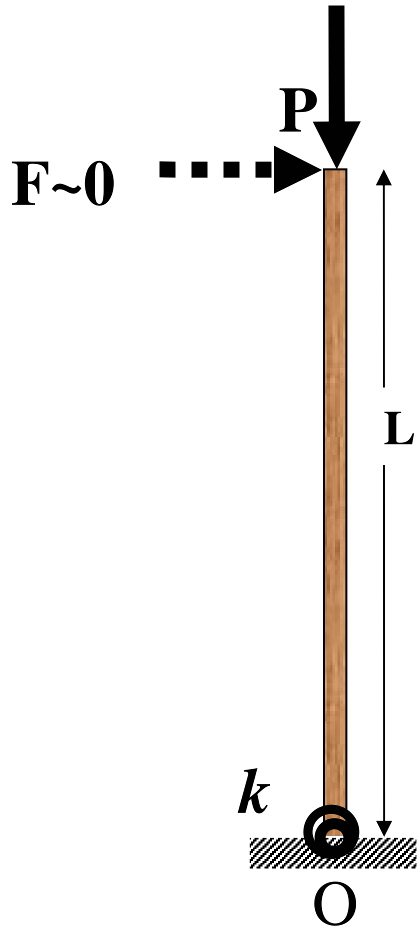
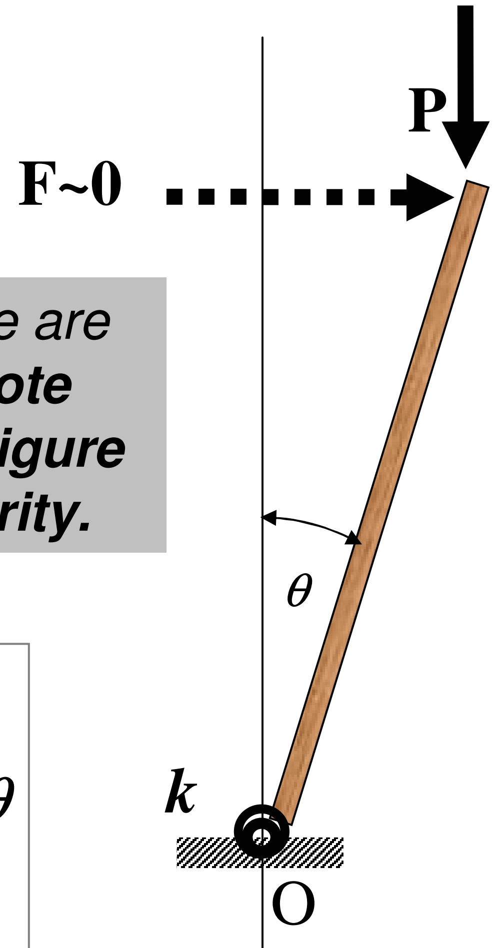
Stability of a column



We first consider a column hinged at the base and loaded with a vertical force P. It is obvious that a slight disturbance in the form of a small horizontal force F can disturb the equilibrium. We consider the case where the hinge has a torsional stiffness k and analyze the stability of this equilibrium .

Will the column come back to its original position in case of a disturbance or not?

Stability of a column



If the disturbance is minimal we are justified in considering F~0. ***Note that the deviation shown in figure is highly exaggerated for clarity.***

$$\sum M_O = 0$$

$$\Rightarrow PL\sin\theta + FL\cos\theta = k\theta$$

$$F = 0, \theta \to 0$$

$$\Rightarrow PL\theta + (0)L = k\theta$$

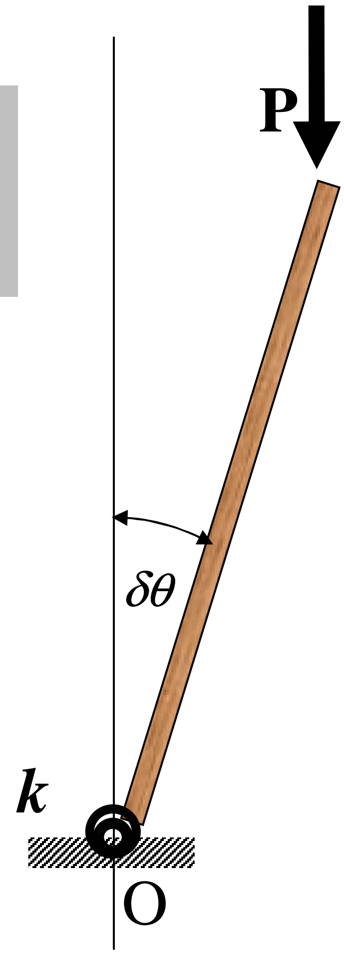
$$\Rightarrow P = \frac{k}{L}$$

$$P > \frac{k}{L} \Rightarrow Stable$$

$$P < \frac{k}{L} \Rightarrow Unstable$$

Stability of a column

We now consider a small disturbance $\delta\theta$ in order to study the nature of equilibrium



$$\sum M_O = 0$$
$$\Rightarrow PL\sin\delta\theta - k\delta\theta = 0$$
$$\Rightarrow PL\delta\theta - k\delta\theta = 0$$
$$\Rightarrow (PL - k)\delta\theta = 0$$

Two possibilities exist

$$PL - k = 0, \delta\theta = 0$$

$$\delta\theta = 0$$

Stable shape – beam is vertical

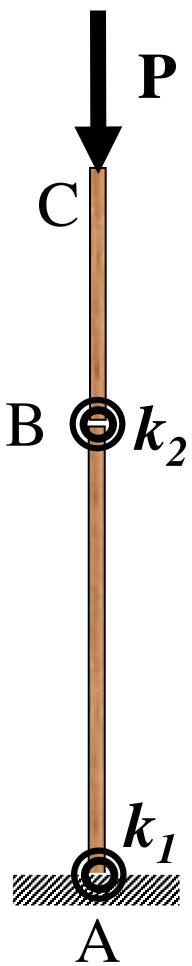
$$P = P_{cr} = \frac{k}{L}$$

Maximum load for which equilibrium is always restored

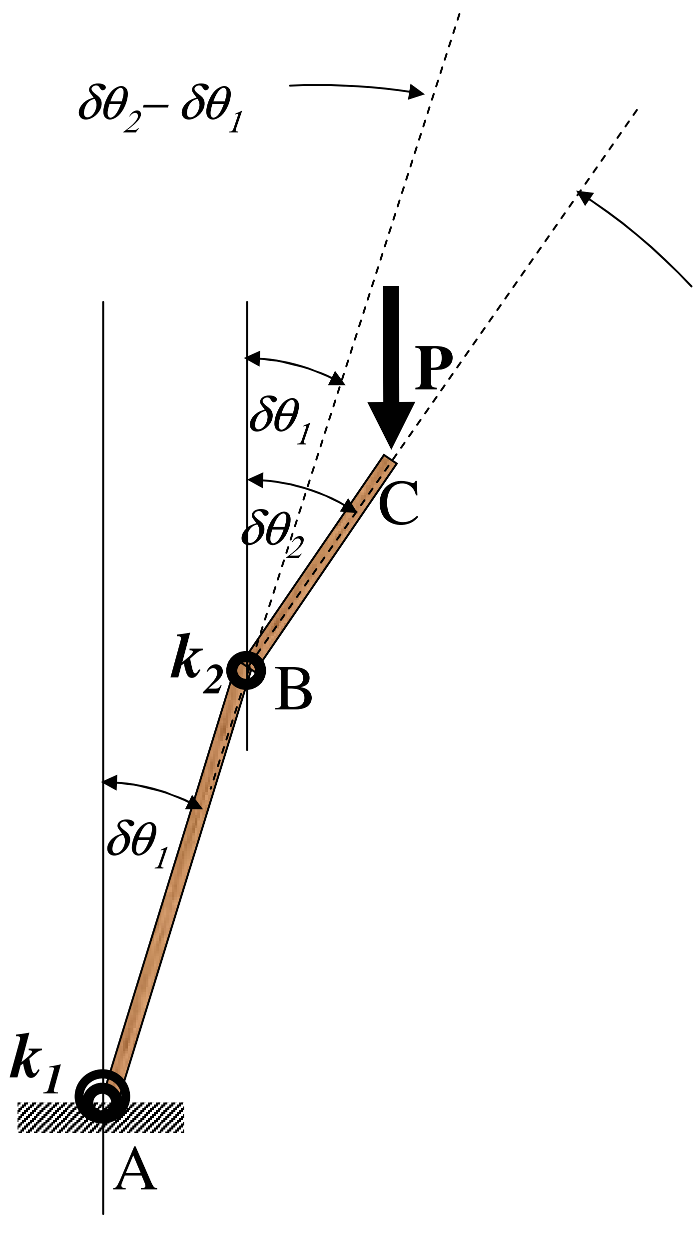
Stability of structures

We consider a 2 DOF system to elaborate and understand this idea

Undisturbed configuration

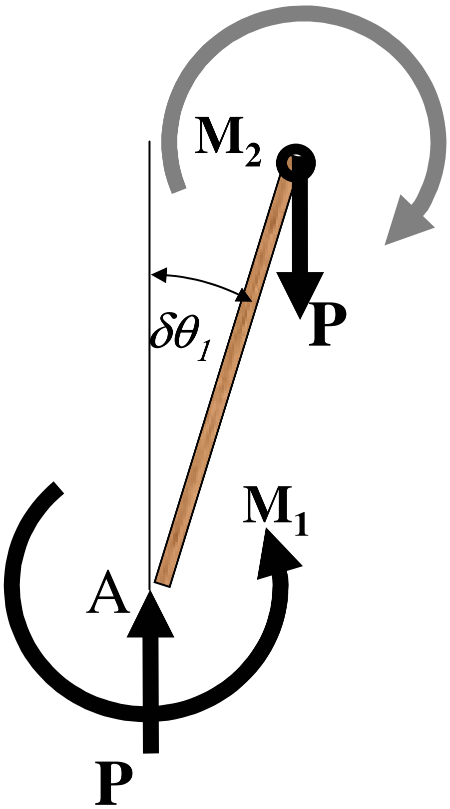


Disturbed configuration

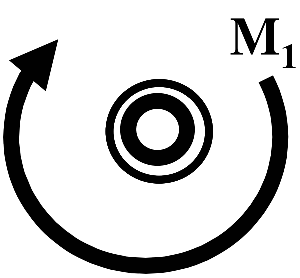


Stability of structures

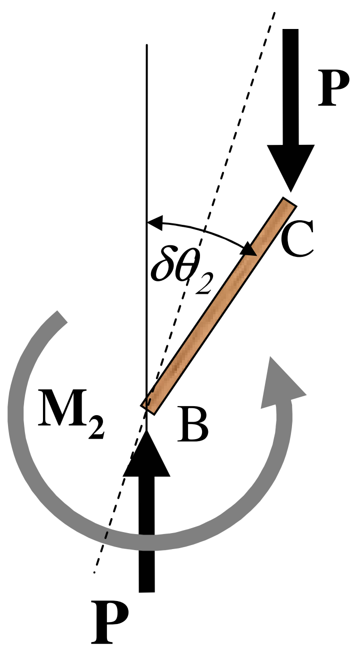
Free body analysis of individual segments



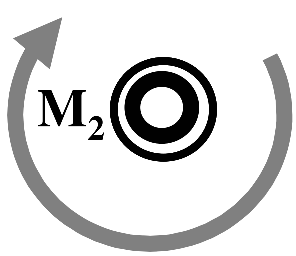
$$\sum M_A = 0$$
$$\Rightarrow PL\sin\delta\theta_2 + M_2 - M_1 = 0$$
$$\Rightarrow PL\delta\theta_1 + M_2 - M_1 = 0$$



$$k_1\delta\theta_1 = M_1$$



$$\sum M_A = 0$$
$$\Rightarrow PL\sin\delta\theta_2 - M_2 = 0$$
$$\Rightarrow PL\delta\theta_2 - M_2 = 0$$



$$k_2\left(\delta\theta_2 - \delta\theta_1\right) = M_2$$

Stability of structures

Equilibrium equations

$$PL\delta\theta_2 - M_2 = 0$$
$$\Rightarrow PL\delta\theta_2 - k_2\left(\delta\theta_2 - \delta\theta_1\right) = 0$$
$$PL\delta\theta_1 + M_2 - M_1$$
$$\Rightarrow PL\delta\theta_1 + k_2\left(\delta\theta_2 - \delta\theta_1\right) - k_1\delta\theta_1 = 0$$

Rearranging in matrix form

$$\begin{bmatrix} k_2 & PL - k_2 \\ PL - k_1 - k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \delta\theta_1 \\ \delta\theta_2 \end{Bmatrix} = 0$$

For non trivial solution

$$\begin{vmatrix} k_2 & PL - k_2 \\ PL - k_1 - k_2 & k_2 \end{vmatrix} = 0$$
$$\Rightarrow P^2L^2 - PL\left(k_1 + 2k_2\right) + k_1k_2 = 0$$

Eigenvalue equation

$$\Rightarrow PL = \frac{k_1 + 2k_2 \pm \sqrt{k_1{}^2 + 4k_2{}^2}}{2}$$

Eigenvalues

Stability of structures

We consider the case where $k_1 = k_2 = k$

$$P_{cr} = \left(3 \pm \sqrt{5}\right)\frac{k}{2L} = \frac{2.618k}{L}, \frac{0.382k}{L}$$

Eigenvalues

What this means is that when initial load is less than 0.382 k/L, a small disturbance will not disturb the equilibrium till the load becomes more than 0.382 k/L. When initial load is between 0.382 k/L and 2.618 k/L a small disturbance will not disturb the equilibrium till the load becomes more than 2.618 k/L. For practical purposes, only the lowest limit is important.

$$P_{cr} = 0.382\frac{k}{L}$$

Eigenvalue first mode

Eigenvector first mode

$$k\delta\theta_1 + \left(PL - k\right)\delta\theta_2 \Rightarrow \begin{Bmatrix} \delta\theta_1 \\ \delta\theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 - \dfrac{PL}{k} \\ 1 \end{Bmatrix}\delta\theta_2 = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix}\delta\theta_2$$

$$P_{cr} = 2.618\frac{k}{L}$$

Eigenvalue second mode

Eigenvector second mode

$$k\delta\theta_1 + \left(PL - k\right)\delta\theta_2 \Rightarrow \begin{Bmatrix} \delta\theta_1 \\ \delta\theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 - \dfrac{PL}{k} \\ 1 \end{Bmatrix}\delta\theta_2 = \begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix}\delta\theta_2$$

*Substituting the critical values gives two **eigenvectors**, whose component ratios are the ratios of the deflections at those critical loads.*