Finite Elements for Engineers

Lecture 3: Isoparametric Formulation

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Isoparametric Formulation

- Equal number of parameters are used to represent the **geometry** and the unknown **variable**
- For example

Geometry $$x(t) = a_0 + a_1 t + a_2 t^2$$

Unknown $$u(t) = b_0 + b_1 t + b_2 t^2$$

Mapping Options

Geometry $$x(\xi) = \sum_{j=1}^{s} \hat{\phi}_j(\xi)\, x_j$$

Unknown $$u(\xi) = \sum_{i=1}^{r} \phi_i(\xi)\, u_i$$

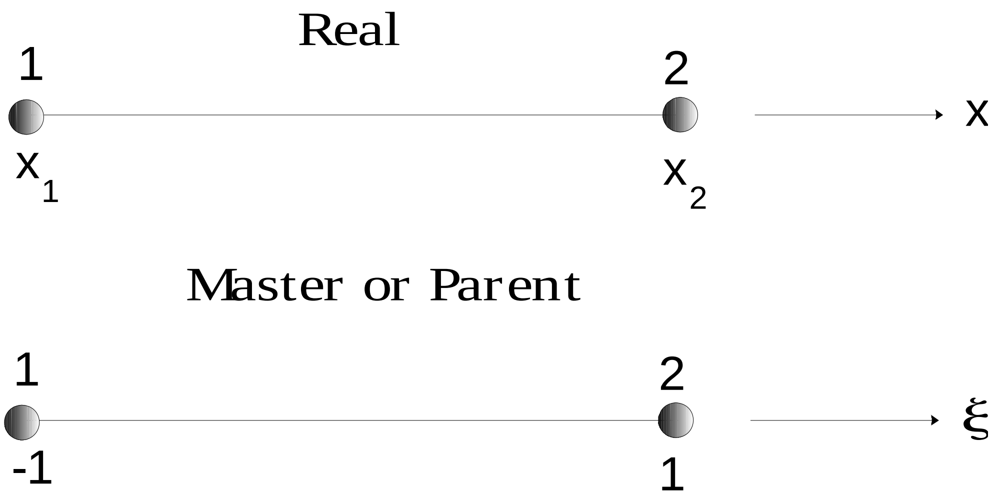
Subparametric $$s < r$$

Isoparametric $$s = r$$

Superparametric $$s > r$$

1D Isoparametric Elements

1D-C^0 Linear Isoparametric Element



$$x(\xi) = \phi_1(\xi)\,x_1 + \phi_2(\xi)\,x_2$$

$$\phi_1 = \frac{1-\xi}{2} \qquad \phi_2 = \frac{1+\xi}{2}$$

$$u(\xi) = \phi_1(\xi)\,u_1 + \phi_2(\xi)\,u_2$$

Derivatives

$$\frac{dx}{d\xi} = \frac{\left(x_2 - x_1\right)}{2} = \frac{L}{2} = J \quad \Rightarrow \quad \frac{d\xi}{dx} = \frac{1}{dx/d\xi} = J^{-1} = \frac{2}{L}$$

1D-C^0 Linear Isoparametric BVP Element

Step 2

$$\sum_{j=1}^{2}\left[\int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx+\int_{\Omega}\phi_i(x)\beta(x)\phi_j(x)dx\right]y_j=$$

$$\int_{\Omega} f(x)\phi_i(x)\,dx-\left[\tau\phi_i\right]^{\Gamma} \qquad i=1,2$$

$$\sum_{j=1}^{2}\left[k_{ij}^{\alpha}+k_{ij}^{\beta}\right]y_j=f_i^{\text{int}}+f_i^{bnd} \qquad i=1,2$$

1D-C^0 Linear Element (cont'd)

Recall

$$\phi_1 = \frac{1-\xi}{2} \qquad \phi_2 = \frac{1+\xi}{2} \qquad J = \frac{dx}{d\xi} = \frac{L}{2}$$

$$\frac{d\phi_1}{d\xi} = -\frac{1}{2} \qquad \frac{d\phi_2}{d\xi} = \frac{1}{2} \qquad \Rightarrow \qquad \frac{d\phi_i}{dx} = \frac{d\phi_i}{d\xi}\frac{d\xi}{dx} = \frac{d\phi_i}{d\xi}\frac{2}{L}$$

$$k_{ij}^{\alpha} = \int_{\Omega} \frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx = \int_{-1}^{1}\frac{d\phi_i}{d\xi}\alpha(x(\xi))\frac{d\phi_j}{d\xi}\frac{4}{L^2}J\,d\xi$$

$$k_{ij}^{\beta} = \int_{\Omega}\phi_i(x)\beta(x)\phi_j(x)\,dx = \int_{-1}^{1}\phi_i(\xi)\beta(x(\xi))\phi_j(\xi)\frac{L}{2}d\xi$$

1D-C^0 Linear Element (cont'd)

Assuming that $\alpha(x)=\hat{\alpha}, \beta(x)=\hat{\beta}, f(x)=\hat{f}$

For example

$$k_{11}^{\alpha}=\int_{-1}^{1}\frac{d\phi_1}{d\xi}\alpha(x(\xi))\frac{d\phi_1}{d\xi}\frac{2}{L}d\xi=\int_{-1}^{1}\frac{1}{4}\hat{\alpha}\frac{2}{L}d\xi=\frac{\hat{\alpha}}{2L}(2)=\frac{\hat{\alpha}}{L}$$

$$k_{12}^{\alpha}=\int_{-1}^{1}\frac{d\phi_1}{d\xi}\alpha(x(\xi))\frac{d\phi_2}{d\xi}\frac{2}{L}d\xi=\int_{-1}^{1}\left(-\frac{1}{4}\right)\hat{\alpha}\frac{2}{L}d\xi=-\frac{\hat{\alpha}}{2L}(2)=-\frac{\hat{\alpha}}{L}$$

$$k_{12}^{\beta}=\int_{-1}^{1}\phi_1(\xi)\,\hat{\beta}\,\phi_2(\xi)\frac{L}{2}d\xi=\frac{\hat{\beta}L}{8}\int_{-1}^{1}(1-\xi^2)d\xi=\frac{\hat{\beta}L}{6}$$

$$f_1^{\text{int}}=\int_{-1}^{1}\hat{f}\phi_1(\xi)\frac{L}{2}d\xi=\frac{\hat{f}L}{4}\int_{-1}^{1}(1-\xi)d\xi=\frac{\hat{f}L}{2}$$

1D-C^0 Linear Element (cont'd)

Element Equations

$$\left[\left[\begin{array}{c|c} \dfrac{\hat{\alpha}}{L}+\dfrac{\hat{\beta}L}{3} & -\dfrac{\hat{\alpha}}{L}+\dfrac{\hat{\beta}L}{6} \\ \hline -\dfrac{\hat{\alpha}}{L}+\dfrac{\hat{\beta}L}{6} & \dfrac{\hat{\alpha}}{L}+\dfrac{\hat{\beta}L}{3} \end{array}\right] - g_1\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right] + h_2\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right]\right]\left\{\begin{array}{c} y_1 \\ \hline y_2 \end{array}\right\} =$$

$$\left\{\begin{array}{c} \dfrac{\hat{f}L}{2} \\ \hline \dfrac{\hat{f}L}{2} \end{array}\right\} + \left\{\begin{array}{c} c_1 \\ \hline -c_2 \end{array}\right\}$$

1D-C^0 Quadratic Isoparametric BVP Element

Step 2

$$\sum_{j=1}^{3}\left[\int_{\Omega}\frac{d\phi_i}{dx}\alpha(x)\frac{d\phi_j}{dx}dx+\int_{\Omega}\phi_i(x)\beta(x)\phi_j(x)dx\right]y_j=$$

$$\int_{\Omega} f(x)\phi_i(x)\,dx-\left[\tau\phi_i\right]^{\Gamma} \qquad i=1,3$$

$$\sum_{j=1}^{3}\left[k_{ij}^{\alpha}+k_{ij}^{\beta}\right]y_j = f_i^{\text{int}}+f_i^{bnd} \qquad i=1,3$$

1D-C^0 Quadratic Isoparametric BVP Element

Always valid element geometry?

$$x = \frac{1}{2}\xi(\xi - 1)x_1 + (1 - \xi^2)x_2 + \frac{1}{2}\xi(\xi + 1)x_3$$

$$J = \frac{dx}{d\xi} = \left(\xi - \frac{1}{2}\right)x_1 + (-2\xi)x_2 + \left(\xi + \frac{1}{2}\right)x_3$$

$$J = \xi(x_1 - 2x_2 + x_3) + \frac{1}{2}(x_3 - x_1)$$

Need Jacobian to be positive throughout the element.

1D-C^0 Quadratic Isoparametric BVP Element

$$J > 0 \quad -1 \le \xi \le 1$$

Hence

$$J(-1) = -\left(\frac{3}{2}\right)x_1 + (2)x_2 - \left(\frac{1}{2}\right)x_3 > 0 \quad \Rightarrow \quad x_2 > x_1 + \frac{L}{4}$$

$$J(+1) = \left(\frac{1}{2}\right)x_1 - (2)x_2 + \left(\frac{3}{2}\right)x_3 > 0 \quad \Rightarrow \quad x_2 < x_1 + \frac{3L}{4}$$

$$x_c - \frac{L}{4} \le x_2 \le x_c + \frac{L}{4} \qquad x_c = \frac{x_1 + x_3}{2}$$

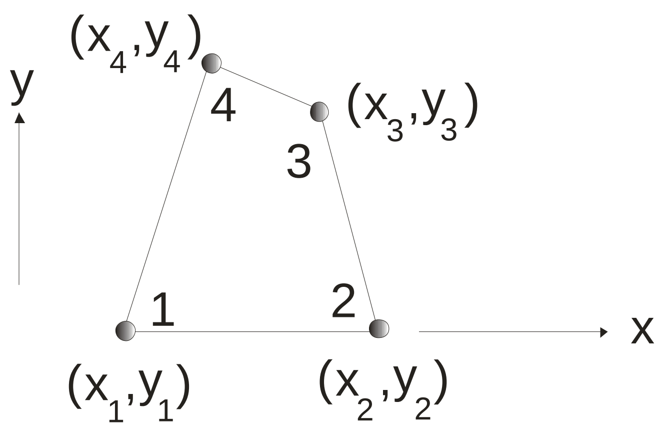
Optimal Flux Computations

Element	Location
1D Linear	$\xi = 0$
1D Quadratic	$\xi_1 = -\frac{1}{\sqrt{3}}$ $\quad \xi_2 = \frac{1}{\sqrt{3}}$

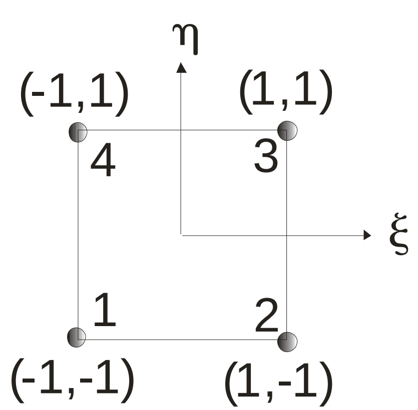
J. Barlow, Optimal stress locations in finite element models, Int. J. Numer. Methods Eng., 10, 243-251 (1976).

2D Isoparametric Elements

Real



Master



Mapping

$$x = \sum_{i=1}^{4} \phi_i(\xi,\eta) x_i \qquad y = \sum_{i=1}^{4} \phi_i(\xi,\eta) y_i$$

Derivatives

$$\frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \xi} x_i \qquad \frac{\partial x}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \eta} x_i$$

$$\frac{\partial y}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \xi} y_i \qquad \frac{\partial y}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial \phi_i}{\partial \eta} y_i$$

2D Isoparametric Elements

Derivatives $u = u(\xi, \eta)$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi} \qquad \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix} u,_{\xi} \\ u,_{\eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u,_{x} \\ u,_{y} \end{Bmatrix} = \mathbf{J}_{2\times 2} \begin{Bmatrix} u,_{x} \\ u,_{y} \end{Bmatrix}$$

$$\begin{Bmatrix} u,_{x} \\ u,_{y} \end{Bmatrix} = \mathbf{\Gamma}_{2\times 2} \begin{Bmatrix} u,_{\xi} \\ u,_{\eta} \end{Bmatrix} \Rightarrow \quad \mathbf{\Gamma} = \mathbf{J}^{-1}$$

Summary

- Isoparametric formulation has several advantages.
  - It ties in nicely with the manner in which shape functions are generated (the use of a master element and the family of low and higher-order elements)
  - It ties in nicely with numerical integration (Gauss Quadrature)
  - It helps detect elements that have bad aspect ratios

Summary

- Isoparametric Formulation
  - Lends itself to automation
  - Easy to write computer programs for the entire family of elements
  - Differences between the elements in a family are (a) the shape functions and (b) the number of integration points required for numerical integration