ELECTRONIC NOTES

CEE598 - Finite Elements for Engineers: Module 2

Part 3: Boundary Value Problems

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Finite Elements for Engineers

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Topic 4: Boundary Value Problems

"All problems are finally scientific problems." George Bernard Shaw

Lesson 1: Two-Dimensional BVP

Objectives: In this lesson we will look at two-dimensional boundary value problems.

- To understand what is meant by 2D BVP.
- To derive the element equations for commonly used low and higher-order isoparametric 2D BVP elements.

Problem Formulation

The two-dimensional BVP dealing with the unknown $u = u(x, y)$, is given by

$$\frac{\partial}{\partial x}\left(\alpha_x(x,y)\frac{\partial u(x,y)}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha_y(x,y)\frac{\partial u(x,y)}{\partial y}\right)+\beta(x,y)u(x,y)$$

$$+ f(x,y) = 0 \qquad \text{(T4L1-1)}$$

with the boundary conditions as

$$u(\hat{x},\hat{y}) = \hat{u} \qquad \text{on } \Gamma_1 \qquad \text{(T4L1-2)}$$

$$\alpha_x \frac{\partial u}{\partial x} n_x + \alpha_y \frac{\partial u}{\partial y} n_y + gu + c = 0 \qquad \text{on } \Gamma_2 \qquad \text{(T4L1-3)}$$

where (n_x, n_y) are the direction cosines of the outward normal to the boundary Γ_2, and g and c are constants. We will use the Galerkin's Method to generate the finite element equations necessary to solve this problem.

Let the trial solution be given as

$$\tilde{u}(x,y) = \sum_{j=1}^{n} \phi_j(x,y)\, u_j \qquad \text{(T4L1-4)}$$

We will drop the tilde (~) from this point onwards to denote the approximate solution.

Step 1: Compute the residual equations for a typical element domain Ω as

$$\iint_\Omega R(x,y,u)\phi_i(x,y)\, dxdy = 0 \qquad i = 1,2,\ldots,n \qquad \text{(T4L1-5a)}$$

Or,
$$\iint_\Omega [\frac{\partial}{\partial x}\left(\alpha_x(x,y)\frac{\partial u(x,y)}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha_y(x,y)\frac{\partial u(x,y)}{\partial y}\right)+\beta(x,y)u(x,y)$$

$$+ f(x,y)]\phi_i(x,y)\, dxdy = 0 \qquad i = 1,2,\ldots,n \qquad \text{(T4L1-5b)}$$

Step 2: Integrate by parts the highest-order derivative

To achieve this objective we need to use the chain rule of differentiation and the Divergence Theorem.

(A) Chain rule of differentiation

$$\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\right)\phi_i = \frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\phi_i\right) - \left(\alpha_x \frac{\partial u}{\partial x}\right)\frac{\partial \phi_i}{\partial x} \qquad \text{(T4L1-6a)}$$

$$\frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\right)\phi_i = \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\phi_i\right) - \left(\alpha_y \frac{\partial u}{\partial y}\right)\frac{\partial \phi_i}{\partial y} \qquad \text{(T4L1-6b)}$$

(B) Divergence Theorem

Let $F = F(x, y)$ and $G = G(x, y)$. Then

$$\iint_\Omega \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) dxdy = \oint_\Gamma \left(Fn_x + Gn_y\right)dS \qquad \text{(T4L1-7)}$$

Using Eqns. (T4L1-5b) and (T4L1-6), we have

$$\iint_\Omega \left(\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\phi_i\right) - \left(\alpha_x \frac{\partial u}{\partial x}\right)\frac{\partial \phi_i}{\partial x} + \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\phi_i\right) - \left(\alpha_y \frac{\partial u}{\partial y}\right)\frac{\partial \phi_i}{\partial y}\right) dxdy$$

$$+\iint_\Omega \left(\beta\, u\, \phi_i + f\, \phi_i\right)dxdy = 0 \qquad i = 1,2,\ldots,n \qquad \text{(T4L1-8)}$$

Using Eqn. (T4L1-7), we have

$$-\left(\iint_\Omega \left\{\frac{\partial u}{\partial x}\alpha_x \frac{\partial \phi_i}{\partial x} + \frac{\partial u}{\partial y}\alpha_y \frac{\partial \phi_i}{\partial y}\right\}dxdy - \iint_\Omega \left\{\beta\, u \phi_i\right\}dxdy\right)$$

$$+\oint_\Gamma \left(\alpha_x \frac{\partial u}{\partial x}n_x + \alpha_y \frac{\partial u}{\partial y}n_y\right)\phi_i\, dS + \iint_\Omega f\,\phi_i dxdy = 0 \qquad i = 1,2,\ldots,n \qquad \text{(T4L1-9)}$$

Rearranging and using Eqn. (T4L1-3)

$$\iint_\Omega \left\{\frac{\partial u}{\partial x}\alpha_x \frac{\partial \phi_i}{\partial x} + \frac{\partial u}{\partial y}\alpha_y \frac{\partial \phi_i}{\partial y} - \beta\, u \phi_i\right\}dxdy$$

$$+\oint_\Gamma \left(gu\phi_i\right)ds = \iint_\Omega f\,\phi_i dxdy - \oint_\Gamma \left(c\phi_i dS\right) \qquad i = 1,2,\ldots,n \qquad \text{(T4L1-10)}$$

Step 3: Substitute the trial solution

$$u = \sum_{j=1}^{n} u_j \phi_j \tag{T4L1-11a}$$

Hence, $\dfrac{\partial u}{\partial x} = \sum_{j=1}^{n} u_j \dfrac{\partial \phi_j}{\partial x}$ and $\dfrac{\partial u}{\partial y} = \sum_{j=1}^{n} u_j \dfrac{\partial \phi_j}{\partial y}$ (T4L1-11b)

Substituting these in Eqn. (T4L1-10) we have

$$\sum_{j=1}^{n} \left(\iint_{\Omega} \left\{ \frac{\partial \phi_i}{\partial x} \alpha_x \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \alpha_y \frac{\partial \phi_j}{\partial y} - \phi_i \beta \phi_j \right\} \; dxdy + \oint_{\Gamma} \phi_i g \phi_j dS \right) \; u_j =$$

$$\iint_{\Omega} f \; \phi_i dxdy - \oint_{\Gamma} \left(c\phi_i dS \right) \qquad i = 1,2,\ldots,n \tag{T4L1-12}$$

We can write the element equations in the matrix form as

$$\left[\mathbf{k}_{n\times n}^{\alpha} + \mathbf{k}_{n\times n}^{\beta} + \mathbf{k}_{n\times n}^{g} \right] \mathbf{u}_{n\times 1} = \mathbf{f}_{n\times 1}^{\text{int}} + \mathbf{f}_{n\times 1}^{bnd} \tag{T4L1-13}$$

where

$$k_{ij}^{\alpha} = \iint_{\Omega} \left\{ \frac{\partial \phi_i}{\partial x} \alpha_x \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \alpha_y \frac{\partial \phi_j}{\partial y} \right\} dxdy \tag{T4L1-14a}$$

$$k_{ij}^{\beta} = -\iint_{\Omega} \phi_i \beta \phi_j \; dxdy \tag{T4L1-14b}$$

$$k_{ij}^{g} = \oint \phi_i g \phi_j \; dS \tag{T4L1-14c}$$

$$f_i^{\text{int}} = \iint_{\Omega} f \; \phi_i dxdy \tag{T4L1-14d}$$

$$f_i^{bnd} = -\oint \left(c\phi_i \; dS \right) \tag{T4L1-14e}$$

Finally, the element flux can be computed as

$$\tau_x = -\alpha_x \frac{\partial u}{\partial x} \tag{T4L1-14f}$$

$$\tau_y = -\alpha_y \frac{\partial u}{\partial y} \tag{T4L1-14g}$$

T3 Element

We will derive the element equations for the three-noded linear triangular element.

Step 4: We will rearrange the nodes of the linear triangular element whose shape functions are given by Eqns. (T2L1-34)-(T2L1-36) so that $\phi_1 = \xi$, $\phi_2 = \eta$ and $\phi_3 = 1 - \xi - \eta$. The Jacobian matrix is given by

$$\mathbf{J}_{2\times 2} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{T4L1-15}$$

Since $x = \sum_{i=1}^{3} \phi_i x_i$ and $y = \sum_{i=1}^{3} \phi_i y_i$,

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial \xi} x_i = x_1 - x_3 = x_{13} \qquad \frac{\partial x}{\partial \eta} = \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial \eta} x_i = x_2 - x_3 = x_{23} \tag{T4L1-16}$$

$$\frac{\partial y}{\partial \xi} = \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial \xi} y_i = y_1 - y_3 = y_{13} \qquad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial \eta} y_i = y_2 - y_3 = y_{23} \tag{T4L1-17}$$

Hence,

$$\mathbf{J}_{2\times 2} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \qquad J = x_{13} y_{23} - x_{23} y_{13} = 2A \tag{T4L1-18}$$

where A is the area of the triangle, and

$$\mathbf{\Gamma}_{2\times 2} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \tag{T4L1-19}$$

$$\begin{Bmatrix} \dfrac{\partial \phi_1}{\partial x} \\ \dfrac{\partial \phi_1}{\partial y} \end{Bmatrix} = \mathbf{\Gamma} \begin{Bmatrix} \dfrac{\partial \phi_1}{\partial \xi} \\ \dfrac{\partial \phi_1}{\partial \eta} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} y_{23} \\ -x_{23} \end{Bmatrix} \tag{T4L1-20}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial \phi_2}{\partial x} \\ \dfrac{\partial \phi_2}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} y_{31} \\ x_{13} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dfrac{\partial \phi_3}{\partial x} \\ \dfrac{\partial \phi_3}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} y_{12} \\ x_{21} \end{Bmatrix} \qquad \text{(T4L1-21)}$$

We are now ready to compute the stiffness matrix and the load vector. To facilitate exact integration (when possible) note the following formula involving area coordinates

$$\iint_{\Omega} \xi^l \eta^m \zeta^n \, dxdy = \frac{l!\,m!\,n!}{(l+m+n+2)!} 2A \qquad \text{(T4L1-22a)}$$

$$\int_i^j \xi^l \eta^m \, dS = \frac{l!\,m!}{(l+m+1)!} L_{ij} \qquad \text{(T4L1-22b)}$$

Consider the following examples with the functions f, α and β evaluated at the centroid of the element

$$k_{11}^{\alpha} = \iint_{\Omega} \left[\left(\frac{y_{23}}{2A} \right) \alpha_x \left(\frac{y_{23}}{2A} \right) + \left(\frac{-x_{23}}{2A} \right) \alpha_y \left(\frac{-x_{23}}{2A} \right) \right] dxdy = \frac{y_{23}^2 \hat{\alpha}_x}{4A} + \frac{x_{23}^2 \hat{\alpha}_y}{4A} \qquad \text{(T4L1-23)}$$

$$f_1^{\text{int}} = \iint_{\Omega} f \phi_1 \, dxdy = \frac{\hat{f}(1!)}{3!}(2A) = \frac{\hat{f} A}{3} \qquad \text{(T4L1-24)}$$

$$k_{12}^{\beta} = \iint_{\Omega} \phi_1 \beta \phi_2 \, dxdy = \hat{\beta} \frac{1!\,1!}{4!} 2A = \frac{A \hat{\beta}}{12} \qquad \text{(T4L1-25)}$$

If, for example, $g \neq 0$ on side 1-2 of the element, then

$$k_{12}^{g} = \int_1^2 \phi_1 \hat{g} \phi_2 \, dS = \hat{g} \frac{1!\,1!}{3!} L_{12} = \frac{\hat{g} L_{12}}{6} \qquad \text{(T4L1-26)}$$

To summarize,

$$\mathbf{k}_{3\times3}^{\alpha} = \frac{\hat{\alpha}_x}{4A}\begin{bmatrix} y_{23}^2 & y_{31}y_{23} & y_{12}y_{23} \\ & y_{31}^2 & y_{12}y_{31} \\ SYM & & y_{12}^2 \end{bmatrix} + \frac{\hat{\alpha}_y}{4A}\begin{bmatrix} x_{23}^2 & x_{31}x_{23} & x_{12}x_{23} \\ & x_{31}^2 & x_{12}x_{31} \\ SYM & & x_{12}^2 \end{bmatrix} \quad \text{(T4L1-27a)}$$

$$\mathbf{k}_{3\times3}^{\beta} = -\frac{A\hat{\beta}}{12}\begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ SYM & & 2 \end{bmatrix} \quad \text{(T4L1-27b)}$$

$$\mathbf{k}_{3\times3}^{g} = \frac{\hat{g}_{12} L_{12}}{6}\begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ SYM & & 0 \end{bmatrix} + \frac{\hat{g}_{23} L_{23}}{6}\begin{bmatrix} 0 & 0 & 0 \\ & 2 & 1 \\ SYM & & 2 \end{bmatrix}$$

$$+\frac{\hat{g}_{31} L_{31}}{6}\begin{bmatrix} 2 & 0 & 1 \\ & 0 & 0 \\ SYM & & 2 \end{bmatrix} \quad \text{(T4L1-27c)}$$

$$\mathbf{f}_{3\times1}^{\text{int}} = \frac{\hat{f}A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \mathbf{f}_{3\times1}^{bnd} = -\frac{\hat{c}_{12} L_{12}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} - \frac{\hat{c}_{23} L_{23}}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} - \frac{\hat{c}_{31} L_{31}}{2}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \quad \text{(T4L1-27d)}$$

Concentrated flux components, if any, should be added to the load vector. To compute the flux, since

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{\Gamma} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} u_1 - u_3 \\ u_2 - u_3 \end{Bmatrix} \tag{T4L1-28}$$

we have

$$\tau_x = -\alpha_x \frac{\partial u}{\partial x} = \frac{-\alpha_x}{2A}\left[y_{23}\left(u_1 - u_3\right) - y_{13}\left(u_2 - u_3\right)\right] \tag{T4L1-29a}$$

$$\tau_y = -\alpha_y \frac{\partial u}{\partial y} = \frac{-\alpha_y}{2A}\left[-x_{23}\left(u_1 - u_3\right) + x_{13}\left(u_2 - u_3\right)\right] \tag{T4L1-29b}$$

Computation of the Element Convective Stiffness Matrix and Load Vector

Planar Elements

Recall that the convective stiffness matrix can be computed using the following equation

$$k_{ij}^{g} = \oint_{\Gamma} \phi_i g \phi_j ds \tag{T4L1-14c}$$

the boundary flux can be computed using the following equation

$$f_i^{bnd} = -\oint_{\Gamma} \left(c\phi_i ds \right) \tag{T4L1-14e}$$

Triangular Elements

Boundary Flux: We will use the second-order element (six-noded triangle) as a means of developing the procedure.

Assume that the flux is applied on side $\overline{142}$. In other words

$$f_i^{bnd} = -\oint_{\Gamma} \left(c\phi_i ds \right) = -\int_{\overline{142}} \left\{ c\phi_i \left(\xi, 0\right) \right\} ds \quad i = 1,4,2 \tag{T4L1-30}$$

Note that on side $\overline{142}$ the shape functions can be simplified as

$$\phi_1\left(\xi,0\right) = \left(1-\xi\right)\left(1-2\xi\right) \tag{T4L1-31a}$$

$$\phi_2\left(\xi,0\right) = \xi\left(2\xi - 1\right) \tag{T4L1-31b}$$

$$\phi_3(\xi,0)=0 \tag{T4L1-31c}$$

$$\phi_4(\xi,0)=4\xi(1-\xi) \tag{T4L1-31d}$$

$$\phi_5(\xi,0)=0 \tag{T4L1-31e}$$

$$\phi_6(\xi,0)=0 \tag{T4L1-31f}$$

In order to evaluate the integral, we should note that

$$ds=\sqrt{dx^2+dy^2} \tag{T4L1-32a}$$

or $$ds=\sqrt{\left(\frac{\partial x}{\partial \xi}d\xi+\frac{\partial x}{\partial \eta}d\eta\right)^2+\left(\frac{\partial y}{\partial \xi}d\xi+\frac{\partial y}{\partial \eta}d\eta\right)^2} \tag{T4L1-32b}$$

or $$ds=\sqrt{\left(J_{11}d\xi+J_{21}\,d\eta\right)^2+\left(J_{12}\,d\xi+J_{22}\,d\eta\right)^2} \tag{T4L1-32c}$$

However, on side $\overline{142}$, $\eta=0$ and $d\eta=0$. Hence, Eqn. (T4L1-32c) can be simplified to

$$ds=J_\Gamma(\xi,0)\,d\xi \tag{T4L1-33}$$

such that

$$J_\Gamma(\xi,0)=\sqrt{\left(J_{11}(\xi,0)\right)^2+\left(J_{12}(\xi,0)\right)^2} \tag{T4L1-34}$$

J_Γ is called the boundary Jacobian and represents the ratio of the differential arc lengths in the two coordinate systems. Using Eqn. (T4L1-34) in (T4L1-30) we have

$$f_i^{bnd}=-\int_0^1\left\{c\phi_i(\xi,0)\right\}J_\Gamma(\xi,0)\,d\xi \quad i=1,4,2 \tag{T4L1-35}$$

To compute the boundary Jacobian we have the following terms.

$$J_{11}(\xi,0)=\frac{\partial x}{\partial \xi}(\xi,0)=\sum_{k=1,4,2}x_k\,\frac{\partial \phi_k(\xi,0)}{\partial \xi} \tag{T4L1-36a}$$

or $$J_{11}(\xi,0)=(4\xi-3)\,x_1-(8\xi-4)\,x_4+(4\xi-1)\,x_2 \tag{T4L1-36b}$$

$$J_{12}(\xi,0)=\frac{\partial y}{\partial \xi}(\xi,0)=\sum_{k=1,4,2} y_k \frac{\partial \phi_k(\xi,0)}{\partial \xi} \tag{T4L1-37a}$$

or $$J_{12}(\xi,0)=(4\xi-3)y_1-(8\xi-4)y_4+(4\xi-1)y_2 \tag{T4L1-37b}$$

In order to use numerical integration using Gauss-Legendre formula, we need to change the limits of the integral from $0 \le \xi \le 1$ to $-1 \le \xi' \le 1$. The following transformation can be used to change the limits of the integration

$$\xi=\frac{1}{2}(\xi'+1) \tag{T4L1-38}$$

Using this transformation in Eqns. (T4L1-31), (T4L1-37) and (T4L1-38) we have

$$\phi_1(\xi',0)=-\frac{1}{2}\xi'(1-\xi') \tag{T4L1-39a}$$

$$\phi_2(\xi',0)=\frac{1}{2}\xi'(1+\xi') \tag{T4L1-39b}$$

$$\phi_4(\xi',0)=(1+\xi')(1-\xi') \tag{T4L1-39c}$$

$$J_{11}(\xi',0)=(2\xi'-1)x_1-4\xi' x_4+(2\xi'+1)x_2 \tag{T4L1-40a}$$

$$J_{12}(\xi',0)=(2\xi'-1)y_1-4\xi' y_4+(2\xi'+1)y_2 \tag{T4L1-40b}$$

Hence,

$$f_i^{bnd}=-\frac{1}{2}\int_{-1}^{1}\left\{c\phi_i(\xi',0)\right\}J_\Gamma(\xi',0)\,d\xi' \tag{T4L1-41a}$$

or $$f_i^{bnd}=-\frac{1}{2}\sum_{l=1}^{n} w_{nl}\left[c\phi_i(\xi',0)J_\Gamma(\xi',0)\right]_{\xi'_{nl}} \quad i=1,4,2 \tag{T4L1-41b}$$

With a constant c, a 3-point rule leads to full integration.

The expressions for side $\overline{361}$ are the same as those for $\overline{142}$ with indices $1,4,2$ replaced with $3,6,1$. Similar comments apply to side $\overline{253}$.

Convective Stiffness Matrix: In a similar manner we can develop the convective stiffness matrix. Once again we will use the second-order element (six-noded triangle) as a means of developing the procedure. Let us assume that convection takes place on side $\overline{142}$. In other words

$$k_{ij}^g = \oint_\Gamma \phi_i g \phi_j ds = \int_{\overline{142}} \left\{ g\phi_i(\xi,0)\phi_j(\xi,0) \right\} ds \qquad i = 1,4,2 \tag{T4L1-42}$$

As shown before with the boundary flux derivation, we have for the convective stiffness term

$$k_{ij}^g = \int_0^1 \left\{ g\phi_i(\xi,0)\phi_j(\xi,0) \right\} J_\Gamma(\xi,0)\, d\xi \quad i = 1,4,2 \tag{T4L1-43}$$

$$k_{ij}^g = \frac{1}{2}\int_{-1}^1 \left\{ g\phi_i(\xi',0)\phi_j(\xi',0) \right\} J_\Gamma(\xi',0)\, d\xi' \tag{T4L1-44}$$

or

$$k_{ij}^g = \frac{1}{2}\sum_{l=1}^n w_{nl} \left[g\phi_i(\xi',0)\phi_j(\xi',0) J_\Gamma(\xi',0) \right]_{\xi'_{nl}} \quad i = 1,4,2 \tag{T4L1-45}$$

With a constant g, a 5-point rule leads to full integration.

Quadrilateral Elements

Now consider the family of quadrilateral elements. As we saw with the first-order triangular element, the applied flux for the first-order quadrilateral element also can reduce to a closed form assuming a constant c. In other words, the load vector for the Q4 element assuming that the flux acts on side 1-2, reduces to

$$\mathbf{f}_{4\times1}^{bnd} = -\frac{cL_{12}}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \tag{T4L1-46}$$

We will now derive the expressions for the quadratic element. Consider the fact that the flux is applied to side $\overline{152}$ of the element. On this side $\eta = -1$ and $d\eta = 0$. The shape functions then reduce to (the others are zero)

$$\phi_1(\xi,-1) = -\frac{1}{2}\xi(1-\xi) \tag{T4L1-47a}$$

$$\phi_2\left(\xi,-1\right)=\frac{1}{2}\xi\left(1+\xi\right) \qquad \text{(T4L1-47b)}$$

$$\phi_5\left(\xi,-1\right)=\left(1+\xi\right)\left(1-\xi\right) \qquad \text{(T4L1-47c)}$$

The boundary Jacobian now reduces to

$$J_\Gamma\left(\xi,-1\right)=\sqrt{\left(J_{11}\left(\xi,-1\right)\right)^2+\left(J_{12}\left(\xi,-1\right)\right)^2} \qquad \text{(T4L1-48)}$$

and

$$f_i^{bnd}=-\int_{-1}^{1}\left\{c\phi_i\left(\xi,-1\right)\right\}J_\Gamma\left(\xi,-1\right)d\xi \quad i=1,5,2 \qquad \text{(T4L1-49)}$$

To compute the boundary Jacobian we have the following terms.

$$J_{11}\left(\xi,-1\right)=\frac{\partial x}{\partial\xi}\left(\xi,-1\right)=\sum_{k=1,5,2}x_k\frac{\partial\phi_k\left(\xi,-1\right)}{\partial\xi} \qquad \text{(T4L1-50a)}$$

or $$J_{11}\left(\xi,-1\right)=\left(\xi-\frac{1}{2}\right)x_1-2\xi x_5+\left(\xi+\frac{1}{2}\right)x_2 \qquad \text{(T4L1-50b)}$$

$$J_{12}\left(\xi,-1\right)=\frac{\partial y}{\partial\xi}\left(\xi,-1\right)=\sum_{k=1,5,2}y_k\frac{\partial\phi_k\left(\xi,-1\right)}{\partial\xi} \qquad \text{(T4L1-51a)}$$

or $$J_{12}\left(\xi,-1\right)=\left(\xi-\frac{1}{2}\right)y_1-2\xi y_5+\left(\xi+\frac{1}{2}\right)y_2 \qquad \text{(T4L1-51b)}$$

Using numerical integration, we have

$$f_i^{bnd}=-\sum_{l=1}^{n}w_{nl}\left[c\phi_i\left(\xi,-1\right)J_\Gamma\left(\xi,-1\right)\right]_{\xi_{nl}} \quad i=1,5,2 \qquad \text{(T4L1-52)}$$

With a constant c, a 3-point rule leads to full integration. Note that this expression is valid for any side of the quadrilateral.

Convective Stiffness Matrix: In a similar manner we can develop the convective stiffness matrix. Once again we will use the second-order element (eight-noded quadrilateral) as a means of developing the procedure. Let us assume that convection takes place on side $\overline{152}$. In other words

$$k_{ij}^{g} = \oint_{\Gamma} \phi_i g \phi_j ds = \int_{\overline{152}} \left\{ g\phi_i\left(\xi,-1\right)\phi_j\left(\xi,-1\right) \right\} ds \qquad i = 1,5,2 \tag{T4L1-53}$$

As shown before with the boundary flux derivation, we have for the convective stiffness term

$$k_{ij}^{g} = \int_{-1}^{1} \left\{ g\phi_i\left(\xi,-1\right)\phi_j\left(\xi,-1\right) \right\} J_{\Gamma}\left(\xi,-1\right) d\xi \quad i = 1,5,2 \tag{T4L1-54}$$

or $$k_{ij}^{g} = \sum_{l=1}^{n} w_{nl} \left[g\phi_i\left(\xi,-1\right)\phi_j\left(\xi,-1\right) J_{\Gamma}\left(\xi,-1\right) \right]_{\xi'_{nl}} \quad i = 1,5,2 \tag{T4L1-55}$$

With a constant g , a 5-point rule leads to full integration.

Lesson 2: Planar Engineering Problems

Objectives: In this lesson we will look the different planar engineering problems.

- To understand the link between the general 2D BVP and specific engineering problems.
- To solve multidisciplinary problems using 2D BVP computer programs.

Engineering Examples

Thin Fin Heat Conduction and Convection Problem

Consider a thin fin as shown in Fig. T4L2-1. The thickness, t, of the plate is small compared to the dimensions of the plate. The temperature is represented as T, thermal conductivity of the plate is k, the convective heat transfer coefficient is h, and the ambient temperature is T_{∞}. $\hat{Q}(x,y)$ represents the interior heat source. Convective heat loss takes place from the top **and** the bottom of the plate. The boundary of the plate (the sides) can be divided into three possible parts. The temperature is held at a specified level T_0 on a part of the boundary, or there is no convective heat loss, or there is heat exchange between the side and the surroundings. The governing differential equation and boundary conditions are as follows (Note that q_0 is zero on the boundary with no convective heat loss).

$$\frac{\partial}{\partial x}(k_x t\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(k_y t\frac{\partial T}{\partial y})-2hT+2hT_\infty+\hat{Q}(x,y)=0$$

$$\text{with } T=T_0 \text{ on } \Gamma_1 \qquad \text{(T4L2-1)}$$

$$k_x t\frac{\partial T}{\partial x}n_x+k_y t\frac{\partial T}{\partial y}n_y=-q_0 \text{ on } \Gamma_2$$

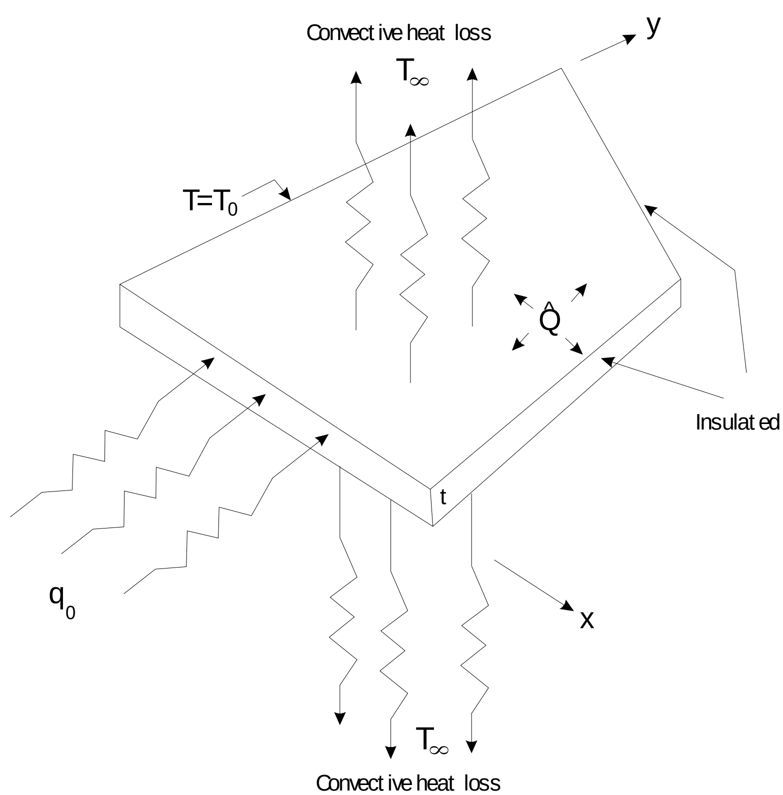

Fig. T4L2-1 Example of Thin Fin heat conduction and convection problem

Long Body Heat Conduction and Convection Problem

Consider a long body as shown in Fig. T4L2-2. The depth of the body, t, is very large compared to the dimensions of the body. The temperature is represented as T , thermal conductivity of the body is k, the convective heat transfer coefficient is h, and the ambient temperature is T_∞. Heat may be generated in a portion or all of the domain of the body. The boundary of the body can be divided into three possible parts. The temperature is held at a specified level T_0 on a part of the boundary, or there is no convective heat loss, or there is heat exchange between the side and the surroundings. The governing differential equation and boundary conditions are as follows (Note that q_0 is zero on the boundary with no convective heat loss).

$$\begin{aligned}
&\frac{\partial}{\partial x}(k_x t\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(k_y t\frac{\partial T}{\partial y})+Q=0\\
&\text{with } T=T_0 \text{ on } \Gamma_1\\
&\qquad k_x t\frac{\partial T}{\partial x}n_x+k_y t\frac{\partial T}{\partial y}n_y=-q_0 \text{ on } \Gamma_2\\
&\qquad k_x t\frac{\partial T}{\partial x}n_x+k_y t\frac{\partial T}{\partial y}n_y+ht(T-T_\infty)=0 \text{ on } \Gamma_3
\end{aligned} \tag{T4L2-2}$$

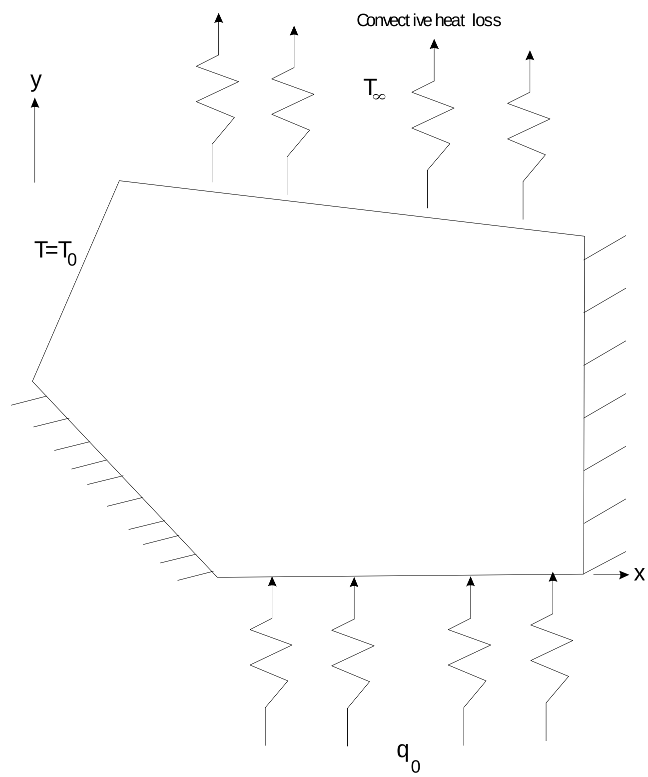

Fig. T4L2-2 Example of Long Body heat conduction and convection problem

Ideal Fluid Flow Problem

Steady (streamlines do not change over time), ideal (fluid has zero viscosity), and incompressible flow in two dimensions can be modeled using the following differential equation and boundary conditions (see Fig. T4L2-3).

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$\text{with } \phi = \phi_0 \text{ on } \Gamma_1$$

$$\frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y = 0 \text{ on } \Gamma_2 \qquad \text{(T4L2-3)}$$

$$\frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y + \alpha\phi + \beta = 0 \text{ on } \Gamma_3$$

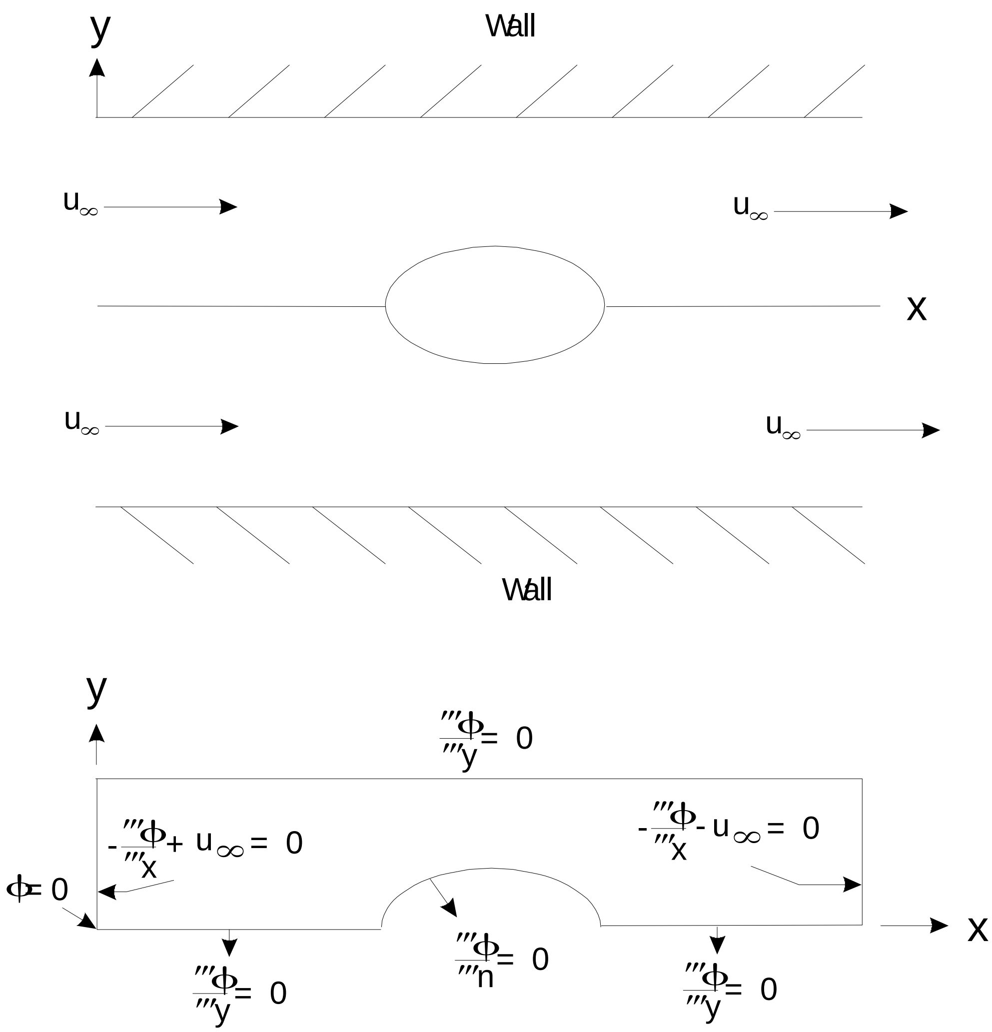

Fig. T4L2-3 Flow of an ideal fluid around an elliptical object where ϕ is the velocity potential, and is given as $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$ where u and v are velocities in the x and y directions. The boundary condition on Γ_2 represents a solid boundary (the flow cannot penetrate the boundary) while the condition on Γ_3 represents the incoming and exiting flow conditions.

Torsion in Bars

The governing differential equation and boundary conditions are (see Fig. T4L3-4)

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + 2 = 0 \qquad \text{(T4L2-4)}$$

with $\psi = 0$ on the boundary

where ψ is the Airy's Stress function, G is the shear modulus, α is the rate of twist, and $\tau_{xz} = G\alpha \frac{\partial \psi}{\partial y}$, $\tau_{yz} = -G\alpha \frac{\partial \psi}{\partial x}$ are the shear stresses in the bar and $M = 2G\alpha \iint_A \psi \, dA$ (M is the torsional moment).

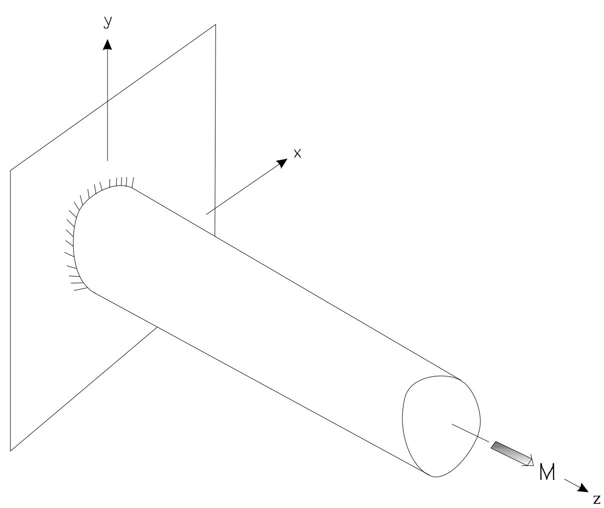

Fig. T4L2-4 Torsion of elastic bars

Electrostatics

In an isotropic dielectric medium, the governing differential equation and boundary conditions are of the form (see Fig. T4L2-5)

$$\varepsilon\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)=-\rho$$
$$\text{with } \Phi=\Phi_0 \text{ on } \Gamma_1 \qquad \text{(T4L2-5)}$$
$$\frac{\partial\Phi}{\partial x}n_x+\frac{\partial\Phi}{\partial y}n_y=0 \text{ on } \Gamma_2$$

where ☞ is the electrostatic potential, //ι is the permittivity, ∠ is the charge density.

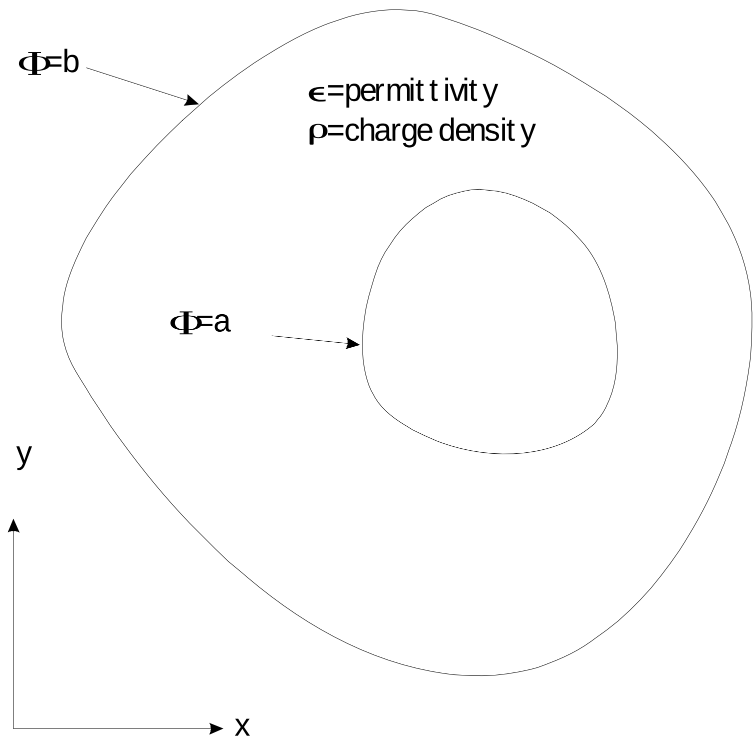

Fig. T4L2-5 Example of Electrostatic problem

Magnetostatics

If u is the magnetic field potential, and ◡ is the permeability, the differential equation and boundary conditions are of the form

$$\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0$$
$$\text{with } u=u_0 \text{ on } \Gamma_1 \qquad \text{(T4L2-6)}$$
$$\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial y}n_y=0 \text{ on } \Gamma_2$$

Finally, a table to illustrate the units used in different areas.

Area	Quantity	FPS	SI
Heat Transfer	Temperature	°F	°C
	Conductivity	Btu/h-ft^2-°F	W/m^2-°C
	Flux	Btu/h-ft^2	W/m^2
	Convective Coef	Btu/h-ft^2-°F	W/m^2-°C
Electromagnetics	Electric Potential		V
	Permeability		H/m
	Permittivity		F/m
	Charge		C
	Magnetic Potential		A
Elasticity	Displacement	ft	m
	Force	lb	N
	Stress	lb/ft^2	N/m^2
Fluid Flow	Absolute Viscosity	lb-s/ft^2	kg/m-s

Illustrative Example 1

Fig. T4L2-6 shows a thin fin where convective heat loss is taking place from the top and the bottom but not from the sides.

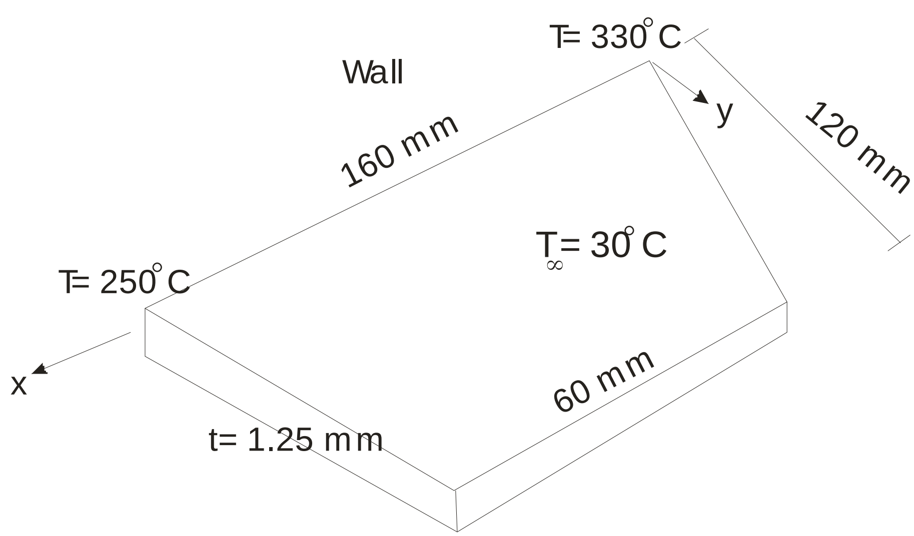

Fig. T4L2-6

The material properties are as follows - $k = 0.20 W/mm - ^\circ C$ and $h = 10^{-5} W/mm^2 - ^\circ C$. The two-element finite element is shown below.

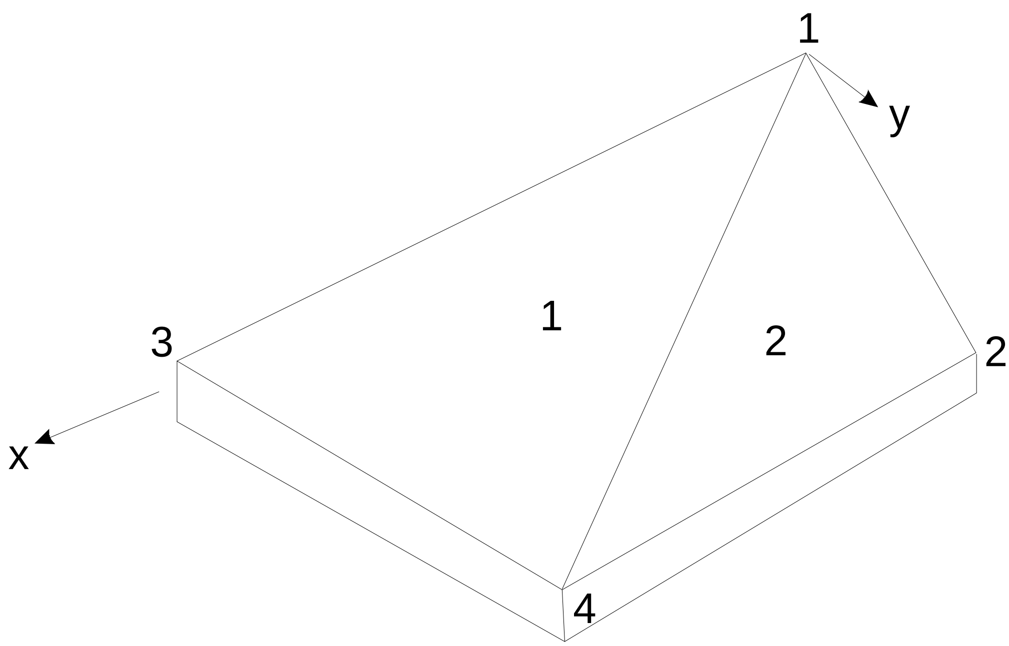

Fig. T4L2-7

The given data can be expressed as shown in the following tables.

Node	X-Coordinate (mm)	Y-Coordinate (mm)	Temperature $\left({}^{\circ}C\right)$
1	0	0	330
2	50	120	
3	160	0	250
4	110	120	

Element	Connectivity			$k_x = k_y$	β	f
	Node 1	Node 2	Node 3			
1	1	3	4	0.25	-2×10^{-5}	6×10^{-4}
2	2	1	4	0.25	-2×10^{-5}	6×10^{-4}

We can now construct the element equations.

Element 1: $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (160,0)$ and $(x_3, y_3) = (110,120)$. Hence, $y_{23} = -120$, $y_{31} = 120$, and $y_{12} = 0$. Also, $x_{23} = 50$, $x_{31} = 110$ and $x_{12} = -160$. $A = 9600$. Finally, $g = 0$ and $c = 0$.

$$\mathbf{k}_{3\times3} = \frac{0.25}{4(9600)}\begin{bmatrix} 120^2 & -120^2 & 0 \\ & 120^2 & 0 \\ SYM & & 0 \end{bmatrix} + \frac{0.25}{4(9600)}\begin{bmatrix} 50^2 & 50(110) & -50(160) \\ & 110^2 & 110(-160) \\ SYM & & 160^2 \end{bmatrix}$$

$$\begin{array}{c} - \frac{2(10^{-5})(9600)}{12}\begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ SYM & & 2 \end{bmatrix} = \begin{bmatrix} 0.14203 & -0.04194 & -0.03608 \\ & 0.20453 & -0.09858 \\ SYM & & 0.19867 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 3 \\ 4 \end{array}$$

The load vector is

$$\mathbf{f}_{3\times1} = \frac{6(10^{-4})(9600)}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1.92 \\ 1.92 \\ 1.92 \end{Bmatrix}$$

Element 2: $(x_1, y_1) = (50,120)$, $(x_2, y_2) = (0,0)$ and $(x_3, y_3) = (110,120)$. Hence, $y_{23} = -120$, $y_{31} = 0$, and $y_{12} = 120$. Also, $x_{23} = -110$, $x_{31} = 60$ and $x_{12} = 50$. $A = 3600$. Finally, $g = 0$ and $c = 0$.

$$\mathbf{k}_{3\times3} = \frac{0.25}{4(3600)}\begin{bmatrix} 120^2 & 0 & -120^2 \\ & 0 & 0 \\ SYM & & 120^2 \end{bmatrix} + \frac{0.25}{4(3600)}\begin{bmatrix} 110^2 & -60(110) & -50(110) \\ & 60^2 & 50(60) \\ SYM & & 50^2 \end{bmatrix}$$

$$-\frac{2(10^{-5})(3600)}{12}\begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ SYM & & 2 \end{bmatrix} = \begin{array}{@{}c@{}c@{}} \begin{bmatrix} 0.47207 & -0.10858 & -0.339498 \\ & 0.07450 & -0.05808 \\ SYM & & 0.30540 \end{bmatrix} & \begin{array}{c} 2 \\ 1 \\ 4 \end{array} \end{array}$$

The load vector is

$$\mathbf{f}_{3\times 1} = \frac{6(10^{-4})(3600)}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.72 \\ 0.72 \\ 0.72 \end{Bmatrix}$$

Assembling the element equations

$$\begin{bmatrix} 0.21653 & -0.10858 & -0.04194 & 0.02200 \\ & 0.47207 & 0 & -0.33949 \\ & & 0.20453 & -0.09858 \\ SYM & & & 0.50407 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 2.64 \\ 0.72 \\ 1.92 \\ 2.64 \end{Bmatrix}$$

Imposing the essential boundary conditions $T_1 = 330$ and $T_3 = 250$. Solving $T_2 = 205.6$ and $T_4 = 178.2$. To compute the flux

$$q_x = -k_x \frac{\partial T}{\partial x} = -\frac{k_x}{2A}\left(y_{23}(T_1 - T_3) - y_{13}(T_2 - T_3)\right) = 0.1\frac{W}{mm^2}$$

$$q_y = -k_y \frac{\partial T}{\partial y} = -\frac{k_y}{2A}\left(-x_{23}(T_1 - T_3) + x_{13}(T_2 - T_3)\right) = 0.1614\frac{W}{mm^2}$$

Hence, the heat loss through side 1-3 of element 1 is

$$q_y(160t) = 32.3W$$

Illustrative Example 2

Consider the torsion of a shaft with a square cross-section. The shear modulus is $30 \times 10^6\, psi$. The rate of twist for the shaft is $0.001\, rad/ft$. We will compute the stress distribution within the shaft cross-section.

Because of the symmetry of the cross-section, we need to look at one-eight of the cross-section. Fig. T4L2-8 shows the finite element model and mesh.

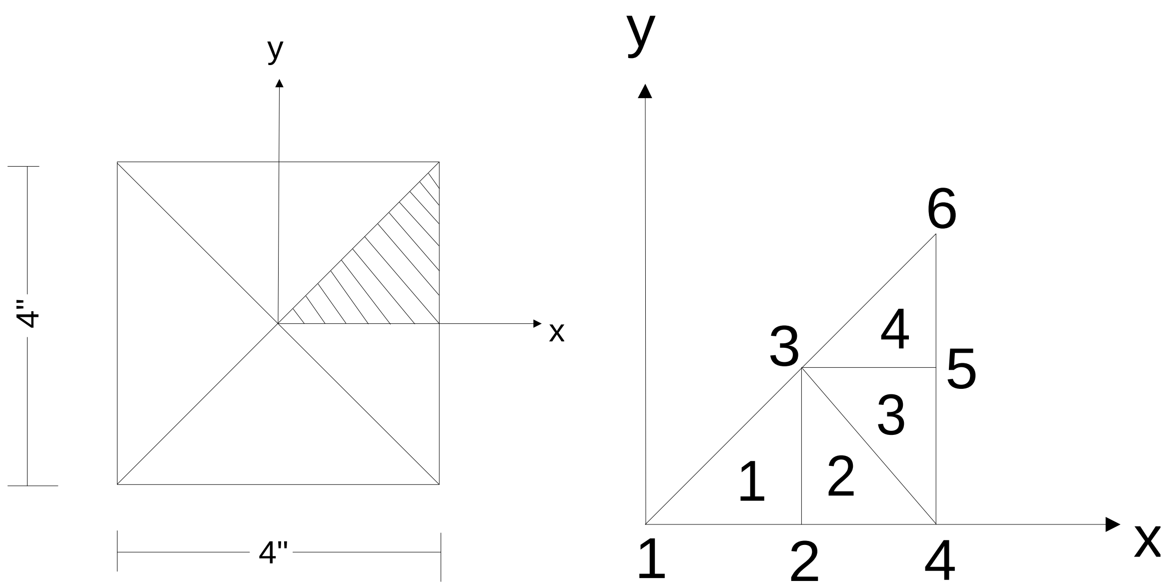

Fig. T4L2-8

We will construct the element equations now.

Element 1: $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (1,0)$ and $(x_3, y_3) = (1,1)$. Hence, $y_{23} = -1$, $y_{31} = 1$, and $y_{12} = 0$. Also, $x_{23} = 0$, $x_{31} = 1$ and $x_{12} = -1$. $A = 0.5$. Finally, $\alpha_x = \alpha_y = 1, \beta = 0$, $g = 0$ and $c = 0$.

$$\mathbf{k}_{3\times3} = \frac{1}{4(0.5)} \begin{bmatrix} 1^2 & (1)(-1) & 0 \\ & 1^2 & 0 \\ SYM & & 0 \end{bmatrix} + \frac{1}{4(0.5)} \begin{bmatrix} 0 & 0 & 0 \\ & 1^2 & (-1)(1) \\ SYM & & 1^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.5 & 0 \\ & 1 & -0.5 \\ SYM & & 0.5 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

The element load vector is given by

$$\mathbf{f}_{3\times 1} = \frac{(2)(0.5)}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{Bmatrix}$$

In a similar manner the other element matrices can be constructed as follows.

$$\mathbf{k}^2 = \begin{bmatrix} 1 & -0.5 & -0.5 \\ & 0.5 & 0 \\ SYM & & 0.5 \end{bmatrix} \begin{matrix} 2 \\ 4 \\ 3 \end{matrix} \qquad \mathbf{k}^3 = \begin{bmatrix} 0.5 & -0.5 & 0 \\ & 1 & -0.5 \\ SYM & & 0.5 \end{bmatrix} \begin{matrix} 4 \\ 5 \\ 3 \end{matrix}$$

$$\mathbf{k}^4 = \begin{bmatrix} 0.5 & -0.5 & 0 \\ & 1 & -0.5 \\ SYM & & 0.5 \end{bmatrix} \begin{matrix} 3 \\ 5 \\ 6 \end{matrix}$$

The global stiffness matrix can be assembled as

$$\mathbf{K}_{6\times6}=\begin{bmatrix} 0.5 & -0.5 & 0 & 0 & 0 & \\ & 2 & -1 & -0.5 & 0 & 0 \\ & & 2 & 0 & -1 & 0 \\ & & & 1 & -0.5 & 0 \\ & & & & 2 & -0.5 \\ SYM & & & & & 0.5 \end{bmatrix}$$

with the load vector as $\mathbf{R}_{6\times1}=\begin{bmatrix}1/3 & 2/3 & 4/3 & 2/3 & 2/3 & 1/3\end{bmatrix}^T$. Imposing the essential boundary conditions we have

$$\begin{bmatrix} 0.5 & -0.5 & 0 \\ & 2 & -1 \\ SYM & & 2 \end{bmatrix}\begin{Bmatrix}\psi_1\\ \psi_2\\ \psi_3\end{Bmatrix}=\begin{Bmatrix}1/3\\ 2/3\\ 4/3\end{Bmatrix}$$

Solving, $\psi_1=2.33\,in^2/rad$, $\psi_2=1.67\,in^2/rad$ and $\psi_3=1.50\,in^2/rad$. As in the previous example, the stresses (equivalent to the flux) in the elements can be computed and the results are as follows.

$$\begin{Bmatrix}\tau_{xz}\\ \tau_{yz}\end{Bmatrix}^1=\begin{Bmatrix}-417\\ 1667\end{Bmatrix}psi \qquad \begin{Bmatrix}\tau_{xz}\\ \tau_{yz}\end{Bmatrix}^2=\begin{Bmatrix}-417\\ 4167\end{Bmatrix}psi$$

$$\begin{Bmatrix}\tau_{xz}\\ \tau_{yz}\end{Bmatrix}^3=\begin{Bmatrix}0\\ 3750\end{Bmatrix}psi \qquad \begin{Bmatrix}\tau_{xz}\\ \tau_{yz}\end{Bmatrix}^4=\begin{Bmatrix}0\\ 3750\end{Bmatrix}psi$$

The torque M now can be computed using

$$M = 2G\alpha\iint_A \psi\, dA = \sum_{i=1}^{4}\left[2G\alpha\iint_A \left(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3\right) dA\right] = 9722.5\, in - lb$$

The total applied torque is eight times the above value or 77,778 in-lb. The theoretical values are - $\tau_{\max} = 6{,}780\, psi$ and $M = 90{,}140\, in - lb$.

Computer Example 1

We will solve the above problem using a computer program. Mesh A is shown in Fig. T2L2-9. It represents a quarter of the actual cross-section.

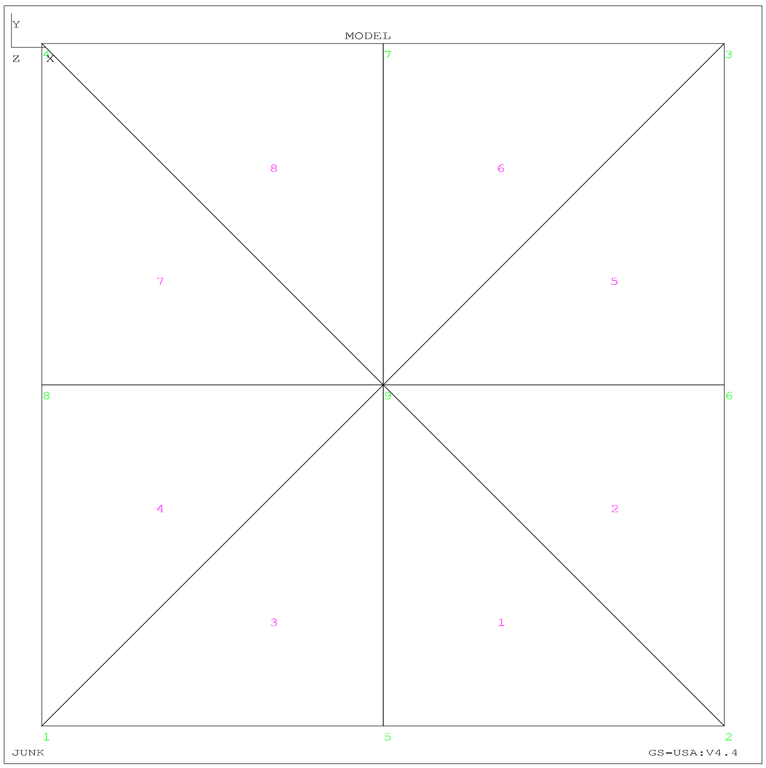

Fig. T2L2-9 Mesh A

The second finest mesh used, Mesh D, is shown in Fig. T2L2-10.

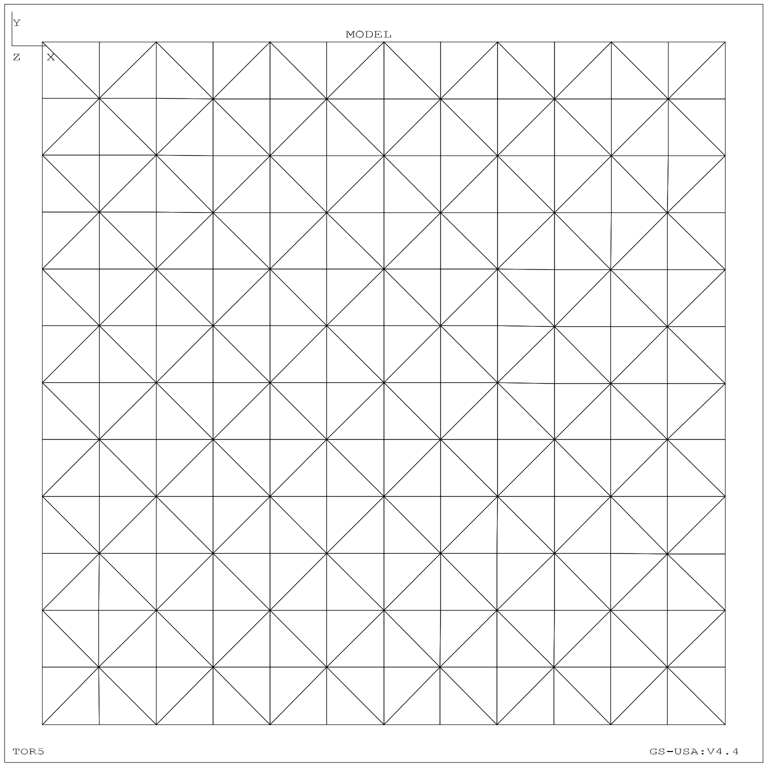

Fig. T2L2-10 Mesh D

The finite element results are shown in the table below.

ID	Number of Elements	$(\tau)_{\max}$ *psi*
Mesh A	8	4168
Mesh B	32	5405
Mesh C	72	5895
Mesh D	288	6350
Mesh E	392	6410

As can be seen from the results, the finite element solution is converging to the analytical solution.

Review Exercises

Problem T4L2-1

TBC

Problem T4L2-2

TBC

Problem T4L2-3

TBC

Problem T4L2-4

TBC

Problem T4L2-5

TBC

Lesson 3: Axisymmetric BVP

Objectives: In this lesson we will look at axisymmetric boundary value problems.

- To understand what is meant by axisymmetric BVP.
- To derive the element equations for commonly used low and higher-order isoparametric axisymmetric BVP elements.

Problem Formulation

What is an axisymmetric problem? It is a two-dimensional problem that deals with solid bodies of revolution. Fig. T4L3-1 shows an axisymmetric body. The z axis is the axis of revolution. It is also known as the axial direction. The r axis represents the radial direction. The geometry (including the entire finite element model[1]) of the body is a function of r and z but is independent of θ. The shaded area shows the part of the body that is used for the axisymmetric analysis.

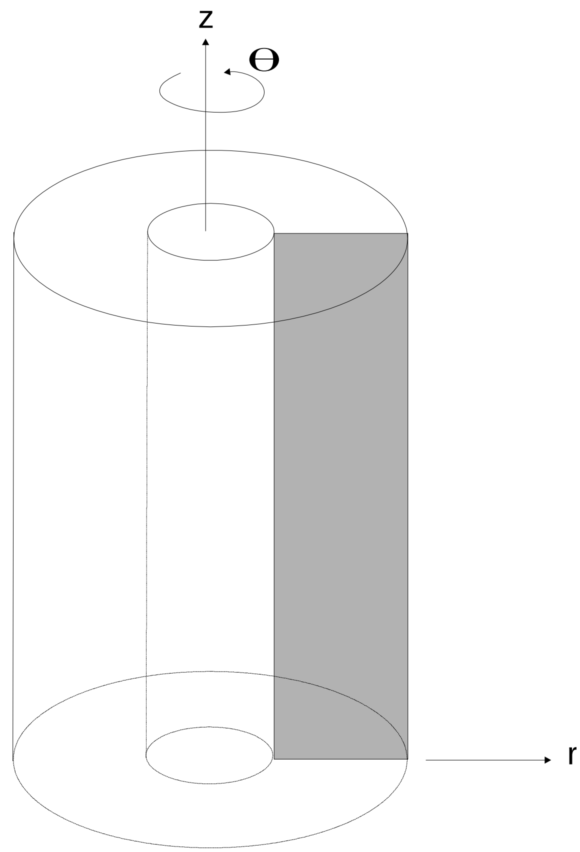

Fig. T4L3-1 Axisymmetric problem showing a solid body of revolution

The axisymmetric BVP dealing with the unknown $u = u(r,z)$, is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha_{rr}(r,z)\frac{\partial u(r,z)}{\partial r}\right) + \frac{\partial}{\partial z}\left(\alpha_{zz}(r,z)\frac{\partial u(r,z)}{\partial z}\right) + \beta(r,z)u(x,y)$$

[1] Axisymmetric problems with non-axisymmetric loads can still be analyzed as an axisymmetric problem. However the problem treatment is left as an exercise.

$$+ f(r,z) = 0 \qquad \text{(T4L3-1)}$$

with the boundary conditions as

$$u(\hat{r},\hat{z}) = \hat{u} \qquad \text{on } \Gamma_1 \qquad \text{(T4L3-2)}$$

$$\alpha_{rr}\frac{\partial u}{\partial r}n_r + \alpha_{zz}\frac{\partial u}{\partial z}n_z + gu + c = 0 \qquad \text{on } \Gamma_2 \qquad \text{(T4L3-3)}$$

where (n_r, n_z) are the direction cosines of the outward normal to the boundary Γ_2, and g and c are constants. We will use the Galerkin's Method to generate the finite element equations necessary to solve this problem.

Let the trial solution be given as

$$\tilde{u}(r,z) = \sum_{j=1}^{n}\phi_j(r,z)\,u_j \qquad \text{(T4L3-4)}$$

We will drop the tilde (~) from this point onwards to denote the approximate solution.

Step 1: Compute the residual equations for a typical element domain Ω as

$$2\pi\iint_{\Omega} R(r,z,u)\phi_i(r,z)\,rdrdz = 0 \qquad i = 1,2,\ldots,n \qquad \text{(T4L3-5a)}$$

Or, $$2\pi\iint_{\Omega}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha_{rr}\frac{\partial u)}{\partial r}\right) + \frac{\partial}{\partial z}\left(\alpha_{zz}\frac{\partial u}{\partial z}\right) + \beta u + f\right]\phi_i(r,z)rdrdz = 0$$

$$i = 1,2,\ldots,n \qquad \text{(T4L3-5b)}$$

Or, $$2\pi\iint_{\Omega}[\frac{\partial}{\partial r}\left(r\alpha_{rr}\frac{\partial u)}{\partial r}\right) + \frac{\partial}{\partial z}\left(r\alpha_{zz}\frac{\partial u}{\partial z}\right) + \beta\, r\, u$$

$$+ fr]\phi_i(r,z)\,dr\,dz = 0 \qquad i = 1,2,\ldots,n \qquad \text{(T4L3-5c)}$$

Step 2: Integrate by parts the highest-order derivative

From this point onwards we will drop the 2π term. To integrate by parts, we need to use the chain rule of differentiation and the Divergence Theorem.

(A) Chain rule of differentiation

$$\frac{\partial}{\partial r}\left(r\alpha_{rr}\frac{\partial u}{\partial r}\right)\phi_i = \frac{\partial}{\partial r}\left(r\alpha_{rr}\frac{\partial u}{\partial r}\phi_i\right) - \left(r\alpha_{rr}\frac{\partial u}{\partial r}\right)\frac{\partial \phi_i}{\partial r} \qquad \text{(T4L3-6a)}$$

$$\frac{\partial}{\partial z}\left(r\alpha_{zz}\frac{\partial u}{\partial z}\right)\phi_i = \frac{\partial}{\partial z}\left(r\alpha_{zz}\frac{\partial u}{\partial z}\phi_i\right) - \left(r\alpha_{zz}\frac{\partial u}{\partial z}\right)\frac{\partial \phi_i}{\partial z} \qquad \text{(T4L3-6b)}$$

(B) Divergence Theorem

Let $F = F(r,z)$ and $G = G(r,z)$. Then

$$\iint_{\Omega}\left(\frac{\partial F}{\partial r} + \frac{\partial G}{\partial z}\right)drdz = \oint_{\Gamma}\left(Fn_r + Gn_z\right)dS \qquad \text{(T4L3-7)}$$

Using Eqns. (T4L3-5b) and (T4L3-6), we have

$$\iint_{\Omega}\left(\frac{\partial}{\partial r}\left(r\alpha_{rr}\frac{\partial u}{\partial r}\phi_i\right) + \frac{\partial}{\partial z}\left(r\alpha_{zz}\frac{\partial u}{\partial z}\phi_i\right)\right)drdz - \iint_{\Omega}\left(\left(r\alpha_{rr}\frac{\partial u}{\partial r}\right)\frac{\partial \phi_i}{\partial r} + \left(r\alpha_{zz}\frac{\partial u}{\partial z}\right)\frac{\partial \phi_i}{\partial z}\right)$$

$$-\iint_{\Omega}\left(\beta\, ur\,\phi_i + f\, r\,\phi_i\right)drdz = 0 \qquad i = 1,2,\ldots,n \qquad \text{(T4L3-8)}$$

Using Eqn. (T4L3-7) and rearranging we have

$$\left(\iint_{\Omega}\left\{\left(r\alpha_{rr}\frac{\partial u}{\partial r}\right)\frac{\partial \phi_i}{\partial r} + \left(r\alpha_{zz}\frac{\partial u}{\partial z}\right)\frac{\partial \phi_i}{\partial z}\right\}dxdy - \iint_{\Omega}\left\{\beta\, r\, u\phi_i\right\}drdz\right)$$

$$= \iint_{\Omega} f\, r\phi_i drdz + \oint_{\Gamma}\left(r\alpha_{rr}\frac{\partial u}{\partial r}n_r + r\alpha_{zz}\frac{\partial u}{\partial z}n_z\right)\phi_i dS \qquad i = 1,2,\ldots,n \qquad \text{(T4L3-9)}$$

Step 3: Substitute the trial solution

$$u = \sum_{j=1}^{n} u_j\phi_j \qquad \text{(T4L3-10a)}$$

Hence, $\frac{\partial u}{\partial r} = \sum_{j=1}^{n} u_j \frac{\partial \phi_j}{\partial r}$ and $\frac{\partial u}{\partial z} = \sum_{j=1}^{n} u_j \frac{\partial \phi_j}{\partial z}$ (T4L3-10b)

Using Eqn. (T4L3-3), and Eqn. (T4L3-10) we have

$$\sum_{j=1}^{n}\left(\iint_{\Omega}\left\{r\frac{\partial\phi_i}{\partial r}\alpha_{rr}\frac{\partial\phi_j}{\partial r}+r\frac{\partial\phi_i}{\partial z}\alpha_{zz}\frac{\partial\phi_j}{\partial z}-r\phi_i\beta\phi_j\right\}\ dxdy+\oint_{\Gamma}r\phi_i g\phi_j dS\right)\ u_j =$$

$$\iint_{\Omega} rf\,\phi_i\ drdz-\oint_{\Gamma}rc\phi_i dS \qquad\qquad i=1,2,\ldots,n \tag{T4L3-11}$$

We can write the element equations in the matrix form as

$$\left[\mathbf{k}^{\alpha}_{n\times n}+\mathbf{k}^{\beta}_{n\times n}+\mathbf{k}^{g}_{n\times n}\right]\mathbf{u}_{n\times 1}=\mathbf{f}^{\text{int}}_{n\times 1}+\mathbf{f}^{bnd}_{n\times 1} \tag{T4L3-12}$$

where

$$k^{\alpha}_{ij}=\iint_{\Omega}\left\{r\frac{\partial\phi_i}{\partial r}\alpha_{rr}\frac{\partial\phi_j}{\partial r}+r\frac{\partial\phi_i}{\partial z}\alpha_{zz}\frac{\partial\phi_j}{\partial z}\right\}drdz \tag{T4L3-13a}$$

$$k^{\beta}_{ij}=-\iint_{\Omega}r\phi_i\beta\phi_j\ drdz \tag{T4L3-13b}$$

$$k^{g}_{ij}=\oint_{\Gamma}r\phi_i g\phi_j\ dS \tag{T4L3-13c}$$

$$f^{\text{int}}_{i}=\iint_{\Omega}f\ r\phi_i drdz \tag{T4L3-13d}$$

$$f^{bnd}_{i}=-\oint_{\Gamma}\left(rc\phi_i\ dS\right) \tag{T4L3-13e}$$

T3 Element

We will derive the element equations for the three-noded linear triangular element.

Step 4: We will rearrange the nodes of the linear triangular element whose shape functions are given by Eqns. (T2L1-34)-(T2L1-36) so that $\phi_1=\xi$, $\phi_2=\eta$ and $\phi_3=1-\xi-\eta$. The Jacobian matrix is given by

$$\mathbf{J}_{2\times 2}=\begin{bmatrix}\dfrac{\partial r}{\partial\xi} & \dfrac{\partial z}{\partial\xi}\\[2mm] \dfrac{\partial r}{\partial\eta} & \dfrac{\partial z}{\partial\eta}\end{bmatrix} \tag{T4L3-15}$$

Since $r = \sum_{i=1}^{3} \phi_i r_i$ and $z = \sum_{i=1}^{3} \phi_i z_i$,

$$\frac{\partial r}{\partial \xi} = \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial \xi} r_i = r_1 - r_3 = r_{13} \qquad \frac{\partial r}{\partial \eta} = \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial \eta} r_i = r_2 - r_3 = r_{23} \tag{T4L3-16}$$

$$\frac{\partial z}{\partial \xi} = \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial \xi} z_i = z_1 - z_3 = z_{13} \qquad \frac{\partial z}{\partial \eta} = \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial \eta} z_i = z_2 - z_3 = z_{23} \tag{T4L3-17}$$

Hence,

$$\mathbf{J}_{2\times 2} = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix} \qquad J = r_{13} z_{23} - r_{23} z_{13} = 2A \tag{T4L3-18}$$

where A is the area of the triangle, and

$$\mathbf{\Gamma}_{2\times 2} = \frac{1}{2A} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & r_{13} \end{bmatrix} \tag{T4L3-19}$$

$$\begin{Bmatrix} \dfrac{\partial \phi_1}{\partial x} \\ \dfrac{\partial \phi_1}{\partial y} \end{Bmatrix} = \mathbf{\Gamma} \begin{Bmatrix} \dfrac{\partial \phi_1}{\partial \xi} \\ \dfrac{\partial \phi_1}{\partial \eta} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & r_{13} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} z_{23} \\ -r_{23} \end{Bmatrix} \tag{T4L3-20}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial \phi_2}{\partial x} \\ \dfrac{\partial \phi_2}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} z_{31} \\ r_{13} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dfrac{\partial \phi_3}{\partial x} \\ \dfrac{\partial \phi_3}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} z_{12} \\ r_{21} \end{Bmatrix} \tag{T4L3-21}$$

We are now ready to compute the stiffness matrix and the load vector. To facilitate exact integration (when possible) note the following formula involving area coordinates

$$\iint_{\Omega} \xi^l \eta^m \zeta^n \, dxdy = \frac{l!\, m!\, n!}{(l+m+n+2)!} 2A \tag{T4L3-22a}$$

$$\int_i^j \xi^l \eta^m \, dS = \frac{l!\,m!}{(l+m+1)!} L_{ij} \tag{T4L3-22b}$$

Consider the following examples with the functions f, α and β evaluated at the centroid of the element ($b_i = z_j - z_k$ etc. and $c_i = -r_j + r_k$ etc. with cyclic i, j, k)

$$k_{ij}^{\alpha} = \iint_{\Omega} \left[r\left(\frac{b_i}{2A}\right)\alpha_{rr}\left(\frac{b_j}{2A}\right) + r\left(\frac{c_i}{2A}\right)\alpha_{zz}\left(\frac{c_j}{2A}\right) \right] drdz$$

$$= \frac{b_i b_j \hat{\alpha}_{rr}}{4A^2} \iint_{\Omega} r drdz + \frac{c_i c_j \hat{\alpha}_{zz}}{4A^2} \iint_{\Omega} r drdz \tag{T4L3-23a}$$

Since we can express $r = \sum_{j=1}^{3} \phi_j r_j$, we have

$$\iint_{\Omega} r \, drdz = \iint_{\Omega} \left(r_1\phi_1 + r_2\phi_2 + r_3\phi_3\right) drdz = \left(\frac{r_1}{3!} + \frac{r_2}{3!} + \frac{r_3}{3!}\right) 2A = \frac{A}{3}\left(r_1 + r_2 + r_3\right) \tag{T4L3-23b}$$

Hence, with $\bar{r} = \frac{1}{3}\left(r_1 + r_2 + r_3\right)$

$$k_{ij}^{\alpha} = \frac{b_i b_j \hat{\alpha}_{rr} + c_i c_j \hat{\alpha}_{zz}}{4A} \bar{r} \tag{T4L3-23c}$$

$$f_1^{\text{int}} = \iint_{\Omega} r f \phi_1 \, dxdy = \frac{\hat{f}A}{12}\left(2r_1 + r_2 + r_3\right) \tag{T4L3-24}$$

$$k_{12}^{\beta} = -\iint_{\Omega} r \phi_1 \beta \phi_2 \, dxdy = -\frac{\bar{r} A \hat{\beta}}{12} \tag{T4L3-25}$$

If, for example, $g \neq 0$ on side 1-2 of the element, then

$$k_{12}^{g} = \int_1^2 r \phi_1 \hat{g} \phi_2 \, dS = \frac{\hat{g} L_{12}}{12}(r_1 + r_2) \tag{T4L3-26}$$

To summarize (we will put back 2π)

$$\mathbf{k}_{3\times3}^{\alpha} = \frac{2\pi\hat{\alpha}_{rr}\bar{r}}{4A}\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ & b_2^2 & b_2b_3 \\ SYM & & b_3^2 \end{bmatrix} + \frac{2\pi\hat{\alpha}_{zz}\bar{r}}{4A}\begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ & c_2^2 & c_2c_3 \\ SYM & & c_3^2 \end{bmatrix} \qquad \text{(T4L3-27a)}$$

$$\mathbf{k}_{3\times3}^{\beta} = -\frac{2\pi\bar{r}A\hat{\beta}}{12}\begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ SYM & & 2 \end{bmatrix} \qquad \text{(T4L3-27b)}$$

$$\mathbf{k}_{3\times3}^{g} = \frac{2\pi\hat{g}_{12}L_{12}}{12}\begin{bmatrix} 3r_1+r_2 & r_1+r_2 & 0 \\ & r_1+3r_2 & 0 \\ SYM & & 0 \end{bmatrix} + \frac{2\pi\hat{g}_{23}L_{23}}{12}\begin{bmatrix} 0 & 0 & 0 \\ & 3r_2+r_3 & r_2+r_3 \\ SYM & & r_2+3r_3 \end{bmatrix}$$

$$+\frac{2\pi\hat{g}_{31}L_{31}}{12}\begin{bmatrix} 3r_1+r_3 & 0 & r_1+r_3 \\ & 0 & 0 \\ SYM & & r_1+3r_3 \end{bmatrix} \qquad \text{(T4L3-27c)}$$

$$\mathbf{f}_{3\times1}^{\text{int}} = \frac{2\pi\hat{f}A}{12}\begin{Bmatrix} 2r_1+r_2+r_3 \\ r_1+2r_2+r_3 \\ r_1+r_2+2r_3 \end{Bmatrix} \qquad \text{(T4L3-27d)}$$

$$\mathbf{f}_{3\times1}^{bnd} = -\frac{2\pi\hat{c}_{12}L_{12}}{6}\begin{Bmatrix} 2r_1+r_2 \\ r_1+2r_2 \\ 0 \end{Bmatrix} - \frac{2\pi\hat{c}_{23}L_{23}}{6}\begin{Bmatrix} 0 \\ 2r_2+r_3 \\ r_2+2r_3 \end{Bmatrix} - \frac{2\pi\hat{c}_{13}L_{13}}{6}\begin{Bmatrix} 2r_1+r_3 \\ 0 \\ r_1+2r_3 \end{Bmatrix} \qquad \text{(T4L3-27e)}$$

Concentrated flux components, if any, should be added to the load vector.

Lesson 4: Axisymmetric Engineering Applications

Objectives: In this lesson we will look the different axisymmetric engineering problems.

- To understand the link between the general axisymmetric BVP and specific engineering problems.
- To solve multidisciplinary problems using axisymmetric BVP computer programs.

Heat Transfer

The axisymmetric heat transfer problem dealing with temperature as the primary unknown $T = T(r,z)$, is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk_{rr}(r,z)\frac{\partial T(r,z)}{\partial r} \right) + \frac{\partial}{\partial z}\left(k_{zz}(r,z)\frac{\partial T(r,z)}{\partial z} \right) + Q(r,z) = 0 \qquad \text{(T4L4-1)}$$

with the boundary conditions as

$$T(\hat{r},\hat{z}) = \hat{T} \quad \text{on } \Gamma_1 \qquad \text{(T4L4-2)}$$

$$(k_{rr}\frac{\partial T}{\partial r}n_r + k_{zz}\frac{\partial T}{\partial z}n_z) = -q_n \quad \text{on } \Gamma_2 \qquad \text{(T4L4-3)}$$

$$k_{rr}\frac{\partial T}{\partial r}n_r + k_{zz}\frac{\partial T}{\partial z}n_z + h(T - T_\infty) = 0 \ \text{ on } \Gamma_3 \qquad \text{(T4L4-4)}$$

where (n_r, n_z) are the direction cosines of the outward normal to the boundary Γ_2 and Γ_3.

Illustrative Example

Fig. T4L4-1 shows a segment of a very long pipe that carries a fluid that is at $200^\circ C$. The thermal conductivity of pipe material is $30\frac{W}{m-^\circ C}$ The outside air is at $25^\circ C$. The convective coefficient of the outside surface is $12\frac{W}{m^2-^\circ C}$. The inner radius of the pipe is 0.1m and the wall thickness is 0.1m. We will solve for the temperature distribution within the pipe and the rate of heat loss from the exterior surface. The 0.5 m length is selected arbitrarily.

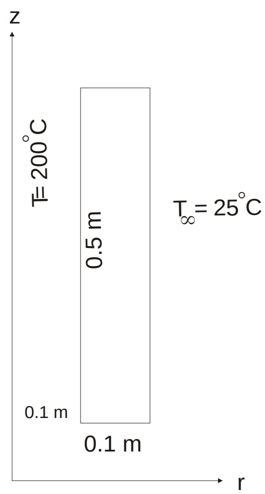

Fig. T4L4-1 Segment of a long pipe

Only two finite element meshes are used to solve the problem (Fig. T4L4-2). The finite element results are given below.

Mesh ID	**Number of elements**	**Avg. Temp. Outer wall $\left({}^{\circ}C\right)$**	**Radial Flux at Outer Wall $\left(W/m^2\right)$**
Mesh A	4	191.1	2548
Mesh B	16	190.9	2270

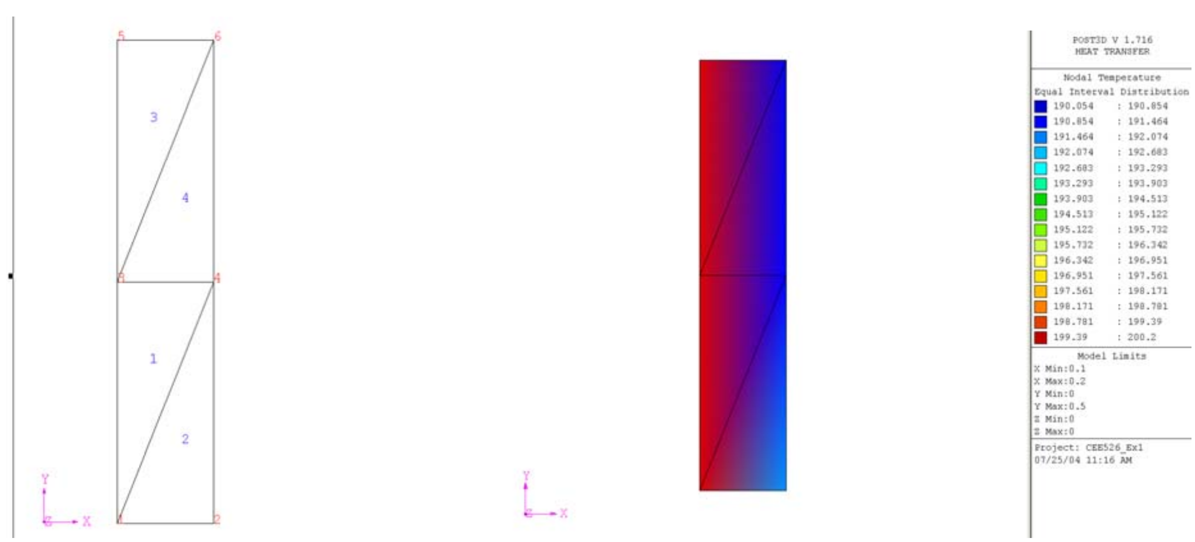

Fig. T4L4-2(a) 4-Element mesh and temperature plot (4 element mesh)

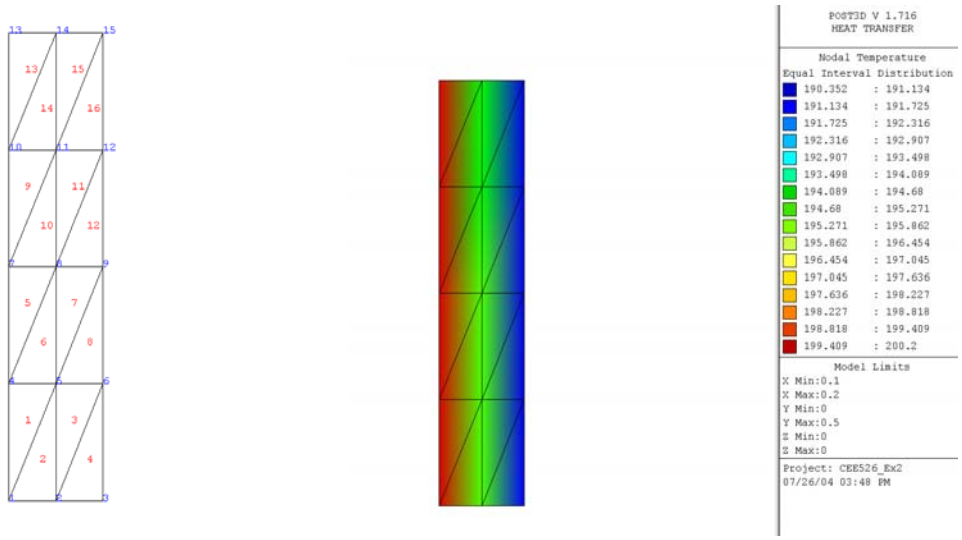

Fig. T4L4-2(b) 16-Element mesh and temperature plot (16 element mesh)

As can be seen from the results, and as we have seen before, the temperature converges much faster than the flux.

Review Exercises

Problem T4L4-1

TBC.

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