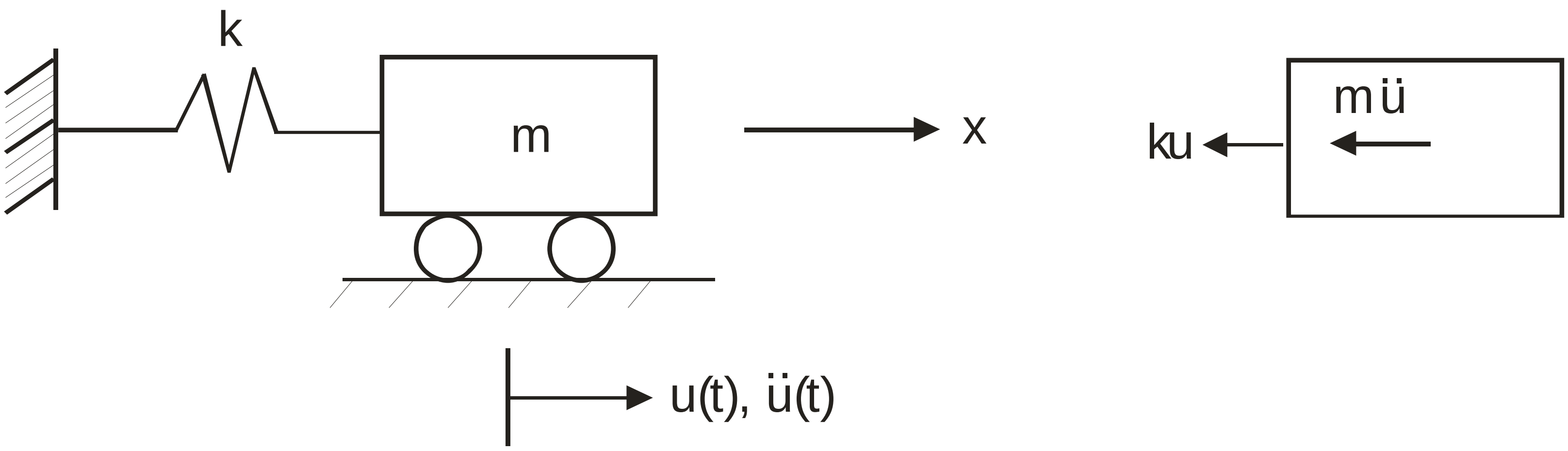
CEE432/CEE532/MAE541
Developing Software for Engineering Applications

Lecture 12:
Eigenproblems

Background

Introduction: Free Vibration



D'Alembert's Principle $$m\ddot{u} + ku = 0$$

Let $\omega^2 = \dfrac{k}{m} \quad \Rightarrow \quad \ddot{u} + \omega^2 u = 0$

Introduction: Free Vibration

Angular Frequency $$\omega = \sqrt{k/m} \quad rad/s$$

Natural Frequency $$f = \omega/2\pi \quad Hz$$

Natural Period $$T = 1/f \quad s$$

Introduction: Free Vibration

Solution

$$u(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

$t = t_0$ $\quad u = u_0$ **Initial displacement**

$\dot{u} = \dot{u}_0$ **Initial velocity**

Solution to free, undamped vibration

$$u = A\cos(\omega t - \alpha)$$

Amplitude $\quad A = \sqrt{u_0^2 + \left(\dfrac{\dot{u}_0}{\omega}\right)^2}$ **Phase Angle** $\quad \alpha = \tan^{-1} \dfrac{\dot{u}_0}{\omega u_0}$

Introduction: Forced Vibration

Harmonic forcing function

$$m\ddot{u} + ku = P\sin\Omega t$$

$$p_m = \frac{P}{m} \Rightarrow \ddot{u} + \omega^2 u = p_m \sin\Omega t$$

Solution (particular solution + general solution)

$$u = C_1 \cos\omega t + C_2 \sin\omega t + C_3 \sin\Omega t$$

$$u = \underbrace{X\sin(\omega t + \phi)}_{\text{Free vib.}} + \underbrace{\left[\frac{1}{1-\left(\Omega/\omega\right)^2}\right]\frac{P}{k}\sin\Omega t}_{\text{Forced vib.}}$$

Introduction: Forced Vibration

$$u = \left[\frac{1}{1-\left(\Omega/\omega\right)^2} \right] \frac{P}{k} \sin \Omega t = \frac{1}{\beta} \frac{P}{k} \sin \Omega t$$

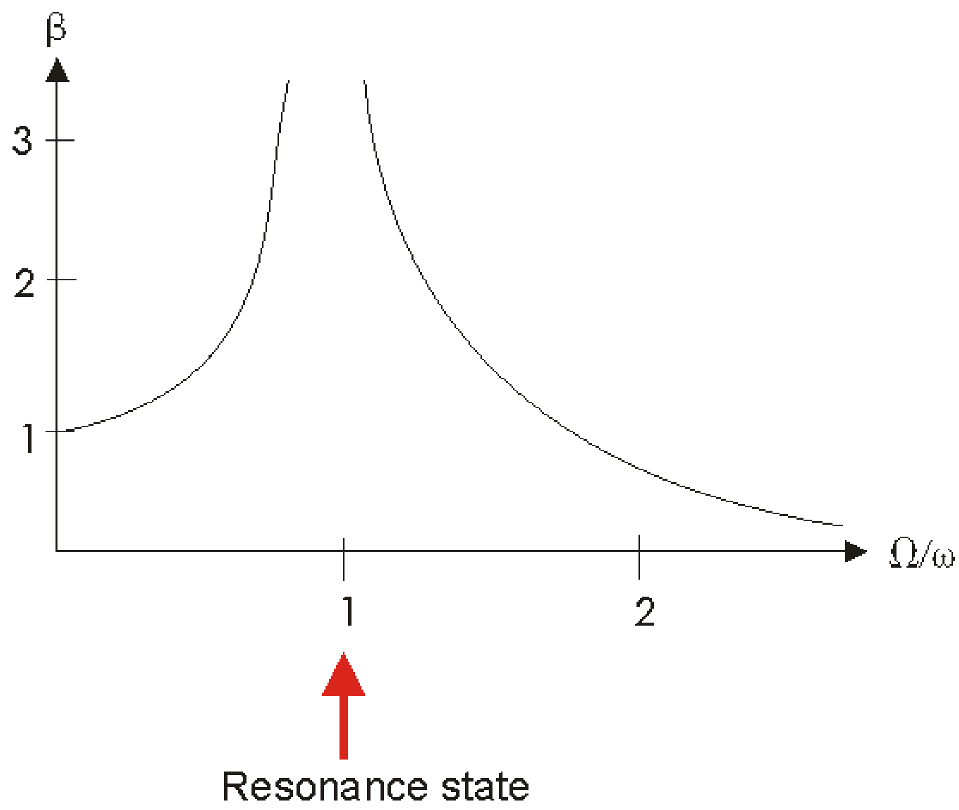
Frequency Ratio

Equivalent Static Deflection

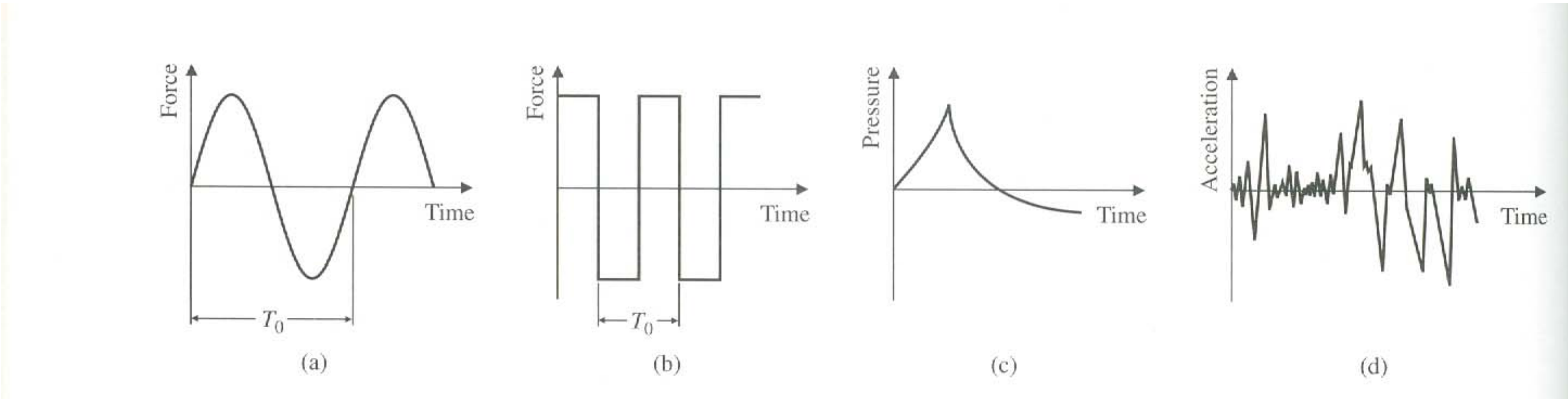
$$r = \frac{\Omega}{\omega}; X_0 = \frac{P}{k}$$

$$X_f = \frac{X_0}{1-r^2} = (DMF) X_0$$

Dynamic Magnification Factor



Dynamic Loads



Simple Harmonic	Periodic, Non-Harmonic	Non-periodic, short duration	Non-periodic, long duration

Examples

- Environmental
  - Wind (non-periodic, long duration)
  - Earthquake (Wave) (non-periodic, long duration)
- Machinery (periodic, harmonic and non-harmonic)
- Vehicular (varied)
- Blast (non-periodic, short duration)

Numerical Techniques

Eigenvalue Analysis

Properties

$$\mathbf{K}_{n\times n}\mathbf{\Phi}_{n\times n} = \mathbf{\Lambda}_{n\times n}\mathbf{M}_{n\times n}\mathbf{\Phi}_{n\times n}$$

- **K** is symmetric and positive definite
- **M** is symmetric $\quad 0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$
- n real eigenvalues

Eigenvalue Analysis

Properties

$$\mathbf{K}\boldsymbol{\varphi}_i = \lambda_i \mathbf{M}\boldsymbol{\varphi}_i$$

$$\boldsymbol{\varphi}_i^T \mathbf{K}\boldsymbol{\varphi}_j = 0$$

$$\boldsymbol{\varphi}_i^T \mathbf{M}\boldsymbol{\varphi}_j = 0$$

$$\boldsymbol{\varphi}_i^T \mathbf{K}\boldsymbol{\varphi}_j = \lambda_i$$

$$\boldsymbol{\varphi}_i^T \mathbf{M}\boldsymbol{\varphi}_i = 1$$

Solution Techniques

Characteristic Polynomial Technique

$$\left[\mathbf{K} - \lambda\mathbf{M}\right]\boldsymbol{\varphi} = 0$$

$$\det\left[\mathbf{K} - \lambda\mathbf{M}\right] = 0$$

The above equation is a polynomial of order $\boldsymbol{n}$. The roots of the polynomial are the eigenvalues.

Example

$$\mathbf{K}_{3\times3}\mathbf{\Phi}_{3\times3} = \mathbf{\Lambda}_{3\times3}\mathbf{M}_{3\times3}\mathbf{\Phi}_{3\times3}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{K}_{3\times3} - \lambda\mathbf{M}_{3\times3}$$

$$\begin{bmatrix} 3-\lambda & 2 & 1 \\ 2 & 2-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Example

$$\det(\mathbf{K}_{3\times 3} - \lambda \mathbf{M}_{3\times 3}) = 0$$

$$\lambda^3 - 6\lambda^2 + 5\lambda - 1 = 0 \quad \Rightarrow \quad \begin{aligned} \lambda_1 &= 0.308 \\ \lambda_2 &= 0.643 \\ \lambda_3 &= 5.049 \end{aligned}$$

$$\lambda_1 = 0.308$$

$$\begin{bmatrix} 3-0.308 & 2 & 1 \\ 2 & 2-0.308 & 1 \\ 1 & 1 & 1-0.308 \end{bmatrix} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Example

Let $\varphi_3 = 1$

$$\begin{bmatrix} 2.692 & 2 \\ 2 & 1.692 \end{bmatrix} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} 0.555 \\ -1.247 \end{Bmatrix}$$

Hence

$$\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} 0.555 \\ -1.247 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.445 \\ 1 \\ -0.802 \end{Bmatrix}$$

Example

$$\mathbf{K}_{3\times3}\mathbf{\Phi}_{3\times3} = \mathbf{\Lambda}_{3\times3}\mathbf{M}_{3\times3}\mathbf{\Phi}_{3\times3}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.591 & 0.737 & 0.328 \\ -1.328 & -0.409 & 0.263 \\ 1.065 & -0.919 & 0.146 \end{bmatrix} =$$

$$\begin{bmatrix} 0.308 & 0 & 0 \\ 0 & 0.643 & 0 \\ 0 & 0 & 5.049 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.591 & 0.737 & 0.328 \\ -1.328 & -0.409 & 0.263 \\ 1.065 & -0.919 & 0.146 \end{bmatrix}$$

Rayleigh-Ritz Analysis

Consider

$$\mathbf{K}\phi = \lambda \mathbf{M}\phi$$

K and **M** are positive definite

Rayleigh Minimum Principle

$$\lambda_1 = \min\left(\rho\left(\phi\right)\right) = \min\left(\frac{\phi^{\mathbf{T}}\mathbf{K}\phi}{\phi^{\mathbf{T}}\mathbf{M}\phi}\right) \quad 0 < \lambda_1 \le \rho\left(\phi\right) \le \lambda_n < \infty$$

Solution Techniques

Inverse Iteration Method

- Step 1: Assume $\mathbf{u}^0$. Set k=0.
- Step 2: Set k=k+1.
- Step 3: Compute $\mathbf{v}^{k-1} = \mathbf{M}\mathbf{u}^{k-1}$
- Step 4: Solve $\mathbf{K}\hat{\mathbf{u}}^{k} = \mathbf{v}^{k-1}$
- Step 5: Let $\hat{\mathbf{v}}^{k} = \mathbf{M}\hat{\mathbf{u}}^{k}$
- Step 6: Estimate $\lambda^k = \dfrac{\hat{\mathbf{u}}^{k^T}\hat{\mathbf{v}}^{k-1}}{\hat{\mathbf{u}}^{k^T}\hat{\mathbf{v}}^{k}}$

Inverse Iteration

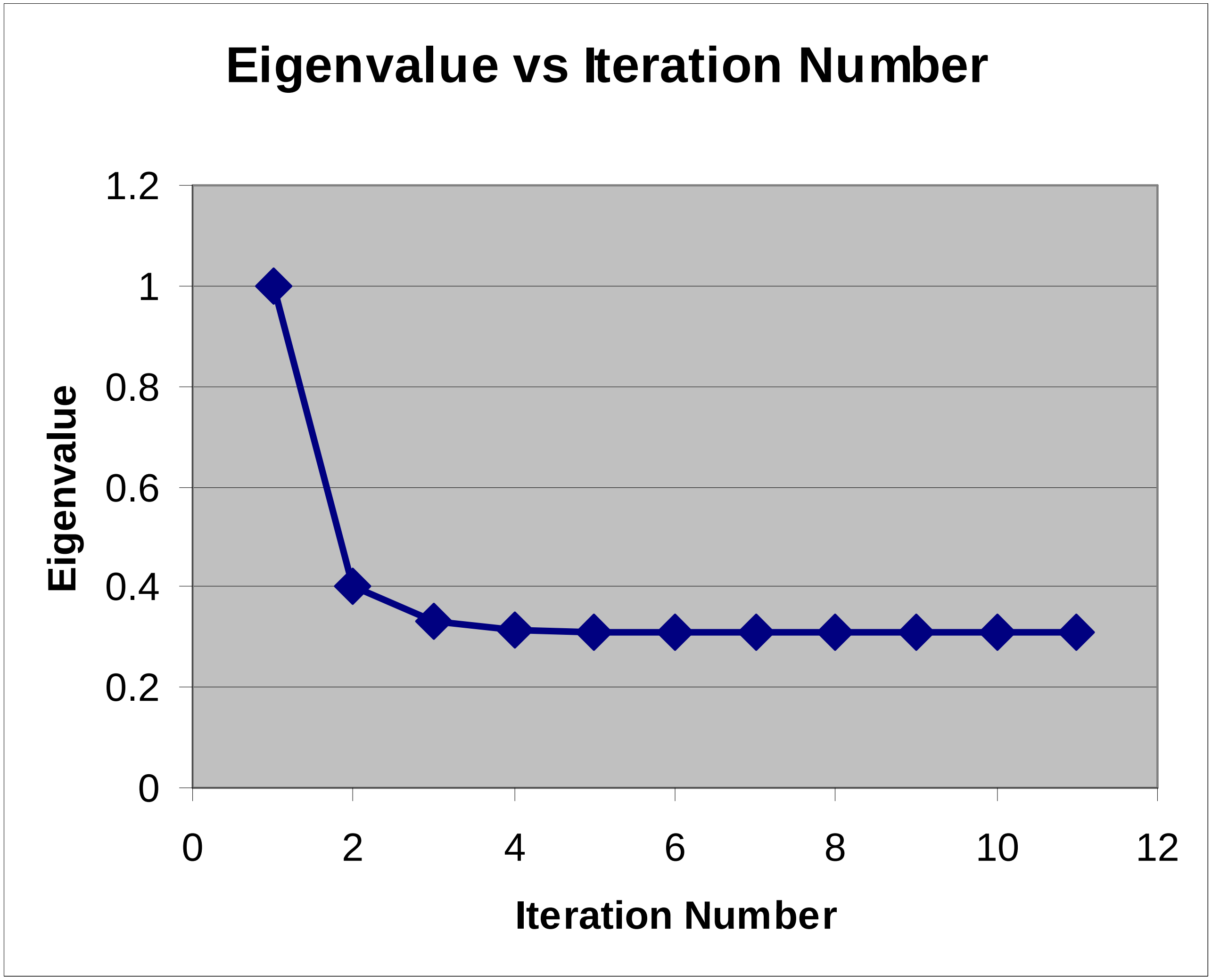
- Step 7: Normalize eigenvector $\mathbf{u}^k = \dfrac{\hat{\mathbf{u}}^k}{\left(\hat{\mathbf{u}}^{k^T} \hat{\mathbf{v}}^k\right)^{1/2}}$
- Step 8: Convergence check $\left|\dfrac{\lambda^k - \lambda^{k-1}}{\lambda^k}\right| \le tolerance$
- Step 9: If not converged, go to Step 2.

Example

$$\mathbf{K}_{3\times3}\mathbf{\Phi}_{3\times3} = \mathbf{\Lambda}_{3\times3}\mathbf{M}_{3\times3}\mathbf{\Phi}_{3\times3}$$

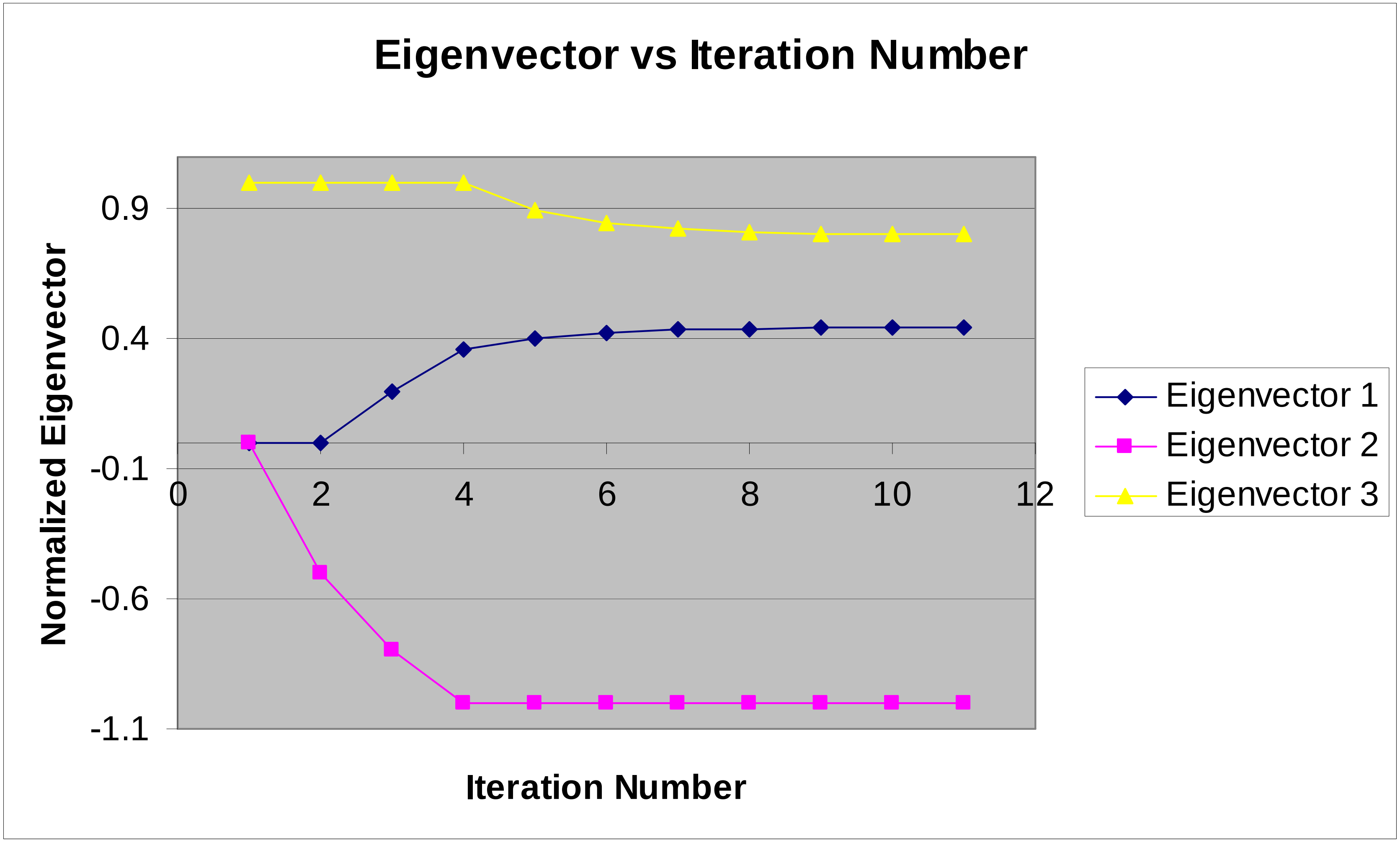
$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \mathbf{\Lambda}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix}$$

Example



Eigenvalue vs Iteration Number
1.2
1
0.8
0.6
0.4
0.2
0
Eigenvalue
0
2
4
6
8
10
12
Iteration Number

Example


Eigenvector vs Iteration Number
Normalized Eigenvector
0.9
0.4
-0.1
-0.6
-1.1
0
2
4
6
8
10
12
Iteration Number
Eigenvector 1
Eigenvector 2
Eigenvector 3

Other Techniques

- Jacobi Method
- Generalized Jacobi Method
- Subspace Iteration Method
- Lanczos Method

Example Program 16.8.1

Shifting

Original problem $\mathbf{K\Phi} = \lambda \mathbf{M\Phi}$

Shift K $\hat{\mathbf{K}} = \mathbf{K} - \rho \mathbf{M}$

New problem $\hat{\mathbf{K}}\mathbf{\Psi} = \mu \mathbf{M\Psi}$

It can be shown that

$$\lambda_i = \rho + \mu_i$$

$$\phi_i = \psi_i$$

Example

Original problem

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \mathbf{\Phi} = \lambda \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{\Phi}$$

$$\det(\mathbf{K} - \lambda \mathbf{M}) = 3\lambda^2 - 18\lambda = 0 \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 6 \end{matrix}$$

New problem $\rho = -2$

$$\begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix} \mathbf{\Phi} = \lambda \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{\Phi}$$

$$\det(\mathbf{K} - \lambda \mathbf{M}) = \lambda^2 - 10\lambda + 16 = 0 \Rightarrow \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 8 \end{matrix}$$

Structural Analysis

Lagrangian, L

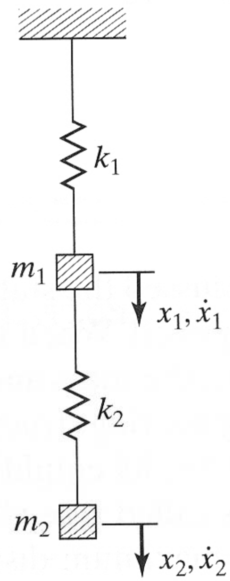
$$L = T - \Pi$$

kinetic energy

Hamilton's Principle (extremize the functional) $$I = \int_{t_1}^{t_2} L\left(D_1, D_2, \ldots, D_n\right) dt$$

Equations of motion $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{D}_i}\right) - \frac{\partial L}{\partial \dot{D}_i} = 0 \qquad i = 1, \ldots, n$$

Equations of Motion



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_1 \dot{x}_2^2$$

$$\Pi = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 \left(x_2 - x_1 \right)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = m_1 \ddot{x}_1 + k_1 x_1 - k_2 \left(x_2 - x_1 \right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} = m_2 \ddot{x}_2 - k_2 \left(x_2 - x_1 \right) = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} \left(k_1 + k_2 \right) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{M}\ddot{\mathbf{D}} + \mathbf{K}\mathbf{D} = \mathbf{0}$$

Mass Matrix

$$T = \frac{1}{2}\int_V \dot{\mathbf{u}}^T \dot{\mathbf{u}} \rho dV$$

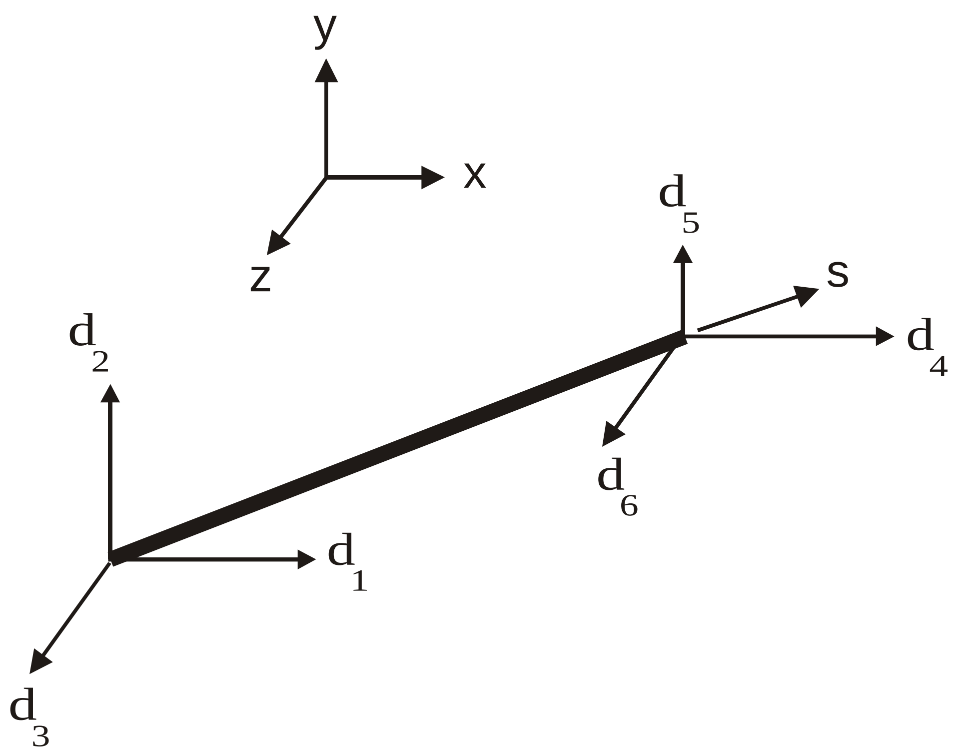
$$\dot{\mathbf{u}} = \begin{bmatrix} \dot{u} & \dot{v} & \dot{w} \end{bmatrix}^T$$

$$\mathbf{u}_{3\times 1} = \mathbf{N}_{3\times n} \mathbf{D}_{n\times 1}$$

$$\dot{\mathbf{u}}_{3\times 1} = \mathbf{N}_{3\times n} \dot{\mathbf{D}}_{n\times 1}$$

$$T = \frac{1}{2}\int_V \dot{\mathbf{u}}^T \dot{\mathbf{u}} \rho dV = \frac{1}{2}\dot{\mathbf{D}}^T \left[\int_e \rho \mathbf{N}^\mathbf{T} \mathbf{N} dV \right] \dot{\mathbf{D}} = \frac{1}{2}\dot{\mathbf{D}}_{1\times n}^T \mathbf{m}_{n\times n} \dot{\mathbf{D}}_{n\times 1}$$

Space Truss Element



Space Truss Element

Consistent mass matrix

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} 1-a & 0 & 0 & a & 0 & 0 \\ 0 & 1-a & 0 & 0 & a & 0 \\ 0 & 0 & 1-a & 0 & 0 & a \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \mathbf{N}_{3\times 6}\mathbf{d}_{6\times 1}$$

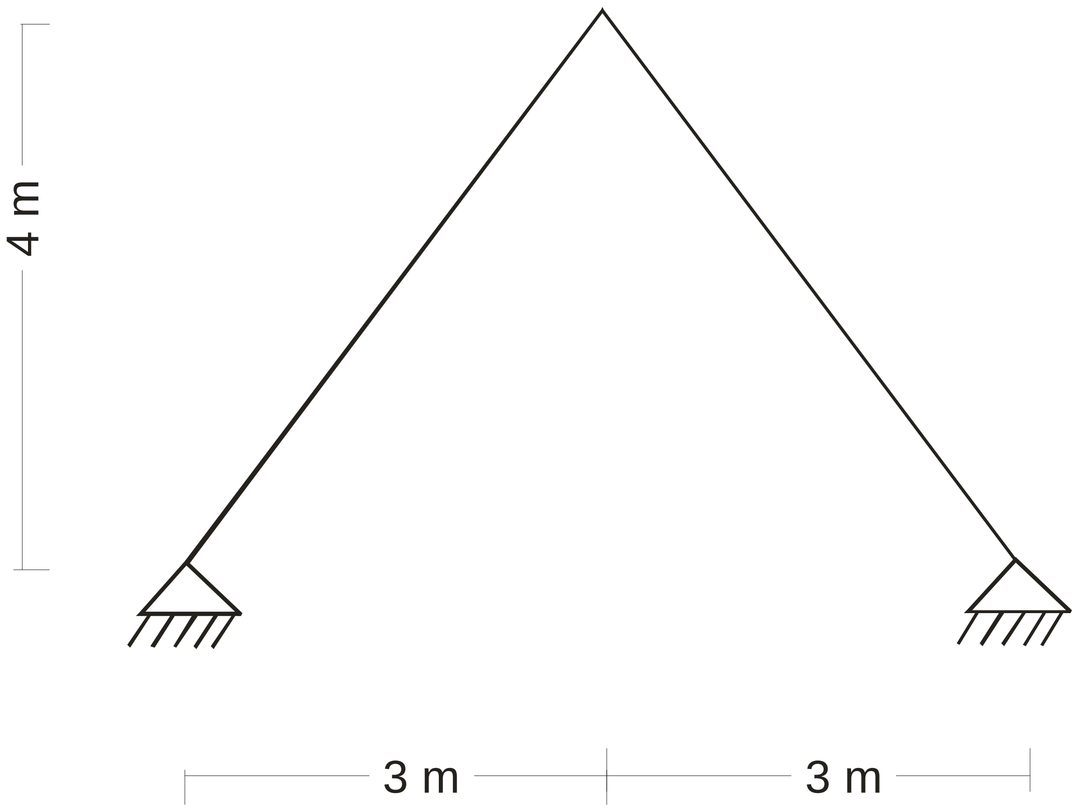
$$\mathbf{m}_{6\times 6} = \int_{\Omega} \rho \mathbf{N}^T \mathbf{N}\, dV$$

Space Truss Element

Consistent mass matrix (global coord. System)

$$\mathbf{m}_{6\times 6} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Example: Ex1



$$E = 200\, GPa$$

$$\rho = 7850\, kg/m^3$$

$$A = 0.01 m^2$$

$$f_1 = 167\, Hz$$

$$f_2 = 223\, Hz$$

Beam Element

Planar beam element (2 dof/element)

$$\mathbf{m}'_{4\times 4} = \frac{\rho A L}{420}\begin{bmatrix} 156 & & SYM & \\ 22L & 4L^2 & & \\ 54 & 13L & 156 & \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

Beam Element

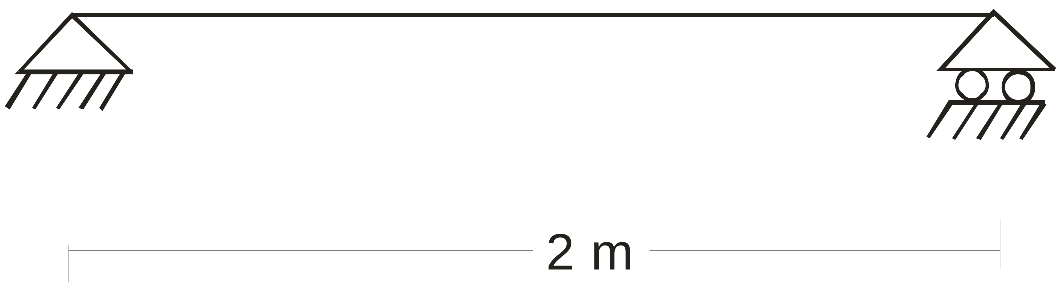
Planar beam element with axial deformation

$$\mathbf{m'}_{6\times6} = \frac{\rho AL}{420}\begin{bmatrix} 140 & & & & & SYM \\ 0 & 156 & & & & \\ 0 & 22L & 4L^2 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & 13L & 0 & 156 & \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix}$$

$$\mathbf{m}_{6\times6} = \mathbf{T}^T_{6\times6}\mathbf{m'}_{6\times6}\,\mathbf{T}_{6\times6}$$

Transformation from local-to-global

Example



$$E = 10^{10}\ Pa$$

$$\rho = 5000\, kg/m^3$$

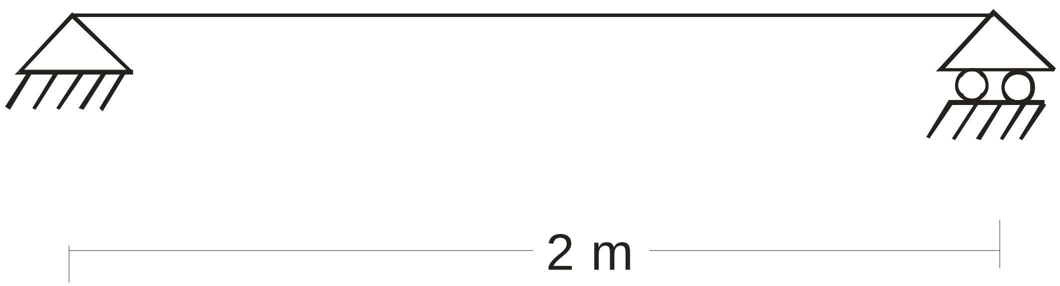
$$A = 0.001 m^2$$

$$I = 0.0001 m^4$$

Analytical solution (bending modes)

$$\omega_n = \frac{(n\pi)^2}{L^2}\sqrt{\frac{EI}{\rho A}} \quad n = 1, 2, 3, \ldots.$$

Example: Ex21 (1 Element)



$$E = 10^{10}\ Pa$$

$$\rho = 5000\, kg/m^3$$

$$A = 0.001 m^2$$

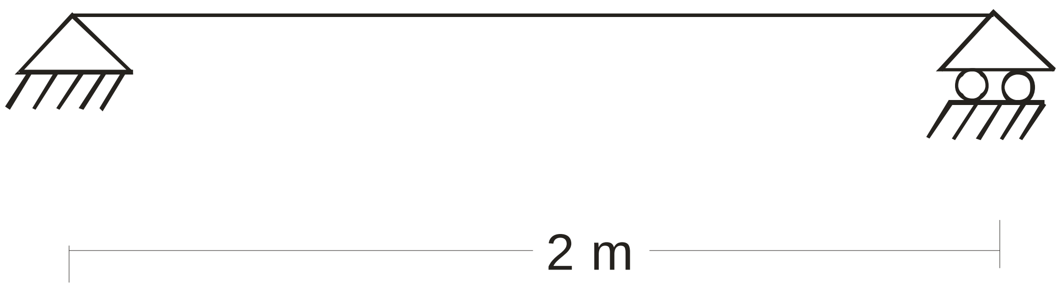
$$I = 0.0001 m^4$$

$$f_1 = 195\, Hz$$

$$f_2 = 195\, Hz$$

$$f_3 = 893\, Hz$$

Example: Ex22 (2 Element)



$$E = 10^{10}\ Pa$$

$$\rho = 5000\, kg/m^3$$

$$A = 0.001 m^2$$

$$I = 0.0001 m^4$$

$$f_1 = 176\, Hz$$

$$f_2 = 181\, Hz$$

$$f_3 = 633\, Hz$$

$$f_4 = 780\, Hz$$

$$f_5 = 1960\, Hz$$

Convergence Study

Mode\ Elements	Frequency (Hz)				
	1	2	4	8	**Exact**
1 (bending)	195	176	176	176	175.6
2	195	181	177	177	
3	893	633	561	538	
4 (bending)	-	780	705	703	702.5
5	-	1960	1020	920	

Notes

- Convergence takes place with increasing number of elements
- Convergence is from above
- Convergence of higher modes takes place more slowly

Lumped Mass Matrix

$$\mathbf{m}'_{6\times 6} = \frac{\rho A L}{2}\begin{bmatrix} 1 & & & & & SYM \\ 0 & 1 & & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\frac{\rho A L}{2}$

$\frac{\rho A L}{2}$

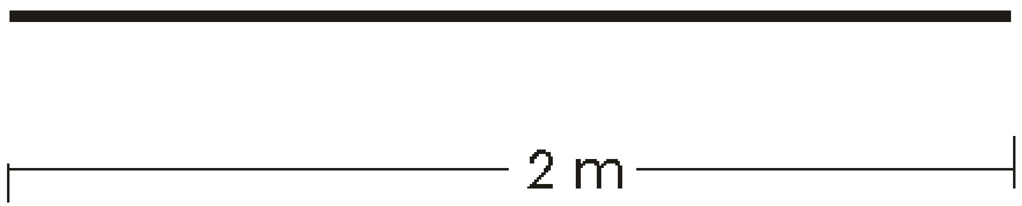
Consistent versus Lumped Mass (8 Elements)

Mode	Frequency (Hz)	
	Consistent	**Lumped**
1	176	176
2	177	176
3	538	523
4	703	702
5	920	849

Notes

- Both consistent and lumped mass matrices are valid formulations
- There are several forms of creating diagonal mass matrices

Floating Beam



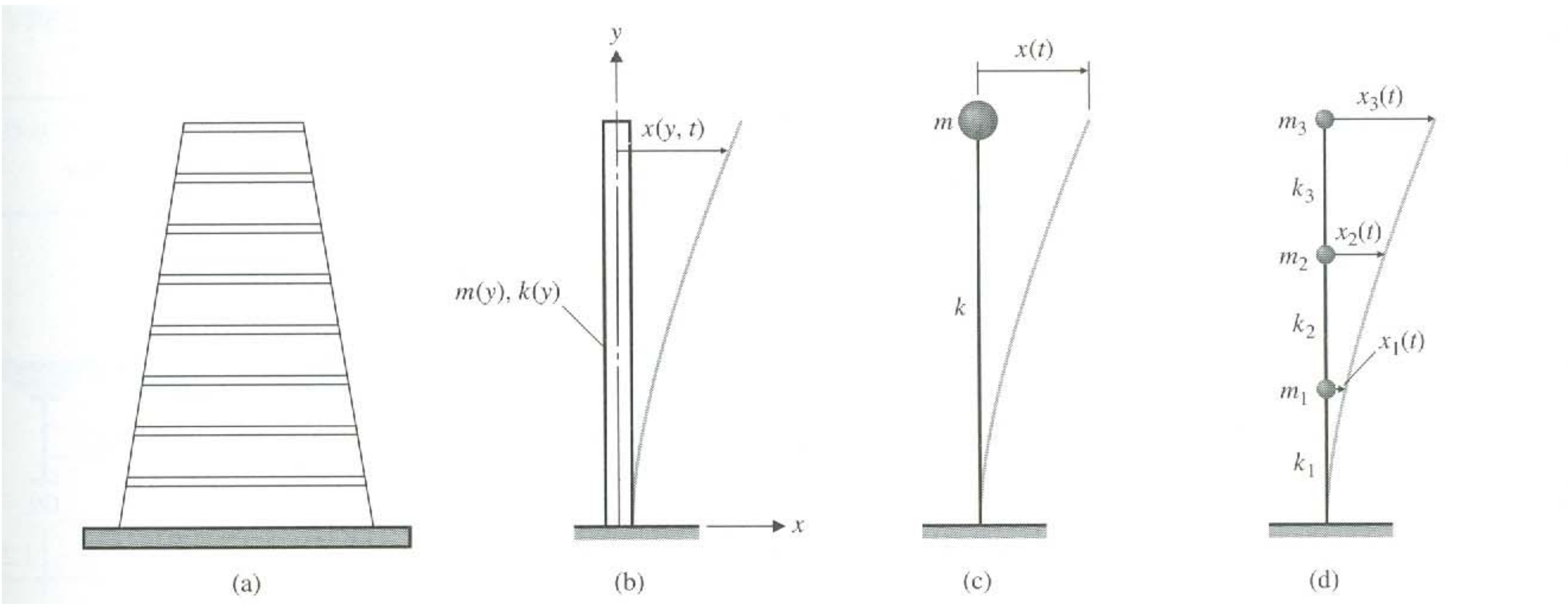
$$E = 10^{10}\ Pa$$

$$\rho = 5000\, kg/m^3$$

$$A = 0.001 m^2$$

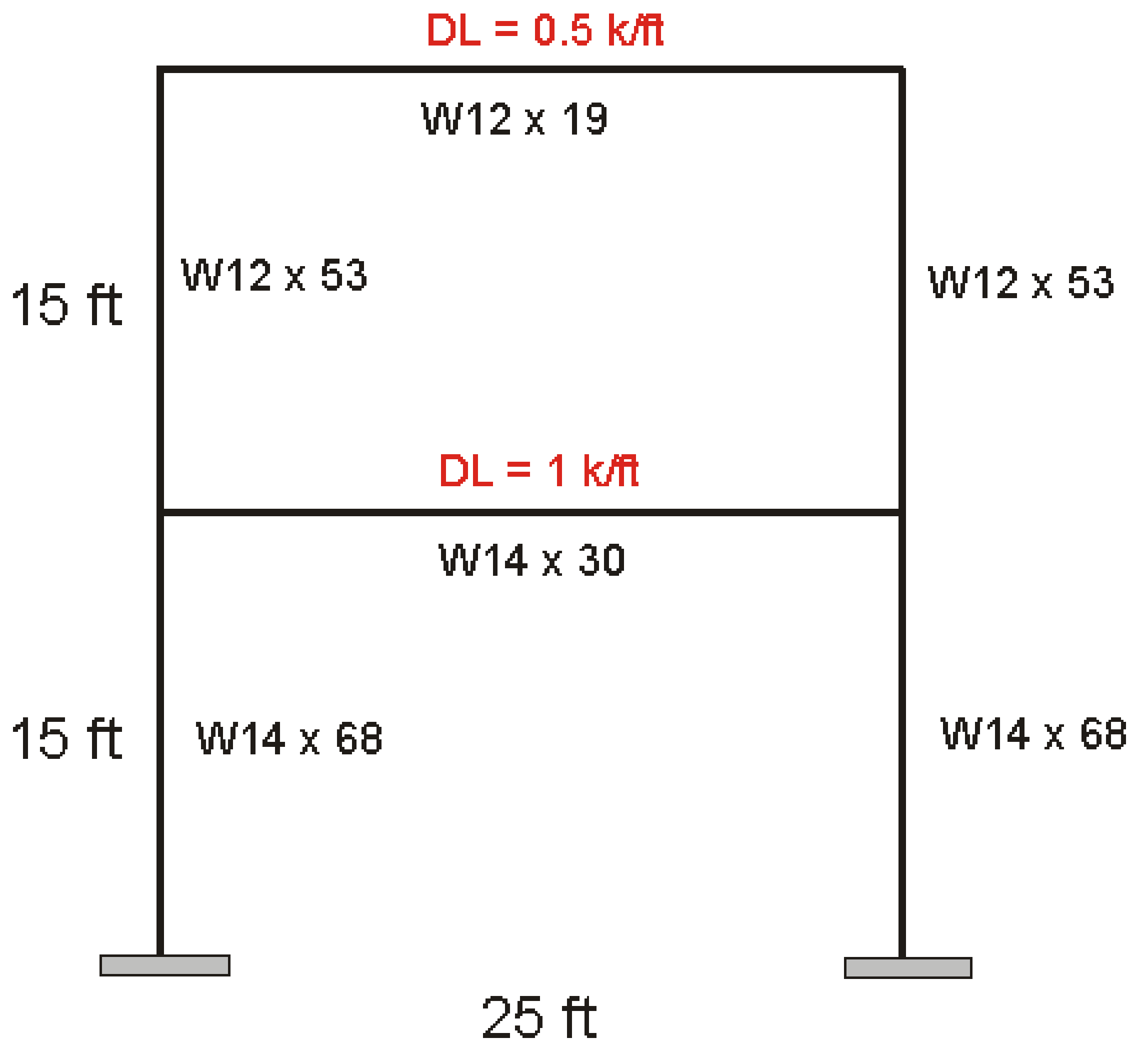
$$I = 0.0001 m^4$$

Frame Analysis



Multi-storied Building | **Continuous Model** | **SDOF Discrete Model** | **3-DOF Discrete Model**

Example



DL = 0.5 k/ft
W12 x 19
15 ft
W12 x 53
W12 x 53
DL = 1 k/ft
W14 x 30
15 ft
W14 x 68
W14 x 68
25 ft

Results

Mode	Eigenvalue	Freq. (Hz)	Time Period (s)
1	2024	7.16	0.14
2	23808.7	24.6	0.04
3	50357.4	35.7	0.028
4	51838.9	36.2	0.028
10	870642	148.5	0.0067
11	943579	154.6	0.0065
12	$1.46357(10^6)$	192.5	0.0052