

TIDAL DISSIPATION IN ROTATING SOLAR-TYPE STARS

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ABSTRACT

We calculate the excitation and dissipation of low-frequency tidal oscillations in uniformly rotating solar-type stars. For tidal frequencies smaller than twice the spin frequency, inertial waves are excited in the convective envelope and are dissipated by turbulent viscosity. Enhanced dissipation occurs over the entire frequency range rather than in a series of very narrow resonant peaks, and is relatively insensitive to the effective viscosity. Hough waves are excited at the base of the convective zone and propagate into the radiative interior. We calculate the associated dissipation rate under the assumption that they do not reflect coherently from the center of the star. Tidal dissipation in a model based on the present Sun is significantly enhanced through the inclusion of the Coriolis force but may still fall short of that required to explain the circularization of close binary stars. However, the dependence of the results on the spin frequency, tidal frequency, and stellar model indicate that a more detailed evolutionary study including inertial and Hough waves is required. We also discuss the case of higher tidal frequencies appropriate to stars with very close planetary companions. The survival of even the closest hot Jupiters can be plausibly explained provided that the Hough waves they generate are not damped at the center of the star. We argue that this is the case because the tide excited by a hot Jupiter in the present Sun would marginally fail to achieve nonlinearity. As conditions at the center of the star evolve, nonlinearity may set in at a critical age, resulting in a relatively rapid inspiral of the hot Jupiter.

Subject headings: binaries: close — planetary systems — stars: oscillations — hydrodynamics — waves

1. INTRODUCTION

1.1. *Tidal interactions involving solar-type stars*

The tidal interaction between a star and an orbiting companion can lead to a significant evolution of the system over astronomical timescales if the orbit is sufficiently close. When the star experiences a periodically varying gravitational potential, an oscillatory tidal disturbance is generated in the fluid. Dissipation of the tide, by whatever mechanism, is directly associated with a secular transfer of angular momentum between the spin and the orbit as well as a loss of energy from the system. All of the spin and orbital parameters therefore evolve as a result of tidal dissipation.

In binary stars the tendency of tidal evolution is usually towards a double synchronous state in which the orbit is circular and both stars spin at the same rate as the orbit (e.g. Hut 1981). In this state each star experiences only a steady tidal distortion and no further dissipation or tidal evolution occurs. Observations of stellar populations of different ages provide clear evidence for the ongoing tidal circularization of close binaries, because in the older samples circular orbits are found for a wider range of orbital periods (Meibom & Mathieu 2005). Measurements of spin are more limited but do provide evidence of tidal synchronization (Meibom, Mathieu, & Stassun 2006).

Many of the known extrasolar planets orbit sufficiently close to their host stars to allow significant tidal evo-

lution. The closest planets, commonly known as “hot Jupiters” or “hot Neptunes”, have circular orbits, probably mainly as a result of the tidal dissipation within the planet rather than that within the star. Although these planets are expected to be tidally synchronized, the host stars, which are typically similar to the Sun in mass and age, are not. The reason is that the orbital moment of inertia of a hot Jupiter or Neptune is at most comparable to the spin moment of inertia of a solar-type star. The tidal torque therefore leads to orbital migration accompanied by a modest change in the spin rate of the star, which is in any case under the control of magnetic braking.

The direction of orbital migration is away from the corotation radius of the star. When the system is young and the star rapidly rotating, the inward migration of a hot Jupiter driven by its interaction with the protoplanetary disk may therefore be halted by the tide it raises in the star (Lin, Bodenheimer, & Richardson 1996). Once the star has spun down, however, tidally driven inspiral threatens the survival of hot Jupiters such as OGLE-TR-56b (Sasselov 2003).

1.2. *Timescales of tidal evolution*

The efficiency of tidal dissipation is often parametrized by a dimensionless quality factor Q (e.g. Goldreich & Soter 1966), which reflects the fact that the star undergoes a forced oscillation and dissipates a small fraction of the associated energy during each oscillation period. Since Q always appears in the theory in the combination

$$Q' = \frac{3Q}{2k_2}, \quad (1)$$

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where k_2 is the potential Love number of order 2 (a dimensionless measure of the interior density profile of the star), it is more convenient to use this combination. (For a homogeneous body without rigidity, $k_2 = 3/2$ and $Q' = Q$.)

The tidal potential experienced by the star can be written as a sum of rigidly rotating components in the form of solid spherical harmonics,

$$\text{Re} [\Psi r^\ell Y_\ell^m(\theta, \phi) e^{-i\omega t}], \quad (2)$$

where (r, θ, ϕ) are spherical polar coordinates in an inertial frame of reference centered on the star, ω is the frequency in that frame, and Ψ is an amplitude. The tidal frequency experienced by the star is

$$\hat{\omega} = \omega - m\Omega, \quad (3)$$

where Ω is its spin frequency.³ Q' is a function of ℓ , m , and $\hat{\omega}$, but can be assumed to be independent of Ψ if linear theory applies. In practice $\ell = 2$ is dominant, and the azimuthal wavenumber can be restricted to $m = 0, 1$, or 2 .

Although the potential applications of tides are very broad, we are concerned here with two problems connected with observations: the circularization of close binary stars and the inward migration of hot Jupiters.

In a binary star with mean motion $n > 0$, tidal dissipation leads to an evolution of the semimajor axis a and eccentricity e at the rates

$$\frac{\dot{a}}{a} = -\frac{9}{2} \frac{M_2}{M_1} \left(\frac{R_1}{a} \right)^5 n \left[\frac{\text{sgn}(2n - 2\Omega_1)}{Q'_{1,2,2n-2\Omega_1}} \right] - \frac{9}{2} \frac{M_1}{M_2} \left(\frac{R_2}{a} \right)^5 n \left[\frac{\text{sgn}(2n - 2\Omega_2)}{Q'_{2,2,2n-2\Omega_2}} \right], \quad (4)$$

$$\begin{aligned} \frac{\dot{e}}{e} = & -\frac{9}{32} \frac{M_2}{M_1} \left(\frac{R_1}{a} \right)^5 n \left[\frac{49 \text{sgn}(3n - 2\Omega_1)}{Q'_{1,2,3n-2\Omega_1}} \right. \\ & + \frac{6}{Q'_{1,0,n}} + \frac{4 \text{sgn}(2\Omega_1 - 2n)}{Q'_{1,2,2n-2\Omega_1}} + \left. \frac{\text{sgn}(2\Omega_1 - n)}{Q'_{1,2,n-2\Omega_1}} \right] \\ & - \frac{9}{32} \frac{M_1}{M_2} \left(\frac{R_2}{a} \right)^5 n \left[\frac{49 \text{sgn}(3n - 2\Omega_2)}{Q'_{2,2,3n-2\Omega_2}} \right. \\ & + \frac{6}{Q'_{2,0,n}} + \frac{4 \text{sgn}(2\Omega_2 - 2n)}{Q'_{2,2,2n-2\Omega_2}} + \left. \frac{\text{sgn}(2\Omega_2 - n)}{Q'_{2,2,n-2\Omega_2}} \right], \end{aligned} \quad (5)$$

where we neglect fractional corrections of second order in eccentricity and obliquity.⁴ Here M and R denote stellar mass and radius, and $Q'_{i,m,\hat{\omega}}$ is the value of Q' of star i for $\ell = 2$, azimuthal wavenumber m , and tidal frequency $\hat{\omega}$. For two identical stars that are already

³ We restrict our attention to uniformly rotating stars. By “frequency” we always mean angular frequency.

⁴ Under the assumption that $1/Q' \propto |\hat{\omega}|$, equation (5) reproduces the result of Darwin (1880) that eccentricity is excited for $\Omega/n > 18/11$ and damped otherwise. Note that the discontinuities in equations (4) and (5) do not occur in practice because $1/Q'$ must vanish as $\hat{\omega}$ tends to zero in the case of a uniformly rotating body.

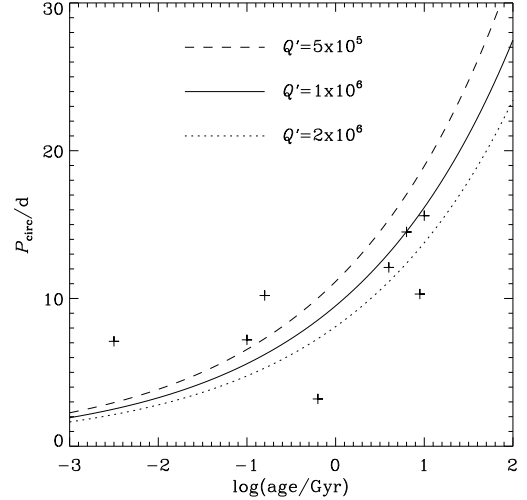


FIG. 1.— Circularization period versus age for the stellar clusters and populations listed in Meibom & Mathieu (2005). The curves show the period for which the circularization timescale (6) is equal to the age for various values of Q' .

synchronized ($\Omega_1 = \Omega_2 = n$), eccentricity is damped on the circularization timescale

$$\begin{aligned} \tau_e = -\frac{e}{\dot{e}} &= \frac{2}{63} \left(\frac{a}{R} \right)^5 \frac{Q'}{n} \\ &\approx 1200 Q' \left(\frac{\bar{\rho}}{\bar{\rho}_\odot} \right)^{5/3} \left(\frac{P}{10 \text{ d}} \right)^{13/3} \text{ yr}, \end{aligned} \quad (6)$$

where P is the orbital period and $\bar{\rho}$ is the mean density ($\bar{\rho}_\odot \approx 1.41 \text{ g cm}^{-3}$). The timescale for synchronization is indeed much shorter. Equation (6) assumes a common value of Q' for the various tidal components, but this assumption may not be so important because the $\hat{\omega} = 3n - 2\Omega$ component dominates unless it has a much larger value of Q' .

The relative contribution of the tide in star 2 is $(Q'_1/Q'_2)(M_1/M_2)^2(R_2/R_1)^5$. For late-type stars $R \propto M^{4/5}$ approximately, giving a ratio of $(Q'_1/Q'_2)(M_2/M_1)^2$. The tide in the secondary star is therefore less important if its Q' is the same.

Fig. 1 reproduces the observational data from the study of Meibom & Mathieu (2005). Although the trend is not particularly clear it is roughly consistent with the frequently quoted value $Q' \approx 10^6$.

To determine the rate of inward migration of a hot Jupiter, we consider a synchronized planet of mass M_p in a circular orbit around a star of mass M_* , in which case

$$\frac{\dot{a}}{a} = -\frac{9}{2} \frac{M_p}{M_*} \left(\frac{R_*}{a} \right)^5 n \left[\frac{\text{sgn}(2n - 2\Omega_*)}{Q'_{*,2,2n-2\Omega_*}} \right]. \quad (7)$$

The inspiral time for a planet into a star similar to the Sun (assuming Q'_* to be independent of frequency) is

$$\begin{aligned} \tau_a = -\frac{2}{13} \frac{a}{\dot{a}} &= \frac{4}{121} \frac{M_*}{M_p} \left(\frac{a}{R_*} \right)^5 \frac{Q'_*}{n} \\ &\approx 0.020 Q'_* \frac{M_*}{M_p} \left(\frac{\bar{\rho}}{\bar{\rho}_\odot} \right)^{5/3} \left(\frac{P}{\text{d}} \right)^{13/3} \text{ yr}. \end{aligned} \quad (8)$$

The extrasolar planet of shortest orbital period found to date is OGLE-TR-56b with $P = 1.21$ d. It orbits a star of mass $1.04 M_{\odot}$ whose age is estimated as 3 ± 1 Gyr (Sasselov 2003). If the value $Q' \approx 10^6$ appropriate for binary stars is employed in this context, the inspiral time is only about 0.036 Gyr.

The observations raise two basic problems for theorists. First, an efficient dissipation mechanism must be found to explain a Q' as small as 10^6 for the binary circularization problem. Second, the same mechanism must not operate so efficiently in the hot Jupiter problem, unless we are extremely fortunate to observe OGLE-TR-56b and others like it.

1.3. Equilibrium and dynamical tides

The analysis of tides in stars was developed principally by Zahn (1966a,b,c, 1970, 1975, 1977). In the theory of the equilibrium tide, the tidal potential induces a time-dependent but quasi-hydrostatic bulge. Owing to the slow oscillation of the bulge a certain velocity field is required within the star. Dissipation occurs in convective zones because an effective viscosity associated with turbulent convection acts on this velocity field. The resulting value of $1/Q'$ is proportional to the effective viscosity, which ought to be reduced when the tidal period exceeds the convective timescale because then only part of the spectrum of turbulent fluctuations damps the tide.

The assumption of a quasi-hydrostatic bulge appears appropriate because the tidal frequency $|\dot{\omega}|$ is typically much smaller than the dynamical frequency $\omega_* = (GM/R^3)^{1/2}$ of the star, which means that fundamental and acoustic modes are driven well below their natural frequencies. However, stars support other types of wave of much lower frequency. The theory of the dynamical tide determines the excitation of such waves by tidal forcing and their dissipation by non-adiabatic processes, which can be effective if the wavelengths are short.

Radiative zones are stably stratified and typically have a Brunt-Väisälä frequency N that is comparable to ω_* and much greater than $|\dot{\omega}|$. Radial motions are strongly inhibited when $|\dot{\omega}| \ll N$. On the other hand $|\dot{\omega}|$ is typically comparable to Ω , so the Coriolis force can be important. In uniformly rotating stars the resulting solutions are called Hough waves and have a short radial wavelength. These reduce to gravity waves in non-rotating stars, as considered in Zahn's work.

Convective zones are usually adiabatically stratified to high accuracy and radial motions are not inhibited. At tidal frequencies $|\dot{\omega}| < 2|\Omega|$ in uniformly rotating stars, convective zones support pure inertial waves, for which the Coriolis force is the restoring force. These have an entirely different character from Hough waves, although a subset of Hough waves are sometimes called "inertial modes" (e.g. Savonije, Papaloizou, & Alberts 1995). In a non-rotating star, or for $2|\Omega| < |\dot{\omega}| \ll \omega_*$, convective zones do not support waves. Even in this case, as discussed below, the tidal response is not described by the equilibrium tide.

1.4. Previous theoretical approaches

Several papers have previously presented calculations of the efficiency of tidal dissipation in solar-type stars under a variety of assumptions. As for the theory of

the equilibrium tide, some controversy has surrounded the correct way to reduce the effective viscosity when the tidal period exceeds the convective timescale, as discussed by Goodman & Oh (1997). The theoretical arguments generally support, at least in an approximate sense, the formula of Goldreich & Keeley (1977) [cf. equation (10) below], which drastically reduces the effective viscosity at high frequency. Recently, however, Penev et al. (2007) find some support for the formula of Zahn (1966b) from an analysis of numerical simulations of turbulent convection. Even with the more optimistic assumption of Zahn (1966b), Q' is too large to explain binary circularization.

Both Terquem et al. (1998) and Goodman & Dickson (1998) extended the theory of the dynamical tide, previously developed by Zahn for early-type stars with convective cores, to solar-type stars with convective envelopes. In this approach, the tidal forcing excites an irrotational flow in the convective zone and low-frequency gravity waves in the radiative interior. Terquem et al. (1998) calculated adiabatic forced gravity waves, which propagate through the entire radiative zone, and determined the dissipation rate by including non-adiabatic effects as a perturbation. Neither radiative damping nor turbulent viscosity is very effective in damping the waves because they are evanescent in the convective zone. The outcome was that tidal dissipation is efficient only when the tidal frequency closely matches that of a global g mode, but the system would evolve very slowly between these resonances. When the dissipation rate is averaged in time, using an appropriately weighted local average in frequency, the resulting value is dominated by the behavior off resonance and corresponds to $Q' \approx 5.1 \times 10^7 (P_{\text{tide}}/10 \text{ d})^{-0.15}$ [based on their equation (37)] where $P_{\text{tide}} = 2\pi/|\dot{\omega}|$ is the tidal period. They used an effective viscosity of the form recommended by Goldreich & Keeley (1977) but which is in fact larger by a factor of $4\pi^2$ in the high-frequency limit and closer to that used by Goldreich & Nicholson (1977).⁵ This value of Q' is substantially larger than that calculated on the basis of the equilibrium tide, $Q' \approx 1.2 \times 10^7 (P_{\text{tide}}/10 \text{ d})^{0.08}$ [based on their equation (36)], as that approximation gives an inaccurate description of the disturbance in the convective zone.

Goodman & Dickson (1998) did not calculate global g modes but considered the possibility that the gravity waves would reflect imperfectly from the center of the star, where the wavelength is very short and nonlinearity can occur because the wave energy is concentrated into an extremely small volume. Imperfect reflection could mean either that the wave reflects from the center with a perturbed phase, or that the wave is substantially dissipated at that location. In either case global g-mode resonances do not occur and tidal dissipation occurs over a continuous range of tidal frequencies. The result is $Q' \approx 1.2 \times 10^9 (P_{\text{tide}}/10 \text{ d})^{8/3}$ [based on their equation (13)]. (When they converted this result into a circularization timescale they apparently underestimated it by a factor of about 20.) The stronger frequency de-

⁵ Such factors are certainly debatable as neither of the cited papers goes beyond simple dimensional estimates. Neither includes the factor of $1/3$ appearing in equation (10) and used by subsequent authors.

pendence in this expression means that this mechanism could be very important for short-period systems.

Neither of these studies included the effect of the Coriolis force on the tidal disturbance, which might be very important if the tidal frequency is not much larger than the spin frequency of the star. Incorporating the Coriolis force is difficult because it leads to wave equations that are not separable in the spherical polar coordinates r and θ . This separability is very useful because the wavelength of the gravity waves can be so short that a two-dimensional calculation with adequate resolution is unfeasible. (The third, azimuthal, direction can always be treated by a separation of variables.) Savonije & Witte (2002) and Witte & Savonije (2002) were able to retain separability by using the “traditional approximation” in which the latitudinal component $-\Omega \sin \theta$ of the angular velocity is neglected in calculating the Coriolis force. Indeed, the traditional approximation is justified in the radiative region because radial motions are suppressed for $|\omega| \ll N$ and the relevant solutions are Hough waves. Savonije & Witte (2002) and Witte & Savonije (2002) developed a highly sophisticated model including the effects of stellar evolution, magnetic braking, and resonance locking. The results of Savonije & Witte (2002) are similar to those of Terquem et al. (1998) off resonance, as they adopt the same effective viscosity, but inclusion of the Coriolis force allows a richer spectrum of resonances with various types of global Hough modes.

1.5. Purpose of this calculation

Unfortunately the traditional approximation is not valid in the convective zone, which is adiabatically stratified to high accuracy. In a previous paper (Ogilvie & Lin 2004, hereafter Paper I) we studied the excitation and dissipation of tidal disturbances in rotating giant planets, which are mostly or fully convective. As noted above, for tidal frequencies $|\omega| < 2|\Omega|$, the convective zone supports pure inertial waves, which are not correctly described by the traditional approximation. The importance of inertial waves for tidal dissipation in giant planets has also been recognized by Wu (2005a,b) and Papaloizou & Ivanov (2005).

Inertial waves have remarkable properties, some of which have been elucidated only since the advent of high-resolution numerical calculations. They propagate along characteristic rays that are inclined to the rotation axis at a certain angle, depending on the wave frequency. This angle is necessarily preserved in reflections of the waves from boundaries, so that a beam is typically either focused or defocused in such a reflection. When propagating within a spherical annulus (rather than a full sphere), the waves are generically focused on to “wave attractors” where intense dissipation occurs (Rieutord & Valdetaro 1997; Rieutord, Georgeot, & Valdetaro 2001). It was demonstrated mathematically by Ogilvie (2005) that the dissipation rate associated with wave attractors in a slightly simplified wave equation is independent of viscosity in the limit of very small Ekman number. In Paper I we found evidence for a similar behavior in the limit of small viscosity, although the numerical solutions indicate that the dissipation is typically concentrated along the rays that emanate from the critical latitude on the inner boundary.

The purpose of the present calculation is to determine

the excitation of inertial waves in the convective zones of solar-type stars and to assess their role in tidal dissipation. This occurs both through the damping of the waves in situ by turbulent viscosity and also through their effect on the excitation of Hough waves in the radiative zone. We aim to determine what range of values of Q' can be obtained for the binary circularization problem and the hot Jupiter problem.

In some sense our calculation is complementary to that of Savonije & Witte (2002). They gave an accurate treatment of the radiative zone but not the convective zone, and assumed that Hough waves reflect coherently from the center of the star. We aim to treat the convective zone more accurately, and assume that the Hough waves do not reflect coherently from the center.

2. NUMERICAL ANALYSIS

2.1. Stellar model

For an accurate and detailed model of the present Sun, we adopt “model S” from Christensen-Dalsgaard et al. (1996). In the mixing-length theory of Böhm-Vitense (1958), the convective energy flux density is related to the convective velocity v by

$$F_c = \frac{1}{3} \frac{8v^3}{\ell} \frac{\rho c_p T}{g\delta} \quad (9)$$

where $\delta = -(\partial \ln \rho / \partial \ln T)_p$ and $\ell = \alpha H_p$ is the mixing length, $H_p = p/\rho g$ being the pressure scaleheight. [See Zahn (1989) for a discussion of the dimensionless numerical prefactors.] The convective timescale is then $\tau = \ell/v$.

The turbulent viscosity based on mixing-length theory is $\frac{1}{3}v\ell$, but this should be reduced when the tidal period exceeds the convective timescale. We take the effective turbulent viscosity to be

$$\nu = \frac{1}{3}v\ell (1 + \hat{\omega}^2\tau^2)^{-1} \quad (10)$$

which is a smoothed version of that of Goldreich & Keeley (1977). In fact, the functional form of equation (10) is suggested by a Maxwellian viscoelastic model in which τ is identified with the relaxation time of the turbulent stress.

Fig. 2 shows the viscosity profile in the Sun for a tidal period of 10 d. This period is chosen as being representative of the binary circularization problem. The viscosity is substantially reduced below the mixing-length value in most of the convective zone. For reasonable spin periods, the dimensionless Ekman number $\nu/2\Omega R^2$ is small but not extremely small (typically of the order of $10^{-5} - 10^{-4}$). This means that inertial waves will be modestly affected by viscosity, and the problem is conveniently accessible to computation with achievable numerical resolution.

Although the interior angular velocity profile of the Sun has been measured, we assume the star to be uniformly rotating. This is partly for technical reasons, as our numerical method can cope easily only with “shel-lular” rotation profiles. In any case it is not clear how the rotation profile would change if the Sun were tidally synchronized at a different rate in a binary system.

2.2. Inertial waves in the convective zone

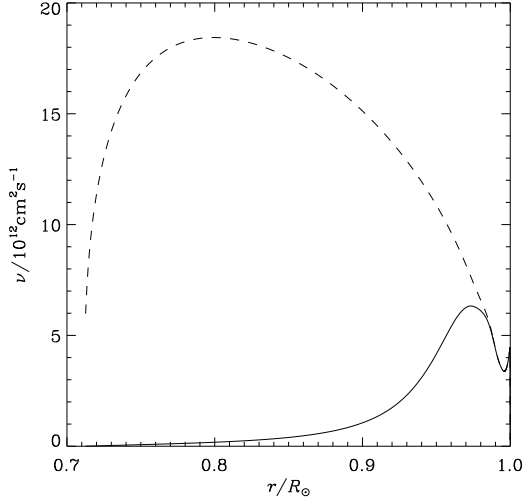


FIG. 2.— Effective viscosity profile in the convective zone of the present Sun. The mixing-length value $\nu = \frac{1}{3}v\ell$ is shown as a dashed line, while the value reduced according to equation (10) for a tidal period of 10 d is shown as a solid line. For a spin period of 10 d a kinematic viscosity of $\nu = 10^{12} \text{ cm}^2 \text{ s}^{-1}$ corresponds to an Ekman number of $Ek = \nu/2\Omega R^2 \approx 1.4 \times 10^{-5}$.

We solve the forced linearized equations in the convective zone as in Paper I. The response consists of an equilibrium tide plus a dynamical tide. The former satisfies a second-order ordinary differential equation in r which we solve numerically. The latter satisfies partial differential equations in r and θ which can be simplified in the low-frequency limit, the simplifications amounting to the Cowling and anelastic approximations. The velocity field is separated into spheroidal and toroidal parts and the equations are converted into a large algebraic system by Galerkin projection on to normalized associated Legendre polynomials (i.e. spherical harmonics) in θ and Chebyshev collocation in r . The linear system is solved by a standard method for block tridiagonal matrices, and the total viscous dissipation rate is calculated. We adopt stress-free impermeable boundary conditions on the dynamical tide at the upper and lower limits of the convective zone, as explained in Paper I.

2.3. Hough waves in the radiative zone

We determine algebraically the excitation of Hough waves in the radiative zone as in Paper I. They are excited at the interface between the radiative and convective zones, partly by tidal forcing in that region and partly by the pressure of the inertial waves acting at the interface. Following Goodman & Dickson (1998), the model assumes that N^2 vanishes in the convective zone and increases initially linearly with distance into the radiative zone. We assume that the waves are not reflected coherently from the center of the star and calculate the resulting energy flux. This is converted into a dissipation rate as described in Paper I. The behavior of waves near the center of the star in the limit $|\hat{\omega}| \gg |\Omega|$ is discussed in Appendix A, where, in a slight refinement of the calculation of Goodman & Dickson (1998), we estimate the conditions under which the waves become nonlinear.

3. RESULTS

3.1. Inertial waves

In the left half of Fig. 3 we plot, as a function of the tidal frequency, the value of Q' resulting from the viscous dissipation of inertial waves in the convective zone of the Sun with a spin period of 10 d. Only the most important azimuthal wavenumber $m = 2$ is considered. Also shown for comparison is the viscous dissipation rate of the irrotational disturbance generated in the convective zone when the Coriolis force is omitted. For frequencies $|\hat{\omega}| < 2|\Omega|$ the dissipation is greatly enhanced by the excitation of inertial waves, but outside this range the Coriolis force has little net effect.

Where the dissipation rate is significantly enhanced, it is relatively insensitive to the viscosity, as found in Paper I. Elsewhere the dissipation rate is directly proportional to the viscosity. The value of Q' obtained when the Coriolis force is omitted is larger than that found by Terquem et al. (1998). This is partly because of the contribution of g-mode resonances to their average value, but mainly because their viscosity is larger than ours by a factor of approximately $4\pi^2$.

The numerical convergence of these results was verified by repeating the calculation with double the resolution in each direction. It was found adequate in most cases to truncate the Legendre and Chebyshev polynomial bases at an order of 100.

3.2. Hough waves

Also shown in Fig. 3 is the same information for Hough waves excited at the interface between the radiative and convective zones. Inclusion of the Coriolis force can either increase or decrease the dissipation rate.

When the Coriolis force is neglected, our results should agree with those of Goodman & Dickson (1998). We find their numerical parameter σ_c to equal -1.19 rather than -0.64 , which we verified by numerically integrating their equation (3). Altogether we find $Q' \approx 3.9 \times 10^8 (P_{\text{tide}}/10 \text{ d})^{8/3}$ when spin is neglected. We also obtain their equation (15) but with $(M_1 + M_2)/M_1$ raised to the power $-5/3$ instead of $+11/6$. This discrepancy suggests that they may have overestimated the circularization rate by a factor of $2^{7/2}$.

3.3. Higher tidal frequencies

In Fig. 4 we plot similar results over a wider range of tidal frequencies. As expected, the Coriolis force is unimportant when $|\hat{\omega}| \gg |\Omega|$. As found by Goodman & Dickson (1998), Hough waves (or gravity waves, in this regime) provide efficient tidal dissipation at high frequencies if they do not reflect coherently from the center of the star.

3.4. Dependence on spin period

In Fig. 5 we present the results for the Sun with a spin period of 30 d. Three times the range of $\hat{\omega}/\Omega$ is plotted, corresponding to the same range of $\hat{\omega}$ as in Fig. 3. As expected, outside the range $|\hat{\omega}| < 2|\Omega|$, Q' is almost independent of the spin frequency Ω at a given tidal frequency $\hat{\omega}$, because the effect of the Coriolis force is weak. Within the frequency range of inertial waves, a smaller Q' is generally achieved in the more rapidly rotating star. Fig. 6 shows the results for a spin period of 3 d. An even

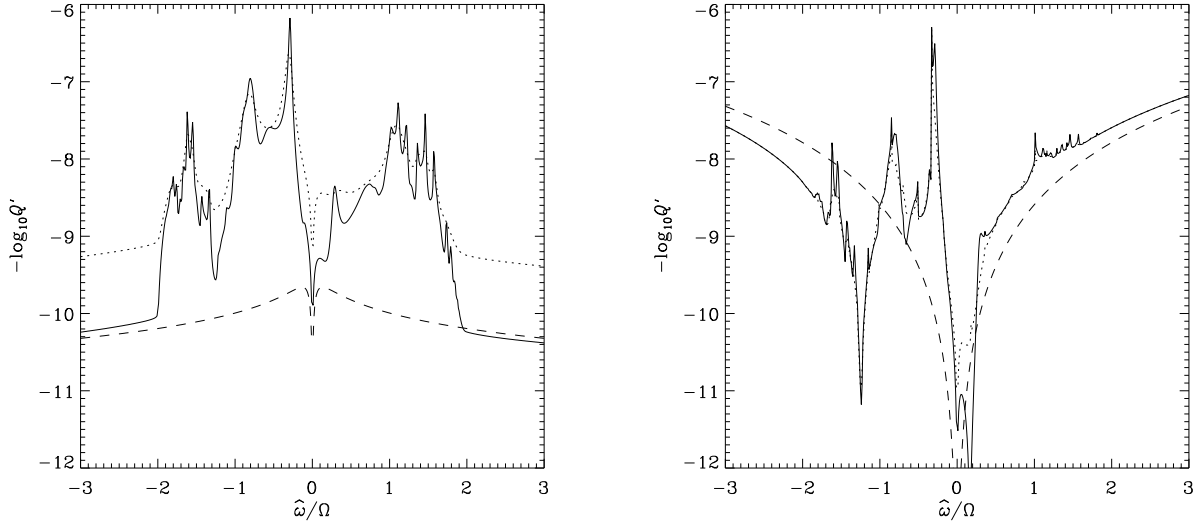


FIG. 3.— Dissipation rate, parametrized as a value of Q' , as a function of tidal frequency. These results refer to tidal forcing by the $\ell = m = 2$ solid harmonic in a model of the Sun with a spin period of 10 d. *Left:* Q' from the viscous dissipation of inertial waves in the convective zone. *Right:* Q' from the excitation of Hough waves in the radiative zone. The dashed lines show the effect of omitting the Coriolis force. In this figure only, the dotted lines show the result of artificially increasing the effective viscosity by a factor of 10.

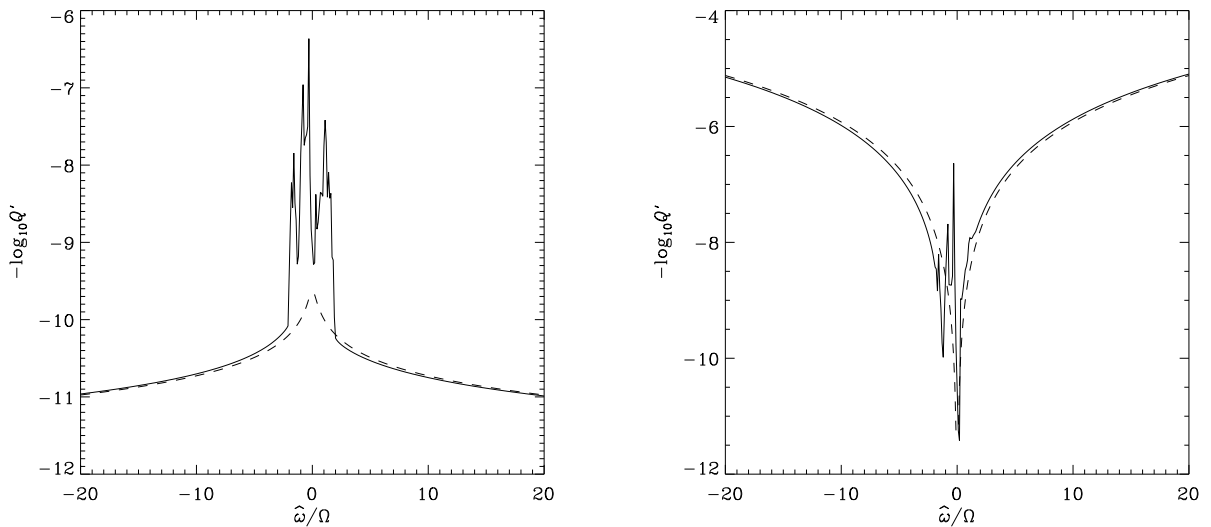


FIG. 4.— Expansion of Fig. 3 to a wider range of tidal frequencies.

smaller Q' results from inertial waves in this case. The scaling is approximately $1/Q' \propto \Omega^2$ for a fixed value of $\hat{\omega}/\Omega$, as suggested by the factor f_Q defined in Paper I. Increasing Ω at fixed $\hat{\omega}/\Omega$ also decreases the Ekman number, resulting in a more structured graph as the system is less controlled by viscosity.

3.5. Lower-mass stars

We also investigated models of lower mass, as the secondaries or even the primaries of “solar-type binaries” may have masses significantly less than $1 M_\odot$. The greater depth and density of the convective envelopes of these stars allows for a more efficient excitation of inertial waves, in the sense that the dimensionless dissipation rate $D_{\text{visc}}/U_{\text{visc}}$ (as defined in Paper I) is generally larger.

However, this effect is offset by the fact that the conversion factor f_Q for these stars is smaller as a result of their greater mean density.

In Fig. 7 we plot the values of Q' resulting from inertial and Hough waves in a stellar model of mass $0.5 M_\odot$, age 5 Gyr, and spin period 10 d. Although the details are different, the qualitative behavior and the range of values of Q' obtained are similar to the case of the Sun.

3.6. Summary

In the frequency range $|\hat{\omega}| < 2|\Omega|$, tidal dissipation in the convective zones of solar-type stars is substantially enhanced through the excitation of inertial waves when the full Coriolis force is included. Here Q' can be decreased by up to four orders of magnitude and has a

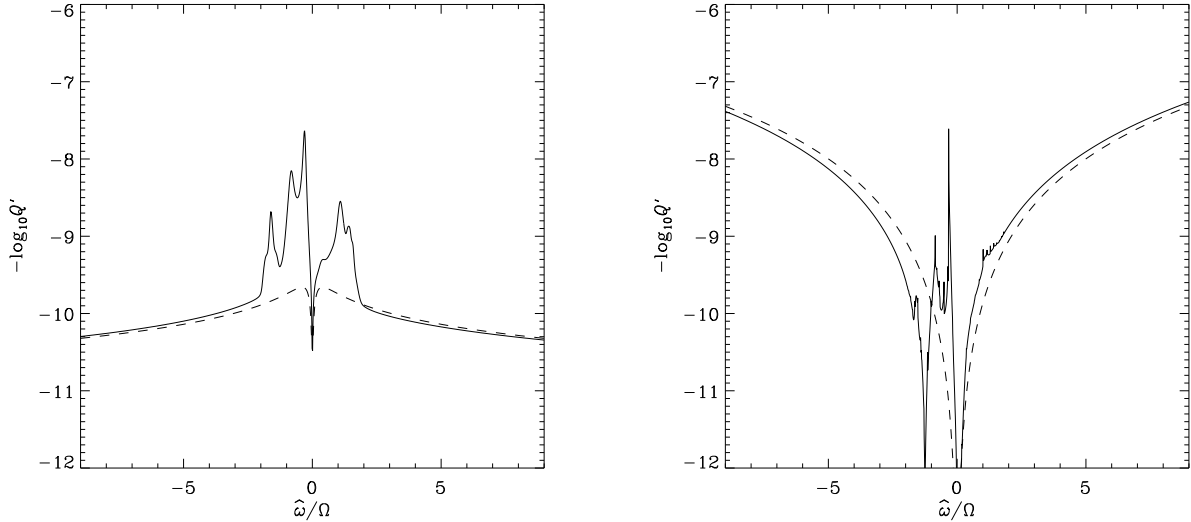


FIG. 5.— As for Fig. 3, but for a spin period of 30 d. Note the wider range of $\hat{\omega}/\Omega$, corresponding to the same range of $\hat{\omega}$ as in Fig. 3.

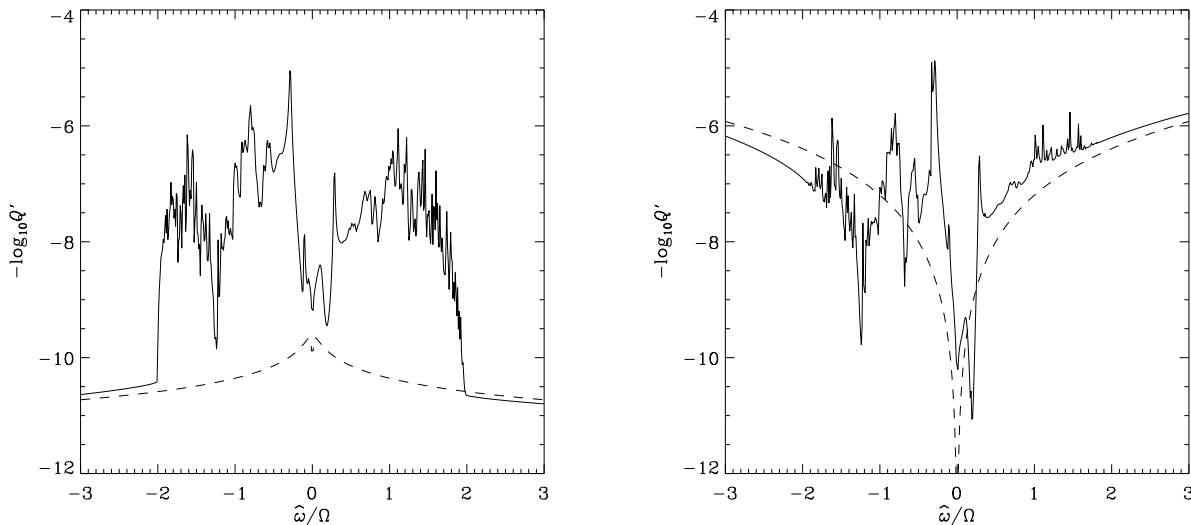


FIG. 6.— As for Fig. 3, but for a spin period of 3 d. Note the different vertical scale.

complicated dependence on tidal frequency. Typically $Q' \propto \Omega^{-2}$ for a fixed ratio $\hat{\omega}/\Omega$. Values as small as 10^6 can be achieved if the star spins more rapidly than the Sun.

If the Hough waves excited at the interface between the radiative and convective zones do not reflect coherently from the center of the star as a result of their nonlinearity, they provide another means of dissipation at all frequencies. The resulting value of Q' scales generally with $|\hat{\omega}|^{-8/3}$ although this is modified, especially in the range $|\hat{\omega}| < 2|\Omega|$, by the inclusion of the Coriolis force. Values as small as 10^6 can be achieved if the tidal period is as short as 1 d. The estimates in Appendix A suggest that, in the case of the present Sun, Hough waves become nonlinear in eccentric binaries but probably not in the hosts of hot Jupiters. Nonlinearity is less likely in

stars that are younger or less massive than the Sun, but is more likely in older stars.

4. COMPARISON WITH OBSERVED SYSTEMS

4.1. Tidally induced orbital migration

4.1.1. Close-in gas giant planets around G dwarfs

One immediate application of the present analysis is to the orbital migration of close-in extrasolar planets. In his study of OGLE-TR-56b, Sasselov (2003) computed the orbital decay timescale $a/|\dot{a}|$ (i) by extrapolating a formula calibrated on the circularization of binary stars, (ii) from equilibrium tidal models calculated using the prescription of Zahn (1966b) for the effective viscosity in the convective envelope, and (iii) by a similar method using the prescription of Goldreich & Keeley (1977). (He set the tidal period equal to the orbital period, thereby un-

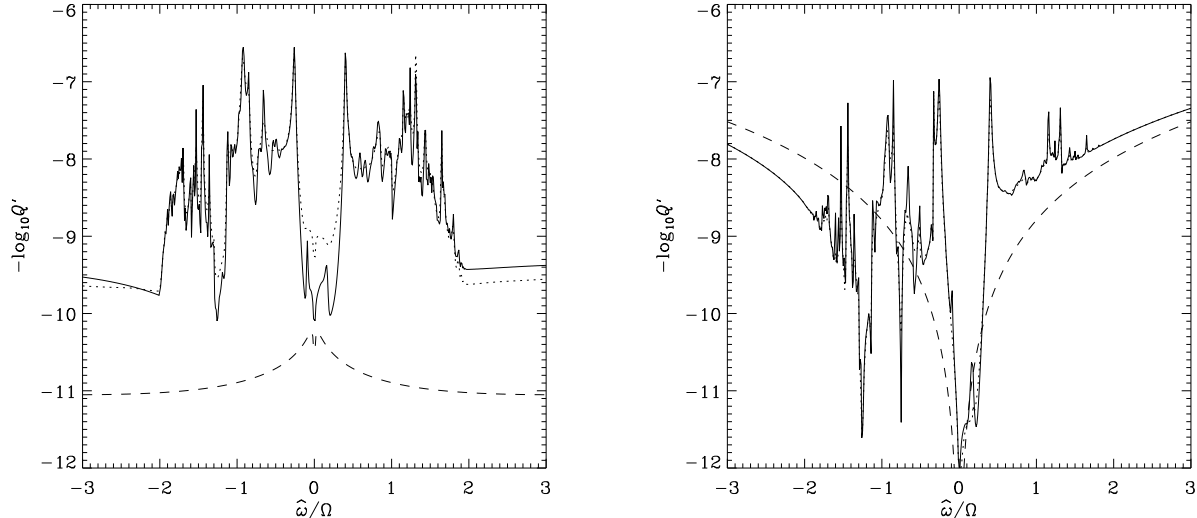


FIG. 7.— As for Fig. 3, but for a star of mass $0.5 M_{\odot}$ and age 5 Gyr.

derestimating the forcing frequency by a factor of approximately 2. The tidal frequency associated with the appropriate $m = 2$ potential component in the hot Jupiter problem is in fact $\hat{\omega} = 2n - 2\Omega$.) He showed that, while the theoretically derived $a/|\dot{a}|$ exceeds 10 Gyr with prescription (ii) and 4000 Gyr with prescription (iii), is it less than 1 Gyr with prescription (i). This discrepancy is similar to that found in equilibrium tidal models for solar-type binary stars (see Section 4.2.1).

To the extent that the mechanism and efficiency of tidal dissipation remain uncertain, prescription (i) should provide a useful empirical estimate. Sasselov (2003) also estimated the age of OGLE-TR-56 as $\tau_* \approx 3 \pm 1$ Gyr, which is considerably longer than his estimate for $a/|\dot{a}|$ based on prescription (i). Note further that, owing to the accelerating nature of inward orbital migration, the inspiral time τ_a is considerably shorter than the present value of $a/|\dot{a}|$ [see equation (8)]. While OGLE-TR-56b was the only known extrasolar planet with a period close to 1 d, a possible interpretation was that it does indeed have $\tau_a \ll \tau_*$ and we observe it with a small probability ($\sim \tau_a/\tau_*$) while it evolves through this period range with a rapidly decreasing τ_a ($\propto P^{13/3}$ under the assumption of constant Q'). However, two additional extrasolar planets, OGLE-TR-113b and OGLE-TR-132b, have been found with only slightly longer orbital periods than OGLE-TR-56b. The values of τ_a obtained for these planets under prescription (i) are also comparable to or shorter than their estimated ages. The period distribution of the known planets does not statistically indicate that these very short-period planets are at the endpoints of their tidal orbital evolution. An alternative interpretation of these data is that the relevant value of Q' of the solar-type host stars may be significantly larger than the one that applies to binary circularization.

Based on the results presented in Section 3, we explore the possibility that this dichotomy in the stellar Q' may be caused by the differences in the tidal forcing frequencies and spin frequencies for the hot Jupiter and binary circularization problems. The shortest orbital pe-

riods of known extrasolar planets are little longer than 1 d, which is an order of magnitude smaller than the circularization periods of mature solar-type binary stars. Owing to selection effects, the spin periods of the known planet-bearing stars are comparable to or longer than that of the Sun. (The absence of significant chromospheric activity, which is generally correlated with rapid stellar spin, has been an important criterion in the selection of target stars for radial-velocity surveys.) In contrast, circularized binary stars appear to attain spin synchronization so that their spin period is roughly 10 d (Meibom, Mathieu, & Stassun 2006). This state is maintained despite the stars' loss of spin angular momentum through magnetic braking because this loss can be compensated by the tidal transfer from the orbit in the case of two massive companions. Extrasolar planets have much less orbital angular momentum and the tidal interaction between the star and the planet cannot compensate for the loss of spin angular momentum through magnetic braking.

The important difference is that the tidal forcing frequency falls within the range of inertial waves in the binary circularization problem and well outside it for the hot Jupiter problem. In a synchronized binary the eccentricity tides have frequencies $\hat{\omega} = \pm\Omega$ (cf. Section 1.2) and can efficiently excite inertial waves, leading to an enhanced tidal dissipation. In contrast, planetary companions in circular orbits are unable to excite inertial waves in their host stars unless $\Omega_* > n/2$. In Fig. 4 the tidal forcing by a planet in a 1 d orbit falls at $\hat{\omega}/\Omega = 18$ and results in $Q' \approx 8.9 \times 10^{10}$ from viscous dissipation in the convective zone; this conclusion depends only weakly on the stellar spin for $|\Omega| \ll n$. On the other hand, Fig. 4 shows that Hough waves would be very important in the tidal evolution of short-period planets, leading to $Q' \approx 1.6 \times 10^5$ for $P = 1$ d, provided that they fail to reflect coherently from the center of the star. The nonlinearity parameter is estimated in Appendix A as $A \approx 320(M_p/M_*)(P/\text{d})^{1/6}$ in the case of the present Sun, which is about 0.3 for OGLE-TR-56b and somewhat less

than unity for hot Jupiters generally. The nonlinearity increases with stellar mass and age. This estimate shows how important it is to determine the fate of Hough waves (or gravity waves, in this regime) approaching the center for various levels of nonlinearity.

In contrast, for binary stars in a nearly synchronous state in which $\hat{\omega} \approx \Omega \approx n$ for the dominant eccentricity tide, the Coriolis force strengthens the energy dissipation associated with both inertial and Hough waves. Over most of the frequency range $|\hat{\omega}| < 2|\Omega|$, the values of Q' associated with inertial waves are 2–3 orders of magnitude smaller than those outside the range. This frequency dependence makes it possible for the dissipation of inertial waves to speed up the circularization of binary stars (see Section 4.2.1) without significantly influencing the orbital evolution of the close-in planets.

Planets are formed in protostellar disks. Classical T Tauri stars are observed to rotate with periods of a few days (Bouvier et al. 1993) and they spin up as they contract on to the zero-age main sequence. These rapid spin rates are sustained for $\tau_* < 100$ Myr (Bouvier, Forestini, & Allain 1997). During this early phase of stellar evolution, the host stars of the very short-period planets may have had $\Omega_* > n/2$, allowing the excitation of inertial waves. A relatively small Q' can occur in this regime owing to the rapid spin (see Fig. 6). In addition, the properties of the convection may have been different. Nevertheless, this phase may contribute little to orbital evolution as it is short-lived.

4.1.2. Close-in brown dwarf around an M dwarf

A precision radial velocity survey also led to the discovery of an $M_p \sin i = 19 M_J$ planet HD 41004Bb on a 1.33 d orbit around a $0.4 M_\odot$ M-dwarf star (Zucker et al. 2003). This period is the shortest known for any brown dwarf companion, and the estimated age of the system is $\tau_* = 1.6 \pm 0.8$ Gyr. Assuming that the M dwarf fails to spin synchronously because of the effects of magnetic braking, we find from equation (8) that $\tau_a \approx \tau_*$ when $Q' \approx 3.7 \times 10^7$. Alternatively, if we apply the frequency-independent Q' -value ($\sim 10^6$), calibrated on the circularization of solar-type binary stars, the extrapolated value of τ_a would be nearly two orders of magnitude smaller than the estimated age.

With its low mass, the host star HD 41004B has a smaller radiative core than the Sun. Since it has no measurable rotation speed (Butler et al. 2006), the tide raised by the brown dwarf companion is likely to be well outside the frequency range of inertial waves where Q' is relatively small. The results in Section 3 indicate that if the spin period of HD 41004B is longer than about 3 d, viscous dissipation in the convective zone provides a Q' much larger than 10^7 and poses no threat to the survival of the brown dwarf. However, if during the early phases of HD 41004B's evolution Ω was substantially larger than its present value, the tidal frequency could have fallen in the range of inertial waves. Provided the resulting low Q' did not last more than 10^8 yr, the amount of orbital decay would still have been negligible.

Hough waves can also be excited in the radiative zone of HD 41004B. Despite the relatively large mass ratio M_p/M_* of this system, the small value of dN/dr at the center of the $0.4 M_\odot$ star means that the Hough waves are unlikely to become nonlinear (see Appendix A). This

is important because if the Hough waves were effectively dissipated they would lead to $Q' < 10^6$ and seriously threaten the survival of the brown dwarf.

4.2. Tidal circularization

4.2.1. Solar-type binary stars

In the introduction, we indicated that the dissipation of tidal disturbances in both the convective and radiative zones of non-rotating solar-type stars fails, by about two orders of magnitude, to match that needed to account for the variation of the circularization period of binary stars as a function of their age (see Fig. 1). These discrepancies are particularly noteworthy since the previously obtained values of Q' might be underestimated as a result of optimistic assumptions. For example, in the application of the Goldreich & Keeley (1977) formula, a factor of $4\pi^2$ was introduced when Terquem et al. (1998) replaced $\hat{\omega}\tau$ with τ/P_{tide} . The discrepancies would be still larger if the prescription (10) were applied. Similarly, Sasselov (2003) set the tidal frequency equal to the orbital frequency and thereby underestimated the suppression of the turbulent viscosity. Finally, Goodman & Dickson (1998) assumed in one section of their paper that Hough waves are unable to reflect coherently from the center and form global standing modes, thereby allowing effective dissipation at all frequencies.

The results presented in this paper show that the Coriolis force can introduce a frequency dependence in and reduce the magnitude of Q' . The timescale for the stars to synchronize is generally much shorter than that required for circularization. The tidal frequency $\hat{\omega}_c = 3n - 2\Omega$ of the principal tidal component involved in circularization is larger than that involved in synchronization ($\hat{\omega}_s = 2n - 2\Omega$). Eccentricity can be excited for sufficiently large spin frequency Ω when $\hat{\omega}_c$ becomes negative. However, during the spin-down of the star towards synchronization, $\hat{\omega}_s$ tends to zero, $\hat{\omega}_c$ becomes positive and tends to Ω , and the eccentricity of the binary is damped. For this tidal frequency, the results in Fig. 3 indicate that the values of Q' associated with inertial and Hough waves are 5.7×10^7 and 1.1×10^8 , respectively. These values are an order of magnitude larger than that obtained for equilibrium and dynamical tides in non-rotating stars if the gravity waves reflect from the stellar center (Terquem et al. 1998). But our values of Q' are still an order of magnitude too large to account for the observational data in Fig. 1. Thus the incorporation of the Coriolis force reduces but does not eliminate the discrepancies between theory and observations.

There are several potential mechanisms that may contribute to the circularization process. Witte & Savonije (2002) made an attempt to take the effects of stellar evolution into account. They suggested that binary systems may undergo a complicated evolution through resonant tidal interactions. In principle a strong tidal dissipation efficiency at $\hat{\omega}_c \approx \Omega$ could lead to effective eccentricity damping after the binary stars are synchronized. But the protracted maintenance of this feature is needed over timescales comparable to τ_e .

The calculations described in this paper are carried out under the assumption of uniform rotation. Differential rotation is directly observed on the surface on the Sun and measured with helioseismological analysis in the

interior. A preliminary study of models with a modest amount of shellular differential rotation indicates that the details of the graph of $Q'(\hat{\omega})$ change significantly when differential rotation is introduced, while the typical range of values of Q' obtained is similar.

It is possible that a detailed study taking into account the effects of stellar evolution (including spin evolution) and differential rotation may be required to account for the tidal evolution of binary stars.

4.2.2. Low-mass binary stars

Most binary stars do not have equal-mass companions. In the solar system, the satellites of gas giants generally dominate the circularization process because they have much lower values of Q' than their host planets and are more intensely perturbed (Goldreich & Soter 1966; Murray & Dermott 1999). We now consider the contribution of a lower-mass companion on the circularization process for binary stars.

We have computed the rates of tidal dissipation via inertial and Hough waves for a $0.5 M_{\odot}$ main-sequence star of age 5 Gyr (Fig. 7). This star is less centrally condensed than a $1 M_{\odot}$ star and the fractional radius occupied by its radiative zone is smaller. The contribution of Hough waves is generally smaller than that of inertial waves unless the tidal forcing frequencies are outside the range of inertial waves. Owing to the different internal structure, the magnitude and frequency-dependence of Q' in this low-mass stellar model differ from that of a $1 M_{\odot}$ star. However, when we take into account the mass-radius relation, the theoretically determined circularization timescale, in the frequency range of inertial waves, is a generally decreasing function of the stellar mass.

The results of these models can be directly compared with some observed systems. The dataset used to derive a circularization period for the Hyades includes Johnson 311 (Griffin et al. 1985; Meibom & Mathieu 2005). This system has a period of 8.5 d and a negligible eccentricity. The mass of both of its two components is $0.5 M_{\odot}$ which make it an ideal system for a case study of low-mass stars. From equation (6), we find that each star must have $Q' \lesssim 1.8 \times 10^5$ in order for the circularization timescale to be less than the age of the Hyades cluster (600 Myr). This observationally calibrated value of Q' is more than two orders of magnitude smaller than that computed for the efficiency of tidal dissipation through both inertial and Hough waves. The discrepancy here is larger than that found for $1 M_{\odot}$ stars. However, Zahn & Bouchet (1989) argued that the circularization of these systems can occur during the Hayashi phase.

4.2.3. Close-in brown dwarfs and gas giant planets

All the theoretical models presented here are for uniformly rotating solar-type stars. These models are complementary to the models we presented in Paper I for gas giant planets. In our previous analysis, we showed that the presence of a solid core in gas giants can lead to a rich spectrum of features in the frequency-dependence of Q' . Since there are extended ranges in the core size and spin periods of gas giant planets, the dispersion in their values of Q' may be large. Nevertheless, the typical magnitude of Q' for gas giants is comparable to that for solar-type stars.

It is useful to make a parallel comparison between the theoretically determined and observationally inferred values of Q' . Owing to the technical challenges of the precision measurement of radial velocities, it is difficult to reduce the uncertainties in planetary orbital eccentricity to less than about 0.05. Many planets with periods less than 5 d have measured eccentricities less than 0.05. There are also supposedly eccentric planets within this period range. However, the apparent eccentricity measurement in such cases may be due to the perturbation of the host star's motion by additional planets in the system.

Finally, we note that the brown dwarf companion HD 41004Bb has a measured eccentricity of 0.08 despite its period of only 1.33 d. The mass ratio of the system is about 0.05. The observed eccentricity can be preserved over the 1.6 Gyr age of the system provided both objects have $Q' \gtrsim 10^7$. The value $Q'_* \gtrsim 3 \times 10^7$ required for the preservation of this brown dwarf implies that the M dwarf is unlikely to have been effective in damping the eccentricity of this system (see the discussions in Section 4.1.2). The lower limit on Q'_p required for the brown dwarf not to damp the eccentricity is an order of magnitude above that derived for the Jovian planet considered in Paper I. The weak dissipation in this case may be due to the different internal structure of brown dwarfs. Hough waves cannot be excited in these fully convective bodies. Without the presence of a core, inertial waves take the form of smooth global modes and are excited only in the vicinity of particular resonant frequencies (Wu 2005a,b).

5. SUMMARY AND DISCUSSION

In this paper, we have calculated the excitation and dissipation of low-frequency tidal oscillations in uniformly rotating solar-type stars and investigated the consequences for the circularization of binary stars and the orbital migration of close planetary companions. Our objectives were, first, to try to account for the efficiency of tidal dissipation inferred from the circularization periods of binary stars and, second, to try to resolve the paradox that gas giant planets with very short periods can be retained despite their intense tidal interaction with their host stars.

The detailed analysis presented here is an extension of our previous investigation. In Paper I, we studied the excitation and dissipation of inertial waves and Hough waves in the convective interior and radiative surface layers of gas giant planets. In the present paper we have examined solar-type stars with convective envelopes and radiative interiors. Again, our calculation allows for two avenues of tidal dissipation. The disturbance excited in the convective zone, which takes the form of inertial waves if the tidal frequency is less than twice the spin frequency, is dissipated by turbulent viscosity. Hough waves are also excited at the interface between the radiative and convective regions and propagate towards the center of the star. We calculate the associated dissipation rate under the assumption that they do not reflect coherently from the center of the star.

The radiative zone presents a reflecting barrier to inertial waves because the stable stratification strongly inhibits radial motions of low frequency. In this way the radiative interior of the star has a similar role to the

solid core in the planetary model. When the inertial waves propagate within a spherical annulus (rather than a full sphere) the tidal response is spatially intricate and enhanced dissipation occurs over extended ranges of frequency. The value of Q' can be reduced by up to four orders of magnitude when the Coriolis force is taken into account.

The values of Q' derived for the binary circularization problem in the case of the present Sun with a spin period of 10 d (Fig. 3) may still be an order of magnitude too large to account for the observed circularization periods of solar-type binary stars. However, the sensitivity of these results to the tidal and spin frequencies, the stellar model, and possible differential rotation means that the mechanisms we describe might be adequate to explain the observations when a detailed evolutionary study is carried out.

The dependence of Q' on the tidal and spin frequencies potentially resolves an outstanding issue with regard to the survival of gas giant planets with very short-period orbits around solar-type stars. These planets cannot supply angular momentum through tidal transfer at a sufficient rate to compensate for the spin-down of the host stars due to magnetic braking. The spin frequencies of these stars are comparable to that of the Sun, which is at least an order of magnitude smaller than the tidal forcing frequency. Consequently inertial waves cannot be excited. In addition we estimate that the Hough waves excited by a hot Jupiter in the present Sun are unlikely to undergo nonlinear damping at the center of the star.

If the Hough waves reflect coherently from the center and form global modes with narrow resonances, their contribution to orbital migration may be negligible. These very short-period planets can therefore be preserved despite their proximity to their host stars. Based on our models, we infer that gas giants with periods less than 1 d period may be preserved around solar-type stars. However, as conditions at the center of the star evolve, nonlinearity may set in at a critical age, resulting in a relatively rapid inspiral of the planet.

It would be valuable to conduct numerical simulations of gravity (or Hough) waves approaching the center of a star to determine at what amplitude the reflection of the wave is affected by nonlinearity (cf. Appendix A). Unfortunately, currently available stellar models do not agree on the value of dN/dr at the center, which is of critical importance.

Recently, a number of new planetary transit candidates were announced with periods less than 1 d in some cases (Sahu et al. 2006). If confirmed, which seems difficult, these systems would present interesting constraints on theories of tidal evolution.

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APPENDIX

GRAVITY WAVES APPROACHING THE CENTER OF THE STAR

Low-frequency gravity waves of small amplitude in a non-rotating star are described by the differential equation (e.g. Goodman & Dickson 1998)

$$\frac{d}{dr} \left[\frac{1}{\rho} \frac{d}{dr} (\rho r^2 \tilde{\xi}_r) \right] = \ell(\ell+1) \left(1 - \frac{N^2}{\omega^2} \right) \tilde{\xi}_r, \quad (\text{A1})$$

where $\xi_r = \text{Re}[\tilde{\xi}_r(r) Y_\ell^m(\theta, \phi) \exp(-i\omega t)]$ is the radial displacement, $\rho(r)$ is the density and $N(r)$ is the Brunt–Väisälä frequency. Very near the center of the star, the density and pressure are nearly uniform while g is linear in r . In this region ρ can be regarded as constant while $N = Cr$ is linear in r . Let $x = N/\omega = (C/\omega)r$. Then the regular solution of the above equation for $\ell \geq 1$ is

$$\tilde{\xi}_r \propto x^{-3/2} J_\nu(\sqrt{\ell(\ell+1)}x), \quad (\text{A2})$$

in terms of the Bessel function of order $\nu = \ell + \frac{1}{2}$. (The second solution, involving Y_ν , diverges as $r \rightarrow 0$.) In particular, for $\ell = 2$,

$$\xi_r = \frac{5A\omega}{8C} x^{-4} \left[6^{-1/2}(1-2x^2) \sin(6^{1/2}x) - x \cos(6^{1/2}x) \right] \sin^2 \theta \cos(2\phi - \omega t), \quad (\text{A3})$$

where A is a dimensionless amplitude. Since the unperturbed entropy profile is proportional to r^2 , the perturbed profile is proportional to $r^2 - 2r\xi_r$. The wave overturns the stratification, according to linear theory, if the quantity

$$\frac{1}{r} \frac{\partial}{\partial r} (r\xi_r) > 1. \quad (\text{A4})$$

The maximum value of this quantity occurs at $r = 0$, $\theta = \pi/2$, and $2\phi - \omega t = 0$, and is equal to A . Therefore A is a measure of the maximum nonlinearity of the wave. If $A < 1$ the entropy profile is never inverted. [In this case the wave may still be disrupted by parametric instabilities, as noted by Goodman & Dickson (1998).]

The energy flux in the incoming part of the wave is

$$F = \frac{5\pi A^2}{72\sqrt{6}} \frac{\rho \omega^8}{C^5}. \quad (\text{A5})$$

Equating this to the energy flux in gravity waves excited at the interface between the radiative and convective zones

in a non-rotating solar model, we estimate

$$A \approx 2500e \left(\frac{P}{d} \right)^{1/6} \quad (\text{A6})$$

for the largest eccentricity tide involved in the circularization of a binary with two synchronized $1 M_{\odot}$ stars, while

$$A \approx 320 \frac{M_p}{M_*} \left(\frac{P}{d} \right)^{1/6} \quad (\text{A7})$$

for the tide excited in a slowly rotating $1 M_{\odot}$ star by a planet of mass M_p and orbital period P . The first estimate is similar to that made by Goodman & Dickson (1998), and implies that the waves always achieve nonlinearity in the binary circularization problem if the eccentricity is measurable. (This remains true when the Coriolis force is taken into account.) The second estimate implies that the waves excited by hot Jupiters are of significant amplitude but probably do not achieve sufficient nonlinearity to disrupt the reflection.

These results depend strongly on the stellar model. For stars of mass $0.5 M_{\odot}$ and age 5 Gyr we find

$$A \approx 2.7e \left(\frac{P}{d} \right)^{1/6} \quad (\text{A8})$$

for binary circularization and

$$A \approx 0.34 \frac{M_p}{M_*} \left(\frac{P}{d} \right)^{1/6} \quad (\text{A9})$$

for hot Jupiters. Therefore even the binary circularization problem is unlikely to give rise to nonlinear tides if both components have $0.5 M_{\odot}$. The main effect causing this difference is that $A \propto (dN/dr)_c^{5/2}$ and $(dN/dr)_c$ is about 7 times larger in the present Sun than in the $0.5 M_{\odot}$ star. In general $(dN/dr)_c$ increases with stellar mass and age, but its value is not very accurately determined by current models of stellar evolution.

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