Finite Elements for Engineers

Lecture 4: Time-Integration Schemes For Forced Vibration Problems

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1D Structural Dynamics

DE

$$\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2}+\mu(x)\frac{\partial u(x,t)}{\partial t}-\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u(x,t)}{\partial x}\right)+\beta(x)u(x,t)=f(x,t)$$

Domain

$$x_a \le x \le x_b \qquad t > t_0$$

BCs

At $x = x_a$ and $t > t_0$

$$u(x_a,t)=u_a(t) \text{ or } \left(-\alpha(x)\frac{\partial u}{\partial x}\right)_{x_a}=\tau_a$$

At $x = x_b$ and $t > t_0$

$$u(x_b,t)=u_b(t) \text{ or } \left(-\alpha(x)\frac{\partial u}{\partial x}\right)_{x_b}=\tau_b$$

ICs

At t_0 $\left(x_a < x < x_b\right)$

$$u(x,t_0)=u_0(x) \quad \text{and} \quad \left(\frac{\partial u(x,t)}{\partial t}\right)_{t_0}=V_0(x)$$

Structural Dynamics

Trial Solution $u(x,t;a) = \sum_{j=1}^{n} a_j(t)\phi_j(x)$

Step 1: Galerkin's Method – Residual Equations

$$\int_{\Omega}\left(\rho(x)\frac{\partial^2 u}{\partial t^2} + \mu(x)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right) + \beta(x)u - f(x,t)\right)\phi_i(x)dx = 0$$

$$i = 1,2,\ldots,n$$

Step 2: Galerkin's Method – Integration of Parts

$$\int_{\Omega}\phi_i(x)\rho(x)\frac{\partial^2 u}{\partial t^2}dx + \int_{\Omega}\phi_i(x)\mu(x)\frac{\partial u}{\partial t}dx + \int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{\partial u}{\partial x}dx + \int_{\Omega}\phi_i(x)\beta(x)u\,dx$$

$$= \int_{\Omega} f(x,t)\phi_i(x)dx - \left[\left(-\alpha(x)\frac{\partial u}{\partial x}\right)\phi_i(x)\right]_{x_1}^{x_n} \qquad i = 1,2,\ldots,n$$

Structural Dynamics

Step 3: Galerkin's Method – Trial Solution

Note

$$\boxed{\frac{\partial u}{\partial x} = \sum_{j=1}^{n} a_j(t)\frac{d\phi_j}{dx}} \quad \boxed{\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \frac{da_j}{dt}\phi_j} \quad \boxed{\frac{\partial^2 u}{\partial t^2} = \sum_{j=1}^{n} \frac{d^2 a_j}{dt^2}\phi_j}$$

$$\sum_{j=1}^{n}\left(\int_{\Omega}\phi_i(x)\rho(x)\phi_j(x)\quad dx\right)\frac{d^2 a_j}{dt^2} + \sum_{j=1}^{n}\left(\int_{\Omega}\phi_i(x)\mu(x)\phi_j(x)dx\right)\frac{da_j}{dt} +$$

$$\sum_{j=1}^{n}\left(\int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx\right)a_j + \sum_{j=1}^{n}\left(\int_{\Omega}\phi_i(x)\beta(x)\phi_j(x)dx\right)a_j$$

$$= \int_{\Omega} f(x,t)\phi_i(x)dx - \left[\tau(x,t;a)\phi_i(x)\right]_{x_1}^{x_n}$$

Structural Dynamics

Step 3: Galerkin's Method – Element Equations

$$\mathbf{m}\left\{\frac{d^2a(t)}{dt^2}\right\}+\mathbf{c}\left\{\frac{da(t)}{dt}\right\}+\mathbf{k}\left\{a(t)\right\}=\left\{f(t)\right\}$$

$$m_{ij}=\int_{\Omega}\phi_i(x)\rho(x)\phi_j(x)\,dx$$

$$c_{ij}=\int_{\Omega}\phi_i(x)\mu(x)\phi_j(x)\,dx$$

$$k_{ij}=\int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx+\int_{\Omega}\phi_i(x)\beta(x)\phi_j(x)\,dx$$

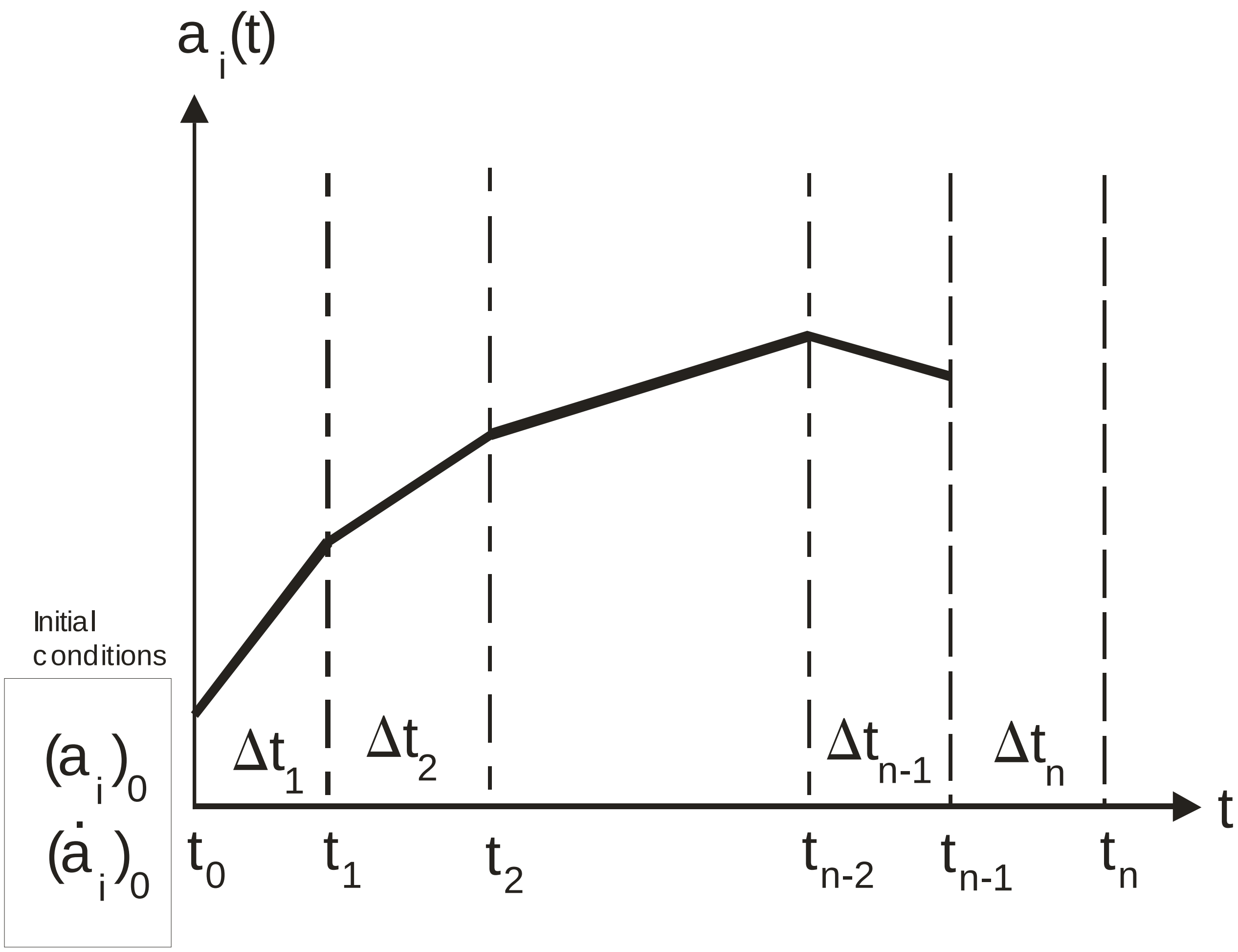
$$f_i(t)=\int_{\Omega}f(x,t)\phi_i(x)dx-\left[\tau(x,t;a)\phi_i(x)\right]_{x_1}^{x_n}$$

Structural Dynamics

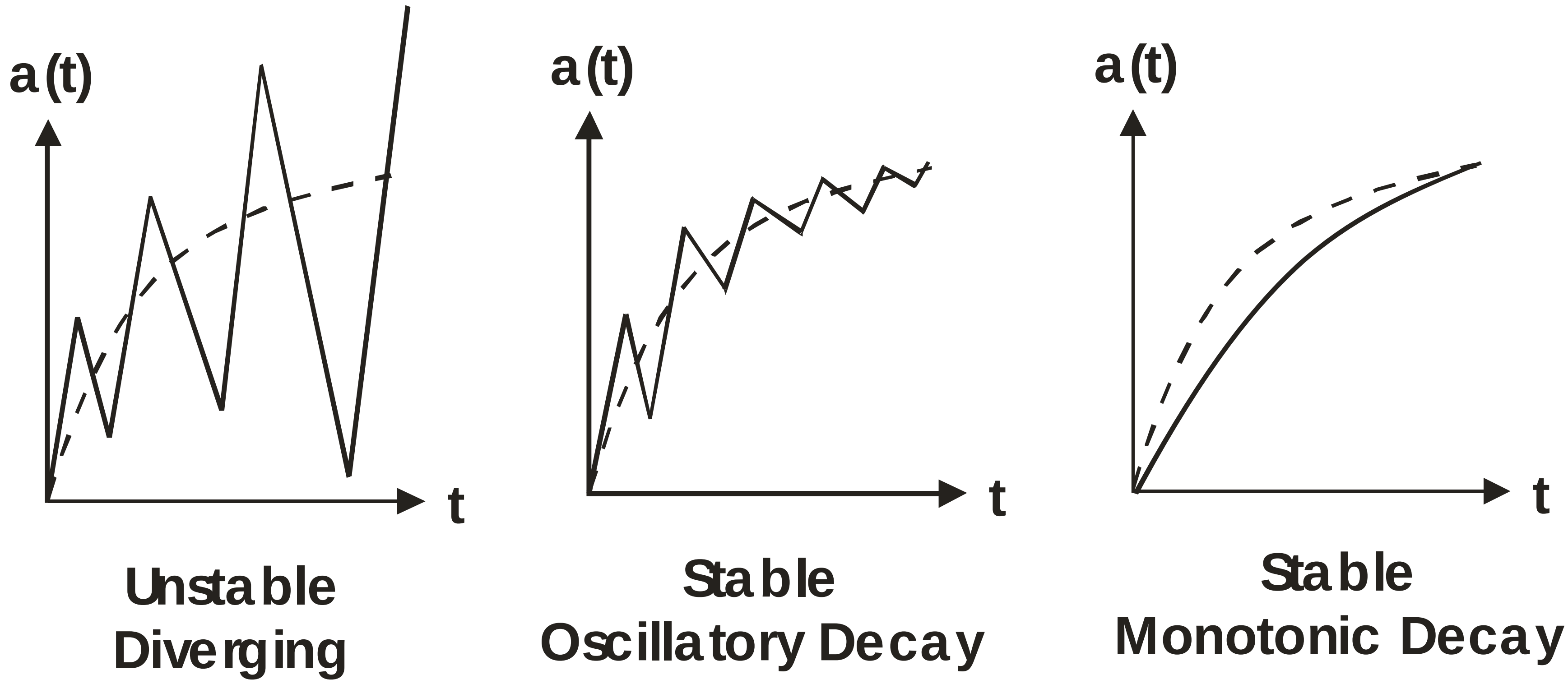
$$\mathbf{M}\left\{\ddot{a}\right\}+\mathbf{C}\left\{\dot{a}\right\}+\mathbf{K}\left\{a\right\}=\left\{\mathbf{F}\right\}$$

- ODEs with constant coefficients
- Direct Integration
  - Time stepping - March forward in time using initial conditions
  - Conditionally or unconditionally stable
- Mode Superposition
  - Uses mode shapes to construct the solution

Numerical Solution



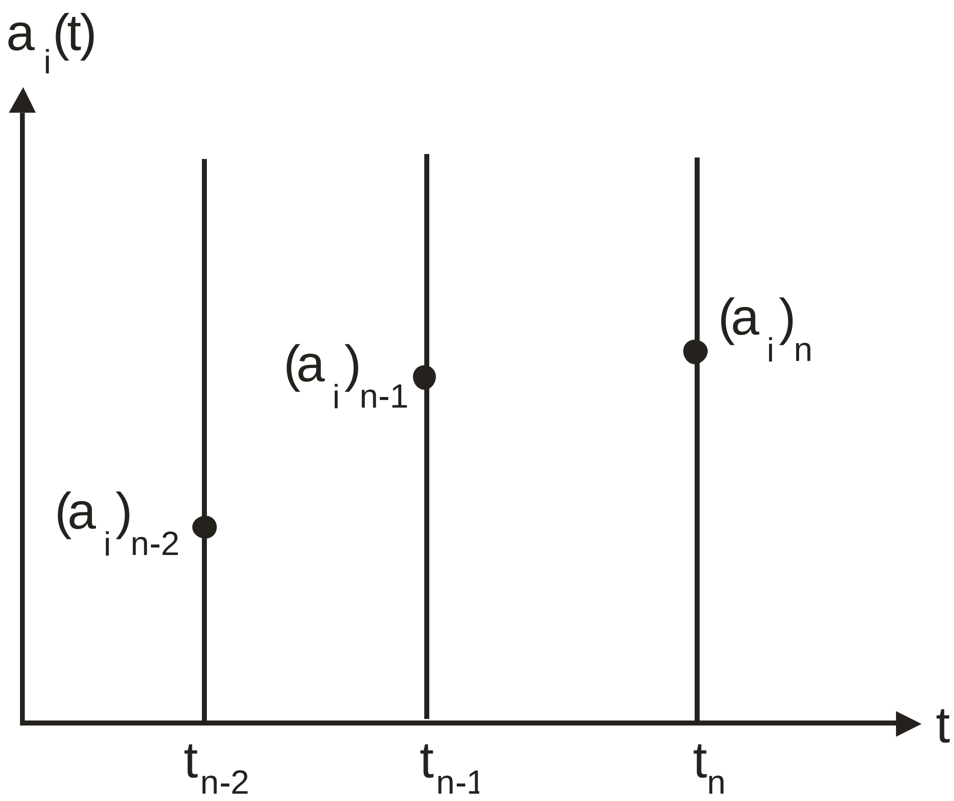
Stability


a(t)
t
Unstable
Diverging
a(t)
t
Stable
Oscillatory Decay
a(t)
t
Stable
Monotonic Decay

Central Difference Method

Evaluate at central time

$$\mathbf{M}\{\ddot{a}\}_{n-1} + \mathbf{C}\{\dot{a}\}_{n-1} + \mathbf{K}\{a\}_{n-1} = \{\mathbf{F}\}_{n-1}$$



2 step procedure

Central Difference Method

Assuming

$$\Delta t = t_n - t_{n-1} = t_{n-1} - t_{n-2}$$

Using difference scheme

$$\{\dot{a}\}_{n-1} \approx \frac{\{a\}_n - \{a\}_{n-2}}{2\Delta t}$$

$$\{\ddot{a}\}_{n-1} \approx \frac{\{\dot{a}\}_{n-1/2} - \{\dot{a}\}_{n-3/2}}{\Delta t} = \frac{\dfrac{\{a\}_n - \{a\}_{n-1}}{\Delta t} - \dfrac{\{a\}_{n-1} - \{a\}_{n-2}}{\Delta t}}{\Delta t}$$

Or

$$\{\ddot{a}\}_{n-1} = \frac{\{a\}_n - 2\{a\}_{n-1} + \{a\}_{n-2}}{\Delta t^2}$$

Central Difference Method

$$\left(\frac{1}{\Delta t^2}\mathbf{M}+\frac{1}{2\Delta t}\mathbf{C}\right)\{a\}_n =$$

$$\{F\}_{n-1}-\left(\mathbf{K}-\frac{2}{\Delta t^2}\mathbf{M}\right)\{a\}_{n-1}-\left(\frac{1}{\Delta t^2}\mathbf{M}-\frac{1}{2\Delta t}\mathbf{C}\right)\{a\}_{n-2}$$

Initial conditions are available for t_0

$$\boxed{\{a\}_{-1}=\{a\}_0-\Delta t\{\dot{a}\}_0+\frac{\Delta t^2}{2}\{\ddot{a}\}_0}$$

$$\left(\frac{1}{\Delta t^2}\mathbf{M}+\frac{1}{2\Delta t}\mathbf{C}\right)\{a\}_1 =$$

$$\{F\}_0-\left(\mathbf{K}-\frac{2}{\Delta t^2}\mathbf{M}\right)\{a\}_0-\left(\frac{1}{\Delta t^2}\mathbf{M}-\frac{1}{2\Delta t}\mathbf{C}\right)\{a\}_{-1}$$

Explicit Scheme

Diagonalize the LHS as follows (uncouples the equations)

$$\mathbf{K}_{eff} = \frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}$$

and solve (requires only matrix multiplication and algebraic division)

$$\mathbf{K}_{eff}\left\{a\right\}_n = \mathbf{F}_{eff}$$

The method is conditionally stable (function of smallest period).

$$\Delta t \le \Delta t_{crit} = \frac{T_n}{\pi}$$

Central Difference: Algorithm

Initial calculations

1. Form $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{C}$.

2. Initialize $\{a\}_0, \{\dot{a}\}_0$ and $\{\ddot{a}\}_0$.

3. Select Δt and calculate $a_0 = \dfrac{1}{\Delta t^2}, a_1 = \dfrac{1}{2\Delta t}, a_2 = 2a_0, a_3 = \dfrac{1}{a_2}$.

4. Calculate $\{a\}_{-1} = \{a\}_0 - \Delta t\{\dot{a}\}_0 + a_3\{\ddot{a}\}_0$.

5. Form $\widehat{\mathbf{M}} = a_0\mathbf{M} + a_1\mathbf{C}$.

6. Decompose $\widehat{\mathbf{M}} = \mathbf{LDL}^{\mathbf{T}}$.

Central Difference: Algorithm

For each time step

1. Calculate effective loads at time t

$$\left\{\widehat{F}\right\}_n = \{F\}_n - (\mathbf{K} - a_2\mathbf{M})\{a\}_n - (a_0\mathbf{M} - a_1\mathbf{C})\{a\}_{n-1}$$

2. Solve for displacements at time $t + \Delta t$

$$\mathbf{LDL^T}\{a\}_{n+1} = \left\{\widehat{F}\right\}_n$$

3. Evaluate accelerations and velocities at time t

$$\{\ddot{a}\} = a_0\left[\{a\}_{n-1} - 2\{a\}_n + \{a\}_{n+1}\right]$$

$$\{\dot{a}\} = a_1\left[-\{a\}_{n-1} + \{a\}_{n+1}\right]$$

Example

Problem (no damping)

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

$$\text{At } t = 0, \quad \begin{Bmatrix} a_1 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ and } \begin{Bmatrix} \dot{a}_1 \\ \dot{a}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solution

Initial acceleration

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

Example

Let $\Delta t = 0.28 \Rightarrow a_0 = 12.8, a_1 = 1.79, a_2 = 25.5, a_3 = 0.0392.$

Initial displacement

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_{-1} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - 0.28\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 0.0392\begin{Bmatrix} 0 \\ 10 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.392 \end{Bmatrix}$$

Effective mass matrix

$$\widehat{\mathbf{M}} = 12.8\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 1.79\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 25.5 & 0 \\ 0 & 12.8 \end{bmatrix}$$

Effective load vector

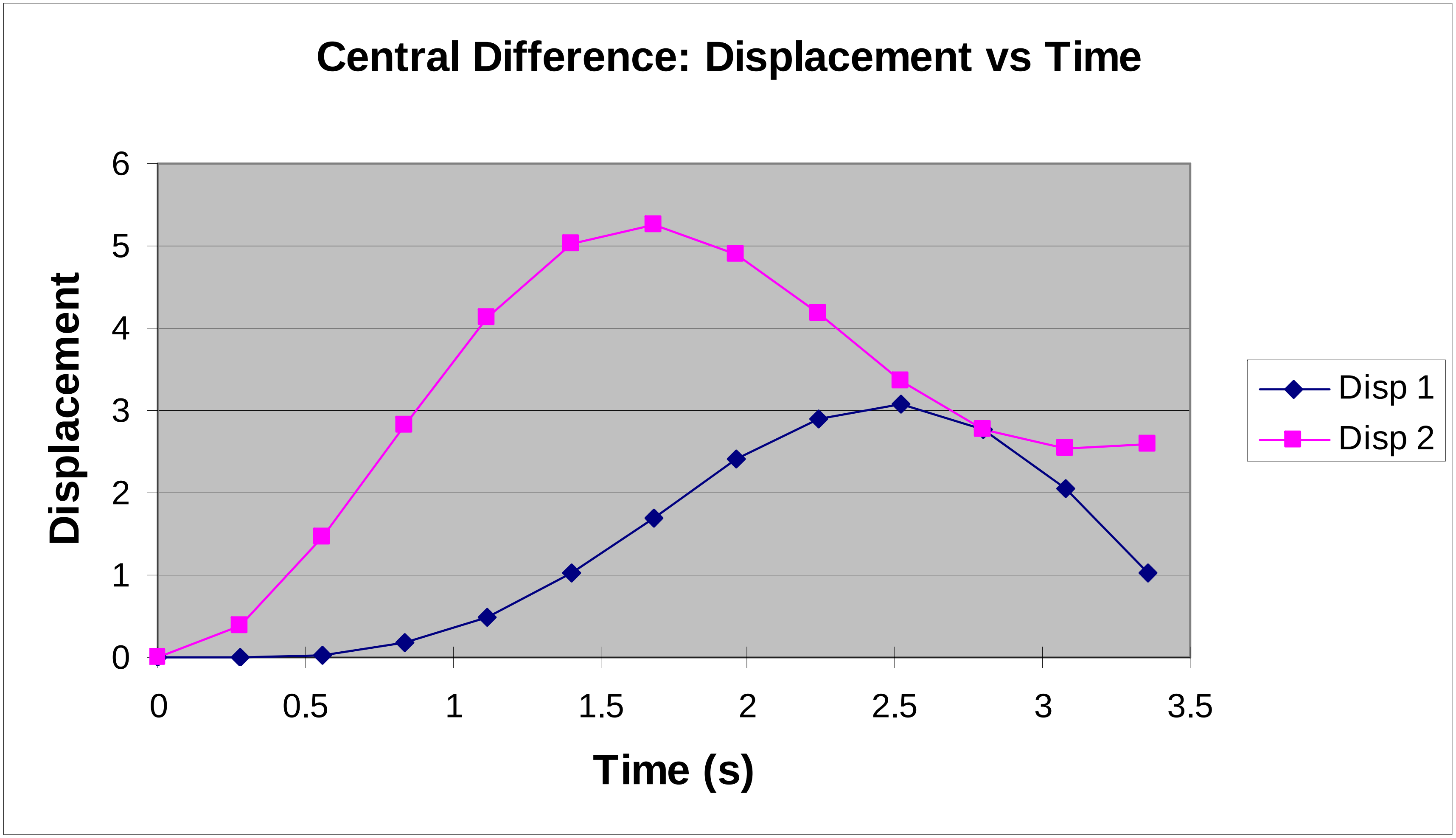
$$\left\{\widehat{F}\right\}_n = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} + \begin{bmatrix} 45.0 & 2 \\ 2 & 21.5 \end{bmatrix}\{a\}_n - \begin{bmatrix} 25.5 & 0 \\ 0 & 12.8 \end{bmatrix}\{a\}_{n-1}$$

Example

Solve these uncoupled equations every time step

$$\begin{bmatrix} 25.5 & 0 \\ 0 & 12.8 \end{bmatrix} \{a\}_{n+1} = \{\widehat{F}\}_n$$

Example

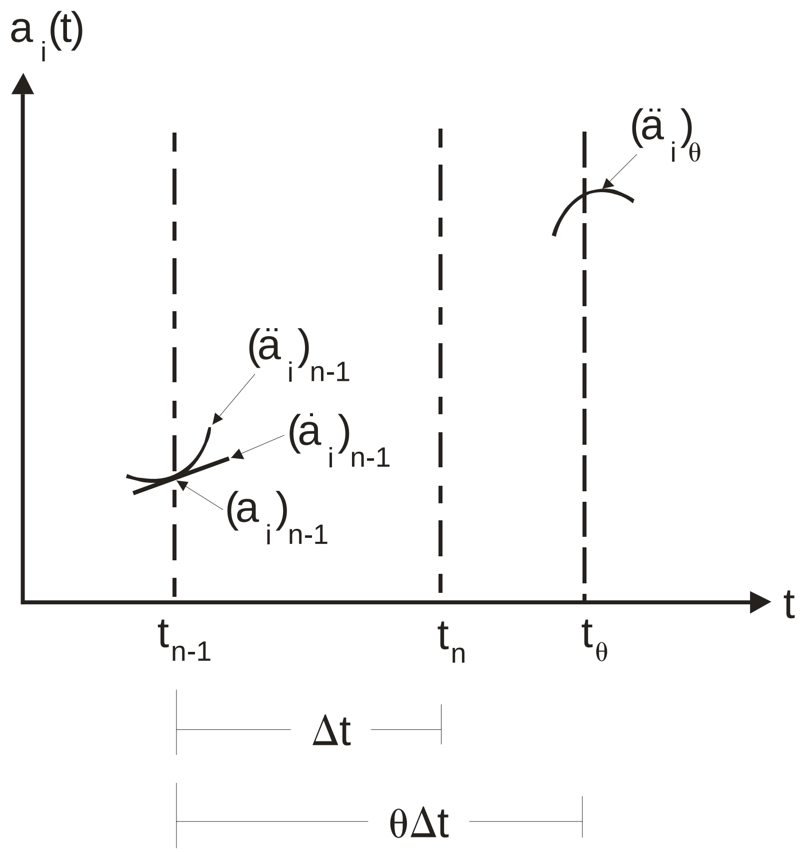


Central Difference: Displacement vs Time
Displacement
Time (s)
0
1
2
3
4
5
6
0
0.5
1
1.5
2
2.5
3
3.5
Disp 1
Disp 2

Wilson-θ Method

One-step Method (assumes linear acceleration)

$$\{a(t)\} = \{c_0\} + \{c_1\}t + \{c_2\}t^2 + \{c_3\}t^3$$



$$t_\theta = t_{n-1} + \theta\,\Delta t$$

$$\theta \geq 1$$

Wilson-θ Method

Interpolating data from t_{n-1} and t_θ

$$\{a(t)\} = \{a\}_{n-1} + \{\dot{a}\}_{n-1}(t - t_{n-1}) + \frac{1}{2}\{\ddot{a}\}_{n-1}(t - t_{n-1})^2 + \frac{1}{6}\{\ddot{a}\}_{n-1}\left(\frac{\{\ddot{a}\}_\theta - \{\ddot{a}\}_{n-1}}{\theta \;\; \Delta t}\right)(t - t_{n-1})^3$$

Differentiating once and twice

$$\{\dot{a}(t)\} = \{\dot{a}\}_{n-1} + \{\ddot{a}\}_{n-1}(t - t_{n-1}) + \frac{1}{2}\left(\frac{\{\ddot{a}\}_\theta - \{\ddot{a}\}_{n-1}}{\theta \;\; \Delta t}\right)(t - t_{n-1})^2$$

$$\{\ddot{a}(t)\} = \{\ddot{a}\}_{n-1} + \left(\frac{\{\ddot{a}\}_\theta - \{\ddot{a}\}_{n-1}}{\theta \;\; \Delta t}\right)(t - t_{n-1})$$

Wilson-θ Method

Evaluating at t_θ

$$\mathbf{M}\{\ddot{a}\}_\theta + \mathbf{C}\{\dot{a}\}_\theta + \mathbf{K}\{a\}_\theta = \{\mathbf{F}\}_\theta$$

$$\{\mathbf{F}\}_\theta = \{\mathbf{F}\}_{n-1} + \theta\left(\{\mathbf{F}\}_n - \{\mathbf{F}\}_{n-1}\right)$$

Substituting

$$\left(\frac{6}{\theta^2\Delta t^2}\mathbf{M} + \frac{3}{\theta\ \ \Delta t}\mathbf{C} + \mathbf{K}\right)\{a\}_\theta = \{\mathbf{F}\}_{n-1} + \theta\left(\{\mathbf{F}\}_n - \{\mathbf{F}\}_{n-1}\right)$$
$$+\left(\frac{6}{\theta^2\Delta t^2}\mathbf{M} + \frac{3}{\theta\ \ \Delta t}\mathbf{C}\right)\{a\}_{n-1} + \left(\frac{6}{\theta\ \ \Delta t}\mathbf{M} + 2\mathbf{C}\right)\{\dot{a}\}_{n-1} + \left(2\mathbf{M} + \frac{\theta\ \ \Delta t}{2}\mathbf{C}\right)\{\ddot{a}\}_{n-1}$$

Wilson-θ Method

- Method is implicit.
- Method is unconditionally stable if $\theta \geq 1.37$
- Since the method is one-step no special startup scheme is needed.

Wilson-θ: Algorithm

Initial calculations

1. Form **K**, **M** and **C**.

2. Initialize $\{a\}_0, \{\dot{a}\}_0$ and $\{\ddot{a}\}_0$.

3. Select Δt and $\theta = 1.4$ and calculate

$$a_0 = \frac{6}{(\theta \Delta t)^2}, a_1 = \frac{3}{\theta\,\Delta t}, a_2 = 2a_1, a_3 = \frac{\theta\,\Delta t}{a_2}$$

$$a_4 = \frac{a_0}{\theta}, a_5 = -\frac{a_2}{\theta}, a_6 = 1 - \frac{3}{\theta}, a_7 = \frac{\Delta t}{2}, a_8 = \frac{\Delta t^2}{6}$$

4. Form $\widehat{\mathbf{K}} = \mathbf{K} + a_0\mathbf{M} + a_1\mathbf{C}$.

5. Decompose $\widehat{\mathbf{K}} = \mathbf{LDL}^{\mathbf{T}}$.

Wilson-θ: Algorithm

For each time step

1. Calculate effective loads at time $t+\theta\Delta t$

$$\left\{\widehat{F}\right\}_{\theta}=\left\{F\right\}_{n}+\theta\left[\left\{F\right\}_{\theta}-\left\{F\right\}_{n}\right]+\mathbf{M}\left[a_0\left\{a\right\}_{n}+a_2\left\{\dot{a}\right\}_{n}+2\left\{\ddot{a}\right\}_{n}\right]$$

$$+\mathbf{C}\left[a_1\left\{a\right\}_{n}+2\left\{\dot{a}\right\}_{n}+a_3\left\{\ddot{a}\right\}_{n}\right]$$

2. Solve for displacements at time $t+\theta\Delta t$

$$\mathbf{LDL}^{\mathbf{T}}\left\{a\right\}_{\theta}=\left\{\widehat{F}\right\}_{\theta}$$

3. Evaluate displacements, accelerations and velocities at time $t+\Delta t$

$$\left\{\ddot{a}\right\}_{n+1}=a_4\left[\left\{a\right\}_{\theta}-\left\{a\right\}_{n}\right]+a_5\left\{\dot{a}\right\}_{n}+a_6\left\{\ddot{a}\right\}_{n}$$

$$\left\{\dot{a}\right\}_{n+1}=\left\{\dot{a}\right\}_{n}+a_7\left[\left\{\ddot{a}\right\}_{n+1}+\left\{\ddot{a}\right\}_{n}\right]$$

$$\left\{a\right\}_{n+1}=\left\{a\right\}_{n}+\Delta t\left\{\dot{a}\right\}_{n}+a_8\left[\left\{\ddot{a}\right\}_{n+1}+2\left\{\ddot{a}\right\}_{n}\right]$$

Example

Problem (no damping)

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

$$\text{At } t = 0, \ \begin{Bmatrix} a_1 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ and } \begin{Bmatrix} \dot{a}_1 \\ \dot{a}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solution

Initial acceleration

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

Example

$$a_0 = 39.0, a_1 = 7.65, a_2 = 15.3, a_3 = 0.196$$

$$a_4 = 27.9, a_5 = -10.9, a_6 = -1.14, a_7 = 0.14, a_8 = 0.0131$$

Effective stiffness matrix

$$\widehat{\mathbf{K}} = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} + 39.0 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 84.1 & -2 \\ -2 & 43.0 \end{bmatrix}$$

For every time step

$$\left\{ \widehat{F} \right\}_\theta = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \left[39.0 \{a\}_n + 15.3 \{\dot{a}\}_n + 2 \{\ddot{a}\}_n \right]$$

$$\widehat{\mathbf{K}} \{a\}_\theta = \left\{ \widehat{F} \right\}_\theta$$

Example

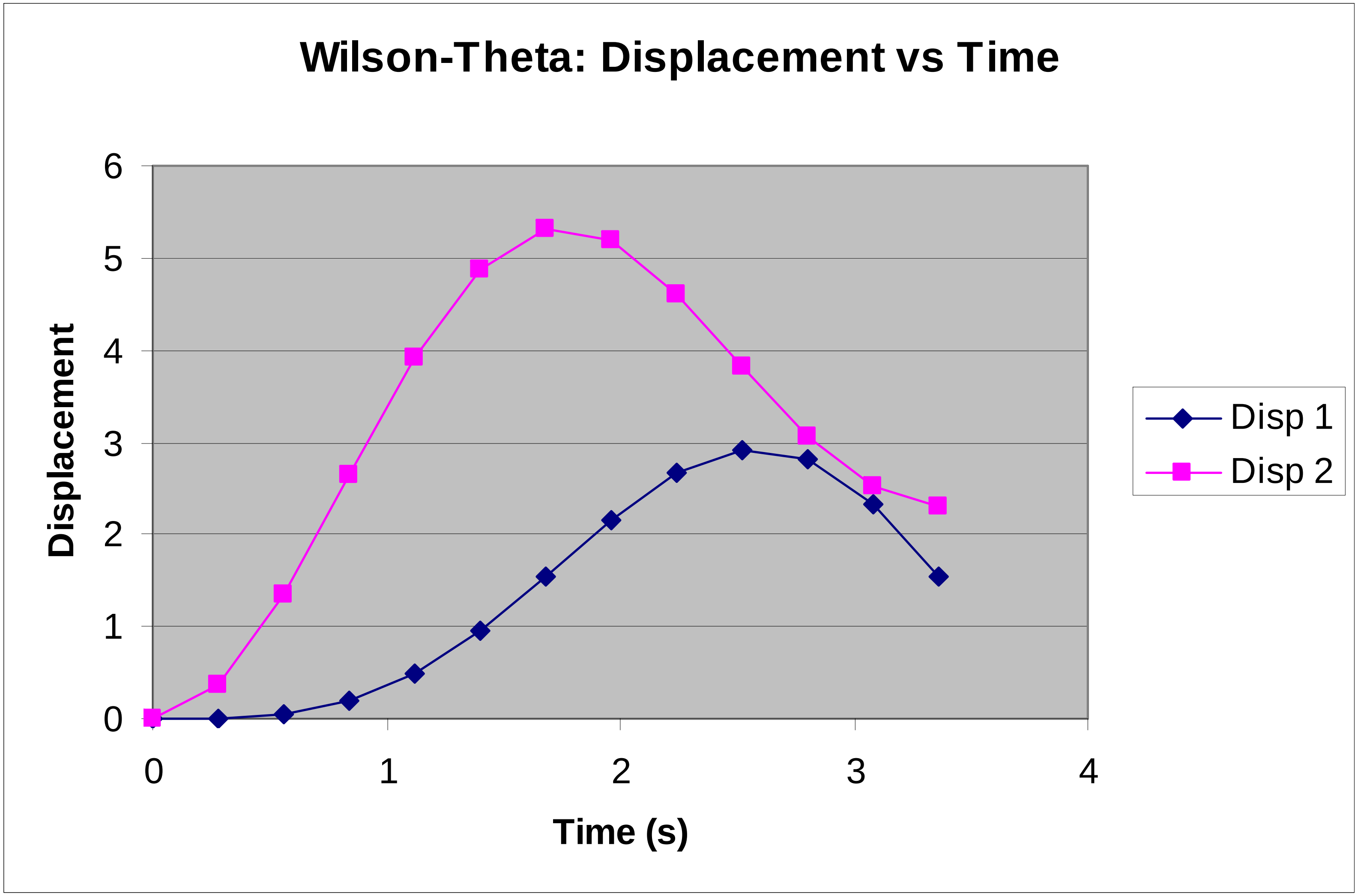
For every time step

$$\{\ddot{a}\}_{n+1} = 27.9\left[\{a\}_{\theta} - \{a\}_{n}\right] - 10.9\{\dot{a}\}_{n} - 1.14\{\ddot{a}\}_{n}$$

$$\{\dot{a}\}_{n+1} = \{\dot{a}\}_{n} + 0.14\left[\{\ddot{a}\}_{n+1} + \{\ddot{a}\}_{n}\right]$$

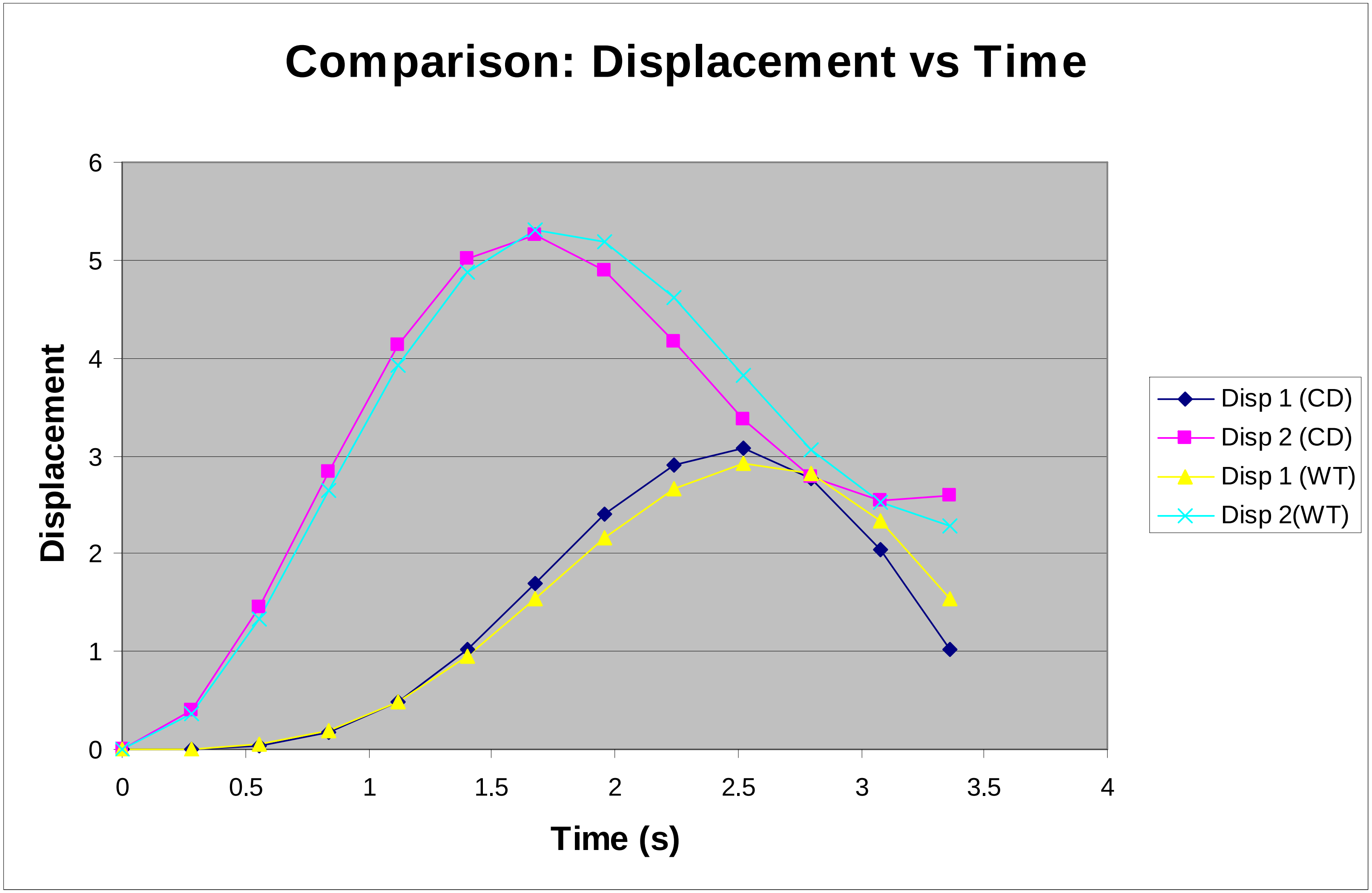
$$\{a\}_{n+1} = \{a\}_{n} + 0.28\{\dot{a}\}_{n} + 0.013\left[\{\ddot{a}\}_{n+1} + 2\{\ddot{a}\}_{n}\right]$$

Example



Wilson-Theta: Displacement vs Time
Displacement
6
5
4
3
2
1
0
0
1
2
3
4
Time (s)
Disp 1
Disp 2

Example



Comparison: Displacement vs Time
Displacement
6
5
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
4
Time (s)
Disp 1 (CD)
Disp 2 (CD)
Disp 1 (WT)
Disp 2(WT)

Mode Superposition

Transformation (Change of Basis)

$$\{a(t)\} = \mathbf{P}_{n\times n}\mathbf{X}_{n\times 1}(t)$$

Substituting into the system equations

$$\mathbf{M}\{\ddot{a}\} + \mathbf{C}\{\dot{a}\} + \mathbf{K}\{a\} = \{\mathbf{F}\}$$

and premultiplying by $\mathbf{P}^T$

$$\widetilde{\mathbf{M}}\ddot{\mathbf{X}}(t) + \widetilde{\mathbf{C}}\dot{\mathbf{X}}(t) + \widetilde{\mathbf{K}}\mathbf{X}(t) = \widetilde{\mathbf{F}}(t) \quad \Rightarrow$$

$$\widetilde{\mathbf{M}} = \mathbf{P}^{\mathbf{T}}\mathbf{M}\mathbf{P}$$

$$\widetilde{\mathbf{C}} = \mathbf{P}^{\mathbf{T}}\mathbf{C}\mathbf{P}$$

$$\widetilde{\mathbf{K}} = \mathbf{P}^{\mathbf{T}}\mathbf{K}\mathbf{P}$$

$$\widetilde{\mathbf{F}} = \mathbf{P}^{\mathbf{T}}\mathbf{F}\mathbf{P}$$

Mode Superposition

There are many ways of selecting a nonsingular **P**. Objective of the transformation is to reduce the half-band width of the original matrices. On approach is to use

$$\mathbf{M}\{\ddot{a}\} + \mathbf{K}\{a\} = 0 \Rightarrow \mathbf{K\Phi} = \mathbf{\Lambda M \Phi}$$

Hence

$$\{a(t)\} = \mathbf{\Phi X}(t)$$

Equilibrium equations

$$\ddot{\mathbf{X}}(t) + \mathbf{\Phi}^T \mathbf{C \Phi} \dot{\mathbf{X}}(t) + \mathbf{\Lambda X}(t) = \mathbf{\Phi}^T \{\mathbf{F}(t)\}$$

Initial conditions

$$\mathbf{X}_0 = \mathbf{\Phi}^T \mathbf{M} \{a\}_0 \qquad \dot{\mathbf{X}}_0 = \mathbf{\Phi}^T \mathbf{M} \{\dot{a}\}_0$$

Example

Problem (no damping)

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} + \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

$$\text{At } t = 0, \ \begin{Bmatrix} a_1 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \text{ and } \begin{Bmatrix} \dot{a}_1 \\ \dot{a}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solution

Generalized Eigenproblem

$$\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \mathbf{\Phi} = \lambda \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{\Phi} \Rightarrow \lambda_1 = 2, \frac{1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}; \lambda_2 = 5, \sqrt{\frac{2}{3}} \begin{Bmatrix} 1/2 \\ -1 \end{Bmatrix}$$

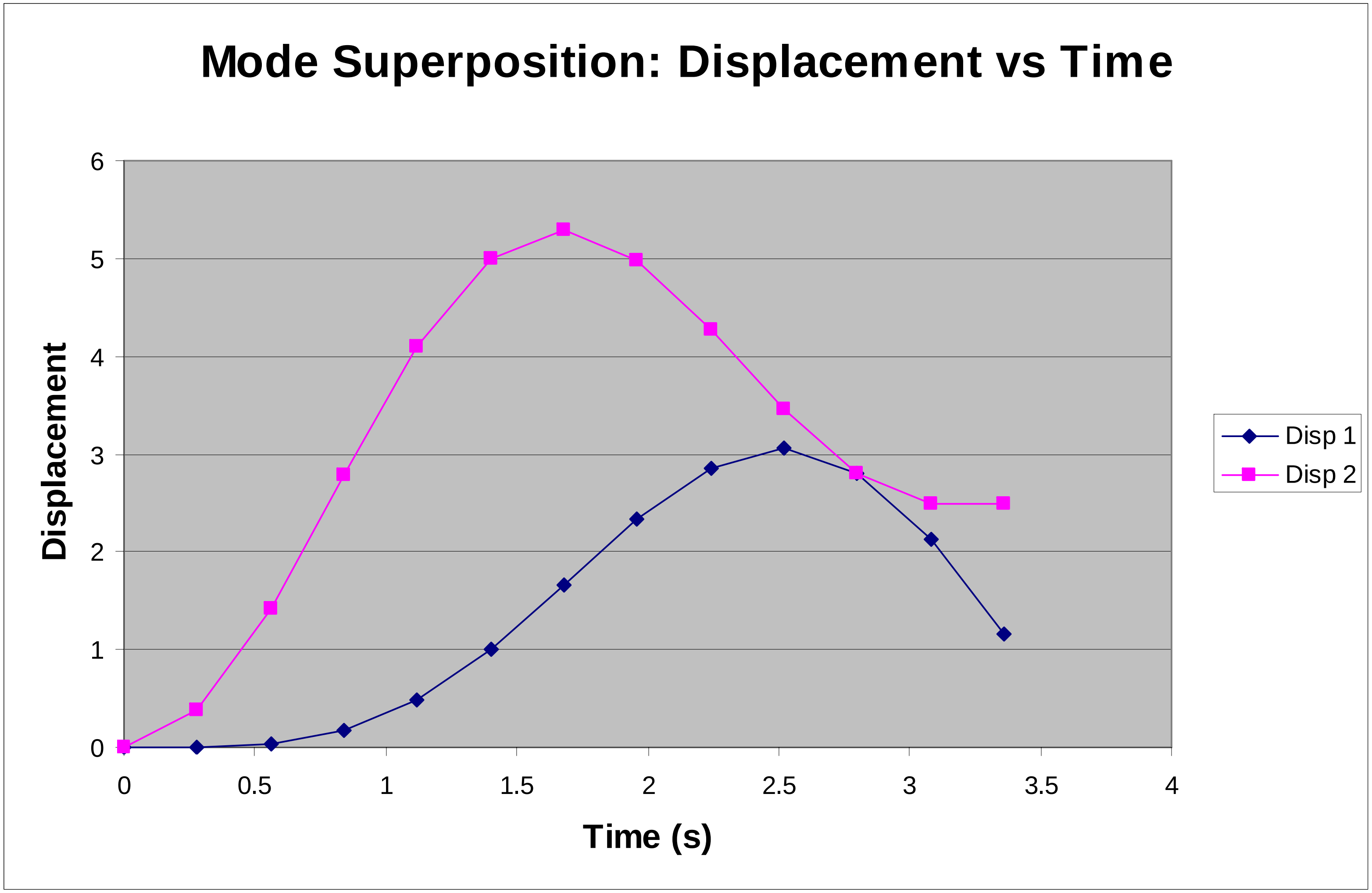
Example

Eigenvector basis equilibrium equations

$$\ddot{\mathbf{X}}(t)+\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}\mathbf{X}(t)=\begin{bmatrix}\dfrac{10}{\sqrt{3}}\\ -10\sqrt{\dfrac{2}{3}}\end{bmatrix}$$

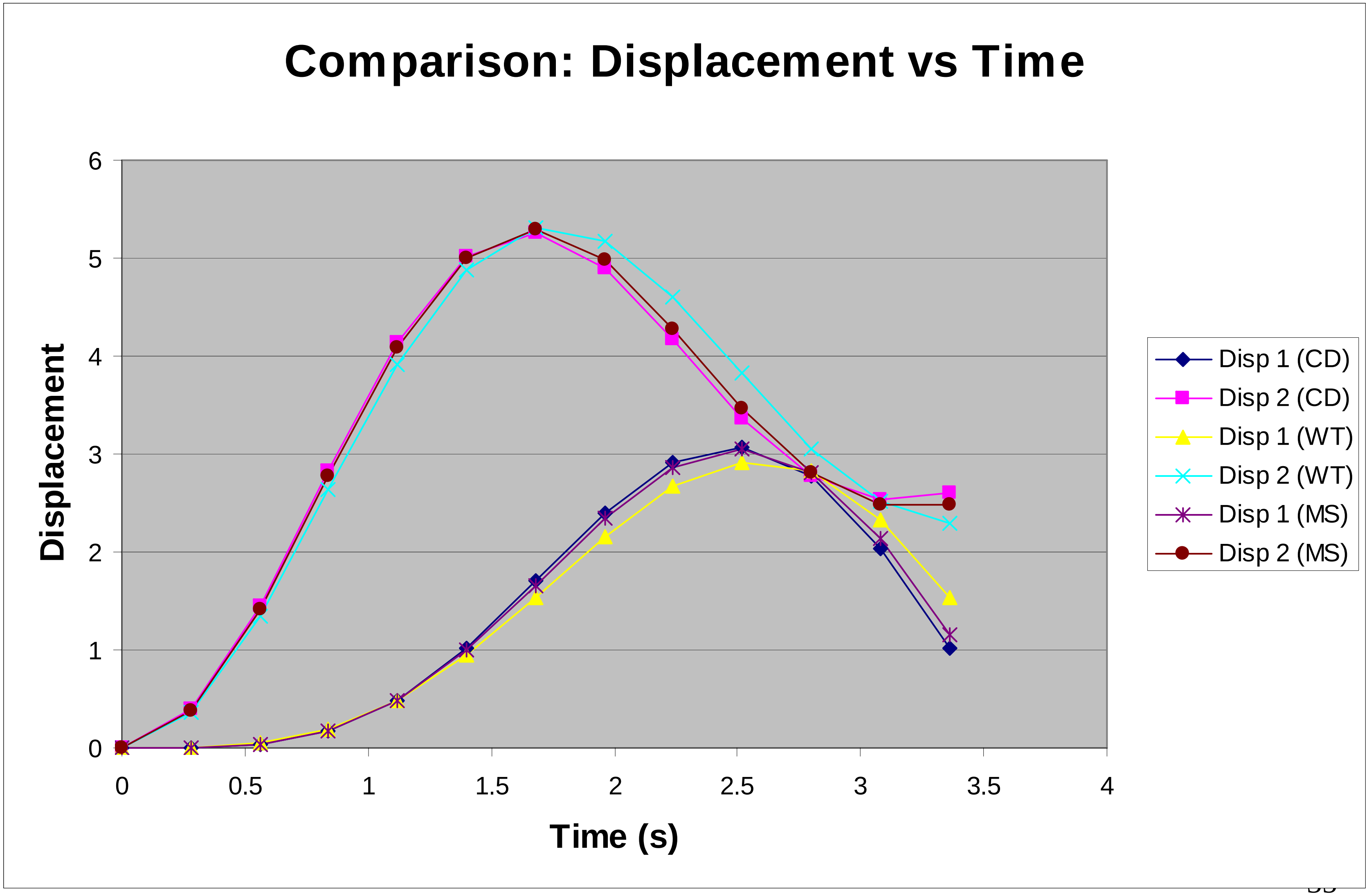
These decoupled equations can be solved using any numerical technique.

Example



Mode Superposition: Displacement vs Time
Displacement
Time (s)
0
1
2
3
4
5
6
0
0.5
1
1.5
2
2.5
3
3.5
4
Disp 1
Disp 2

Example



Comparison: Displacement vs Time
Displacement
Time (s)
0
1
2
3
4
5
6
0
0.5
1
1.5
2
2.5
3
3.5
4
Disp 1 (CD)
Disp 2 (CD)
Disp 1 (WT)
Disp 2 (WT)
Disp 1 (MS)
Disp 2 (MS)

Damping

- Is not similar to stiffness and mass. Can be thought of as contributing to overall energy dissipation. Assume that total damping in the structure is the sum of individual damping in each mode.

$$\phi_i^T \mathbf{C} \phi_j = 2\omega_i \xi_i \delta_{ij}$$

ξ_i : modal damping parameter

Damping

Hence mode superposition equilibrium equations become

$$\ddot{x}_i(t) + 2\omega_i \xi_i \dot{x}_i(t) + \omega_i^2 x_i(t) = f_i(t)$$

Rayleigh Damping: Sometimes two different modes are selected and

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$

Example

Problem

Given $\omega_1 = 2$ and $\omega_2 = 3$.

Also $\xi_1 = 0.02$ and $\xi_2 = 0.10$. Compute α and β.

Solution

$$\phi_i^T \left[\alpha \mathbf{M} + \beta \mathbf{K} \right] \phi_i = 2\omega_i \xi_i \qquad \Rightarrow \qquad \alpha + 4\beta = 0.08$$

$$\text{Or, } \alpha + \beta \omega_i^2 = 2\omega_i \xi_i \qquad \qquad \alpha + 9\beta = 0.60$$

Hence

$$\mathbf{C} = -0.336\mathbf{M} + 0.104\mathbf{K}$$

Summary

- Central Difference Method is an explicit, conditionally stable method.
- Wilson-θ is unconditionally stable for $\theta >$ 1.37.
- Other Time-Integration Methods
  - Houbolt
  - Newmark
- Direct Integration: Computational effort is proportional to the number of time steps.

Summary

- Direct Integration: Short duration simulation.
- Mode Superposition: Longer duration simulation. However requires solution of an eigenproblem.

Further Reading

- Bathe, *Finite Element Procedures*, Prentice-Hall.