Finite Elements For Engineers

Lecture 7: Convergence and Modeling Issues

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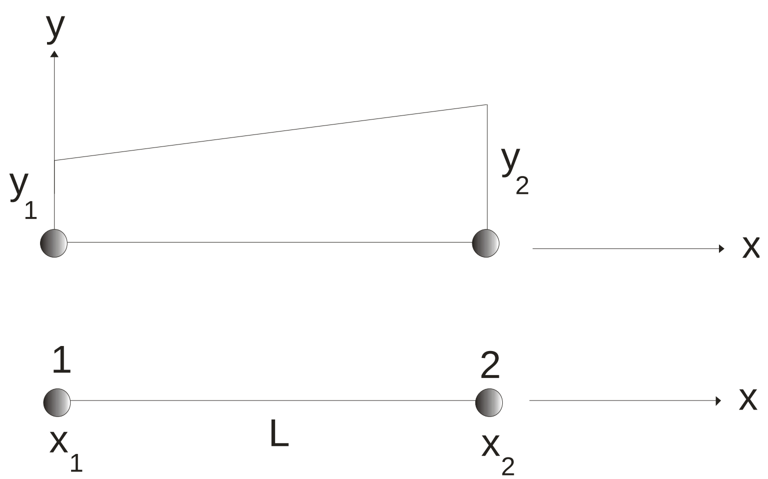
Higher-Order Elements (p-Convergence)

- Better accuracy with higher-order interpolation
- For the same level of accuracy compared to lower order elements
  - Requires less number of elements (and nodes)
  - However element generation is more expensive
  - **k** will usually have more non-zero elements

Elements and Interpolation Order

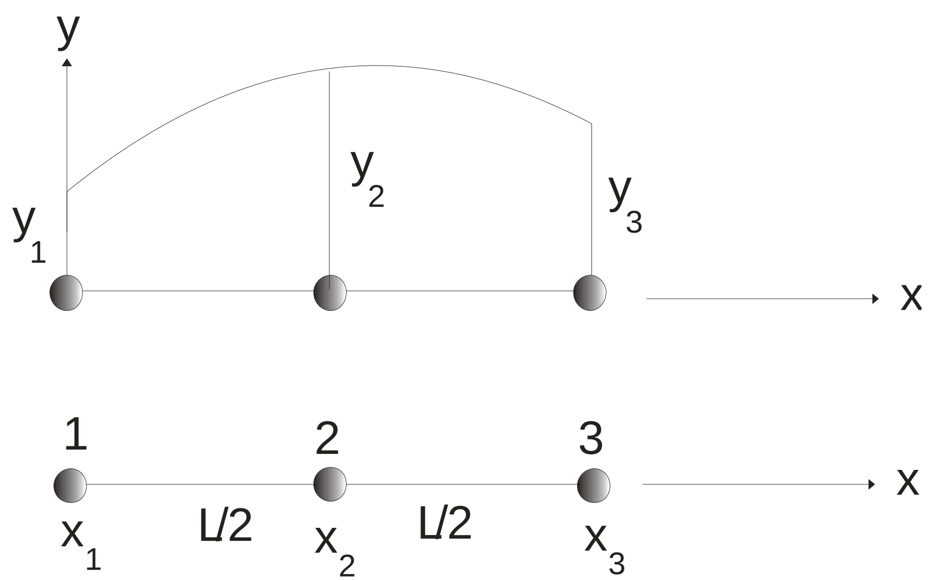
Linear Interpolation

$$\tilde{y}(x) = a_1 + a_2 x = \phi_1(x) y_1 + \phi_2(x) y_2$$



Quadratic Interpolation

$$\tilde{y}(x) = a_1 + a_2 x + a_3 x^2 = \phi_1(x) y_1 + \phi_2(x) y_2 + \phi_3(x) y_3$$



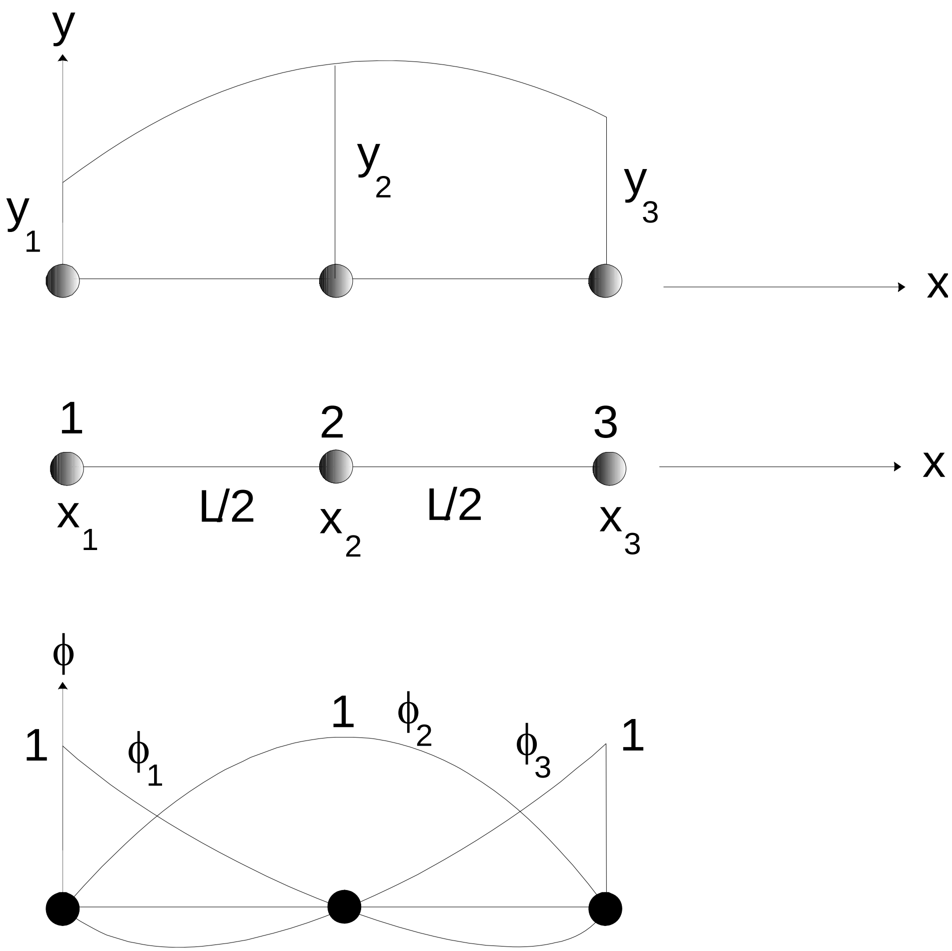
Properties of Shape Functions

- They are the interpolating functions (interpolation using the nodal value)
- If the number of DOF per node is one, then we need as many nodes as the number of unknown coefficients
- The shape functions must satisfy the following relationships

$$\phi_i(x_j) = \delta_{ij} \qquad \sum_i \phi_i(x) = 1$$

1D-C^0 Quadratic Element

$$\tilde{y}(x) = a_1 + a_2 x + a_3 x^2 = \phi_1(x) y_1 + \phi_2(x) y_2 + \phi_3(x) y_3$$



$$\phi_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

$$\phi_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

$$\phi_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

1D-C^0 Quadratic Element

Galerkin Step 4

Typical Stiffness Term

$$k_{ij} = \int_{\Omega} \frac{d\phi_i}{dx} \alpha(x) \frac{d\phi_j}{dx} dx + \int_{\Omega} \phi_i(x) \beta(x) \phi_j(x) dx$$

Assume for the time being that these are constants

$$\alpha(x) = \alpha \qquad \beta(x) = \beta$$

1D-C^0 Quadratic Element

$$\left[\begin{bmatrix} \frac{7\alpha}{3L} & -\frac{8\alpha}{3L} & \frac{\alpha}{3L} \\ -\frac{8\alpha}{3L} & \frac{16\alpha}{3L} & -\frac{8\alpha}{3L} \\ \frac{\alpha}{3L} & -\frac{8\alpha}{3L} & \frac{7\alpha}{3L} \end{bmatrix} + \begin{bmatrix} \frac{4\beta L}{30} & \frac{2\beta L}{30} & -\frac{\beta L}{30} \\ \frac{2\beta L}{30} & \frac{16\beta L}{30} & \frac{2\beta L}{30} \\ -\frac{\beta L}{30} & \frac{2\beta L}{30} & \frac{4\beta L}{30} \end{bmatrix} - g_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right.$$

$$\left. + h_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} \frac{fL}{6} \\ \frac{4fL}{6} \\ \frac{fL}{6} \end{Bmatrix} + \begin{Bmatrix} c_1 \\ 0 \\ -c_3 \end{Bmatrix}$$

1D-C^0 Quadratic Element

Element Flux $\tilde{\tau} = -\alpha(x)\dfrac{d\tilde{y}}{dx}$

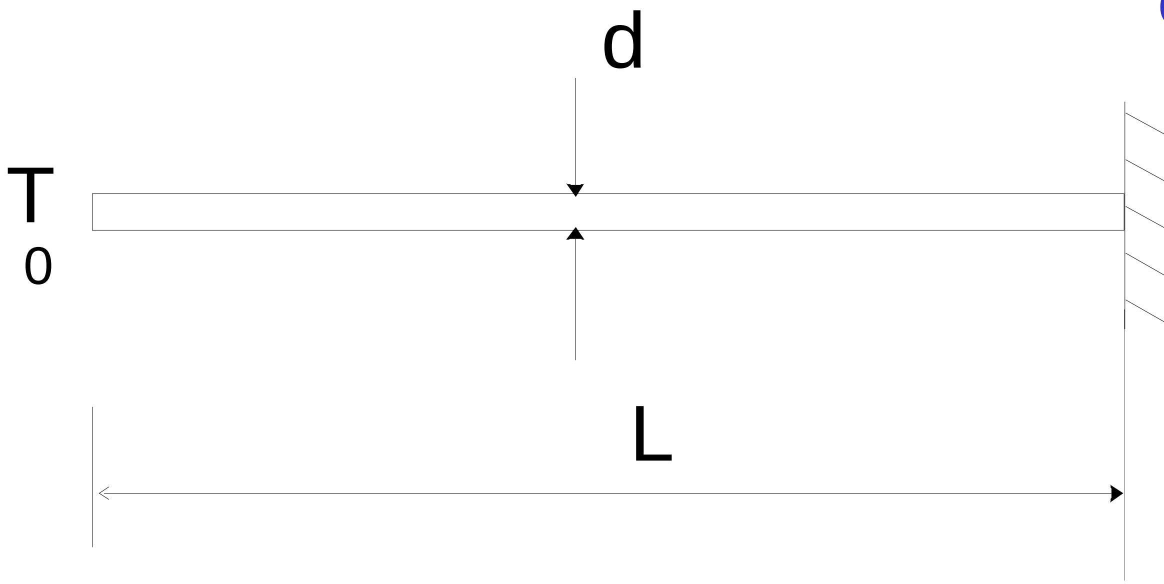
$$\tilde{\tau} = -\frac{\alpha}{L^2}\left[x\left(4y_1 - 8y_2 + 4y_3\right) + x_1\left(4y_2 - 2y_3\right) - 2x_2\left(y_1 + y_3\right) - 2x_3\left(y_1 + 2y_2\right)\right]$$

Mesh Refinement (h and p-convergence)

- **h-**convergence refers to the process of refining the FE mesh (adding more elements)
- **p-**convergence refers to the process of increasing the interpolation order of the elements in the FE mesh

Example T4L3-2

Compute temperature and flux.



Conduction and Convection

$$T_0 = 150^\circ F$$

$$T_\infty = 80^\circ F$$

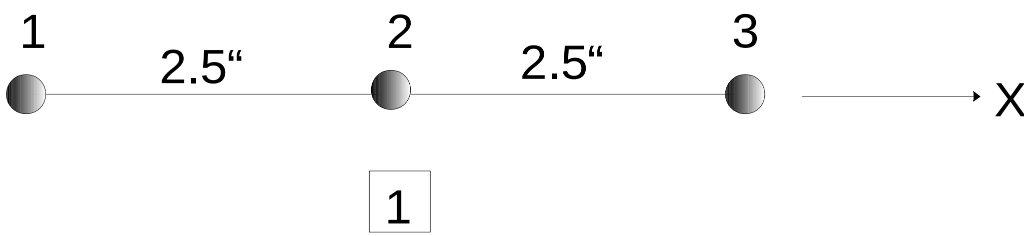
$$h = 6 \frac{BTU}{h \cdot ft^2 \cdot^\circ F}$$

$$k = 24.8 \frac{BTU}{h \cdot ft \cdot^\circ F}$$

Example T4L3-2

Units: BTU, hr, ft, F

Discretization: FE Mesh



Element 1

$$\begin{bmatrix} 181.7 & -116.38 & -1.4256 \\ -116.38 & 488.33 & -116.38 \\ -1.4256 & -116.38 & 181.7 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 5111.3 \\ 20445 \\ 5111.3 \end{Bmatrix}$$

Example T4L3-2

Imposition of EBC $T_1 = 150$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 488.33 & -116.38 \\ 0 & -116.38 & 181.7 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 150 \\ 37902 \\ 5325.2 \end{Bmatrix}$$

Solution

$$\{T_1, T_2, T_3\} = \{150,\ \ 99.8,\ \ 93.3\}^{\circ} F$$

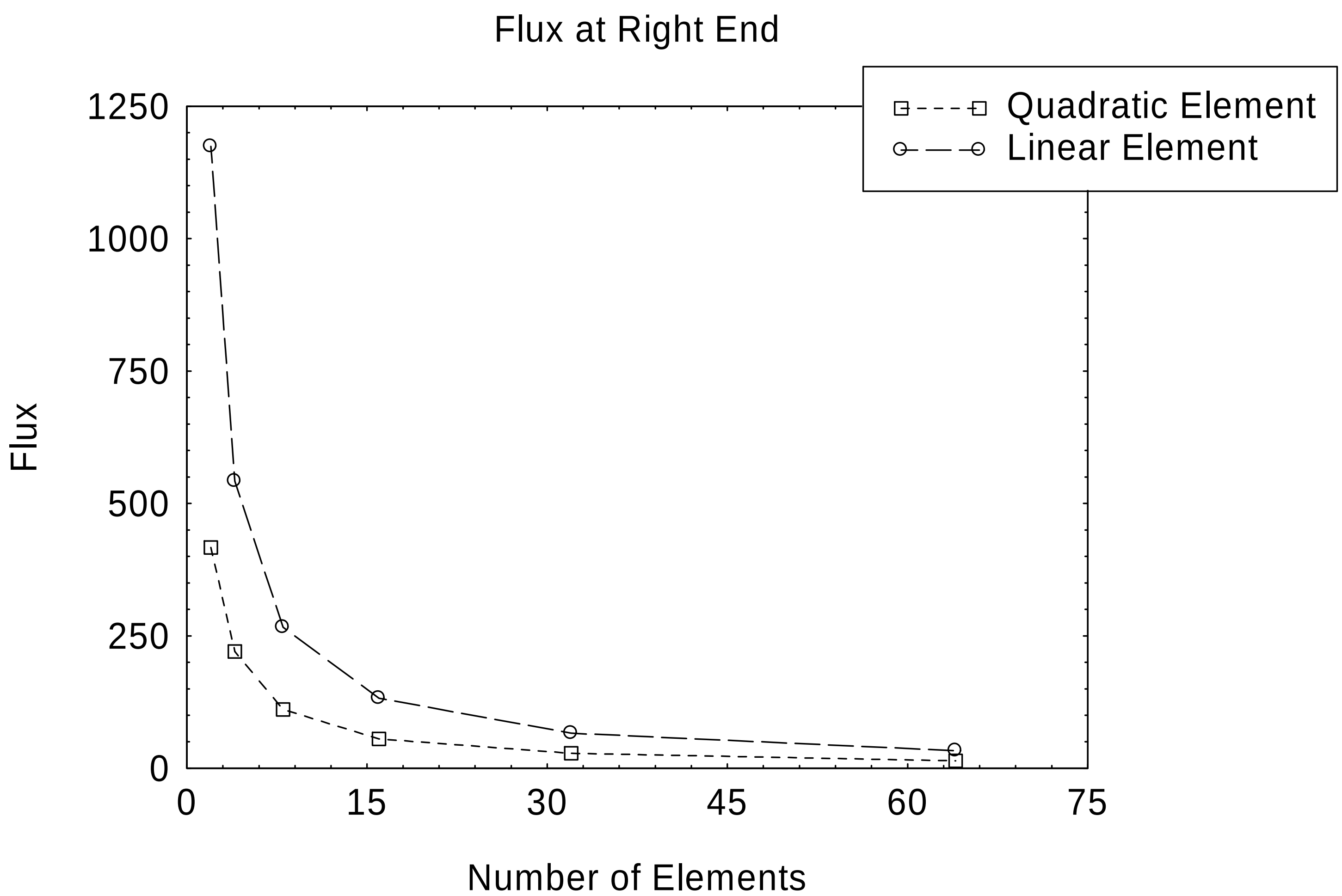
Derived Variables

$$\tau(x = 0.0879') = 6382 \frac{BTU}{h \cdot ft^2} \qquad \tau(x = 0.3281') = 383.1 \frac{BTU}{h \cdot ft^2}$$

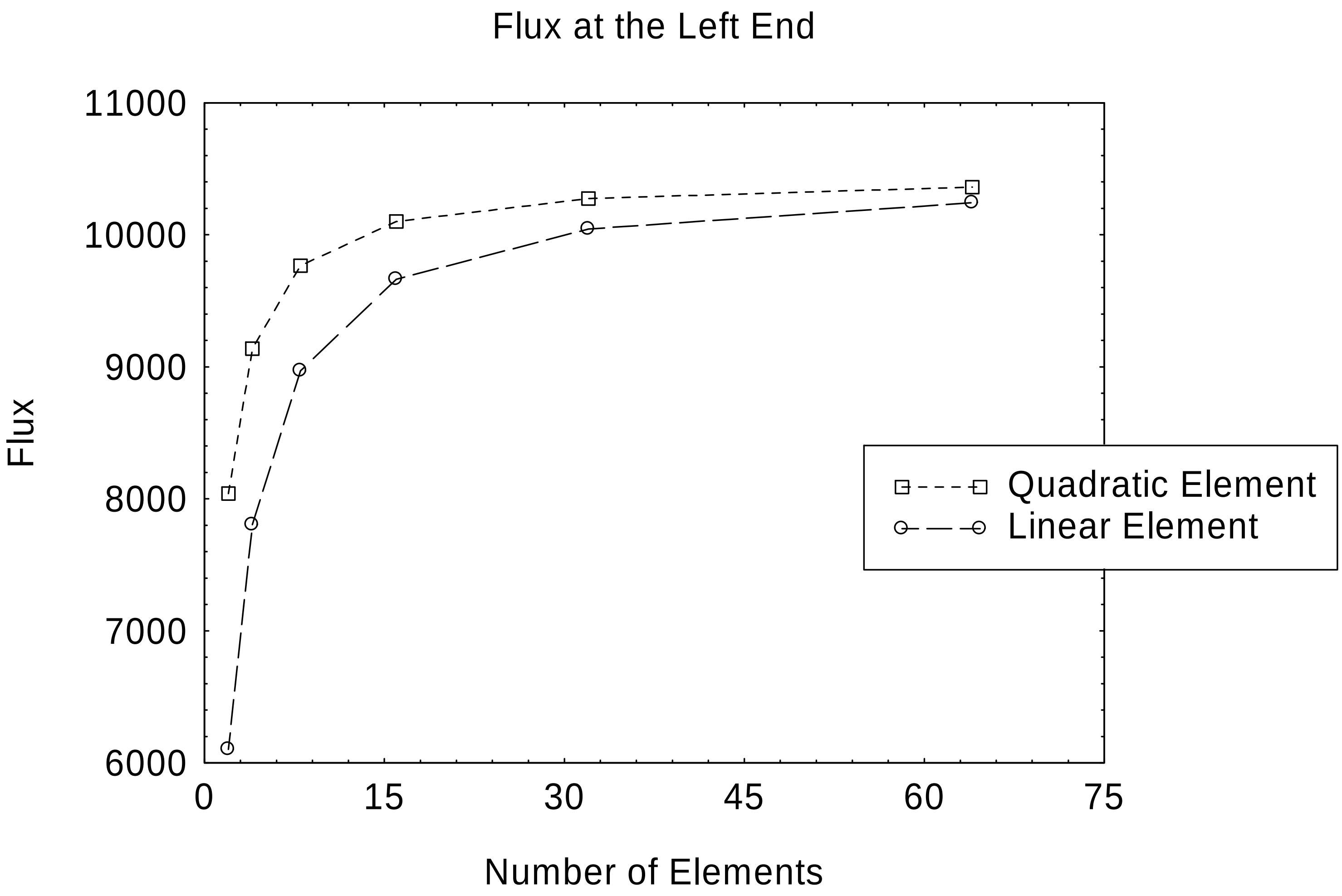
Comparison

	Temperature $(^\circ F)$		$Flux\left(\dfrac{BTU}{h \cdot ft^2}\right)$	
Node	Linear Element	Quadratic Element	Linear Elements	Quadratic Element
1	150	150		
2	98.9	99.8	6102	6382
3	89.0	93.3	1174	383

Flux Convergence Study



Flux Convergence Study



Summary

- Element concept with trial solution makes it easy to generate the element equations including imposition of the BCs
- The element equations are of the form

$$\mathbf{k}_{n\times n}\mathbf{d}_{n\times 1} = \mathbf{f}_{n\times 1}$$

- Both **h** and **p**-convergence make it possible to obtain efficient solutions to general FE problems

Further Reading

- See if you can generate the element equations for the 1D-C^0 cubic element using a symbolic computer program (e.g. Maple)

Thermal Loading & Stresses

Initial Strain

$$\varepsilon_0 = \alpha \Delta T$$

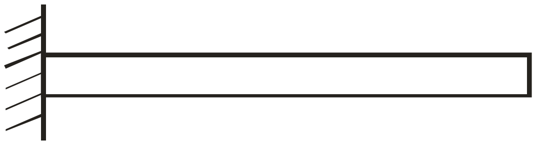
Stress-Strain Relationship

$$\sigma = E\left(\varepsilon - \varepsilon_0\right)$$

Equivalent Load Vector

$$\mathbf{f}_{2\times 1} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = AE\alpha\,\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Example T4L2-2(a)



The length of the bar is 2 m. The cross-sectional area is 0.001m^2. The bar is initially at 25^{0}C. The temperature is increased to 100^{0}C. Compute the stress in the bar.

$$E = 200\,GPa$$

$$\alpha = 11.7 \times 10^{-6}\,\frac{m}{m - {}^\circ C}$$

Units: m, kg, N, C

One-element solution.

Example T4L2-2(a)

$$\frac{(0.001)\left(200\times10^{9}\right)}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}D_1\\ D_2\end{Bmatrix}=(0.001)\left(200\times10^{9}\right)\left(11.7\times10^{-6}\right)(75)\begin{Bmatrix}-1\\ 1\end{Bmatrix}$$

$$\begin{bmatrix}10^8 & -10^8\\ -10^8 & 10^8\end{bmatrix}\begin{Bmatrix}D_1\\ D_2\end{Bmatrix}=\begin{Bmatrix}-175500\\ 175500\end{Bmatrix}$$

$$10^8 D_2 = 175500 \Rightarrow D_2 = 0.001755\,\mathrm{m}$$

$$\varepsilon = \frac{D_2 - D_1}{L} = \frac{0.001755}{2} = 0.0008775$$

$$\varepsilon_0 = \left(11.7\times10^{-6}\right)(75) = 0.0008775$$

$$\sigma = E\left(\varepsilon - \varepsilon_0\right) = 200\times10^{9}\left(0.0008775 - 0.0008775\right) = 0$$

Using the Windows-based 1DBVP Program

1DBVP Program Terminology

- Positive coordinate system points to the right
- FE mesh contains one or more segments
- A segment has the same properties but may contain one or more elements
- The leftmost segment (or the first segment) has the first node and the first element (or node 1 and element 1)
- You create the segment data; 1DBVP creates the nodes, elements and loads

Solid Mechanics

DE

$$-\frac{d}{dx}\left(\alpha(x)\frac{dy(x)}{dx}\right)+\beta(x)y(x)=f(x)$$

Left Mixed BC $\tau = c_a y + d_a$

Right Mixed BC $\tau = c_b y + d_b$

DE

$$-\frac{d}{dx}\left(A(x)E(x)\frac{du(x)}{dx}\right)=w(x)A(x)$$

$$\tau = F = -A(x)E(x)\frac{du(x)}{dx}$$

Solid Mechanics

$$\alpha(x) = A(x)E(x) \equiv F$$

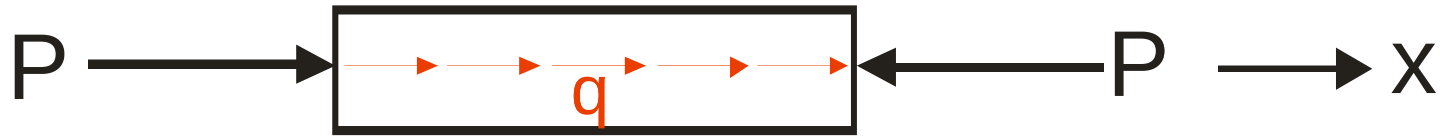
$$\beta(x) = 0$$

$$f(x) = w(x)A(x) \equiv F/L$$

$$c_a = c_b = 0$$

$$d_a = d_b \equiv F$$

Examples



$$\beta(x) = 0$$

$$\alpha(x) = A(x)E(x) \equiv F$$

$$f(x) = q \equiv F/L$$

$$\frac{AE}{L}\left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array}\right]\left\{\begin{array}{c} u_1 \\ \hline u_2 \end{array}\right\} = \left\{\begin{array}{c} P \\ \hline -P \end{array}\right\} + \left\{\begin{array}{c} qL/2 \\ \hline qL/2 \end{array}\right\}$$

Heat Transfer

DE

$$-\frac{d}{dx}\left(\alpha(x)\frac{dy(x)}{dx}\right)+\beta(x)y(x)= f(x)$$

Left Mixed BC $\tau = c_a y + d_a$

Right Mixed BC $\tau = c_b y + d_b$

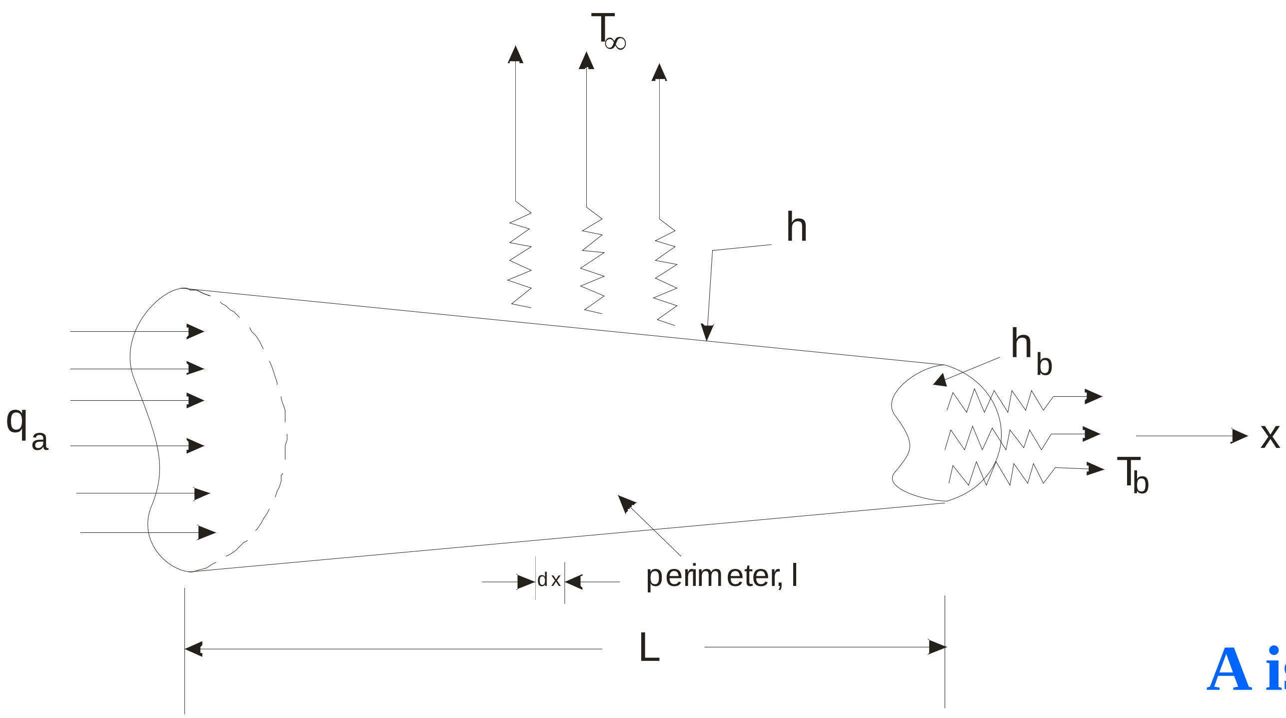
Comparing (assuming A is a constant)

$$\alpha(x) = k(x) \equiv \frac{E}{tLT}$$

$$\beta(x) = \frac{hl}{A} \equiv \frac{E}{tL^3T}$$

$$f(x) \equiv Q(x) + \frac{hl}{A}T_\infty \equiv \frac{E}{tL^3}$$

Heat Transfer

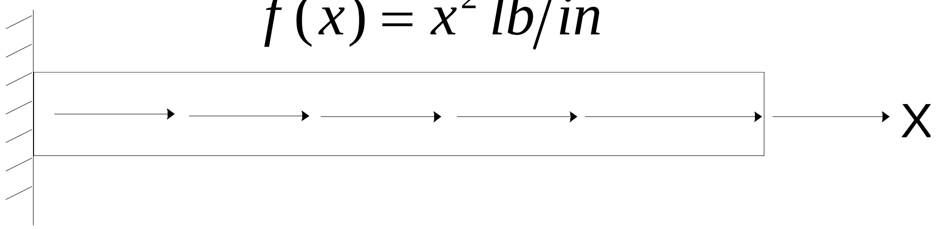


A is a NOT constant

$$\left[\begin{bmatrix} \dfrac{\bar{k}}{L}+\dfrac{\overline{hl}L}{3A} & -\dfrac{\bar{k}}{L}+\dfrac{\overline{hl}L}{6A} \\ -\dfrac{\bar{k}}{L}+\dfrac{\overline{hl}L}{6A} & \dfrac{\bar{k}}{L}+\dfrac{\overline{hl}L}{3A} \end{bmatrix} + h_b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{L}{2} \begin{Bmatrix} \dfrac{\overline{hl}}{A} T_\infty \\ \dfrac{\overline{hl}}{A} T_\infty \end{Bmatrix} + \begin{Bmatrix} q_a \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ h_b T_b \end{Bmatrix}$$

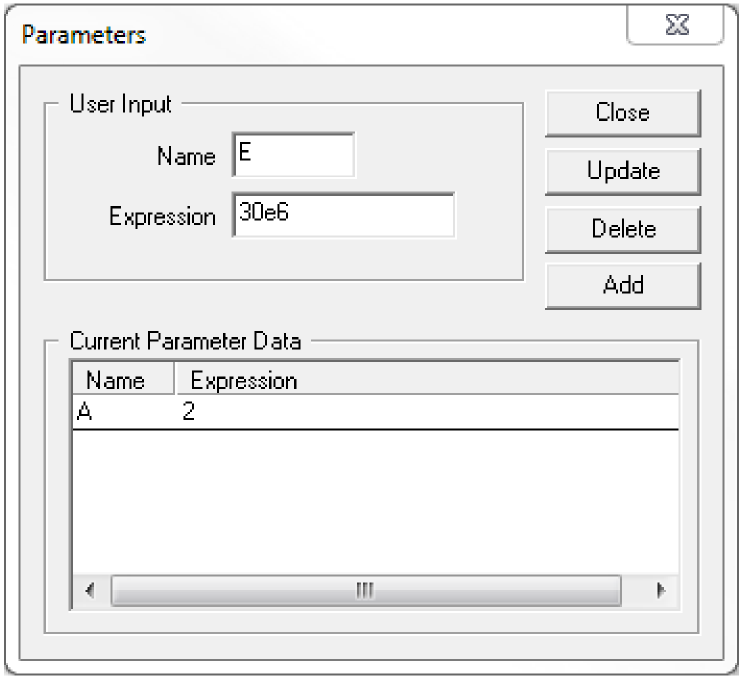
Problem T4L2-1

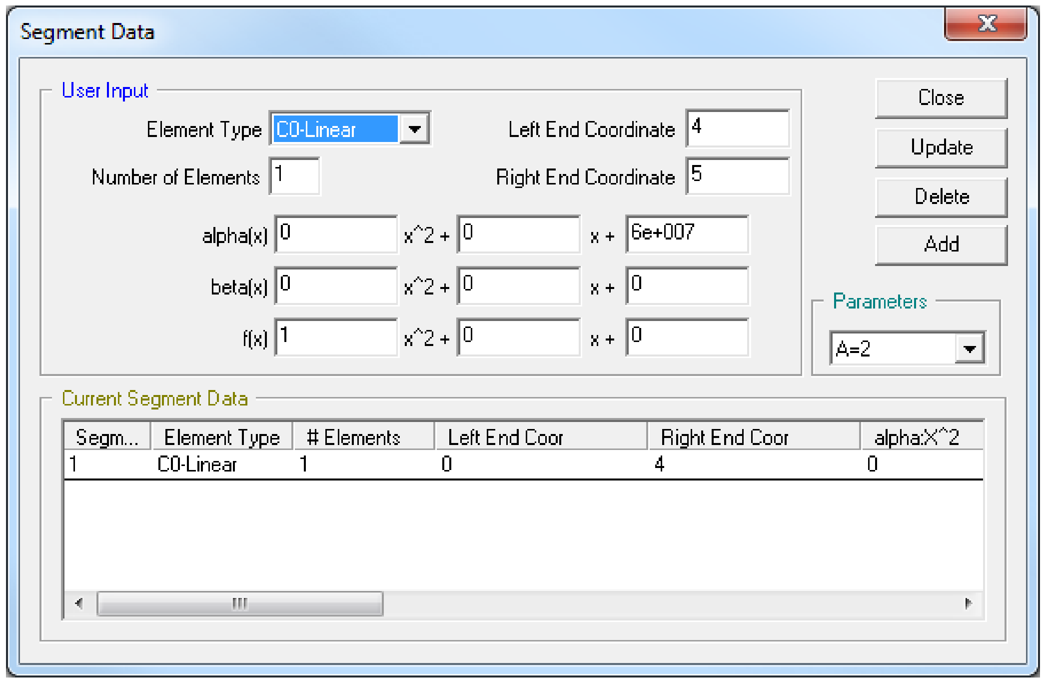
$$f(x) = x^2 \; lb/in$$



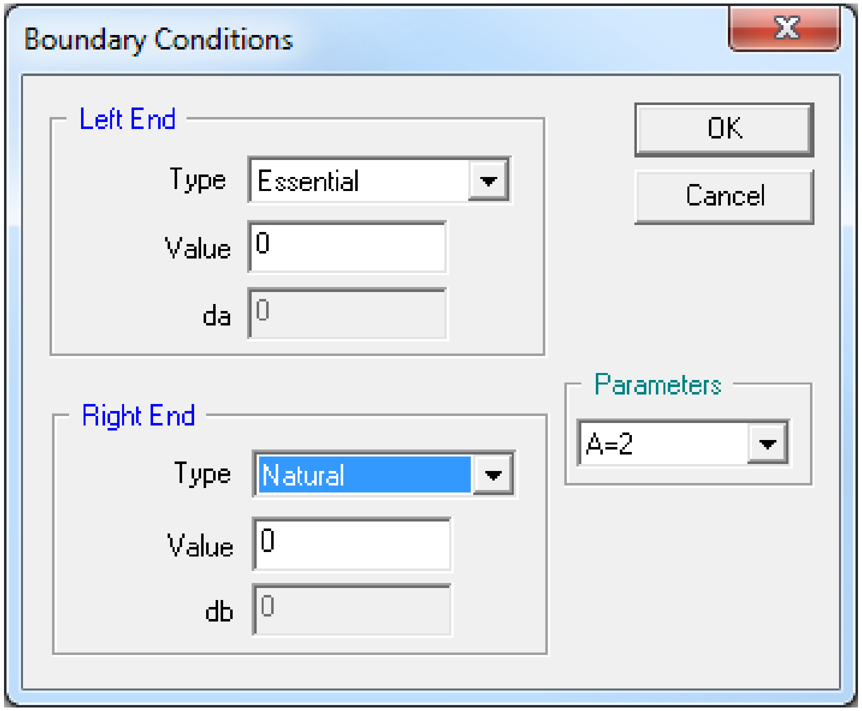
The 4" long steel bar with a 2 in^2 cross-sectional area is loaded by the given surface traction. Find the displacement and force distribution in the bar.

Problem T4L2-1





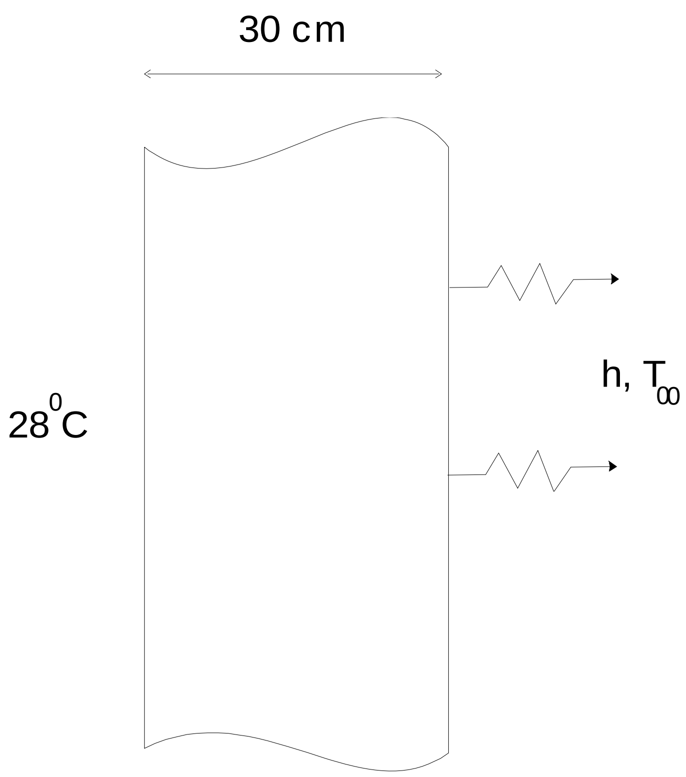
Problem T4L2-1



Problem T4L2-1

Mesh	Tip Disp. (in)	Root Force (lb)	Tip Force (lb)
1-element	$1.067(10^{-6})$	16	16
2-elements	$1.067(10^{-6})$	20.67	11.33
4-elements	$1.067(10^{-6})$	21.25	6.75
8-elements	$1.067(10^{-6})$	21.32	3.68
16-elements	$1.067(10^{-6})$	21.33	1.92

Problem T4L3-1



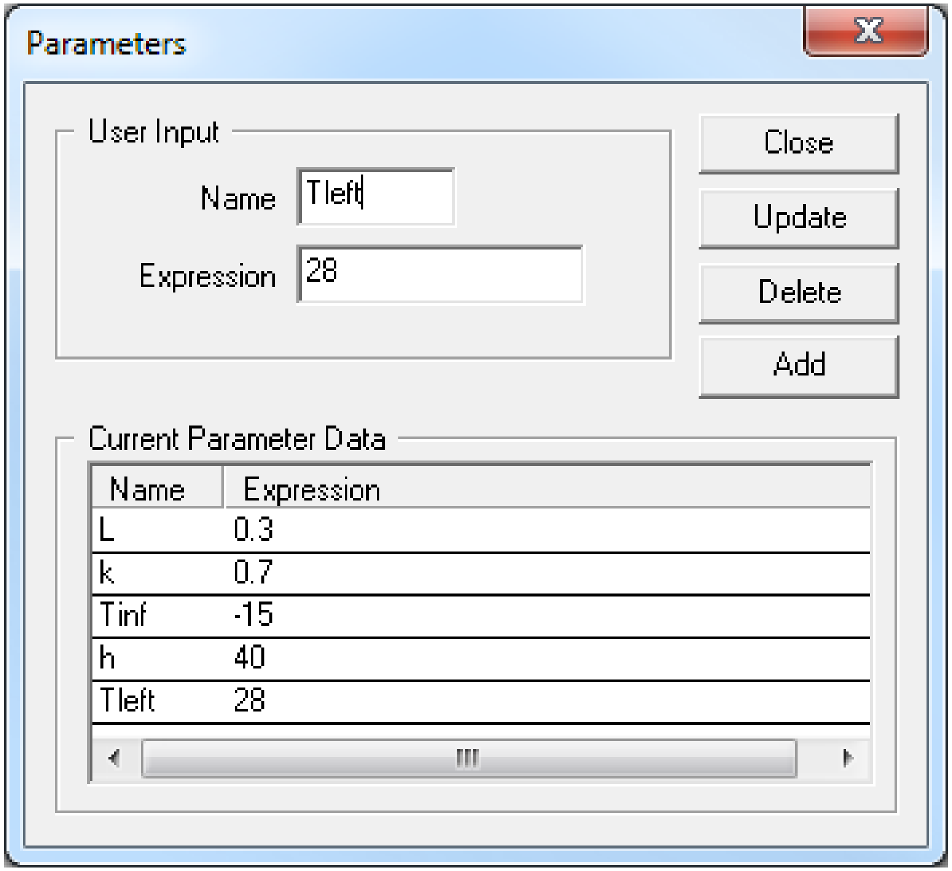
Brick wall: Determine the steady-state temperature distribution within the wall and also the heat flux through the wall.

$$k = 0.7\, W/m \cdot {}^\circ C$$

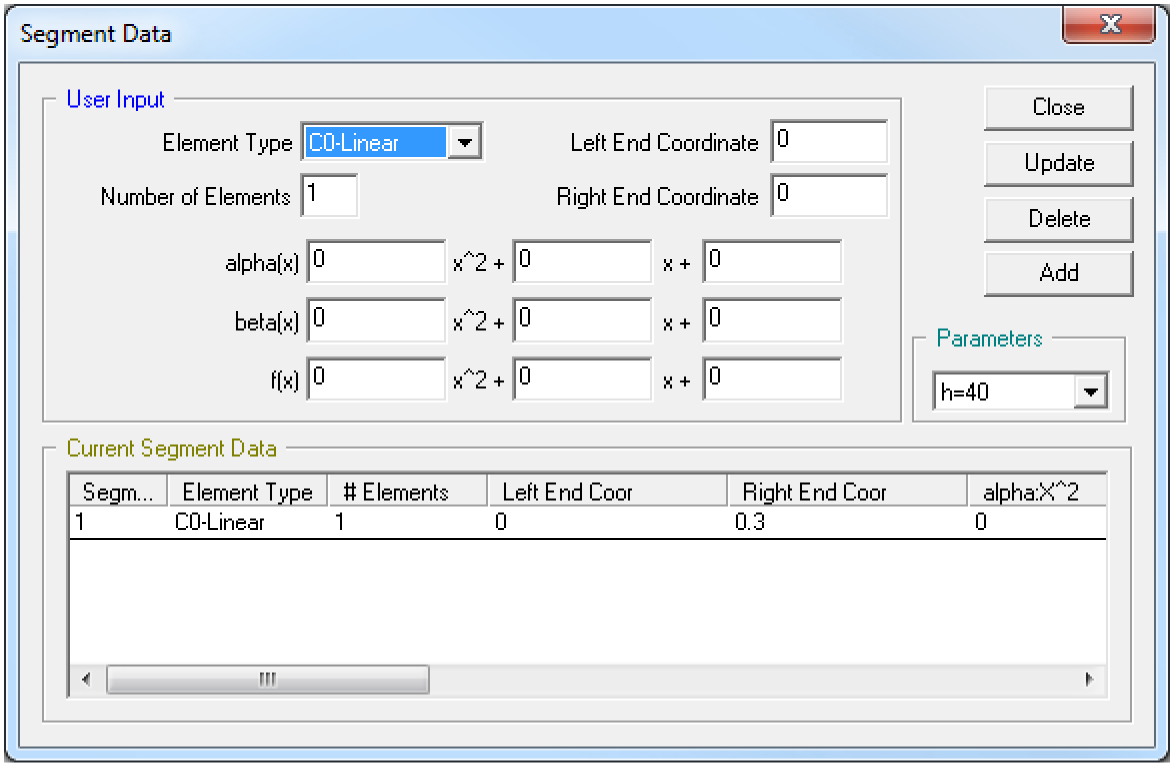
$$T_\infty = -15^\circ C$$

$$h = 40\, W/m^2 \cdot {}^\circ C$$

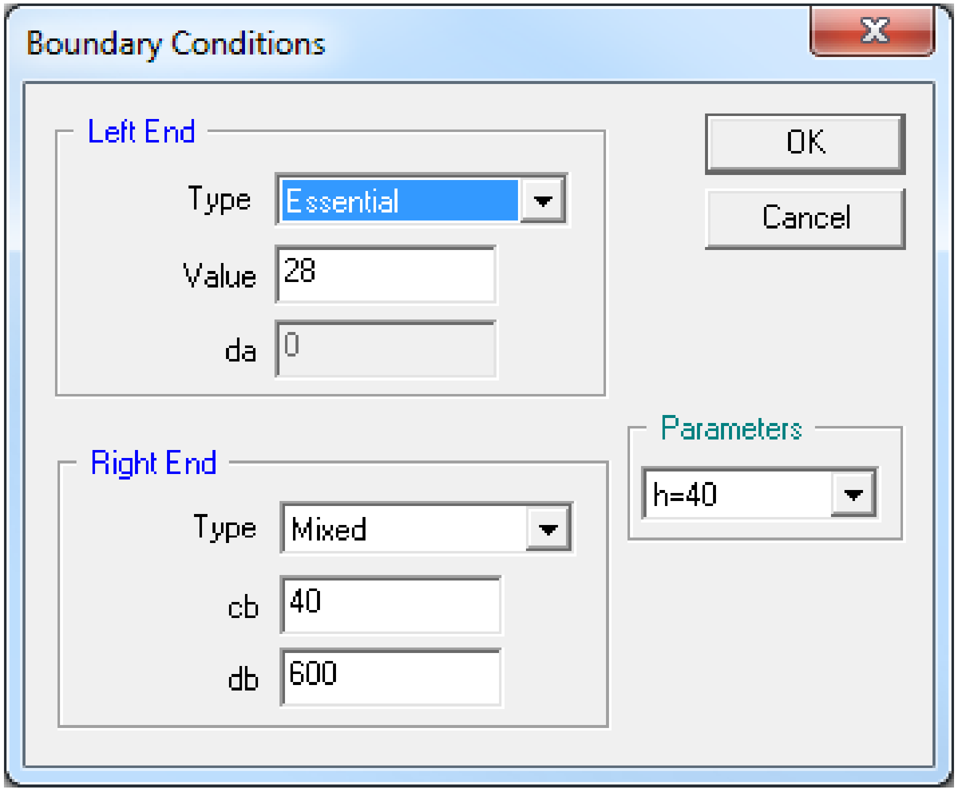
Problem T4L3-1



Problem T4L3-1



Problem T4L3-1



Problem T4L3-1

Mesh	Right Temp. (C)	Right Flux (W/m^2)	
1-element	-12.63	94.8	
2-elements	-12.63	94.8	
4-elements	-12.63	94.8	