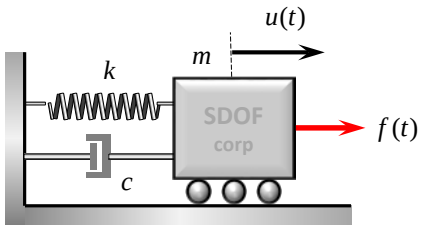
CEE 536—Structural Dynamics

Numerical Methods

ordinary differential equations



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Course Notes

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Contents.

Numerical Methods

Why numerical methods?

Finding classical solutions to differential equations is difficult in all but the simplest circumstances. That is why we resort to numerical methods.

In numerical integration we discretize the problem to find the response at certain points in time (and then "connect the dots" to approximate the continuous response history. If we are trying to approximate a function $u(t)$ then we will find a sequence of values

$$\{u_0, u_1, u_2, u_3, ..., u_{N-1}, u_N\}$$

Where $u_i = u(t_i)$ is meant to represent the value of the exact function of time at that particular point in time. In the discrete world derivatives are approximated by difference and integrals are approximated by simple formulas for area under the curve (like trapezoids). The key is that you can only use the values at the discrete points in the approximation.

There are two main things to think about when applying numerical methods to the solution of differential equations: (1) the formula for the method itself and (2) the performance of the method. There are *lots* of method available for integrating ordinary differential equations. We will consider two or three of those (the best ones!). The integration formulas are very easy to implement in a computer program so any equation of motion we need to solve can be done fairly easily (including nonlinear effects).

Numerical solutions introduce approximation error and we need to understand what happens to our results in order to be competent users of numerical methods. On the simple end of the spectrum is the idea that as we take a finer and finer time discretization (smaller time steps) we should get closer and closer to the exact solution. On the more complicated end we have the problem of potential numerical instability of the integration scheme. Herein lies the key difference among the available numerical methods.

In these notes we will study in detail the performance of a few numerical integrators.

Numerical Methods

Road map

We will first take a look at the simplest differential equation—a first-order ordinary differential equation (the one-dimensional heat equation). This warm-up exercise will help to set the stage for the analysis of the more complex second-order problems.

What we will do is study the numerical integrator in the context of a *test problem*. The test problem will be one in which we can actually solve the differential equation classical. With both a classical and numerical solution we can make comparisons to aid our judgments of the performance of the numerical integrator. Of course, we can apply the numerical scheme to problems for which we cannot get a classical solution.

Along the way we will encounter *difference equations*, which we will be able to solve exactly for the *test problem*. The solution of difference equations is very similar to solving differential equations classically.

So here is the order of business:

1. First order equations
    1. Classical solution
    2. Generalized trapezoidal rule
2. Second-order equations
    1. Classical solution
    2. Central difference method
    3. Generalized trapezoidal rule
    4. Newmark's method

For each method we will analyze the difference equation exactly to see what it tells us about the method.

What we really want to know boils down to two main things: (1) how accurate is the method and (2) is the method numerically stable. For accuracy, we will discover what physical parameters affect the accuracy. For stability we will find out what restrictions, if any, we must place on the size of the time step to assure stability.

Numerical computations are vital to the engineering simulations required to do structural dynamics. This study should give insight into how best to use them.

Numerical Methods

First-order equation, classical solution

Consider the first-order, initial-value problem

$$\dot{u}(t) + c\,u(t) = 0$$
$$u(0) = u_o$$

This equation comes up in one-dimensional heat transfer, population growth, and other application. The classical solution is in the form of exponentials. We write the function and its derivative as

$$u = Ae^{\lambda t}$$
$$\dot{u} = \lambda Ae^{\lambda t}$$

Substituting these into the governing equation gives

$$\left[\lambda + c\right] Ae^{\lambda t} = 0$$

This equation gives rise to the so-called *characteristic equation* of the differential equation (by setting the term in [.] equal to zero).

Since the exponential term can never be zero and since $A = 0$ is a trivial solution to the problem, the only interesting solution happens when

$$\lambda = -c$$

Substituting this value and using the initial condition $u(0) = u_o$ yields the *classical solution* to the differential equation

$$u(t) = u_o e^{-ct}$$

Clearly, for positive c the nature of this response is exponential decay. The physical constant c determines how fast the decay from the initial value is. We will use this exact solution as a basis to judge our numerical solution to the problem. Note that for discrete time points at spacing h the time is $t = nh$ and the solution can be written as

$$u_n = u(nh) = u_o e^{-cnh} = u_o \left(e^{-ch}\right)^n$$

We will use this exact result as a comparison solution for our numerical methods.

Numerical Methods

First-order equation
Generalized trapezoidal rule

The discrete version of the first-order differential equation can be expressed as

$$v_n + c\,u_n = 0$$

where

$$u_n = u(t_n), \quad v_n = \dot{u}(t_n)$$

represent the values of the function u and its derivative at the specific time t_n. The *generalized trapezoidal rule* (GT) gives

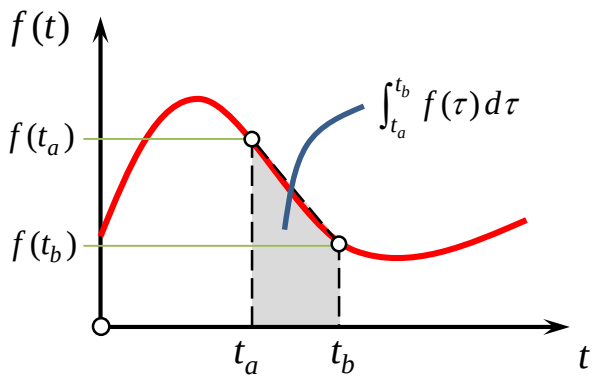
$$u_{n+1} = u_n + h\left[\beta\, v_n + (1-\beta) v_{n+1}\right]$$

where $h = t_{n+1} - t_n$ is the size of the time step. By substitution, eliminate the v_n and v_{n+1} terms

$$u_{n+1} - u_n - h\left[\beta\left(-cu_n\right) + (1-\beta)\left(-cu_{n+1}\right)\right] = 0$$

Rearranging terms we get

$$\left(1 + ch\left(1-\beta\right)\right) u_{n+1} - \left(1 - ch\beta\right) u_n = 0$$



Trapezoidal rule. The *trapezoidal rule* uses the observation that the trapezoid created by two adjacent function values creates an area give by

$$Area = \tfrac{1}{2}(t_b - t_a)\left[\, f(t_a) + f(t_b)\right]$$

The integral is simply the area under the curve, so this trapezoidal area is an approximation to that area. Hence, the integral can be approximated as

$$\int_{t_a}^{t_b} f(\tau)\, d\tau \approx \tfrac{1}{2}(t_b - t_a)\left[\, f(t_a) + f(t_b)\right]$$

Numerical Methods

Difference equations

The result of applying a numerical integration rule to a differential equation gives what is called a *difference equation.* It turns out that we can solve difference equations exactly.

Consider the simple difference equation

$$C_1 u_{n+1} - C_0 u_n = 0$$

where C_0 and C_1 are constants. To solve the difference equation let us investigate a solution of the form

$$u_n = A z^n$$

where A and z are as yet to be determined constants. The counter n plays the role of t as it indexes the state variables. Plug the approximation into the difference equation:

$$C_1 \left(A z^{n+1} \right) - C_0 \left(A z^n \right) = 0$$

$$A z^n \left[C_1 z - C_0 \right] = 0$$

There are a few ways this equation can be satisfied. First note that z^n can never be zero (unless z is zero). If A=0 then all u_n=0 and the solution is simply zero. The only other way the solution can be satisfied is if

$$C_1 z - C_0 = 0$$

This is called the *characteristic equation*. We can use it to fine the value of z that works:

$$z = \frac{C_0}{C_1}$$

The interesting observation is that this is the only value of z that is consistent with the difference equation.

Since we have an initial value problem we know the value u_o as given data. We can use the initial conditions to determine A. In fact, it is pretty straightforward to see that since

$$u_n = A \left(\frac{C_0}{C_1} \right)^n \quad \rightarrow \quad u_0 = A \left(\frac{C_0}{C_1} \right)^0 = A$$

Hence,

$$u_n = u_o \left(\frac{C_0}{C_1} \right)^n$$

Numerical Methods

First order equation

Exact solution to the difference equations

For the difference equation for the one-dimensional heat problem we have

$$\left(1+ch\left(1-\beta\right)\right)u_{n+1}-\left(1-ch\beta\right)u_n=0$$

So, by simple association we have

$$C_1=1+ch\left(1-\beta\right)$$
$$C_0=1-ch\beta$$

In the notation of the previous page. Hence,

$$z=\frac{C_0}{C_1}=\frac{1-ch\beta}{1+ch\left(1-\beta\right)}$$

Therefore, the exact numerical solution to this problem is

$$u_n=u_o\left[\frac{1-ch\beta}{1+c\,h(1-\beta)}\right]^n$$

Recall that the exact classical solution is

$$u_n=u_o\left(e^{-ch}\right)^n$$

So the numerical solution is obviously different form the exact solution. We can notice a few things about the numerical solution.

First, the numerical solution depends upon the physical parameter c and the numerical analysis parameters h and β. Second, we can see that for time steps where

$$0<h<\frac{1}{c\beta}$$

The solution always decays for positive values of c (like the classical solution). That is good. But also note that for

$$h>\frac{1}{c\beta}$$

The solution changes sign *every* time step. That is bad. We refer to such a phenomenon as a *numerical instability*.

Numerical Methods

Order of accuracy

One way to investigate the accuracy of a numerical integrator is to examine the difference between the Taylor series expansions of the exact solution and the approximate result. For our first-order test problem we have

$$u_n = u_o \left[e^{-ch} \right]^n \qquad \text{Exact}$$

$$u_n = u_o \left[\frac{1 - ch\beta}{1 + c\,h(1-\beta)} \right]^n \qquad \text{GT method}$$

The two functions in [•] are, therefore, supposed to be the same. We will judge how close to "same" they are by looking at their Taylor series. As a warm-up note the following series

$$e^{-x} = 1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 - \cdots$$

$$\frac{1 - \frac{1}{2}x}{1 + \frac{1}{2}x} = 1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{4}x^3 + \tfrac{1}{8}x^4 - \cdots$$

which are easily verified (or looked up).

By making the association $x=ch$ we can see that the difference between the Taylor series for the exact and approximate expressions (which we will call *error*) is (for the specific case β=1/2)

$$error = \left\{ e^{-ch} - \frac{1 - \frac{1}{2}ch}{1 + \frac{1}{2}ch} \right\} = \tfrac{1}{12}(ch)^3 + O(h^4)$$

Where the notation $O(h^n)$ means "on the order of h^n." And what that means is that the additional terms are all of polynomial order higher than the n^{th} power. This is important for small values of h as higher and higher powers produces smaller and smaller numbers.

So, the *trapezoidal rule* basically matches the first three terms of the Taylor series of the exact solution. We call that a *second-order method* (because the highest order matched is h^2).

For other values of β we have

$$error = \left(\beta - \tfrac{1}{2}\right)x^2 - \left\{\tfrac{1}{6} - (1-\beta)^2\right\}x^3 + O\left(h^4\right)$$

Hence, the generalized trapezoidal rule is second order accurate only for β = 1/2.

Numerical Methods

Implementation of a numerical integrator

We do not implement the numerical method in the form of the power law. One reason is because we do not have to. But the main reason is that we cannot get *z* for nonlinear problems, so we generally do not have that formula.

The implementation has two pieces: (1) the governing equation and (2) the time integration scheme (e.g., the trapezoidal rule). The two together provide the information we need to advance the solution from the initial conditions.

In the case of the heat equation we have

$$v_{n+1} + c\,u_{n+1} = 0$$

and

$$u_{n+1} = u_n + h\left[\beta\, v_n + (1-\beta) v_{n+1}\right]$$

As we advance the solution we would consider that the state at step *n* is completely know so our aim is to compute the state at step *n*+1. We have, then, two equation in two unknowns.

One way to organize the computation is to say that we want to solve the equations simultaneously. Since the equation in this case are linear we can do that by simple substitution. However, if the governing equation is nonlinear then we need to use a method like Newton's method to solve the equations.

We can also imagine a simplification in which we use the old u_n in the governing equation to estimate

$$v_{n+1} = -c\,u_n$$

Now, with the next v_{n+1} known we can use the trapezoidal integrator to compute the next

$$u_{n+1} = u_n + h\left[\beta\, v_n + (1-\beta) v_{n+1}\right]$$

This approach is called an *explicit method* because the computation is completely sequential and non-iterative. Explicit methods are generally prone to numerical stability problems and are usually less accurate than *implicit methods* (i.e., ones where the approximation using the old information is not used).

Numerical Methods

Second-order equations
Classical solution, differences

Consider the second-order initial-value problem (an SDOF un-damped oscillator under free vibration)

$$\ddot{u}(t) + \omega^2 u(t) = 0$$
$$u(0) = u_o$$
$$\dot{u}(0) = v_o$$

As we know the classical solution is

$$u(t) = u_o \cos \omega t + \frac{v_o}{\omega} \sin \omega t$$

We will use this solution as a guide to understanding the behavior of our numerical solutions. We will consider three different time integration schemes: (1) the *central difference method* (CD), (2) the *generalized trapezoidal rule* (GT), and (3) *Newmark's method* (which is the most commonly used method in structural dynamics.

To set up our discussion of the numerical integrators we adopt the notation

$$u_n = u(t_n), \quad v_n = \dot{u}(t_n), \quad a_n = \ddot{u}(t_n)$$

What this notation implies is that the value we compute from the discrete equations is not exactly the same as the value of the exact function at that time, but it is meant to be an approximation of it. In this view, the displacement, velocity, and acceleration are not time derivatives of each other, but rather independent variables that we will relate using different numerical schemes. For example, the *central difference* approximations for the first and second derivative are

$$v_n = \frac{u_{n+1} - u_{n-1}}{2h} \qquad a_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}$$

These formulas make intuitive sense. The first derivative is the secant defined by the points to the right and left of point *n*. The second derivative is the difference between to two secants on the right and left side of point *n*. Other difference can be used to approximate the derivative, but these two enjoy simplicity and symmetry.

Numerical Methods

Central difference method
Characteristic equation

Substituting the difference equation for the acceleration into the equation of motion we transform the differential equation into a difference equation

$$\left(u_{n+1} - 2u_n + u_{n-1}\right) + \omega^2 h^2 u_n = 0$$

Rearranging terms we can put this difference equation into the form

$$u_{n+1} - \left(2 - \omega^2 h^2\right) u_n + u_{n-1} = 0$$

Once again, we assume that the solution of the difference equation is in the form

$$u_n = A z^n$$

Substituting this expression into the difference equation we get.

$$A z^{n+1} - \left(2 - \omega^2 h^2\right) A z^n + A z^{n-1} = 0$$

$$\left[z^2 - \left(2 - \omega^2 h^2\right) z + 1 \right] A z^{n-1} = 0$$

Hence, the *characteristic equation* for the *central difference integrator* for the single-degree-of-freedom, undamped oscillator is

$$z^2 - \left(2 - \omega^2 h^2\right) z + 1 = 0$$

The value(s) of z that come from this equation are the only values consistent with the difference equation.

Solving the quadratic equation gives roots

$$\begin{aligned} z &= 1 - \tfrac{1}{2}\omega^2 h^2 \pm \tfrac{1}{2}\sqrt{\left(2 - \omega^2 h^2\right)^2 - 4} \\ &= 1 - \tfrac{1}{2}\omega^2 h^2 \pm \tfrac{1}{2}\sqrt{4 - 4\omega^2 h^2 + \omega^4 h^4 - 4} \\ &= 1 - \tfrac{1}{2}\omega^2 h^2 \pm i\omega h\sqrt{1 - \tfrac{1}{4}\omega^2 h^2} \end{aligned}$$

It is evident that there are two roots to the characteristic equation and for small values of ωh they are complex conjugates of each other.

$$z = 1 - \tfrac{1}{2}\omega^2 h^2 \pm i\omega h\sqrt{1 - \tfrac{1}{4}\omega^2 h^2}$$

Numerical Methods

Spectral radius and
Characteristic angle

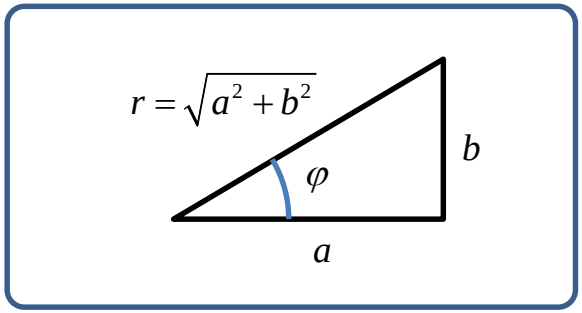
Because the roots of the characteristic equation come out as complex conjugates it will prove convenient to put them into Euler's form for complex numbers. We can write the value of z in the following equivalent forms

$$\begin{aligned} z &= 1-\tfrac{1}{2}\omega^2h^2 \pm i\omega h\sqrt{1-\tfrac{1}{4}\omega^2h^2} \\ &= a+ib \\ &= r\left(\cos\varphi + i\sin\varphi\right) \end{aligned}$$

(Note: we use a and b because they are standard for complex numbers. We use a for acceleration, too, but it always has a subscript for the step number). Note that the constants a and b in this case have the values

$$\begin{aligned} a &= 1-\tfrac{1}{2}\omega^2h^2 \\ b &= \omega h\sqrt{1-\tfrac{1}{4}\omega^2h^2} \end{aligned}$$

Note that r is often called the *spectral radius*.



The relationship can be seen in the triangle above. The *modulus* of the complex number is

$$\begin{aligned} r^2 &= a^2+b^2 \\ &= \left(1-\tfrac{1}{2}\omega^2h^2\right)^2 + \left(\omega h\sqrt{1-\tfrac{1}{4}\omega^2h^2}\right)^2 \\ &= 1-\omega^2h^2+\tfrac{1}{4}\omega^4h^4+\omega^2h^2-\tfrac{1}{4}\omega^4h^4 \\ &= 1 \end{aligned}$$

and the angle φ is defined through the relationship

$$\tan\varphi = \frac{b}{a} = \frac{\omega h\sqrt{1-\tfrac{1}{4}\omega^2h^2}}{1-\tfrac{1}{2}\omega^2h^2}$$

The implication of the importance of the values of r and φ will become evident in the following pages.

Numerical Methods

Central difference method
Alternative form of general solution

Because the characteristic equation had two roots, the general solution to the difference equation must be the sum of two terms

$$u_n = A(z_1)^n + B(z_2)^n$$
$$= A(a+ib)^n + B(a-ib)^n$$

Note that this implies that we now have two constants, A and B, in our general solution. We can determine the values of those constants using the initial conditions of the problem

$$u(0) = u_o \qquad \dot{u}(0) = v_o$$

We can rewrite the general solution in the following (more convenient) form

$$u_n = r^n\left[A(\cos n\varphi + i\sin n\varphi) + B(\cos n\varphi - i\sin n\varphi)\right]$$
$$= r^n\left[(A+B)\cos n\varphi + i(A-B)\sin n\varphi\right]$$

Let us look for the pattern in exponentiation of the expression of the complex number

$$z = r(\cos\varphi + i\sin\varphi)$$

First, raise it to the second power

$$z^2 = r^2(\cos\varphi + i\sin\varphi)^2$$
$$= r^2(\cos\varphi + i\sin\varphi)(\cos\varphi + i\sin\varphi)$$
$$= r^2(\cos^2\varphi + 2i\sin\varphi\cos\varphi - \sin^2\varphi)$$
$$= r^2(\cos^2\varphi - \sin^2\varphi + 2i\sin\varphi\cos\varphi)$$

Now use trig identities to get

$$z^2 = r^2(\cos 2\varphi + i\sin 2\varphi)$$

The pattern continues to give the nth term

$$z^n = r^n(\cos n\varphi + i\sin n\varphi)$$

This result will be central to our analysis.

Numerical Methods

Central difference method
Initial conditions

Recall the expression for the displacement at step n

$$u_n = A(a+ib)^n + B(a-ib)^n$$

Where a and b were defined previously. Let us get the constants A and B from the initial conditions. Using the above equation we find (using n=0 and n=1)

$$\begin{aligned} u_o &= A + B \\ u_1 &= A(a+ib) + B(a-ib) \\ &= (A+B)a + i(A-B)b \\ &= u_o\, a + i(A-B)b \end{aligned}$$

From the discrete equation for velocity and the equation of motion we have

$$\begin{aligned} u_1 - u_{-1} &= 2hv_o \\ u_1 + u_{-1} &= \left(2 - h^2\omega^2\right)u_o \end{aligned}$$

One thing that is a little strange about these equations is that the introduce the displacement at time step −1 (i.e., u_{-1}). That is clearly a fictitious quantity, but if we let it into our equations we will be able to work it out. In particular, if we add the two equations together and solving for u_1, we get

$$u_1 = hv_o + \left(1 - \tfrac{1}{2}h^2\omega^2\right)u_o = h\,v_o + a\,u_o$$

Now equating the two expressions for u_1 (one from the initial condition, one from our general solution)

$$u_1 = a\,u_o + i(A-B)b = hv_o + a\,u_o$$

We can solve for $i(A–B)$ to get

$$i(A-B) = \frac{hv_o}{b} = \frac{hv_o}{\omega h\left(1 - \tfrac{1}{4}\omega^2h^2\right)}$$

Thus, the final expression for the exact solution to the difference equation is

$$u_n = u_o \cos n\varphi + \frac{v_o}{\omega\sqrt{1 - \tfrac{1}{4}\omega^2h^2}} \sin n\varphi$$

Numerical Methods

Central difference method
Interpretation of results

From the previous page, the exact solution to the difference equation is

$$u_n = u_o \cos n\varphi + \frac{v_o}{\omega\sqrt{1-\frac{1}{4}\omega^2 h^2}} \sin n\varphi$$

One thing you might have noticed is that the general solution had the complex unit *i* in the solution, but the final expression does not. What this means is that *A* and *B* are also complex numbers and that their sum was purely real and their difference was purely imaginary. Hence, the final solution for displacement is purely real.

This outcome is very similar to what happens in the classical solution to the differential equation and is directly related to the connection between complex exponentials and the sinusoidal functions through Euler's identity.

There are some very evident similarities to the classical solution to the differential equation

$$u(t_n) = u_o \cos \omega t_n + \frac{v_o}{\omega} \sin \omega t_n$$

First note that both are oscillatory and there are distinct similarities between the coefficients of the sine and cosine terms. Noting that $t_n = hn$ we can write the numerical solution as

$$u_n = u_o \cos \tilde{\omega} t_n + \frac{v_o}{\omega\sqrt{1-\frac{1}{4}\omega^2 h^2}} \sin \tilde{\omega} t_n$$

where

$$\tilde{\omega} \equiv \frac{\varphi}{h}$$

is the *apparent frequency* of the numerical method. We can also observe that the solution is well behaved as long as

$$1-\tfrac{1}{4}\omega^2 h^2 > 0 \quad \Rightarrow \quad h < \frac{2}{\omega}$$

This is the *stability limit* of the numerical method and it defines the largest time step which gives a response consistent with the classical solution.

Numerical Methods

Generalized trapezoidal rule

The discrete version of the equation of motion is

$$a_n + \omega^2 u_n = 0$$

The *generalized trapezoidal rule* for the second order system has the form

$$v_{n+1} = v_n + h\left[\beta a_n + (1-\beta) a_{n+1}\right]$$
$$u_{n+1} = u_n + h\left[\beta v_n + (1-\beta) v_{n+1}\right]$$

Substituting the first into the second gives an alternative form

$$u_{n+1} = u_n + h v_n + h^2 \beta (1-\beta) a_n + h^2 (1-\beta)^2 a_{n+1}$$

Substituting the accelerations from the equation of motion at the two time steps gives

$$D_1 u_{n+1} - D_2 u_n - h v_n = 0$$

$$D_1 = 1 + (1-\beta)^2 h^2 \omega^2, \quad D_2 = 1 - \beta(1-\beta) h^2 \omega^2$$

At two adjacent time steps we can write

$$D_1 u_{n+1} - D_2 u_n - h v_n = 0$$
$$D_1 u_n - D_2 u_{n-1} - h v_{n-1} = 0$$

The difference of these equations gives

$$D_1 u_{n+1} - \left(D_1 + D_2\right) u_n + D_2 u_{n-1} - h\left(v_n - v_{n-1}\right) = 0$$

Now, from the original numerical integration equation and the equation of motion we can eliminate the velocity terms to get

$$D_1 u_{n+1} - \left(D_1 + D_2 - h^2 (1-\beta)\omega^2\right) u_n + \left(D_2 + h^2 \beta \omega^2\right) u_{n-1} = 0$$

Simplifying, we get

$$D_1 u_{n+1} - 2 D_2 u_n + D_3 u_{n-1} = 0$$

$$D_1 = 1 + (1-\beta)^2 h^2 \omega^2$$
$$D_2 = 1 - \beta(1-\beta) h^2 \omega^2$$
$$D_3 = 1 + \beta^2 h^2 \omega^2$$

Numerical Methods

Generalized trapezoidal rule
Solution to difference equation

Solve the difference equation

$$a_n = Az^n$$

$$D_1 u_{n+1} - 2D_2 u_n + D_3 u_{n-1} = 0$$

$$\left[D_1 z^2 - 2D_2 z + D_3\right] Az^{n-1} = 0$$

$$z = \frac{D_2 \pm \sqrt{D_2^2 - D_1 D_3}}{D_1}$$

Simplify the term under the radical

$$\left(\sqrt{(\bullet)}\right)^2 = D_2^2 - D_1 D_3$$

$$= \left(1 - \omega^2 h^2 \beta(1-\beta)\right)^2$$

$$-\left(1 + \omega^2 h^2 (1-\beta)^2\right)\left(1 + \omega^2 h^2 \beta^2\right)$$

$$= 1 - 2\omega^2 h^2 \beta(1-\beta) + \omega^4 h^4 \beta^2 (1-\beta)^2$$

$$-\left(1 + \omega^2 h^2 (1-\beta)^2 + \omega^2 h^2 \beta^2 + \omega^4 h^4 \beta^2 (1-\beta)^2\right)$$

$$= -2\omega^2 h^2 \beta(1-\beta) - \omega^2 h^2 (1-\beta)^2 - \omega^2 h^2 \beta^2$$

$$= -\omega^2 h^2 \left(2\beta - 2\beta^2 + \left[1 - 2\beta + \beta^2\right] + \beta^2\right)$$

$$= -\omega^2 h^2$$

The roots take the form

$$z = \frac{D_2 \pm \sqrt{D_2^2 - D_1 D_3}}{D_1}$$

$$= \left(\frac{1 - \omega^2 h^2 \beta(1-\beta)}{1 + \omega^2 h^2 (1-\beta)^2}\right) \pm i\left(\frac{\omega h}{1 + \omega^2 h^2 (1-\beta)^2}\right)$$

In standard form

$$z = r\left(\cos\varphi + i\sin\varphi\right) = a + ib$$

$$a = \left(\frac{1 - \omega^2 h^2 \beta(1-\beta)}{1 + \omega^2 h^2 (1-\beta)^2}\right)$$

$$b = \left(\frac{\omega h}{1 + \omega^2 h^2 (1-\beta)^2}\right)$$

Note that since the term under the radical is always negative there is no possibility of a time step that does not yield an oscillatory solution. Thus, this scheme is *unconditionally stable*.

Note that in this case the amplitude r is not unity. That means that it is possible for the amplitude of the solution to either grow or decay with time.

Numerical Methods

Generalized trapezoidal rule
Initial conditons

Note that the root the characteristic equation is complex and can, therefore, be put in the form

$$z^n = r^n\left(\cos n\varphi + i\sin n\varphi\right) = \left(a + ib\right)^n$$

where the constants a and b are

$$a = \frac{1-\omega^2h^2\beta(1-\beta)}{1+\omega^2h^2(1-\beta)^2}$$

$$b = \frac{\omega h}{1+\omega^2h^2(1-\beta)^2}$$

$$r = \sqrt{a^2+b^2}$$

We can immediately identify the angle φ from

$$\tan\varphi = \left(\frac{\omega h}{1-\omega^2h^2\beta(1-\beta)}\right)$$

First, we can compute the acceleration at time steps 0 and 1 from the equation of motion

$$a_o = -\omega^2 u_o$$

$$a_1 = -\omega^2 u_1$$

Next, get velocity at time step 1 from the from the generalized trapezoidal rule equations

$$v_1 = v_o + h\left[\beta a_o + (1-\beta)a_1\right]$$

and the displacement at time step1 from the from the generalized trapezoidal rule equations

$$\begin{aligned} u_1 &= u_o + h\left[\beta v_o + (1-\beta)v_1\right] \\ &= u_o + h\left[\beta v_o + (1-\beta)\left\{v_o + h\left[\beta a_o + (1-\beta)a_1\right]\right\}\right] \\ &= \left(1-\omega^2h^2\beta(1-\beta)\right)u_o + hv_o - \omega^2h^2(1-\beta)^2u_1 \\ &= \frac{\left(1-\omega^2h^2\beta(1-\beta)\right)u_o + hv_o}{1+\omega^2h^2(1-\beta)^2} \end{aligned}$$

So, the displacement at time step 1 is

$$u_1 = \frac{\left(1-\omega^2h^2\beta(1-\beta)\right)u_o + hv_o}{1+\omega^2h^2(1-\beta)^2} = au_o + b\frac{v_o}{\omega}$$

Numerical Methods

Generalized trapezoidal rule
initial conditons

We can also compute the displacement at time steps 0 and 1 from our solution to the difference equation

$$u_n = A(a+ib)^n + B(a-ib)^n$$

For the values $n = 0$ and $n = 1$ we have

$$\begin{aligned} u_o &= A + B \\ u_1 &= A(a+ib) + B(a-ib) \\ &= (A+B)a + i(A-B)b \\ &= u_o a + i(A-B)b \end{aligned}$$

Comparing this with

$$u_1 = \frac{\left(1-\omega^2 h^2 \beta(1-\beta)\right)u_o + h v_o}{1+\omega^2 h^2 (1-\beta)^2} = a u_o + b\frac{v_o}{\omega}$$

gives

$$i(A-B) = \frac{v_o}{\omega}$$

Substituting back into the spectral form of the solution to the difference equation we get

$$u_n = r^n\left[u_o \cos n\varphi + \frac{v_o}{\omega}\sin n\varphi\right]$$

Note that the apparent frequency is

$$\tilde{\omega} = \frac{\varphi}{h}$$

where φ can be computed from

$$\tan\varphi = \left(\frac{\omega h}{1-\omega^2 h^2 \beta(1-\beta)}\right)$$

For small values of ωh the value of φ is small so that $\tan\varphi \approx \varphi$. Then

$$\tilde{\omega} = \frac{\varphi}{h} \approx \frac{\omega}{1-\omega^2 h^2 \beta(1-\beta)}$$

Which converges to the actual natural frequency as the time step gets small.

Numerical Methods

Generalized trapezoidal rule
Spectral radius for $\beta = 1/2$

Let's try to compute the amplitude to see if we can determine the growth or decay in the numerical solution

$$z^n = r^n\left(\cos n\varphi + i\sin n\varphi\right) = \left(a + ib\right)^n$$

where the constants a and b are

$$a = \frac{1-\omega^2h^2\beta(1-\beta)}{1+\omega^2h^2(1-\beta)^2}$$

$$b = \frac{\omega h}{1+\omega^2h^2(1-\beta)^2}$$

$$r = \sqrt{a^2+b^2}$$

What we know is this: If r is greater than one, then the solution grows with time because the oscillation is being multiplied by a quantity that is greater than one and getting larger. Conversely, if r is less than one then the solution decays with time. So the size of r seems to be of fundamental importance to the character of the solution.

Let us compute r and take a look specifically at the value of β=1/2.

$$D_1^2 = 1+\omega^2h^2(1-\beta)^2$$

$$D_1^2a^2 = \left(1-\omega^2h^2\beta(1-\beta)\right)^2$$

$$= 1-2\omega^2h^2\beta(1-\beta)+\omega^4h^4\beta^2(1-\beta)^2$$

$$D_1^2b^2 = \left(\omega h\right)^2$$

$$D_1^2r^2 = 1-2\omega^2h^2\beta(1-\beta)+\omega^4h^4\beta^2(1-\beta)^2+\omega^2h^2$$

$$= 1-\omega^2h^2\left[2\beta-2\beta^2-1\right]+\omega^4h^4\beta^2(1-\beta)^2$$

$$r^2 = \frac{1-\omega^2h^2\left[2\beta-2\beta^2-1\right]+\omega^4h^4\beta^2(1-\beta)^2}{\left(1+\omega^2h^2(1-\beta)^2\right)^2}$$

$$r^2(\beta=\tfrac{1}{2}) = \frac{1+\frac{1}{2}\omega^2h^2+\frac{1}{16}\omega^4h^4}{\left(1+\frac{1}{4}\omega^2h^2\right)^2}$$

$$= \frac{\left(1+\frac{1}{4}\omega^2h^2\right)^2}{\left(1+\frac{1}{4}\omega^2h^2\right)^2}$$

$$= 1$$

For β=1/2 the solution retains its amplitude. For other values, the spectral radius is not one, so we can expect growth or decay.

Numerical Methods

Newmark's Method

In a famous paper from 1959 Nathan Newmark proposed the following integrator

$$v_{n+1} = v_n + h\left[\gamma\, a_n + (1-\gamma)\, a_{n+1}\right]$$
$$u_{n+1} = u_n + h\, v_n + h^2\left[\beta\, a_n + (\tfrac{1}{2}-\beta)\, a_{n+1}\right]$$

where, h is the size of the time step and the constants γ and β are the *numerical integration* parameters (Newmark's parameters). Note that the β in Newmark's method is not the same as the β in the Generalized Trapezoidal Rule. The nature of the integrator is determined by the values selected for these parameters (like β was for the generalized trapezoidal rule in our previous discussion).

It should be evident that Newmark's method might be identical to the generalized trapezoidal rule for certain (but not all) values of γ and β. So this method represents a generalization of the generalized trapezoidal rule.

To compare the numerical integration parameters of the generalized trapezoidal rule and Newmark's method recall that the generalized trapezoidal equations are

$$v_{n+1} = v_n + h\left[\beta\, a_n + (1-\beta)\, a_{n+1}\right]$$
$$u_{n+1} = u_n + h\, v_n + h^2\beta\,(1-\beta)\, a_n + h^2(1-\beta)^2\, a_{n+1}$$

The two methods will be the same if the coefficients of the method match exactly. This happens when

$$\beta_N = \beta_{GT}\,(1-\beta_{GT})$$
$$\tfrac{1}{2} - \beta_N = (1-\beta_{GT})^2$$
$$\gamma_N = \beta_{GT}$$

Solving for the constant gives

$$1 - 2\beta_{GT}\,(1-\beta_{GT}) = 2(1-\beta_{GT})^2$$
$$1 - 2\beta_{GT} + 2\beta_{GT}^2 = 2 - 4\beta_{GT} + 2\beta_{GT}^2$$
$$0 = 1 - 2\beta_{GT}$$

The methods are the same when

$$\beta_{GT} = \tfrac{1}{2} \quad \rightarrow \quad \beta_N = \tfrac{1}{4}, \quad \gamma_N = \tfrac{1}{2}$$