ELECTRONIC NOTES

CEE598 - Finite Elements for Engineers: Module 3

Part 1: Solid Mechanics

S. D. Rajan
Department of Civil Engineering
Arizona State University
Tempe, AZ 85287-5306

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Finite Elements for Engineers

Department of Civil Engineering
Arizona State University
Tempe, AZ 85287-5306
Phone 602.965.1712 • Fax 602.965.0557
e-mail s.rajan@asu.edu

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Introduction

"Change is inevitable, growth is optional." Anon.

This course is the last one in a three-part series or modules titled "Finite Elements for Engineers" that meets the mathematics requirements for the Master of Engineering (M. Eng.) degree in the College of Engineering at Arizona State University.

Who should take this course?

Finite elements has become the defacto industry standard for solving multi-disciplinary engineering problems that can be described by equations of calculus. Applications cut across several industries by virtue of the applications – solid mechanics (civil, aerospace, automotive, mechanical, biomedical, electronic), fluid mechanics (geotechnical, aerospace, electronic, environmental, hydraulics, biomedical, chemical), heat transfer (automotive, aerospace, electronic, chemical), acoustics (automotive, mechanical, aerospace), electromagnetics (electronic, aerospace) and many, many more.

Course Objectives

- To extend the Isoparametric Formulation to handle three-dimensional stress analysis.
- To understand the basics of FEM-based structural dynamics including modal (eigenvalue) analysis and time-dependent response computations.

Prerequisites

- Modules 1 and 2.

Instructor-Student Interaction

To successfully meet the course objectives it is necessary that the students avail themselves of all the resources – discussion forums, e-mail, chat rooms, libraries. Keep the instructor and teaching assistant informed of all your concerns. The web pages connected with this course will contain instructions on how to communicate with the instructor regarding the questions you may have or turning in the assignments etc.

Syllabus

"A man with one watch knows what time it is. A man with two watches is never sure." Segal's Law.

Outline (Lesson Plan)

- Isoparametric Three-Dimensional Elasticity Problems
- Fundamentals of Structural Dynamics
- Plates and Shells

Notation

"As complexity rises, precise statements lose meaning, and meaningful statements lose precision."
Lotfi Zadeh

Vectors

$\mathbf{a}_{n\times 1}$	column vector with n rows
a_i	element i of vector $\mathbf{a}$
$\mathbf{b}_{1\times m}$	row vector with m columns

Matrices

$\mathbf{A}_{m\times n}$	matrix with m rows and n columns
A_{ij}	element row i and column j of matrix

Others

y'	Derivative of y (or, $\dfrac{dy}{dx}$)
L	(Units of) length
F	(Units of) force
M	(Units of) mass
t	(Units of) time
T	(Units of) temperature
E	(Units of) energy

Lesson Plan

"Problems cannot be solved by the same level of thinking that created them." A. Einstein.

Module 3 is divided into two major topics. Each topic has several lessons designed to focus on the critical issues. With each lesson there is a set of objectives. I have also listed the relevant pages from the list of textbooks that appear in the syllabus. There are several review problems at the end of every topic. Solutions to most problems are also provided. Note that the set of problems represents the minimal set needed to understand the material. You should solve more problems from some of the referenced texts.

Topic 1 looks at extending the isoparametric ideas introduced in Module 2 to handle three-dimensional elasticity problems.

Topic 1: Three-Dimensional Stress Analysis

Lesson 1: Introduction.
Lesson 2: Finite Element Formulation.
Review Exercises

Structural dynamics problems are of different types. In this topic we will learn how to generate the element equations suitable for modal analysis and dynamic analysis. We will also look at numerical schemes for solving the eigenvalue problems and time-dependent structural dynamics problems.

Topic 2: Structural Dynamics

Lesson 1: Overview.
Lesson 2: Generating element equations for truss and frames.
Lesson 3: Eigenvalue Analysis.
Lesson 4: Time-Integration Schemes.
Review Exercises

Topic 1: 3D Stress Analysis

"The past is of no importance. The present is of no importance. It is with the future that we have to deal. For the past is what man should not have been. The present is what man ought not to be. The future is what artists are." Oscar Wilde

Lesson 1: Introduction

Objectives: In this lesson we will look at three-dimensional elasticity problems and the different types of three-dimensional finite elements. The major objectives are listed below.

- To understand what is meant by three-dimensional elasticity problems.
- To understand the different types of three-dimensional elements.

Overview: So far we have seen the following classes of problems and elements – Truss, Frame, Plane Elasticity, and Axisymmetric Elasticity. The assumptions behind these elements should be noted so that they can be used to model the appropriate class of problems.

When the structural system is such that (a) the overall dimensions in the three directions cannot be ignored, or in other words, when all the x-y-z dimensions are important, and (b) simplifications cannot be made with respect to the stress or strain distribution, loads and boundary conditions, the structure should be thought of as a three-dimensional elasticity problem. At a point in the structure, the displacement field has three components

$$u = u(x, y, z) \qquad v = v(x, y, z) \qquad w = w(x, y, z) \tag{T1L1-1}$$

Corresponding to this displacement field, the strain and stress tensors are complete.

$$\boldsymbol{\varepsilon}_{6\times1} = \left[\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}\right]^T \tag{T1L1-2a}$$

$$\boldsymbol{\sigma}_{6\times1} = \left[\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\right]^T \tag{T1L1-2b}$$

For isotropic, linear material behavior, the stress-stress relationship can be expressed as

$$\boldsymbol{\sigma}_{6\times1} = \mathbf{D}_{6\times6}\boldsymbol{\varepsilon}_{6\times1} \tag{T1L1-3}$$

where

$$\mathbf{D}_{6\times6} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & 0.5-\nu & 0 & 0 \\ & SYM & & & 0.5-\nu & 0 \\ & & & & & 0.5-\nu \end{bmatrix} \tag{T1L1-4}$$

The strain-displacement relations are given by

$$\boldsymbol{\varepsilon}_{6\times1} = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]^T \tag{T1L1-5}$$

Element Geometries & Shape Functions: Similar to two-dimensional domains that can be discretized into triangular and quadrilateral subdomains, three-dimensional domains can be split into tetrahedral and hexahedral subdomains. They can also be split into wedges, prisms and pyramids, though these elements are less popular.

Hexahedral Elements

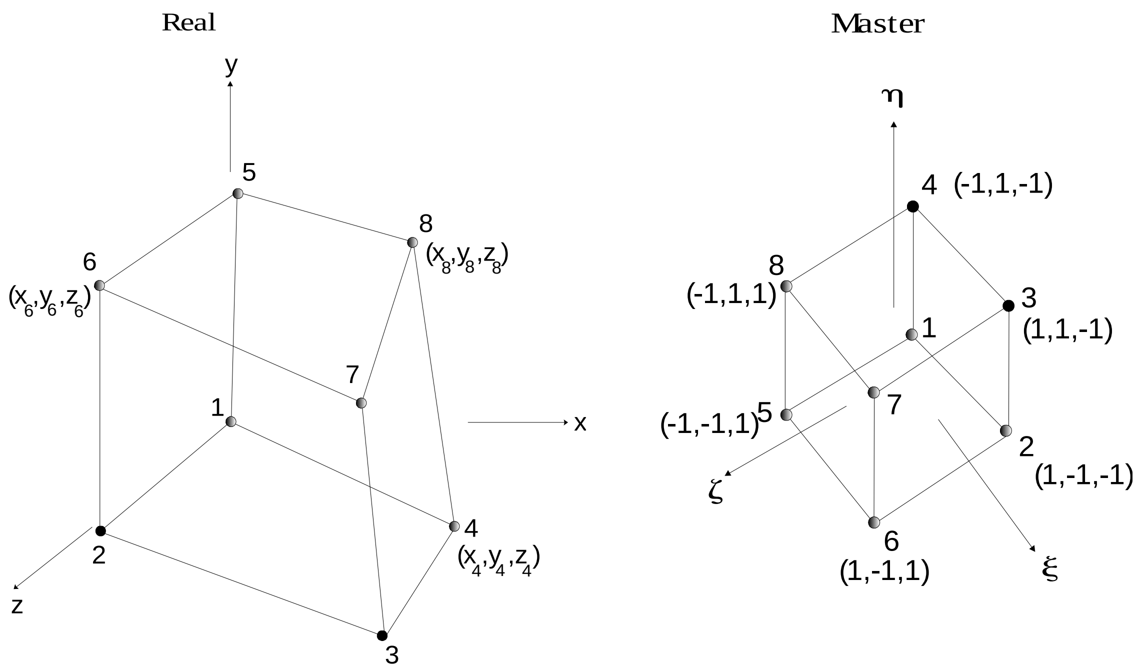

Fig. T1L1-1 Eight-noded hexahedral element

The lowest-order hexahedral isoparametric element has eight nodes and the assumed displacement field is expressed as

$$u = a_1 + a_2\xi + a_3\eta + a_4\zeta + a_5\xi\eta + a_6\eta\zeta + a_7\xi\zeta + a_8\xi\eta\zeta \quad \text{(T1L1-6a)}$$

$$v = b_1 + b_2\xi + b_3\eta + b_4\zeta + b_5\xi\eta + b_6\eta\zeta + b_7\xi\zeta + b_8\xi\eta\zeta \quad \text{(T1L1-6b)}$$

$$w = c_1 + c_2\xi + c_3\eta + c_4\zeta + c_5\xi\eta + c_6\eta\zeta + c_7\xi\zeta + c_8\xi\eta\zeta \quad \text{(T1L1-6c)}$$

The shape functions can be generated using the same strategy as discussed with two-dimensional quadrilateral elements. The nodes in the three-dimensional grid are labeled by their three indices (I, J,K) as

$$\phi_i \equiv \phi_{IJK} = l_I^n(\xi)\, l_J^m(\eta) l_K^o(\zeta) \quad \text{(T1L1-7)}$$

As an example, Node 1:

$$\phi_1 = N_{000} = l_0^1(\xi) l_0^1(\eta) l_0^1(\zeta) = \frac{(\xi - \xi_1)}{(\xi_0 - \xi_1)} \frac{(\eta - \eta_1)}{(\eta_0 - \eta_1)} \frac{(\zeta - \zeta_1)}{(\zeta_0 - \zeta_1)} = \frac{(\xi - 1)}{(-1-1)} \frac{(\eta - 1)}{(-1-1)} \frac{(\zeta - 1)}{(-1-1)}$$
$$= \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta) \qquad \text{(T1L1-8)}$$

Similarly, the other shape functions can be generated. The higher-order hexahedral elements are similar to the quadrilateral elements – they are either Lagrange or Serendipity.

In a manner similar to the one-dimensional element, if $u(x, y, z)$ represents the unknown, then

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \xi} \qquad \text{(T1L1-9a)}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \eta} \qquad \text{(T1L1-9b)}$$

$$\frac{\partial u}{\partial \zeta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \zeta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \zeta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \zeta} \qquad \text{(T1L1-9b)}$$

or,
$$\begin{Bmatrix} u,_{\xi} \\ u,_{\eta} \\ u,_{\zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} u,_{x} \\ u,_{y} \\ u,_{z} \end{Bmatrix} = \mathbf{J}_{3\times3} \begin{Bmatrix} u,_{x} \\ u,_{y} \\ u,_{z} \end{Bmatrix} \qquad \text{(T1L1-10)}$$

And,
$$\begin{Bmatrix} u,_{x} \\ u,_{y} \\ u,_{z} \end{Bmatrix} = \mathbf{\Gamma}_{3\times3} \begin{Bmatrix} u,_{\xi} \\ u,_{\eta} \\ u,_{\zeta} \end{Bmatrix} \qquad \text{where} \quad \mathbf{\Gamma} = \mathbf{J}^{-1} \qquad \text{(T1L1-11)}$$

Numerical integration of functions that involve three independent natural coordinates are handled in a manner similar to one-dimensional functions.

$$\int_e^f \int_c^d \int_a^b F(x, y, z)\, dx\, dy\, dz = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} F\big(x(\xi,\eta,\zeta), y(\xi,\eta,\zeta), z(\xi,\eta,\zeta)\big) |J|\, d\xi\, d\eta\, d\zeta$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} w_i w_j w_k f\left(\xi_i, \eta_j, \zeta_k\right) \tag{T1L1-12a}$$

where $$f\left(\xi_i, \eta_j, \zeta_k\right) = F\left(x(\xi,\eta,\zeta), y(\xi,\eta,\zeta), z(\xi,\eta,\zeta)\right) \left|J\right| \tag{T1L1-12b}$$

$$\left|J\right| = \det(\mathbf{J}) \qquad \text{where } \mathbf{J}_{3\times 3} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \tag{T1L1-12c}$$

The values of the weights and natural coordinates are the same as shown in Table T2L2-1 (Module 2) except that ξ in the table refers to ξ, η and ζ.

Tetrahedral Elements

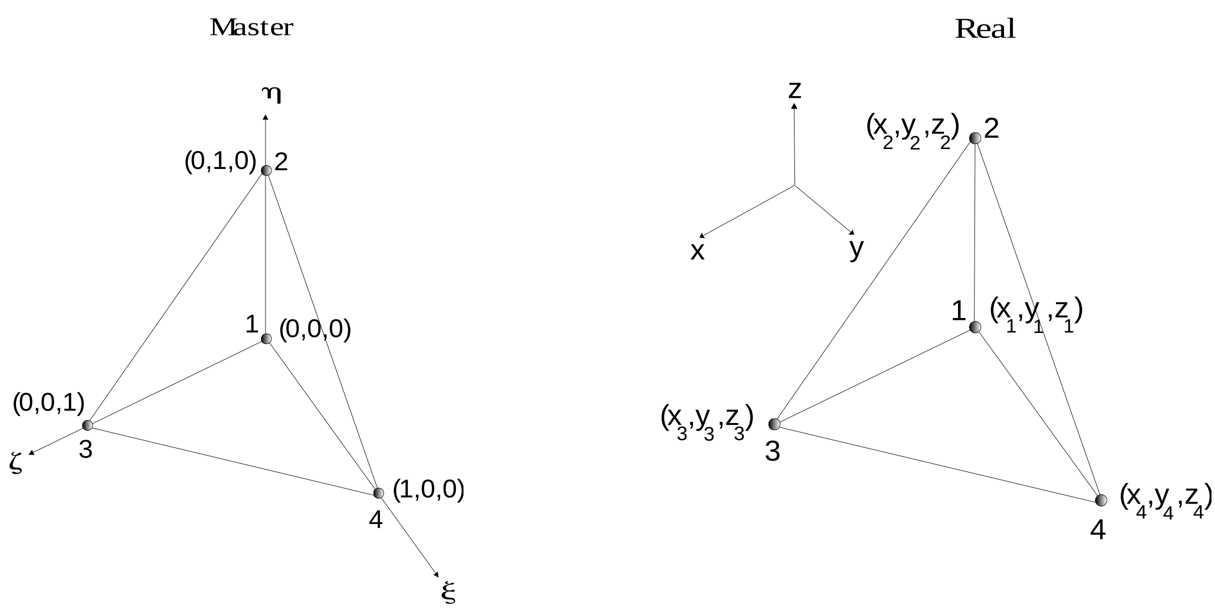

Fig. T1L1-3 Four-noded tetrahedral element

The lowest-order tetrahedral isoparametric element has four nodes and the assumed displacement field is expressed as

$$u = a_1 + a_2\xi + a_3\eta + a_4\zeta \quad \text{(T1L1-13a)}$$

$$v = b_1 + b_2\xi + b_3\eta + b_4\zeta \quad \text{(T1L1-13b)}$$

$$w = c_1 + c_2\xi + c_3\eta + c_4\zeta \quad \text{(T1L1-13c)}$$

where $(\xi, \eta, \zeta, \varsigma)$ are the volume coordinates such that

$$\xi + \eta + \zeta + \varsigma = 1 \quad \text{(T1L1-14)}$$

The shape functions can be generated using the same strategy as discussed with two-dimensional triangular elements. Denoting a typical node i by (I,J,K,L) corresponding to the node's area coordinates $(\xi, \eta, \zeta, \varsigma)$, the shape function for that node can be written as

$$\phi_i = l_I^I(\xi)\, l_J^J(\eta)\, l_K^K(\zeta) l_L^L(\varsigma) \quad \text{(T1L1-15)}$$

and $$I + J + K + L = M \quad \text{(T1L1-16)}$$

For the four-noded tetrahedral element $I, J, K, L = 0,1$ and $M = 1$.

Node 1: $\phi_1 = l_0^0(\xi)\, l_0^0(\eta)\, l_0^0(\zeta)\, l_1^1(\varsigma) = (1)(1)(1)\dfrac{(\varsigma - \varsigma_0)}{(\varsigma_1 - \varsigma_0)} = \dfrac{(\varsigma - 0)}{(1-0)} = 1 - \xi - \eta - \zeta$ (T1L1-17)

Similarly, the other shape functions can be generated. The higher-order tetrahedral elements are similar to the triangular elements - the quadratic element has 10 nodes etc.

Numerical integration in volume coordinates requires a procedure similar to integration with area coordinates.

$$\int_0^1 \int_0^{1-\zeta} \int_0^{1-\eta} F(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta = \frac{1}{6}\sum_{i=1}^{n} w_i F(\xi_i, \eta_i, \zeta_i) \quad \text{(T1L1-18)}$$

The weights and locations are given in Table T1L1-1.

Table T1L1-1 Gauss points and weights using Volume Coordinates

Order, *n*	**Weight**	**Location**
1	1.0	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$
2	0.25	$(\alpha, \beta, \beta, \beta)$
	0.25	$(\beta, \alpha, \beta, \beta)$
	0.25	$(\beta, \beta, \alpha, \beta)$

	0.25	$(\beta, \beta, \beta, \alpha)$
3	γ	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$
	δ	$\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$
	δ	$\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right)$
	δ	$\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6}\right)$
	δ	$\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{3}\right)$

where $\alpha = 0.58541020$, $\beta = 0.13819660$, $\gamma = -4/5$ and $\delta = 9/20$.

Wedge Elements

The shape functions for the wedge elements can be created using a combination of area coordinates (ξ_1, ξ_2, ξ_3) and natural (intrinsic) coordinate $-1 \le \zeta \le 1$. The first and the second-order elements are shown in Figs. T1L1-4(a) and (b).

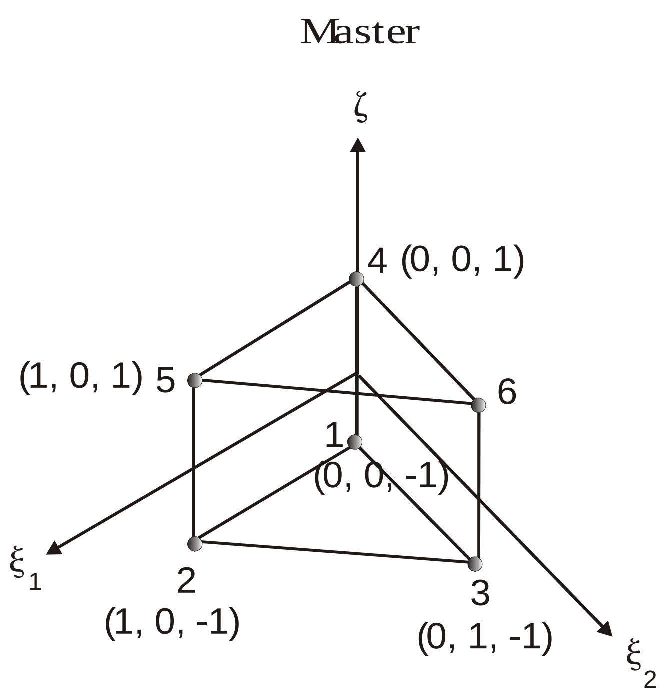

Fig. T1L1-4(a) Six-noded wedge element

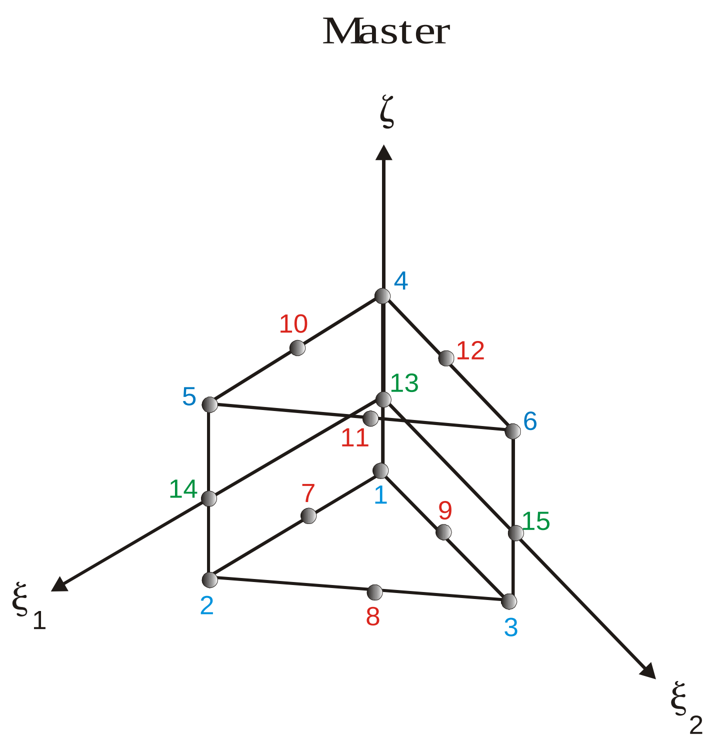

Fig. T1L1-4(b) Fifteen-noded wedge element

The shape functions for the first-order element are as follows.

$$\begin{aligned}
\phi_1 &= \frac{1}{2}\left(1-\xi_1-\xi_2\right)\left(1-\zeta\right) \\
\phi_2 &= \frac{1}{2}\xi_1\left(1-\zeta\right) \\
\phi_3 &= \frac{1}{2}\xi_2\left(1-\zeta\right) \\
\phi_4 &= \frac{1}{2}\left(1-\xi_1-\xi_2\right)\left(1+\zeta\right) \\
\phi_5 &= \frac{1}{2}\xi_1\left(1+\zeta\right) \\
\phi_6 &= \frac{1}{2}\xi_2\left(1+\zeta\right)
\end{aligned} \qquad \text{(T1L1-19)}$$

The shape functions for the second-order element are as follows.

$$\phi_1 = \frac{1}{2}\xi_3\left[\left(2\xi_3-1\right)\left(1-\zeta\right)-\left(1-\zeta^2\right)\right]$$
$$\phi_2 = \frac{1}{2}\xi_1\left[\left(2\xi_1-1\right)\left(1-\zeta\right)-\left(1-\zeta^2\right)\right]$$
$$\phi_3 = \frac{1}{2}\xi_2\left[\left(2\xi_2-1\right)\left(1-\zeta\right)-\left(1-\zeta^2\right)\right]$$
$$\phi_4 = \frac{1}{2}\xi_3\left[\left(2\xi_3-1\right)\left(1+\zeta\right)-\left(1-\zeta^2\right)\right] \qquad \text{(T1L1-20a)}$$
$$\phi_5 = \frac{1}{2}\xi_1\left[\left(2\xi_1-1\right)\left(1+\zeta\right)-\left(1-\zeta^2\right)\right]$$
$$\phi_6 = \frac{1}{2}\xi_2\left[\left(2\xi_2-1\right)\left(1+\zeta\right)-\left(1-\zeta^2\right)\right]$$

$$\phi_7 = 2\xi_3\xi_1\left(1-\zeta\right)$$
$$\phi_8 = 2\xi_1\xi_2\left(1-\zeta\right)$$
$$\phi_9 = 2\xi_2\xi_3\left(1-\zeta\right)$$
$$\phi_{10} = 2\xi_3\xi_1\left(1+\zeta\right) \qquad \text{(T1L1-20b)}$$
$$\phi_{11} = 2\xi_1\xi_2\left(1+\zeta\right)$$
$$\phi_{12} = 2\xi_2\xi_3\left(1+\zeta\right)$$

$$\phi_{13} = \xi_3\left(1-\zeta^2\right)$$
$$\phi_{14} = \xi_1\left(1-\zeta^2\right) \qquad \text{(T1L1-20c)}$$
$$\phi_{15} = \xi_2\left(1-\zeta^2\right)$$

Node	$\left(\xi_1,\xi_2,\xi_3\right)$	ζ	ϕ_i
1	(0,0,1)	-1	$\frac{1}{2}\xi_3\left[\left(2\xi_3-1\right)\left(1-\zeta\right)-\left(1-\zeta^2\right)\right]=\frac{1}{2}(1)\left[(1)(2)-0\right]=1$
2	(1,0,0)	-1	$\frac{1}{2}\xi_1\left[\left(2\xi_1-1\right)\left(1-\zeta\right)-\left(1-\zeta^2\right)\right]=\frac{1}{2}(1)\left[(1)(2)-0\right]=1$
3	(0,1,0)	-1	$\frac{1}{2}\xi_2\left[\left(2\xi_2-1\right)\left(1-\zeta\right)-\left(1-\zeta^2\right)\right]=\frac{1}{2}(1)\left[(2-1)(2)-0\right]=1$

4	(0,0,1)	1	$\frac{1}{2}\xi_3\left[\left(2\xi_3-1\right)\left(1+\zeta\right)-\left(1-\zeta^2\right)\right]=\frac{1}{2}(1)\left[(2-1)(2)-0\right]=1$
5	(1,0,0)	1	$\frac{1}{2}\xi_1\left[\left(2\xi_1-1\right)\left(1+\zeta\right)-\left(1-\zeta^2\right)\right]=\frac{1}{2}(1)\left[(2-1)(2)-0\right]=1$
6	(0,1,0)	1	$\frac{1}{2}\xi_2\left[\left(2\xi_2-1\right)\left(1+\zeta\right)-\left(1-\zeta^2\right)\right]=\frac{1}{2}(1)\left[(2-1)(2)-0\right]=1$
7	$\left(\frac{1}{2},0,\frac{1}{2}\right)$	-1	$2\xi_3\xi_1\left(1-\zeta\right)=2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(2)=1$
8	$\left(\frac{1}{2},\frac{1}{2},0\right)$	-1	$2\xi_1\xi_2\left(1-\zeta\right)=2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(2)=1$
9	$\left(0,\frac{1}{2},\frac{1}{2}\right)$	-1	$2\xi_2\xi_3\left(1-\zeta\right)=2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(2)=1$
10	$\left(\frac{1}{2},0,\frac{1}{2}\right)$	1	$2\xi_3\xi_1\left(1+\zeta\right)=2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(2)=4$
11	$\left(\frac{1}{2},\frac{1}{2},0\right)$	1	$2\xi_1\xi_2\left(1+\zeta\right)=2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(2)=4$
12	$\left(0,\frac{1}{2},\frac{1}{2}\right)$	1	$2\xi_2\xi_3\left(1+\zeta\right)=2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(2)=1$
13	(0,0,1)	0	$\xi_3\left(1-\zeta^2\right)=(1)(1-0)=1$
14	(1,0,0)	0	$\xi_1\left(1-\zeta^2\right)=(1)(1-0)=1$
15	(0,1,0)	0	$\xi_2\left(1-\zeta^2\right)=(1)(1-0)=1$

Lesson 2: Finite Element Formulation

Objectives: In this lesson we will look at specific elements that can be used to solve three-dimensional elasticity problems. The major objectives are listed below.

- To generate the element equations for different solid elements.
- To solve simple problems using the three-dimensional elements.

We will use the Variational Technique, specifically the Theorem of Minimum Potential Energy, to generate the element equations. The body force, traction vector and elemental volume are given by

$$\mathbf{f}_{3\times1} = \left[f_x, f_y, f_z\right]^{\mathbf{T}} \qquad \mathbf{T}_{3\times1} = \left[T_x, T_y, T_z\right]^{\mathbf{T}} \qquad dV = dxdydz \tag{T1L2-1}$$

The body force has units force per unit volume, while the traction has the units force per unit area.

Step 1: Assume the displacement field as

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} \phi_1 & 0 & 0 & \phi_2 & .. & 0 & 0 \\ 0 & \phi_1 & 0 & 0 & .. & \phi_n & 0 \\ 0 & 0 & \phi_1 & 0 & .. & 0 & \phi_n \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ .. \\ d_{3n-2} \\ d_{3n-1} \\ d_{3n} \end{Bmatrix} \tag{T1L2-2a}$$

or, $$\mathbf{u}_{3\times1} = \left[u, v, w\right]^{\mathbf{T}} = \mathbf{\Phi}_{3\times3n}\mathbf{d}_{3n\times1} \tag{T1L2-2b}$$

where $\mathbf{\Phi}$ is the matrix of shape functions, $\phi_1, \phi_2, ...\phi_n$ and $\mathbf{d}$ is the vector of nodal displacements. Note that there are three displacements at every node and that $3n$ is the total number of degrees of freedom in the element.

Step 2: The strain-displacement relationship can be expressed as (using Eqn. T1L1-5)

$$\boldsymbol{\varepsilon}_{6\times1}=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1\\0&1&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&1&0\\0&0&1&0&0&0&1&0&0\end{bmatrix}_{6\times9}\begin{Bmatrix}\frac{\partial u}{\partial x}\\\frac{\partial u}{\partial y}\\\frac{\partial u}{\partial z}\\\frac{\partial v}{\partial x}\\\frac{\partial v}{\partial y}\\\frac{\partial v}{\partial z}\\\frac{\partial w}{\partial x}\\\frac{\partial w}{\partial y}\\\frac{\partial w}{\partial z}\end{Bmatrix}_{9\times1}=\mathbf{L}_{6\times9}\mathbf{a}_{9\times1} \qquad \text{(T1L2-3a)}$$

$$\mathbf{a}_{9\times1} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{bmatrix}_{9\times9} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial \zeta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \\ \dfrac{\partial v}{\partial \zeta} \\ \dfrac{\partial w}{\partial \xi} \\ \dfrac{\partial w}{\partial \eta} \\ \dfrac{\partial w}{\partial \zeta} \end{Bmatrix}_{9\times1}$$

$$= \mathbf{M}_{9\times9}\mathbf{b}_{9\times1} \qquad \text{(T1L2-3b)}$$

$$\mathbf{b}_{9\times1} = \begin{bmatrix} \phi_{1,\xi} & 0 & 0 & \phi_{2,\xi} & \ldots & 0 & 0 \\ \phi_{1,\eta} & 0 & 0 & \phi_{2,\eta} & .. & 0 & 0 \\ \phi_{1,\zeta} & 0 & 0 & \phi_{2,\zeta} & .. & 0 & 0 \\ 0 & \phi_{1,\xi} & 0 & 0 & .. & \phi_{n,\xi} & 0 \\ 0 & \phi_{1,\eta} & 0 & 0 & .. & \phi_{n,\eta} & 0 \\ 0 & \phi_{1,\zeta} & 0 & 0 & .. & \phi_{n,\zeta} & 0 \\ 0 & 0 & \phi_{1,\xi} & 0 & .. & 0 & \phi_{n,\xi} \\ 0 & 0 & \phi_{1,\eta} & 0 & .. & 0 & \phi_{n,\eta} \\ 0 & 0 & \phi_{1,\zeta} & 0 & .. & 0 & \phi_{n,\zeta} \end{bmatrix}_{9\times2n} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ .. \\ v_n \\ w_n \end{Bmatrix}_{3n\times1} = \mathbf{N}_{9\times3n}\mathbf{d}_{3n\times1} \tag{T1L2-3c}$$

Hence,

$$\boldsymbol{\varepsilon}_{6\times1} = \mathbf{L}_{6\times9}\mathbf{M}_{9\times9}\mathbf{N}_{9\times3n}\mathbf{d}_{3n\times1} = \mathbf{O}_{6\times9}\mathbf{N}_{9\times3n}\mathbf{d}_{3n\times1} = \mathbf{B}_{6\times3n}\mathbf{d}_{3n\times1} \tag{T1L2-3d}$$

where $$\mathbf{O}_{6\times9} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & 0 & 0 & 0 & \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \end{bmatrix} \tag{T1L2-3e}$$

and $\mathbf{B}$ is known as the strain-displacement matrix.

Step 3: The total strain energy per element is

$$U(\mathbf{d}) = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\ dV = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T \mathbf{D}\boldsymbol{\varepsilon}\ dV = \frac{1}{2}\mathbf{d}^{\mathbf{T}}_{1\times 3n}\mathbf{k}_{3n\times 3n}\mathbf{d}_{3n\times 1} \qquad \text{(T1L2-4a)}$$

where $$\mathbf{k}_{3n\times 3n} = \int_V \mathbf{B}^{\mathbf{T}}\mathbf{D}\mathbf{B}\ dV \qquad \text{(T1L2-4b)}$$

is the element stiffness matrix. Hence, the total potential energy for the entire structure is

$$\Pi(\mathbf{D}) = \sum_{i=1}^{e}\left[\frac{1}{2}\mathbf{d}^{\mathbf{T}}_{1\times 3n}\mathbf{k}_{3n\times 3n}\mathbf{d}_{3n\times 1} - \mathbf{d}^{\mathbf{T}}_{1\times 3n}\mathbf{f}_{3n\times 1} - \mathbf{d}^{\mathbf{T}}_{1\times 3n}\mathbf{T}_{3n\times 1}\right]_i - \mathbf{D}^{T}_{1\times N}\mathbf{P}_{N\times 1} \qquad \text{(T1L2-5)}$$

where $\mathbf{d}^{\mathbf{T}}_{1\times 3n}\mathbf{f}_{3n\times 1}$, $\mathbf{d}^{\mathbf{T}}_{1\times 3n}\mathbf{T}_{3n\times 1}$ and $\mathbf{D}^{T}_{1\times N}\mathbf{P}_{N\times 1}$ represent the work potential due to the body forces, surface tractions and concentrated forces. If initial strains are included then we must include the effect as $\mathbf{d}^{\mathbf{T}}\left[\int_A \mathbf{B}^{\mathbf{T}}\mathbf{D}\boldsymbol{\varepsilon}_0\ dV\right]$ as the work potential due to the initial strains.

We now have all the ingredients to compute the element equations for different types of solid elements. The process is no different than the one we saw for the plane elasticity elements.

Review Exercises

Problem T1L1-1

(i) Derive the shape functions for the (a) 8-noded and (b) 27-noded hexahedral elements.

(ii) Derive the shape functions for the (a) 4-noded and (b) 10-noded tetrahedral elements.

Problem T1L2-1

Derive the stiffness matrices for the elements listed in Problem T1L1-1.

Problem T1L2-2

TBC

Problem T1L2-3

TBC

Topic 2: Structural Dynamics

"I adore simple pleasures. They are the last refuge of the complex." Oscar Wilde

Lesson 1: Overview

Objectives: In this lesson we will look at free and forced vibrations of structural systems.

- To understand what is meant by eigenproblems.
- To understand what is meant by structural dynamics problems.
- To understand and apply the finite element procedure to solve modal analysis and structural dynamics problems.

Preliminaries

Consider the spring-mass system shown in Fig. T2L1-1 that is initially at rest and where the displacement $u = u(t)$.

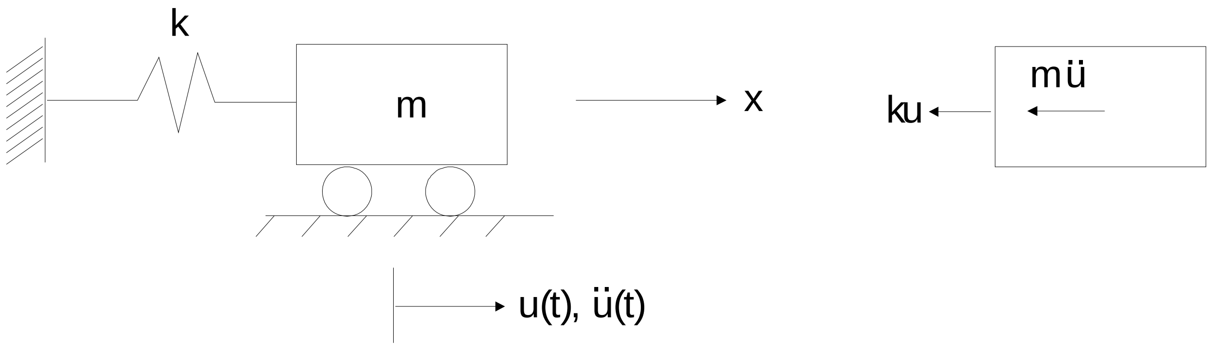

Fig. T2L1-1 System and its free-body diagram

Let us assume that the system is perturbed somehow. From the FBD, using the D'Alembert's Principle

$$-ku - m\ddot{u} = 0 \qquad \text{(T2L1-1)}$$

or, $$m\ddot{u} + ku = 0 \qquad \text{(T2L1-2)}$$

Let, $\omega^2 = \dfrac{k}{m}$. Substituting in the above equation we have

$$\ddot{u} + \omega^2 u = 0 \qquad \text{(T2L1-3)}$$

Solution to the above differential equation is of the form (sum of two harmonics)

$$u(t) = C_1 \cos\omega t + C_2 \sin\omega t \qquad \text{(T2L1-4)}$$

The term $\omega = \sqrt{k/m}$ is the angular frequency (expressed in rad/s), the term $f = \omega/2\pi$ is the natural frequency (expressed in Hz) and $T = 1/f$ is the natural period (expressed in s). To obtain the constants in Eqn. (T2L1-4) we need the initial conditions.

At $t = t_0$, $u = u_0$ which is the initial displacement, and

$\dot{u} = \dot{u}_0$ which is the initial velocity.

Using these initial conditions, $C_1 = u_0$ and $C_2 = \dfrac{\dot{u}_0}{\omega}$. Hence,

$$u = u_0 \cos \omega t + \frac{\dot{u}_0}{\omega} \sin \omega t \qquad \text{(T2L1-5)}$$

Or, $$u = A\cos(\omega t - \alpha) \qquad \text{(T2L1-6)}$$

where $A = \sqrt{u_0^2 + \left(\frac{\dot{u}_0}{\omega}\right)^2}$ which is the amplitude of vibration and $\alpha = \tan^{-1} \frac{\dot{u}_0}{\omega u_0}$ is the phase angle. Eqns. (T2L1-5) and (T2L1-6) represent the solution to free, undamped vibration.

Consider a harmonic forcing function

$$m\ddot{u} + ku = P \sin \Omega t \qquad \text{(T2L1-7)}$$

or, $$\ddot{u} + \omega^2 u = p_m \sin \Omega t \qquad \text{(T2L1-8)}$$

where $p_m = \frac{P}{m}$ (force per unit mass). The total solution consists of a general solution for the homogenous part and a particular solution that satisfies the whole equation. The particular solution is

$$u = C_3 \sin \Omega t \qquad \text{(T2L1-9)}$$

Substituting into Eqn. (T2L1-8), $C_3 = \frac{p_m}{\omega^2 - \Omega^2}$. The total solution is then

$$u = C_1 \cos \omega t + C_2 \sin \omega t + C_3 \sin \Omega t \qquad \text{(T2L1-10)}$$

where the first two terms arise from the free vibration and the last term arises from the forced vibration. Hence,

$$u = \left[\frac{1}{1 - \left(\Omega / \omega\right)^2}\right] \frac{P}{k} \sin \Omega t \qquad \text{(T2L1-11)}$$

The first part of the RHS is the "magnification factor" $(1/\beta)$ and the second part is the equivalent "static" load, and the solution represents the steady state forced response.

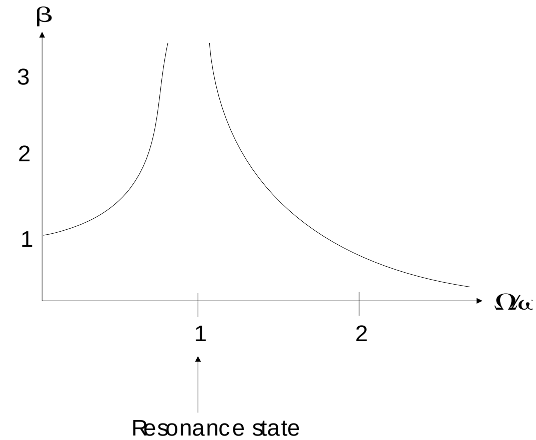

Fig. T2L1-2 Plot of β vs Ω/ω

One-Dimensional Eigenproblem

The governing differential equation is of the form

$$-\frac{d}{dx}\left\{\alpha(x)\frac{du(x)}{dx}\right\}+\beta(x)u(x)-\lambda\gamma(x)u(x)=0 \qquad x_a < x < x_b \qquad \text{(T2L1-12)}$$

with the boundary conditions as

At x_a: $u(x_a)=0$ or $\tau(x_a)=0$ (T2L1-13a)

At x_b: $u(x_b)=0$ or $\tau(x_b)=0$ (T2L1-13b)

A few points are in order when we compare this differential equation to the one-dimensional BVP. First, there is no driving force, i.e. $f(x)=0$. Second, there is an additional term, $-\lambda\gamma(x)u(x)$ where $\gamma(x)$ describes a physical property of the system (usually mass or mass density) and the scalar λ is called the eigenvalue. Third, there are several solutions called eigensolutions to this problem. The eigensolutions consist of pairs of eigenfunction $u(x)$ and eigenvalue λ. Both of these are unknowns. Lastly, in the absence of driving forces, the condition of the system changes. This is the resonant or natural state where the internal energy oscillates back and forth between different forms e.g. kinetic and potential, without energy exchange with the surroundings.

Once again we will use the Galerkin's Approach to solve the problem.

Step 1: Residual Equations

In a typical element with the approximate solution as $u = \tilde{u}(x)$ (dropping the tilde notation for convenience)

$$\int_{\Omega} \left[-\frac{d}{dx}\left\{ \alpha(x) \frac{du(x)}{dx} \right\} + \beta(x)u(x) - \lambda\gamma(x)u(x) \right] \phi_i(x)\, dx = 0$$

$$i = 1,2,\ldots,n \qquad \text{(T2L1-14)}$$

Step 2: Integrate by parts the highest order derivative

$$\int_{\Omega} \frac{d\phi_i(x)}{dx} \alpha(x) \frac{d\tilde{u}}{dx} dx + \int_{\Omega} \phi_i(x)\beta(x)\tilde{u}\, dx - \lambda \int_{\Omega} \phi_i(x)\gamma(x)\tilde{u}\, dx =$$

$$- \left[\left\{ -\alpha(x) \frac{d\tilde{u}}{dx} \right\} \phi_i(x) \right]_{x_1}^{x_n} \qquad \text{(T2L1-15)}$$

where x_1 and x_n are the coordinates of the ends of the element. The last term must vanish since the boundary conditions either are essential or homogenous (meaning zero valued) natural BC's (see T2L1-13).

Step 3: Trial solution

Let the trial solution be represented as $\tilde{u}(x,a) = \sum_{j=1}^{n} a_j \phi_j(x)$. Hence

$$\sum_{j=1}^{n} \left\{ \int_{\Omega} \frac{d\phi_i(x)}{dx} \alpha(x) \frac{d\phi_j(x)}{dx} dx + \int_{\Omega} \phi_i(x)\beta(x)\phi_j(x) dx \right\} a_j$$

$$- \lambda \sum_{j=}^{n} \left\{ \int_{\Omega} \phi_i(x)\gamma(x)\phi_j(x)\, dx \right\} a_j = 0 \qquad i = 1,2,\ldots,n \qquad \text{(T2L1-16)}$$

Writing the above equation in a compact form

$$\mathbf{k}_{n\times n}\mathbf{a}_{n\times 1} - \lambda \mathbf{m}_{n\times n}\mathbf{a}_{n\times 1} = \mathbf{0} \qquad \text{(T2L1-17)}$$

Step 4: Element equations for the $1D - C^0$ *linear element*

Considering the C^0 linear element we have

$$\phi_1(x) = \frac{x_2 - x}{x_2 - x_1} \qquad \text{and} \qquad \phi_2(x) = \frac{x - x_1}{x_2 - x_1} \tag{T2L1-18}$$

The terms in $\mathbf{k}$ were evaluated in Module 1. We will handle the mass matrix here.

$$m_{11} = \int_{x_1}^{x_2} \phi_1(x)\gamma(x)\phi_1(x)\, dx = \int_{x_1}^{x_2} \frac{x_2 - x}{x_2 - x_1}\gamma(x)\frac{x_2 - x}{x_2 - x_1}\, dx = \frac{\overline{\gamma}L}{3} = m_{22}$$

$$m_{12} = \int_{x_1}^{x_2} \phi_1(x)\gamma(x)\phi_2(x)\, dx = \frac{\overline{\gamma}L}{6} = m_{21}$$

Hence

$$\mathbf{m}_{2\times 2} = \frac{\overline{\gamma}L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

There is no flux to compute since we are solving an eigenproblem. Once the element equations are assembled into the system equations, we obtain the system eigenproblem as

$$\mathbf{K\Phi} = \mathbf{\Lambda M \Phi} \tag{T2L1-19}$$

that can then be solved for the eigenvalues $\mathbf{\Lambda}$ and the corresponding eigenvectors $\mathbf{\Phi}$.

Lesson 2: Element Equations for Truss

Objectives: In this lesson we will look at deriving the element equations for the truss element. The major objectives are listed below.

- To understand the dynamic behavior of the truss element.
- To derive the mass matrix for the truss element.

We saw the truss element is Module 2 and generated the stiffness matrix. In the previous section we derived the general expression for the mass matrix as

$$m_{ij} = \int_{\Omega} \phi_i(x)\gamma(x)\phi_j(x)\,dx \tag{T2L2-1}$$

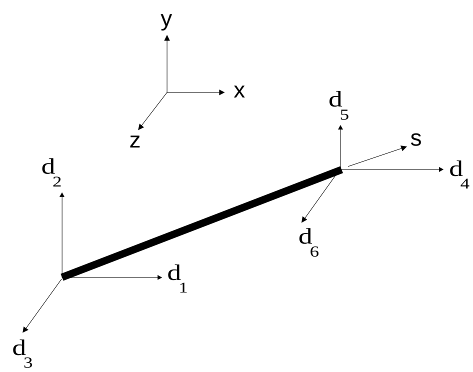

Fig. T2L2-1 Global degrees-of-freedom for a space truss element

Note that we will generate the global element mass matrix and bypass the local-to-global transformation. For some elements, the equivalent mass matrix is invariant with respect to the orientation and position of the coordinate axes. Examples include the truss element and the tetrahedral elements. The global displacements at a point in the element can be written as $(a = s/L)$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} 1-a & 0 & 0 & a & 0 & 0 \\ 0 & 1-a & 0 & 0 & a & 0 \\ 0 & 0 & 1-a & 0 & 0 & a \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \mathbf{A}_{3\times6}\mathbf{d}_{6\times1} \tag{T2L2-2}$$

We can rewrite Eqn. (T2L2-1) as $\mathbf{m}_{6\times6} = \int_{\Omega} \gamma \mathbf{A}^T \mathbf{A} dV$. Substituting and integrating

$$\mathbf{m}_{6\times6} = \frac{\gamma AL}{6} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \tag{T2L2-3}$$

Note that the mass matrix is symmetric. The off-diagonal terms represent the dynamic coupling between the different degrees of freedom.

Lesson 3: Element Equations for Frame

Objectives: In this lesson we will look at deriving the element equations for the beam or frame element. The major objectives are listed below.

- To understand the dynamic behavior of the frame element.
- To derive the mass and geometic stiffness matrices element.

The governing differential equation for the transverse vibrations of a beam with no axial loads is given by

$$\frac{\partial}{\partial x^2}\left(EI\,\frac{\partial w}{\partial x^2}\right) = -\bar{\rho}\,\frac{\partial^2 w}{\partial t^2} \qquad \text{(T2L3-1)}$$

where $w = w(x,t)$ is the transverse displacement and $\bar{\rho}$ is the mass per unit length. For free vibrations, we can write the transverse displacements as

$$w(x,t) = W(x)\sin\omega t \qquad \text{(T2L3-2)}$$

Substituting Eqn. (T2L3-2) into (T2L3-1)

$$\frac{d^2}{dx^2}\left[EI\,\frac{d^2W}{dx^2}\right] - \bar{\rho}\omega^2 W = 0 \qquad \text{(T2L3-3)}$$

The Galerkin's Method can now be used to generate the element equations.

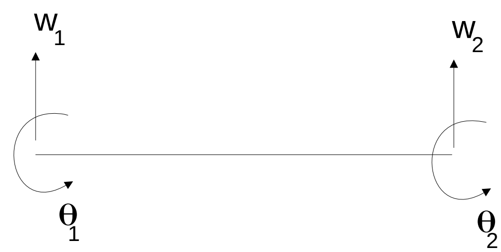

Fig. T2L3-1

Step 1: Residual Equations

$$\int_{x_1}^{x_2}\left[(EIW'')'' - \bar{\rho}\omega^2 W\right]\phi_i(x)\,dx = 0 \qquad i = 1,2,3,4 \qquad \text{(T2L3-4)}$$

Step 2: Integrating by parts

$$\int_{x_1}^{x_2}\left[(EIW'')'\phi_i'' - \omega^2\bar{\rho}W\phi_i\right]dx = \left[(EIW'')\phi_i'\right]_{x_1}^{x_2} - \left[(EIW'')'\phi_i\right]_{x_1}^{x_2} \qquad \text{(T2L3-5)}$$

Step 3: Trial Solution

$$\text{Let } W = \sum_{j=1}^{4} a_j\phi_j \qquad \text{(T2L3-6)}$$

Substituting the trial solution in Eqn. (T2L3-5) yields

$$\left[\mathbf{k}_{4\times4} - \omega^2 \mathbf{m}_{4\times4}\right]\mathbf{a}_{4\times1} = \begin{Bmatrix} -V(x_1) \\ M(x_1) \\ V(x_2) \\ -M(x_2) \end{Bmatrix} \tag{T2L3-7}$$

with the sign convention described as $EI\dfrac{\partial^2 w}{\partial x^2} = -M$ and $\dfrac{d}{dx}\left[EI\dfrac{\partial^2 w}{\partial x^2}\right] = -V$.

Using the shape functions for the beam element, the mass matrix is obtained as

$$\mathbf{m}'_{4\times4} = \frac{\rho AL}{420}\begin{bmatrix} 156 & & SYM & \\ 22L & 4L^2 & & \\ 54 & 13L & 156 & \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \tag{T2L3-8}$$

with $\overline{\rho} = \rho A$ and ρ is the mass density.

When the axial effects are combined with the transverse effects, the mass matrix for the one-dimensional element is added to the mass matrix for the (transverse) beam element to obtain

$$\mathbf{m}'_{6\times6} = \frac{\rho AL}{420}\begin{bmatrix} 140 & & & SYM & & \\ 0 & 156 & & & & \\ 0 & 22L & 4L^2 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & 13L & 0 & 156 & \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \tag{T2L3-9}$$

The important point to note is that mass matrix is in the local coordinate system and must be transformed into the global coordinate system as

$$\mathbf{m}_{6\times6} = \mathbf{T}^T_{6\times6}\mathbf{m}'_{6\times6}\,\mathbf{T}_{6\times6} \tag{T2L3-10}$$

similar to the manner in which the stiffness matrix is transformed.

Euler Buckling Analysis

The governing differential equation for an axially loaded bar is given by

$$\frac{d^2}{dx^2}\left[EI\frac{d^2W}{dx^2}\right]+\frac{d}{dx}\left[P\frac{dW}{dx}\right]=0 \qquad \text{(T2L3-11)}$$

where P is the axial force and $W=W(x)$ is the deflection from the straight configuration. The basic objective of this analysis is to find the critical force P_{cr} necessary to cause the system to buckle.

Fig. T2L1-1 Euler buckling problem

We will use the Galerkin's Method to derive the element equations.

Step 1: Residual equations

$$\int_{x_1}^{x_2}\left[(EIW'')''+(PW')'\right]\phi_i(x)\,dx=0 \qquad i=1,2,3,4 \qquad \text{(T2L3-12)}$$

Step 2: Integrate by parts

$$\int_{x_1}^{x_2}(EIW'')''\phi_i(x)dx=\left[(EIW'')'\phi_i\right]_{x_1}^{x_2}-\left[(EIW'')\phi_i'\right]_{x_1}^{x_2}+\int_{x_1}^{x_2}(EIW'')\phi_i''\,dx \qquad \text{(T2L3-13a)}$$

$$\int_{x_1}^{x_2}(PW')'\phi_i dx=\left[PW\phi_i\right]_{x_1}^{x_2}-\int_{x_1}^{x_2}(PW')\phi_i'\,dx \qquad \text{(T2L3-13b)}$$

Hence,

$$\int_{x_1}^{x_2}(EIW'')\phi_i''\,dx-\int_{x_1}^{x_2}(PW')\phi_i'\,dx=\left[(EIW'')\phi_i'\right]_{x_1}^{x_2}-\left[\{(EIW'')'+PW'\}\phi_i\right]_{x_1}^{x_2} \qquad \text{(T2L3-14)}$$

Step 3: Substitute trial solution $W=\sum_{j=1}^{4}a_j\phi_j$

$$\sum_j \left[\int_{x_1}^{x_2} \phi_j'' EI \phi_i'' \, dx \right] a_j - \sum_j \left[\int_{x_1}^{x_2} \phi_j' P \phi_i' \, dx \right] a_j$$

$$= \left[(EIW'') \phi_i' \right]_{x_1}^{x_2} - \left[\{ (EIW'')' + PW' \} \phi_i \right]_{x_1}^{x_2} \qquad \text{(T2L3-15)}$$

Let,

$$k_{ij} = \int_{x_1}^{x_2} \phi_i'' EI \phi_j'' \, dx \qquad EIW'' = -M \qquad \text{(T2L3-16a)}$$

$$\kappa_{ij} = \int_{x_1}^{x_2} \phi_i' \phi_j' \, dx \qquad (EIW'')' + PW' = -M' + PW' = -V \qquad \text{(T2L3-16b)}$$

Hence, the element equations become

$$[\mathbf{k} - P\,\boldsymbol{\kappa}]\mathbf{a} = \begin{Bmatrix} [V\phi_1 - M\phi_1'] \\ [V\phi_2 - M\phi_2'] \\ [V\phi_3 - M\phi_3'] \\ [V\phi_4 - M\phi_4'] \end{Bmatrix}_{x_1}^{x_2} = \begin{Bmatrix} -V(x_1) \\ M(x_1) \\ V(x_2) \\ -M(x_2) \end{Bmatrix} \qquad \text{(T2L3-17)}$$

where $\boldsymbol{\kappa}$ is called the geometric stiffness matrix.

Step 4: Element Equations

Using the shape functions ϕ_i for the beam element we have the geometric stiffness matrix as

$$\boldsymbol{\kappa}_{4\times4} = \begin{bmatrix} \frac{6}{5L} & & SYM & \\ \frac{1}{10} & \frac{2L}{15} & & \\ -\frac{6}{5L} & -\frac{1}{10} & \frac{6}{5L} & \\ \frac{1}{10} & -\frac{L}{30} & -\frac{1}{10} & \frac{2L}{15} \end{bmatrix} \qquad \text{(T2L3-18)}$$

It should be noted that the geometric stiffness is in the local coordinate system. To use the matrix in the global coordinate system we augment the matrix as follows

$$\boldsymbol{\kappa}'_{6\times6} = \begin{bmatrix} 0 & & & SYM & & \\ 0 & \frac{6}{5L} & & & & \\ 0 & \frac{1}{10} & \frac{2L}{15} & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -\frac{6}{5L} & -\frac{1}{10} & 0 & \frac{6}{5L} & \\ 0 & \frac{1}{10} & -\frac{L}{30} & 0 & -\frac{1}{10} & \frac{2L}{15} \end{bmatrix} \qquad \text{(T2L3-19)}$$

and the matrix is transformed to the global coordinate system as usual

$$\boldsymbol{\kappa}_{6\times6} = \mathbf{T}^T_{6\times6}\boldsymbol{\kappa}'_{6\times6}\,\mathbf{T}_{6\times6} \qquad \text{(T2L3-20)}$$

Lesson 4: Eigenvalue Analysis

Objectives: In this lesson we will look at the eigenvalue problem. The major objectives are listed below.

- To understand what is meant by Modal and Buckling Analysis.
- To learn the simple numerical techniques to solve for the eigenvalues and eigenvectors of structural systems.

Modal Analysis

The term modal analysis refers to the computation of the mode shapes. In other words, the solution to the eigenproblem

$$\mathbf{K}_{n\times n}\mathbf{\Phi}_{n\times n} = \mathbf{\Lambda}_{n\times n}\mathbf{M}_{n\times n}\mathbf{\Phi}_{n\times n} \qquad \text{(T2L4-1)}$$

generates the eigenpairs – the eigenvalues and their corresponding eigenvectors. The structural stiffness matrix $\mathbf{K}$ is symmetric and positive definite whereas the structural mass matrix $\mathbf{M}$ is symmetric. The general properties of the eigenvalue problem as described above are as follows.

- There are n real eigenvalues and eigenvectors such that

$$0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n \qquad \text{(T2L4-2)}$$

- The eigenvector $\boldsymbol{\varphi}_i$ corresponding to the eigenvalue λ_i is such that

$$\mathbf{K}\boldsymbol{\varphi}_i = \lambda_i \mathbf{M}\boldsymbol{\varphi}_i \qquad \text{(T2L4-3)}$$

- The eigenvectors are such that

$$\boldsymbol{\varphi}_i^T \mathbf{K}\boldsymbol{\varphi}_j = 0 \qquad i \ne j \qquad \text{(T2L4-4a)}$$

$$\boldsymbol{\varphi}_i^T \mathbf{M}\boldsymbol{\varphi}_j = 0 \qquad i \ne j \qquad \text{(T2L4-4b)}$$

- The eigenvectors are generally normalized so that

$$\boldsymbol{\varphi}_i^T \mathbf{M}\boldsymbol{\varphi}_i = 1 \qquad \text{(T2L4-5a)}$$

$$\boldsymbol{\varphi}_i^T \mathbf{K}\boldsymbol{\varphi}_i = \lambda_i \qquad \text{(T2L4-5b)}$$

Other normalization schemes are possible such as making the largest entry in the eigenvector equal to unity. We will now look at two methods to compute the eigenpairs. These methods have limitations in terms of the size of the problem that can be handled efficiently. It should be noted that solving an eigenproblem is computationally more expensive than solving algebraic equations.

Characteristic Polynomial Technique

The method is quite simple. Eqn. (T2L4-1) can be rewritten as

$$\left[\mathbf{K} - \lambda\mathbf{M}\right]\boldsymbol{\varphi} = 0 \qquad \text{(T2L4-6)}$$

requiring that for a nontrivial eigenvector

$$\det[\mathbf{K} - \lambda\mathbf{M}] = 0 \tag{T2L4-7}$$

Expansion of Eqn. (T2L4-7) leads to a polynomial of degree n. Solving for the roots of the polynomial leading to the eigenvalues, is generally difficult. We will use this technique only for very small problems.

Vector Iteration Methods

These iteration methods use the properties of the Rayleigh quotient. The Rayleigh quotient $Q(\mathbf{v})$ is defined as

$$Q(\mathbf{v}) = \frac{\mathbf{v}^{\mathbf{T}}\mathbf{K}\mathbf{v}}{\mathbf{v}^{\mathbf{T}}\mathbf{M}\mathbf{v}} \tag{T2L4-8}$$

where $\mathbf{v}$ is an arbitrary vector. The basic property of the Rayleigh quotient is that

$$\lambda_1 \le Q(\mathbf{v}) \le \lambda_n \tag{T2L4-9}$$

Inverse Iteration Method: The Inverse Iteration Method is used to evaluate the lowest eigenvalues. The process starts with an assumed (initial guess) $\mathbf{u}^0$. The algorithm is as follows.

Step 1: Assume the initial guess for the eigenvector as $\mathbf{u}^0$. With k as the iteration counter, set $k = 0$.

Step 2: Set $k = k + 1$.

Step 3: Compute $\mathbf{v}^{k-1}$ using $\mathbf{v}^{k-1} = \mathbf{M}\mathbf{u}^{k-1}$.

Step 4: Solve $\mathbf{K}\hat{\mathbf{u}}^k = \mathbf{v}^{k-1}$ for $\hat{\mathbf{u}}^k$.

Step 5: Let $\hat{\mathbf{v}}^k = \mathbf{M}\hat{\mathbf{u}}^k$.

Step 6: Estimate eigenvalue $\lambda^k = \dfrac{\hat{\mathbf{u}}^{k^T}\hat{\mathbf{v}}^{k-1}}{\hat{\mathbf{u}}^{k^T}\hat{\mathbf{v}}^k}$.

Step 7: Normalize eigenvector $\mathbf{u}^k = \dfrac{\hat{\mathbf{u}}^k}{\left(\hat{\mathbf{u}}^{k^T}\hat{\mathbf{v}}^k\right)^{1/2}}$

Step 8: Check for convergence using $\left|\frac{\lambda^k - \lambda^{k-1}}{\lambda^k}\right| \le tolerance$. If this condition is satisfied then $\mathbf{u}^k$ is the eigenvector $\boldsymbol{\varphi}$. Otherwise go to Step 2.

The two most powerful techniques to solve the eigenproblem are the Subspace Iteration and the Lanczos Methods. The Subspace Iteration method uses the Generalized Jacobi Method when solving for all the eigenvalues in the subspace. You may wish to study these methods using the reference texts.

Lesson 5: Time-Integration Schemes

Objectives: In this lesson we will look at the forced vibration problem. The major objectives are listed below.

- To understand what is meant by Time-Integration Schemes.
- To learn the simple numerical techniques to solve for time-dependent response of structural systems.

Structural Dynamics

The governing differential equation, boundary and initial conditions for the one-dimensional structural dynamics problem are

DE: $$\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2}+\mu(x)\frac{\partial u(x,t)}{\partial t}-\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u(x,t)}{\partial x}\right)+\beta(x)u(x,t)=f(x,t) \quad \text{(T2L5-1)}$$

Domain: $x_a \le x \le x_b \qquad t > t_0$

BC's: At $x = x_a$ and $t > t_0$

$$u(x_a,t)=u_a(t) \quad \text{or} \quad \left(-\alpha(x)\frac{\partial u}{\partial x}\right)_{x_a}=\tau_a \quad \text{(T2L5-2a)}$$

At $x = x_b$ and $t > t_0$

$$u(x_b,t)=u_b(t) \quad \text{or} \quad \left(-\alpha(x)\frac{\partial u}{\partial x}\right)_{x_b}=\tau_b \quad \text{(T2L5-2b)}$$

IC's: At t_0 $(x_a < x < x_b)$

$$u(x,t_0)=u_0(x) \quad \text{and} \quad \left(\frac{\partial u(x,t)}{\partial t}\right)_{t_0}=V_0(x) \quad \text{(T2L5-3)}$$

As stated before, $u(x,t)$ represents the displacement, $\partial u/\partial t$ is the velocity and $\partial^2 u/\partial t^2$ is the acceleration, ρ is the mass per unit volume, $\rho\,\partial^2 u/\partial t^2$ is the inertial force per unit volume, μ is viscosity and $\mu\,\partial u/\partial t$ is the damping force. The trial solution is assumed to be

$$u(x,t;a)=\sum_{j=1}^{n}a_j(t)\phi_j(x) \quad \text{(T2L5-4)}$$

and the usual steps in the Galerkin's Method can be applied as follows.

Step 1: Residual Equations

$$\int_\Omega\left(\rho(x)\frac{\partial^2 u}{\partial t^2}+\mu(x)\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right)+\beta(x)u-f(x,t)\right)\phi_i(x)dx=0 \qquad i=1,2,\ldots,n$$

(T2L5-5)

Step 2: Integrate by parts

$$\int_{\Omega} \phi_i(x)\rho(x)\frac{\partial^2 u}{\partial t^2}\,dx + \int_{\Omega} \phi_i(x)\mu(x)\frac{\partial u}{\partial t}\,dx + \int_{\Omega} \frac{d\phi_i}{dx}\alpha(x)\frac{\partial u}{\partial x}\,dx + \int_{\Omega} \phi_i(x)\beta(x)u\,dx$$

$$= \int_{\Omega} f(x,t)\phi_i(x)dx - \left[\left(-\alpha(x)\frac{\partial u}{\partial x}\right)\phi_i(x)\right]_{x_1}^{x_n} \qquad i = 1,2,...,n \tag{T2L5-6}$$

Step 3: Use the trial solution

Note that $$\frac{\partial u}{\partial x} = \sum_{j=1}^{n} a_j(t)\frac{d\phi_j}{dx} \tag{T2L5-7a}$$

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \frac{da_j}{dt}\phi_j \tag{T2L5-7b}$$

$$\frac{\partial^2 u}{\partial t^2} = \sum_{j=1}^{n} \frac{d^2 a_j}{dt^2}\phi_j \tag{T2L5-7c}$$

Substituting into Eqn. (T2L5-6) yields

$$\sum_{j=1}^{n}\left(\int_{\Omega} \phi_i(x)\rho(x)\phi_j(x)\,dx\right)\frac{d^2 a_j}{dt^2} + \sum_{j=1}^{n}\left(\int_{\Omega} \phi_i(x)\mu(x)\phi_j(x)\,dx\right)\frac{da_j}{dt} + \sum_{j=1}^{n}\left(\int_{\Omega} \frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}\,dx\right)a_j$$

$$+ \sum_{j=1}^{n}\left(\int_{\Omega} \phi_i(x)\beta(x)\phi_j(x)dx\right)a_j = \int_{\Omega} f(x,t)\phi_i(x)dx - \left[\tau(x,t;a)\phi_i(x)\right]_{x_1}^{x_n}$$

$$i = 1,2,....,n \tag{T2L5-8}$$

The element equations can be written as for a typical element

$$\mathbf{m}\left\{\frac{d^2 a(t)}{dt^2}\right\} + \mathbf{c}\left\{\frac{da(t)}{dt}\right\} + \mathbf{k}\left\{a(t)\right\} = \left\{f(t)\right\} \tag{T2L5-9}$$

or, $$\mathbf{M}\left\{\ddot{a}\right\} + \mathbf{C}\left\{\dot{a}\right\} + \mathbf{K}\left\{a\right\} = \left\{\mathbf{F}\right\} \tag{T2L5-9a}$$

at the system level, where

$$m_{ij} = \int_{\Omega} \phi_i(x)\rho(x)\phi_j(x)\,dx \tag{T2L5-10a}$$

$$c_{ij} = \int_{\Omega} \phi_i(x)\mu(x)\phi_j(x)\,dx \tag{T2L5-10b}$$

$$k_{ij} = \int_{\Omega} \frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}\,dx + \int_{\Omega} \phi_i(x)\beta(x)\phi_j(x)\,dx \tag{T2L5-10c}$$

$$f_i(t) = \int_{\Omega} f(x,t)\phi_i(x)dx - \left[\tau(x,t;a)\phi_i(x)\right]_{x_1}^{x_n} \tag{T2L5-10d}$$

Eqn. (T2L5-10a)-(10d) describes the elements of the mass matrix, damping matrix, stiffness matrix and the load vector respectively. We have seen all the terms except the damping matrix term that is very similar to the $\mathbf{k}^{\beta}$ matrix. The mass and damping matrices can be diagonalized by 'lumping' the mass and the damping.

Eqn. (T2L5-9) describes the equation of motion – a differential equation whose solution must be handled differently (than the methods used to solve linear algebraic equations). We will look at the solution techniques next.

Numerical Techniques

The numerical techniques used to solve Eqn. (T2L5-9a) can be generally divided into two types – direct integration and mode superposition. We will see the differences between the two techniques later. The Direct Integration schemes are time-stepping methods that use the initial boundary conditions to march forward into time. The solution techniques are either *conditionally stable* or *unconditionally stable*. The former implies that the solution will diverge unless the time-step used is less than a certain value. The latter implies that the solution will converge irrespective of the chosen time step – this however does not imply that the answers are correct or acceptable!

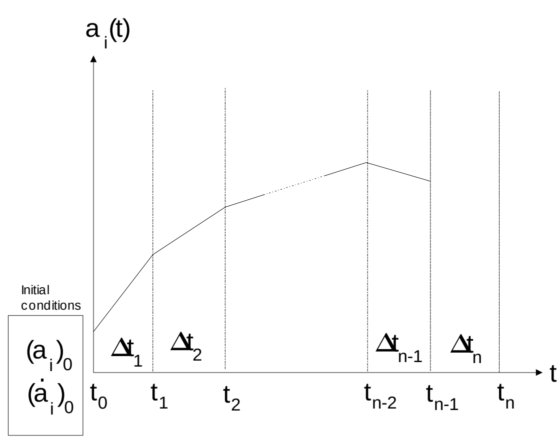

Fig. T2L5-1 Notation used for recurrence relations

We will look at two different solution techniques – the Central Difference method and the Wilson-Theta method.

Central Difference Method

This is a two-step method. The system equations (T2L5-9) are evaluated at the central time, t_{n-1}

$$\mathbf{M}\left\{\ddot{a}\right\}_{n-1} + \mathbf{C}\left\{\dot{a}\right\}_{n-1} + \mathbf{K}\left\{a\right\}_{n-1} = \left\{\mathbf{F}\right\}_{n-1} \qquad \text{(T2L5-11)}$$

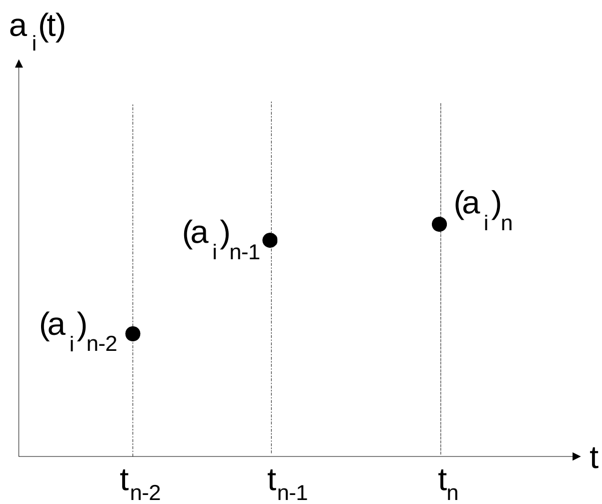

Fig. T2L5-2 Data used in the central difference recurrence relation; all three values are interpolated

The two derivatives are then approximated by central difference expressions, and it is assumed that both steps are of the same length, i.e. $\Delta t = t_n - t_{n-1} = t_{n-1} - t_{n-2}$. Hence,

$$\left\{\dot{a}\right\}_{n-1} \approx \frac{\{a\}_n - \{a\}_{n-2}}{2\Delta t} \qquad \text{(T2L5-12)}$$

$$\left\{\ddot{a}\right\}_{n-1} \approx \frac{\left\{\dot{a}\right\}_{n-1/2} - \left\{\dot{a}\right\}_{n-3/2}}{\Delta t} = \frac{\dfrac{\{a\}_n - \{a\}_{n-1}}{\Delta t} - \dfrac{\{a\}_{n-1} - \{a\}_{n-2}}{\Delta t}}{\Delta t}$$

$$= \frac{\{a\}_n - 2\{a\}_{n-1} + \{a\}_{n-2}}{\Delta t^2} \qquad \text{(T2L5-13)}$$

Substituting Eqns. (T2L5-12) & (T2L5-13) into (T2L5-11) yields the central difference recurrence relation (we will place all the known terms on the RHS)

$$\left(\frac{1}{\Delta t^2}\mathbf{M}+\frac{1}{2\Delta t}\mathbf{C}\right)\{a\}_n =$$

$$\{F\}_{n-1}-\left(\mathbf{K}-\frac{2}{\Delta t^2}\mathbf{M}\right)\{a\}_{n-1}-\left(\frac{1}{\Delta t^2}\mathbf{M}-\frac{1}{2\Delta t}\mathbf{C}\right)\{a\}_{n-2} \qquad \text{(T2L5-14)}$$

The above equation is a system of linear algebraic equations. Since this method is a two-step method and the initial conditions are available only for time t_0, a starting procedure is needed. One possibility is to use the relation

$$\{a\}_{-1}=\{a\}_0-\Delta t\left\{\dot{a}\right\}_0 \qquad \text{(T2L5-15)}$$

and then substitute the values $\{a\}_{-1}$ and $\{a\}_0$ into Eqn. (T2L5-14) evaluated at $n=1$. The numerical efficiency of the system can be increased as follows. The LHS of Eqn. (T2L5-14) can be rewritten as

$$\mathbf{K}_{eff}=\frac{1}{\Delta t^2}\mathbf{M}+\frac{1}{2\Delta t}\mathbf{C} \qquad \text{(T2L5-16)}$$

Since the mass and the damping matrices can be diagonalized using the lumping technique, the equations in (T2L5-14) can be decoupled. The method is made *explicit* so to speak. The explicit central difference technique is conditionally stable and requires that

$$\Delta t \le \Delta t_{crit}=\frac{2}{\omega_{\max}} \qquad \text{(T2L5-17)}$$

where $f_{\max}=\omega_{\max}/2\pi$ is the maximum natural frequency of the FE model.

The Wilson Method

This is a one-step method in which the solution is approximated by a cubic polynomial

$$\{a(t)\}=\{c_0\}+\{c_1\}t+\{c_2\}t^2+\{c_3\}t^3 \qquad \text{(T2L5-18)}$$

The polynomial is defined over the current step as well as over part of the next step. The parameter θ measures how far beyond t_n the interval of definition extends

$$t_\theta=t_{n-1}+\theta\,\Delta t \qquad \theta\ge 1 \qquad \text{(T2L5-19)}$$

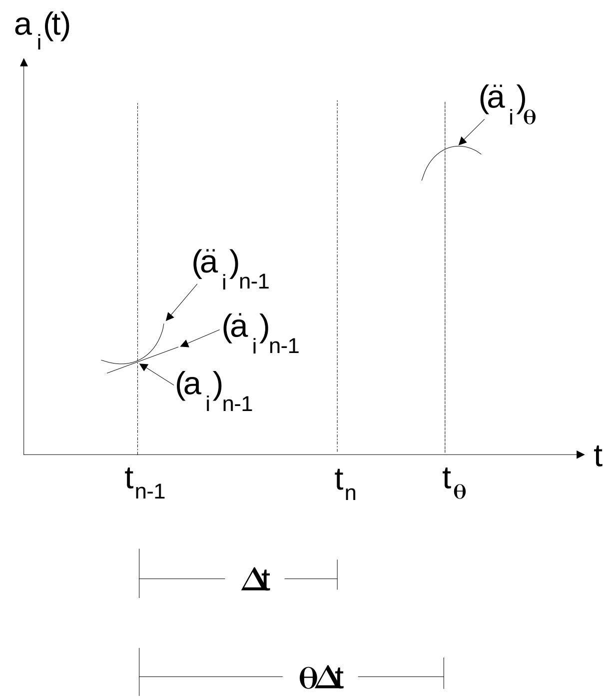

Fig. T2L5-3 Data used in the Wilson recurrence relation; all four values are interpolated

The coefficients $\{c_i\}, i = 1,2,3,4$, are determined by interpolating $\{a(t)\}$, $\left\{\dot{a}(t)\right\}$ and $\left\{\ddot{a}(t)\right\}$ at t_{n-1}, and $\left\{\ddot{a}(t)\right\}$ at t_θ. Therefore Eqn. (T2L5-18) becomes

$$\{a(t)\} = \{a\}_{n-1} + \left\{\dot{a}\right\}_{n-1}(t - t_{n-1}) + \frac{1}{2}\left\{\ddot{a}\right\}_{n-1}(t - t_{n-1})^2 + \frac{1}{6}\left\{\ddot{a}\right\}_{n-1}\left(\frac{\left\{\ddot{a}\right\}_\theta - \left\{\ddot{a}\right\}_{n-1}}{\theta\ \Delta t}\right)(t - t_{n-1})^3$$

(T2L5-20)

This is the basic approximating polynomial from which all the remaining relations are derived. Differentiating Eqn. (T2L5-20) once and twice yields

$$\left\{\dot{a}(t)\right\} = \left\{\dot{a}\right\}_{n-1} + \left\{\ddot{a}\right\}_{n-1}(t - t_{n-1}) + \frac{1}{2}\left(\frac{\left\{\ddot{a}\right\}_\theta - \left\{\ddot{a}\right\}_{n-1}}{\theta\ \Delta t}\right)(t - t_{n-1})^2 \qquad \text{(T2L5-21)}$$

$$\left\{\ddot{a}(t)\right\} = \left\{\ddot{a}\right\}_{n-1} + \left(\frac{\left\{\ddot{a}\right\}_{\theta} - \left\{\ddot{a}\right\}_{n-1}}{\theta\,\Delta t}\right)(t - t_{n-1}) \qquad \text{(T2L5-22)}$$

The system equations are now evaluated at time t_θ

$$\mathbf{M}\left\{\ddot{a}\right\}_{\theta} + \mathbf{C}\left\{\dot{a}\right\}_{\theta} + \mathbf{K}\left\{a\right\}_{\theta} = \left\{\mathbf{F}\right\}_{\theta} \qquad \text{(T2L5-23)}$$

where $\left\{\mathbf{F}\right\}_{\theta}$ is linearly interpolated

$$\left\{\mathbf{F}\right\}_{\theta} = \left\{\mathbf{F}\right\}_{n-1} + \theta\left(\left\{\mathbf{F}\right\}_{n} - \left\{\mathbf{F}\right\}_{n-1}\right) \qquad \text{(T2L5-24)}$$

Expressions for $\left\{\ddot{a}\right\}_{\theta}$ and $\left\{\dot{a}\right\}_{\theta}$ are obtained by evaluating Eqns. (T2L5-21) and (T2L5-22) at t_θ, solving the former for $\left\{\ddot{a}\right\}_{\theta}$, and substituting into the latter for $\left\{\dot{a}\right\}_{\theta}$. These expressions may then be substituted into Eqn. (T2L5-23) to yield

$$\left(\frac{6}{\theta^2\Delta t^2}\mathbf{M} + \frac{3}{\theta\,\Delta t}\mathbf{C} + \mathbf{K}\right)\left\{a\right\}_{\theta} = \left\{\mathbf{F}\right\}_{n-1} + \theta\left(\left\{\mathbf{F}\right\}_{n} - \left\{\mathbf{F}\right\}_{n-1}\right)$$

$$+\left(\frac{6}{\theta^2\Delta t^2}\mathbf{M} + \frac{3}{\theta\,\Delta t}\mathbf{C}\right)\left\{a\right\}_{n-1} + \left(\frac{6}{\theta\,\Delta t}\mathbf{M} + 2\mathbf{C}\right)\left\{\dot{a}\right\}_{n-1} + \left(2\mathbf{M} + \frac{\theta\,\Delta t}{2}\mathbf{C}\right)\left\{\ddot{a}\right\}_{n-1} \qquad \text{(T2L5-25)}$$

Eqn. (T2L5-25) is the Wilson recurrence relation. After solving it for $\left\{a\right\}_{\theta}$, the latter is substituted into the equation for $\left\{\ddot{a}\right\}_{\theta}$. Then $\left\{\ddot{a}\right\}_{\theta}$ is substituted into Eqns. (T2L5-20)-(22) evaluated at time t_n to obtain $\left\{a\right\}_{n}$, $\left\{\dot{a}\right\}_{n}$ and $\left\{\ddot{a}\right\}_{n}$.

The Wilson method is obviously implicit. It is unconditionally stable for linear problems only if $\theta \geq 1.37$; a value of $\theta = 1.40$ is normally used. Because it is only one step, no special starting procedure is needed. Thus the first step is obtained using $n = 1$ in Eqn. (T2L5-25). On the RHS $\left\{a\right\}_{0}$ and $\left\{\dot{a}\right\}_{0}$ are given, and $\left\{\ddot{a}\right\}_{0}$ may be calculated from the system equations at time t_0.

Review Exercises

Problem T2L2-1

TBC

Problem T2L3-1

TBC

Problem T2L3-2

TBC

Problem T2L3-3

Calculate the critical buckling load P_{cr} for a rod clamped at one end and hinged at the other using a one-element model. Compare with the exact solution.

Problem T2L5-1

Consider a single degree-of-freedom oscillator with no damping. Derive the governing differential equation. Using $m = k = 1, f = 0$, with the initial conditions as $u_0 = 0, \dot{u}_0 = 1$ obtain the exact solution. Calculate the transient response using the central difference method. What is the value of Δt_{Crit}? First use $\Delta t = 05\Delta t_{Crit}$. Repeat the calculations using $\Delta t = 2t_{Crit}$. Now calculate the solution using one other technique.

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