Finite Elements for Engineers

Lecture 3: Eigenvalue Analysis

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Eigenvalue Analysis

Properties

$$\mathbf{K}_{n\times n}\mathbf{\Phi}_{n\times n} = \mathbf{\Lambda}_{n\times n}\mathbf{M}_{n\times n}\mathbf{\Phi}_{n\times n}$$

- **K** is symmetric and positive definite
- **M** is symmetric

$$0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$$

- *n* real eigenvalues

Eigenvalue Analysis

Properties

$$\mathbf{K}\boldsymbol{\varphi}_i = \lambda_i \mathbf{M}\boldsymbol{\varphi}_i$$

$$\boldsymbol{\varphi}_i^T \mathbf{K}\boldsymbol{\varphi}_j = 0$$

$$\boldsymbol{\varphi}_i^T \mathbf{M}\boldsymbol{\varphi}_j = 0$$

$$\boldsymbol{\varphi}_i^T \mathbf{K}\boldsymbol{\varphi}_j = \lambda_i$$

$$\boldsymbol{\varphi}_i^T \mathbf{M}\boldsymbol{\varphi}_i = 1$$

Solution Techniques

Characteristic Polynomial Technique

$$\left[\mathbf{K} - \lambda \mathbf{M}\right]\boldsymbol{\varphi} = 0$$

$$\det\left[\mathbf{K} - \lambda \mathbf{M}\right] = 0$$

The above equation is a polynomial of order ***n***. The roots of the polynomial are the eigenvalues.

Example

$$\mathbf{K}_{3\times3}\mathbf{\Phi}_{3\times3} = \mathbf{\Lambda}_{3\times3}\mathbf{M}_{3\times3}\mathbf{\Phi}_{3\times3}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{K}_{3\times3} - \lambda\mathbf{M}_{3\times3}$$

$$\begin{bmatrix} 3-\lambda & 2 & 1 \\ 2 & 2-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Example

$$\det(\mathbf{K}_{3\times 3} - \lambda \mathbf{M}_{3\times 3}) = 0$$

$$\lambda^3 - 6\lambda^2 + 5\lambda - 1 = 0 \quad \Rightarrow \quad \begin{aligned} \lambda_1 &= 0.308 \\ \lambda_2 &= 0.643 \\ \lambda_3 &= 5.049 \end{aligned}$$

$$\lambda_1 = 0.308$$

$$\begin{bmatrix} 3-0.308 & 2 & 1 \\ 2 & 2-0.308 & 1 \\ 1 & 1 & 1-0.308 \end{bmatrix} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Example

Let $\varphi_3 = 1$

$$\begin{bmatrix} 2.692 & 2 \\ 2 & 1.692 \end{bmatrix} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \begin{Bmatrix} 0.555 \\ -1.247 \end{Bmatrix}$$

Hence

$$\begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \begin{Bmatrix} 0.555 \\ -1.247 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -0.445 \\ 1 \\ -0.802 \end{Bmatrix}$$

Example

$$\mathbf{K}_{3\times3}\mathbf{\Phi}_{3\times3} = \mathbf{\Lambda}_{3\times3}\mathbf{M}_{3\times3}\mathbf{\Phi}_{3\times3}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0.591 & 0.737 & 0.328 \\ -1.328 & -0.409 & 0.263 \\ 1.065 & -0.919 & 0.146 \end{bmatrix} =$$

$$\begin{bmatrix} 0.308 & 0 & 0 \\ 0 & 0.643 & 0 \\ 0 & 0 & 5.049 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.591 & 0.737 & 0.328 \\ -1.328 & -0.409 & 0.263 \\ 1.065 & -0.919 & 0.146 \end{bmatrix}$$

Rayleigh-Ritz Analysis

Consider

$$\mathbf{K}\phi = \lambda \mathbf{M}\phi \qquad \text{$\mathbf{K}$ and $\mathbf{M}$ are positive definite}$$

Rayleigh Minimum Principle

$$\lambda_1 = \min\left(\rho\left(\phi\right)\right) = \min\left(\frac{\phi^{\mathbf{T}}\mathbf{K}\phi}{\phi^{\mathbf{T}}\mathbf{M}\phi}\right) \quad 0 < \lambda_1 \leq \rho\left(\phi\right) \leq \lambda_n < \infty$$

Solution Techniques

Inverse Iteration Method

- Step 1: Assume $\mathbf{u}^0$. Set k=0.
- Step 2: Set k=k+1.
- Step 3: Compute $\mathbf{v}^{k-1} = \mathbf{M}\mathbf{u}^{k-1}$
- Step 4: Solve $\mathbf{K}\hat{\mathbf{u}}^{k} = \mathbf{v}^{k-1}$
- Step 5: Let $\hat{\mathbf{v}}^{k} = \mathbf{M}\hat{\mathbf{u}}^{k}$
- Step 6: Estimate $\lambda^k = \dfrac{\hat{\mathbf{u}}^{k^T}\hat{\mathbf{v}}^{k-1}}{\hat{\mathbf{u}}^{k^T}\hat{\mathbf{v}}^{k}}$

Inverse Iteration

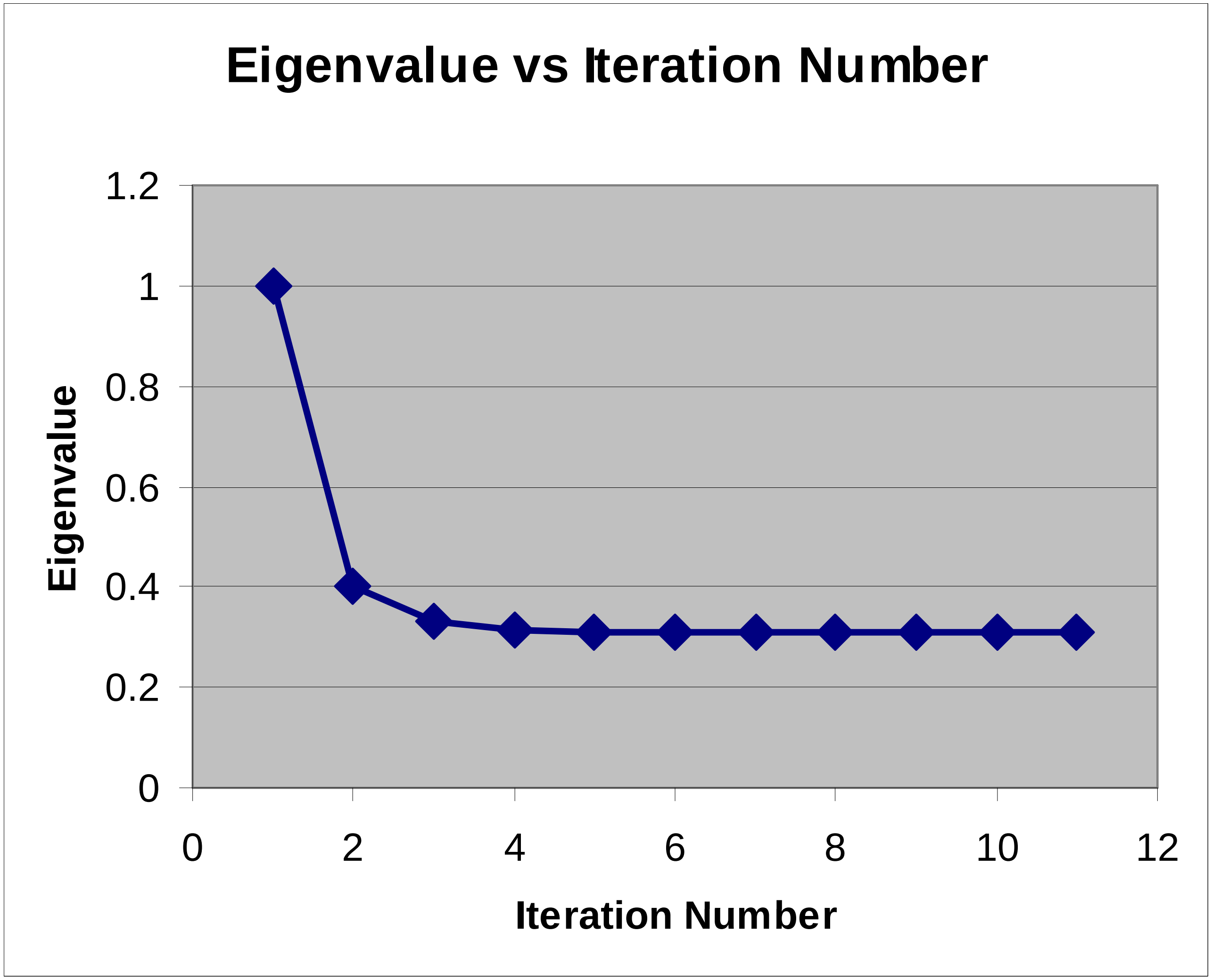
- Step 7: Normalize eigenvector $\mathbf{u}^k = \dfrac{\hat{\mathbf{u}}^k}{\left(\hat{\mathbf{u}}^{k^T}\hat{\mathbf{v}}^k\right)^{1/2}}$
- Step 8: Convergence check $\left|\dfrac{\lambda^k - \lambda^{k-1}}{\lambda^k}\right| \le tolerance$
- Step 9: If not converged, go to Step 2.

Example

$$\mathbf{K}_{3\times3}\mathbf{\Phi}_{3\times3} = \mathbf{\Lambda}_{3\times3}\mathbf{M}_{3\times3}\mathbf{\Phi}_{3\times3}$$

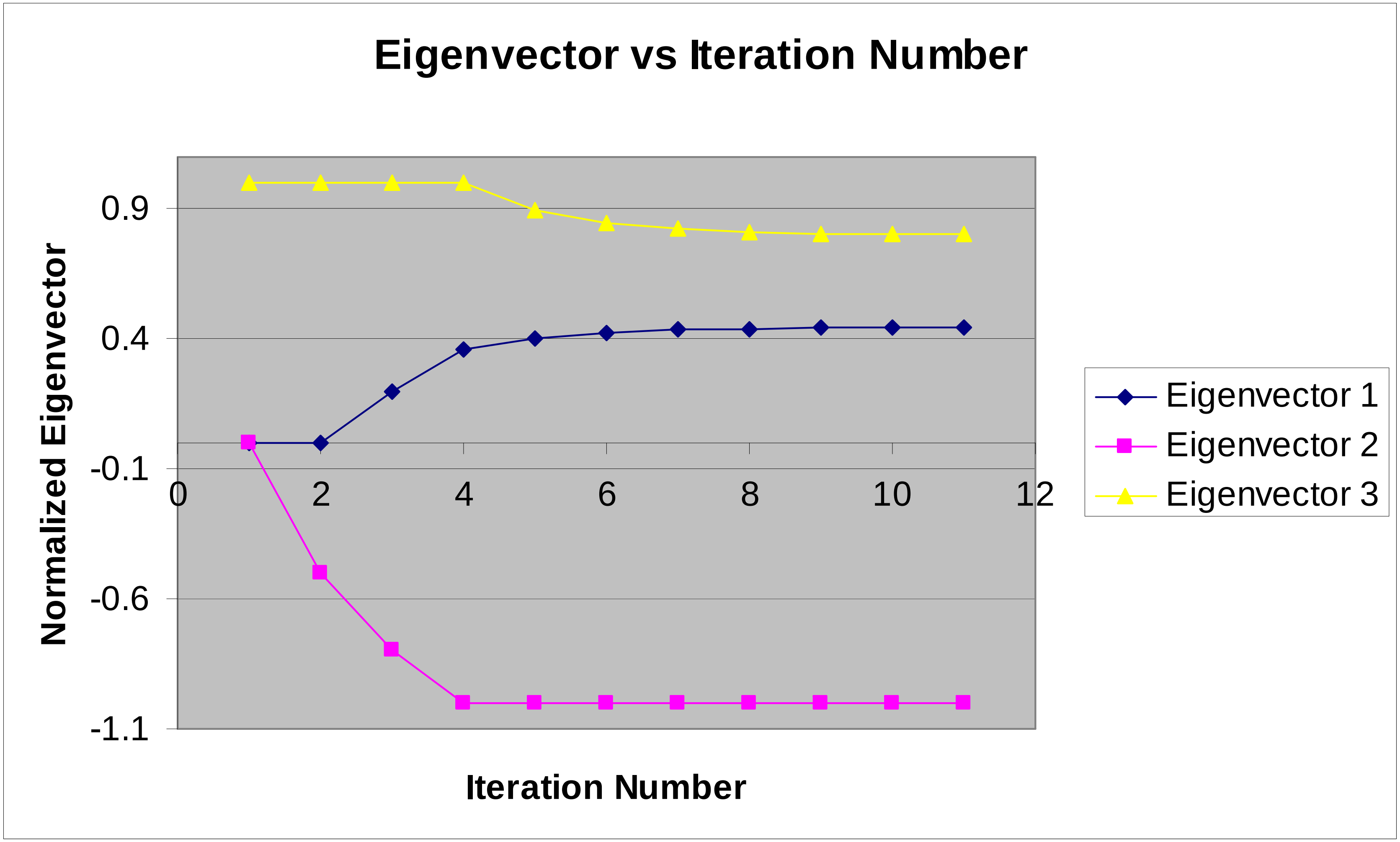
$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} = \mathbf{\Lambda} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix}$$

Example



Eigenvalue vs Iteration Number
1.2
1
0.8
0.6
0.4
0.2
0
Eigenvalue
0
2
4
6
8
10
12
Iteration Number

Example



Eigenvector vs Iteration Number
Normalized Eigenvector
0.9
0.4
-0.1
-0.6
-1.1
0
2
4
6
8
10
12
Iteration Number
Eigenvector 1
Eigenvector 2
Eigenvector 3

Transformation Methods

Recall

$$\boldsymbol{\Phi}^T \mathbf{K} \boldsymbol{\Phi} = \boldsymbol{\Lambda}$$

$$\boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} = \mathbf{I}$$

Iteratively

$$\mathbf{K}_1 = \mathbf{K} \qquad \mathbf{M}_1 = \mathbf{M}$$

$$\mathbf{K}_2 = \mathbf{P}_1^T \mathbf{K}_1 \mathbf{P}_1 \qquad \mathbf{M}_2 = \mathbf{P}_1^T \mathbf{M}_1 \mathbf{P}_1$$

$$\mathbf{K}_3 = \mathbf{P}_2^T \mathbf{K}_2 \mathbf{P}_2 \qquad \mathbf{M}_3 = \mathbf{P}_2^T \mathbf{M}_2 \mathbf{P}_2$$

$$\ldots\ldots \qquad \ldots\ldots$$

$$\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}_k \qquad \mathbf{M}_{k+1} = \mathbf{P}_k^T \mathbf{M}_k \mathbf{P}_k$$

$$\left.\begin{array}{l} \mathbf{K}_{k+1} \to \boldsymbol{\Lambda} \\ \mathbf{M}_{k+1} \to \mathbf{I} \end{array}\right. \quad \text{as } k \to \infty \qquad \boldsymbol{\Phi} = \mathbf{P}_1 \mathbf{P}_2 \ldots \mathbf{P}_l$$

Transformation Methods

In practice

$$\mathbf{K}_{\mathbf{k+1}} \rightarrow diag(K_r) \qquad \text{as } k \rightarrow \infty$$
$$\mathbf{M}_{\mathbf{k+1}} \rightarrow diag(M_r)$$

$$\mathbf{\Lambda} = diag\left(\frac{K_r^{(l+1)}}{M_r^{(l+1)}}\right)$$

$$\mathbf{\Phi} = \mathbf{P}_1\mathbf{P}_2\ldots\mathbf{P}_l\, diag\left(\frac{1}{\sqrt{M_r^{(l+1)}}}\right)$$

Jacobi Method for Standard Eigenproblem

- **K** needs to be symmetric
- Can be used to compute negative, zero or positive eigenvalues

$$\mathbf{K\Phi} = \lambda \mathbf{\Phi}$$

k^th Step

$$\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}_k$$

$$\mathbf{P}_k^T \mathbf{P}_k = \mathbf{I}$$

Threshold Jacobi Method

$$\mathbf{P}_k = \begin{bmatrix} 1 & & & & & & \\ & .. & & & & & \\ & & \cos\theta & & & -\sin\theta & \\ & & & 1 & & & \\ & & & & .. & & \\ & & \sin\theta & & & \cos\theta & \\ & & & & & .. & \\ & & & & & & 1 \end{bmatrix}$$

$$k_{ij}^{(k)} = 0$$

For $k_{ii}^{(k)} \neq k_{jj}^{(k)}$

$$\tan 2\theta = \frac{2k_{ij}^{(k)}}{k_{ii}^{(k)} - k_{jj}^{(k)}}$$

For $k_{ii}^{(k)} = k_{jj}^{(k)}$

$$\theta = \frac{\pi}{4}$$

Applied only if $\left| k_{ij}^{(k)} \right| > tol$

Jacobi Method

$$k_{ij}^{(k+1)} = \left(\sin\theta\cos\theta\right)\left(k_{jj} - k_{ii}\right) + \left(\cos^2\theta - \sin^2\theta\right)k_{ij} = 0$$

Simplifying

$$\left(\sin 2\theta\right)\frac{\left(k_{jj} - k_{ii}\right)}{2} + \left(\cos 2\theta\right)k_{ij} = 0$$

$$\tan 2\theta = \frac{2k_{ij}}{k_{ii} - k_{jj}}$$

Jacobi Method for Generalized Eigenproblem

$$
\mathbf{P}_k = \begin{bmatrix}
1 & & & & & & & \\
& \ddots & & & & & & \\
& & 1 & & & \alpha & & \\
& & & 1 & & & & \\
& & & & \ddots & & & \\
& & \beta & & & 1 & & \\
& & & & & & \ddots & \\
& & & & & & & 1
\end{bmatrix}
$$

See page 384 of the text book on how to obtain these values.

Subspace Iteration Method

- Useful for computing the lowest few eigenpairs.
- Establish q starting iteration vectors, $q > p$ where p is the number of required eigenpairs.
- Use simultaneous inverse iteration and the q vectors and Ritz analysis to extract the "best" eigenpair approximations from the q iteration vectors.
- After convergence, use Sturm Sequence check to verify that the required eigenpairs have been calculated.

Subspace Iteration Method

- Set $q = \min(2p, p+8)$
- Assume initial eigenvector guess $\mathbf{X}_0$
- For $k = 1,2,\ldots.$
- Solve $\mathbf{K}\overline{\mathbf{X}}_{k+1} = \mathbf{M}\mathbf{X}_k$
- Project $\mathbf{K}$ and $\mathbf{M}$ into the subspace

$$\mathbf{K}_{k+1} = \overline{\mathbf{X}}_{k+1}^T \mathbf{K} \overline{\mathbf{X}}_k$$

$$\mathbf{M}_{k+1} = \overline{\mathbf{X}}_{k+1}^T \mathbf{M} \overline{\mathbf{X}}_k$$

- Solve the reduced eigensystem

$$\mathbf{K}_{k+1}\mathbf{Q}_{k+1} = \mathbf{\Lambda}_{k+1}\mathbf{M}_{k+1}\mathbf{Q}_{k+1}$$

- Find an improved approximation $\mathbf{X}_{k+1} = \overline{\mathbf{X}}_{k+1}\mathbf{Q}_{k+1}$

$$\mathbf{\Lambda}_{k+1} \to \mathbf{\Lambda} \text{ and } \mathbf{X}_{k+1} \to \mathbf{\Phi}$$

Shifting

Original problem $\mathbf{K\Phi} = \lambda\mathbf{M\Phi}$

Shift K $\hat{\mathbf{K}} = \mathbf{K} - \rho\mathbf{M}$

New problem $\hat{\mathbf{K}}\mathbf{\Psi} = \mu\mathbf{M\Psi}$

It can be shown that

$$\lambda_i = \rho + \mu_i$$

$$\phi_i = \psi_i$$

Example

Original problem

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \mathbf{\Phi} = \lambda \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{\Phi}$$

$$\det(\mathbf{K} - \lambda \mathbf{M}) = 3\lambda^2 - 18\lambda = 0 \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 6 \end{matrix}$$

New problem $\rho = -2$

$$\begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix} \mathbf{\Phi} = \lambda \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{\Phi}$$

$$\det(\mathbf{K} - \lambda \mathbf{M}) = \lambda^2 - 10\lambda + 16 = 0 \Rightarrow \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 8 \end{matrix}$$

Guyan Reduction

- Computational efficiency is the motivation
- Omit degrees-of-freedom where applied and inertial forces are negligible
- Transform

$$\mathbf{M\ddot{D}} + \mathbf{KD} = \mathbf{F}$$

to

$$\mathbf{KD} = \mathbf{F}$$

by grouping inertial and applied forces

Guyan Reduction

$$\begin{bmatrix} \mathbf{K}_{rr} & \mathbf{K}_{ro} \\ \mathbf{K}_{ro}^T & \mathbf{K}_{oo} \end{bmatrix} \begin{Bmatrix} \mathbf{D}_r \\ \mathbf{D}_o \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_r \\ \mathbf{F}_o \end{Bmatrix} \Rightarrow \mathbf{F}_o \text{ small}$$

Setting $\mathbf{F}_o$ to zero, second equation yields

$$\mathbf{D}_o = -\mathbf{K}_{oo}^{-1}\mathbf{K}_{ro}^T\mathbf{D}_r$$

Strain energy can be expressed as

$$U = \frac{1}{2}\begin{bmatrix} \mathbf{D}_r^T & \mathbf{D}_o^T \end{bmatrix} \begin{bmatrix} \mathbf{K}_{rr} & \mathbf{K}_{ro} \\ \mathbf{K}_{ro}^T & \mathbf{K}_{oo} \end{bmatrix} \begin{Bmatrix} \mathbf{D}_r \\ \mathbf{D}_o \end{Bmatrix}$$

Guyan Reduction

$$U = \frac{1}{2}\mathbf{D}_r^T \mathbf{K}_r \mathbf{D}_r \Rightarrow \mathbf{K}_r = \mathbf{K}_{rr} - \mathbf{K}_{ro}\mathbf{K}_{oo}^{-1}\mathbf{K}_{ro}^T$$

Similarly, kinetic energy can be expressed as

$$V = \frac{1}{2}\dot{\mathbf{D}}^T \mathbf{M}\dot{\mathbf{D}} = \frac{1}{2}\dot{\mathbf{D}}_r{}^T \mathbf{M}_r \dot{\mathbf{D}}_r$$

$$\mathbf{M}_r = \mathbf{M}_{rr} - \mathbf{M}_{ro}\mathbf{K}_{oo}^{-1}\mathbf{K}_{ro}^T - \mathbf{K}_{ro}\mathbf{K}_{oo}^{-1}\mathbf{M}_{ro}^T + \mathbf{K}_{ro}\mathbf{K}_{oo}^{-1}\mathbf{M}_{oo}\mathbf{K}_{oo}^{-1}\mathbf{K}_{ro}^T$$

Guyan Reduction

Solve a smaller system

$$\mathbf{K}_r \mathbf{D}_r = \lambda \mathbf{M}_r \mathbf{D}_r$$

Then recover

$$\mathbf{D}_o = -\mathbf{K}_{oo}^{-1} \mathbf{K}_{ro}^{T} \mathbf{D}_r$$

Further Reading

- From the textbook
  - Chapter 11
- Bathe, *Finite Element Procedures*, Prentice-Hall.