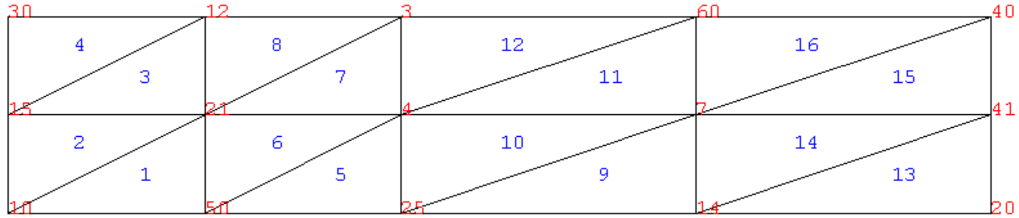
Finite Elements for Engineers

Lecture 2: The Six Major Steps via Direct Stiffness Method

S. D. Rajan

Step 1: Discretization

- Break the problem domain into a collection of simple shapes
  - One-dimensional
  - Two-dimensional (triangle, quadrilateral)
  - Three-dimensional (tetrahedron, hexahedron, wedge, pyramid)

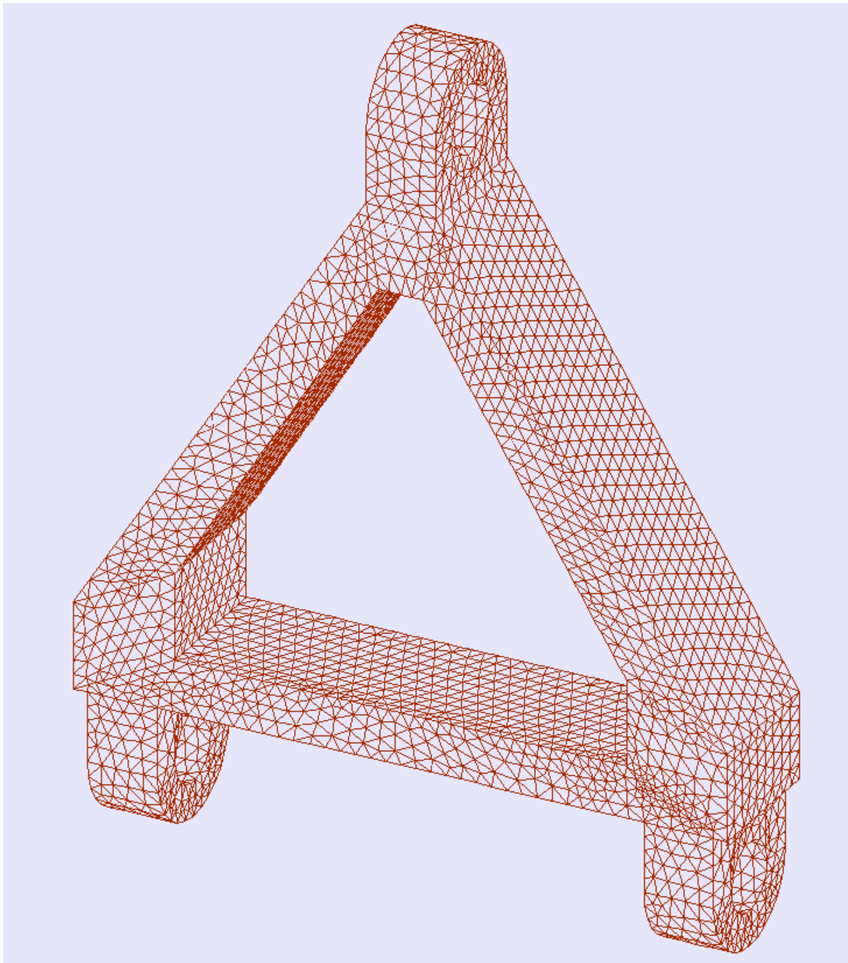


2D Example

(Collection of triangles)

3D Example

(Collection of tetrahedra)



Step 2: Element Equations

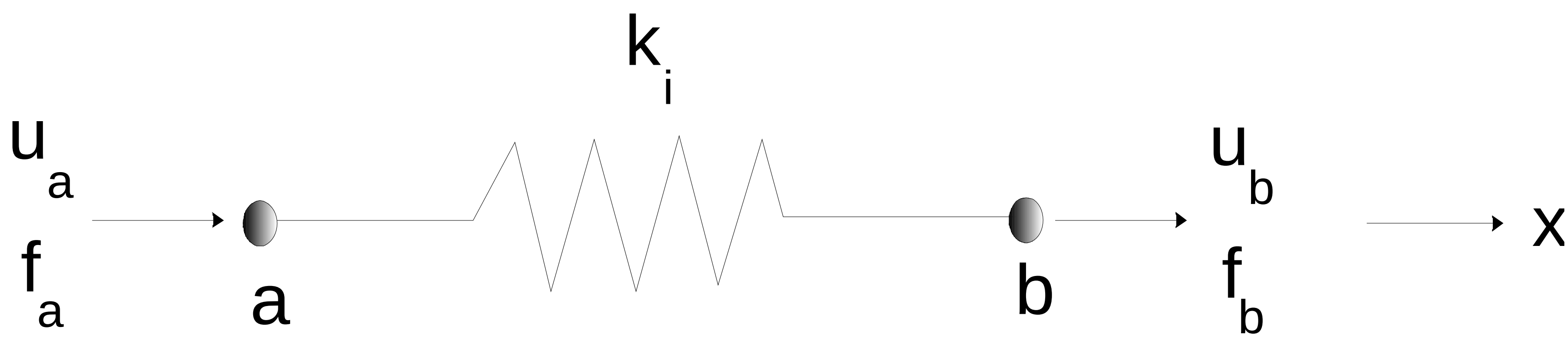
- Relate the properties of the system to the primary unknowns
- Cantilever Beam Example: Relate displacements to beam dimensions, thickness, material properties, how the beam is supported and loaded.

Direct Stiffness Method

M: Mass (slg, kg)
L: Length (ft, m)
T: Time (s)
t: Temperature (F, C)

F: Force (**ML/T^2**) (lb, N)
E: Energy (**ML^2/T^2**) (Btu, J)
P: Power (**ML^2/T^3**) (HP, W)

Linear Springs (Hooke's Law)



$$k_i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_a \\ u_b \end{Bmatrix} = \begin{Bmatrix} f_a \\ f_b \end{Bmatrix}$$

Dimensional Analysis:(**F/L**)(**L**) = **F**

Derivation

$$\overset{\rightarrow +}{\sum} F_x = 0 = k_i\left(u_a - u_b\right) = f_a$$

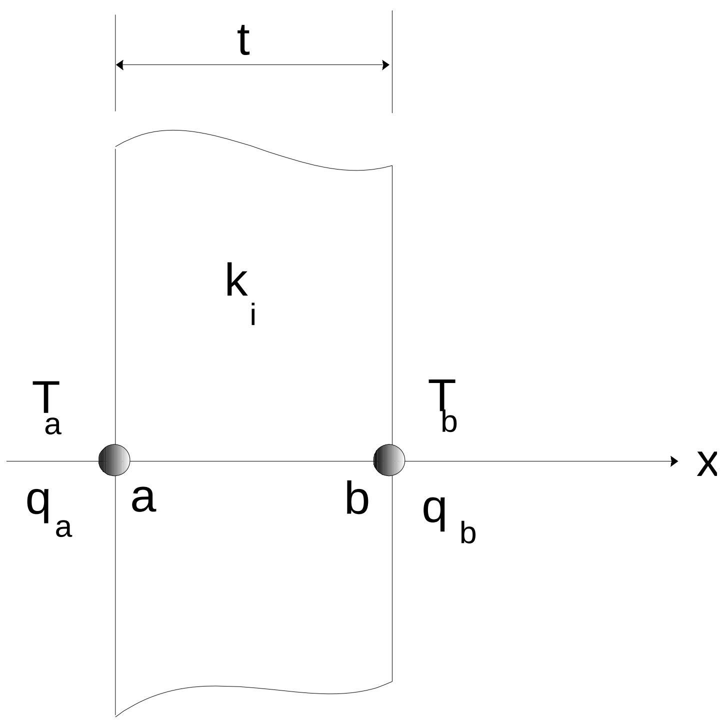
$$\overset{\rightarrow +}{\sum} F_x = 0 = k_i\left(u_b - u_a\right) = f_b$$

$$\Rightarrow \quad k_i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_a \\ u_b \end{Bmatrix} = \begin{Bmatrix} f_a \\ f_b \end{Bmatrix}$$

Constitutive Relationship
(Hooke's Law)

Equilibrium

1D Heat Flow (Fourier's Law)



$$\frac{k_i A}{t}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_a \\ T_b \end{Bmatrix} = \begin{Bmatrix} q_a \\ q_b \end{Bmatrix}$$

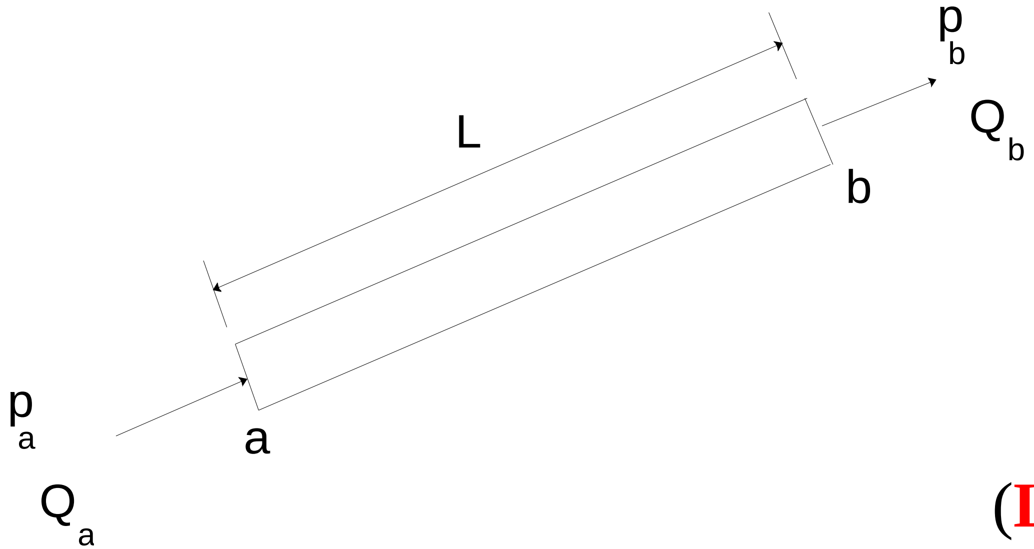
Dimensional Analysis

(**P/Lt**)(**L^2/L**)(**t**) = **P**

$$\frac{k_i}{t}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_a \\ T_b \end{Bmatrix} = \begin{Bmatrix} \tau_a \\ \tau_b \end{Bmatrix}$$

(**P/Lt**)(**1/L**)(**t**) = (**P/L^2**)

Pipe Flow (Darcy's Law)

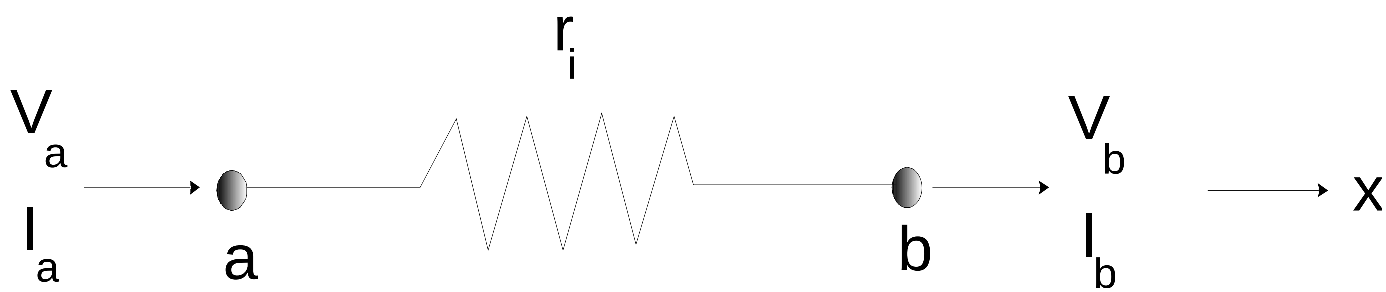


Dimensional Analysis:

(**L⁴/(LFT/L²)**)(**F/L²**) = **L³/T**

$$\frac{\pi D^4}{128 L\mu}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} p_a \\ p_b \end{Bmatrix} = \begin{Bmatrix} Q_a \\ Q_b \end{Bmatrix}$$

Electrical Network (Ohm’s Law)



$$\frac{1}{r_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} V_a \\ V_b \end{Bmatrix} = \begin{Bmatrix} I_a \\ I_b \end{Bmatrix}$$

C: Current (A)

$$V = \frac{W}{A}$$

$$\Omega = \frac{V}{A}$$

Element Equations

$$\mathbf{k}_{2\times2}\mathbf{d}_{2\times1} = \mathbf{f}_{2\times1}$$

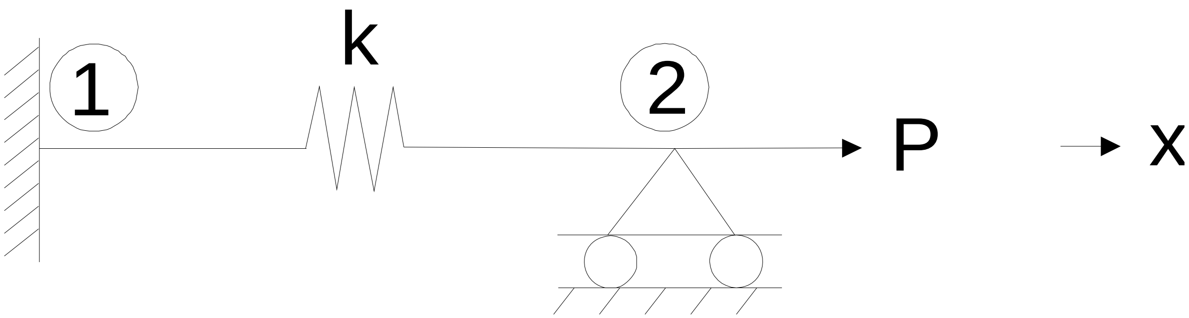
Step 3: Assembly

- "Property" of the system is the sum of the "properties" of all the elements
- However this is not an algebraic sum
- In general we need to assemble

$$\mathbf{k}_{2\times 2}\mathbf{d}_{2\times 1} = \mathbf{f}_{2\times 1} \rightarrow \mathbf{K}_{n\times n}\mathbf{D}_{n\times 1} = \mathbf{F}_{n\times 1}$$

Example 1

1-Element Example



$$k = 300\, lb/in$$

$$P = 30\, lb$$

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Element Equations

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 30 \end{Bmatrix}$$

System Equations
Cannot solve!

Step 4: Boundary Conditions

Since D_1=0, we can modify the two equations as follows. This is NOT an approximation.

$$\begin{bmatrix} 1 & 0 \\ 0 & 300 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \end{Bmatrix}$$

This is known as the Elimination Technique of imposing essential boundary conditions (EBCs).

Step 5: Solution System Equations

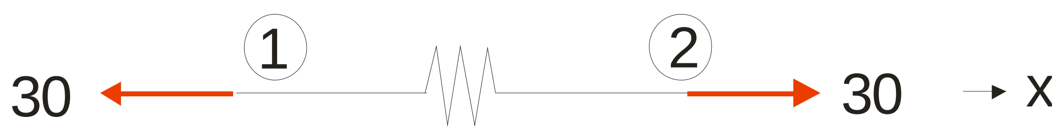
Solving the two equations, we have the solution as follows.

$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.1'' \end{Bmatrix}$$

Step 6: Derived Variables

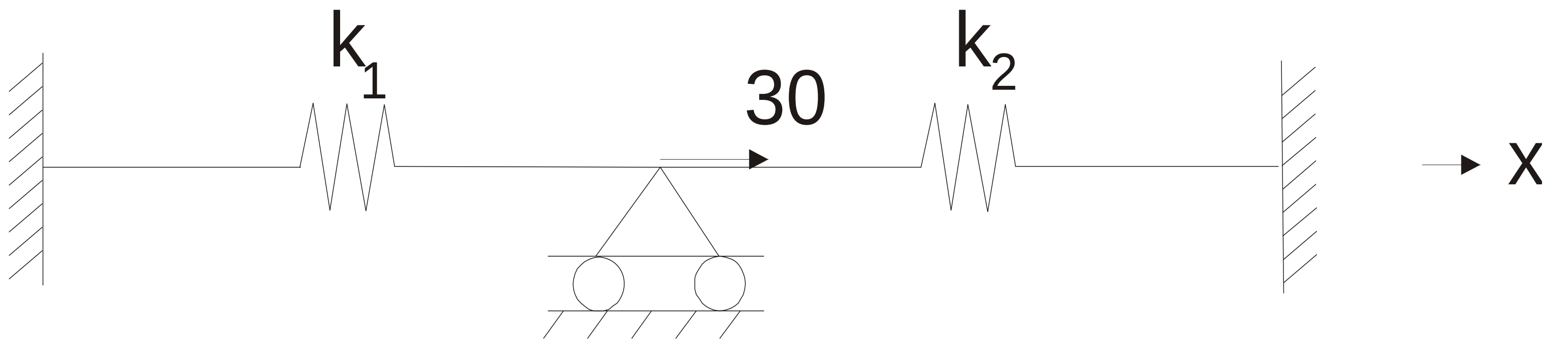
Now we can compute the force in the spring using the element equations as follows.

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.1 \end{Bmatrix} = \begin{Bmatrix} -30 \\ 30 \end{Bmatrix} lb$$



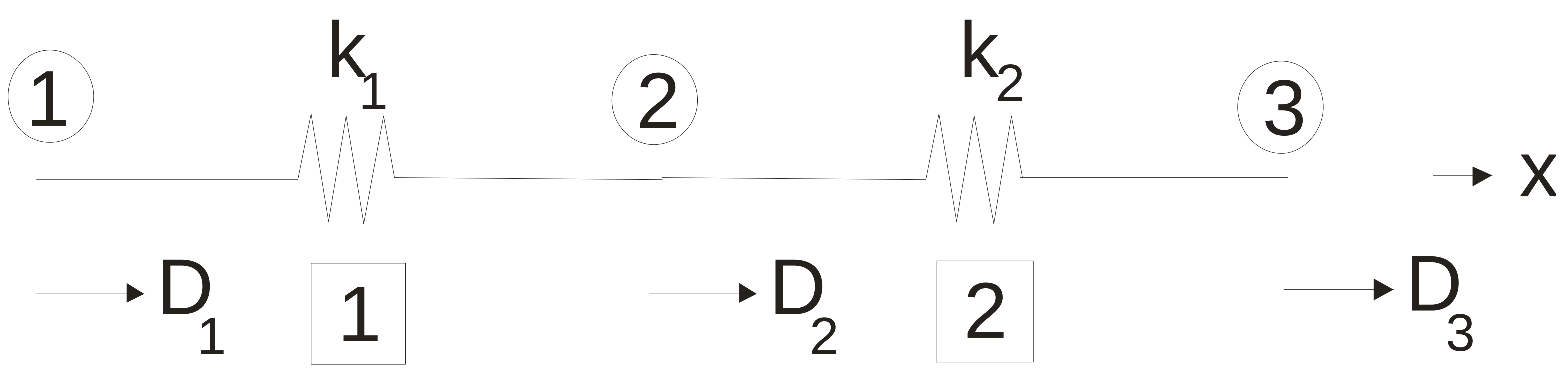
Example 2

2-Element Example



$$k_1 = 300\, lb/in$$

$$k_2 = 200\, lb/in$$



Example (Step 2)

Element 1

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{\,1} \\ f_2^{\,1} \end{Bmatrix}$$

Element 2

$$\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{\,2} \\ f_2^{\,2} \end{Bmatrix}$$

Example (Step 3)

Element 1

$$\begin{bmatrix} 300 & -300 & 0 \\ -300 & 300 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \\ 0 \end{Bmatrix}$$

Element 2

$$\begin{bmatrix} 300 & -300 & 0 \\ -300 & 300+200 & -200 \\ 0 & -200 & 200 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 \end{Bmatrix}$$

Example (Step 4)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \\ 0 \end{Bmatrix}$$

Example (Step 5)

Solving the three equations, we have the solution as follows.

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.06'' \\ 0 \end{Bmatrix}$$

Example (Step 6)

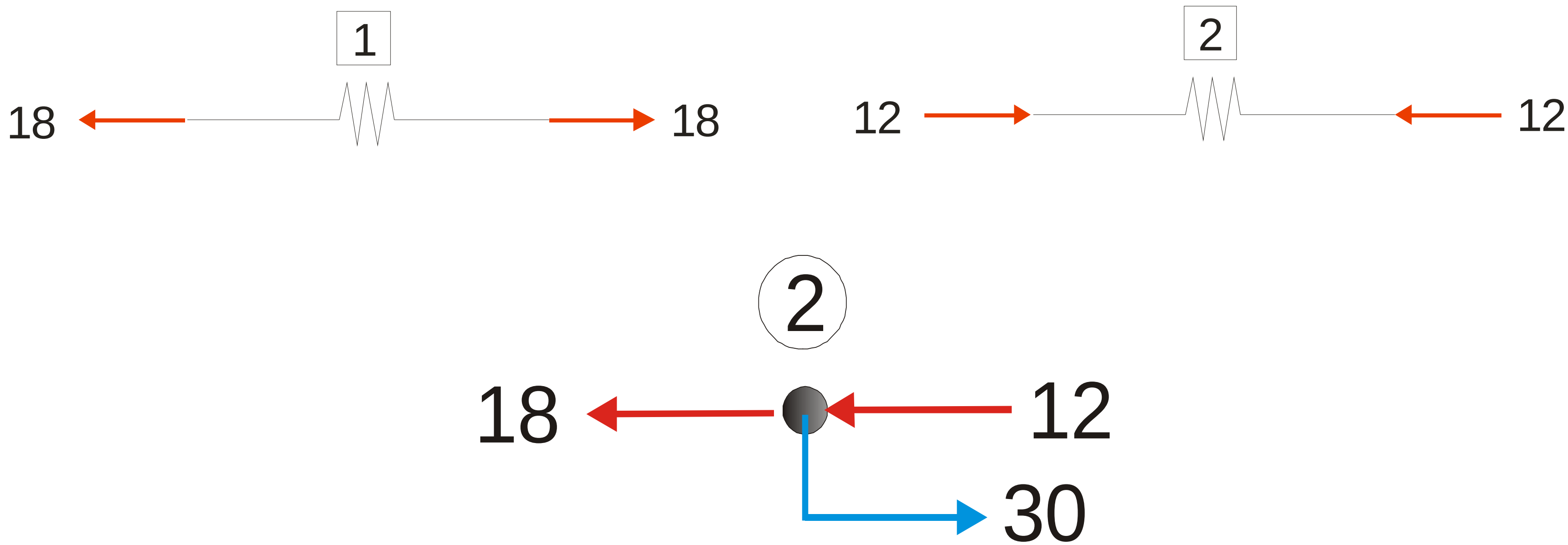
Element 1

$$\begin{bmatrix} 300 & -300 \\ -300 & 300 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.06 \end{Bmatrix} = \begin{Bmatrix} -18 \\ 18 \end{Bmatrix} lb$$

Element 2

$$\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.06 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 12 \\ -12 \end{Bmatrix} lb$$

Example (Equilibrium Check)



Review & Reflection

Solution Steps

- Choose consistent problem units
- Select a (global) coordinate system
- Label the nodes and elements
- Identify and label the nodal unknowns
- Identify the “boundary conditions”

Solution Steps

- Loop thro’ all elements
- Form the element equations
- Assemble into the system equations
- End loop
- Impose essential boundary conditions
- Solve the equations
- Obtain the secondary unknowns
- Check the solution for correctness

Theoretical Notes

- Element stiffness matrix **k** is symmetric but rank deficient
- Structural stiffness matrix **K** is symmetric
- **K** is rank deficient before imposing EBC
- **K** is positive definite after imposing EBC

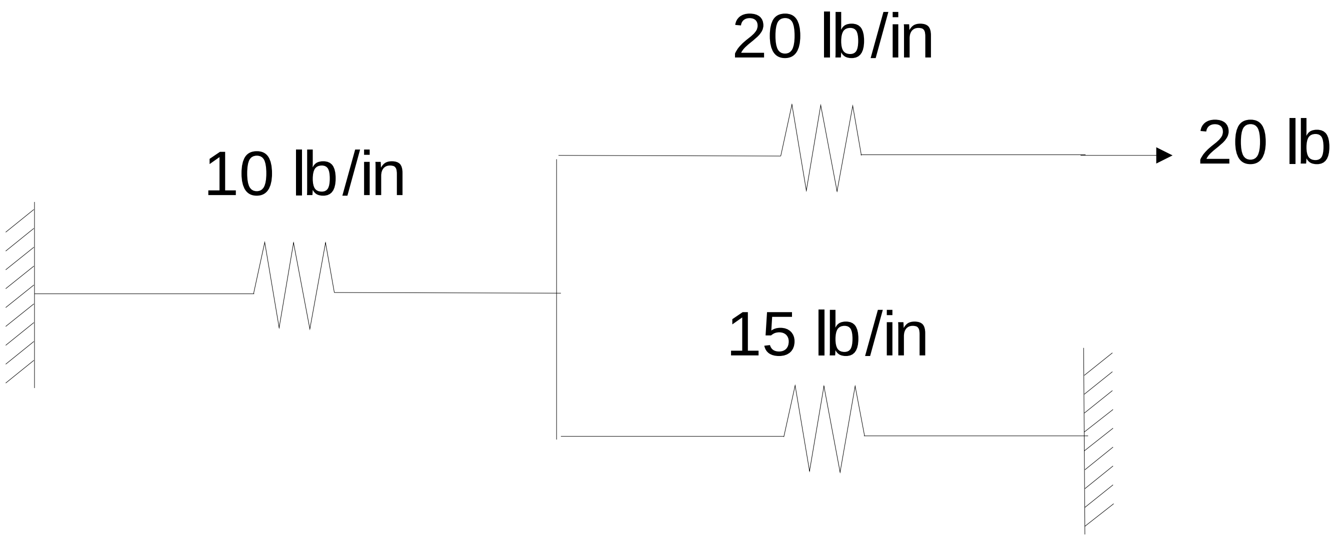
The Six Steps

- Step 1: Discretization
- Step 2: Element equations
- Step 3: Assembly
- Step 4: Imposition of boundary conditions
- Step 5: Solution of the system equations
- Step 6: Computation of secondary unknowns

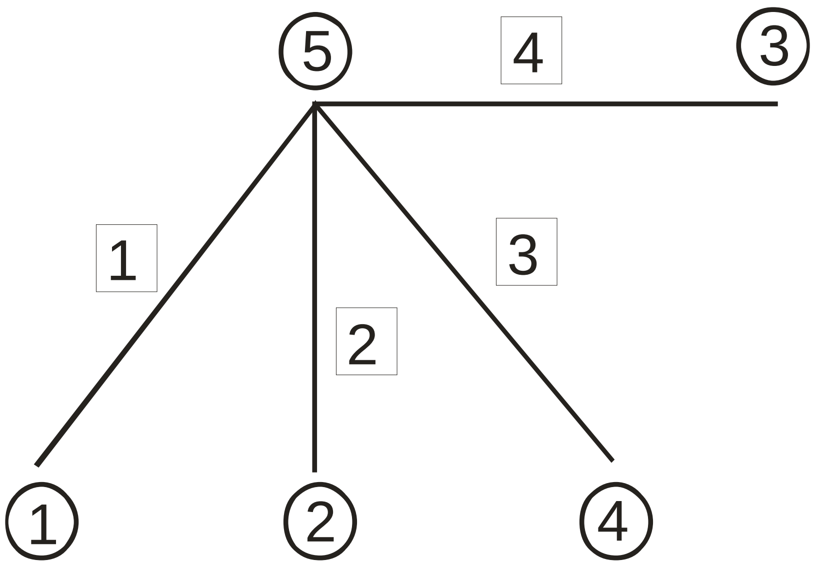
Symbolic Assembly

- Given the FE model with the node and element numbers, we should be able to generate the structure and form of the system stiffness matrix **K**

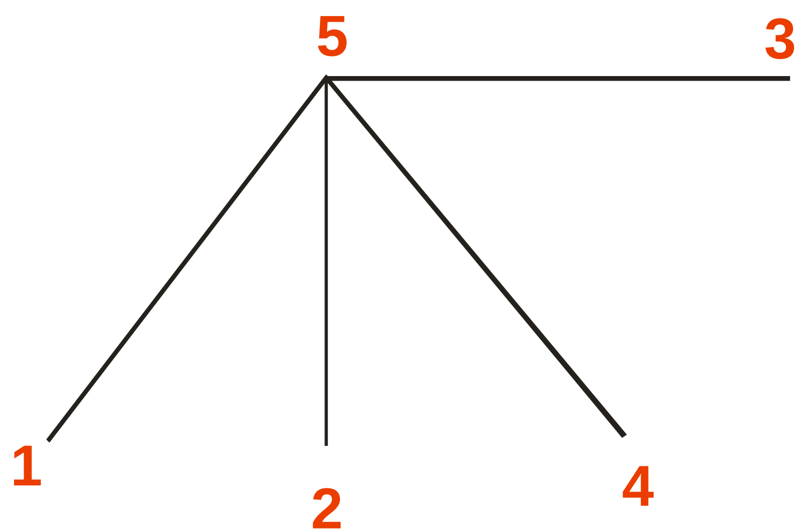
Example



Example



Elements and Nodes



Nodal Unknowns
1 DOF/node

Assembled **K**

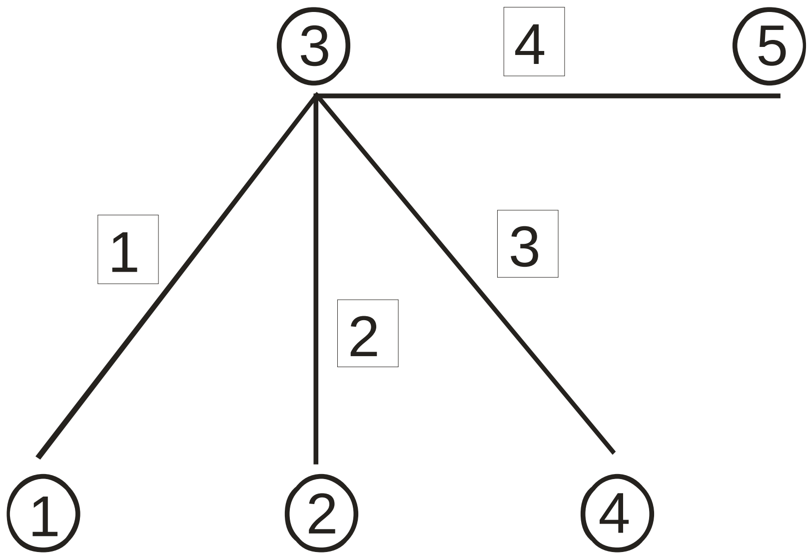
Upper triangular, banded matrix

Rectangular storage

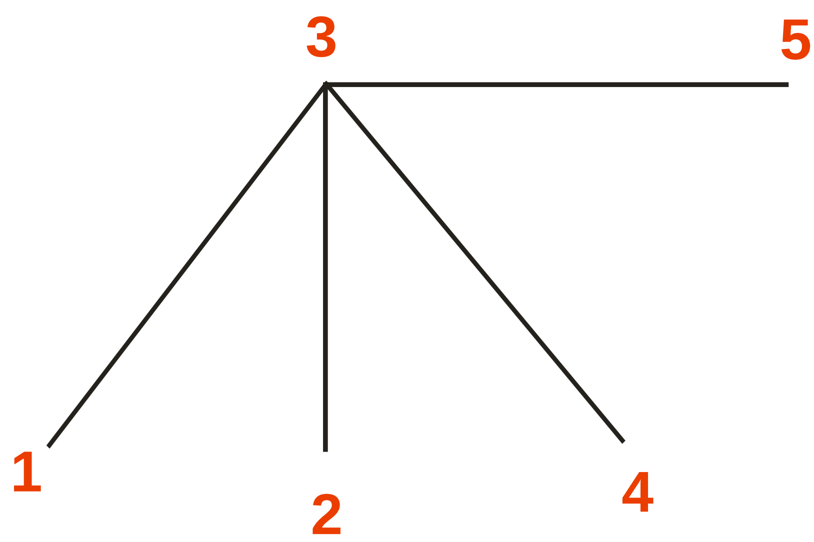
$$\begin{bmatrix} K_{11} & & & & K_{15} \\ & K_{22} & & & K_{25} \\ & & K_{33} & & K_{35} \\ & & & K_{44} & K_{55} \\ Sym & & & & K_{55} \end{bmatrix} \Rightarrow \begin{bmatrix} K_{11} & 0 & 0 & 0 & K_{15} \\ K_{22} & 0 & 0 & K_{25} & 0 \\ K_{33} & 0 & K_{35} & 0 & 0 \\ K_{44} & K_{55} & 0 & 0 & 0 \\ K_{55} & 0 & 0 & 0 & 0 \end{bmatrix}$$

HBW=5 !

Is there a better solution?



Elements and Nodes



Nodal Unknowns
1 DOF/node

Assembly

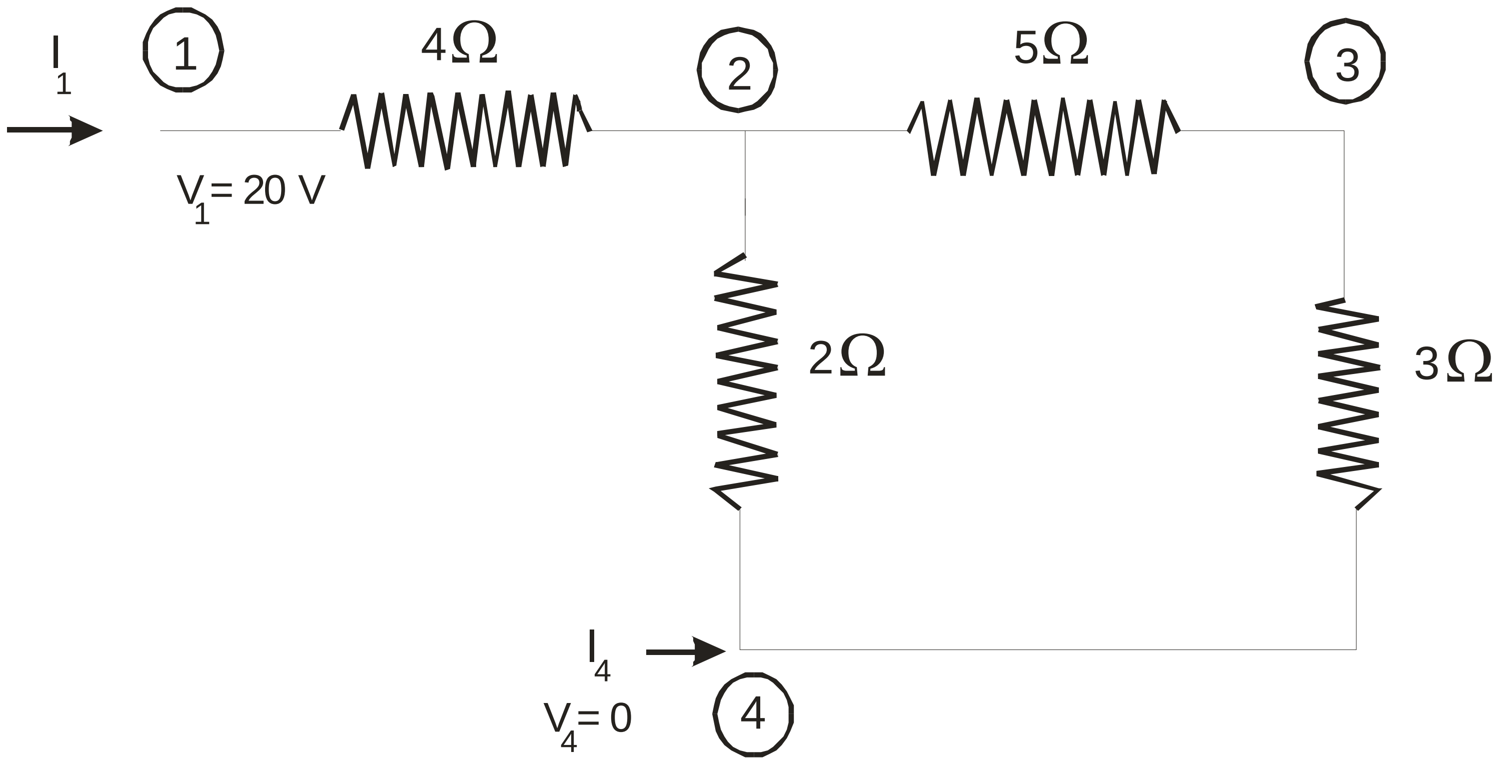
Upper triangular, banded matrix

Rectangular storage

$$\begin{bmatrix} K_{11} & & K_{13} & & \\ & K_{22} & K_{23} & & \\ & & K_{33} & K_{34} & K_{35} \\ & & & K_{44} & \\ Sym & & & & K_{55} \end{bmatrix} \Rightarrow \begin{bmatrix} K_{11} & 0 & K_{13} \\ K_{22} & K_{23} & 0 \\ K_{33} & K_{34} & K_{35} \\ K_{44} & 0 & 0 \\ K_{55} & 0 & 0 \end{bmatrix}$$

HBW=3

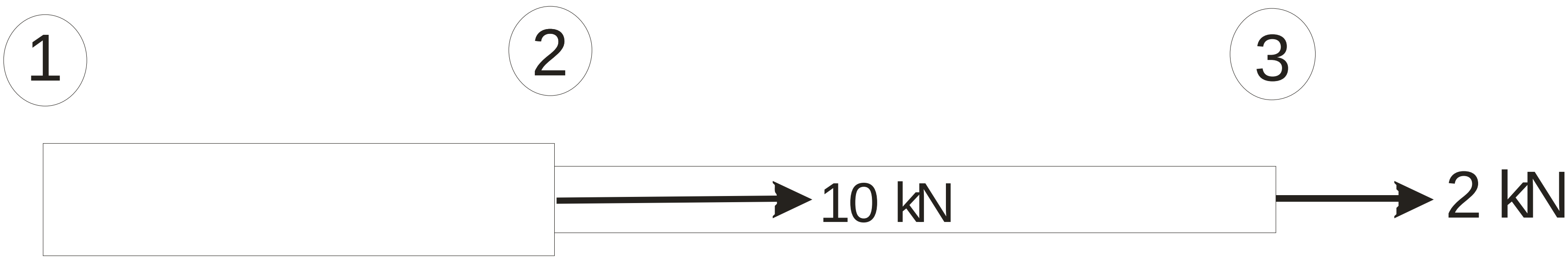
Example



(1) Is K_{13} zero?

(2) How many nonzero terms in the upper triangular part of **K**?

Example

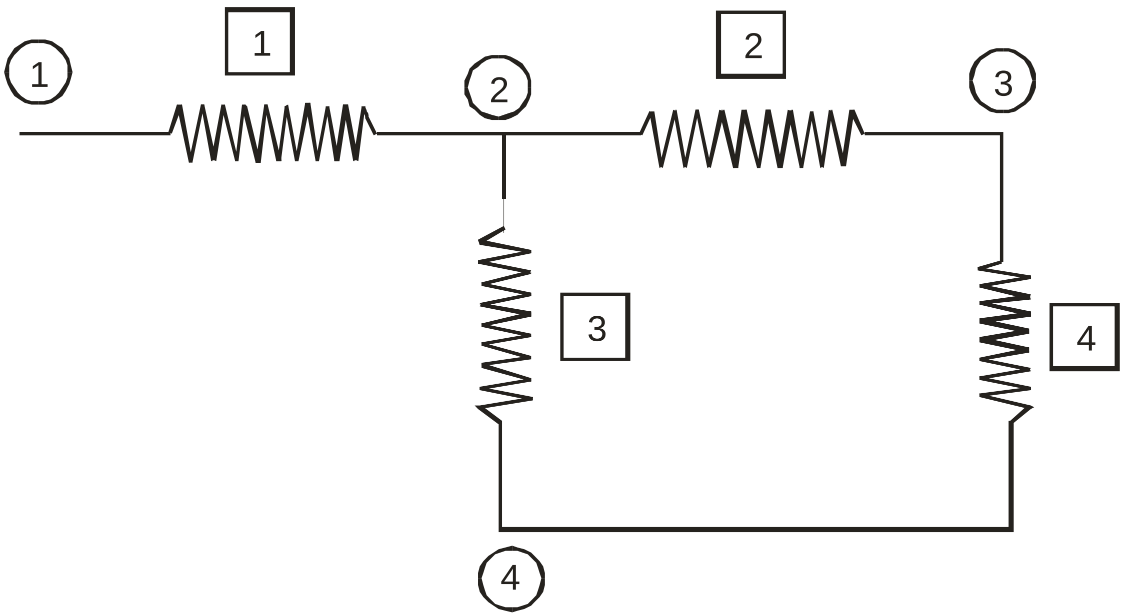


(1) What is the size of K?

(2) How many zero elements in K?

(3) How many nonzero elements in F?

Assembly Process



Element 3

$$\begin{array}{c|cc} & 4 & 2 \\ \hline 4 & k_{11} & k_{12} \\ 2 & k_{21} & k_{22} \end{array}$$

In general

$$\begin{array}{c|cc} & i & j \\ \hline i & k_{11} & k_{12} \\ j & k_{21} & k_{22} \end{array}$$

Reflection

- Assembling a symmetric, rank deficient **k** into **K** yields a symmetric, rank deficient **K**
- Applying BCs (EBCs) is needed to make **K** symmetric and nonsingular
- The location of the nonzero entries in **K** is a function of the node numbers (not element numbers)

Suggested Problems

- T2L2-1
- T2L2-2
- T2L2-3
- T2L2-4