Finite Elements for Engineers

Lecture 7: Additional Considerations for Solid Mechanics Problems

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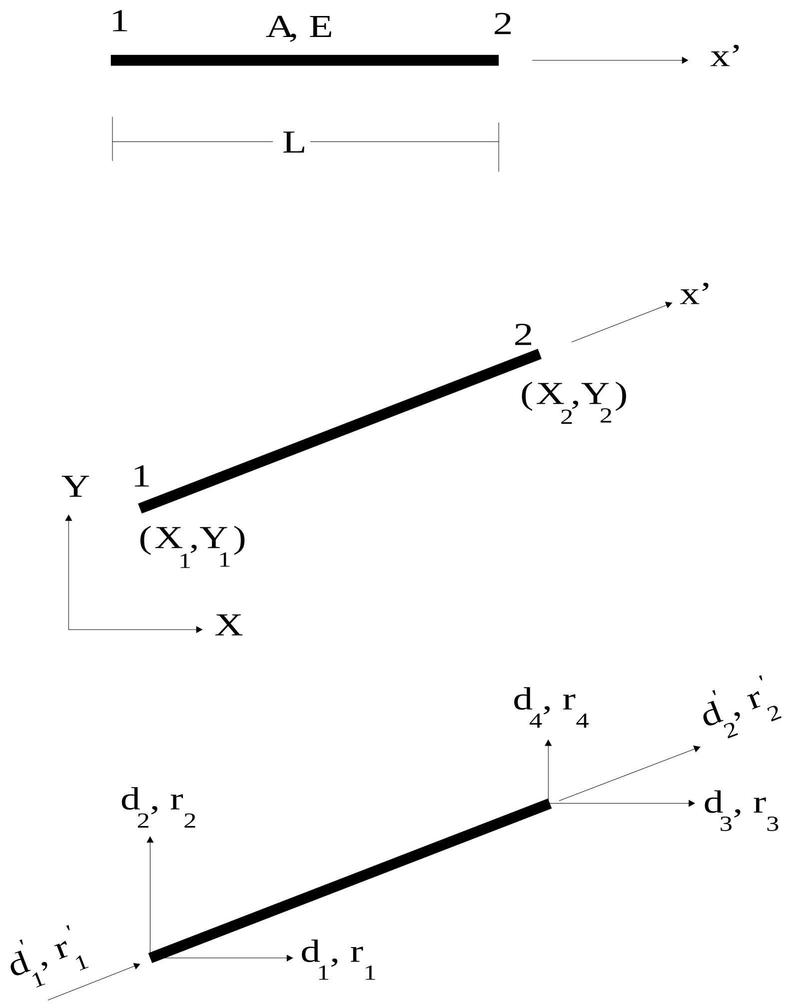
Truss and Beam Elements

Planar Truss

- Straight element with prismatic, slender cross-section
- All connections are pins
- All forces are applied at the nodes
- Small displacements and strains
- As a result
  - Elements are either in tension or compression

Planar Truss Elements



Displacement Field

$$u(\xi) = \phi_1(\xi)\, d_1' + \phi_2(\xi)\, d_2' = \frac{1-\xi}{2} d_1' + \frac{1+\xi}{2} d_2'$$

Strain-Displacement Field

$$\frac{du}{d\xi} = \frac{d}{d\xi}\begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}\begin{Bmatrix} d_1' \\ d_2' \end{Bmatrix}$$

$$\frac{du}{d\xi} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} d_1' \\ d_2' \end{Bmatrix}$$

Planar Truss Element

Strain-Displacement Relationship

$$x'(\xi) = \phi_1(\xi)x_1' + \phi_2(\xi)x_2' = \frac{1-\xi}{2}x_1' + \frac{1+\xi}{2}x_2'$$

$$\frac{dx'}{d\xi} = \frac{1}{2}\left(x_2' - x_1'\right) = \frac{L}{2}$$

$$\varepsilon = \frac{du}{dx'} = \frac{d\xi}{dx'}\frac{du}{d\xi} = \frac{2}{L}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\mathbf{d}'_{2\times 1} = \mathbf{B}_{1\times 2}\,\mathbf{d}'_{2\times 1}$$

Planar Truss Element

Strain Energy

$$U = \int_V U_0 \, dV = \int_0^L \frac{1}{2} \varepsilon \sigma A dx = \int_{-1}^{1} \frac{1}{2} \left[\mathbf{B}_{1\times 2} \mathbf{d}'_{2\times 1} \right]^T E \left[\mathbf{B}_{1\times 2} \mathbf{d}'_{2\times 1} \right] A \frac{L}{2} d\xi$$

$$U = \left[\mathbf{d}'\right]^T_{1\times 2} \left[\mathbf{k}'\right]_{2\times 2} \left[\mathbf{d}'\right]_{2\times 1}$$

where $$\left[\mathbf{k}'\right]_{2\times 2} = \int_{-1}^{1} \mathbf{B}^T_{2\times 1} \left[\frac{AEL}{4} \right]_{1\times 1} \mathbf{B}_{1\times 2} \, d\xi = \frac{AE}{L} \left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array} \right]$$

Element Equations

$$\frac{AE}{L} \left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array} \right] \left\{ \begin{array}{c} d'_1 \\ \hline d'_2 \end{array} \right\} = \left\{ \begin{array}{c} r'_1 \\ \hline r'_2 \end{array} \right\} \quad \Rightarrow \quad \mathbf{k}'_{2\times 2} \mathbf{d}'_{2\times 1} = \mathbf{f}'_{2\times 1}$$

Planar Truss Element

Local-to-Global Displacement Transformation

$$\mathbf{d}'_{2\times1} = \begin{bmatrix} l_{x'} & m_{x'} & 0 & 0 \\ 0 & 0 & l_{x'} & m_{x'} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \mathbf{T}_{2\times4}\mathbf{d}_{4\times1}$$

$$l_{x'} = \frac{X_2 - X_1}{L} \qquad m_{x'} = \frac{Y_2 - Y_1}{L}$$

Planar Truss Element

Global-to-Local Force Transformation

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{bmatrix} l_{x'} & 0 \\ m_{x'} & 0 \\ 0 & l_{x'} \\ 0 & m_{x'} \end{bmatrix} \begin{Bmatrix} f_1^{'} \\ f_2^{'} \end{Bmatrix} \Rightarrow \mathbf{f}_{4\times 1} = \mathbf{T}_{4\times 2}^{\mathbf{T}} \ \mathbf{f}_{2\times 1}^{'}$$

Planar Truss Element

Element Equations in Global Coordinate System

$$\mathbf{k}'_{2\times 2}\mathbf{d}'_{2\times 1} = \mathbf{f}'_{2\times 1}$$

$$\mathbf{d}'_{2\times 1} = \mathbf{T}_{2\times 4}\mathbf{d}_{4\times 1} \quad \Rightarrow \quad \mathbf{k}_{4\times 4}\mathbf{d}_{4\times 1} = \mathbf{f}_{4\times 1}$$

$$\mathbf{f}_{4\times 1} = \mathbf{T}^{T}_{4\times 2}\mathbf{f}'_{2\times 1}$$

where

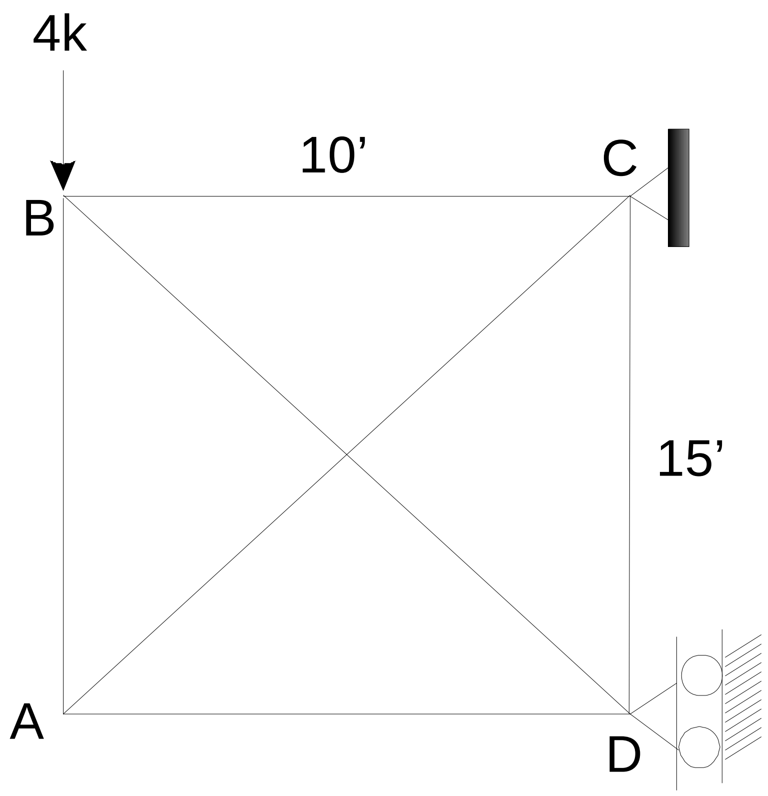
$$\mathbf{k}_{4\times 4} = \mathbf{T}^{\mathbf{T}}_{4\times 2}\mathbf{k}'_{2\times 2}\mathbf{T}_{2\times 4}$$

Planar Truss Analysis

Step 6: Computing element forces

$$f_1^{'} = \frac{AE}{L}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

Example



$$E = 30(10^6)\,psi$$

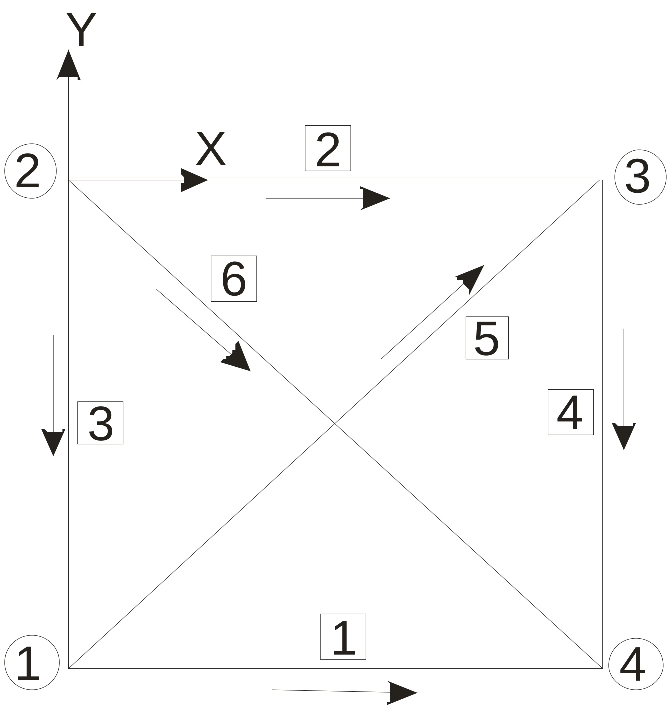
$$A = 1.2\,in^2$$

Compute nodal displacements, element forces and support reactions.

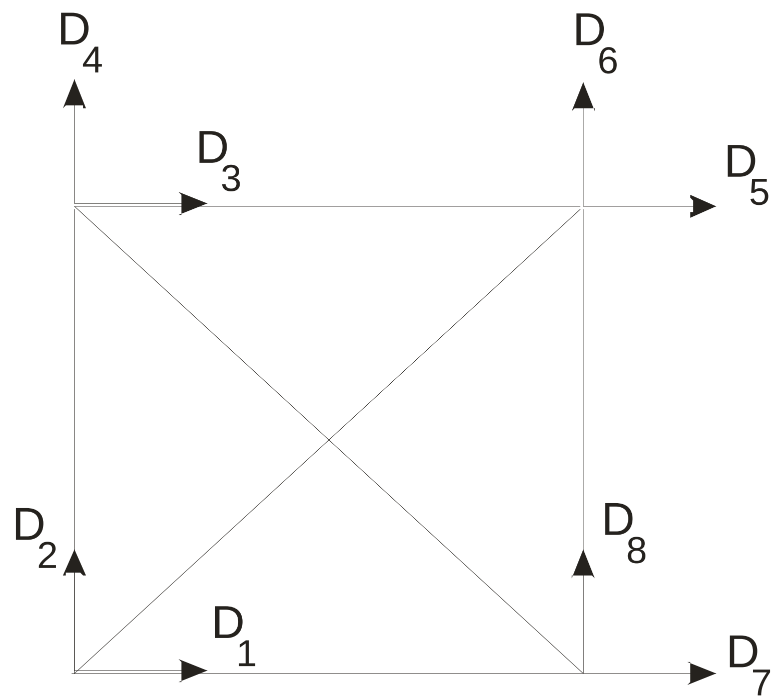
Units: lb, in

Example

FE Model



System Unknowns



Example

Member	(x_1, y_1)	(x_2, y_2)	L	l	m	$\frac{AE}{L}$
1	(0,-180)	(120,-180)	120	1	0	$3(10^5)$
2	(0,0)	(120,0)	120	1	0	$3(10^5)$
3	(0,0)	(0,-180)	180	0	-1	$2(10^5)$
4	(120,0)	(120,-180)	180	0	-1	$2(10^5)$
5	(0,-180)	(120,0)	216.333	0.5547	0.832051	$1.664(10^5)$
6	(0,0)	(120,-180)	216.333	0.5547	-0.832051	$1.664(10^5)$

Example

Note $\mathbf{k}_{4\times4} = \mathbf{T}^{\mathrm{T}}_{4\times2}\mathbf{k}'_{2\times2}\mathbf{T}_{2\times4}$

$$\mathbf{k}_{4\times4} = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Example

Element 1

$$
\begin{array}{cccccc}
 & 1 & 2 & 7 & 8 & \\
10^5 & \multicolumn{4}{c}{\left[\begin{matrix} 3 & 0 & \times & 0 \\ 0 & 0 & \times & 0 \\ \times & \times & \times & \times \\ 0 & 0 & \times & 0 \end{matrix}\right]} & \begin{Bmatrix} D_1 \\ D_2 \\ D_7 \\ D_8 \end{Bmatrix}
\end{array}
$$

Element 2

$$
\begin{array}{cccccc}
 & 3 & 4 & 5 & 6 & \\
10^5 & \multicolumn{4}{c}{\left[\begin{matrix} 3 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{matrix}\right]} & \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix}
\end{array}
$$

Example

Step 4: System Equations (after BCs)

$$(10)^5 \begin{bmatrix} 3.5120 & 0.7679 & 0 & 0 & 0 \\ 0.7679 & 3.1520 & 0 & -2 & 0 \\ 0 & 0 & 3.5120 & -0.7679 & 0.7679 \\ 0 & -2 & -0.7679 & 3.1520 & -1.1520 \\ 0 & 0 & 0.7679 & -1.1520 & 3.1520 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -4000 \\ 0 \end{Bmatrix}$$

Example

Step 5

$$\{D_1, D_2, D_3, D_4, D_5, D_8\} = 10^{-3}\{4.44367, -20.3232, 4.44367, -30.3232, -10\}\ in$$

Step 6

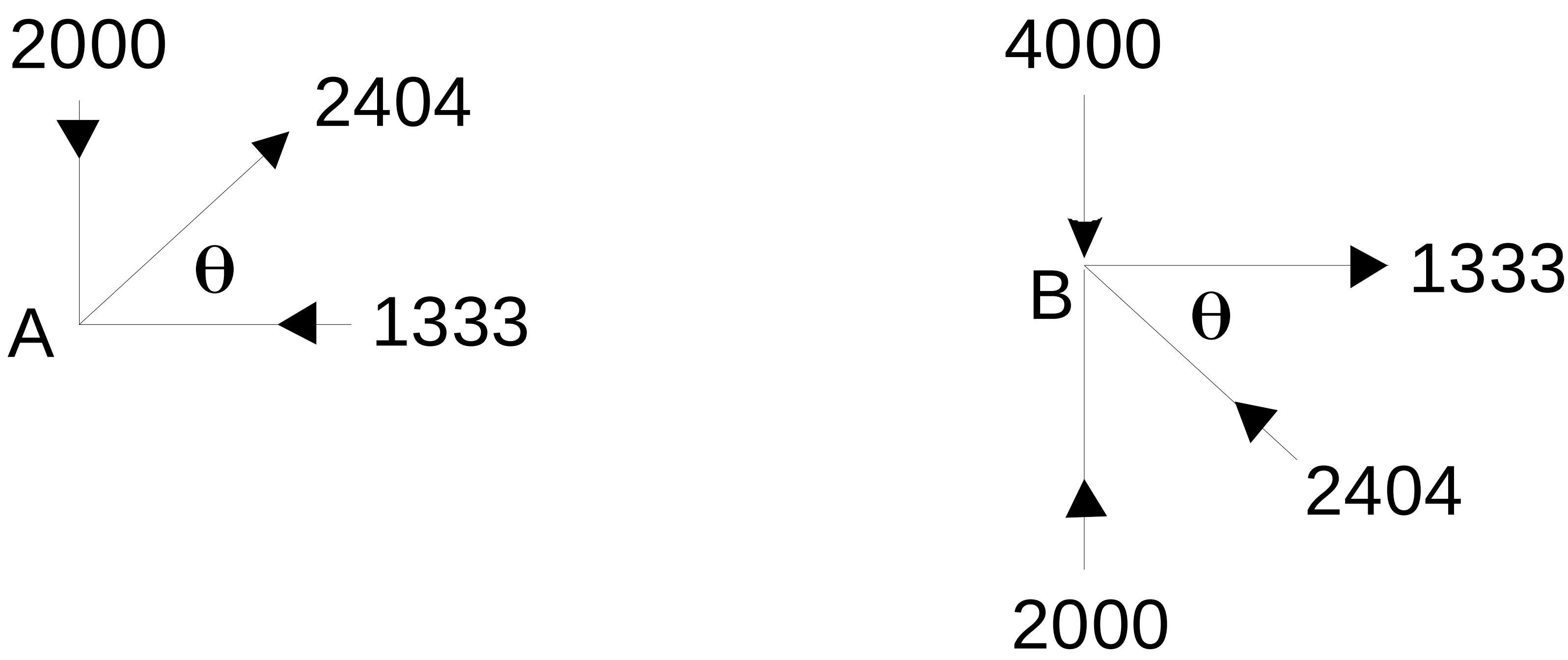
$$f_1^{'} = 3(10^5)[1 \quad 0 \quad -1 \quad 0][D_1 \quad D_2 \quad D_7 \quad D_8]^{\mathbf{T}} = 1333\,\text{lb}$$

....

$$f_4^{'} = 2(10^5)[0 \quad -1 \quad 0 \quad 1][D_5 \quad D_6 \quad D_7 \quad D_8]^{\mathbf{T}} = -2000\,\text{lb}$$

Example

Equilibrium Check



Space Truss Element

Element Equations in Global Coordinate System

$$\mathbf{k}'_{2\times2}\mathbf{d}'_{2\times1} = \mathbf{f}'_{2\times1}$$

$$\mathbf{d}'_{2\times1} = \mathbf{T}_{2\times6}\mathbf{d}_{6\times1} \quad \Rightarrow \quad \mathbf{k}_{6\times6}\mathbf{d}_{6\times1} = \mathbf{f}_{6\times1}$$

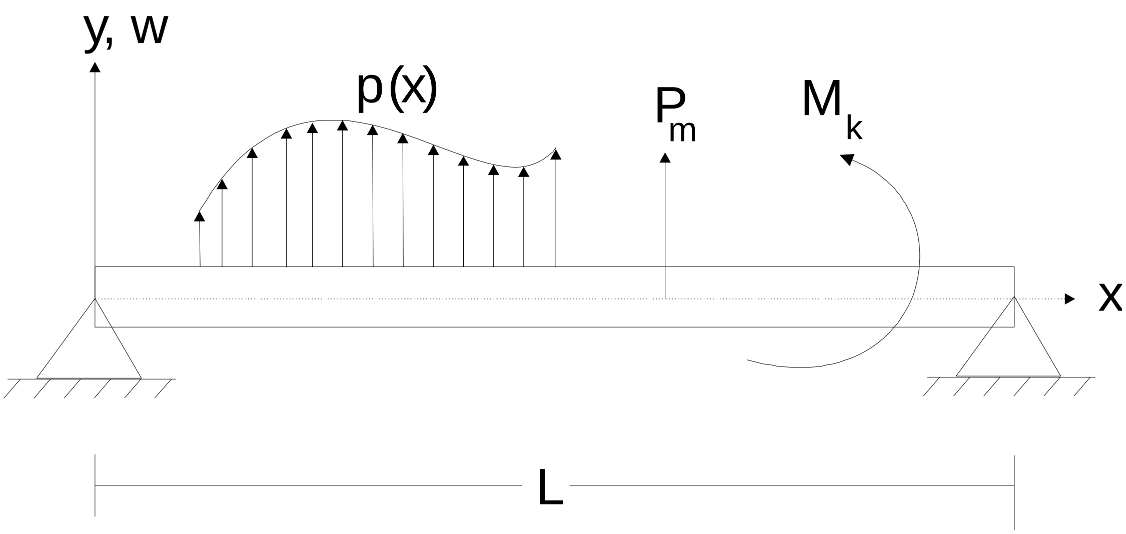
$$\mathbf{f}_{6\times1} = \mathbf{T}^{T}_{6\times2}\mathbf{f}'_{2\times1}$$

where

$$\mathbf{k}_{6\times6} = \mathbf{T}^{\mathbf{T}}_{6\times2}\mathbf{k}'_{2\times2}\mathbf{T}_{2\times6}$$

Planar Beam Element

Euler-Bernoulli Beam



- Prismatic
- Slender
- Plane sections remain plane
- Small deflections

Strain Energy

$$\sigma_x = -\frac{M_z y}{I_z} \qquad \sigma_x = E\varepsilon_x \qquad \frac{d^2 w(x)}{dx^2} = \frac{M_z}{EI_z}$$

Planar Beam Element

Strain Energy

$$U = \int_V U_0 \, dV = \int_0^L \int_A \frac{1}{2} \varepsilon \sigma \, dA \, dx = \frac{1}{2} \int_0^L \left[\frac{M^2}{EI^2} \int_A y^2 \, dA \right] dx$$

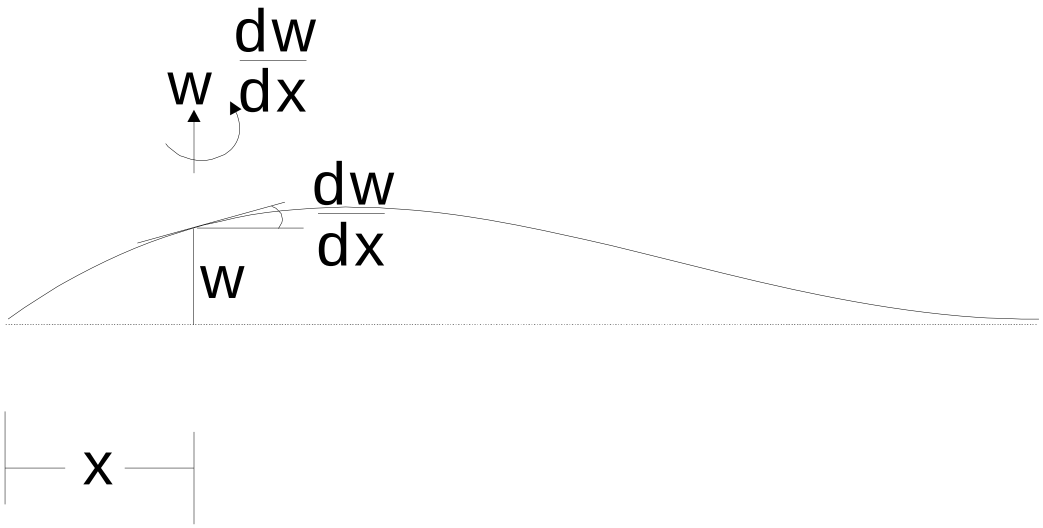
$$I = \int_A y^2 \, dA \quad \Rightarrow \quad U = \frac{1}{2} \int_0^L EI \left(\frac{d^2 w}{dx^2} \right)^2 dx$$

Total Potential Energy

$$\Pi = \frac{1}{2} \int_0^L EI \left(\frac{d^2 w}{dx^2} \right)^2 dx - \int_0^L p w \, dx - \sum_m P_m w_m - \sum_k M_k \frac{dw}{dx}$$

Planar Beam Element

Deflections and DOF



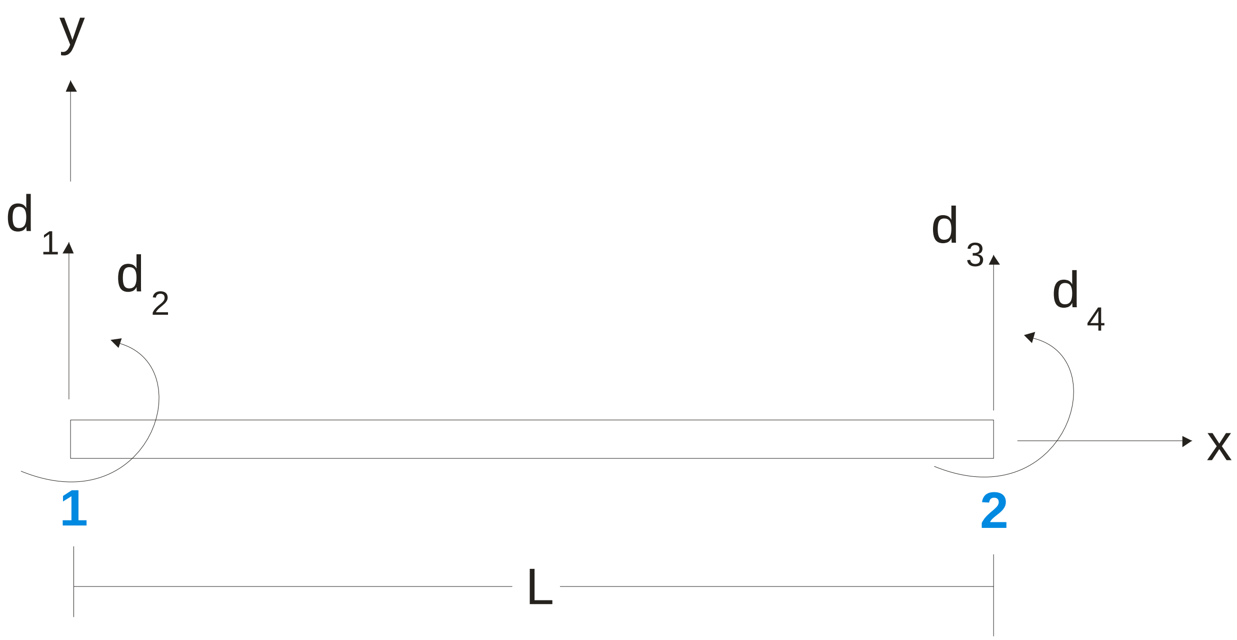
Hermite Cubics

$$H_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3 \qquad i = 1,..,4$$

$$H_i^{'} = b_i + 2c_i\xi + 3d_i\xi^2 \qquad i = 1,..,4$$

Planar Beam Element

Typical Element



Assumed displacement field

$$w(\xi) = H_1 w_1 + H_2 \left(\frac{dw}{d\xi} \right)_1 + H_3 w_2 + H_4 \left(\frac{dw}{d\xi} \right)_2$$

Planar Beam Element

Geometry

$$x = \frac{1-\xi}{2} x_1 + \frac{1+\xi}{2} x_2$$

$$\frac{dw}{d\xi} = \frac{dw}{dx}\frac{dx}{d\xi} = \frac{dw}{dx}\frac{L}{2}$$

Displacement Field

$$w(\xi) = H_1 d_1 + \frac{L}{2} H_2 d_2 + H_3 d_3 + \frac{L}{2} H_4 d_4 = \mathbf{H}_{1\times 4}\mathbf{d}_{4\times 1}$$

Planar Beam Element

Strain Energy

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx \quad \Rightarrow \quad U = \frac{1}{2}\mathbf{d}_{1\times4}^{\mathbf{T}}\mathbf{k}_{4\times4}\mathbf{d}_{4\times1}$$

Stiffness Matrix

$$\mathbf{k}_{4\times4} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ SYM & & 12 & -6L \\ & & & 4L^2 \end{bmatrix}$$

General Beam Element

Stiffness Matrix

$$\mathbf{k}'_{6\times6} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ SYM & & & \frac{EA}{L} & 0 & 0 \\ & & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ & & & & & \frac{4EI}{L} \end{bmatrix}$$

Local-To-Global Transformations

$$\mathbf{k}'_{6\times6}\mathbf{d}'_{6\times1} = \mathbf{f}'_{6\times1}$$

$$\mathbf{d}'_{6\times1} = \mathbf{T}_{6\times6}\mathbf{d}_{6\times1} \Rightarrow \mathbf{k}_{6\times6}\mathbf{d}_{6\times1} = \mathbf{f}_{6\times1}$$

$$\mathbf{f}_{6\times1} = \mathbf{T}^{T}_{6\times6}\mathbf{f}'_{6\times1}$$

where

$$\mathbf{k}_{6\times6} = \mathbf{T}^{\mathbf{T}}_{6\times6}\mathbf{k}'_{6\times6}\mathbf{T}_{6\times6}$$

Element Loads

Equivalent Nodal Forces

$$\mathbf{f}'_{6\times1} = \int_0^L p(x) H_i(x)\,dx = \int_{-1}^{1} p(x(\xi)) H_i(\xi) J d\xi$$

Example (Uniform distributed loading)

$$\mathbf{f}'_{6\times1} = p\int_{-1}^{1} H_i(\xi)\, J\, d\xi \Rightarrow \left[0, \frac{pL}{2}, \frac{pL^2}{12}, 0, \frac{pL}{2}, -\frac{pL^2}{12}\right]^{\mathbf{T}}$$

Local-to-global transformation

$$\mathbf{f}_{6\times1} = \mathbf{T}^{\mathbf{T}}_{6\times1}\mathbf{f}'_{6\times1}$$

Element Nodal Forces

Step 6

$$\mathbf{d'}_{6\times1} = \mathbf{T}_{6\times6}\,\mathbf{d}_{6\times1}$$

$$\mathbf{f}'_{6\times1} = \mathbf{k'}_{6\times6}\,\mathbf{d'}_{6\times1} - \sum_i \left(\mathbf{f}'_{6\times1}\right)_i$$

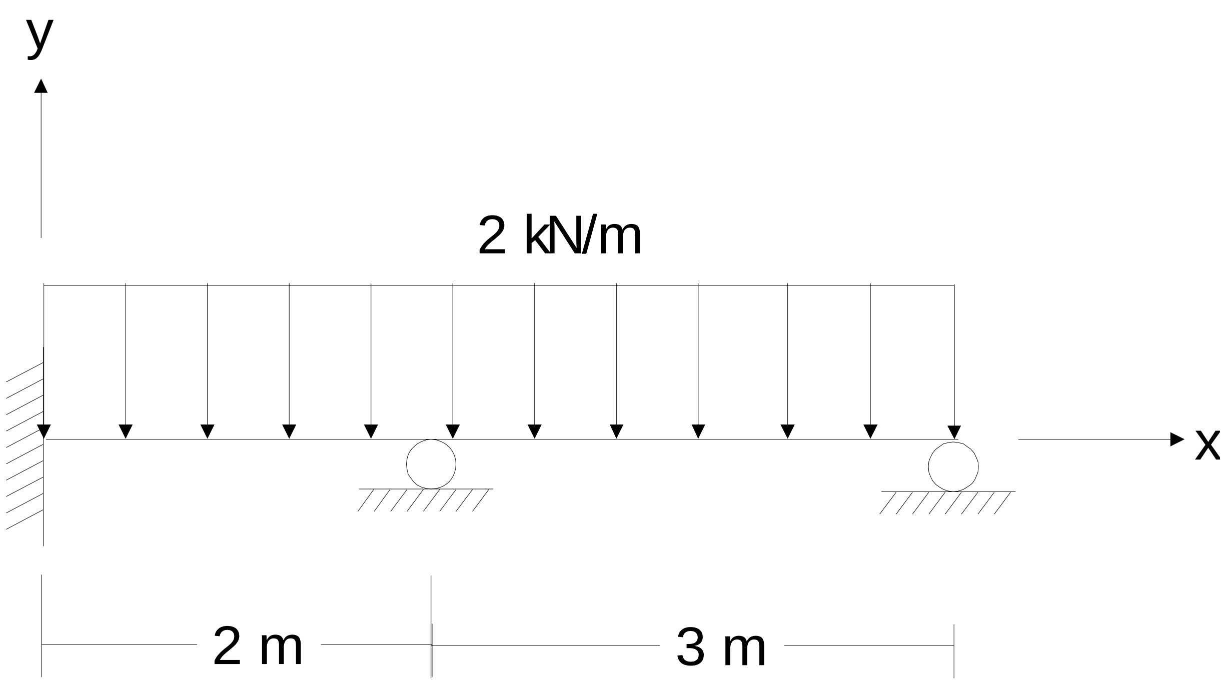
Example

Compute element forces and support reactions

$$E = 2(10)^{11}\,Pa$$

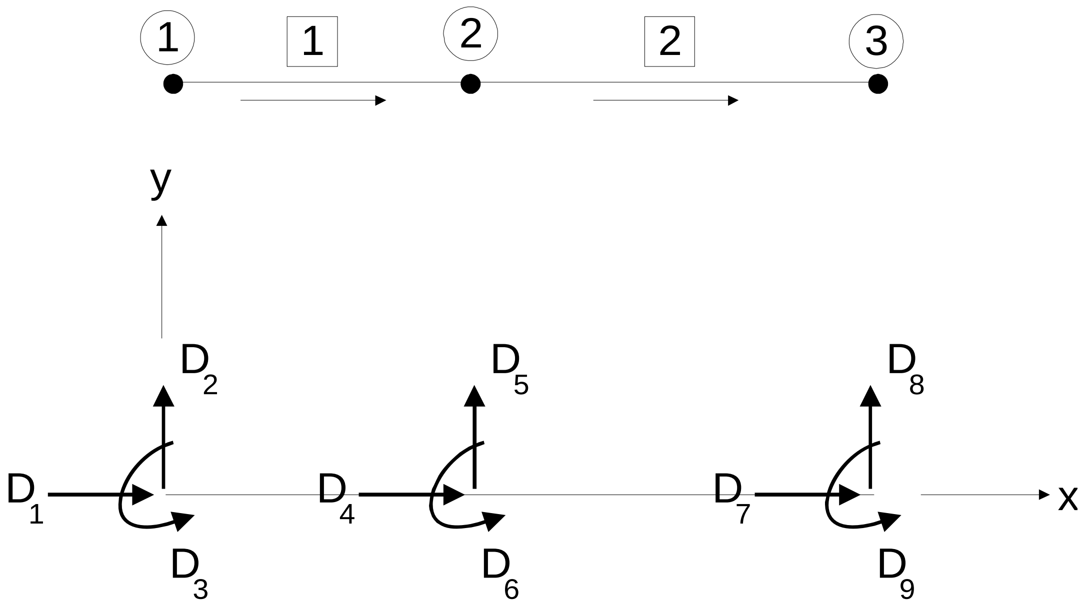
$$A = 0.01\,m^2$$

$$I = 0.0001\,m^4$$



Example

FE Model



Example

Step 4: System Equations after BCs

$$(10^7)\begin{bmatrix} 166.7 & 0 & -66.7 & 0 \\ 0 & 6.67 & 0 & 1.33 \\ -66.7 & 0 & 66.7 & 0 \\ 0 & 1.33 & 0 & 2.67 \end{bmatrix}\begin{Bmatrix} D_4 \\ D_6 \\ D_7 \\ D_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -833.33 \\ 0 \\ 1500 \end{Bmatrix}$$

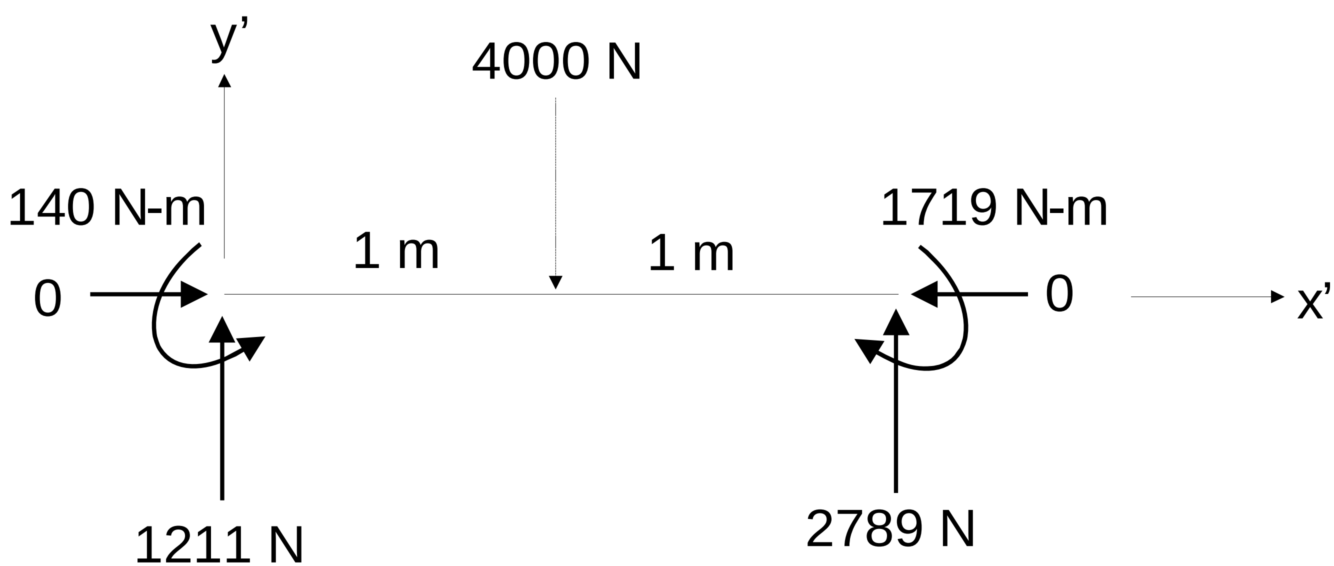
Example

Step 5: Solution

$$\{D_4, D_6, D_7, D_9\} = \{0,\text{-}2.63092(10^{-5})\,\mathrm{rad}, 0, 6.92851(10^{-5})\,\mathrm{rad}\}$$

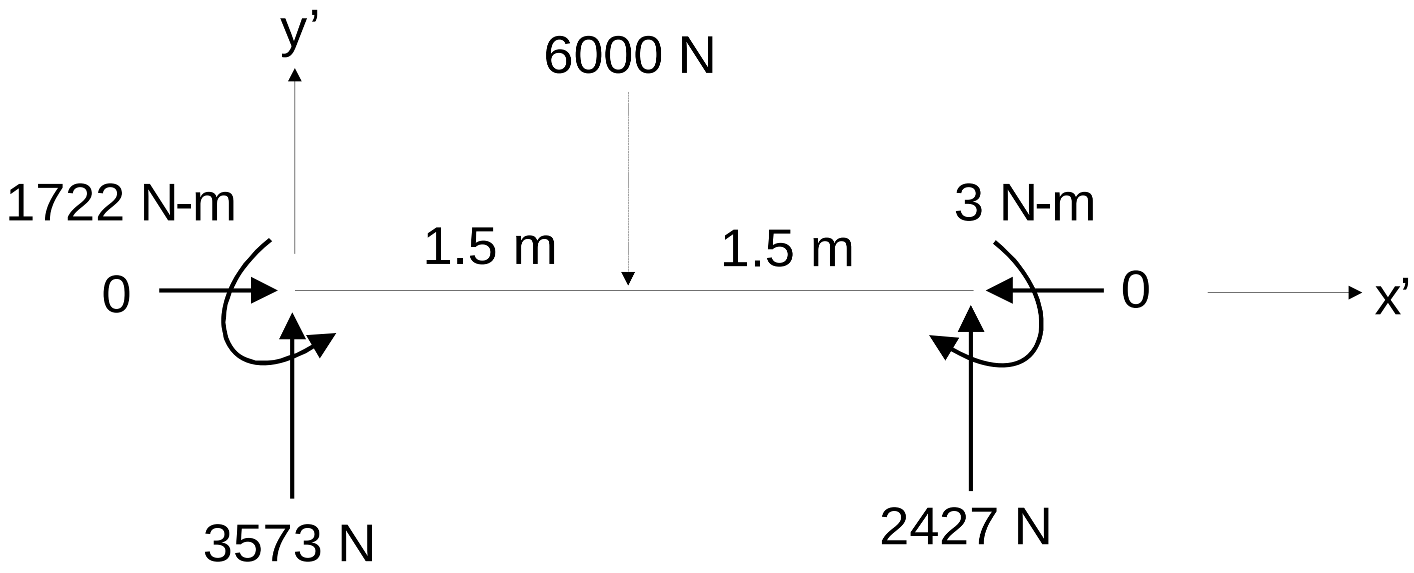
Step 6

Element 1



Example

Element 2



Shape Functions

Cartesian Coordinate System

$$\phi_1 = 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$$

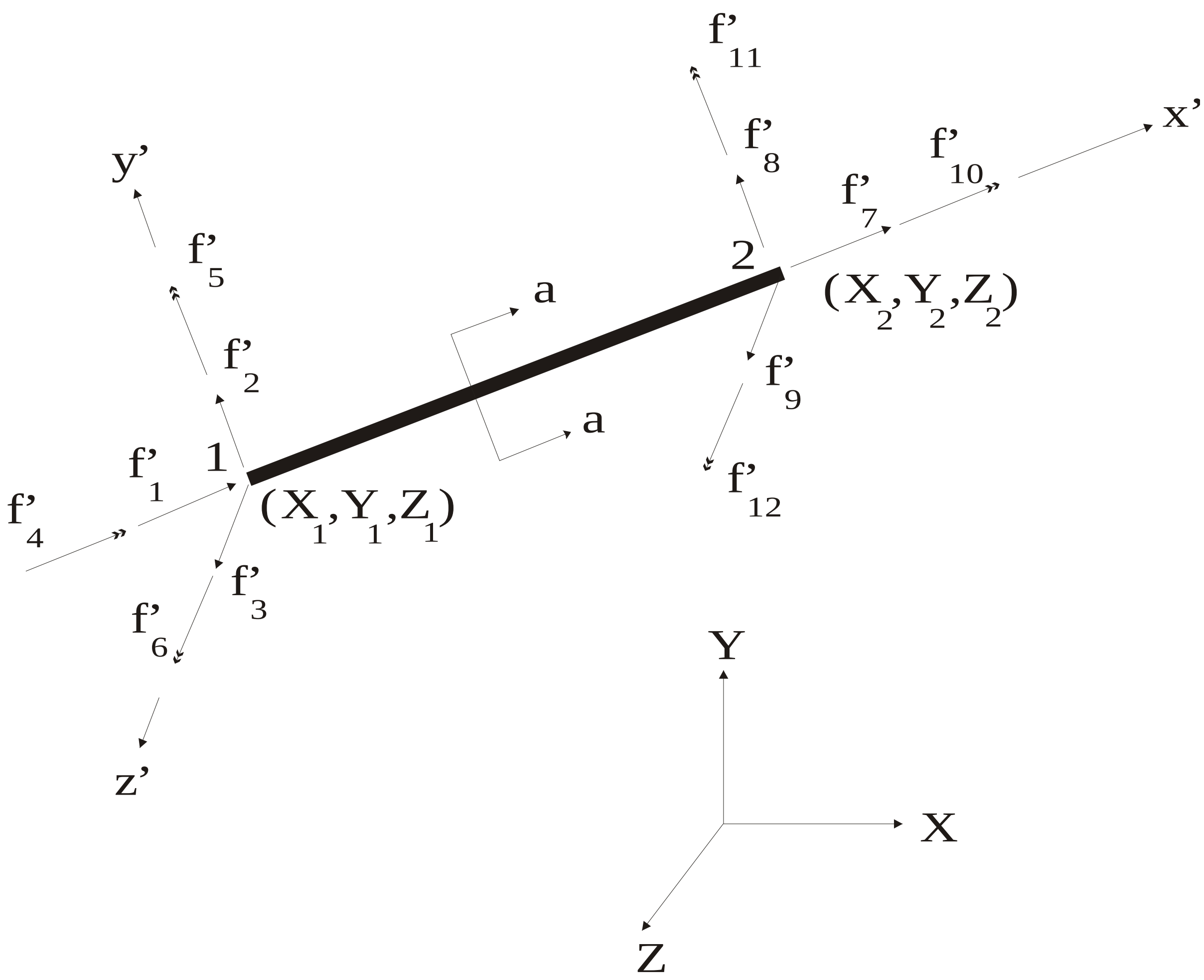
$$\phi_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$$

$$\phi_2 = x - \frac{2x^2}{L} + \frac{x^3}{L^2}$$

$$\phi_4 = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

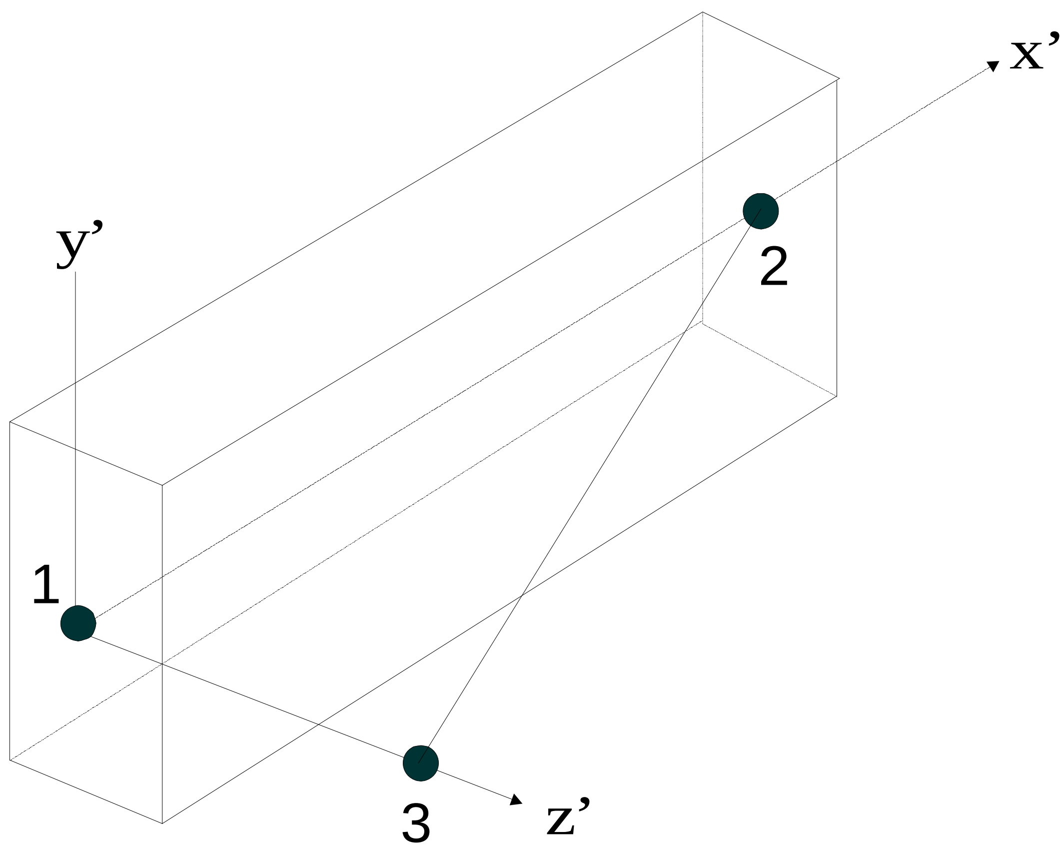
Space Beam Element

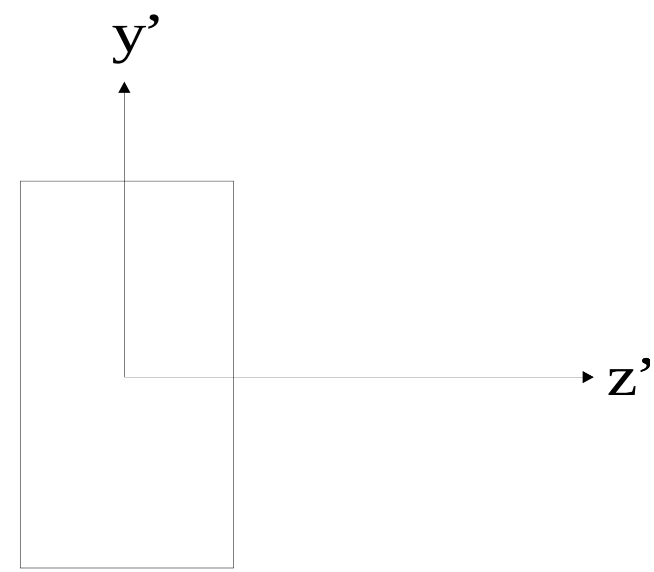
Nodal DOF



Space Beam Element

Element Orientation





View a-a

Space Beam Element

Stiffness Matrix

$$\mathbf{k'}_{12\times12} = \left[\begin{array}{c|c} \mathbf{k_{11}} & \mathbf{k_{12}} \\ \hline \mathbf{k_{21}} & \mathbf{k_{22}} \end{array}\right]_{12\times12}$$

Space Beam Element

$$\mathbf{k}_{11} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ & & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\ & & & \frac{GJ}{L} & 0 & 0 \\ & SYM & & & \frac{4EI_y}{L} & 0 \\ & & & & & \frac{4EI_z}{L} \end{bmatrix}$$

Space Beam Element

$$
\mathbf{k}_{22} = \begin{bmatrix}
\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\
 & & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\
 & & & \frac{GJ}{L} & 0 & 0 \\
 & SYM & & & \frac{4EI_y}{L} & 0 \\
 & & & & & \frac{4EI_z}{L}
\end{bmatrix}
$$

Space Beam Element

$$\mathbf{k}_{12} = \mathbf{k}_{21}^{\mathbf{T}} = \begin{bmatrix} -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\ 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \end{bmatrix}$$

Space Beam Element

Local-to-global Transformation

$$\mathbf{T}_{12\times 12} = \left[\begin{array}{c|c|c|c} \mathbf{\Lambda} & & & \\ \hline & \mathbf{\Lambda} & & \\ \hline & & \mathbf{\Lambda} & \\ \hline & & & \mathbf{\Lambda} \end{array}\right] \qquad \mathbf{\Lambda}_{3\times 3} = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \\ l_{z'} & m_{z'} & n_{z'} \end{bmatrix}$$

Space Beam Element

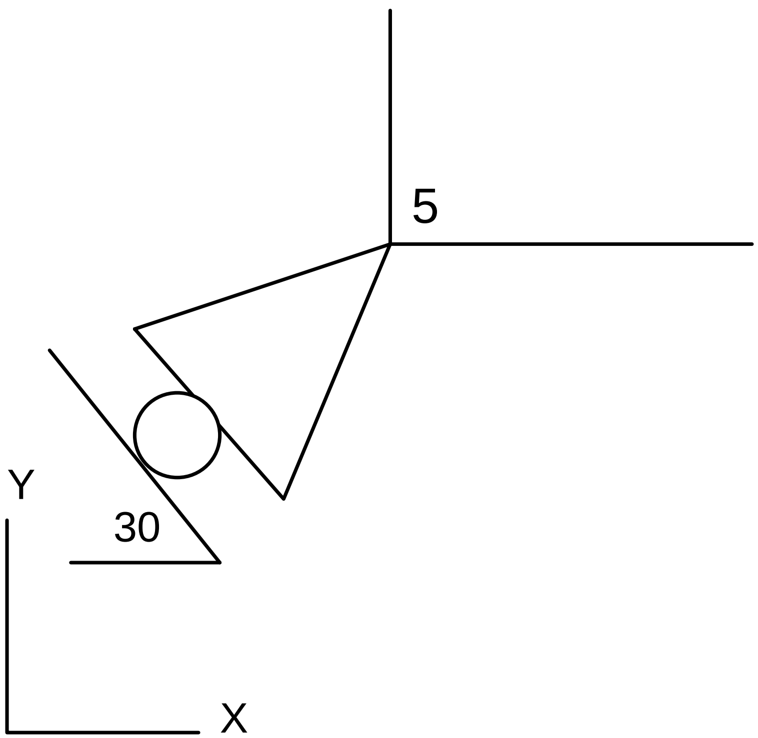
$$\mathbf{e}_{x'} = \left[l_{x'} \quad m_{x'} \quad n_{x'} \right] \Rightarrow l_{x'} = \frac{X_2 - X_1}{L}, \quad m_{x'} = \frac{Y_2 - Y_1}{L}, \quad n_{x'} = \frac{Z_2 - Z_1}{L}$$

$$\mathbf{e}_{13} = \frac{X_3 - X_1}{L_{13}} \hat{i} + \frac{Y_3 - Y_1}{L_{13}} \hat{j} + \frac{Z_3 - Z_1}{L_{13}} \hat{k}$$

$$\mathbf{e}_{y'} = \left\lfloor l_{y'} \quad m_{y'} \quad n_{y'} \right\rfloor \Rightarrow \mathbf{e}_{y'} = \mathbf{e}_{13} \times \mathbf{e}_{x'}$$

$$\mathbf{e}_{z'} = \left[l_{z'} \quad m_{z'} \quad n_{z'} \right] \Rightarrow \mathbf{e}_{z'} = \mathbf{e}_{13}$$

Handling Constraints



$$c_i D_i + c_j D_j = c$$

Handling Constraints

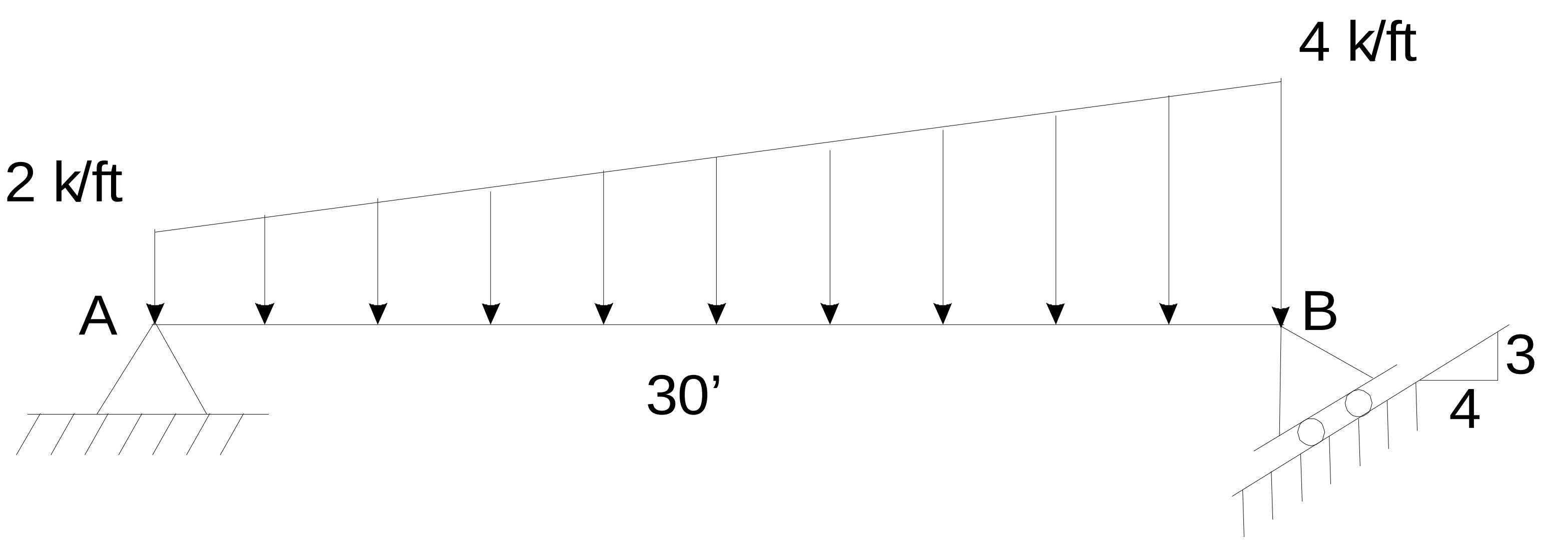
$$\Pi(\mathbf{D}) = \frac{1}{2}\mathbf{D}^{\mathbf{T}}\mathbf{K}\mathbf{D} - \mathbf{D}^{\mathbf{T}}\mathbf{F} + \frac{1}{2}C\left(c_i D_i + c_j D_j - c\right)^2$$

$$\partial\Pi/\partial D = 0 \Rightarrow \begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix} \to \begin{bmatrix} K_{ii} + Cc_i^2 & K_{ij} + Cc_i c_j \\ K_{ji} + Cc_i c_j & K_{jj} + Cc_j^2 \end{bmatrix}$$

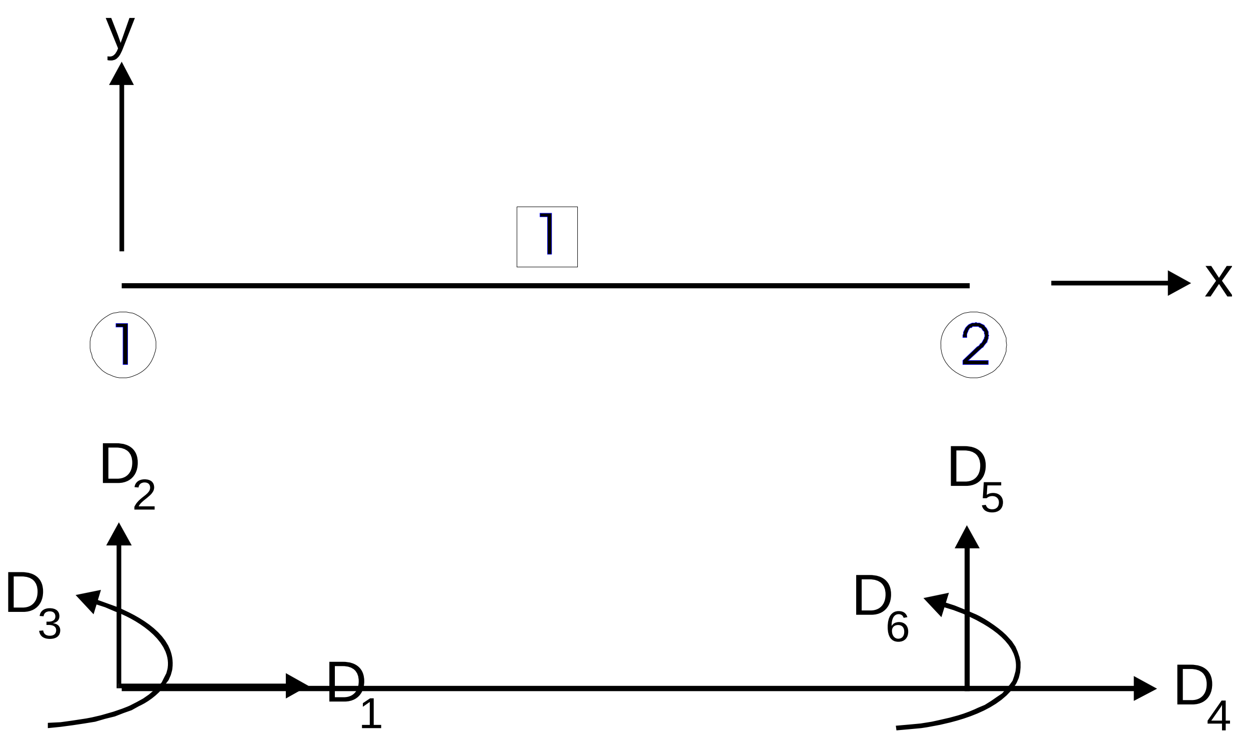
$$C = 10^4 \max\left|K_{pq}\right| \qquad \begin{Bmatrix} F_i \\ F_j \end{Bmatrix} \to \begin{Bmatrix} F_i + Ccc_i \\ F_j + Ccc_j \end{Bmatrix}$$

$$1 \le p, q \le n$$

Example



Example



$$\frac{D_5}{D_4} = \frac{3}{4} \Rightarrow 3D_4 - 4D_5 = 0$$

Example

$$10^2 \begin{bmatrix} 93.827 & 0 & -4.6914 & 46.914 \\ & 101.33 & 0 & 0 \\ & & 0.3128 & -4.6914 \\ Sym & & & 93.827 \end{bmatrix} \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} -210 \\ 0 \\ -51 \\ 240 \end{Bmatrix}$$

$$C = (10^4)(1.0133 \times 10^4) = 1.0133(10^8)$$

$$10^2 \begin{bmatrix} 93.827 & 0 & -4.6914 & 46.914 \\ & 9.1201(10^6) & -1.216(10^7) & 0 \\ & & 1.6213(10^7) & -4.6914 \\ Sym & & & 93.827 \end{bmatrix} \begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} -210 \\ 0 \\ -51 \\ 240 \end{Bmatrix}$$

Summary

- Use of Theorem of Minimum Potential Energy to generate element equations for
  - Truss Element
  - Frame/Beam Element
- Some unique characteristics
  - Elements are essentially 1D
  - Need local-to-global transformation
  - Integrals were evaluated exactly (but could have used numerical integration)

Material Modeling

Material Modeling

Stress-Strain Relationship (36 constants)

$$\sigma_i = C_{ij}\varepsilon_j \qquad i, j = 1, \ldots, 6$$

C_{ij}: stiffness matrix!

Material with strain energy density function

$$C_{ij} = C_{ji}$$

Anisotropic Stress-Strain Relationship (21 constants)

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & & & & & \\ C_{12} & C_{22} & & & Sym & \\ C_{13} & C_{23} & C_{33} & & & \\ C_{14} & C_{24} & C_{34} & C_{44} & & \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}$$

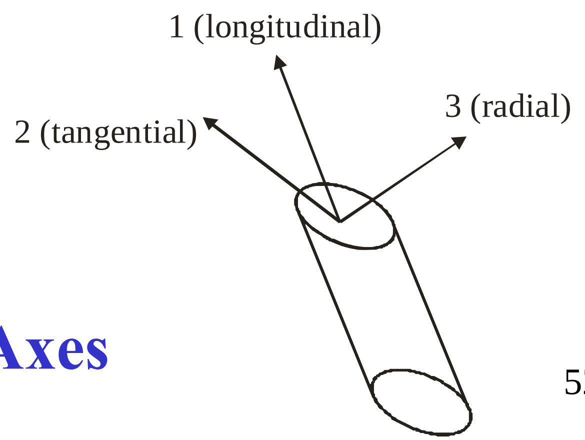
Material Modeling

Two Planes of symmetry (Orthotropic. 9 constants)

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & & & & & \\ C_{12} & C_{22} & & & Sym & \\ C_{13} & C_{23} & C_{33} & & & \\ 0 & 0 & 0 & C_{44} & & \\ 0 & 0 & 0 & 0 & C_{55} & \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}$$

(1) Normal stresses are independent of shearing strains.

(2) Some piezoelectric material and 2-ply composites



Principal Material Axes

Material Modeling

One plane (say 1-2) is isotropic (Transversely Isotropic. 5 constants)

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & & & & & \\ C_{12} & C_{11} & & & Sym & \\ C_{13} & C_{13} & C_{33} & & & \\ 0 & 0 & 0 & C_{44} & & \\ 0 & 0 & 0 & 0 & C_{44} & \\ 0 & 0 & 0 & 0 & 0 & \dfrac{(C_{11}-C_{12})}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}$$

(1) Some piezoelectric material

(2) Composites with all fibers being parallel

Material Modeling

Infinite planes of material symmetry (Isotropic. 2 constants)

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & & & & & \\ C_{12} & C_{11} & & & Sym & \\ C_{12} & C_{12} & C_{11} & & & \\ 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} & & \\ 0 & 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} & \\ 0 & 0 & 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}$$

Material Modeling

Strain-Stress Relationship (Orthotropic)

$$\varepsilon_i = S_{ij}\sigma_j \qquad i, j = 1, \ldots, 6$$

S_{ij}: compliance matrix

$$\left[S_{ij}\right] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

Poisson's Ratio

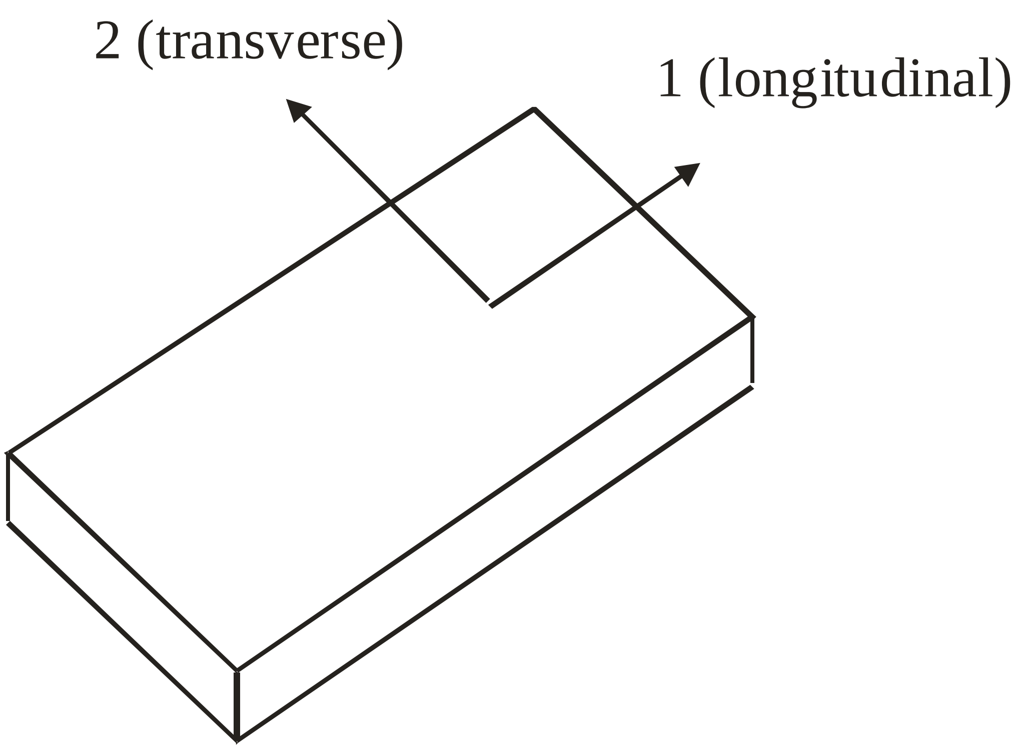
$$\nu_{ij} = -\frac{\varepsilon_j}{\varepsilon_i} \Leftarrow \text{Apply}\, \sigma_i = \sigma$$

Symmetry

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \qquad i, j = 1, 2, 3$$

Plane Stress Analysis

Geometry



Plane Stress

Stress-Strain Relationship (4 independent constants)

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\nu_{21}} & \dfrac{E_1\nu_{21}}{1-\nu_{12}\nu_{21}} & 0 \\ \dfrac{E_2\nu_{12}}{1-\nu_{12}\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}$$

$$\boldsymbol{\sigma}_{\mathbf{3\times1}} = \mathbf{D}^{m}_{3\times3}\boldsymbol{\varepsilon}_{3\times1}$$

Plane Stress

Stress & Strain Relationships (PMD vs Global)

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} l_1^2 & m_1^2 & 2l_1m_1 \\ m_1^2 & l_1^2 & -2l_1m_1 \\ -2l_1m_1 & l_1m_1 & l_1^2 - m_1^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{Bmatrix} = \begin{bmatrix} l_1^2 & m_1^2 & 2l_1m_1 \\ m_1^2 & l_1^2 & -2l_1m_1 \\ -2l_1m_1 & l_1m_1 & l_1^2 - m_1^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix}$$

Plane Stress

Material Matrix Relationship (PMD vs Global)

$$D_{11} = D_{11}^m \cos^4\theta + 2\left(D_{12}^m + 2D_{33}^m\right)\cos^2\theta\sin^2\theta + D_{22}^m \sin^4\theta$$

$$D_{12} = \left(D_{11}^m + D_{22}^m - 4D_{33}^m\right)\cos^2\theta\sin^2\theta + D_{12}^m\left(\cos^4\theta + \sin^4\theta\right)$$

$$D_{13} = \left(D_{11}^m - D_{12}^m - 2D_{33}^m\right)\sin\theta\cos^3\theta + \left(D_{12}^m - D_{22}^m + 2D_{33}^m\right)\sin^3\theta\cos\theta$$

$$D_{22} = D_{11}^m \sin^4\theta + 2\left(D_{12}^m + 2D_{33}^m\right)\cos^2\theta\sin^2\theta + D_{22}^m \cos^4\theta$$

$$D_{23} = \left(D_{11}^m - D_{12}^m - 2D_{33}^m\right)\sin^3\theta\cos\theta + \left(D_{12}^m - D_{22}^m + 2D_{33}^m\right)\sin\theta\cos^3\theta$$

$$D_{33} = \left(D_{11}^m + D_{22}^m - 2D_{12}^m - 2D_{33}^m\right)\cos^2\theta\sin^2\theta + D_{33}^m\left(\cos^4\theta + \sin^4\theta\right)$$

Plane Stress (Alt. Form)

Material Matrix Relationship (PMD vs Global)

$$D_{11} = D_{11}^m l^4 + 2\left(D_{12}^m + 2D_{33}^m\right) l^2 m^2 + D_{22}^m m^4$$

$$D_{12} = \left(D_{11}^m + D_{22}^m - 4D_{33}^m\right) l^2 m^2 + D_{12}^m \left(l^4 + m^4\right)$$

$$D_{13} = \left(D_{11}^m - D_{12}^m - 2D_{33}^m\right) l^3 m + \left(D_{12}^m - D_{22}^m + 2D_{33}^m\right) l m^3$$

$$D_{22} = D_{11}^m m^4 + 2\left(D_{12}^m + 2D_{33}^m\right) l^2 m^2 + D_{22}^m l^4$$

$$D_{23} = \left(D_{11}^m - D_{12}^m - 2D_{33}^m\right) l m^3 + \left(D_{12}^m - D_{22}^m + 2D_{33}^m\right) l^3 m$$

$$D_{33} = \left(D_{11}^m + D_{22}^m - 2D_{12}^m - 2D_{33}^m\right) l^2 m^2 + D_{33}^m \left(l^4 + m^4\right)$$

Plane Stress

- Construct direction cosines (l, m)
- Form $\mathbf{D^m}$ and using $\mathbf{D^m}$ construct $\mathbf{D}$
- Form $\mathbf{k}$ as usual

- Compute global strain and stress components
- Form $\mathbf{T}$
- Compute PMD strain and stress components