Finite Elements For Engineers

Lecture 6: One-Dimensional Boundary Value Problem

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1D BVP

DE

$$-\frac{d}{dx}\left(\alpha(x)\frac{dy(x)}{dx}\right)+\beta(x)y(x)=f(x) \qquad x_a < x < x_b$$

BCs

$$x = x_a \Rightarrow y = y_a \quad or \quad \tau = c_a y + d_a$$

$$x = x_b \Rightarrow y = y_b \quad or \quad \tau = c_b y + d_b$$

1D BVP

Galerkin Step 1

$$\tilde{y}(x;a) = \sum_{j=1}^{n} y_j \phi_j(x)$$

$$\int_{\Omega} \left[-\frac{d}{dx}\left(\alpha(x) \frac{dy(x)}{dx} \right) + \beta(x) y(x) - f(x) \right] \phi_i(x)\, dx = 0 \qquad i = 1,2,..,n$$

Galerkin Step 2

$$\int_{\Omega} \left[\alpha(x) \frac{dy}{dx} \frac{d\phi_i}{dx} + \beta(x) y(x) \phi_i \right] dx = \int_{\Omega} f(x) \phi_i dx - \left[\tau \phi_i \right]^{\Gamma}$$

$$i = 1,2,..,n$$

1D BVP

Galerkin Step 3

$$\sum_{j=1}^{n}\left[\int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx+\int_{\Omega}\phi_i(x)\beta(x)\phi_j(x)\right]y_j=$$

$$\int_{\Omega} f(x)\phi_i(x)dx-\left[\tau\phi_i\right]^{\Gamma} \qquad i=1,2.,,n$$

Typical Stiffness Term

$$k_{ij}=\int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx+\int_{\Omega}\phi_i(x)\beta(x)\phi_j(x)\,dx$$

1D BVP

Galerkin Step 4 (2-noded linear interpolation element)

$$\phi_1 = \frac{x_2 - x}{L} \qquad \phi_2 = \frac{x - x_1}{L}$$

Example

$$k_{11} = \int_{x_1}^{x_2} \left(-\frac{1}{x_2 - x_1} \right) \alpha(x) \left(-\frac{1}{x_2 - x_1} \right) dx + \int_{x_1}^{x_2} \left(\frac{x_2 - x}{x_2 - x_1} \right) \beta(x) \left(\frac{x_2 - x}{x_2 - x_1} \right) dx$$

To evaluate the integral, let

$$\overline{\alpha} = \alpha \left(x = \frac{x_1 + x_2}{2} \right) \qquad \overline{\beta} = \beta \left(x = \frac{x_1 + x_2}{2} \right)$$

1D BVP

Galerkin Step 4 (cont'd)

$$\begin{bmatrix} \dfrac{\overline{\alpha}}{L}+\dfrac{\overline{\beta}L}{3} & -\dfrac{\overline{\alpha}}{L}+\dfrac{\overline{\beta}L}{6} \\ -\dfrac{\overline{\alpha}}{L}+\dfrac{\overline{\beta}L}{6} & \dfrac{\overline{\alpha}}{L}+\dfrac{\overline{\beta}L}{3} \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\overline{f}L}{2} \\ \dfrac{\overline{f}L}{2} \end{Bmatrix} - \begin{Bmatrix} \left[\tau\phi_1\right]^{\Gamma} \\ \left[\tau\phi_2\right]^{\Gamma} \end{Bmatrix}$$

NBC require special treatment

$$\left[\tau\phi_i\right]^{\Gamma} = \left[\tau\phi_i\right]_{x_2} - \left[\tau\phi_i\right]_{x_1} = \left[(cy+d)\phi_i\right]_{x_2} - \left[(cy+d)\phi_i\right]_{x_1} \qquad i=1,2$$

1D BVP

Element Equations

$$\left[\left[\begin{array}{c|c} \dfrac{\bar{\alpha}}{L}+\dfrac{\bar{\beta}L}{3} & -\dfrac{\bar{\alpha}}{L}+\dfrac{\bar{\beta}L}{6} \\ \hline -\dfrac{\bar{\alpha}}{L}+\dfrac{\bar{\beta}L}{6} & \dfrac{\bar{\alpha}}{L}+\dfrac{\bar{\beta}L}{3} \end{array}\right] - c_1\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right] + c_2\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right]\right]\left\{\begin{array}{c} y_1 \\ \hline y_2 \end{array}\right\} =$$

$$\left\{\begin{array}{c} \dfrac{\bar{f}L}{2} \\ \hline \dfrac{\bar{f}L}{2} \end{array}\right\} + \left\{\begin{array}{c} d_1 \\ \hline -d_2 \end{array}\right\}$$

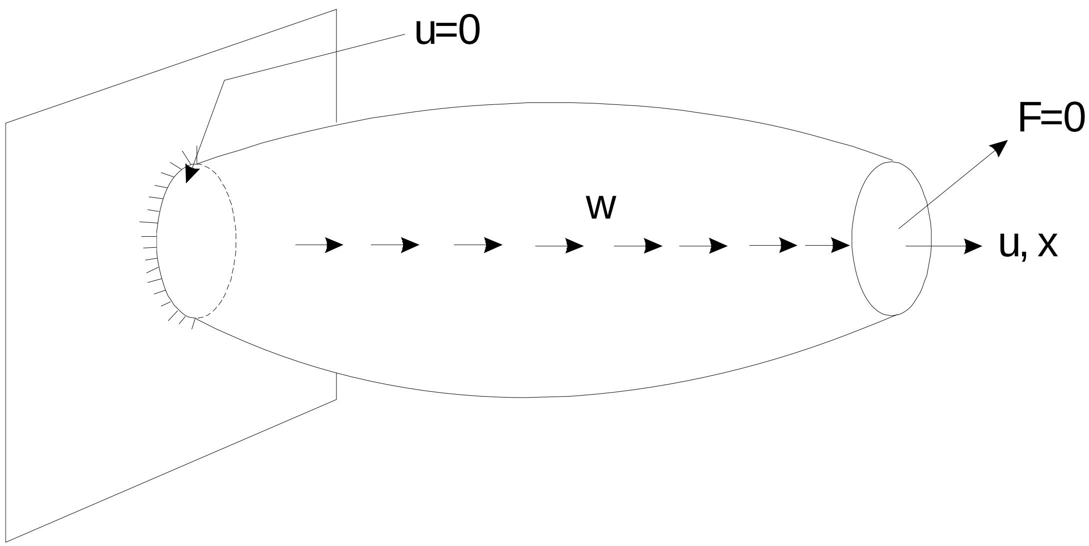
Element Flux

$$\tau = -\alpha\frac{dy}{dx} = -\frac{\bar{\alpha}}{L}\left(y_2 - y_1\right)$$

Summary

- 1D BVP derivation using Galerkin's Method yields the same equations as the Direct Stiffness Method
- **k** is symmetric and rank deficient

Solid Mechanics



DE $$-\frac{d}{dx}\left(A(x)E(x)\frac{du(x)}{dx}\right)=w(x)A(x)$$

EBC $$u=c$$

NBC $$\overline{X}=n_xF_x=n_xAE\frac{du}{dx}$$

Solid Mechanics

Element Equations

$$\frac{\overline{AE}}{L}\left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array}\right]\left\{\begin{array}{c} u_1 \\ \hline u_2 \end{array}\right\} = \left\{\begin{array}{c} \dfrac{\overline{wAL}}{2} \\ \hline \dfrac{\overline{wAL}}{2} \end{array}\right\} + \left\{\begin{array}{c} F_1 \\ \hline F_2 \end{array}\right\}$$

Element loads **Nodal loads**

Relationship with 1D-BVP

Original

$$-\frac{d}{dx}\left(A(x)E(x)\frac{du(x)}{dx}\right)=w(x)A(x)$$

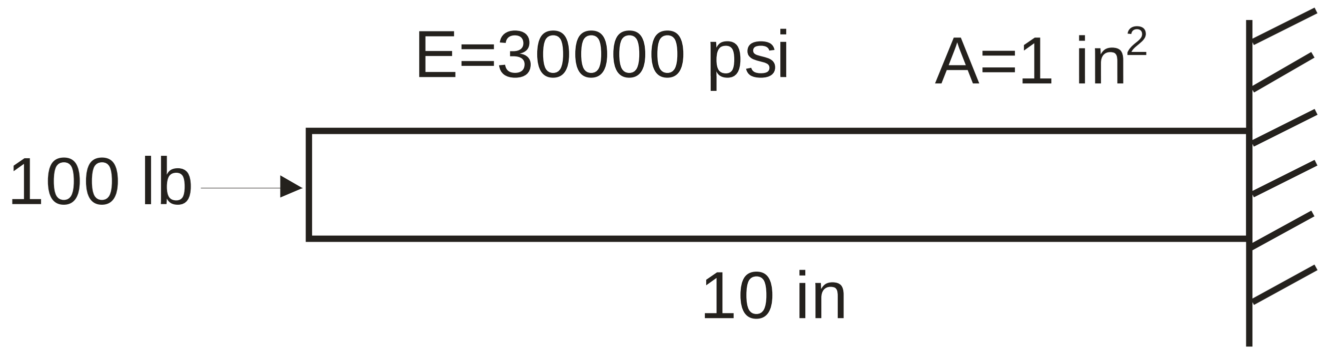
LHS: $\frac{1}{L}\left[L^2\left(\frac{F}{L^2}\right)\left(\frac{L}{L}\right)\right]=\frac{F}{L}$ RHS: $\frac{F}{L^3}\left(L^2\right)=\frac{F}{L}$

Modified

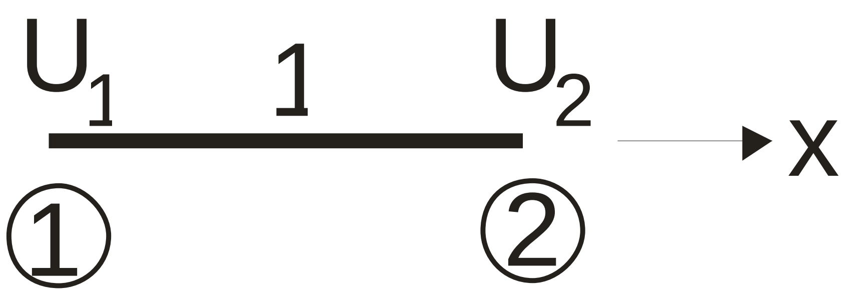
$$-\frac{d}{dx}\left(\underbrace{A(x)E(x)}\frac{du(x)}{dx}\right)=w(x)$$

$$-\frac{d}{dx}\left(\alpha(x)\frac{dy(x)}{dx}\right)+\beta(x)y(x)=f(x)$$

Example 1



Units: lb, in



Example 1

Element 1

$$\begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Imposition of EBC $U_2 = 0$ **Imposition of NBC** $F_1 = 100$

$$3000 U_1 = 100$$

Solution $U_1 = 0.0333333 in$

Strain $\varepsilon = \dfrac{U_2 - U_1}{L} = -0.00333333$

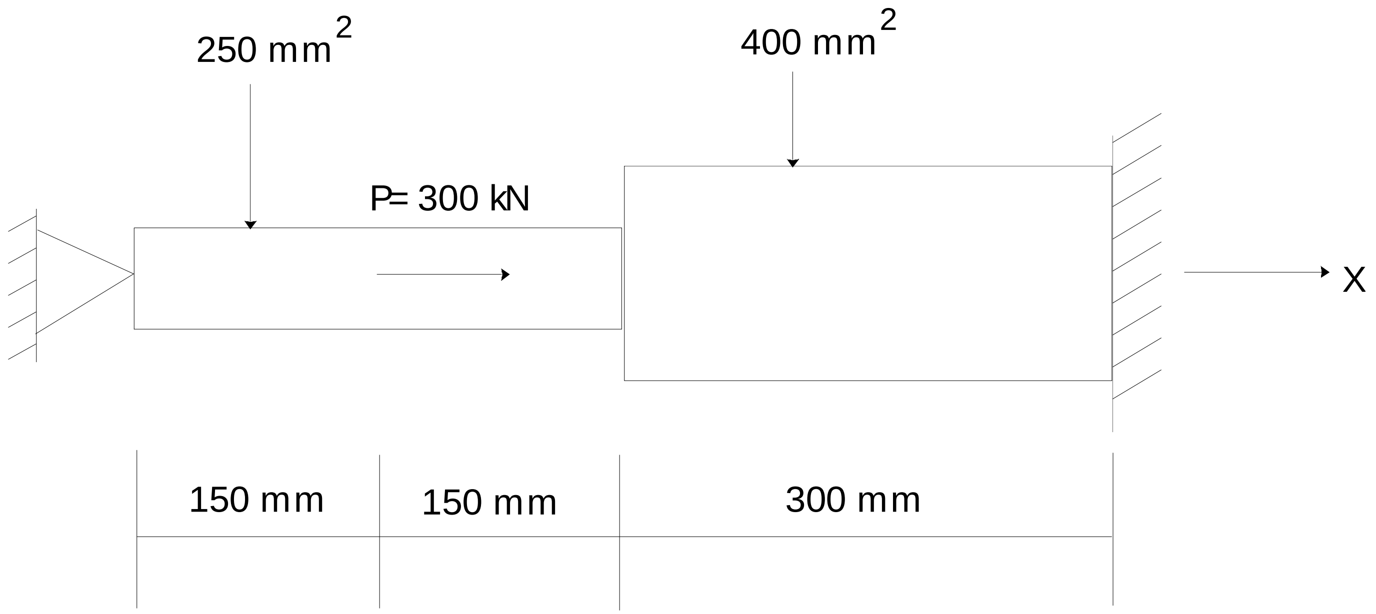
Stress $\sigma = E\varepsilon = 30000\left(\text{-}0.00333333\right) = -100\ psi$

Force $F = A\sigma = \left(1\right)(-100) = -100\ \text{lb}$

Example T4L2-1

Compute displacements, strains and stresses

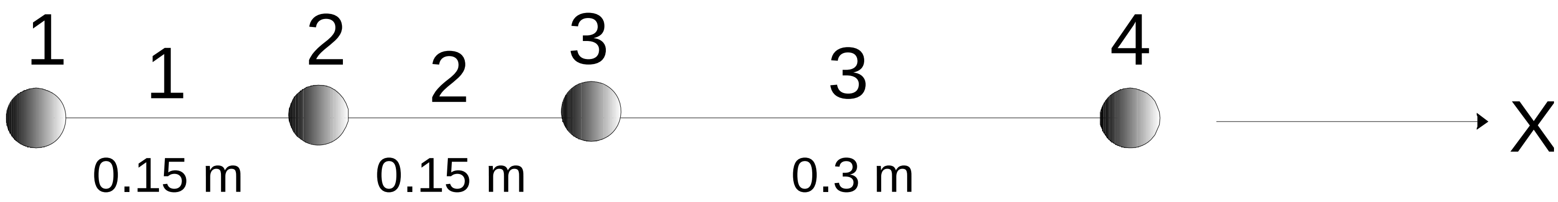
$$E = 200\,GPa$$



Example T4L2-1

Units: N, m

Discretization: FE Mesh



Element 1

$$\begin{bmatrix} 3.333(10^8) & -3.333(10^8) \\ -3.333(10^8) & 3.333(10^8) \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ -F_2^1 \end{Bmatrix}$$

Element 2

$$\begin{bmatrix} 3.333(10^8) & -3.333(10^8) \\ -3.333(10^8) & 3.333(10^8) \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1^2 \\ F_2^2 \end{Bmatrix}$$

Example T4L2-1

Element 3

$$\begin{bmatrix} 2.667(10^8) & -2.667(10^8) \\ -2.667(10^8) & 2.667(10^8) \end{bmatrix} \begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1^3 \\ F_2^3 \end{Bmatrix}$$

Assembly (System Equations)

$$10^8 \begin{bmatrix} 3.333 & -3.333 & 0 & 0 \\ -3.333 & 6.667 & -3.333 & 0 \\ 0 & -3.333 & 6 & -6.667 \\ 0 & 0 & -6.667 & 6.667 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1^1 \\ 300(10^3) \\ 0 \\ F_2^3 \end{Bmatrix}$$

Example T4L2-1

Imposition of EBC $U_1 = U_4 = 0$

$$10^8 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6.667 & -3.333 & 0 \\ 0 & -3.333 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 300(10^3) \\ 0 \\ 0 \end{Bmatrix}$$

Solution

$$\{U_1, U_2, U_3, U_4\} = \{0, 6.23, 3.46, 0\} \times 10^{-4} m$$

Example T4L2-1

Derived Variables

Element 1

Strain $$\varepsilon = \frac{U_2 - U_1}{L} = 0.00415333$$

Stress $$\sigma = E\varepsilon = 200\times10^9\left(0.00415333\right) = 831\,MPa$$

Force $$F = A\sigma = \left(250\times10^{-6}\right)831(10^6) = 207667\text{ N}$$

Reaction at left support must be 208 kN

Example T4L2-1

Element 3

Strain $$\varepsilon = \frac{U_4 - U_3}{L} = -0.00115333$$

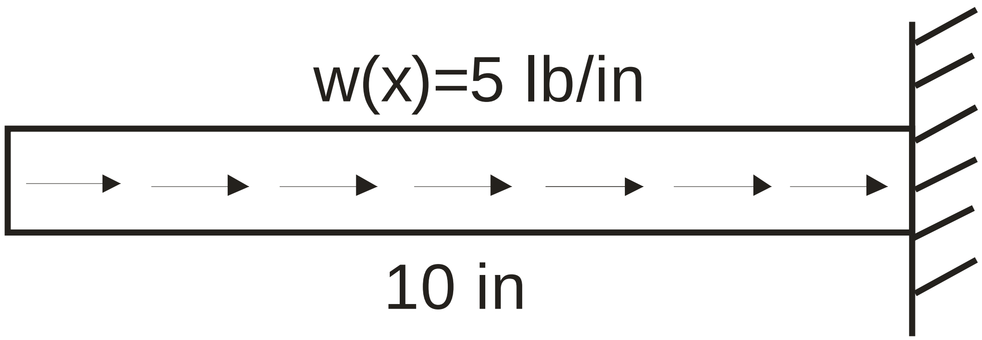
Stress $$\sigma = E\varepsilon = 200\times10^9\left(-0.00115333\right) = -231\,MPa$$

Force $$F = A\sigma = \left(400\times10^{-6}\right)\left(-231\times10^6\right) = -92266.7\text{ N}$$

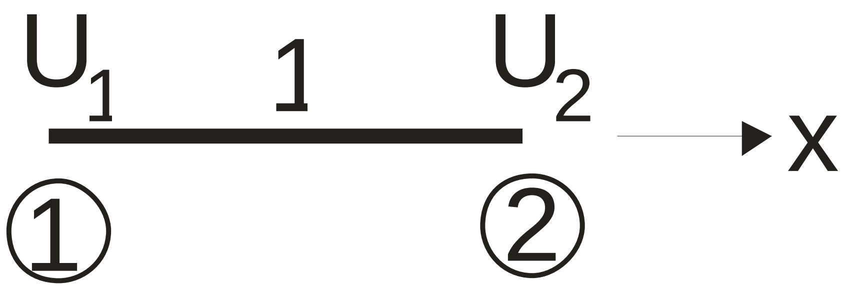
Reaction at right support must be 92 kN

Example 1

E=30000 psi   A=1 in^2



Units: lb, in



Example 1

Element 1 and System Equations

$$\begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 25 \\ 25 \end{Bmatrix}$$

Imposition of EBC $U_2 = 0$ $\quad 3000 U_1 = 25$

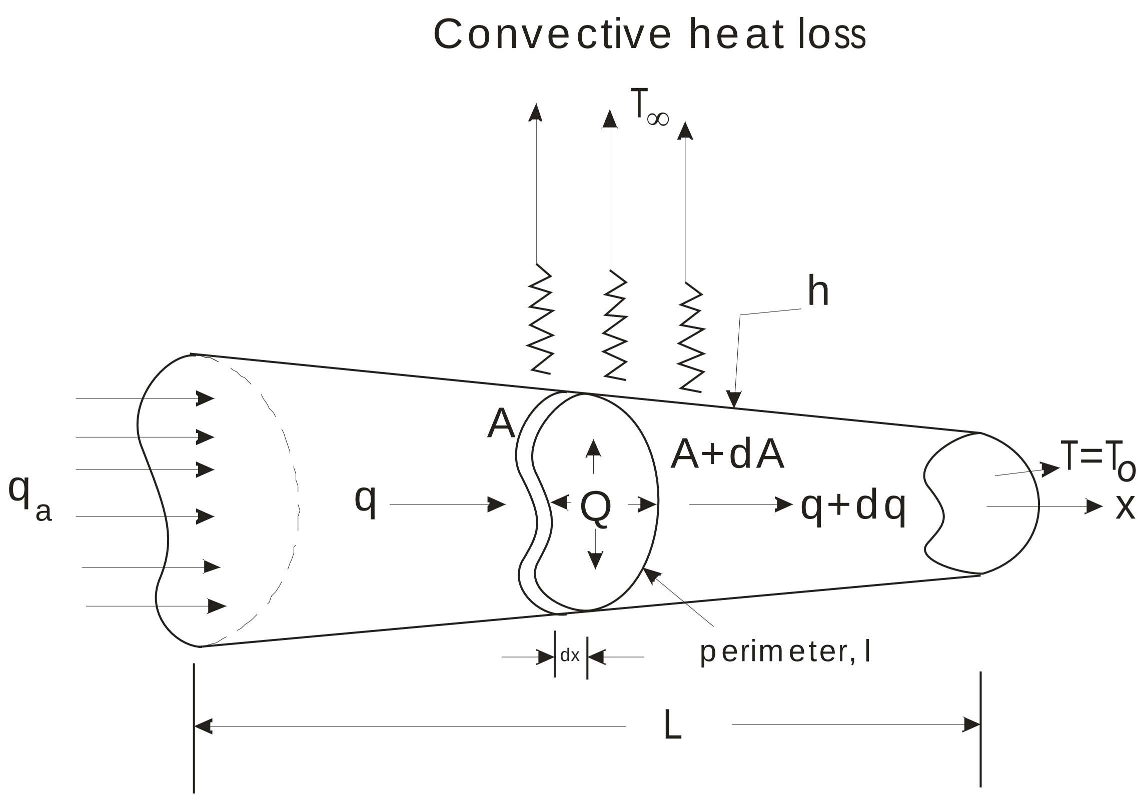
Solution $U_1 = 0.00833333 in$

Strain $$\varepsilon = \frac{U_2 - U_1}{L} = -0.00083333$$

Stress $$\sigma = E\varepsilon = 30000\left(\text{-}0.00083333\right) = -25\, psi$$

Force $$F = A\sigma = \left(1\right)(-25) = -25 \text{ lb}$$

Heat Transfer



DE $$-\frac{d}{dx}\left(A(x)k(x)\frac{dT(x)}{dx}\right)+h(x)l(x)T(x)=Q(x)A(x)+h(x)l(x)T_{\infty}$$

Heat Transfer

EBC $T = \hat{T}$

NBC $q_x n_x = -q_S$

Mixed $q_x n_x = h(T_S - T_\infty)$

Sign convention: Heat flowing into a surface is positive.

$$q_x n_x + q_y n_y + q_z n_z = -q_S$$

Sign convention: Free convection from surface S

$$q_x n_x + q_y n_y + q_z n_z = h(T_S - T_\infty)$$

Possible BCs

Left end $\left(n_x = -1\right)$

$T = T_a$ **EBC**

$q = q_a$

$q = 0$ **NBC**

Mixed

$$q = -h_a T + h_a T_a^{\infty}$$

Right end $\left(n_x = 1\right)$

$T = T_b$ **EBC**

$q = q_b$

$q = 0$ **NBC**

Mixed

$$q = h_b T - h_b T_b^{\infty}$$

Heat Transfer

Element Equations

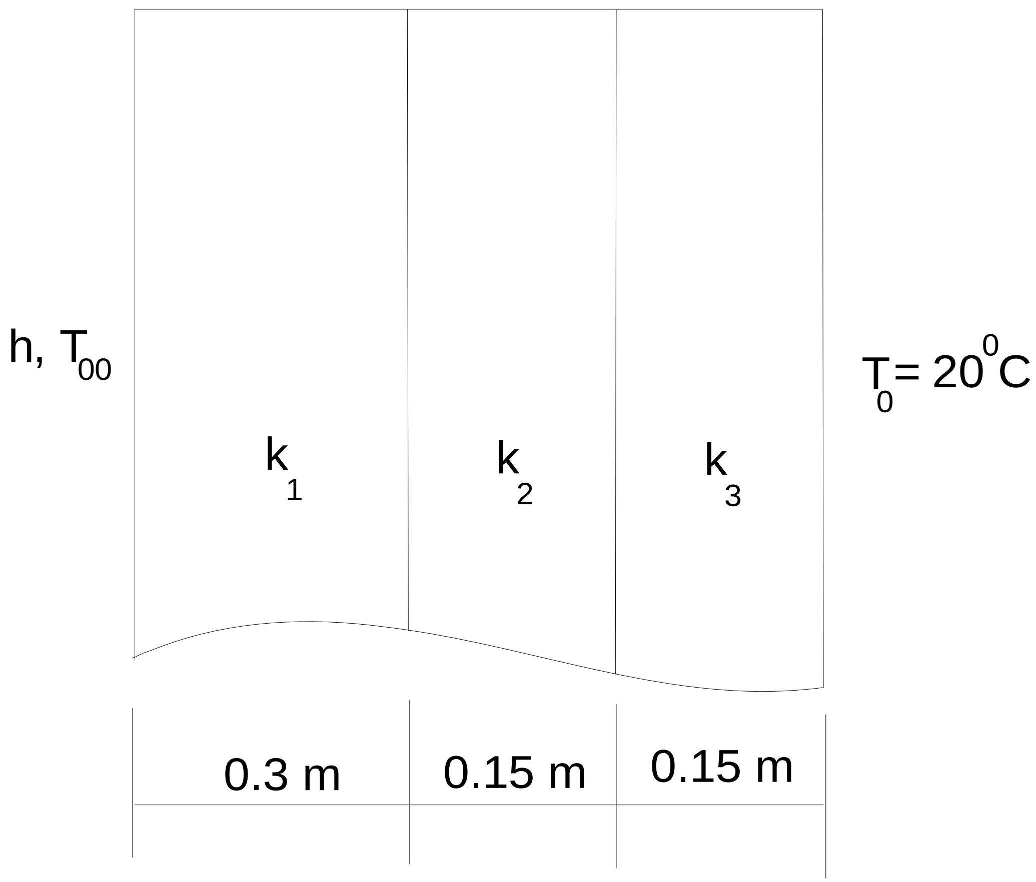
$$\left[\begin{bmatrix} \dfrac{\bar{k}}{L}+\dfrac{\overline{hlL}}{3A} & -\dfrac{\bar{k}}{L}+\dfrac{\overline{hlL}}{6A} \\ -\dfrac{\bar{k}}{L}+\dfrac{\overline{hlL}}{6A} & \dfrac{\bar{k}}{L}+\dfrac{\overline{hlL}}{3A} \end{bmatrix} + h_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + h_2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} =$$

$$\frac{L}{2}\begin{Bmatrix} \overline{Q}+\dfrac{\overline{hl}}{A}T_\infty \\ \overline{Q}+\dfrac{\overline{hl}}{A}T_\infty \end{Bmatrix} + \begin{Bmatrix} q_1 \\ -q_2 \end{Bmatrix} + \begin{Bmatrix} h_1T_\infty^1 \\ h_2T_\infty^2 \end{Bmatrix}$$

Example T4L3-1

Compute temperature and flux.

Conduction and Convection



$$k_1 = 20 \frac{W}{m \cdot {}^\circ C}$$

$$k_2 = 30 \frac{W}{m \cdot {}^\circ C}$$

$$k_3 = 50 \frac{W}{m \cdot {}^\circ C}$$

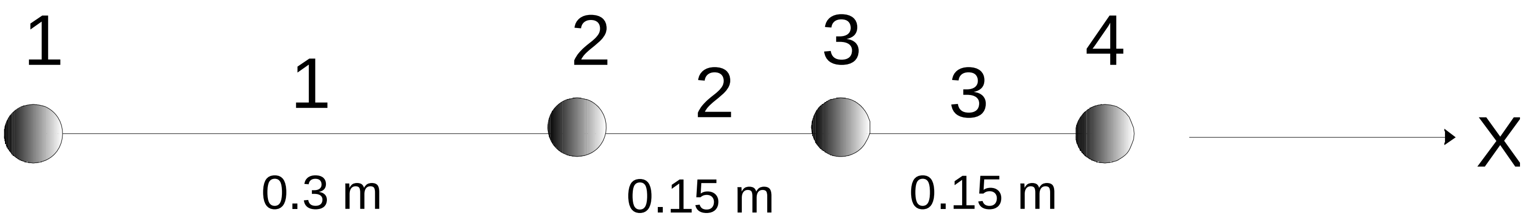
$$T_\infty = 800^\circ C$$

$$h = 25 \frac{W}{m^2 \cdot {}^\circ C}$$

Example T4L3-1

Units: W, m, C

Discretization: FE Mesh



Element 1

$$
\left[\begin{array}{c|c} \dfrac{20}{0.3}+25 & -\dfrac{20}{0.3} \\ \hline -\dfrac{20}{0.3} & \dfrac{20}{0.3} \end{array}\right]\left\{\begin{array}{c} T_1 \\ \hline T_2 \end{array}\right\} = \left\{\begin{array}{c} 25(800) \\ \hline 0 \end{array}\right\}
$$

Example T4L3-1

Element 2

$$\left[\begin{array}{c|c} \dfrac{30}{0.15} & -\dfrac{30}{0.15} \\ \hline -\dfrac{30}{0.15} & \dfrac{30}{0.15} \end{array}\right]\left\{\begin{array}{c} T_2 \\ \hline T_3 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hline 0 \end{array}\right\}$$

Element 3

$$\left[\begin{array}{c|c} \dfrac{50}{0.15} & -\dfrac{50}{0.15} \\ \hline -\dfrac{50}{0.15} & \dfrac{50}{0.15} \end{array}\right]\left\{\begin{array}{c} T_3 \\ \hline T_4 \end{array}\right\} = \left\{\begin{array}{c} 0 \\ - \\ 0 \end{array}\right\}$$

Example T4L3-1

Assembly (System Equations)

$$\begin{bmatrix} 91.6667 & -66.6667 & 0 & 0 \\ -66.6667 & 266.6667 & -200.0 & 0 \\ 0 & -200.0 & 533.3333 & -333.3333 \\ 0 & 0 & -333.3333 & 333.3333 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 20,000 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Imposition of EBC $T_4 = 20$

$$\begin{bmatrix} 91.6667 & -66.6667 & 0 & 0 \\ -66.6667 & 266.6667 & -200.0 & 0 \\ 0 & -200.0 & 533.3333 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 20,000 \\ 0 \\ 6666.6667 \\ 20 \end{Bmatrix}$$

Example T4L3-1

Solution

$$\{T_1,\ \ T_2,\ \ T_3,\ \ T_4\} = \{304.8,\ \ 119.1,\ \ 57.1,\ \ 20\}\ ^{\circ}C$$

Derived Variables

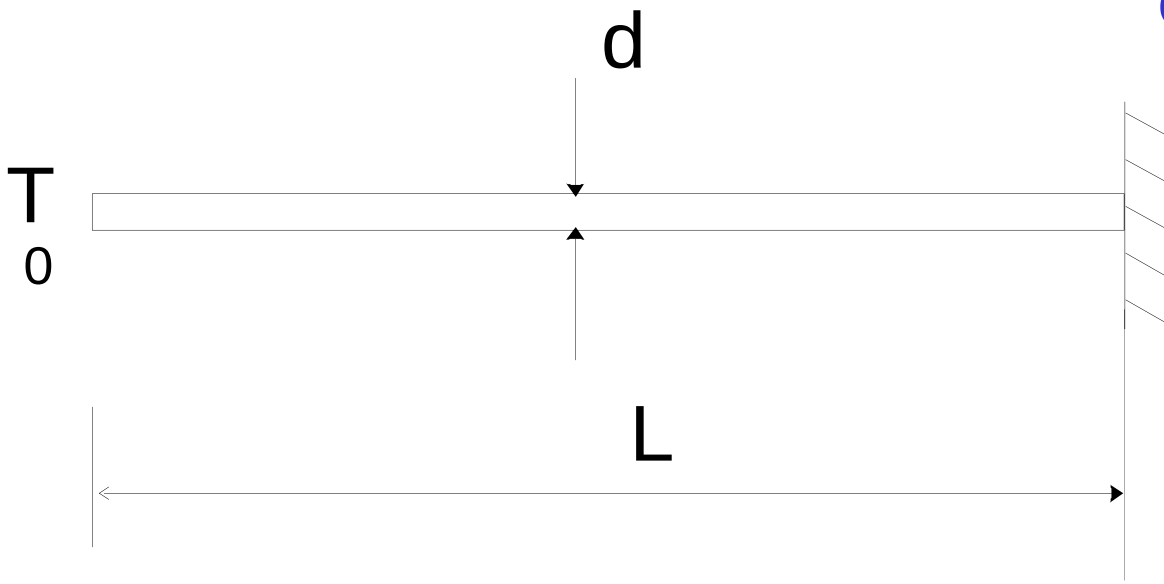
Element 1 $$\tau = -\frac{k_1}{L_1}(T_2 - T_1) = -\frac{20}{0.3}(119.05 - 304.76) = 12381\frac{W}{m^2}$$

Element 2 $$\tau = -\frac{k_2}{L_2}(T_3 - T_2) = -\frac{30}{0.15}(57.14 - 119.05) = 12381\frac{W}{m^2}$$

Element 3 $$\tau = -\frac{k_3}{L_3}(T_4 - T_3) = -\frac{50}{0.15}(20.0 - 57.1) = 12381\frac{W}{m^2}$$

Example T4L3-2

Compute temperature and flux.



Conduction and Convection

$$T_0 = 150^\circ F$$

$$T_\infty = 80^\circ F$$

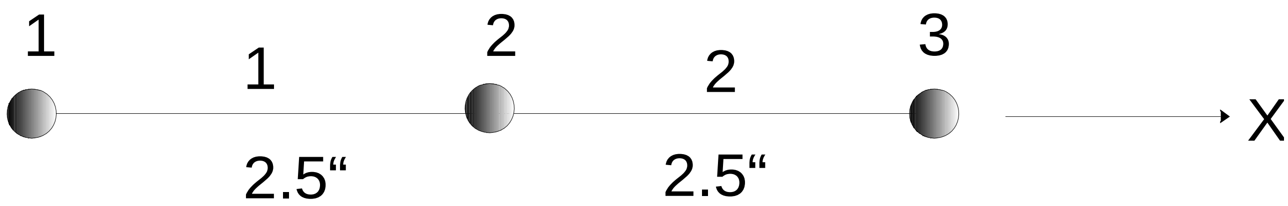
$$h = 6\frac{BTU}{h \cdot ft^2 \cdot^\circ F}$$

$$k = 24.8\frac{BTU}{h \cdot ft \cdot^\circ F}$$

Example T4L3-2

Units: BTU, hr, ft, F

Discretization: FE Mesh



Element 1 and 2

$$\left[\begin{array}{c|c} 183.12 & -87.285 \\ \hline -87.285 & 183.12 \end{array}\right]\left\{\begin{array}{c} T_1 \\ \hline T_2 \end{array}\right\} = \left\{\begin{array}{c} 7667 \\ \hline 7667 \end{array}\right\}$$

Example T4L3-2

Assembly (System Equations)

$$\begin{bmatrix} 183.12 & -87.285 & 0 \\ -87.285 & 366.25 & -87.285 \\ 0 & -87.285 & 183.12 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 7667 \\ 15334 \\ 7667 \end{Bmatrix}$$

Imposition of EBC $T_1 = 150$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 366.245 & -87.285 \\ 0 & -87.285 & 183.12 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 150 \\ 28427 \\ 7667 \end{Bmatrix}$$

Example T4L3-2

Solution

$$\left\{ T_1, T_2, T_3 \right\} = \left\{ 150, 98.8, 89.0 \right\}^{\circ} F$$

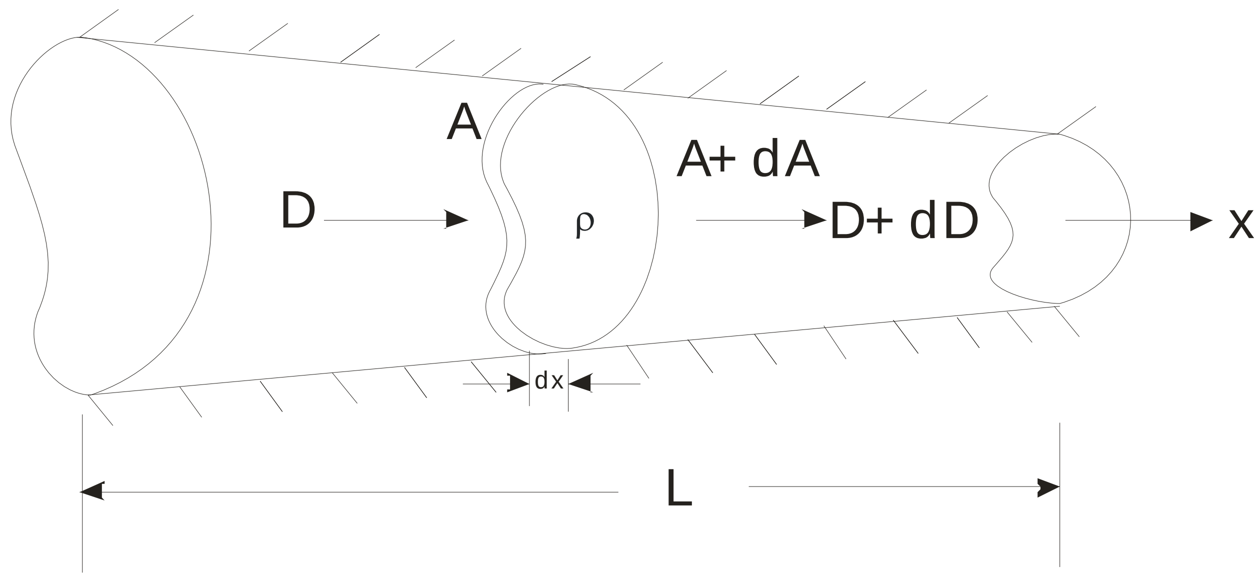
Derived Variables

Element 1 $$\tau = -\frac{k_1}{L_1}(T_2 - T_1) = -\frac{24.8}{0.208}(98.82 - 150) = 6102 \frac{BTU}{h \cdot ft^2}$$

Element 2 $$\tau = -\frac{k_2}{L_2}(T_3 - T_2) = -\frac{24.8}{0.208}(88.97 - 98.82) = 1174 \frac{BTU}{h \cdot ft^2}$$

Electrostatics

Dielectric rod with lateral surface electrically insulated



DE $$-\frac{d}{dx}\left(\varepsilon(x)A(x)\frac{d\Phi(x)}{dx}\right)=\rho(x)A(x)$$

EBC $$\Phi(x=x_a,x_b)=\Phi_a \ or \ \Phi_b$$

NBC $$\hat{D}(x)=-\varepsilon(x)A(x)\frac{d\Phi(x)}{dx}=c$$

Summary

- The general 1D BVP differential equation describes a number of engineering problems
- The basic element equations (Step 2) are derived from Galerkin's Method
- Some terms (or parameters) for a specific engineering problem are zero when compared to the general 1DBVP derivation

Summary

- Most of the general comments on the solution steps applicable to Direct Stiffness Method are also applicable here.
- EBCs are exactly satisfied
- NBCs are satisfied only in the limit
- One must carry out a convergence analysis to study the problem solution