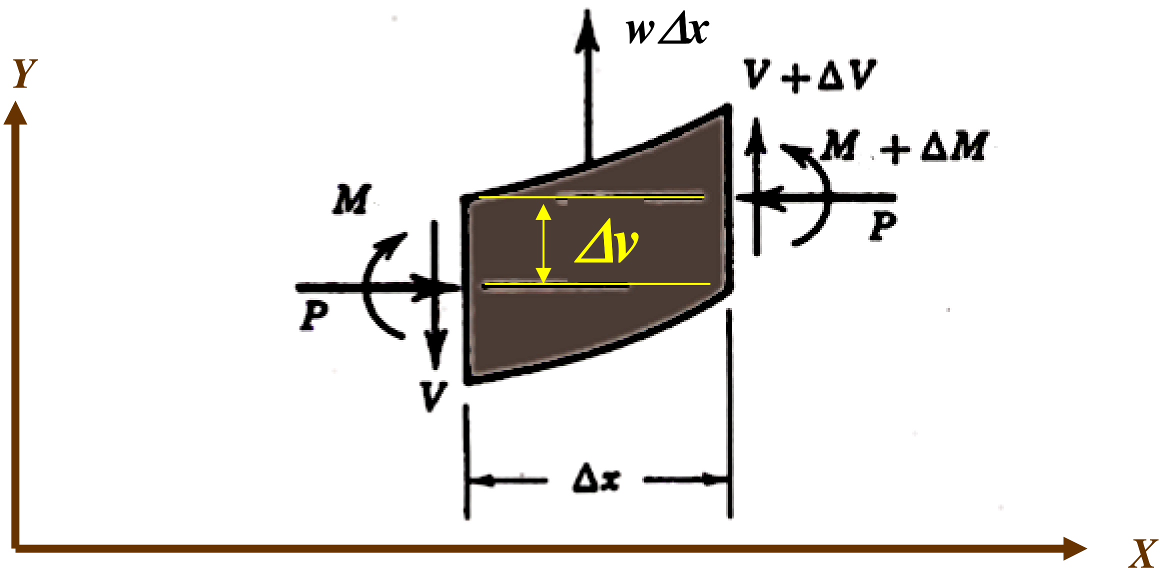
Stability of beam column



General Equation of a Beam Column

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 v}{dx^2}\right)+\frac{d}{dx}\left(P\frac{dv}{dx}\right)=w$$

Where EI and P are constants

$$EI\frac{d^4 v}{dx^4}+P\frac{d^2 v}{dx^2}=w$$

We consider the stability of the second case only

Stability of beam column

$$EI\frac{d^4v}{dx^4}+P\frac{d^2v}{dx^2}=w$$

The above equation is a ***4th order linear ODE*** *whose solution has two parts –*

- ***homogeneous solution or free response***

and the

- ***particular integral or forced response .***

Whether the beam column will buckle (become unstable) or not (remain stable) due to the effect of orthogonal loading can be diagnosed by considering the free or homogeneous solution subject to given boundary conditions.

The forced response or the particular integral will give the steady state deformed shape of the beam column that we shall observe ***provided that shape is stable.***

Stability of beam column

Homogeneous equation

$$EI\frac{d^4v}{dx^4} + P\frac{d^2v}{dx^2} = 0$$

Solution

$$EI\frac{d^4v}{dx^4} + P\frac{d^2v}{dx^2} = 0$$

$$\Rightarrow \frac{d^4v}{dx^4} + \left(\frac{P}{EI}\right)\frac{d^2v}{dx^2} = 0$$

$$\Rightarrow \frac{d^4v}{dx^4} + k^2\frac{d^2v}{dx^2} = 0, \qquad k=\sqrt{\frac{P}{EI}}$$

$$\frac{d^4v}{dx^4} + k^2\frac{d^2v}{dx^2} = 0 \Rightarrow \frac{d^2}{dx^2}\left(\frac{d^2v}{dx^2} + k^2v\right) = 0$$

$$Substituing\ m=\frac{d}{dx} \Rightarrow m^2\left(m^2 + k^2\right)v = 0$$

$$m = 0,0,\pm ik$$

Hence solution is

$$v=A\sin(kx) + B\cos(kx) + Cx + D$$

Stability of beam column

Beam column hinged at both ends

$$v(0)=0,\frac{d^2v}{dx^2}(0)=0$$
$$v(L)=0,\frac{d^2v}{dx^2}(L)=0$$

Solution

$$v = A\sin(kx) + B\cos(kx) + Cx + D$$
$$\frac{dv}{dx} = kA\cos(kx) - kB\sin(kx) + C$$
$$\frac{d^2v}{dx^2} = -k^2A\sin(kx) - k^2B\cos(kx)$$
$$\frac{d^3v}{dx^3} = -k^3A\cos(kx) + k^3B\sin(kx)$$

Stability of beam column

Solution for Beam column hinged at both ends

$$v(0)=0 \Rightarrow A\,sin(0)+B\,cos(0)+C(0)+D=0$$

$$\frac{d^2v}{dx^2}(0) \Rightarrow -k^2A\,sin(0)-k^2B\,cos(0)=0$$

$$v(L)=0 \Rightarrow A\,sin(kL)+B\,cos(kL)+CL+D=0$$

$$\frac{d^2v}{dx^2}(L)=0 \Rightarrow -k^2A\,sin(kL)-k^2B\,cos(kL)=0$$

In matrix form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -k^2 & 0 & 0 \\ sin(kL) & cos(kL) & 0 & 0 \\ 1 & 1 & L & 1 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = 0$$

Stability of beam column

Solution for Beam column hinged at both ends

For Nontrivial Solution

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & -k^2 & 0 & 0 \\ sin(kL) & cos(kL) & 0 & 0 \\ 1 & 1 & L & 1 \end{vmatrix} = 0$$

$$\Rightarrow sin(kL) = 0 \Rightarrow kL = n\pi \Rightarrow k = \frac{n\pi}{L}$$

Mode shape

$$\therefore v = Asin(kx) = Asin\left(\frac{n\pi}{L}x\right)$$

For first buckling mode, i.e. n=1

$$k = \frac{\pi}{L} \Rightarrow \sqrt{\frac{P}{EI}} = \frac{\pi}{L}$$

Critical load *for a beam column hinged at both ends*

$$\therefore P_{cr} = \frac{\pi^2 EI}{L^2}$$

Critical load *for higher modes*

$$P_{cr} = \frac{n^2\pi^2 EI}{L^2}$$

Stability of beam column

Beam column : one end fixed, other end free

$$v(0)=0, \frac{dv}{dx}(0)=0$$

$$Moment = \frac{d^2v}{dx^2}(L)=0$$

$$Shear = EI\frac{d^3v}{dx^3}(L)+P\frac{dv}{dx}(L)=0$$

For Nontrivial Solution

$$cos(kL)=0$$

For buckling

$$kL=(2n-1)\frac{\pi}{2}$$

Mode shape

$$v = B(1-cos\,kx) = B\left[1-cos\left\{(2n-1)\frac{\pi x}{2L}\right\}\right]$$

***Critical load** for a fixed free beam column*

$$P_{cr}=(2n-1)^2\frac{\pi^2 EI}{2L^2}$$

First mode

$$v = B\left[1-cos\left(\frac{\pi x}{2L}\right)\right], P_{cr}=\frac{\pi^2 EI}{2L^2}$$

Stability of beam column

Beam column : one end fixed, other end hinged

$$v(0)=0,\frac{dv}{dx}(0)=0,v(L)=0,\frac{d^2v}{dx^2}(L)=0$$

$$-sin(kL)+kLcos(kL)=0$$

$$or \quad tan(kL)=kL$$

$$kL=4.4934,7.7253,10.9041,...$$

Nontrivial Solution

For buckling

For first mode, i.e. n = 1

$$kL=4.493$$

Mode shape

$$v=A\left[sin(kx)-\frac{k}{L}cos(kx)-kx+\frac{k}{L}\right]$$

***Critical load** for a fixed free beam column*

$$P_{cr}=20.19\frac{EI}{L^2},$$

$$59.68\frac{EI}{L^2},118.9\frac{EI}{L^2},....$$

Stability of beam column

Beam column : both ends fixed

$$v(0)=0,\frac{dv}{dx}(0)=0,v(L)=0,\frac{dv}{dx}(L)=0$$

$$2\{cos(kL)-1\}+kLsin(kL)=0$$

Nontrivial Solution

$$or \quad sin\left(\frac{kL}{2}\right)\left\{\frac{kL}{2}cos\left(\frac{kL}{2}\right)-sin\left(\frac{kL}{2}\right)\right\}=kL$$

$$kL=2n\pi, tan\left(\frac{kL}{2}\right)=\frac{kL}{2},.... \Rightarrow kL=0,8.9868,....$$

Critical load *for a fixed fixed beam column*

$$kL=2n\pi$$

$$\Rightarrow v=B\left[cos\left(\frac{2n\pi x}{L}\right)-1\right]\Rightarrow P_{cr}=\frac{4n^2\pi^2 EI}{L^2},$$

$$tan\left(\frac{kL}{2}\right)=\frac{kL}{2}\Rightarrow P_{cr}=\frac{8.18\pi^2 EI}{L^2},$$