Finite Elements for Engineers

Lecture 6: Diffusion Problems

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1D Mixed Initial BVP

DE

$$\mu(x)\frac{\partial u(x,t)}{\partial t}-\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u(x,t)}{\partial t}\right)+\beta(x)u(x,t)=f(x,t)$$

Domain

$$x_a \le x \le x_b \qquad t > t_0$$

BCs

At $x = x_a$ and $t > t_0$

$$u(x_a,t)=u_a(t) \text{ or } \left(-\alpha(x)\frac{\partial u}{\partial x}\right)_{x_a}=\tau_a$$

At $x = x_b$ and $t > t_0$

$$u(x_b,t)=u_b(t) \text{ or } \left(-\alpha(x)\frac{\partial u}{\partial x}\right)_{x_b}=\tau_b$$

ICs

$$\text{At } t_0 \quad \left(x_a < x < x_b\right)$$

$$u(x,t_0)=u_0(x)$$

1D Mixed Initial BVP

Trial Solution $$u(x,t,a)=\sum_{j=1}^{n} a_j(t)\phi_j(x)$$

Step 1: Galerkin's Method – Residual Equations

$$\int_{\Omega}\left[\mu(x)\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right)+\beta(x)u-f(x,t)\right]\phi_i(x)dx=0 \quad i=1,2,\ldots,n$$

Step 2: Galerkin's Method – Integration of Parts

$$\int_{\Omega}\phi_i(x)\mu(x)\frac{\partial u}{\partial t}dx+\int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{\partial u}{\partial x}dx+\int_{\Omega}\phi_i(x)\beta(x)u(x,t,a)dx$$

$$=\int_{\Omega}f(x,t)\phi_i(x)dx-\left[\left(-\alpha(x)\frac{\partial u}{\partial x}\right)\phi_i(x)\right]_{x_1}^{x_n}$$

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Step 3: Galerkin's Method – Element Equations

$$\mathbf{c}\left\{\frac{da(t)}{dt}\right\}+\mathbf{k}\left\{a(t)\right\}=\left\{f(t)\right\}$$

Capacity Matrix: $c_{ij}=\int_{\Omega}\phi_i(x)\mu(x)\phi_j(x)\,dx$

$$k_{ij}=\int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx+\int_{\Omega}\phi_i\beta(x)\phi_j dx$$

$$f_i\left(t\right)=\int_{\Omega}f(x,t)\phi_i(x)dx-\left[\tau\phi_i\right]_{x_1}^{x_n}$$

1D Mixed Initial BVP

Step 4: 1D-C^0 linear element

$$\mathbf{c}_{2\times 2} = \frac{\hat{\mu} L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

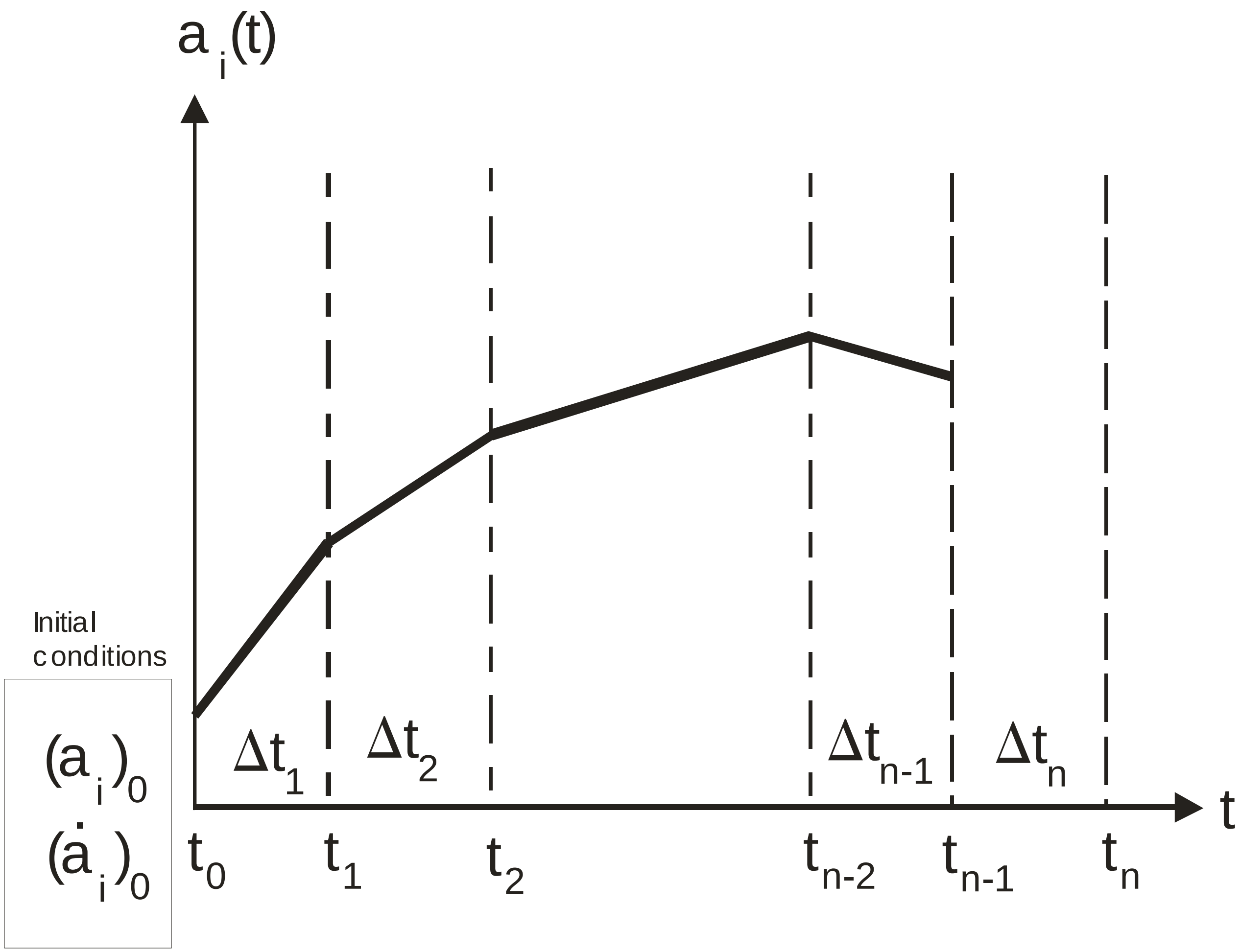
$$\text{where } \hat{\mu} = \mu(x_c) = \mu\left(\frac{x_1 + x_2}{2}\right)$$

Diffusion Problem

System Equations

$$\mathbf{C}\left\{\frac{da(t)}{dt}\right\}+\mathbf{K}\left\{a(t)\right\}=\left\{F(t)\right\}$$

Numerical Solution



Linear One-Step Method

Recurrence relation

$$\mathbf{P}\mathbf{a}_n + \mathbf{Q}\mathbf{a}_{n-1} = p\mathbf{F}_n + q\mathbf{F}_{n-1}$$

n=1

$$\mathbf{P}\mathbf{a}_1 = p\mathbf{F}_1 + q\mathbf{F}_0 - \mathbf{Q}\mathbf{a}_0 \Rightarrow \text{solve for } \mathbf{a}_1$$

n=2

$$\mathbf{P}\mathbf{a}_2 = p\mathbf{F}_2 + q\mathbf{F}_1 - \mathbf{Q}\mathbf{a}_1 \Rightarrow \text{solve for } \mathbf{a}_2$$

Backward Difference

Quantities are evaluated at *forward* end of time step

$$\mathbf{C}\left\{\frac{da}{dt}\right\}_n + \mathbf{K}\{a\}_n = \mathbf{F}_n$$

Time derivative approximation

$$\left\{\frac{da}{dt}\right\}_n = \frac{\{a\}_n - \{a\}_{n-1}}{\Delta t_n} \qquad \Delta t_n = t_n - t_{n-1}$$

Substituting

$$\left(\frac{1}{\Delta t_n}\mathbf{C} + \mathbf{K}\right)\mathbf{a}_n = \mathbf{F}_n + \frac{1}{\Delta t_n}\mathbf{C}\,\mathbf{a}_{n-1} \Rightarrow \boxed{\mathbf{K}_{eff}\,\mathbf{a}_n = \mathbf{F}_{eff}}$$

Comparing with the general form (implicit method)

$$\mathbf{P} = \frac{1}{\Delta t_n}\mathbf{C} + \mathbf{K}; \mathbf{Q} = -\frac{1}{\Delta t_n}\mathbf{C}; p = 1; q = 0$$

Mid Difference

Quantities are evaluated at *center* of time step

$$\mathbf{C}\left\{\frac{da}{dt}\right\}_{n-1/2} + \mathbf{K}\left\{a\right\}_{n-1/2} = \mathbf{F}_{n-1/2}$$

Time derivative approximation

$$\left\{\frac{da}{dt}\right\}_{n-1/2} = \frac{\left\{a\right\}_n - \left\{a\right\}_{n-1}}{\Delta t_n} \text{ where } \Delta t_n = t_n - t_{n-1}; \left\{a\right\}_{n-1/2} = \frac{\left\{a\right\}_{n-1} + \left\{a\right\}_n}{2}$$

Substituting (implicit method)

$$\left(\frac{1}{\Delta t_n}\mathbf{C} + \frac{1}{2}\mathbf{K}\right)\mathbf{a}_n = \mathbf{F}_{n-1/2} + \left(\frac{1}{\Delta t_n}\mathbf{C} - \frac{1}{2}\mathbf{K}\right)\mathbf{a}_{n-1} \text{ where } \mathbf{F}_{n-1/2} = \frac{\mathbf{F}_{n-1} + \mathbf{F}_n}{2}$$

$$\Rightarrow \boxed{\mathbf{K}_{eff}\mathbf{a}_n = \mathbf{F}_{eff}}$$

Forward Difference

Quantities are evaluated at *backward* end of time step

$$\mathbf{C}\left\{\frac{da}{dt}\right\}_{n-1} + \mathbf{K}\left\{a\right\}_{n-1} = \mathbf{F}_{n-1}$$

Time derivative approximation

$$\left\{\frac{da}{dt}\right\}_{n-1/2} = \frac{\left\{a\right\}_{n} - \left\{a\right\}_{n-1}}{\Delta t_n} \text{ where } \Delta t_n = t_n - t_{n-1}$$

Substituting

$$\left(\frac{1}{\Delta t_n}\mathbf{C}\right)\mathbf{a}_n = \mathbf{F}_{n-1} + \left(\frac{1}{\Delta t_n}\mathbf{C} - \mathbf{K}\right)\mathbf{a}_{n-1}$$

Forward Difference

Method can be made explicit

$$\boxed{CL_{ii} = \sum_{j=1}^{n} C_{ij} \quad i = 1, 2, \ldots, n} \qquad \boxed{CL_{ij} = 0 \quad i \neq j}$$

$$\mathbf{a}_n = \mathbf{a}_{n-1} + \Delta t_n \left[CL\right]^{-1} \left(\mathbf{F}_{n-1} - \mathbf{K}\mathbf{a}_{n-1}\right)$$

$$\text{where } \left[CL\right]^{-1} = \begin{bmatrix} 1/CL_{11} & & & \\ & 1/CL_{22} & & \\ & & \ddots & \\ & & & 1/CL_{nn} \end{bmatrix}$$

θ-Method

In general

$$\mathbf{C}\left\{\frac{da}{dt}\right\}_{\theta} + \mathbf{K}\{a\}_{\theta} = \mathbf{F}_{\theta}$$

$$\text{where } \theta = \frac{t - t_{n-1}}{\Delta t_n}, \Delta t_n = t_n - t_{n-1}, 0 < \theta < 1$$

Using

$$\{a\}_{\theta} \cong (1-\theta)\{a\}_{n-1} + \theta\{a\}_n$$

$$\{F\}_{\theta} \cong (1-\theta)\{F\}_{n-1} + \theta\{F\}_n$$

$$\Rightarrow \left\{\frac{da}{dt}\right\}_{\theta} = \frac{1}{\Delta t_n}\frac{d\{a\}_{\theta}}{d\theta} = \frac{\{a\}_n - \{a\}_{n-1}}{\Delta t_n}$$

θ-Method

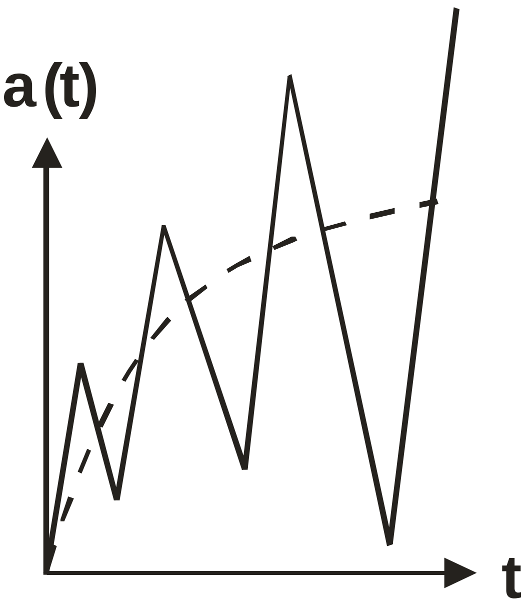
Hence

$$\left(\frac{1}{\Delta t_n}\mathbf{C}+\theta\,\mathbf{K}\right)\mathbf{a}_n=(1-\theta)\mathbf{F}_{n-1}+\theta\,\mathbf{F}_n+\left(\frac{1}{\Delta t_n}\mathbf{C}-(1-\theta)\mathbf{K}\right)\mathbf{a}_{n-1}$$

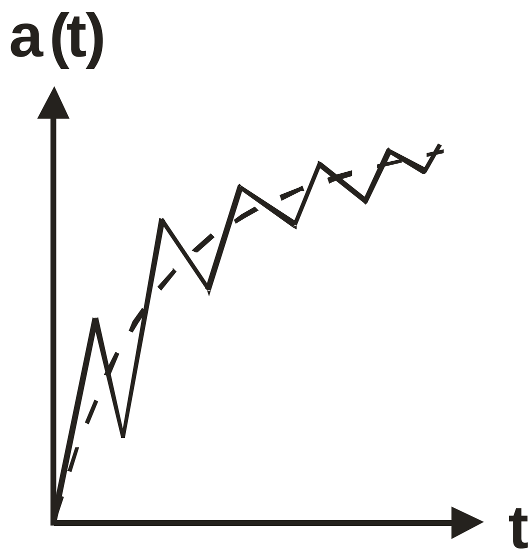
Or $$\mathbf{K}_{eff}\,\mathbf{a}_n=\mathbf{F}_{eff}$$

θ	Method
0	Forward Difference
½	Mid-Difference
1	Backward Difference

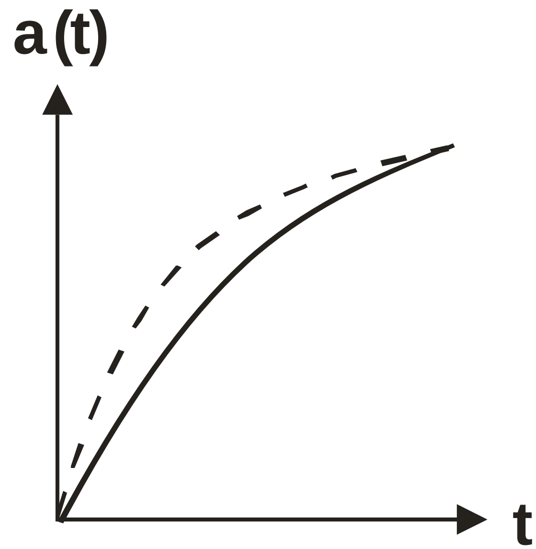
Stability



Unstable
Diverging



Stable
Oscillatory Decay



Stable
Monotonic Decay

Stability Analysis

Free response analysis as $t \rightarrow \infty$

Single DOF system

$$C\frac{da(t)}{dt} + Ka(t) = 0$$

$$\left(\frac{1}{\Delta t}C + \theta K\right)a_n = (1-\theta)F_n + \theta F_n + \left(\frac{1}{\Delta t}C - (1-\theta)K\right)a_{n-1} = 0$$

$$\frac{a_n}{a_{n-1}} = \frac{1-(1-\theta)\lambda\Delta t}{1+\theta \quad \lambda \quad \Delta t} \text{ where } \lambda = K/C$$

Stability Analysis

Multiple DOF system

$$\mathbf{C}\left\{\frac{da(t)}{dt}\right\} + \mathbf{K}\left\{a(t)\right\} = \{0\}$$

Using mode superposition

$$\mathbf{C}\left(\sum_{j=1}^{NDOF}\frac{dA_j(t)}{dt}\boldsymbol{\varphi}_j\right) + \mathbf{K}\left(\sum_{j=1}^{NDOF}A_j(t)\boldsymbol{\varphi}_j\right) = \mathbf{F}(t)$$

For stability

$$\left|\frac{(A_i)_n}{(A_i)_{n-1}}\right| < 1 \quad i = 1, 2, \ldots, NDOF$$

Stability Analysis

For stability

$$\frac{-2}{2\theta - 1} < \lambda_i \Delta t$$

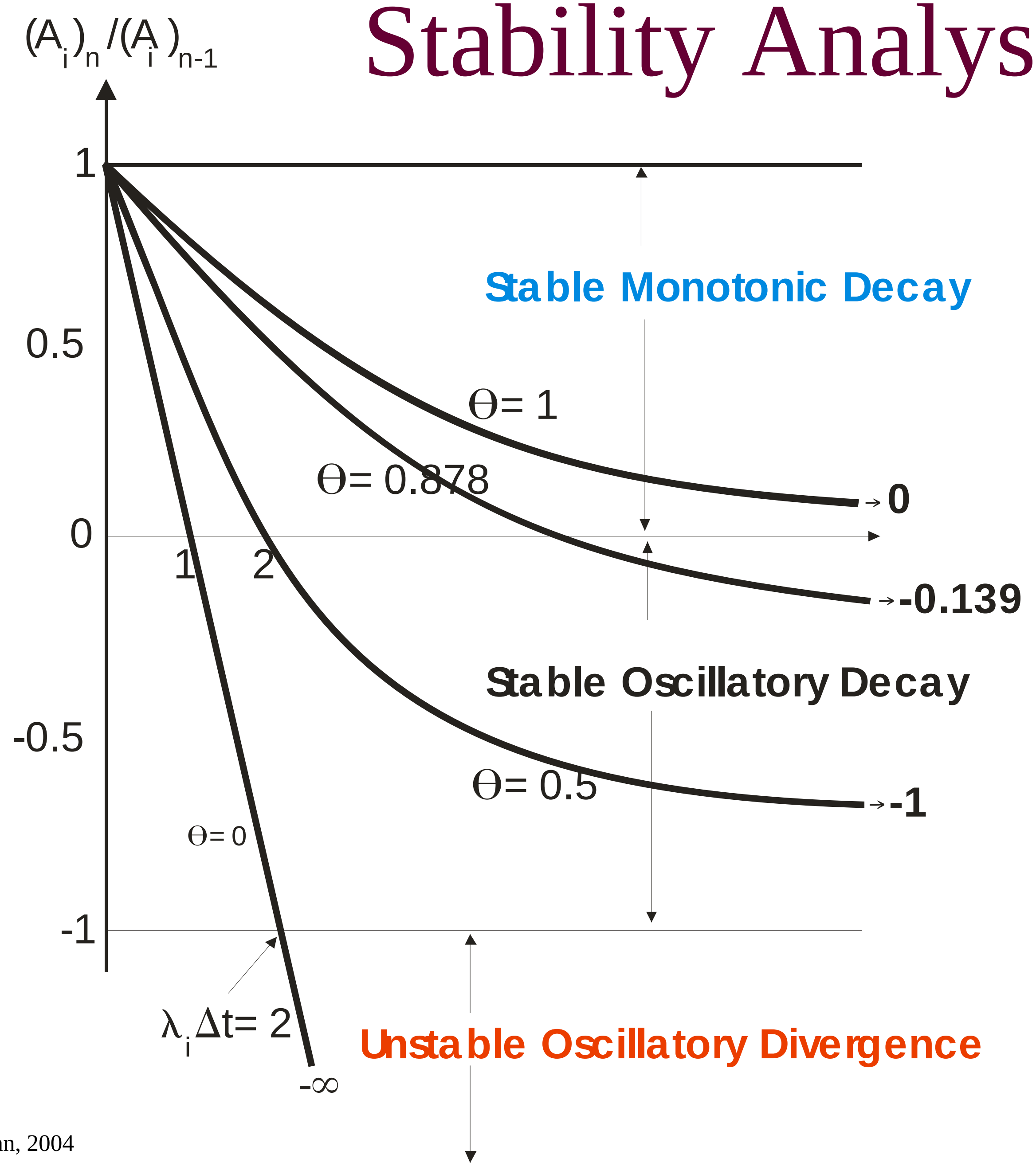
Or

$$\text{For } 0 \le \theta \le 1/2: \quad \lambda_i \Delta t < \frac{2}{1-2\theta}$$

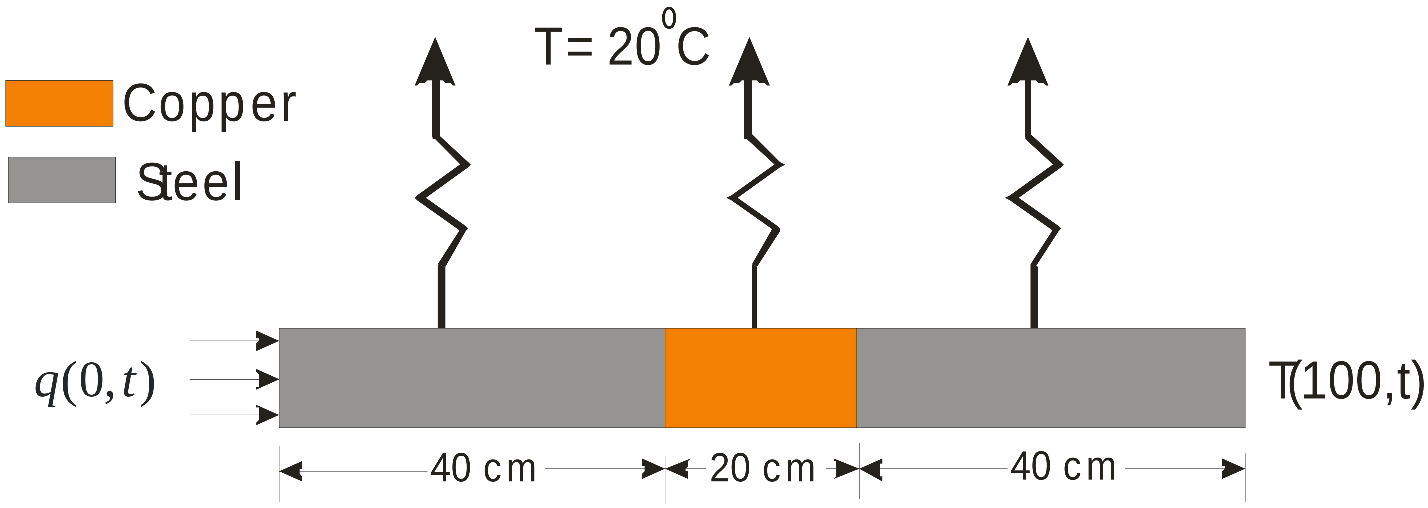
$$\text{For } \theta \ge 1/2: \quad \lambda_i \Delta t \text{ can have any positive value}$$

$$\Delta t < \Delta t_{Crit} = \frac{2}{1-2\theta} \frac{1}{\lambda_{\max}}$$

Stability Analysis

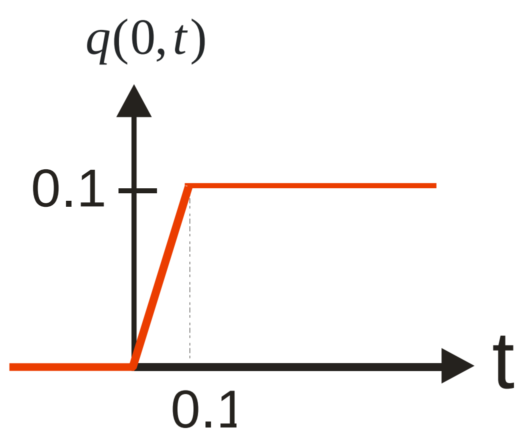


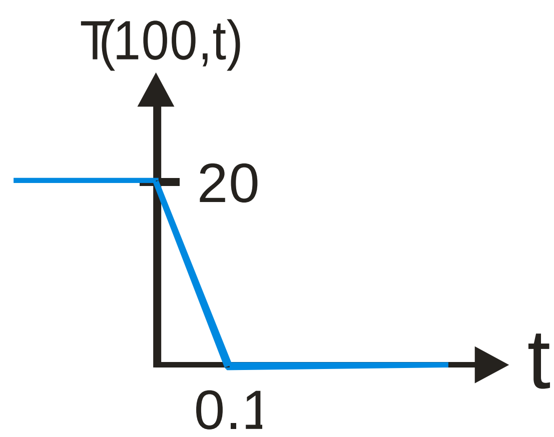
Example



IC $T(x,0) = 20^{\circ} C \quad 0 \le x \le 100$

BC

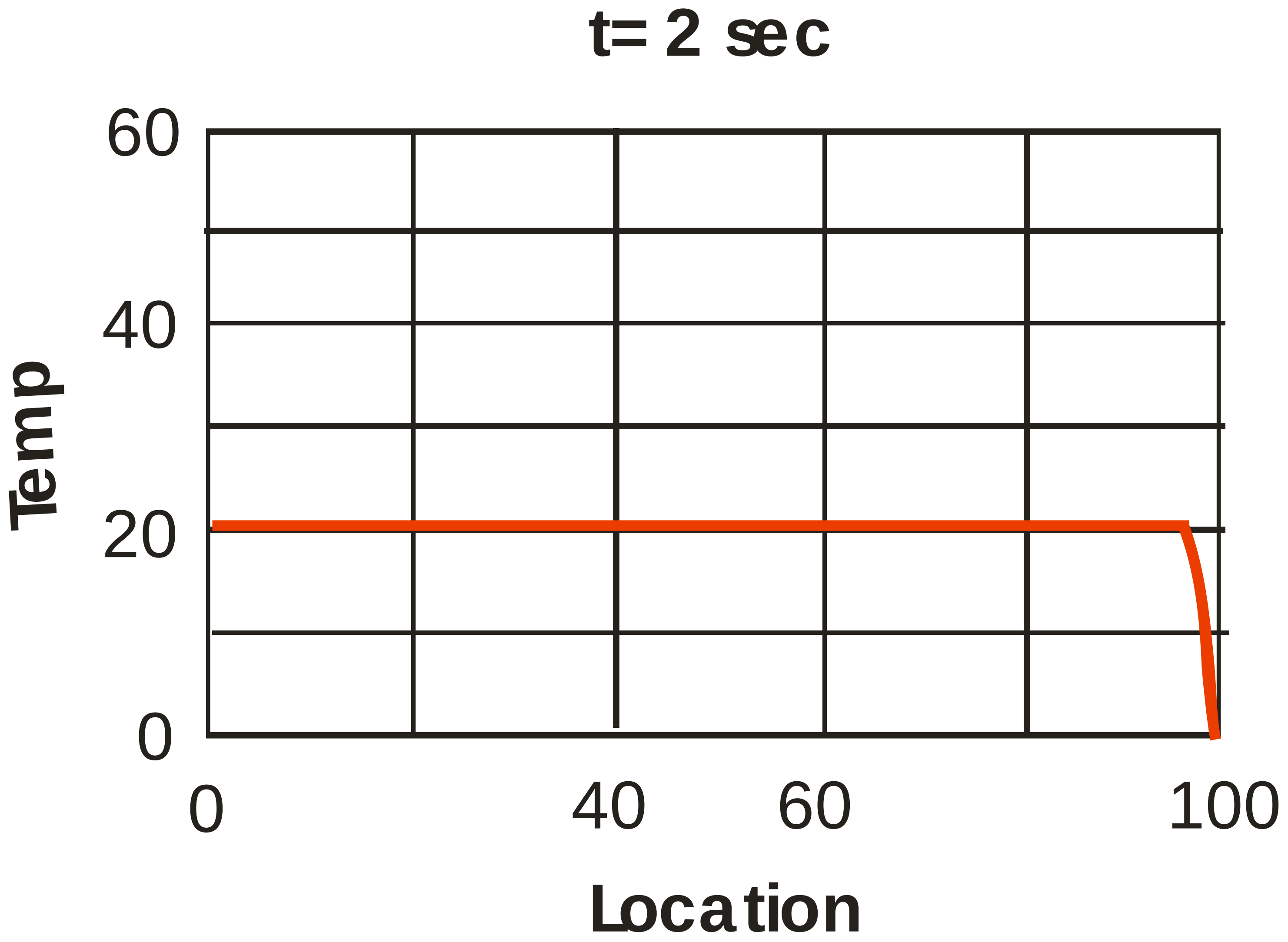




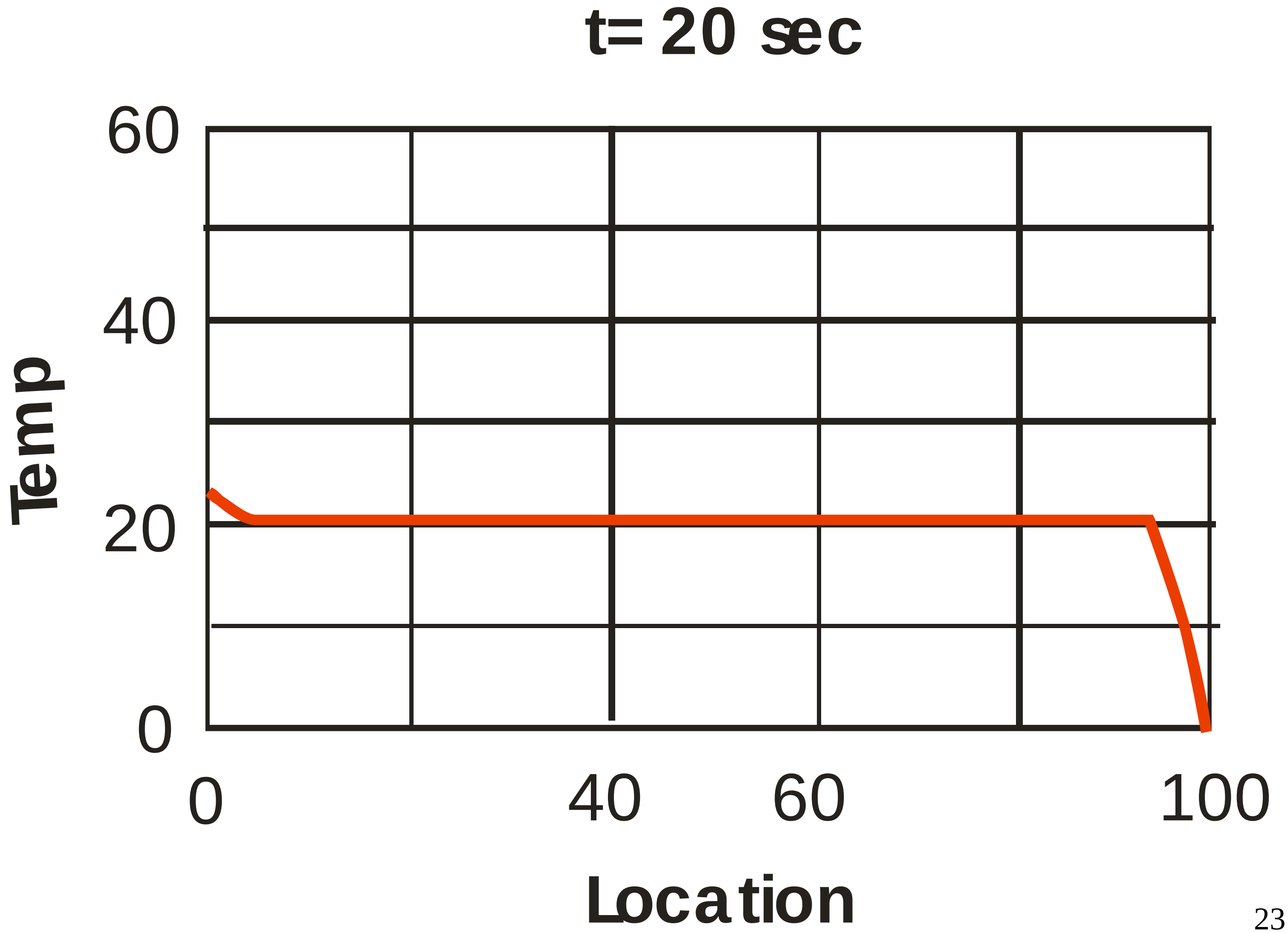
Example

Interval	θ	Δt, sec	# of steps	Time Span
1	0	0.05	2	0 – 0.1
2	0	0.05	38	0.1 – 2
3	2/3	1	18	2 – 20
4	2/3	10	18	20 – 200
5	2/3	100	18	200 – 2000
6	2/3	500	16	2000 – 10000
7	1	10^6	1	$10^4 - 1.01\text{x}10^6$
8	1	10^6	1	$1.01\text{x}10^6 - 2.01\text{x}10^6$

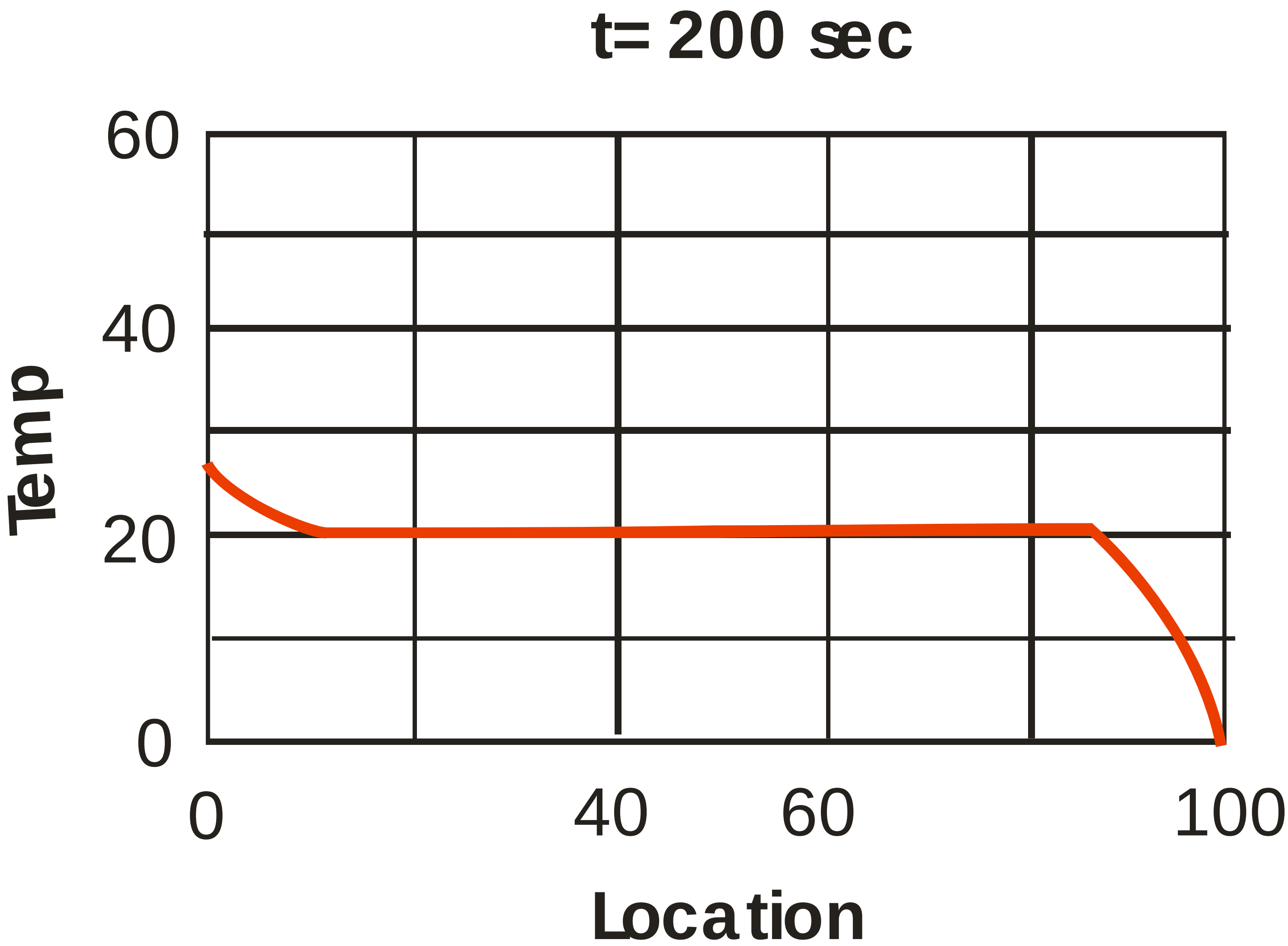
Example



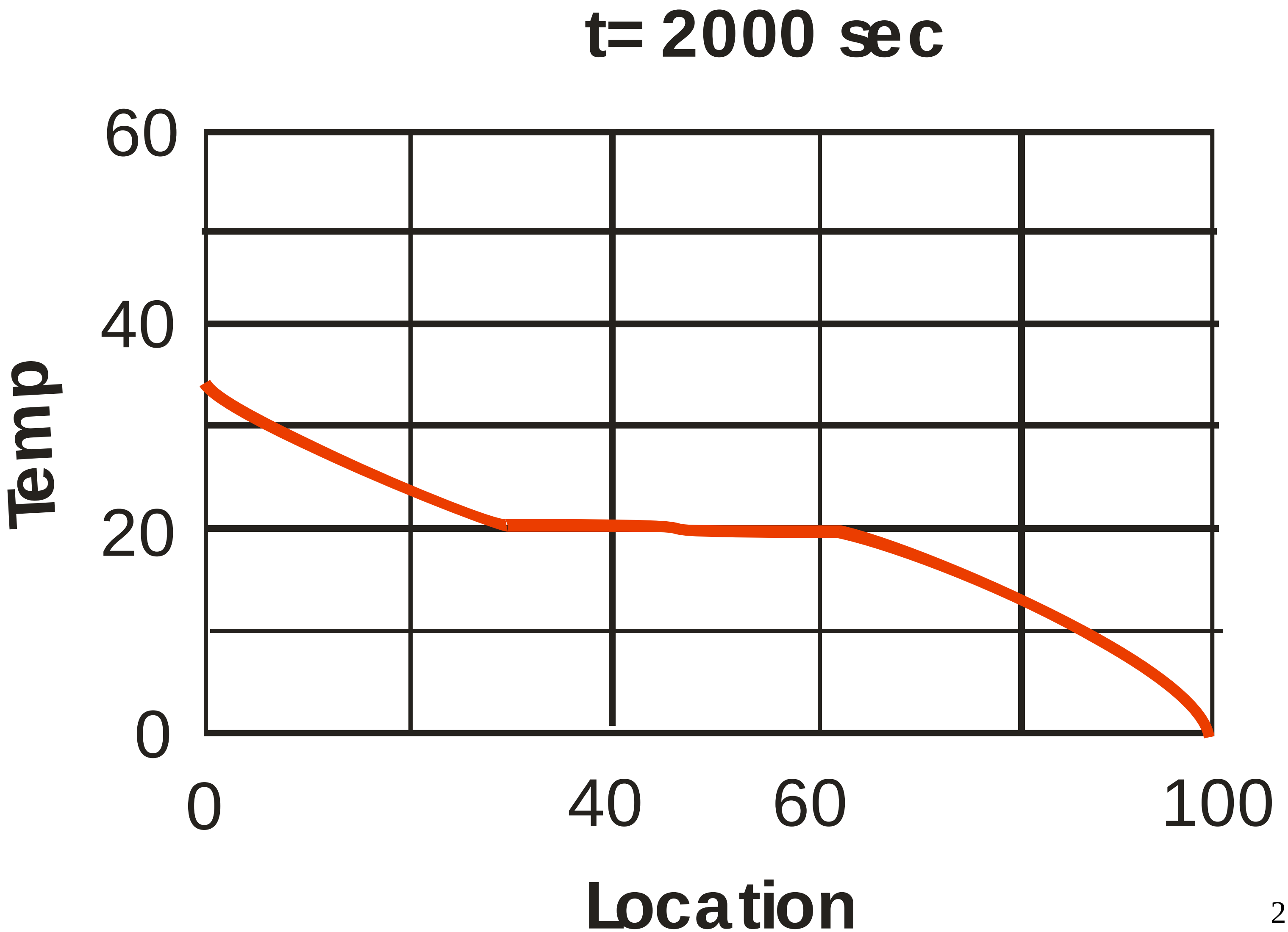
Example



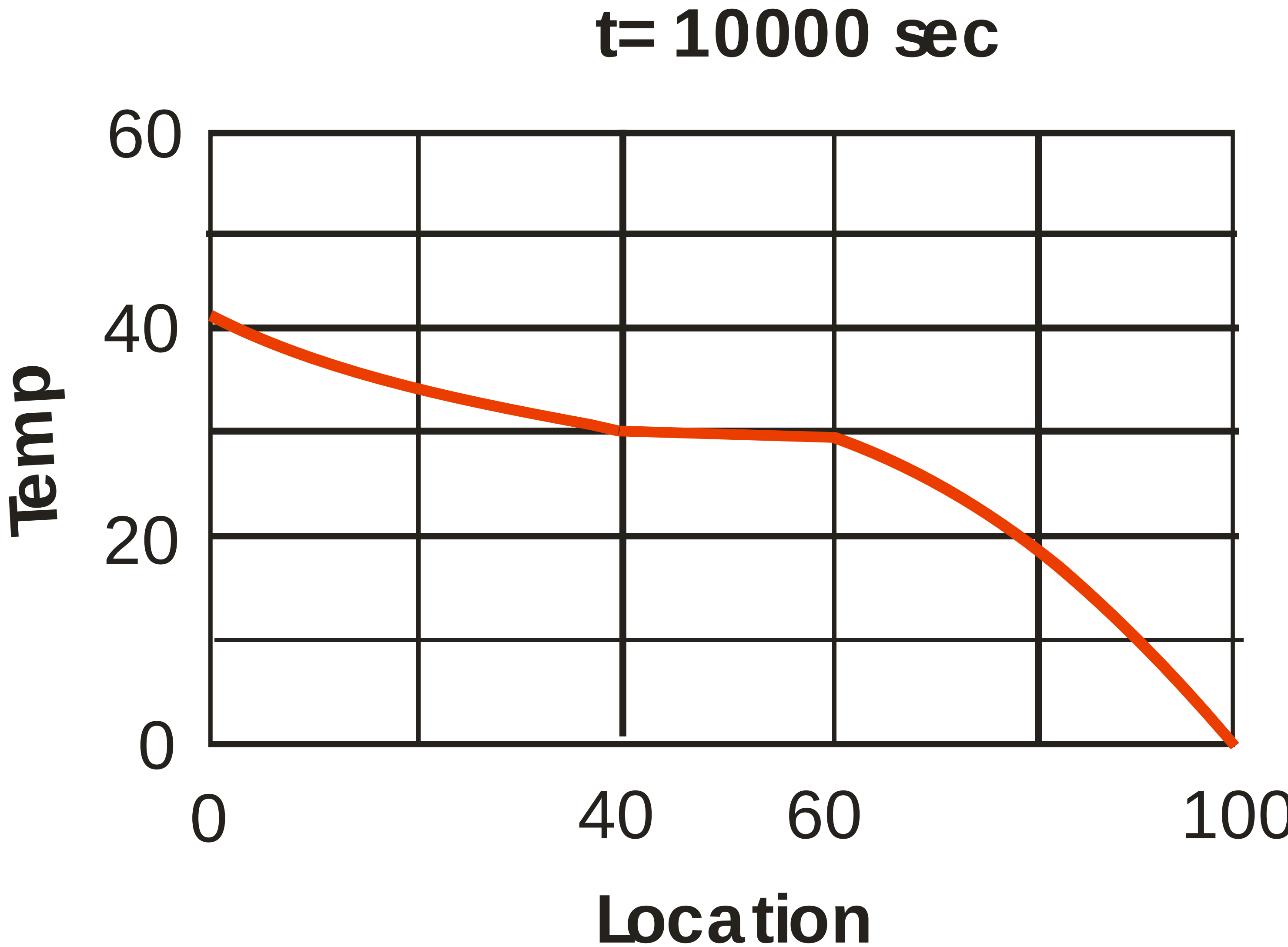
Example



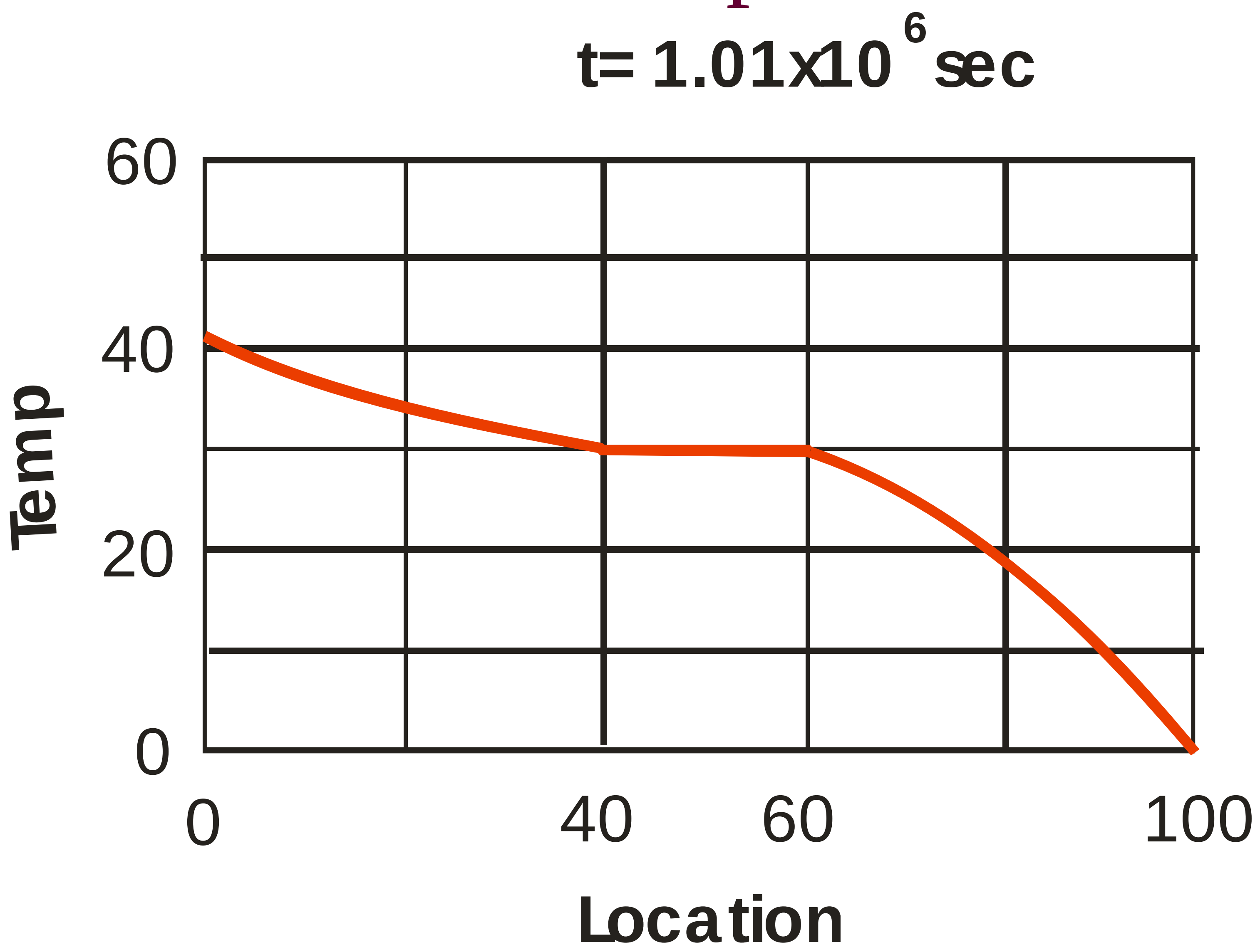
Example



Example



Example



Summary

- Accuracy of Backward Difference is $O(\Delta t)$
- Accuracy of Mid-Difference is $O(\Delta t^2)$
- Accuracy of Forward Difference is $O(\Delta t)$
- The θ Method is a general method that captures all the above cases
  - Compare with Crank-Nicholson Method, a finite difference scheme