Finite Elements for Engineers

Lecture 2: Structural Dynamics

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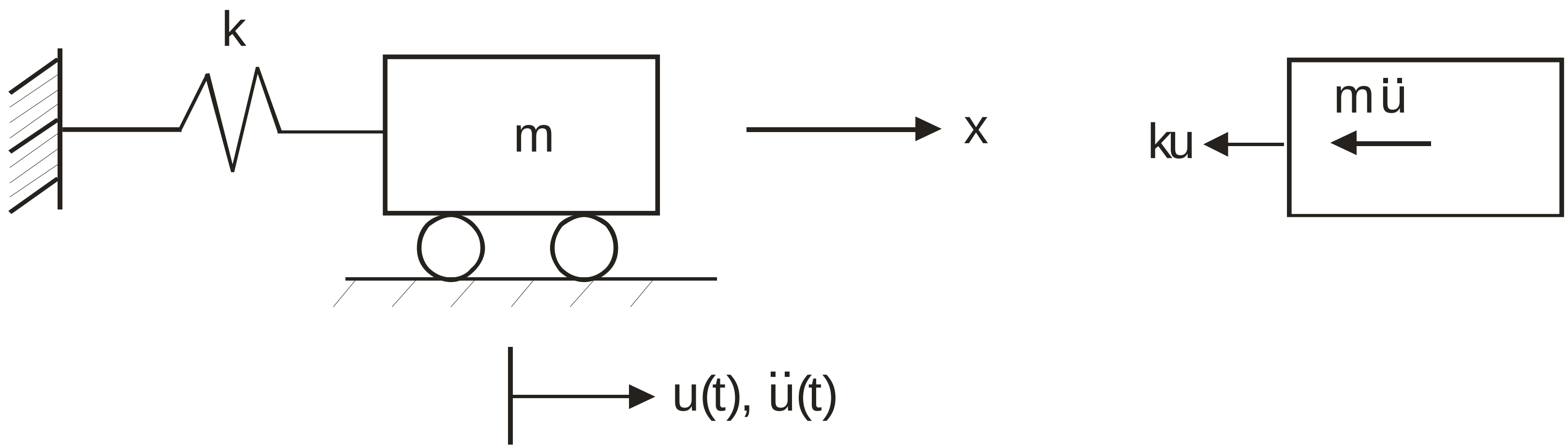
Introduction: Free Vibration



D'Alembert's Principle $$m\ddot{u} + ku = 0$$

Let $\omega^2 = \dfrac{k}{m} \quad \Rightarrow \quad \ddot{u} + \omega^2 u = 0$

Introduction: Free Vibration

Angular Frequency $\omega = \sqrt{k/m} \quad rad/s$

Natural Frequency $f = \omega/2\pi \quad Hz$

Natural Period $T = 1/f \quad s$

Introduction: Free Vibration

Solution

$$u(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

$t = t_0$ $\quad$ $u = u_0$ $\quad$ **Initial displacement**

$\dot{u} = \dot{u}_0$ $\quad$ **Initial velocity**

Solution to free, undamped vibration

$$u = A\cos(\omega t - \alpha)$$

Amplitude $\quad A = \sqrt{u_0^2 + \left(\dfrac{\dot{u}_0}{\omega}\right)^2}$ $\quad$ **Phase Angle** $\quad \alpha = \tan^{-1} \dfrac{\dot{u}_0}{\omega u_0}$

Introduction: Forced Vibration

Harmonic forcing function

$$m\ddot{u} + ku = P \sin \Omega t$$

$$p_m = \frac{P}{m} \Rightarrow \ddot{u} + \omega^2 u = p_m \sin \Omega t$$

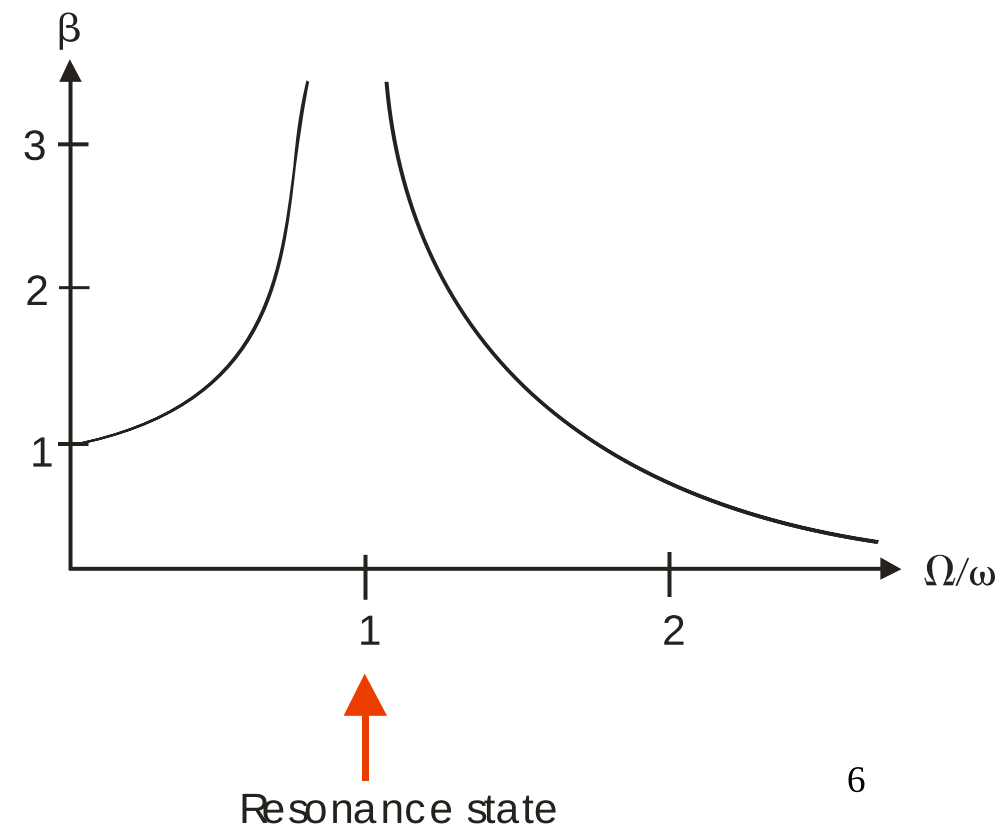
Solution (particular solution + general solution)

$$u = C_3 \sin \Omega t$$

$$u = C_1 \cos \omega t + C_2 \sin \omega t + C_3 \sin \Omega t$$

Introduction: Forced Vibration

$$u = \left[\frac{1}{1 - \left(\Omega / \omega \right)^2} \right] \frac{P}{k} \sin \Omega t = \frac{1}{\beta} \frac{P}{k} \sin \Omega t$$



1D Eigenproblem

DE

$$-\frac{d}{dx}\left\{\alpha(x)\frac{du(x)}{dx}\right\}+\beta(x)u(x)-\lambda\gamma(x)u(x)=0 \quad x_a < x < x_b$$

BCs

$$x_a \Rightarrow u(x_a)=0 \quad or \quad \tau(x_a)=0$$

$$x_b \Rightarrow u(x_b)=0 \quad or \quad \tau(x_b)=0$$

Galerkin's Method

Galerkin Step 1: Residual Equations

$$\int_{\Omega}\left[-\frac{d}{dx}\left\{\alpha(x)\frac{du(x)}{dx}\right\}+\beta(x)u(x)-\lambda\gamma(x)u(x)\right]\phi_i(x)dx=0 \quad i=1,2,\ldots,n$$

Galerkin Step 2: Integration by parts

$$\int_{\Omega}\frac{d\phi_i(x)}{dx}\alpha(x)\frac{d\tilde{u}}{dx}dx+\int_{\Omega}\phi_i(x)\beta(x)\tilde{u}\,dx-\lambda\int_{\Omega}\phi_i(x)\gamma(x)\tilde{u}\,dx=$$

$$-\cancel{\left[\left\{-\alpha(x)\frac{d\tilde{u}}{dx}\right\}\phi_i(x)\right]_{x_1}^{x_n}}$$

Galerkin's Method

Galerkin Step 3: Use of trial solution

$$\tilde{u}(x,y) = \sum_{j=1}^{n} \phi_j(x,y) u_j$$

$$\sum_{j=1}^{n} \left\{ \int_{\Omega} \frac{d\phi_i(x)}{dx} \alpha(x) \frac{d\phi_j(x)}{dx} dx + \int_{\Omega} \phi_i(x) \beta(x) \phi_j(x) dx \right\} a_j$$

$$- \lambda \sum_{j=}^{n} \left\{ \int_{\Omega} \phi_i(x) \gamma(x) \phi_j(x)\, dx \right\} a_j = 0 \quad i = 1,2,\ldots,n$$

$$\mathbf{k}_{n\times n} \mathbf{a}_{n\times 1} - \lambda \mathbf{m}_{n\times n} \mathbf{a}_{n\times 1} = \mathbf{0}$$

1D-C^0 Linear Element

Galerkin Step 4

$$\phi_1(x) = \frac{x_2 - x}{x_2 - x_1} \qquad \phi_2(x) = \frac{x - x_1}{x_2 - x_1}$$

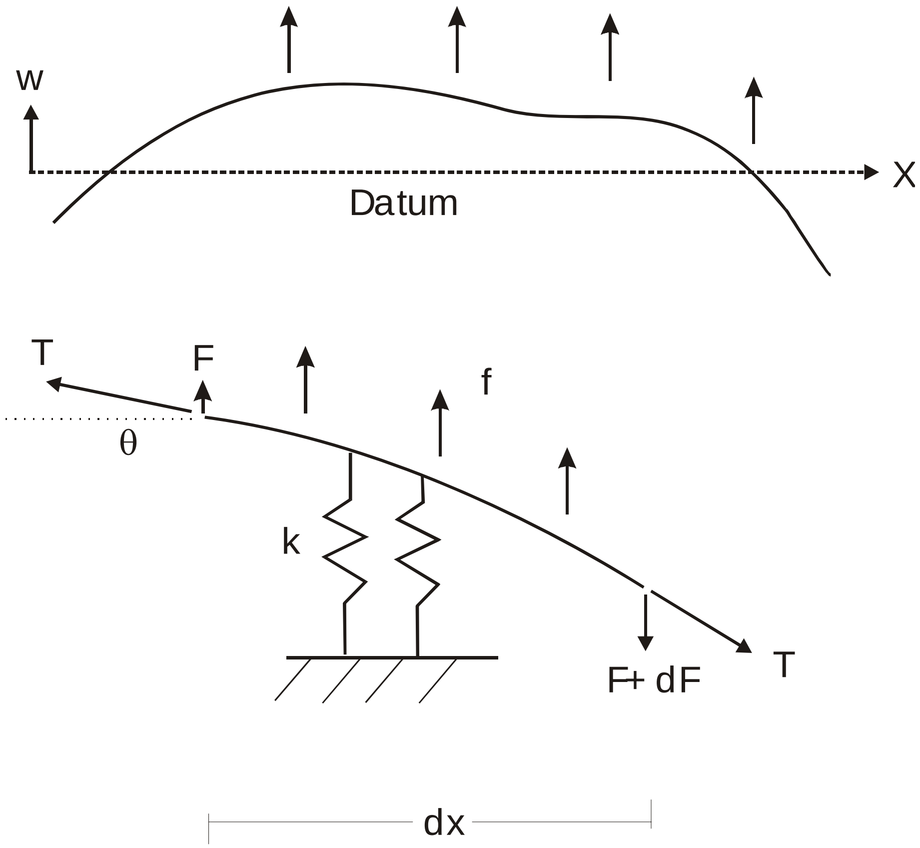
$$\left[\frac{\overline{\alpha}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\overline{\beta L}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right] \mathbf{a} - \lambda \frac{\overline{\gamma} L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{a} = \mathbf{0}$$

System Equations

Eigenproblem

$$\mathbf{K}_{n\times n}\mathbf{\Phi}_{n\times n} = \mathbf{\Lambda}_{n\times n}\mathbf{M}_{n\times n}\mathbf{\Phi}_{n\times n}$$

Free Vibration of a Cable



Transverse Equilibrium

$$\frac{\partial F(x,t)}{\partial x} + k(x)w(x,t) = f(x) - \rho(x)\frac{\partial^2 w(x,t)}{\partial t^2}$$

Free motion

$$\frac{\partial F(x,t)}{\partial x} + k(x)w(x,t) = -\rho(x)\frac{\partial^2 w(x,t)}{\partial t^2}$$

Constitutive relation

$$F(x,t) = -T(x)\frac{\partial w(x,t)}{\partial x}$$

Free Vibration

$$-\frac{\partial}{\partial x}\left[T(x)\frac{\partial w(x,t)}{\partial x}\right]+k(x)w(x,t)=-\rho(x)\frac{\partial^2 w(x,t)}{\partial t^2}$$

Harmonic solution

$$w(x,t)=W(x)\sin\omega t$$

Governing DE

$$-\frac{d}{dx}\left[T(x)\frac{dW(x)}{dx}\right]+k(x)W(x)-\omega^2\rho(x)W(x)=0$$

BCs

$$W(0)=0$$

$$W(L)=0$$

Free Vibration

General 1D Eigenproblem

$$-\frac{d}{dx}\left[\alpha(x)\frac{du(x)}{dx}\right]+\beta(x)u(x)-\lambda\gamma(x)u(x)=0$$

Free vibration of a cable

$$-\frac{d}{dx}\left[T(x)\frac{dW(x)}{dx}\right]+k(x)W(x)-\omega^2\rho(x)W(x)=0$$

Example

Problem data

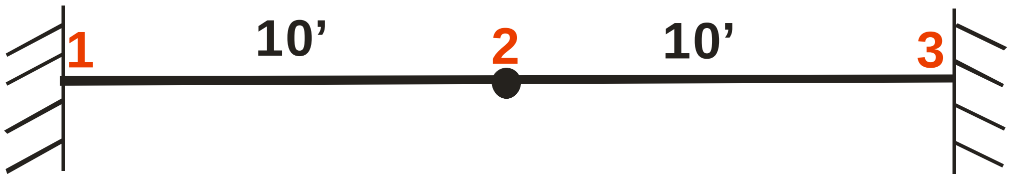
$$L = 20\ ft$$

$$T = 100\,lb$$

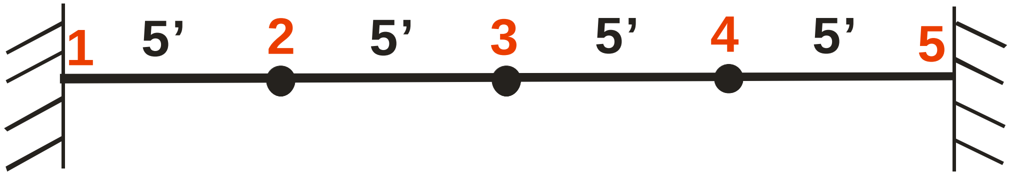
$$\rho = 0.01553\,slug/ft$$

$$k = 0$$

2-Element mesh



4-element mesh



2-Element Mesh

Element 1

$$\frac{100}{10}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} - \lambda\frac{(0.01553)(10)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = 0$$

Element 2

$$\frac{100}{10}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} W_2 \\ W_3 \end{Bmatrix} - \lambda\frac{(0.01553)(10)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} W_2 \\ W_3 \end{Bmatrix} = 0$$

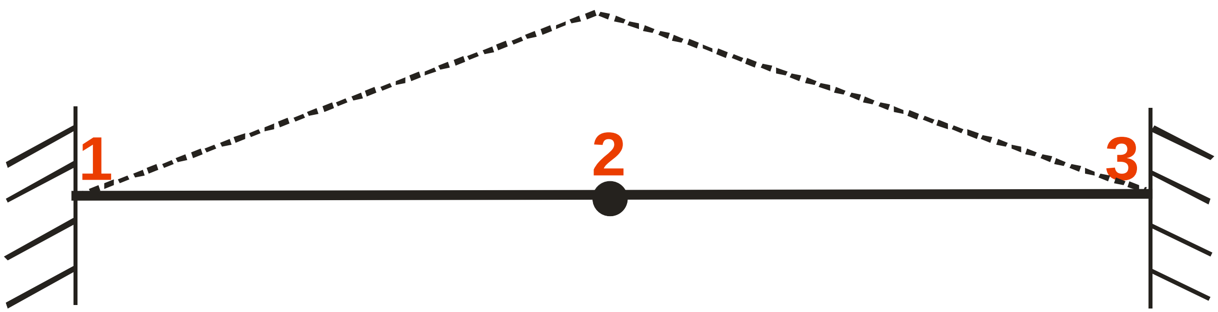
2-Element Model

System Equation

$$[20]\{W_2\} - \lambda[0.103533]\{W_2\} = 0$$

Solution

$$\det(20 - 0.103533\lambda) = 0 \Rightarrow \lambda_1 = 193$$



4-Element Model

Element 1

$$\frac{100}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} - \lambda \frac{(0.01553)(5)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = 0$$

$$20\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} - \lambda (0.0129417)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = 0$$

Element 2, 3, 4

$$20\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} W_n \\ W_{n+1} \end{Bmatrix} - \lambda (0.0129417)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} W_n \\ W_{n+1} \end{Bmatrix} = 0$$

4-Element Solution

System Equations

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix} - \lambda \begin{bmatrix} 0.0517668 & 0.0129417 & 0 \\ 0.0129417 & 0.0517668 & 0.0129417 \\ 0 & 0.0129417 & 0.0517668 \end{bmatrix} \begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix} = 0$$

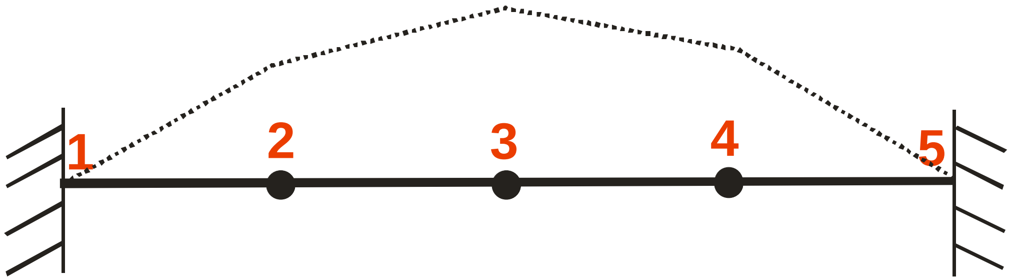
Solution

$$\lambda_1 = 169.38 \qquad \lambda_2 = 782.8 \qquad \lambda_3 = 2067.1$$

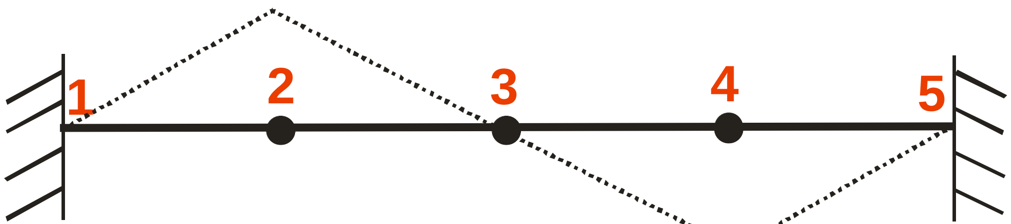
$$\phi_1 = \begin{Bmatrix} 0.146 \\ 0.207 \\ 0.146 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} 0.112 \\ 0 \\ -0.112 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0.061 \\ -0.086 \\ 0.061 \end{Bmatrix}$$

4-Element Solution

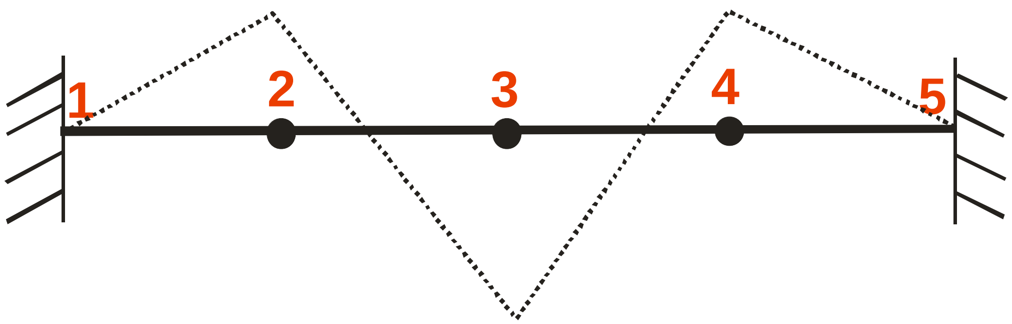
Mode 1



Mode 2



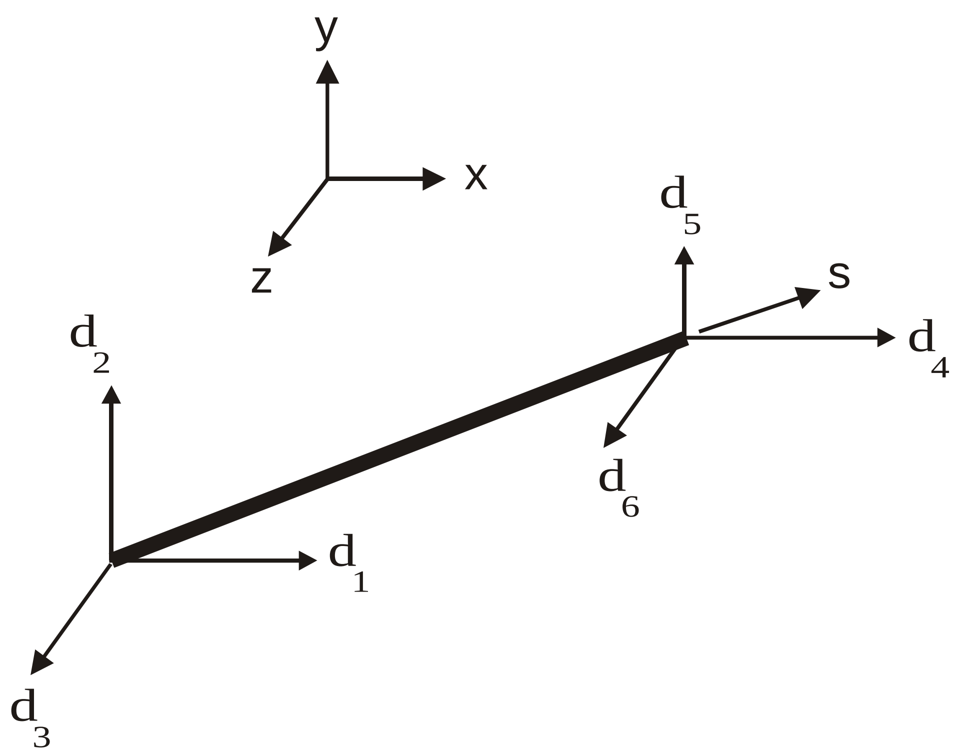
Mode 3



Convergence Study

# of elements	Eigenvalue 1	Eigenvalue 2
4	169.38	782.8
6	162.5	695.4
18	159.3	642.1
24	159.0	639.3
30	159.0	637.7

Truss Element



Consistent mass matrix

$$m_{ij} = \int_{\Omega} \phi_i(x)\gamma(x)\phi_j(x)\,dx$$

Truss Element

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} 1-a & 0 & 0 & a & 0 & 0 \\ 0 & 1-a & 0 & 0 & a & 0 \\ 0 & 0 & 1-a & 0 & 0 & a \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \mathbf{A}_{3\times 6}\mathbf{d}_{6\times 1}$$

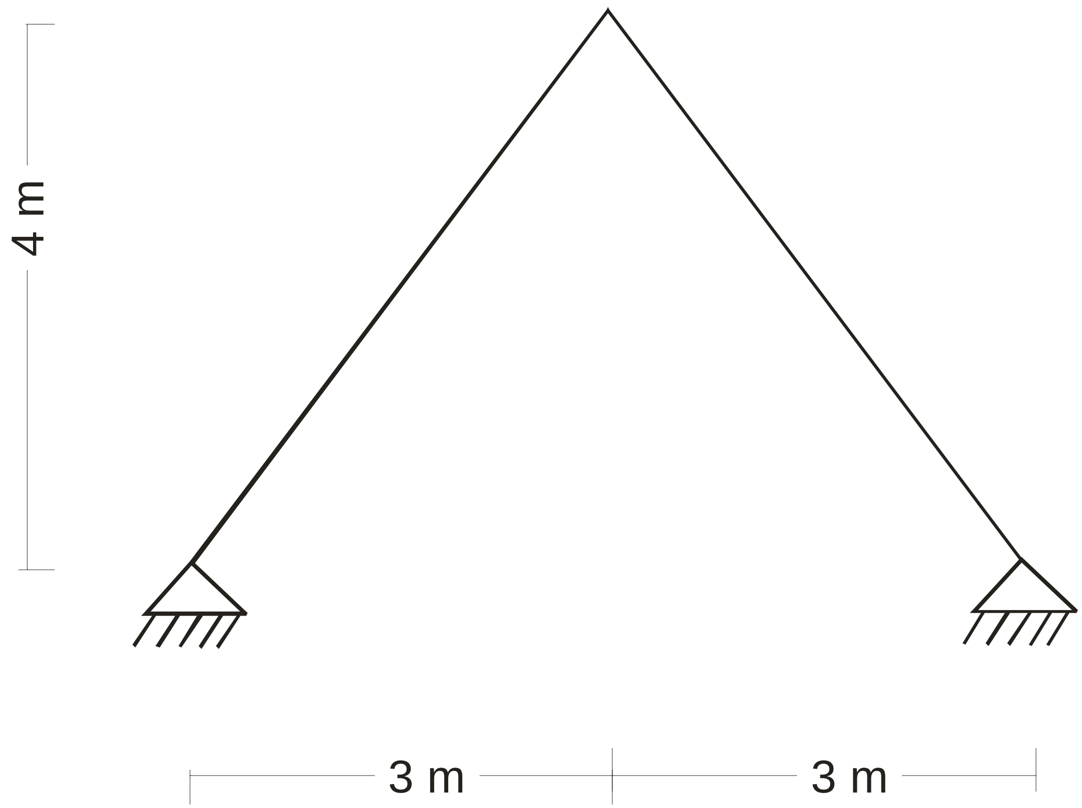
$$\mathbf{m}_{6\times 6} = \int_{\Omega} \gamma \mathbf{A}^T \mathbf{A} dV$$

Truss Element

Consistent mass matrix

$$\mathbf{m}_{6\times 6} = \frac{\gamma AL}{6}\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Example: Ex1



$$E = 200\,GPa$$

$$\rho = 7850\,kg/m^3$$

$$A = 0.01m^2$$

$$f_1 = 167\,Hz$$

$$f_2 = 223\,Hz$$

Beam Element

DE

$$\frac{\partial}{\partial x^2}\left(EI\,\frac{\partial w}{\partial x^2}\right) = -\bar{\rho}\,\frac{\partial^2 w}{\partial t^2}$$

Solution

$$w(x,t) = W(x)\sin\omega t$$

Substituting

$$\frac{d^2}{dx^2}\left[EI\,\frac{d^2 W}{dx^2}\right] - \bar{\rho}\omega^2 W = 0$$

Beam Element

Galerkin Step 1: Residual Equations

$$\int_{x_1}^{x_2} \left[\left(EIW'' \right)'' - \overline{\rho}\omega^2 W \right] \phi_i(x)\, dx = 0 \quad i = 1, 2, 3, 4$$

Galerkin Step 2: Integration by parts

$$\int_{x_1}^{x_2} \left[\left(EIW'' \right)' \phi_i'' - \omega^2 \overline{\rho} \;\; W \phi_i \right] dx = \left[\left(EIW'' \right) \phi_i' \right]_{x_1}^{x_2} - \left[\left(EIW'' \right)' \phi_i \right]_{x_1}^{x_2}$$

Beam Element

Galerkin Step 3: Use of trial solution

$$W = \sum_{j=1}^{4} a_j \phi_j$$

$$\left[\mathbf{k}_{4\times4} - \omega^2 \mathbf{m}_{4\times4} \right] \mathbf{a}_{4\times1} = \begin{Bmatrix} -V(x_1) \\ M(x_1) \\ V(x_2) \\ -M(x_2) \end{Bmatrix}$$

Sign Convention

$$EI \frac{\partial^2 w}{\partial x^2} = -M$$

$$\frac{d}{dx}\left[EI \frac{\partial^2 w}{\partial x^2} \right] = -V$$

Beam Element

Planar beam element (2 dof/element)

$$\mathbf{m}'_{4\times 4} = \frac{\rho A L}{420}\begin{bmatrix} 156 & & SYM & \\ 22L & 4L^2 & & \\ 54 & 13L & 156 & \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

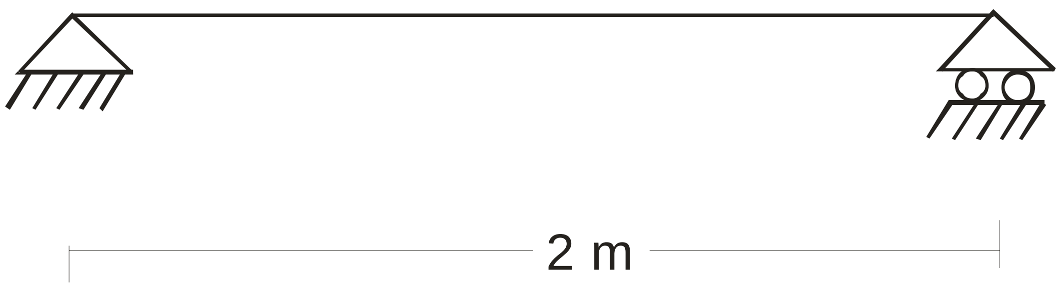
Beam Element

Planar beam element with axial deformation

$$\mathbf{m}'_{6\times 6} = \frac{\rho AL}{420}\begin{bmatrix} 140 & & & & & SYM \\ 0 & 156 & & & & \\ 0 & 22L & 4L^2 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & 13L & 0 & 156 & \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix}$$

$$\mathbf{m}_{6\times 6} = \mathbf{T}^T_{6\times 6}\mathbf{m}'_{6\times 6}\,\mathbf{T}_{6\times 6}$$

Example



$$E = 10^{10}\ Pa$$

$$\rho = 5000\, kg/m^3$$

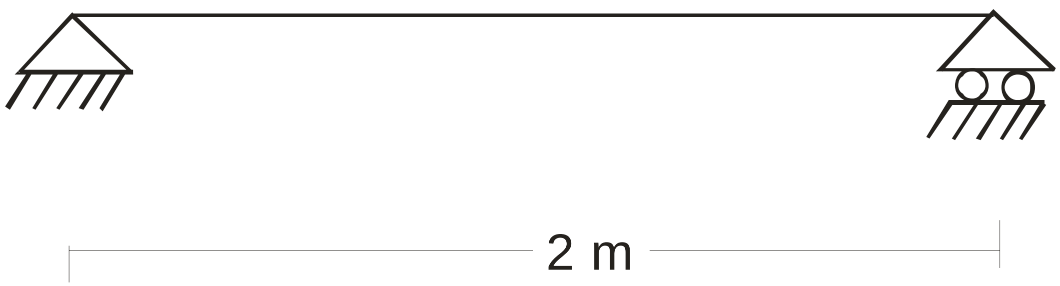
$$A = 0.001 m^2$$

$$I = 0.0001 m^4$$

Analytical solution (bending modes)

$$\omega_n = \frac{(n\pi)^2}{L^2}\sqrt{\frac{EI}{\rho A}} \quad n = 1, 2, 3, \ldots.$$

Example: Ex21 (1 Element)



$$E = 10^{10}\ Pa$$

$$\rho = 5000\, kg/m^3$$

$$A = 0.001 m^2$$

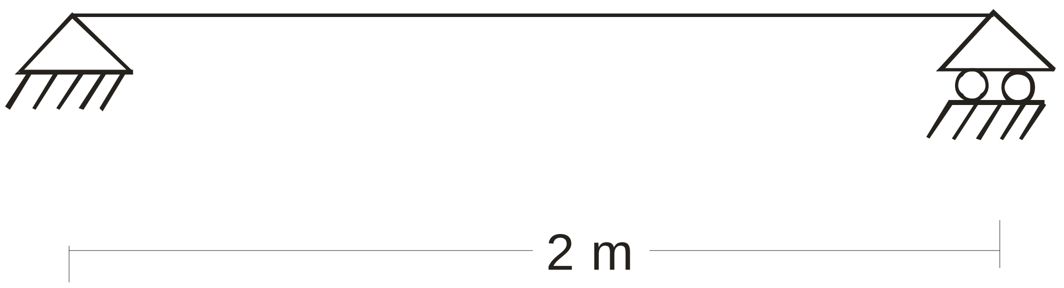
$$I = 0.0001 m^4$$

$$f_1 = 195\, Hz$$

$$f_2 = 195\, Hz$$

$$f_3 = 893\, Hz$$

Example: Ex22 (2 Element)



$$E = 10^{10}\ Pa$$

$$\rho = 5000\, kg/m^3$$

$$A = 0.001 m^2$$

$$I = 0.0001 m^4$$

$$f_1 = 176\, Hz$$

$$f_2 = 181\, Hz$$

$$f_3 = 633\, Hz$$

$$f_4 = 780\, Hz$$

$$f_5 = 1960\, Hz$$

Convergence Study

Mode\ Elements	Frequency (Hz)				
	1	2	4	8	**Exact**
1	195	176	176	176	175.6
2	195	181	177	177	
3	893	633	561	538	
4	-	780	705	703	702.5
5	-	1960	1020	920	

Lumped Mass Matrix

$$\mathbf{m}'_{6\times 6} = \frac{\rho AL}{2}\begin{bmatrix} 1 & & & & & SYM \\ 0 & 1 & & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consistent versus Lumped Mass (8 Elements)

Mode	Frequency (Hz)	
	Consistent	Lumped
1	176	176
2	177	176
3	538	523
4	703	702
5	920	849

Summary

- **K** is symmetric and positive definite
- **M** is symmetric and possibly positive definite
- Eigenvalues are real and positive
- Eigenvalues converge from above

Further Reading

- From the textbook
  - Chapter 11