CEE432/CEE532 Developing Software for Engineering Applications

Lecture 15: Introduction to Finite Element Method

What is Finite Element Method?

- Numerical Method
- (Usually) Approximate Solution
- Solves algebraic, differential and integral equations
- Converts the original problem into a set of algebraic equations or an eigenproblem

Finite Element Solutions

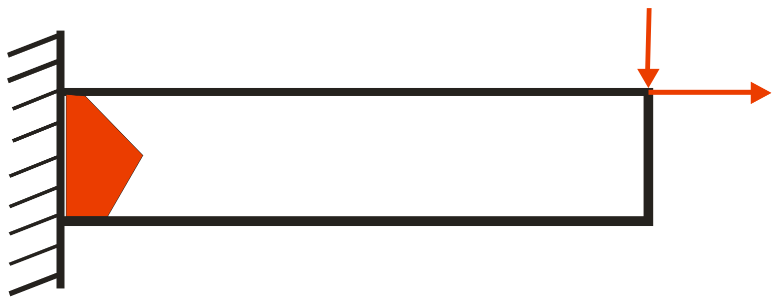
- Direct Stiffness Method
- Variational Technique: Theorem of Minimum Potential Energy
- Weak Formulation: Method of Weighted Residuals

The Six-Step Process

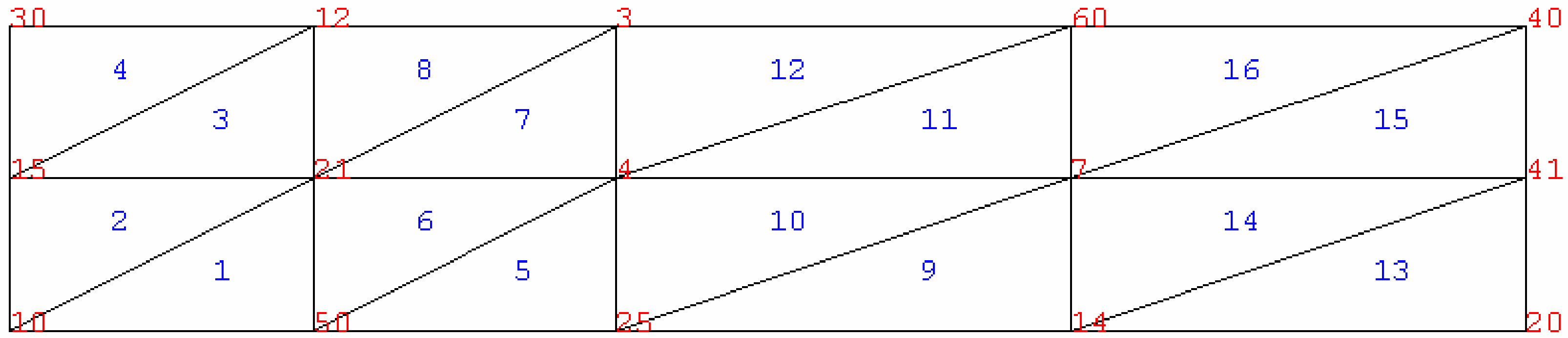
- Step 1: Discretization
- Step 2: Element equations
- Step 3: Assembly of system equations
- Step 4: Imposition of boundary conditions
- Step 5: Solution of system equations
- Step 6: Computation of secondary quantities

Example

- Steel cantilever beam (10” x 2” x 0.1”)
- Loading
  - Mechanical: Concentrated load at tip of beam
  - Thermal: Temperature change
- Objective: Compute displacements, strain and stress distribution

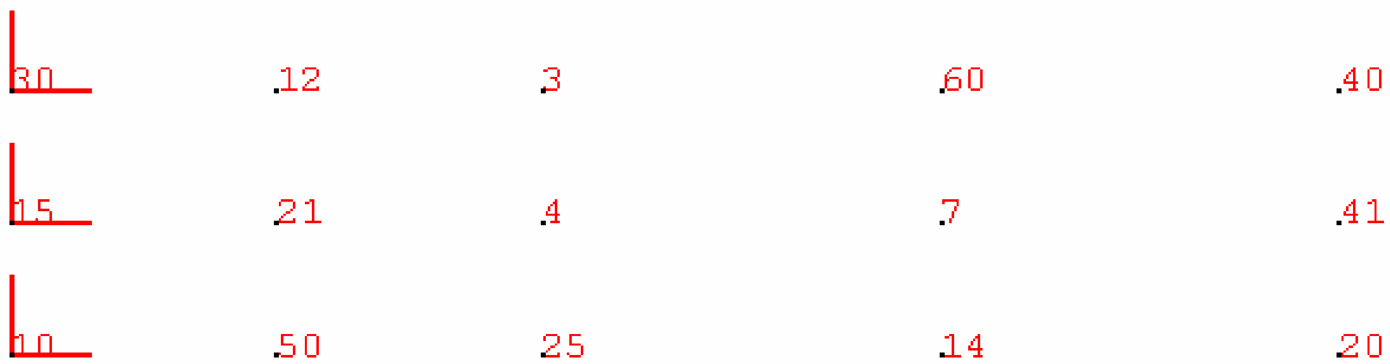


FE Model Details: Nodes and Elements

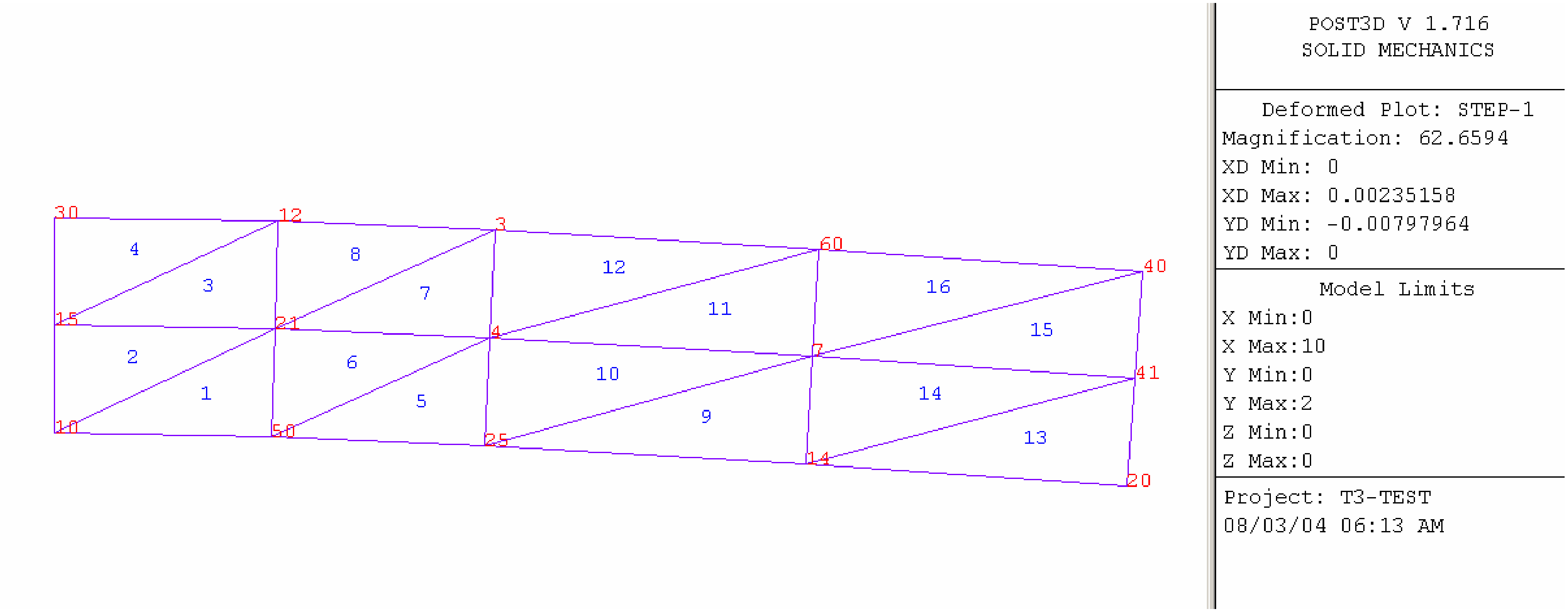


FE Model Details: Loads

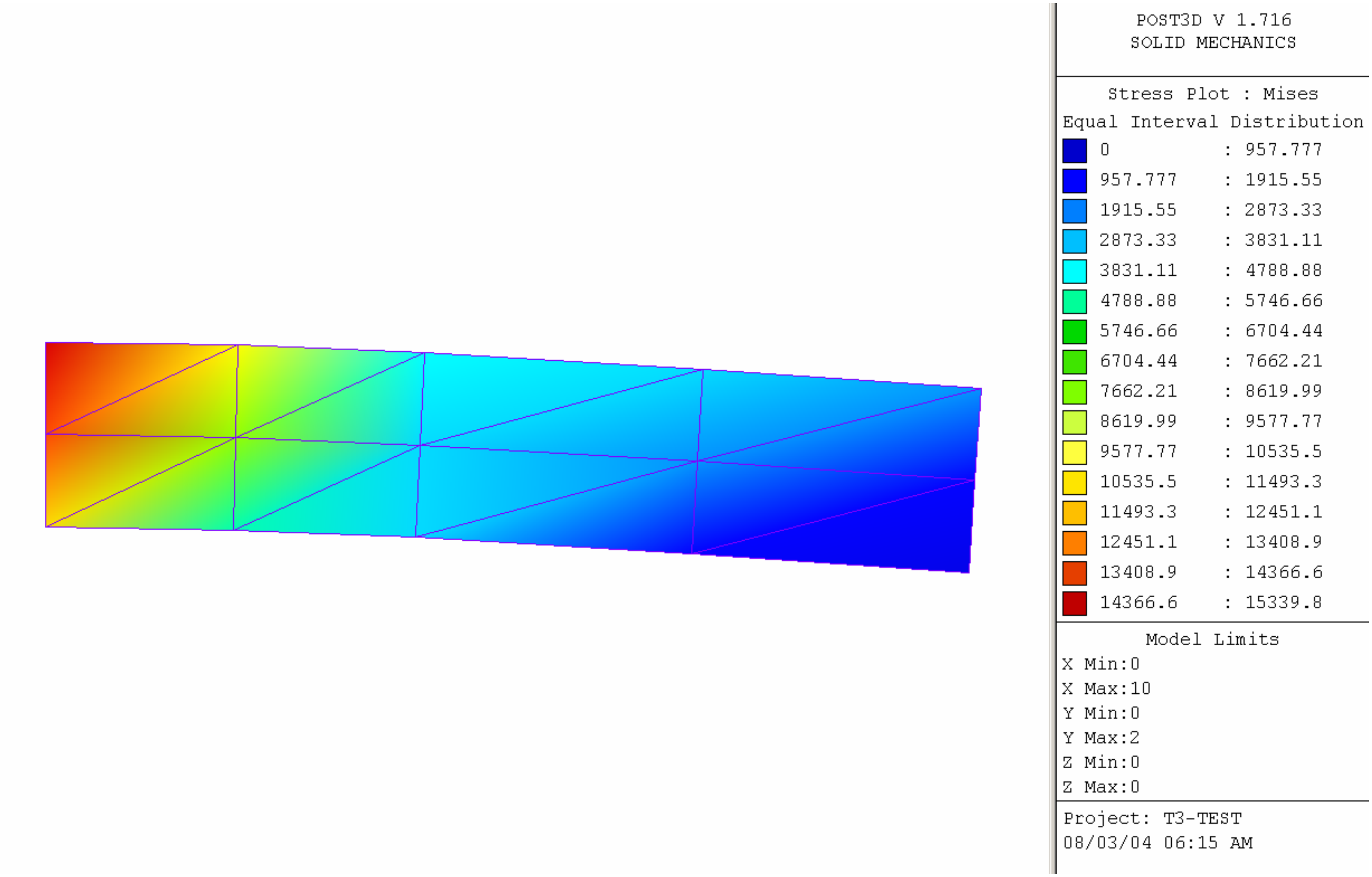
FE Model Details: Nodal Boundary Conditions



FE Results: Deformed Shape

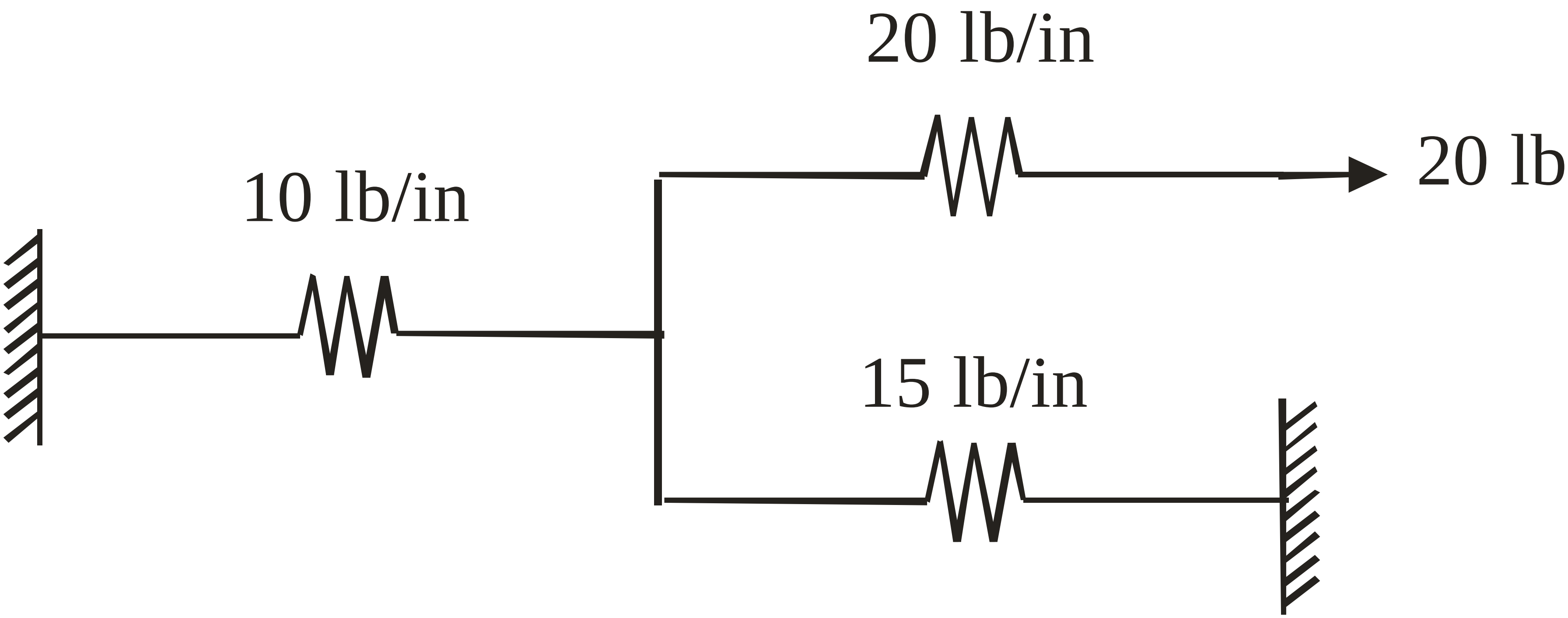


FE Results: Stress Distribution



Direct Stiffness Method

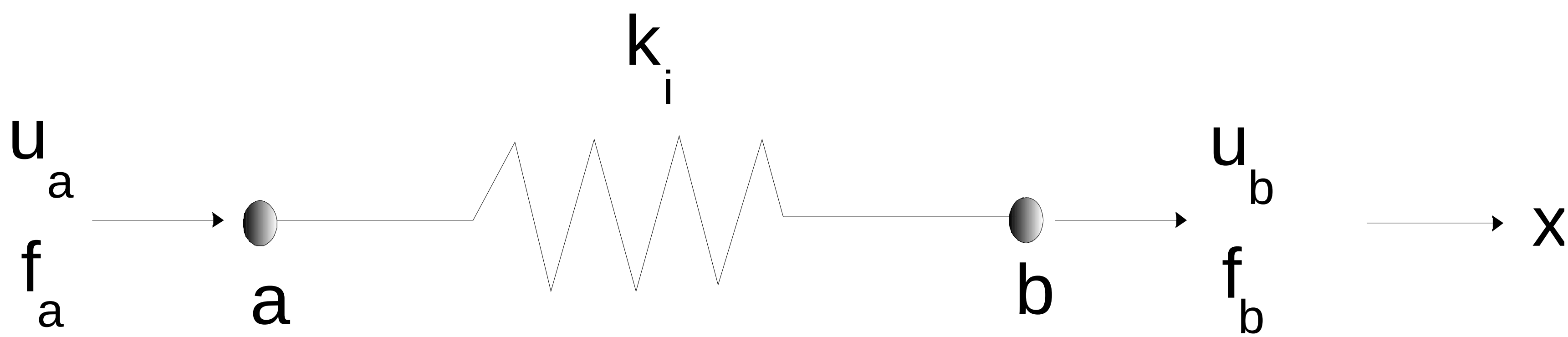
FE Analysis of a System of Springs



FE Analysis of a System of Springs

- Step 1: Discretization
- Step 2: Element Equations
  - Constitutive Relationship: Hooke’s Law
  - System Property: Equilibrium

Step 2: Element Equations



$$k_i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_a \\ u_b \end{Bmatrix} = \begin{Bmatrix} f_a \\ f_b \end{Bmatrix}$$

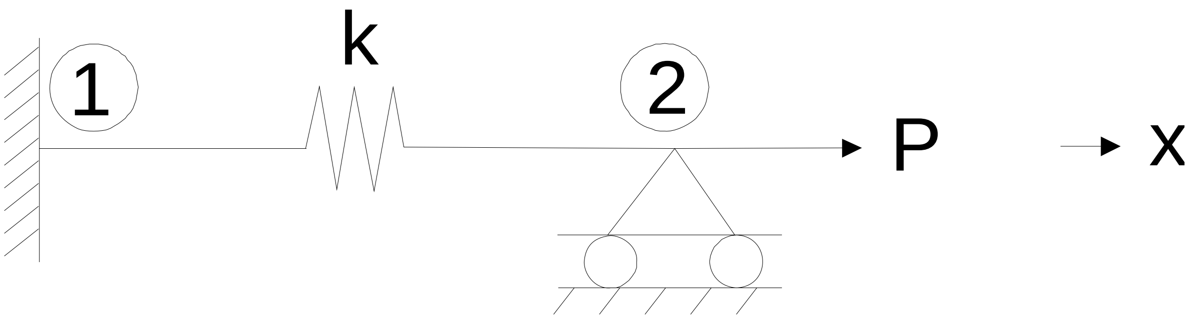
Dimensional Analysis:(F/L)(L) = F

Step 3: Assembly

- "Property" of the system is the sum of the "properties" of all the elements
- However this is not an algebraic sum
- In general we need to assemble

$$\mathbf{k}_{2\times 2}\mathbf{d}_{2\times 1} = \mathbf{f}_{2\times 1} \rightarrow \mathbf{K}_{n\times n}\mathbf{D}_{n\times 1} = \mathbf{F}_{n\times 1}$$

1-Element Example



$$k = 300\,lb/in$$

$$P = 30\,lb$$

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Element Equations

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 30 \end{Bmatrix}$$

System Equations
Cannot solve!

Step 4: Boundary Conditions

Since $D_1=0$, we can modify the two equations as follows. This is NOT an approximation.

$$\begin{bmatrix} 1 & 0 \\ 0 & 300 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \end{Bmatrix}$$

This is known as the Elimination Technique of imposing essential boundary conditions (EBCs).

Step 5: Solution System Equations

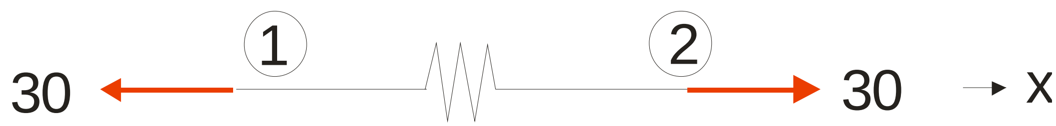
Solving the two equations, we have the solution as follows.

$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.1'' \end{Bmatrix}$$

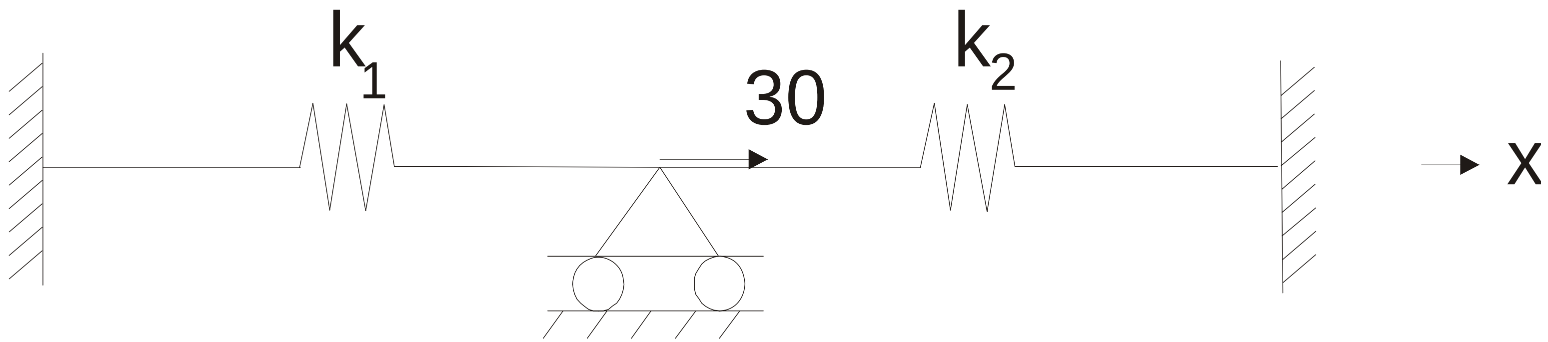
Step 6: Derived Variables

Now we can compute the force in the spring using the element equations as follows.

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.1 \end{Bmatrix} = \begin{Bmatrix} -30 \\ 30 \end{Bmatrix} lb$$

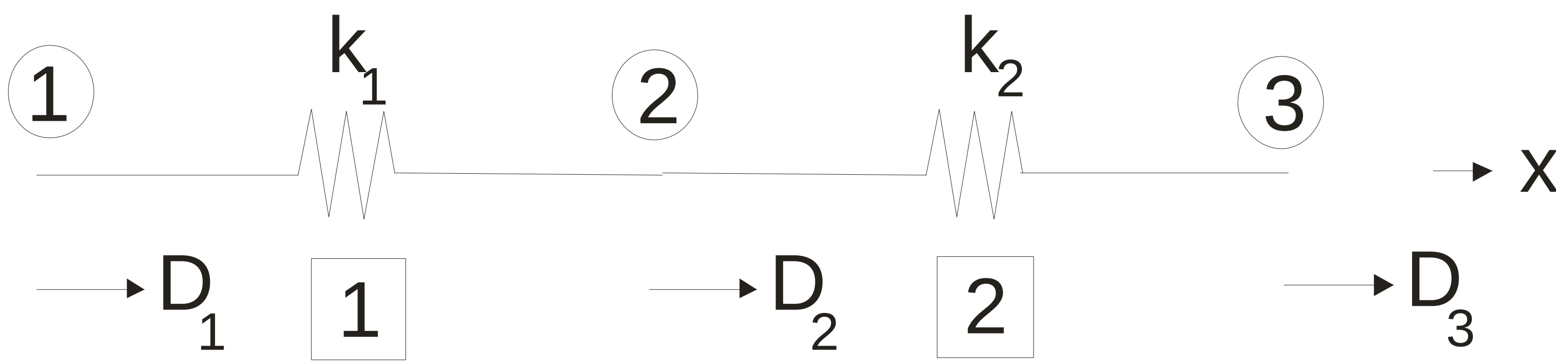


2-Element Example



$$k_1 = 300\, lb/in$$

$$k_2 = 200\, lb/in$$



Example (Step 2)

Element 1

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{\,1} \\ f_2^{\,1} \end{Bmatrix}$$

Element 2

$$\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{\,2} \\ f_2^{\,2} \end{Bmatrix}$$

Example (Step 3)

Element 1

$$\begin{bmatrix} 300 & -300 & 0 \\ -300 & 300 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{\,1} \\ f_2^{\,1} \\ 0 \end{Bmatrix}$$

Element 2

$$\begin{bmatrix} 300 & -300 & 0 \\ -300 & 300+200 & -200 \\ 0 & -200 & 200 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{\,1} \\ f_2^{\,1} + f_1^{\,2} \\ f_2^{\,2} \end{Bmatrix}$$

Example (Step 4)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \\ 0 \end{Bmatrix}$$

Example (Step 5)

Solving the three equations, we have the solution as follows.

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.06'' \\ 0 \end{Bmatrix}$$

Example (Step 6)

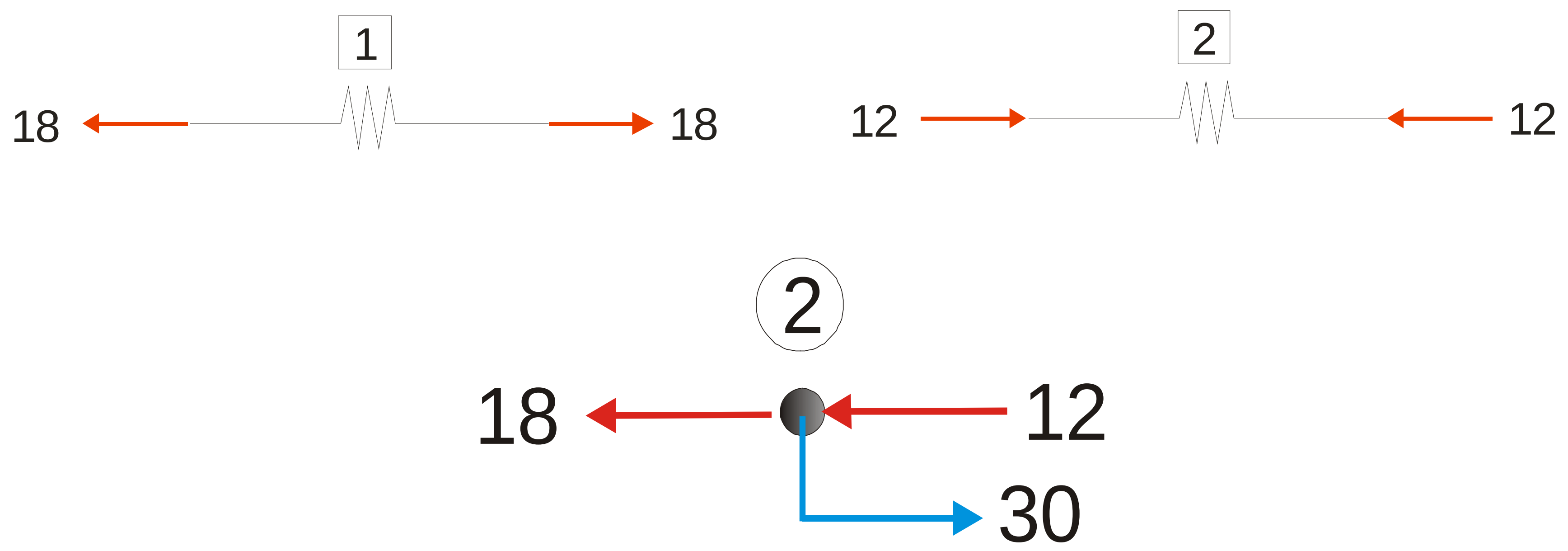
Element 1

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.06 \end{Bmatrix} = \begin{Bmatrix} -18 \\ 18 \end{Bmatrix} lb$$

Element 2

$$\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.06 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 12 \\ -12 \end{Bmatrix} lb$$

Example (Equilibrium Check)



Review: Solution Steps

- Choose consistent problem units
- Select a (global) coordinate system
- Label the nodes and elements
- Identify and label the nodal unknowns
- Identify the “boundary conditions”

Review: Solution Steps

- Loop thro' all elements
- Form the element equations
- Assemble into the system equations
- End loop
- Impose essential boundary conditions
- Solve the equations
- Obtain the secondary unknowns
- Check the solution for correctness

Summary

- Element stiffness matrix **k** is symmetric but rank deficient
- Structural stiffness matrix **K** is symmetric
- **K** is rank deficient before imposing EBC
- **K** is positive definite after imposing EBC

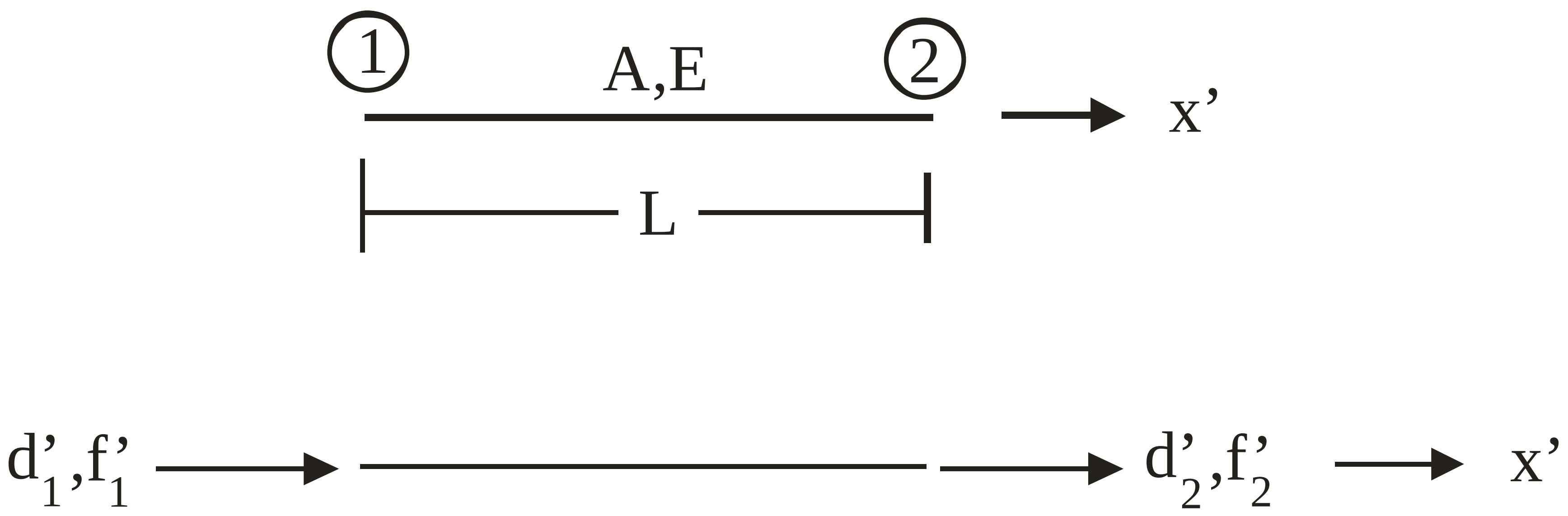
Planar Truss Analysis

Planar Truss

- Straight element with prismatic, slender cross-section
- All connections are pins
- All forces are applied at the nodes
- Small displacements and strains
- As a result
  - Elements are either in tension or compression

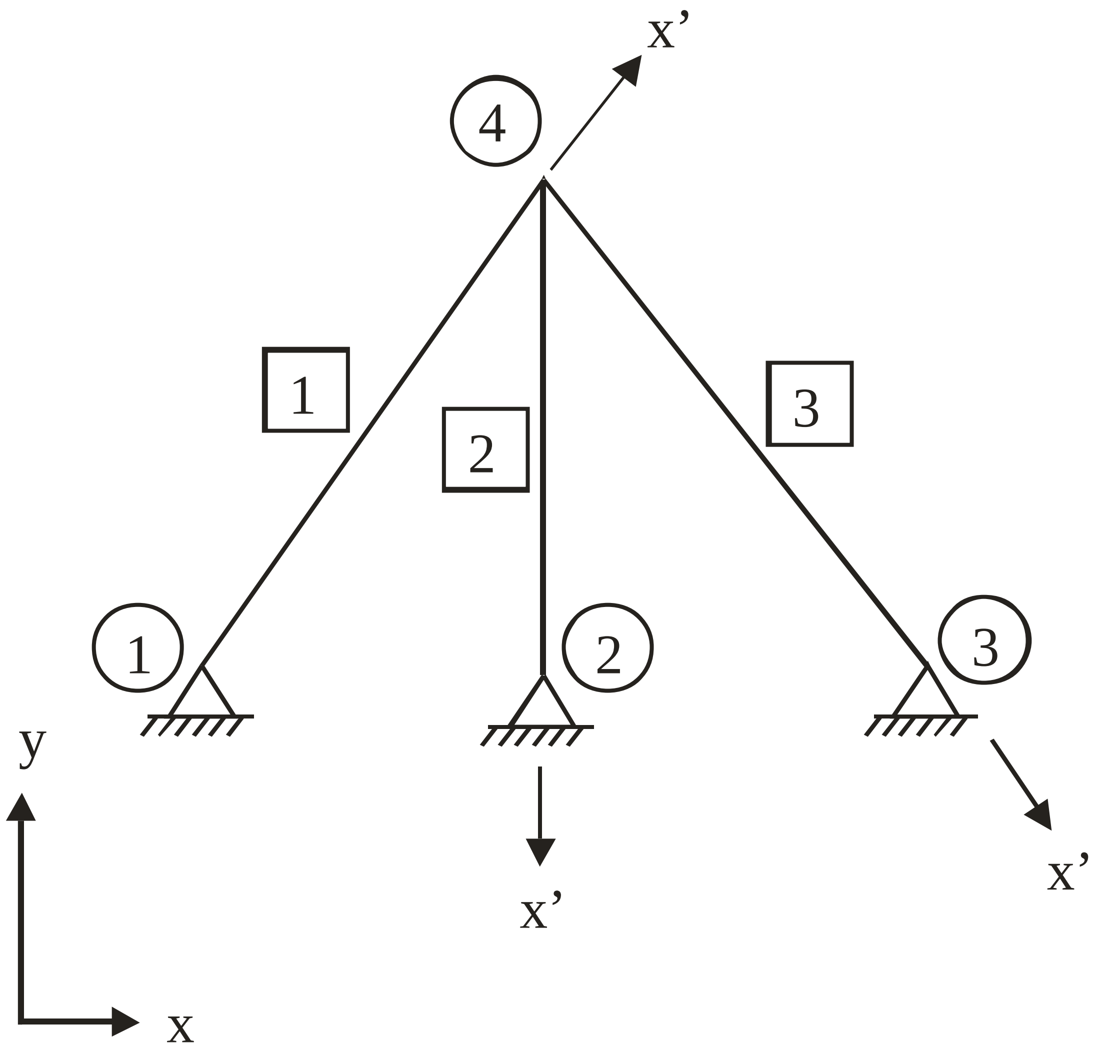
Planar Truss Elements

Element Configuration (local coordinate system)



Planar Truss Element

3 local coordinate systems. 1 global coordinate system.



Planar Truss Element

Hooke's Law

$$d = \frac{f}{AE/L} = \frac{fL}{AE}$$

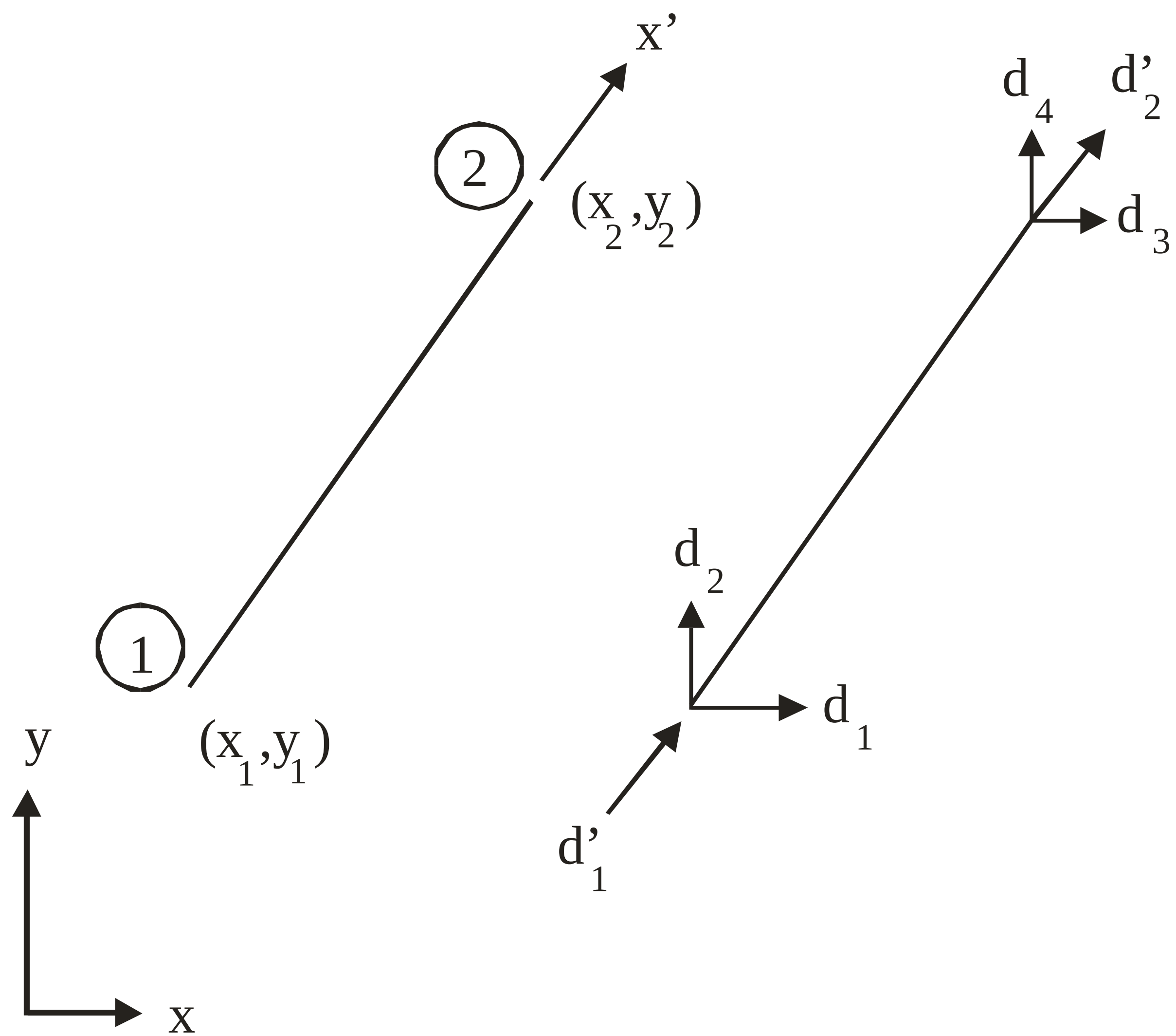
Nodal equilibrium equations

$$f_1^{'} = \frac{AE}{L} d_1^{'} - \frac{AE}{L} d_2^{'} \qquad f_2^{'} = -\frac{AE}{L} d_1^{'} + \frac{AE}{L} d_2^{'}$$

Element Equations

$$\frac{AE}{L}\left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array}\right]\left\{\begin{array}{c} d_1^{'} \\ \hline d_2^{'} \end{array}\right\} = \left\{\begin{array}{c} f_1^{'} \\ \hline f_2^{'} \end{array}\right\} \Rightarrow \mathbf{k}^{'}_{2\times 2}\mathbf{d}^{'}_{2\times 1} = \mathbf{f}^{'}_{2\times 1}$$

Planar Truss Element



Planar Truss Element

Local-to-Global Displacement Transformation

$$\left(d_1^{'}\right)^2 = \left(d_1\right)^2 + \left(d_2\right)^2$$

$$d_1^{'} = \frac{d_1}{d_1^{'}} d_1 + \frac{d_2}{d_1^{'}} d_2 = l d_1 + m d_2$$

$$d_2^{'} = \frac{d_3}{d_2^{'}} d_3 + \frac{d_4}{d_2^{'}} d_4 = l d_3 + m d_4$$

Direction cosines

$$L = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

$$l = \frac{x_2 - x_1}{L} \qquad m = \frac{y_2 - y_1}{L}$$

Planar Truss Element

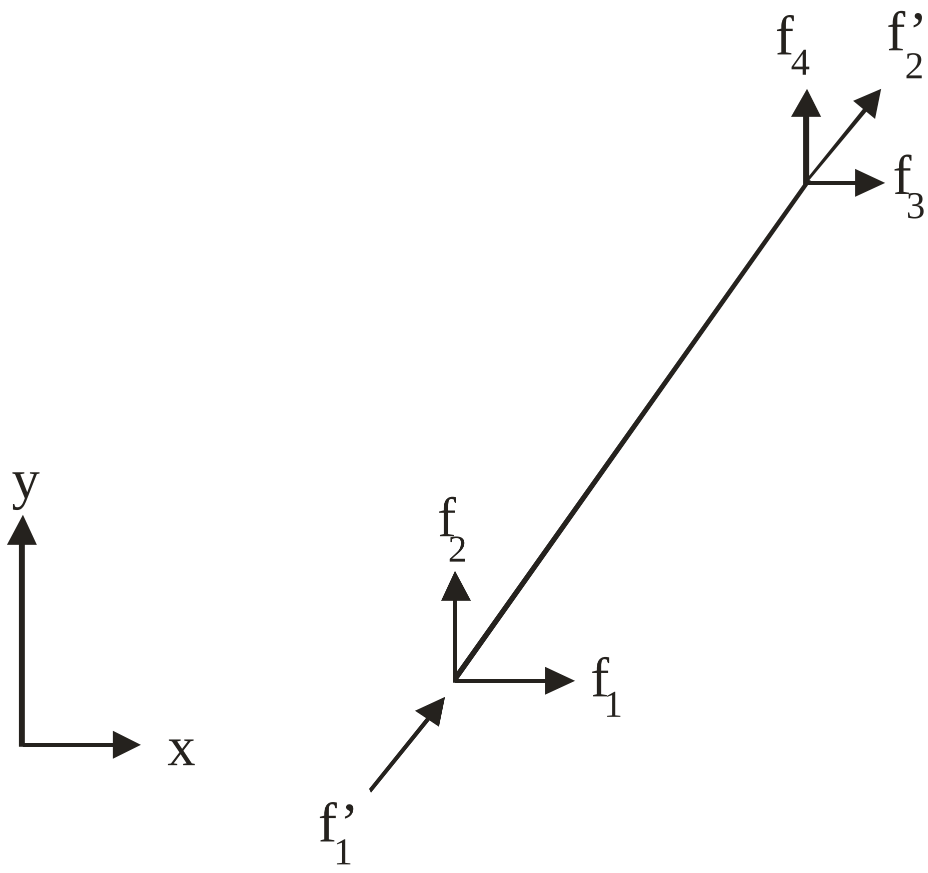
Local-to-Global Displacement Transformation

$$\begin{Bmatrix} d_1' \\ d_2' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

$$\mathbf{d}'_{2\times1} = \mathbf{T}_{2\times4}\mathbf{d}_{4\times1}$$

Planar Truss Element

Global-to-Local Force Transformation



$$f_1 = l f_1' \qquad f_2 = m f_1'$$

$$f_3 = l f_2' \qquad f_4 = m f_2'$$

Planar Truss Element

Global-to-Local Force Transformation

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{Bmatrix} f_1' \\ f_2' \end{Bmatrix} \qquad \mathbf{f}_{4\times1} = \mathbf{T}^{\mathbf{T}}_{4\times2}\mathbf{f}'_{2\times1}$$

Planar Truss Element

Element Equations in Global Coordinate System

$$\mathbf{k}'_{2\times2}\mathbf{d}'_{2\times1} = \mathbf{f}'_{2\times1}$$

$$\mathbf{d}'_{2\times1} = \mathbf{T}_{2\times4}\mathbf{d}_{4\times1} \quad \Rightarrow \quad \mathbf{k}_{4\times4}\mathbf{d}_{4\times1} = \mathbf{f}_{4\times1}$$

$$\mathbf{f}_{4\times1} = \mathbf{T}^{T}_{4\times2}\mathbf{f}'_{2\times1}$$

where

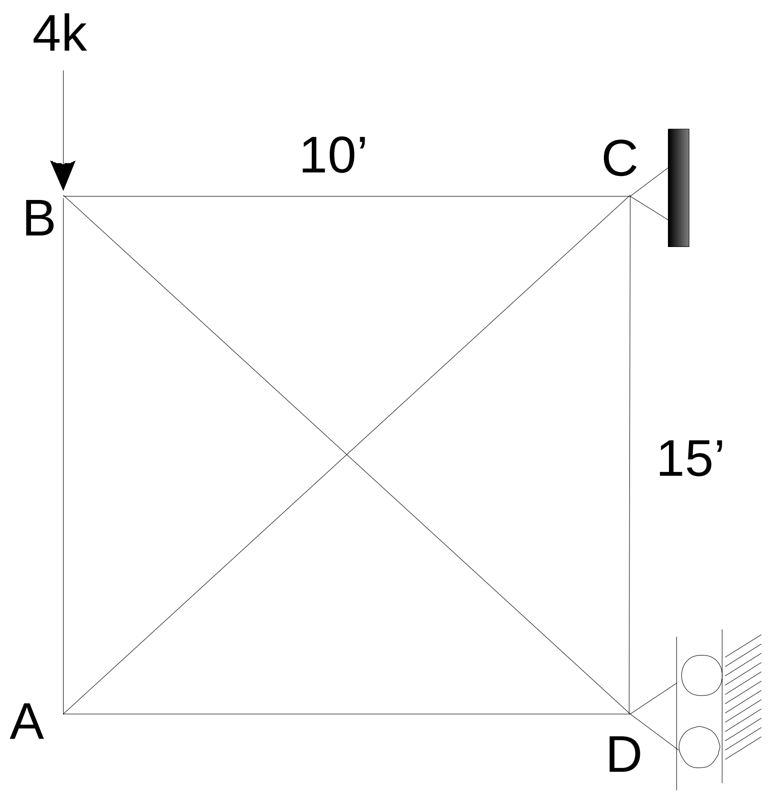
$$\mathbf{k}_{4\times4} = \mathbf{T}^{\mathbf{T}}_{4\times2}\mathbf{k}'_{2\times2}\mathbf{T}_{2\times4}$$

Planar Truss Analysis

Step 6: Computing element forces

$$f_1' = \frac{AE}{L}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

Example



$$E = 30(10^6)\,psi$$

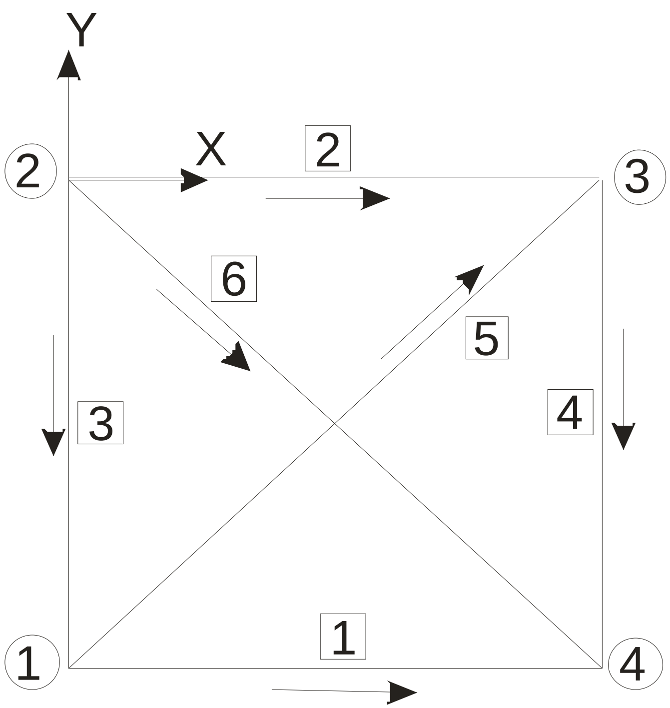
$$A = 1.2\,in^2$$

Compute nodal displacements, element forces and support reactions.

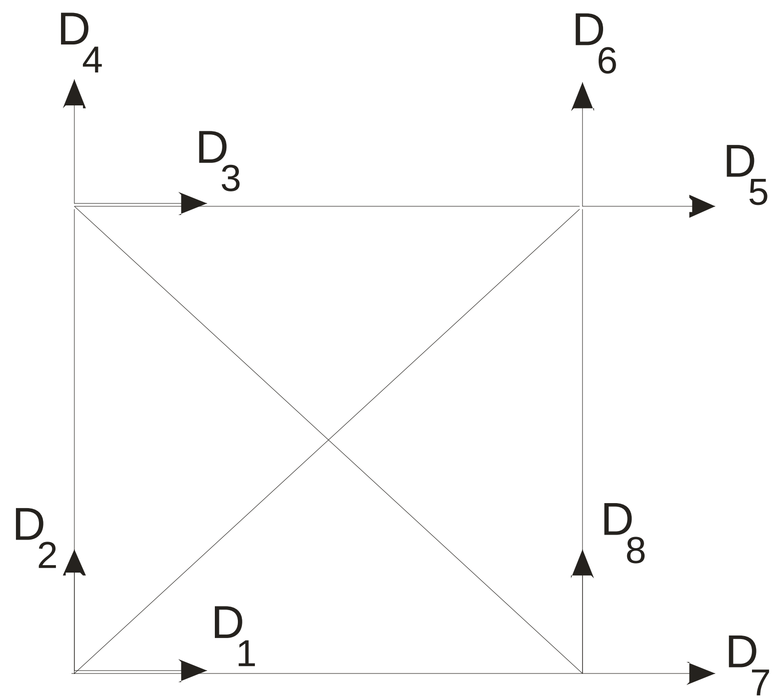
Units: lb, in

Example

FE Model



System Unknowns



Example

Member	(x_1, y_1)	(x_2, y_2)	L	l	m	$\frac{AE}{L}$
1	(0,-180)	(120,-180)	120	1	0	$3(10^5)$
2	(0,0)	(120,0)	120	1	0	$3(10^5)$
3	(0,0)	(0,-180)	180	0	-1	$2(10^5)$
4	(120,0)	(120,-180)	180	0	-1	$2(10^5)$
5	(0,-180)	(120,0)	216.333	0.5547	0.832051	$1.664(10^5)$
6	(0,0)	(120,-180)	216.333	0.5547	-0.832051	$1.664(10^5)$

Example

Note $$\mathbf{k}_{4\times 4} = \mathbf{T}^{\mathrm{T}}_{4\times 2}\mathbf{k}'_{2\times 2}\mathbf{T}_{2\times 4}$$

$$\mathbf{k}_{4\times 4} = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Example

Element 1

$$10^5 \begin{array}{c} \boxed{\begin{array}{cccc} 1 & 2 & 7 & 8 \end{array}} \\ \begin{bmatrix} 3 & 0 & \times & 0 \\ 0 & 0 & \times & 0 \\ \times & \times & \times & \times \\ 0 & 0 & \times & 0 \end{bmatrix} \end{array} \begin{Bmatrix} D_1 \\ D_2 \\ D_7 \\ D_8 \end{Bmatrix}$$

Element 2

$$10^5 \begin{array}{c} \boxed{\begin{array}{cccc} 3 & 4 & 5 & 6 \end{array}} \\ \begin{bmatrix} 3 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \end{array} \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}$$

Example

Step 4: System Equations (after BCs)

$$(10)^5 \begin{bmatrix} 3.5120 & 0.7679 & 0 & 0 & 0 \\ 0.7679 & 3.1520 & 0 & -2 & 0 \\ 0 & 0 & 3.5120 & -0.7679 & 0.7679 \\ 0 & -2 & -0.7679 & 3.1520 & -1.1520 \\ 0 & 0 & 0.7679 & -1.1520 & 3.1520 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -4000 \\ 0 \end{Bmatrix}$$

Example

Step 5

$$\{D_1, D_2, D_3, D_4, D_5, D_8\} = 10^{-3}\{4.44367, -20.3232, 4.44367, -30.3232, -10\}\ in$$

Step 6

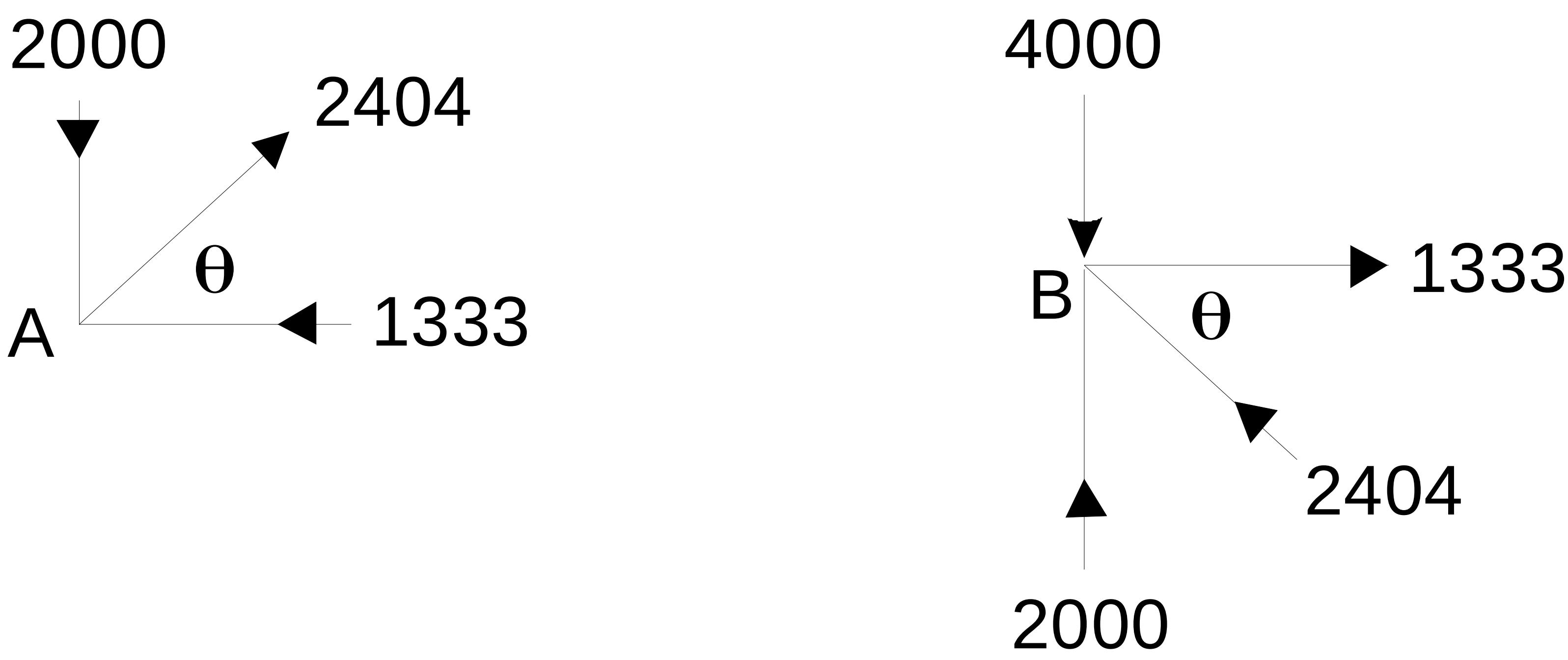
$$f_1^{'} = 3(10^5)[1 \quad 0 \quad -1 \quad 0][D_1 \quad D_2 \quad D_7 \quad D_8]^{\mathbf{T}} = 1333\,\mathrm{lb}$$

....

$$f_4^{'} = 2(10^5)[0 \quad -1 \quad 0 \quad 1][D_5 \quad D_6 \quad D_7 \quad D_8]^{\mathbf{T}} = -2000\,\mathrm{lb}$$

Example

Equilibrium Check



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