Finite Elements for 2-D Problems

General Formula for the Stiffness Matrix

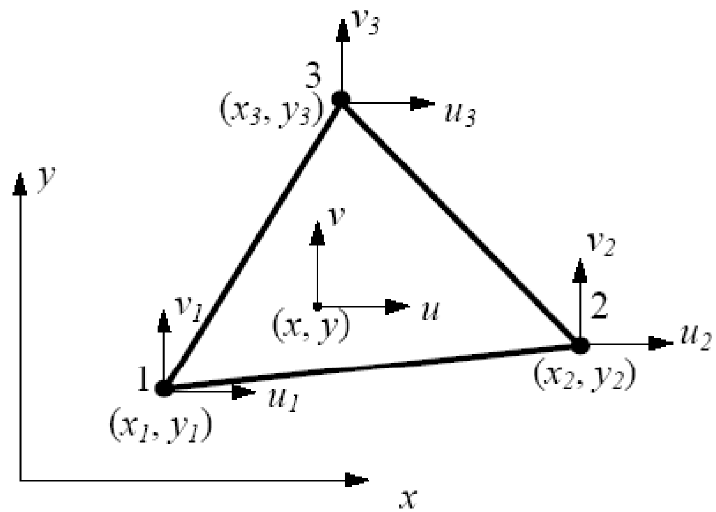
Displacements (*u, v) in a plane element are interpolated* from nodal displacements (*ui, vi) using shape functions Ni as* follows,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots \\ 0 & N_1 & 0 & N_2 & \cdots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{Bmatrix} \quad \text{or} \quad \mathbf{u} = \mathbf{N}\mathbf{d}$$

where N is the *shape function matrix, u the displacement vector* and d the *nodal displacement vector. Here we have assumed* that *u depends on the nodal values of u only, and v on nodal* values of *v only.* Most commonly employed 2-D elements are linear or quadratic triangles and quadrilaterals.

Constant Strain Triangle (CST or T3)

This is the simplest 2-D element, which is also called ***linear triangular element.***



For this element, we have three nodes at the vertices of the triangle, which are numbered around the element in the counterclockwise direction. Each node has two degrees of freedom (can move in the *x and y directions). The* displacements *u and v are assumed to be linear functions within* the element, that is,

$$u = b_1 + b_2 x + b_3 y, \quad v = b_4 + b_5 x + b_6 y$$

where *bi (i = 1, 2, ..., 6) are constants. From these, the strains* are found to be,

$$\varepsilon_x = b_2, \quad \varepsilon_y = b_6, \quad \gamma_{xy} = b_3 + b_5$$

which are constant throughout the element.

The displacements should satisfy the following six equations,

$$
\begin{aligned}
u_1 &= b_1 + b_2 x_1 + b_3 y_1 \\
u_2 &= b_1 + b_2 x_2 + b_3 y_2 \\
&\vdots \\
v_3 &= b_4 + b_5 x_3 + b_6 y_3
\end{aligned}
$$

Solving these equations, we can find the coefficients *b1, b2, ...,* and *b6 in terms of nodal displacements and coordinates.*

The displacements can be expressed as

$$
\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}
$$

The shape functions (linear functions in *x and y) are*

$$N_1 = \frac{1}{2A}\{(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\}$$

$$N_2 = \frac{1}{2A}\{(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\}$$

$$N_3 = \frac{1}{2A}\{(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\}$$

and

$$A = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$ is the area of the triangle.

The strain-displacement relations are written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \mathbf{Bd} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

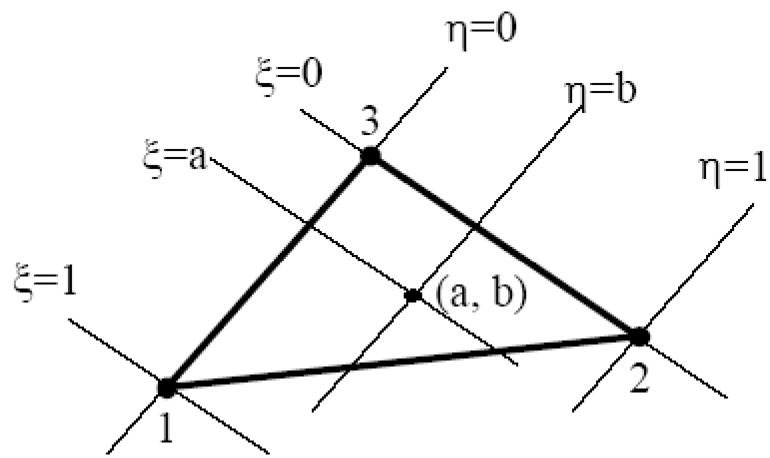
where *xij = xi - xj and yij = yi - yj (i, j = 1, 2, 3). Again, we see* constant strains within the element. From stress-strain relation, we see that stresses obtained using the CST element are also constant.

The element stiffness matrix for the CST element,

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV = tA(\mathbf{B}^T \mathbf{E} \mathbf{B})$$

in which *t is the thickness of the element. Notice that* **k** for CST is a 6 by 6 *symmetric matrix.*

The Natural Coordinates



We introduce the *natural coordinates* (ξ, η) on the triangle, then *the shape functions can be represented simply by,*

$$N_1 = \xi, \quad N_2 = \eta, \quad N_3 = 1 - \xi - \eta$$

Notice that,

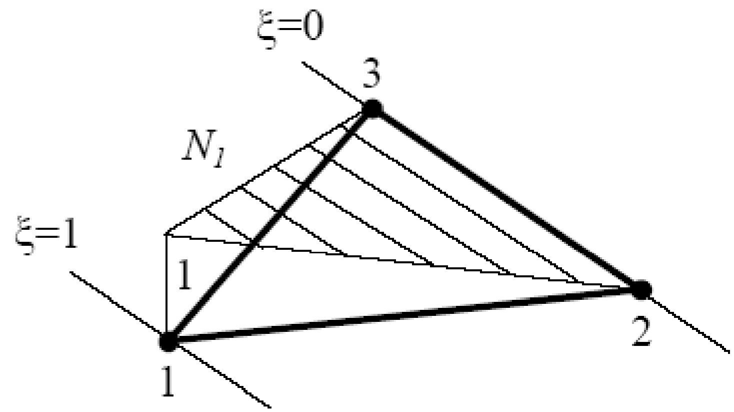
$$N_1 + N_2 + N_3 = 1$$

which ensures that the rigid body translation is represented by the chosen shape functions. Also, as in the 1-D case,

$$N_i = \begin{cases} 1, & \text{at node i;} \\ 0, & \text{at the other nodes} \end{cases}$$

and varies linearly within the element.

The plot for shape function *N1 is shown in the following figure. N2 and N3 have* similar features.



We have two coordinate systems for the element: the global coordinates (*x, y) and the natural coordinates* (ξ, η). The relation between the two is given by

$$\begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 \\ y &= N_1 y_1 + N_2 y_2 + N_3 y_3 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= x_{13}\xi + x_{23}\eta + x_3 \\ y &= y_{13}\xi + y_{23}\eta + y_3 \end{aligned}$$

where *xij = xi - xj and yij = yi - yj (i, j = 1, 2, 3)* as defined earlier.

Displacement *u or v on the element can be viewed as* functions of (*x, y) or* (ξ, η) *.*

Using the chain rule for derivatives, we have,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix}$$

where **J** is called the *Jacobian matrix* of the transformation***,*** and is expressed as

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}, \qquad \mathbf{J}^{-1} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

where $\det \mathbf{J} = x_{13} y_{23} - x_{23} y_{13} = 2A$ *and A is the area of* the triangular element.

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} u_1 - u_3 \\ u_2 - u_3 \end{Bmatrix}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} v_1 - v_3 \\ v_2 - v_3 \end{Bmatrix}$$

Using the relations $\varepsilon = \mathbf{Du} = \mathbf{DNd} = \mathbf{Bd}$, we obtain the strain-displacement matrix,

$$\mathbf{B} = \frac{1}{2A} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Applications of the CST Element:

· Use in areas where the strain gradient is small.

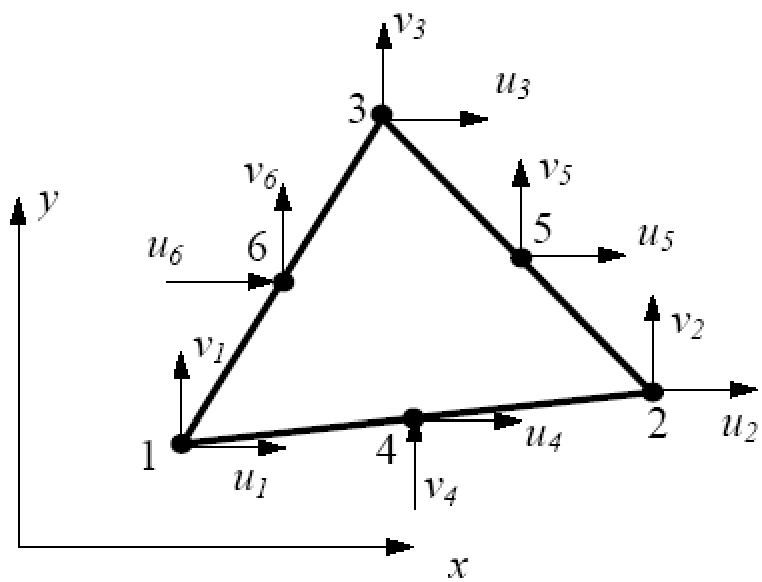
· Use in mesh transition areas (fine mesh to coarse mesh).

· Avoid using CST in stress concentration or other crucial areas in the structure, such as edges of holes and corners.

· Recommended for quick and preliminary FE analysis of 2-D problems.

Linear Strain Triangle (LST or T6)

This element is also called ***quadratic triangular element.***



There are six nodes on this element: three corner nodes and three mid-side nodes. Each node has two degrees of freedom (DOF) as before. The displacements (*u, v) are assumed to be* quadratic functions of (*x, y),*

$$u = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_6 y^2$$
$$v = b_7 + b_8 x + b_9 y + b_{10} x^2 + b_{11} xy + b_{12} y^2$$

where *bi (i = 1, 2, ..., 12) are constants.*

The strains are found to be,

$$\varepsilon_x = b_2 + 2b_4 x + b_5 y$$
$$\varepsilon_y = b_9 + b_{11} x + 2b_{12} y$$
$$\gamma_{xy} = (b_3 + b_8) + (b_5 + 2b_{10})x + (2b_6 + b_{11})y$$

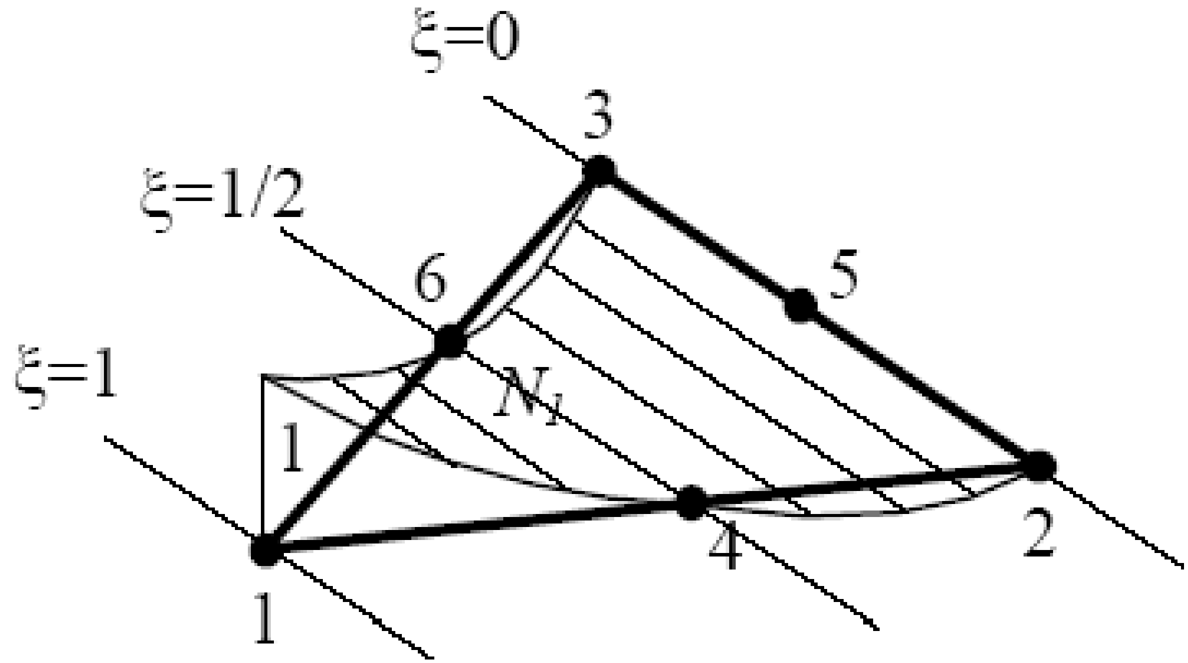
which are linear functions. Thus, we have the “linear strain triangle” (LST), which provides better results than the CST.

In the natural coordinate system we defined earlier, the six shape functions for the LST element are,

$$N_1 = \xi(2\xi - 1)$$
$$N_2 = \eta(2\eta - 1)$$
$$N_3 = \zeta(2\zeta - 1)$$
$$N_4 = 4\xi\eta$$
$$N_5 = 4\eta\zeta$$
$$N_6 = 4\zeta\xi$$

in which $\zeta = 1 - \xi - \eta$

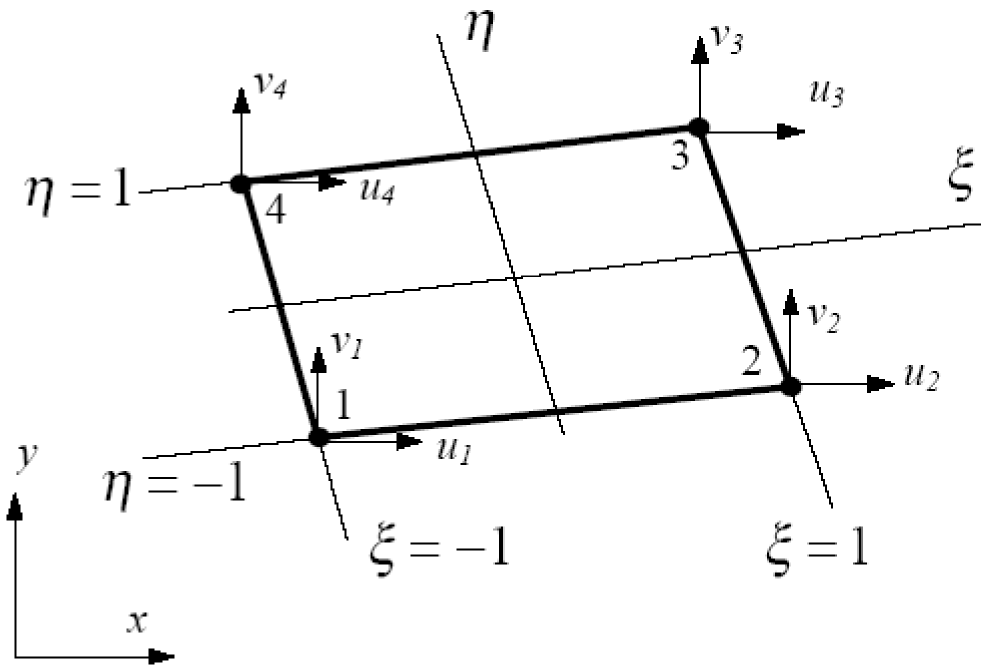
Each of these six shape functions represents a quadratic form on the element as shown in the following figure.



Displacements can be written as,

$$u = \sum_{i=1}^{6} N_i u_i, \qquad v = \sum_{i=1}^{6} N_i v_i$$

Linear Quadrilateral Element (Q4)



There are four nodes at the corners of the quadrilateral shape. In the natural coordinate system (ξ, η)*, the four shape* functions are,

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \qquad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta), \qquad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$\sum_{i=1}^{4} N_i = 1$ at any point inside the element.

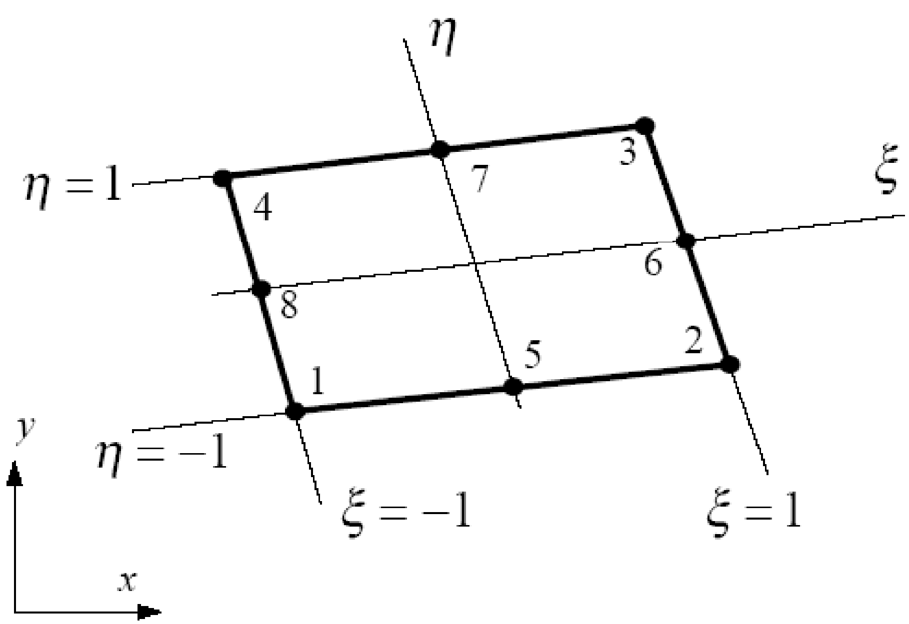
The displacement field is given by

$$u = \sum_{i=1}^{4} N_i u_i, \qquad v = \sum_{i=1}^{4} N_i v_i$$

which are bilinear functions over the element.

Quadratic Quadrilateral Element (Q8)

This is the most widely used element for 2-D problems due to its high accuracy in analysis and flexibility in modeling.



There are eight nodes for this element, four corners nodes and four mid-side nodes.

In the natural coordinate system (ξ,η) the eight shape functions are,

$$N_1 = \frac{1}{4}(1-\xi)(\eta-1)(\xi+\eta+1) \qquad N_5 = \frac{1}{2}(1-\eta)(1-\xi^2)$$

$$N_2 = \frac{1}{4}(1+\xi)(\eta-1)(\eta-\xi+1) \qquad N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \qquad N_7 = \frac{1}{2}(1+\eta)(1-\xi^2)$$

$$N_4 = \frac{1}{4}(\xi-1)(\eta+1)(\xi-\eta+1) \qquad N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

Again, we have $\sum_{i=1}^{8} N_i = 1$ at any point inside the element.

The displacement field is given by

$$u = \sum_{i=1}^{8} N_i u_i, \qquad v = \sum_{i=1}^{8} N_i v_i$$

which are quadratic functions over the element. Strains and stresses over a quadratic quadrilateral element are quadratic functions, which are better representations.

Notes:

· Q4 and T3 are usually used together in a mesh with linear elements.

· Q8 and T6 are usually applied in a mesh composed of quadratic elements.

· Quadratic elements are preferred for stress analysis, because of their high accuracy and the flexibility in modeling complex geometry, such as curved boundaries.

Stress Calculation

The stress in an element is determined by the following relation,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \mathbf{E} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \mathbf{EBd}$$

where **B** is the strain-nodal displacement matrix and d is the nodal displacement vector which is known for each element once the global FE equation has been solved.

Stresses can be evaluated at any point inside the element (such as the center) or at the nodes. Contour plots are usually used in FEA software packages (during post-process) for users to visually inspect the stress results.

The von Mises Stress:

The von Mises stress is the *effective or equivalent stress for* 2-D and 3-D stress analysis.

$$\sigma_e = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

in which $\sigma_1, \sigma_2 \text{ and } \sigma_3$ *and are the three principle stresses at the* considered point in a structure.

For 2-D problems, the two principle stresses in the plane are determined by

$$\sigma_1^{\ P} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2^{\ P} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Thus, we can also express the von Mises stress in terms of the stress components in the *xy coordinate system.*

For plane stress conditions, we have,

$$\sigma_e = \sqrt{(\sigma_x + \sigma_y)^2 - 3(\sigma_x \sigma_y - \tau_{xy}^2)}$$