SUPPLEMENTARY NOTES

CEE598 - Finite Elements for Engineers: Module 3

Part 2: Boundary-Value Problems

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MASTER OF ENGINEERING

Finite Elements for Engineers

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Introduction

"Change is inevitable, growth is optional." Anon.

This course is the last one in a three-part series or modules titled "Finite Elements for Engineers" that meets the mathematics requirements for the Master of Engineering (M. Eng.) degree in the College of Engineering at Arizona State University.

Who should take this course?

Finite elements has become the defacto industry standard for solving multi-disciplinary engineering problems that can be described by equations of calculus. Applications cut across several industries by virtue of the applications – solid mechanics (civil, aerospace, automotive, mechanical, biomedical, electronic), fluid mechanics (geotechnical, aerospace, electronic, environmental, hydraulics, biomedical, chemical), heat transfer (automotive, aerospace, electronic, chemical), acoustics (automotive, mechanical, aerospace), electromagnetics (electronic, aerospace) and many, many more.

Course Objectives

- To extend the Isoparametric Formulation to handle three-dimensional FE analysis.
- To understand the basics of the diffusion (or, initial boundary value problems) including eigenvalue analysis using the FE technique.

Prerequisites

- Modules 1 and 2.

Instructor-Student Interaction

To successfully meet the course objectives it is necessary that the students avail themselves of all the resources – discussion forums, e-mail, chat rooms, libraries. Keep the instructor and teaching assistant informed of all your concerns. The web pages connected with this course will contain instructions on how to communicate with the instructor regarding the questions you may have or turning in the assignments etc.

Syllabus

"A man with one watch knows what time it is. A man with two watches is never sure." Segal's Law.

Outline (Lesson Plan)

- Isoparametric Three-Dimensional Elasticity Problems
- Fundamentals of Structural Dynamics
- Plates and Shells

Notation

"As complexity rises, precise statements lose meaning, and meaningful statements lose precision."
Lotfi Zadeh

Vectors

$\mathbf{a}_{n\times 1}$	column vector with n rows
a_i	element i of vector $\mathbf{a}$
$\mathbf{b}_{1\times m}$	row vector with m columns

Matrices

$\mathbf{A}_{m\times n}$	matrix with m rows and n columns
A_{ij}	element row i and column j of matrix

Others

y'	Derivative of y (or, $\frac{dy}{dx}$)
L	(Units of) length
F	(Units of) force
M	(Units of) mass
t	(Units of) time
T	(Units of) temperature
E	(Units of) energy

Lesson Plan

"Problems cannot be solved by the same level of thinking that created them." A. Einstein.

Module 3 is divided into two major topics. Each topic has several lessons designed to focus on the critical issues. With each lesson there is a set of objectives. I have also listed the relevant pages from the list of textbooks that appear in the syllabus. There are several review problems at the end of every topic. Solutions to most problems are also provided. Note that the set of problems represents the minimal set needed to understand the material. You should solve more problems from some of the referenced texts.

Topic 1 looks at extending the isoparametric ideas introduced in Module 2 to handle three-dimensional elasticity problems.

Topic 1: Three-Dimensional BVP

Lesson 1: Introduction.
Lesson 2: Finite Element Formulation.
Review Exercises

Time-dependent problems are of different types. In this topic we will learn how to generate the element equations suitable for eigenvalue analysis and time-dependent analysis. We will also look at numerical schemes for solving these problems.

Topic 2: Diffusion Problem

Lesson 1: Overview.
Lesson 2: Eigenvalue Analysis.
Lesson 3: Diffusion Problems
Lesson 3: Time-Integration Schemes.
Review Exercises

Topic 1: 3D BVP

"The past is of no importance. The present is of no importance. It is with the future that we have to deal. For the past is what man should not have been. The present is what man ought not to be. The future is what artists are." Oscar Wilde

Lesson 1: Introduction

Objectives: In this lesson we will look at three-dimensional boundary-value problems and the different types of three-dimensional finite elements. The major objectives are listed below.

- To understand what is meant by three-dimensional BVP.
- To understand the different types of three-dimensional elements.

Overview: So far we have seen the following classes of problems – one-dimensional, two-dimensional and axisymmetric boundary-value problems. The assumptions behind these problems should be noted so that they can be used to model the appropriate class of problems.

When the structural system is such that (a) the overall dimensions in the three directions cannot be ignored, or in other words, when all the x-y-z dimensions are important, and (b) simplifications cannot be made with respect to the distribution of the unknown, loads and boundary conditions, the problem should be thought of as a three-dimensional boundary-value problem. At a point in the structure, the unknown is expressed as

$$u = u(x, y, z) \qquad \text{(T1L1-1)}$$

Lesson 2: Finite Element Formulation

Objectives: In this lesson we will look at specific elements that can be used to solve three-dimensional BVP. The major objectives are listed below.

- To generate the element equations for different solid elements.
- To solve simple problems using the three-dimensional elements.

The governing differential equation is given as

$$\frac{\partial}{\partial x}\left(\alpha_x(x,y,z)\frac{\partial u(x,y,z)}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha_y(x,y,z)\frac{\partial u(x,y,z)}{\partial y}\right)+\frac{\partial}{\partial z}\left(\alpha_z(x,y,z)\frac{\partial u(x,y,z)}{\partial z}\right)$$

$$+\beta(x,y,z)u(x,y,z)+f(x,y,z)=0 \qquad \text{(T1L2-1)}$$

with the boundary conditions as

$$u(\hat{x},\hat{y},\hat{z})=\hat{u} \quad \text{on } \Gamma_1 \qquad \text{(T1L2-2)}$$

$$\alpha_x\frac{\partial u}{\partial x}n_x+\alpha_y\frac{\partial u}{\partial x}n_y+\alpha_z\frac{\partial u}{\partial z}n_z+gu+c=0 \text{ on } \Gamma_2 \qquad \text{(T1L2-3)}$$

where $\left(n_x,n_y,n_z\right)$ are the direction cosines of the outward normal to the boundary Γ_2, and g and c are constants. We will use the Galerkin's Method to generate the finite element equations necessary to solve this problem.

Let the trial solution be given as

$$\tilde{u}(x,y,z)=\sum_{j=1}^{n}\phi_j(x,y,z)u_j \qquad \text{(T1L2-4)}$$

We will drop the tilde (~) from this point onwards to denote the approximate solution.

Step 1: Compute the residual equations for a typical element domain Ω as

$$\iiint_\Omega R(x,y,z,u)\phi_i(x,y,z)\,dxdydz=0 \qquad i=1,2,\ldots,n \qquad \text{(T1L2-5a)}$$

Or,

$$\iiint_\Omega [\frac{\partial}{\partial x}\left(\alpha_x\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha_y\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\alpha_z\frac{\partial u}{\partial z}\right)+\beta(x,y,z)u(x,y,z)$$

$$+f(x,y,z)]\phi_i(x,y,z)\,dxdydz=0 \qquad i=1,2,\ldots,n \qquad \text{(T1L2-5b)}$$

Step 2: Integrate by parts the highest-order derivative

To achieve this objective we need to use the chain rule of differentiation and the Divergence Theorem.

(A) Chain rule of differentiation

$$\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\right)\phi_i = \frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\phi_i\right) - \left(\alpha_x \frac{\partial u}{\partial x}\right)\frac{\partial \phi_i}{\partial x} \tag{T1L2-6a}$$

$$\frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\right)\phi_i = \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\phi_i\right) - \left(\alpha_y \frac{\partial u}{\partial y}\right)\frac{\partial \phi_i}{\partial y} \tag{T1L2-6b}$$

$$\frac{\partial}{\partial z}\left(\alpha_z \frac{\partial u}{\partial z}\right)\phi_i = \frac{\partial}{\partial z}\left(\alpha_z \frac{\partial u}{\partial z}\phi_i\right) - \left(\alpha_z \frac{\partial u}{\partial z}\right)\frac{\partial \phi_i}{\partial z} \tag{T1L2-6c}$$

(B) Divergence Theorem

Let $F = F(x,y,z)$, $G = G(x,y,z)$ and $H = H(x,y,z)$. Then

$$\iiint_\Omega \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) dxdydz = \iint_\Gamma \left(Fn_x + Gn_y + Hn_z\right) dS \tag{T1L2-7}$$

Using Eqns. (T4L1-5b) and (T1L2-6), we have

$$\iiint_\Omega \left(\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\phi_i\right) - \left(\alpha_x \frac{\partial u}{\partial x}\right)\frac{\partial \phi_i}{\partial x} + \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\phi_i\right) - \left(\alpha_y \frac{\partial u}{\partial y}\right)\frac{\partial \phi_i}{\partial y}\right) dxdydz$$

$$+\iiint_\Omega \left(\frac{\partial}{\partial z}\left(\alpha_z \frac{\partial u}{\partial z}\phi_i\right) - \left(\alpha_z \frac{\partial u}{\partial z}\right)\frac{\partial \phi_i}{\partial z}\right) dxdydz + \iiint_\Omega \left(\beta\, u\, \phi_i + f\, \phi_i\right) dxdydz = 0$$

$$i = 1,2,\ldots,n \tag{T1L2-8}$$

Using Eqn. (T1L2-7), we have

$$-\left(\iiint_\Omega \left\{\frac{\partial u}{\partial x}\alpha_x \frac{\partial \phi_i}{\partial x} + \frac{\partial u}{\partial y}\alpha_y \frac{\partial \phi_i}{\partial y} + \frac{\partial u}{\partial z}\alpha_z \frac{\partial \phi_i}{\partial z}\right\} dxdydz - \iiint_\Omega \left\{\beta\, u \phi_i\right\} dxdydz\right)$$

$$+\iint_\Gamma \left(\alpha_x \frac{\partial u}{\partial x} n_x + \alpha_y \frac{\partial u}{\partial y} n_y + \alpha_z \frac{\partial u}{\partial z} n_z\right)\phi_i\, dS + \iiint_\Omega f\, \phi_i dxdydz = 0 \qquad i = 1,2,\ldots,n \tag{T1L2-9}$$

Rearranging and using Eqn. (T1L2-3)

$$\iiint_\Omega \left\{\frac{\partial u}{\partial x}\alpha_x \frac{\partial \phi_i}{\partial x} + \frac{\partial u}{\partial y}\alpha_y \frac{\partial \phi_i}{\partial y} + \frac{\partial u}{\partial z}\alpha_z \frac{\partial \phi_i}{\partial z} - \beta\, u \phi_i\right\} dxdydz$$

$$+\iint_{\Gamma} \left(g u \phi_i \right) dS = \iiint_{\Omega} f\, \phi_i dxdydz - \iint_{\Gamma} \left(c \phi_i dS \right) \quad i = 1,2,...,n \qquad \text{(T1L2-10)}$$

Step 3: Substitute the trial solution

$$u = \sum_{j=1}^{n} u_j \phi_j \qquad \text{(T1L2-11a)}$$

Hence, $\dfrac{\partial u}{\partial x} = \sum_{j=1}^{n} u_j \dfrac{\partial \phi_j}{\partial x}, \dfrac{\partial u}{\partial y} = \sum_{j=1}^{n} u_j \dfrac{\partial \phi_j}{\partial y}$ and $\dfrac{\partial u}{\partial z} = \sum_{j=1}^{n} u_j \dfrac{\partial \phi_j}{\partial z}$ (T1L2-11b)

Substituting these in Eqn. (T1L2-10) we have

$$\sum_{j=1}^{n} \left(\iiint_{\Omega} \left\{ \frac{\partial \phi_i}{\partial x} \alpha_x \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \alpha_y \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z} \alpha_z \frac{\partial \phi_j}{\partial z} - \phi_i \beta \phi_j \right\} dxdydz + \iint_{\Gamma} \phi_i g \phi_j dS \right) u_j =$$

$$\iiint_{\Omega} f\, \phi_i dxdydz - \iint_{\Gamma} \left(c \phi_i dS \right) \quad i = 1,2,...,n \qquad \text{(T1L2-12)}$$

We can write the element equations in the matrix form as

$$\left[\mathbf{k}_{n\times n}^{\alpha} + \mathbf{k}_{n\times n}^{\beta} + \mathbf{k}_{n\times n}^{g} \right] \mathbf{u}_{n\times 1} = \mathbf{f}_{n\times 1}^{\text{int}} + \mathbf{f}_{n\times 1}^{bnd} \qquad \text{(T1L2-13)}$$

where

$$k_{ij}^{\alpha} = \iiint_{\Omega} \left\{ \frac{\partial \phi_i}{\partial x} \alpha_x \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \alpha_y \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z} \alpha_z \frac{\partial \phi_j}{\partial z} \right\} dxdydz \qquad \text{(T1L2-14a)}$$

$$k_{ij}^{\beta} = -\iiint_{\Omega} \phi_i \beta \phi_j dxdydz \qquad \text{(T1L2-14b)}$$

$$k_{ij}^{g} = \iint_{\Gamma} \phi_i g \phi_j dS \qquad \text{(T1L2-14c)}$$

$$f_i^{\text{int}} = \iiint_{\Omega} f\, \phi_i dxdydz \qquad \text{(T1L2-14d)}$$

$$f_i^{bnd} = -\iint_{\Gamma} \left(c \phi_i dS \right) \qquad \text{(T1L2-14e)}$$

We now have all the ingredients to compute the element equations for different types of solid elements. The process is no different than the one we saw for the two-dimensional BVP.

Computation of the Element Convective Stiffness Matrix and Load Vector for Solid Elements

Recall that

(a) the convective stiffness matrix can be computed using the following equation

$$k_{ij}^{g} = \iint_{\Gamma} \phi_i g \phi_j dS \qquad \text{(T1L2-14c)}$$

(b) the boundary flux can be computed using the following equation

$$f_i^{bnd} = -\iint_{\Gamma} \left(c\phi_i dS \right) \qquad \text{(T1L2-14e)}$$

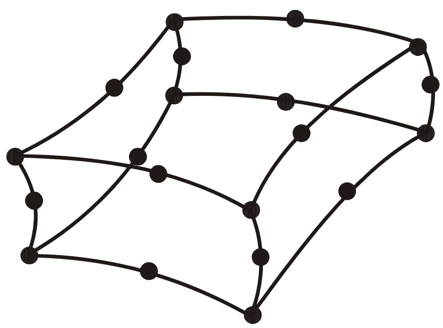
The basic idea in the evaluation of the two integrals is to use integrals over parametrized surfaces. Note that a surface is an object in 3-dimensional space that *locally* looks like a plane. In other words, the surface is given by a vector-valued function $\mathbf{P}$ (describing the *x*, *y*, and *z* coordinates of points on the surface) that can be expressed as a function of *two* parameters, say ξ and η. The key idea behind all the computations is summarized in the formula

$$\frac{\partial P}{\partial \xi} \times \frac{\partial P}{\partial \eta} d\xi\, d\eta = \mathbf{n}\, dS \qquad \text{(T1L2-15)}$$

where $\mathbf{n}$ is the outward unit normal to the surface S. Since $\mathbf{P}$ is vector-valued, $\dfrac{\partial P}{\partial \xi}$ and $\dfrac{\partial P}{\partial \eta}$ are vectors. Their cross-product is a vector with two important properties: it is normal to the surface parametrized by $\mathbf{P}$, and its length gives the scale factor between area in the parameter space and the corresponding area on the surface. Thus, taking lengths on both sides of the above formula above gives

$$\left\| \frac{\partial P}{\partial \xi} \times \frac{\partial P}{\partial \eta} \right\| d\xi\, d\eta = dS \qquad \text{(T1L2-16)}$$

Eqn. (T1L2-16) enables us to compute the area of a parametrized surface, or to integrate any function along the surface with respect to the surface area. Consider the 20-noded hexahedral element as shown in Fig. 1 below.



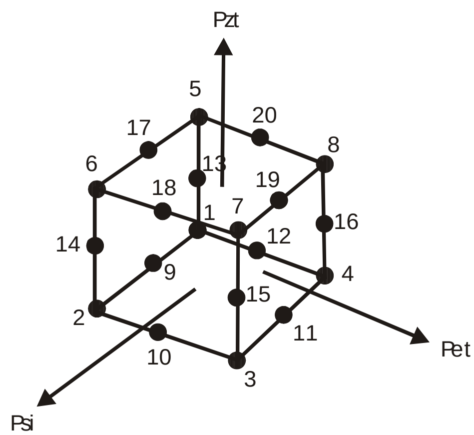


Fig. 1 20-Noded hexahedral "serendipity" element

Convective element loading: Consider the face 1-9-2-12-10-4-11-3 where $\zeta = -1$. Assuming that c is a constant we have

$$f_i^{bnd} = -\iint_\Gamma \left(c\phi_i dS \right) = -c \iint_\Gamma \phi_i \left\| \frac{\partial P}{\partial \xi} \times \frac{\partial P}{\partial \eta} \right\| d\xi \, d\eta \tag{T1L2-17}$$

where

$$P = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad \frac{\partial P}{\partial \xi} = \begin{Bmatrix} J_{11} \\ J_{12} \\ J_{13} \end{Bmatrix} \qquad \frac{\partial P}{\partial \eta} = \begin{Bmatrix} J_{21} \\ J_{22} \\ J_{23} \end{Bmatrix} \tag{T1L2-18}$$

$$\mathbf{J}_{3\times3} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ 0 & 0 & 0 \end{bmatrix}_{3\times3} \tag{T1L2-19}$$

where J_{ij} represents the components of $\mathbf{J}_{3\times3}$ (jacobian matrix) and $\| \; \|$ represents the length of the vector. Using numerical integration, we have

$$f_i^{bnd} = -\sum_{k=1}^{n} \sum_{l=1}^{n} w_k w_l \, c \left[\phi_i \left\| \frac{\partial P}{\partial \xi} \times \frac{\partial P}{\partial \eta} \right\| \right]_{(\xi_k, \eta_l)} \tag{T1L2-20}$$

The above equation can be suitably modified for integration over tetrahedral domains.

Convective Stiffness Matrix: In a similar manner we can develop the convective stiffness matrix for face 1-9-2-12-10-4-11-3 where $\zeta = -1$. Assuming that g is a constant we have

$$k_{ij}^{g} = \iint_{\Gamma} \phi_i g \phi_j dS = g \iint_{\Gamma} \phi_i \phi_j \left\| \frac{\partial P}{\partial \xi} \times \frac{\partial P}{\partial \eta} \right\| d\xi \, d\eta \qquad \text{(T1L2-21)}$$

Using numerical integration, we have

$$k_{ij}^{g} = \sum_{k=1}^{n} \sum_{l=1}^{n} w_k w_l g \left[\phi_i \phi_j \left\| \frac{\partial P}{\partial \xi} \times \frac{\partial P}{\partial \eta} \right\| \right]_{(\xi_k, \eta_l)} \qquad \text{(T1L2-22)}$$

As before, the above equation can be suitably modified for integration over tetrahedral domains.

Review Exercises

Problem T1L1-1

(i) Derive the shape functions for the (a) 8-noded and (b) 27-noded hexahedral elements.

(ii) Derive the shape functions for the (a) 4-noded and (b) 10-noded tetrahedral elements.

Problem T1L2-1

Derive the stiffness matrices for the elements listed in Problem T1L1-1.

Topic 2: Diffusion Problems

"I adore simple pleasures. They are the last refuge of the complex." Oscar Wilde

Lesson 1: Overview

Objectives: In this lesson we will look at eigenproblems.

- To understand what is meant by eigenproblems.
- To understand and apply the finite element procedure to solve these problems.

One-Dimensional Eigenproblem

The governing differential equation is of the form

$$-\frac{d}{dx}\left\{\alpha(x)\frac{du(x)}{dx}\right\}+\beta(x)u(x)-\lambda\gamma(x)u(x)=0 \qquad x_a<x<x_b \tag{T2L1-12}$$

with the boundary conditions as

$$\text{At } x_a:\ u(x_a)=0 \quad \text{or} \quad \tau(x_a)=0 \tag{T2L1-13a}$$

$$\text{At } x_b:\ u(x_b)=0 \quad \text{or} \quad \tau(x_b)=0 \tag{T2L1-13b}$$

A few points are in order when we compare this differential equation to the one-dimensional BVP. First, there is no driving force, i.e. $f(x)=0$. Second, there is an additional term, $-\lambda\gamma(x)u(x)$ where $\gamma(x)$ describes a physical property of the system (usually mass or mass density) and the scalar λ is called the eigenvalue. Third, there are several solutions called eigensolutions to this problem. The eigensolutions consist of pairs of eigenfunction $u(x)$ and eigenvalue λ. Both of these are unknowns. Lastly, in the absence of driving forces, the condition of the system changes. This is the resonant or natural state where the internal energy oscillates back and forth between different forms e.g. kinetic and potential, without energy exchange with the surroundings.

Once again we will use the Galerkin's Approach to solve the problem.

Step 1: Residual Equations

In a typical element with the approximate solution as $u=\tilde{u}(x)$ (dropping the tilde notation for convenience)

$$\int_\Omega\left[-\frac{d}{dx}\left\{\alpha(x)\frac{du(x)}{dx}\right\}+\beta(x)u(x)-\lambda\gamma(x)u(x)\right]\phi_i(x)\,dx=0 \qquad i=1,2,\ldots,n \tag{T2L1-14}$$

Step 2: Integrate by parts the highest order derivative

$$\int_\Omega\frac{d\phi_i(x)}{dx}\alpha(x)\frac{d\tilde{u}}{dx}dx+\int_\Omega\phi_i(x)\beta(x)\tilde{u}\,dx-\lambda\int_\Omega\phi_i(x)\gamma(x)\tilde{u}\,dx=$$

$$-\left[\left\{-\alpha(x)\frac{d\tilde{u}}{dx}\right\}\phi_i(x)\right]_{x_1}^{x_n} \tag{T2L1-15}$$

where x_1 and x_n are the coordinates of the ends of the element. The last term must vanish since the boundary conditions either are essential or homogenous (meaning zero valued) natural BC's (see T2L1-13).

Step 3: Trial solution

Let the trial solution be represented as $\tilde{u}(x,a) = \sum_{j=1}^{n} a_j \phi_j(x)$. Hence

$$\sum_{j=1}^{n}\left\{\int_{\Omega}\frac{d\phi_i(x)}{dx}\alpha(x)\frac{d\phi_j(x)}{dx}\,dx + \int_{\Omega}\phi_i(x)\beta(x)\phi_j(x)dx\right\}a_j$$

$$-\lambda\sum_{j=}^{n}\left\{\int_{\Omega}\phi_i(x)\gamma(x)\phi_j(x)\,dx\right\}a_j = 0 \qquad i = 1,2,\ldots,n \qquad \text{(T2L1-16)}$$

Writing the above equation in a compact form

$$\mathbf{k}_{n\times n}\mathbf{a}_{n\times 1} - \lambda\mathbf{m}_{n\times n}\mathbf{a}_{n\times 1} = \mathbf{0} \qquad \text{(T2L1-17)}$$

Step 4: Element equations for the $1D - C^0$ *linear element*

Considering the C^0 linear element we have

$$\phi_1(x) = \frac{x_2 - x}{x_2 - x_1} \qquad \text{and} \qquad \phi_2(x) = \frac{x - x_1}{x_2 - x_1} \qquad \text{(T2L1-18)}$$

The terms in $\mathbf{k}$ were evaluated in Module 1. We will handle the mass matrix here.

$$m_{11} = \int_{x_1}^{x_2}\phi_1(x)\gamma(x)\phi_1(x)\,dx = \int_{x_1}^{x_2}\frac{x_2 - x}{x_2 - x_1}\gamma(x)\frac{x_2 - x}{x_2 - x_1}dx = \frac{\bar{\gamma}L}{3} = m_{22}$$

$$m_{12} = \int_{x_1}^{x_2}\phi_1(x)\gamma(x)\phi_2(x)\,dx = \frac{\bar{\gamma}L}{6} = m_{21}$$

Hence

$$\mathbf{m}_{2\times 2} = \frac{\bar{\gamma}L}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

There is no flux to compute since we are solving an eigenproblem. Once the element equations are assembled into the system equations, we obtain the system eigenproblem as

$$\mathbf{K\Phi} = \mathbf{\Lambda M \Phi} \qquad \text{(T2L1-19)}$$

that can then be solved for the eigenvalues $\mathbf{\Lambda}$ and the corresponding eigenvectors $\mathbf{\Phi}$.

Lesson 2: Diffusion Problems

Objectives: In this lesson we will look at the initial-boundary value problems. The major objectives are listed below.

- To understand what is meant by IBVP.
- To derive the element equations for an IBVP.

Overview

Let us first look at a pure initial value problem. The differential equation is given as

$$c\frac{du(t)}{dt} + ku(t) = f(t) \qquad t > t_0 \tag{T2L2-1}$$

with the initial condition as

$$u(t_0) = u_0 \tag{T2L2-2}$$

This is an example of a first-order DE. Unlike the 1D BVP, the initial condition is required only at $t = t_0$.

Free Response: The free response corresponds to the case when $f(t) = 0$. Hence

$$c\frac{du(t)}{dt} + ku(t) = 0 \tag{T2L2-3}$$

The solution to the free response problem is of the form

$$u(t) = Ae^{-\lambda t} \tag{T2L2-4}$$

Substituting (T2L3-4) into (T2L3-3), we have $A(-c\lambda + k)^{e^{-\lambda t}} = 0$. Or, $k - \lambda c = 0$ and

$$\lambda = \frac{k}{c} \tag{T2L2-5}$$

This is the characteristic value or eigenvalue. Substituting (T2L2-5) into (T2L2-4)

$$u(t) = Ae^{-(k/c)t} \tag{T2L2-6}$$

The constant A is determined by using the initial condition (Eqn. T2L3-2)

$$u(t) = u_0 e^{-(k/c)(t-t_0)} \tag{T2L3-7}$$

Graphing the function will indicate the exponential decay characteristic of the solution.

Steady-State Load: If the applied load has a steady state value, $f \equiv \text{constant}$, then

$$u(t) = \frac{f}{k} + \left(u_0 - \frac{f}{k}\right)e^{-(k/c)(t-t_0)} \tag{T2L2-8}$$

As $t \to \infty$, $u(t) \to \dfrac{f}{k}$, the steady-state solution corresponding to $\dfrac{du}{dt} = 0$ in Eqn. (T2L2-1). A few notes are in order. First, if the loads are held steady for any interval of time, the solution decays exponentially during that interval. Second, if they are held indefinitely, the solution ultimately settles down exponentially to a final equilibrium state (steady-state). These are the characteristics of the diffusion problem. When multiple degrees-of-freedom, $u_1(t), u_2(t), ... u_n(t)$, are involved then

$$\mathbf{c}\left\{\frac{du(t)}{dt}\right\} + \mathbf{k}\{u(t)\} = \{f(t)\} \qquad \text{(T2L2-9)}$$

The solution to this problem must be obtained using a time-stepping scheme.

Mixed Initial BVP

The governing equation for a one-dimensional problem involving spatial and time variables is

$$\mu(x)\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u(x,t)}{\partial t}\right) + \beta(x)u(x,t) = f(x,t) \qquad \text{(T2L2-10)}$$

with the problem domain being

$$x_a < x < x_b \quad \text{and} \quad t > t_0 \qquad \text{(T2L2-11)}$$

The boundary conditions are

$$\text{For } t > t_0, \quad u(x_a, t) = u_a(t) \qquad \text{(T2L2-12a)}$$

$$\left[-\alpha(x)\frac{\partial u(x,t)}{\partial x}\right]_{x_a} = \tau_a(t) \qquad \text{(T2L2-12b)}$$

with similar conditions at $x = x_b$. The initial conditions are

$$\text{At } t_0 \ (x_a < x < x_b) \quad u(x, t_0) = u_0(x) \qquad \text{(T2L2-13)}$$

A few notes about the problem formulation. First, the terms are the same as a typical BVP except for the additional term - $\mu(x)\dfrac{\partial u(x,t)}{\partial t}$. Second, the derivatives are partial derivatives since $u = u(x,t)$. Third, the interior load $f = f(x,t)$. Fourth, the boundary conditions are functions of both x and t. Lastly, though this problem appears to be a two-dimensional problem involving space and time, only the spatial variable is treated using finite elements and the time domain is handled using finite difference.

Element Equations

We will use the Galerkin's Method to generate the element equations. Using the concept of separation of variables (Method of Kantorovich), the assumed solution is of the form

$$u(x,t,a) = \sum_{j=1}^{n} a_j(t)\phi_j(x) \tag{T2L2-14}$$

This form has certain advantages. First, if $u = \sum_j a_j \phi_j(x,t)$, then the steady state solution must be computed first. Second, the same procedure used to solve a BVP can be used since $a_j(t)$ is treated the same way as a_j in BVP. Hence the final resulting equations are ordinary differential equations rather than algebraic equations. In other words, the FE procedure transforms the IBVP into a pure initial value problem.

Step 1: Residual Equations

$$\int_\Omega \left[\mu(x)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right) + \beta(x)u - f(x,t) \right]\phi_i(x)dx = 0 \quad i = 1,2,\ldots,n \tag{T2L2-15}$$

Step 2: Integrate by parts

$$\int_\Omega \phi_i(x)\mu(x)\frac{\partial u}{\partial t}\,dx + \int_\Omega \frac{d\phi_i}{dx}\alpha(x)\frac{\partial u}{\partial x}\,dx + \int_\Omega \phi_i(x)\beta(x)u(x,t,a)\,dx$$

$$= \int_\Omega f(x,t)\phi_i(x)dx - \left[\left(-\alpha(x)\frac{\partial u}{\partial x}\right)\phi_i(x)\right]_{x_1}^{x_n} \tag{T2L2-16}$$

Step 3: Substitute trial functions

$$\mathbf{c}\left\{\frac{da(t)}{dt}\right\} + \mathbf{k}\{a(t)\} = \{f(t)\} \tag{T2L2-17}$$

where

$$c_{ij} = \int_\Omega \phi_i(x)\mu(x)\phi_j(x)\,dx \text{ is the capacity matrix} \tag{T2L2-18}$$

$$k_{ij} = \int_\Omega \frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}\,dx + \int_\Omega \phi_i\,\beta(x)\phi_j\,dx \tag{T2L2-19}$$

$$f_i(t) = \int_{\Omega} f(x,t)\phi_i(x)dx - [\tau\phi_i]_{x_1}^{x_n} \tag{T2L2-20}$$

Step 4: We will now derive the equations for the $1D - C^0$ element

The capacity matrix is similar to the mass matrix derived earlier and

$$\mathbf{c} = \frac{\hat{\mu} L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{T2L2-21}$$

where $\hat{\mu} = \mu(x_c) = \mu(\frac{x_1 + x_2}{2})$ (T2L2-22)

Step 5: The assembly of the element equations will yield

$$\mathbf{C}\left\{\frac{da(t)}{dt}\right\} + \mathbf{K}\{a(t)\} = \{F(t)\} \tag{T2L2-23}$$

The system of ODE can be solved for $\{a(t)\}$ using time-stepping methods.

Heat Transfer Analysis

The governing differential equation for a time-dependent heat transfer analysis can be generalized from Eqn. (T2L2-10) where

(a) $\mu(x) = \rho(x)c(x)$ with $\rho(x)$ is mass density and $c(x)$ is specific heat.

(b)

Lesson 3: Time-Integration Scheme

Objectives: In this lesson we will look at the forced vibration problem. The major objectives are listed below.

- To understand what is meant by Time-Integration Schemes.
- To learn the simple numerical techniques to solve for time-dependent response of structural systems.

Numerical Techniques

The numerical techniques used to solve Eqn. (T2L2-23) are known as time-stepping methods. The solution techniques are either *conditionally stable* or *unconditionally stable*. The former implies that the solution will diverge unless the time-step used is less than a certain value. The latter implies that the solution will converge irrespective of the chosen time step – this however does not imply that the answers are correct or acceptable!

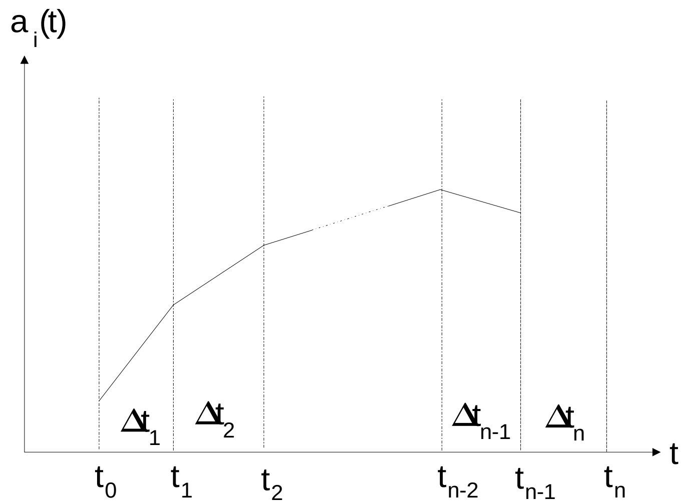


Fig. T2L3-1 Notation used for recurrence relations

The time-stepping methods have the following characteristics.

(1) The time domain is discretized beginning at t_0 with uniform or nonuniform time steps Δt_i.

(2) Instead of obtaining $a = a(t)$ over the continuous domain of t, discrete values for $a(t)$ are obtained for different t_i.

(3) Recurrence relations are used to obtain $\{a\}_n, n = 1,2,\ldots$. These relations are approximations to the differential equations.

Linear One-Step Method

The term linear indicates the nature of the recurrence relation. In this method, for the n^{th} step

$$\mathbf{P}\mathbf{a}_n + \mathbf{Q}\mathbf{a}_{n-1} = p\mathbf{F}_n + q\mathbf{F}_{n-1} \qquad \text{(T2L3-1)}$$

where $\mathbf{a}_n$ is the vector of unknowns and the rest of the quantities are known. For the first step $(n = 1)$

$$\mathbf{Pa}_1 = p\mathbf{F}_1 + q\mathbf{F}_0 - \mathbf{Qa}_0 \qquad \text{(T2L3-2)}$$

where $\mathbf{a}_0$ represents the initial conditions. The equations are solved for $\mathbf{a}_1$. Similarly, for the second step $(n=2)$

$$\mathbf{Pa}_2 = p\mathbf{F}_2 + q\mathbf{F}_1 - \mathbf{Qa}_1 \qquad \text{(T2L3-3)}$$

The equations are solved for $\mathbf{a}_2$. The process is repeated for all the time steps.

One-Step Methods

There are several one-step methods that we will take a look at. Some of these are the Backward Difference Method (Backward Euler Rule), Mid-Difference Method (Crank-Nicolson Method), Forward Difference Method (Euler's Rule) and the $\theta-$Method. The finite difference methods, as the aforementioned methods, are generally known as will be discussed next.

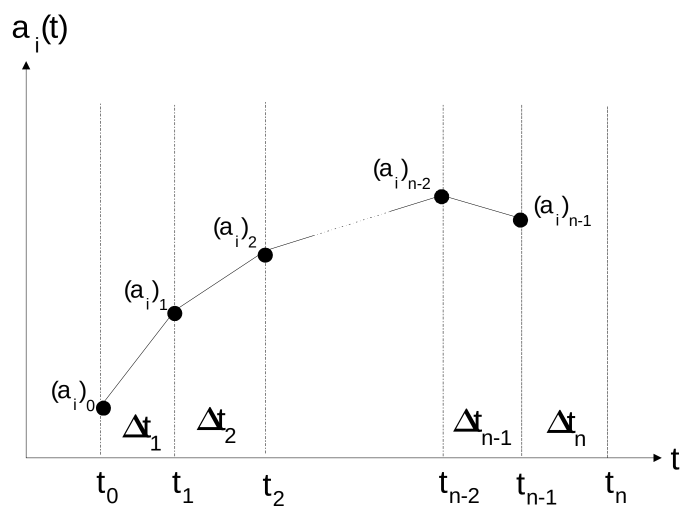


Fig. T2L3-2 Finite difference notation (solution known at t_{n-1}, compute solution at t_n)

Backward Difference

All quantities are evaluated at *forward* end of time step.

$$\mathbf{C}\left\{\frac{da}{dt}\right\}_n + \mathbf{K}\{a\}_n = \mathbf{F}_n \qquad \text{(T2L3-4)}$$

The time derivative in the above equation is approximated by a backward difference

$$\left\{\frac{da}{dt}\right\}_n = \frac{\{a\}_n - \{a\}_{n-1}}{\Delta t_n} \text{ where } \Delta t_n = t_n - t_{n-1} \qquad \text{(T2L3-5)}$$

For the i^{th} degree-of-freedom

$$\left(\frac{da_i}{dt}\right)_n = \frac{(a_i)_n - (a_i)_{n-1}}{\Delta t_n} \qquad i = 1,2,.... \tag{T2L3-6}$$

Substituting Eqn. (T2L3-5) into (T2L3-4)

$$\left(\frac{1}{\Delta t_n}\mathbf{C} + \mathbf{K}\right)\mathbf{a}_n = \mathbf{F}_n + \frac{1}{\Delta t_n}\mathbf{C}\,\mathbf{a}_{n-1} \tag{T2L3-7}$$

Comparing with the general form of the recurrence relation

$$\mathbf{P} = \frac{1}{\Delta t_n}\mathbf{C} + \mathbf{K}\,;\ \mathbf{Q} = -\frac{1}{\Delta t_n}\mathbf{C}\,;\ p = 1;\ \ q = 0 \tag{T2L3-8}$$

Rewriting Eqn. (T2L3-7) we have

$$\mathbf{K}_{eff}\ \mathbf{a}_n = \mathbf{F}_{eff} \tag{T2L3-9}$$

indicating that the method is IMPLICIT because the unknowns are coupled. The accuracy of the method is $O(\Delta t)$.

Mid-Difference

All quantities are evaluated at *center* of time step.

$$\mathbf{C}\left\{\frac{da}{dt}\right\}_{n-1/2} + \mathbf{K}\{a\}_{n-1/2} = \mathbf{F}_{n-1/2} \tag{T2L3-10}$$

The time derivative in the above equation is approximated as

$$\left\{\frac{da}{dt}\right\}_{n-1/2} = \frac{\{a\}_n - \{a\}_{n-1}}{\Delta t_n} \text{ where } \Delta t_n = t_n - t_{n-1} \tag{T2L3-11}$$

$$\{a\}_{n-1/2} = \frac{\{a\}_{n-1} + \{a\}_n}{2} \tag{T2L3-12}$$

Substituting

$$\left(\frac{1}{\Delta t_n}\mathbf{C} + \frac{1}{2}\mathbf{K}\right)\mathbf{a}_n = \mathbf{F}_{n-1/2} + \left(\frac{1}{\Delta t_n}\mathbf{C} - \frac{1}{2}\mathbf{K}\right)\mathbf{a}_{n-1} \tag{T2L3-13}$$

where $\mathbf{F}_{n-1/2} = \dfrac{\mathbf{F}_{n-1} + \mathbf{F}_n}{2}$ (T2L3-14)

or, $\mathbf{K}_{eff}\ \mathbf{a}_n = \mathbf{F}_{eff}$ (T2L3-15)

indicating that the method is IMPLICIT because the unknowns are coupled. The accuracy of the method is $O(\Delta t^2)$.

Forward Difference

All quantities are evaluated at *backward* end of time, t_{n-1}.

$$\mathbf{C}\left\{\frac{da}{dt}\right\}_{n-1} + \mathbf{K}\{a\}_{n-1} = \mathbf{F}_{n-1} \qquad \text{(T2L3-16)}$$

The time derivative in the above equation is approximated as

$$\left\{\frac{da}{dt}\right\}_{n-1/2} = \frac{\{a\}_n - \{a\}_{n-1}}{\Delta t_n} \text{ where } \Delta t_n = t_n - t_{n-1} \qquad \text{(T2L3-17)}$$

Substituting

$$\left(\frac{1}{\Delta t_n}\mathbf{C}\right)\mathbf{a}_n = \mathbf{F}_{n-1} + \left(\frac{1}{\Delta t_n}\mathbf{C} - \mathbf{K}\right)\mathbf{a}_{n-1} \qquad \text{(T2L3-18)}$$

or, $\mathbf{K}_{eff}\ \mathbf{a}_n = \mathbf{F}_{eff}$ (T2L3-19)

Only $\mathbf{C}$ appears on the left-hand side and it can be diagonalized. Hence the equations can be uncoupled and the method is EXPLICIT. The accuracy of the method is usually $O(\Delta t)$.

One way of lumping the capacity matrix is to use

$$CL_{ii} = \sum_{j=1}^{n} C_{ij} \quad i = 1,2...,n \qquad \text{(T2L3-20a)}$$

and $CL_{ij} = 0 \qquad i \neq j$ (T2L3-20b)

With this lumping technique, substituting Eqn. (T2L3-20) in (T2L3-18)

$$\mathbf{a}_n = \mathbf{a}_{n-1} + \Delta t_n [CL]^{-1}(\mathbf{F}_{n-1} - \mathbf{K}\mathbf{a}_{n-1}) \qquad \text{(T2L3-21)}$$

where $$[CL]^{-1} = \begin{bmatrix} 1/CL_{11} & & & \\ & 1/CL_{22} & & \\ & & .. & \\ & & & 1/CL_{nn} \end{bmatrix} \quad \text{(T2L3-22)}$$

The θ -Method

This method is the most general and the other methods discussed before are special cases.

$$\mathbf{C}\left\{\frac{da}{dt}\right\}_\theta + \mathbf{K}\{a\}_\theta = \mathbf{F}_\theta \quad \text{(T2L3-23)}$$

where $\theta = \dfrac{t - t_{n-1}}{\Delta t_n}$, $\Delta t_n = t_n - t_{n-1}$, $0 < \theta < 1$ (T2L3-24)

$\{a(t)\}$ and $\{F(t)\}$ may be approximated by polynomials

$$\{a\}_\theta \cong (1-\theta)\{a\}_{n-1} + \theta\{a\}_n \quad \text{(T2L3-25a)}$$

$$\{F\}_\theta \cong (1-\theta)\{F\}_{n-1} + \theta\{F\}_n \quad \text{(T2L3-25b)}$$

Differentiating Eqn. (T2L3-25)

$$\left\{\frac{da}{dt}\right\}_\theta = \frac{1}{\Delta t_n}\frac{d\{a\}_\theta}{d\theta} = \frac{\{a\}_n - \{a\}_{n-1}}{\Delta t_n} \quad \text{(T2L3-26)}$$

Substituting into Eqn. (T2L3-23)

$$\left(\frac{1}{\Delta t_n}\mathbf{C} + \theta\,\mathbf{K}\right)\mathbf{a}_n = (1-\theta)\mathbf{F}_{n-1} + \theta\,\mathbf{F}_n + \left(\frac{1}{\Delta t_n}\mathbf{C} - (1-\theta)\mathbf{K}\right)\mathbf{a}_{n-1} \quad \text{(T2L3-27)}$$

or, $\mathbf{K}_{eff}\,\mathbf{a}_n = \mathbf{F}_{eff}$ (T2L3-28)

Comparing with the previous three methods

$\theta = 0$ Forward Difference

$\theta = 1/2$ Mid-Difference

$\theta = 1$ Forward Difference

Stability Considerations

Whether a solution technique is stable or not can be answered by analyzing the free response of the system. Behavior of the solution as time $t \to \infty$ can be one of three types as shown below.

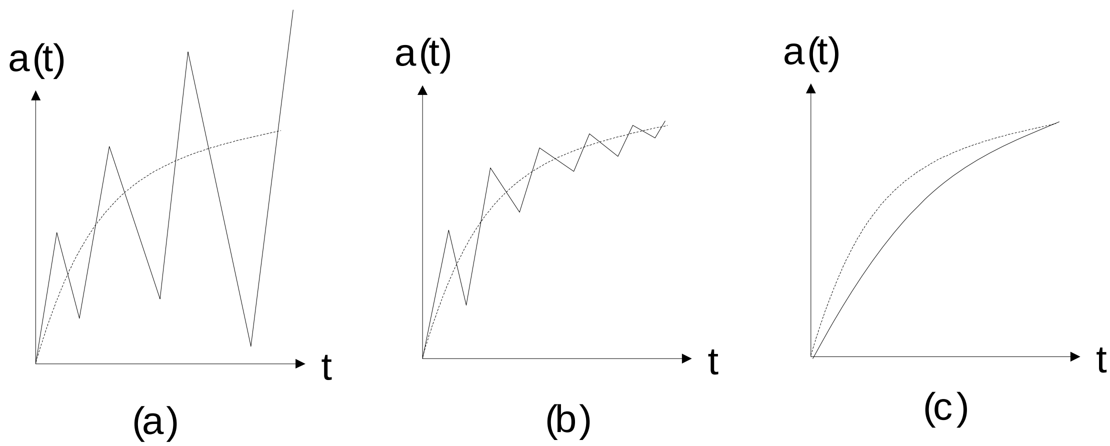


Fig. T2L3-3 (a) Unstable – Oscillatory divergence (b) Stable – Oscillatory decay (c) Stable – Monotonic decay

Let us consider a one degree-of-freedom system.

$$C\frac{da(t)}{dt} + Ka(t) = F(t) \qquad \text{(T2L3-29)}$$

For free response

$$C\frac{da(t)}{dt} + Ka(t) = 0 \qquad \text{(T2L3-30)}$$

The exact solution is given as

$$a(t) = Ae^{-\lambda t} \text{ where } \lambda = K/C \qquad \text{(T2L3-31)}$$

The recurrence relation is

$$\left(\frac{1}{\Delta t}C + \theta K\right)a_n = (1-\theta)F_n + \theta F_n + \left(\frac{1}{\Delta t}C - (1-\theta)K\right)a_{n-1} = 0 \qquad \text{(T2L3-32)}$$

Multiplying both sides by $\Delta t/c$ and using $\lambda = K/C$

$$\frac{a_n}{a_{n-1}} = \frac{1-(1-\theta)\lambda\Delta t}{1+\theta\,\lambda\,\Delta t} \tag{T2L3-33}$$

For multiple degree-of-freedom system, the response of any linear system can be represented by a linear superposition of its modes.

$$\mathbf{C}\left\{\frac{da(t)}{dt}\right\} + \mathbf{K}\{a(t)\} = \{0\} \tag{T2L3-34}$$

The exact solution is of the form $\{a(t)\} = \boldsymbol{\varphi}\, e^{-\lambda t}$ (T2L3-35)

Substituting Eqn. (T2L3-35) into (T2L3-34)

$$(-\lambda\mathbf{C} + \mathbf{K})\boldsymbol{\varphi} e^{-\lambda t} = \mathbf{0} \tag{T2L3-36}$$

Nontrivial solutions for $\boldsymbol{\varphi}$ exist if and only if

$$\det(-\lambda\mathbf{C} + \mathbf{K}) = 0 \tag{T2L3-37}$$

This is the same as an eigenanalysis and $\lambda_i > 0$ since $\mathbf{C}$ and $\mathbf{K}$ are positive definite. The general solution to Eqn. (T2L3-34) is a linear superposition (sum) of solutions of the form (T2L3-35), one for each mode

$$\{a(t)\} = \sum_{j=1}^{NDOF} A_j \boldsymbol{\varphi}_j e^{-\lambda_j t} \tag{T2L3-38}$$

where the constants A_j are determined from initial conditions, and since $\lambda_j > 0$, the solution contains exponentially decaying modes. When $\{F(t)\} \neq \{0\}$,

$$\{a(t)\} = \sum_{j=1}^{NDOF} A_j(t) \boldsymbol{\varphi}_j \tag{T2L3-39}$$

As the solution indicates, time dependence is no longer exponential. Instead each mode has a general time-varying amplitude $A_j(t)$. Substituting Eqn. (T2L3-39) into (T2L3-34) with $\{F(t)\} \neq 0$

$$\mathbf{C}\left(\sum_{j=1}^{NDOF} \frac{dA_j(t)}{dt} \boldsymbol{\varphi}_j\right) + \mathbf{K}\left(\sum_{j=1}^{NDOF} A_j(t)\boldsymbol{\varphi}_j\right) = \mathbf{F}(t) \tag{T2L3-40}$$

Premultiplying Eqn. (T2L3-40) by $\boldsymbol{\varphi}_i^T$ and using $\boldsymbol{\varphi}_i^T \mathbf{K} \boldsymbol{\varphi}_j = \lambda_i \delta_{ij}$ and $\boldsymbol{\varphi}_i^T \mathbf{C} \boldsymbol{\varphi}_j = \delta_{ij}$

$$\frac{dA_i(t)}{dt} + \lambda_i A_i(t) = f_i(t) \qquad i = 1,2,\ldots, NDOF \tag{T2L3-41}$$

where $f_i(t) = \boldsymbol{\varphi}_i^T \mathbf{F}(t)$. The recurrence relation is

$$\left(\frac{1}{\Delta t} + \theta \lambda_i\right)(A_i)_n = (1-\theta)(f_i)_{n-1} + \theta (f_i)_n$$

$$+\left(\frac{1}{\Delta t} - (1-\theta)\lambda_i\right)(A_i)_{n-1} \qquad i = 1,2,\ldots, NDOF \tag{T2L3-42}$$

For a free response system

$$\frac{(A_i)_n}{(A_i)_{n-1}} = \frac{1-(1-\theta)\lambda_i \Delta t}{1+\theta \lambda_i \Delta t} \tag{T2L3-43}$$

For stability

$$\left|\frac{(A_i)_n}{(A_i)_{n-1}}\right| < 1 \qquad i = 1,2,\ldots, NDOF \tag{T2L3-44}$$

This is necessary to ensure that successive values will not continuously grow larger. Or,

$$\frac{(A_i)_n}{(A_i)_{n-1}} < +1 \text{ and } \frac{(A_i)_n}{(A_i)_{n-1}} > -1 \tag{T2L3-45}$$

Using Eqn. (T2L3-43) and (44)

$$-1 < \frac{1-(1-\theta)\lambda_i \Delta t}{1+\theta \lambda_i \Delta t} < +1 \tag{T2L3-46}$$

From $\dfrac{1-(1-\theta)\lambda_i \Delta t}{1+\theta \lambda_i \Delta t} < 1 \Rightarrow -\lambda_i \Delta t < 0$ .Or, $\lambda_i \Delta t > 0$. Note this condition is always true.

From $-1 < \dfrac{1-(1-\theta)\lambda_i \Delta t}{1+\theta \lambda_i \Delta t} \Rightarrow -1 + \theta \lambda_i \Delta t < 1-(1-\theta)\lambda_i \Delta t$. Or,

$$\frac{-2}{2\theta - 1} < \lambda_i \Delta t \tag{T2L3-47}$$

This is always true if $\theta \ge 1/2$. For $\theta < 1/2$, need $\lambda_i \Delta t < \dfrac{2}{1-2\theta}$. Hence

For $0 \le \theta \le 1/2$: $\lambda_i \Delta t < \dfrac{2}{1-2\theta}$ (T2L3-48a)

For $\theta \ge 1/2$: $\lambda_i \Delta t$ can have any positive value (T2L3-48b)

The former requirement is for conditional stability and the latter for unconditional stability.

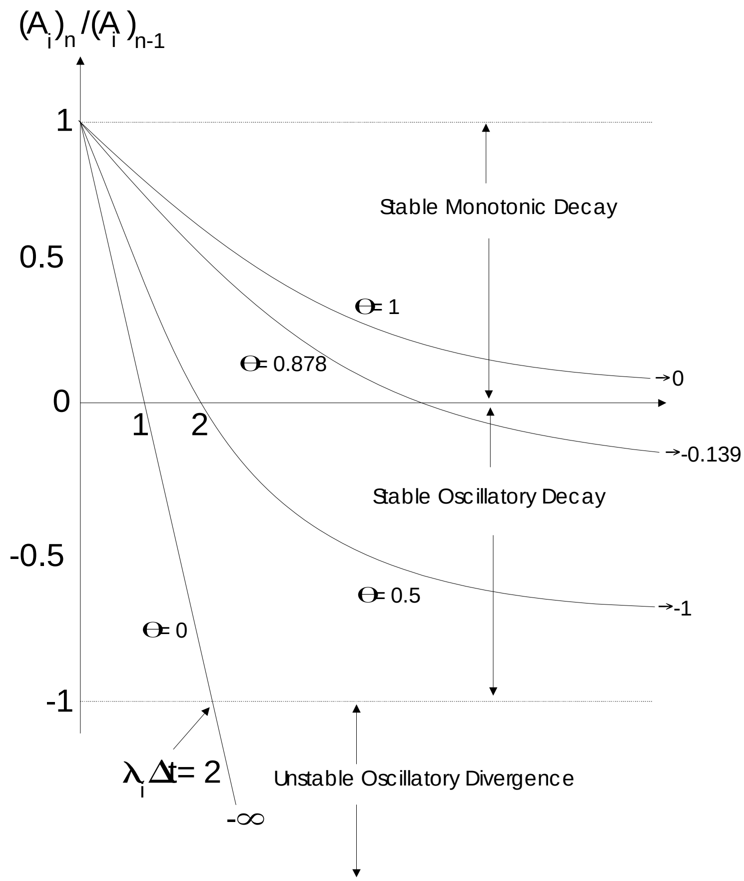


Fig. T2L3-4 Solution type

Note that for conditional stability, for $\theta = 0 \Rightarrow \dfrac{(A_i)_n}{(A_i)_{n-1}} = 1 - \lambda_i \Delta t$. Or, $\lambda_i \Delta t < \dfrac{2}{1-2\theta}$ must be satisfied by all λ_i's. Or, $\Delta t < \Delta t_{Crit} = \dfrac{2}{1-2\theta}\dfrac{1}{\lambda_{\max}}$. Without proof,

$\lambda_{\max} \le \lambda_{\max}^{(e)}$ (T2L3-49a)

$$\lambda_{\max} \approx \frac{d\pi^2}{\left((\mu/\alpha)\delta^2\right)_{\min}^{(e)}} \tag{T2L3-49b}$$

where the superscript $^{(e)}$ indicates over all elements in the mesh, d is the spatial dimension of the problem, μ and α are from the DE and δ is the distance between two adjacent nodes. Using the above relations

$$\Delta t_{Crit} \cong \frac{2}{d(1-2\theta)\pi^2}\left((\mu/\alpha)\delta^2\right)_{\min}^{(e)} \qquad 0 \le \theta \le 1/2$$

For forward difference $(\theta = 0)$

$$\Delta t_{Crit} \cong \frac{2}{d\pi^2}\left((\mu/\alpha)\delta^2\right)_{\min}^{(e)}$$

The mid-difference method shows oscillatory decay and the backward difference shows monotonic decay.

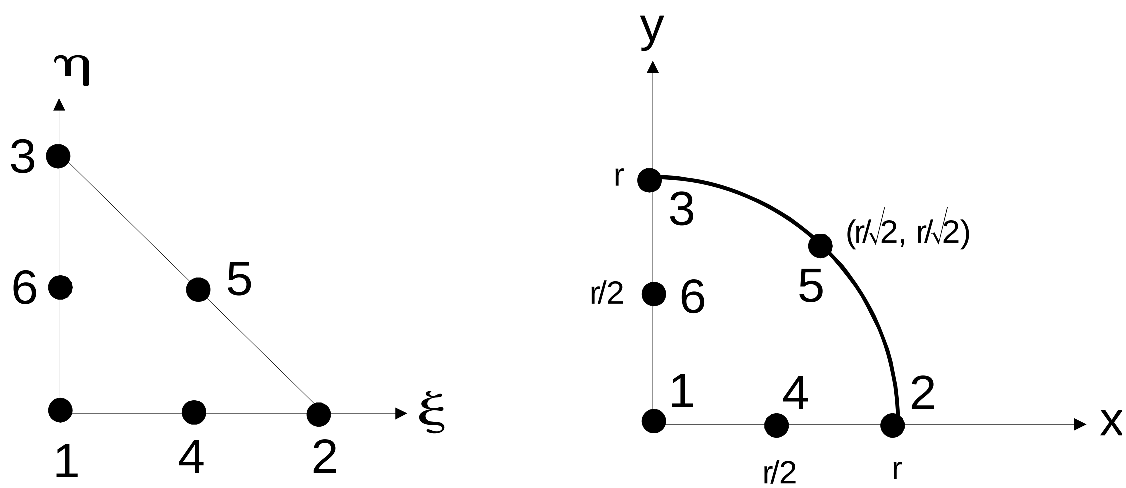
Review Exercises

Problem T2L2-1

The parent element for a C^0-quadratic isoparametric triangle is mapped onto the two different elements shown below. Nodes 2, 5, and 3 lie on a 90^0 circular arc of radius r. Nodes 4, 5, and 6 are at the midpoints of their sides.

(i) Derive the stiffness matrix and the mass matrix for the 90^0 circular arc triangular element.

(ii) Derive the load vector for the 90^0 circular arc triangular element.



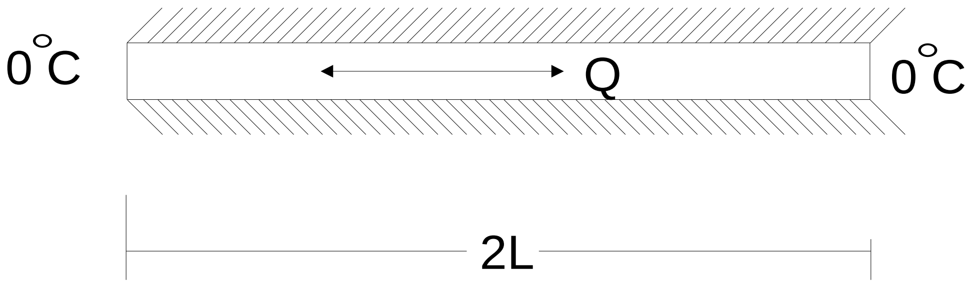
Problem T2L2-2

Calculate the first nonzero cut-off frequency for the propagation of radially symmetric sound waves through a circular cylindrical tube of radius a with rigid walls. Use the 90^0 circular arc triangular element. Model one quadrant of the cross-section with one element.

Problem T2L3-1

A thin cylindrical rod with the lateral surface insulated is initially at $0^0 C$. An interior heat source Q, uniform along the length, is turned on at $t = 0$ and remains on indefinitely. The end faces are maintained at $0^0 C$.

The problem has bilateral symmetry about the center so only half the rod needs to be analyzed. We will use a uniform mesh of two C^0-linear elements.



Use the forward difference method to calculate several steps of the transient solution to the heat conduction problem. Assume that $L = 10cm$, $\rho = 10\, gm/cm^3$, $c = 0.1\, cal/gm - ^0C$, $k = 1\, cal/\sec - cm - ^0C$, and $Q = 2\, cal/cm^3 - \sec$.

(a) Estimate the critical step for stable solution.

(b) Calculate five steps with $\Delta t = 10$, $\Delta t = 15$ and $\Delta t = 20$. For each time step ascertain whether the solution is stable or not.

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