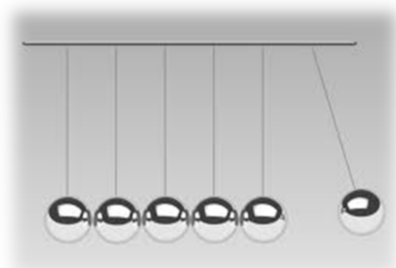
CEE 212—Dynamics

Vibrations

Classical analysis of linear systems



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Vibrations

elastic systems

Objectives. In this module we extend the ideas of dynamics to include a very common type of response: vibration. The forces developed in deformable systems generally provide restoring forces to the system that result in oscillations. In fact, any linear system that has a restoring force will exhibit this characteristic. This mode of behavior is particularly important for civil and mechanical systems where deformations under force are expected to remain small.

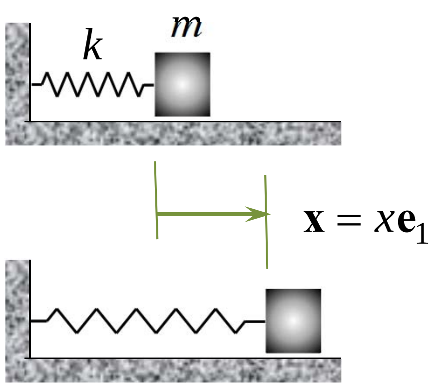
We go through the complete derivation of the classical solution to the linear second order differential equation to show that the character of the solution is oscillatory. Further, we find that the system has a *natural frequency of vibration.* This fact is probably *the* big deal of the module. This property—generally the ratio of stiffness to mass—is a very important and useful characterization of the dynamic properties of the system.

We show how the systems we have been analyzing throughout the course can be examined through the lens of vibration through a process of *linearization* of the equations of motion. We provide an invitation to this concept through the pendulum. Any system can be linearized and the engineering question is whether or not the motions are small enough to justify that assumption.

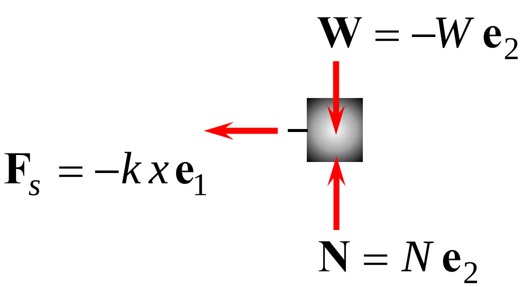
We finally extend the ideas to a system with two degrees of freedom. The classical solution is a bit more involved, but it can be solved completely and analytically. Generalization of this concept is one of the fundamental building blocks of structural dynamics and earthquake engineering. The big idea from the multiple degrees of freedom is that the system has more than one *natural frequency* and that it can oscillate at those frequencies, but only in certain *mode shapes*.

Vibration

Single degree of freedom



Geometry



Free body diagram

Single degree of freedom system. Consider the spring-mass system shown. Let us formulate the equations of motion. Assume that the mass slides without friction on the base. The kinematics are

$$\mathbf{x} = x\,\mathbf{e}_1$$
$$\dot{\mathbf{x}} = \dot{x}\,\mathbf{e}_1$$
$$\ddot{\mathbf{x}} = \ddot{x}\,\mathbf{e}_1$$

Balance of linear momentum can be written from the free body diagram

$$\mathbf{F}_s + \mathbf{W} + \mathbf{N} = m\mathbf{a}$$
$$-k\,x\mathbf{e}_1 - W\,\mathbf{e}_2 + N\,\mathbf{e}_2 = m\ddot{x}\mathbf{e}_1$$

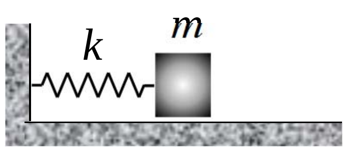
Dotting with $\mathbf{e}_1$ and $\mathbf{e}_2$, in turn gives

$$m\ddot{x} + kx = 0$$
$$N = W$$

The first equation is our equation of motion. The second, is a static equation because motion is prevented in the vertical direction. The first equation needs to be integrated in time to get a solution to the problem.

Vibration

Single degree of freedom



$$m\ddot{x} + kx = 0$$
$$x(0) = x_o$$
$$\dot{x}(0) = v_o$$

Solutions are possible *only* if

$$\omega = \sqrt{\frac{k}{m}}$$

This is called *the natural frequency of vibration* of the system. This is a feature of linear systems.

Solving the differential equation. The equation of motion for this system is *a linear second order ordinary differential equation*. We can solve such an equation analytically. The governing equations (with initial conditions) are shown at left. What we know about differential equations of this type is that the solution is in the form of trigonometric functions. Let us consider the solution to be

$$x(t) = B_1 \sin \omega t + B_2 \cos \omega t$$
$$\dot{x}(t) = \omega\left(B_1 \cos \omega t - B_2 \sin \omega t\right)$$
$$\ddot{x}(t) = -\omega^2\left(B_1 \sin \omega t + B_2 \cos \omega t\right) = -\omega^2 x(t)$$

Let's plug these results into the governing differential equation to see if it can satisfy it.

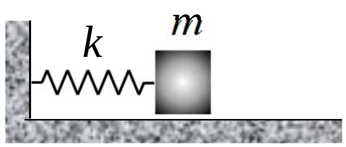
$$m\ddot{x} + kx = 0$$
$$m\left(-\omega^2 x\right) + kx = 0$$
$$\left(k - \omega^2 m\right)x(t) = 0$$

To have solutions other than $x(t) = 0$ it is evident that the only possibility is

$$\boxed{k - \omega^2 m = 0}$$

Vibration

Single degree of freedom



$$m\ddot{x} + kx = 0$$
$$x(0) = x_o$$
$$\dot{x}(0) = v_o$$

A *classical solution* to a differential equation is one that satisfies the governing equations exactly at all points in time. They can generally be written in terms of *named functions* (e.g., polynomials, trigonometric functions, etc.). Classical solutions are great for insight and for verification of computer programs.

Implementing initial conditions. The solution to the equations has only been found in general terms. To be a complete solution the function of time must also satisfy the *initial conditions* of the problem. Substituting the initial conditions gives

$$x(0) = x_o = B_2$$
$$\dot{x}(0) = v_o = \omega B_1$$

It is interesting to note that we have two constants and we have two initial conditions. That is more than a coincidence. A well-posed problem will always have this balance. If we solve for the constant and substitute back into the general equation we get the final solution.

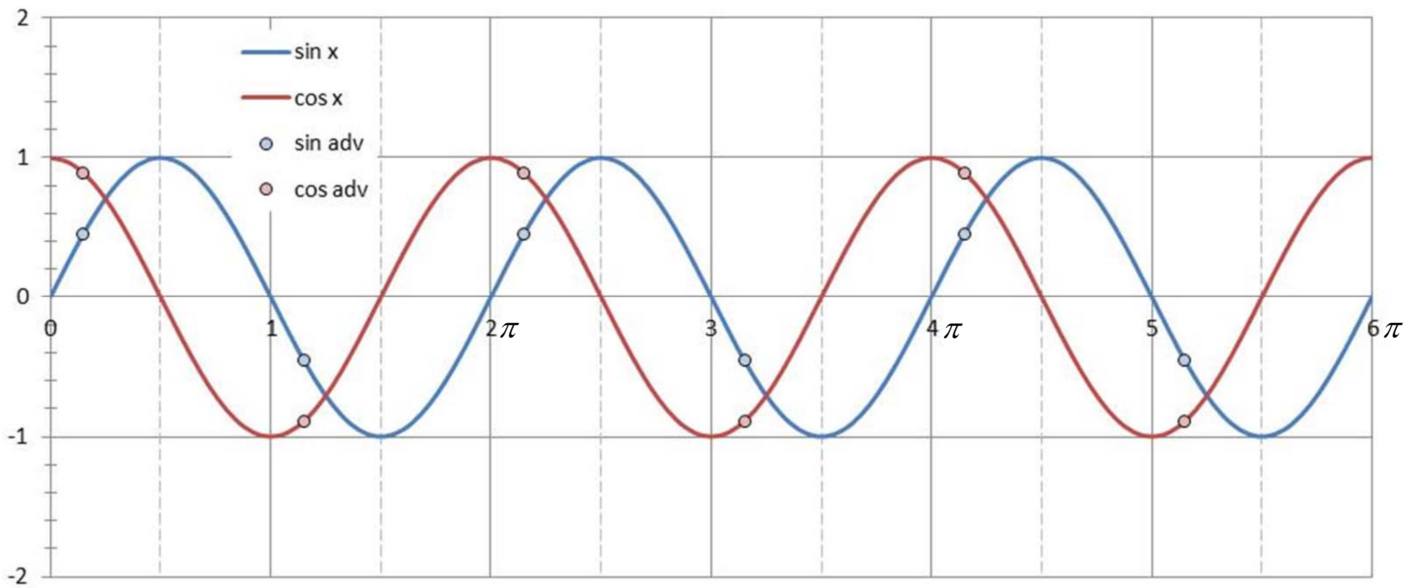
$$x(t) = \frac{v_o}{\omega}\sin\omega t + x_o\cos\omega t$$

Now the position is a completely known function of time (contrast that with a numerical solution where we generate the solution as a sequence of points along the response curve). As a check we can substitute this solution back into the differential equation and initial conditions to verify that all equations are satisfied at all points in time.

Vibration

Oscillation and the sinusoids

Oscillation and the sinusoids. The sine and cosine functions are important because the embody the notion of oscillation. They are the most basic periodic functions (i.e., functions that return to the same value after some period).

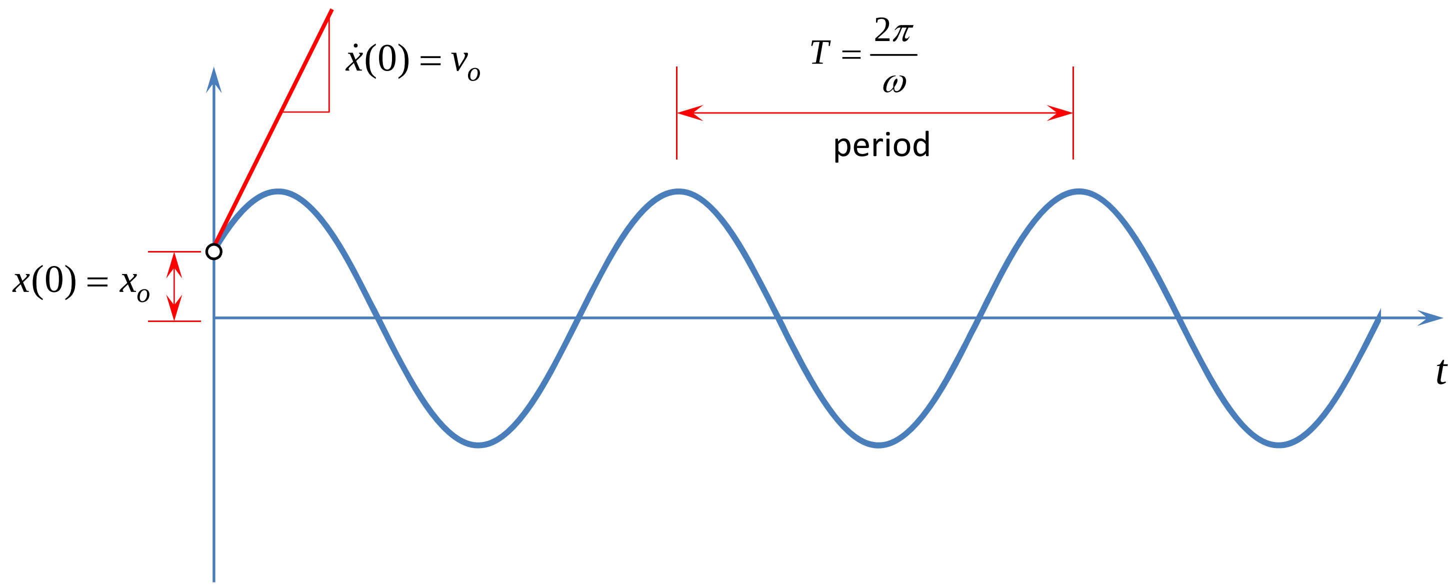


In this figure we show the sine function and the cosine function superimposed on each other. The blue and pink dots are values of the functions at values of *x* that are advanced from each other by a value π, 2π, 3π, etc. You can see the repeating patters of the values that result. One of the key features of the sine and cosine is that one is zero where the other one isn't. That gives them the ability to satisfy any arbitrary initial conditions in the vibration problem

Vibration

Frequency and period

Frequency and period. The frequency is the most natural constant to emerge from solving the differential equation. The *period* is the most natural way to describe the time it takes to return to the same point of response. The period is also the measure from peak to peak of the oscillation.

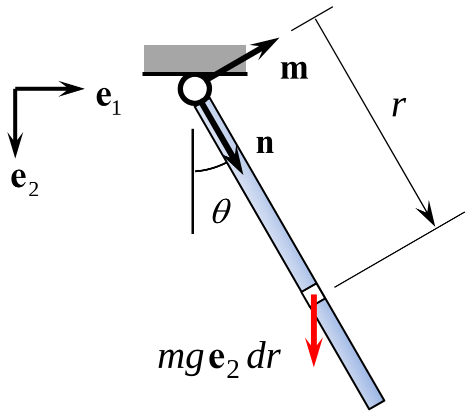


$$x(t) = \frac{v_o}{\omega} \sin \omega t + x_o \cos \omega t$$

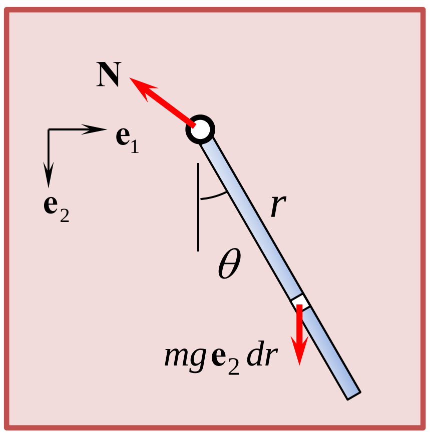
The above graph is a plot of the complete solution to the spring/mass vibration problem. Note that the response starts at time zero with the initial position and has a slope equal to the initial velocity. The response from there on is a pure sinusoid.

Vibrations

The pendulum



$$\mathbf{n} = \sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2$$
$$\mathbf{m} = \cos\theta\,\mathbf{e}_1 - \sin\theta\,\mathbf{e}_2$$



The pendulum. One of the most basic examples of a vibrating system is the pendulum. The pendulum has no elastic elements but oscillates under the force of gravitational restoring force. The position, velocity, and acceleration are

$$\mathbf{x}(r,t) = r\,\mathbf{n}(\theta(t))$$
$$\dot{\mathbf{x}} = r\,\dot{\mathbf{n}} = r\dot{\theta}\mathbf{m}$$
$$\ddot{\mathbf{x}} = r\left(\ddot{\theta}\mathbf{m} - \dot{\theta}^2\mathbf{n}\right)$$

Where the unit vectors $\mathbf{n}(t)$ and $\mathbf{m}(t)$ have the explicit expressions given at left.

Balance of linear and angular momentum imply that:

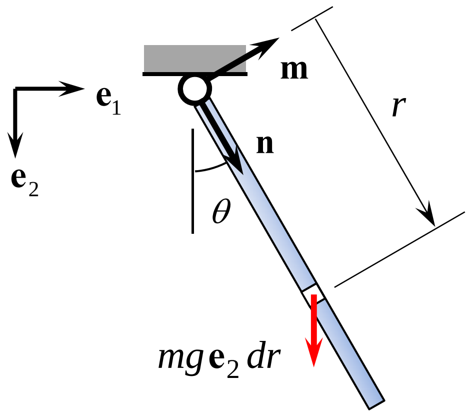
$$\mathbf{N} + \int_0^{\ell} mg\mathbf{e}_2 dr = \int_0^{\ell} m\ddot{\mathbf{x}}\,dr$$

$$\int_0^{\ell} r\,\mathbf{n}\times mg\,\mathbf{e}_2\,dr = \int_0^{\ell} r\,\mathbf{n}\times m\ddot{\mathbf{x}}\,dr$$

where $\mathbf{N}$ is the reaction force at the pin. These two equations should be sufficient to determine the motion of the system.

Vibrations

The pendulum



Pendulum equations. Substituting the expression for the acceleration in the second equation and carrying out the indicated integrations gives

$$\ddot{\theta} + \left(\frac{3g}{2L}\right)\sin\theta = 0$$

This equation involves only the angle θ, but it is a nonlinear differential equation. We can make some headway if we consider only motions with only small values of θ so that $\cos\theta = 1$ and $\sin\theta = \theta$. With such an assumption, the equation reduces to

$$\ddot{\theta} + \left(\frac{3g}{2\ell}\right)\theta = 0$$

Let $\omega^2 \equiv \dfrac{3g}{2\ell}$

This is an ordinary differential equation with constant coefficients. It has the general solution

$$\theta(t) = A\cos\omega t + B\sin\omega t \qquad \dot{\theta}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

where A and B are constants that must be determined from the initial conditions. Let the initial conditions be

$$\theta(0) = \theta_o \qquad \dot{\theta}(0) = \omega_o$$

With these conditions the solution to the equation is

$$\theta(t) = \theta_o\cos\omega t + \frac{\omega_o}{\omega}\sin\omega t$$

The interesting feature of this motion is that it is oscillatory, gravity supplies the restoring force.

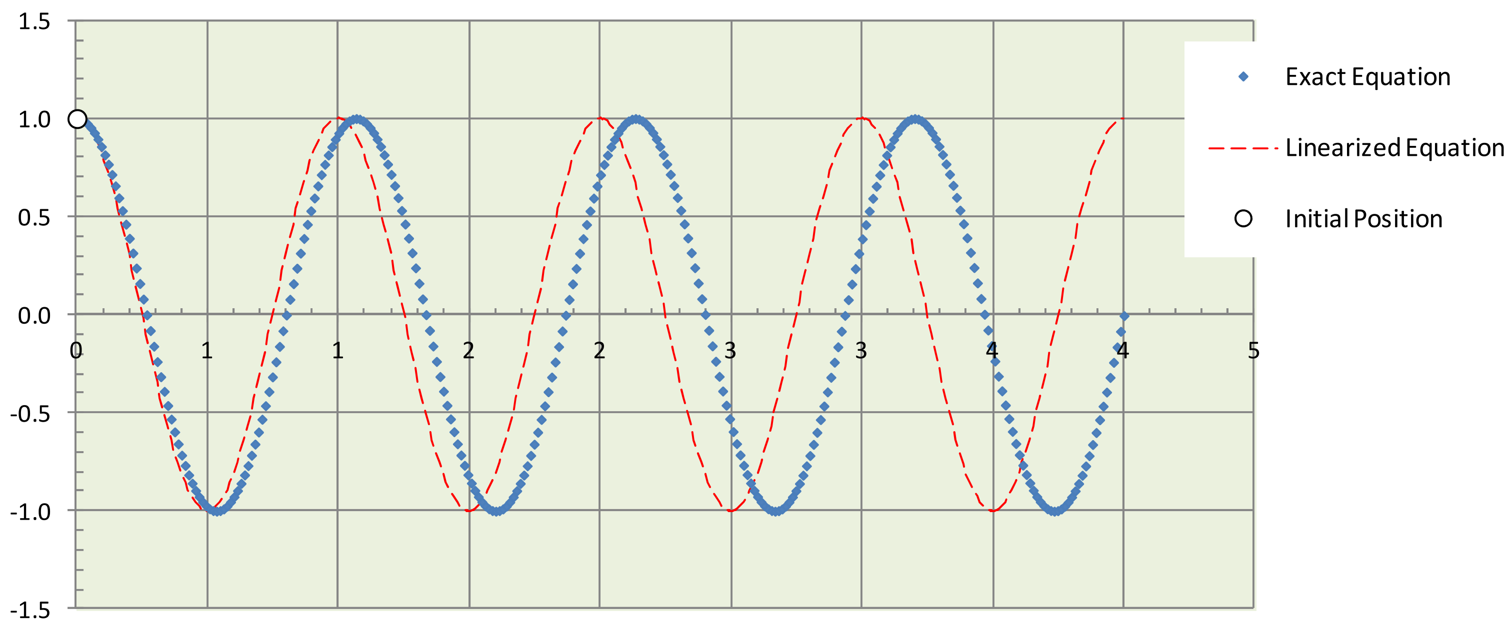
Vibrations

The pendulum

$$\ddot{\theta} + \left(\frac{3g}{2\ell}\right)\sin\theta = 0$$

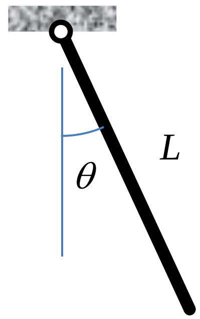
$$\ddot{\theta} + \left(\frac{3g}{2\ell}\right)\theta = 0$$

Pendulum Response. The response of the pendulum is an oscillatory motion. Any system that has a restoring force will exhibit this sort of motion because there is always a force to resist the motion—pulling it back to some preferred spot. This graph shows how the nonlinear equations of motion have a solution that is different from the linear equations of motion. However, both are oscillatory. For small values of the initial conditions the two systems give a very similar response, justifying the concept of linearization of the equations of motion.



Vibration

Linear Systems



$$\ddot{\theta} + \left(\frac{3g}{2L}\right)\sin\theta = 0$$

$$\ddot{\theta} + \left(\frac{3g}{2L}\right)\theta = 0$$

The result of linearization is a Linear Ordinary Differential Equation with Constant Coefficients!!

Linearization. The governing differential equation of motion for a dynamical system is generally not linear. The simple pendulum is a case in point. It should be fairly clear that the pendulum exhibits oscillatory motion and, hence, can be considered a vibrating system. We can gain additional insight through the process of linearization of the equations of motion.

Linearization means to make linear. Linearity is a property of polynomials. Taylor's theorem suggests that any function can be expressed as a polynomial through its Taylor Series expansion.

$$f(x) = f(a) + (x-a)f'(a) + \tfrac{1}{2!}(x-a)^2 f''(a) + \cdots + \tfrac{1}{n!}(x-a)^n f^n(a) + \cdots$$

To linearize, then, means to expand in a Taylor Series and then lop off all terms beyond the linear term. Since we know the series expression for many functions linearization is generally pretty straightforward process. For example, for the trigonometric functions we use the approximations

$$\sin\theta \approx \theta$$
$$\cos\theta \approx 1$$

Vibration

Linear Systems

$$\ddot{\theta}+\left(\frac{3g}{2L}\right)\theta=0$$

Possibilities:

$$e^{i\omega t}\neq 0$$

$$A=0$$

$$\left[-\omega^2+\left(\frac{3g}{2L}\right)\right]=0$$

A=0 means no motion. Thus, motion is possible only if

$$\omega=\pm\sqrt{\frac{3g}{2L}}$$

This value is called the *natural frequency* of the system.

Linear ODEs with Constant Coefficients. The linearized pendulum equation is an example of a linear ordinary differential equation with constant coefficients (i.e., they do not depend upon time). The solution to this type of equation is always an exponential function. The process by which we figure out which exact exponential function we need is instructive.

Let us assume that the function we seek has the form of an exponential with an unknown magnitude and an unknown exponent (we know it is a function of t we just don't know what the rate is). Take derivatives, see what happens

$$\theta(t)=Ae^{i\omega t}$$

$$\dot{\theta}(t)=i\omega Ae^{i\omega t}$$

$$\ddot{\theta}(t)=\left(i\omega\right)^2 Ae^{i\omega t}=-\omega^2 Ae^{i\omega t}$$

Substituting this result into the original differential equation gives

$$-\omega^2 Ae^{i\omega t}+\left(\frac{3g}{2L}\right)Ae^{i\omega t}=0$$

$$\left[-\omega^2+\left(\frac{3g}{2L}\right)\right]Ae^{i\omega t}=0$$

Because the exponential can never be zero the two conditions (at left) are possible.

Vibration

Linear Systems

$$\ddot{\theta} + \left(\frac{3g}{2L}\right)\theta = 0$$

Initial conditions determine the values of the constants. To wit,

$$\theta(0) = \theta_o = B_2$$
$$\dot{\theta}(0) = \dot{\theta}_o = \omega B_1$$

Solving for B_1 and B_2 in terms of the initial angle and initial angular velocity gives the *complete solution*

$$\theta(t) = \frac{\dot{\theta}_o}{\omega}\sin\omega t + \theta_o \cos\omega t$$

The nature of the solution shows why we call the value ω in the solution the "frequency."

The complete solution. The interesting thing about assuming the form of exponentials is that when we implement that assumption into the governing differential equation it results in a quadratic equation for a quantity that we call the *natural frequency*. We will now see why we call it that. The reason we get a quadratic equation is because the differential equation has order two (the highest derivative is a second derivative). You can see from the algebra how this turns out to yield a quadratic.

The truth about quadratic equations is that they always yield two roots. That means that there are really two exponential functions that will work. We take the general solution as the sum of the two (each with its, as yet, unknown coefficient)

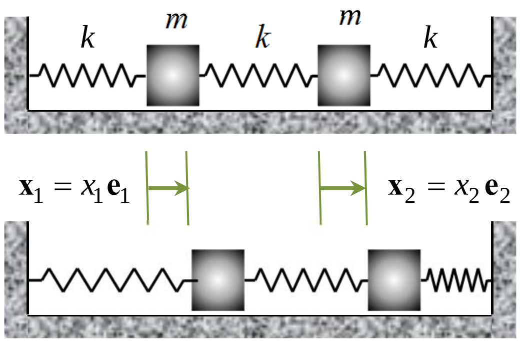
$$\begin{aligned}\theta(t) &= A_1 e^{i\omega t} + A_2 e^{-i\omega t} \\ &= B_1 \sin\omega t + B_2 \cos\omega t\end{aligned}$$

The identity that allows us to convert from complex exponentials and trigonometric functions is due to Euler and is called *Euler's Formula*. You can really retrace your steps assuming trigonometric functions at the start to come to the same place. The derivatives are

$$\dot{\theta}(t) = \omega\left(B_1 \cos\omega t - B_2 \sin\omega t\right)$$
$$\ddot{\theta}(t) = -\omega^2\left(B_1 \sin\omega t + B_2 \cos\omega t\right) = -\omega^2\theta(t)$$

Vibration

Multiple degrees of freedom



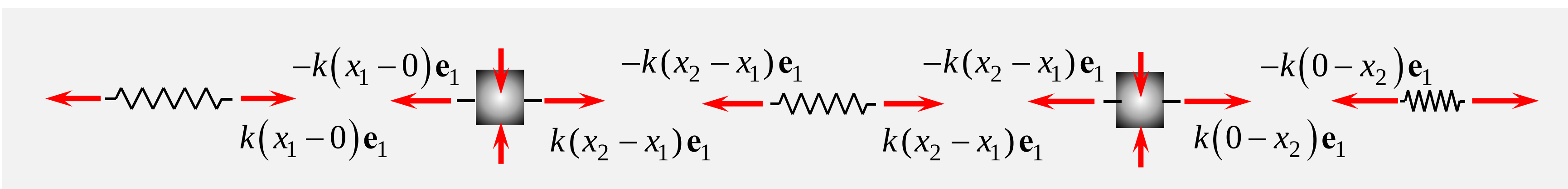
Geometry

Two degrees of freedom. It is interesting to examine what happens when the system has more than a single degree of freedom. The one pictured at left has two degrees of freedom and the governing equations. There are two masses and hence we will keep track of two positions. The kinematics are:

$$
\begin{aligned}
\mathbf{x}_1 &= x_1\,\mathbf{e}_1 & \qquad \mathbf{x}_2 &= x_2\,\mathbf{e}_1 \\
\dot{\mathbf{x}}_1 &= \dot{x}_1\,\mathbf{e}_1 & \qquad \dot{\mathbf{x}}_2 &= \dot{x}_2\,\mathbf{e}_1 \\
\ddot{\mathbf{x}}_1 &= \ddot{x}_1\,\mathbf{e}_1 & \qquad \ddot{\mathbf{x}}_2 &= \ddot{x}_2\,\mathbf{e}_1
\end{aligned}
$$

Balance of linear momentum comes from the free body diagrams of the masses. We can determine the spring forces by considering the amount of stretch of each spring due to the motion of the masses:

Free body diagrams



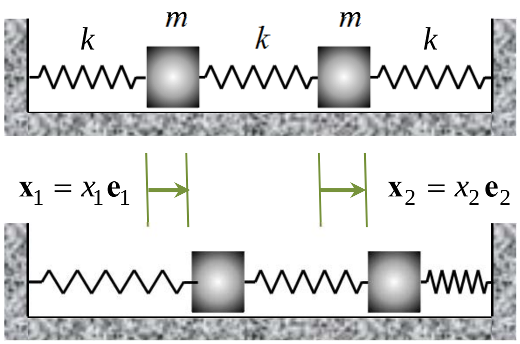
$$
\begin{aligned}
\mathbf{e}_1\cdot\left(\mathbf{F}_s+\mathbf{W}+\mathbf{N}\right) &= \mathbf{e}_1\cdot\left(m\mathbf{a}\right) \\
-k\,x_1+k\left(x_2-x_1\right) &= m\ddot{x}_1 \\
-k\left(x_2-x_1\right)-k\,x_2 &= m\ddot{x}_2
\end{aligned}
$$

The equations of motion are, therefore:

$$
\boxed{
\begin{aligned}
m\ddot{x}_1+2k\,x_1-k\,x_2 &= 0 \\
m\ddot{x}_2-k\,x_1+2k\,x_2 &= 0
\end{aligned}
}
$$

Vibration

Multiple degrees of freedom



Note that by assuming that the time part of the solution is in the form of trigonometric functions we get

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

That is like the single degree of freedom system. We need to see if this can lead to a solution for the two degree of freedom system.

Two degrees of freedom. We can put the equations of motion for the two degree of freedom system into the form:

$$\boxed{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

The key feature of the system is that you cannot simply integrate one or the other of these equations because both variables are present in both equations. Let us use the same reasoning as before to explore the system of equations.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix}\left(A\sin\omega t + B\cos\omega t\right)$$

where φ_1, φ_2, A, and B, are constants. Taking derivatives gives

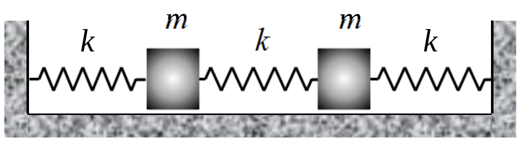
$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \omega\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix}\left(A\cos\omega t - B\sin\omega t\right)$$

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\omega^2\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix}\left(A\sin\omega t + B\cos\omega t\right)$$

We will substitute these results into the equation of motion to see if it is possible to satisfy the differential equation.

Vibration

Multiple degrees of freedom



Solve the *characteristic equation*:

$$4k^2 - 4km\omega^2 + m^2\omega^4 - k^2 = 0$$
$$m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$$

This equation can be written as

$$\left(m\omega^2 - k\right)\left(m\omega^2 - 3k\right) = 0$$

Thus, there are four values of ω that work. Positive and negative of

$$\omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3k}{m}}$$

Corresponding to two *natural frequencies* of the system.

Two degrees of freedom. Substituting the assumed functions into the equations of motion give:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

which can be written as

$$\begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

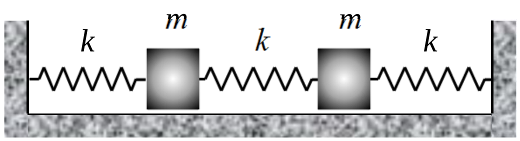
The situation is a little different from the single degree of freedom system, but it is also very similar. We want a solution that is not the zero solution. Recall from linear algebra that the only way this system of algebraic equations can have a non-trivial solution is if the determinant of the coefficient matrix is zero. Hence,

$$\det\begin{bmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{bmatrix} = \left(2k - \omega^2 m\right)^2 - k^2 = 0$$

This equation is called the *characteristic equation*. What this means is that there may be a value of *w* that works. If it does exist then we have a general solution to the governing differential equation. We solve this equation in the box at left.

Vibration

Multiple degrees of freedom



Mode shapes. Since we got two different values for ω we have to include both possibilities in our solution. There still remains a required condition for solution that we have not yet attended to. This conditions is related to the *characteristic equation*. Recall that we sought values of w that made the system of equations singular. That implies that there is a dependency of the two equations. This consideration gives rise to the idea of *mode shapes*.

The following function satisfies the governing differential equation (check it if you don't believe it)

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} \left(A_1 \sin \omega_1 t + B_1 \cos \omega_1 t \right) + \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} \left(A_2 \sin \omega_2 t + B_2 \cos \omega_2 t \right)$$

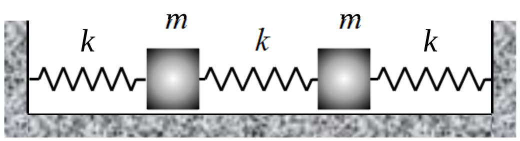
Now we have 8 constants, but we have only 4 initial conditions (2 for each mass). There must be some other condition that we have not accounted for. If we plug our candidate solutions back in to the governing equations we find that the constant must satisfy

$$\begin{bmatrix} 2k - \omega_1^2 m & -k \\ -k & 2k - \omega_1^2 m \end{bmatrix} \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \qquad \begin{bmatrix} 2k - \omega_2^2 m & -k \\ -k & 2k - \omega_2^2 m \end{bmatrix} \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The values that satisfy these equations are called the *mode shapes* because these equations only determine how the two components must be in proportion to each other. The equations are not sufficient to find the actual magnitudes of both. Because they multiply the two other unknown constants we can simply select a scaling value arbitrarily.

Vibration

Multiple degrees of freedom



Mode shapes. Let us implement the *mode shape* relationship into the governing equations. If we go back to the equations that let to the *characteristic equation* and substitute the values of ω that we found, then we can see the implication. The equations are singular and hence the values of the solution φ are not completely free to be anything they want. They have to exist in a certain shape.

Mode 1.

$$\begin{bmatrix} 2k-\omega_1^2 m & -k \\ -k & 2k-\omega_1^2 m \end{bmatrix} \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \xrightarrow{\omega_1^2 = \frac{k}{m}} \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \varphi_{11} \\ \varphi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \boxed{\varphi_{11} = \varphi_{21}}$$

Mode 2.

$$\begin{bmatrix} 2k-\omega_2^2 m & -k \\ -k & 2k-\omega_2^2 m \end{bmatrix} \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \xrightarrow{\omega_2^2 = \frac{3k}{m}} \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \boxed{\varphi_{12} = -\varphi_{22}}$$

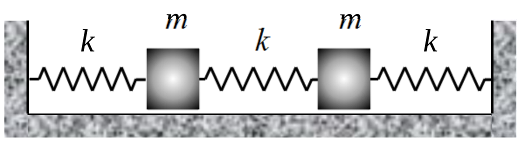
With these relationships established we can write our general equation as

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \left(A_1 \sin \omega_1 t + B_1 \cos \omega_1 t \right) + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \left(A_2 \sin \omega_2 t + B_2 \cos \omega_2 t \right)$$

Now all that remains is the implementation of the initial conditions for the problem to determine the four remaining constants.

Vibration

Multiple degrees of freedom



Initial Conditions. To get the remaining constants we must implement the initial conditions. We know the position and velocity of each mass at time $t = 0$. That is enough information to find the four constants. It should be evident that these ideas carry over to any number of masses (with a bit of additional algebraic complexity).

The position and velocity are given by the general solution that we have found. Specifically,

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \left(A_1 \sin \omega_1 t + B_1 \cos \omega_1 t \right) + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \left(A_2 \sin \omega_2 t + B_2 \cos \omega_2 t \right)$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \omega_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \left(A_1 \cos \omega_1 t - B_1 \sin \omega_1 t \right) + \omega_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \left(A_2 \cos \sin \omega_2 t - B_2 \sin \omega_2 t \right)$$

At $t = 0$ the position and velocity are equal to the initial position and initial velocity. Hence, we have

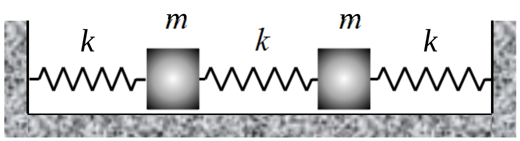
$$\begin{Bmatrix} x_{o1} \\ x_{o2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \left(B_1 \right) + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \left(B_2 \right), \qquad \begin{Bmatrix} v_{o1} \\ v_{o2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \left(\omega_1 A_1 \right) + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \left(\omega_2 A_2 \right)$$

Solving these for the constants gives

$$\begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} x_{o1} + x_{o2} \\ x_{o1} - x_{o2} \end{Bmatrix} \qquad \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \frac{1}{2\omega_1 \omega_2} \begin{Bmatrix} \omega_2 \left(v_{o1} + v_{o2} \right) \\ \omega_1 \left(v_{o1} - v_{o2} \right) \end{Bmatrix}$$

Vibration

MATLAB and the eigenvalue problem



For the present example the quantities are defined as follows:

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$\ddot{\mathbf{x}} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}$$

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

MATLAB and the eigenvalue problem. Our two degree of freedom system has equations of motion of the form:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

We can write this in matrix form as

$$\boxed{\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}}$$

The assumption of periodic motion can be written as

$$\mathbf{x} = \boldsymbol{\varphi}\left(A\sin\omega t + B\cos\omega t\right)$$

$$\ddot{\mathbf{x}} = -\omega^2\boldsymbol{\varphi}\left(A\sin\omega t + B\cos\omega t\right)$$

The equations of motion then give rise to the following *eigenvalue problem*:

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\boldsymbol{\varphi} = \mathbf{0}$$

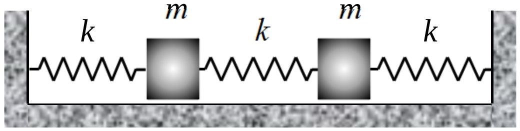
MATLAB solves this eigenvalue problem with the command

$$[\mathbf{V}, \mathbf{D}] = \mathrm{eig}\left(\mathbf{K}, \mathbf{M}\right)$$

The columns of **V** are the eigenvectors $\boldsymbol{\varphi}$ and the diagonal elements of the matrix **D** are the eigenvalues ω^2.

Vibration

General case



Evaluate at time t=0

$$\mathbf{x}(0) = \mathbf{x}_o = \sum_{i=1}^{N} \boldsymbol{\varphi}_i B_i$$

$$\dot{\mathbf{x}}(0) = \mathbf{v}_o = \sum_{i=1}^{N} \omega_i \boldsymbol{\varphi}_i A_i$$

Pre-multiply by $\boldsymbol{\varphi}_j^T \mathbf{M}$, note orthogonality, and solve to get

$$B_j = \frac{\boldsymbol{\varphi}_j^T \mathbf{M} \mathbf{x}_o}{\boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_j}$$

$$A_j = \frac{1}{\omega_j} \frac{\boldsymbol{\varphi}_j^T \mathbf{M} \mathbf{v}_o}{\boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_j}$$

The general case. For the general case the equations of motion of free (un-damped) vibration are:

$$\boxed{\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}}$$

where **K** and **M** are N by N matrices (and at least **M** is positive definite). There are N eigenvalues and eigenvectors ω_i and $\boldsymbol{\varphi}_i$. The general solution (and its derivatives) is

$$\mathbf{x} = \sum_{i=1}^{N} \boldsymbol{\varphi}_i \left(A_i \sin \omega_i t + B_i \cos \omega_i t \right)$$

$$\dot{\mathbf{x}} = \sum_{i=1}^{N} \omega_i \boldsymbol{\varphi}_i \left(A_i \cos \omega_i t - B_i \sin \omega_i t \right)$$

$$\ddot{\mathbf{x}} = -\sum_{i=1}^{N} \omega_i^2 \boldsymbol{\varphi}_i \left(A_i \sin \omega_i t + B_i \cos \omega_i t \right)$$

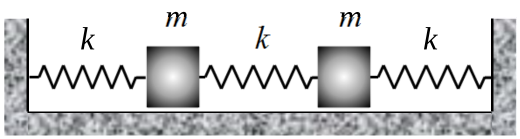
The constants A_i and B_i can be determined by the initial conditions (as shown at left), using the important condition of *orthogonality of the eigenvectors*

$$\boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_j = 0 \quad \text{if } \omega_i \neq \omega_j$$

There are always exactly the same number of initial conditions as unknown constants (i.e., 2N).

Vibration

General case



Why CP7 then?

Note that this classical analysis does not include external forces, nor does it allow for nonlinearity (e.g., yielding). It is fairly straightforward to extend the ideas to a damped system (with velocity-proportional damping) and to certain external load forms. However, it is not generally possible to extend this approach to non-linear response. Hence, the numerical solution of the vibration problem is generally used in practice for extreme loads like earthquakes and other rare, but intense, natural or man-made events

The general case (cont'd). Substituting the values of the constants we get the final expression:

$$\mathbf{x}(t) = \sum_{i=1}^{N} \boldsymbol{\varphi}_i \left(\frac{1}{\omega_i} \left(\frac{\boldsymbol{\varphi}_i^T \mathbf{M} \mathbf{v}_o}{\boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_i} \right) \sin \omega_i t + \left(\frac{\boldsymbol{\varphi}_i^T \mathbf{M} \mathbf{x}_o}{\boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_i} \right) \cos \omega_i t \right)$$

This is the classical solution to the differential equation of vibration. It can be used to verify the numerical calculations in CP7.

This is really the foundation of much of structural dynamics because many structures are designed to respond linearly (at least over a range of loads that are expected to occur frequently). You can see the fundamental role of the eigenvalue problem in solving this equation.

There are a few observations that one can make from the above equation. First, if the initial displacement or velocities are in the same ratios as one of the eigenvectors then the response shape will be in that shape and will oscillate in the associated natural frequency (because of orthogonality of the eigenvectors). If not then the response will be a complex mix of different shapes vibrating at different frequencies to make up the total response (by superposition since these terms are summed).

Vibrations

Summary

Summary. In this module we stepped back a bit from the general formulation and general problem solving that we have pursued throughout the course. The primary objective of the course was to show that we can systematically formulate the equations of motion for a dynamical system and that the equations of motion are, almost always, nonlinear. We developed numerical methods to deal with solving nonlinear differential equations.

That said, there are certain problems that are amenable to classical solutions methods (i.e., solving differential equations in closed form). Those problems fall into the category of linear second order ordinary differential equations with constant coefficients. Small amplitude vibration of linear systems are just such systems. In this module we learned how to set up and solve those differential equations. We also learned that nonlinear equations can be linearized and those linear equations can be treated with the methods presented herein.

Classical solutions bring tremendous insight into the behavior of oscillating systems and they provide useful verification problems for computational methods (even for nonlinear problems because we can often specify the conditions such that small motions result).

Along the way we have run into the concepts of *natural frequency (and period) of vibration*, *natural modes of vibration,* and *linearization.* These concepts provide the building blocks for many other ideas in mechanics and are particularly important to structural dynamics and earthquake engineering.