# Finite Elements for Engineers

## Lecture 1: Variational Techniques

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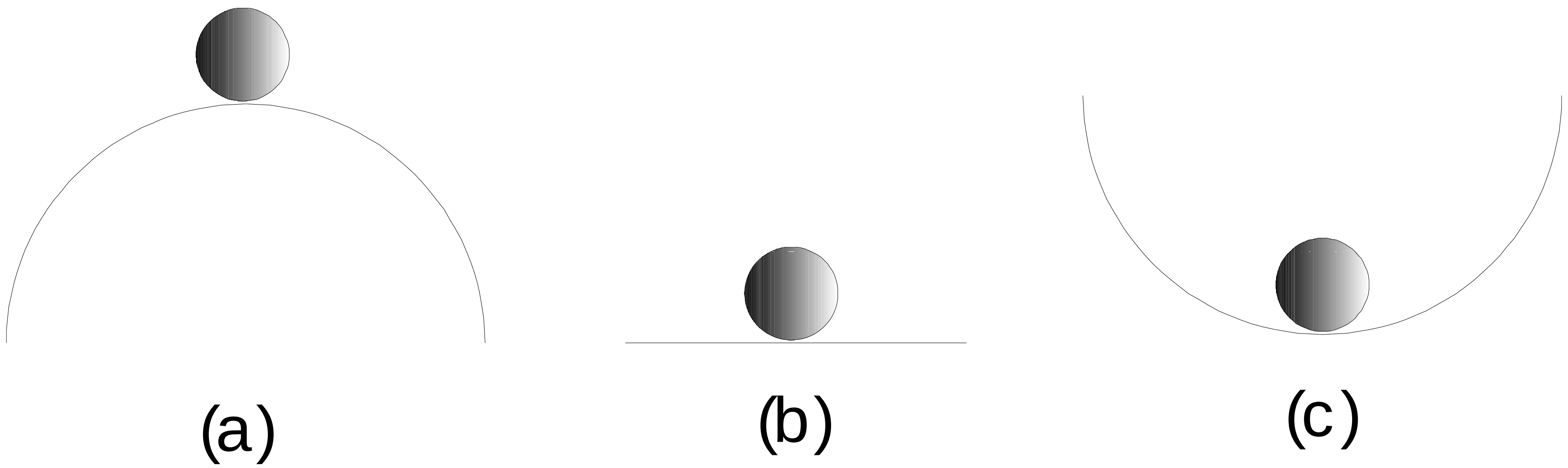
# Introduction

- Generating element equations
  - Direct Stiffness Method: Law plus constitutive relationship
  - Galerkin's Method: PDE
- In Variational Techniques the starting points is an integral equation
- However, these, as usual, are converted into algabraic equations

# Thm. of Min. Potential Energy

**For a <u>conservative</u> system, amongst all <u>admissible configurations</u> those that satisfy the equations of <u>equilibrium</u> make the potential energy stationary with respect to small variations of displacement. If the stationary condition is a <u>minimum</u>, the equilibrium state is <u>stable</u>.**

# Equilibrium States




(a) Unstable (b) Neutral (c) Stable

# Thm. of Min. Potential Energy

$$\Pi = \int_V U_0 \, dV - \int_V \mathbf{f}^T \mathbf{F} \, dV - \int_S \mathbf{f}^T \Phi \, dS - \mathbf{D}^{\mathbf{T}} \mathbf{P}$$

$$\frac{\partial \Pi(\mathbf{D})}{\partial \mathbf{D}} = 0 \quad i = 1,\ 2, \ldots\ n$$

$$U_0 = \frac{1}{2} \{\varepsilon\}^T \mathbf{E} \{\varepsilon\} - \{\varepsilon\}^T \mathbf{E} \{\varepsilon_0\} + \{\varepsilon\}^T \{\sigma_0\}$$

$$\{\sigma\} = \mathbf{E}\{\varepsilon\} - \mathbf{E}\{\varepsilon_0\} + \{\sigma_0\}$$

# Rayleigh-Ritz Technique

- Step 1: Assume a trial solution for the displacement (valid for the entire problem domain). This solution satisfies the boundary conditions.
- Step 2: Write the expression for the total potential energy in terms of the trial solution
- Step 3: Minimize the PE with respect to the undetermined coefs in the trial solution. Solve the resulting equations.

# R-R Technique: Example


**Step 1**

$$u(x) = a_0 + a_1 x \quad 0 < x < L$$

$$u(x=0) = 0 \Rightarrow u(x=0) = 0 = a_0 \Rightarrow u(x) = a_1 x$$

**Step 2**

$$\varepsilon_x = \frac{du}{dx} = a_1$$

$$\Pi(a_1) = \int_V U_0\, dV - PD = \int_0^L \frac{1}{2}(a_1)(E)(a_1)\, A dx - P(a_1 L)$$

$$\Pi(a_1) = \frac{1}{2} a_1^2 EAL - Pa_1 L$$

# Example (cont'd)

**Step 3**

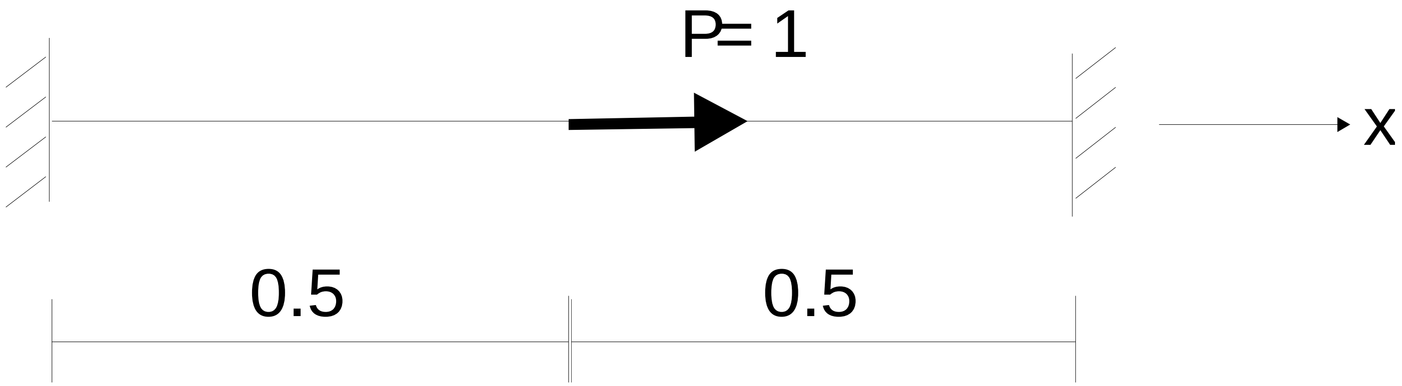
$$\frac{d\Pi}{da_1} = 0 = a_1 EAL - PL \quad \Rightarrow \quad a_1 = \frac{P}{AE}$$

$$u(x) = \frac{Px}{AE}$$

$$\varepsilon_x = \frac{du}{dx} = \frac{P}{AE}$$

$$\sigma = E\varepsilon_x = \frac{P}{A}$$

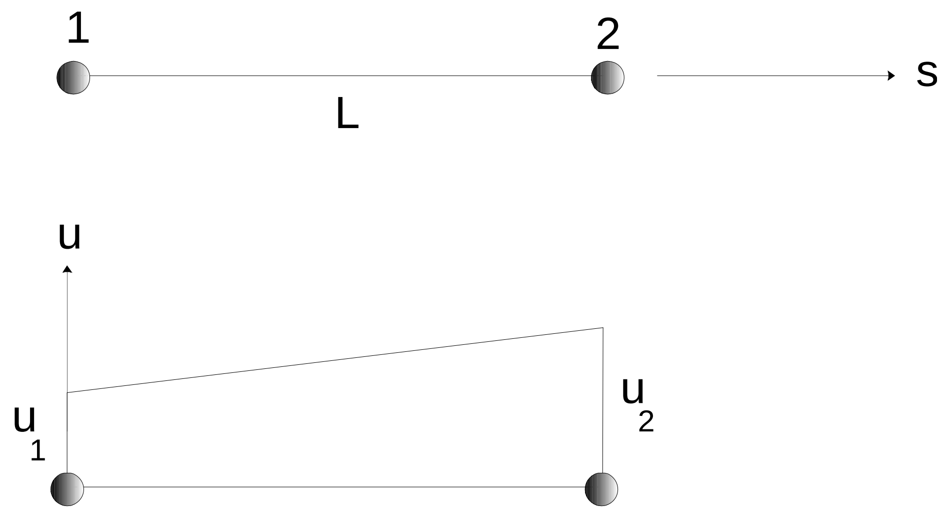
# Example




Can we get the exact (mechanics of materials solution) answer to this problem using the Rayleigh-Ritz Method?

# R-R with (Finite) Elements

**Linear Element**




**Trial Solution**

$$u(s) = a_1 + a_2 s = \frac{L-s}{L} u_1 + \frac{s}{L} u_2 = \phi_1 u_1 + \phi_2 u_2$$

# R-R Finite Elements

## Strain-Displacement Relation

$$\varepsilon = \frac{du}{ds} = -\frac{1}{L}u_1 + \frac{1}{L}u_2 = \frac{1}{L}(u_2 - u_1)$$

$$\varepsilon = \left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor_{1\times 2} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_{2\times 1} = \mathbf{B}_{1\times 2}\mathbf{d}_{2\times 1}$$

## Stress-Strain Relation

$$\sigma = E\varepsilon$$

## Strain Energy

$$U = \int_V U_0 \, dV = \frac{1}{2}\int_V \varepsilon\sigma \, dV = \frac{1}{2}\varepsilon \, E \, \varepsilon \, A L$$

# R-R Finite Elements

**Strain Energy**

$$U = \frac{1}{2}\mathbf{d}_{1\times2}^{T}\mathbf{B}_{2\times1}^{T}\left(EAL\right)_{1\times1}\mathbf{B}_{1\times2}\mathbf{d}_{2\times1}$$

$$U = \frac{1}{2}\mathbf{d}_{1\times2}^{T}\,\mathbf{k}_{2\times2}\,\mathbf{d}_{2\times1}$$

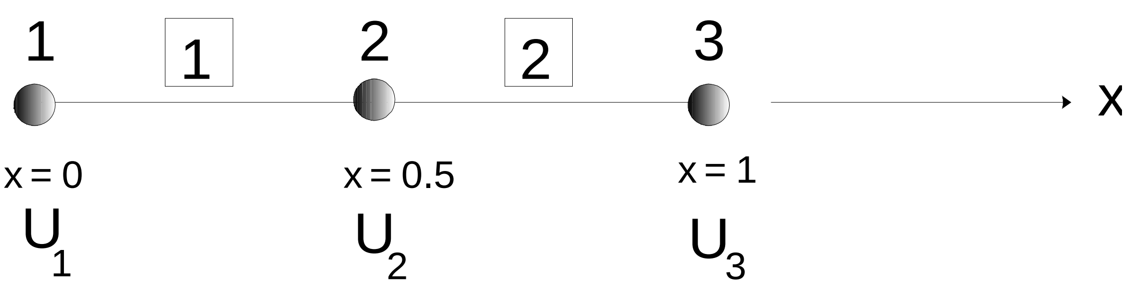
$$k_{2\times2} = \mathbf{B}_{2\times1}^{T}\left(EAL\right)_{1\times1}\mathbf{B}_{1\times2} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

**Total Potential Energy**

$$\Pi_e(\mathbf{d}) = \frac{1}{2}\mathbf{d}_{1\times2}^{T}\,\mathbf{k}_{2\times2}\,\mathbf{d}_{2\times1} + \text{work potential}$$

# Example


**Step 3 (2-Element Solution)**




$$\mathbf{D}_{3\times 1} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$

# Example

**Total P.E.**

$$\Pi(\mathbf{D}) = \frac{1}{2}\mathbf{D}^T_{1\times 3}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{D}_{3\times 1} + \frac{1}{2}\mathbf{D}^T_{1\times 3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix}\mathbf{D}_{3\times 1} - (1)U_2$$

**Stationary Point**

$$\frac{\partial \Pi}{\partial U_1} = 0 = 2U_1 - 2U_2$$

$$\frac{\partial \Pi}{\partial U_2} = 0 = -2U_1 + 4U_2 - 2U_3 - 1 \quad \Rightarrow \quad \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\frac{\partial \Pi}{\partial U_3} = 0 = -2U_2 + 2U_3$$

# Example

**Step 4: BCs** $$U_1 = U_3 = 0$$

**Step 5: Solution** $$4U_2 = 1 \Rightarrow U_2 = 0.25$$

**Step 6: Secondary Quantities**

**Element 1** $$\varepsilon = \frac{du}{ds} = \frac{U_2 - U_1}{L} = \frac{0.25}{0.5} = 0.5 \qquad f = AE\varepsilon = 0.5$$

**Element 2** $$\varepsilon = \frac{du}{ds} = \frac{U_3 - U_2}{L} = \frac{-0.25}{0.5} = -0.5 \qquad f = AE\varepsilon = -0.5$$

# Summary

- Variational Technique as the starting point
  - Theorem of Min. Potential Energy
  - Rayleigh-Ritz Method
  - Element equations exactly the same as those from Direct Stiffness and Galerkin's Method
- The element concept fits in nicely in the usual six-step process