Finite Elements For Engineers

Lecture 5: The Element Concept

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Overview

- The Classical Approach is powerful
- However it cannot be used to solve more general problems
  - Trial function applies to the entire problem domain
  - Boundary conditions are cumbersome to enforce (we need an automated procedure)

Example (Galerkin's Method)

DE $$\frac{d}{dx}\left(x\frac{dy(x)}{dx}\right)=\frac{2}{x^2} \qquad 1 \le x \le 2$$

BCs $$y(x=1)=2 \qquad \left(-x\frac{dy}{dx}\right)_{x=2}=\frac{1}{2}$$

Trial Solution

$$\tilde{y}(x;a)=\phi_0(x)+\sum_{i=1}^{n}a_i\phi_i(x)$$

Moving Towards the Element Concept!

Galerkin Step 1

$$\int_{x_a}^{x_b} \left[\frac{d}{dx}\left(x \frac{d\,\tilde{y}(x)}{dx} \right) - \frac{2}{x^2} \right] \phi_i(x)\,dx = 0 \qquad i = 1,\ldots,n$$

Galerkin Step 2

$$\int_{x_a}^{x_b} x \frac{d\,\tilde{y}}{dx}\frac{d\phi_i}{dx}\,dx = -\int_{x_a}^{x_b} \frac{2}{x^2}\,\phi_i\,dx - \left[\left(-x \frac{d\,\tilde{y}}{dx} \right) \phi_i \right]_{x_a}^{x_b} \qquad i = 1,\ldots,n$$

Example

Galerkin Step 3

$$\sum_{j=1}^{n}\left(\int_{x_a}^{x_b}\frac{d\phi_i}{dx}x\frac{d\phi_j}{dx}dx\right)a_j=-\int_{x_a}^{x_b}\frac{2}{x^2}\phi_i\,dx-\left[\left(-x\frac{d\tilde{y}}{dx}\right)\phi_i\right]_{x_a}^{x_b}-\int_{x_a}^{x_b}\frac{d\phi_i}{dx}x\frac{d\phi_0}{dx}dx$$

$$i=1,..,n$$

Let

$$K_{ij}=\int_{x_a}^{x_b}\frac{d\phi_i}{dx}\quad x\quad\frac{d\phi_j}{dx}\quad dx\qquad\qquad i,j=1,..,n$$

$$F_i=-\int_{x_a}^{x_b}\frac{2}{x^2}\phi_i dx-\left[\left(-x\frac{d\tilde{y}}{dx}\right)\phi_i\right]-\int_{x_a}^{x_b}\frac{d\phi_i}{dx}x\frac{d\phi_0}{dx}dx$$

Example

Galerkin Step 3 (Cont'd)

$$\begin{bmatrix} K_{11} & K_{12} & . & . & K_{1n} \\ K_{21} & K_{22} & . & . & K_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ K_{n1} & K_{n2} & . & . & K_{nn} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ . \\ . \\ F_n \end{Bmatrix}$$

Element Equations

$$\mathbf{K}_{n\times n}\mathbf{a}_{n\times 1} = \mathbf{F}_{n\times 1}$$

Example

Galerkin Step 4

Quadratic Trial Soln

$$\tilde{y} = \sum_{j=1}^{n} a_j \phi_j(x) = a_1 + a_2 x + a_3 x^2 \implies$$

$$\phi_1(x) = 1$$

$$\phi_2(x) = x$$

$$\phi_3(x) = x^2$$

$$\tilde{\tau} = -x \frac{d\,\tilde{y}}{dx} = -a_2 x - 2a_3 x^2$$

Example Terms

$$K_{23} = \int_{x_a}^{x_b} (1)(x)(2x)dx = \frac{2}{3}\left(x_b^3 - x_a^3\right)$$

$$F_2^{\text{int}} = -\int_{x_a}^{x_b} \frac{2}{x^2} x \quad dx = -2\ln\frac{x_b}{x_a}$$

$$F_2^{bnd} = \left(-x\frac{d\,\tilde{y}}{dx}\right)_{x_a} x_a - \left(-x\frac{d\,\tilde{y}}{dx}\right)_{x_b} x_b$$

Example

Galerkin Step 5

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(x_b^2 - x_a^2\right) & \frac{2}{3}\left(x_b^3 - x_a^3\right) \\ 0 & \frac{2}{3}\left(x_b^3 - x_a^3\right) & \left(x_b^4 - x_a^4\right) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 2\left(\frac{1}{x_b} - \frac{1}{x_a}\right) \\ -2\ln\frac{x_b}{x_a} \\ -2(x_b - x_a) \end{Bmatrix} + \begin{Bmatrix} \tilde{\tau}\,|_{x_a} - \tilde{\tau}\,|_{x_b} \\ \tilde{\tau}\,|_{x_a}\, x_a - \tilde{\tau}\,|_{x_b}\, x_b \\ \tilde{\tau}\,|_{x_a}\, x_a^2 - \tilde{\tau}\,|_{x_b}\, x_b^2 \end{Bmatrix}$$

Example

Galerkin Step 6

$$x_a = 1 \qquad x_b = 2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{14}{3} \\ 0 & \frac{14}{3} & 15 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} -1 \\ -2\ln 2 \\ -2 \end{Bmatrix} + \begin{Bmatrix} \tilde{\tau}\,|_{x=1} - \tilde{\tau}\,|_{x=2} \\ \tilde{\tau}\,|_{x=1} - \tilde{\tau}\,|_{x=2}\,(2) \\ \tilde{\tau}\,|_{x=1} - \tilde{\tau}\,|_{x=2}\,(4) \end{Bmatrix}$$

Example

Galerkin Step 7

NBC $\left(-x\dfrac{dy}{dx}\right)_{x=2} = \dfrac{1}{2}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{3}{2} & \dfrac{14}{3} \\ 0 & \dfrac{14}{3} & 15 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} -1 \\ -2\ln 2 \\ -2 \end{Bmatrix} + \begin{Bmatrix} \tilde{\tau}\,|_{x=1} - \dfrac{1}{2} \\ \tilde{\tau}\,|_{x=1} - 1 \\ \tilde{\tau}\,|_{x=1} - 2 \end{Bmatrix}$$

Example

Galerkin Step 7 (cont'd)

EBC $y(x=1)=2$

$$a_1 + a_2 + a_3 = 2 \quad \Rightarrow \quad a_3 = 2 - a_1 - a_2$$

$$\begin{bmatrix} 0 & 0 \\ -\frac{14}{3} & \frac{3}{2}-\frac{14}{3} \\ -15 & \frac{14}{3}-15 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \tilde{\tau}\,|_{x=1} - \frac{3}{2} \\ \tilde{\tau}\,|_{x=1} - 2\ln 2 - \frac{31}{3} \\ \tilde{\tau}\,|_{x=1} - 34 \end{Bmatrix}$$

Example

Galerkin Step 7 (cont'd)

Eqn. (1)-Eqn. (3)

Eqn. (2)-Eqn. (3)

$$\begin{bmatrix} 15 & \frac{31}{3} \\ \frac{31}{3} & \frac{43}{6} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \frac{65}{2} \\ \frac{71}{3} - 2\ln 2 \end{Bmatrix}$$

Galerkin Step 8

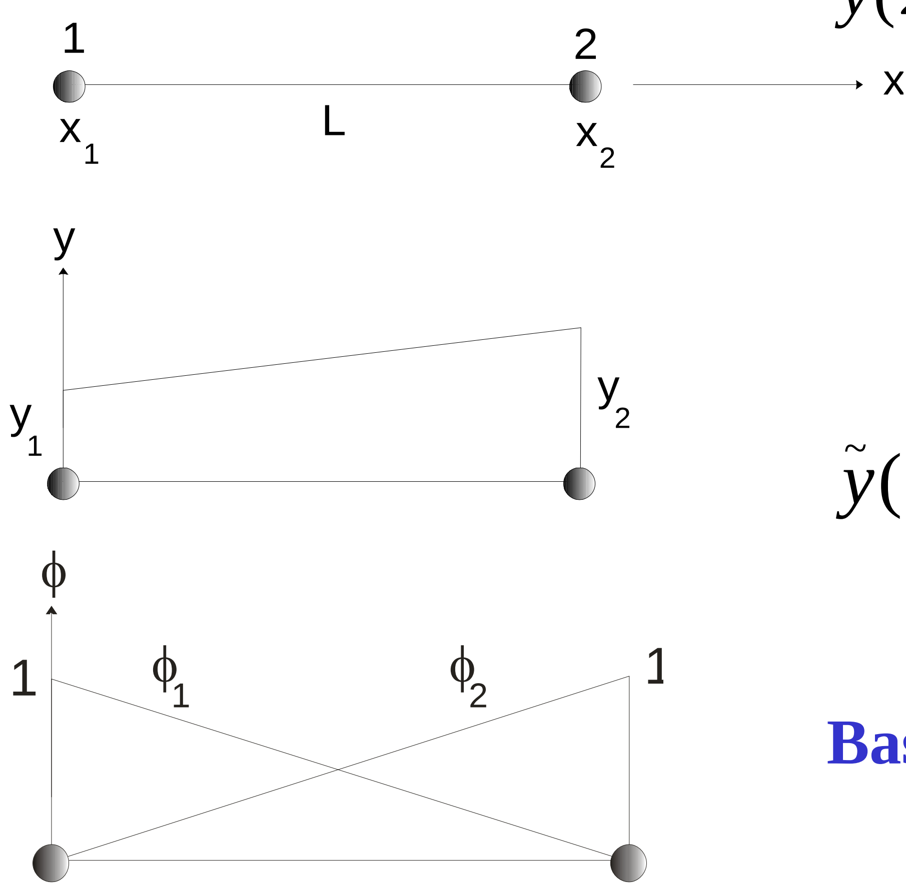
$$a_1 = 3.719 \qquad a_2 = -2.254 \quad \Rightarrow \quad a_3 = 0.535$$

Solution

$$\tilde{y} = 3.719 - 2.254x + 0.535x^2 \qquad \tilde{\tau} = 2.254x - 1.070x^2$$

The Element Approach: Improvement!

Step 4



$$\tilde{y}(x) = a_1 + a_2 x = \phi_1(x) y_1 + \phi_2(x) y_2$$

End Conditions

$$y(x = x_1) = y_1$$

$$y(x = x_2) = y_2$$

$$\tilde{y}(x) = \frac{x_2 - x}{L} y_1 + \frac{x - x_1}{L} y_2$$

Basic Idea: Interpolation

Element Equations

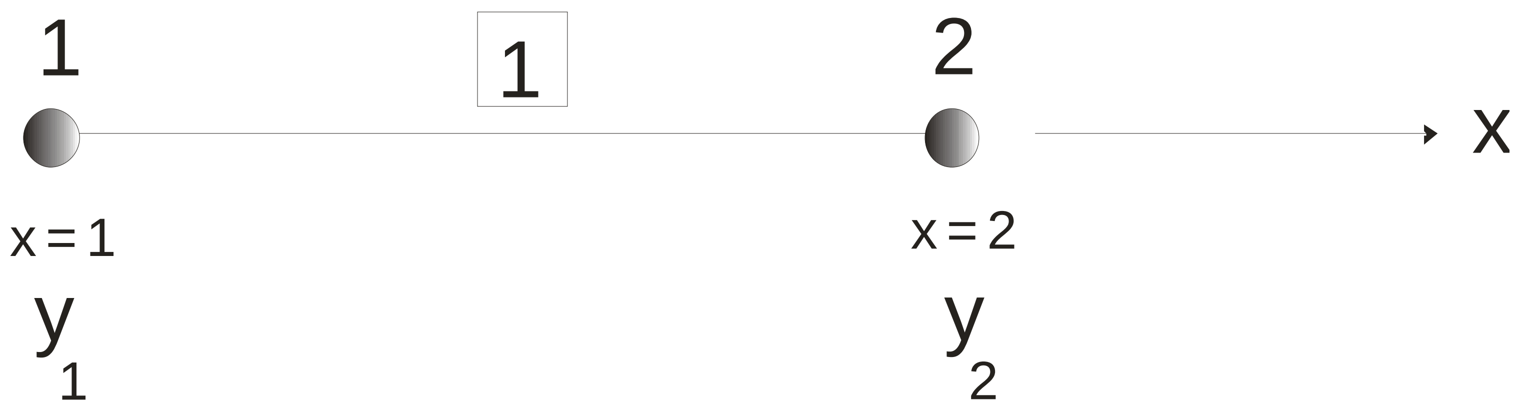
Step 5

$$\frac{1}{2L}\begin{bmatrix} (x_1+x_2) & -(x_1+x_2) \\ -(x_1+x_2) & (x_1+x_2) \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{2}{x_1}+\dfrac{2}{L}\ln\dfrac{x_2}{x_1} \\ \dfrac{2}{x_2}-\dfrac{2}{L}\ln\dfrac{x_2}{x_1} \end{Bmatrix} + \begin{Bmatrix} \tilde{\tau}\,|_{x=1} \\ -\tilde{\tau}\,|_{x=2} \end{Bmatrix}$$

$$\tilde{\tau} = -x\frac{d\,\tilde{y}}{dx} = \frac{x}{x_2-x_1}(y_1-y_2)$$

1-Element Solution

FE Model



1-Element Solution

Step 6 $x_1 = 1 \qquad x_2 = 2$

$$\frac{1}{2}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} -2+2\ln 2 \\ 1-2\ln 2 \end{Bmatrix} + \begin{Bmatrix} \tilde{\tau}\,|_{x=1} \\ -\tilde{\tau}\,|_{x=2} \end{Bmatrix}$$

Step 7 **NBC** $\tau_{x=2} = \dfrac{1}{2}$

$$\frac{1}{2}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} -2+2\ln 2 \\ 1-2\ln 2 \end{Bmatrix} + \begin{Bmatrix} \tilde{\tau}\,|_{x=1} \\ -\dfrac{1}{2} \end{Bmatrix}$$

1-Element Solution

Step 7 (cont'd) **EBC** $y(x=1) = y_1 = 2$

Elimination Technique

$$\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{3}{2} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ \dfrac{7}{2} - 2\ln 2 \end{Bmatrix}$$

Step 8 $y_1 = 2$ $y_2 = 1.409$

Element Solution

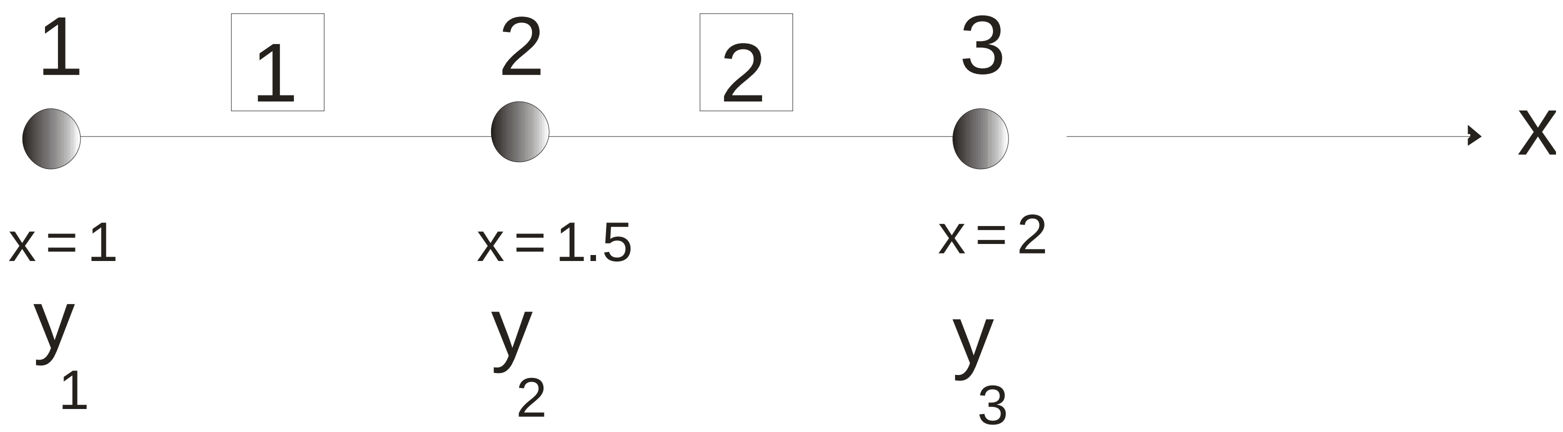
$$\tilde{y} = 2.591 - 0.591x$$

$$\tilde{\tau} = 0.591x$$

Solution is not good!

2-Element Solution

FE Model



2-Element Solution

Step 6

Element 1 $x_1 = 1$ $x_2 = 1.5$

$$\frac{1}{2(0.5)}\begin{bmatrix} (1+1.5) & -(1+1.5) \\ -(1+1.5) & (1+1.5) \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{2}{1}+\frac{2}{0.5}\ln\frac{1.5}{1} \\ \frac{2}{1.5}-\frac{2}{0.5}\ln\frac{1.5}{1} \end{Bmatrix} + \begin{Bmatrix} \left(\tilde{\tau}\,|_{x=1}\right)_1 \\ \left(-\tilde{\tau}\,|_{x=1.5}\right)_1 \end{Bmatrix}$$

Element 2 $x_1 = 1.5$ $x_2 = 2$

$$\frac{1}{2(0.5)}\begin{bmatrix} (1.5+2) & -(1.5+2) \\ -(1.5+2) & (1.5+2) \end{bmatrix}\begin{Bmatrix} y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{2}{1.5}+\frac{2}{0.5}\ln\frac{2}{1.5} \\ \frac{2}{2}-\frac{2}{0.5}\ln\frac{2}{1.5} \end{Bmatrix} + \begin{Bmatrix} \left(\tilde{\tau}\,|_{x=1.5}\right)_2 \\ \left(-\tilde{\tau}\,|_{x=2}\right)_2 \end{Bmatrix}$$

2-Element Solution

After Assembly

$$\begin{bmatrix} 2.5 & -2.5 & 0 \\ -2.5 & 6.0 & -3.5 \\ 0 & -3.5 & 3.5 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} -2 + 4\ln\dfrac{3}{2} \\ 4\ln\dfrac{8}{9} \\ 1 - 4\ln\dfrac{4}{3} \end{Bmatrix} + \begin{Bmatrix} \left(\tilde{\tau}\,\big|_{x=1}\right)_1 \\ 0 \\ \left(-\tilde{\tau}\,\big|_{x=2}\right)_2 \end{Bmatrix}$$

Note inter-element flux continuity

$$\left(\tilde{\tau}\,\big|_{x=1.5}\right)_1 = \left(\tilde{\tau}\,\big|_{x=1.5}\right)_2$$

2-Element Solution

Step 7 **NBC** $\tau_{x=2} = \dfrac{1}{2}$

$$\begin{bmatrix} 2.5 & -2.5 & 0 \\ -2.5 & 6.0 & -3.5 \\ 0 & -3.5 & 3.5 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} -2 + 4\ln\dfrac{3}{2} \\ 4\ln\dfrac{8}{9} \\ 1 - 4\ln\dfrac{4}{3} \end{Bmatrix} + \begin{Bmatrix} \tilde{\tau}\,\big|_{x=1} \\ 0 \\ -\dfrac{1}{2} \end{Bmatrix}$$

2-Element Solution

Step 7 (cont'd) **EBC** $y(x=1)=y_1=2$

Elimination Technique

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6.0 & -3.5 \\ 0 & -3.5 & 6.0 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4\ln\dfrac{8}{9}+5 \\ \dfrac{1}{2}-4\ln\dfrac{4}{3} \end{Bmatrix}$$

Step 8

$$y_1 = 2 \qquad y_2 = 1.551 \qquad y_3 = 1.365$$

2-Element Solution

Element 1

$$\left(\tilde{y}(x)\right)_1 = 2\left(\frac{1.5-x}{0.5}\right) + 1.551\left(\frac{x-1}{0.5}\right) = -0.898x + 2.898$$

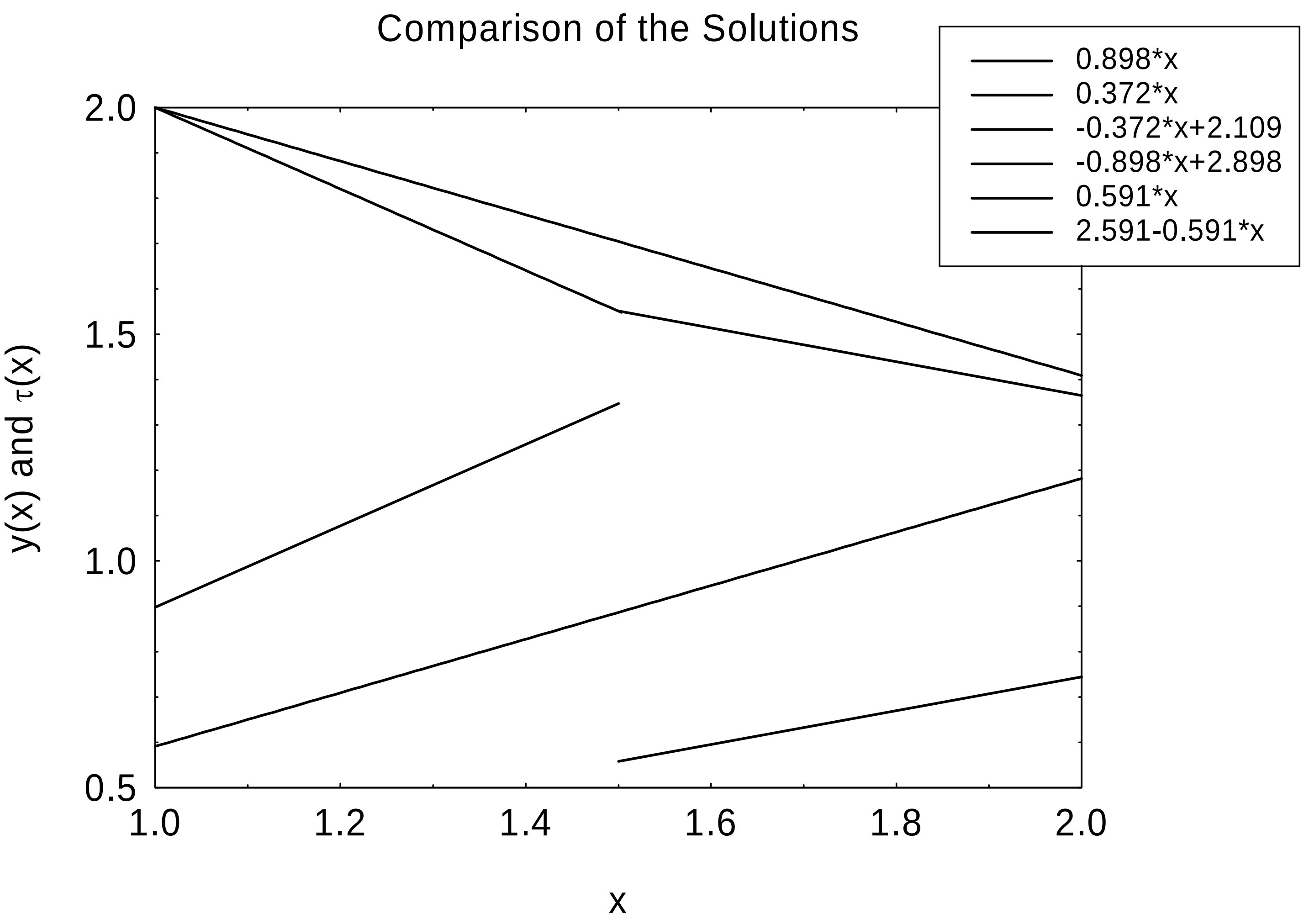
$$\left(\tilde{\tau}(x)\right)_1 = 0.898x$$

Element 2

$$\left(\tilde{y}(x)\right)_2 = 1.551\left(\frac{2.0-x}{0.5}\right) + 1.365\left(\frac{x-1.5}{0.5}\right) = -0.372x + 2.109$$

$$\left(\tilde{\tau}(x)\right)_2 = 0.372x$$

Comparison



Comparison of the Solutions
0.898*x
0.372*x
-0.372*x+2.109
-0.898*x+2.898
0.591*x
2.591-0.591*x
2.0
1.5
1.0
0.5
1.0
1.2
1.4
1.6
1.8
2.0
y(x) and τ(x)
x

Summary (Galerkin's Method)

- Step 1: Assume the trial solution in its general form
- Step 2: Integrate by parts the highest derivative term
- Step 3: Rewrite the equations putting stiffness terms on the left and force terms on the right

Summary (Galerkin's Method)

- Step 4: Assume the exact form of the trial solution. This will enable the generation of the element equations.

$$\mathbf{k}_{n\times n}\mathbf{u}_{n\times 1} = \mathbf{f}_{n\times 1}$$

k is symmetric and singular.

Summary

- Central to the element concept is the idea of **interpolation** using nodal values
- The order of the interpolation determines how many nodal conditions are needed
- Converging solution is usually obtained by using more nodes and elements in the model