Finite Elements for Engineers

Lecture 4: Plane Elasticity Problems

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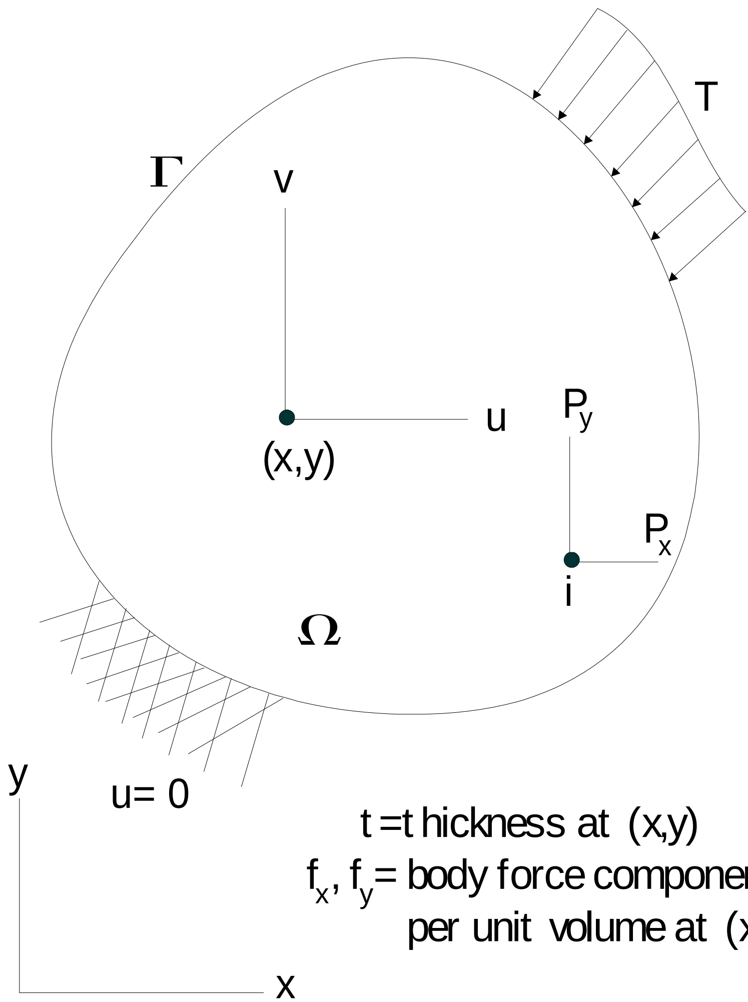
Plane Elasticity Problems



t =t hickness at (x,y)
f_x, f_y= body force component s per unit volume at (x,y)

Displacement Field

$$\mathbf{u}_{2\times 1} = [u, v]^{\mathbf{T}}$$

Stress State

$$\boldsymbol{\sigma} = \left[\sigma_x, \sigma_y, \tau_{xy}\right]^{\mathbf{T}}$$

Strain State

$$\boldsymbol{\varepsilon} = \left[\varepsilon_x, \varepsilon_y, \gamma_{xy}\right]^{\mathbf{T}}$$

Loading Terms

$$\mathbf{f}_{2\times 1} = \left[f_x, f_y\right]^T \quad \mathbf{T}_{2\times 1} = \left[T_x, T_y\right]^T$$

Types of Plane Elasticity Problems

- Plane Stress
  - Body lies in the X-Y plane
  - Thin (t << L)
  - Loading is in the X-Y plane

$$\tau_{xz} = \tau_{yz} = \sigma_z = 0$$

$$\Rightarrow \quad \begin{aligned} \gamma_{xz} &= \gamma_{yz} = 0 \\ \varepsilon_z &\neq 0 \end{aligned}$$

Types of Plane Elasticity Problems

- Plane Strain
  - Body is infinitely long in the z-direction
  - Geometry, loading and boundary conditions are NOT functions of z

$$\gamma_{xz} = \gamma_{yz} = \varepsilon_z = 0$$

$$\Rightarrow \quad \begin{aligned} &\tau_{xz} = \tau_{yz} = 0 \\ &\sigma_z \neq 0 \end{aligned}$$

Plane Elasticity Problems

Strain-Displacement Relations

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{3\times1} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}_{3\times1} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}_{3\times2} \begin{Bmatrix} u \\ v \end{Bmatrix}_{2\times1} = \mathbf{\Lambda}_{3\times2} \quad \mathbf{u}_{2\times1}$$

$$\varepsilon_{3\times1} = \mathbf{\Lambda}_{3\times2} \quad \mathbf{u}_{2\times1}$$

Plane Elasticity Problems

Stress-Strain Relations

$$\boldsymbol{\sigma}_{3\times1} = \mathbf{D}_{3\times3}\left[\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0\right]_{3\times1}$$

Plane Stress

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \left[\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \right] \left\{ \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \alpha \Delta T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \right\}$$

Plane Strain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \left[\frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1}{2}-\nu \end{bmatrix} \right] \left\{ \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - (1+\nu)(\alpha \Delta T) \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \right\}$$

Plane Elasticity Problems

Step 1: Assumed displacement field

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & .. & \phi_n & 0 \\ 0 & \phi_1 & 0 & \phi_2 & .. & 0 & \phi_n \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ .. \\ d_{2n-1} \\ d_{2n} \end{Bmatrix}$$

$$\mathbf{u}_{2\times 1} = [u, v]^{\mathrm{T}} = \mathbf{\Phi}_{2\times 2n} \mathbf{d}_{2n\times 1}$$

Plane Elasticity Problems

Step 2: Strain-displacement relationship

$$\boldsymbol{\varepsilon}_{3\times1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{3\times4} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix}_{4\times1} = \mathbf{L}_{3\times4}\mathbf{a}_{4\times1}$$

Plane Elasticity Problems

$$\mathbf{a}_{4\times1} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{11} & \Gamma_{12} \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \end{bmatrix}_{4\times4} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}_{4\times1} = \mathbf{M}_{4\times4}\mathbf{b}_{4\times1}$$

Plane Elasticity Problems

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}_{4\times 1} = \begin{bmatrix} \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \dots & \phi_{n,\xi} & 0 \\ \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & .. & \phi_{n,\eta} & 0 \\ 0 & \phi_{1,\xi} & 0 & \phi_{2,\xi} & .. & 0 & \phi_{n,\xi} \\ 0 & \phi_{1,\eta} & 0 & \phi_{2,\eta} & .. & 0 & \phi_{n,\eta} \end{bmatrix}_{4\times 2n} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ .. \\ u_n \\ v_n \end{Bmatrix}_{2n\times 1} = \mathbf{N}_{4\times 2n}\mathbf{d}_{2n\times 1}$$

Plane Elasticity Problems

Strain-displacement relationship

$$\boldsymbol{\varepsilon}_{3\times1} = \mathbf{L}_{3\times4}\mathbf{M}_{4\times4}\mathbf{N}_{4\times2n}\mathbf{d}_{2n\times1} = \mathbf{O}_{3\times4}\mathbf{N}_{4\times2n}\mathbf{d}_{2n\times1} = \mathbf{B}_{3\times2n}\mathbf{d}_{2n\times1}$$

$$\mathbf{O}_{3\times4} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{11} & \Gamma_{12} \end{bmatrix}$$

$$\boldsymbol{\varepsilon}_{3\times1} = \mathbf{B}_{3\times2n}\mathbf{d}_{2n\times1}$$

Plane Elasticity Problems

Step 3: Total strain energy (per element)

$$U(\mathbf{d}) = \frac{1}{2}\int_A \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\, t\, dA = \frac{1}{2}\int_A \boldsymbol{\varepsilon}^T \mathbf{D}\boldsymbol{\varepsilon}\, t\, dA = \frac{1}{2}\mathbf{d}^{\mathbf{T}}_{1\times 2n}\mathbf{k}_{2n\times 2n}\mathbf{d}_{2n\times 1}$$

$$\mathbf{k}_{2n\times 2n} = \int_A \mathbf{B}^{\mathbf{T}}\mathbf{DB}\, t\, dA$$

Plane Elasticity Problems

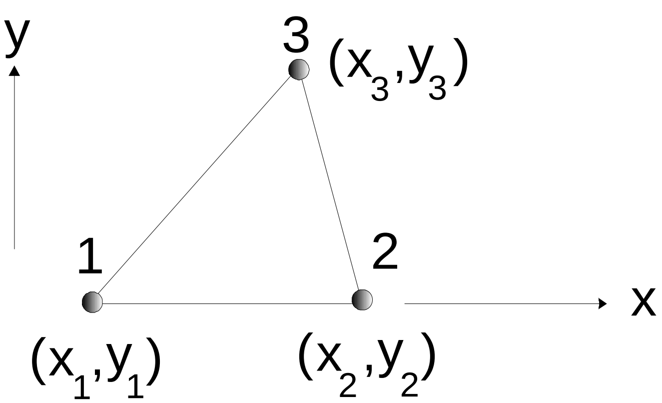
Total Potential Energy

$$\Pi(\mathbf{D}) = \sum_{i=1}^{e} \left[\frac{1}{2} \mathbf{d}^{\mathbf{T}}_{1\times 2n} \mathbf{k}_{2n\times 2n} \mathbf{d}_{2n\times 1} - \mathbf{d}^{\mathbf{T}}_{1\times 2n} \mathbf{f}_{2n\times 1} - \mathbf{d}^{\mathbf{T}}_{1\times 2n} \mathbf{T}_{2n\times 1} + \mathbf{d}^{\mathbf{T}} \left[\int_A \mathbf{B}^{\mathbf{T}} \mathbf{D} \boldsymbol{\varepsilon}_0 \, t dA \right] \right]_i$$

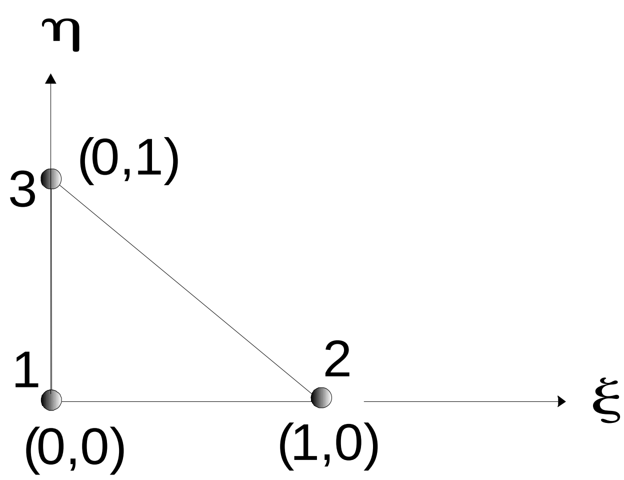
$$-\mathbf{D}^{T}_{1\times N} \mathbf{P}_{N\times 1}$$

Now we can customize the expressions for the components of the element equations – stiffness and load terms, for various elements.

Constant Strain Triangular Element



Real



Master or Parent

Assumed displacement field

$$u = a_1 + a_2\xi + a_3\eta$$

$$v = b_1 + b_2\xi + b_3\eta$$

Shape functions

$$\phi_1 = 1 - \xi - \eta$$

$$\phi_2 = \xi$$

$$\phi_3 = \eta$$

CST Element

Computing Jacobian and its inverse

$$x = \phi_1 x_1 + \phi_2 x_2 + \phi_3 x_3 \qquad y = \phi_1 y_1 + \phi_2 y_2 + \phi_3 y_3$$

$$\mathbf{J}_{2\times 2} = \begin{bmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1) & (y_3 - y_1) \end{bmatrix} = \begin{bmatrix} x_{21} & y_{21} \\ x_{31} & y_{31} \end{bmatrix} \Rightarrow \det(J) = x_{21} y_{31} - x_{31} y_{21}$$

$$\mathbf{\Gamma}_{2\times 2} = \frac{1}{\det(J)} \begin{bmatrix} y_{31} & -y_{21} \\ -x_{31} & x_{21} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{31} & -y_{21} \\ -x_{31} & x_{21} \end{bmatrix}$$

CST Element

Strain-displacement relations

$$\mathbf{B}_{3\times6} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{11} & \Gamma_{12} \end{bmatrix}_{3\times4} \begin{bmatrix} \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \phi_{3,\xi} & 0 \\ \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & \phi_{3,\eta} & 0 \\ 0 & \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \phi_{3,\xi} \\ 0 & \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & \phi_{3,\eta} \end{bmatrix}_{4\times6}$$

$$\mathbf{B}_{3\times6} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}_{3\times6}$$

CST Element

Element Stiffness Matrix

$$\mathbf{k}_{6\times6} = \int_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, t\, dA = tA\,\mathbf{B}^{T}_{6\times3}\mathbf{D}_{3\times3}\mathbf{B}_{3\times6}$$

Body forces

$$\mathbf{f}^{B}_{6\times1} = \frac{tA}{3}\left[f_x, f_y, f_x, f_y, f_x, f_y \right]^{\mathrm{T}}$$

Surface Tractions

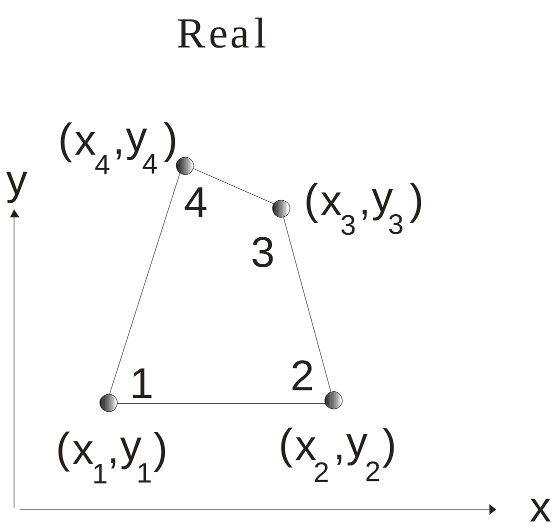
$$\mathbf{T}_{6\times1} = \frac{tL_{1-2}}{6}\left[2T_{x1} + T_{x2}, 2T_{y1} + T_{y2}, T_{x1} + 2T_{x2}, T_{y1} + 2T_{y2}, 0, 0\right]^{\mathrm{T}}$$

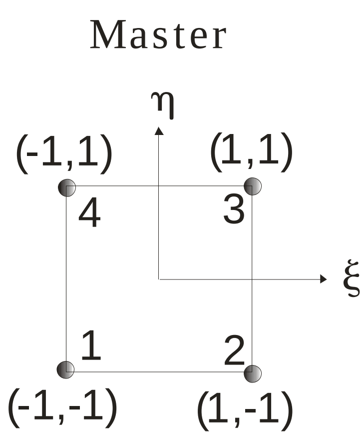
Thermal Load Vector

$$\mathbf{f}^{Th}_{6\times1} = tA\mathbf{B}^{\mathrm{T}}\mathbf{D}\boldsymbol{\varepsilon}_0$$

$$\mathbf{k}_{6\times6}\mathbf{d}_{6\times1} = \mathbf{f}^{Th}_{6\times1} + \mathbf{T}_{6\times1} + \mathbf{f}^{B}_{6\times1}$$

Linear Quadrilateral Element





Assumed displacement field

$$u = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$

$$v = b_1 + b_2\xi + b_3\eta + b_4\xi\eta$$

Shape Functions

$$\phi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \qquad i = 1,2,3,4$$

Jacobian

$$x = \sum_{i=1}^{4} \phi_i x_i \qquad y = \sum_{i=1}^{4} \phi_i y_i$$

Linear Quadrilateral Element

$$\mathbf{J}_{2\times 2} = \frac{1}{4}\left[\begin{array}{c|c} \eta\left(x_1 - x_2 + x_3 - x_4\right) & \eta\left(y_1 - y_2 + y_3 - y_4\right) \\ +\left(-x_1 + x_2 + x_3 - x_4\right) & +\left(-y_1 + y_2 + y_3 - y_4\right) \\ \hline \xi\left(x_1 - x_2 + x_3 - x_4\right) & \xi\left(y_1 - y_2 + y_3 - y_4\right) \\ +\left(-x_1 - x_2 + x_3 + x_4\right) & +\left(-y_1 - y_2 + y_3 + y_4\right) \end{array}\right]$$

$$\mathbf{\Gamma}_{2\times 2} = \frac{1}{\det(J)}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

$$dxdy = \det(J)\, d\xi d\eta$$

Linear Quadrilateral Element

Strain-Displacement Method

$$\mathbf{B}_{3\times 8} = \mathbf{O}_{3\times 4}\mathbf{N}_{4\times 8}$$

$$\mathbf{B}_{3\times 8} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{11} & \Gamma_{12} \end{bmatrix}_{3\times 4} \begin{bmatrix} \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \phi_{3,\xi} & 0 & \phi_{4,\xi} & 0 \\ \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & \phi_{3,\eta} & 0 & \phi_{3,\eta} & 0 \\ 0 & \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \phi_{3,\xi} & 0 & \phi_{4,\xi} \\ 0 & \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & \phi_{3,\eta} & 0 & \phi_{4,\eta} \end{bmatrix}_{4\times 8}$$

Linear Quadrilateral Element

Element Stiffness Matrix

$$\mathbf{k}_{8\times 8} = \int_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, t\, dA = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, t \det(J)\, d\xi d\eta$$

$$\mathbf{k}_{8\times 8} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, t \det(J)\, d\xi d\eta = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j f(\xi_i, \eta_j)$$

where $f(\xi_i, \eta_j) = \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, t \det(J)\Big|_{\xi_i, \eta_j}$

Algorithm for Programming

(1) Clear $\mathbf{k}_{8\times 8}$ to zero.

(2) Enter the *i* loop to integrate in the ξ -direction. Set values for w_i and ξ_i .

(3) Enter the *j* loop to integrate in the η -direction. Set values for w_j and η_j .

(4) At the current Gauss point $\left(\xi_i, \eta_j\right)$, compute the following.

(a) The shape functions ϕ_k and the derivatives of the shape functions $\dfrac{\partial \phi_k}{\partial \xi}, \dfrac{\partial \phi_k}{\partial \eta}$. If necessary, use the shape functions to compute the thickness, t_{ij} at the current point.

(b) Construct the jacobian matrix, $\mathbf{J}_{2\times 2}$, $\det(J)$ and the inverse $\mathbf{\Gamma}_{2\times 2}$.

(c) Form the strain-displacement matrix $\mathbf{B}_{3\times 8}$ using Eqn. (T3L2-9d).

Algorithm for Programming

(5) At the current point, compute the product $\mathbf{T}_{3\times3} = w_i w_j\, t_{ij} \det(J)\mathbf{D}_{3\times3}$.

(6) Now compute the triple product $\mathbf{B}^{\mathbf{T}}_{8\times3}\mathbf{T}_{3\times3}\mathbf{B}_{3\times8}$ and update $\mathbf{k}_{8\times8}$.

(7) Increment *j*.

(8) Increment *i*.

Handling Higher-Order Elements

- Same old procedure
  - Generate shape functions for the assumed displacement field
  - Using the appropriate Gauss-Quadrature Rule, generate **B, J, det(J) and**

$$\mathbf{k}_{2n\times 2n} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{T}_{2n\times 3}\mathbf{D}_{3\times 3}\mathbf{B}_{3\times 2n}\, t \det(J)\, d\xi\, d\eta = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j f(\xi_i, \eta_j)$$

Computing Strains & Stresses

Step 6

Strains $$\boldsymbol{\varepsilon}_{3\times1} = \mathbf{B}_{3\times2n}\mathbf{d}_{2n\times1}$$

Stresses $$\boldsymbol{\sigma}_{3\times1} = \mathbf{D}_{3\times3}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0\right)_{3\times1}$$

Computing Element Loads

Basic Idea $$\Pi = \int_V U_0 \, dV - \int_V \mathbf{f}^T \mathbf{F} \, dV - \int_S \mathbf{f}^T \Phi \, dS - \mathbf{D}^\mathbf{T} \mathbf{P}$$

Traction Load Vector $$\mathbf{f}_i^{sur} = t \oint_\Gamma \phi_i \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} ds$$

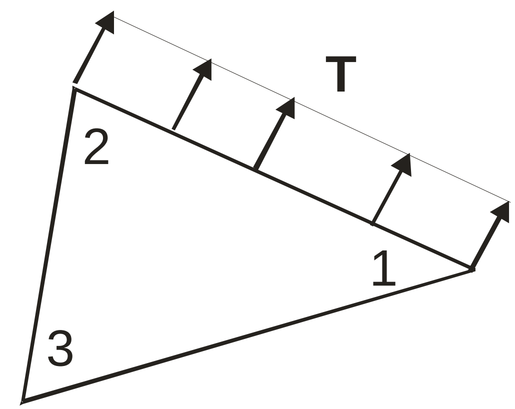
Body Forces $$\mathbf{f}_i^{body} = t \iint \phi_i \begin{Bmatrix} B_x \\ B_y \end{Bmatrix} dA$$

Thermal Loads $$\mathbf{f}^{ther} = t \iint \mathbf{B}^\mathbf{T} \mathbf{D} \boldsymbol{\varepsilon}_\mathbf{0} dA$$

Surface Tractions

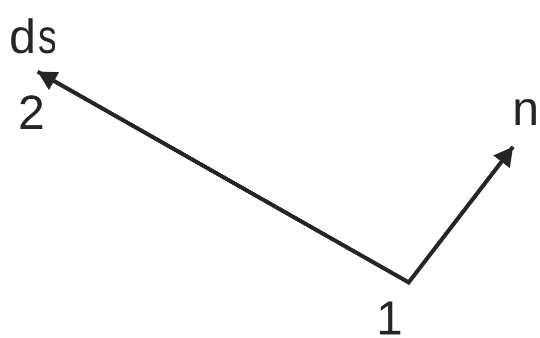
Assume constant surface traction normal to surface

$$\begin{Bmatrix} T_x \\ T_y \end{Bmatrix} = T\mathbf{n}$$



$$\mathbf{f}_i^{sur} = t\oint_{\Gamma} \phi_i T\mathbf{n}\,ds$$

$$\mathbf{n}ds = \mathbf{ds} \times \mathbf{z} = \begin{Bmatrix} dx \\ dy \\ 0 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} dy \\ -dx \\ 0 \end{Bmatrix}$$



Element Loads

Note

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta = J_{11} d\xi + J_{21}\, d\eta$$

$$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta = J_{12}\, d\xi + J_{22}\, d\eta$$

Substituting

$$\mathbf{f}_i^{sur} = tT \oint_{\Gamma} \phi_i \left(dy\, \hat{i} - dx\, \hat{j} \right) = tT \oint_{\Gamma} \phi_i \left(\left[J_{12} d\xi + J_{22} d\eta \right] \hat{i} - \left[J_{11} d\xi + J_{21} d\eta \right] \hat{j} \right)$$

Element Loads: T3 Element

Side 1-2 $\eta = 0 \quad d\eta = 0$ $\qquad \phi_1 = 1 - \xi$

$$\phi_2 = \xi \qquad \phi_3 = 0$$

$$\mathbf{f}_i^{sur} = t\oint_{\Gamma} \phi_i T \mathbf{n} ds = t\int_{\overline{12}} \phi_i T \mathbf{n} ds \quad i = 1, 2$$

Hence

$$f_{i,x}^{sur} = tT\int_0^1 \phi_i J_{12}\, d\xi$$

$$f_{i,y}^{sur} = -tT\int_0^1 \phi_i J_{11}\, d\xi$$

Element Loads: T3 Element

$$J_{11}(\xi,0)=\frac{\partial x}{\partial \xi}(\xi,0)=\sum_{k=1,2} x_k \frac{\partial \phi_k(\xi,0)}{\partial \xi}=x_2-x_1=x_{21}$$

$$J_{12}(\xi,0)=\frac{\partial y}{\partial \xi}(\xi,0)=\sum_{k=1,2} y_k \frac{\partial \phi_k(\xi,0)}{\partial \xi}=y_2-y_1=y_{21}$$

Change in coordinates

$$\xi=\frac{1}{2}(\xi'+1) \quad \Rightarrow$$

$$\phi_1(\xi',0)=\frac{1}{2}(1-\xi')$$

$$\phi_2(\xi',0)=\frac{1}{2}(1+\xi')$$

$$\phi_3(\xi',0)=0$$

Element Loads: T3 Element

$$f_{i,x}^{sur} = tT\int_0^1 \phi_i J_{12}\, d\xi = tTy_{21}\int_0^1 \phi_i\, d\xi$$

$$f_{i,y}^{sur} = -tT\int_0^1 \phi_i J_{11}\, d\xi = -tTx_{21}\int_0^1 \phi_i\, d\xi$$

$$f_{1,x}^{sur} = \frac{1}{2}tTy_{21}\int_{-1}^1 \phi_1\, d\xi' = \frac{1}{4}tTy_{21}\int_{-1}^1 \left(1-\xi'\right) d\xi' = \frac{tTy_{21}}{2}$$

$$f_{1,y}^{sur} = -\frac{1}{2}tTx_{21}\int_{-1}^1 \phi_1\, d\xi' = -\frac{1}{2}tTx_{21}\int_{-1}^1 \left(1-\xi'\right) d\xi' = -\frac{tTx_{21}}{2}$$

$$f_{2,x}^{sur} = \frac{1}{2}tTy_{21}\int_{-1}^1 \phi_2\, d\xi' = \frac{1}{4}tTy_{21}\int_{-1}^1 \left(1+\xi'\right) d\xi' = \frac{tTy_{21}}{2}$$

$$f_{2,y}^{sur} = -\frac{1}{2}tTx_{21}\int_{-1}^1 \phi_2\, d\xi' = -\frac{1}{2}tTx_{21}\int_{-1}^1 \left(1+\xi'\right) d\xi' = -\frac{tTx_{21}}{2}$$

Element Loads: T3 Element

$$\mathbf{f}_{6\times 1} = \frac{tT}{2}\begin{Bmatrix} y_{21} \\ -x_{21} \\ y_{21} \\ -x_{21} \\ 0 \\ 0 \end{Bmatrix}$$

Sign convention for T?

Integration with Area Coordinates

Order, n	Weight	Location
1	1.0	(1/3, 1/3)
2	1/3	(2/3, 1/6)
	1/3	(1/6, 2/3)
	1/3	(1/6, 1/6)

Element Formulation: G-Q Rule

Element	Stiffness	Stress/Strain
T3	1	1
T6	3	3
Q4	2 x 2	1
Q8/Q9	3 x 3	2 x 2

Analysis & Retrospection

- Equilibrium is usually not satisfied within elements.
- Equilibrium is usually not satisfied between elements.
- Equilibrium of nodal forces and moments is satisfied.
- Compatibility may or may not be satisfied along element boundaries.

Analysis & Retrospection

- Compatibility is satisfied within elements.
- Compatibility is enforced at the nodes.

Summary

- We can now see the power of isoparametric formulation
- The same procedure will work for other classes of problems such as axisymmetric problems and three-dimensional elasticity problems
- Model building is going to take time