Finite Elements for Engineers

Lecture 1: 3D Stress Analysis

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Introduction

Displacement Field

$$u = u(x, y, z)$$

$$v = v(x, y, z)$$

$$w = w(x, y, z)$$

Stress and Strain Tensors

$$\boldsymbol{\varepsilon}_{6\times1} = \left[\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}\right]^T$$

$$\boldsymbol{\sigma}_{6\times1} = \left[\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\right]^T$$

Stress-Strain Relationship

$$\boldsymbol{\sigma}_{6\times1} = \mathbf{D}_{6\times6}\boldsymbol{\varepsilon}_{6\times1}$$

$$\mathbf{D}_{6\times6} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & 0.5-\nu & 0 & 0 \\ & SYM & & & 0.5-\nu & 0 \\ & & & & & 0.5-\nu \end{bmatrix}$$

Strain-Displacement Relations

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

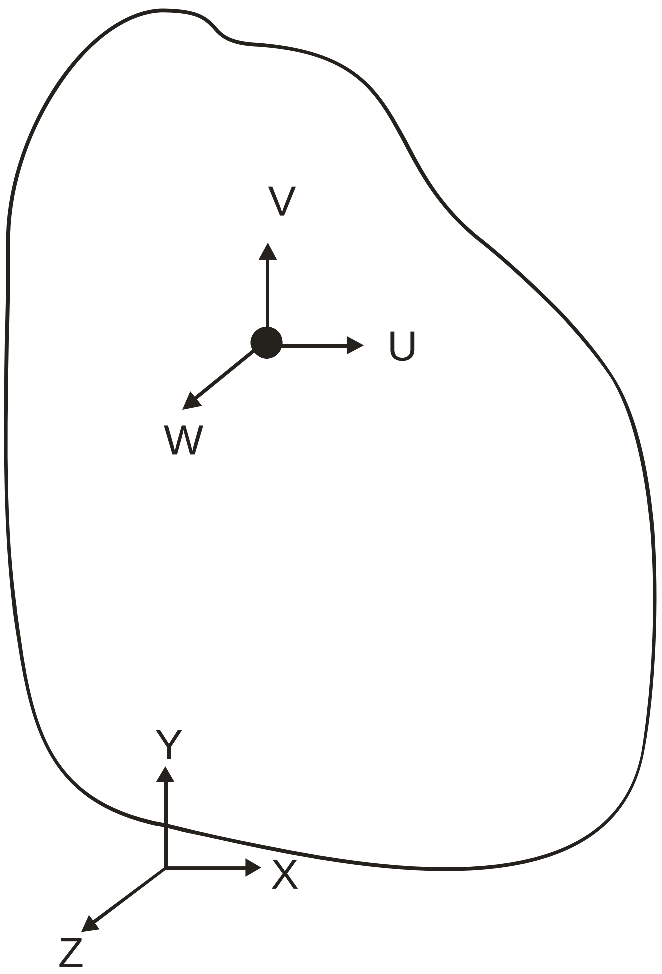
$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Assumed Displacement Field

$$u = \sum_{i=1}^{n} \phi_i \, u_i$$

$$v = \sum_{i=1}^{n} \phi_i \, v_i$$

$$w = \sum_{i=1}^{n} \phi_i \, w_i$$



Chain Rule Differentiation

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \eta}$$

$$\frac{\partial u}{\partial \zeta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \zeta}$$

Jacobian

$$\begin{Bmatrix} u,_{\xi} \\ u,_{\eta} \\ u,_{\zeta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} u,_{x} \\ u,_{y} \\ u,_{z} \end{Bmatrix} = \mathbf{J}_{3\times 3} \begin{Bmatrix} u,_{x} \\ u,_{y} \\ u,_{z} \end{Bmatrix}$$

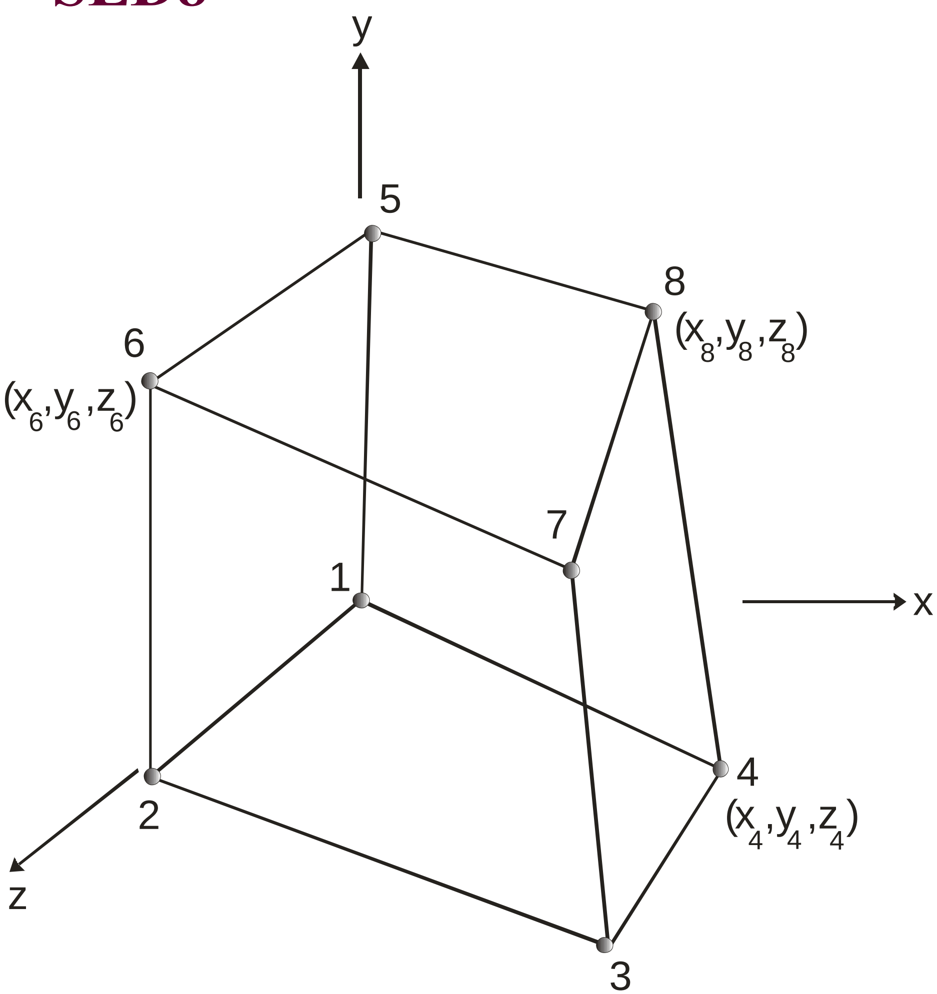
Jacobian

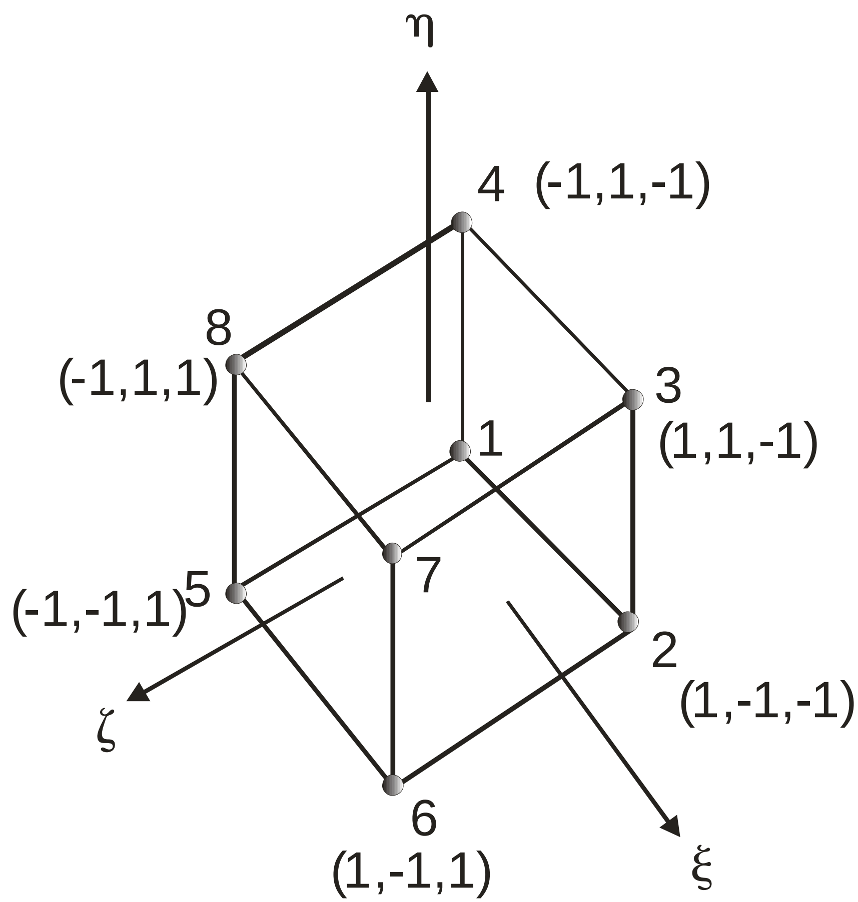
$$\begin{Bmatrix} u,_x \\ u,_y \\ u,_z \end{Bmatrix} = \mathbf{\Gamma}_{3\times 3} \begin{Bmatrix} u,_\xi \\ u,_\eta \\ u,_\zeta \end{Bmatrix}$$

$$\mathbf{\Gamma} = \mathbf{J}^{-1}$$

First-Order Hexahedral Element

SLD8





First-Order Hexahedral Element

Assumed Displacement Field

$$u = a_1 + a_2\xi + a_3\eta + a_4\zeta + a_5\xi\eta + a_6\eta\zeta + a_7\xi\zeta + a_8\xi\eta\zeta$$

$$v = b_1 + b_2\xi + b_3\eta + b_4\zeta + b_5\xi\eta + b_6\eta\zeta + b_7\xi\zeta + b_8\xi\eta\zeta$$

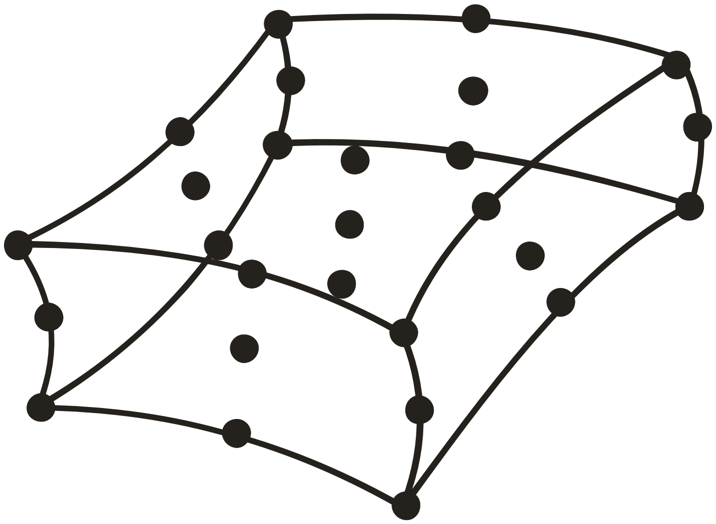
$$w = c_1 + c_2\xi + c_3\eta + c_4\zeta + c_5\xi\eta + c_6\eta\zeta + c_7\xi\zeta + c_8\xi\eta\zeta$$

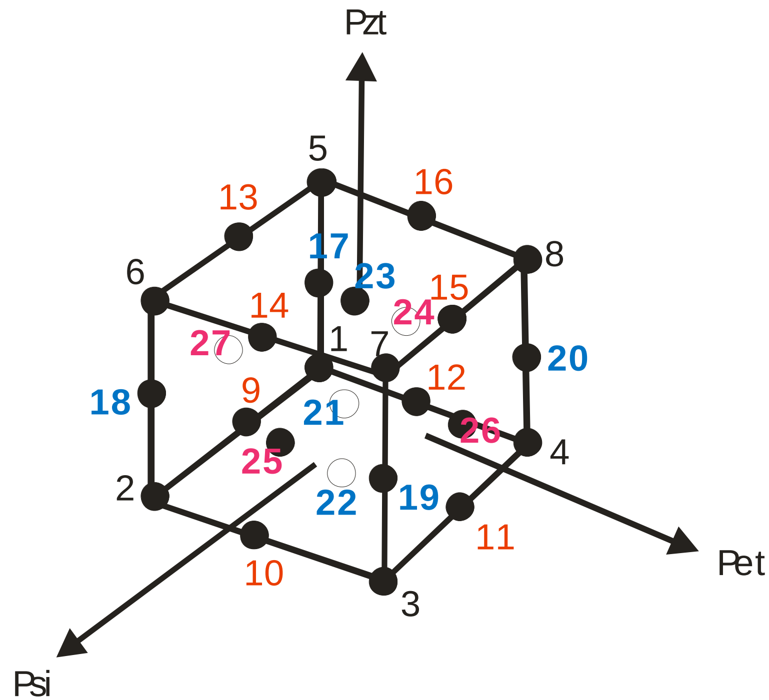
Shape Functions

$$\phi_i = \frac{1}{8}\left(1-\xi\xi_i\right)\left(1-\eta\eta_i\right)\left(1-\zeta\zeta_i\right)$$

Second-Order Lagrange Element

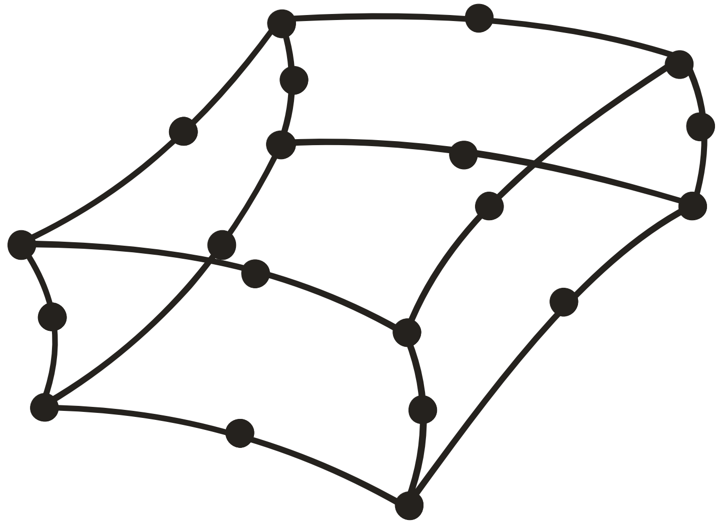
SLD27

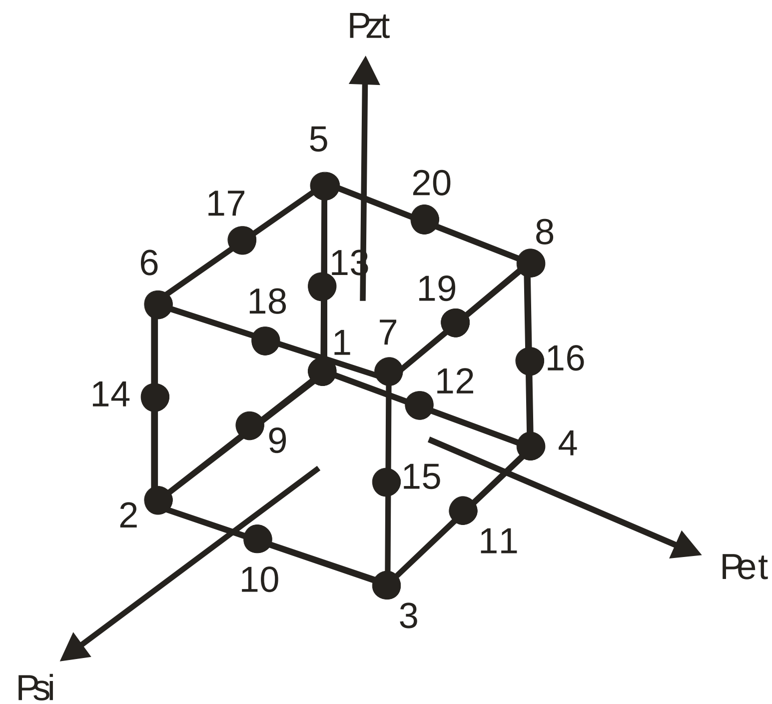




Second-Order Serendipity Element

SLD20





Numerical Integration

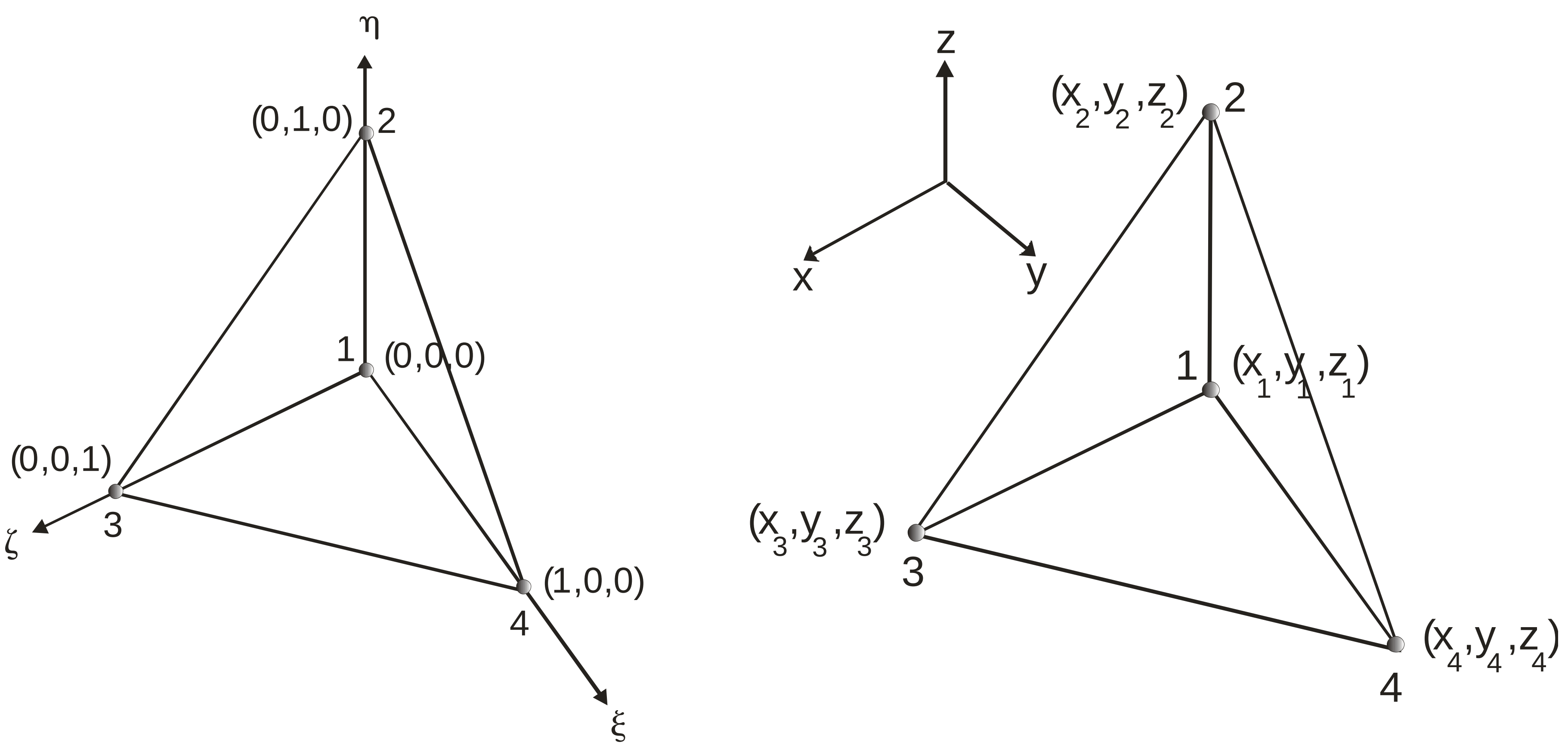
$$I = \int_e^f \int_c^d \int_a^b F\left(x, y, z\right) dx dy dz$$

$$I = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} F\left(x(\xi,\eta,\zeta), y(\xi,\eta,\zeta), z(\xi,\eta,\zeta)\right) \left|J\right| d\xi d\eta d\zeta$$

$$I = \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} w_i w_j w_k f\left(\xi_i, \eta_j, \zeta_k\right)$$

First-Order Tetrahedral Element

TET4



First-Order Tetrahedral Element

Assumed Displacement Field

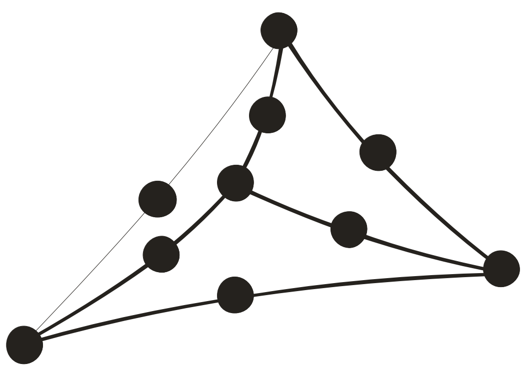
$$u = a_1 + a_2\xi + a_3\eta + a_4\zeta$$

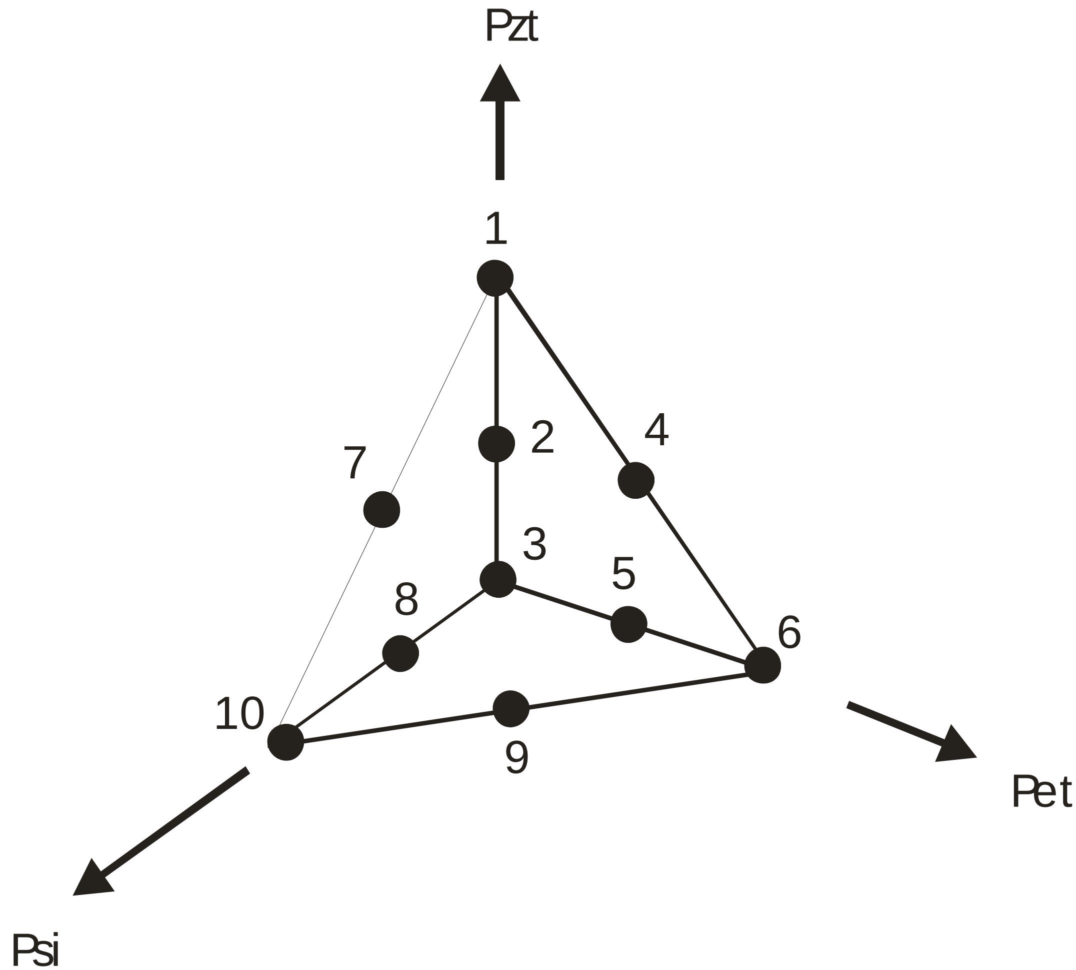
$$v = b_1 + b_2\xi + b_3\eta + b_4\zeta$$

$$w = c_1 + c_2\xi + c_3\eta + c_4\zeta$$

Second-Order Tetrahedral Element

TET10



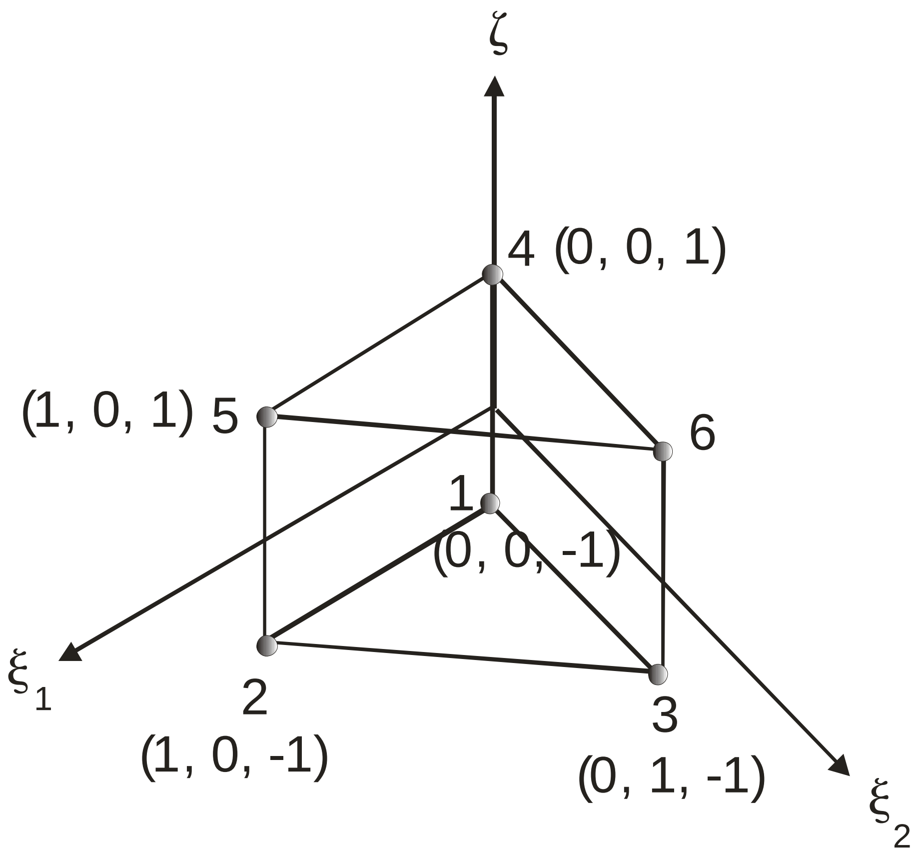


Numerical Integration

Volume Coordinates

$$\int_0^1 \int_0^{1-\zeta} \int_0^{1-\eta} F(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta = \frac{1}{6}\sum_{i=1}^{n} w_i F(\xi_i,\eta_i,\zeta_i)$$

First-Order Wedge Element



WGE6

Shape Functions

$$\phi_1 = \frac{1}{2}\left(1-\xi_1-\xi_2\right)\left(1-\zeta\right)$$

$$\phi_2 = \frac{1}{2}\xi_1\left(1-\zeta\right)$$

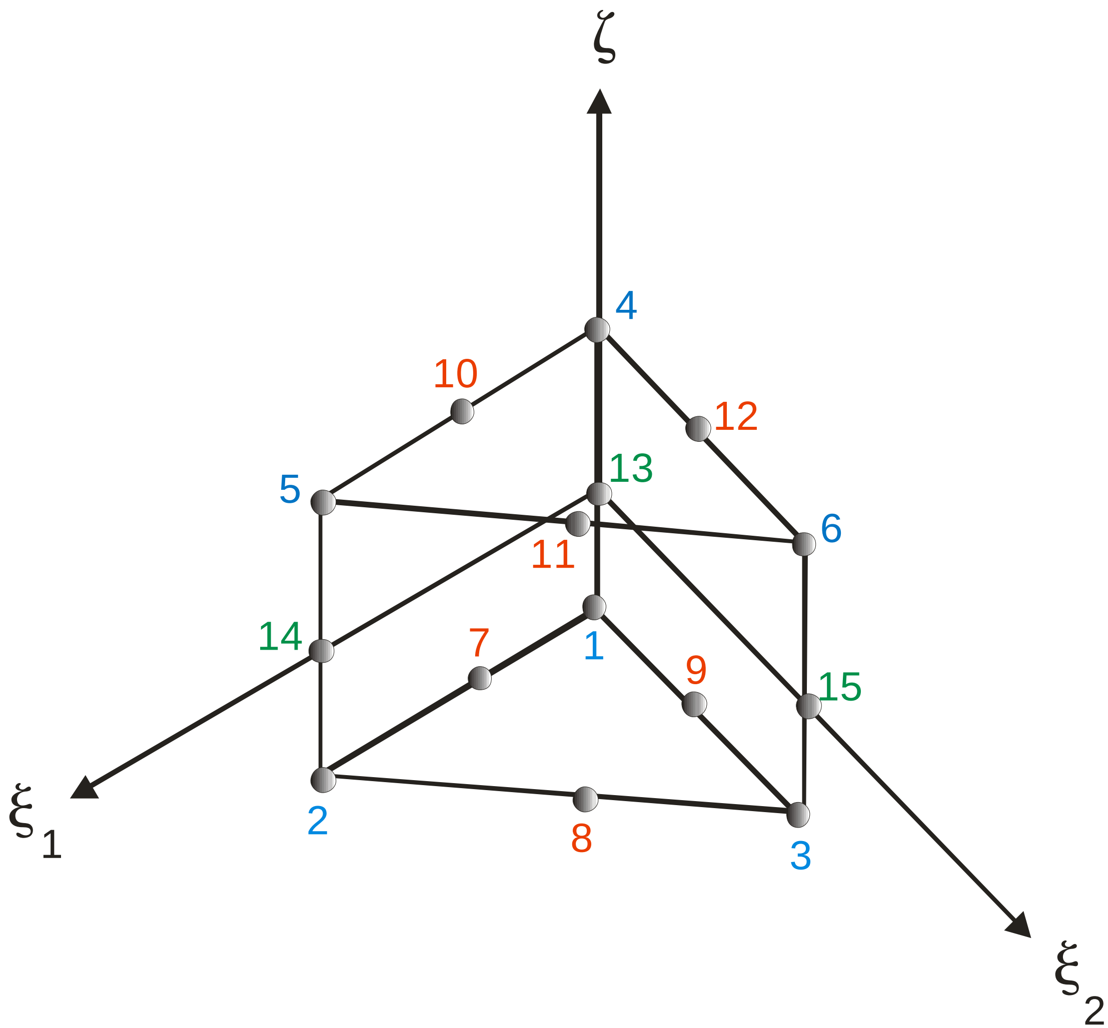
$$\phi_3 = \frac{1}{2}\xi_2\left(1-\zeta\right)$$

$$\phi_4 = \frac{1}{2}\left(1-\xi_1-\xi_2\right)\left(1+\zeta\right)$$

$$\phi_5 = \frac{1}{2}\xi_1\left(1+\zeta\right)$$

$$\phi_6 = \frac{1}{2}\xi_2\left(1+\zeta\right)$$

Second-Order Wedge Element



WGE15

Shape Functions

$$\phi_1 = \frac{1}{2}\xi_3\left[\left(2\xi_3 - 1\right)\left(1-\zeta\right) - \left(1-\zeta^2\right)\right]$$

$$\phi_2 = \frac{1}{2}\xi_1\left[\left(2\xi_1 - 1\right)\left(1-\zeta\right) - \left(1-\zeta^2\right)\right]$$

$$\phi_3 = \frac{1}{2}\xi_2\left[\left(2\xi_2 - 1\right)\left(1-\zeta\right) - \left(1-\zeta^2\right)\right]$$

$$\phi_4 = \frac{1}{2}\xi_3\left[\left(2\xi_3 - 1\right)\left(1+\zeta\right) - \left(1-\zeta^2\right)\right]$$

$$\phi_5 = \frac{1}{2}\xi_1\left[\left(2\xi_1 - 1\right)\left(1+\zeta\right) - \left(1-\zeta^2\right)\right]$$

$$\phi_6 = \frac{1}{2}\xi_2\left[\left(2\xi_2 - 1\right)\left(1+\zeta\right) - \left(1-\zeta^2\right)\right]$$

$$\phi_7 = 2\xi_3\xi_1\left(1-\zeta\right)$$

$$\phi_8 = 2\xi_1\xi_2\left(1-\zeta\right)$$

$$\phi_9 = 2\xi_2\xi_3\left(1-\zeta\right)$$

$$\phi_{10} = 2\xi_3\xi_1\left(1+\zeta\right)$$

$$\phi_{11} = 2\xi_1\xi_2\left(1+\zeta\right)$$

$$\phi_{12} = 2\xi_2\xi_3\left(1+\zeta\right)$$

$$\phi_{13} = 2\xi_3\left(1-\zeta^2\right)$$

$$\phi_{14} = 2\xi_1\left(1-\zeta^2\right)$$

$$\phi_{15} = 2\xi_2\left(1-\zeta^2\right)$$

Element Equations

Assumed displacement field

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} \phi_1 & 0 & 0 & \phi_2 & .. & 0 & 0 \\ 0 & \phi_1 & 0 & 0 & .. & \phi_n & 0 \\ 0 & 0 & \phi_1 & 0 & .. & 0 & \phi_n \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ .. \\ d_{3n-2} \\ d_{3n-1} \\ d_{3n} \end{Bmatrix}$$

$$\mathbf{u}_{3\times 1} = \mathbf{\Phi}_{3\times 3n} \mathbf{d}_{3n\times 1}$$

Element Equations

$$\boldsymbol{\varepsilon}_{6\times1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{6\times9} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial z} \end{Bmatrix}_{9\times1}$$

$$\boldsymbol{\varepsilon}_{6\times1} = \mathbf{L}_{6\times9}\mathbf{a}_{9\times1}$$

Element Equations

$$\mathbf{a}_{9\times1} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{bmatrix}_{9\times9} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \\ \frac{\partial v}{\partial \zeta} \\ \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \\ \frac{\partial w}{\partial \zeta} \end{Bmatrix}_{9\times1}$$

$$\mathbf{a}_{6\times1} = \mathbf{M}_{6\times9}\mathbf{b}_{9\times1}$$

Element Equations

$$\mathbf{b}_{9\times1} = \begin{bmatrix} \phi_{1,\xi} & 0 & 0 & \phi_{2,\xi} & \ldots & 0 & 0 \\ \phi_{1,\eta} & 0 & 0 & \phi_{2,\eta} & .. & 0 & 0 \\ \phi_{1,\zeta} & 0 & 0 & \phi_{2,\zeta} & .. & 0 & 0 \\ 0 & \phi_{1,\xi} & 0 & 0 & .. & \phi_{n,\xi} & 0 \\ 0 & \phi_{1,\eta} & 0 & 0 & .. & \phi_{n,\eta} & 0 \\ 0 & \phi_{1,\zeta} & 0 & 0 & .. & \phi_{n,\zeta} & 0 \\ 0 & 0 & \phi_{1,\xi} & 0 & .. & 0 & \phi_{n,\xi} \\ 0 & 0 & \phi_{1,\eta} & 0 & .. & 0 & \phi_{n,\eta} \\ 0 & 0 & \phi_{1,\zeta} & 0 & .. & 0 & \phi_{n,\zeta} \end{bmatrix}_{9\times2n} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ .. \\ v_n \\ w_n \end{Bmatrix}_{3n\times1}$$

$$\mathbf{b}_{9\times1} = \mathbf{N}_{9\times3n}\mathbf{d}_{3n\times1}$$

Element Equations

$$\boldsymbol{\varepsilon}_{6\times1} = \mathbf{L}_{6\times9}\mathbf{M}_{9\times9}\mathbf{N}_{9\times3n}\mathbf{d}_{3n\times1} = \mathbf{O}_{6\times9}\mathbf{N}_{9\times3n}\mathbf{d}_{3n\times1} = \mathbf{B}_{6\times3n}\mathbf{d}_{3n\times1}$$

$$\mathbf{O}_{6\times9} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & 0 & 0 & 0 & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \end{bmatrix}$$

Element Equations

$$\mathbf{k}_{3n\times 3n}\,\mathbf{d}_{3n\times 1} = \mathbf{f}_{3n\times 1}$$

$$\mathbf{k}_{3n\times 3n} = \int_V \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,dV$$

Summary

- Hexahedral elements: Element shape functions can be generated in the same manner as with quadrilateral elements
- Tetrahedral elements: Element shape functions can be generated in the same manner as with triangular elements
- Numerical integration can be carried out using natural coordinates and volume coordinates

Further Reading

- From the textbook
  - Chapter 9