CEE432/CEE532/MAE541
Developing Software for Engineering Applications

Lecture 16: More on Truss Analysis and Variational Technique

Space Truss Analysis

Comparison with Planar Truss

- Differences
  - 3 nodal coordinates
  - 3 degrees-of-freedom per node
- Similarities
  - All other behavior characteristics are the same

Space Truss Element

$$\mathbf{k}'_{2\times 2}\mathbf{d}'_{2\times 1} = \mathbf{f}'_{2\times 1}$$

Element Equations in Global Coordinate System

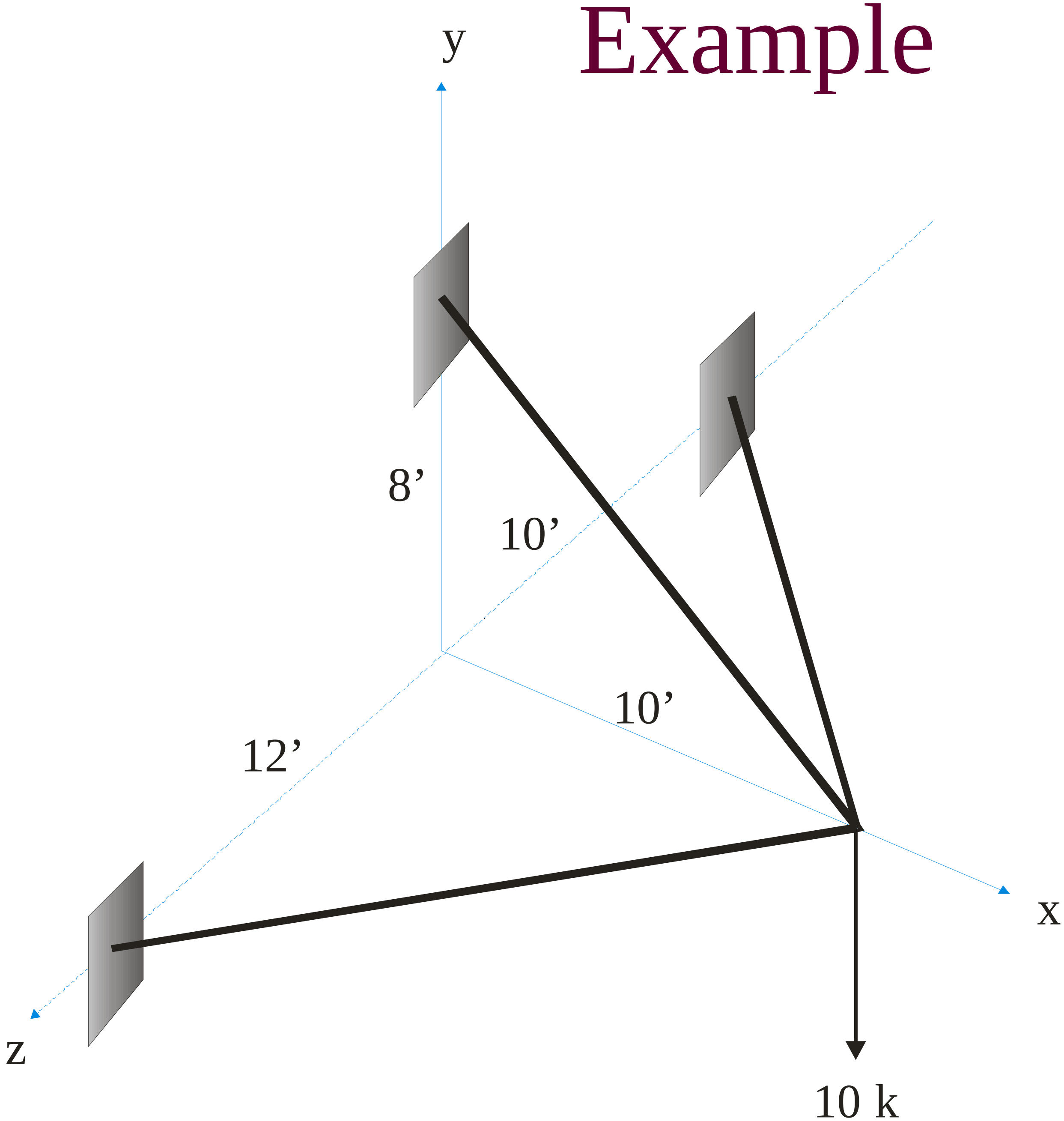
$$\mathbf{d}'_{2\times 1} = \mathbf{T}_{2\times 6}\mathbf{d}_{6\times 1} \quad \Rightarrow \quad \mathbf{k}_{6\times 6}\mathbf{d}_{6\times 1} = \mathbf{f}_{6\times 1}$$

$$\mathbf{f}_{6\times 1} = \mathbf{T}^{T}_{6\times 2}\mathbf{f}'_{2\times 1}$$

where

$$\mathbf{k}_{6\times 6} = \mathbf{T}^{\mathrm{T}}_{6\times 2}\mathbf{k}'_{2\times 2}\mathbf{T}_{2\times 6}$$

Example

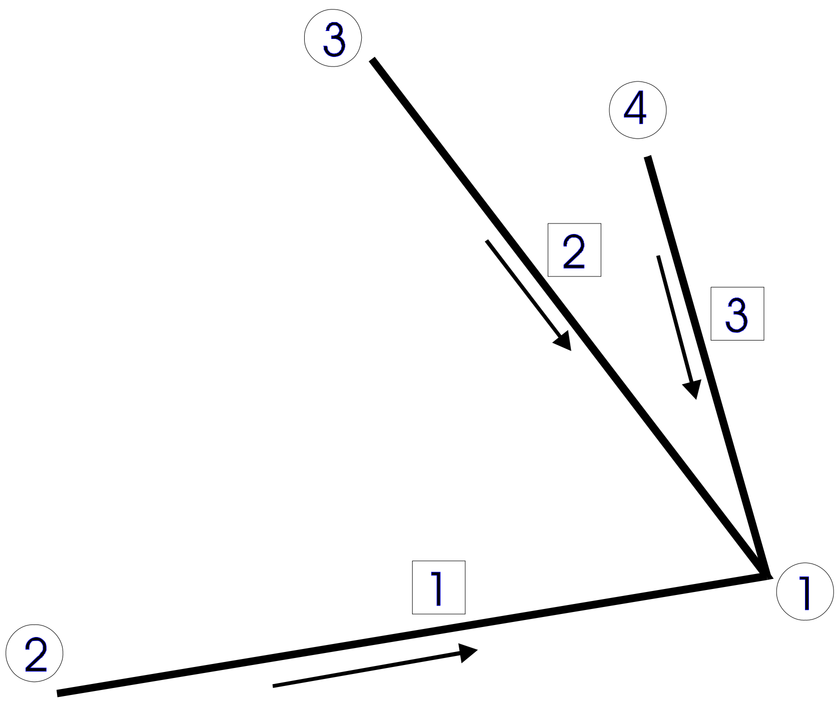


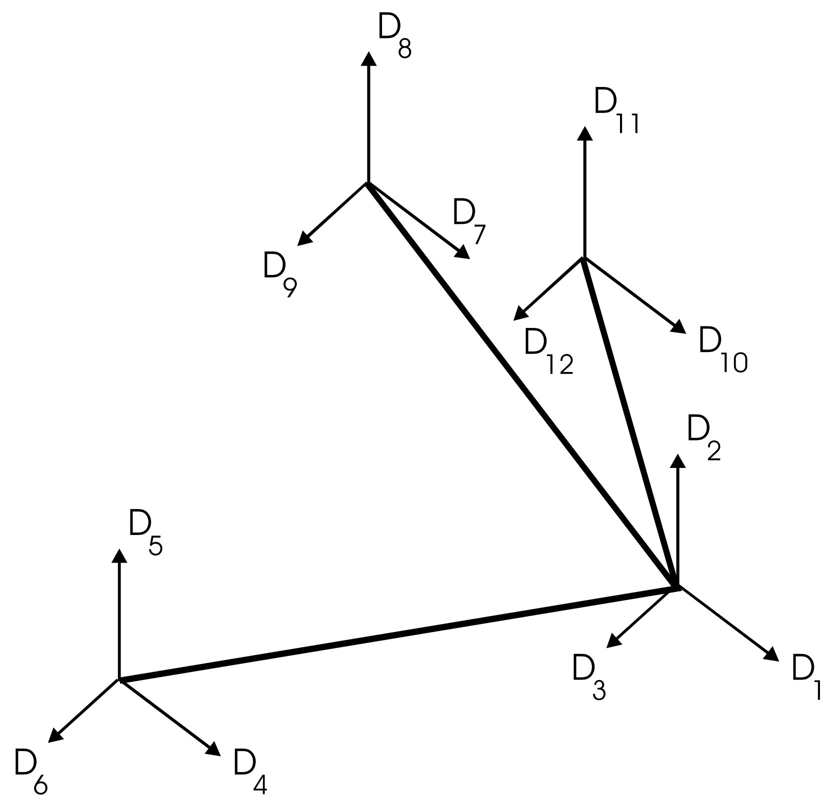
$E = 29000$ ksi

$A = 2.5$ in^2

Example

FE Model





Example

Element 1

$$10^5 \begin{bmatrix} 1.5852 & 0 & -1.9022 & -1.5852 & 0 & 1.9022 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 2.2826 & 1.9022 & 0 & -2.2826 \\ & & & 1.5852 & 0 & -1.9022 \\ & & & & 0 & 0 \\ Sym & & & & & 2.2826 \end{bmatrix} \begin{Bmatrix} D_4 \\ D_5 \\ D_6 \\ D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \\ f_3^1 \\ f_4^1 \\ f_5^1 \\ f_6^1 \end{Bmatrix}$$

Example

Element 2

$$10^5 \begin{bmatrix} 2.8767 & -2.3013 & 0 & -2.8767 & 2.3013 & 0 \\ & 1.8411 & 0 & 2.3013 & -1.8411 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 2.8767 & -2.3013 & 0 \\ & & & & 1.8411 & 0 \\ Sym & & & & & 0 \end{bmatrix} \begin{Bmatrix} D_7 \\ D_8 \\ D_9 \\ D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} f_1^2 \\ f_2^2 \\ f_3^2 \\ f_4^2 \\ f_5^2 \\ f_6^2 \end{Bmatrix}$$

Example

Element 3

$$10^5 \begin{bmatrix} 2.1361 & 0 & 2.1361 & -2.1361 & 0 & -2.1361 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 2.1361 & -2.1361 & 0 & -2.1361 \\ & & & 2.1361 & 0 & 2.1361 \\ & & & & 0 & 0 \\ Sym & & & & & 2.1361 \end{bmatrix} \begin{Bmatrix} D_{10} \\ D_{11} \\ D_{12} \\ D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} f_1^3 \\ f_2^3 \\ f_3^3 \\ f_4^3 \\ f_5^3 \\ f_6^3 \end{Bmatrix}$$

Example

Steps 3 and 4: System Equations after BC

$$10^5 \begin{bmatrix} 6.5979 & -2.3013 & 2.3386 \\ -2.3013 & 1.8411 & 0 \\ 2.3386 & 0 & 4.4187 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10^4 \\ 0 \end{Bmatrix}$$

Step 5: Nodal Displacements

$$D_1 = -3.3703(10^{-2})\,in$$

$$D_2 = -9.6445(10^{-2})\,in$$

$$D_3 = 1.7838(10^{-3})\,in$$

Example

Element 1

$$f' = 386778\begin{bmatrix} 0.64 & 0 & -0.768 & -0.64 & 0 & 0.768 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0337 \\ -0.09644 \\ 0.00178 \end{Bmatrix} = 8875\,lb\,(C)$$

Element 2

$$f' = 471775\begin{bmatrix} 0.781 & -0.625 & 0 & -0.781 & 0.625 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0337 \\ -0.09644 \\ 0.00178 \end{Bmatrix} = -16008\,lb\,(T)$$

Example

Element 3

$$f^{'} = 427210\begin{bmatrix} 0.707 & 0 & -0.707 & -0.707 & 0 & 0.707 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.0337 \\ -0.09644 \\ 0.00178 \end{Bmatrix} = 9642\, lb(C)$$

Theorem of Minimum Potential Energy

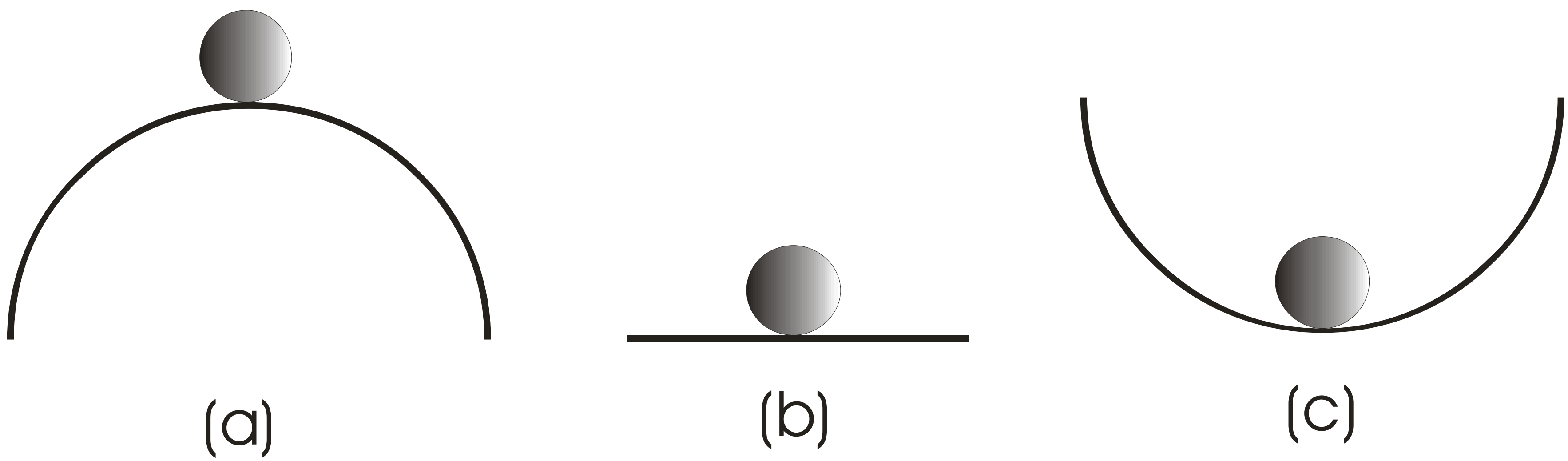
Theorem

- The Theorem of Minimum Potential Energy states that for a conservative system, amongst all admissible configurations those that satisfy the equations of equilibrium make the potential energy stationary with respect to small variations of displacement. If the stationary condition is a minimum, the equilibrium state is stable.

Theorem

$$\Pi = \Pi\left(D_1, D_2, ..., D_n\right) = \Pi\left(\mathbf{D}\right)$$

$$\frac{\partial \Pi}{\partial \mathbf{D}} = 0 \quad i = 1, \;\; 2, \ldots \;\; n$$



Theorem

Total Potential Energy

$$\Pi = \text{strain energy} + \text{work potential}$$

$$\Pi = \int_V U_0 \, dV - \int_V \mathbf{f}^T \mathbf{F} \, dV - \int_S \mathbf{f}^T \Phi \, dS - \mathbf{D}^\mathbf{T} \mathbf{P}$$

Theorem

Strain Energy Density

$$U_0 = \frac{1}{2}\{\varepsilon\}^T \mathbf{E}\{\varepsilon\} - \{\varepsilon\}^T \mathbf{E}\{\varepsilon_0\} + \{\varepsilon\}^T \{\sigma_0\}$$

Stress-strain Relationship

$$\{\sigma\} = \mathbf{E}\{\varepsilon\} - \mathbf{E}\{\varepsilon_0\} + \{\sigma_0\}$$

Strain-displacement Relationship

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

Example

Problem Statement

Consider a bar of constant cross-section, length and modulus of elasticity subjected to a constant axial force at the right tip and fixed at the left end. Compute the tip displacement and the state of stress in the bar.

Solution

Assumed solution $$u(x) = a_0 + a_1 x$$

EBC $$u(x=0) = 0$$

Example

Assumed Displacement

$$u(x) = a_0 + a_1 x$$

$$u(x=0) = 0$$

$$u(x=0) = 0 = a_0$$

$$u(x) = a_1 x$$

Strain-Disp.

$$\varepsilon_x = \frac{du}{dx} = a_1$$

Stress-Strain

$$\sigma_x = E\varepsilon_x$$

Total Potential Energy

$$\Pi(a_1) = \int_V U_0 \, dV - PD$$

$$\Pi(a_1) = \int_0^L \frac{1}{2}(a_1)(E)(a_1)A dx - P(a_1 L)$$

$$\Pi(a_1) = \frac{1}{2} a_1^2 EAL - Pa_1 L$$

Minimization

$$\frac{d\Pi}{da_1} = 0 = a_1 EAL - PL \Rightarrow a_1 = \frac{P}{AE}$$

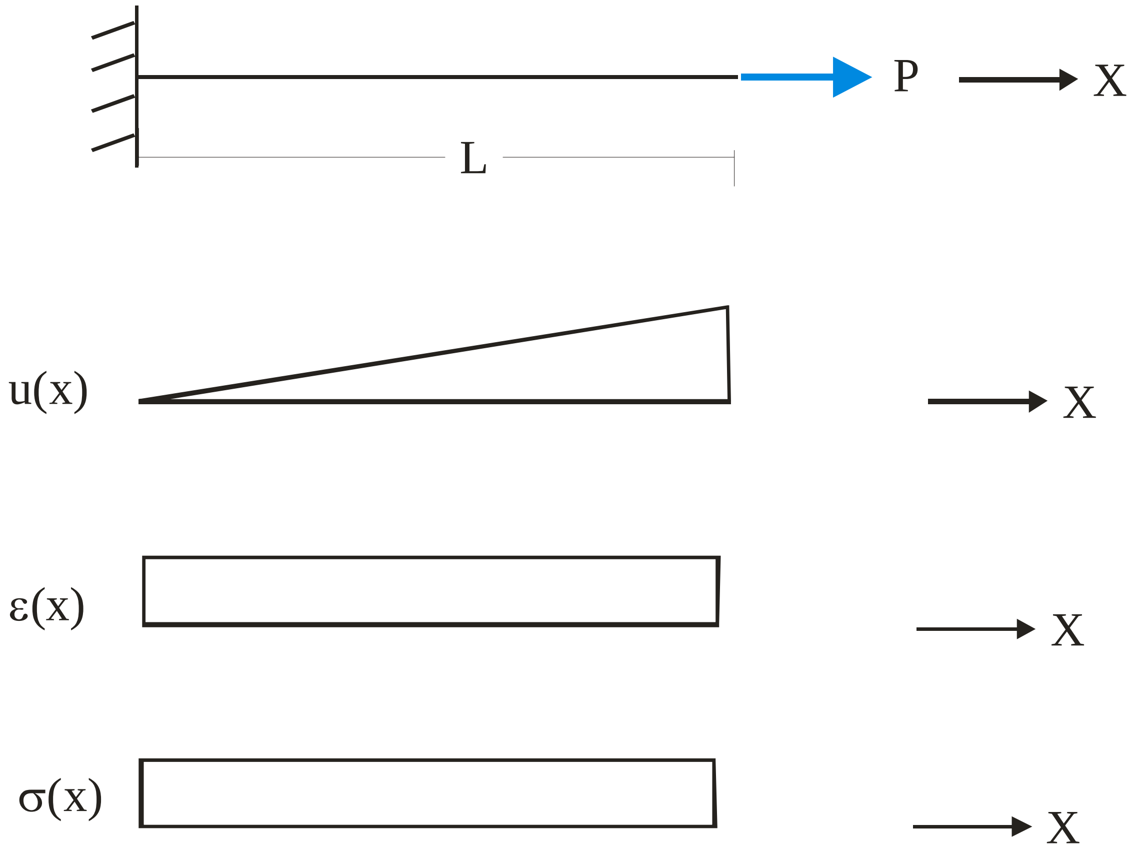
Example

Final Solution

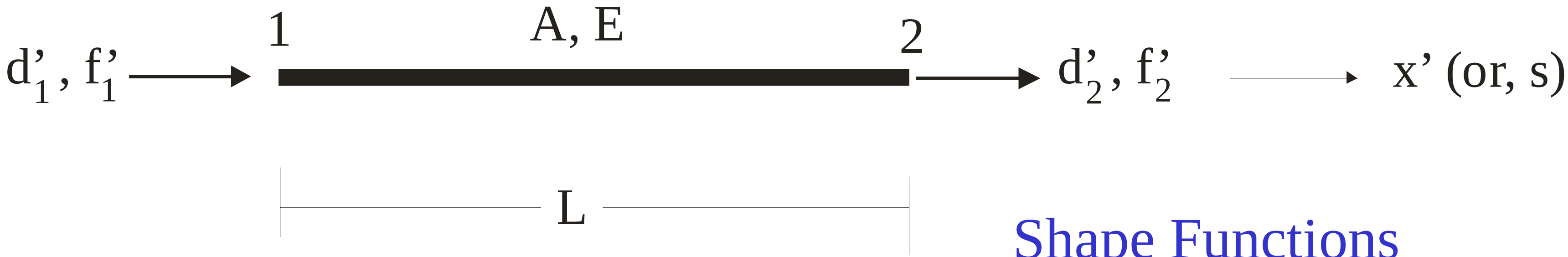
$$u(x) = \frac{Px}{AE}$$

$$\varepsilon_x = \frac{du}{dx} = \frac{P}{AE}$$

$$\sigma = E\varepsilon_x = \frac{P}{A}$$



Truss Analysis



Shape Functions

Assumed displacement

$$u(s) = \phi_1(s) d_1^{'} + \phi_2(s) d_2^{'} = \frac{L-s}{L} d_1^{'} + \frac{s}{L} d_2^{'}$$

Strain-Disp.

$$\varepsilon = \frac{du}{ds} = \frac{d}{ds}\begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{Bmatrix} d_1^{'} \\ d_2^{'} \end{Bmatrix} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} d_1^{'} \\ d_2^{'} \end{Bmatrix} = \mathbf{B}_{1\times 2}\, \mathbf{d}^{'}_{2\times 1}$$

Truss Analysis

Stress-Strain

$$\sigma_x = E\varepsilon_x$$

Total Potential Energy

$$U = \int_V U_0 \, dV = \int_0^L \frac{1}{2}\varepsilon_x \sigma_x A ds = \int_0^L \frac{1}{2}\left[\mathbf{B}_{1\times 2}\,\mathbf{d}'_{2\times 1}\right]^T E\left[\mathbf{B}_{1\times 2}\,\mathbf{d}'_{2\times 1}\right] A\, ds$$

$$U = \left[\mathbf{d}'\right]^T_{1\times 2}\left[\int_0^L \mathbf{B}^T_{2\times 1}(EA)_{1\times 1}\mathbf{B}_{1\times 2}\, ds\right]\left[\mathbf{d}'\right]_{2\times 1} = \left[\mathbf{d}'\right]^T_{1\times 2}\left[\mathbf{k}'\right]_{2\times 2}\left[\mathbf{d}'\right]_{2\times 1}$$

$$\left[\mathbf{k}'\right]_{2\times 2} = \int_0^L \mathbf{B}^T_{2\times 1}(EA)_{1\times 1}\mathbf{B}_{1\times 2}\, ds = \frac{AE}{L}\left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array}\right]$$

Truss Analysis

Minimization

$$W = -[\mathbf{d}']^T_{2\times1}[\mathbf{f}']_{2\times1}$$

$$\Pi(\mathbf{d}') = U + W = \frac{1}{2}[\mathbf{d}']^T_{1\times2}[\mathbf{k}']_{2\times2}[\mathbf{d}']_{2\times1} - [\mathbf{d}']^T_{2\times1}[\mathbf{f}']_{2\times1}$$

$$\frac{\partial\Pi}{\partial\mathbf{d}'} = 0 \Rightarrow \frac{AE}{L}\left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array}\right]\left\{\begin{array}{c} d_1' \\ \hline d_2' \end{array}\right\} = \left\{\begin{array}{c} f_1' \\ \hline f_2' \end{array}\right\}$$

$$or, [\mathbf{k}']_{2\times2}[\mathbf{d}']_{2\times1} = [\mathbf{f}']_{2\times1}$$

Truss Analysis

Step 6: Secondary unknowns

$$\varepsilon = \frac{du}{ds} = \frac{d}{ds}\left(\phi_1 d_1^{'} + \phi_2 d_2^{'}\right) = \frac{d_2^{'} - d_1^{'}}{L}$$

$$\sigma = E\varepsilon$$

$$N = \sigma A$$

Thermal Loading

Theory

$$\varepsilon_0 = \alpha \, \Delta T$$

$$\left(\mathbf{f}_t^{'}\right)_{2\times 1} = EA\varepsilon_0 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\mathbf{f}_t = \mathbf{T}^{\mathrm{T}}\mathbf{f}_t^{'}$$

$$\sigma = E(\varepsilon - \varepsilon_0)$$

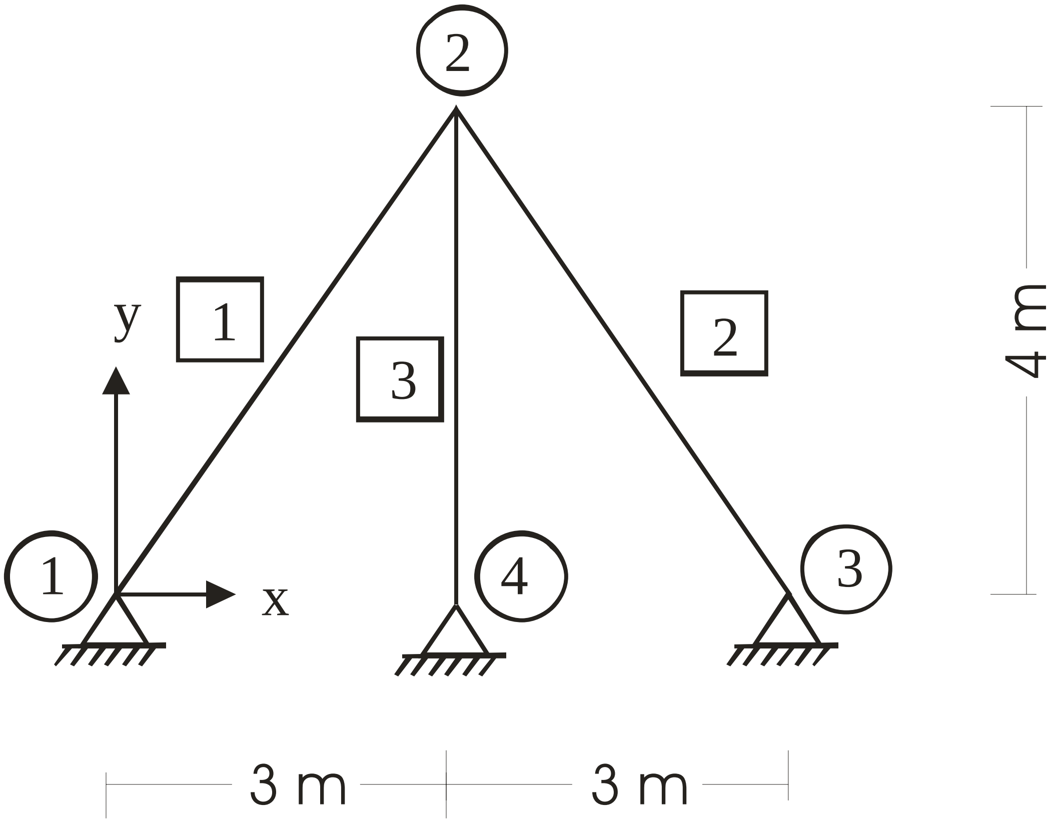
$$or, \sigma = E\left[\frac{d_2^{'} - d_1^{'}}{L} - \alpha(\Delta T)\right]$$

Implementation

1. Compute thermal load vector for each element with change in temperature. Add to **F**.

2. For each element with temperature change, subtract the initial strain.

Example



$A = 0.01 \text{ m}^2$

$E = 200$ GPa

$\alpha = 1.2(10^{-5})$ m/m-^{0}C

$\Delta T_1 = 50^0$C

Units: N, m

Example

Solution without Element 3

$$4\left(10^8\right)\begin{bmatrix} 0.72 & 0 \\ 0 & 1.28 \end{bmatrix}\begin{Bmatrix} D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_4 \end{Bmatrix}$$

Thermal Load Vector

$$\left(\mathbf{f}_t^{'}\right)_{2\times 1} = EA\varepsilon_0 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\left(\mathbf{f}_t^{'}\right)_{2\times 1} = \left(2\times 10^{11}\right)(0.01)\left(1.2\times 10^{-5}\right)(50)\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\left(\mathbf{f}_t^{'}\right)_{2\times 1} = \begin{Bmatrix} -1200000 \\ 1200000 \end{Bmatrix} N$$

$$\Rightarrow \mathbf{f}_t = \mathbf{T}^{\mathrm{T}}\mathbf{f}_t^{'} = \begin{Bmatrix} lf_1^{'} \\ mf_1^{'} \\ lf_2^{'} \\ mf_2^{'} \end{Bmatrix} = \begin{Bmatrix} -720000 \\ -960000 \\ 720000 \\ 960000 \end{Bmatrix}$$

Example

Nodal Displacements

$$10^8 \begin{bmatrix} 2.88 & 0 \\ 0 & 5.12 \end{bmatrix} \begin{Bmatrix} D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} 720000 \\ 960000 \end{Bmatrix}$$

$$\begin{Bmatrix} D_3 \\ D_4 \end{Bmatrix} = 10^{-4} \begin{Bmatrix} 25 \\ 18.75 \end{Bmatrix} m$$

Example

Element 1

$$\mathbf{d}'_{2\times1} = \mathbf{T}_{2\times4}\mathbf{d}_{4\times1}$$

$$\mathbf{d}'_{2\times1} = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 25\left(10^{-4}\right) \\ 18.75\left(10^{-4}\right) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30\left(10^{-4}\right) \end{Bmatrix} m$$

$$\varepsilon = \frac{d'_2 - d'_1}{L} = \frac{30\times10^{-4}}{5} = 6\times10^{-4}$$

$$\sigma = E\left(\varepsilon - \varepsilon_0\right) = 2\left(10^{11}\right)\left(6\times10^{-4} - 6\times10^{-4}\right) = 0$$

Example

Element 2

$$\mathbf{d}^{'}_{2\times1} = \mathbf{T}_{2\times4}\mathbf{d}_{4\times1}$$

$$\mathbf{d}^{'}_{2\times1} = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 \\ 0 & 0 & 0.6 & -0.8 \end{bmatrix} \begin{Bmatrix} 25\left(10^{-4}\right) \\ 18.75\left(10^{-4}\right) \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} m$$

$$\varepsilon = \frac{d_2^{'} - d_1^{'}}{L} = 0$$

$$\sigma = E\left(\varepsilon - \varepsilon_0\right) = 0$$