ELECTRONIC NOTES

CEE598 - Finite Elements for Engineers: Module 2

Part 2: Solid Mechanics

S. D. Rajan
Department of Civil Engineering
Arizona State University
Tempe, AZ 85287-5306

Initial Version: Fall 1998
Revised: Fall 2011

Finite Elements for Engineers

Department of Civil Engineering
Arizona State University
Tempe, AZ 85287-5306
Phone 480.965.1712 • Fax 480.965.0557
e-mail s.rajan@asu.edu

Table of Contents

Topic 3: Problems in Solid Mechanics

"Life is a bridge. Cross over it, but build no house on it." An old Indian proverb

Lesson 1: Discrete Structures – Truss and Beam Elements

Objectives: In this lesson we will look at two commonly used finite elements – the truss and beam elements.

- To understand the truss element formulation.
- To understand the beam element formulation.

Planar Truss Element

The truss element is used to model the truss behavior. The basic assumptions for a system or component to be a truss are as follows.

(a) The members are straight, prismatic and slender.

(b) The members are connected to each other via pins.

(c) The loads are concentrated forces applied at the joints.

As a consequence of these assumptions, the truss members have just an axial force and demonstrate a one-dimensional elemental behavior. While the element geometry and behavior are one-dimensional, the element can be located in a two-dimensional or three-dimensional space. We will derive the element equations using the Theorem of Minimum Potential Energy[1].

With reference to Fig. T3L1-1, the displacement, $u(\xi)$ in the truss element can be assumed as a linear polynomial with the strain, ε , in the element being constant.

$$u(\xi) = \phi_1(\xi)\, d_1^{'} + \phi_2(\xi)\, d_2^{'} = \frac{1-\xi}{2} d_1^{'} + \frac{1+\xi}{2} d_2^{'} \tag{T3L1-1a}$$

$$\frac{du}{d\xi} = \frac{d}{d\xi}\begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{Bmatrix} d_1^{'} \\ d_2^{'} \end{Bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} d_1^{'} \\ d_2^{'} \end{Bmatrix} \tag{T3L1-1b}$$

The geometry of the element is given by

$$x'(\xi) = \phi_1(\xi)\, x_1^{'} + \phi_2(\xi)\, x_2^{'} = \frac{1-\xi}{2} x_1^{'} + \frac{1+\xi}{2} x_2^{'} \tag{T3L1-2a}$$

$$\frac{dx'}{d\xi} = \frac{1}{2}\left(x_2^{'} - x_1^{'}\right) = \frac{L}{2} \tag{T3L1-2b}$$

The strain, ε , in the element can be expressed as

$$\varepsilon = \frac{du}{dx'} = \frac{d\xi}{dx'}\frac{du}{d\xi} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \mathbf{d}^{'}_{2\times 1} = \mathbf{B}_{1\times 2}\, \mathbf{d}^{'}_{2\times 1} \tag{T3L1-3}$$

The term $\mathbf{B}$ is usually called the strain-displacement "matrix" and we will see it used with a variety of finite elements. The stress-strain relationship is given as (a scalar relationship)

---

[1] A check of the topics that we have discussed so far would show that the element discussed in Module 1-Topic 4-Lesson 2 is in fact the truss element.

$$\sigma = E\varepsilon \qquad \text{(T3L1-4)}$$

Hence the strain energy in the truss element can be written as

$$U = \int_V U_0\, dV = \int_0^L \frac{1}{2}\varepsilon\,\sigma A dx = \int_{-1}^{1} \frac{1}{2}\left[\mathbf{B}_{1\times2}\mathbf{d}^{'}_{2\times1}\right]^T E\left[\mathbf{B}_{1\times2}\mathbf{d}^{'}_{2\times1}\right] A\frac{L}{2}d\xi \qquad \text{(T3L1-5)}$$

Simplifying,

$$U = \frac{1}{2}\left[\mathbf{d}'\right]^T_{1\times2}\left[\mathbf{k}'\right]_{2\times2}\left[\mathbf{d}'\right]_{2\times1} \qquad \text{(T3L1-6a)}$$

where $\left[\mathbf{k}'\right]_{2\times2} = \int_{-1}^{1}\mathbf{B}^T_{2\times1}\left[\frac{AEL}{2}\right]_{1\times1}\mathbf{B}_{1\times2}\,d\xi = \frac{AE}{L}\left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1\end{array}\right]$ (T3L1-6b)

is the element stiffness matrix.

The work potential takes place due to concentrated forces acting at the ends of the element and can be written as

$$W = -\left[\mathbf{d}'\right]^T_{2\times1}\left[\mathbf{f}'\right]_{2\times1} \qquad \text{(T3L1-7)}$$

Using the Theorem of Minimum Potential Energy, we have

$$\Pi(\mathbf{d}') = U + W$$

and minimizing Π we have the element equations expressed as

$$\frac{AE}{L}\left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1\end{array}\right]\left\{\begin{array}{c} d_1^{'} \\ \hline d_2^{'}\end{array}\right\} = \left\{\begin{array}{c} r_1^{'} \\ \hline r_2^{'}\end{array}\right\} \qquad \text{(T3L1-8a)}$$

Or, $\quad \mathbf{k}^{'}_{2\times2}\mathbf{d}^{'}_{2\times1} = \mathbf{f}^{'}_{2\times1}$ (T3L1-8b)

where A, E and L are the element cross-sectional area, modulus of elasticity and the length respectively; $d_1^{'}, d_2^{'}$ and $f_1^{'}, f_2^{'}$ are the element nodal displacement and element nodal forces respectively, along the x' (or, axial) direction at nodes 1 and 2. The situation is depicted in Fig. T3L1-1. A few comments are in order.

(a) The right-hand side is different than the Module 1-Eqn. (T4L2-6a). The first term on the RHS is not present in a truss since element loads are not permitted. The second term on the RHS is equivalent to the $\mathbf{f}'$ vector.

(b) This is the first problem we have seen so far where the primary unknown is a vector. The primary unknown in elasticity problems will typically be nodal displacements.

(c) The behavior of the truss element is fundamentally a one-dimensional phenomenon. However, since the different elements in a truss can have different orientations, we need to define the element behavior in a local coordinate system, x'. Quantities that are appropriately described in a local coordinate system are denoted as *primed* $(')$ quantities. To describe the behavior of the truss system that is now a collection of two or more truss elements, we need to define a global coordinate system $X-Y$ that is same for all the elements. Quantities that are appropriately described in a global coordinate system are denoted without any primes $(')$.

(d) Since the truss element is located in the $X-Y$ plane, there are two displacements and two force components at each node of the element. In other words, there are two degrees of freedom at each node leading to a total of four degrees-of-freedom per element.

(e) The displacement in a typical element is linear. Hence the strain and stress in each element are constants.

(f) The $\mathbf{k'}, \mathbf{d'}$ and $\mathbf{f'}$ are the element stiffness matrix, element nodal displacement vector and element nodal force vector in the local coordinate system.

(g) Note the manner in which the displacements (and forces) are numbered. The x-displacement at a node is numbered first followed by the y-displacement.

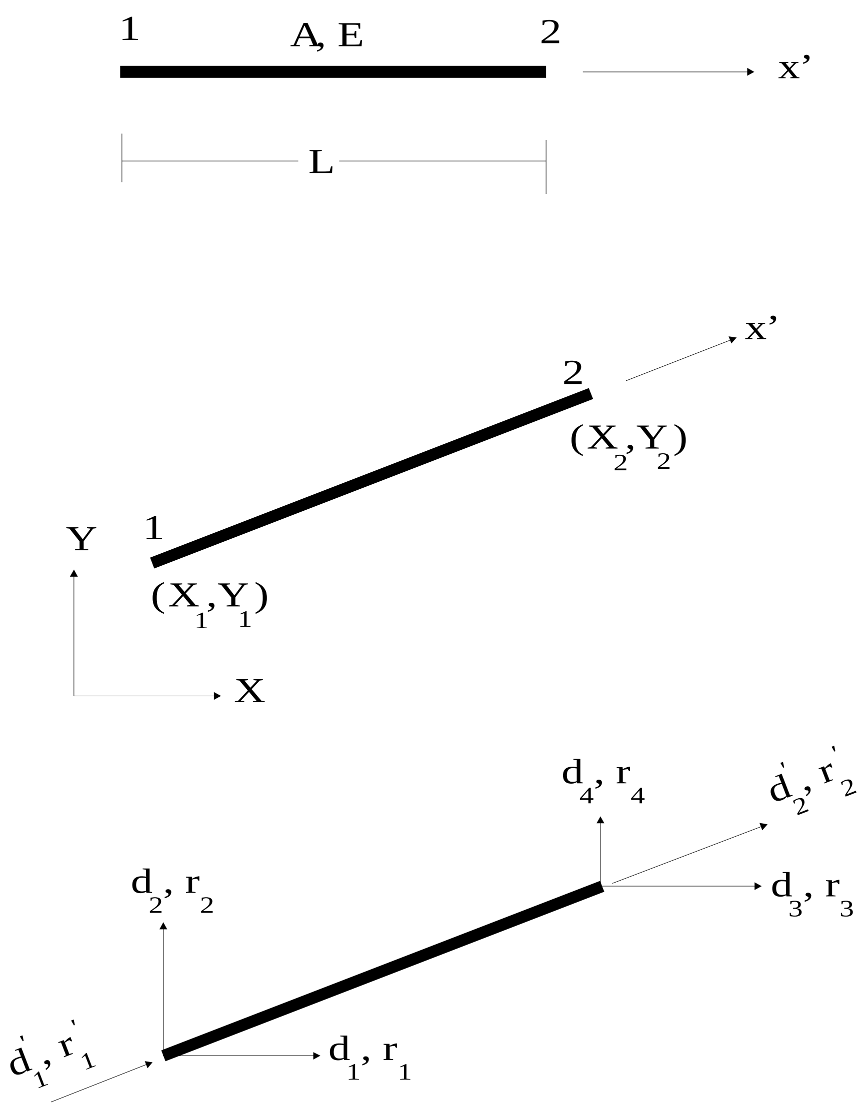


Fig. T3L1-1 Planar Truss Element description

Our next task is to transform Eqn. (T3L1-8b) from the local coordinate system to the common reference frame – the global coordinate system. This can be done by first relating the local and global displacements and forces. Note that the following are valid for small displacements.

$$d_1' = \sqrt{(d_1)^2 + (d_2)^2} \qquad \text{(T3L1-9a)}$$

Or, $$(d_1')^2 = (d_1)^2 + (d_2)^2 \qquad \text{(T3L1-9b)}$$

Or, $$d_1' = \frac{d_1}{d_1'} d_1 + \frac{d_2}{d_1'} d_2 \qquad \text{(T3L1-9c)}$$

Or, $d_1^{'} = l_{x'} d_1 + m_{x'} d_2$ (T3L1-9d)

where $(l_{x'}, m_{x'})$ are the direction cosines of the x' coordinate system with respect to the global coordinate system. Similarly, we can write the equation for the other local displacement as

$$d_2^{'} = l_{x'} d_3 + m_{x'} d_4 \qquad \text{(T3L1-9e)}$$

Note that the direction cosines can be computed as

$$l_{x'} = \frac{X_2 - X_1}{L}, \quad m_{x'} = \frac{Y_2 - Y_1}{L}, \quad L = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} \qquad \text{(T3L1-9f)}$$

Combining Eqns. (T3L1-9d) and (T3L1-9e), we have

$$\mathbf{d}_{2\times1}^{'} = \begin{bmatrix} l_{x'} & m_{x'} & 0 & 0 \\ 0 & 0 & l_{x'} & m_{x'} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} = \mathbf{T}_{2\times4} \mathbf{d}_{4\times1} \qquad \text{(T3L1-10)}$$

Similarly, we can relate the nodal forces as

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{bmatrix} l_{x'} & 0 \\ m_{x'} & 0 \\ 0 & l_{x'} \\ 0 & m_{x'} \end{bmatrix} \begin{Bmatrix} f_1^{'} \\ f_2^{'} \end{Bmatrix} \Rightarrow \mathbf{f}_{4\times1} = \mathbf{T}_{4\times2}^{\mathbf{T}} \; \mathbf{f}_{2\times1}^{'} \qquad \text{(T3L1-11)}$$

Substituting Eqns. (T3L1-10) and (T3L1-11) into (T3L1-8b) we have

$$\mathbf{k}_{4\times4} \mathbf{d}_{4\times1} = \mathbf{f}_{4\times1} \qquad \text{(T3L1-12a)}$$

where $\mathbf{k}_{4\times4} = \mathbf{T}_{4\times2}^{\mathbf{T}} \mathbf{k}_{2\times2}^{'} \mathbf{T}_{2\times4}$ (T3L1-12b)

is the element stiffness matrix in the global coordinate system. As in the case of other solid mechanics examples, Eqn. (T3L1-12a) is the equilibrium-compatibility equation for a typical element.

After the structural equations are solved for the nodal displacements, the strain, ε, stress, σ and axial force, N, in a typical element is computed by first using Eqn. (T3L1-10) to obtain $\mathbf{d'}$ and then

$$\varepsilon = \frac{du}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi}\left(\phi_1 d_1' + \phi_2 d_2'\right) = \frac{2}{L}\left[\frac{1}{2}\left(d_2' - d_1'\right)\right] = \frac{d_2' - d_1'}{L} \qquad \text{(T3L1-13a)}$$

$$\sigma = E\varepsilon \qquad \text{(T3L1-13b)}$$

$$N = \sigma A \qquad \text{(T3L1-13c)}$$

When temperature effects are considered, additional calculations need to be done. First, the load vector must be modified by the addition of the thermal loads as

$$\varepsilon_0 = \alpha \Delta T \qquad \text{(T3L1-14a)}$$

$$\left(\mathbf{f}_t'\right)_{2\times1} = EA\varepsilon_0 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \text{ and } \left(\mathbf{f}_t\right)_{4\times1} = \mathbf{T}_{4\times2}^{\mathbf{T}} \left(\mathbf{f}_t'\right)_{2\times1} \qquad \text{(T3L1-14b)}$$

where ε_0 represents the initial strains, α is the coefficient of thermal expansion, ΔT is the temperature change in the element, $\left(\mathbf{f}_t'\right)_{2\times1}$ is the thermal load vector in the local coordinate system and $\left(\mathbf{f}_t\right)_{4\times1}$ is in the global coordinate system. Second, after the displacements are computed, the stress in the element is now

$$\sigma = E(\varepsilon - \varepsilon_0) \qquad \text{(T3L1-14c)}$$

Example 1: Thermal loading on a truss

Solve for the nodal displacements and member forces for the truss shown below. The modulus of elasticity is $30(10^6)\ psi$, the cross-sectional area of each member is $1.2\,in^2$, the coefficient of thermal expansion is 1/150000 per $^\circ F$, and the temperature change in members AB and BC is $100^\circ F$.

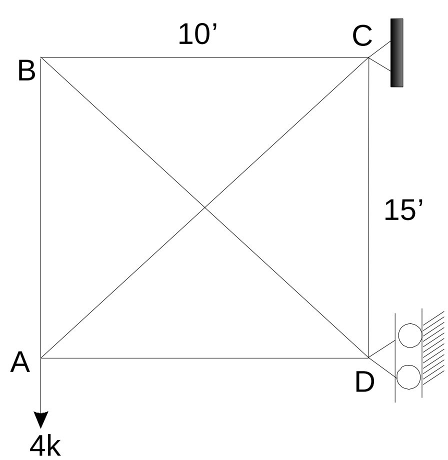


Fig. E1 (a)

Solution

Step 1: The problem units are lb, in. The model is shown in Fig. E1(b).

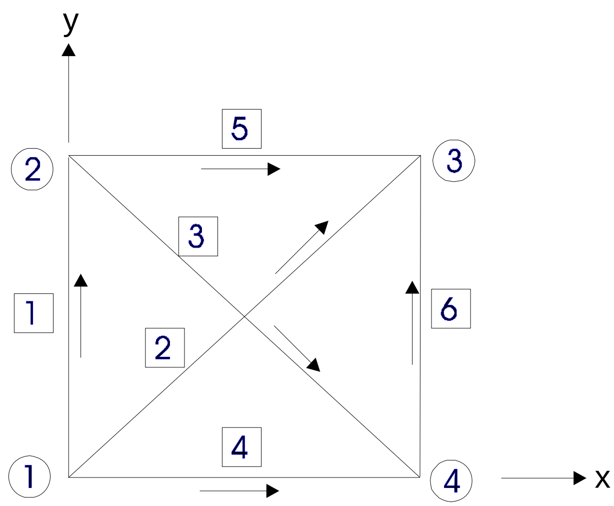


Fig. E1(b)

The global degrees-of-freedom is shown in Fig. E1(c).

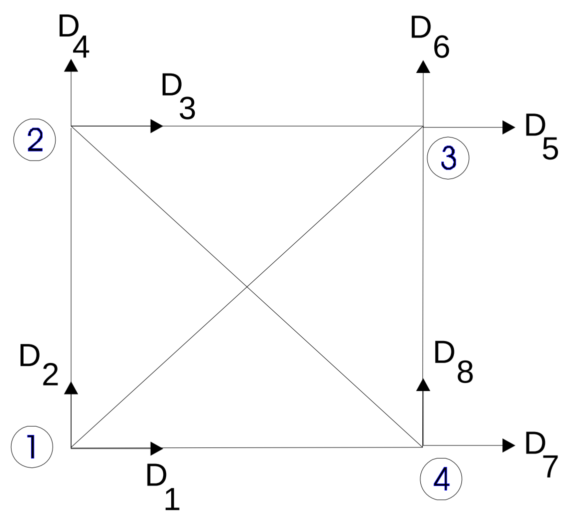


Fig. E1(c)

Th details of the stiffness matrix generation will not be presented here. However, we will focus on the generation of the load vector due to the temperature changes.

Element 1

Using the given data

$$\left(l,m\right) = (0.0,1.0)$$

$$EA = 36(10^6)\ psi$$

$$\varepsilon_0 = \alpha\left(\Delta T\right) = 6.66667(10^{-4})$$

Hence, $\left(\mathbf{q}_t^{'}\right)_{2\times1} = EA\varepsilon_0 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = 24000 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -24000 \\ 24000 \end{Bmatrix}$

Since, $\mathbf{T}_{2\times4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, using Eqn. (T3L1-14b) we have

$$\left(\mathbf{q}_t\right)_{4\times1} = \begin{Bmatrix} 0 \\ -24000 \\ 0 \\ 24000 \end{Bmatrix} lb = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

Element 2

In a similar manner, we have

$$\left(\mathbf{q}_t\right)_{4\times1} = \begin{Bmatrix} -24000 \\ 0 \\ 24000 \\ 0 \end{Bmatrix} lb = \begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

Taking these two load vectors, we can construct the structural nodal load vector as

$$\mathbf{F}_{8\times1} = \begin{Bmatrix} 0 \\ -24000 \\ -24000 \\ 24000 \\ 24000 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \text{and using the external load we have } \mathbf{F}_{8\times1} = \begin{Bmatrix} 0 \\ -28000 \\ -24000 \\ 24000 \\ 24000 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Step 2: Assembly of the System Equations

$$10^5\begin{bmatrix} 3.512 & 0.76805 & 0 & 0 & 0 \\ & 3.1521 & 0 & -2 & 0 \\ & & 3.512 & -0.76805 & 0.76805 \\ Sym & & & 3.5121 & -1.1521 \\ & & & & 3.1521 \end{bmatrix}\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -28000 \\ -24000 \\ 24000 \\ 0 \end{Bmatrix}$$

Step 3: Solution of the System Equations

Solving the system equilibrium equations, we have

$D_1 = 0.0186''$ $\quad$ $D_2 = -0.0851''$ $\quad$ $D_3 = -0.0703''$

$D_4 = 0.0130''$ $\quad$ $D_8 = 0.0219''$

Step 4: Element nodal forces

Now we can compute the net force in each member by multiplying the stress with the element area. We will show the calculations for elements 1 and 2 that have a nonzero second term.

Element 1

Since, $\mathbf{d}_{4\times1} = \{0.0186, -0.0851, -0.0703, 0.0130\}^T$

$$\mathbf{d}'_{2\times1} = \mathbf{T}_{2\times4}\mathbf{d}_{4\times1} = \begin{Bmatrix} -0.0851 \\ 0.0130 \end{Bmatrix} in$$

$$\sigma = 30(10^6)\left[\frac{0.0130-(-0.0851)}{180} - 6.6667(10^{-4})\right] = -3650\ psi$$

The negative sign indicates that the element is in compression.

$$f = \sigma A = 3650(1.2) = 4380\, lb\ \text{(C)}$$

Element 2

Since, $\mathbf{d}_{4\times1} = \{-0.0703, 0.0130, 0, 0\}^T$

$$\mathbf{d}'_{2\times1} = \mathbf{T}_{2\times4}\mathbf{d}_{4\times1} = \begin{Bmatrix} -0.0703 \\ 0 \end{Bmatrix} in$$

$$\sigma = 30(10^6)\left[\frac{0-(-0.0703)}{120} - 6.6667(10^{-4})\right] = -2425\, psi$$

$$f = \sigma A = 2425(1.2) = 2910\, lb \text{ (C)}$$

Space Truss Element

The space truss element has only minor differences compared to the planar truss element. The equations in the local coordinate system do not change. There are three degrees-of-freedom per node and six degrees-of-freedom per element in the global coordinate system.

$$\mathbf{d}'_{2\times1} = \mathbf{T}_{2\times6}\mathbf{d}_{6\times1} \tag{T3L1-15}$$

$$\mathbf{f}_{6\times1} = \mathbf{T}^{\mathbf{T}}_{6\times2}\ \mathbf{f}'_{2\times1} \tag{T3L1-16}$$

$$\mathbf{k}_{6\times6}\mathbf{d}_{6\times1} = \mathbf{f}_{6\times1} \tag{T3L1-17a}$$

$$\mathbf{k}_{6\times6} = \mathbf{T}^{\mathbf{T}}_{6\times2}\mathbf{k}'_{2\times2}\mathbf{T}_{2\times6} \tag{T3L1-17b}$$

$$l_{x'} = \frac{X_2 - X_1}{L}, \quad m_{x'} = \frac{Y_2 - Y_1}{L}, \quad n_{x'} = \frac{Z_2 - Z_1}{L} \tag{T3L1-18a}$$

$$L = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \tag{T3L1-18b}$$

$$\mathbf{T}_{2\times6} = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{x'} & m_{x'} & n_{x'} \end{bmatrix} \tag{T3L1-19}$$

Planar Beam Element

The beam behavior illustrated in this section is one that includes axial, shear and moment effects. In that sense, the element is a beam-frame element in the traditional parlance. The basic assumptions are as follows.

(a) The member is prismatic and slender.

(b) The deflections are small.

(c) Plane sections remain plane.

This beam is also referred to as the Euler-Bernoulli beam. Fig. T3L1-2 shows a simply-supported beam subjected to transverse loads.

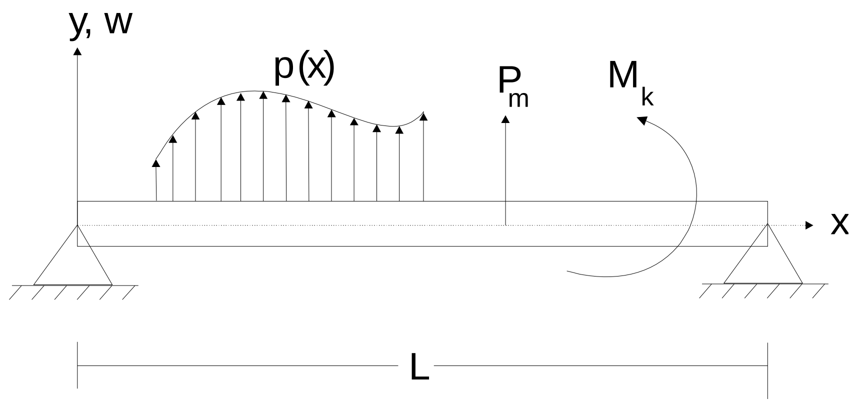


Fig. T3L1-2 Planar beam subjected to different loads

From elementary beam theory (compression is negative)

$$\sigma_x = -\frac{M_z y}{I_z} \tag{T3L1-20a}$$

$$\sigma_x = E\varepsilon_x \tag{T3L1-20b}$$

$$\frac{d^2 w(x)}{dx^2} = \frac{M_z}{EI_z} \tag{T3L1-20c}$$

where M_z is the moment, E is the modulus of elasticity, w is the transverse deflection of the centroidal axis, and I_z is the moment of inertia about the centroidal axis. We will drop the subscripts in the next step. The strain energy in the beam is given by (neglecting shear strain energy)

$$U = \int_V U_0 \, dV = \int_0^L \int_A \frac{1}{2} \varepsilon\, \sigma \, dA \, dx = \frac{1}{2}\int_0^L \left[\frac{M^2}{EI^2} \int_A y^2 \, dA \right] dx \tag{T3L1-21}$$

Noting that $I = \int_A y^2 \, dA$, we have

$$U = \frac{1}{2}\int_0^L EI \left(\frac{d^2 w}{dx^2} \right)^2 dx \tag{T3L1-22}$$

The total potential energy in the beam is given by

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L pw\,dx - \sum_m P_m w_m - \sum_k M_k \frac{dw}{dx} \qquad \text{(T3L1-23)}$$

We are now ready to build a typical finite element and computer its potential energy. Fig. T3L1-3 shows the two degrees of freedom at any point on the beam (and the beam element).

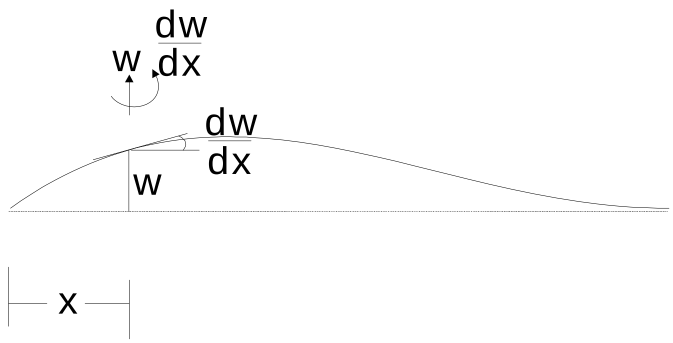


Fig. T3L1-3 Deformation of the neutral axis showing the two dof at any point

Fig. T3L1-4 shows a typical beam element. It is described by two nodes and four degrees-of-freedom. Since the degrees-of-freedom involve the transverse displacement and its derivative, the Lagrange interpolation scheme cannot be used. The Hermite interpolation scheme must be used.

Assuming that the displacement varies as a cubic polynomial, a typical shape function is of the form

$$H_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3 \qquad i = 1,..,4 \qquad \text{(T3L1-24a)}$$

$$H_i^{'} = b_i + 2c_i\xi + 3d_i\xi^2 \qquad i = 1,..,4 \qquad \text{(T3L1-24b)}$$

To find the four shape functions, we must use the end conditions.

i	$H_i(\xi=-1)$	$H_i^{'}(\xi=-1)$	$H_i(\xi=+1)$	$H_i^{'}(\xi=+1)$
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1

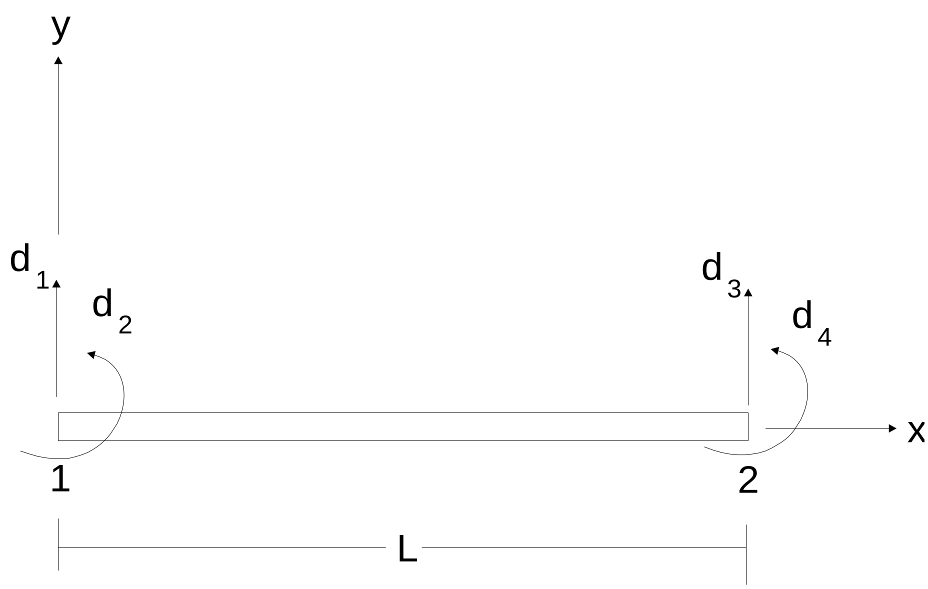


Fig. T3L1-4 Typical beam element description

Using these conditions, the four shape functions are computed as

$$H_1 = \frac{1}{4}(1-\xi)^2(2+\xi) \qquad H_1' = \frac{1}{4}(-3+3\xi^2) \tag{T3L1-25a}$$

$$H_2 = \frac{1}{4}(1-\xi)^2(1+\xi) \qquad H_2' = \frac{1}{4}(-1-2\xi+3\xi^2) \tag{T3L1-25b}$$

$$H_3 = \frac{1}{4}(1+\xi)^2(2-\xi) \qquad H_3' = \frac{1}{4}(3-3\xi^2) \tag{T3L1-25c}$$

$$H_4 = \frac{1}{4}(1+\xi)^2(\xi-1) \qquad H_4' = \frac{1}{4}(-1+2\xi+3\xi^2) \tag{T3L1-25d}$$

A cubic polynomial has four coefficients. The beam element as assumed here has four degrees of freedom. Hence the choice of a cubic polynomial for the transverse displacement.

$$w(\xi) = H_1 w_1 + H_2\left(\frac{dw}{d\xi}\right)_1 + H_3 w_2 + H_4\left(\frac{dw}{d\xi}\right)_2 \tag{T3L1-26}$$

The geometry is interpolated as

$$x = \frac{1-\xi}{2}x_1 + \frac{1+\xi}{2}x_2 \tag{T3L1-27a}$$

$$dx = \frac{L}{2} d\xi \tag{T3L1-27b}$$

This element is a $1D - C^1$ cubic subparametric element. The element is C^1 because the displacement and the <u>first</u> derivative of the displacement (slope) are both continuous across element boundaries. Using the preceding equations, we have

$$\frac{dw}{d\xi} = \frac{dw}{dx}\frac{dx}{d\xi} = \frac{dw}{dx}\frac{L}{2} \tag{T3L1-28}$$

$$w(\xi) = H_1 d_1 + \frac{L}{2} H_2 d_2 + H_3 d_3 + \frac{L}{2} H_4 d_4 = \mathbf{H}_{1\times4}\mathbf{d}_{4\times1} \tag{T3L1-29}$$

where $\mathbf{H}_{1\times4} = \left\{ H_1, \frac{L}{2} H_2, H_3, \frac{L}{2} H_4 \right\}$.

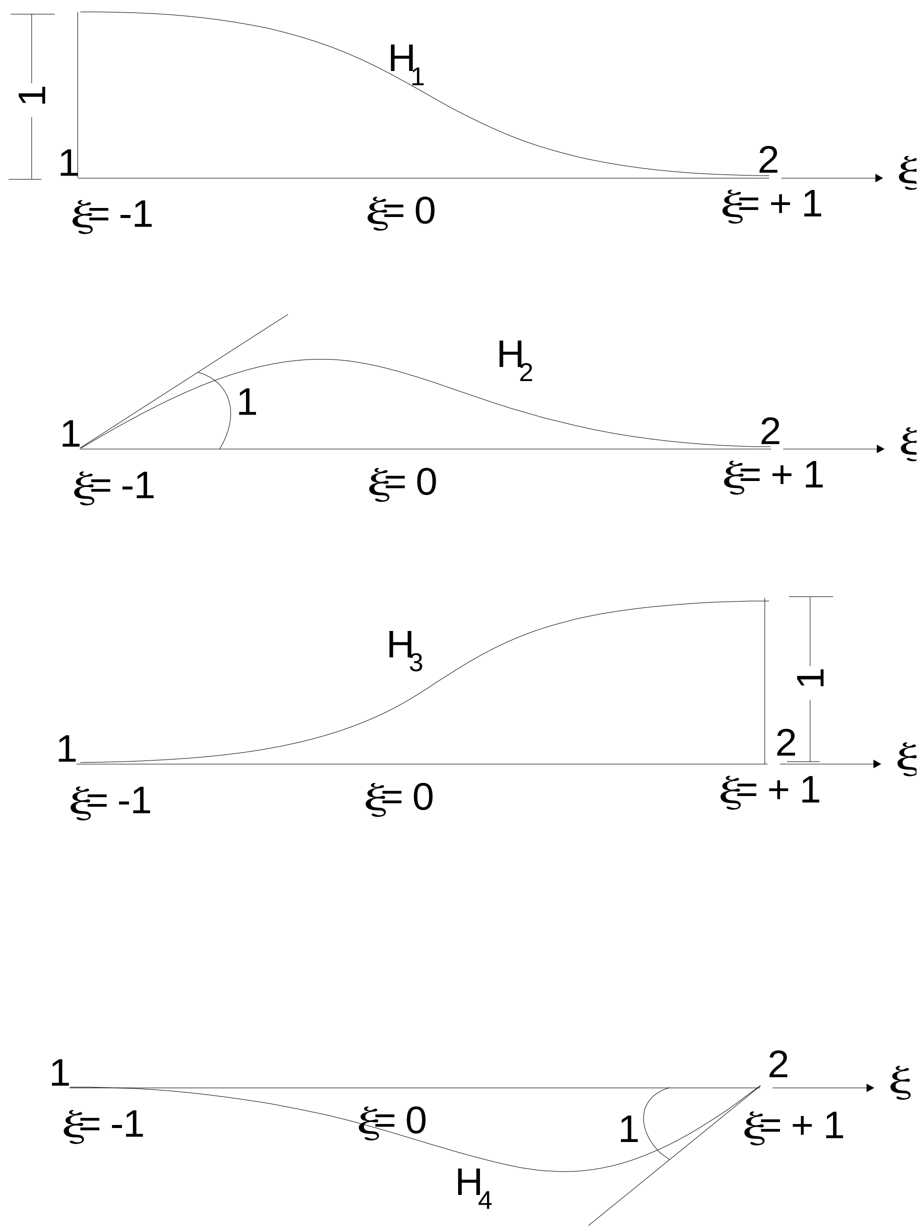


Fig. T3L1-5 Hermite cubic shape functions

Now using Eqn. (T3L1-22) to compute the strain energy in a typical element, we note that

$$\frac{dw}{dx} = \frac{2}{L}\frac{dw}{d\xi} \quad \text{and} \quad \frac{d^2w}{dx^2} = \frac{4}{L^2}\frac{d^2w}{d\xi^2} \tag{T3L1-30}$$

and
$$\left(\frac{d^2w}{dx^2}\right)^2 = \mathbf{d}^T \frac{16}{L^4}\left(\frac{d^2\mathbf{H}}{d\xi^2}\right)^{\mathbf{T}}\left(\frac{d^2\mathbf{H}}{d\xi^2}\right)\mathbf{d} \tag{T3L1-31}$$

$$\left(\frac{d^2\mathbf{H}}{d\xi^2}\right)=\left[\frac{3}{2}\xi,\frac{-1+3\xi}{2}\frac{L}{2},-\frac{3}{2}\xi,\frac{1+3\xi}{2}\frac{L}{2}\right] \quad \text{(T3L1-32)}$$

Substituting these expressions into Eqn. (T3L1-22), we obtain

$$U=\frac{1}{2}\mathbf{d}^{\mathbf{T}}\left[\frac{8EI}{L^3}\int_{-1}^{1}\begin{bmatrix}\frac{9}{4}\xi^2 & \frac{3}{8}\xi(-1+3\xi)L & -\frac{9}{4}\xi^2 & \frac{3}{8}\xi(1+3\xi)L \\ & \left(\frac{-1+3\xi}{4}\right)^2L^2 & -\frac{3}{8}\xi(-1+3\xi)L & \frac{-1+9\xi^2}{16}L^2 \\ SYM & & \frac{9}{4}\xi^2 & -\frac{3}{8}\xi(1+3\xi)L \\ & & & \left(\frac{1+3\xi}{4}\right)^2L^2\end{bmatrix}d\xi\right]\mathbf{d}$$

$$=\frac{1}{2}\mathbf{d}^{\mathbf{T}}_{1\times 4}\mathbf{k}_{4\times 4}\mathbf{d}_{4\times 1} \quad \text{(T3L1-33)}$$

Hence, $$\mathbf{k}_{4\times 4}=\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ SYM & & 12 & -6L \\ & & & 4L^2\end{bmatrix} \quad \text{(T3L1-34)}$$

While the prime notation has not been used, the above derivation is for the quantities in the local coordinate system. Moreover, it does **not** include axial effects. The inclusion of axial effects is quite simple. Fig. T3L1-6 shows the general beam element. With the assumptions made at the beginning of the section, the axial effects are independent of the bending effects. The general beam element is the linear superposition of the axial behavior captured by truss element and the bending behavior captured by the beam element. The following should be noted about the element description.

(a) The element lies in the $X-Y$ plane. The coordinate systems are such that the local z' and the global Z coincide. They are obtained by taking the cross product of the local x' and y' axes. To find the direction cosines of the x' axis we can employ the following expressions

$$l_{x'}=\frac{X_2-X_1}{L},\quad m_{x'}=\frac{Y_2-Y_1}{L},\quad L=\sqrt{(X_2-X_1)^2+(Y_2-Y_1)^2} \quad \text{(T3L1-35)}$$

Similarly, the direction cosines of the y' axis can be written as

$$l_{y'} = -m_{x'}, \quad m_{y'} = l_{x'} \tag{T3L1-36}$$

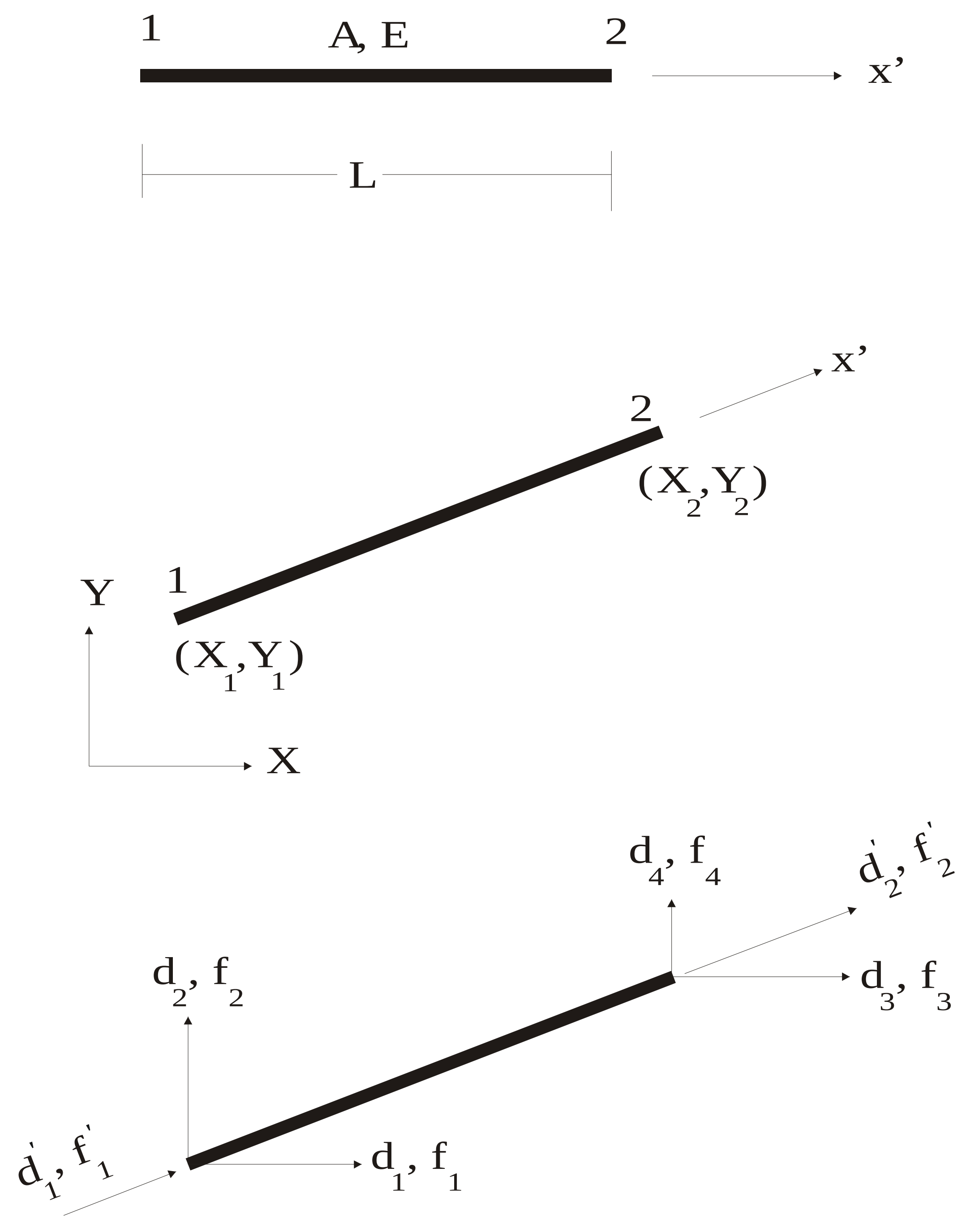


Fig. T3L1-6 General beam element description

(b) There are six degrees-of-freedom in the element in the local $\left(d_1^{'}, d_2^{'}, d_3^{'}, d_4^{'}, d_5^{'}, d_6^{'}\right)$ and the global $\left(d_1, d_2, d_3, d_4, d_5, d_6\right)$ coordinate systems. The three nodal forces at each node in the local coordinate system refer to the axial force $\left(f_1^{'}, f_4^{'}\right)$, the shear force $\left(f_2^{'}, f_5^{'}\right)$ and the bending moment $\left(f_3^{'}, f_6^{'}\right)$. The global forces, in general, cannot be classified. The local and global displacements and forces are related to each other as follows.

$$\mathbf{d}^{'}_{6\times1} = \begin{bmatrix} l_{x'} & m_{x'} & 0 & 0 & 0 & 0 \\ l_{y'} & m_{y'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{x'} & m_{x'} & 0 \\ 0 & 0 & 0 & l_{y'} & m_{y'} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \mathbf{T}_{6\times6}\mathbf{d}_{6\times1} \tag{T3L1-37}$$

$$\mathbf{f}_{6\times1} = \mathbf{T}^{\mathbf{T}}_{6\times6}\mathbf{f}_{6\times1} \tag{T3L1-38}$$

(c) The element local stiffness matrix is obtained by combining Eqn. (T3L1-6b) and (T3L1-34) and is given as

$$\mathbf{k}'_{6\times6} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ & & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ SYM & & & \frac{EA}{L} & 0 & 0 \\ & & & & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ & & & & & \frac{4EI}{L} \end{bmatrix} \tag{T3L1-39}$$

The element global stiffness matrix is obtained similar to the truss element

$$\mathbf{k}_{6\times6} = \mathbf{T}^{\mathbf{T}}_{6\times6}\mathbf{k'}_{6\times6}\,\mathbf{T}_{6\times6} \tag{T3L1-40}$$

The equivalent nodal forces due to loads acting on the element can be found as follows. For example, if a uniformly distributed load of intensity p acts in the positive y' direction, the equivalent nodal forces are computed as

$$\mathbf{f}'_{6\times1} = \int_0^L p(x)H_i(x)dx = \int_{-1}^{1} p(x(\xi))\,H_i(\xi)\,J\,d\xi = p\int_{-1}^{1} H_i(\xi)\,J\,d\xi \tag{T3L1-41a}$$

Using the expressions for $\mathbf{H}_{1\times4}$ from Eqn. (T3L1-29), we have

$$\mathbf{f}'_{6\times1} = \left[0, \frac{pL}{2}, \frac{pL^2}{12}, 0, \frac{pL}{2}, -\frac{pL^2}{12}\right]^{\mathbf{T}} \tag{T3L1-41b}$$

These equivalent nodal forces can then be transformed from the local to the global coordinate system

$$\mathbf{f}_{6\times1} = \mathbf{T}^{\mathbf{T}}_{6\times6}\mathbf{f}'_{6\times1} \tag{T3L1-41c}$$

and added to the system nodal force vector.

(d) Once the system or global equilibrium equations are solved for the nodal displacements, the member nodal forces can be computed as

$$\mathbf{d'}_{6\times1} = \mathbf{T}_{6\times6}\,\mathbf{d}_{6\times1} \tag{T3L1-42}$$

$$\mathbf{f}'_{6\times1} = \mathbf{k'}_{6\times6}\,\mathbf{d'}_{6\times1} - \sum_i \left(\mathbf{f}'_{6\times1}\right)_i \tag{T3L1-43}$$

where the summation is over all the element loads acting on the element. The last term is necessary to satisfy element equilibrium since the element is subjected to element loads. Note that the strains and stresses can be computed only if the cross-sectional shape is known.

Example 2: Planar Frame

Consider the frame shown below. The modulus of elasticity is $2(10^{11})\,Pa$, the cross-sectional area is $0.01m^2$ and the moment of inertia is $0.0001m^4$ for both the members. Compute the member nodal forces.

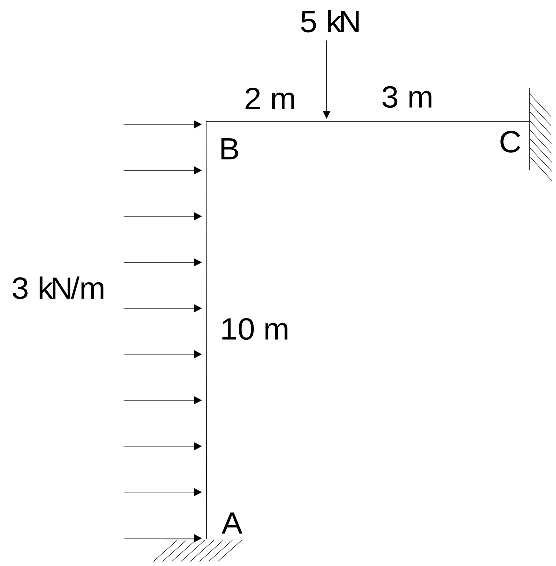


Fig. E2(a)

Solution

Step 1: The problem units are N, m. We will select the origin of the coordinate system at A. The node and element numbers are shown below. We will also number the global degrees-of-freedom at the nodes. As can be seen from the figure, there are a total of nine degrees-of-freedom in the frame. However, the boundary conditions of the frame are such that

$$D_1 = D_2 = D_3 = D_7 = D_8 = D_9 = 0$$

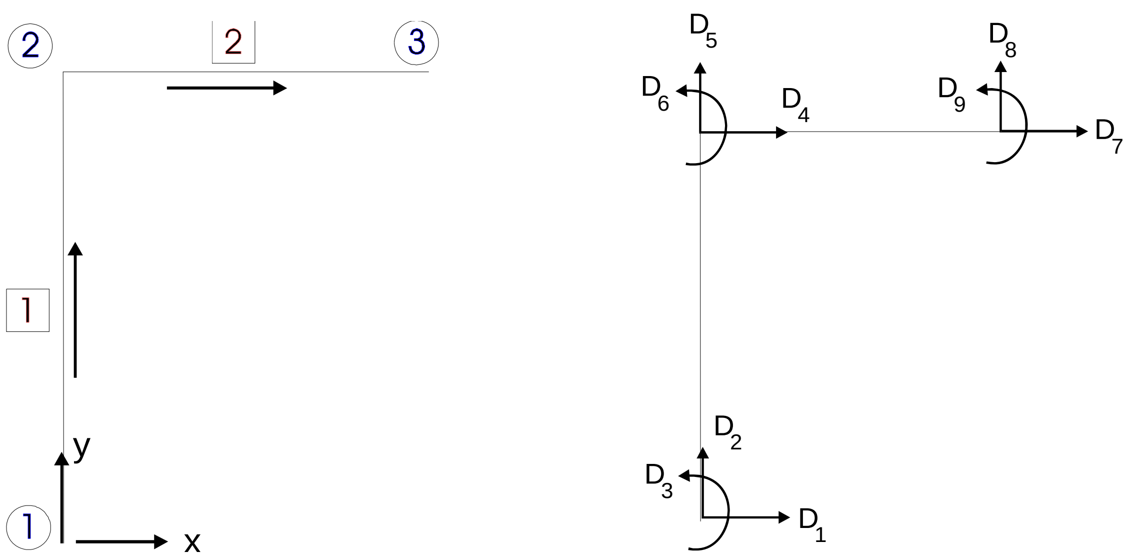


Fig. E2(b)

Element Load on Element 1: With $w = -3000\,N/m$ (see Fig. 6.2.3-9)

$$\mathbf{q}'_{6\times1} = \left\{0, \frac{wL}{2}, \frac{wL^2}{12}, 0, \frac{wL}{2}, -\frac{wL^2}{12}\right\} = \{0,15000,-25000,0,15000,25000\}$$

Element Load on Element 2: With $P = -5000\,N$, $a = 2\,m, b = 3\,m$

$$\mathbf{q}'_{6\times1} = \left\{0, \frac{Pb^2(L+2a)}{L^3}, \frac{Pab^2}{L^2}, 0, \frac{Pa^2(L+2b)}{L^3}, -\frac{Pa^2b}{L^2}\right\} = \{0,-3240,-3600,0,-1760,2400\}$$

These loads need to be transformed to the global coordinate system.

Step 2: The element equilibrium equations

We can use the results from Step 1 to generate the element equilibrium equations for each element.

Element 1

$$10^5\begin{bmatrix} 24 & 0 & -12 & -2.4 & 0 & -12 \\ 0 & 2000 & 0 & 0 & -2000 & 0 \\ -12 & 0 & 80 & 12 & 0 & 40 \\ -2.4 & 0 & 12 & 2.4 & 0 & 12 \\ 0 & -2000 & 0 & 0 & 2000 & 0 \\ -12 & 0 & 40 & 12 & 0 & 80 \end{bmatrix}\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 15000 \\ 0 \\ -25000 \\ 15000 \\ 0 \\ 25000 \end{Bmatrix}$$

Element 2

$$10^5\begin{bmatrix} 4000 & 0 & 0 & -4000 & 0 & 0 \\ 0 & 19.2 & 48 & 0 & -19.2 & 48 \\ 0 & 48 & 160 & 0 & -48 & 80 \\ -4000 & 0 & 0 & 4000 & 0 & 0 \\ 0 & -19.2 & -48 & 0 & 19.2 & -48 \\ 0 & 48 & 80 & 0 & -48 & 160 \end{bmatrix}\begin{Bmatrix} D_4 \\ D_5 \\ D_6 \\ D_7 \\ D_8 \\ D_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -3240 \\ -3600 \\ 0 \\ -1760 \\ 2400 \end{Bmatrix}$$

Step 3: Assembly of the system equations $\mathbf{K}_{3\times3}\mathbf{D}_{3\times1} = \mathbf{F}_{3\times1}$

We will assemble only the effective equations.

$$10^5\begin{bmatrix} 4002.4 & 0 & 12 \\ 0 & 2019.2 & 48 \\ 12 & 48 & 240 \end{bmatrix}\begin{Bmatrix} D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 15000 \\ -3240 \\ 21400 \end{Bmatrix}$$

Note that both the element stiffness matrix and the system stiffness matrix are symmetric.

Step 4: Solution of the equilibrium equations

Solving the three equations, we have

$$D_4 = 3.48(10^{-5})\, m$$

$$D_5 = -3.74(10^{-5})\, m$$

$$D_6 = 8.97(10^{-4})\, rad$$

Step 5: Computation of element nodal forces

Element 1

$$\mathbf{f}'_{6\times 1} = \{7476, 16085, 28631, -7476, 13915, -17779\} \text{ N}$$

Element 2

$$\mathbf{f}'_{6\times 1} = \{13915, 7476, 17779, -13915, -2476, 4600\} \text{ N}$$

The element FBDs are shown below.

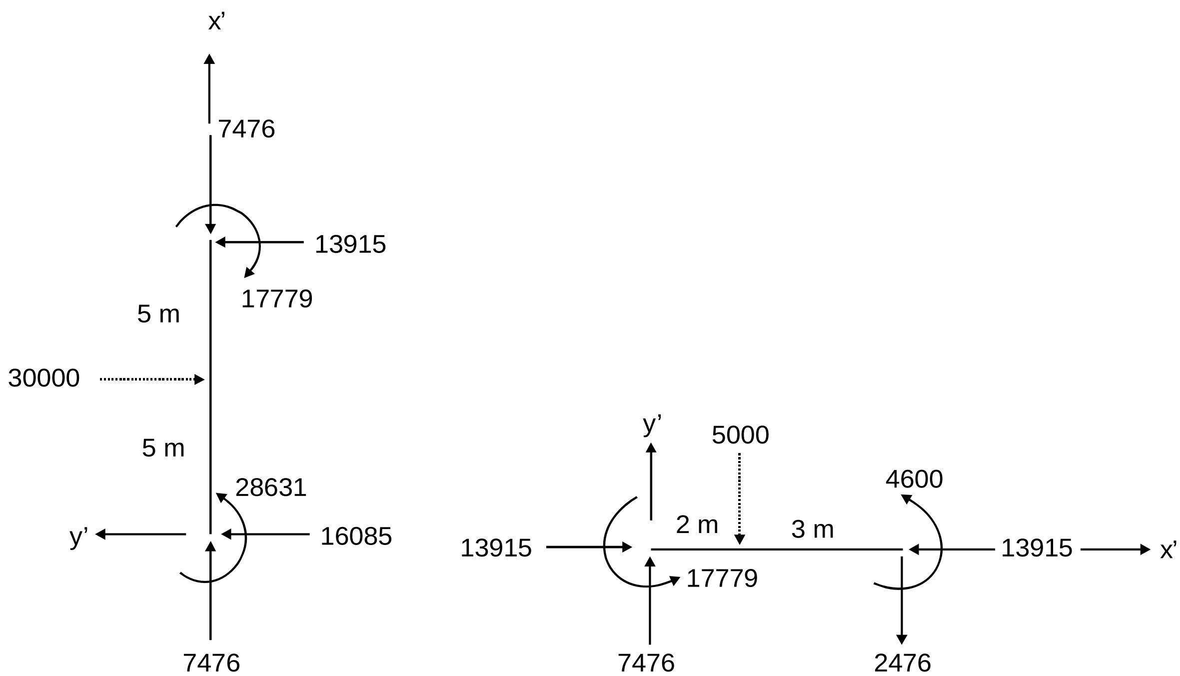


Fig. E2(c)

Space Beam Element

The space beam element is quite a bit different than the planar beam element. There are twelve degrees-of-freedom in the element with six degrees-of-freedom per node. The element is shown in Fig. T3L1-7. The composite behavior of the element is a superposition of the following effects -

(a) Axial deformation along x',

(b) Bending about the y' and z' axes, and

(c) Torsional deformation (rotation) about the x' axis.

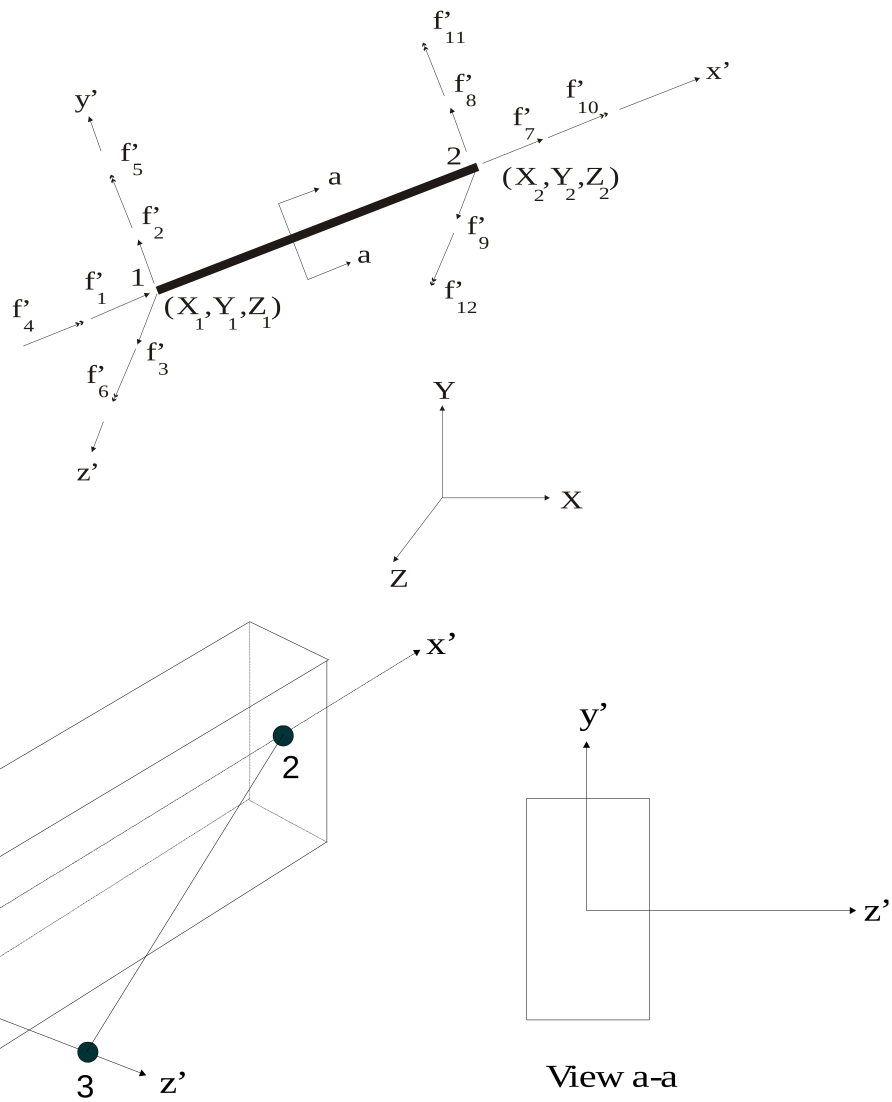


Fig. T3L1-7 (a) Orientation (b) Local coordinate system (c) View a-a

The x' axis is the centroidal axial axis. The y' and z' axes are the principal axes with the $x'-z'$ plane as the major principal plane of bending and $x'-y'$ plane as the minor principal plane of bending. $\left(f_1^{'},\ f_7^{'}\right)$ are the axial forces, $\left(f_4^{'},\ f_8^{'}\right)$ are the torsional moments, $\left(f_2^{'},\ f_8^{'}\right)$ are the shear forces in the y' direction, $\left(f_3^{'},\ f_9^{'}\right)$ are the shear forces in the z' direction, $\left(f_5^{'},\ f_{11}^{'}\right)$ are the bending moments about the y' axis, and $\left(f_6^{'},\ f_{12}^{'}\right)$ are the bending moments about the z' axis. To define the orientation of the element, we need an additional point. Node 3 is known as the reference point. In this formulation, the purpose of the specifying the third point is to define the (major) principal plane of bending. While point 3 can be placed anywhere on the principal plane, we will place point 3 on the z' axis so that $1-3$ points in the positive z' direction.

The element stiffness matrix can now be computed and expressed as follows.

$$\mathbf{k'}_{12\times12} = \left[\begin{array}{c|c} \mathbf{k}_{11} & \mathbf{k}_{12} \\ \hline \mathbf{k}_{21} & \mathbf{k}_{22} \end{array}\right]_{12\times12} \qquad \text{where} \qquad \text{(T3L1-45)}$$

$$\mathbf{k}_{11} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ & & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\ & & & \frac{GJ}{L} & 0 & 0 \\ & SYM & & & \frac{4EI_y}{L} & 0 \\ & & & & & \frac{4EI_z}{L} \end{bmatrix} \qquad \text{(T3L1-46a)}$$

$$\mathbf{k}_{22} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\ & & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ & & & \frac{GJ}{L} & 0 & 0 \\ & SYM & & & \frac{4EI_y}{L} & 0 \\ & & & & & \frac{4EI_z}{L} \end{bmatrix} \qquad \text{(T3L1-46b)}$$

$$\mathbf{k}_{12} = \mathbf{k}_{21}^{\mathrm{T}} = \begin{bmatrix} -\frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\ 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \end{bmatrix} \qquad \text{(T3L1-47)}$$

The local-to-global transformation matrix $\mathbf{T}_{12\times12}$ can be constructed as

$$\mathbf{T}_{12\times12} = \left[\begin{array}{c|c|c|c} \mathbf{\Lambda} & & & \\ \hline & \mathbf{\Lambda} & & \\ \hline & & \mathbf{\Lambda} & \\ \hline & & & \mathbf{\Lambda} \end{array}\right] \qquad \mathbf{\Lambda}_{3\times3} = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \\ l_{z'} & m_{z'} & n_{z'} \end{bmatrix} \qquad \text{(T3L1-48)}$$

$$L = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \qquad \text{(T3L1-49a)}$$

$$\mathbf{e}_{x'} = [l_{x'} \quad m_{x'} \quad n_{x'}] \Rightarrow l_{x'} = \frac{X_2 - X_1}{L}, \quad m_{x'} = \frac{Y_2 - Y_1}{L}, \quad n_{x'} = \frac{Z_2 - Z_1}{L} \qquad \text{(T3L1-49b)}$$

$$\mathbf{e}_{13} = \frac{X_3 - X_1}{L_{13}}\hat{i} + \frac{Y_3 - Y_1}{L_{13}}\hat{j} + \frac{Z_3 - Z_1}{L_{13}}\hat{k} \qquad \text{(T3L1-49c)}$$

$$L_{13} = \sqrt{(X_3 - X_1)^2 + (Y_3 - Y_1)^2 + (Z_3 - Z_1)^2} \qquad \text{(T3L1-49d)}$$

$$\mathbf{e}_{y'} = [l_{y'} \quad m_{y'} \quad n_{y'}] \Rightarrow \mathbf{e}_{y'} = \mathbf{e}_{13} \times \mathbf{e}_{x'} \qquad \text{(T3L1-49e)}$$

$$\mathbf{e}_{z'} = [l_{z'} \quad m_{z'} \quad n_{z'}] \Rightarrow \mathbf{e}_{z'} = \mathbf{e}_{13} \qquad \text{(T3L1-49f)}$$

The rest of the computations including computation of the equivalent nodal forces, nodal forces (or, element stress resultants) etc. are carried out in a manner similar to those described for the planar beam element.

Multi-Point Constraints

Skew supports, or inclined rollers (see Fig. T3L1-8) can be handled as a special case of what is generally referred to as multi-point constraints (MPCs). Consider a case where there exists a known relationship between two different degrees-of-freedom D_i and D_j. Let the equation that represents the relationship be

$$c_i D_i + c_j D_j = c \qquad \text{(T3L1-50)}$$

where c_i, c_j and c are known constants. The total potential energy for a structure with the MPC is

$$\Pi(\mathbf{D}) = \frac{1}{2}\mathbf{D}^{\mathbf{T}}\mathbf{K}\mathbf{D} - \mathbf{D}^{\mathbf{T}}\mathbf{F} + \frac{1}{2}C\left(c_i D_i + c_j D_j - c\right)^2 \qquad \text{(T3L1-51)}$$

where C is a large number.

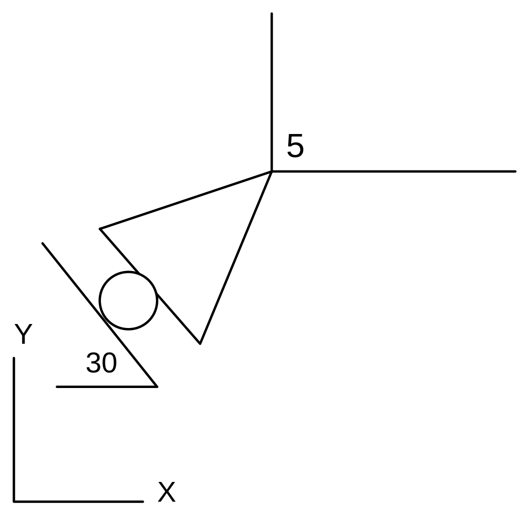


Fig. T3L1-8 Skew roller support

Note that Π takes on a minimum value when $c_i D_i + c_j D_j - c$ is zero (or, numerically very small). Using the Theorem of Minimum Potential Energy, $\partial\Pi/\partial D = 0$ yields the usual equilibrium equations except for the rows and columns dealing with D_i and D_j. The usual and the modified terms are shown below.

$$\begin{bmatrix} K_{ii} & K_{ij} \\ \\ K_{ji} & K_{jj} \end{bmatrix} \rightarrow \begin{bmatrix} K_{ii} + Cc_i^2 & K_{ij} + Cc_i c_j \\ \\ K_{ji} + Cc_i c_j & K_{jj} + Cc_j^2 \end{bmatrix} \tag{T3L1-52}$$

$$\begin{Bmatrix} F_i \\ \\ F_j \end{Bmatrix} \rightarrow \begin{Bmatrix} F_i + Ccc_i \\ \\ F_j + Ccc_j \end{Bmatrix} \tag{T3L1-53}$$

Note that the modified equations $\mathbf{KD} = \mathbf{F}$ containing terms from Eqns. (T3L1-52) and (T3L1-53) are still symmetric and positive definite. The only question left to answer is what is the suitable value for the large number C. A popular choice that seems to work effectively, is to make the constant a function of the largest element in the structural stiffness matrix.

$$C = 10^4 \max\left|K_{pq}\right|,\ 1 \le p, q \le n \tag{T3L1-54}$$

Example 3: Simply supported beam with skew support

A simply-supported beam $(E = 273600\, k/ft^2)$ with an inclined roller support at one end is shown Fig. E3(a). The cross-section is rectangular of width $8"$ by height $20"$. Compute all the support reactions.

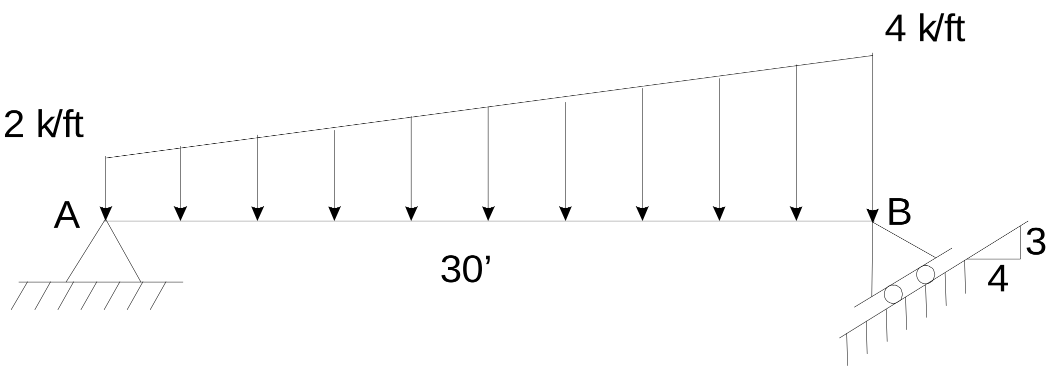


Fig. E3(a)

Solution

Step 1: The problem units are k, ft. The model is shown below.

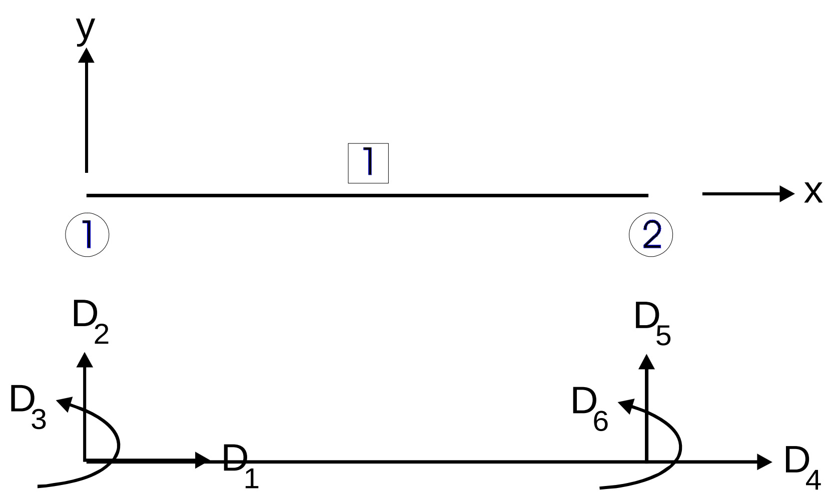


Fig. E3(b)

The boundary conditions are as follows - $D_1 = D_2 = 0$. The inclined or skew support is such that

$$\frac{D_5}{D_4} = \frac{3}{4} \Rightarrow 3D_4 - 4D_5 = 0 \tag{1}$$

Step 2: Element equilibrium equations

The element equilibrium equations are as follows.

$$10^2\begin{bmatrix} 101.33 & 0 & 0 & -101.33 & 0 & 0 \\ 0 & 0.3128 & 4.6914 & 0 & -0.3128 & 4.6914 \\ 0 & 4.6914 & 93.827 & 0 & -4.6914 & 46.914 \\ -101.33 & 0 & 0 & 101.33 & 0 & 0 \\ 0 & -0.3128 & -4.6914 & 0 & 0.3128 & -4.6914 \\ 0 & 4.6914 & 46.914 & 0 & -4.6914 & 93.827 \end{bmatrix}\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -39 \\ -210 \\ 0 \\ -51 \\ 240 \end{Bmatrix}$$

Step 3: Assembly of the system equations

$$10^2\begin{bmatrix} 93.827 & 0 & -4.6914 & 46.914 \\ & 101.33 & 0 & 0 \\ & & 0.3128 & -4.6914 \\ Sym & & & 93.827 \end{bmatrix}\begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} -210 \\ 0 \\ -51 \\ 240 \end{Bmatrix}$$

These equations cannot be solved until we impose the constraint equation, Eqn. (1). Note that $\max\left|K_{pq}\right| = 1.0133(10^4)$. Hence

$$C = (10^4)(1.0133\times 10^4) = 1.0133(10^8)$$

Using Eqns. (T3L1-52) and (T3L1-53), we have $c_i = c_4 = 3$, $c_j = c_5 = -4$ and $c = 0$. Hence

$$10^2\begin{bmatrix} 93.827 & 0 & -4.6914 & 46.914 \\ & 9.1201(10^6) & -1.216(10^7) & 0 \\ & & 1.6213(10^7) & -4.6914 \\ Sym & & & 93.827 \end{bmatrix}\begin{Bmatrix} D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} -210 \\ 0 \\ -51 \\ 240 \end{Bmatrix}$$

Step 4: Solution of the system equations

Solving, we obtain

$$D_3 = -0.04699\,rad \qquad D_4 = -0.0037\ ft$$

$$D_5 = -0.00278\ ft \qquad D_6 = 0.0489\,rad$$

Step 5: Element nodal forces

The member nodal forces are obtained in the usual manner using Eqns. (6.4.2-22) and (6.4.2-23). Using the equations, we have

$$\mathbf{f}'_{6\times1} = \left\{37.5\,k, 40\,k, 0, -37.5\,k, 50\,k, 0\right\}$$

The element FBD is shown below.

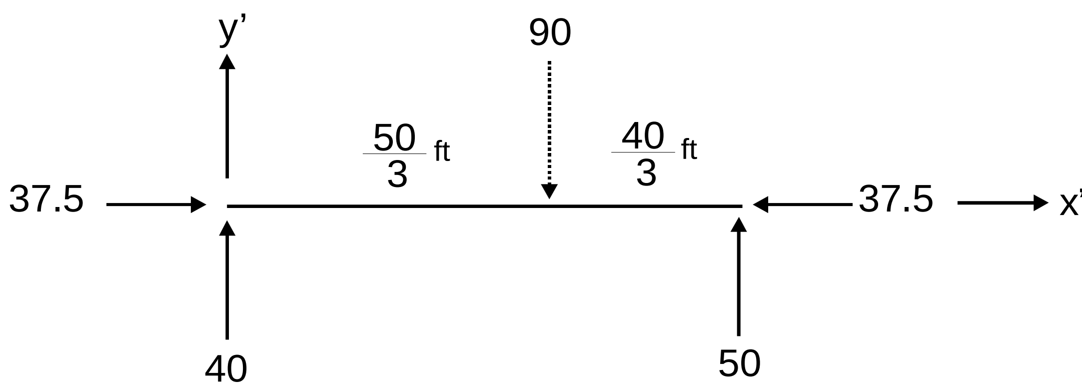
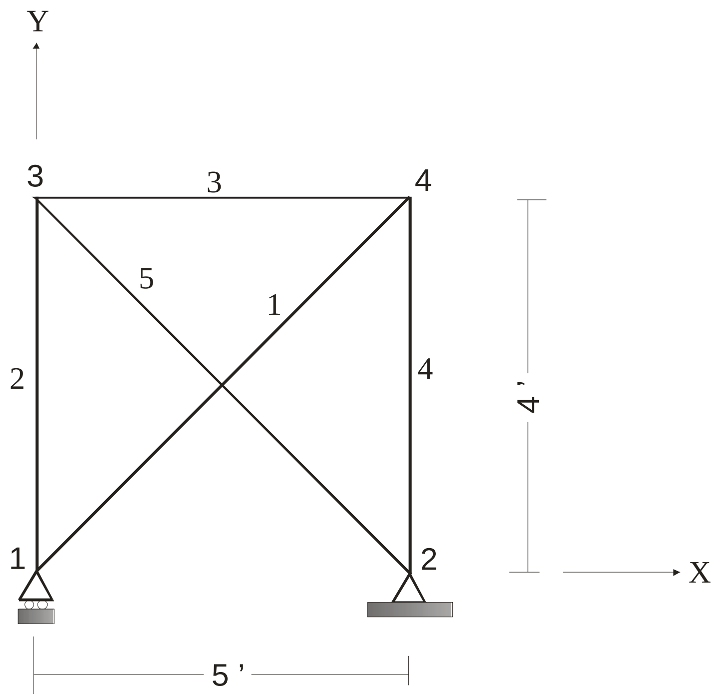


Fig. E3(c)

Review Exercises

Problem T3L1-1

Figure shows a planar truss. The modulus of elasticity is $29(10^6)$ psi and the cross-sectional area is $2\,in^2$ for all the members.



Compute all the member forces and support reactions for the following two conditions. Comment on the solutions.

(a) There is a (vertical) support settlement at node 2 equal to 0.5”.

(b) There is a (vertical) support settlement at node 2 equal to 0.5” but change the support at node 1 from a roller support to a pin support.

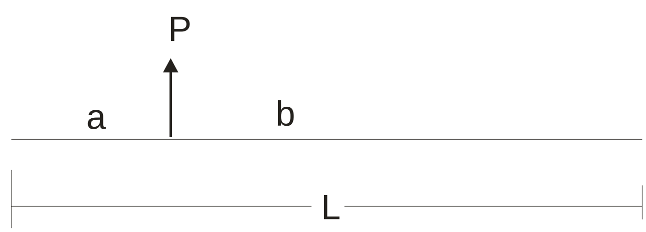
Problem T3L1-2

For the planar (general) beam element derive the element stiffness matrix if either end of the member has a moment release hinge.

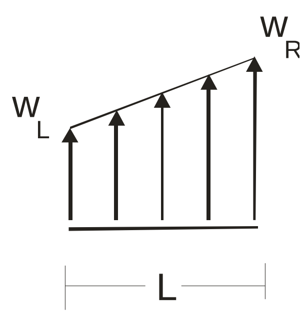
Problem T3L1-3

Using the shape functions, derive the equivalent nodal forces $\{0, q_1, q_2, 0, q_3, q_4\}$ for the following cases.

(a) Planar beam element subjected to a concentrated force P acting at a distance a from the start node.

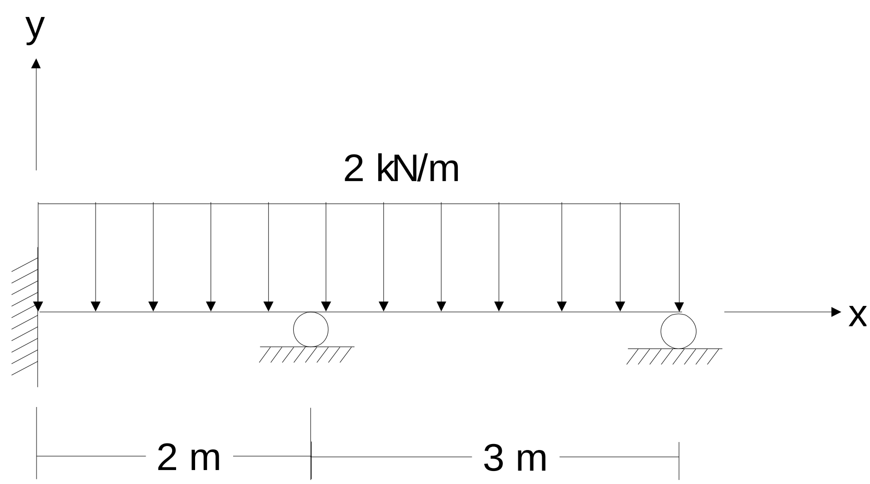


(b) Planar beam element subjected to a linear varying load with intensity w_L at the start node and w_R at the end node.



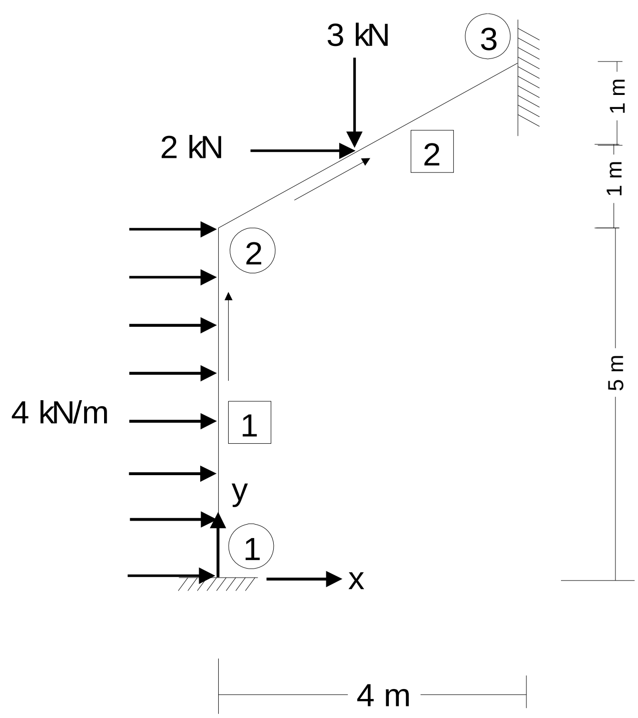
Problem T3L1-4

Fig. shows a continuous beam. The material is steel, $200\,GPa$ and the cross-sectional properties are such that $A = 0.01 m^2$ and $I = 0.0001 m^4$. Solve for the support reactions and member nodal forces.



Problem T3L1-5

The frame shown below is made of steel. The two members have the following cross-sectional properties - $A = 0.01 m^2$ and $I = 10^{-4} m^4$. Compute the support reactions. Draw the shear force and bending moment diagrams.



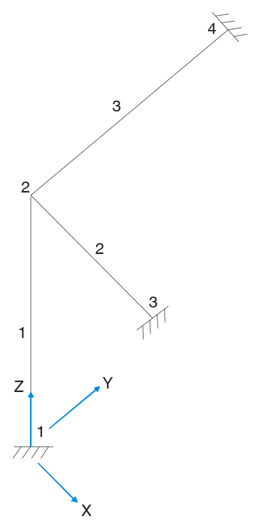
Problem T3L1-6

Fig. shows a steel space (Modulus of elasticity = $29(10^6)\ psi$) frame.

Node	X-Coordinate (ft)	Y-Coordinate (ft)	Z-Coordinate (ft)
1	0	0	0
2	0	0	20
3	20	0	20
4	0	30	20

Member	Rectangular Cross-section dimensions (H x W)
1	16" x 10"
2, 3	12" x 8"

There a uniformly distributed loading in the positive Z direction equal to 5000 lb/ft. acting on element 3. Solve for the member forces.

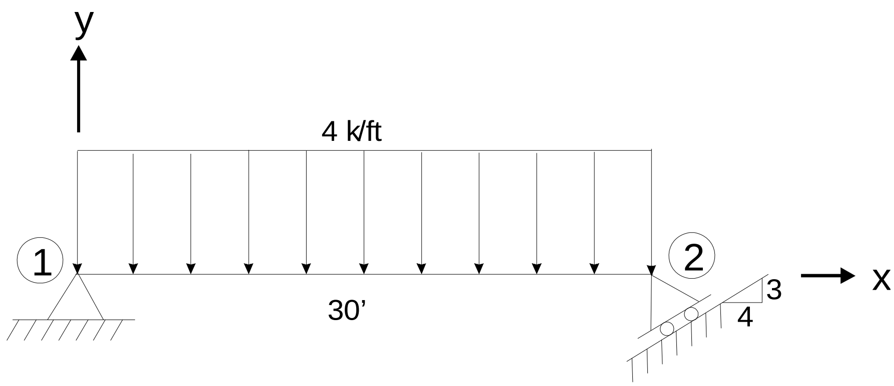


Problem T3L1-7

Compute the load vector due to a uniform temperature change in a beam element.

Problem T3L1-8

Compute the nodal displacements and the support reactions of the beam shown below. The member is built using AISC W36x210. Ignore self-weight.


y
4 k/ft
1
2
3
4
x
30’

Lesson 2: Plane Elasticity

Objectives: In this lesson we will look at plane elasticity problems.

- To understand what is meant by plane stress and plane strain problems.
- To derive the element equations for commonly used low and higher-order isoparametric plane elasticity elements.
- To solve plane elasticity problems using plane elasticity computer programs.

Plane Elasticity Problems

Plane elasticity problems involve a body whose behavior can be captured by its geometry in a plane (see Fig. T3L2-1). The displacement vector $\mathbf{u}$ is given by

$$\mathbf{u}_{2\times1} = [u, v]^{\mathbf{T}} \tag{T3L2-1}$$

where u and v are the x and y components.

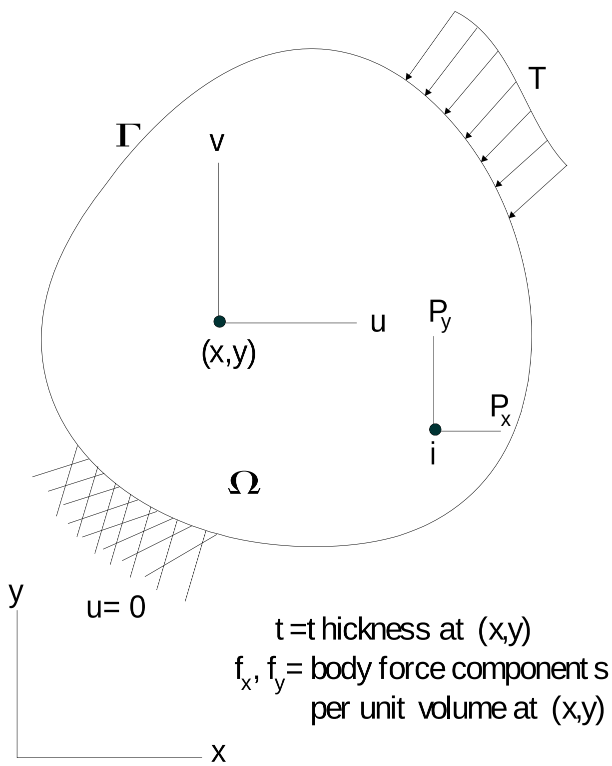


Fig. T3L2-1 Plane elasticity problem

The state of stress and strain at a point is given by

$$\boldsymbol{\sigma} = \left[\sigma_x, \sigma_y, \tau_{xy}\right]^{\mathbf{T}} \tag{T3L2-2}$$

$$\boldsymbol{\varepsilon} = \left[\varepsilon_x, \varepsilon_y, \gamma_{xy}\right]^{\mathbf{T}} \tag{T3L2-3}$$

The body force, traction vector and elemental volume are given by

$$\mathbf{f}_{2\times1} = \left[f_x,\ f_y\right]^{\mathbf{T}} \qquad \mathbf{T}_{2\times1} = \left[T_x, T_y\right]^{\mathbf{T}} \qquad dV = t\, dA \tag{T3L2-4}$$

where t is the thickness in the z-direction. The body force has units force per unit volume, while the traction has the units force per unit area.

Strain-Displacement Relations: The relationship between displacements and strains is expressed as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_{3\times1} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix}_{3\times1} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}_{3\times2} \begin{Bmatrix} u \\ v \end{Bmatrix}_{2\times1} = \mathbf{\Lambda}_{3\times2} \quad \mathbf{u}_{2\times1} \tag{T3L2-5}$$

Plane Stress: A thin isotropic planar body subjected to in-plane loading on its edge surface is said to be in plane stress. The out-of-plane stress components are zero, i.e. $\tau_{xz} = \tau_{yz} = \sigma_z = 0$. The Hooke's Law for plane stress is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \left[\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \right] \left\{ \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \alpha\Delta T \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \right\} \tag{T3L2-6a}$$

or, $$\boldsymbol{\sigma}_{3\times1} = \mathbf{D}_{3\times3}\left[\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0\right]_{3\times1} \tag{T3L2-6b}$$

Plane Strain: If a long isotropic body of uniform cross section is subjected to transverse loading along its length, it is said to be in plane strain condition. The out-of-plane strain components are zero, i.e. $\gamma_{xz} = \gamma_{yz} = \varepsilon_z = 0$. The Hooke's Law for plane strain is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \left[\frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1}{2}-\nu \end{bmatrix} \right] \left\{ \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - (1+\nu)(\alpha\Delta T) \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \right\} \tag{T3L2-7a}$$

or, $$\boldsymbol{\sigma}_{3\times1} = \mathbf{D}_{3\times3}\left[\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0\right]_{3\times1} \tag{T3L2-7b}$$

Having set the background for plane elasticity problems, we can now use the Theorem of Minimum Potential Energy to compute the element equations.

Step 1: Assume the displacement field as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & .. & \phi_n & 0 \\ 0 & \phi_1 & 0 & \phi_2 & .. & 0 & \phi_n \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ .. \\ d_{2n-1} \\ d_{2n} \end{Bmatrix} \quad \text{(T3L2-8a)}$$

or, $$\mathbf{u}_{2\times 1} = [u, v]^{\mathrm{T}} = \mathbf{\Phi}_{2\times 2n} \mathbf{d}_{2n\times 1} \quad \text{(T3L2-8b)}$$

where $\mathbf{\Phi}$ is the matrix of shape functions, $\phi_1, \phi_2, ...\phi_n$ and $\mathbf{d}$ is the vector of nodal displacements. Note that there are two displacements at every node and that $2n$ is the total number of degrees of freedom in the element.

Step 2: The strain-displacement relationship can be expressed as (using Eqn. T3L2-5 and T3L2-8b)

$$\boldsymbol{\varepsilon}_{3\times 1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{3\times 4} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix}_{4\times 1} = \mathbf{L}_{3\times 4} \mathbf{a}_{4\times 1} \quad \text{(T3L2-9a)}$$

$$\mathbf{a}_{4\times1} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ \Gamma_{21} & \Gamma_{22} & 0 & 0 \\ 0 & 0 & \Gamma_{11} & \Gamma_{12} \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \end{bmatrix}_{4\times4} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}_{4\times1} = \mathbf{M}_{4\times4}\mathbf{b}_{4\times1} \qquad \text{(T3L2-9b)}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}_{4\times1} = \begin{bmatrix} \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & ... & \phi_{n,\xi} & 0 \\ \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & .. & \phi_{n,\eta} & 0 \\ 0 & \phi_{1,\xi} & 0 & \phi_{2,\xi} & .. & 0 & \phi_{n,\xi} \\ 0 & \phi_{1,\eta} & 0 & \phi_{2,\eta} & .. & 0 & \phi_{n,\eta} \end{bmatrix}_{4\times2n} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ .. \\ u_n \\ v_n \end{Bmatrix}_{2n\times1} = \mathbf{N}_{4\times2n}\mathbf{d}_{2n\times1} \qquad \text{(T3L2-9c)}$$

Hence,

$$\boldsymbol{\varepsilon}_{3\times1} = \mathbf{L}_{3\times4}\mathbf{M}_{4\times4}\mathbf{N}_{4\times2n}\mathbf{d}_{2n\times1} = \mathbf{O}_{3\times4}\mathbf{N}_{4\times2n}\mathbf{d}_{2n\times1} = \mathbf{B}_{3\times2n}\mathbf{d}_{2n\times1} \qquad \text{(T3L2-9d)}$$

where $$\mathbf{O}_{3\times4} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{11} & \Gamma_{12} \end{bmatrix} \qquad \text{(T3L2-9e)}$$

and $\mathbf{B}$ is known as the strain-displacement matrix and $\boldsymbol{\Gamma}$ is given by Eqn. (T2L3-13).

Step 3: The total strain energy per element is

$$U(\mathbf{d}) = \frac{1}{2}\int_A \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\, t\, dA = \frac{1}{2}\int_A \boldsymbol{\varepsilon}^T \mathbf{D}\boldsymbol{\varepsilon}\, t\, dA = \frac{1}{2}\mathbf{d}^{\mathbf{T}}_{1\times 2n}\mathbf{k}_{2n\times 2n}\mathbf{d}_{2n\times 1} \tag{T3L2-10a}$$

where $$\mathbf{k}_{2n\times 2n} = \int_A \mathbf{B}^{\mathbf{T}}\mathbf{D}\mathbf{B}\, t\, dA \tag{T3L2-10b}$$

is the element stiffness matrix. Hence, the total potential energy for the entire structure is

$$\Pi(\mathbf{D}) = \sum_{i=1}^{e}\left[\frac{1}{2}\mathbf{d}^{\mathbf{T}}_{1\times 2n}\mathbf{k}_{2n\times 2n}\mathbf{d}_{2n\times 1} - \mathbf{d}^{\mathbf{T}}_{1\times 2n}\mathbf{f}_{2n\times 1} - \mathbf{d}^{\mathbf{T}}_{1\times 2n}\mathbf{T}_{2n\times 1}\right]_i - \mathbf{D}^T_{1\times N}\mathbf{P}_{N\times 1} \tag{T3L2-11}$$

where $\mathbf{d}^{\mathbf{T}}_{1\times 2n}\mathbf{f}_{2n\times 1}, \mathbf{d}^{\mathbf{T}}_{1\times 2n}\mathbf{T}_{2n\times 1}$ and $\mathbf{D}^T_{1\times N}\mathbf{P}_{N\times 1}$ represent the work potential due to the body forces, surface tractions and concentrated forces. If initial strains are included then we must include the effect as $\mathbf{d}^{\mathbf{T}}\left[\int_A \mathbf{B}^{\mathbf{T}}\mathbf{D}\boldsymbol{\varepsilon}_0\, t dA\right]$ as work potential due to initial strains.

We now have all the ingredients to compute the element equations for different types of elements.

Constant Strain Triangular (CST) Element

To recap, the real and the parent elements for the lowest-order triangular element are shown below.

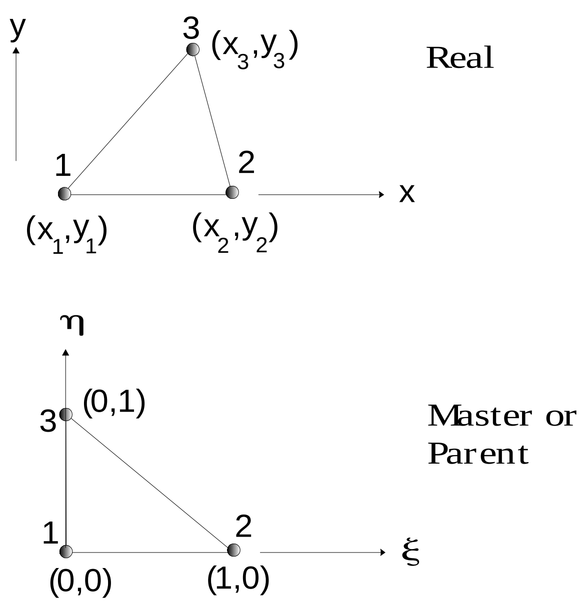


Fig. T3L2-1 CST Element

The assumed displacement field is of the form

$$u = a_1 + a_2\xi + a_3\eta \qquad\qquad v = b_1 + b_2\xi + b_3\eta \tag{T3L2-12}$$

With this assumed displacement field, it is clear that the strains (and, hence stresses) within the element are constants. Hence the element is known as the constant strain triangular element or CST for short. The shape functions are as follows.

$$\phi_1 = 1 - \xi - \eta \qquad\qquad \phi_2 = \xi \qquad\qquad \phi_3 = \eta \tag{T3L2-13}$$

Since, $x = \phi_1 x_1 + \phi_2 x_2 + \phi_3 x_3 \qquad\qquad y = \phi_1 y_1 + \phi_2 y_2 + \phi_3 y_3$ (T3L2-14a)

$$\mathbf{J}_{2\times2} = \begin{bmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1) & (y_3 - y_1) \end{bmatrix} = \begin{bmatrix} x_{21} & y_{21} \\ x_{31} & y_{31} \end{bmatrix} \text{ and } \det(J) = x_{21}y_{31} - x_{31}y_{21} \tag{T3L2-14b}$$

$$\boldsymbol{\Gamma}_{2\times2} = \frac{1}{\det(J)}\begin{bmatrix} y_{31} & -y_{21} \\ -x_{31} & x_{21} \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} y_{31} & -y_{21} \\ -x_{31} & x_{21} \end{bmatrix} \tag{T3L2-14c}$$

The strain-displacement matrix can be computed as

$$\mathbf{B}_{3\times6} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{11} & \Gamma_{12} \end{bmatrix}_{3\times4} \begin{bmatrix} \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \phi_{3,\xi} & 0 \\ \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & \phi_{3,\eta} & 0 \\ 0 & \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \phi_{3,\xi} \\ 0 & \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & \phi_{3,\eta} \end{bmatrix}_{4\times6} \tag{T3L2-14d}$$

Simplifying

$$\mathbf{B}_{3\times6} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}_{3\times6} \tag{T3L2-15}$$

It should be noted that the strain-displacement matrix is a constant. Hence using Eqn. (T3L2-10b) and assuming that the thickness is a constant within the element

$$\mathbf{k}_{6\times6} = \int_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, t\, dA = tA\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \tag{T3L2-16}$$

We will look at the computation of element loads later in this lesson. Once the nodal displacements are computed, the element strains are evaluated

$$\boldsymbol{\varepsilon}_{3\times1} = \mathbf{B}_{3\times2n}\mathbf{d}_{2n\times1} \tag{T3L2-17}$$

and the element stresses using

$$\boldsymbol{\sigma}_{3\times1} = \mathbf{D}_{3\times3}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0\right)_{3\times1} \tag{T3L2-18}$$

These strains and stresses are the same at any point in the element (triangle).

Linear Quadrilateral Element

The assumed displacement field is of the form

$$u = a_1 + a_2\xi + a_3\eta + a_4\xi\eta \qquad\qquad v = b_1 + b_2\xi + b_3\eta + b_4\xi\eta \tag{T3L2-19a}$$

With this assumed displacement field, it is clear that the strains (and, hence stresses) within the element are NOT constants. The shape functions are as follows.

$$\phi_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \qquad\qquad i = 1,2,3,4 \tag{T3L2-19b}$$

where (ξ_i, η_i) are the natural coordinates of node i.

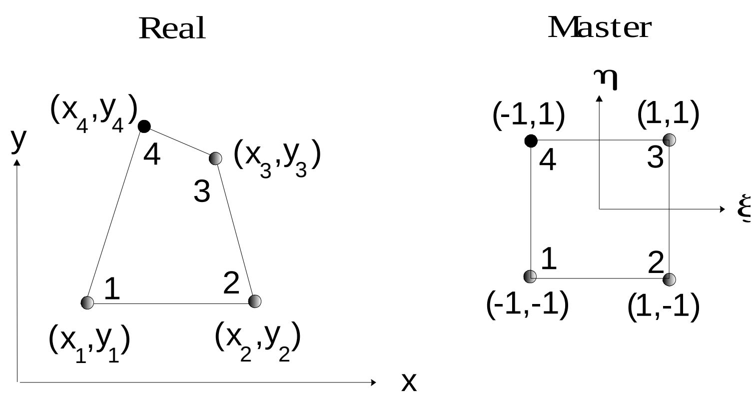


Fig. T3L2-2

To compute the jacobian, we note that

$$x = \sum_{i=1}^{4} \phi_i x_i \qquad y = \sum_{i=1}^{4} \phi_i y_i \tag{T3L2-20a}$$

$$\mathbf{J}_{2\times2} = \frac{1}{4}\left[\begin{array}{c|c} \eta(x_1 - x_2 + x_3 - x_4) & \eta(y_1 - y_2 + y_3 - y_4) \\ +(-x_1 + x_2 + x_3 - x_4) & +(-y_1 + y_2 + y_3 - y_4) \\ \hline \xi(x_1 - x_2 + x_3 - x_4) & \xi(y_1 - y_2 + y_3 - y_4) \\ +(-x_1 - x_2 + x_3 + x_4) & +(-y_1 - y_2 + y_3 + y_4) \end{array}\right] \tag{T3L2-20b}$$

$$\mathbf{\Gamma}_{2\times2} = \frac{1}{\det(J)}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \tag{T3L2-20c}$$

$$dxdy = \det(J)\, d\xi d\eta \tag{T3L2-21}$$

The difficulty with evaluating the strain-displacement matrix is that the intermediate terms such as the inverse of the jacobian are more complex than seen before. Writing the terms in a symbolic fashion is extremely messy. We shall now use the concepts associated with numerical integration to help evaluate the terms.

We resort to Eqns. (T3L2-9d,9e) to develop the strain-displacement matrix as

$$\mathbf{B}_{3\times8} = \mathbf{O}_{3\times4}\mathbf{N}_{4\times8} \tag{T3L2-22a}$$

$$\mathbf{B}_{3\times6} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{11} & \Gamma_{12} \end{bmatrix}_{3\times4} \begin{bmatrix} \varphi_{1,\xi} & 0 & \varphi_{2,\xi} & 0 & \varphi_{3,\xi} & 0 & \varphi_{4,\xi} & 0 \\ \varphi_{1,\eta} & 0 & \varphi_{2,\eta} & 0 & \varphi_{3,\eta} & 0 & \varphi_{4,\eta} & 0 \\ 0 & \varphi_{1,\xi} & 0 & \varphi_{2,\xi} & 0 & \varphi_{3,\xi} & 0 & \varphi_{4,\xi} \\ 0 & \varphi_{1,\eta} & 0 & \varphi_{2,\eta} & 0 & \varphi_{3,\eta} & 0 & \varphi_{4,\eta} \end{bmatrix}_{4\times8} \tag{T3L2-22b}$$

We can now express the stiffness matrix as

$$\mathbf{k}_{8\times8} = \int_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, t\, dA = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, t \det(J)\, d\xi d\eta \tag{T3L2-23}$$

The integrand cannot be evaluated analytically. The numerical evaluation is of the form

$$\mathbf{k}_{8\times8} = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, t \det(J)\, d\xi d\eta = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j\, f(\xi_i, \eta_j) \tag{T3L2-24a}$$

where $f(\xi_i, \eta_j) = \mathbf{B}^{\mathbf{T}}\mathbf{D}\mathbf{B}\, t \det(J)\Big|_{\xi_i, \eta_j}$ (T3L2-24b)

The algorithm to compute the element stiffness matrix is given below.

Algorithm:

(1) Clear $\mathbf{k}_{8\times8}$ to zero.

(2) Enter the *i* loop to integrate in the ξ -direction. Set values for w_i and ξ_i .

(3) Enter the *j* loop to integrate in the η -direction. Set values for w_j and η_j .

(4) At the current Gauss point (ξ_i, η_j), compute the following.

   (a) The shape functions ϕ_k and the derivatives of the shape functions $\dfrac{\partial \phi_k}{\partial \xi}, \dfrac{\partial \phi_k}{\partial \eta}$. If necessary, use the shape functions to compute the thickness, t_{ij} at the current point.

   (b) Construct the jacobian matrix, $\mathbf{J}_{2\times2}$, $\det(J)$ and the inverse $\mathbf{\Gamma}_{2\times2}$.

   (c) Form the strain-displacement matrix $\mathbf{B}_{3\times8}$ using Eqn. (T3L2-9d).

(5) At the current point, compute the product $\mathbf{T}_{3\times3} = w_i w_j\, t_{ij} \det(J)\mathbf{D}_{3\times3}$.

(6) Now compute the triple product $\mathbf{B}^{\mathbf{T}}_{8\times3}\mathbf{T}_{3\times3}\mathbf{B}_{3\times8}$ and update $\mathbf{k}_{8\times8}$.

(7) Increment *j*.

(8) Increment *i*.

In the above algorithm the apt question to be asked is "What is the appropriate order of integration?" Note that the integrand is not a polynomial but an irrational fraction involving polynomials. Numerical experience has shown that the numerical integration order that leads to the correct volume of the element is also acceptable for the element stiffness. Hence for this element, the one-point Gauss Quadrature rule can be applied[2].

[2] There are several references that discuss the merits of different strategies. Instead of using exact integration, the strategies suggest the use of **reduced integration** for most elements and **higher-order integration** for distorted elements.

The body forces, surface tractions and thermal load vector can be evaluated as we did with the CST element. Once the nodal displacements are obtained, the element strains and stresses can be obtained using the following equations

$$\boldsymbol{\varepsilon}_{3\times1} = \mathbf{B}_{3\times2n}\mathbf{d}_{2n\times1} \tag{T3L2-25a}$$

$$\boldsymbol{\sigma}_{3\times1} = \mathbf{D}_{3\times3}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0)_{3\times1} \tag{T3L2-25b}$$

There is a big difference between the quadrilateral element and the CST element with respect to strains and stresses. The strains and stresses in the quadrilateral element are not constants. The appropriate question to ask is "Where should the strains and stresses be computed?" The points that lead to the least error in computing stresses are called the Barlow[3] points. For this quadrilateral element, the point $(\xi = \eta = 0)$ is the optimal point.

Computation of the Element Load Vector

Recall that the total potential energy of a linearly elastic system is given by

$$\Pi = \int_V U_0\, dV - \int_V \mathbf{f}^T \mathbf{F}\, dV - \int_S \mathbf{f}^T \Phi\, dS - \mathbf{D}^\mathbf{T}\mathbf{P} \tag{T3L2-26}$$

We can use this to compute the following expressions.

(a) The traction load vector can be computed using the following equation

$$\mathbf{f}_i^{sur} = t\oint_\Gamma \phi_i \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} ds \tag{T3L2-27}$$

(b) The body force vector can be computed using the following equation

$$\mathbf{f}_i^{body} = t\iint \phi_i \begin{Bmatrix} B_x \\ B_y \end{Bmatrix} dA \tag{T3L2-28}$$

(c) The thermal load vector can be computed using the following equation

$$\mathbf{f}^{ther} = t\iint \mathbf{B}^\mathbf{T}\mathbf{D}\boldsymbol{\varepsilon}_\mathbf{0} dA \tag{T3L2-29}$$

For constant traction, the load vector can be expressed using outward unit normal vector such that

---

[3] J. Barlow, "Optimal Stress Locations in Finite Element Models", *Int J Num Meth Eng*, **10**: 243-251, 1976.

$$\begin{Bmatrix} T_x \\ T_y \end{Bmatrix} = T\mathbf{n} \tag{T3L2-30}$$

By using (T3L2-27), we can rewrite the traction load vector as

$$\mathbf{f}_i^{sur} = t\oint_\Gamma \phi_i T\mathbf{n}\,ds \tag{T3L2-31}$$

Note that the outward normal can be obtained using the cross-product

$$\mathbf{n}ds = \mathbf{ds}\times\mathbf{z} = \begin{Bmatrix} dx \\ dy \\ 0 \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} dy \\ -dx \\ 0 \end{Bmatrix} \tag{T3L2-32}$$

where

$$dx = \frac{\partial x}{\partial \xi}d\xi + \frac{\partial x}{\partial \eta}d\eta = J_{11}d\xi + J_{21}\,d\eta \tag{T3L2-33a}$$

$$dy = \frac{\partial y}{\partial \xi}d\xi + \frac{\partial y}{\partial \eta}d\eta = J_{12}\,d\xi + J_{22}\,d\eta \tag{T3L2-33b}$$

Hence

$$\mathbf{f}_i^{sur} = tT\oint_\Gamma \phi_i\left(dy\,\hat{i} - dx\,\hat{j}\right) = tT\oint_\Gamma \phi_i\left(\left[J_{12}d\xi + J_{22}d\eta\right]\hat{i} - \left[J_{11}d\xi + J_{21}d\eta\right]\hat{j}\right) \tag{T3L2-34}$$

Triangular Elements

Surface traction: Consider the case of triangular elements as shown in Fig. T3L2-11. We will use the second-order element (six-noded triangle) as a means of developing the procedure.

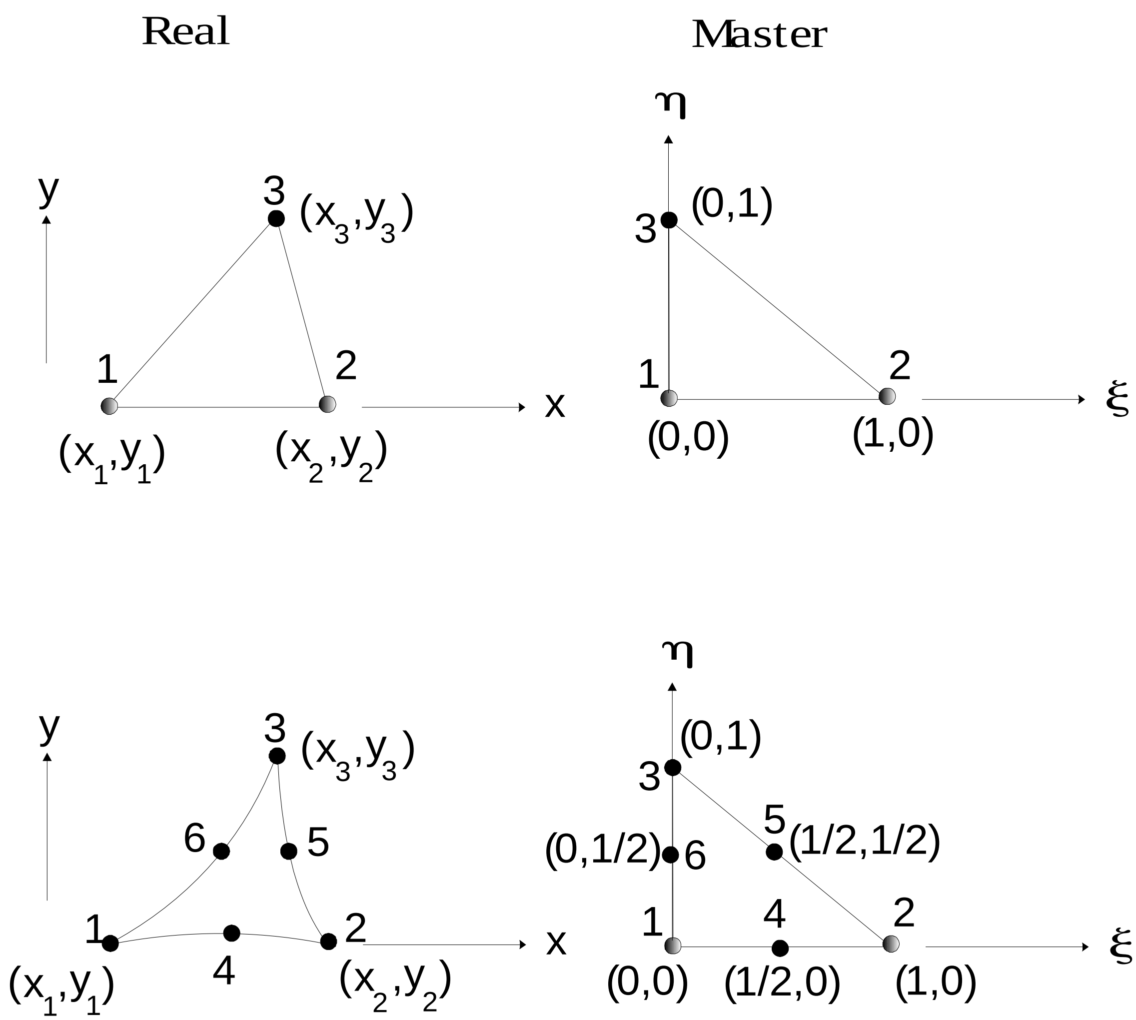


Fig. T3L2-11 Linear and Quadratic triangular elements

Assume that the normal traction is applied on side $\overline{142}$. In other words

$$\mathbf{f}_i^{sur} = t\oint_{\Gamma} \phi_i T\mathbf{n}ds = t\int_{\overline{142}} \phi_i T\mathbf{n}ds \quad i = 1,4,2 \tag{T3L2-35}$$

Using Eqn.(T3L2-33) and Eqn.(T3L2-34), and since on side $\overline{142}$, $\eta = 0$ and $d\eta = 0$, the above equation collapses

$$f_{i,x}^{sur} = t\int_0^1 \phi_i T J_{12}\, d\xi \tag{T3L2-36a}$$

$$f_{i,y}^{sur} = -t\int_0^1 \phi_i T J_{11}\, d\xi \tag{T3L2-36b}$$

Note that on side $\overline{142}$ the shape functions can be simplified as

$$\phi_1(\xi,0)=(1-\xi)(1-2\xi) \qquad \text{(T3L2-37a)}$$

$$\phi_2(\xi,0)=\xi(2\xi-1) \qquad \text{(T3L2-37b)}$$

$$\phi_3(\xi,0)=0 \qquad \text{(T3L2-37c)}$$

$$\phi_4(\xi,0)=4\xi(1-\xi) \qquad \text{(T3L2-37d)}$$

$$\phi_5(\xi,0)=0 \qquad \text{(T3L2-37e)}$$

$$\phi_6(\xi,0)=0 \qquad \text{(T3L2-37f)}$$

The Jacobian terms can be computed as

$$J_{11}(\xi,0)=\frac{\partial x}{\partial \xi}(\xi,0)=\sum_{k=1,4,2} x_k \frac{\partial \phi_k(\xi,0)}{\partial \xi} \qquad \text{(T3L2-38a)}$$

or $$J_{11}(\xi,0)=(4\xi-3)x_1-(8\xi-4)x_4+(4\xi-1)x_2 \qquad \text{(T3L2-38b)}$$

$$J_{12}(\xi,0)=\frac{\partial y}{\partial \xi}(\xi,0)=\sum_{k=1,4,2} y_k \frac{\partial \phi_k(\xi,0)}{\partial \xi} \qquad \text{(T3L2-39a)}$$

or $$J_{12}(\xi,0)=(4\xi-3)y_1-(8\xi-4)y_4+(4\xi-1)y_2 \qquad \text{(T3L2-39b)}$$

In order to use numerical integration using Gauss-Legendre formula, we need to change the limits of the integral from $0\le\xi\le1$ to $-1\le\xi'\le1$. The following transformation can be used to change the limits of the integration

$$\xi=\frac{1}{2}(\xi'+1) \qquad \text{(T3L2-40)}$$

Using this transformation in Eqns. (T3L2-37), (T3L2-38) and (T3L2-39) we have

$$\phi_1(\xi',0)=-\frac{1}{2}\xi'(1-\xi') \qquad \text{(T3L2-41a)}$$

$$\phi_2(\xi',0)=\frac{1}{2}\xi'(1+\xi') \qquad \text{(T3L2-42b)}$$

$$\phi_4(\xi',0) = (1+\xi')(1-\xi') \tag{T3L2-43c}$$

$$J_{11}(\xi',0) = (2\xi'-1)x_1 - 4\xi' x_4 + (2\xi'+1)x_2 \tag{T3L2-44a}$$

$$J_{12}(\xi',0) = (2\xi'-1)y_1 - 4\xi' y_4 + (2\xi'+1)y_2 \tag{T3L2-44b}$$

Hence,

$$f_{i,x}^{sur} = \frac{1}{2}t\int_{-1}^{1}\phi_i(\xi',0)TJ_{12}(\xi',0)\,d\xi' = \frac{1}{2}t\sum_{l=1}^{n} w_{nl}\left[T\phi_i(\xi',0)J_{12}(\xi',0)\right]_{\xi'_{nl}} \tag{T3L2-45a}$$

$$f_{i,x}^{sur} = -\frac{1}{2}t\int_{-1}^{1}\phi_i(\xi',0)TJ_{11}(\xi',0)\,d\xi' = -\frac{1}{2}t\sum_{l=1}^{n} w_{nl}\left[T\phi_i(\xi',0)J_{11}(\xi',0)\right]_{\xi'_{nl}} \tag{T3L2-45b}$$

With a constant T, a 3-point rule leads to full integration. The expressions for side $\overline{361}$ are the same as those for $\overline{142}$ with indices $1,4,2$ replaced with $3,6,1$. Similar comments apply to side $\overline{253}$.

Body force: In a similar manner we can develop the body force vector. Once again we will use the second-order element (six-noded triangle) as a means of developing the procedure.

$$\mathbf{f}_i^{body} = t\iint \phi_i \begin{Bmatrix} B_x \\ B_y \end{Bmatrix} dA \tag{T3L2-46}$$

Numerically,

$$f_{i,x}^{body} = t\sum_{l=1}^{n} w_{nl}\left[B_x\phi_i(\xi,\eta)\right]_{(\xi_{nl},\eta_{nl})} \tag{T3L2-47a}$$

$$f_{i,y}^{body} = t\sum_{l=1}^{n} w_{nl}\left[B_y\phi_i(\xi,\eta)\right]_{(\xi_{nl},\eta_{nl})} \tag{T3L2-47b}$$

Thermal loading:

$$\mathbf{f}^{ther} = t\iint \mathbf{B}^{\mathbf{T}}\mathbf{D}\boldsymbol{\varepsilon}_{\mathbf{0}}\,dA \tag{T3L2-48}$$

In the above expression, ΔT can be computed as

$$\Delta T = \sum_{i=1}^{n}\phi_i(\Delta T)_i \tag{T3L2-49}$$

using the shape functions and the temperature changes at the element nodes. The above integration can be evaluated numerically.

Quadrilateral Elements

Now consider the family of quadrilateral elements.

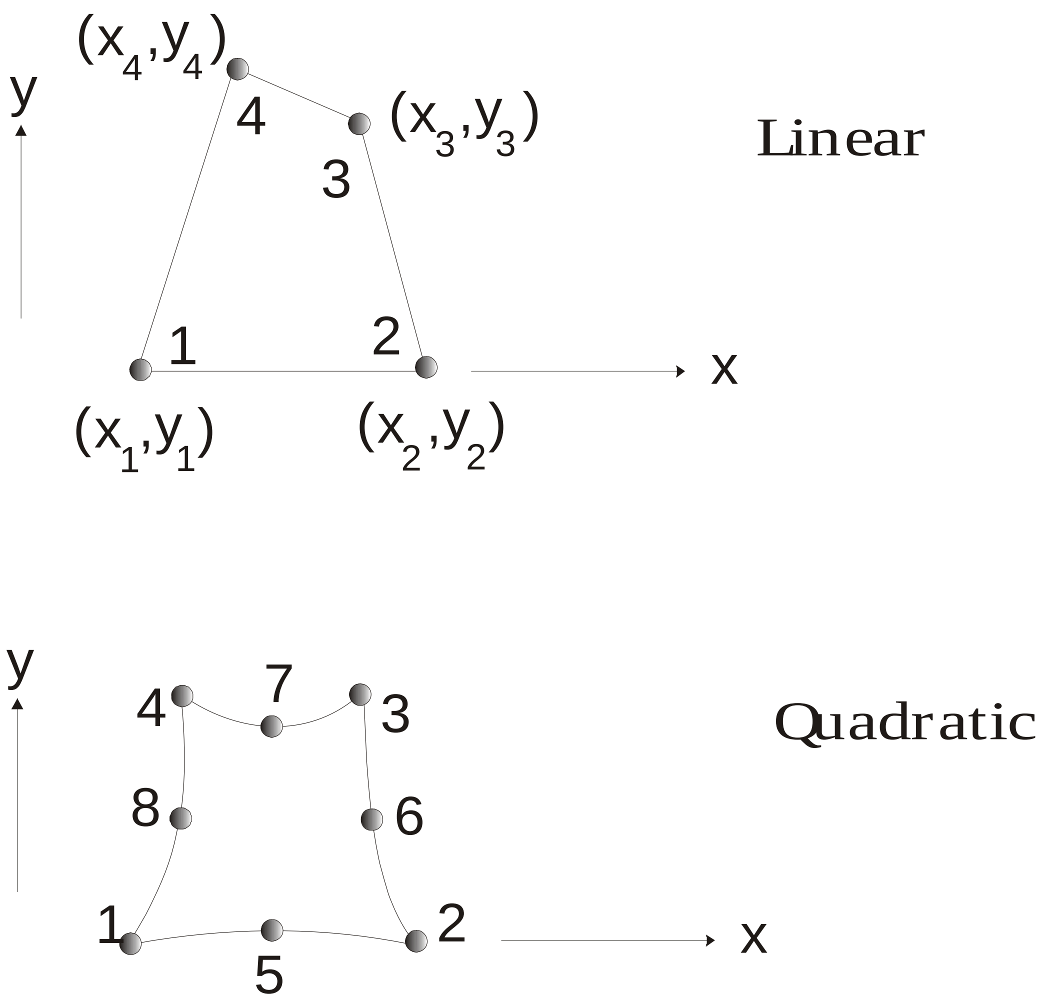


Fig. T3L2-12 Linear and Quadratic quadrilateral "serendipity" elements

We will now derive the expressions for the quadratic element. Consider the fact that the normal traction is applied to side $\overline{152}$ of the element. On this side $\eta=-1$ and $d\eta=0$. The shape functions then reduce to (the others are zero)

$$\phi_1(\xi,-1)=-\frac{1}{2}\xi(1-\xi) \qquad \text{(T3L2-50a)}$$

$$\phi_2\left(\xi,-1\right)=\frac{1}{2}\xi\left(1+\xi\right) \qquad \text{(T3L2-50b)}$$

$$\phi_5\left(\xi,-1\right)=\left(1+\xi\right)\left(1-\xi\right) \qquad \text{(T3L2-50c)}$$

The traction load vector becomes

$$f_{i,x}^{sur}=t\int_{-1}^{1}\phi_i(\xi,-1)TJ_{12}(\xi,-1)\,d\xi \quad i=1,5,2 \qquad \text{(T3L2-51a)}$$

$$f_{i,y}^{sur}=-t\int_{0}^{1}\phi_i(\xi,-1)TJ_{11}(\xi,-1)\,d\xi \qquad \text{(T3L2-51b)}$$

with

$$J_{11}\left(\xi,-1\right)=\frac{\partial x}{\partial \xi}\left(\xi,-1\right)=\sum_{k=1,5,2}x_k\frac{\partial\phi_k\left(\xi,-1\right)}{\partial\xi} \qquad \text{(T3L2-52a)}$$

or $$J_{11}\left(\xi,-1\right)=\left(\xi-\frac{1}{2}\right)x_1-2\xi x_5+\left(\xi+\frac{1}{2}\right)x_2 \qquad \text{(T3L2-52b)}$$

$$J_{12}\left(\xi,-1\right)=\frac{\partial y}{\partial \xi}\left(\xi,-1\right)=\sum_{k=1,5,2}y_k\frac{\partial\phi_k\left(\xi,-1\right)}{\partial\xi} \qquad \text{(T3L2-53a)}$$

or $$J_{12}\left(\xi,-1\right)=\left(\xi-\frac{1}{2}\right)y_1-2\xi y_5+\left(\xi+\frac{1}{2}\right)y_2 \qquad \text{(T3L2-53b)}$$

Using numerical integration, we have

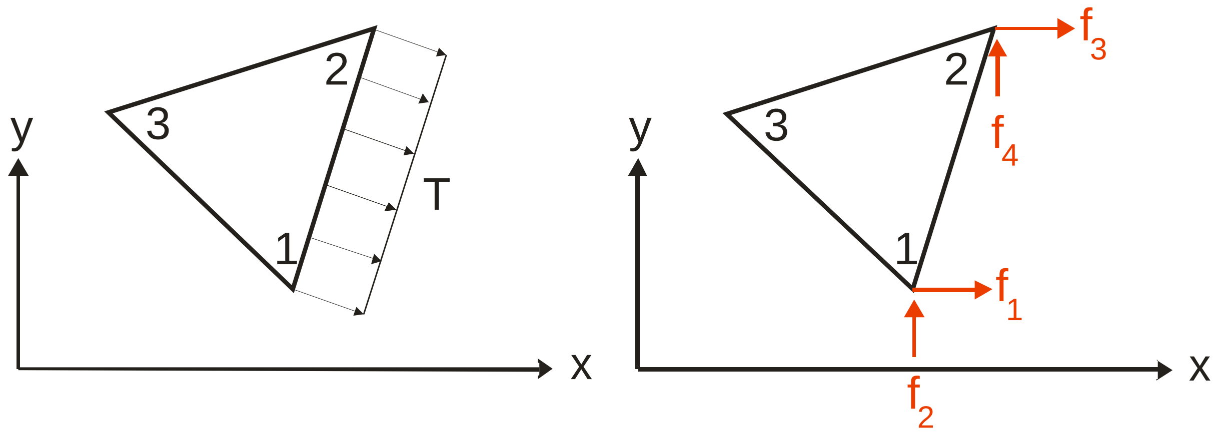
$$f_{i,x}^{sur}=t\sum_{l=1}^{n}w_{nl}\left[T\phi_i\left(\xi,-1\right)J_{12}\left(\xi,-1\right)\right]_{\xi_{nl}} \quad i=1,5,2 \qquad \text{(T3L2-54a)}$$

$$f_{i,y}^{sur}=-t\sum_{l=1}^{n}w_{nl}\left[T\phi_i\left(\xi,-1\right)J_{11}\left(\xi,-1\right)\right]_{\xi_{nl}} \qquad \text{(T3L2-54b)}$$

With a constant T, a 3-point rule leads to full integration. Note that this expression is valid for any side of the quadrilateral. Comparing Eqn. (T3L2-54) to (T3L2-47) shows that the two expressions are identical (the boundary Jacobian for the triangle is twice that of the quadrilateral but is compensated with the $1/2$ factor). The body force and thermal loadings are easily computed in a way similar in triangular case.

Illustrative Example

Compute the equivalent nodal forces for the following data (a) T=100 psi, t=0.1 in, (x_1, y_1)=(7", 3") and (x_2, y_2)=(7", 7"), and (b) T=100 psi, t=0.1 in, (x_1, y_1)=(7", 3") and (x_2, y_2)=(10", 7").



For the T3 element, the shape functions are as follows.

$$\phi_1 = 1 - \xi - \eta \qquad \phi_2 = \xi \qquad \phi_3 = \eta$$

For side 1-2 of the element, these shape functions and their derivatives are as follows.

$$\phi_1 = 1 - \xi \qquad \phi_2 = \xi \qquad \phi_3 = 0$$

$$\phi_{1,\xi} = -1 \qquad \phi_{2,\xi} = 1 \qquad \phi_{3,\xi} = 0$$

$$J_{11}(\xi,0) = \frac{\partial x}{\partial \xi}(\xi,0) = \sum_{k=1,2} x_k \frac{\partial \phi_k(\xi,0)}{\partial \xi} = x_2 - x_1$$

$$J_{12}(\xi,0) = \frac{\partial y}{\partial \xi}(\xi,0) = \sum_{k=1,2} y_k \frac{\partial \phi_k(\xi,0)}{\partial \xi} = y_2 - y_1$$

Eqns. (T3L2-36a), (T3L2-36b) can now be reduced to the following.

$$f_{i,x}^{sur} = tT\int_0^1 \phi_i J_{12}\, d\xi = tT\int_0^1 \phi_i (y_2 - y_1)\, d\xi$$

$$f_{i,y}^{sur} = -tT\int_0^1 \phi_i J_{11}\, d\xi = -tT\int_0^1 \phi_i (x_2 - x_1)\, d\xi = tT\int_0^1 \phi_i (x_1 - x_2)\, d\xi$$

Substituting the shape functions and integrating, we have the following

$$\mathbf{f}_{6\times1} = \frac{tT}{2}\left[\left(y_2 - y_1\right) \quad \left(x_1 - x_2\right) \quad \left(y_2 - y_1\right) \quad \left(x_1 - x_2\right) \quad 0 \quad 0 \right]^T$$

(a) Substituting the numerical values we have

$$\mathbf{f}_{6\times1} = \left[20 \quad 0 \quad 20 \quad 0 \quad 0 \quad 0\right]^T lb$$

(b) Substituting the numerical values we have

$$\mathbf{f}_{6\times1} = \left[20 \quad -15 \quad 20 \quad -15 \quad 0 \quad 0\right]^T lb$$

Higher-Order Elements

The treatment of higher-order elements is no different than the manner in which the lower-order elements were treated. The Linear Strain Triangular (LST) element and the Quadratic Quadrilateral (Lagrange and Serendipity) element are the next in the order of triangular and quadrilateral elements. The basic procedure shown with respect to the Linear Quadrilateral element applies.

Illustrative Example

Fig. T3L2-8 shows a metallic disk with a circular hole. The properties are as follows - $E = 30(10^6)\ psi$, $\nu = 0.3$ and $t = 0.25\,in$. The objective of this analysis is to compute the principal stresses at the points along AB as well as find the vertical displacement under the load.

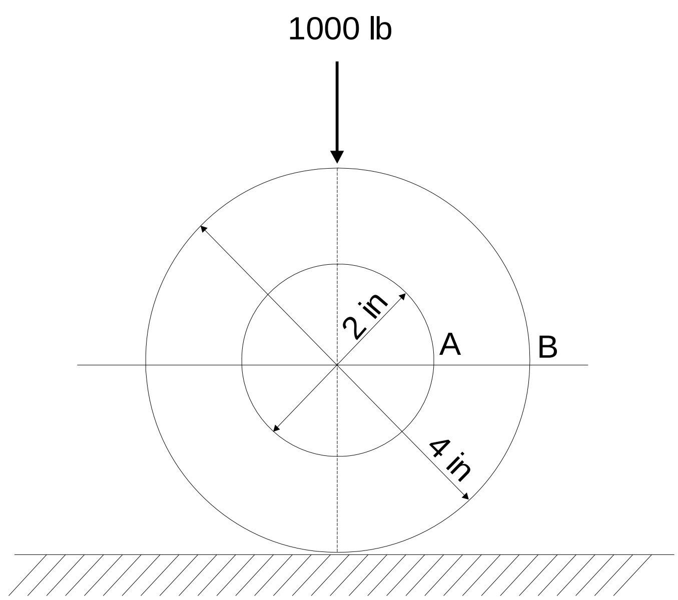


Fig. T3L2-8

The model that can be used to solve the problem is shown below. The idea of symmetry is used in constructing the half-model. Since the points on the disk on the vertical line of symmetry will displace exactly downwards, a roller support is needed to achieve the proper boundary conditions on the vertical line of symmetry. For a static analysis, all rigid body modes must be prevented. Hence, both the X and the Y displacements must be suppressed at the point of contact of the disk with the rigid planar surface. This is a plane stress problem.

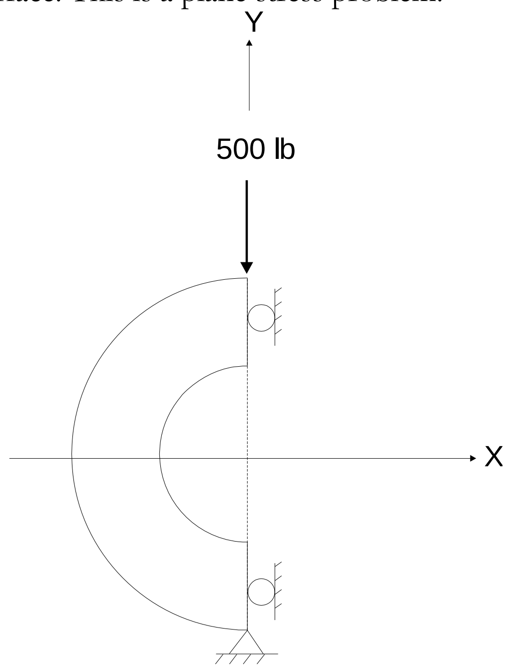


Fig. T3L2-9

A typical finite element mesh containing CST elements is shown below.

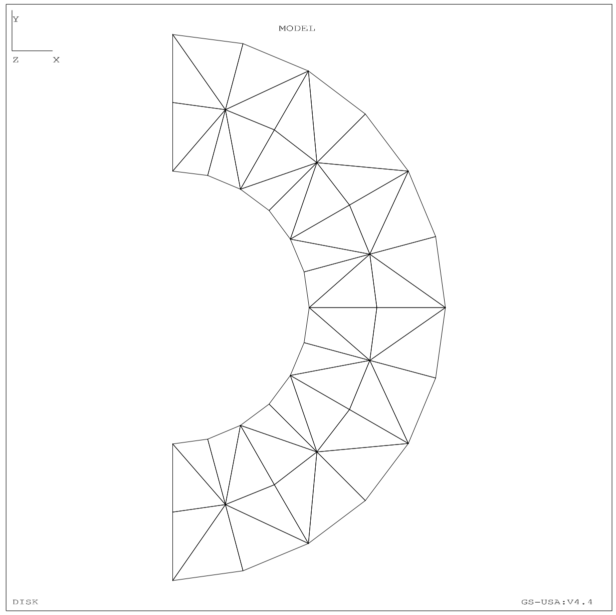


Fig. T3L2-10 Model ID 1

Model ID	**Number of Elements**	$\left(\sigma_{principal}\right)_A$, **psi**	$\left(\sigma_{principal}\right)_B$, **psi**	**Vertical defl. C (in)**
1	48	(0, -7258)	(1790, -1277)	-9.8439E-04
2	132	(0, -8982)	(2616, -818)	-1.1408E-03

The model should be refined further as we have done in the previous problems, so that the proper convergence characteristics can be established.

Notes on the Solution

At this stage it is appropriate to examine the characteristics of the solution process. The following points can be made based on the theory.

(1) *Equilibrium is usually not satisfied within elements*. Stresses must satisfy the differential equations of equilibrium

$$\sigma_{i,i} + \tau_{ij,j} + F_i = 0$$

In the absence of body forces, these equations are satisfied provided the stresses are constant. Certainly, the CST element satisfies this requirement but not the linear quadrilateral element.

(2) *Equilibrium is usually not satisfied between elements.* The stresses in an element are functions of the displacements associated only with that element. Consider the two CST elements shown below. In general $(\sigma_x)_1 \neq (\sigma_x)_2$.

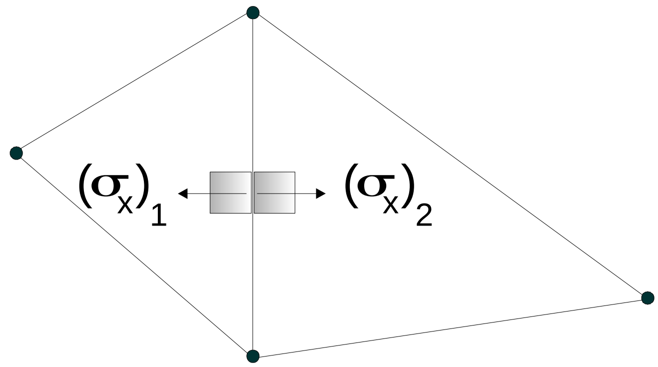


Fig. T3L2-3

Also at the free edges of a structure, the normal stress and the shear stress are not zero. For a proper solution, their magnitude should be small in comparison with the greatest stresses in the mesh.

(3) *Equilibrium of nodal forces and moments is satisfied.* Since the system or global equations that refer to the nodal degree of freedom are solved, equilibrium of nodal forces and moments are enforced and satisfied. In other words,

$$\mathbf{KD} - \mathbf{R} = \mathbf{0}$$

provided the truncation and round-off errors are negligible.

(4) *Compatibility may or may not be satisfied along element boundaries.* This is a function of the types of elements that share common boundaries. Consider the two elements shown below.

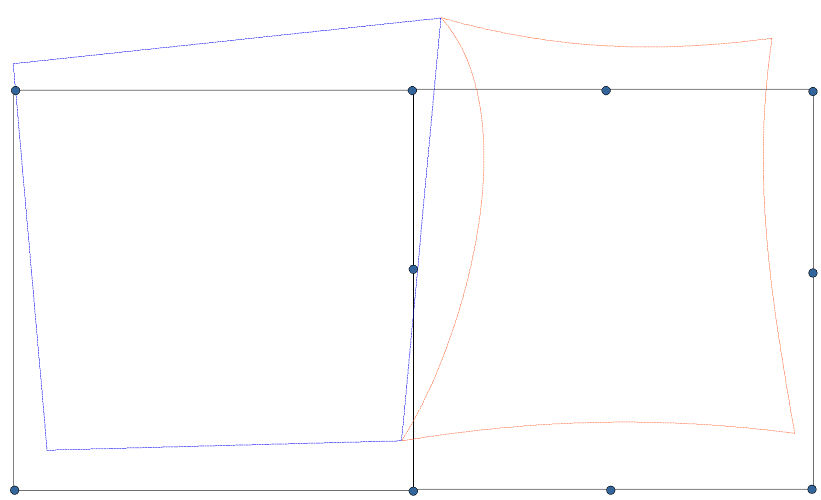


Fig. T3L2-4

The left element is described by edges that deform in a linear fashion. The right element is described by edges that deform in a quadratic fashion (see dashed line). Hence there is an incompatibility between the two elements as illustrated by their deformed shapes.

(5) *Compatibility is satisfied within elements*. This is obvious and true if the assumed displacements are continuous within the element.

(6) *Compatibility is enforced at the nodes*. This is obvious since the displacements at the nodes are unique.

Review Exercises

Problem T3L2-1

Figs. (a) and (b) show two finite element models made up of finite elements that have one degree-of-freedom per node. For each model compute (i) the size of the system stiffness matrix, $\mathbf{K}$, and (ii) the half-band width of $\mathbf{K}$. Show $\mathbf{K}$ in a symbolic form (mark 0 where the stiffness matrix has no contribution from any element and an X where one or more elements contributes a term). Renumber the nodes to minimize the half-band width.

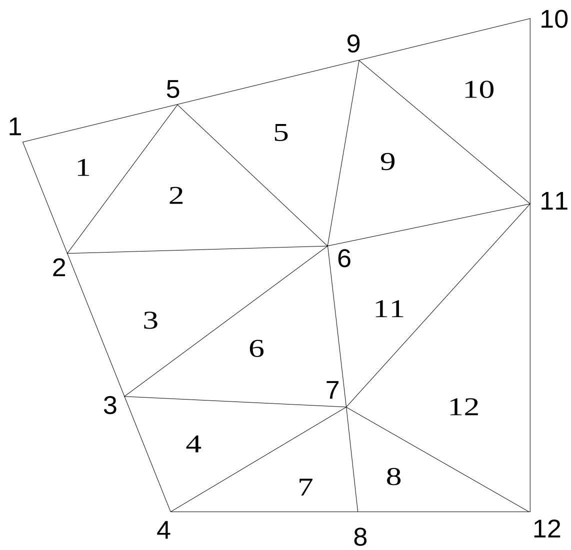


Fig. (a)

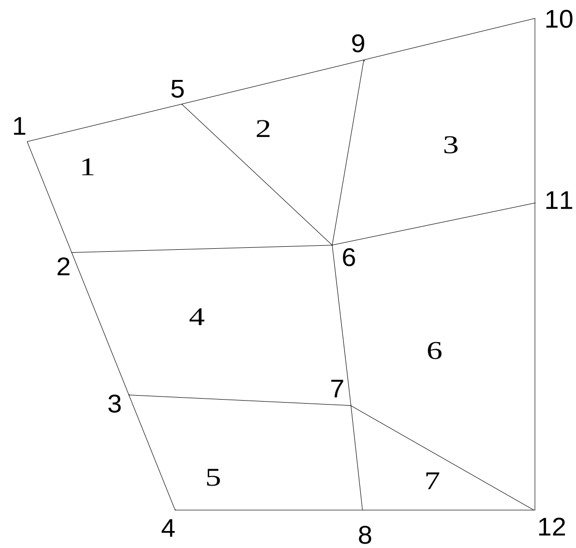


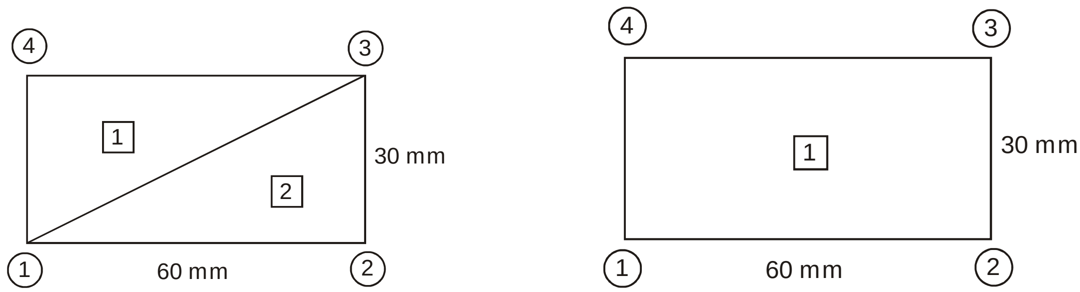
Fig. (b)

Problem T3L2-2

An aluminum cantilever beam is subjected to a tip loading of 10 kN (node 3, negative Y direction). It is modeled as shown in the two FE models below. Assume nodes 1 and 4 are completed restrained. Use $E = 70 GPa$, $\nu = 0.33$ and $t = 10 mm$. Compute nodal displacements, strains, stresses and support reactions.

(a) Use two CST elements.

(b) Use one linear quadrilateral element.



Problem T3L2-3

TBC

Problem T3L2-4

TBC

Problem T3L2-5

TBC

Problem T3L2-6

TBC

Solve the following problems using a computer software.

Problem T3L2-7

TBC

Problem T3L2-8

TBC

Problem T3L2-9

Solve Problem 5.16 from text T1 using a computer program.

Problem T3L2-10

Solve Problem 5.17 from text T1 using a computer program.

Problem T3L2-11

TBC

Problem T3L2-12

TBC

Lesson 3: Axisymmetric Problems

Objectives: In this lesson we will look at axisymmetric problems.

- To understand what is meant by axisymmetric analysis.
- To derive the element equations for commonly used low and higher-order isoparametric axisymmetric elements.
- To solve axisymmetric problems using axisymmetric computer programs.

Axisymmetric Elasticity Problems

What is an axisymmetric problem? It is a two-dimensional problem that deals with solid bodies of revolution. Fig. T3L3-1 shows an axisymmetric body. The z axis is the axis of revolution. It is also known as the axial direction. The r axis represents the radial direction. The geometry (including the entire finite element model[4]) of the body is a function of r and z but is independent of θ. The shaded area shows the part of the body that is used for the axisymmetric analysis.

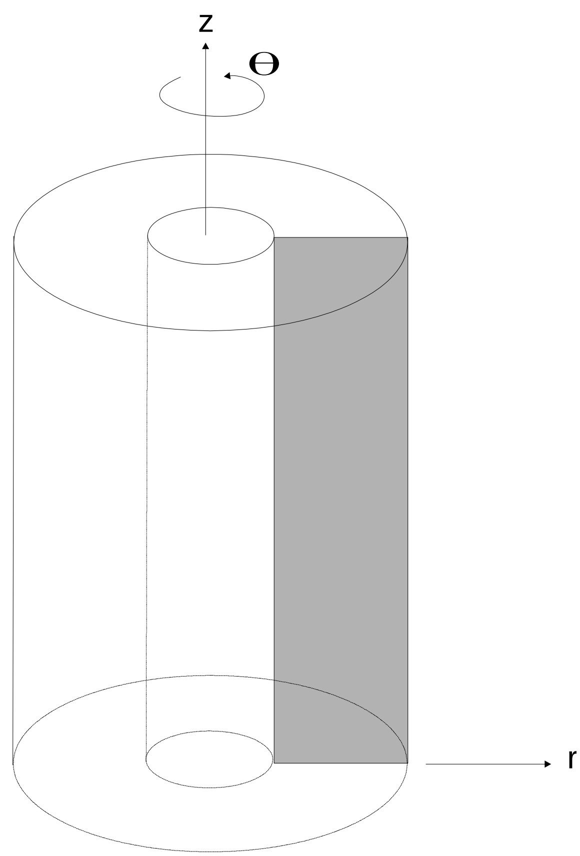


Fig. T3L3-1 Axisymmetric problem showing a solid body of revolution

The three components of displacement along the (r,θ,z), respectively, are

$$u = u(r,z) \qquad \text{(T3L3-1a)}$$

$$v = 0 \qquad \text{(T3L3-1b)}$$

[4] Axisymmetric problems with non-axisymmetric loads can still be analyzed as an axisymmetric problem. However the problem treatment is left as an exercise.

$$w = w(r, z) \tag{T3L3-1c}$$

Hence, $\mathbf{u} = [u, w]^{\mathbf{T}}$ (T3L3-1d)

The material matrix relating the stress to the strain is as follows

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \tau_{rz} \\ \sigma_\theta \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & \nu \\ & 1-\nu & 0 & \nu \\ & & \frac{1}{2}-\nu & 0 \\ SYM & & & 1-\nu \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \gamma_{rz} \\ \varepsilon_\theta \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \alpha \\ 0 \\ \alpha \end{Bmatrix} \Delta T \right) \tag{T3L3-2}$$

The major difference between a planar problem and an axisymmetric problem is that while in the former either the out-of-plane stress or strain is zero in an axisymmetric problem both the components σ_θ and ε_θ are nonzero. These two components are commonly referred to as hoop stress and strain. The strain-displacement relationship is given by

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \gamma_{rz} \\ \varepsilon_\theta \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial r \\ \partial w/\partial z \\ \partial u/\partial z + \partial w/\partial r \\ u/r \end{Bmatrix} \tag{T3L3-3}$$

The total potential energy can be expressed as

$$\Pi = \frac{1}{2}\int_0^{2\pi}\int_A \boldsymbol{\sigma}^{\mathbf{T}}\boldsymbol{\varepsilon}\, r\, dA d\theta - \int_0^{2\pi}\int_A \mathbf{u}^{\mathbf{T}}\mathbf{f}\, r\, dA d\theta - \int_0^{2\pi}\int_\Gamma \mathbf{u}^{\mathbf{T}}\mathbf{T}\, r\, dl\, d\theta - \sum_i \mathbf{u}_i^{\mathbf{T}}\mathbf{P}_i \tag{T3L3-4}$$

Since the variables are independent of θ

$$\Pi = 2\pi\left(\frac{1}{2}\int_A \boldsymbol{\sigma}^{\mathbf{T}}\boldsymbol{\varepsilon}\, r\, dA - \int_A \mathbf{u}^{\mathbf{T}}\mathbf{f}\, r\, dA - \int_\Gamma \mathbf{u}^{\mathbf{T}}\mathbf{T}\, r\, dl \right) - \sum_i \mathbf{u}_i^{\mathbf{T}}\mathbf{P}_i \tag{T3L3-5}$$

Linear Triangular Element

The assumed displacement field is of the form

$$u = a_1 + a_2\xi + a_3\eta \qquad\qquad w = b_1 + b_2\xi + b_3\eta \tag{T3L3-6a}$$

With a slightly different ordering of the nodes in the parent element, the shape functions can be written as

$$\phi_1 = \xi \qquad \phi_2 = \eta \qquad \phi_3 = 1 - \xi - \eta \tag{T3L3-6b}$$

Since, $r = \phi_1 r_1 + \phi_2 r_2 + \phi_3 r_3 \qquad z = \phi_1 z_1 + \phi_2 z_2 + \phi_3 z_3$ (T3L3-7)

$$\mathbf{J}_{2\times 2} = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix} \quad \text{and} \quad \det(J) = r_{13}z_{23} - r_{23}z_{13} \tag{T3L3-8}$$

$$\mathbf{\Gamma}_{2\times 2} = \frac{1}{\det(J)} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & r_{13} \end{bmatrix} \tag{T3L3-9}$$

Note that $(r - z)$ for this element is the same as $(x - y)$ for the CST element. The strain-displacement matrix can be computed as

$$\mathbf{B}_{4\times 6} = \begin{bmatrix} \mathbf{B}^1_{3\times 6} \\ \hline \mathbf{B}^2_{1\times 6} \end{bmatrix} \tag{T3L3-10a}$$

$$\mathbf{B}^1_{3\times 6} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & 0 & 0 \\ 0 & 0 & \Gamma_{21} & \Gamma_{22} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{11} & \Gamma_{12} \end{bmatrix}_{3\times 4} \begin{bmatrix} \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \phi_{3,\xi} & 0 \\ \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & \phi_{3,\eta} & 0 \\ 0 & \phi_{1,\xi} & 0 & \phi_{2,\xi} & 0 & \phi_{3,\xi} \\ 0 & \phi_{1,\eta} & 0 & \phi_{2,\eta} & 0 & \phi_{3,\eta} \end{bmatrix}_{4\times 6} \tag{T3L3-11a}$$

$$\mathbf{B}^2_{1\times 6} = \begin{bmatrix} \dfrac{\phi_1}{r} & 0 & \dfrac{\phi_2}{r} & 0 & \dfrac{\phi_3}{r} & 0 \end{bmatrix}_{1\times 6} \tag{T3L3-11b}$$

Simplifying

$$\mathbf{B}_{4\times6} = \begin{bmatrix} \frac{z_{23}}{\det(J)} & 0 & \frac{z_{31}}{\det(J)} & 0 & \frac{z_{12}}{\det(J)} & 0 \\ 0 & \frac{r_{32}}{\det(J)} & 0 & \frac{r_{13}}{\det(J)} & 0 & \frac{r_{21}}{\det(J)} \\ \frac{r_{32}}{\det(J)} & \frac{z_{23}}{\det(J)} & \frac{r_{13}}{\det(J)} & \frac{z_{31}}{\det(J)} & \frac{r_{21}}{\det(J)} & \frac{z_{12}}{\det(J)} \\ \frac{\phi_1}{r} & 0 & \frac{\phi_2}{r} & 0 & \frac{\phi_3}{r} & 0 \end{bmatrix} \tag{T3L3-12}$$

The element stiffness matrix can now be computed as

$$\mathbf{k}_{6\times6} = 2\pi \iint_A \mathbf{B}^{\mathbf{T}}_{6\times4} \mathbf{D}_{4\times4} \mathbf{B}_{4\times6}\, r dA \tag{T3L3-13}$$

Two points are in order with respect to the strain-displacement matrix $\mathbf{B}$. First, the circumferential strain ε_θ is not constant within the element. In the CST element where we used the same shape functions, the strains were constant within the element. Second, the last row contains the factor $1/r$. This makes the circumferential strain ε_θ (and hence stress) infinite at $r = 0$. However, it should be noted that elements that have nodes at that location are subjected to the EBC

$$u(0, z) = 0 \tag{T3L3-14}$$

Now, can Eqn. (T3L3-13) be simplified so that we can obtain a closed form solution for the stiffness terms which we were able to do with the CST element? One such procedure is to evaluate the stiffness at the centroid of the element $\bar{r} = \dfrac{r_1 + r_2 + r_3}{3}$. With this strategy

$$\mathbf{k}_{6\times6} = 2\pi \bar{r} A \mathbf{B}^{\mathbf{T}}_{6\times4} \mathbf{D}_{4\times4} \mathbf{B}_{4\times6} \tag{T3L3-15}$$

In a similar fashion the nodal forces due to body force can be computed as

$$\mathbf{f}_{6\times1} = \frac{2\pi \bar{r} A}{3} \left[\overline{f_r}, \overline{f_z}, \overline{f_r}, \overline{f_z}, \overline{f_r}, \overline{f_z} \right]^{\mathbf{T}} \tag{T3L3-16}$$

where $\overline{f_r}, \overline{f_z}$ are the body force components (force per unit volume) in the r and z directions. The nodal forces due to a uniform surface traction on side $1-2$ of the triangle are

$$\mathbf{T}_{6\times1} = \frac{2\pi L_{1-2}}{6}\left[aT_r, aT_z, bT_r, bT_z, 0, 0\right]^{\mathbf{T}} \qquad \text{(T3L3-17)}$$

where T_r and T_z are the surface traction (in force per unit area) acting in the r and z directions at nodes 1 and 2, L_{1-2} is the length of side $1-2$, and $a = 2r_1 + r_2$ and $b = r_1 + 2r_2$. The thermal load vector due to a temperature change in the element is given as

$$\mathbf{f}_{6\times1} = 2\pi \bar{r} A \overline{\mathbf{B}}^{\mathbf{T}} \mathbf{D}\overline{\boldsymbol{\varepsilon}_0} \qquad \text{(T3L3-18)}$$

where $\overline{\boldsymbol{\varepsilon}_0} = \left[\alpha\Delta T, \alpha\Delta T, 0, \alpha\Delta T\right]^{\mathbf{T}}$

Once the nodal displacements are computed, the element strains are evaluated at the centroid of the element

$$\boldsymbol{\varepsilon}_{4\times1} = \overline{\mathbf{B}}_{4\times6}\mathbf{d}_{6\times1} \qquad \text{(T3L3-19)}$$

and the element stresses using

$$\boldsymbol{\sigma}_{4\times1} = \mathbf{D}_{4\times4}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0\right)_{4\times1} \qquad \text{(T3L3-20)}$$

Illustrative Example

Fig. T3L3-2 shows a long circular cylinder subjected to an internal pressure of $2\,MPa$. The material properties are as follows - $E = 200\,GPa$ and $\nu = 0.3$. A few points should be noted about the corresponding finite element model. First, due to the nature of the axisymmetric problem, only "one half" of the model needs to be constructed. The FE model is shown in Fig. T3L3-3. Second, the axial direction z is taken as the Y direction. The radial direction r is denoted X. Lastly, the structure has no rigid body modes in the X direction or a rotation about the Z axis. To prevent the rigid body mode in the Y direction, (minimum of) a single point must be selected and its displacement suppressed in that direction. The general rule of thumb is to select a point where the response quantities are **not** important[5].

The objective of the analysis is to compute the largest principal stress and the radial displacement in the model. The model dimension in the axial direction is arbitrarily selected to be 50 mm.

---

[5] St. Venant's Principle states that while different statically equivalent solutions may differ significantly close to a support, the solutions are valid in regions that are sufficiently far away from supports.

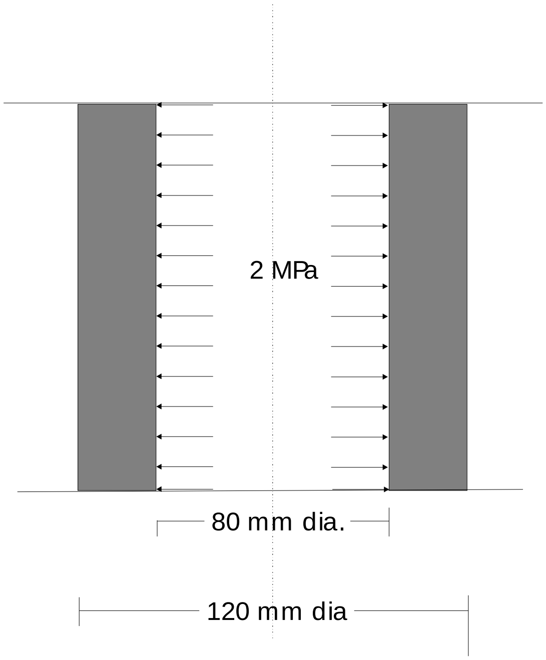


Fig. T3L3-2 Long circular cylinder

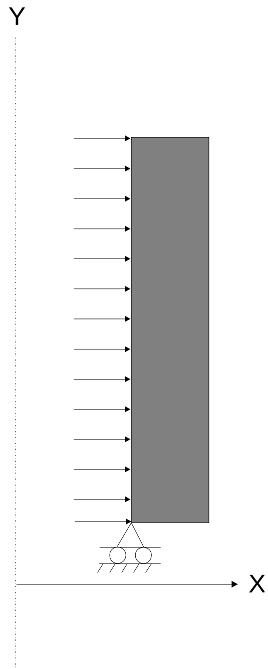


Fig. T3L3-3 Finite element model

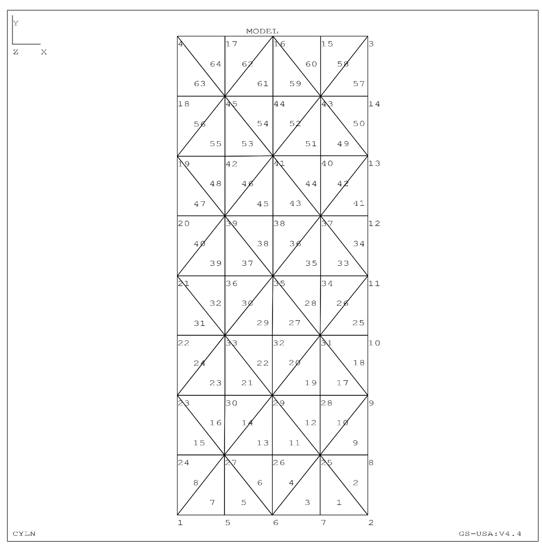


Fig. T3L3-4 Model ID 1 FE mesh

The results from the FE analysis are shown below.

Model ID	Number of Elements	$(\sigma_{principal})_{\max}$, Pa	Max. Radial defl. (m)
1	45	(4.954E+06, -1.689E+06)	1.1622E-06
2	168	(5.033E+06, -1.780E+06)	1.1611E-06

The model should be refined further as we have done in the previous problems, so that the proper convergence characteristics can be established.

Review Exercises

Problem T3L3-1

TBC

Problem T3L3-2

TBC

Problem T3L3-3

TBC

Concluding Remarks

In this topic we saw the isoparametric formulation and derivation of the element equations for solving solid mechanics problems. Specifically, we saw the (straight) truss and beam elements that can be used in two and three dimensions. Next, we saw the power of the isoparametric formulation with plane elasticity and axisymmetric elasticity elements. We also saw the used of these elements in solving more realistic problems than those used in Module 1. However, the basic ideas are still the same. We discretize the domain into suitable finite elements starting with a coarse mesh and gradually refine the mesh to study the effects of the mesh size and density on key response quantities.

There are several topics that have not been mentioned in this topic. I will list some of the topics that you may wish to explore.

- Make a list of all the low and higher-order elements suitable for solving static, linear solid mechanics problems. Write down the total number of degrees freedom per element, the number of Gauss points needed to generate the element stiffness and the element load vector, and the number of Gauss points needed to evaluate the strain and stress tensors.

- We did not formally tackle the issue of model and mesh generation. Explore the topics of geometric modeling and mesh generation.

- Resolve the example problems in several different ways. (a) All the example problems were solved using a uniform mesh. Resolve the problems using a non-uniform mesh. (b) All the example problems were solved using the lowest order elements. Resolve the problems using higher-order elements.

- Can you think of a way of moving from a region of low-order elements to high-order elements without losing element compatibility?

- Some problems can be solved very effectively using symmetric conditions. We saw an example in an earlier lesson. Sometimes lines or planes of symmetry are not aligned with the global X and Y axes. How can we use single point constraints to handle these problems? What is multi-point constraint and where can they be used?

- There exists a class of problems involving semi-infinite domains. Examples include problems in pavement analysis and geotechnical engineering. How can semi-infinite elements be formulated and used?

- To make the one-to-one mapping work for isoparametric elements, we saw why it was necessary to have a positive jacobian. However, there is a class of problems for which singularity is desirable. Numerical fracture mechanics requires the use of singular finite elements to capture the stress singularity that exist at crack tips.

- Review the relationship between the stress components and stress invariants used in failure criteria such as principal stress, von Mises failure criterion, Tresca failure criterion etc.

- The two most compute time intensive steps in the FE methodology are the ones involving generation of element and system equations, and the solution of these equations. Efficiency of the latter can be increased by going from full matrices to banded, skyline or sparse storage schemes. Study the different types of equation solvers. How would you monitor the quality of the numerical solution with respect to the quality of the stiffness matrix, and the truncation and round-off errors?

- Once the element stresses are computed at the appropriate Gauss points, it is possible to extrapolate them to the element nodes. Study the different ways one could generate the nodal stresses. Can these be used to generate error indicators?

- We have looked at linear, elastic, isotropic, homogenous material behavior. Where exactly in the formulation and generation of the element equations are these assumptions used? Can you think of how these assumptions can be relaxed to tackle other classes of problems? What implementation issues arise in these classes of problems?

Index