ELECTRONIC NOTES

CEE598 - Finite Elements for Engineers: Module 2

Part 1: Introduction

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Finite Elements for Engineers

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Introduction

"Change is inevitable, growth is optional." Anon.

This course is the second in a three-part series or modules titled "Finite Elements for Engineers" that meets the mathematics requirements for the Master of Engineering (M. Eng.) degree in the College of Engineering at Arizona State University.

Who should take this course?

Finite elements has become the defacto industry standard for solving multi-disciplinary engineering problems that can be described by equations of calculus. Applications cut across several industries by virtue of the applications – solid mechanics (civil, aerospace, automotive, mechanical, biomedical, electronic), fluid mechanics (geotechnical, aerospace, electronic, environmental, hydraulics, biomedical, chemical), heat transfer (automotive, aerospace, electronic, chemical), acoustics (automotive, mechanical, aerospace), electromagnetics (electronic, aerospace) and many, many more.

Course Objectives

- To understand the Variational Technique.
- To understand Isoparametric Formulation.
- To understand and apply the Variational and Galerkin's Method in solving two and three-dimensional engineering problems.

Prerequisites

- Mathematics – Linear Algebra, Numerical Analysis, Partial and/or ordinary differential equations.
- Knowledge of undergraduate core material from (at least two of) solid mechanics, fluid mechanics, heat transfer and electromagnetics. Knowledge in one area should be at an advanced level (aligned with the undergraduate major).
- Knowledge of a high level programming language and use of computer-based tools.
- Module 1

Instructor-Student Interaction

To successfully meet the course objectives it is necessary that the students avail themselves of all the resources – discussion forums, e-mail, chat rooms, libraries. Keep the instructor and teaching assistant informed of all your concerns. The web pages connected with this course will contain instructions on how to communicate with the instructor regarding the questions you may have or turning in the assignments etc.

Computer Programs and Computer Aids

You are also free to use, when stated, commercial programs that you may have access to.

Syllabus

"A man with one watch knows what time it is. A man with two watches is never sure." Segal's Law.

Outline (Lesson Plan)

- Variational Techniques
- Basics of Isoparametric Formulation
- Problems in Solid Mechanics
- Boundary Value Problems

Notation

"As complexity rises, precise statements lose meaning, and meaningful statements lose precision."
Lotfi Zadeh

Vectors

$\mathbf{a}_{n\times 1}$ column vector with n rows
a_i element i of vector $\mathbf{a}$
$\mathbf{b}_{1\times m}$ row vector with m columns

Matrices

$\mathbf{A}_{m\times n}$ matrix with m rows and n columns
A_{ij} element row i and column j of matrix

Others

y' Derivative of y (or, $\frac{dy}{dx}$)
L (Units of) length
F (Units of) force
M (Units of) mass
t (Units of) time
T (Units of) temperature
E (Units of) energy

Lesson Plan

"Problems cannot be solved by the same level of thinking that created them." A. Einstein.

Module 2 is divided into four topics. Each topic has several lessons designed to focus on the critical issues. With each lesson there is a set of objectives. I have also listed the relevant pages from the list of textbooks that appear in the syllabus. There are several review problems at the end of every topic. Solutions to most problems are also provided. Note that the set of problems represents the minimal set needed to understand the material. You should solve more problems from some of the referenced texts.

Topic 1 looks at the second major FE formulation – the Variational techniques. We looked at the Method of Weighted Residuals in Module 1. The Variational techniques provide specific advantages especially for solving problems in the solid mechanics area.

Topic 1: Variational Techniques

Lesson 1: Ritz Method.
Lesson 2: Finite Element Formulation.
Review Exercises

Modern FE numerical techniques are based on the isoparametric formulation for constructing the element equations. The formulation makes it easy to construct the element equations in the spatial domain and provides a convenient path from low-order to higher-order finite elements.

Topic 2: Basics of Isoparametric Formulation

Lesson 1: Generating Shape Functions.
Lesson 2: Numerical Integration.
Lesson 3: Isoparametric Finite Elements.
Review Exercises

Topics 3 and 4 form the backbone of Module 2. We will look at using FE software to solve practical engineering problems.

Topic 3: Problems in Solid Mechanics

Lesson 1: Discrete Structures – Truss and Beam Elements.
Lesson 2: Plane Elasticity.
Lesson 3: Axisymmetric Problems.
Review Exercises

Topic 4: Boundary Value Problems

Lesson 1: Two-Dimensional BVP.
Lesson 2: Planar Engineering Applications.
Lesson 3: Axisymmetric BVP.
Lesson 4: Axisymmetric Engineering Applications.
Review Exercises

Topic 1: Variational Techniques

"I would remind you that extremism in the defense of liberty is no vice! And let me remind you also that moderation in the pursuit of justice is no virtue!" Barry Goldwater

"Extremism in the pursuit of the Presidency is an unpardonable vice. Moderation in the affairs of the nation is the highest virtue." Lyndon Johnson

Lesson 1: Ritz Method

Objectives: In this lesson we will look at the Ritz Method to generate approximate solutions – another approach to deriving the finite element equations. The major objectives are listed below.

- To understand what is meant by the Ritz Variational Method.
- To couple the Ritz Method with the Theorem of Minimum Potential Energy.

What is the Ritz Variational Method?

In module 1, we saw the Galerkin's Method in detail. The differential equations describing the problem were handled directly and were used to generate the element equations. The Ritz Variational Method handles the solution quite differently. Instead of using the original differential equations, the solution is dependent on the existence of an appropriate variational principle.

The relationship between the one-dimensional boundary value problem[1]

$$-\frac{d}{dx}\left(\alpha(x)\frac{dy(x)}{dx}\right)+\beta(x)y(x)=f(x) \qquad a\le x\le b \tag{T1L1-1}$$

$$-\alpha\left.\frac{dy}{dx}\right|_{x=a}+gy(a)=c_a \qquad x=a \tag{T1L1-2}$$

$$-\alpha\left.\frac{dy}{dx}\right|_{x=b}+hy(b)=c_b \qquad x=b \tag{T1L1-3}$$

and its corresponding functional[2]

$$I(y)=\frac{1}{2}\int_a^b\left[\alpha\left(\frac{dy}{dx}\right)^2+\beta y^2\right]dx+\frac{gy(b)^2}{2}+\frac{hy(a)^2}{2}$$
$$-\int_a^b fy\,dx-c_by(b)-c_ay(a) \tag{T1L1-4}$$

is that the stationary value given by

$$\delta I(y)=0 \tag{T1L1-5}$$

is the boundary value problem described by Eqns. (T1L1-1) to (T1L1-3).

In the area of solid mechanics, one of the variational principles is the Theorem of Minimum Potential Energy with the potential energy being the functional.

Theorem of Minimum Potential Energy (Solid Mechanics)

The Theorem of Minimum Potential Energy states that for a <u>conservative</u> system, amongst all <u>admissible configurations</u> those that satisfy the equations of <u>equilibrium</u> make the potential

[1] It is suggested that you read a text (e.g. T2) that describes the weak formulation, if you are interested in a formal link between a differential equation and its corresponding variational principle. Understanding this link, while desirable, is not essential in following the material that follows.

[2] A functional is a function of functions.

energy stationary with respect to small variations of displacement. If the stationary condition is a minimum, the equilibrium state is stable.

Pay particular attention to the underlined terms to understand the applicability and limitations of the theorem.

Consider the following situation. Let Π denote the total potential energy of the system. Let the potential energy be a function of a set of displacements $\mathbf{D} = \{D_1, D_2, ..., D_n\}$. If the displacements satisfy the boundary conditions such that the system is in stable equilibrium, then the following conditions must be satisfied

$$\frac{\partial \Pi}{\partial \mathbf{D}} = 0 \quad i = 1, 2, ... n \tag{T1L1-6}$$

and can be used to compute the displacements. Figs. T1L1-1(a)-(c) show the state of equilibrium of a sphere resting on different surfaces.

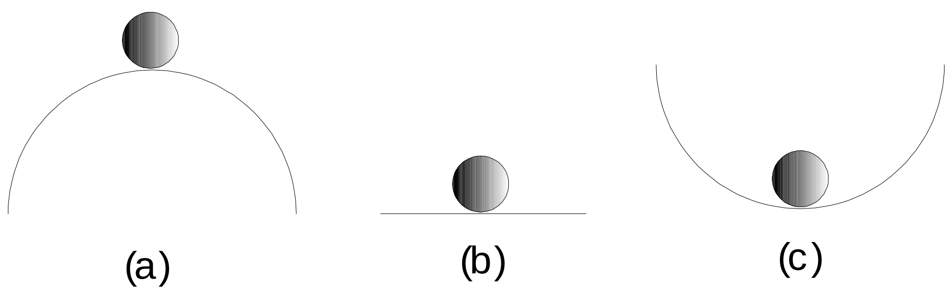


Fig. T1L1-1(a) Unstable equilibrium (b) Neutral equilibrium (c) Stable equilibrium

The unstable equilibrium corresponds to the state of highest potential energy of the sphere, the neutral equilibrium state shows no change in the potential energy when the sphere is perturbed slightly, and the stable equilibrium state corresponds to the state of lowest potential energy of the sphere (in the immediate neighborhood). The same comments apply to structural systems. We are interested in finding the equilibrium state that has the minimum potential energy so that the structural system is stable.

Total Potential Energy

The total potential energy of a linearly elastic system is given by

$$\Pi = \text{strain energy} + \text{work potential}$$

$$= \int_V U_0 \, dV - \int_V \mathbf{f}^T \mathbf{F} \, dV - \int_S \mathbf{f}^T \Phi \, dS - \mathbf{D}^\mathbf{T} \mathbf{P} \tag{T1L1-7}$$

where

U_0 strain energy per unit volume

$\mathbf{f}$ displacement field, e.g. $\{u \quad v \quad w\}$ in three dimensions

$\mathbf{F}$ body forces per unit volume

Φ surface tractions per unit area

D nodal displacements
P concentrated forces

The strain energy density is given by

$$U_0 = \frac{1}{2}\{\varepsilon\}^T \mathbf{E}\{\varepsilon\} - \{\varepsilon\}^T \mathbf{E}\{\varepsilon_0\} + \{\varepsilon\}^T\{\sigma_0\} \tag{T1L1-8}$$

where

$\{\varepsilon\}$ strain components
$\{\varepsilon_0\}$ initial strain components
$\{\sigma_0\}$ initial stress components
E material matrix relating strains and stresses

$$\{\sigma\} = \mathbf{E}\{\varepsilon\} - \mathbf{E}\{\varepsilon_0\} + \{\sigma_0\} \tag{T1L1-9}$$

Illustrative Example T1L1-1

Consider a bar of constant cross-section A, length L and modulus of elasticity E subjected to a constant axial force P at the right tip and fixed at the left end. Compute the tip displacement and the state of stress in the bar.

Solution: Let the tip displacement be D. This is the sole unknown or degree-of-freedom in this problem. Using Eqns. (T1L1-7)-(T1L1-8), we have

$$\varepsilon_x = D/L$$

$$dV = A\,dx$$

$$U = \int_0^L \left\{ \frac{1}{2}\left(\frac{D}{L}\right) E \left(\frac{D}{L}\right) \right\} A\,dx$$

$$\Pi(D) = \int_V U_0\, dV - PD = \frac{D^2 EA}{2L} - PD \tag{T1L1-10}$$

Using the Theorem of Minimum Potential Energy via Eqn. (T1L1-6), we have

$$\frac{d\Pi}{dD} = 0 = \frac{DEA}{L} - P \tag{T1L1-11}$$

or,

$$D = \frac{PL}{AE} \tag{T1L1-12}$$

Hence,

$$\varepsilon_x = D/L = \frac{P}{AE} \tag{T1L1-13}$$

and

$$\sigma = E\varepsilon_x = \frac{P}{A} \tag{T1L1-14}$$

Let's examine the process and the solution. We assumed that the entire problem could be described by a single unknown D at the tip of the bar. Is this correct? With the displacement at the left end assumed to be zero (since it is fixed) and the right end displacement as D, the net effect of the assumptions is that the bar has a linear displacement field, i.e. a linear function with a value zero at $x=0$ and D at $x=L$ describes the deformation of the bar. This assumption is certainly valid for this problem but is not true if the loading on the bar is changed for example. How can we overcome this *ad hoc* nature and formalize the solution process? We will do so the same way we did in Module 1. We will assume the solution, i.e. the displacement field in this case, and proceed to use the Theorem of Minimum Potential Energy. This is the **Rayleigh-Ritz** Technique.

Rayleigh-Ritz Technique

The first step is to assume the form of the solution. The assumed form will have one or more unknown parameters or degrees-of-freedom. The assumed form must be able to satisfy the essential boundary conditions for the problem. Note that this step is no different than the trial solution that we assumed in the MWR process. The next step is to use this assumed form of the solution and construct the total potential energy. The third step is to use Eqn. (T1L1-6) and generate the equations which then can be solved for the unknown parameters.

Illustrative Example T1L1-2

Resolve Example T1L1-1.

Solution: Let the displacement field $\mathbf{f}=u$ (a single displacement component) in the bar be described by a linear function

$$u(x)=a_0+a_1x \qquad \text{(T1L1-15)}$$

The essential boundary condition for this problem is

$$u(x=0)=0 \qquad \text{(T1L1-16)}$$

Substituting Eqn. (T1L1-16) in (T1L1-15), we have

$$u(x=0)=0=a_0 \qquad \text{(T1L1-17)}$$

Hence,

$$u(x)=a_1x \qquad \text{(T1L1-18)}$$

The axial strain ε_x is given by

$$\varepsilon_x=\frac{du}{dx}=a_1 \qquad \text{(T1L1-19)}$$

Substituting Eqns. (T1L1-18) and (T1L1-19) into Eqn. (T1L1-7)

$$\Pi(a_1)=\int_V U_0\,dV-PD=\int_0^L \frac{1}{2}(a_1)(E)(a_1)\,A\,dx-P(a_1L)$$

$$=\frac{1}{2}a_1^2EAL-Pa_1L \qquad \text{(T1L1-20)}$$

Using the Theorem of Minimum Potential Energy via Eqn. (T1L1-6), we have

$$\frac{d\Pi}{da_1} = 0 = a_1 EAL - PL \tag{T1L1-21}$$

or, $$a_1 = \frac{P}{AE} \tag{T1L1-22}$$

Hence,

$$u(x) = \frac{Px}{AE}$$

$$\varepsilon_x = \frac{du}{dx} = \frac{P}{AE} \tag{T1L1-23}$$

and

$$\sigma = E\varepsilon_x = \frac{P}{A} \tag{T1L1-24}$$

The results are the same as Example 1. However, we have solved a simple problem so far where the solution is smooth. Let us look at a slightly different problem.

Illustrative Example T1L1-3

Consider a bar of unit length that is fixed at both ends and is loaded by a unit point at the center of the bar. Assume that $AE = 1$. Find the displacement and the stresses in the bar.

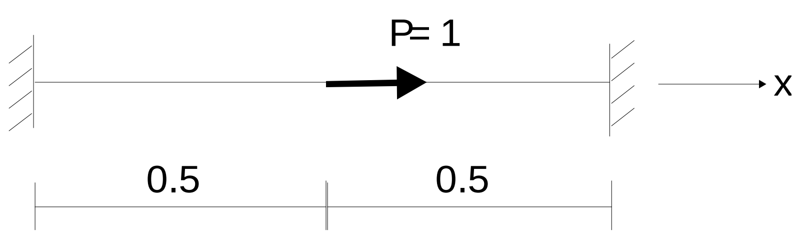


Fig. T1L1-2

Solution: Again we assume a polynomial as the solution as[3]

$$u(x) = a_0 + a_1 x + a_2 x^2 \tag{T1L1-25}$$

The essential boundary conditions for this problem are

$$u(x=0) = 0$$
$$u(x=1) = 0 \tag{T1L1-26}$$

Substituting Eqn. (T1L1-26) into (T1L1-25), we have

$$u(x=0) = 0 = a_0$$
$$u(x=1) = 0 = a_1 + a_2 \Rightarrow a_2 = -a_1 \tag{T1L1-27}$$

Hence,

[3] This is the lowest order polynomial that can be assumed as the solution.

$$u(x) = a_1 x - a_1 x^2 \tag{T1L1-28}$$

and $$\frac{du}{dx} = a_1 - 2a_1 x \tag{T1L1-29}$$

Now constructing the total potential energy, we have

$$\Pi(a_1) = \int_V U_0\, dV - PD = \int_0^1 \frac{1}{2}\left(a_1 - 2a_1 x\right)^2 dx - (1)(0.5a_1 - 0.25a_1)$$

$$= \frac{a_1^2}{6} - 0.25a_1 \tag{T1L1-30}$$

Using the Theorem of Minimum Potential Energy via Eqn. (T1L1-6), we have

$$\frac{d\Pi}{da_1} = 0 = \frac{a_1}{3} - 0.25 \tag{T1L1-31}$$

or, $$a_1 = 0.75 \tag{T1L1-32}$$

Hence,

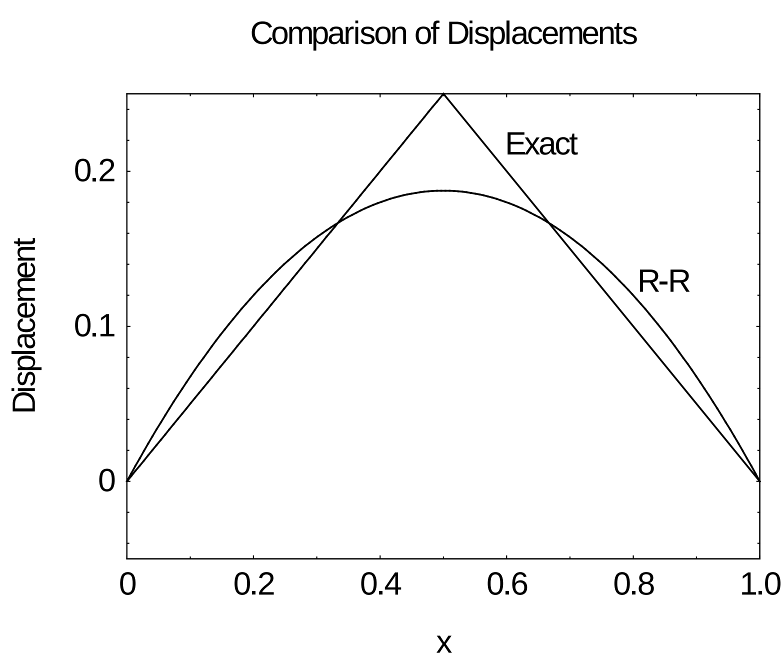
$$u(x) = 0.75\left(x - x^2\right) \tag{T1L1-33}$$

$$\varepsilon_x = \frac{du}{dx} = 0.75 - 1.5x \tag{T1L1-34}$$

and

$$\sigma = E\varepsilon_x = 0.75 - 1.5x \tag{T1L1-35}$$

Fig. T1L1-3 shows the comparison between the RR solution and the exact (mechanics of materials) solution.



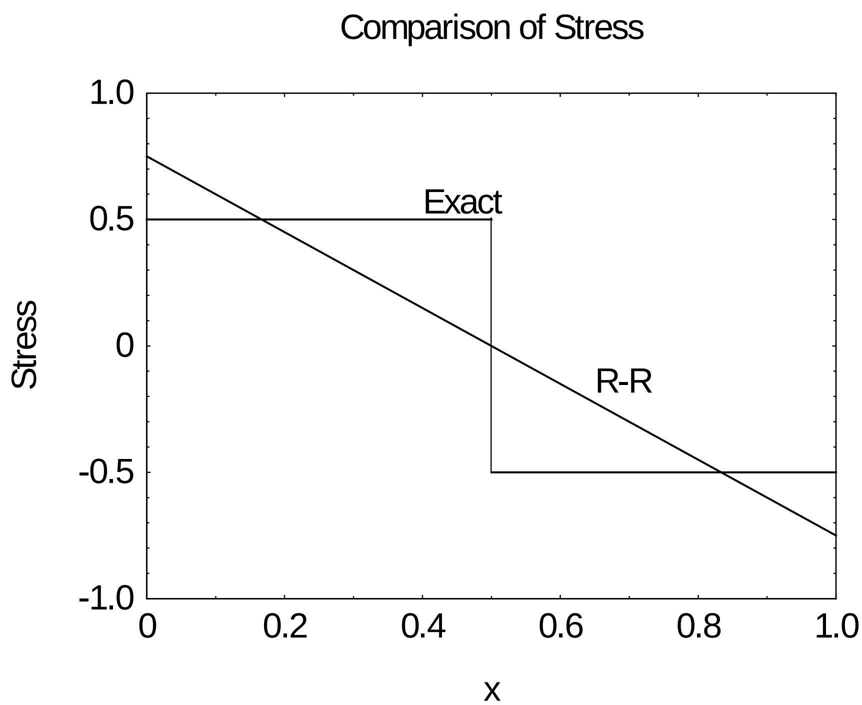


Fig. T1L1-3 Comparison of results

The exact displacement is piecewise linear and the exact stress is discontinuous. Hence it should be clear that increasing the order of the polynomial while yielding better solutions will never yield the exact solution to this problem.

Lesson 2: Finite Element Formulation

Objectives: In this lesson we will mesh the concepts associated with the Ritz Method with the finite element concepts seen in Module 1.

- To understand the limitations of the classical Ritz Method.
- To understand how to couple the finite element ideas with the Ritz Method.

Rayleigh-Ritz Based Finite Elements

The R-R Technique while powerful, has severe limitations in being a practical tool. First, the assumed solution is valid for the entire problem domain. As we saw in the last lesson, very simple solutions such as piecewise linear cannot be obtained. Second, the assumed solution lacks physical meaning. For example, if the assumed solution is a polynomial, what do the two coefficients represent?

Two elegant modifications can be made to the above procedure. The Rayleigh-Ritz concept (of assuming an approximate solution) can be used over an element instead of the entire problem domain. And the assumed solution can be transformed to the unknown nodal values using the concept of interpolation.

Illustrative Example T1L2-1

Resolve Example T1L1-3.

Solution: The basic approach now will be to discretize the domain into finite elements. Let us use two elements – one from $x = 0$ to $x = 0.5$ and the other from $x = 0.5$ to $x = 1.0$.

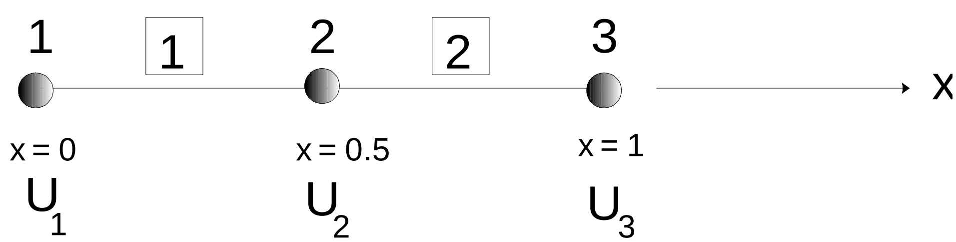


Fig. T1L2-1

We could assume the following trial solution for element 1 as

$$u_1(x) = a_1 + a_2 x \quad \text{(T1L2-1)}$$

and for element 2 as

$$u_2(x) = b_1 + b_2 x \quad \text{(T1L2-2)}$$

To ensure that the displacement is continuous at the element interface, i.e. at $x = 0.5$, we could enforce the following constraint

$$u_1(x = 0.5) = u_2(x = 0.5) \quad \text{(T1L2-3)}$$

The approach is laborious especially as the size of the problem increases. Instead we can convert the trial solution to a form involving the nodal values via the concept of interpolation (as we saw in Module 1 with the MWR approach). A typical element we can assume (s is a local coordinate system with the same sense as x)

$$u(s) = a_1 + a_2 s \tag{T1L2-4}$$

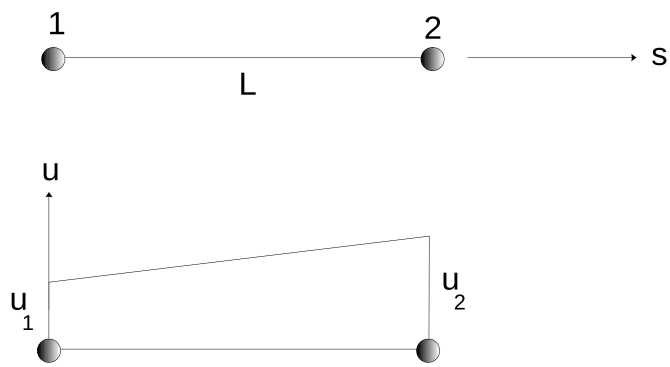


Fig. T1L2-2 A typical element

Using the nodal conditions, $u(s=0)=u_1$ and $u(s=L)=u_2$, we can rewrite the above equation as

$$u(s) = a_1 + a_2 s = \frac{L-s}{L}u_1 + \frac{s}{L}u_2 = \phi_1 u_1 + \phi_2 u_2 \tag{T1L2-5}$$

where ϕ_1 and ϕ_2 are the shape functions that we saw in Topic 4 in Module 1. Hence, the strain and stress in the element can be expressed as

$$\begin{aligned}\varepsilon = \frac{du}{ds} &= -\frac{1}{L}u_1 + \frac{1}{L}u_2 = \frac{1}{L}(u_2 - u_1) \\ &= \left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor_{1\times 2} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_{2\times 1} = \mathbf{B}_{1\times 2}\mathbf{d}_{2\times 1}\end{aligned} \tag{T1L2-6}$$

$$\sigma = E\varepsilon \tag{T1L2-7}$$

$\mathbf{B}_{1\times 2}$ is known as the strain-displacement matrix since it relates strain, ε (left-hand side) to the displacements, $\mathbf{d}_{2\times 1}$ (right-hand side). Note that the strain and the stress within the element are constants. Hence, the strain energy in a typical element can be written as

$$U = \int_V U_0 \, dV = \frac{1}{2}\int_V \varepsilon\sigma \, dV = \frac{1}{2}\varepsilon \, E \, \varepsilon \, AL \tag{T1L2-8a}$$

Substituting Eqns. (T1L2-6) and (T1L2-7) into (T1L2-8a), we have

$$U = \frac{1}{2}\mathbf{d}_{1\times2}^T \mathbf{B}_{2\times1}^T \left(EAL\right)_{1\times1} \mathbf{B}_{1\times2} \mathbf{d}_{2\times1} \tag{T1L2-8b}$$

$$U = \frac{1}{2}\mathbf{d}_{1\times2}^T\ \mathbf{k}_{2\times2}\ \mathbf{d}_{2\times1} \tag{T1L2-8c}$$

where $\mathbf{k}_{2\times2}$ is the element stiffness matrix given as

$$k_{2\times2} = \mathbf{B}_{2\times1}^T \left(EAL\right)_{1\times1} \mathbf{B}_{1\times2} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{T1L2-8d}$$

This matrix is the same as derived in Module 1 (Topic 4 Lesson 2).

Since there are no body forces or surface tractions in this problem, the work potential term need not be computed. In general, however, we have to compute the work potential. The total potential energy in a typical element is given by

$$\Pi_e(\mathbf{d}) = \frac{1}{2}\mathbf{d}_{1\times2}^T\ \mathbf{k}_{2\times2}\ \mathbf{d}_{2\times1} + \text{work potential} \tag{T1L2-9}$$

Now using the numerical data for the problem, we have the following.

Element 1: $EA = 1,\ L = 0.5,\ \mathbf{d}_{1\times2}^T = \left\{U_1, U_2\right\}$.
Element 2: $EA = 1,\ L = 0.5,\ \mathbf{d}_{1\times2}^T = \left\{U_2, U_3\right\}$.

Hence the total potential energy in the system can be written as

$$\Pi(\mathbf{D}) = \frac{1}{2}\mathbf{D}_{1\times3}^T \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{D}_{3\times1} + \frac{1}{2}\mathbf{D}_{1\times3}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \mathbf{D}_{3\times1}$$

$$-(1)U_2 \tag{T1L2-10}$$

where $\mathbf{D}^T = \left\{U_1 \quad U_2 \quad U_3\right\}$ is the vector of (system) nodal displacements, the first term is due to the strain energy in element 1, the second term is due to the strain energy in element 2 and the last term is the work potential due to the concentrated force $P = 1$ acting at $x = 0.5$.

Using the Theorem of Minimum Potential Energy by finding the stationary point of $\Pi(\mathbf{D})$

$$\frac{\partial \Pi}{\partial U_1} = 0 = 2U_1 - 2U_2$$

$$\frac{\partial \Pi}{\partial U_2} = 0 = -2U_1 + 4U_2 - 2U_3 - 1 \qquad \text{(T1L2-11)}$$

$$\frac{\partial \Pi}{\partial U_3} = 0 = -2U_2 + 2U_3$$

The three equations can be written in the matrix form as

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \qquad \text{(T1L2-12)}$$

or, $$\mathbf{K}_{3\times3}\mathbf{D}_{3\times1} = \mathbf{F}_{3\times1} \qquad \text{(T1L2-13)}$$

These are the system equations. The process of obtaining these equations was a bit involved. We could have generated the elements equations and gone through the assembly process as we did in Module 1. The process was employed here to show the equivalence between the MWR and the Variational technique.

Now imposing the boundary conditions, $U_1 = U_3 = 0$, we have a effectively single equation to solve

$$4U_2 = 1 \Rightarrow U_2 = 0.25 \qquad \text{(T1L2-14)}$$

which is the exact solution! Now the strains and stresses can be computed in each element using the equations developed earlier (T1L2-6 and T1L2-7).

Concluding Remarks

In this topic we saw a different approach to solving the engineering problem. If a functional or a variational principle can be found, the stationary point gives us the solution to the problem. For the types of the problems addressed in this course, the solution obtained using the Variational Technique is the same as the solution obtained from the Weighted Residual Method. The starting point is the major difference between the two approaches. In the case of MWR, the differential equation is the starting point. In the case of Variational Method, a functional is the starting point. Beyond this point, the implementation of the two approaches is the same. The element concept applies in the same manner as do the rest of the steps.

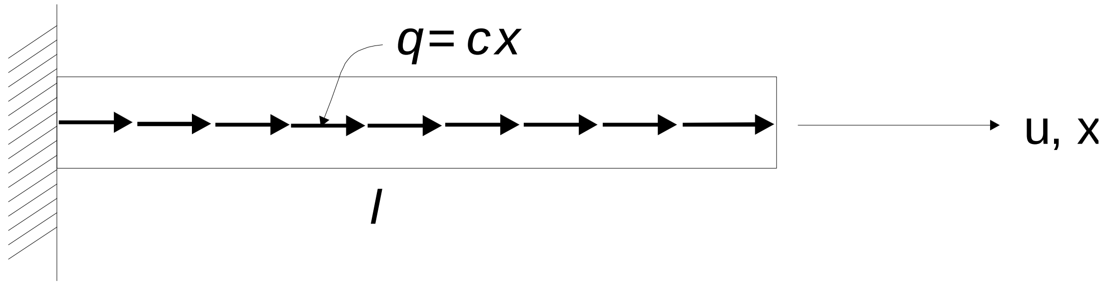
In the area of solid mechanics, most commonly encountered problems can be solved using the Theorem of Minimum Potential Energy. The key is to compute the components of the potential energy – the strain energy and the work potential.

Variational Principles do exist in other engineering areas and can be used in a similar manner.

Review Exercises

Problem T1L1-1

Figure shows a bar (constant cross-sectional area A, modulus of elasticity E and length l) subjected to an axial loading, $q = cx$ where c is a constant.



Using the Rayleigh-Ritz Method compute the axial displacement assuming

(a) the axial displacement is a linear function,

(b) the axial displacement is a quadratic function, and

(c) the axial displacement is a cubic function.

Compare the solutions to the exact solution by plotting the displacements and the stresses.

Problem T1L2-1

Resolve Problem T1L1-1 using the finite element method.

(a) Use a two-element mesh.

(b) Use a four-element mesh.

(c) Use an eight-element mesh.

Plot the solutions and compare them to the exact solution.

Topic 2: Basics of the Isoparametric Formulation

"I have nothing new to teach the world. Truth and Non-violence are as old as the hills. All I have done is to try experiments in both on as vast a scale as I could.." Mohandas Gandhi

Lesson 1: Generating Shape Functions

Objectives: In this lesson we will look at expanding the concept of interpolation as applied to elements in one, two and three dimensions.

- To understand the concept of shape functions.
- To understand the natural coordinate system and mapping.
- To understand how to generate the shape functions.

Shape Functions and Interpolation

As we saw with one-dimensional problems in Module 1, one of the fundamental ideas in finite elements is the concept of interpolation. Once the form of the unknown is assumed, e.g. a quadratic polynomial, the interpolation function is constructed using shape functions and the corresponding nodal values.

The element geometry is linked directly to the spatial domain of the problem and the order of the trial solution. The discretization of the spatial domain into finite elements involves simple shapes whose properties can be evaluated easily. In two dimensions, the simple shapes are triangle and quadrilateral. In three dimensions, the simple shapes are tetrahedron and hexahedron (other shapes such as wedge, prism, pyramid are less commonly used).

Standard Process: So far we have assumed the solution for a 1D problem in the form of a polynomial

$$y(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n \qquad \text{(T2L1-1)}$$

where n is the number of degrees-of-freedom in the element. The requirements of the assumed solution (or, trial solution) are as follows.

(1) The polynomial is continuous and complete, i.e. it contains all the terms … constant, linear, and so on. The family of one-dimensional elements is shown below.

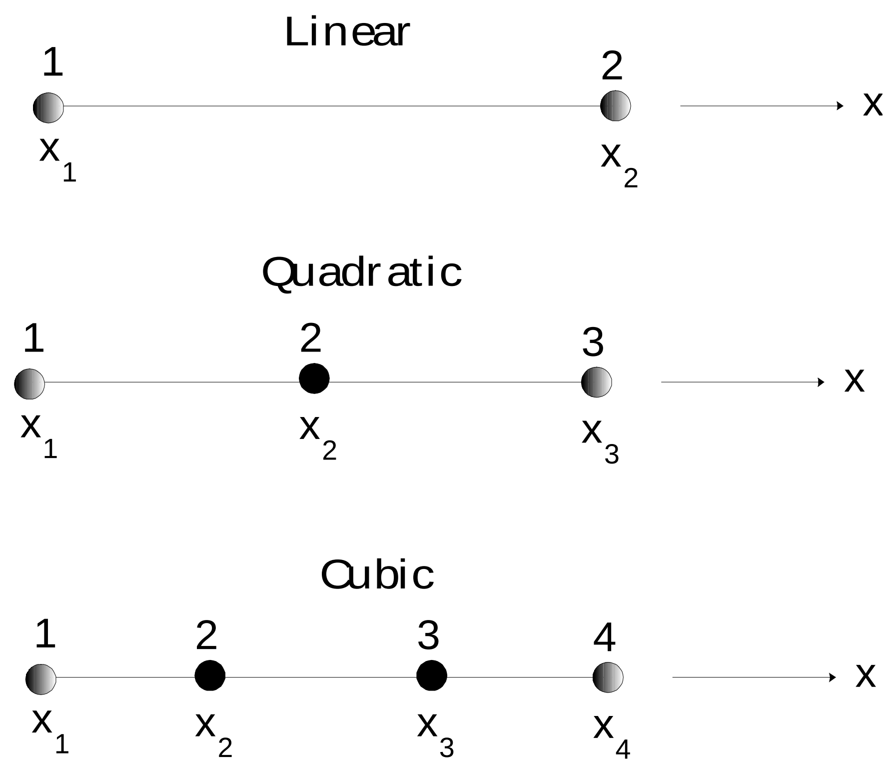


Fig. T2L1-1(a) Family of linear, quadratic and cubic one-dimensional elements

(2) The number of DOFs is equal to the number of nodal conditions. For the type of elements that we have seen so far, this is equal to the number of nodes in the element[4]. In other words,

$$y(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = \phi_1 y_1 + \phi_2 y_2 + ... + \phi_n y_n \qquad \text{(T2L1-2)}$$

where ϕ_i are the shape functions.

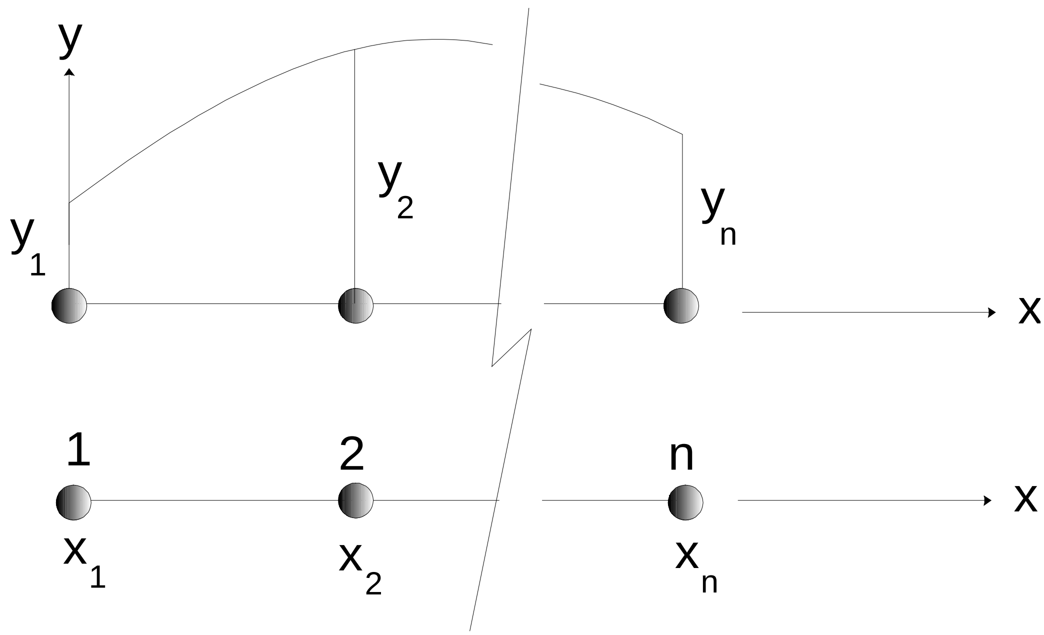


Fig. T2L1-1(b) Trial solution and mesh for $1D - C^0$ problems

The standard process of using the nodal conditions to compute the coefficients $a_0 ... a_n$ involves constructing the equations as

$$\begin{aligned} y_1 &= a_0 + a_1 x_1 + ... + a_n x_1^n \\ y_2 &= a_0 + a_1 x_2 + ... + a_n x_2^n \\ &.... \\ y_n &= a_0 + a_1 x_n + ... + a_n x_n^n \end{aligned} \qquad \text{(T2L1-3)}$$

[4] The shape functions associated with the trial solutions that we will see in this lesson, apply to elements with simple degrees-of-freedom involving just the unknown. These elements are C^0 elements implying that the unknown variable is continuous across element boundaries. Notationally, a C^m element is one where the variable and all its derivative upto order *m* are continuous across element boundaries.

Or, $$\begin{Bmatrix} y_1 \\ y_2 \\ .. \\ y_n \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & .. & x_1^n \\ 1 & x_2 & .. & x_2^n \\ .. & .. & .. & .. \\ 1 & x_n & .. & x_n^n \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ .. \\ a_n \end{Bmatrix} \quad \text{(T2L1-4)}$$

Or, $$\mathbf{y} = \mathbf{C}\,\mathbf{a} \quad \text{(T2L1-5)}$$

From the above equations,

$$\mathbf{a} = \mathbf{C}^{-1}\mathbf{y} \quad \text{(T2L1-6)}$$

We could rewrite Eqns. (T2L1-2) as

$$y = \mathbf{Pa} = \mathbf{PC}^{-1}\mathbf{y} \quad \text{(T2L1-7)}$$

where $P = \left\{1, x, ..., x^n\right\}$ (T2L1-8)

Hence, the shape functions can be computed as

$$y = \mathbf{Ny} = \mathbf{PC}^{-1}\mathbf{y}$$

Or, $$\mathbf{N} = \{\phi_1, \phi_2, ..., \phi_n\} = \mathbf{PC}^{-1} \quad \text{(T2L1-9)}$$

The problems with the above procedure are that computing $\mathbf{PC}^{-1}$ symbolically can be tedious and sometimes, $\mathbf{C}^{-1}$, may not exist.

Improved Process: To understand the deficiency in the standard process, let us develop the trial functions for two-dimensional elements. To carry out the task, we need to look at Pascal's triangle (see Fig. T2L1-2). The number of terms needed for different order polynomials can be found by examining the terms in the triangle. For example, a complete linear function has three terms, a complete cubic function has 10 terms etc.

				1						Constant
			x		y					Linear
		x^2		xy		y^2				Quadratic
	x^3		x^2y		xy^2		y^3			Cubic
x^4		x^3y		x^2y^2		xy^3		y^4		Quartic

Fig. T2L1-2 Pascal Triangle

Now let's look at the family of triangular elements (Fig. T2L1-3). The variation of the unknown, u, in the linear[5] element can be expressed as

$$u(x, y) = a_0 + a_1 x + a_2 y \tag{T2L1-10}$$

There are three coefficients and three nodes in the triangle. Note that we have used all the terms contained in the linear function (see Pascal's triangle). Should we first decide on the function and then construct the triangle with its nodes or vice-versa? The answer will become clear as we look at the quadratic triangular element. For a complete quadratic function we need a total of six terms (see Pascal's triangle).

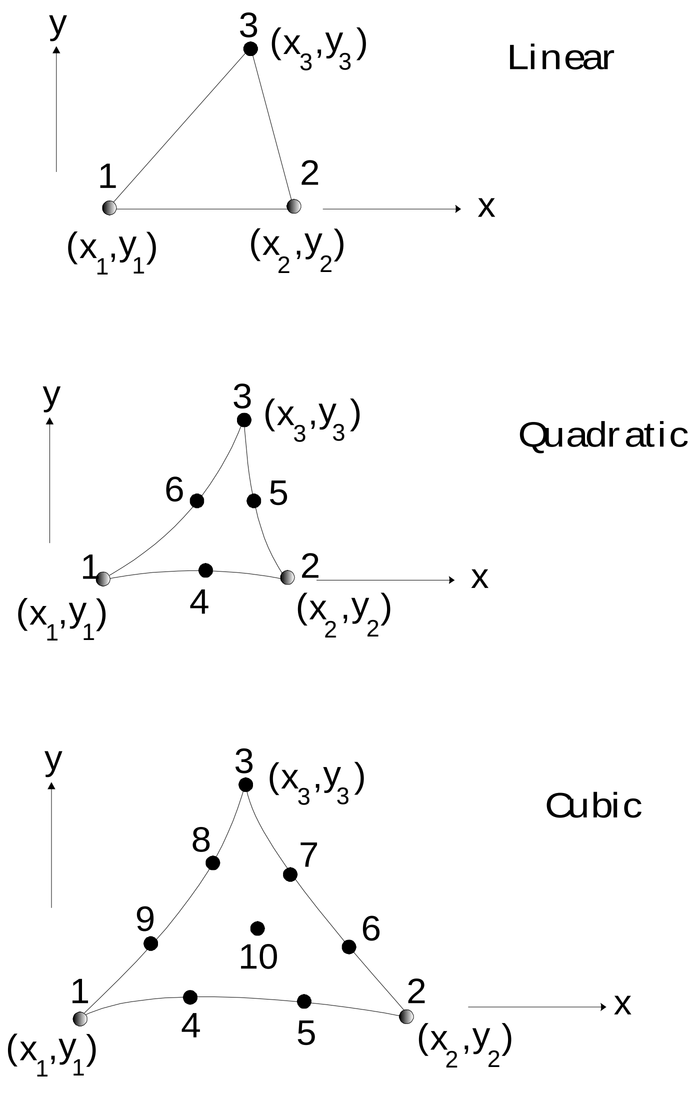


Fig. T2L1-3 Family of triangular elements

[5] The element should be called bilinear since it is linear in both x and y.

Hence,

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 \qquad \text{(T2L1-11)}$$

An examination of the element shows that the edges are quadratic since a quadratic function passes through three points. The next element in the family of triangular elements is the cubic element. From the Pascal's triangle, the cubic function contains a total of 10 terms. Hence

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^3 \qquad \text{(T2L1-12)}$$

To place 10 nodes in the triangle without losing symmetry, we need four nodes on each edge of the triangle. This will ensure that the edges of the triangle are cubic. The last node needs to be placed inside the triangle. There are two basic requirements when constructing the trial solution in multi-dimensions.

(1) The function must have *geometric isotropy*. In other words, in two-dimensional problems, the function must be symmetric with respect to both x and y. If this condition is not satisfied, the final solution will be dependent on the orientation of the coordinate system.

(2) The function should preferably be *complete*. An incomplete polynomial is acceptable provided no lower order terms are omitted. In other words, if a cubic polynomial is assumed, then the constant, linear and the quadratic terms cannot be omitted. If the lower order terms are omitted, convergence is generally poor[6].

Now let's look at the family of quadrilateral elements (Fig. T2L1-4). The variation of the unknown, u, in the linear element can be expressed as

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy \qquad \text{(T2L1-13)}$$

There are four coefficients and four nodes in the quadrilateral. The trial solution made up of an incomplete polynomial. The last term is selected so as to preserve geometric isotropy. In addition, we want to include all the lower-order terms before bringing in the higher-order terms. For the quadratic element, the solution can be assumed as

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + a_6 x^2 y + a_7 xy^2 \qquad \text{(T2L1-14)}$$

Again, the function is incomplete but the selection of the last two terms ensures geometric isotropy with the inclusion of all the lower-order terms. The variation of the unknown on the edges is quadratic. Similar comments apply to the cubic element in which the unknown can be expressed as

$$u(x,y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2 + a_6 x^2 y + a_7 xy^2 + a_8 x^3 + a_9 y^3 + a_{10} x^3 y + a_{11} xy^3 \qquad \text{(T2L1-15)}$$

[6] Another way of expressing this idea is that the shape functions should be selected with the highest complete polynomial for a given number of degrees of freedom.

These elements are described as "serendipity" quadrilateral elements. There is another family of quadrilateral elements called the "Lagrange" elements that are shown in Fig. T2L1-5.

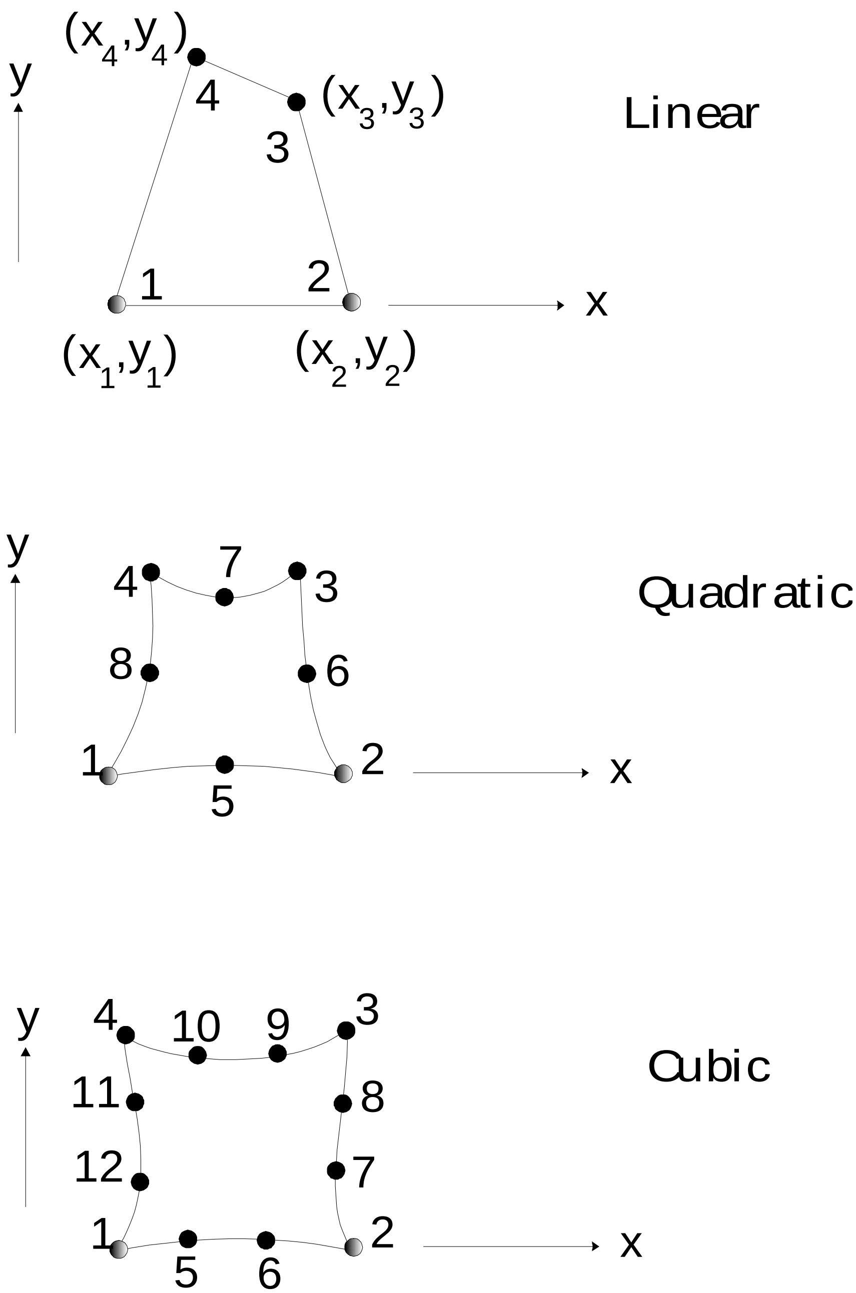


Fig. T2L1-4 Family of "serendipity" quadrilateral elements

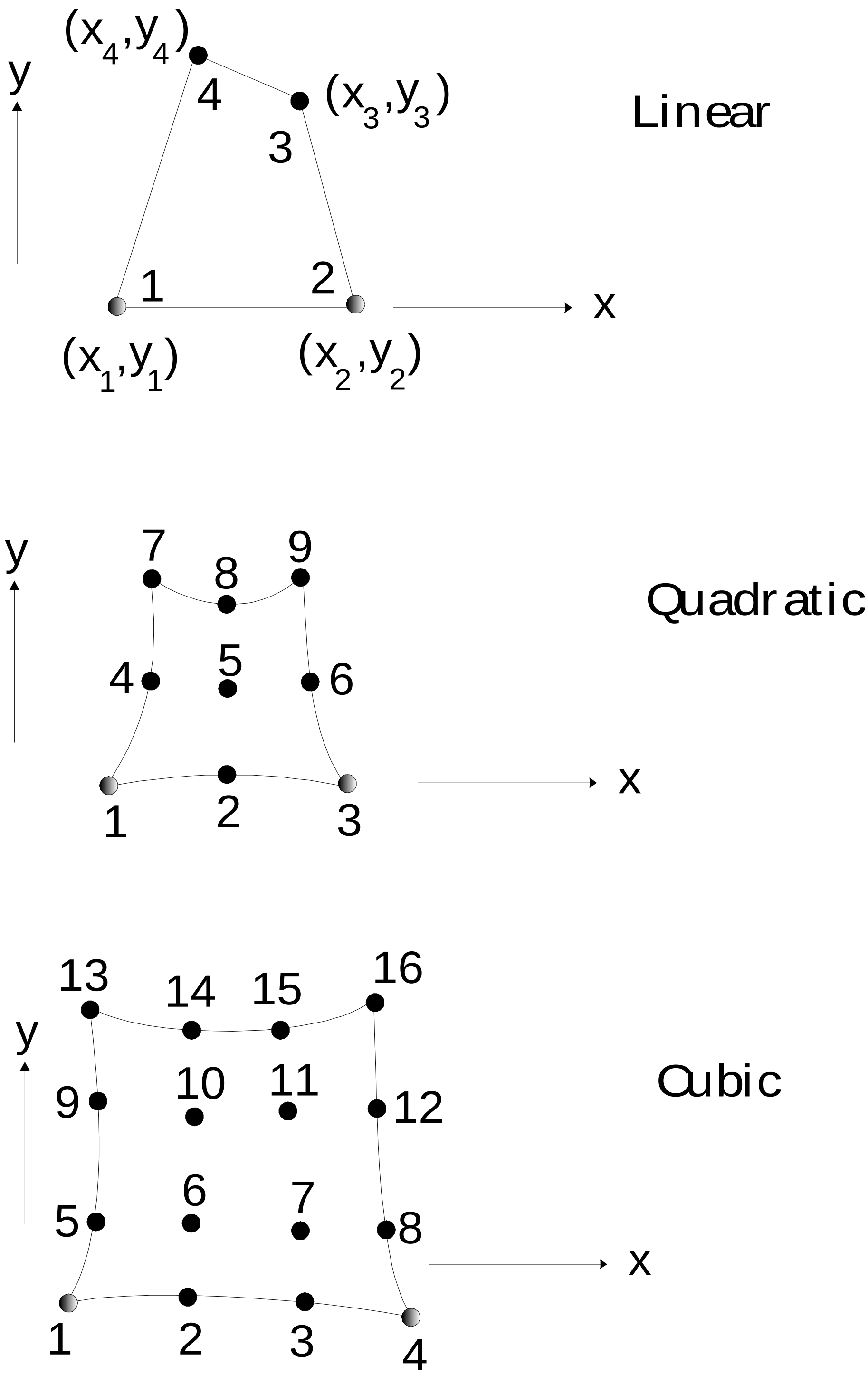


Fig. T2L1-5 Family of "Lagrange" quadrilateral elements

Natural Coordinates

The finite elements in a one-dimensional problem or multi-dimensional problems can have any coordinates and dimensions. As we will see now, there are tremendous advantages to create *parent or master elements* in natural coordinates (where the coordinates are between –1 and 1, or between 0 and 1). These master elements can then be linked to the *real elements* through appropriate mapping.

Let us first look at the one-dimensional problems. The one-dimensional linear element is shown in Fig. T2L1-6. The length of the real element is $(x_2 - x_1)$ and the length of the parent element is 2.

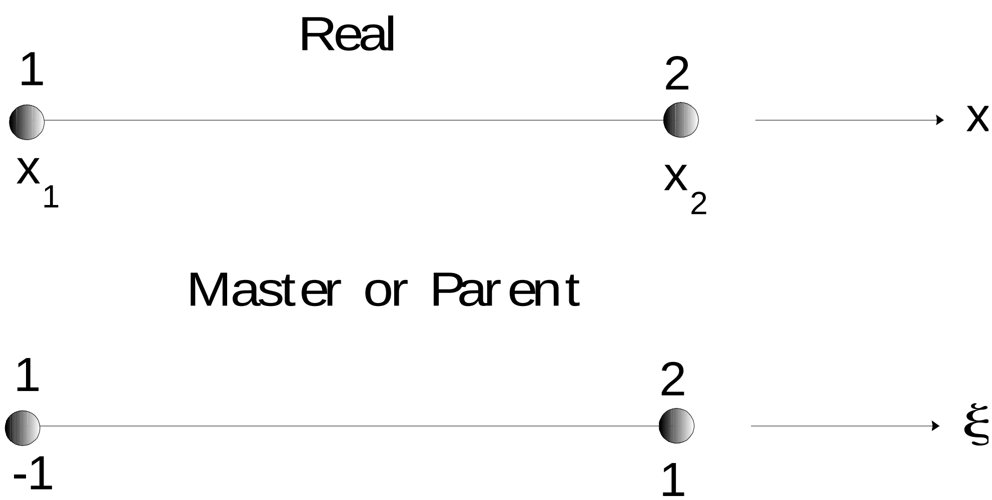


Fig. T2L1-6 Real and Parent $1D-C^0$ Linear Element

To carry out the one to one coordinate transformation we need to set up a one-to-one mapping

$$x = x(\xi) \tag{T2L1-16}$$

We will examine the details of the mapping later in this topic. In the case of quadrilateral elements, a similar situation can be set up as shown in Fig. T2L1-7. We need two natural coordinates, $\xi - \eta$ (analogous to x and y). Both these natural coordinates vary between –1 and 1. To handle the coordinate transformations associated with this element we need to set up a one-to-one mapping of the form

$$x = x(\xi, \eta) \tag{T2L1-17a}$$
$$y = y(\xi, \eta) \tag{T2L1-17b}$$

The situation for triangular elements is different than the two examples that we have seen so far. Fig. T2L1-8 shows the real and parent elements for the bilinear triangular element. The natural coordinates, $\xi - \eta - \zeta$ are called area coordinates. These natural coordinates vary between 0 and 1. Only two of the three are independent since

$$\xi + \eta + \zeta = 1 \qquad \text{(T2L1-17c)}$$

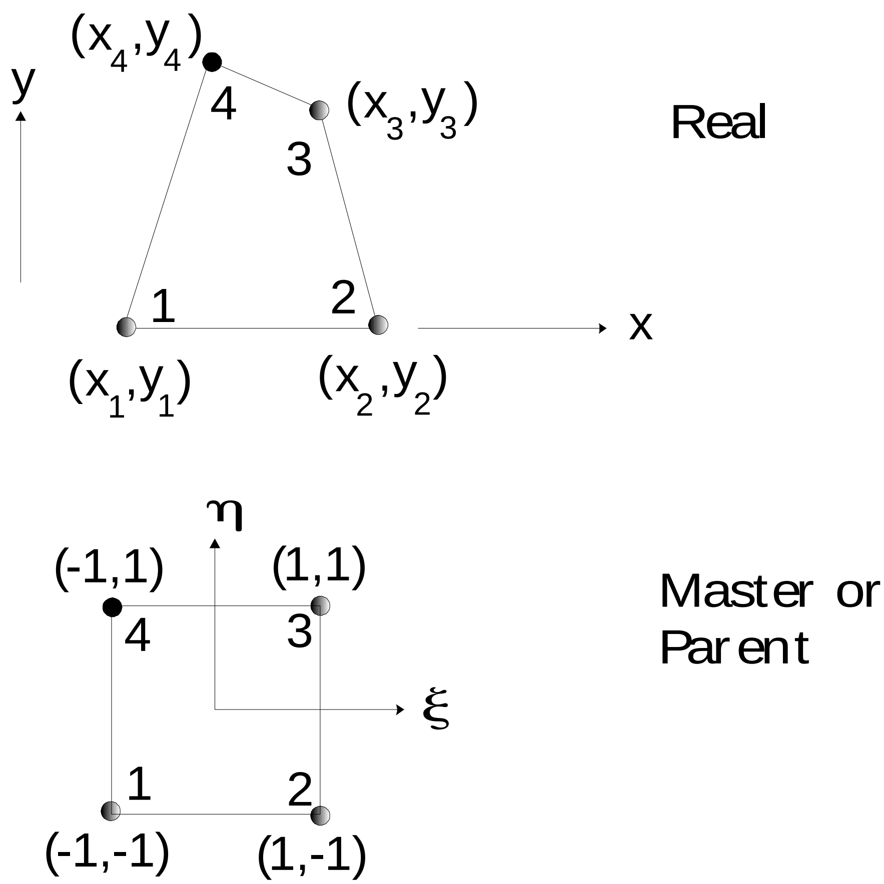


Fig. T2L1-7 Real and Parent $2D - C^0$ Bilinear Quadrilateral Element

The mapping using area coordinates are of the form

$$x = x(\xi, \eta) \qquad \text{(T2L1-18a)}$$

$$y = y(\xi, \eta) \qquad \text{(T2L1-18b)}$$

We can finally start the process of defining the shape functions in a seamless manner for low and higher-order finite elements in one, two and three dimensions. The shape functions will be defined in terms of the natural coordinates (not, cartesian coordinates).

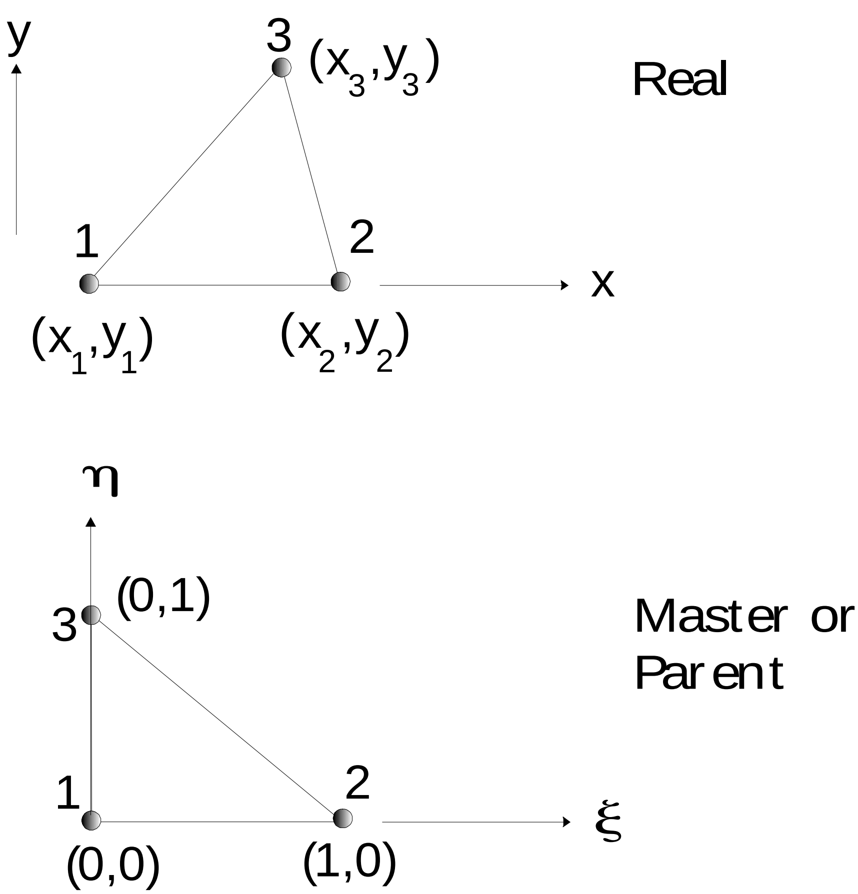


Fig. T2L1-8 Real and Parent $2D-C^0$ Bilinear Triangular Element

Generating Shape Functions

Consider the one-dimensional parent element shown in Fig. T2L1-9. The shape function for node 1 is shown. Note that the shape function has a unit value at node 1 and is zero at all other nodes. Polynomials in one coordinate having this property are known as Lagrange polynomials. The polynomial can be constructed as[7]

$$l_k^n(\xi)=\frac{(\xi-\xi_0)(\xi-\xi_1)\ldots(\xi-\xi_{k-1})(\xi-\xi_{k+1})\ldots(\xi-\xi_n)}{(\xi_k-\xi_0)(\xi_k-\xi_1)\ldots(\xi_k-\xi_{k-1})(\xi_k-\xi_{k+1})\ldots(\xi_k-\xi_n)} \quad \text{(T2L1-19)}$$

This polynomial has a unit value at ξ_k and passes through *n* points.

[7] Note that, by definition, $l_0^0=1$.

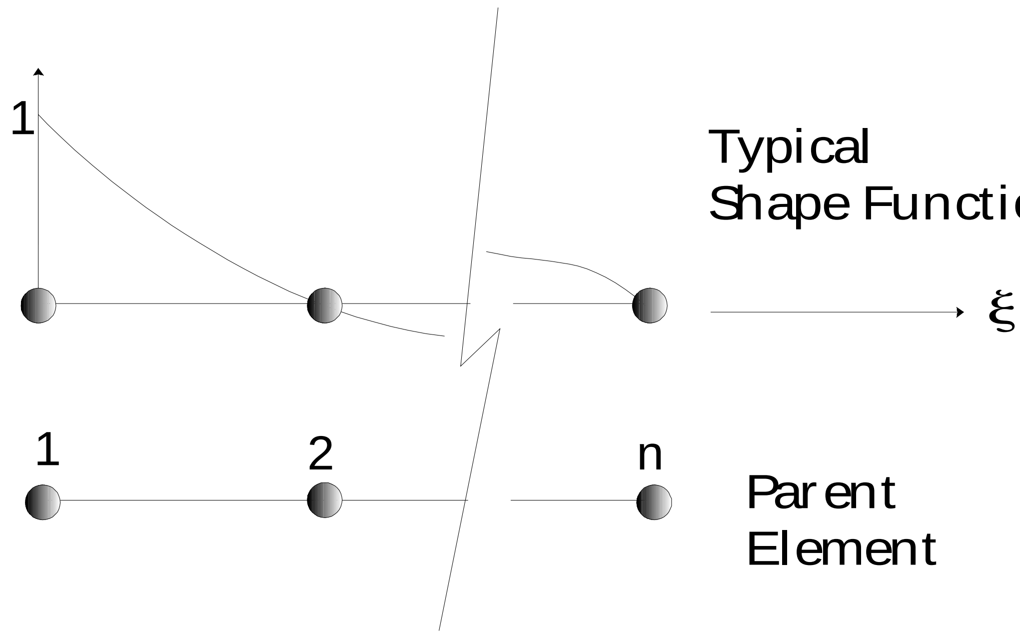


Fig. T2L1-9 Shape function for node 1

We will use Eqn. (T2L1-19) to generate the shape functions for the linear and the quadratic one-dimensional elements.

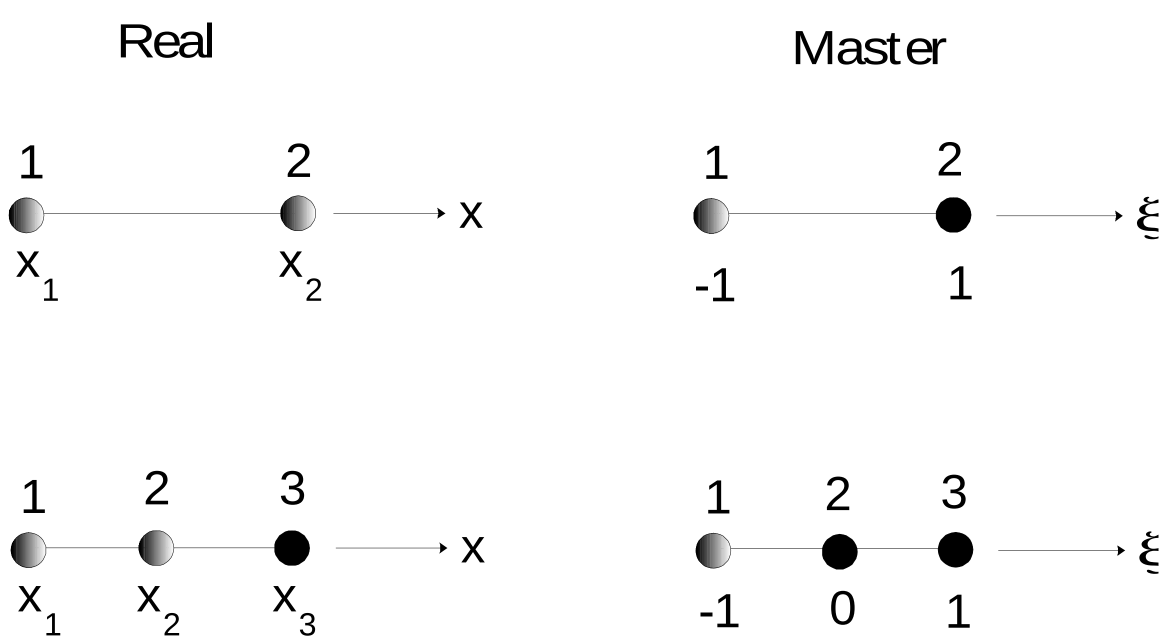


Fig. T2L1-10 Linear and Quadratic 1D Elements

Linear Element: $n = 1$, $k = 0,1$

Node 1: $\phi_1 = l_0^1 = \dfrac{(\xi - \xi_1)}{(\xi_0 - \xi_1)} = \dfrac{(\xi - 1)}{(-1-1)} = \dfrac{1-\xi}{2}$ (T2L1-20)

Node 2: $\phi_2 = l_1^1 = \dfrac{(\xi - \xi_0)}{(\xi_1 - \xi_0)} = \dfrac{(\xi - (-1))}{(1-(-1))} = \dfrac{1+\xi}{2}$ (T2L1-21)

Quadratic Element: $n = 2$, $k = 0,1,2$

Node 1: $\phi_1 = l_0^2 = \dfrac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_0 - \xi_1)(\xi_0 - \xi_2)} = \dfrac{(\xi - 0)(\xi - 1)}{(-1-0)(-1-1)} = \dfrac{1}{2}\xi(\xi - 1)$ (T2L1-22)

Node 2: $\phi_2 = l_1^2 = \dfrac{(\xi - \xi_0)(\xi - \xi_2)}{(\xi_1 - \xi_0)(\xi_1 - \xi_2)} = \dfrac{(\xi - (-1))(\xi - 1)}{(0-(-1))(0-1)} = 1 - \xi^2$ (T2L1-23)

Node 3: $\phi_3 = l_2^2 = \dfrac{(\xi - \xi_0)(\xi - \xi_1)}{(\xi_2 - \xi_0)(\xi_2 - \xi_1)} = \dfrac{(\xi - (-1))(\xi - 0)}{(1-(-1))(1-0)} = \dfrac{1}{2}\xi(\xi + 1)$ (T2L1-24)

In a similar manner, the shape functions can be generated for the higher-order one-dimensional elements – cubic, quartic etc.

Now we will look at the Lagrange quadrilateral elements formed as a grid as shown in Fig. T2L1-11. The linear and quadratic elements are shown in Fig. T2L1-12.

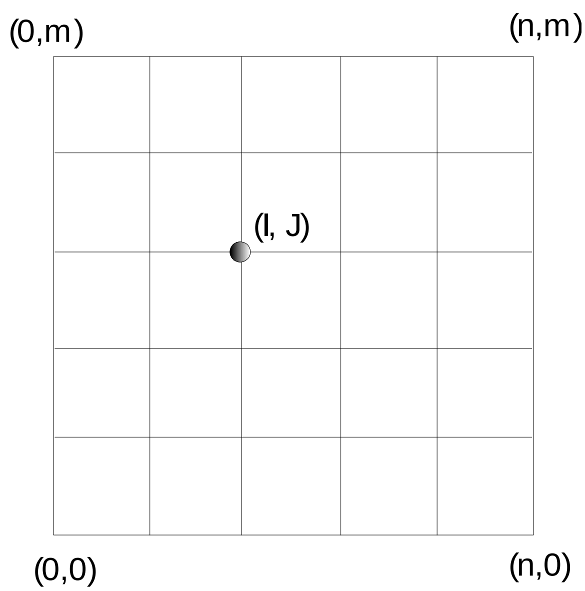


Fig. T2L1-11 Lagrangian element

The nodes in the grid are labeled by their row and column node numbers, (I, J) as

$$\phi_i \equiv \phi_{IJ} = l_I^n(\xi)\, l_J^m(\eta) \qquad \text{(T2L1-25)}$$

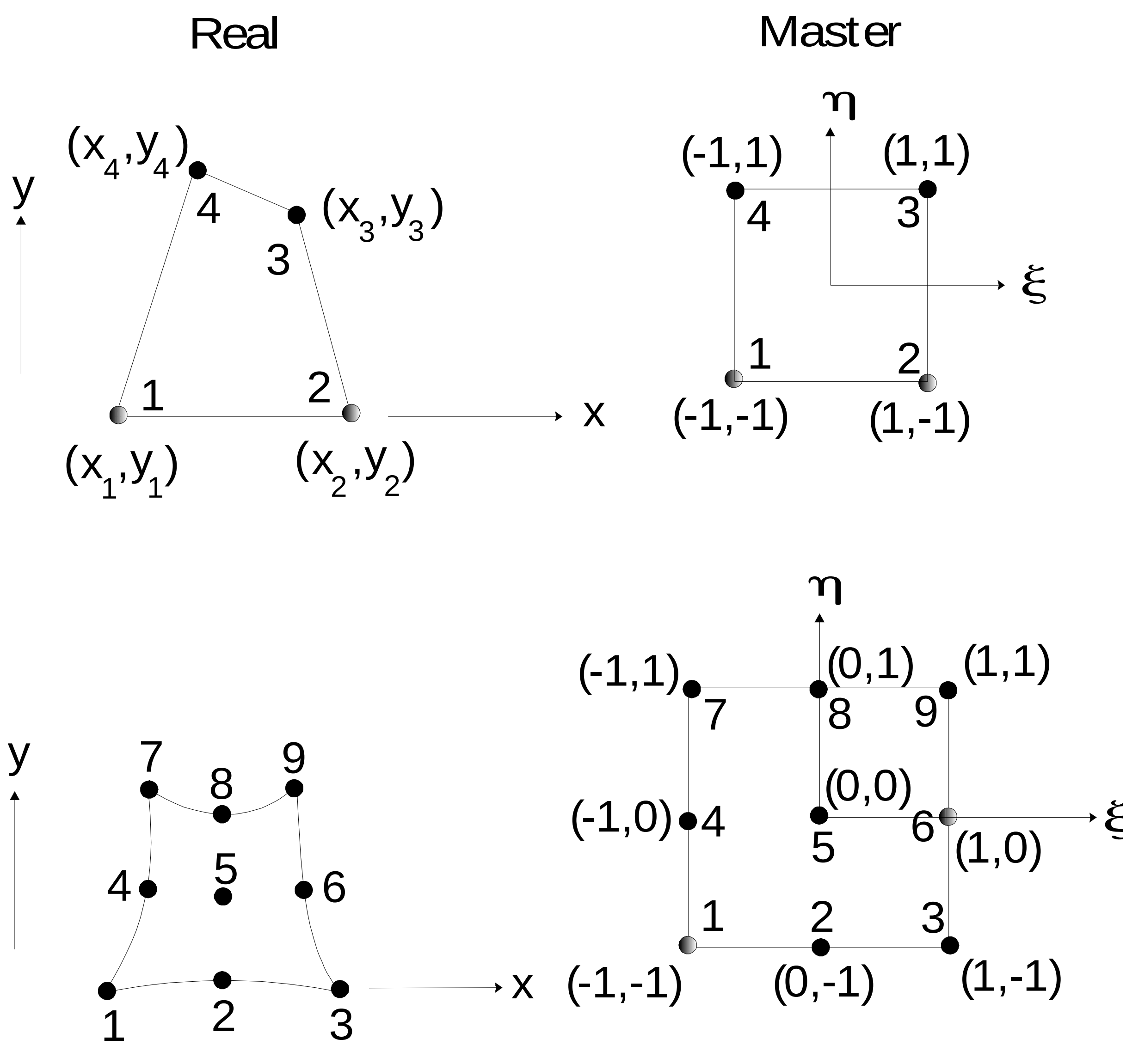


Fig. T2L1-12 Linear and Quadratic Lagrange Quadrilateral elements

Linear Element: $n = 1$, $m = 1$, $I,J = 0,1$

Node 1: $$\phi_1 = N_{00} = l_0^1(\xi) l_0^1(\eta) = \frac{(\xi - \xi_1)}{(\xi_0 - \xi_1)} \frac{(\eta - \eta_1)}{(\eta_0 - \eta_1)} = \frac{(\xi - 1)}{(-1-1)} \frac{(\eta - 1)}{(-1-1)}$$
$$= \frac{1}{4}(1-\xi)(1-\eta) \qquad \text{(T2L1-26)}$$

Node 2: $$\phi_2 = N_{10} = l_1^1(\xi) l_0^1(\eta) = \frac{(\xi - \xi_0)}{(\xi_1 - \xi_0)} \frac{(\eta - \eta_1)}{(\eta_0 - \eta_1)} = \frac{(\xi - (-1))}{(1-(-1))} \frac{(\eta - 1)}{(-1-1)}$$
$$= \frac{1}{4}(1+\xi)(1-\eta) \qquad \text{(T2L1-27)}$$

Node 3: $$\phi_3 = N_{11} = l_1^1(\xi) l_1^1(\eta) = \frac{(\xi - \xi_0)}{(\xi_1 - \xi_0)} \frac{(\eta - \eta_0)}{(\eta_1 - \eta_0)} = \frac{(\xi - (-1))}{(1-(-1))} \frac{(\eta - (-1))}{(1-(-1))}$$
$$= \frac{1}{4}(1+\xi)(1+\eta) \qquad \text{(T2L1-28)}$$

Node 4: $$\phi_4 = N_{01} = l_0^1(\xi) l_1^1(\eta) = \frac{(\xi - \xi_1)}{(\xi_0 - \xi_1)} \frac{(\eta - \eta_0)}{(\eta_1 - \eta_0)} = \frac{(\xi - 1)}{(-1-1)} \frac{(\eta - (-1))}{(1-(-1))}$$
$$= \frac{1}{4}(1-\xi)(1+\eta) \qquad \text{(T2L1-29)}$$

Quadratic Element: $n = 2$, $m = 2$, $I,J = 0,1,2$

Node 1: $$\phi_1 = N_{00} = l_0^2(\xi) l_0^2(\eta) = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_0 - \xi_1)(\xi_0 - \xi_2)} \frac{(\eta - \eta_1)(\eta - \eta_2)}{(\eta_0 - \eta_1)(\eta_0 - \eta_2)}$$
$$= \frac{(\xi - 0)(\xi - 1)}{(-1-0)(-1-1)} \frac{(\eta - 0)(\eta - 1)}{(-1-0)(-1-1)} = \frac{1}{4}\xi\eta(1-\xi)(1-\eta) \qquad \text{(T2L1-30)}$$

Node 5: $$\phi_5 = N_{11} = l_1^2(\xi) l_1^2(\eta) = \frac{(\xi - \xi_0)(\xi - \xi_2)}{(\xi_1 - \xi_0)(\xi_1 - \xi_2)} \frac{(\eta - \eta_0)(\eta - \eta_2)}{(\eta_1 - \eta_0)(\eta_1 - \eta_2)}$$
$$= \frac{(\xi + 1)(\xi - 1)}{(0+1)(0-1)} \frac{(\eta + 1)(\eta - 1)}{(0+1)(0-1)} = (1-\xi^2)(1-\eta^2) \qquad \text{(T2L1-31)}$$

Node 6: $$\phi_6 = N_{21} = l_2^2(\xi) l_1^2(\eta) = \frac{(\xi - \xi_0)(\xi - \xi_1)}{(\xi_2 - \xi_0)(\xi_2 - \xi_1)} \frac{(\eta - \eta_0)(\eta - \eta_2)}{(\eta_1 - \eta_0)(\eta_1 - \eta_2)}$$
$$= \frac{(\xi + 1)(\xi - 0)}{(1+1)(1-0)} \frac{(\eta + 1)(\eta - 1)}{(0+1)(0-1)} = \frac{1}{2}\xi(1+\xi)(1-\eta^2) \qquad \text{(T2L1-32)}$$

Now we will look at the triangular elements formed as a grid as shown in Fig. T2L1-13. The linear and quadratic elements are shown in Fig. T2L1-14.

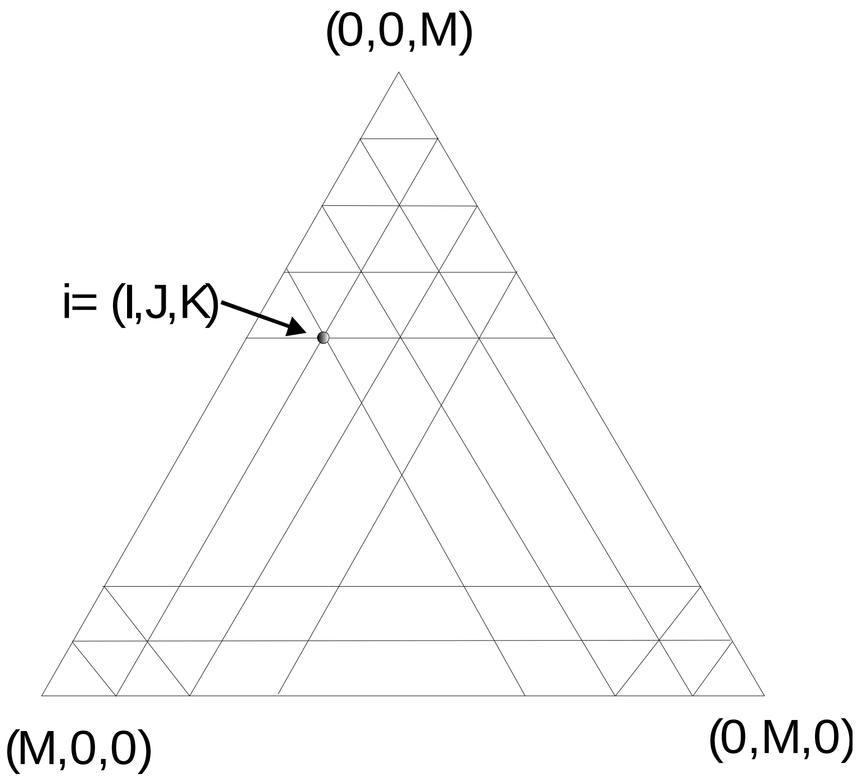


Fig. T2L1-13 Triangular element

Denoting a typical node *i* by (I,J,K) corresponding to the node's area coordinates (ξ,η,ζ), the shape function for that node can be written as

$$\phi_i = l_I^I(\xi)\, l_J^J(\eta)\, l_K^K(\zeta) \qquad \text{(T2L1-33a)}$$

and $$I + J + K = M \qquad \text{(T2L1-33b)}$$

Linear Element: $I, J, K = 0,1$ and $M = 1$

Node 1: $$\phi_1 = l_0^0(\xi)\, l_0^0(\eta)\, l_1^1(\zeta) = (1)(1)\frac{(\zeta-\zeta_0)}{(\zeta_1-\zeta_0)} = \frac{(\zeta-0)}{(1-0)} = 1-\xi-\eta \qquad \text{(T2L1-34)}$$

Node 2: $$\phi_2 = l_1^1(\xi)\, l_0^0(\eta)\, l_0^0(\zeta) = \frac{(\xi-\xi_0)}{(\xi_1-\xi_0)}(1)(1) = \frac{(\xi-0)}{(1-0)} = \xi \qquad \text{(T2L1-35)}$$

Node 3: $\phi_3 = l_0^0(\xi)\, l_1^1(\eta)\, l_0^0(\zeta) = (1)\dfrac{(\eta-\eta_0)}{(\eta_1-\eta_0)}(1) = \dfrac{(\eta-0)}{(1-0)} = \eta$ (T2L1-36)

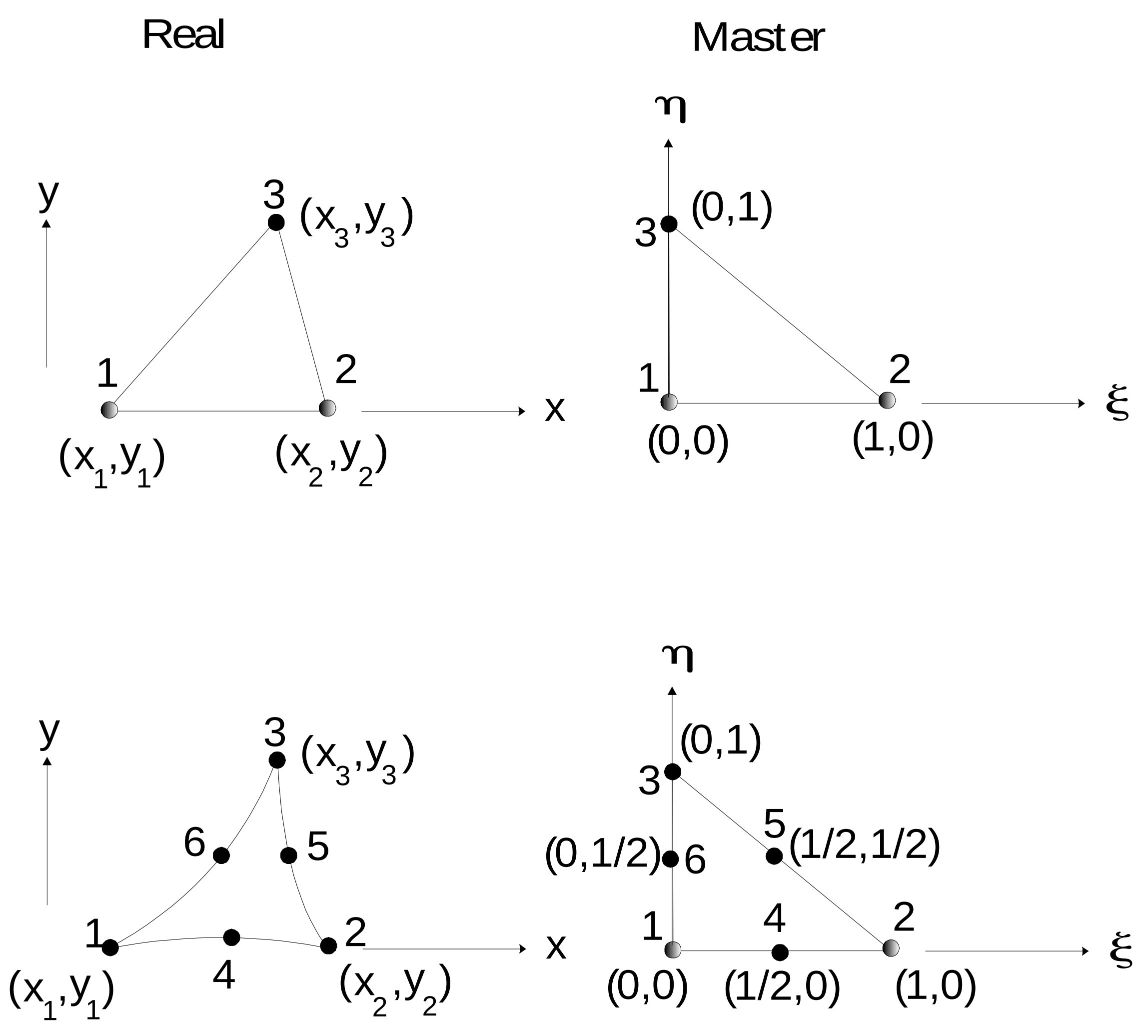


Fig. T2L1-14 Linear and Quadratic triangular elements

Quadratic Element: $I, J, K = 0,1,2$ and $M = 2$

Node 1: $\phi_1 = l_0^0(\xi)\, l_0^0(\eta)\, l_2^2(\zeta) = (1)(1)\dfrac{(\zeta-\zeta_0)(\zeta-\zeta_1)}{(\zeta_2-\zeta_0)(\zeta_2-\zeta_1)} = \dfrac{(\zeta-0)(\zeta-1/2)}{(1-0)(1-1/2)}$

$= \zeta(2\zeta-1) = (1-\xi-\eta)(1-2\xi-2\eta)$ (T2L1-37)

$$\text{Node 4: } \phi_4 = l_1^1(\xi)\, l_0^0(\eta)\, l_1^1(\zeta) = \frac{(\xi - \xi_0)}{(\xi_1 - \xi_0)}(1)\frac{(\zeta - \zeta_0)}{(\zeta_1 - \zeta_0)} = \frac{(\xi - 0)}{(1/2 - 0)}\frac{(\zeta - 0)}{(1/2 - 0)}$$
$$= 4\xi\zeta = 4\xi(1 - \xi - \eta) \qquad \text{(T2L1-38)}$$

$$\text{Node 6: } \phi_6 = l_0^0(\xi) l_1^1(\eta) l_1^1(\zeta) = (1)\frac{(\eta - \eta_0)}{(\eta_1 - \eta_0)}\frac{(\zeta - \zeta_0)}{(\zeta_1 - \zeta_0)} = (1)\frac{(\eta - 0)}{(1/2 - 0)}\frac{(\zeta - 0)}{(1/2 - 0)}$$

$$= 4\eta\zeta = 4\eta\left(1 - \xi - \eta\right) \qquad \text{(T2L1-39)}$$

Hermite Cubics: For elements involving the first derivative of the unknown as a degree of freedom at the node, Hermite Cubic interpolation is required. These elements are known as C^1 elements. The only example of this interpolation scheme will be used with the beam element. Hence the discussions on Hermite Cubics will be deferred to Topic 3.

Lesson 2: Numerical Integration

Objectives: In this lesson we will study numerical integration.

- To understand the numerical integration procedure suitable for finite elements.
- To implement the Gaussian Quadrature procedure.

Numerical Integration

Consider the problem of evaluating the integral

$$\int_a^b F(x)\,dx$$

and assume that

(a) $F(x)$ is dependent on a jacobian, and

(b) $F(x)$ is known at a few discrete points.

The basic idea in numerical integration is to construct another function $P(x)$ (usually a polynomial) that is a suitable approximation of $F(x)$ and is simple to integrate. The interpolating polynomial of degree *n*, denoted P_n, is such that it interpolates the integrand at *(n+1)* points in the interval $[a,b]$. While there exist errors, $E = F(x) - P_n(x)$, the error may not always be of the same sign so that the overall error is small.

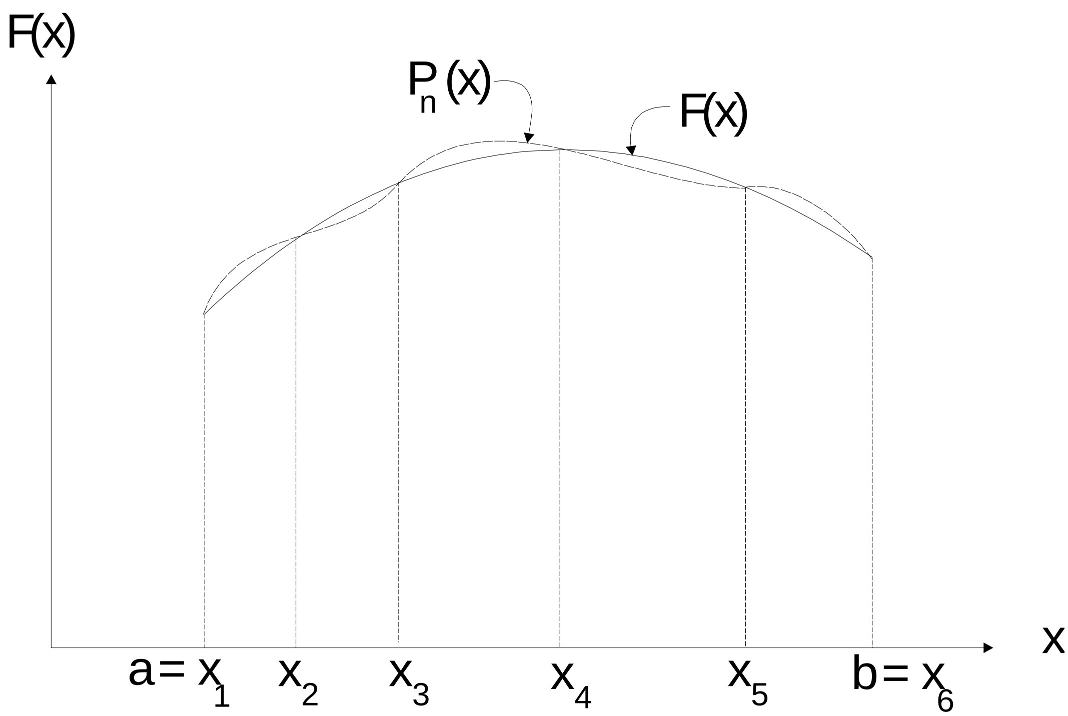


Fig. T2L2-1 Original function and its polynomial approximation

The traditional Newton-Cotes techniques such as Trapezoidal Rule or Simpson's Rule are not as efficient or accurate as the Gauss-Legendre Quadrature for linear finite element problems. We will discuss the latter method in this section.

Gauss-Legendre Quadrature: The base points x_i and the weights w_i are chosen so that the sum of the *(n+1)* appropriately weighted values of the function yields the integral exactly when $F(x)$ is a polynomial of degree *(2n+1)* or less.

$$\int_a^b F(x)\,dx = \int_{-1}^{1} \hat{F}(\xi)\,d\xi = \sum_{i=1}^{n} w_i\,\hat{F}(\xi_i) \qquad \text{(T2L2-1)}$$

where ξ_i are the base points (or, roots of the Legendre polynomial $P_{n+1}(\xi)$), and

$$F(x)dx = F(x(\xi)) \cdot \frac{dx}{d\xi} d\xi = F(x(\xi)) \cdot J(\xi) d\xi = \hat{F}(\xi) d\xi \qquad \text{(T2L2-2)}$$

where J is the Jacobian. The following two points should be noted

(a) Gauss-Legendre is more efficient because it requires fewer base points to achieve the same level of accuracy as the Newton-Cotes methods, and

(b) The error is zero if the $(2n+2)^{th}$ derivative of the integrand vanishes. Or, a polynomial of degree *n* is integrated exactly by employing $(n+1)/2$ Gauss points.

Table T2L2-1 Gauss points and weights

Order, *n*	**Weight**	**Location**
1	2.0	0.0
2	1.0	0.57735 02691
	1.0	-0.57735 02691
3	0.55555 55555	0.77459 66692
	0.55555 55555	-0.77459 66692
	0.88888 88888	0.0
4	0.3478548451	±0.8611363116
	0.6521451549	±0.3399810436
5	0.2369268851	±0.0.9061798459
	0.4786286705	±0.5384693101
	0.5688888889	0.0
6	0.1713244924	±0.9324695142
	0.3607615730	±0.6612093865
	0.4679139346	±0.2386191861

Example 1: Evaluate $\int_{-1}^{1} x^4 dx$.

Solution: The exact answer is 0.4. We have $F(x) = F(\xi) = x^4$. No Jacobian is needed.

Using $n = 1$: $w_1 = 2.0$ and $\xi_1 = 0.0$. Hence $I = (2.0)(0.0)^4 = 0.0$.

Using $n=2$: $(w_1,\xi_1)=(1.0, 0.5773502691)$ and $(w_1,\xi_1)=(1.0, -0.5773502691)$.

Hence $I=(1.0)(0.5773502691)^4+(1.0)(-0.5773502691)^4=0.222222222$.

Using $n=3$: $(w_1,\xi_1)=(0.5555555555, 0.7745966692)$

$(w_2,\xi_2)=(0.5555555555, -0.7745966692)$

$(w_3,\xi_3)=(0.8888888888, 0.0)$

Hence $I=(0.5555555555)(0.7745966692)^4+(0.5555555555)(-0.7745966692)^4$

$+(0.8888888888)(0.0)^4=0.4$

The original function is a polynomial of degree 4 and can be integrated exactly by Gauss Quadrature rule $\frac{(n+1)}{2}=\frac{4+1}{2}=3$ (note that the rounding takes place to the next highest integer). The results bear out the rule.

Example 2: Evaluate $\int_{-1}^{1}\left[\frac{2x-1}{x^2-6x+13}\right]dx$.

Solution: The integrand is not a polynomial. The exact solution is -0.1119. No Jacobian is needed.

Order, n	w_i	ξ_i	$f(\xi_i)$	$w_i f(\xi_i)$
2	1.0	0.57735 02691	0.015675	0.015675
	1.0	-0.57735 02691	-0.128276	-0.128276
			TOTAL	-0.112601
3	0.55555 55555	0.77459 66692	0.0613458	0.034081
	0.55555 55555	-0.77459 66692	-0.1397	-0.0776113
	0.88888 88888	0.0	-0.0769231	-0.0683761
			TOTAL	-0.111906

Two-Dimensional Functions: Functions that involve two independent natural coordinates are handled in a manner similar to one-dimensional functions.

$$\int_c^d \int_a^b F(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} F\left(x(\xi,\eta), y(\xi,\eta)\right)\left|J\right| d\xi\, d\eta$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} w_i w_j f\left(\xi_i,\eta_j\right) \qquad \text{(T2L2-3a)}$$

where $f\left(\xi_i,\eta_j\right) = F\left(x(\xi,\eta), y(\xi,\eta)\right)\left|J\right|$ (T2L2-3b)

$$\left|J\right| = \det(\mathbf{J}) \quad \text{where } \mathbf{J}_{2\times 2} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \qquad \text{(T2L2-3c)}$$

The values of the weights and natural coordinates are the same as shown in Table T2L2-1 except that ξ in the table refers to both ξ and η.

Example 1: Evaluate $\int_{-1}^{1}\int_{-1}^{1} x^2\, dx\, dy$

Solution: The exact answer is $4/3$. Using the $n=2$ rule, we have four Gauss points. No Jacobian is needed.

ξ_i	η_j	w_i	w_j	$f(\xi_i,\eta_j)$	$w_i w_j f(\xi_i,\eta_j)$
-0.5773502691	-0.57735 02691	1.0	1.0	1/3	1/3
0.5773502691	-0.57735 02691	1.0	1.0	1/3	1/3
0.5773502691	0.57735 02691	1.0	1.0	1/3	1/3
-0.5773502691	0.57735 02691	1.0	1.0	1/3	1/3
				TOTAL	4/3

The answer obtained is the exact answer. Once again, the appropriate quadrature order is $\frac{(n+1)}{2}=\frac{(2+1)}{2}=2$ and using the rule leads to the exact answer.

Area Coordinates: The use of area coordinates requires a different numerical integration procedure.

$$\int_0^1 \int_0^{1-\eta} F(\xi,\eta)\,d\xi\,d\eta = \frac{1}{2}\sum_{i=1}^{n} w_i F(\xi_i,\eta_i) \qquad \text{(T2L2-4)}$$

The weights and locations are given in Table T2L2-2.

Table T2L2-2 Gauss points and weights using Area Coordinates

Number of points, *n*	Weight	Multiplicity	Location
1	1.0	1	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$
2	$\frac{1}{3}$	3	$\left(\frac{2}{3},\frac{1}{6},\frac{1}{6}\right)$
3	$\frac{1}{3}$	3	$\left(\frac{1}{2},\frac{1}{2},0\right)$
4	$-\frac{9}{16}$	1	$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$
	$\frac{25}{48}$	3	$\left(\frac{3}{5},\frac{1}{5},\frac{1}{5}\right)$
6	$\frac{1}{6}$	6	$\left(0.6590276223, 0.2319333685, 0.1090390090\right)$

We will look at examples using area coordinates later in this topic.

Lesson 3: Isoparametric Finite Elements

Objectives: In this lesson we will look at isoparametric finite elements.

- To understand what is meant by isoparametric formulation.
- To understand the link between isoparametric formulation, numerical integration and element equations.
- To derive the element equations for low and high-order elements to solve the one-dimensional BVP.

Why do we need isoparametric formulation?

First what is isoparametric formulation? The term isoparametric means that equal numbers of parameters are used to represent the geometry and the unknown variable. In isoparametric formulation, the same shape functions are used to handle the geometric variable and the unknown variable. The real power and advantage of the isoparametric formulation will be seen when we start deriving the element equations for two and three-dimensional finite elements.

One-Dimensional Elements: In this section we will look at a simple example to show how the concept works with one-dimensional elements.

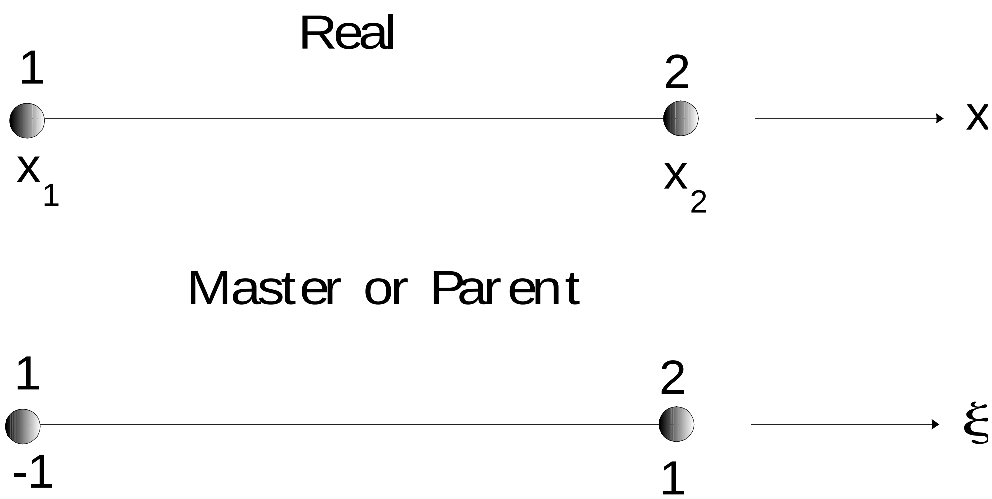


Fig. T2L3-1 One-dimensional C^0-linear element

The unknown, u, over the element can be expressed as

$$u(\xi) = \phi_1(\xi)\, u_1 + \phi_2(\xi)\, u_2 \qquad \text{(T2L3-1)}$$

The geometry or spatial variable, x, can be expressed as

$$x(\xi) = \phi_1(\xi)\, x_1 + \phi_2(\xi)\, x_2 \qquad \text{(T2L3-2)}$$

In general, the unknown can be written as

$$u(\xi) = \sum_{i=1}^{r} \phi_i(\xi)\, u_i \qquad \text{(T2L3-3)}$$

and the spatial variable can be written as

$$x(\xi) = \sum_{j=1}^{s} \hat{\phi}_j(\xi)\, x_j \qquad \text{(T2L3-4)}$$

Elements are

(i) subparametric when $s < r$,

(ii) isoparametric , when $s = r$ and

(iii) superparametric when $s > r$.

Subparametric elements are valid but superparametric elements are usually not valid. An example of subparametric element is a triangle with six nodes and straight sides. The unknown can vary quadratically but the geometry is defined by the three vertex nodes. In this course we will only deal with isoparametric elements.

Now going back to Eqns. (T2L3-1) and (T2L3-2), the unknown and the spatial variable can be related to each other through Eqn. (T2L3-2) as

$$x = \frac{(1-\xi)}{2} x_1 + \frac{(1+\xi)}{2} x_2 \tag{T2L3-5}$$

$$\frac{dx}{d\xi} = \frac{(x_2 - x_1)}{2} = \frac{L}{2} = J \tag{T2L3-6}$$

$$\frac{d\xi}{dx} = \frac{1}{dx/d\xi} = J^{-1} = \frac{2}{L} \tag{T2L3-7}$$

where J is the jacobian. The jacobian represents the scale factor between the two coordinate systems. Why are these relationships important? We need the jacobian and the inverse of the jacobian so that element equations can be generated.

Two-Dimensional Elements: We will start with the simplest quadrilateral Lagrange element.

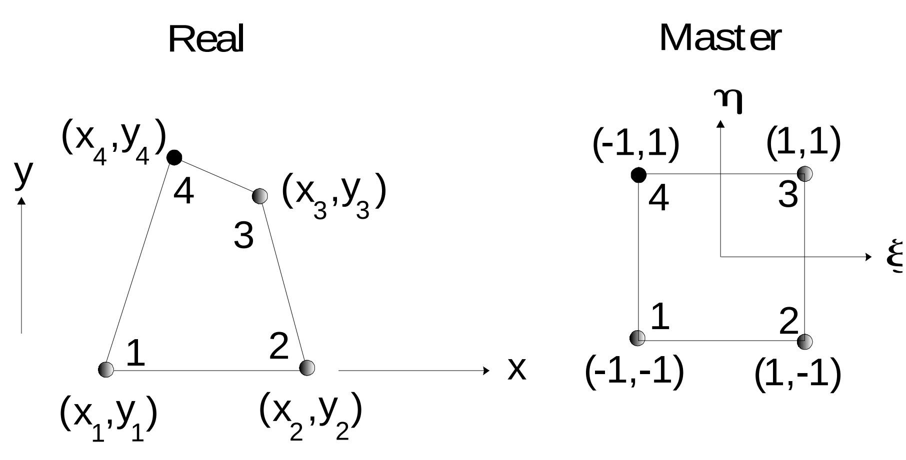


Fig. T2L3-2

Using Eqns. (T2L1-26)-(T2L1-29), we have

$$x = \frac{1}{4}(1-\xi)(1-\eta)\,x_1 + \frac{1}{4}(1+\xi)(1-\eta)\,x_2 + \frac{1}{4}(1+\xi)(1+\eta)\,x_3$$

$$+\frac{1}{4}(1-\xi)(1+\eta)\,x_4 \qquad \text{(T2L3-8)}$$

$$y = \frac{1}{4}(1-\xi)(1-\eta)\,y_1 + \frac{1}{4}(1+\xi)(1-\eta)\,y_2 + \frac{1}{4}(1+\xi)(1+\eta)\,y_3$$

$$+\frac{1}{4}(1-\xi)(1+\eta)\,y_4 \qquad \text{(T2L3-9)}$$

$$\frac{\partial x}{\partial \xi} = \frac{1}{4}\left[\eta\left(x_1 - x_2 + x_3 - x_4\right) + \left(-x_1 + x_2 + x_3 - x_4\right)\right] \qquad \text{(T2L3-10a)}$$

$$\frac{\partial x}{\partial \eta} = \frac{1}{4}\left[\xi\left(x_1 - x_2 + x_3 - x_4\right) + \left(-x_1 - x_2 + x_3 + x_4\right)\right] \qquad \text{(T2L3-10b)}$$

$$\frac{\partial y}{\partial \xi} = \frac{1}{4}\left[\eta\left(y_1 - y_2 + y_3 - y_4\right) + \left(-y_1 + y_2 + y_3 - y_4\right)\right] \qquad \text{(T2L3-10c)}$$

$$\frac{\partial y}{\partial \eta} = \frac{1}{4}\left[\xi\left(y_1 - y_2 + y_3 - y_4\right) + \left(-y_1 - y_2 + y_3 + y_4\right)\right] \qquad \text{(T2L3-10b)}$$

In a manner similar to the one-dimensional element, if $u(x, y)$ represents the unknown, then

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi} \qquad \text{(T2L3-11a)}$$

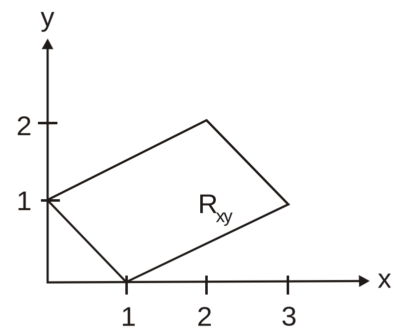
$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta} \qquad \text{(T2L3-11b)}$$

or,

$$\begin{Bmatrix} u,_{\xi} \\ u,_{\eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u,_x \\ u,_y \end{Bmatrix} = \mathbf{J}_{2\times 2} \begin{Bmatrix} u,_x \\ u,_y \end{Bmatrix} \qquad \text{(T2L3-12)}$$

And, $\begin{Bmatrix} u,_x \\ u,_y \end{Bmatrix} = \boldsymbol{\Gamma}_{2\times 2} \begin{Bmatrix} u,_\xi \\ u,_\eta \end{Bmatrix}$ where $\boldsymbol{\Gamma} = \mathbf{J}^{-1}$ (T2L3-13)

Example: Evaluate $I = \iint\limits_{R_{xy}} (x+y)^3\, dxdy$



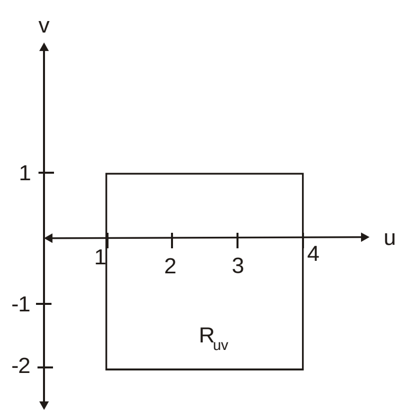
Traditional Approach: The sides of the domain are not parallel to the axes making it difficult to set the limits of integration. However, the sides are such that

$$x + y = c_1 \qquad \text{and} \qquad x - 2y = c_2$$

Therefore, we can introduce two new variables and set up a mapping such that

$$u = x + y \qquad \text{and} \qquad v = x - 2y$$

The transformed domain is shown below.



Hence, $I = \iint\limits_{R_{xy}} (x+y)^3\, dxdy = \iint\limits_{R_{uv}} (u)^3 \det(J)\, dudv$

$$\det(J) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{1}{3}$$

Substituting, $I = \iint_{R_{uv}} (u)^3 \det(J) du dv = \int_{-2}^{1}\int_{1}^{4} \frac{u^3}{3} du dv = \frac{765}{12} = 63.75$

Isoparametric Approach: We will use the quadrilateral element to construct the integration domain. The real element is such that $(x_1, y_1) = (1,0)$, $(x_2, y_2) = (3,1)$, $(x_3, y_3) = (2,2)$ and $(x_4, y_4) = (0,1)$. The jacobian can be constructed as

$$\mathbf{J}_{2\times 2} = \frac{1}{4}\begin{bmatrix} 4 & 2 \\ -2 & 2 \end{bmatrix} \qquad \det(\mathbf{J}) = \frac{3}{4}$$

(a) We will use the one-point rule first.

$$i = j = 1 \qquad w_i = w_j = 2.0 \qquad (\xi_i, \eta_j) = (0,0)$$

$$x + y = \left(\sum_{i=1}^{4} \phi_i x_i\right) + \left(\sum_{i=1}^{4} \phi_i y_i\right) = \left(\frac{1}{4}(1+3+2+0) + \frac{1}{4}(0+1+2+1)\right) = \frac{5}{2}$$

$$I = (2)(2)\left(\frac{5}{2}\right)^3 \left(\frac{3}{4}\right) = 46.875$$

(b) Now the two point rule. The details of the calculations are shown below.

ξ_i	η_j	w_i	w_j	ϕ_1	ϕ_2	ϕ_3	ϕ_4	x	y	$(x+y)^3$
-0.57735	-0.57735	1.0	1.0	0.622008	0.166667	0.044658	0.166667	1.21133	0.42265	4.362508
0.57735	-0.57735	1.0	1.0	0.166667	0.622008	0.166667	0.044658	2.36603	1	38.13748
0.57735	0.57735	1.0	1.0	0.044658	0.166667	0.622008	0.166667	1.78868	1.57735	38.13748
-0.57735	0.57735	1.0	1.0	0.166667	0.044658	0.166667	0.622008	0.63398	1	4.362508
								$I = \det(J)\sum(x+y)^3$		63.75

In the next section we will see the derivation of the 1D elements that we saw in Module 1 but we will derive the element equations using the isoparametric formulation. In the next two topics we will see the derivation of isoparametric finite elements for multi-dimensional elements in the areas of solid mechanics and other areas.

Isoparametric Linear $1D-C^0$ BVP Element

We will continue with the derivation of the element equations that we started with Fig. T2L3-1. For $1D-C^0$ BVP Element [from Module 1 Eqn. (T4L1-61)]

$$\sum_{j=1}^{n}\left[\int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx+\int_{\Omega}\phi_i(x)\beta(x)\phi_j(x)\,dx\right]y_j=$$

$$\int_{\Omega}f(x)\phi_i(x)\,dx-\left[\tau\phi_i\right]^{\Gamma} \qquad i=1,2.,,n \tag{T2L3-14}$$

To evaluate the first term on the LHS, we note that

$$\frac{d\phi_i}{dx}=\frac{d\phi_i}{d\xi}\frac{d\xi}{dx}=\frac{d\phi_i}{d\xi}\frac{2}{L} \tag{T2L3-15}$$

using Eqn. (T2L3-7). Hence the first term can be written as

$$k_{ij}^{\alpha}=\int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx=\int_{-1}^{1}\frac{d\phi_i}{d\xi}\alpha(x(\xi))\frac{d\phi_j}{d\xi}\frac{4}{L^2}J\,d\xi$$

$$=\int_{-1}^{1}\frac{d\phi_i}{d\xi}\alpha(x(\xi))\frac{d\phi_j}{d\xi}\frac{2}{L}d\xi \tag{T2L3-16}$$

Similarly the second term can be written as

$$k_{ij}^{\beta}=\int_{\Omega}\phi_i(x)\beta(x)\phi_j(x)\,dx=\int_{-1}^{1}\phi_i(\xi)\beta(x(\xi))\phi_j(\xi)\frac{L}{2}d\xi \tag{T2L3-17}$$

and the interior load vector can be written as

$$f_i^{\text{int}}=\int_{\Omega}f(x)\phi_i(x)\,dx=\int_{-1}^{1}f(\xi)\,\phi_i(\xi)\frac{L}{2}d\xi \tag{T2L3-18}$$

We will now evaluate at a few representative terms to illustrate the procedure without using numerical integration[8]. Let us assume that $\alpha(x) = \hat{\alpha}$, $\beta(x) = \hat{\beta}$ and $f(x) = \hat{f}$ are constants. The relevant expressions are as follows

$$\phi_1 = \frac{1-\xi}{2} \text{ and } \phi_2 = \frac{1+\xi}{2} \qquad \text{(T2L3-19)}$$

$$\frac{d\phi_1}{d\xi} = -\frac{1}{2} \text{ and } \frac{d\phi_2}{d\xi} = \frac{1}{2} \qquad \text{(T2L3-20)}$$

and $$J = \frac{dx}{d\xi} = \frac{L}{2} \qquad \text{(T2L3-21)}$$

Using these expressions

$$k_{11}^{\alpha} = \int_{-1}^{1} \frac{d\phi_1}{d\xi} \alpha(x(\xi)) \frac{d\phi_1}{d\xi} \frac{2}{L} d\xi = \int_{-1}^{1} \frac{1}{4} \hat{\alpha} \frac{2}{L} d\xi = \frac{\hat{\alpha}}{2L}(2) = \frac{\hat{\alpha}}{L}$$

$$k_{12}^{\alpha} = \int_{-1}^{1} \frac{d\phi_1}{d\xi} \alpha(x(\xi)) \frac{d\phi_2}{d\xi} \frac{2}{L} d\xi = \int_{-1}^{1} \left(-\frac{1}{4}\right) \hat{\alpha} \frac{2}{L} d\xi = -\frac{\hat{\alpha}}{2L}(2) = -\frac{\hat{\alpha}}{L}$$

$$k_{12}^{\beta} = \int_{-1}^{1} \phi_1(\xi) \hat{\beta} \phi_2(\xi) \frac{L}{2}\, d\xi = \frac{\hat{\beta} L}{8} \int_{-1}^{1} (1-\xi^2)\, d\xi = \frac{\hat{\beta} L}{6}$$

$$f_1^{\text{int}} = \int_{-1}^{1} \hat{f}\, \phi_1(\xi) \frac{L}{2} d\xi = \frac{\hat{f} L}{4} \int_{-1}^{1} (1-\xi)\, d\xi = \frac{\hat{f} L}{2}$$

and the results are identical to the expressions obtained in Module 1.

Isoparametric Quadratic $1D - C^0$ BVP Element

The power of the isoparametric formulation is evident here if we note that Eqns. (T2L3-16)-(18) are valid for any element. The element dependence is with respect to the shape functions and their derivatives. We will not derive the element matrix components but instead will focus on the validity of the element formulation.

---

[8] Integration is straightforward for this element. The need and power of numerical integration will be evident with two-dimensional elements.

As pointed out earlier, the key concept in isoparametric formulation is the one-to-one mapping. For the quadratic element

$$x = \frac{1}{2}\xi(\xi - 1)x_1 + (1 - \xi^2)x_2 + \frac{1}{2}\xi(\xi + 1)x_3 \qquad \text{(T2L3-22a)}$$

Hence

$$J = \frac{dx}{d\xi} = \left(\xi - \frac{1}{2}\right)x_1 + (-2\xi)x_2 + \left(\xi + \frac{1}{2}\right)x_3 \qquad -1 \le \xi \le 1 \qquad \text{(T2L3-22b)}$$

For the mapping to be acceptable , we need $J > 0$. Since the jacobian is a function of ξ (and the nodal coordinates), the condition states that the jacobian should be positive throughout the element <u>including</u> the end nodes. If $J < 0$ it means that a positive $d\xi$ maps into a negative dx and the mapping is double-valued (not unique). The placement of the interior node and hence the value of x_2 determines whether the mapping is valid. Let us see when this occurs.

From Eqn. (T2L3-22b) we note that J is linear function of ξ. Therefore, $J > 0$ in $-1 \le \xi \le 1$ if only if $J > 0$ is satisfied at the ends of the interval, i.e. at $\xi = \pm 1$.

At $\xi = -1$ $\qquad J(-1) = -\left(\frac{3}{2}\right)x_1 + (2)x_2 - \left(\frac{1}{2}\right)x_3 > 0 \quad \Rightarrow \quad x_2 > x_1 + \frac{L}{4}$

At $\xi = +1$ $\qquad J(+1) = \left(\frac{1}{2}\right)x_1 - (2)x_2 + \left(\frac{3}{2}\right)x_3 > 0 \quad \Rightarrow \quad x_2 < x_1 + \frac{3L}{4}$

Or, $\qquad x_c - \frac{L}{4} \le x_2 \le x_c + \frac{L}{4}$ where $x_c = \frac{x_1 + x_3}{2}$. This implies that the center node must lie within $L/4$ of the center of the element.

Concluding Remarks

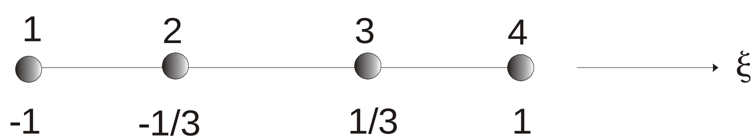
In this topic we saw how to derive the shape functions for low and high-order finite elements in one and two-dimensions. We saw the use of Gauss Quadrature to integrate functions numerically in one-dimension, and in two-dimensions over square and triangular domains. The concept of natural (and area) coordinates ties nicely with numerical integration. Finally, we saw the concept of one-to-one mapping between cartesian and natural coordinates illustrated using the linear and quadratic 1D element. The element equations including the stiffness matrix and the load vector can be derived seamlessly using the isoparametric formulation. We did not see the power of the isoparametric formulation. But that revelation will take place in the next topic with two-dimensional elements.

Review Exercises

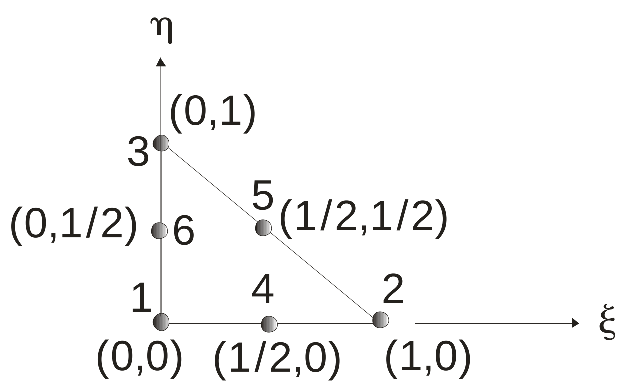
Problem T2L1-1

Derive the shape functions for the following elements. In each case, write the shape functions in their simplest polynomial form, e.g. NOT as $\frac{1}{4}(\xi-1)(2\xi+1)$ but as $\frac{1}{2}\xi^2-\frac{1}{4}\xi-\frac{1}{4}$. The master element is shown for each element.

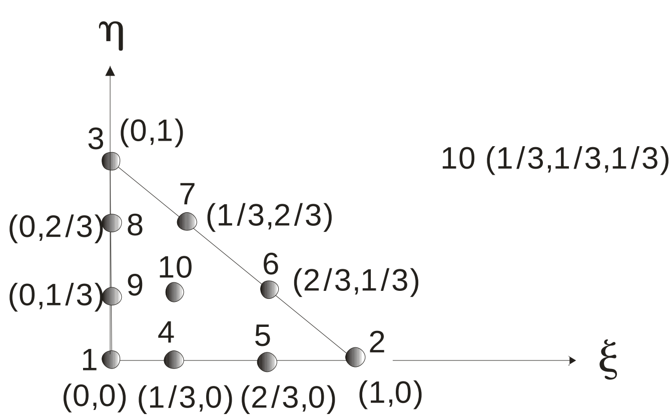
(a) $1D-C^0$ cubic element.



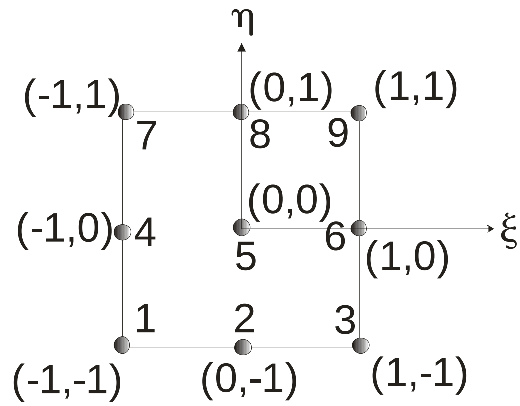
(b) $2D-C^0$ six-noded triangular element.



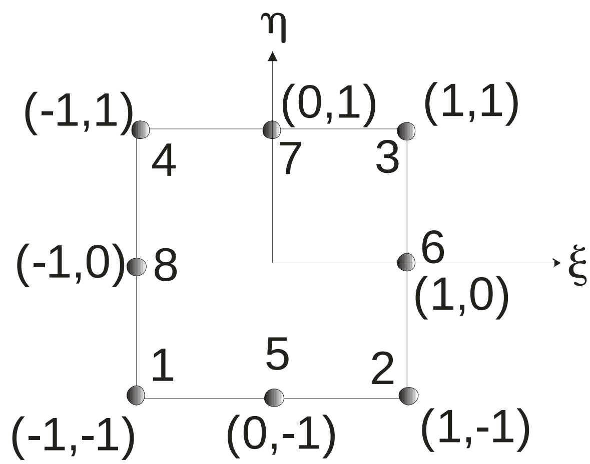
(c) $2D-C^0$ ten-noded triangular element.



(d) $2D-C^0$ nine-noded quadrilateral element.



(e) $2D-C^0$ eight-noded quadrilateral element.



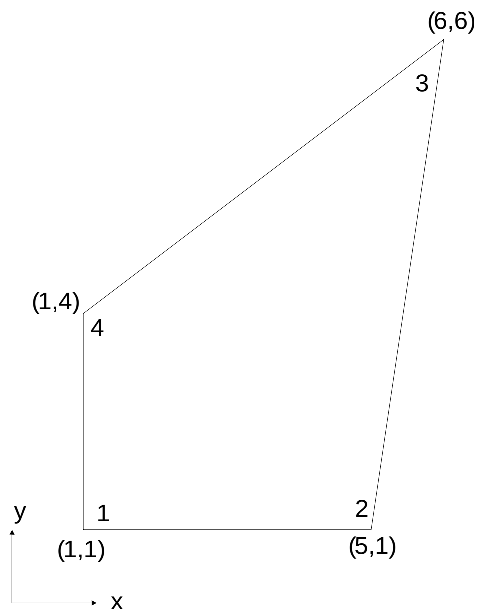
Problem T2L2-1

Evaluate the following integrals using numerical integration. Whenever possible compare with the exact answer.

(a) $\int_1^7 \frac{1}{x} dx$

(b) $\int_{-1}^{1}\int_{-1}^{1} \left(x^2 + xy^2\right) dxdy$

(c) $\iint_A \left(x^2 + xy^2\right) dxdy$ where A denotes the area given in the figure below.



Problem T2L3-1

Derive the element equations for the following elements suitable for solving the one-dimensional boundary value problems. Assume that $\alpha(x)=\hat{\alpha}$, $\beta(x)=\hat{\beta}$ and $f(x)=\hat{f}$ are constants.

(a) $1D-C^0$ quadratic element.

(b) $1D-C^0$ cubic element.

How would you change the procedure if the terms given above are not assumed to be constants but are known functions of x?

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