

Finite Elements For Engineers

Lecture 7: Convergence and Modeling Issues

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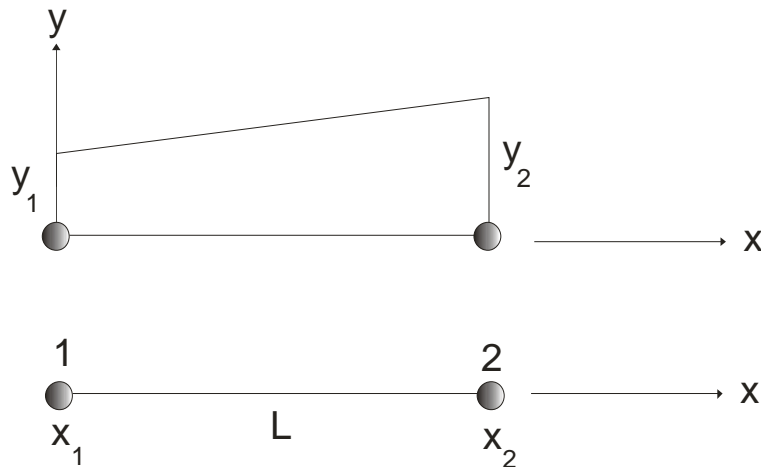
Higher-Order Elements (p-Convergence)

- Better accuracy with higher-order interpolation
- For the same level of accuracy compared to lower order elements
 - Requires less number of elements (and nodes)
 - However element generation is more expensive
 - \mathbf{k} will usually have more non-zero elements

Elements and Interpolation Order

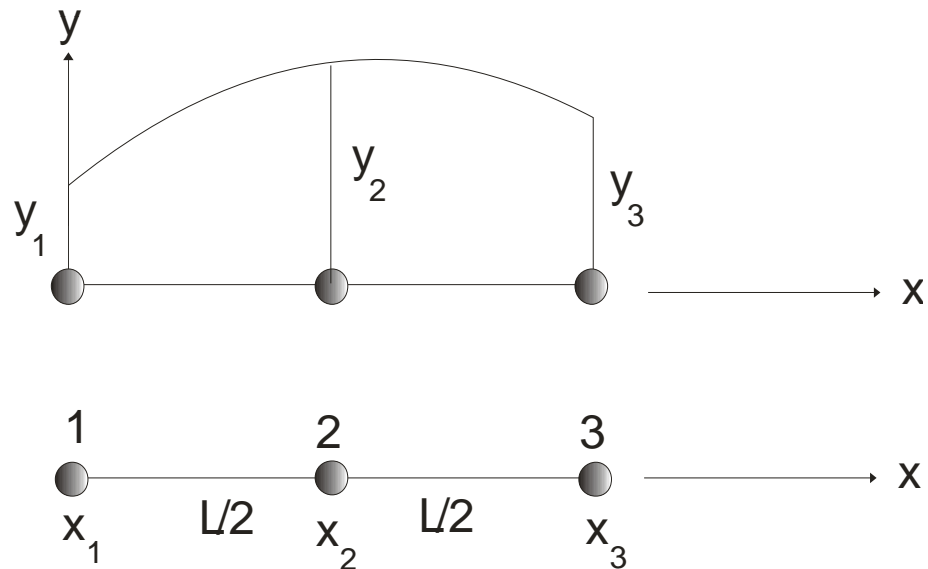
Linear Interpolation

$$\tilde{y}(x) = a_1 + a_2x = \phi_1(x)y_1 + \phi_2(x)y_2$$



Quadratic Interpolation

$$\tilde{y}(x) = a_1 + a_2x + a_3x^2 = \phi_1(x)y_1 + \phi_2(x)y_2 + \phi_3(x)y_3$$



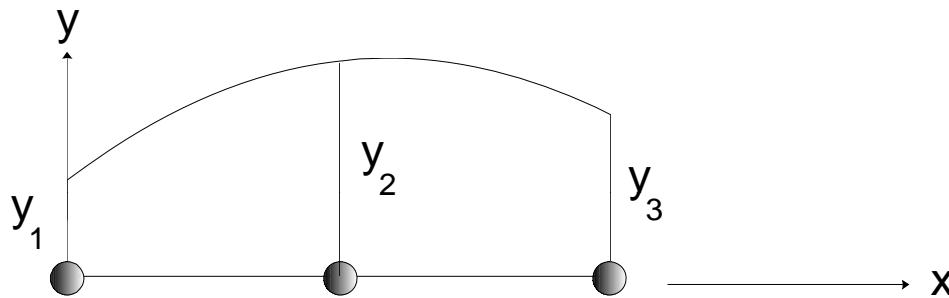
Properties of Shape Functions

- They are the interpolating functions (interpolation using the nodal value)
- If the number of DOF per node is one, then we need as many nodes as the number of unknown coefficients
- The shape functions must satisfy the following relationships

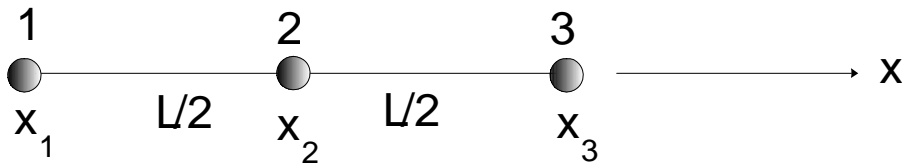
$$\phi_i(x_j) = \delta_{ij} \qquad \sum_i \phi_i(x) = 1$$

1D-C⁰ Quadratic Element

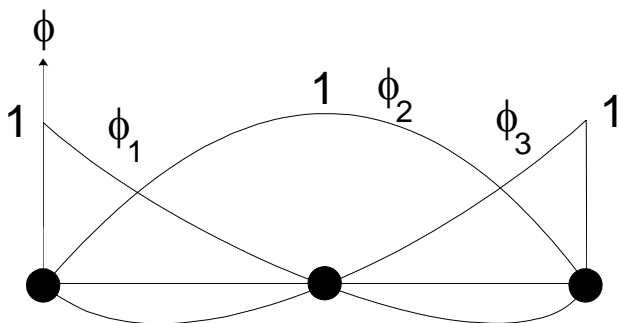
$$\tilde{y}(x) = a_1 + a_2x + a_3x^2 = \phi_1(x)y_1 + \phi_2(x)y_2 + \phi_3(x)y_3$$



$$\phi_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$



$$\phi_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$



$$\phi_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

1D-C⁰ Quadratic Element

Galerkin Step 4

Typical Stiffness Term

$$k_{ij} = \int_{\Omega} \frac{d\phi_i}{dx} \alpha(x) \frac{d\phi_j}{dx} dx + \int_{\Omega} \phi_i(x) \beta(x) \phi_j(x) dx$$

Assume for the time being that these are constants

$$\alpha(x) = \alpha \quad \beta(x) = \beta$$

1D-C⁰ Quadratic Element

$$\begin{bmatrix} \frac{7\alpha}{3L} & -\frac{8\alpha}{3L} & \frac{\alpha}{3L} \\ -\frac{8\alpha}{3L} & \frac{16\alpha}{3L} & -\frac{8\alpha}{3L} \\ \frac{\alpha}{3L} & -\frac{8\alpha}{3L} & \frac{7\alpha}{3L} \end{bmatrix} + \begin{bmatrix} \frac{4\beta L}{30} & \frac{2\beta L}{30} & -\frac{\beta L}{30} \\ \frac{2\beta L}{30} & \frac{16\beta L}{30} & \frac{2\beta L}{30} \\ -\frac{\beta L}{30} & \frac{2\beta L}{30} & \frac{4\beta L}{30} \end{bmatrix} - g_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ h_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} \frac{fL}{6} \\ \frac{4fL}{6} \\ \frac{fL}{6} \end{Bmatrix} + \begin{Bmatrix} c_1 \\ 0 \\ -c_3 \end{Bmatrix}$$

1D-C⁰ Quadratic Element

Element Flux $\tilde{\tau} = -\alpha(x) \frac{d\tilde{y}}{dx}$

$$\tilde{\tau} = -\frac{\alpha}{L^2} \left[x(4y_1 - 8y_2 + 4y_3) + x_1(4y_2 - 2y_3) - 2x_2(y_1 + y_3) - 2x_3(y_1 + 2y_2) \right]$$

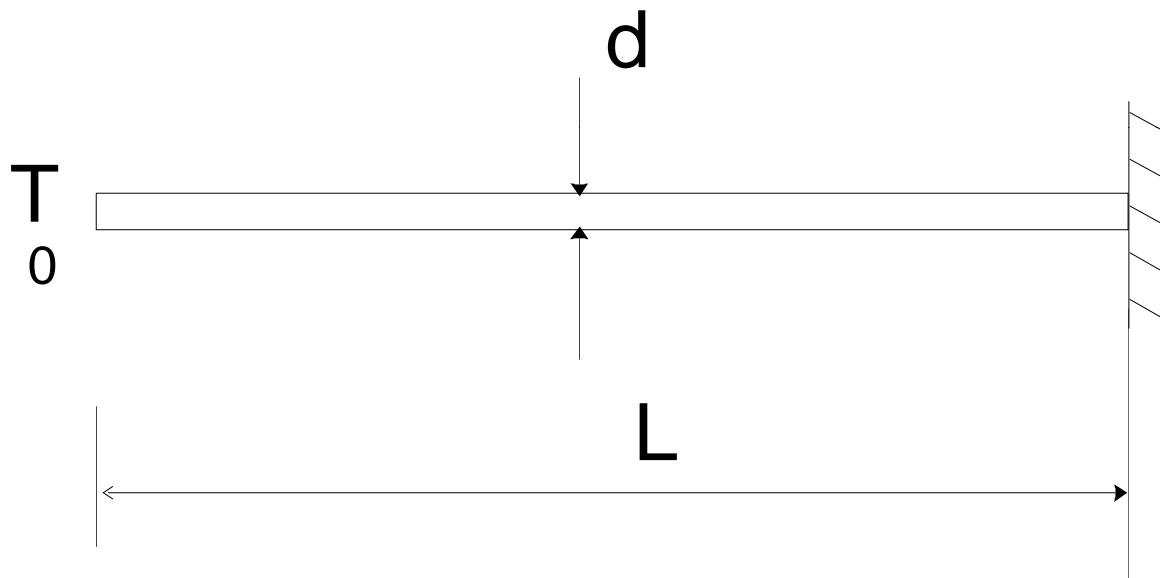
Mesh Refinement (h and p-convergence)

- **h**-convergence refers to the process of refining the FE mesh (adding more elements)
- **p**-convergence refers to the process of increasing the interpolation order of the elements in the FE mesh

Example T4L3-2

Compute temperature and flux.

Conduction and
Convection



$$T_0 = 150^\circ F$$

$$T_\infty = 80^\circ F$$

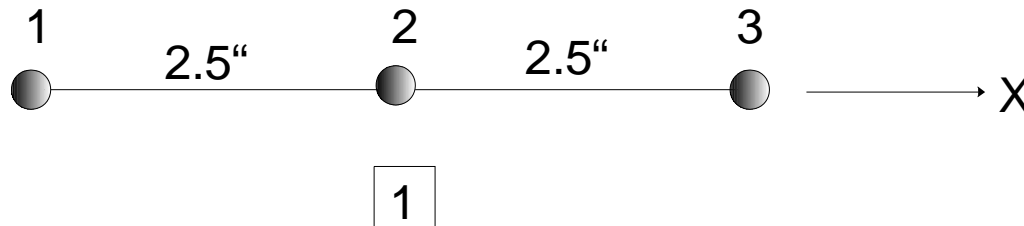
$$h = 6 \frac{BTU}{h \cdot ft^2 \cdot ^\circ F}$$

$$k = 24.8 \frac{BTU}{h \cdot ft \cdot ^\circ F}$$

Example T4L3-2

Units: BTU, hr, ft, F

Discretization: FE Mesh



Element 1

$$\begin{bmatrix} 181.7 & -116.38 & -1.4256 \\ -116.38 & 488.33 & -116.38 \\ -1.4256 & -116.38 & 181.7 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 5111.3 \\ 20445 \\ 5111.3 \end{Bmatrix}$$

Example T4L3-2

Imposition of EBC $T_1 = 150$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 488.33 & -116.38 \\ 0 & -116.38 & 181.7 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 150 \\ 37902 \\ 5325.2 \end{Bmatrix}$$

Solution

$$\{T_1, T_2, T_3\} = \{150, 99.8, 93.3\}^\circ F$$

Derived Variables

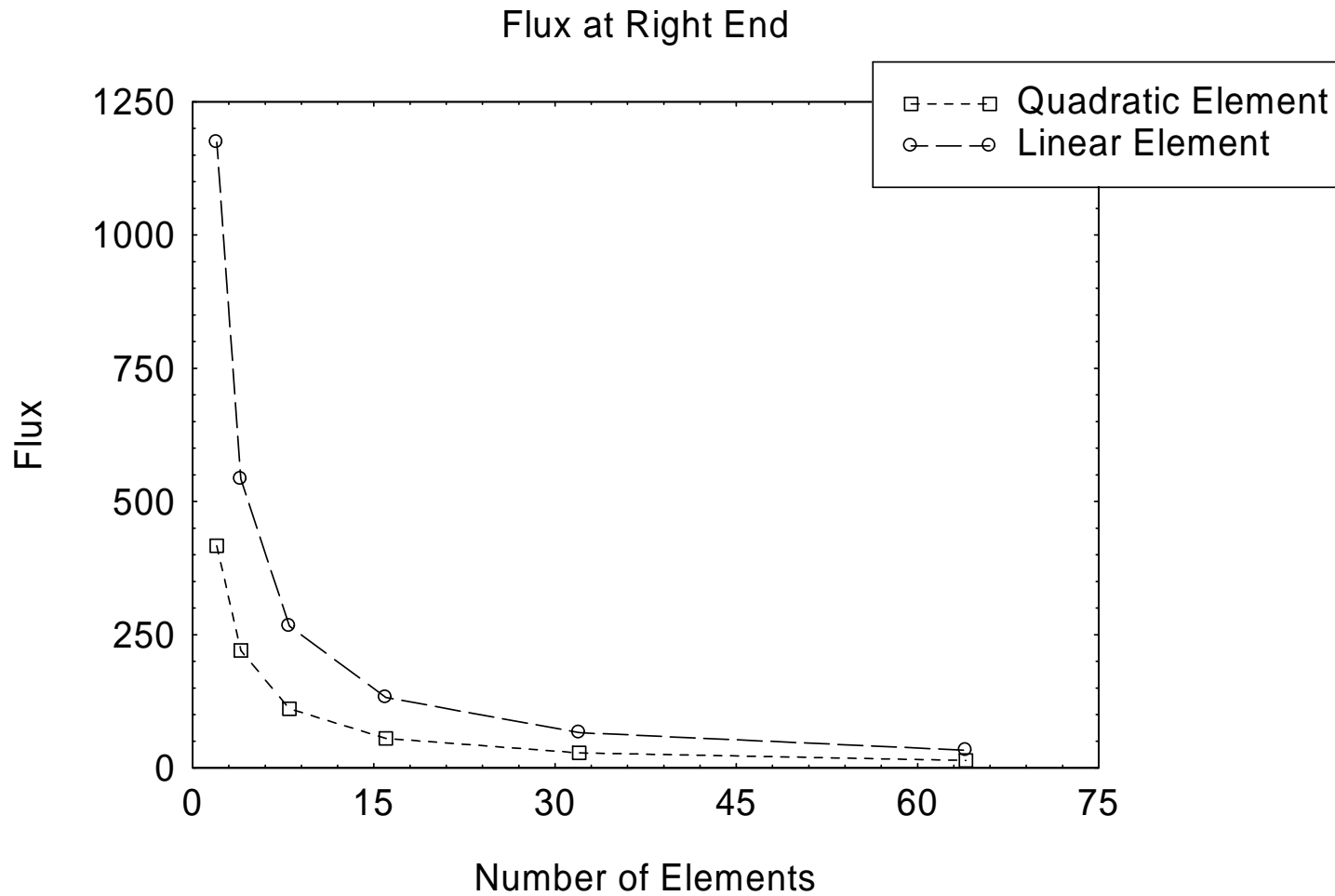
$$\tau(x = 0.0879') = 6382 \frac{BTU}{h \cdot ft^2}$$

$$\tau(x = 0.3281') = 383.1 \frac{BTU}{h \cdot ft^2}$$

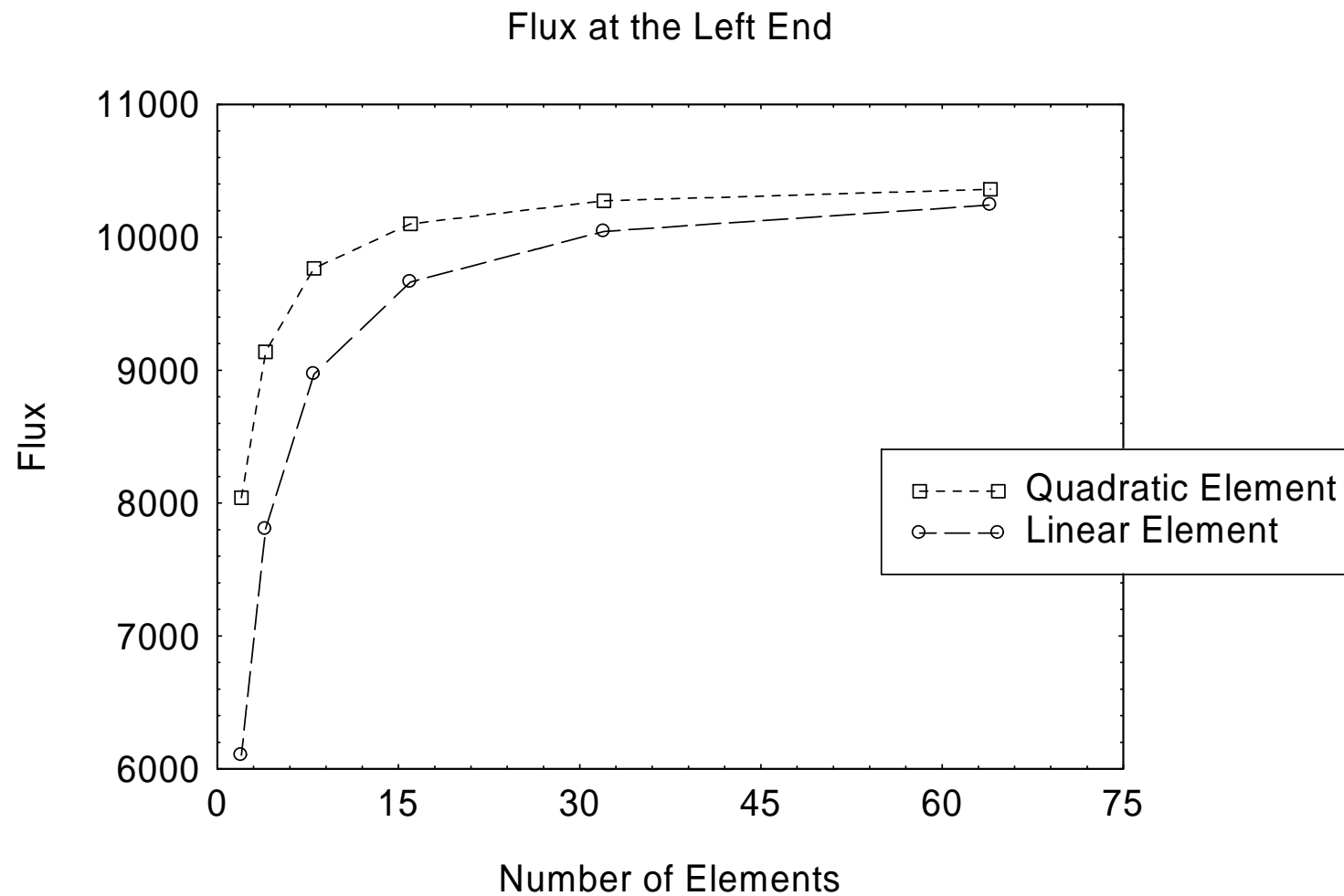
Comparison

| | <i>Temperature</i> $(^{\circ}F)$ | | <i>Flux</i> $\left(\frac{BTU}{h \cdot ft^2} \right)$ | |
|------|-------------------------------------|----------------------|---|----------------------|
| Node | Linear Element | Quadratic Element | Linear Elements | Quadratic Element |
| 1 | 150 | 150 | | |
| 2 | 98.9 | 99.8 | 6102 | 6382 |
| 3 | 89.0 | 93.3 | 1174 | 383 |

Flux Convergence Study



Flux Convergence Study



Summary

- Element concept with trial solution makes it easy to generate the element equations including imposition of the BCs
- The element equations are of the form

$$\mathbf{k}_{n \times n} \mathbf{d}_{n \times 1} = \mathbf{f}_{n \times 1}$$

- Both **h** and **p**-convergence make it possible to obtain efficient solutions to general FE problems

Further Reading

- See if you can generate the element equations for the 1D- C^0 cubic element using a symbolic computer program (e.g. Maple)

Thermal Loading & Stresses

Initial Strain

$$\varepsilon_0 = \alpha \Delta T$$

Stress-Strain Relationship

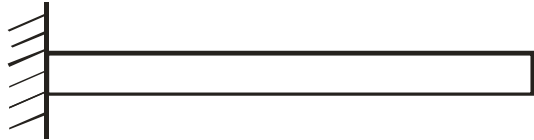
$$\sigma = E(\varepsilon - \varepsilon_0)$$

Equivalent Load Vector

$$\mathbf{f}_{2 \times 1} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = AE\alpha \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$



Example T4L2-2(a)



The length of the bar is 2 m. The cross-sectional area is 0.001m^2 . The bar is initially at 25°C . The temperature is increased to 100°C . Compute the stress in the bar.

$$E = 200\text{ GPa}$$

$$\alpha = 11.7 \times 10^{-6} \frac{\text{m}}{\text{m} - ^\circ\text{C}}$$

Units: m, kg, N, C

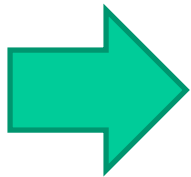
One-element solution.

Example T4L2-2(a)

$$\frac{(0.001)(200 \times 10^9)}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = (0.001)(200 \times 10^9)(11.7 \times 10^{-6})(75) \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$



$$\begin{bmatrix} 10^8 & -10^8 \\ -10^8 & 10^8 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} -175500 \\ 175500 \end{Bmatrix}$$



$$10^8 D_2 = 175500 \Rightarrow D_2 = 0.001755 \text{ m}$$

$$\varepsilon = \frac{D_2 - D_1}{L} = \frac{0.001755}{2} = 0.0008775$$

$$\varepsilon_0 = (11.7 \times 10^{-6})(75) = 0.0008775$$

$$\sigma = E(\varepsilon - \varepsilon_0) = 200 \times 10^9 (0.0008775 - 0.0008775) = 0$$

Using the Windows-based 1DBVP Program

1DBVP Program Terminology

- Positive coordinate system points to the right
- FE mesh contains one or more segments
- A segment has the same properties but may contain one or more elements
- The leftmost segment (or the first segment) has the first node and the first element (or node 1 and element 1)
- You create the segment data; 1DBVP creates the nodes, elements and loads

Solid Mechanics

DE

$$-\frac{d}{dx}\left(\alpha(x)\frac{dy(x)}{dx}\right) + \beta(x)y(x) = f(x)$$

Left Mixed
BC

$$\tau = c_a y + d_a$$

Right
Mixed BC

$$\tau = c_b y + d_b$$

DE

$$-\frac{d}{dx}\left(A(x)E(x)\frac{du(x)}{dx}\right) = w(x)A(x)$$

$$\tau = F = -A(x)E(x)\frac{du(x)}{dx}$$

Solid Mechanics

$$\alpha(x) = A(x)E(x) \equiv F$$

$$\beta(x) = 0$$

$$f(x) = w(x)A(x) \equiv F/L$$

$$c_a = c_b = 0$$

$$d_a = d_b \equiv F$$

Examples



$$\beta(x) = 0$$

$$\alpha(x) = A(x)E(x) \equiv F$$

$$f(x) = q \equiv F/L$$

$$\frac{\overline{AE}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P \\ -P \end{Bmatrix} + \begin{Bmatrix} qL/2 \\ qL/2 \end{Bmatrix}$$

Heat Transfer

DE

$$-\frac{d}{dx} \left(\alpha(x) \frac{dy(x)}{dx} \right) + \beta(x) y(x) = f(x)$$

Left Mixed
BC

$$\tau = c_a y + d_a$$

Right

Mixed BC

$$\tau = c_b y + d_b$$

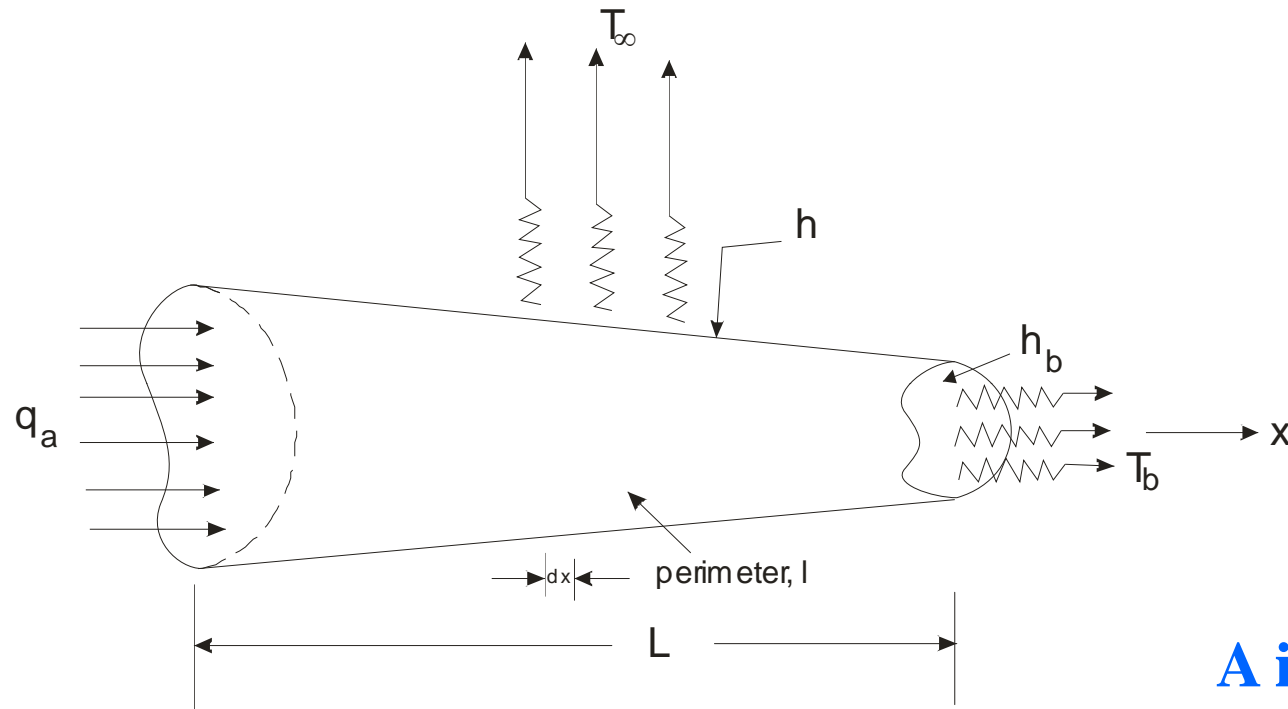
Comparing (assuming A is a constant)

$$\alpha(x) = k(x) \equiv \frac{E}{tLT}$$

$$\beta(x) = \frac{hl}{A} \equiv \frac{E}{tL^3T}$$

$$f(x) \equiv Q(x) + \frac{hl}{A} T_\infty \equiv \frac{E}{tL^3}$$

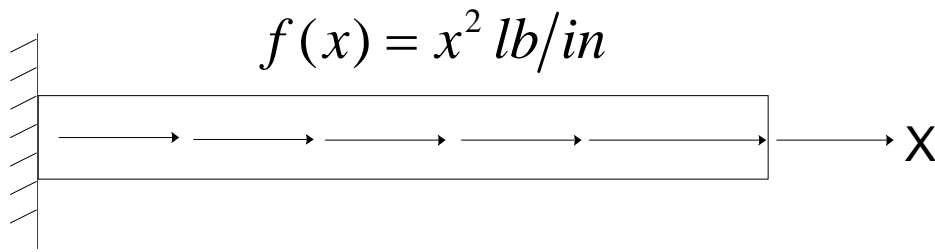
Heat Transfer



A is a NOT constant

$$\left[\left[\begin{array}{c|c} \frac{\bar{k}}{L} + \frac{\bar{h}lL}{3A} & -\frac{\bar{k}}{L} + \frac{\bar{h}lL}{6A} \\ \hline -\frac{\bar{k}}{L} + \frac{\bar{h}lL}{6A} & \frac{\bar{k}}{L} + \frac{\bar{h}lL}{3A} \end{array} \right] + h_b \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array} \right] \right] \left\{ \begin{array}{c} T_1 \\ T_2 \end{array} \right\} = \frac{L}{2} \left\{ \begin{array}{c} \frac{\bar{h}l}{A} T_\infty \\ \frac{\bar{h}l}{A} T_\infty \end{array} \right\} + \left\{ \begin{array}{c} q_a \\ 0 \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ h_b T_b \end{array} \right\}$$

Problem T4L2-1



The 4" long steel bar with a 2 in² cross-sectional area is loaded by the given surface traction. Find the displacement and force distribution in the bar.

Problem T4L2-1

Parameters

User Input

Name:

Expression:

Close

Update

Delete

Add

Current Parameter Data

| Name | Expression |
|------|------------|
| A | 2 |

Segment Data

User Input

Element Type:

Left End Coordinate:

Number of Elements:

Right End Coordinate:

alpha(x): x² + x +

beta(x): x² + x +

f(x): x² + x +

Parameters:

Close

Update

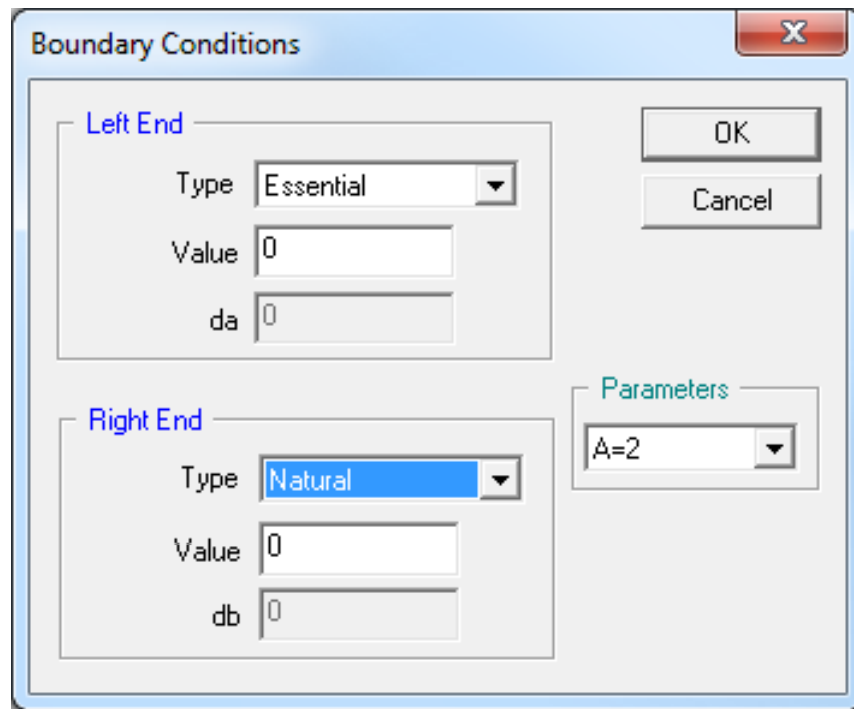
Delete

Add

Current Segment Data

| Segm... | Element Type | # Elements | Left End Coor | Right End Coor | alpha:X^2 |
|---------|--------------|------------|---------------|----------------|-----------|
| 1 | C0-Linear | 1 | 0 | 4 | 0 |

Problem T4L2-1



The image shows a software dialog box titled "Boundary Conditions" with a standard Windows-style title bar (minimize, maximize, close buttons). The dialog is divided into several sections. On the left, there are two main sections: "Left End" and "Right End". The "Left End" section has a "Type" dropdown menu set to "Essential", a "Value" text box containing "0", and a "da" text box containing "0". The "Right End" section has a "Type" dropdown menu set to "Natural", a "Value" text box containing "0", and a "db" text box containing "0". To the right of these sections, there are "OK" and "Cancel" buttons. Further to the right, there is a "Parameters" section with a dropdown menu set to "A=2".

Boundary Conditions

Left End

Type: Essential

Value: 0

da: 0

Right End

Type: Natural

Value: 0

db: 0

Parameters

A=2

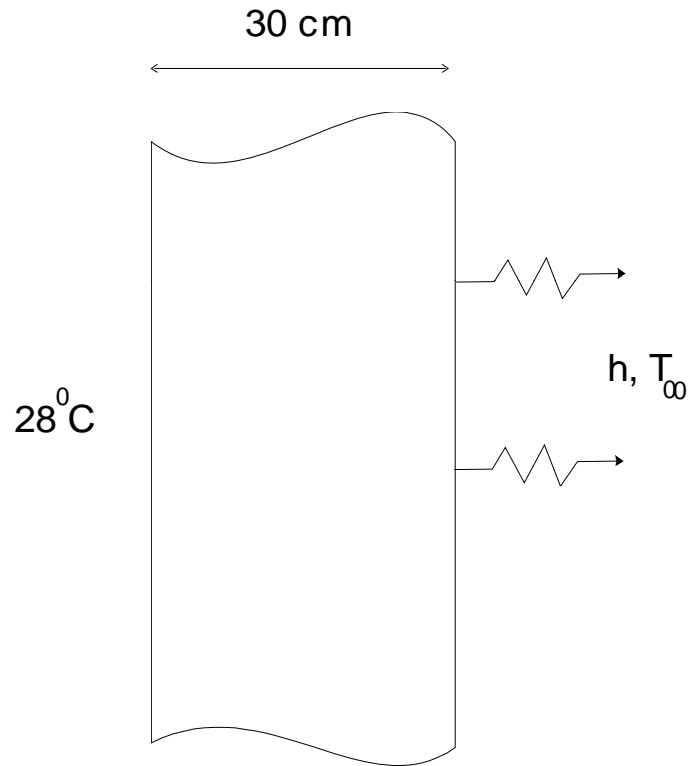
OK

Cancel

Problem T4L2-1

| Mesh | Tip Disp. (in) | Root Force (lb) | Tip Force (lb) |
|-------------|--------------------------|--------------------|-------------------|
| 1-element | 1.067(10 ⁻⁶) | 16 | 16 |
| 2-elements | 1.067(10 ⁻⁶) | 20.67 | 11.33 |
| 4-elements | 1.067(10 ⁻⁶) | 21.25 | 6.75 |
| 8-elements | 1.067(10 ⁻⁶) | 21.32 | 3.68 |
| 16-elements | 1.067(10 ⁻⁶) | 21.33 | 1.92 |

Problem T4L3-1



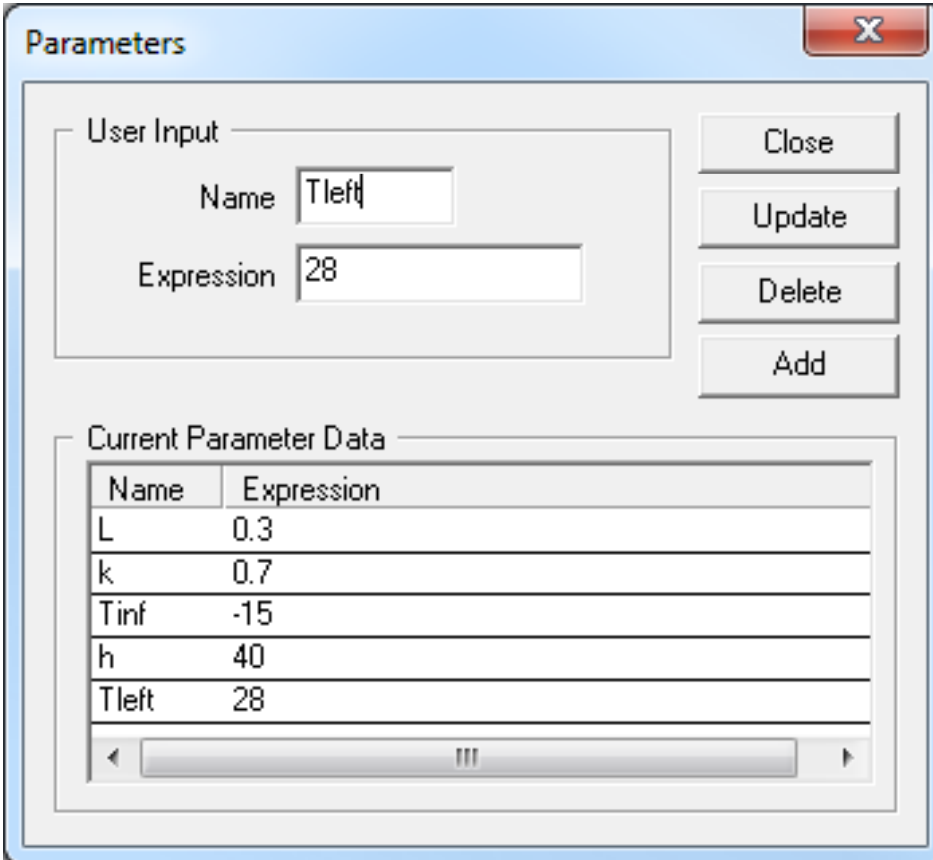
Brick wall: Determine the steady-state temperature distribution within the wall and also the heat flux through the wall.

$$k = 0.7 \text{ W/m} \cdot ^{\circ}\text{C}$$

$$T_{\infty} = -15^{\circ}\text{C}$$

$$h = 40 \text{ W/m}^2 \cdot ^{\circ}\text{C}$$

Problem T4L3-1



A screenshot of a software window titled "Parameters" with a standard Windows-style title bar (blue background, red close button). The window is divided into two main sections. The top section, labeled "User Input", contains two text input fields: "Name" with the value "Tleft" and "Expression" with the value "28". To the right of these fields are four buttons stacked vertically: "Close", "Update", "Delete", and "Add". The bottom section, labeled "Current Parameter Data", contains a table with two columns: "Name" and "Expression". The table lists five parameters: L (0.3), k (0.7), Tinf (-15), h (40), and Tleft (28). Below the table is a horizontal scrollbar.

| Name | Expression |
|-------|------------|
| L | 0.3 |
| k | 0.7 |
| Tinf | -15 |
| h | 40 |
| Tleft | 28 |

Problem T4L3-1

Segment Data [X]

User Input

Element Type: C0-Linear Left End Coordinate:

Number of Elements: Right End Coordinate:

alpha(x): x² + x +

beta(x): x² + x +

f(x): x² + x +

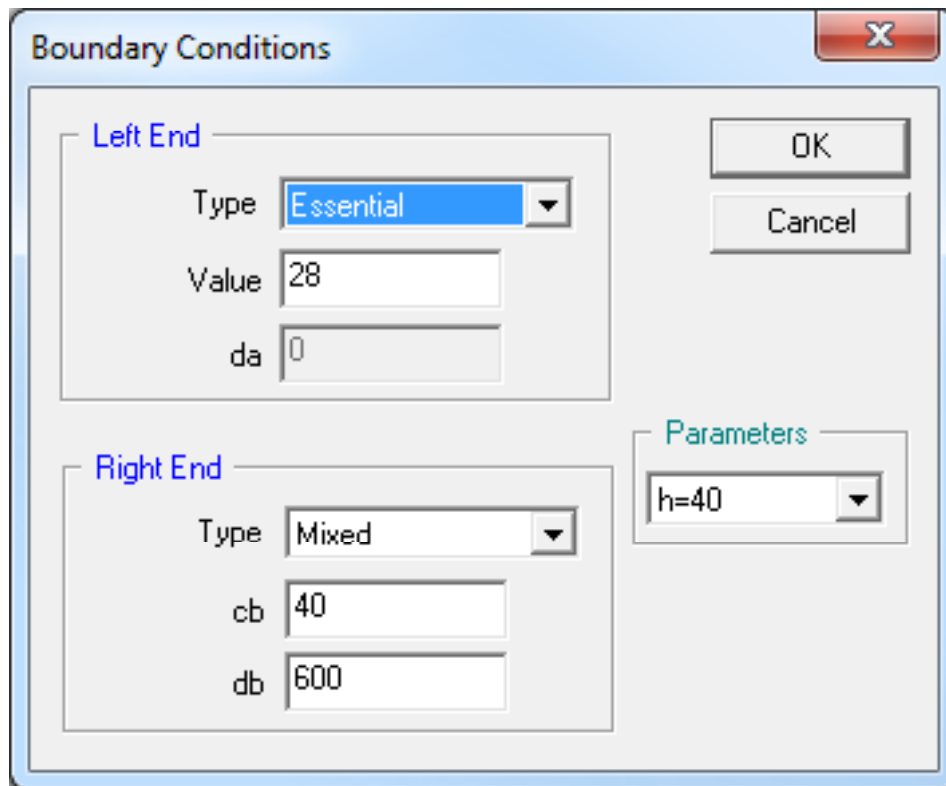
Parameters: h=40

Current Segment Data

| Segm... | Element Type | # Elements | Left End Coor | Right End Coor | alpha:X ² |
|---------|--------------|------------|---------------|----------------|----------------------|
| 1 | C0-Linear | 1 | 0 | 0.3 | 0 |

III

Problem T4L3-1



The image shows a software dialog box titled "Boundary Conditions" with a close button (X) in the top right corner. The dialog is divided into two main sections: "Left End" and "Right End".

Left End:

- Type: Essential (selected in a dropdown menu)
- Value: 28 (text input field)
- da: 0 (text input field)

Right End:

- Type: Mixed (selected in a dropdown menu)
- cb: 40 (text input field)
- db: 600 (text input field)

Parameters:

- h=40 (selected in a dropdown menu)

At the bottom right of the dialog are two buttons: "OK" and "Cancel".

Problem T4L3-1

| Mesh | Right Temp. (C) | Right Flux (W/m ²) | |
|------------|--------------------|-----------------------------------|--|
| 1-element | -12.63 | 94.8 | |
| 2-elements | -12.63 | 94.8 | |
| 4-elements | -12.63 | 94.8 | |