Finite Elements for Engineers

Lecture 5: 3D Boundary Value Problems

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3D BVP

DE

$$\frac{\partial}{\partial x}\left(\alpha_x(x,y,z)\frac{\partial u(x,y,z)}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha_y(x,y,z)\frac{\partial u(x,y,z)}{\partial y}\right)+\frac{\partial}{\partial z}\left(\alpha_z(x,y,z)\frac{\partial u(x,y,z)}{\partial z}\right)$$
$$+\beta(x,y,z)u(x,y,z)+f(x,y,z)=0$$

BCs

$$u(\hat{x},\hat{y},\hat{z})=\hat{u} \text{ on } \Gamma_1$$

$$\alpha_x\frac{\partial u}{\partial x}n_x+\alpha_y\frac{\partial u}{\partial x}n_y+\alpha_z\frac{\partial u}{\partial z}n_z+gu+c=0 \text{ on } \Gamma_2$$

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Trial Solution

Step 1: Galerkin's Method – Residual Equations

$$\iiint_{\Omega} [\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\alpha_z \frac{\partial u}{\partial z}\right) + \beta(x,y,z)u(x,y,z)$$

$$+ f(x,y,z)]\phi_i(x,y,z)dxdydz = 0 \qquad i = 1,2,...,n$$

Divergence Theorem

$$\iiint_{\Omega} \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) dxdydz = \iint_{\Gamma} \left(Fn_x + Gn_y + Hn_z\right) dS$$

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Step 2: Galerkin's Method – Integration of Parts

$$\iiint_{\Omega} \left\{ \frac{\partial u}{\partial x}\alpha_x \frac{\partial \phi_i}{\partial x} + \frac{\partial u}{\partial y}\alpha_y \frac{\partial \phi_i}{\partial y} + \frac{\partial u}{\partial z}\alpha_z \frac{\partial \phi_i}{\partial z} - \beta u \phi_i \right\} dxdydz$$

$$+ \iint_{\Gamma} \left(g u \phi_i \right) dS = \iiint_{\Omega} f \phi_i dxdydz - \iint_{\Gamma} \left(c \phi_i dS \right) \quad i = 1, 2, \ldots, n$$

Step 3: Using the trial solution

$$\sum_{j=1}^{n} \left(\iiint_{\Omega} \left\{ \frac{\partial \phi_i}{\partial x}\alpha_x \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\alpha_y \frac{\partial \phi_j}{\partial y} + \frac{\partial \phi_i}{\partial z}\alpha_z \frac{\partial \phi_j}{\partial z} - \phi_i \beta \phi_j \right\} dxdydz + \iint_{\Gamma} \phi_i g \phi_j dS \right) u_j =$$

$$\iiint_{\Omega} f \phi_i dxdydz - \iint_{\Gamma} \left(c \phi_i dS \right) \qquad i = 1, 2, \ldots, n$$

3D BVP

Element Equations

$$\left[\mathbf{k}^{\alpha}_{n\times n}+\mathbf{k}^{\beta}_{n\times n}+\mathbf{k}^{g}_{n\times n}\right]\mathbf{u}_{n\times 1}=\mathbf{f}^{\text{int}}_{n\times 1}+\mathbf{f}^{bnd}_{n\times 1}$$

$$k^{\alpha}_{ij}=\iiint_{\Omega}\left\{\frac{\partial\phi_i}{\partial x}\alpha_x\frac{\partial\phi_j}{\partial x}+\frac{\partial\phi_i}{\partial y}\alpha_y\frac{\partial\phi_j}{\partial y}+\frac{\partial\phi_i}{\partial z}\alpha_z\frac{\partial\phi_j}{\partial z}\right\}dxdydz$$

$$k^{\beta}_{ij}=-\iiint_{\Omega}\phi_i\beta\phi_j dxdydz$$

$$k^{g}_{ij}=\iint_{\Gamma}\phi_i g\phi_j dS$$

$$f^{\text{int}}_i=\iiint_{\Omega}f\phi_i dxdydz \qquad f^{bnd}_i=-\iint_{\Gamma}\left(c\phi_i dS\right)$$

3D BVP

Convective Stiffness and Element Load Vector

$$k_{ij}^{g} = \iint_{\Gamma} \phi_i g \phi_j dS$$

$$f_i^{bnd} = -\iint_{\Gamma} \left(c\phi_i dS \right)$$

$$\frac{\partial P}{\partial \xi} \times \frac{\partial P}{\partial \eta} d\xi \, d\eta = \mathbf{n} \, dS$$

$$P = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad \frac{\partial P}{\partial \xi} = \begin{Bmatrix} J_{11} \\ J_{12} \\ J_{13} \end{Bmatrix} \quad \frac{\partial P}{\partial \eta} = \begin{Bmatrix} J_{21} \\ J_{22} \\ J_{23} \end{Bmatrix}$$

$$\left\| \frac{\partial P}{\partial \xi} \times \frac{\partial P}{\partial \eta} \right\| d\xi \, d\eta = dS$$

3D BVP

Element Load Vector

$$f_i^{bnd} = -\sum_{k=1}^{n}\sum_{l=1}^{n} w_k w_l\, c \left[\phi_i \left\| \frac{\partial P}{\partial \xi} \times \frac{\partial P}{\partial \eta} \right\| \right]_{(\xi_k, \eta_l)}$$

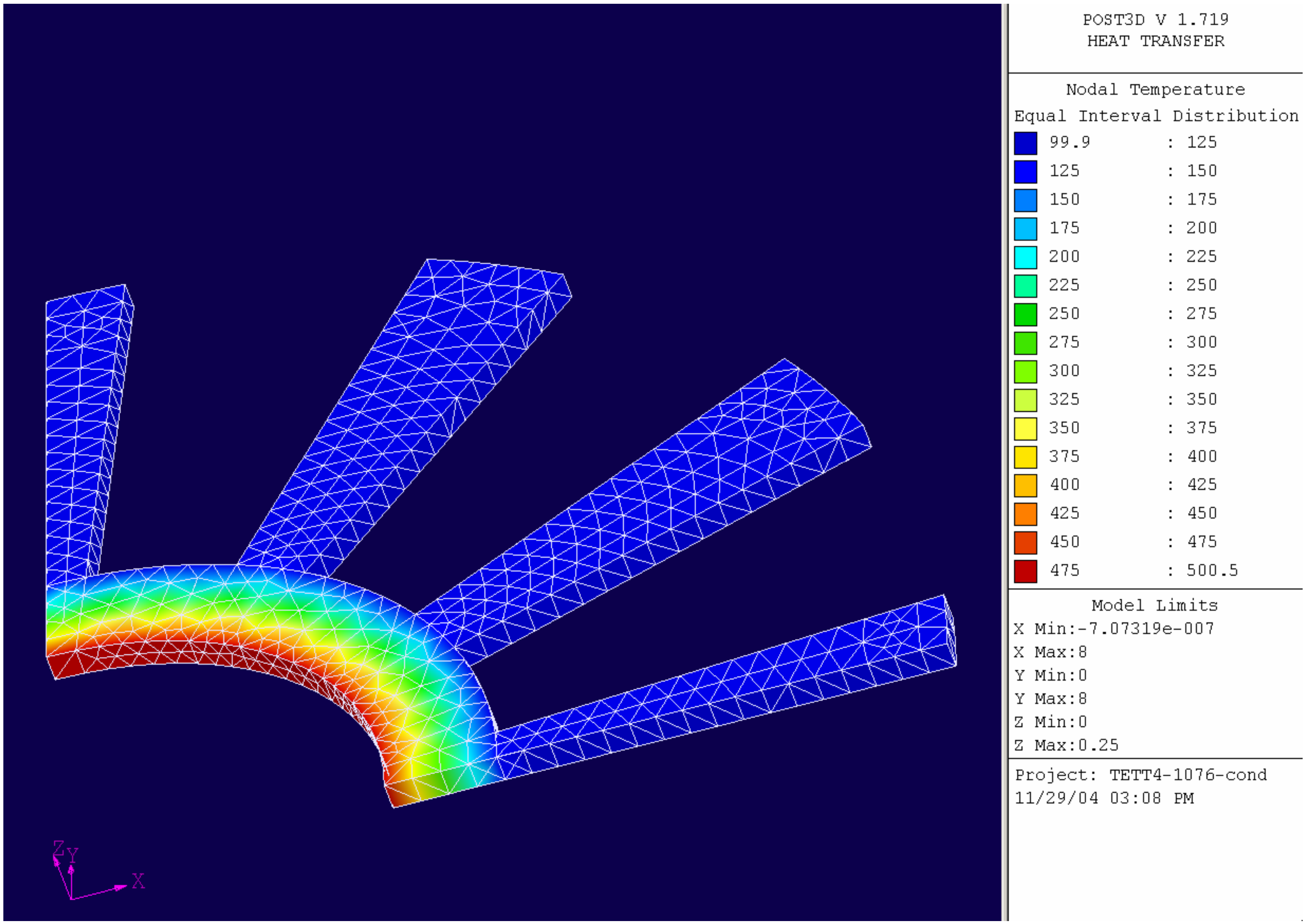
Convective Stiffness

$$k_{ij}^{g} = \sum_{k=1}^{n}\sum_{l=1}^{n} w_k w_l\, g \left[\phi_i \phi_j \left\| \frac{\partial P}{\partial \xi} \times \frac{\partial P}{\partial \eta} \right\| \right]_{(\xi_k, \eta_l)}$$

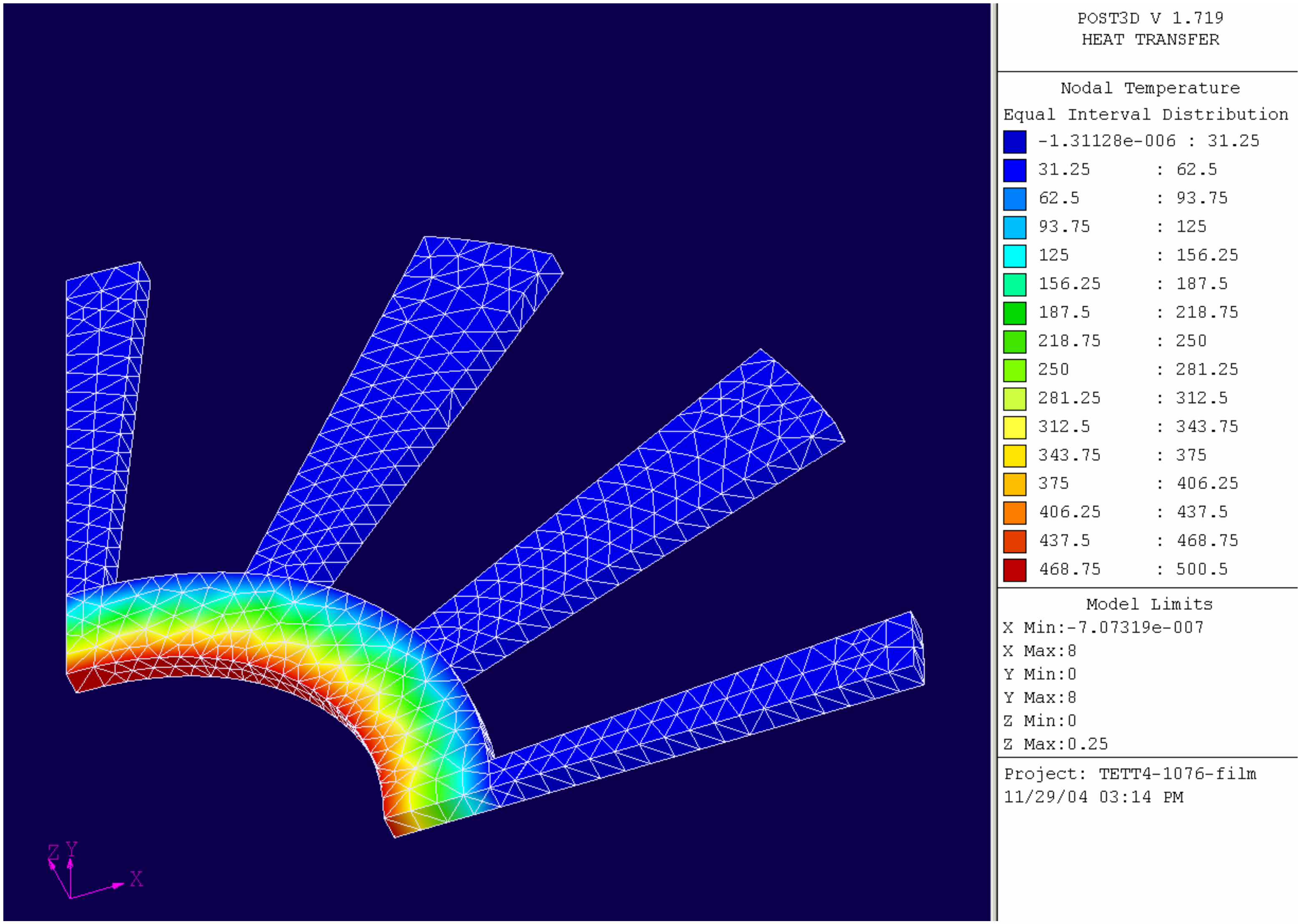
With volume coordinates

$$\int_V \xi^l \eta^m \zeta^n \varsigma^o \, dV = \frac{l!m!n!o!6V}{(l+m+n+o+3)!}$$

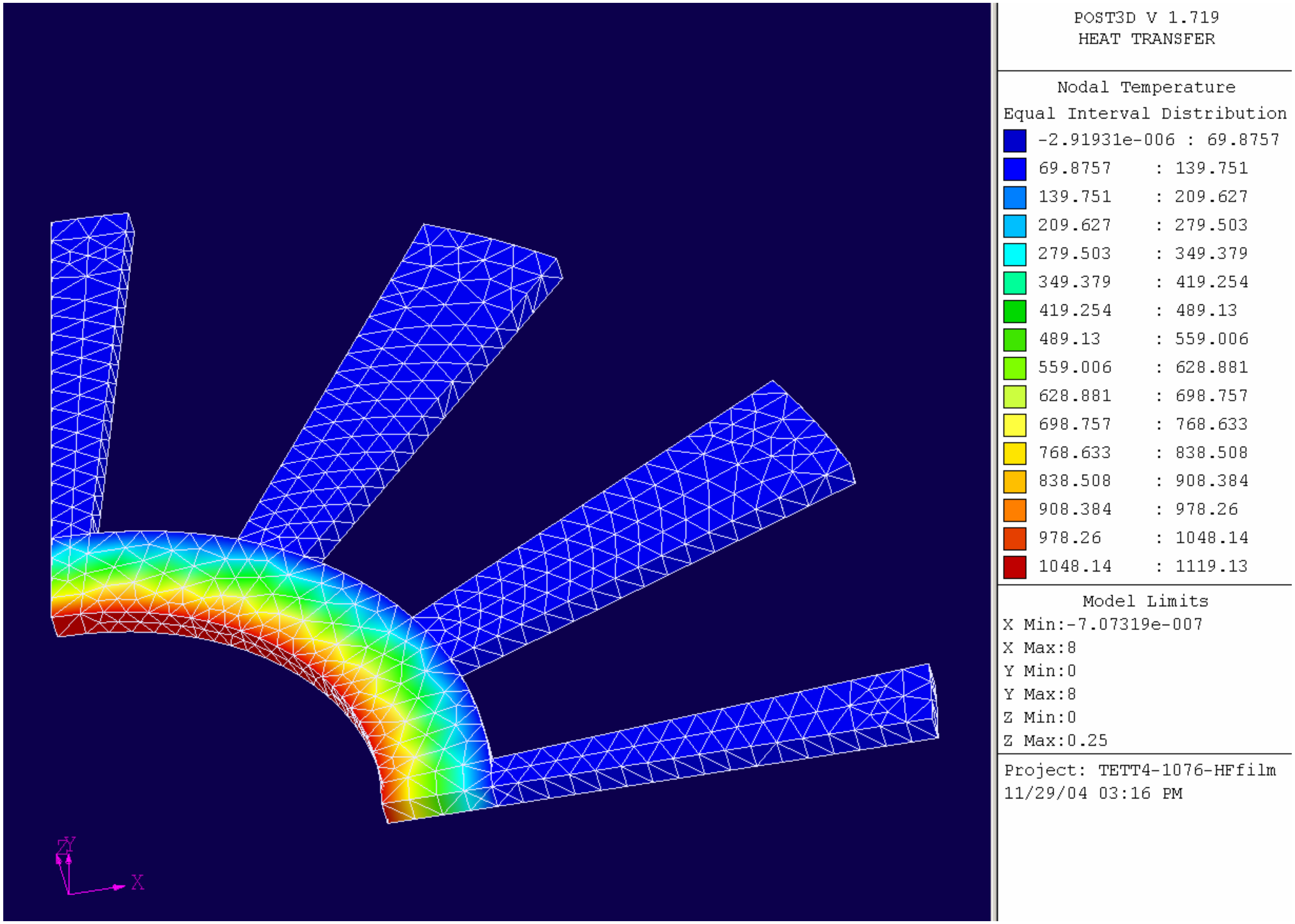
Example: Conduction Only



Example: Cond + Conv



Example: Cond + Conv + Flux



Summary

- The basic ideas from 1D and 2D BVP carry over.
  – Chain rule of differentiation
  – Divergence Theorem
  – Integration by parts
- The element shape functions for the 3D elements (hexahedral, tetrahedral and wedge) are the same as before.

Summary

- Numerical integration can be applied (as before) using natural and volume coordinates.
- Computation of boundary-related terms (stiffness and load) requires special treatment.