Finite Elements for Engineers

Lecture 8: 2D Boundary Value Problems

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2D BVP

DE

$$\frac{\partial}{\partial x}\left(\alpha_x(x,y)\frac{\partial u(x,y)}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha_y(x,y)\frac{\partial u(x,y)}{\partial y}\right)+\beta(x,y)u(x,y)$$

$$+f(x,y)=0$$

BCs

$$u(\hat{x},\hat{y})=\hat{u}$$

$$\alpha_x\frac{\partial u}{\partial x}n_x+\alpha_y\frac{\partial u}{\partial y}n_y+gu+c=0$$

2D BVP

Galerkin Step 1: Residual Equations

$$\tilde{u}(x,y) = \sum_{j=1}^{n} \phi_j(x,y)\, u_j$$

$$\iint_{\Omega} R(x,y,u)\phi_i(x,y)\, dxdy = 0 \qquad i = 1,2,\ldots,n$$

$$\iint_{\Omega} [\frac{\partial}{\partial x}\left(\alpha_x(x,y)\frac{\partial u(x,y)}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha_y(x,y)\frac{\partial u(x,y)}{\partial y}\right) + \beta(x,y)u(x,y)$$

$$+ f(x,y)]\phi_i(x,y)\, dxdy = 0 \qquad i = 1,2,\ldots,n$$

2D BVP

Concepts

Chain Rule of Differentiation

$$\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\right)\phi_i = \frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\phi_i\right) - \left(\alpha_x \frac{\partial u}{\partial x}\right)\frac{\partial \phi_i}{\partial x}$$

$$\frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\right)\phi_i = \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\phi_i\right) - \left(\alpha_y \frac{\partial u}{\partial y}\right)\frac{\partial \phi_i}{\partial y}$$

2D BVP

Divergence Theorem

$$F = F(x, y)$$

$$G = G(x, y)$$

$$\iint_{\Omega} \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dxdy = \oint_{\Gamma} \left(Fn_x + Gn_y \right) dS$$

2D BVP

Galerkin Step 2: Integration by parts

$$\iint_{\Omega} \left\{ \frac{\partial u}{\partial x} \alpha_x \frac{\partial \phi_i}{\partial x} + \frac{\partial u}{\partial y} \alpha_y \frac{\partial \phi_i}{\partial y} - \beta u \phi_i \right\} dxdy$$

$$+ \oint_{\Gamma} \left(g u \phi_i \right) ds = \iint_{\Omega} f \phi_i dxdy - \oint_{\Gamma} \left(c \phi_i dS \right) \qquad i = 1,2,\ldots,n$$

2D BVP

Galerkin Step 3: Use of trial solution

$$\sum_{j=1}^{n}\left(\iint_{\Omega}\left\{\frac{\partial\phi_i}{\partial x}\alpha_x\frac{\partial\phi_j}{\partial x}+\frac{\partial\phi_i}{\partial y}\alpha_y\frac{\partial\phi_j}{\partial y}-\phi_i\beta\phi_j\right\}dxdy+\oint_{\Gamma}\phi_i g\phi_j dS\right)u_j=$$

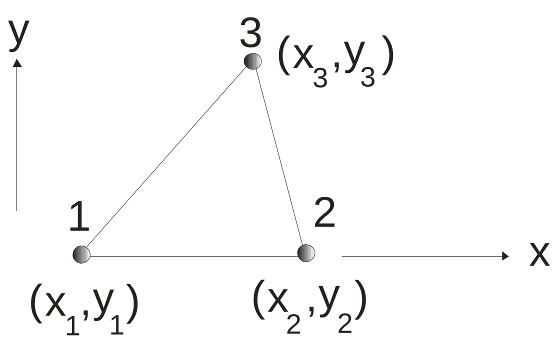
$$\iint_{\Omega} f\,\phi_i dxdy-\oint_{\Gamma}\left(c\phi_i dS\right) \qquad i=1,2,\ldots,n$$

$$\left[\mathbf{k}^{\alpha}_{n\times n}+\mathbf{k}^{\beta}_{n\times n}+\mathbf{k}^{g}_{n\times n}\right]\mathbf{u}_{n\times 1}=\mathbf{f}^{\text{int}}_{n\times 1}+\mathbf{f}^{bnd}_{n\times 1}$$

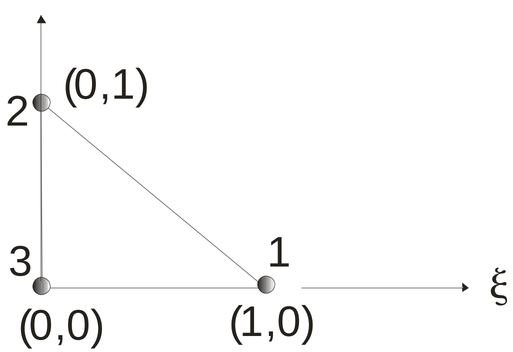
Linear Triangular Element

Galerkin Step 4

Real



Master or Parent



Shape Functions

$$\phi_1 = \xi \qquad \phi_2 = \eta \qquad \phi_3 = 1 - \xi - \eta$$

Jacobian

$$\mathbf{J}_{2\times 2} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

$$x = \phi_1 x_1 + \phi_2 x_2 + \phi_3 x_3$$

$$y = \phi_1 y_1 + \phi_2 y_2 + \phi_3 y_3$$

Linear Triangular Element

Computing Derivatives

$$\begin{Bmatrix} \dfrac{\partial \phi_1}{\partial x} \\ \dfrac{\partial \phi_1}{\partial y} \end{Bmatrix} = \mathbf{\Gamma} \begin{Bmatrix} \dfrac{\partial \phi_1}{\partial \xi} \\ \dfrac{\partial \phi_1}{\partial \eta} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} y_{23} \\ -x_{23} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial \phi_2}{\partial x} \\ \dfrac{\partial \phi_2}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} y_{31} \\ x_{13} \end{Bmatrix} \qquad \begin{Bmatrix} \dfrac{\partial \phi_3}{\partial x} \\ \dfrac{\partial \phi_3}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{Bmatrix} y_{12} \\ x_{21} \end{Bmatrix}$$

Linear Triangular Element

Example

$$k_{ij}^{\alpha} = \iint_{\Omega} \left\{ \frac{\partial \phi_i}{\partial x} \alpha_x \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \alpha_y \frac{\partial \phi_j}{\partial y} \right\} dxdy$$

Exact Integration

$$\iint_{\Omega} \xi^l \eta^m \zeta^n \, dxdy = \frac{l!\,m!\,n!}{(l+m+n+2)!} 2A$$

$$\int_i^j \xi^l \eta^m \, dS = \frac{l!\,m!}{(l+m+1)!} L_{ij}$$

Linear Triangular Element

Sample Terms

$$k_{11}^{\alpha} = \iint_{\Omega} \left[\left(\frac{y_{23}}{2A} \right) \alpha_x \left(\frac{y_{23}}{2A} \right) + \left(\frac{-x_{23}}{2A} \right) \alpha_y \left(\frac{-x_{23}}{2A} \right) \right] dxdy = \frac{y_{23}^2 \hat{\alpha}_x}{4A} + \frac{x_{23}^2 \hat{\alpha}_y}{4A}$$

$$k_{12}^{\beta} = \iint_{\Omega} \phi_1 \beta \phi_2 \quad dxdy = \hat{\beta} \frac{1! \;\; 1!}{4!} 2A = \frac{A\hat{\beta}}{12}$$

$$k_{12}^{g} = \int_1^2 \phi_1 \hat{g} \phi_2 \quad dS = \hat{g} \frac{1! \;\; 1!}{3!} L_{12} = \frac{\hat{g} L_{12}}{6}$$

$$f_1^{\text{int}} = \iint_{\Omega} f \phi_1 \, dxdy = \frac{\hat{f}(1!)}{3!}(2A) = \frac{\hat{f} A}{3}$$

Linear Triangular Element

$$\mathbf{k}^{\alpha}_{3\times 3} = \frac{\hat{\alpha}_x}{4A}\begin{bmatrix} y_{23}^2 & y_{31}y_{23} & y_{12}y_{23} \\ & y_{31}^2 & y_{12}y_{31} \\ SYM & & y_{12}^2 \end{bmatrix} + \frac{\hat{\alpha}_y}{4A}\begin{bmatrix} x_{23}^2 & x_{31}x_{23} & x_{12}x_{23} \\ & x_{31}^2 & x_{12}x_{31} \\ SYM & & x_{12}^2 \end{bmatrix}$$

$$\mathbf{k}^{\beta}_{3\times 3} = -\frac{\hat{A}\beta}{12}\begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ SYM & & 2 \end{bmatrix}$$

Linear Triangular Element

$$\mathbf{k}^{g}_{3\times3} = \frac{\hat{g}_{12}L_{12}}{6}\begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ SYM & & 0 \end{bmatrix} + \frac{\hat{g}_{23}L_{23}}{6}\begin{bmatrix} 0 & 0 & 0 \\ & 2 & 1 \\ SYM & & 2 \end{bmatrix}$$

$$+\frac{\hat{g}_{31}L_{31}}{6}\begin{bmatrix} 2 & 0 & 1 \\ & 0 & 0 \\ SYM & & 2 \end{bmatrix}$$

Linear Triangular Element

$$\mathbf{f}_{3\times 1}^{\text{int}} = \frac{\hat{f}A}{3}\begin{Bmatrix}1\\1\\1\end{Bmatrix}$$

$$\mathbf{f}_{3\times 1}^{bnd} = -\frac{\hat{c}_{12}L_{12}}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix} - \frac{\hat{c}_{23}L_{23}}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix} - \frac{\hat{c}_{31}L_{31}}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix}$$

Element Flux

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{\Gamma} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} u_1 - u_3 \\ u_2 - u_3 \end{Bmatrix}$$

$$\tau_x = -\alpha_x \frac{\partial u}{\partial x} = \frac{-\alpha_x}{2A}\left[y_{23}\left(u_1 - u_3\right) - y_{13}\left(u_2 - u_3\right) \right]$$

$$\tau_y = -\alpha_y \frac{\partial u}{\partial y} = \frac{-\alpha_y}{2A}\left[-x_{23}\left(u_1 - u_3\right) + x_{13}\left(u_2 - u_3\right) \right]$$

Other Elements

- Isoparametric Formulation
  - Different shape functions
  - Different sizes for the matrices
- The generic equations are evaluated numerically (numerical integration)

Convective Stiffness and Load Vector

Convective Stiffness

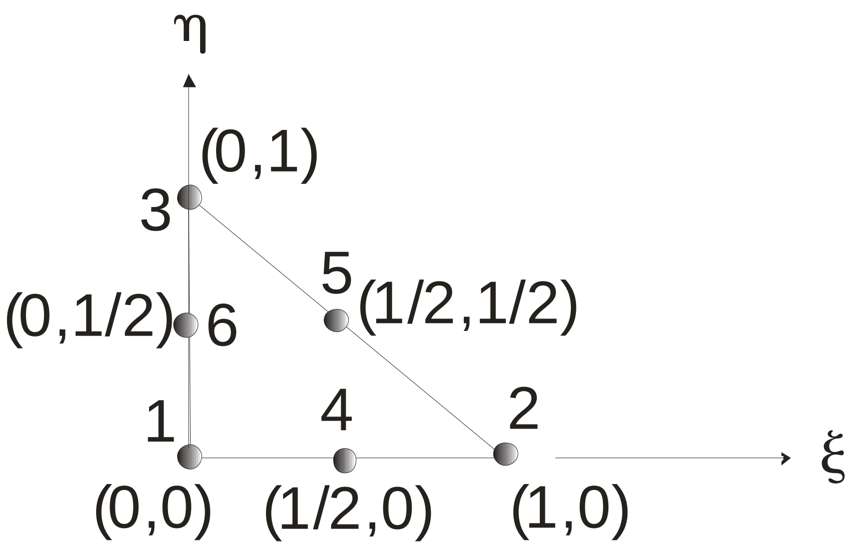
$$k_{ij}^{g} = \oint_{\Gamma} \phi_i g \phi_j ds$$

Boundary Flux (Load Vector)

$$f_i^{bnd} = -\oint_{\Gamma} \left(c\phi_i ds \right)$$

Load Vector: T6

Example (T6): Convection on side 1-4-2



$$f_i^{bnd} = -\oint_{\Gamma} \left(c\phi_i ds \right) = -\int_{\overline{142}} \left\{ c\phi_i \left(\xi, 0 \right) \right\} ds \quad i = 1, 4, 2$$

Load Vector: T6

Example (T6): Convection on side 1-4-2

$$\eta = 0$$

$$d\eta = 0$$

$$\phi_1(\xi,0) = (1-\xi)(1-2\xi)$$

$$\phi_2(\xi,0) = \xi(2\xi-1)$$

$$\phi_4(\xi,0) = 4\xi(1-\xi)$$

Note

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{\left(\frac{\partial x}{\partial \xi}d\xi + \frac{\partial x}{\partial \eta}d\eta\right)^2 + \left(\frac{\partial y}{\partial \xi}d\xi + \frac{\partial y}{\partial \eta}d\eta\right)^2}$$

Load Vector: T6

$$ds = \sqrt{\left(J_{11}d\xi + J_{21}\,d\eta\right)^2 + \left(J_{12}\,d\xi + J_{22}\,d\eta\right)^2}$$

$$ds = J_{\Gamma}\left(\xi,0\right)d\xi = \sqrt{\left(J_{11}\left(\xi,0\right)\right)^2 + \left(J_{12}\left(\xi,0\right)\right)^2}\,d\xi$$

Simplifying

$$f_i^{bnd} = -\int_0^1 \left\{c\phi_i\left(\xi,0\right)\right\} J_{\Gamma}\left(\xi,0\right)d\xi \quad i = 1,4,2$$

Load Vector: T6

$$J_{11}(\xi,0)=\frac{\partial x}{\partial \xi}(\xi,0)=(4\xi-3)x_1-(8\xi-4)x_4+(4\xi-1)x_2$$

$$J_{12}(\xi,0)=\frac{\partial y}{\partial \xi}(\xi,0)=(4\xi-3)y_1-(8\xi-4)y_4+(4\xi-1)y_2$$

Transforming the coordinate system

$$\phi_1(\xi',0)=-\frac{1}{2}\xi'(1-\xi')$$

$$\xi=\frac{1}{2}(\xi'+1) \quad \Rightarrow \quad \phi_2(\xi',0)=\frac{1}{2}\xi'(1+\xi')$$

$$\phi_4(\xi',0)=(1+\xi')(1-\xi')$$

$$-1\le\xi'\le1$$

$$J_{11}(\xi',0)=(2\xi'-1)x_1-4\xi'x_4+(2\xi'+1)x_2$$

$$J_{12}(\xi',0)=(2\xi'-1)y_1-4\xi'y_4+(2\xi'+1)y_2$$

Load Vector: T6

Hence

$$f_i^{bnd} = -\frac{1}{2}\int_{-1}^{1}\left\{c\phi_i\left(\xi',0\right)\right\}J_\Gamma\left(\xi',0\right)d\xi'$$

$$f_i^{bnd} = -\frac{1}{2}\sum_{l=1}^{n} w_{nl}\left[c\phi_i\left(\xi',0\right)J_\Gamma\left(\xi',0\right)\right]_{\xi'_{nl}} \quad i = 1,4,2$$

Convective Stiffness Matrix: T6

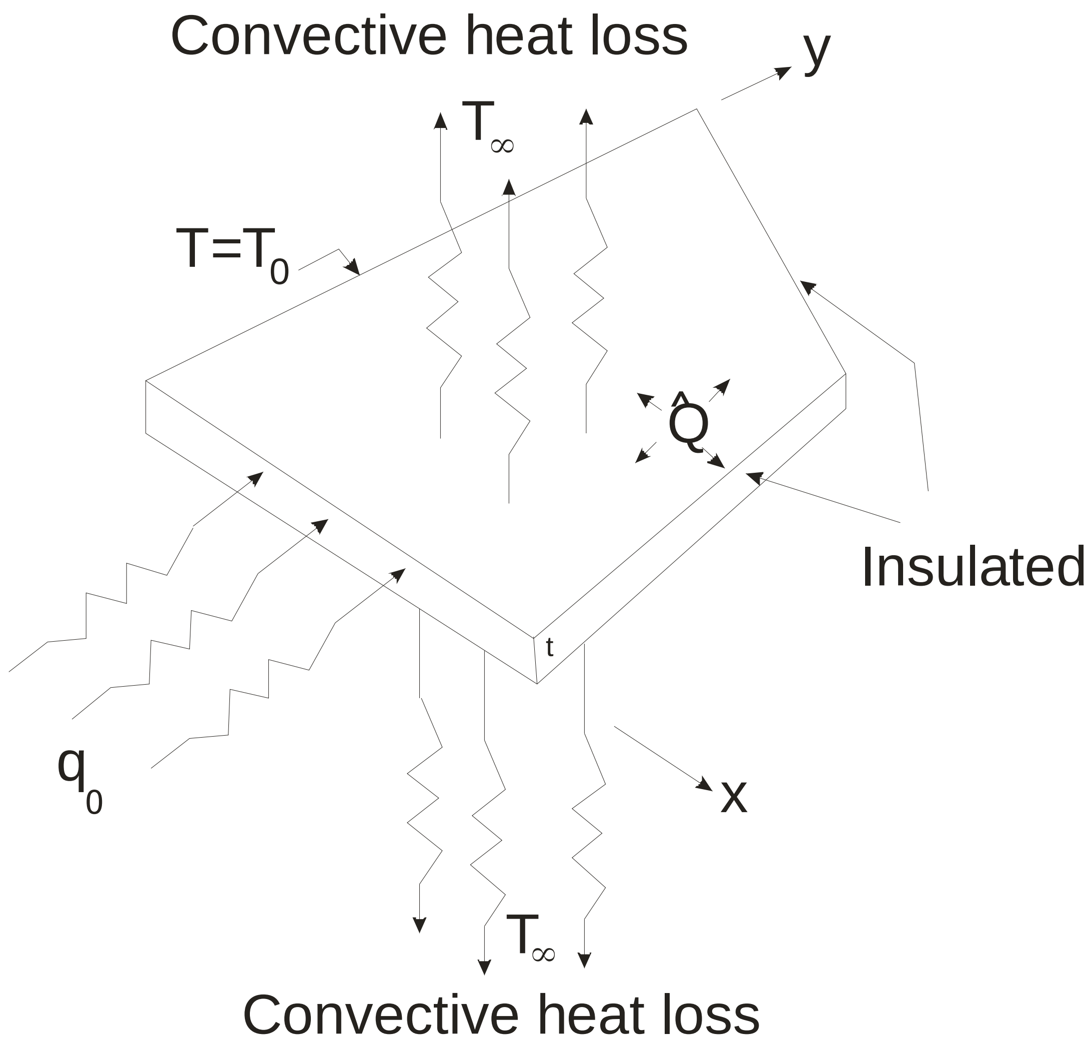
$$k_{ij}^{g} = \oint_{\Gamma} \phi_i g \phi_j ds = \int_{\overline{142}} \left\{ g\phi_i\left(\xi,0\right)\phi_j\left(\xi,0\right)\right\} ds \qquad i = 1,4,2$$

As before

$$k_{ij}^{g} = \int_{0}^{1} \left\{ g\phi_i\left(\xi,0\right)\phi_j\left(\xi,0\right)\right\} J_{\Gamma}\left(\xi,0\right) d\xi \quad i = 1,4,2$$

$$k_{ij}^{g} = \frac{1}{2}\sum_{l=1}^{n} w_{nl}\left[g\phi_i\left(\xi',0\right)\phi_j\left(\xi',0\right) J_{\Gamma}\left(\xi',0\right)\right]_{\xi'_{nl}} \quad i = 1,4,2$$

Heat Transfer Problem: Thin Fin



Similar to a plane stress problem

Thin Fin Problem

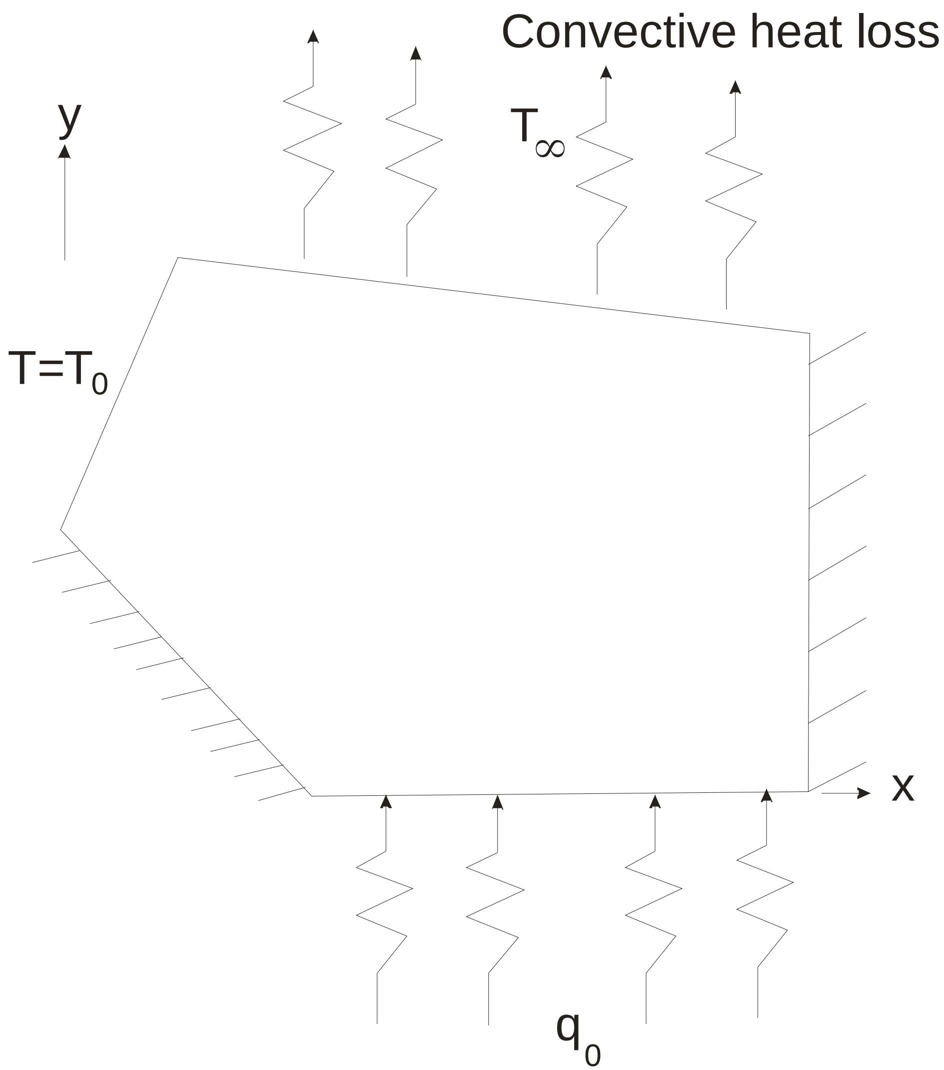
DE and BCs

$$\frac{\partial}{\partial x}(k_x t \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k_y t \frac{\partial T}{\partial y}) - 2hT + 2hT_\infty + \hat{Q}(x,y) = 0$$

with $T = T_0$ on Γ_1

$$k_x t \frac{\partial T}{\partial x} n_x + k_y t \frac{\partial T}{\partial y} n_y = -q_0 \text{ on } \Gamma_2$$

Heat Transfer Problem: Long Body



Similar to a plane strain problem

Long Body Problem

DE and BCs

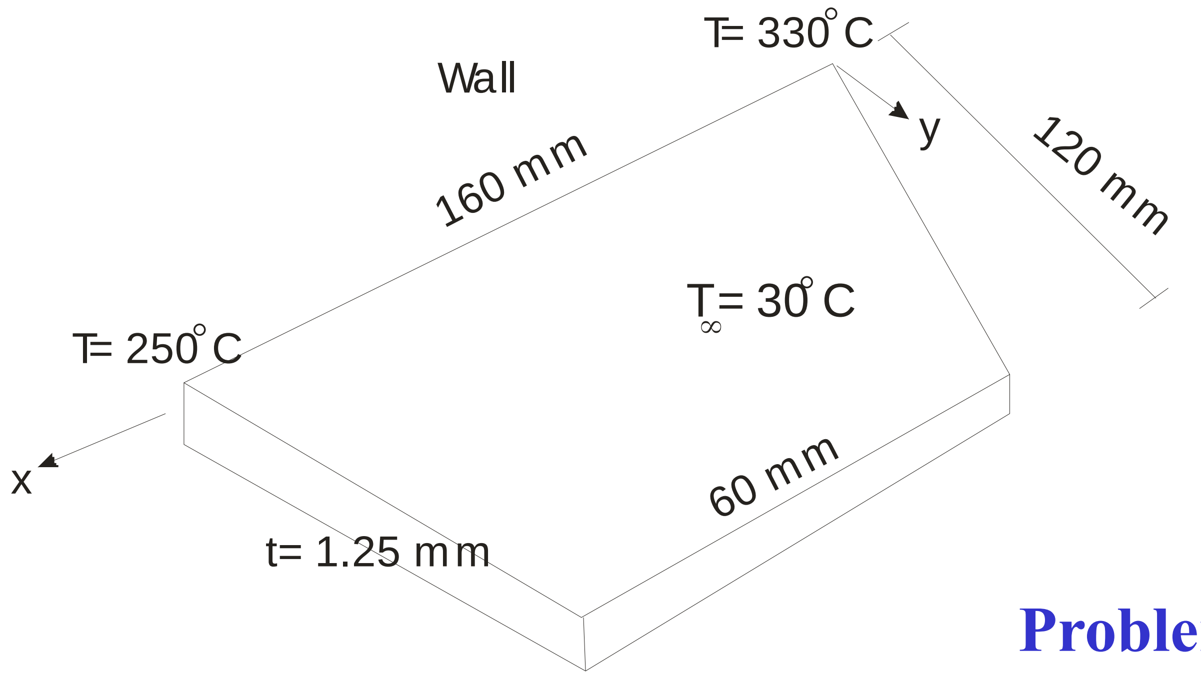
$$\frac{\partial}{\partial x}(k_x t \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(k_y t \frac{\partial T}{\partial y}) + Q = 0$$

with $T = T_0$ on Γ_1

$$k_x t \frac{\partial T}{\partial x} n_x + k_y t \frac{\partial T}{\partial y} n_y = -q_0 \text{ on } \Gamma_2$$

$$k_x t \frac{\partial T}{\partial x} n_x + k_y t \frac{\partial T}{\partial y} n_y + ht(T - T_\infty) = 0 \text{ on } \Gamma_3$$

Example



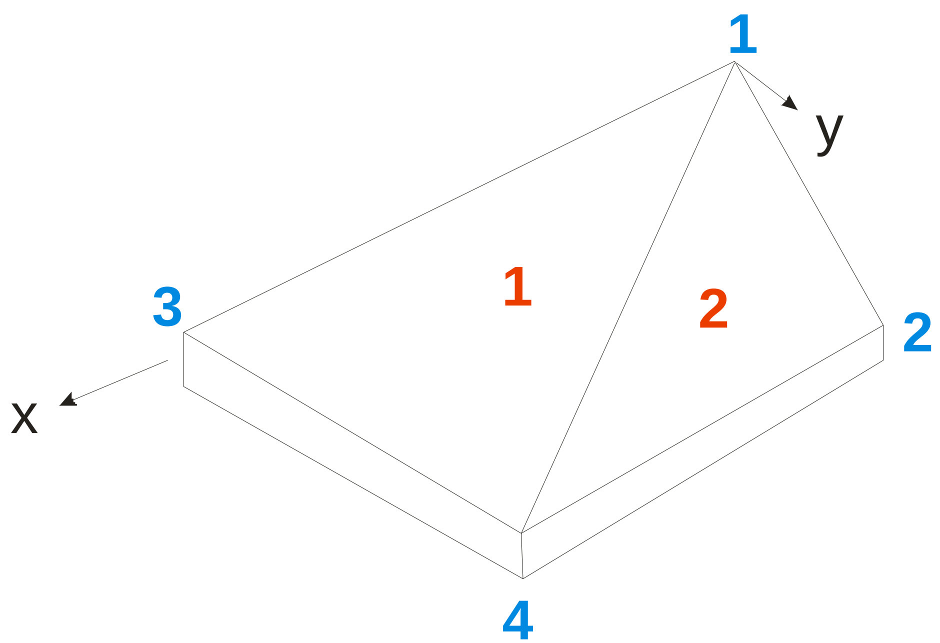
Thin Fin Problem

Problem Data

$$k = 0.20\, W / mm -^{\circ} C$$

$$h = 10^{-5}\, W / mm^{2} -^{\circ} C$$

Example



Element 1: 1-3-4

Element 2: 2-1-4

Problem Data

$$\alpha_x = k_x t$$

$$\alpha_y = k_y t$$

$$\beta = -2h$$

$$f = 2hT_\infty + \hat{Q}$$

$$g = 0$$

$$c = q_0$$

Example

Element 1

$$\mathbf{k}_{3\times3} = \begin{array}{c} \begin{bmatrix} 0.14203 & -0.04194 & -0.03608 \\ & 0.20453 & -0.09858 \\ SYM & & 0.19867 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 3 \\ 4 \end{array}$$

$$\mathbf{f}_{3\times1} = \frac{6(10^{-4})(9600)}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1.92 \\ 1.92 \\ 1.92 \end{Bmatrix} \begin{array}{c} 1 \\ 3 \\ 4 \end{array}$$

Example

Element 2

$$\mathbf{k}_{3\times 3} = \begin{array}{c} \begin{bmatrix} 0.47207 & -0.10858 & -0.339498 \\ & 0.07450 & -0.05808 \\ SYM & & 0.30540 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 1 \\ 4 \end{array}$$

$$\mathbf{f}_{3\times 1} = \frac{6(10^{-4})(3600)}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.72 \\ 0.72 \\ 0.72 \end{Bmatrix} \begin{array}{c} 2 \\ 1 \\ 4 \end{array}$$

Example

System Equations Before BCs

$$\begin{bmatrix} 0.21653 & -0.10858 & -0.04194 & 0.02200 \\ & 0.47207 & 0 & -0.33949 \\ & & 0.20453 & -0.09858 \\ SYM & & & 0.50407 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 2.64 \\ 0.72 \\ 1.92 \\ 2.64 \end{Bmatrix}$$

Example

EBCs $T_1 = 330^\circ C$, $T_3 = 250^\circ C$

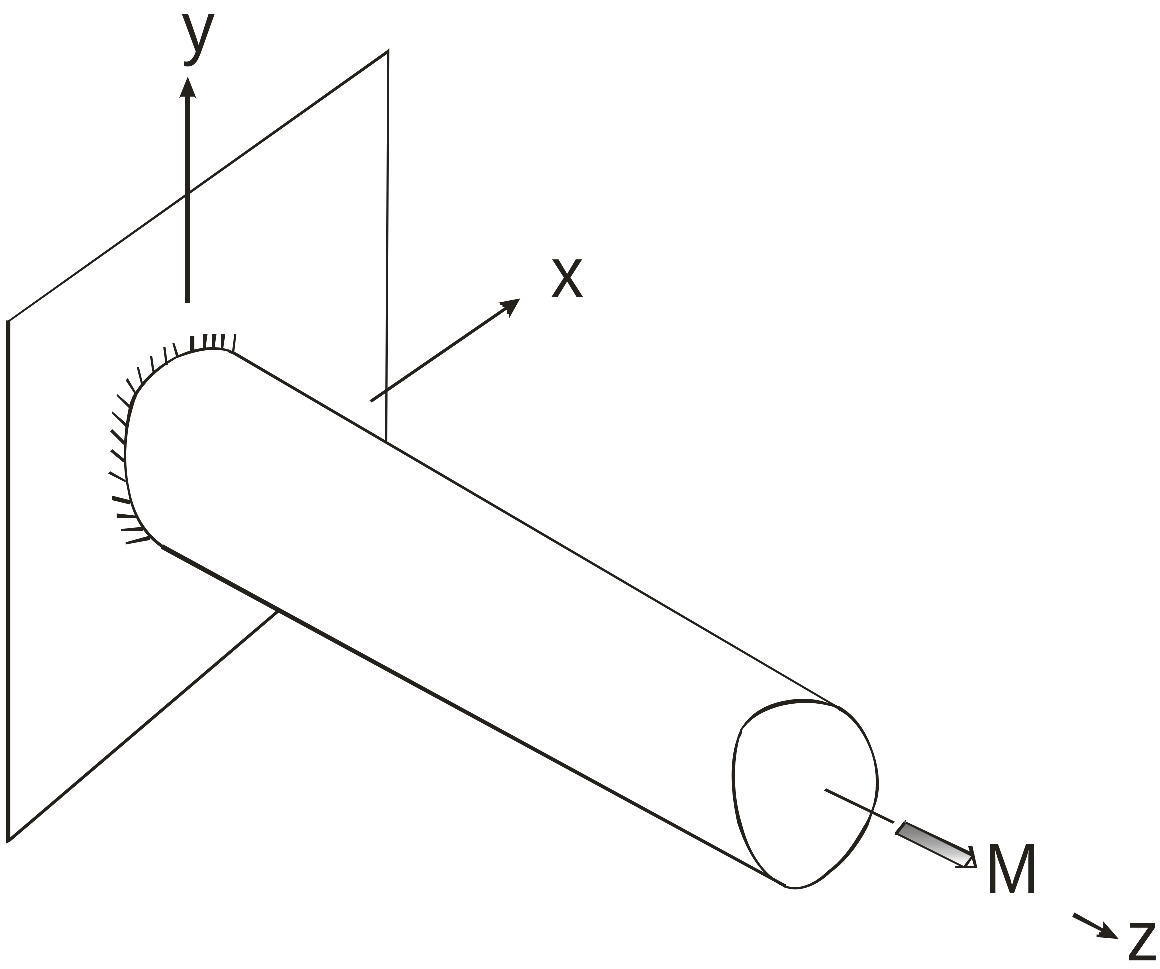
Solving $T_2 = 205.6^\circ C$, $T_4 = 178.2^\circ C$

Element Flux: Element 1

$$q_x = -k_x \frac{\partial T}{\partial x} = -\frac{k_x}{2A}\left(y_{23}(T_1 - T_3) - y_{13}(T_2 - T_3)\right) = 0.1 \frac{W}{mm^2}$$

$$q_y = -k_y \frac{\partial T}{\partial y} = -\frac{k_y}{2A}\left(-x_{23}(T_1 - T_3) + x_{13}(T_2 - T_3)\right) = 0.1614 \frac{W}{mm^2}$$

Torsion in Bars



Torsion in Bars

DE and BCs

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + 2 = 0$$

Airy's Stress Function

with $\psi = 0$ on the boundary

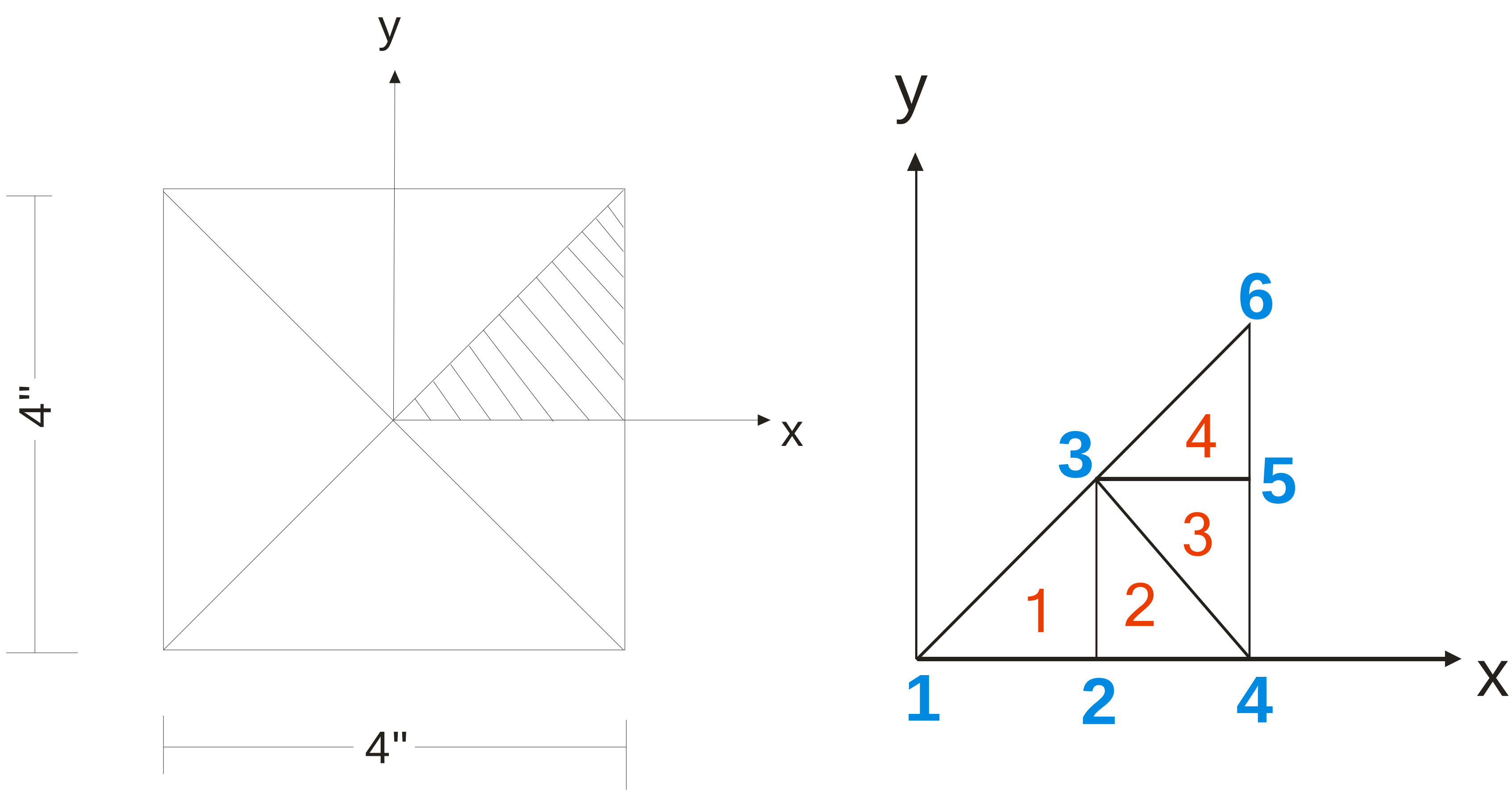
Note that

$$\tau_{xz} = G\alpha \frac{\partial \psi}{\partial y}$$

$$\tau_{yz} = -G\alpha \frac{\partial \psi}{\partial x}$$

$$M = 2G\alpha \iint_A \psi \, dA$$

Example



Example

Element 1

$$\mathbf{k}_{3\times 3} = \begin{array}{c} \begin{bmatrix} 0.5 & -0.5 & 0 \\ & 1 & -0.5 \\ SYM & & 0.5 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$\mathbf{f}_{3\times 1} = \frac{(2)(0.5)}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Example

System Equations Before BCs

$$\begin{bmatrix} 0.5 & -0.5 & 0 & 0 & 0 & \\ & 2 & -1 & -0.5 & 0 & 0 \\ & & 2 & 0 & -1 & 0 \\ & & & 1 & -0.5 & 0 \\ & & & & 2 & -0.5 \\ SYM & & & & & 0.5 \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{Bmatrix} = \begin{Bmatrix} 1/3 \\ 2/3 \\ 4/3 \\ 2/3 \\ 2/3 \\ 1/3 \end{Bmatrix}$$

Example

System Equations After BCs

$$\begin{bmatrix} 0.5 & -0.5 & 0 \\ & 2 & -1 \\ SYM & & 2 \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix} = \begin{Bmatrix} 1/3 \\ 2/3 \\ 4/3 \end{Bmatrix}$$

Solution

$$\psi_1 = 2.33 \quad in^2/rad$$

$$\psi_2 = 1.67 \quad in^2/rad$$

$$\psi_3 = 1.50 \quad in^2/rad$$

Example

Element Flux

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^1 = \begin{Bmatrix} -417 \\ 1667 \end{Bmatrix} psi \qquad \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^2 = \begin{Bmatrix} -417 \\ 4167 \end{Bmatrix} psi$$

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^3 = \begin{Bmatrix} 0 \\ 3750 \end{Bmatrix} psi \qquad \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^4 = \begin{Bmatrix} 0 \\ 3750 \end{Bmatrix} psi$$

Example

Torque

$$M = 2G\alpha \iint_A \psi \, dA = \sum_{i=1}^{4} \left[2G\alpha \iint_A \left(\phi_1 \psi_1 + \phi_2 \psi_2 + \phi_3 \psi_3 \right) dA \right] = 9722.5 \, in - lb$$

Applied Torque = 8(9722.5) = 77,778 in-lb

Theoretical Results

$$\tau_{\max} = 6{,}780 \, psi$$

$$M = 90{,}140 \, in - lb$$

Example

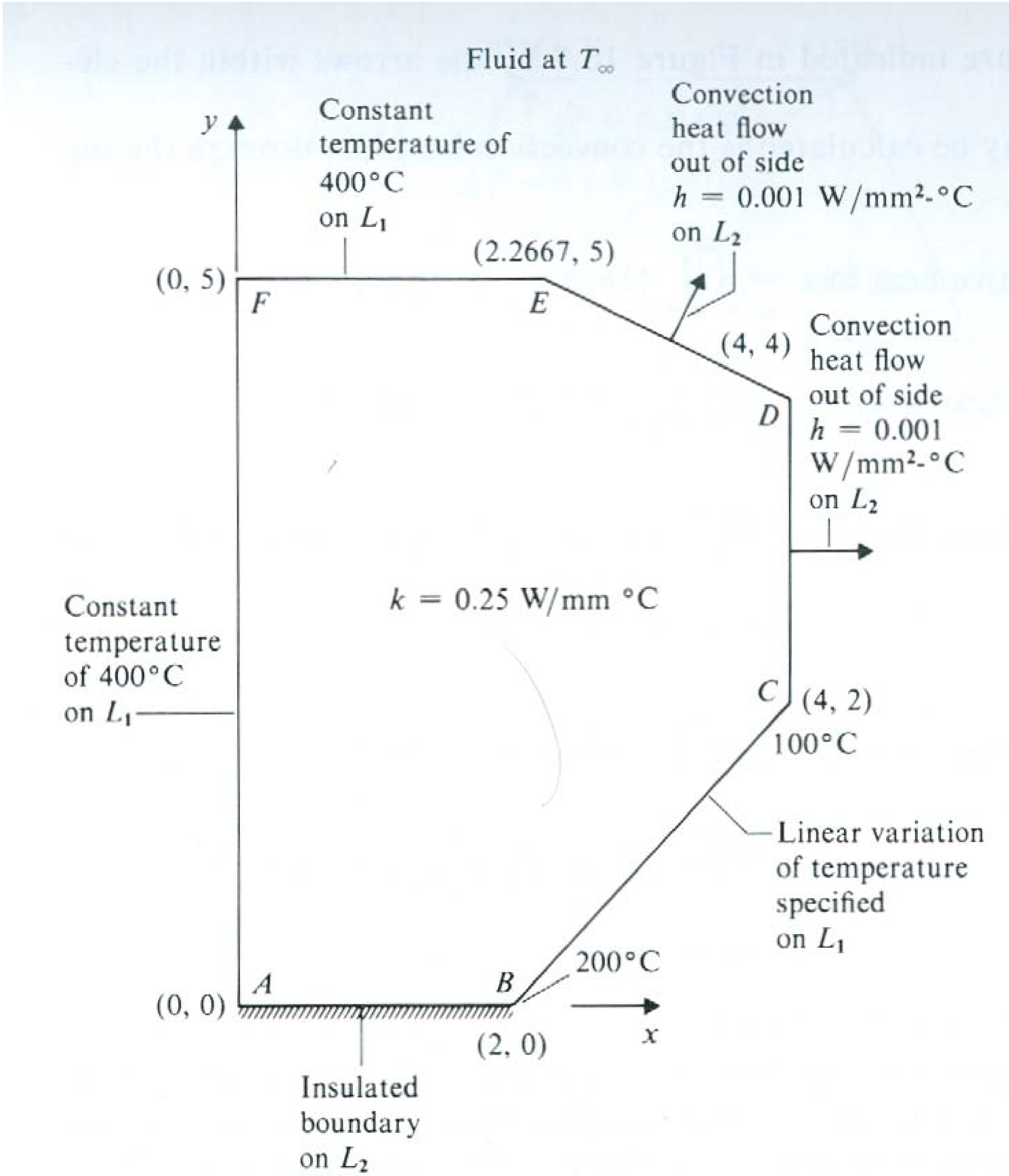
ID	Number of Elements	$\tau_{\max}(psi)$
Mesh A	8	4168
Mesh B	32	5405
Mesh C	72	5895
Mesh D	288	6350
Mesh E	392	6410

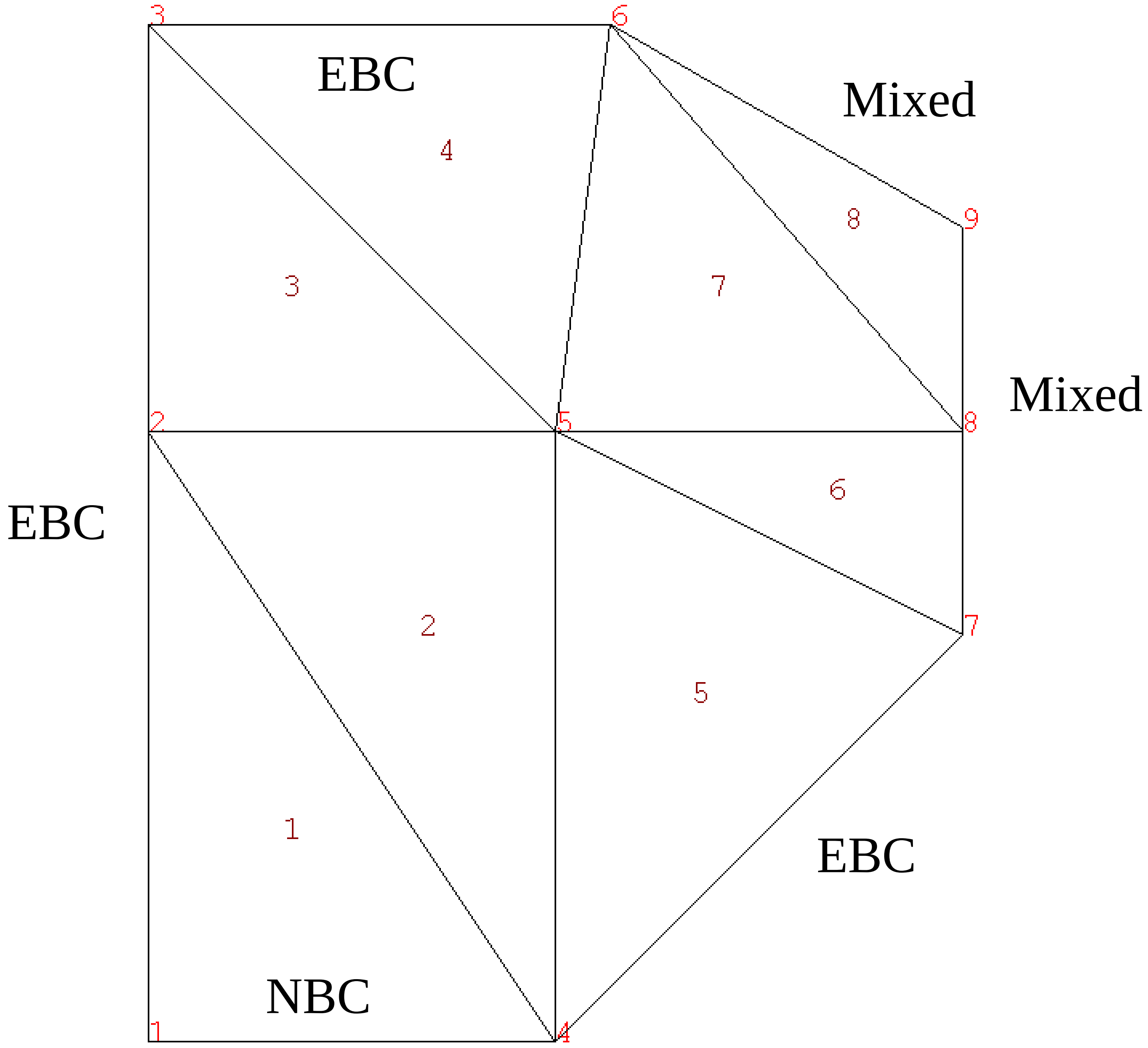
Summary

- Creating BVP element equations is the similar to elasticity problems
  - Same shape functions
  - Isoparametric formulation
  - Numerical integration
- Scalar unknowns
- System equations are still symmetric and positive definite

Numerical Examples

Long Body Example

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6
EBC
Mixed
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Mixed
2
5
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2
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1
EBC
NBC
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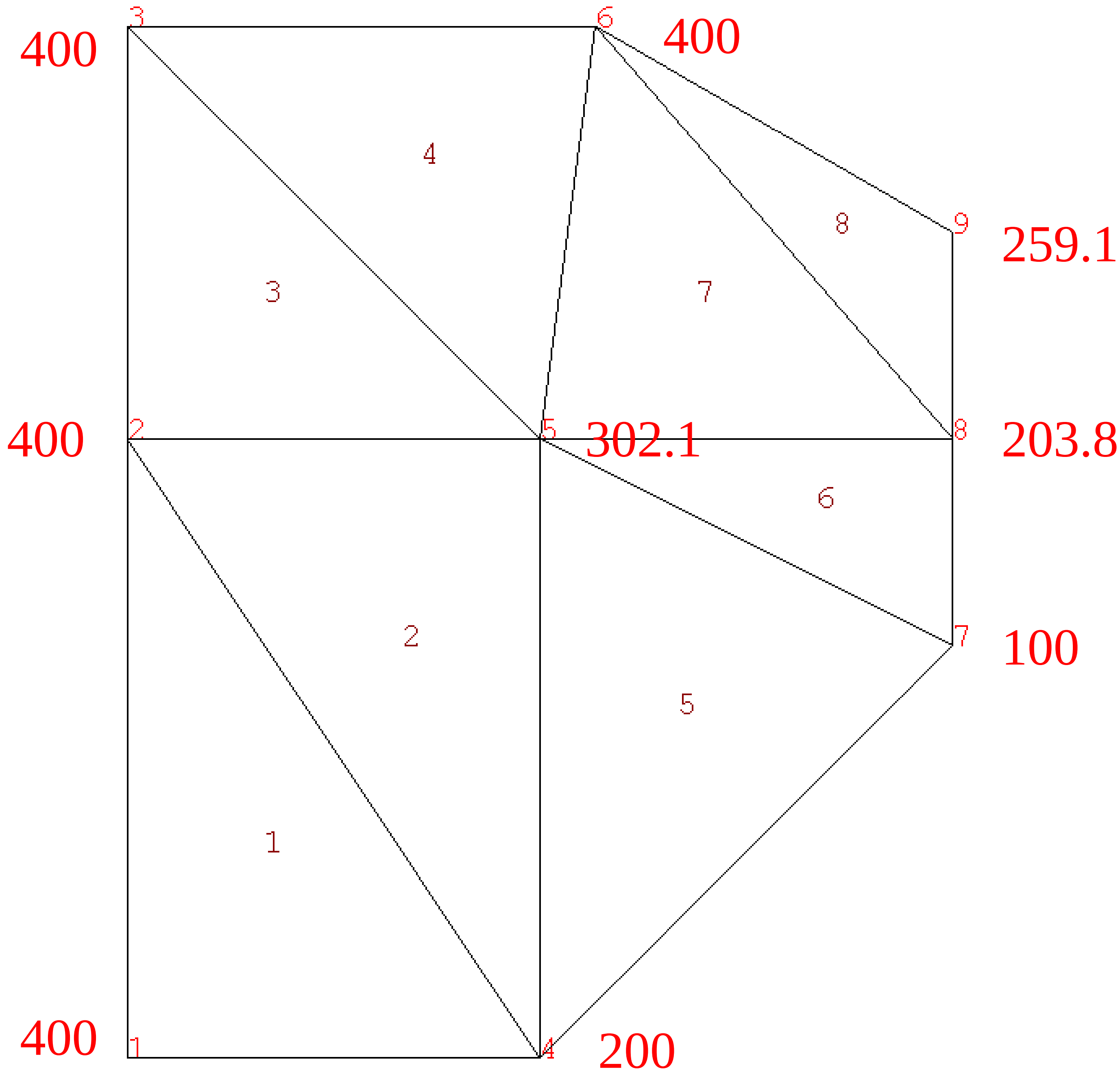

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259.1
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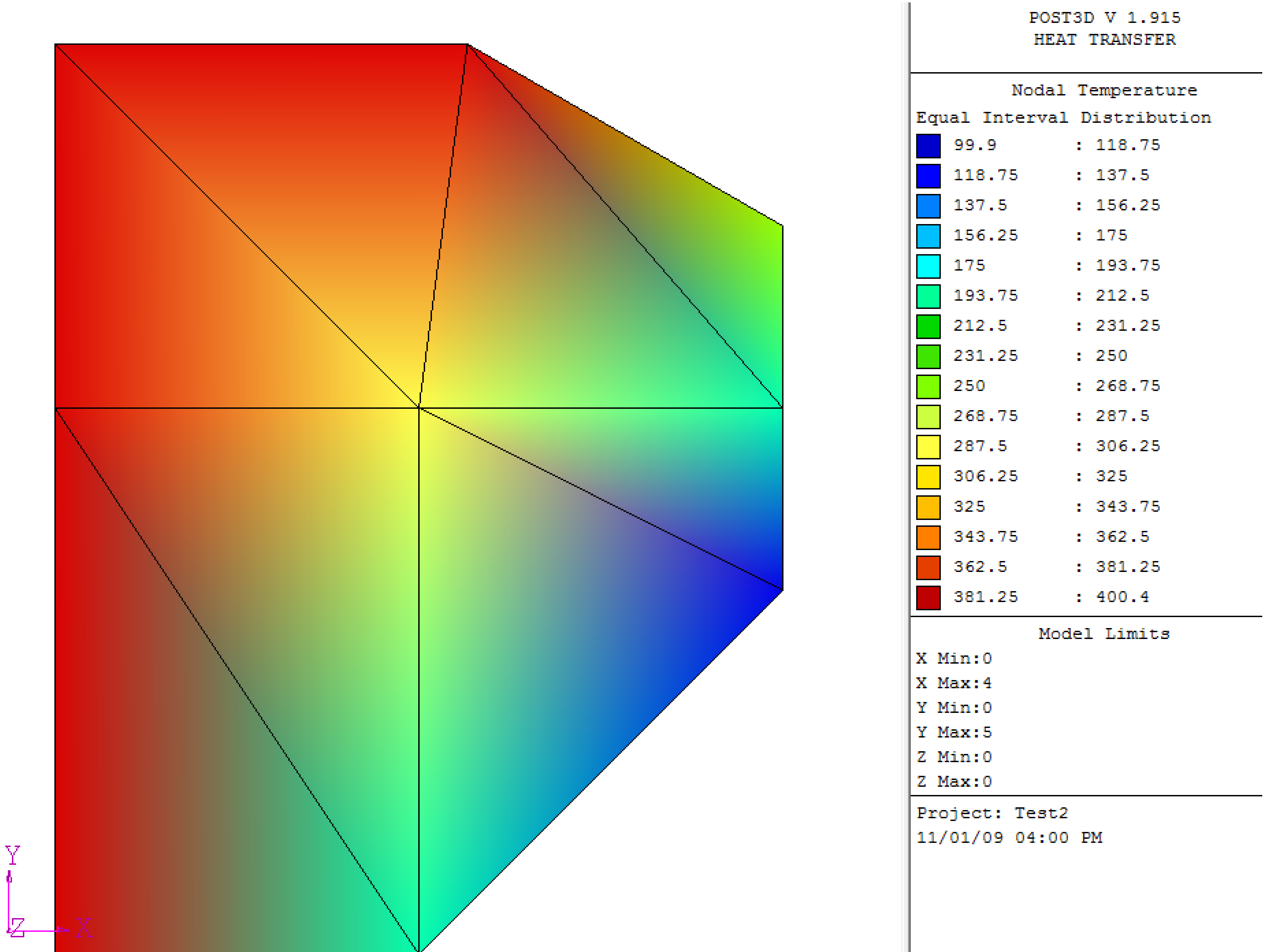
200


Y
Z
X
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HEAT TRANSFER
Nodal Temperature
Equal Interval Distribution
99.9 : 118.75
118.75 : 137.5
137.5 : 156.25
156.25 : 175
175 : 193.75
193.75 : 212.5
212.5 : 231.25
231.25 : 250
250 : 268.75
268.75 : 287.5
287.5 : 306.25
306.25 : 325
325 : 343.75
343.75 : 362.5
362.5 : 381.25
381.25 : 400.4
Model Limits
X Min:0
X Max:4
Y Min:0
Y Max:5
Z Min:0
Z Max:0
Project: Test2
11/01/09 04:00 PM

Example

10.1. Consider a brick wall (Fig. P10.1) of thickness $L = 30$ cm, $k = 0.7$ W/m · °C. The inner surface is at 28°C and the outer surface is exposed to cold air at −15°C. The heat-transfer coefficient associated with the outside surface is $h = 40$ W/m^2 · °C. Determine the steady-state temperature distribution within the wall and also the heat flux through the wall. Use a two-element model, and obtain the solution by hand calculations. Assume one-dimensional flow. Then prepare input data and run program HEAT1D.

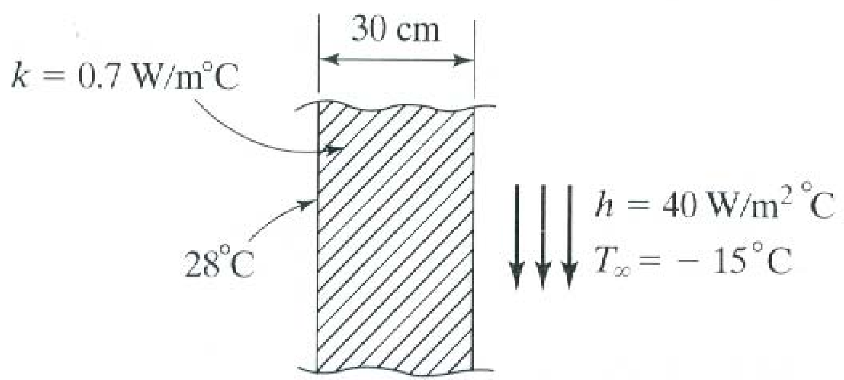
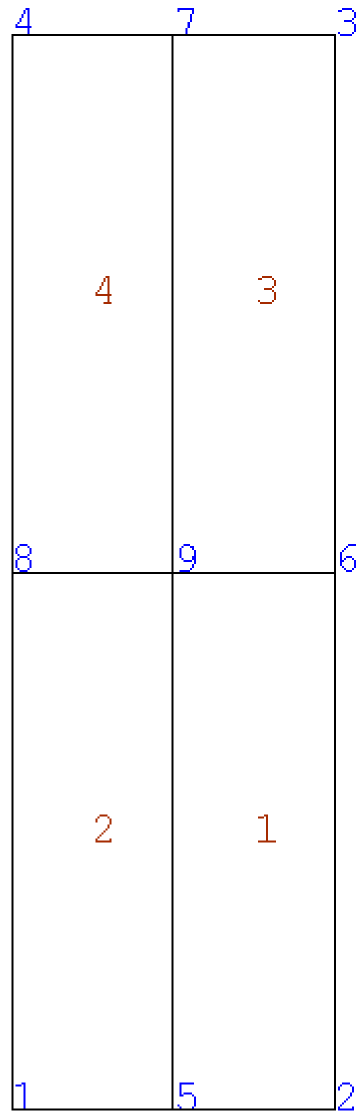
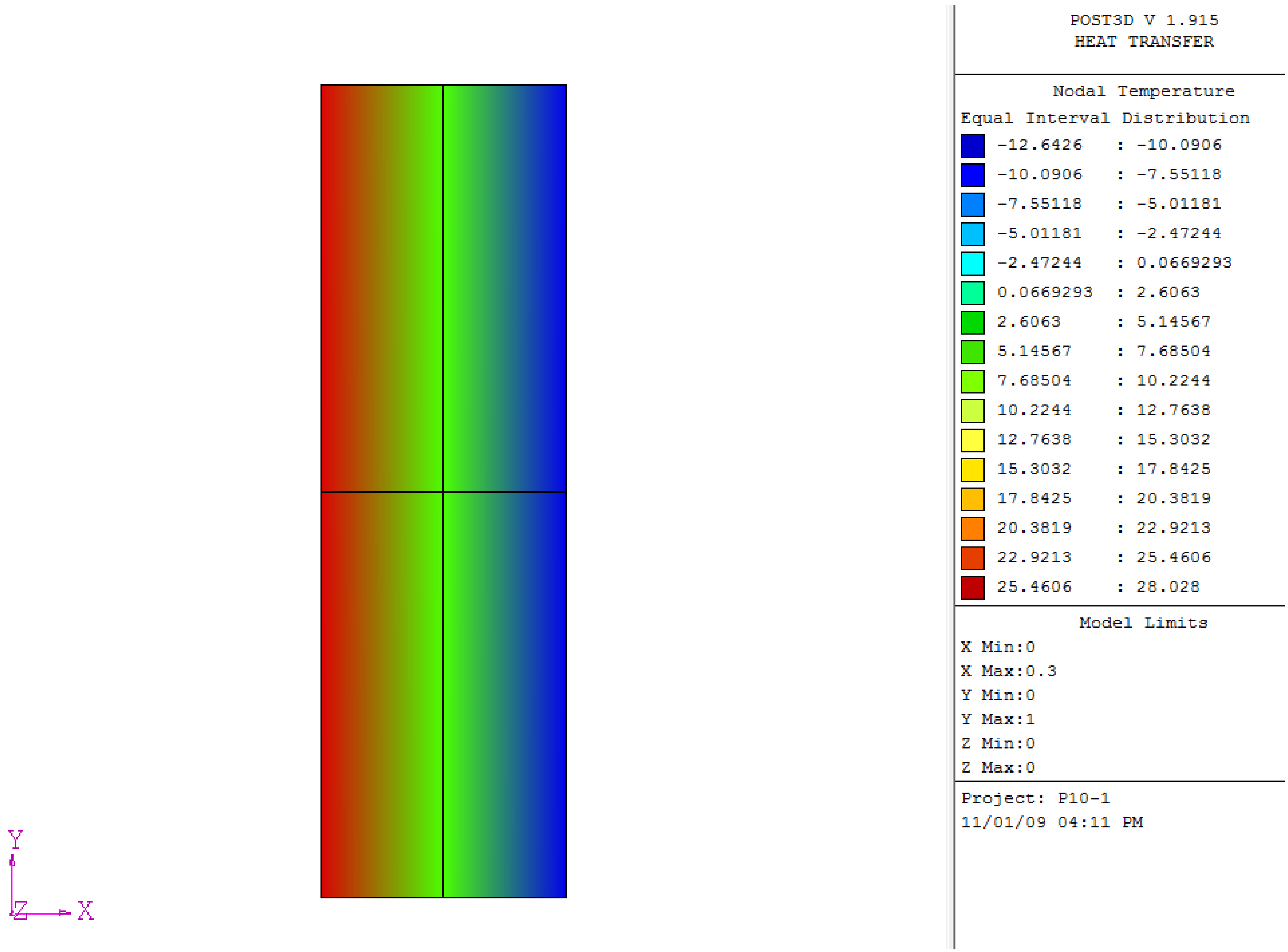
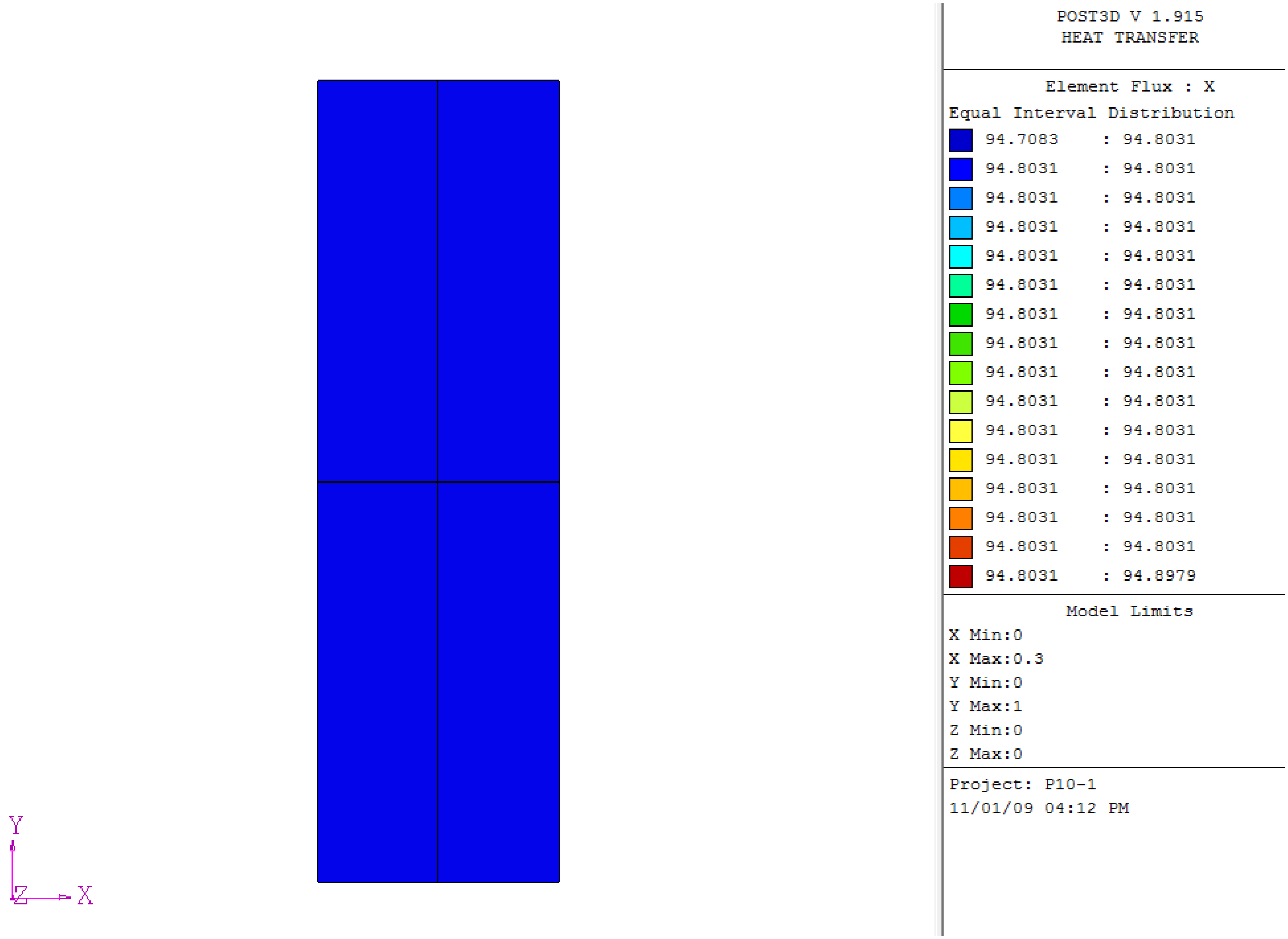
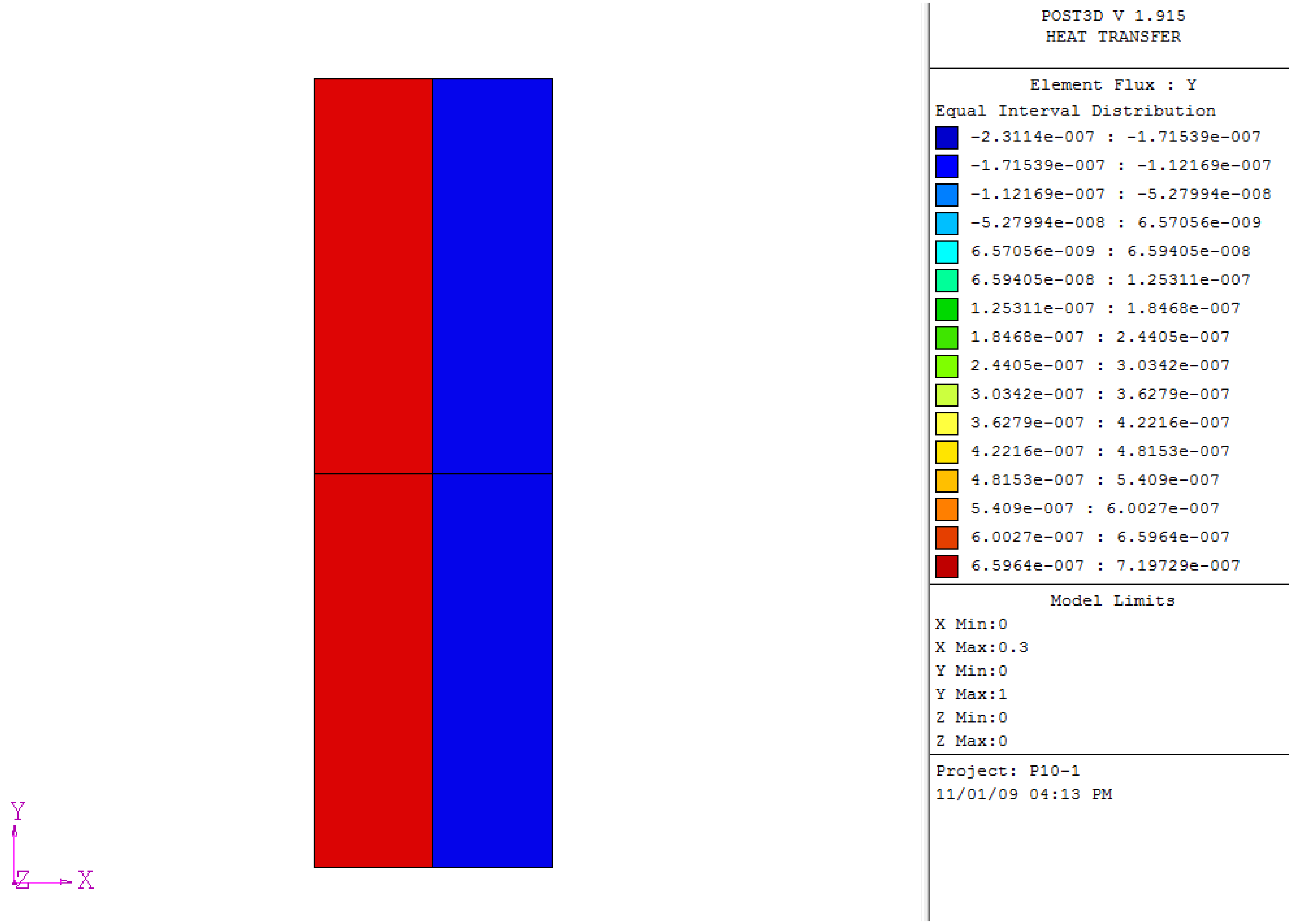


FIGURE P10.1


4
7
3
4
3
8
9
6
2
1
1
5
2


POST3D V 1.915
HEAT TRANSFER
Nodal Temperature
Equal Interval Distribution
-12.6426 : -10.0906
-10.0906 : -7.55118
-7.55118 : -5.01181
-5.01181 : -2.47244
-2.47244 : 0.0669293
0.0669293 : 2.6063
2.6063 : 5.14567
5.14567 : 7.68504
7.68504 : 10.2244
10.2244 : 12.7638
12.7638 : 15.3032
15.3032 : 17.8425
17.8425 : 20.3819
20.3819 : 22.9213
22.9213 : 25.4606
25.4606 : 28.028
Model Limits
X Min:0
X Max:0.3
Y Min:0
Y Max:1
Z Min:0
Z Max:0
Project: P10-1
11/01/09 04:11 PM
Y
Z
X


POST3D V 1.915
HEAT TRANSFER
Element Flux : X
Equal Interval Distribution
94.7083 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8031
94.8031 : 94.8979
Model Limits
X Min:0
X Max:0.3
Y Min:0
Y Max:1
Z Min:0
Z Max:0
Project: P10-1
11/01/09 04:12 PM
Y
X
Z


POST3D V 1.915
HEAT TRANSFER
Element Flux : Y
Equal Interval Distribution
-2.3114e-007 : -1.71539e-007
-1.71539e-007 : -1.12169e-007
-1.12169e-007 : -5.27994e-008
-5.27994e-008 : 6.57056e-009
6.57056e-009 : 6.59405e-008
6.59405e-008 : 1.25311e-007
1.25311e-007 : 1.8468e-007
1.8468e-007 : 2.4405e-007
2.4405e-007 : 3.0342e-007
3.0342e-007 : 3.6279e-007
3.6279e-007 : 4.2216e-007
4.2216e-007 : 4.8153e-007
4.8153e-007 : 5.409e-007
5.409e-007 : 6.0027e-007
6.0027e-007 : 6.5964e-007
6.5964e-007 : 7.19729e-007
Model Limits
X Min:0
X Max:0.3
Y Min:0
Y Max:1
Z Min:0
Z Max:0
Project: P10-1
11/01/09 04:13 PM
Y
Z
X

Problem 10.10

10.10. A long steel tube (Fig. P10.10a) with inner radius $r_1 = 3$ cm and outer radius $r_1 = 5$ cm and $k = 20$ W/m · °C has its inner surface heated at a rate $q_0 = -100\ 000$ W/m^2. (The minus sign indicates that heat flows into the body.) Heat is dissipated by convection from the outer surface into a fluid at temperature $T_\infty = 120$°C and $h = 400$ W/m^2 · °C. Considering the eight-element, nine-node finite element model shown in Fig. P10.6b, determine the following:

(a) The boundary conditions for the model.

(b) The temperatures T_1, T_2 at the inner and outer surfaces, respectively. Use program HEAT2D.

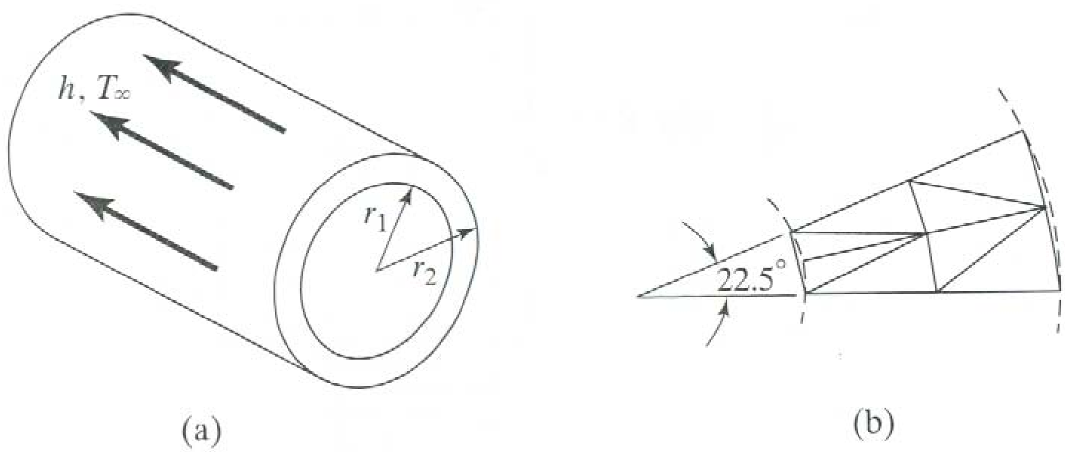


FIGURE P10.10

Problem 10.14

10.14. A large industrial furnace is supported on a long column of fireclay brick, which is 1×1 m on a side (Fig. P10.11). During steady-state operation, installation is such that three surfaces of the column are maintained at 600 °K while the remaining surface is exposed to an airstream for which $T_\infty = 300$ °K and $h = 12\ \mathrm{W/m^2 \cdot {}^\circ K}$. Determine, using program HEAT2D, the temperature distribution in the column and the heat rate to the airstream per unit length of column. Take $k = 1\ \mathrm{W/m \cdot {}^\circ K}$.

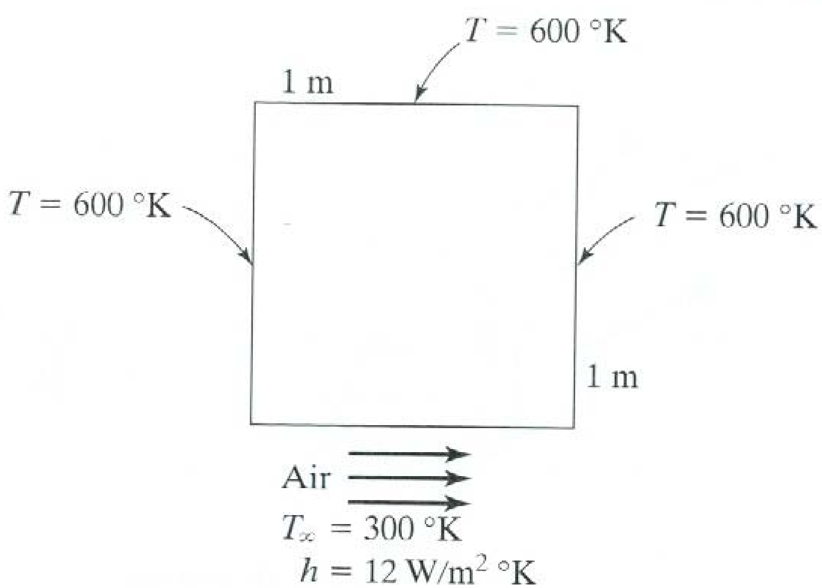
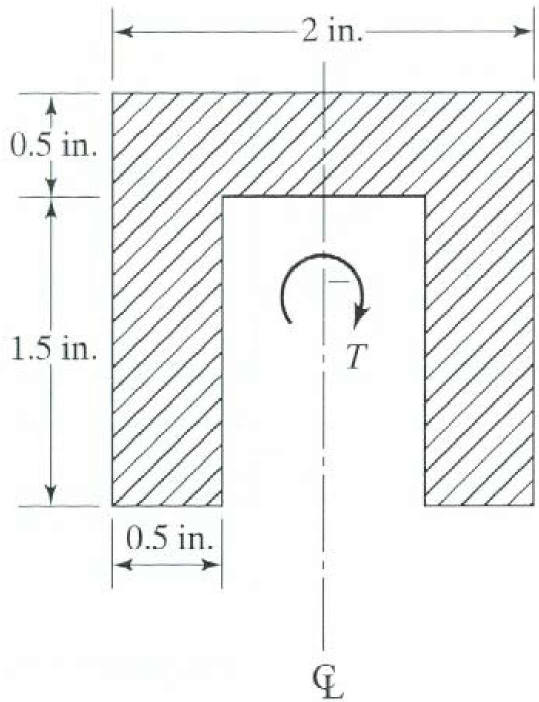


FIGURE P10.14

Problem 10.19

10.19. The cross section of the steel beam in Fig. P10.19 is subjected to a torque $T = 5000$ in/lb. Determine, using program TORSION, the angle of twist and the location and magnitude of the maximum shearing stresses.



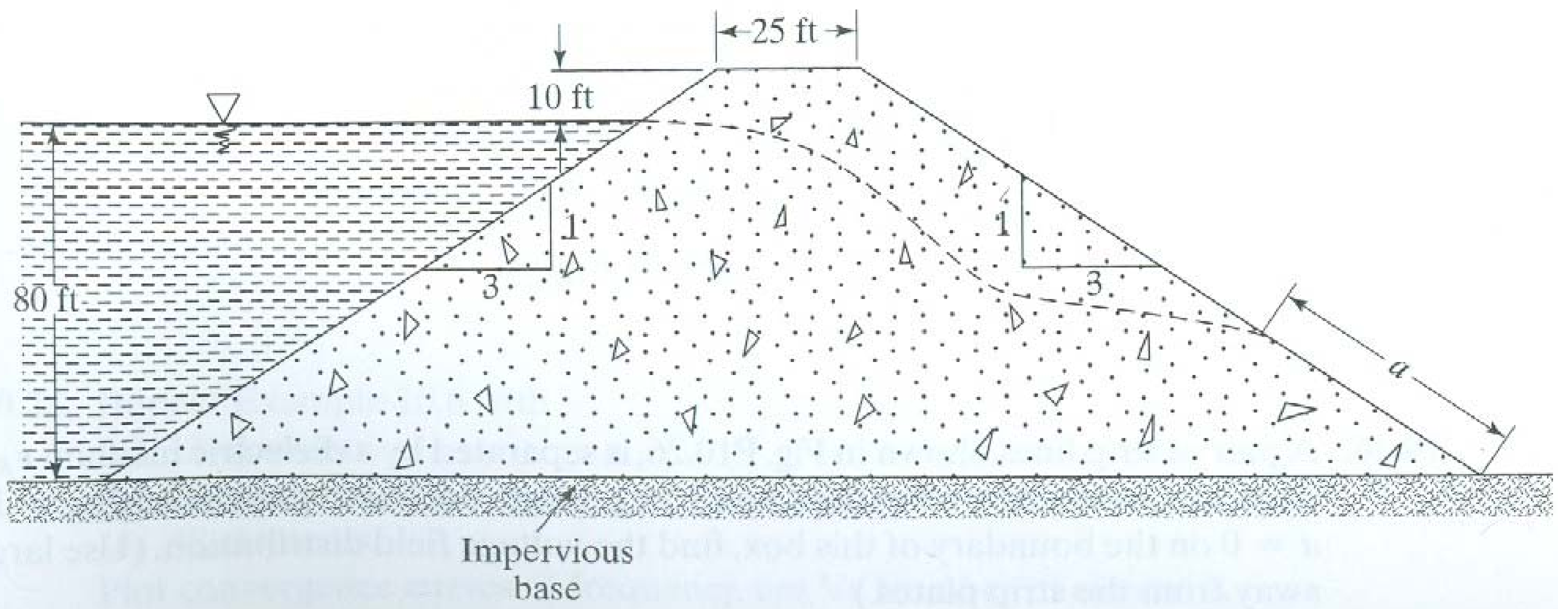
Problem 10.23

10.23. For the dam section shown in Fig. P10.23, $k = 0.003$ ft/min. Determine the following:

(a) The line of seepage.

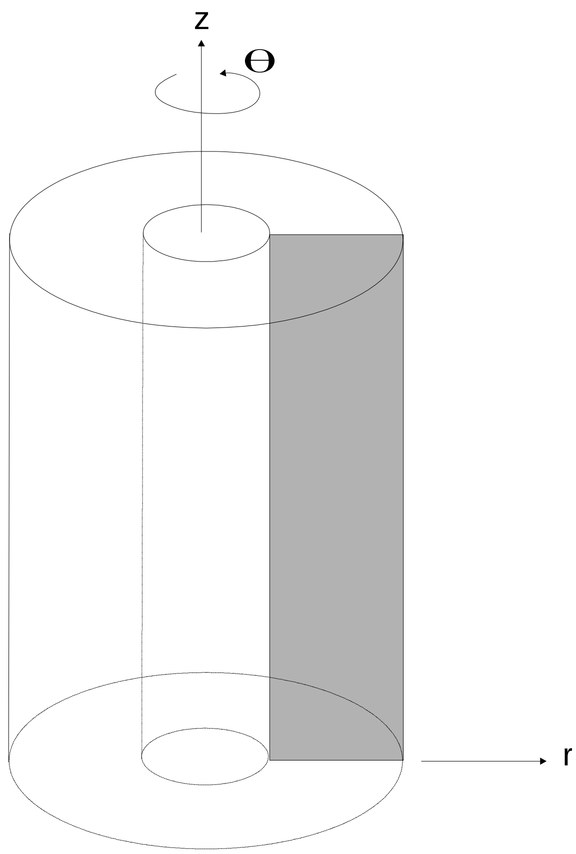
(b) The quantity of seepage per 100-ft length of the dam.

(c) The length of the surface of seepage a.



Axisymmetric Problems

Axisymmetric Problem



- Entire FE model (geometry, properties, boundary conditions) are functions of **r** and **z**.
- Cylindrical coordinate system (r, z, θ)
- **r** is the radial direction
- **z** is the axial direction

Axisymmetric BVP

DE

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha_{rr}(r,z)\frac{\partial u(r,z)}{\partial r}\right)+\frac{\partial}{\partial z}\left(\alpha_{zz}(r,z)\frac{\partial u(r,z)}{\partial z}\right)+\beta u(r,z)+f(r,z)=0$$

BCs

$$u\left(\hat{r},\hat{z}\right)=\hat{u} \quad on\ \Gamma_1$$

$$\alpha_{rr}\frac{\partial u}{\partial r}n_r+\alpha_{zz}\frac{\partial u}{\partial z}n_z+gu+c=0 \quad on\ \Gamma_2$$

Axisymmetric BVP

Galerkin Step 1: Residual Equations

$$\tilde{u}(r,z)=\sum_{j=1}^{n}\phi_j(r,z)\,u_j$$

For i=1,2,…,n

$$2\pi\iint_{\Omega} R(r,z,u)\phi_i(r,z)\,rdrdz=0$$

$$2\pi\iint_{\Omega}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha_{rr}\frac{\partial u)}{\partial r}\right)+\frac{\partial}{\partial z}\left(\alpha_{zz}\frac{\partial u}{\partial z}\right)+\beta u+f\right]\phi_i(r,z)rdrdz=0$$

$$2\pi\iint_{\Omega}[\frac{\partial}{\partial r}\left(r\alpha_{rr}\frac{\partial u)}{\partial r}\right)+\frac{\partial}{\partial z}\left(r\alpha_{zz}\frac{\partial u}{\partial z}\right)+\beta\, r\, u+fr]\phi_i(r,z)drdz=0$$

Axisymmetric BVP

Chain Rule of Differentiation

$$\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\right)\phi_i = \frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\phi_i\right) - \left(\alpha_x \frac{\partial u}{\partial x}\right)\frac{\partial \phi_i}{\partial x}$$

$$\frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\right)\phi_i = \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\phi_i\right) - \left(\alpha_y \frac{\partial u}{\partial y}\right)\frac{\partial \phi_i}{\partial y}$$

Axisymmetric BVP

Divergence Theorem

$$F = F(x, y)$$

$$G = G(x, y)$$

$$\iint_{\Omega} \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \right) dxdy = \oint_{\Gamma} \left(Fn_x + Gn_y \right) dS$$

Axisymmetric BVP

Galerkin Step 2: Integration by parts

$$\left(\iint_{\Omega} \left\{ \left(r\alpha_{rr} \frac{\partial u}{\partial r} \right) \frac{\partial \phi_i}{\partial r} + \left(r\alpha_{zz} \frac{\partial u}{\partial z} \right) \frac{\partial \phi_i}{\partial z} \right\} dx\, dy - \iint_{\Omega} \left\{ \beta r u \phi_i \right\} drdz \right)$$

$$= \iint_{\Omega} f\; r\phi_i drdz + \oint_{\Gamma} \left(r\alpha_{rr} \frac{\partial u}{\partial r} n_r + r\alpha_{zz} \frac{\partial u}{\partial z} n_z \right) \phi_i dS \qquad i = 1, 2, .., n$$

Axisymmetric BVP

Galerkin Step 3: Use of trial solution

$$\sum_{j=1}^{n}\left(\iint_{\Omega}\left\{r\frac{\partial\phi_i}{\partial r}\alpha_{rr}\frac{\partial\phi_j}{\partial r}+r\frac{\partial\phi_i}{\partial z}\alpha_{zz}\frac{\partial\phi_j}{\partial z}-r\phi_i\beta\phi_j\right\}dxdy+\oint_{\Gamma}r\phi_i g\phi_j dS\right)u_j=$$

$$\iint_{\Omega}rf\,\phi_i\,drdz-\oint_{\Gamma}rc\phi_i dS \qquad i=1,2,\ldots,n$$

$$\left[\mathbf{k}^{\alpha}_{n\times n}+\mathbf{k}^{\beta}_{n\times n}+\mathbf{k}^{g}_{n\times n}\right]\mathbf{u}_{n\times 1}=\mathbf{f}^{\text{int}}_{n\times 1}+\mathbf{f}^{bnd}_{n\times 1}$$

Axisymmetric BVP

Summary

$$k_{ij}^{\alpha} = \iint_{\Omega} \left\{ r \frac{\partial \phi_i}{\partial r} \alpha_{rr} \frac{\partial \phi_j}{\partial r} + r \frac{\partial \phi_i}{\partial z} \alpha_{zz} \frac{\partial \phi_j}{\partial z} \right\} drdz$$

$$k_{ij}^{\beta} = -\iint_{\Omega} r\phi_i \beta \phi_j \, drdz$$

$$k_{ij}^{g} = \oint_{\Gamma} r\phi_i g \phi_j \, dS$$

$$f_i^{\text{int}} = \iint_{\Omega} f \; r\phi_i drdz$$

$$f_i^{bnd} = -\oint_{\Gamma} \left(rc\phi_i \, dS \right)$$

Linear Triangular Element

Example

$$k_{ij}^{\alpha} = \iint_{\Omega} \left[r\left(\frac{b_i}{2A}\right)\alpha_{rr}\left(\frac{b_j}{2A}\right) + r\left(\frac{c_i}{2A}\right)\alpha_{zz}\left(\frac{c_j}{2A}\right) \right] drdz$$

$$= \frac{b_i b_j \hat{\alpha}_{rr}}{4A^2} \iint_{\Omega} rdrdz + \frac{c_i c_j \hat{\alpha}_{zz}}{4A^2} \iint_{\Omega} rdrdz$$

Note

$$r = \sum_{j=1}^{3} \phi_j r_j$$

$$\iint_{\Omega} r\, drdz = \iint_{\Omega} \left(r_1\phi_1 + r_2\phi_2 + r_3\phi_3 \right) drdz = \frac{A}{3}\left(r_1 + r_2 + r_3 \right)$$

Linear Triangular Element

Sample Terms

$$\bar{r} = \frac{1}{3}\left(r_1 + r_2 + r_3\right)$$

$$k_{ij}^{\alpha} = \frac{b_i b_j \hat{\alpha}_{rr} + c_i c_j \hat{\alpha}_{zz}}{4A}\bar{r}$$

$$f_1^{\text{int}} = \iint_{\Omega} rf\,\phi_1\,dxdy = \frac{\hat{f}A}{12}\left(2r_1 + r_2 + r_3\right)$$

$$k_{12}^{\beta} = -\iint_{\Omega} r\phi_1\beta\phi_2\,dxdy = -\frac{\bar{r}A\hat{\beta}}{12}$$

$$k_{12}^{g} = \int_1^2 r\phi_1\hat{g}\phi_2\,dS = \frac{\hat{g}L_{12}}{12}(r_1 + r_2)$$

Linear Triangular Element

$$\mathbf{k}_{3\times3}^{\alpha} = \frac{2\pi\hat{\alpha}_{rr}\bar{r}}{4A}\begin{bmatrix} b_1^2 & b_1b_2 & b_1b_3 \\ & b_2^2 & b_2b_3 \\ SYM & & b_3^2 \end{bmatrix} + \frac{2\pi\hat{\alpha}_{zz}\bar{r}}{4A}\begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ & c_2^2 & c_2c_3 \\ SYM & & c_3^2 \end{bmatrix}$$

$$\mathbf{k}_{3\times3}^{\beta} = -\frac{2\pi\bar{r}\hat{A}\beta}{12}\begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ SYM & & 2 \end{bmatrix}$$

Linear Triangular Element

$$\mathbf{k}^g_{3\times 3} = \frac{2\pi \hat{g}_{12} L_{12}}{12}\begin{bmatrix} 3r_1 + r_2 & r_1 + r_2 & 0 \\ & r_1 + 3r_2 & 0 \\ SYM & & 0 \end{bmatrix}$$

$$+ \frac{2\pi \hat{g}_{23} L_{23}}{12}\begin{bmatrix} 0 & 0 & 0 \\ & 3r_2 + r_3 & r_2 + r_3 \\ SYM & & r_2 + 3r_3 \end{bmatrix}$$

$$+ \frac{2\pi \hat{g}_{31} L_{31}}{12}\begin{bmatrix} 3r_1 + r_3 & 0 & r_1 + r_3 \\ & 0 & 0 \\ SYM & & r_1 + 3r_3 \end{bmatrix}$$

Linear Triangular Element

$$\mathbf{f}_{3\times1}^{\text{int}} = \frac{2\pi\ \hat{f}A}{12}\begin{Bmatrix} 2r_1 + r_2 + r_3 \\ r_1 + 2r_2 + r_3 \\ r_1 + r_2 + 2r_3 \end{Bmatrix}$$

$$\mathbf{f}_{3\times1}^{bnd} = -\frac{2\pi\, \hat{c}_{12}\, L_{12}}{6}\begin{Bmatrix} 2r_1 + r_2 \\ r_1 + 2r_2 \\ 0 \end{Bmatrix} - \frac{2\pi\, \hat{c}_{23}\, L_{23}}{6}\begin{Bmatrix} 0 \\ 2r_2 + r_3 \\ r_2 + 2r_3 \end{Bmatrix} - \frac{2\pi\, \hat{c}_{13}\, L_{13}}{6}\begin{Bmatrix} 2r_1 + r_3 \\ 0 \\ r_1 + 2r_3 \end{Bmatrix}$$

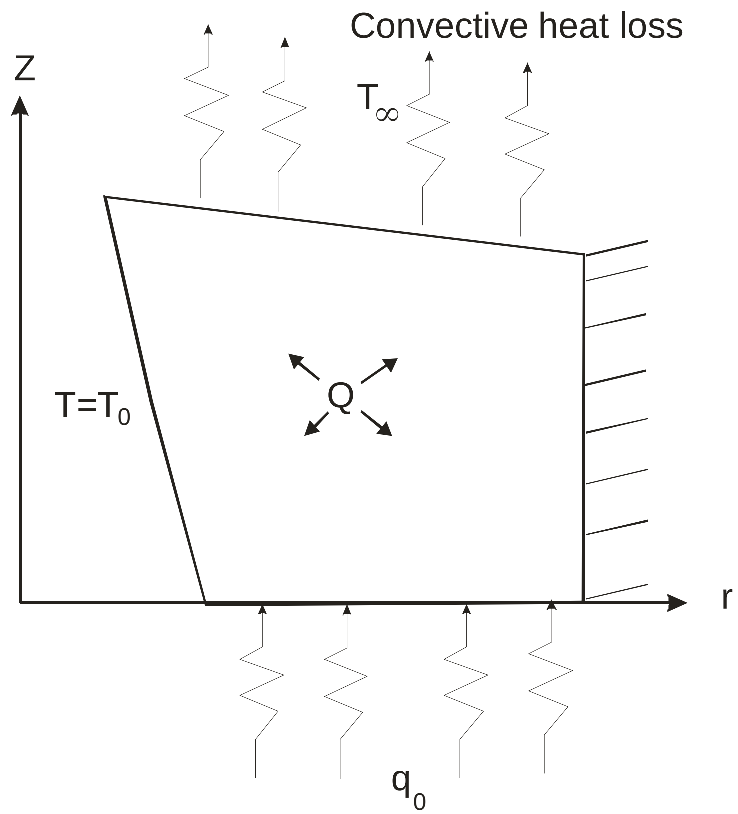
Element Flux

$$\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{\Gamma} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & r_{13} \end{bmatrix} \begin{Bmatrix} u_1 - u_3 \\ u_2 - u_3 \end{Bmatrix}$$

$$\tau_r = -\alpha_r \frac{\partial u}{\partial r} = \frac{-\alpha_r}{2A} \left[z_{23} \left(u_1 - u_3 \right) - z_{13} \left(u_2 - u_3 \right) \right]$$

$$\tau_z = -\alpha_z \frac{\partial u}{\partial z} = \frac{-\alpha_z}{2A} \left[-r_{23} \left(u_1 - u_3 \right) + r_{13} \left(u_2 - u_3 \right) \right]$$

Heat Transfer Problems



BCs

$$T\left(\hat{r},\hat{z}\right)=\hat{T}\ \text{ on } \Gamma_1$$

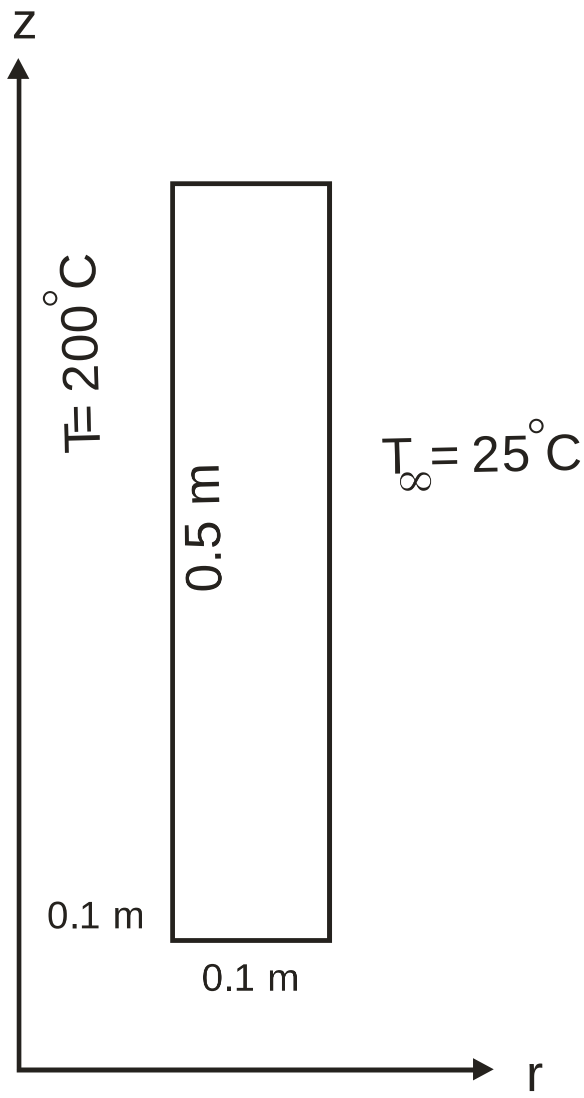
$$\left(k_{rr}\frac{\partial T}{\partial r}n_r+k_{zz}\frac{\partial T}{\partial z}n_z\right)=-q_n\ \text{ on } \Gamma_2$$

$$k_{rr}\frac{\partial T}{\partial r}n_r+k_{zz}\frac{\partial T}{\partial z}n_z+h(T-T_\infty)=0\ \text{on } \Gamma_2$$

DE

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk_{rr}(r,z)\frac{\partial T(r,z)}{\partial r}\right)+\frac{\partial}{\partial z}\left(k_{zz}(r,z)\frac{\partial T(r,z)}{\partial z}\right)+Q(r,z)=0$$

Example



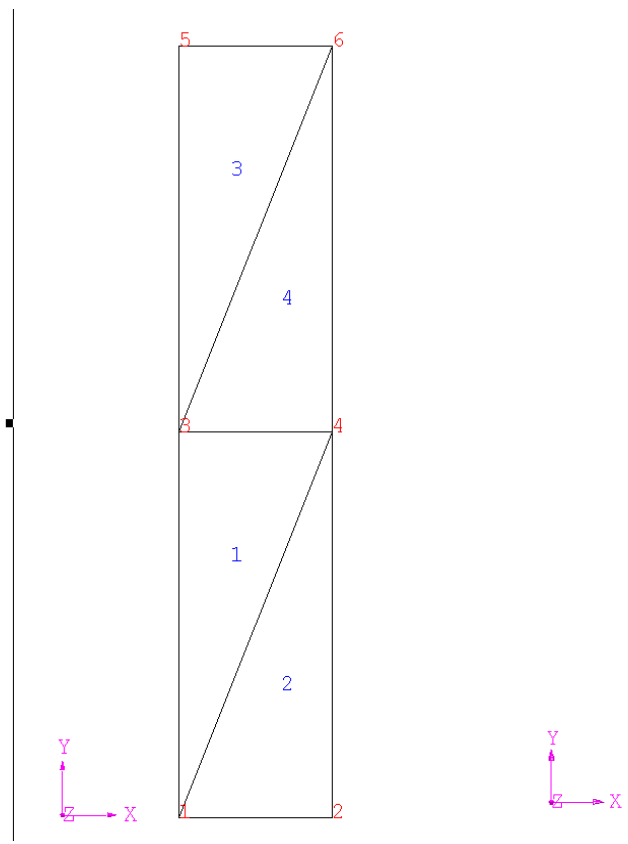
Problem Data

$$k = 30 \frac{W}{m - ^\circ C}$$

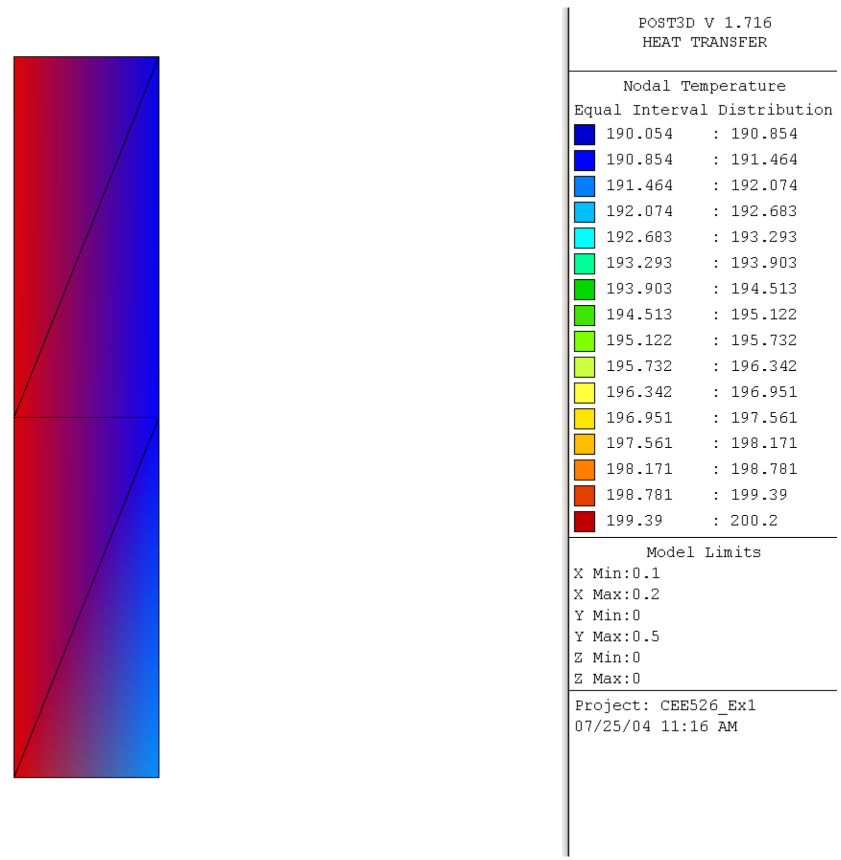
$$h = 12 \frac{W}{m^2 - ^\circ C}$$

Example

FE Mesh

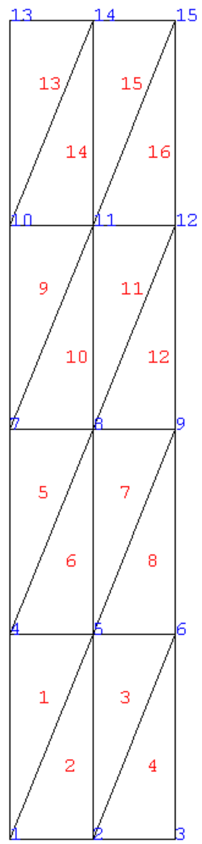


Solution: Temp Distribution

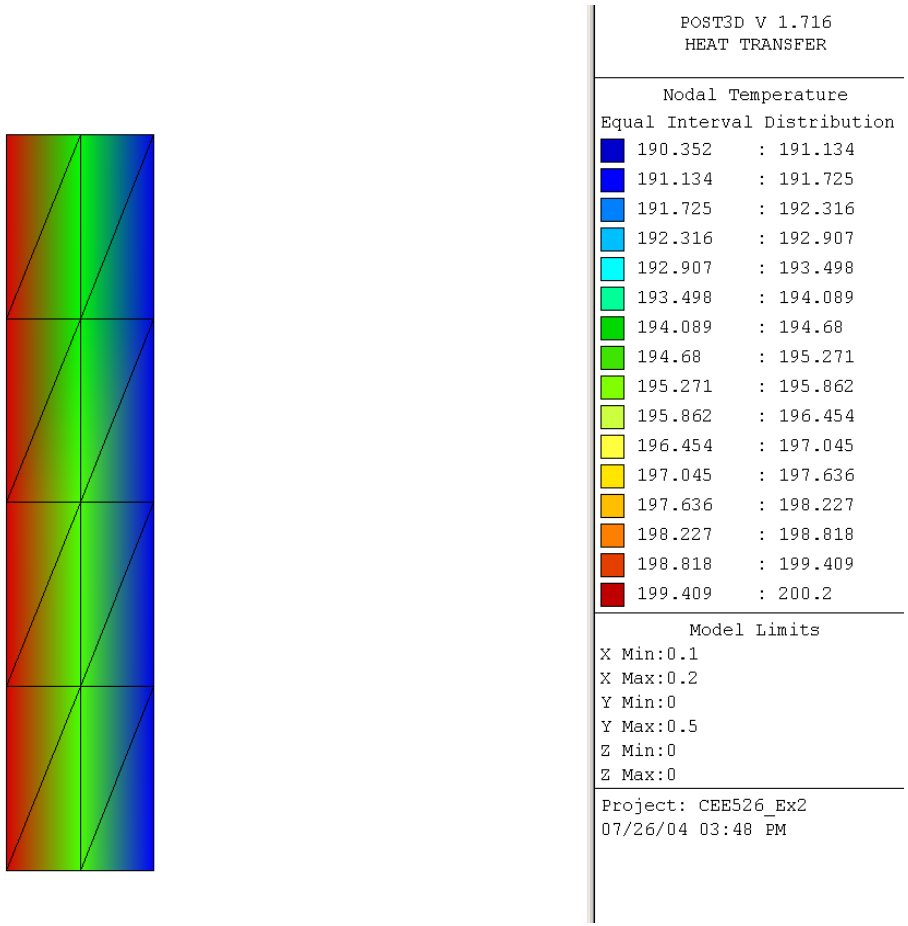


Example

FE Mesh



Solution: Temp Distribution



Example

Mesh ID	**Number of elements**	**Avg. Temp. Outer Wall** $\left({}^{\circ}C\right)$	**Radial Flux Outer Wall** $\left(W/m^2\right)$
Mesh A	4	191.1	2548
Mesh B	16	190.9	2270

Programming Project: Option 2

- What needs to be programmed?
  - Input? Output?
- Theory?
- Algorithm?
- Program organization?
- Debugging?
- Test Cases?
- Documentation?