Finite Elements For Engineers

Lecture 4: Galerkin's Method

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Overview

- In the Direct Stiffness Method, the starting point was a typical element and using the physics of the problem, we generated the element equations
- In the Galerkin's Method, the starting point is a differential equation

What is a Well-Posed Differential Equation?

DE

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x) \qquad a < x < b$$

BCs

BC at x=a

BC at x=b

Solution

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x} + C_0 F(x) + C_1 F'(x) + \ldots..$$

Types of Boundary Conditions

- Essential or Dirichlet

$$y = c$$

- Natural or Neumann

$$y' = c$$

- Mixed or Robin

$$ay' + by = c$$

Generating Element Equations (Step 2)

- Three distinct operations
  - Assume a trial solution
  - Apply optimizing criterion
  - Estimate the error
- Assume that we are solving a DE

$$\Im(y) - F(x) = 0$$

A Trial Solution

Usually a polynomial with undetermined coefficients

$$\tilde{y}(x;a) = \phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + \ldots\ldots + a_n\phi_n(x)$$

Trial or Basis Functions $\phi_i(x)$

Unknown coefs a_i

$\phi_0(x)$ Satisfies some or all BCs

Residual (Error) $R(x;a) = \Im(\tilde{y}) - F(x) \neq 0$

Optimizing Criterion

- This criterion is used to generate appropriate equations that will allow us to determine the values of the (unknown) coefficients
- Options
  - Method of Weighted Residuals
  - Ritz Variational Method

Method of Weighted Residuals

- Idea is to make the residual as small as possible in order to minimize the error
- Converts the DE to linear algebraic equations
- Several Flavors
  - Collocation Method
  - Subdomain Method
  - Least-Squares Method
  - Galerkin's Method

Galerkin's Method

$$\int_{\Omega} R(x;a) \quad \phi_i(x)dx = 0 \qquad\qquad i = 1,..,n$$

(1) The weighted average of the residual over the entire domain is set to zero.

(2) There are as many equations as there are undetermined coefficients.

Example (Galerkin's Method)

DE $$\frac{d}{dx}\left(x\frac{dy(x)}{dx}\right)=\frac{2}{x^2} \qquad 1\le x\le 2$$

BCs $$y(x=1)=2 \qquad \left(-x\frac{dy}{dx}\right)_{x=2}=\frac{1}{2}$$

Trial Solution

$$\tilde{y}=a_1+a_2x+a_3x^2+a_4x^3$$

Derived Term: Flux $$\tau(x)=-x\frac{dy}{dx}$$

Classical Solution

Satisfy BCs

$$\tilde{y}(x=1) = 2 = a_1 + a_2(1) + a_3(1)^2 + a_4(1)^3$$

$$\left(-x\frac{d\,\tilde{y}}{dx}\right)_{x=2} = \frac{1}{2} = -2a_2 - 8a_3 - 24a_4$$

Simplifying

$$a_1 = 2 - a_2 - a_3 - a_4$$

$$a_2 = -\frac{1}{4} - 4a_3 - 12a_4$$

Finally

$$\tilde{y} = 2 - \frac{1}{4}(x-1) + a_3(x-1)(x-3) + a_4(x-1)(x^2+x-11)$$

Classical Solution

Comparing $$\tilde{y}(x;a) = \phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x)$$

We have $$\phi_0(x) = 2 - \frac{1}{4}(x-1)$$

$$\phi_1(x) = (x-1)(x-3)$$

$$\phi_2(x) = (x-1)(x^2 + x - 11)$$

The residual is

$$R(x;a) = \frac{d}{dx}\left(x\frac{d\,\tilde{y}(x)}{dx} \right) - \frac{2}{x^2} = -\frac{1}{4} + 4(x-1)a_1 + 3(3x^2-4)a_2 - \frac{2}{x^2}$$

Classical Solution

Optimizing criterion

$$\int_1^2 \left(-\frac{1}{4} + 4(x-1)a_1 + 3(3x^2-4)a_2 - \frac{2}{x^2} \right)(x-1)(x-3)\, dx = 0$$

$$\int_1^2 \left(-\frac{1}{4} + 4(x-1)a_1 + 3(3x^2-4)a_2 - \frac{2}{x^2} \right)(x-1)(x^2+x-11)\, dx = 0$$

Simplifying

$$\begin{bmatrix} \frac{5}{3} & \frac{41}{5} \\ \frac{41}{5} & \frac{81}{2} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} -\frac{29}{6} + 8\ln 2 \\ -\frac{211}{16} + 24\ln 2 \end{Bmatrix} \Rightarrow \quad \begin{aligned} a_1 &= 2.138 \\ a_2 &= -0.348 \end{aligned}$$

Classical Solution

Substituting in the trial solution

$$\tilde{y}(x;a) = 2 - \frac{1}{4}(x-1) + 2.138(x-1)(x-3) - 0.348(x-1)(x^2+x-11)$$

Or

$$\tilde{y}(x;a) = -0.348x^3 + 2.138x^2 - 4.629x + 4.839$$

And

$$\tilde{\tau}(x;a) = 1.043x^3 - 4.276x^2 + 4.629x$$

Questions

- Why is the trial solution cubic?
- Are there other options?
- Why is the application of the boundary conditions cumbersome in this Classical Approach?
- Can we check the accuracy?
- Can we improve the accuracy?

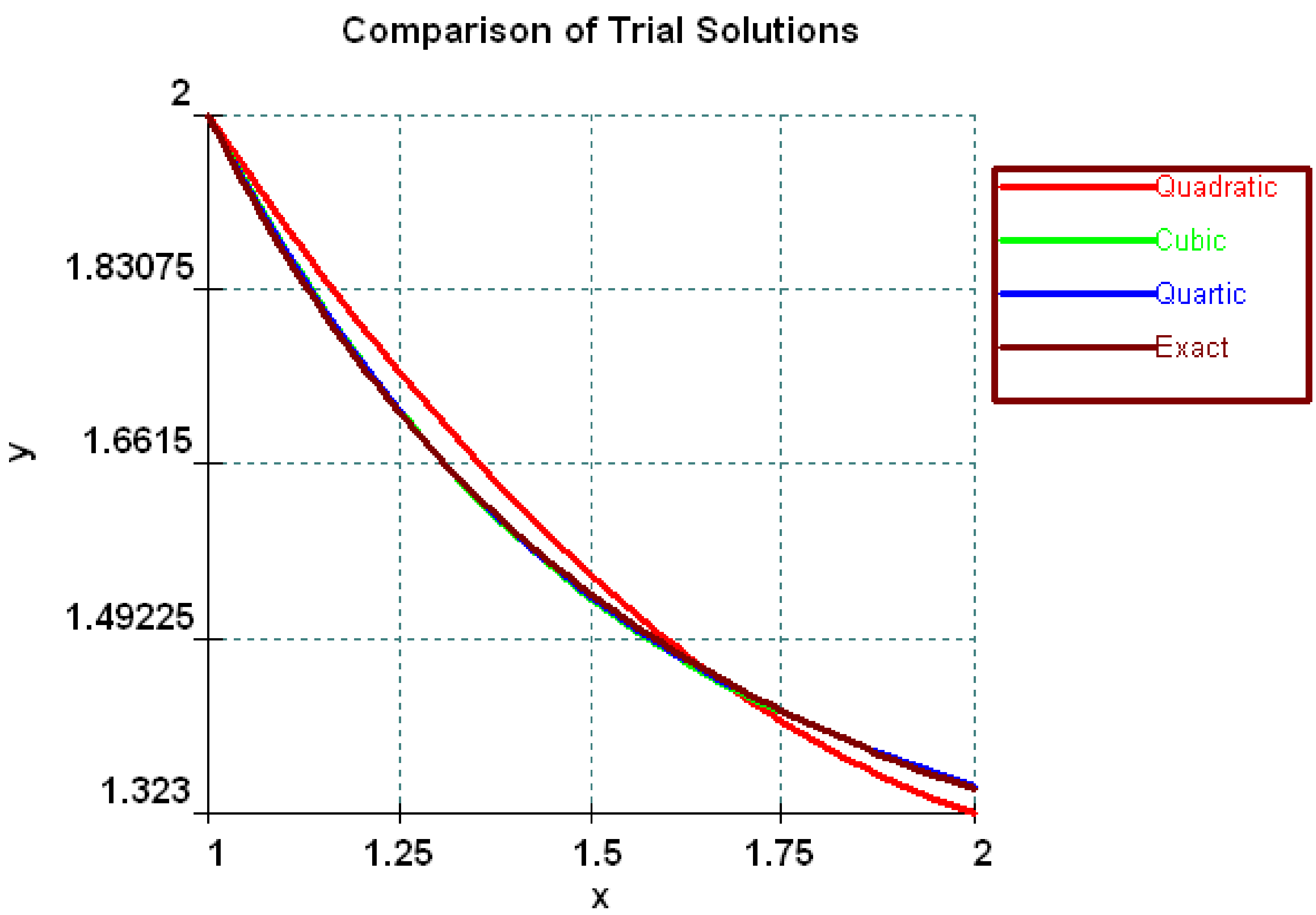
Convergence

Quadratic $$\tilde{y}(x;a) = 0.427x^2 - 1.958x + 3.531$$

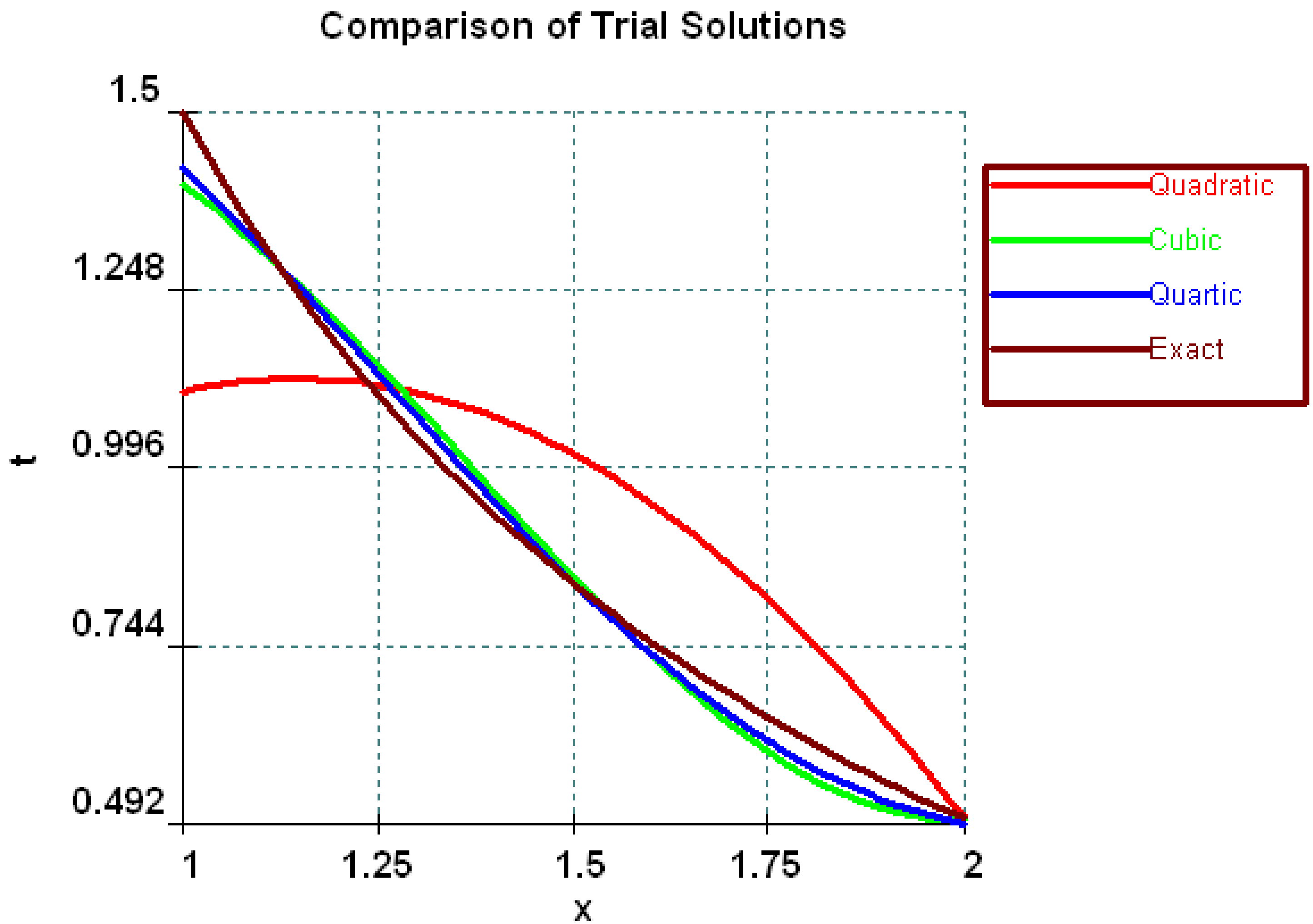
Cubic $$\tilde{y}(x;a) = -0.348x^3 + 2.138x^2 - 4.629x + 4.839$$

Quartic $$\tilde{y}(x;a) = 0.0864x^4 - 0.888x^3 + 3.3725x^2 - 5.848x + 5.277$$

Comparison of y(x)



Comparison of τ(x)



Summary

- The classical Galerkin's Method shows promise. It converts the original DE to a set of algebraic equations.
- The number of undetermined parameters in the trial solution must be greater than the number of BCs. The free parameters are also called degrees of freedom.
- Enforcement of boundary conditions is a problem.

Summary

- We need to carry out a convergence study to find out the quality of the solution.
- A “good” match of the primary unknown does not guarantee a similar quality match of derived quantities (usually they will be worse off).
- We need an improved approach.

Further Reading

- This is an excellent text but is unfortunately out of print - Burnett, *Finite Element Analysis – From Concepts to Applications*, Addison-Wesley.