Rates of Return

IE 201 – Chapter 6

Based on Dr. Darabi's and Dr. Haghighi's Notes

UIC ENGINEERING

Chapter 6: Rates of Return

6.1 Internal Rate of Return Calculations

6.2 External Rate of Return Calculations

Methods of Comparing Economic Alternatives

1. The present worth (PW) method converts all cash flows to a single sum equivalent at time zero using i = MARR. Recall: MARR = Minimum attractive rate of return

2. The benefit-cost ratio (B/C) method determines the ratio of the present worth of benefits and savings to the present worth of investments and costs using i = MARR. Alternative: calculate benefits minus costs ($B - C$).

3. The discounted payback period (DPBP) method determines how long it takes for the cumulative present worth to be positive using i = MARR.

4. The capitalized worth (CW) method determines the present worth (using i = MARR) when the planning horizon is infinitely long.

Chapter 4

5. The annual worth (AW) method converts all cash flows to an equivalent uniform annual series of cash flows over the planning horizon using i = MARR.

6. The future worth (FW) method converts all cash flows to a single sum equivalent at the end of the planning horizon using i = MARR.

Chapter 5

7. The internal rate of return (IRR) method determines the interest rate that yields a future worth (or present worth or annual worth) of zero.

8. The external rate of return (ERR) method determines the interest rate that equates the future worth of the invested capital to the future worth of recovered capital (when the latter is computed using the MARR.)

Chapter 6

Rates of Return

- **Rates of Return (ROR)** are also measures of economic worth (% vs $)
  - Previously, i was always given – in this chapter, the goal is to solve for the interest rate
- One of the most popular methods among individuals (i.e. in personal investment decision-making) and corporations
- ROR are generally used as a supplement to one of the traditional "worth" methods (i.e. present, future, or annual worth)
- Incremental analysis is required when using rates of return (it is not a rank-based approach)
- Can obtain the same result as the traditional ranking methods (e.g. PW)
- When using internal ROR, multiple solutions can occur when comparing mutually exclusive alternatives (we will learn how to deal with them)

Sec. 6.1 – Internal Rate of Return

- **Internal Rate of Return (IRR):** The interest rate that makes the future worth, the present worth, and the annual worth equal to 0
- Often considered a "project's rate of return" (analogous to an investment's interest rate)
- We will focus on FW, but using both PW and AW is also acceptable
- Also referred to as the discounted cash flow rate of return, the cash flow rate of return, the rate of return (ROR), the return on investment (ROI), and the true rate of return

FW = Revenues – Costs = 0

Revenues = Costs

The IRR is the minimum rate of return such that the costs and revenues are equal
(i.e. you break even on your investment)

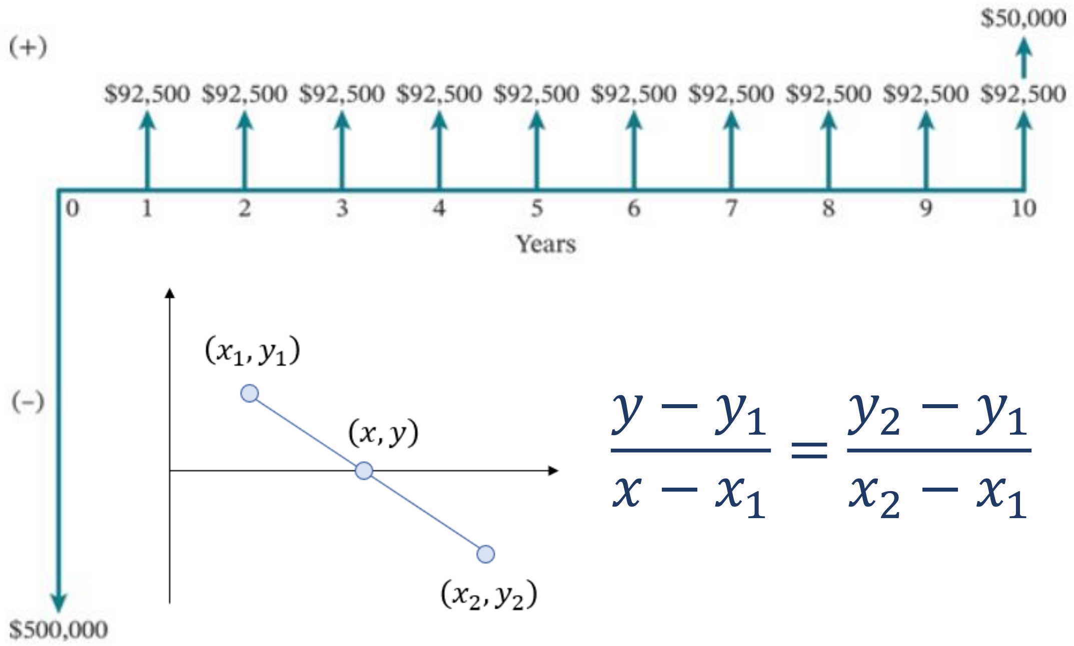
A project that reinvests its profits at the IRR is self-sustaining

IRR Calculations: Single Alternative

- Let the IRR = i^* – our goal is to determine its value
- Set the FW equal to zero and solve for the interest rate, $i*$
- If $i*$ > MARR, select the project; otherwise, select the DN option
- Note: When finding $i*$, you are actually solving for the root(s) of an n-degree polynomial (there could be up to n roots). Solution methods include:
  - Linear interpolation (the equation will be provided to you)
  - Your graphing calculator (be clear in your work what your input values are)
  - Excel (see the book for examples with tutorials)

IRR for Single Alternative: Example 1

Find the IRR. Do you select the following project if MARR = 10%?



$$\text{FW}(i^*) = 0$$

$$-500{,}000(F|P\ i^*, 10) + 92{,}500(F|A\ i^*, 10) + 50{,}000 = 0$$

How to solve for i^*?

Use linear interpolation and the TVOM tables (or formulas):

1) Choose a starting value for i^* and plug it in to the FW equation
   - If FW > 0, your i is too small – try a bigger one
   - If FW < 0, your i is too big – try a smaller one

2) Once you have two values of i^*(one with a positive FW and one with a negative FW), interpolate them to find i^*:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

	$x = i^*$	y = FW
1	12%	$120,333.45
2	15%	-$94,685.90

There is an i in between 12% and 15% that makes FW = 0.
Find it by interpolating: $x = i^*$, $y = 0$

$$\frac{0 - 120{,}333.45}{i^* - 12} = \frac{-94{,}685.90 - 120{,}333.45}{15 - 12}$$

i^* = 13.68% > 10% = MARR

The project is selected

You can also use the IRR function in Excel (see the book for examples) or a graphing calculator

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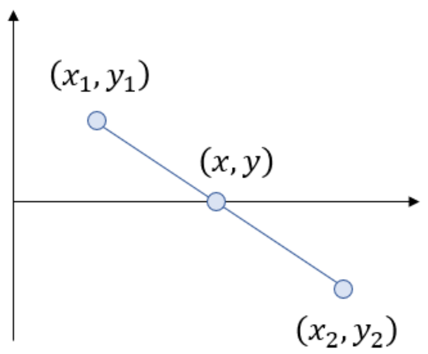
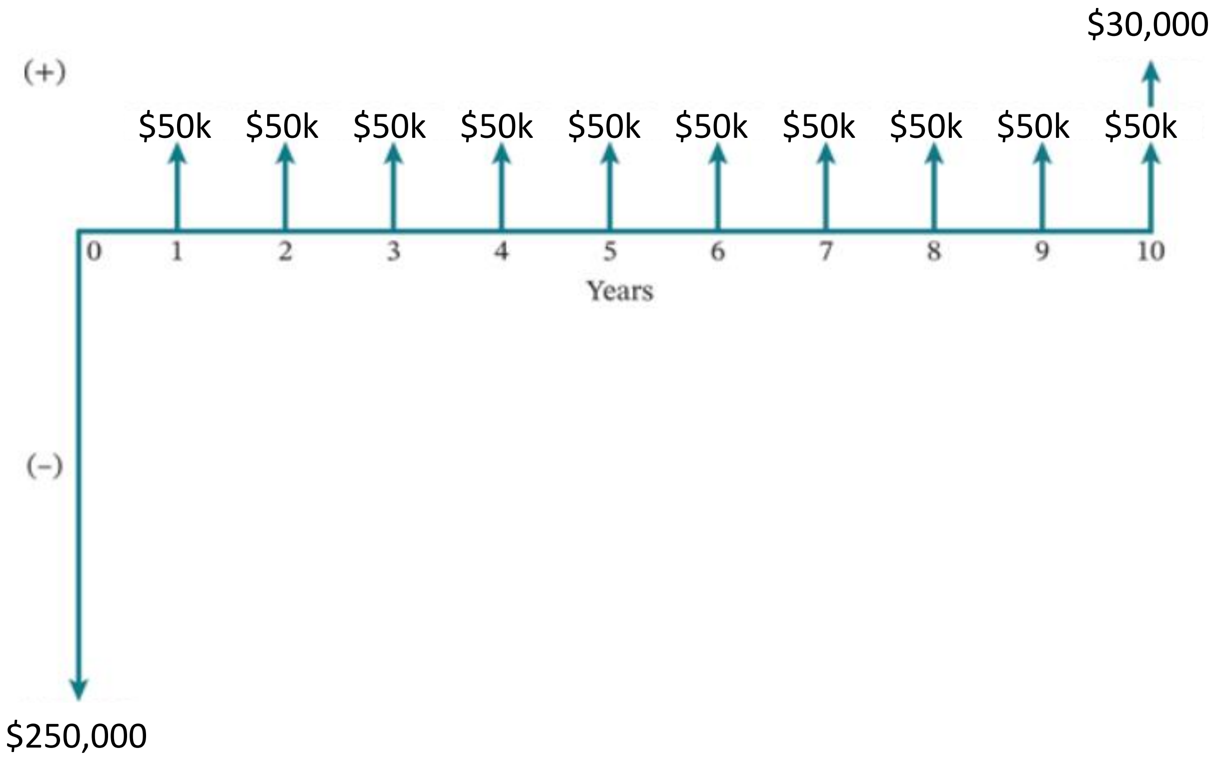
IRR for Single Alternative: Example 2

Find the IRR. Do you select the following project if MARR = 16%?

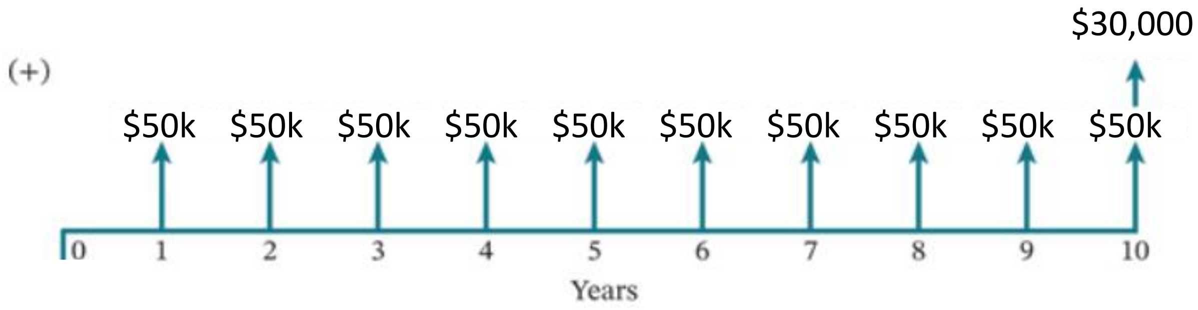


$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$



IRR for Single Alternative: Example 2

Find the IRR. Do you select the following project if MARR = 16%?



$$\text{FW}(i^*) = 0$$
$$-250{,}000(F|P\ i^*, 10) + 50{,}000(F|A\ i^*, 10) + 30{,}000 = 0$$

1. Try i = 15%:

$$\text{FW}_{15\%} = -250{,}000(F|P\ 15\%, 10) + 50{,}000(F|A\ 15\%, 10) + 30{,}000$$
$$\text{FW}_{15\%} = -250{,}000(4.04556) + 50{,}000(20.30372) + 30{,}000 = 33{,}796$$

2. The FW is positive. Try i = 18%:

$$\text{FW}_{18\%} = -250{,}000(F|P\ 18\%, 10) + 50{,}000(F|A\ 18\%, 10) + 30{,}000$$
$$\text{FW}_{18\%} = -250{,}000(5.23384) + 50{,}000(23.52131) + 30{,}000 = -102{,}394.50$$

	i^*	FW
1	15%	$33,796
2	18%	-$102,394.50

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

3. Interpolate to find the value of i that leads to FW=0: $x = i^*, y = 0$

Solving for i^*:

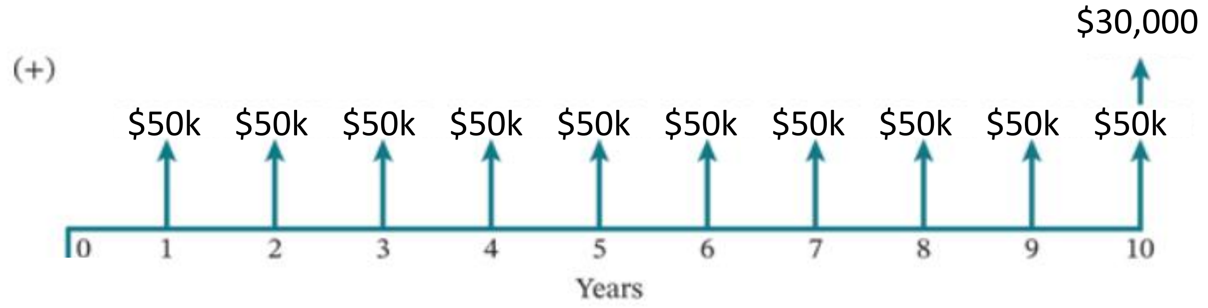
$$i^* = x_1 - y_1\left(\frac{x_2 - x_1}{y_2 - y_1}\right) = 15 - 33{,}796\left(\frac{18 - 15}{-102{,}394.50 - 33{,}796}\right) = 15.74\%$$

Would you select this project?

A) Yes B) No

IRR for Single Alternative: Example 2

Find the IRR. Do you select the following project if MARR = 12%?



$\mathrm{FW}(i^*) = 0$

$-250{,}000(F|P\ i^*, 10) + 50{,}000(F|A\ i^*, 10) + 30{,}000 = 0$

1. Try i = 15%:

$\mathrm{FW}_{15\%} = -250{,}000(F|P\ 15\%, 10) + 50{,}000(F|A\ 15\%, 10) + 30{,}000$

$\mathrm{FW}_{15\%} = -250{,}000(4.04556) + 50{,}000(20.30372) + 30{,}000 = 33{,}796$

2. Try i = 18%:

$\mathrm{FW}_{18\%} = -250{,}000(F|P\ 18\%, 10) + 50{,}000(F|A\ 18\%, 10) + 30{,}000$

$\mathrm{FW}_{18\%} = -250{,}000(5.23384) + 50{,}000(23.52131) + 30{,}000 = -102{,}394.50$

	i^*	FW
1	15%	$33,796
2	18%	-$102,394.50

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

3. Interpolate to find the value of i that leads to FW=0: $x = i^*, y = 0$

Solving for i^*:

$$i^* = x_1 - y_1\left(\frac{x_2 - x_1}{y_2 - y_1}\right) = 15 - 33{,}796\left(\frac{18 - 15}{-102{,}394.50 - 33{,}796}\right) = 15.74\%$$

Would you select this project?

A) Yes B) No

$i^* = 15.74\% < \mathrm{MARR} = 16\%$: The project is **not** selected

IRR Calculations – Multiple Roots

- **Descartes' Rule of Signs** (applied to IRR analysis)**:** Indicates there will be *at most* as many positive rates of return as there are <u>sign changes</u> in the cash flow profile
- For example, a cash flow profile with one change of sign (usually one or more negative cash flows followed by one or more positive cash flows) will have at most one positive IRR value; three sign changes will lead to at most three positive IRR values
- Most cash flow profiles encountered in practice, however, will have one unique IRR, despite multiple changes in sign

(Rules like these are no longer crucial now that we have computers)

IRR Calculations (Multiple Roots): Example

What is maximum number of positive RORs that the following cash flow can have?

Cash Flow Profile	
EOY	CF
0	– $4,000
1	$16,400
2	– $22,320
3	$10,080

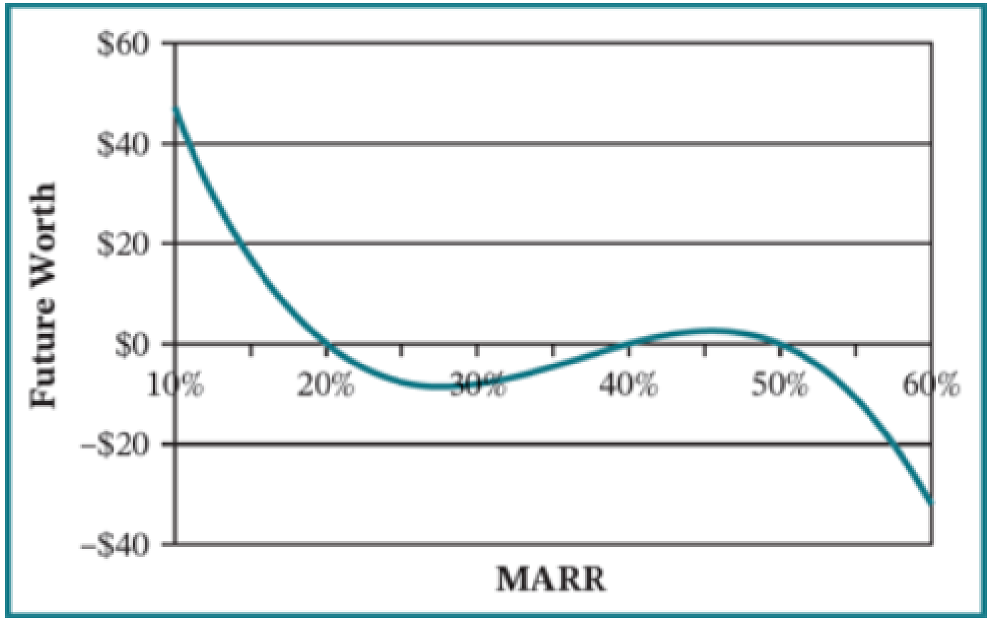
- ✓ Three sign changes, so the maximum number of positive roots is 3
- ✓ You can find 3 rates of returns i^* that lead to FW=0

$$\text{FW} = -4{,}000(F|P\ i^*,3) + 16{,}400(F|P\ i^*,2) - 22{,}320(F|P\ i^*,1) + 10{,}080 = 0$$

Solve for i^*: i^* = 20%, 40%, and 50%

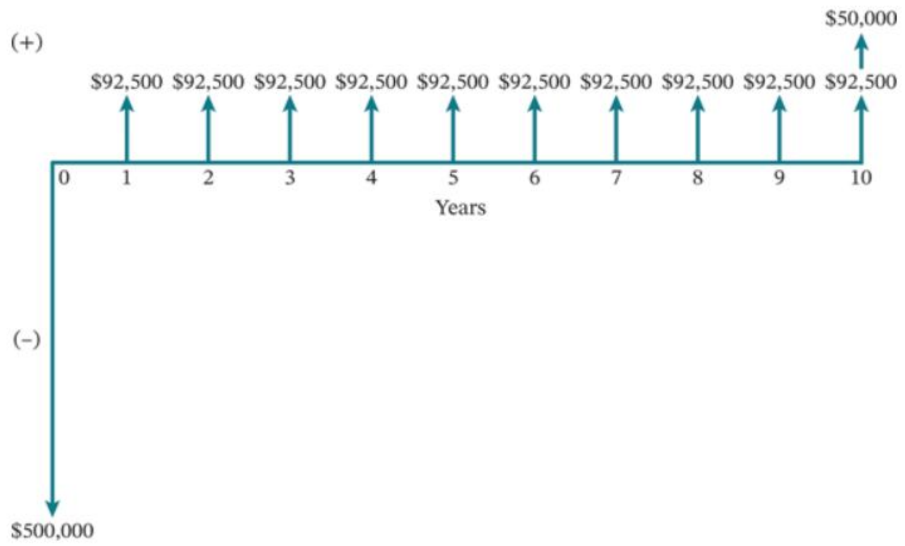
The plot of FW vs ROR crosses the horizontal axis at 3 points – the rates that make the FW function equal to zero

We'll learn later in this chapter how to deal with cash flows that yield multiple RORs



Norstrom's Criterion

- If the *cumulative* cash flow series begins with a negative value and changes <u>only once</u> to a positive value, then there exists <u>exactly one</u> real positive-valued internal rate of return
  - If your cash flow profile abides by this requirement, finding the IRR is simplified – you know there will be only a single solution
  - This cash flow profile is typical of many projects – a large initial investment, followed by annual revenues
  - **Example:** Purchase an SMP machine →



(Rules like these are no longer crucial now that we have computers)

IRR Calculations – Multiple Alternatives

- You **cannot** rank multiple alternatives based on their IRR!
  - If IRR(Alternative A) = 15% and IRR(Alternative B) = 12%, you **cannot** say that Alternative A is better since it has a higher IRR
- **Example:** Project A has an IRR of 10%, Project B has an IRR of 15%. Which do you choose?
  - It is impossible to say from the rates alone
  - Would you rather earn 10% of $1B, or 15% of $1M?
  - The IRR is a **relative** measure of economic worth, so incremental analysis is necessary when multiple projects are being considered

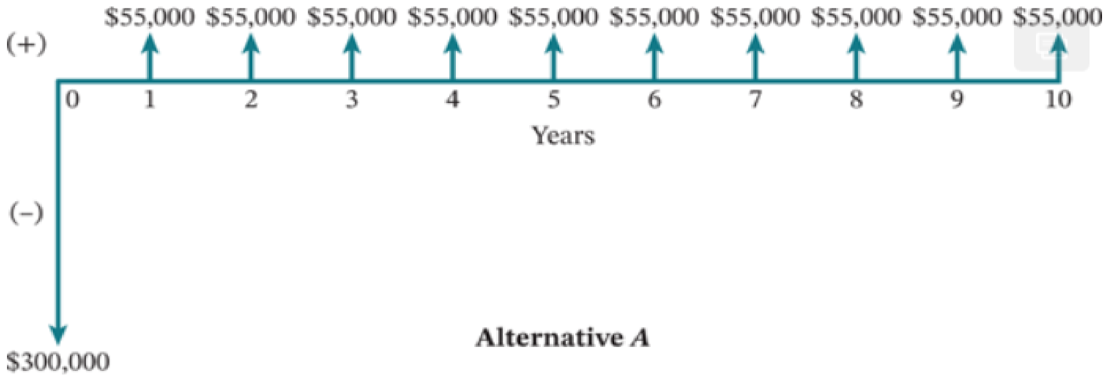
IRR Calculations – Multiple Alternatives

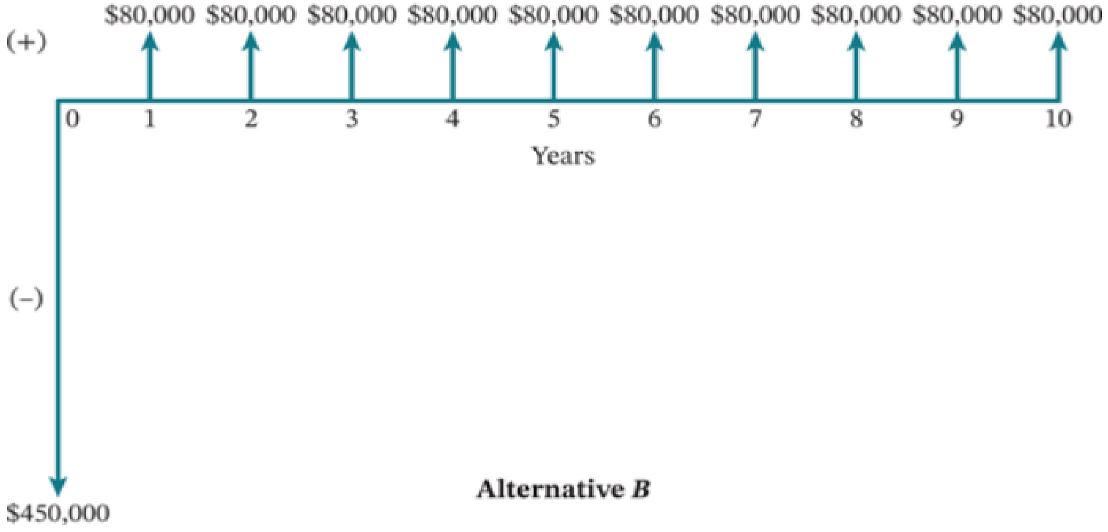
- Rate of return methods must be used **incrementally** when comparing mutually exclusive investment alternatives:
- Let the investment with the lowest initial cost be Alternative 1 (or the *base alternative*), the investment with the next lowest cost is Alternative 2, and so on
- Analyze each alternative in order of increasing initial cost:
  - For Alternatives 2 and beyond, the rate of return is determined for the additional increment of investment (above the base alternative), rather than the entire cost
  - If the incremental rate of return exceeds the MARR, the larger-cost alternative is preferred to the lower-cost alternative (i.e. there is value in investing the additional incremental capital for a subsequent alternative when its rate of return is equal to or exceeds the MARR)
  - The alternative that is selected becomes Alternative 1, the next alternative on the list becomes Alternative 2, and the process continues until all alternatives have been considered

This is similar to the process we used for B/C analysis!

IRR Calculations (Multiple Alternatives): Example

Which alternative should be selected? (DN exists and MARR = 10%)





IRR Calculations (Multiple Alternatives): Example

Which alternative should be selected? (DN exists and MARR = 10%)

1. Rank alternatives from lowest to highest initial cost: DN → A → B, so Alternative 1 = DN, Alternative 2 = A
2. Look at the incremental cash flow of Alt. 2 – Alt. 1 to find the incremental IRR (set PW, FW, or AW equal to 0 and solve for $i_{A-\mathrm{DN}}$)

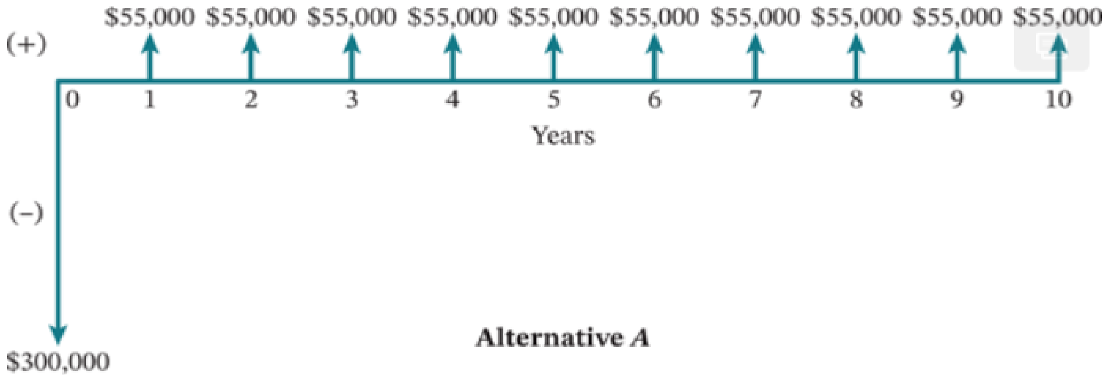
$$\mathrm{PW}(\mathrm{Alt.}\,2 - \mathrm{Alt.}\,1) = \mathrm{PW}(A - \mathrm{DN}) =$$
$$= -300{,}000 + 55000(P|A\ i_{A-\mathrm{DN}}, 10) = 0$$

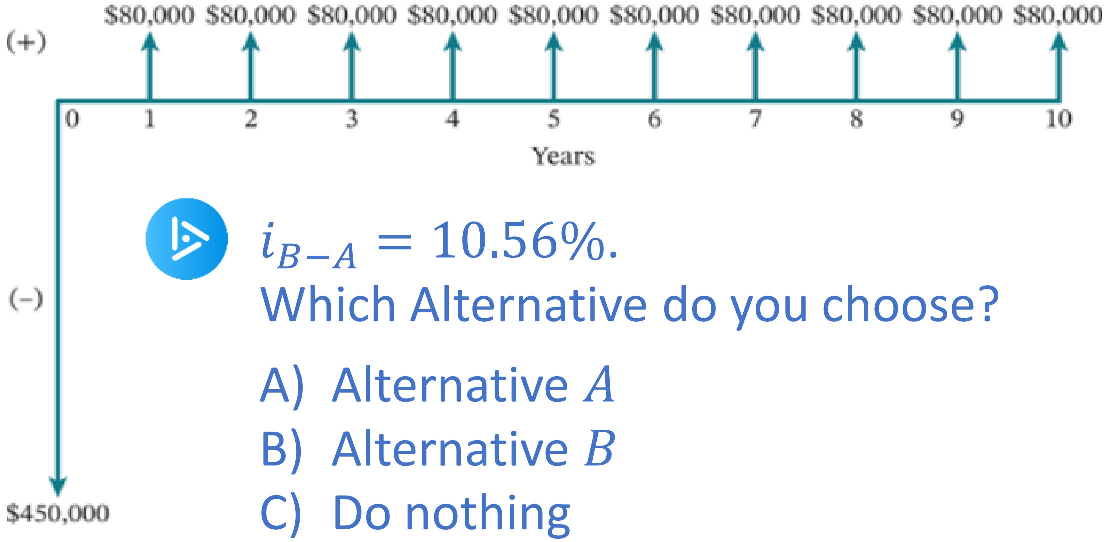
Solve for $i_{A-\mathrm{DN}}$: $i_{A-\mathrm{DN}} = 12.87\%$

3. Compare the incremental IRR to MARR: 12.87 > 10 so Alt. 2 is preferred over Alt. 1 (A over DN)
4. The winning alternative becomes Alternative 1, and the next alternative on the list is Alternative 2
   Alternative 1 = A, Alternative 2 = B
5. Repeat Step 2 to find i_{B-A}

$$\mathrm{PW}(\mathrm{Alt.}\,2 - \mathrm{Alt.}\,1) = \mathrm{PW}(B - A)$$
$$= (-450{,}000 - (-300{,}000)) + (80{,}000 - 55{,}000)(P|A\ i_{B-A}, 10) = 0$$

Solve for i_{B-A}: $i_{B-A} = 10.56\%$





$i_{B-A} = 10.56\%$.
Which Alternative do you choose?

A) Alternative A
B) Alternative B
C) Do nothing

IRR Calculations (Multiple Alternatives): Example

Which alternative should be selected? (DN exists and MARR = 10%)

1. Rank alternatives from lowest to highest initial cost: DN → A → B, so Alternative 1 = DN, Alternative 2 = A
2. Look at the incremental cash flow of Alt. 2 – Alt. 1 to find the incremental IRR (set PW, FW, or AW equal to 0 and solve for $i_{A-\mathrm{DN}}$)

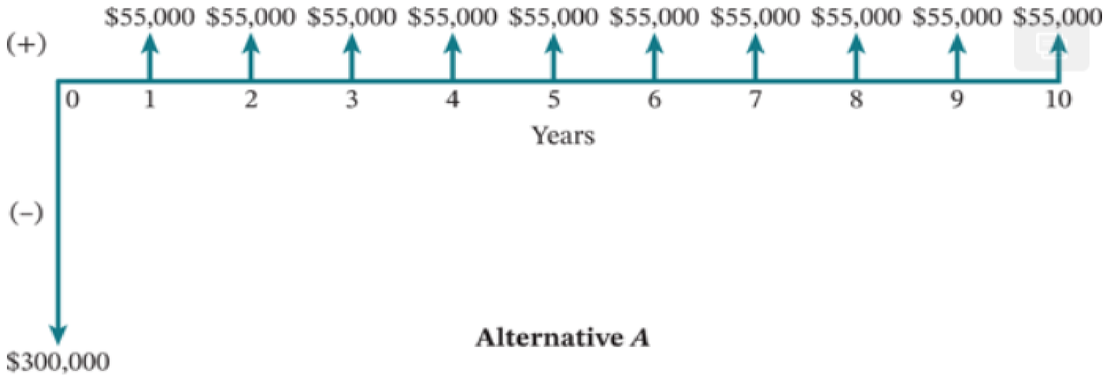
$$\mathrm{PW}(\mathrm{Alt.}\,2-\mathrm{Alt.}\,1)=\mathrm{PW}(A-\mathrm{DN})=$$
$$=-300{,}000+55000(P|A\ i_{A-\mathrm{DN}},10)=0$$

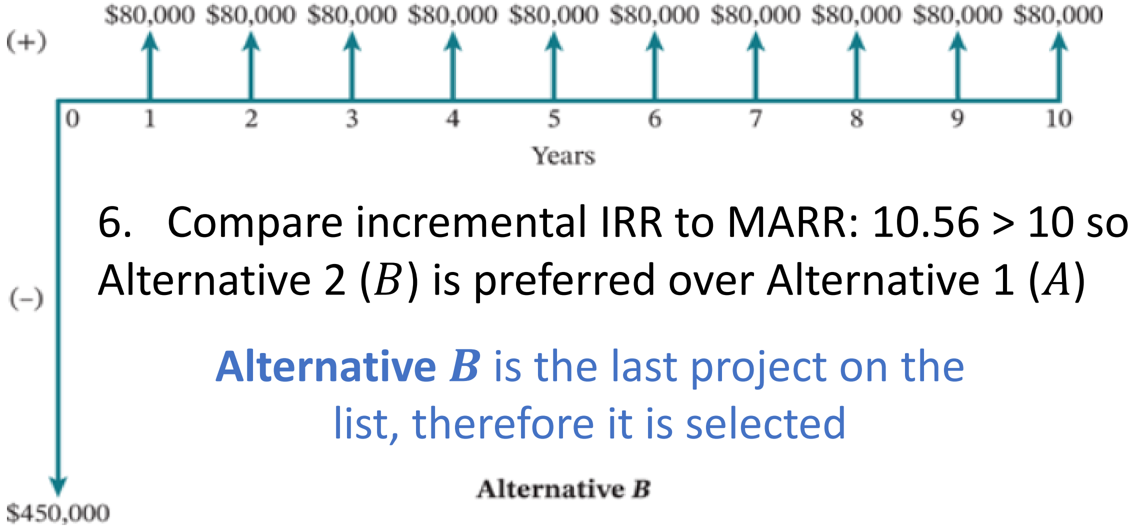
Solve for $i_{A-\mathrm{DN}}$: $i_{A-\mathrm{DN}}=12.87\%$

3. Compare the incremental IRR to MARR: 12.87 > 10 so Alt. 2 is preferred over Alt. 1 (A over DN)
4. The winning alternative becomes Alternative 1, and the next alternative on the list is Alternative 2 Alternative 1 = A, Alternative 2 = B
5. Repeat Step 2 to find i_{B-A}

$$\mathrm{PW}(\mathrm{Alt.}\,2-\mathrm{Alt.}\,1)=\mathrm{PW}(B-A)$$
$$=(-450{,}000-(-300{,}000))+(80{,}000-55{,}000)(P|A\ i_{B-A},10)=0$$

Solve for i_{B-A}: $i_{B-A}=10.56\%$





6. Compare incremental IRR to MARR: 10.56 > 10 so Alternative 2 (B) is preferred over Alternative 1 (A)

Alternative B is the last project on the list, therefore it is selected

Check other examples in the book!

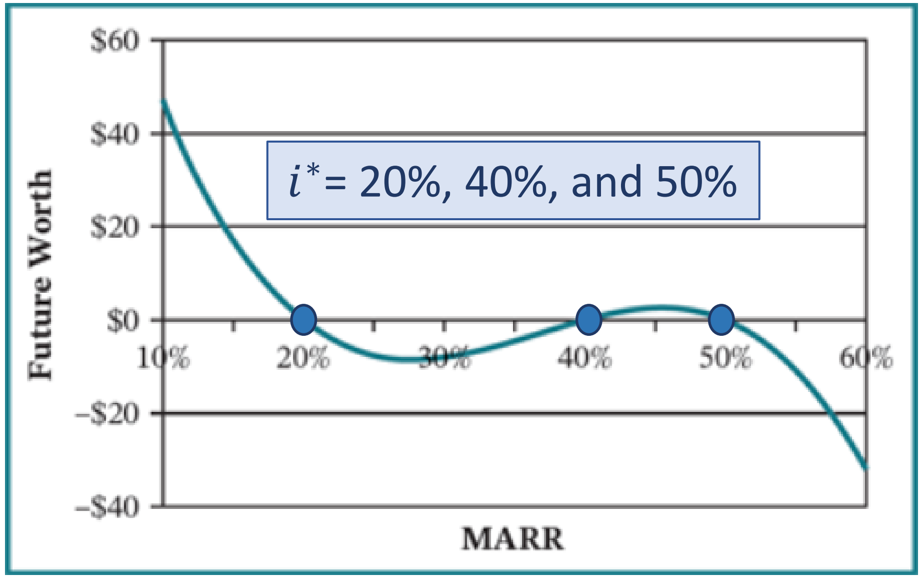
Method 8: External Rate of Return

Motivation: Recall Descartes' Rule of Signs:

There will be at most as many positive rates of return as there are sign changes in the cash flow profile

Cash Flow Profile	
EOY	CF
0	– $4,000
1	$16,400
2	– $22,320
3	$10,080

- If a project has multiple IRRs, how do we evaluate that project? (i.e. which IRR do we compare with MARR?)
- To avoid this problem, we can define and calculate another rate of return called the **External Rate of Return** or ERR, because there will only be one ERR (even if the cash flow profile has multiple IRRs)



Sec. 6.2 – External Rate of Return

- **External Rate of Return (ERR):** The interest rate that makes the absolute value of the future worth of negative-valued cash flows equal to the future worth of positive-valued cash flows that are reinvested at the MARR
  - IRR method: revenues are reinvested into the same project at i^* (project revenues go towards project costs)
  - ERR method: revenues are invested elsewhere at MARR

$$\sum_{t=0}^{n} \underbrace{R_t(1+r)^{n-t}}_{\text{Revenues:}} = \sum_{t=0}^{n} \underbrace{C_t(1+i')^{n-t}}_{\text{Costs:}}$$

R_t: revenues
C_t: |costs|
r = reinvestment rate
i': External Rate of Return

- Two features of the ERR:
  - There exists a unique solution (no multiple rates of return)
  - The ERR is always *between* the IRR and the MARR
    - If IRR = MARR, then IRR = ERR = MARR

External Rate of Return – Single alterative

- r is the reinvestment rate, i' is the ERR
- Solve for i': if i' > MARR, select the project; otherwise, Do Nothing

$$\sum_{t=0}^{n} R_t(1+r)^{n-t} = \sum_{t=0}^{n} C_t(1+i')^{n-t}$$

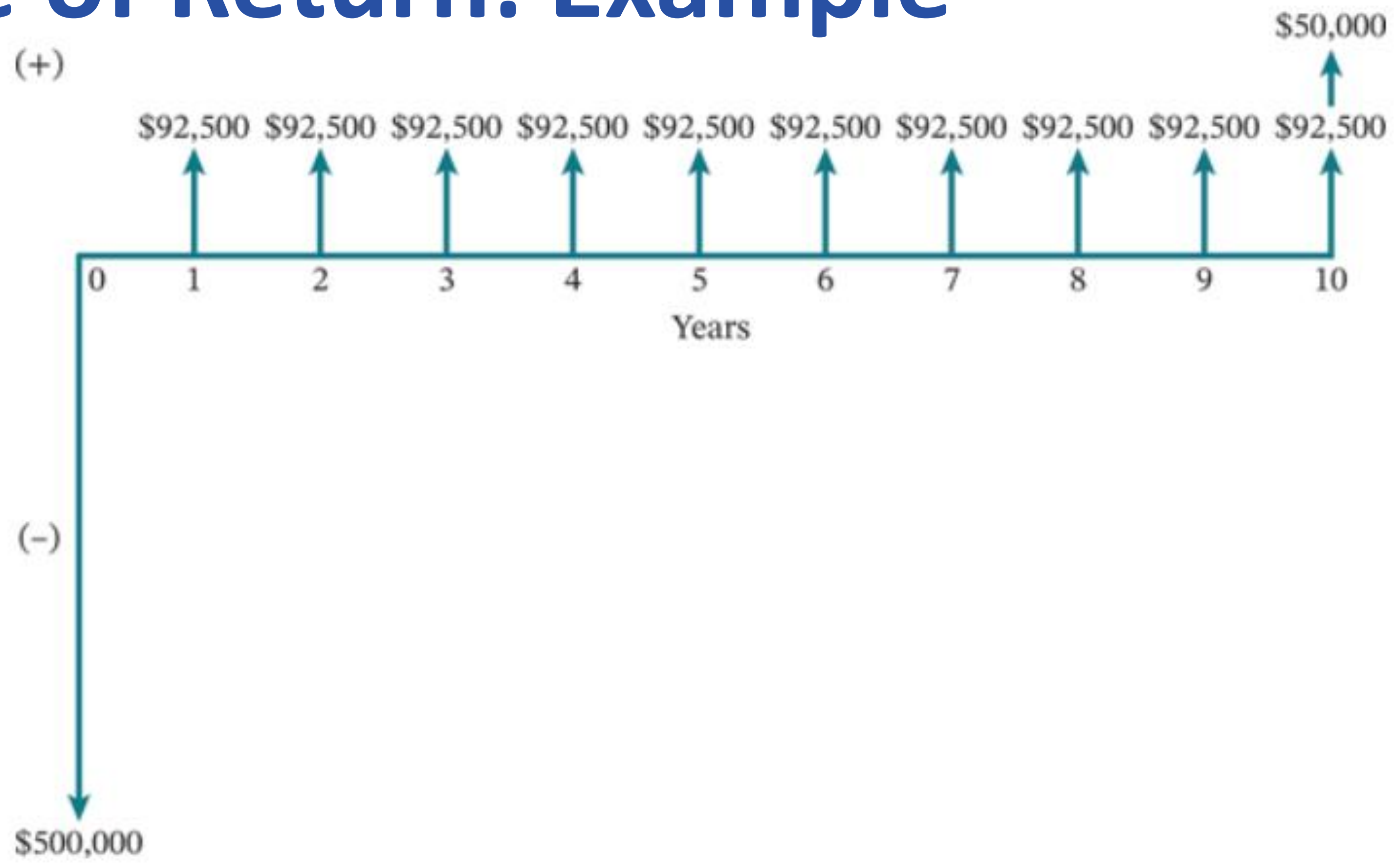
Future worth of positive-valued cash flows that are reinvested at MARR = Absolute value of the future worth of negative-valued cash flows with rate of ERR

External Rate of Return: Example

Do you select this project?

(MARR = 10%)

$$\sum_{t=0}^{n} R_t(1+r)^{n-t} = \sum_{t=0}^{n} C_t(1+i')^{n-t}$$

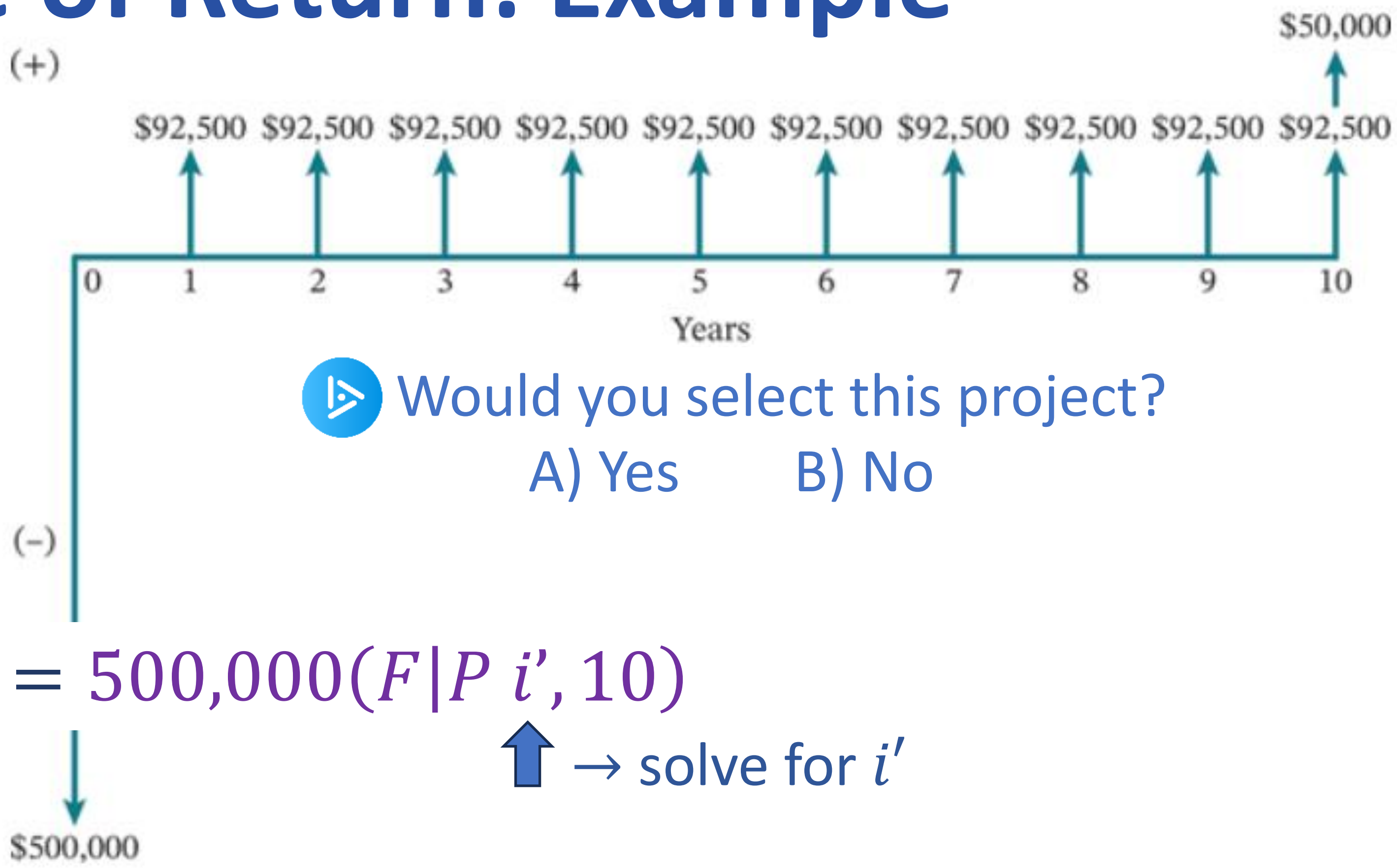


External Rate of Return: Example

Do you select this project?

(MARR = 10%)

$$\sum_{t=0}^{n} R_t(1+r)^{n-t} = \sum_{t=0}^{n} C_t(1+i')^{n-t}$$



Would you select this project?

A) Yes B) No

$$92{,}500(F|A\ 10\%, 10) + 50{,}000 = 500{,}000(F|P\ i', 10)$$

$\rightarrow$ solve for i'

$$92{,}500(15.93742) + 50{,}000 = 500{,}000(1+i')^{10}$$

$$3.0484 = (1+i')^{10} \quad \rightarrow \quad (3.0484)^{1/10} = 1 + i'$$

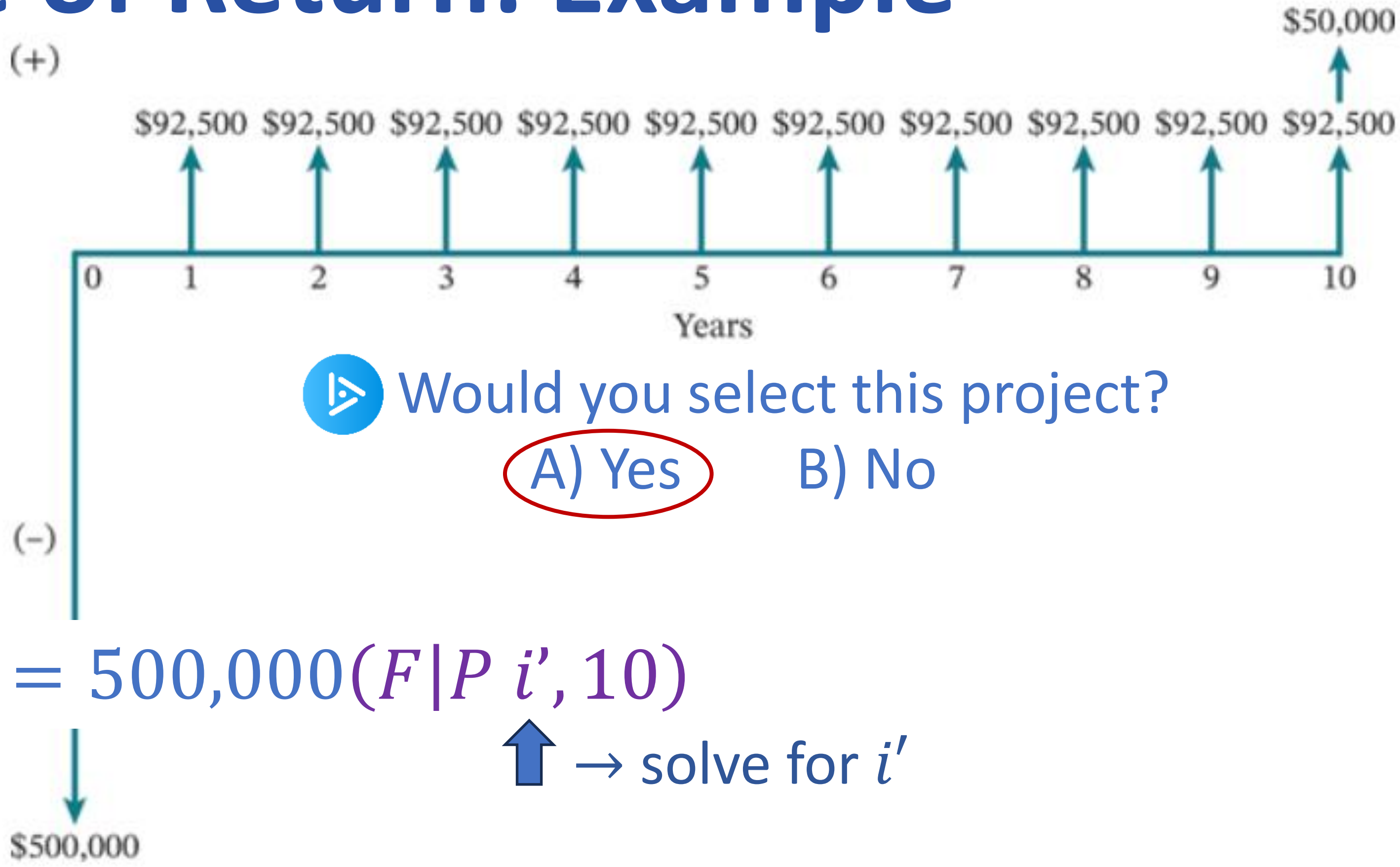
$$i' = 1.1179 - 1 \quad \rightarrow \quad i' = 11.79\%$$

External Rate of Return: Example

Do you select this project?

(MARR = 10%)

$$\sum_{t=0}^{n} R_t(1+r)^{n-t} = \sum_{t=0}^{n} C_t(1+i')^{n-t}$$



Would you select this project?

A) Yes B) No

$$92{,}500(F|A\ 10\%, 10) + 50{,}000 = 500{,}000(F|P\ i', 10)$$

$\rightarrow$ solve for i'

$$92{,}500(15.93742) + 50{,}000 = 500{,}000(1+i')^{10}$$

$$3.0484 = (1+i')^{10} \quad \rightarrow \quad (3.0484)^{1/10} = 1 + i'$$

$$i' = 1.1179 - 1 \quad \rightarrow \quad i' = 11.79\%$$

ERR = i' = 11.79%

> MARR = 10%

The project is selected

ERR Calculations – Multiple Alternatives

- When mutually exclusive investment alternatives exist, the ERR method can be used to select the economically preferred one. However, **like the internal rate of return method, it must be applied incrementally**
- Although there are multiple alternatives and each has its own external rate of return (i_j'), **there is a common reinvestment rate** (r)
- Mathematically, for alternative j, the following equality must hold:

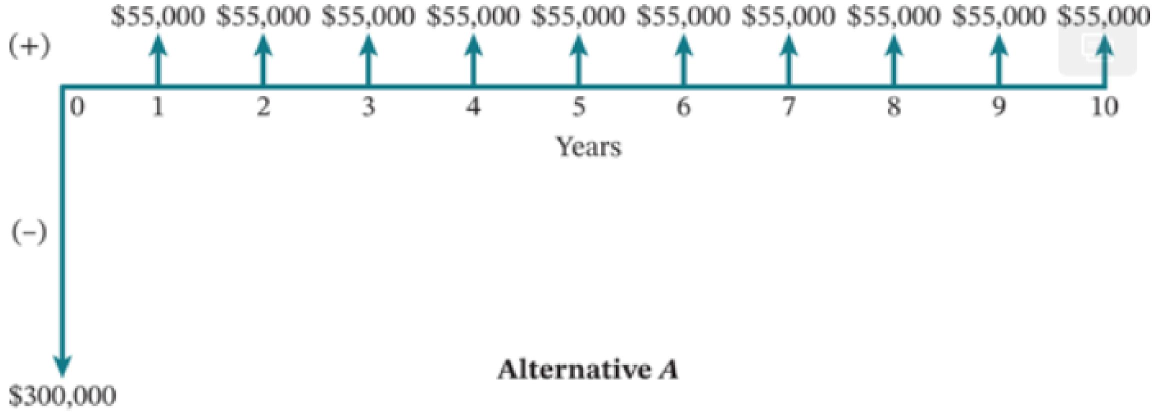
$$\sum_{t=0}^{n} R_{jt}(1+r)^{n-t} = \sum_{t=0}^{n} C_{jt}\left(1+i_j'\right)^{n-t}$$

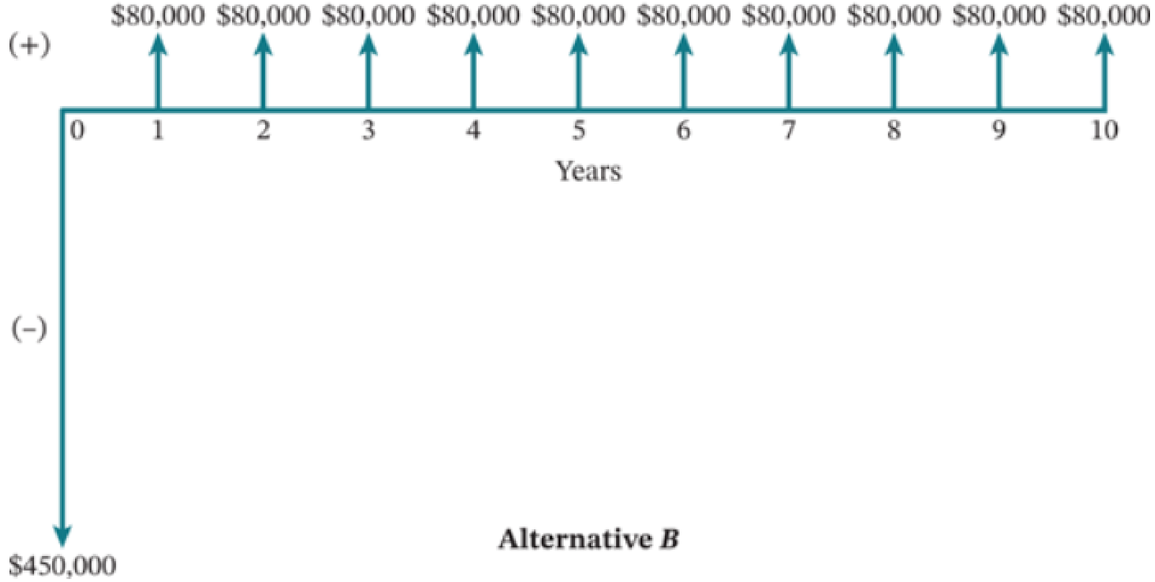
where R_{jt} denotes the positive-valued cash flows and C_{jt} denotes the absolute value of the negative-valued cash flows in the cash flow series for alternative j

ERR Calculations (Multiple Alternatives): Example

Which alternative should be selected? (DN exists and MARR = 10%)

$$\sum_{t=0}^{n} R_{jt}(1+r)^{n-t} = \sum_{t=0}^{n} C_{jt}\left(1+i'_j\right)^{n-t}$$





ERR Calculations (Multiple Alternatives): Example

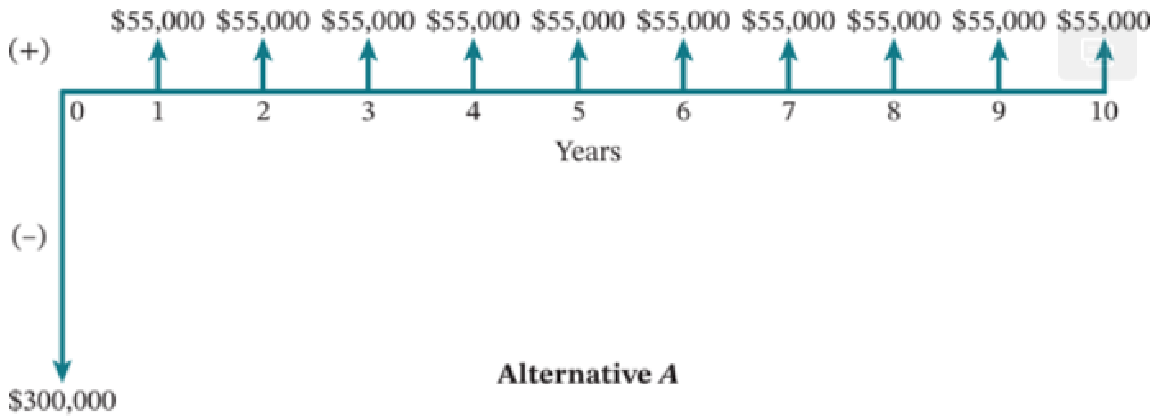
Which alternative should be selected? (DN exists and MARR = 10%)

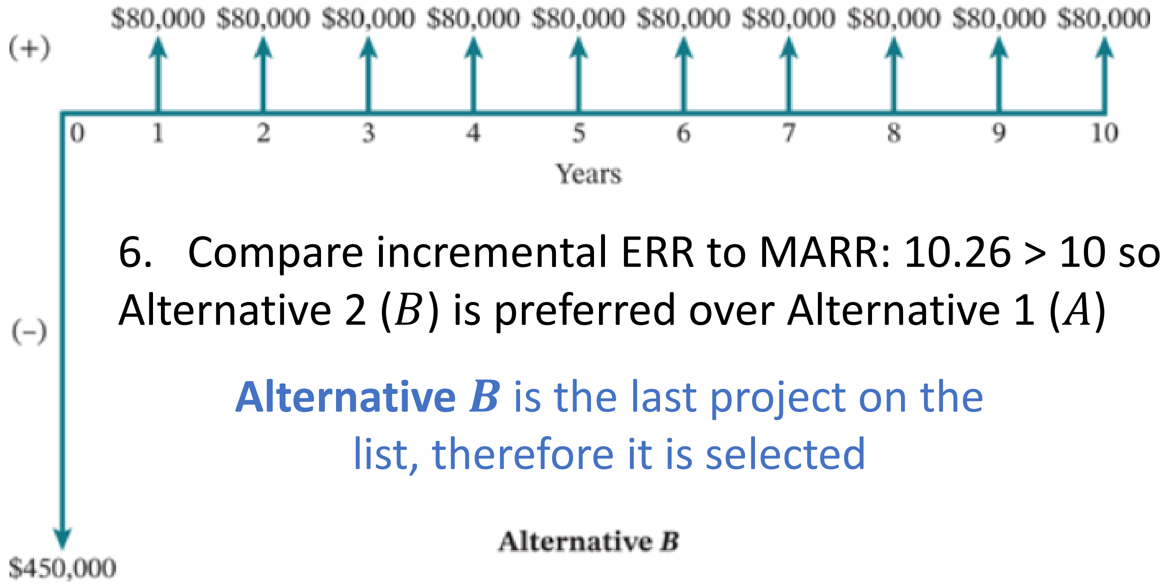
1. Rank alternatives from lowest to highest initial cost: DN → A → B, so Alternative 1 = DN, Alternative 2 = A
2. Look at the incremental cash flow of Alt. 2 – Alt. 1 to find the incremental ERR

$$300{,}000(1+i'_A)^{10} = 55{,}000(F|A\ 10\%, 10)$$
$$(1+i'_A)^{10} = 55{,}000(15.93742)/300{,}000$$
$$i'_A = 11.31814\% \ > \ \text{MARR} = 10\%$$

3. Compare the incremental ERR to MARR: 11.32 > 10 so Alt. 2 is preferred over Alt. 1 (A over DN)
4. The winning alternative becomes Alternative 1, and the next alternative on the list is Alternative 2
   Alternative 1 = A, Alternative 2 = B
5. Repeat Step 2 to find i'_{B-A}

$$150{,}000(1+i'_{B-A})^{10} = 25{,}000(F|A\ 10\%, 10)$$
$$(1+i'_{B-A})^{10} = 25{,}000(15.93742)/150{,}000$$
$$i'_{B-A} = 10.26219\% \ > \ \text{MARR} = 10\%$$





6. Compare incremental ERR to MARR: 10.26 > 10 so Alternative 2 (B) is preferred over Alternative 1 (A)

Alternative B is the last project on the list, therefore it is selected