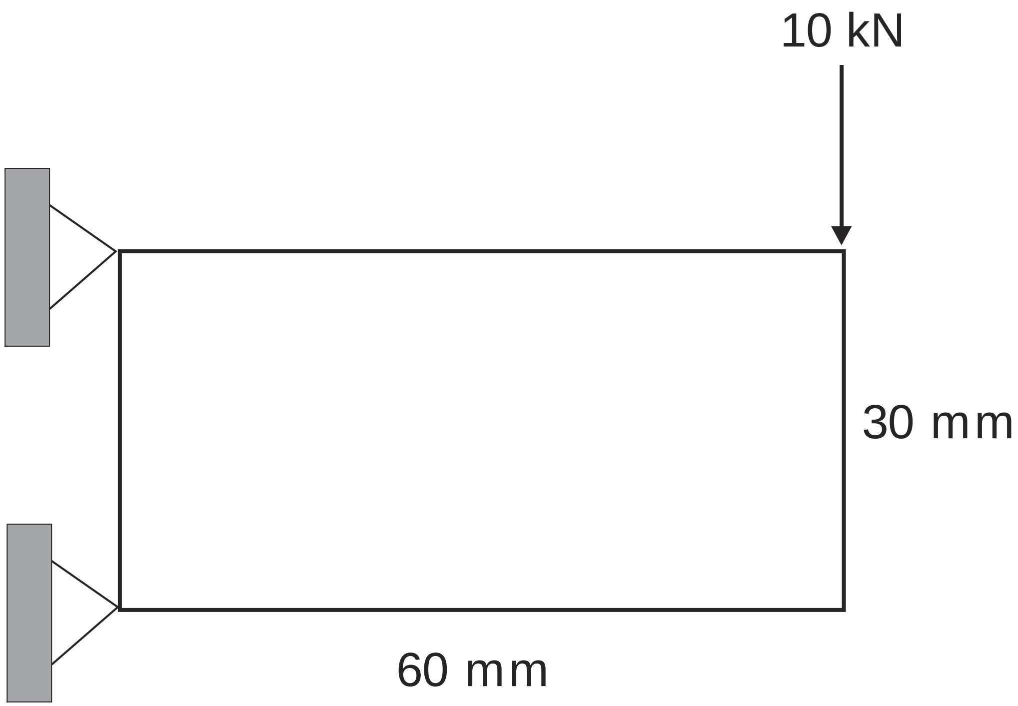
Summary

- Axisymmetric analysis provides an efficient 2D solution to a 3D problem (*r* is *x* and *z* is *y*).
- The analysis has several similarities with plane elasticity analysis. The same isoparametric procedure applies.
- Hoop strain and stress are generated in addition to the other three components.

Programming Project: Option 1

- What needs to be programmed?
  - Input? Output?
- Theory?
- Algorithm?
- Program organization?
- Debugging?
- Test Cases?
- Documentation?

Problem T3L2-2

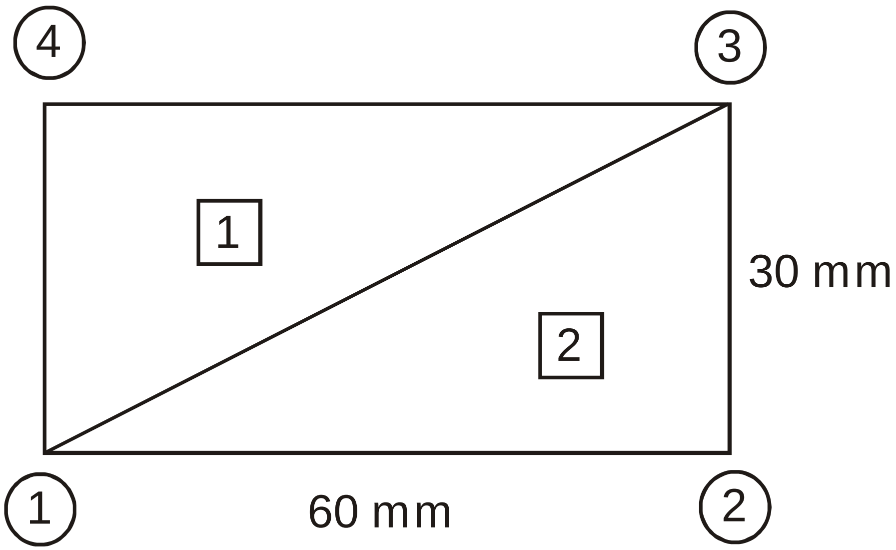


E = 70 GPa
ν = 0.33
t = 10 mm

Compute nodal displacements, strains, stresses and support reactions.

Problem T3L2-2(a): CST Solution

Units: m, N



Element 1: 1-3-4
Element 2: 1-2-3
Plane stress problem.

CST Solution

Step 2: Element equations

```
ELEMENT STIFFNESS MATRIX. ELEMENT :    1
 ROW :    1
    2.6316E+08   0.0000E+00   0.0000E+00  -1.3158E+08  -2.6316E+08   1.3158E+08
 ROW :    2
    0.0000E+00   7.8555E+08  -1.2962E+08   0.0000E+00   1.2962E+08  -7.8555E+08
 ROW :    3
    0.0000E+00  -1.2962E+08   1.9639E+08   0.0000E+00  -1.9639E+08   1.2962E+08
 ROW :    4
   -1.3158E+08   0.0000E+00   0.0000E+00   6.5789E+07   1.3158E+08  -6.5789E+07
 ROW :    5
   -2.6316E+08   1.2962E+08  -1.9639E+08   1.3158E+08   4.5954E+08  -2.6119E+08
 ROW :    6
    1.3158E+08  -7.8555E+08   1.2962E+08  -6.5789E+07  -2.6119E+08   8.5134E+08


 ELEMENT STIFFNESS MATRIX. ELEMENT :    2
 ROW :    1
    1.9639E+08   0.0000E+00  -1.9639E+08   1.2962E+08   0.0000E+00  -1.2962E+08
 ROW :    2
    0.0000E+00   6.5789E+07   1.3158E+08  -6.5789E+07  -1.3158E+08   0.0000E+00
 ROW :    3
   -1.9639E+08   1.3158E+08   4.5954E+08  -2.6119E+08  -2.6316E+08   1.2962E+08
 ROW :    4
    1.2962E+08  -6.5789E+07  -2.6119E+08   8.5134E+08   1.3158E+08  -7.8555E+08
 ROW :    5
    0.0000E+00  -1.3158E+08  -2.6316E+08   1.3158E+08   2.6316E+08   0.0000E+00
 ROW :    6
   -1.2962E+08   0.0000E+00   1.2962E+08  -7.8555E+08   0.0000E+00   7.8555E+08
```

CST Solution

Step 4: System equations: $K_{4x4}D_{4x1} = F_{4x1}$

$$10^8 \begin{bmatrix} 4.5954 & -2.6119 & -2.6316 & 1.2962 \\ & 8.5134 & 1.3158 & -7.8555 \\ & & 4.5954 & 0 \\ sym & & & 8.5134 \end{bmatrix} \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -10000 \end{Bmatrix}$$

Step 5: Solution

```
                NODAL DISPLACEMENTS
                 LC       X DISP                 Y DISP
                           (    )                 (    )
NODE :    1      1      0.0000E+00             0.0000E+00
NODE :    2      1     -2.2359E-05            -1.2056E-04
NODE :    3      1      2.1716E-05            -1.1959E-04
NODE :    4      1      0.0000E+00             0.0000E+00
```

Step 6: Secondary unknowns

CST Solution

ELEMENT STRAINS FOR MATERIAL GROUP: 1

	ELM	LC	SP	EX	EY	EXY
	1	1	1	3.619E-04	0.000E+00	-1.993E-03
	2	1	1	-3.727E-04	3.249E-05	-5.402E-04
MIN				-3.727E-04	0.000E+00	-1.993E-03
ELEMENT				2	1	1
MAX				3.619E-04	3.249E-05	-5.402E-04
ELEMENT				1	2	2

ELEMENT STRESSES (/ ^2) MATERIAL GROUP: 1

	ELM	LC	SP	SX	SY	SXY
	1	1	1	2.843E+07	9.382E+06	-5.245E+07
	2	1	1	-2.843E+07	-7.108E+06	-1.422E+07
MIN				-2.843E+07	-7.108E+06	-5.245E+07
ELEMENT				2	2	1
MAX				2.843E+07	9.382E+06	-1.422E+07
ELEMENT				1	1	2

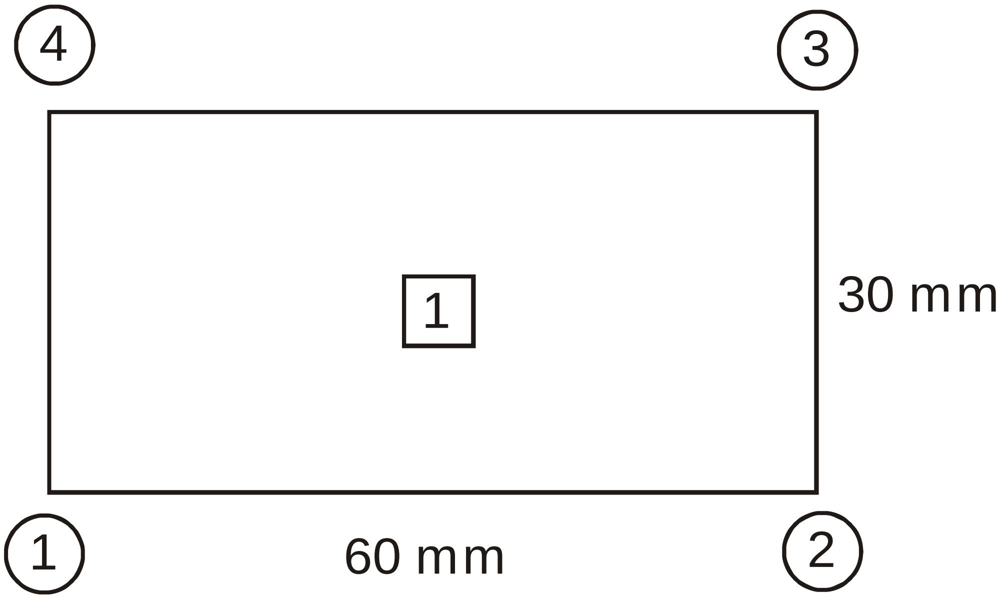
CST Solution

```
                  STRESS EQUIVALENTS. MATERIAL GROUP:  1
 ELM  LC  SP      S11         S22         S33        TRESCA     VONMISES
  1   1   1   7.222E+07  -4.398E+00  -3.440E+07   1.066E+08   9.425E+07
  2   1   1   1.843E+00  -7.103E-01  -3.554E+07   3.554E+07   3.554E+07
MIN           1.843E+00  -4.398E+00  -3.554E+07   3.554E+07   3.554E+07
ELEMENT               2           1           2           2           2
MAX           7.222E+07  -7.103E-01  -3.440E+07   1.066E+08   9.425E+07
ELEMENT               1           2           1           1           1
```

```
                         NODAL REACTIONS
                       LC      X REAC               Y REAC
                               (   )                (   )
      NODE :   1       1     2.0000E+04          -6.8236E+02
      NODE :   4       1    -2.0000E+04           1.0682E+04
  MIN                       -2.0000E+04          -6.8236E+02
  NODE                                4                    1
  MAX                        2.0000E+04           1.0682E+04
  NODE                                1                    4
```

Problem T3L2-2: Q4 Solution

Units: m, N



Element 1: 1-2-3-4

Plane stress problem

Q4 Solution

Step 2: Element equations

```
ELEMENT STIFFNESS MATRIX. ELEMENT :    1
ROW :   1
   3.0636E+08   1.3060E+08  -4.3205E+07  -9.8193E+05  -1.5318E+08  -1.3060E+08
  -1.0998E+08   9.8193E+05
ROW :   2
   1.3060E+08   5.6756E+08   9.8193E+05   2.1799E+08  -1.3060E+08  -2.8378E+08
  -9.8193E+05  -5.0177E+08
ROW :   3
  -4.3205E+07   9.8193E+05   3.0636E+08  -1.3060E+08  -1.0998E+08  -9.8193E+05
  -1.5318E+08   1.3060E+08
ROW :   4
  -9.8193E+05   2.1799E+08  -1.3060E+08   5.6756E+08   9.8193E+05  -5.0177E+08
   1.3060E+08  -2.8378E+08
ROW :   5
  -1.5318E+08  -1.3060E+08  -1.0998E+08   9.8193E+05   3.0636E+08   1.3060E+08
  -4.3205E+07  -9.8193E+05
ROW :   6
  -1.3060E+08  -2.8378E+08  -9.8193E+05  -5.0177E+08   1.3060E+08   5.6756E+08
   9.8193E+05   2.1799E+08
ROW :   7
  -1.0998E+08  -9.8193E+05  -1.5318E+08   1.3060E+08  -4.3205E+07   9.8193E+05
   3.0636E+08  -1.3060E+08
ROW :   8
   9.8193E+05  -5.0177E+08   1.3060E+08  -2.8378E+08  -9.8193E+05   2.1799E+08
  -1.3060E+08   5.6756E+08
```

Q4 Solution

Step 4: System equations: $K_{4x4}D_{4x1} = F_{4x1}$

$$10^8\begin{bmatrix} 3.0636 & -1.3060 & -1.0998 & -0.009819 \\ & 5.6756 & 0.009819 & -5.0177 \\ & & 3.0636 & 1.3060 \\ sym & & & 5.6756 \end{bmatrix}\begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -10000 \end{Bmatrix}$$

Step 5: Solution

```
                NODAL DISPLACEMENTS
                 LC      X DISP              Y DISP
                         (   )               (   )
NODE :    1      1    0.0000E+00          0.0000E+00
NODE :    2      1   -6.1928E-05         -2.0148E-04
NODE :    3      1    6.8636E-05         -2.1165E-04
NODE :    4      1    0.0000E+00          0.0000E+00
```

Step 6: Secondary unknowns

Q4 Solution

```
          ELEMENT STRAINS FOR MATERIAL GROUP:  1
       ELM  LC  SP      EX          EY          EXY
        1   1   1   5.591E-05  -1.694E-04  -1.267E-03
MIN                 5.591E-05  -1.694E-04  -1.267E-03
ELEMENT                  1           1           1
MAX                 5.591E-05  -1.694E-04  -1.267E-03
ELEMENT                  1           1           1
```

```
        ELEMENT STRESSES (   /   ^2) MATERIAL GROUP:  1
        ELM  LC  SP      SX           SY           SXY
         1   1   1  -1.699E+00  -1.186E+07  -3.333E+07
MIN                 -1.699E+00  -1.186E+07  -3.333E+07
ELEMENT                   1            1            1
MAX                 -1.699E+00  -1.186E+07  -3.333E+07
ELEMENT                   1            1            1
```

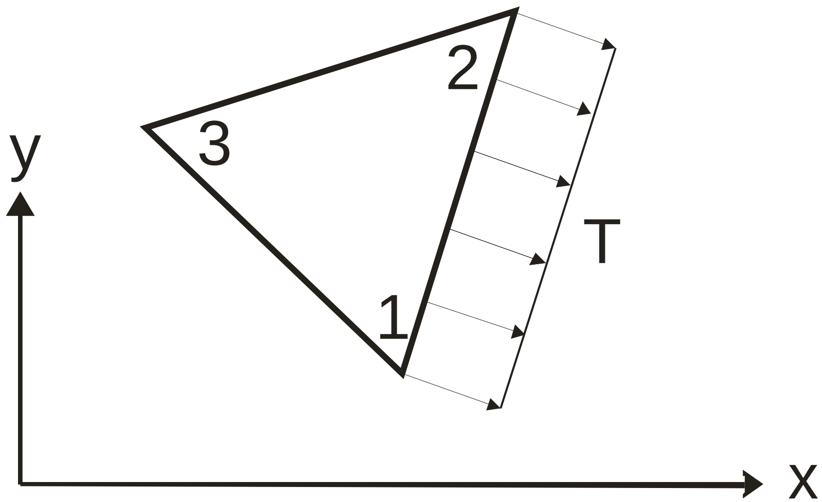
Q4 Solution

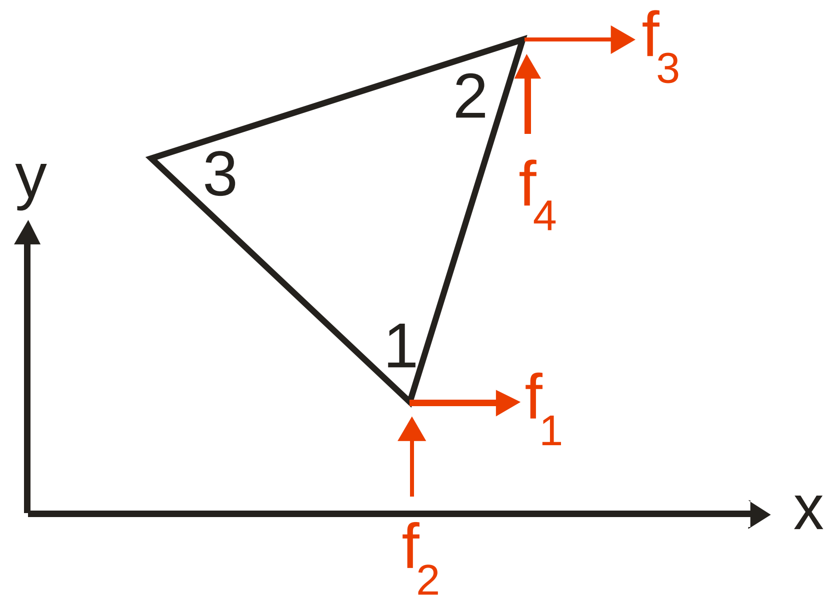
```
                 STRESS EQUIVALENTS. MATERIAL GROUP:  1
 ELM  LC  SP      S11         S22         S33         TRESCA      VONMISES
  1   1   1   2.793E+07  -1.397E+00  -3.979E+07   6.771E+07   5.894E+07
MIN           2.793E+07  -1.397E+00  -3.979E+07   6.771E+07   5.894E+07
ELEMENT               1           1           1           1           1
MAX           2.793E+07  -1.397E+00  -3.979E+07   6.771E+07   5.894E+07
ELEMENT               1           1           1           1           1
```

```
                          NODAL REACTIONS
                        LC      X REAC                Y REAC
                                (   )                 (   )
        NODE :   1       1     2.0000E+04           7.1154E+03
        NODE :   4       1    -2.0000E+04           2.8846E+03
    MIN                       -2.0000E+04           2.8846E+03
    NODE                               4                    4
    MAX                        2.0000E+04           7.1154E+03
    NODE                               1                    1
```

Surface Traction

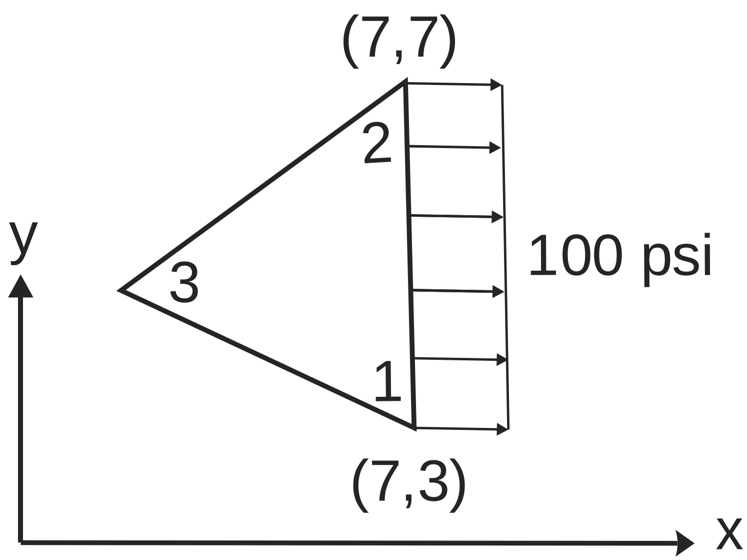
t=0.1 in





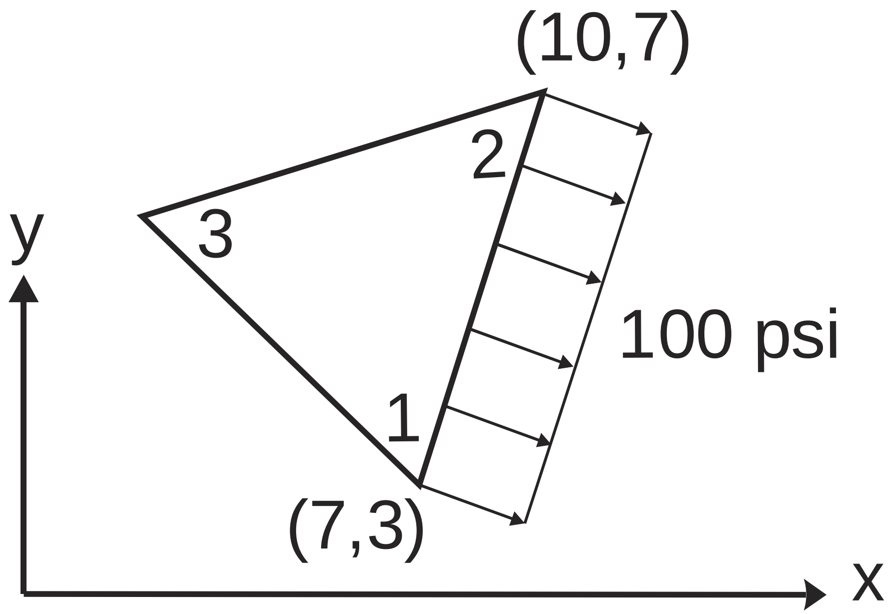
$$\mathbf{f}_{6\times 1} = \frac{tT}{2}\left[\left(y_2 - y_1\right) \quad \left(x_1 - x_2\right) \quad \left(y_2 - y_1\right) \quad \left(x_1 - x_2\right) \quad 0 \quad 0 \right]^T$$

Example 1



$$\mathbf{f}_{6\times1} = \left\{ \begin{array}{|c|} \hline 20 \\ \hline 0 \\ 20 \\ 0 \\ 0 \\ 0 \end{array} \right\} lb$$

Example 2



$$\mathbf{f}_{6\times1} = \left\{ \begin{array}{|c|} \hline 20 \\ \hline -15 \\ 20 \\ -15 \\ 0 \\ 0 \end{array} \right\} lb$$