THE PRECESSION AND NUTATION OF DEFORMABLE BODIES*

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Abstract. The aim of the present paper will be to derive from the fundamental equations of hydrodynamics the explicit form of the Eulerian equations which govern the motion about the centre of gravity of self-gravitating bodies, consisting of compressible fluid of arbitrary viscosity, in an arbitrary external field of force. If the problem is particularized so that the external field of force represents the attraction of the sun and the moon, this motion would represent the luni-solar precession and nutation of a fluid viscous earth; if, on the other hand, the external field of force were governed by the earth (and the sun), the motion would define the physical librations of the moon regarded as a deformable body. The same equations are, moreover, equally applicable to the phenomena of precession and nutation of rotating fluid components in close binary systems, distorted by mutual tidal action; and the present paper contains the first formulation of the effects of viscosity on such phenomena.

1. Introduction

The differential equations which govern the motions of self-gravitating bodies about their centres of mass – whether such motions are free of forced – have been in use since the early days of the history of rational mechanics; and the investigators of their solutions bearing on the precession and nutation of the earth, or the physical librations of the moon, included (to name only the greatest) Newton, Euler, Lagrange, and Laplace. All these investigators assumed, however, in common that the structure of the body moving about its centre of gravity in external field of force can be treated as *rigid* (i.e., under the assumption that the relative positions of all mass particles constituting such bodies are not influenced by the motion of the body as a whole); and if so, the external form (as well as the moments of inertia) of such a body are fixed and independent of the time. It was not till in the second half of the 19th century that it has been gradually realized that planetary bodies of the mass of the earth (or even of that of the moon) cannot be regarded as completely incompressible or rigid; for the observed period of the terrestrial free precession required definite departures from these conditions for its explanation. Moreover, observations have disclosed that (at least in the case of the earth) the form of our globe responds periodically to a fluctuating external field of force through bodily tides.

In double-star astronomy, the fluid components of close binary systems exhibit the effects of deformability – under the influence of their axial rotation and mutual tidal action – to a much greater extent than any similar phenomena in the solar

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system; and tidal bulges of such stars sweeping around the equator in the orbital period of the pair often attain heights comparable with the absolute dimensions of the respective components. As, moreover, such interaction to observable phenomena the existence of which is borne out by extensive photometric observations of such systems, it is clearly desirable to develop a mathematical theory which could serve as a basis for an interpretation of such rotational effects. The aim of the present investigation will be to develop an outline of a theory which could lend itself for this purpose.

A mathematical theory of the motion of *deformable* bodies about their centre of mass in an external field of force represents a classical problem, the investigation of which was, however, slow to advance (cf. LIOUVILLE, 1858; GYLDÉN, 1873; DARWIN, 1879; OPPENHEIMER, 1885; POINCARÉ, 1910) in the past hundred years, and is still far from being properly solved for the precession and nutation of the earth; while its bearing on the physical librations of the lunar globe has not even been considered. Moreover, in stellar astronomy, the problem of the precession and nutation of fluid components of close binary systems has so far been barely opened up (cf. KOPAL, 1959).

The aim of the present investigation will be to treat the problem anew by a different method than has been done so far: for whereas all previous investigators mentioned above – from Liouville to Poincaré – did so by starting from the Lagrangian equations of mechanics, we propose to depart from the fundamental equations of *hydrodynamics* of viscous flow, in which the three velocity components u, v, w will be systematically expressed in terms of the independent rotations, about the three respective axes x, y, z with angular velocities ω_x, ω_y, ω_z. We shall carry out this task without resorting at any stage to any kind of linearization in terms of the dependent variables, or any other approximation which could influence the results. In particular, no limit will be imposed on the magnitude of the viscosity μ of our fluid, nor on the way in which $\mu(x, y, z)$ can vary inside our configuration. Only one simplifying hypothesis will be made which will restrict the generality of our work: namely, an assumption that all three angular velocity components $\omega_{x,y,z}$ do not depend on relative position, and are functions of the time only. A more general treatment of the problem in which $\omega \equiv \omega(x, y, z; t)$ is being postponed for subsequent investigation.

2. Equations of the Problem

As is well known, the Eulerian fundamental equations of hydrodynamics governing the motion of compressible viscous fluids can be expressed in rectangular coordinates in the symmetrical form

$$\rho \frac{Du}{Dt} = \rho \frac{\partial \Omega}{\partial x} - \frac{\partial P}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}, \tag{2.1}$$

$$\rho \frac{Dv}{Dt} = \rho \frac{\partial \Omega}{\partial y} - \frac{\partial P}{\partial y} + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}, \tag{2.2}$$

$$\rho \frac{Dw}{Dt} = \rho \frac{\partial \Omega}{\partial z} - \frac{\partial P}{\partial z} + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}, \tag{2.3}$$

where u, v, w denote the velocity components of fluid motion, at the time t, in the direction of increasing coordinates x, y, z, respectively;

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \tag{2.4}$$

representing the Lagrangian time-derivative (following the motion); ρ stands for the local density of the fluid; P, for its pressure; Ω, for the total potential (internal as well as external) of all forces acting upon it; and

$$\sigma_{xx} = \tfrac{2}{3}\mu \left\{ 3 \frac{\partial u}{\partial x} - \Delta \right\}, \tag{2.5}$$

$$\sigma_{yy} = \tfrac{2}{3}\mu \left\{ 3 \frac{\partial v}{\partial y} - \Delta \right\}, \tag{2.6}$$

$$\sigma_{zz} = \tfrac{2}{3}\mu \left\{ 3 \frac{\partial w}{\partial z} - \Delta \right\}, \tag{2.7}$$

$$\sigma_{xy} = \mu \left\{ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right\} = \sigma_{yx}, \tag{2.8}$$

$$\sigma_{yz} = \mu \left\{ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right\} = \sigma_{zy}, \tag{2.9}$$

$$\sigma_{zx} = \mu \left\{ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right\} = \sigma_{xz}, \tag{2.10}$$

are the respective components of the viscous stress tensor, where μ denotes the coefficient of viscosity, and

$$\Delta \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{2.11}$$

the divergence of the velocity vector of the fluid.

As is well known, Equations (2.1)–(2.3) safeguard the conservation of momentum of the underlying dynamical problem; and as such represent only one-half of the system necessary for a complete specification of the six dependent variables

$$u,\ v,\ w\,;$$
$$\rho, P, \Omega\,;$$

of our problem. Of the remaining three equations, two can be adjoined with relative ease: namely, the equation of continuity

$$\frac{D\rho}{Dt} + \rho\Delta = 0 \tag{2.12}$$

safeguarding the conservation of mass, and the Poisson equation

$$\nabla^2 \Omega = -4\pi G \rho, \tag{2.13}$$

which must be satisfied by the gravitational potential (G denoting the constant of gravitation).

The sole remaining equation required to render the solution of our system determinate (for an appropriate set of boundary conditions) must be derived from the principle of the conservation of energy, in the form of an 'equation of state' relating P and ρ; but its explicit formulation will be postponed until a latter stage of our analysis.

3. The Components of Velocities and Accelerations

In order to apply the system of equations set up in the preceding section for the study of the motion of a self-gravitating body about its centre of gravity, consider the transformation of rectangular coordinates between an *inertial* (fixed) system of *space* axes x, y, z, and a *rotating* system of *body* (primed) axes x', y', z', possessing the same origin, but with the primed axes rotated with respect to the space axes by the Eulerian angles ϕ, θ, ψ, in accordance with a scheme illustrated on the accompanying Figure 1.

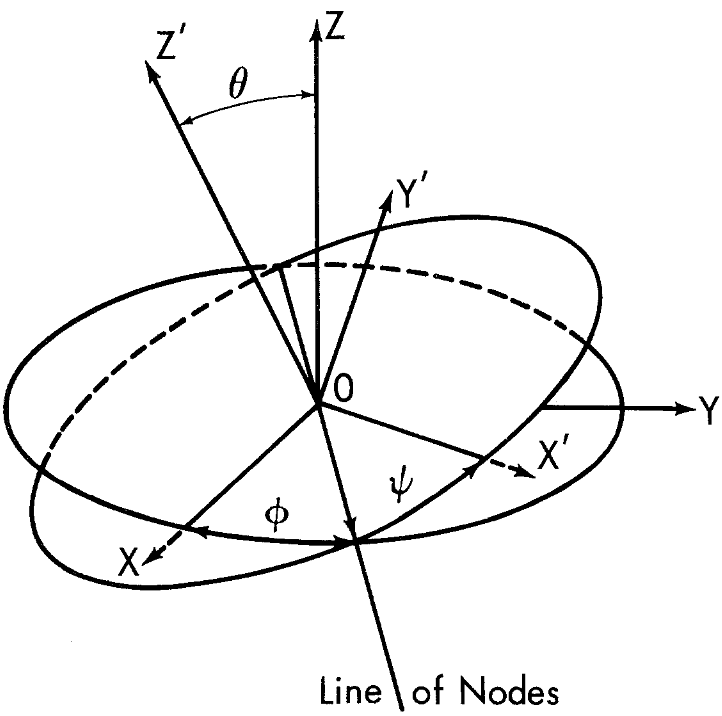

Fig. 1. Definition of Eulerian angles.

As is well known, the transformation of coordinates from the space to the body axes is governed by the following matrix equation

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}, \tag{3.1}$$

where the coefficients a_{ik}, expressed in terms of the Eulerian angles assume the explicit forms

$$\left.\begin{aligned} a_{11} &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi, \\ a_{12} &= -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi, \\ a_{13} &= \sin\theta\sin\phi \quad ; \end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned} a_{21} &= \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi, \\ a_{22} &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi, \\ a_{23} &= -\sin\theta\cos\phi \quad ; \end{aligned}\right\} \tag{3.3}$$

$$\left.\begin{aligned} a_{31} &= \sin\psi\sin\theta, \\ a_{32} &= \cos\psi\sin\theta, \\ a_{33} &= \cos\theta. \end{aligned}\right\} \tag{3.4}$$

In order to obtain the corresponding *space* velocity-components u, v, w, let us differentiate Equations (3.1) with respect to the time. With dots denoting hereafter ordinary (total) derivatives with respect to t, we find that

$$\dot{x} = u = \dot{a}_{11}x' + \dot{a}_{12}y' + \dot{a}_{13}z' + a_{11}\dot{x}' + a_{12}\dot{y}' + a_{13}\dot{z}', \tag{3.5}$$

$$\dot{y} = v = \dot{a}_{21}x' + \dot{a}_{22}y' + \dot{a}_{23}z' + a_{21}\dot{x}' + a_{22}\dot{y}' + a_{23}\dot{z}', \tag{3.6}$$

$$\dot{z} = w = \dot{a}_{31}x' + \dot{a}_{32}y' + \dot{a}_{33}z' + a_{31}\dot{x}' + a_{32}\dot{y}' + a_{33}\dot{z}'; \tag{3.7}$$

whereas the *body* velocity-components u', v', w' follow from

$$\begin{aligned} \dot{x}' = u' &= \dot{a}_{11}x + \dot{a}_{21}y + \dot{a}_{31}z \\ &+ a_{11}\dot{x} + a_{21}\dot{y} + a_{31}\dot{z}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \dot{y}' = v' &= \dot{a}_{12}x + \dot{a}_{22}y + \dot{a}_{32}z \\ &+ a_{12}\dot{x} + a_{22}\dot{y} + a_{32}\dot{z}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \dot{z}' = w' &= \dot{a}_{13}x + \dot{a}_{23}y + \dot{a}_{33}z \\ &+ a_{13}\dot{x} + a_{23}\dot{y} + a_{33}\dot{z}; \end{aligned} \tag{3.10}$$

where

$$\left.\begin{aligned} \dot{a}_{11} = \quad a_{12}\dot{\psi} - a_{21}\dot{\phi} + a_{31}\dot{\theta}\sin\phi &= a_{31}\omega_y - a_{21}\omega_z \\ &= a_{12}\omega_{z'} - a_{13}\omega_{y'}, \end{aligned}\right\} \tag{3.11}$$

$$\left.\begin{aligned} \dot{a}_{12} = -a_{11}\dot{\psi} - a_{22}\dot{\phi} + a_{32}\dot{\theta}\sin\phi &= a_{32}\omega_y - a_{22}\omega_z \\ &= a_{13}\omega_{x'} - a_{11}\omega_{z'}, \end{aligned}\right\} \tag{3.12}$$

$$\left.\begin{aligned} \dot{a}_{13} = \quad - a_{23}\dot{\phi} + a_{33}\dot{\theta}\sin\phi &= a_{33}\omega_y - a_{23}\omega_z \\ &= a_{11}\omega_{y'} - a_{12}\omega_{x'}; \end{aligned}\right\} \tag{3.13}$$

$$\left.\begin{aligned}\dot{a}_{21} &= \quad a_{22}\dot{\psi} + a_{11}\dot{\phi} - a_{31}\dot{\theta}\cos\phi = a_{11}\omega_z - a_{31}\omega_x \\ &= a_{22}\omega_{z'} - a_{23}\omega_{y'},\end{aligned}\right\} \tag{3.14}$$

$$\left.\begin{aligned}\dot{a}_{22} &= -a_{21}\dot{\psi} + a_{12}\dot{\phi} - a_{32}\dot{\theta}\cos\phi = a_{12}\omega_z - a_{32}\omega_x \\ &= a_{23}\omega_{x'} - a_{21}\omega_{z'},\end{aligned}\right\} \tag{3.15}$$

$$\left.\begin{aligned}\dot{a}_{23} &= \qquad + a_{13}\dot{\phi} - a_{33}\dot{\theta}\cos\phi = a_{13}\omega_z - a_{33}\omega_x \\ &= a_{21}\omega_{y'} - a_{22}\omega_{x'};\end{aligned}\right\} \tag{3.16}$$

$$\left.\begin{aligned}\dot{a}_{31} &= a_{32}\dot{\psi} \qquad + \dot{\theta}\sin\psi\cos\theta = a_{21}\omega_x - a_{11}\omega_y \\ &= a_{32}\omega_{z'} - a_{33}\omega_{y'},\end{aligned}\right\} \tag{3.17}$$

$$\left.\begin{aligned}\dot{a}_{32} &= -a_{31}\dot{\psi} \qquad + \dot{\theta}\cos\psi\cos\theta = a_{22}\omega_x - a_{12}\omega_y \\ &= a_{33}\omega_{x'} - a_{31}\omega_{z'},\end{aligned}\right\} \tag{3.18}$$

$$\left.\begin{aligned}\dot{a}_{33} &= \qquad - \dot{\theta}\sin\theta \qquad = a_{23}\omega_x - a_{13}\omega_y \\ &= a_{31}\omega_{y'} - a_{32}\omega_{x'}\end{aligned}\right\} \tag{3.19}$$

where, taking advantage of the fact that

$$\left.\begin{aligned}a_{11}\dot{a}_{11} + a_{12}\dot{a}_{12} + a_{13}\dot{a}_{13} &= 0 \\ a_{21}\dot{a}_{21} + a_{22}\dot{a}_{22} + a_{23}\dot{a}_{23} &= 0 \\ a_{31}\dot{a}_{31} + a_{32}\dot{a}_{32} + a_{33}\dot{a}_{33} &= 0\end{aligned}\right\} \tag{3.20}$$

and

$$\left.\begin{aligned}a_{11}\dot{a}_{11} + a_{21}\dot{a}_{21} + a_{31}\dot{a}_{31} &= 0 \\ a_{12}\dot{a}_{12} + a_{22}\dot{a}_{22} + a_{32}\dot{a}_{32} &= 0 \\ a_{13}\dot{a}_{13} + a_{23}\dot{a}_{23} + a_{33}\dot{a}_{33} &= 0\end{aligned}\right\} \tag{3.21}$$

the respective angular velocities of rotation are given by

$$\left.\begin{aligned}\omega_x &= +(a_{21}\dot{a}_{31} + a_{22}\dot{a}_{32} + a_{23}\dot{a}_{33}) \\ &= -(a_{31}\dot{a}_{21} + a_{32}\dot{a}_{22} + a_{33}\dot{a}_{23}),\end{aligned}\right\} \tag{3.22}$$

$$\left.\begin{aligned}\omega_y &= +(a_{31}\dot{a}_{11} + a_{32}\dot{a}_{12} + a_{33}\dot{a}_{13}) \\ &= -(a_{11}\dot{a}_{31} + a_{12}\dot{a}_{32} + a_{13}\dot{a}_{33}),\end{aligned}\right\} \tag{3.23}$$

$$\left.\begin{aligned}\omega_z &= +(a_{11}\dot{a}_{21} + a_{12}\dot{a}_{22} + a_{13}\dot{a}_{23}) \\ &= -(a_{21}\dot{a}_{11} + a_{22}\dot{a}_{12} + a_{23}\dot{a}_{13}),\end{aligned}\right\} \tag{3.24}$$

with respect to the space axes; or

$$\left.\begin{aligned}\omega_{x'} &= +(a_{13}\dot{a}_{12} + a_{23}\dot{a}_{22} + a_{33}\dot{a}_{32}) \\ &= -(a_{12}\dot{a}_{13} + a_{22}\dot{a}_{23} + a_{32}\dot{a}_{33}),\end{aligned}\right\} \tag{3.25}$$

$$\left.\begin{aligned}\omega_{y'} &= +(a_{11}\dot{a}_{13} + a_{21}\dot{a}_{23} + a_{31}\dot{a}_{33}) \\ &= -(a_{13}\dot{a}_{11} + a_{23}\dot{a}_{21} + a_{33}\dot{a}_{31}),\end{aligned}\right\} \tag{3.26}$$

$$\left.\begin{aligned}\omega_{z'} &= +(a_{12}\dot{a}_{11} + a_{22}\dot{a}_{21} + a_{32}\dot{a}_{31}) \\ &= -(a_{11}\dot{a}_{12} + a_{21}\dot{a}_{22} + a_{31}\dot{a}_{32}),\end{aligned}\right\} \tag{3.27}$$

with respect to the body axes; the pairs of alternative equations arising from the fact that, by a time-differentiation of the relations $a_{ij}a_{ik}=\delta_{jk}$ it follows that $a_{ij}\dot{a}_{ik}+a_{ik}\dot{a}_{ij}=0$.

Inserting in the Equations (3.22)–(3.27) from (3.11)–(3.19) it follows that, in terms of the Eulerian angles,

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi\,, \tag{3.28}$$

$$\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi\,, \tag{3.29}$$

$$\omega_z = \qquad + \dot{\psi}\cos\theta \qquad + \dot{\phi}\,; \tag{3.30}$$

while

$$\omega_{x'} = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\,, \tag{3.31}$$

$$\omega_{y'} = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\,, \tag{3.32}$$

$$\omega_{z'} = \dot{\phi}\cos\theta \qquad + \dot{\psi}\,, \tag{3.33}$$

as could be also directly verified by an application of the inverse of the transformation (3.1), in accordance with which

$$\left.\begin{aligned} \omega_{x'} &= a_{11}\omega_x + a_{21}\omega_y + a_{31}\omega_z\,,\\ \omega_{y'} &= a_{12}\omega_x + a_{22}\omega_y + a_{32}\omega_z\,,\\ \omega_{z'} &= a_{13}\omega_x + a_{23}\omega_y + a_{33}\omega_z\,. \end{aligned}\right\} \tag{3.34}$$

With the aid of the preceding results the Equations (3.5)–(3.7) or (3.8)–(3.10) for the velocity-components with respect to the space or body axes can be reduced to the forms

$$u = z\omega_y - y\omega_z + u_0' \tag{3.35}$$

$$v = x\omega_z - z\omega_x + v_0' \tag{3.36}$$

$$w = y\omega_x - x\omega_y + w_0' \tag{3.37}$$

or

$$u' = -\,z'\omega_{y'} + y'\omega_{z'} + u_0\,, \tag{3.38}$$

$$v' = -\,x'\omega_{z'} + z'\omega_{x'} + v_0\,, \tag{3.39}$$

$$w' = -\,y'\omega_{x'} + x'\omega_{y'} + w_0\,, \tag{3.40}$$

where

$$\left.\begin{aligned} u_0 &= a_{11}u + a_{21}v + a_{31}w\\ v_0 &= a_{12}u + a_{22}v + a_{32}w\\ w_0 &= a_{13}u + a_{23}v + a_{33}w \end{aligned}\right\} \tag{3.41}$$

are the *space* velocity components in the direction of the *rotating* axes x', y', z'; and

$$\left.\begin{aligned} u_0' &= a_{11}u' + a_{12}v' + a_{13}w'\\ v_0' &= a_{21}u' + a_{22}v' + a_{23}w'\\ w_0' &= a_{31}u' + a_{32}v' + a_{33}w' \end{aligned}\right\} \tag{3.42}$$

are the *body* velocity components in the direction of the *fixed* axes x, y, z.

In order to specify the appropriate forms of the components of *acceleration*, let us differentiate the foregoing Equations (3.35)–(3.40) for the velocity components with respect to the time. Doing so we find that those with respect to the *space* axes assume the forms

$$\dot{u} = w\omega_y + z\dot{\omega}_y - v\omega_z - y\dot{\omega}_z + \dot{u}'_0, \tag{3.43}$$

$$\dot{v} = u\omega_z + x\dot{\omega}_z - w\omega_x - z\dot{\omega}_x + \dot{v}'_0, \tag{3.44}$$

$$\dot{w} = v\omega_x + y\dot{\omega}_x - u\omega_y - x\dot{\omega}_y + \dot{w}'_0, \tag{3.45}$$

where the velocity components u, v, w have already been given by Equations (3.35)–(3.37); and where, by differentiation of (3.42),

$$\begin{aligned}\dot{u}'_0 = a_{11}\dot{u}' + a_{12}\dot{v}' + a_{13}\dot{w}' \\ + \dot{a}_{11}u' + \dot{a}_{12}v' + \dot{a}_{13}w',\end{aligned} \tag{3.46}$$

$$\begin{aligned}\dot{v}'_0 = a_{21}\dot{u}' + a_{22}\dot{v}' + a_{23}\dot{w}' \\ + \dot{a}_{21}u' + \dot{a}_{22}v' + \dot{a}_{23}w',\end{aligned} \tag{3.47}$$

$$\begin{aligned}\dot{w}'_0 = a_{31}\dot{u}' + a_{32}\dot{v}' + a_{33}\dot{w}' \\ + \dot{a}_{31}u' + \dot{a}_{32}v' + \dot{a}_{33}w'.\end{aligned} \tag{3.48}$$

The first three terms in each of these expressions represent obviously the body accelerations with respect to the space axes; and we shall abbreviate them as

$$\left.\begin{aligned}a_{11}\dot{u}' + a_{12}\dot{v}' + a_{13}\dot{w}' = (\dot{u})'_0, \\ a_{21}\dot{u}' + a_{22}\dot{v}' + a_{23}\dot{w}' = (\dot{v})'_0, \\ a_{31}\dot{u}' + a_{32}\dot{v}' + a_{33}\dot{w}' = (\dot{w})'_0.\end{aligned}\right\} \tag{3.49}$$

Since, moreover, by insertion from (3.11)–(3.13) and (3.42),

$$\begin{aligned}\dot{a}_{11}u' + \dot{a}_{12}v' + \dot{a}_{13}w' &= (a_{31}\omega_y - a_{21}\omega_z)\,u' \\ &\quad + (a_{32}\omega_y - a_{22}\omega_z)\,v' \\ &\quad + (a_{33}\omega_y - a_{23}\omega_z)\,w' \\ &= \omega_y(a_{31}u' + a_{32}v' + a_{33}w') \\ &\quad - \omega_z(a_{21}u' + a_{22}v' + a_{23}w') \\ &= \omega_y w'_0 - \omega_z v'_0\,;\end{aligned} \tag{3.50}$$

and, similarly,

$$\dot{a}_{21}u' + \dot{a}_{22}v' + \dot{a}_{23}w' = \omega_z u'_0 - \omega_x w'_0 \tag{3.51}$$

while

$$\dot{a}_{31}u' + \dot{a}_{32}v' + \dot{a}_{33}w' = \omega_x v'_0 - \omega_y u'_0, \tag{3.52}$$

Equations (3.43)–(3.45) can be rewritten in a more explicit form

$$\begin{aligned}\dot{u} = -\,x(\omega_y^2 + \omega_z^2) + y(\omega_x\omega_y - \dot{\omega}_z) + z(\omega_x\omega_z + \dot{\omega}_y) \\ + (\dot{u})'_0 + 2(w'_0\omega_y - v'_0\omega_z),\end{aligned} \tag{3.53}$$

$$\begin{aligned}\dot{v} = -\,y(\omega_z^2 + \omega_x^2) + z(\omega_y\omega_z - \dot{\omega}_x) + x(\omega_x\omega_y + \dot{\omega}_z) \\ + (\dot{v})'_0 + 2(u'_0\omega_z - w'_0\omega_x),\end{aligned} \tag{3.54}$$

and

$$\dot{w} = - z(\omega_x^2 + \omega_y^2) + x(\omega_x\omega_z - \dot{\omega}_y) + y(\omega_y\omega_z + \dot{\omega}_x) + (\dot{w})'_0 + 2(v'_0\omega_x - u'_0\omega_y). \tag{3.55}$$

The foregoing equations refer to accelerations with respect to the inertial system of space axes. Those with respect to the (rotating) *body* axes can be obtained by an analogous process from the equations

$$\dot{u}' = - w'\omega_{y'} - z'\dot{\omega}_{y'} + v'\omega_{z'} + y'\dot{\omega}_{z'} + \dot{u}_0, \tag{3.56}$$

$$\dot{v}' = - u'\omega_{z'} - x'\dot{\omega}_{z'} + w'\omega_{x'} + z'\dot{\omega}_{x'} + \dot{v}_0, \tag{3.57}$$

$$\dot{w}' = - v'\omega_{x'} - y'\dot{\omega}_{x'} + u'\omega_{y'} + x'\dot{\omega}_{y'} + \dot{w}_0, \tag{3.58}$$

equivalent to (3.43)–(3.45); which on being treated in the same way as the latter can eventually be reduced to the form

$$\begin{aligned}\dot{u}' = &- x'(\omega_{y'}^2 + \omega_{z'}^2) + y'(\omega_{x'}\omega_{y'} + \dot{\omega}_{z'}) + z'(\omega_{x'}\omega_{z'} - \dot{\omega}_{y'}) \\ &+ (\dot{u})_0 - 2(w_0\omega_{y'} - v_0\omega_{z'}),\end{aligned} \tag{3.59}$$

$$\begin{aligned}\dot{v}' = &- y'(\omega_{z'}^2 + \omega_{x'}^2) + z'(\omega_{y'}\omega_{z'} + \dot{\omega}_{x'}) + x'(\omega_{x'}\omega_{y'} - \dot{\omega}_{z'}) \\ &+ (\dot{v})_0 - 2(u_0\omega_{z'} - w_0\omega_{x'}),\end{aligned} \tag{3.60}$$

$$\begin{aligned}\dot{w}' = &- z'(\omega_{x'}^2 + \omega_{y'}^2) + x'(\omega_{x'}\omega_{z'} + \dot{\omega}_{y'}) + y'(\omega_{y'}\omega_{z'} - \dot{\omega}_{x'}) \\ &+ (\dot{w})_0 - 2(v_0\omega_{x'} - u_0\omega_{y'}),\end{aligned} \tag{3.61}$$

where the space velocity components u_0, v_0, w_0 in the direction of increasing x', y', z' continue to be given by Equations (3.41); while the corresponding components of the accelerations are given by

$$\left.\begin{aligned}(\dot{u})_0 &= a_{11}\dot{u} + a_{21}\dot{v} + a_{31}\dot{w}, \\ (\dot{v})_0 &= a_{12}\dot{u} + a_{22}\dot{v} + a_{32}\dot{w}, \\ (\dot{w})_0 &= a_{13}\dot{u} + a_{23}\dot{v} + a_{33}\dot{w}.\end{aligned}\right\} \tag{3.62}$$

If, in particular, we consider the restricted case of a rotation about the z-axis only (so that $\omega_x = \omega_y = 0$), Equations (3.53)–(3.55) will reduce to the system

$$\left.\begin{aligned}\boxed{\dot{u}} &= (\dot{u})'_0 - 2v\omega_z + x\omega_z^2 - y\dot{\omega}_z, \\ \boxed{\dot{v}} &= (\dot{v})'_0 + 2u\omega_z + y\omega_z^2 + x\dot{\omega}_z, \\ \boxed{\dot{w}} &= (\dot{w})'_0;\end{aligned}\right\} \tag{3.63}$$

while Equations (3.59)–(3.61) will likewise reduce to

$$\left.\begin{aligned}\dot{u}' &= \boxed{(\dot{u})_0} + 2v'\omega_{z'} + x'\omega_{z'}^2 + y'\dot{\omega}_{z'}, \\ \dot{v}' &= \boxed{(\dot{v})_0} - 2u'\omega_{z'} + y'\omega_{z'}^2 - x'\dot{\omega}_{z'}, \\ \dot{w}' &= \boxed{(\dot{w})_0}.\end{aligned}\right\} \tag{3.64}$$

It is the accelerations in the cartouches of the two systems – referred as they are to the inertial space axes – which should be identified with the Lagrangian time-derivatives

$$\frac{DV}{Dt}$$

on the left-hand sides of the Equations (2.2)–(2.3) of motion if these are referred to the inertial or rotating axes of coordinates.

A closing note concerning the time differentiation of the coordinates or velocities should be added in this place. As

$$x \equiv x(t), \qquad y \equiv y(t), \qquad z \equiv z(t), \tag{3.65}$$

it follows that

$$\left.\begin{aligned} \dot{x} \equiv u &= \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial x}{\partial t}, \\ \dot{y} \equiv v &= \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial y}{\partial t}, \\ \dot{z} \equiv w &= \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial t}, \end{aligned}\right\} \tag{3.66}$$

i.e., the ordinary (total) and partial derivatives of the coordinates with respect to the time are obviously identical. This is, however, no longer true of the time-differentiation of the velocities – whether linear or angular. As

$$\left.\begin{aligned} u &\equiv u(x, y, z; t) \\ v &\equiv v(x, y, z; t) \\ w &\equiv w(x, y, z; t) \end{aligned}\right\} \tag{3.67}$$

or

$$\omega_{x,y,z} \equiv \omega_{x,y,z}(x, y, z; t), \tag{3.68}$$

where the coordinates (3.65) are themselves functions of the time. In consequence,

$$\begin{aligned} \dot{u} \equiv \frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} \\ &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \end{aligned} \tag{3.69}$$

by virtue of (3.66); and similarly for $\dot{v}$ and $\dot{w}$. Likewise,

$$\dot{\omega} \equiv \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + w\frac{\partial \omega}{\partial z} \tag{3.70}$$

for $\omega \equiv \omega_{x,y,z}$.

For coordinate systems referred to the rotating body axes similar relations hold good; care being merely taken to replace the unprimed coordinates or velocity components by the primed ones.

4. Formation of the Eulerian Equations for Precession and Nutation

In Section 2 of this paper we set up the general form of the equations governing the motion of compressible viscous fluids in rectangular coordinates; and in Section 3 we expressed its velocity components in terms of arbitrary rotations about three rectangular axes. The aim of the present section will be to combine the fundamental Equations (2.1)–(2.3) rewritten in terms of the angular variables $\omega_{x,y,z}$ introduced in Section 3 in a form suitable for their subsequent solution.

In order to embark on this task, let us multiply Equations (3.53)–(3.55) by x, y, z and form their following differences:

$$\begin{aligned} y\dot{w} - z\dot{v} = (y^2 + z^2)\,\dot{\omega}_x &+ (y^2 - z^2)\,\omega_y\omega_z \\ - xy(\dot{\omega}_y - \omega_x\omega_z) &- xz(\dot{\omega}_z + \omega_x\omega_y) \\ &- yz(\omega_y^2 - \omega_z^2) \\ + \{y(\dot{w})_0' - z(\dot{v})_0'\} &+ 2y\{v_0'\omega_x - u_0'\omega_y\} \\ &- 2z\{u_0'\omega_z - w_0'\omega_x\}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} z\dot{u} - x\dot{w} = (z^2 + x^2)\,\dot{\omega}_y &+ (z^2 - x^2)\,\omega_x\omega_z \\ - yz(\dot{\omega}_z - \omega_y\omega_x) &- yx(\dot{\omega}_x + \omega_y\omega_z) \\ &- zx(\omega_z^2 - \omega_x^2) \\ + \{z(\dot{u})_0' - x(\dot{w})_0'\} &+ 2z\{w_0'\omega_y - v_0'\omega_z\} \\ &- 2x\{v_0'\omega_x - u_0'\omega_y\}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} x\dot{v} - y\dot{u} = (x^2 + y^2)\,\dot{\omega}_z &+ (x^2 - y^2)\,\omega_x\omega_y \\ - zx(\dot{\omega}_x - \omega_y\omega_z) &- zy(\dot{\omega}_y + \omega_x\omega_z) \\ &- xy(\omega_x^2 - \omega_y^2), \\ + \{x(\dot{v})_0' - y(\dot{u})_0'\} &+ 2x\{u_0'\omega_z - w_0'\omega_x\} \\ &- 2y\{w_0'\omega_y - v_0'\omega_z\}. \end{aligned} \tag{4.3}$$

If so, however, Equations (2.1)–(2.3) can be combined accordingly to yield

$$y\dot{w} - z\dot{v} + \frac{1}{\rho}\left\{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right\}P - \left\{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right\}\Omega = y\mathscr{H} - z\mathscr{G}, \tag{4.4}$$

$$z\dot{u} - x\dot{w} + \frac{1}{\rho}\left\{z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right\}P - \left\{z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right\}\Omega = z\mathscr{F} - x\mathscr{H}, \tag{4.5}$$

$$x\dot{v} - y\dot{u} + \frac{1}{\rho}\left\{x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right\}P - \left\{x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right\}\Omega = x\mathscr{G} - y\mathscr{F}, \tag{4.6}$$

where

$$\rho\mathscr{F} = \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z}, \tag{4.7}$$

$$\rho\mathscr{G} = \frac{\partial\sigma_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z}, \tag{4.8}$$

$$\rho \mathscr{H} = \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}, \tag{4.9}$$

represent the effects of viscosity.

In order to proceed further, let us rewrite the foregoing expressions in terms of the respective velocity components. Inserting for the components σ_{ij} of the viscous stress tensor from (2.5)–(2.10) we find the expressions on the right-hand sides of Equations (4.7)–(4.9) to assume the more explicit forms

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \mu \nabla^2 u &+ \frac{\mu}{3} \frac{\partial \Delta}{\partial x} \\ &+ 2\left\{\frac{\partial u}{\partial x} - \frac{\Delta}{3}\right\} \frac{\partial u}{\partial x} \\ &+ \left\{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right\} \frac{\partial \mu}{\partial y} \\ &+ \left\{\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right\} \frac{\partial \mu}{\partial z},\end{aligned} \tag{4.10}$$

$$\begin{aligned}\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \mu \nabla^2 v &+ \frac{\mu}{3} \frac{\partial \Delta}{\partial y} \\ &+ 2\left\{\frac{\partial v}{\partial y} - \frac{\Delta}{3}\right\} \frac{\partial \mu}{\partial y} \\ &+ \left\{\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right\} \frac{\partial \mu}{\partial z} \\ &+ \left\{\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right\} \frac{\partial \mu}{\partial x},\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \mu \nabla^2 w &+ \frac{\mu}{3} \frac{\partial \Delta}{\partial z} \\ &+ 2\left\{\frac{\partial w}{\partial z} - \frac{\Delta}{3}\right\} \frac{\partial \mu}{\partial z} \\ &+ \left\{\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right\} \frac{\partial \mu}{\partial x} \\ &+ \left\{\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right\} \frac{\partial \mu}{\partial y},\end{aligned} \tag{4.12}$$

where Δ denotes, as before, the divergence (2.11) of the velocity vector; and ∇^2 stands for the Laplacean operator.

Next, let us insert for the velocity components u, v, w from (3.35)–(3.37); by doing so we find that

$$\nabla^2 u = 2\nabla^2\omega_y - y\nabla^2\omega_z + \nabla^2 u_0' + 2\left\{\frac{\partial\omega_y}{\partial z} - \frac{\partial\omega_z}{\partial y}\right\}, \tag{4.13}$$

$$\nabla^2 v = x\nabla^2\omega_z - z\nabla^2\omega_x + \nabla^2 v_0' + 2\left\{\frac{\partial\omega_z}{\partial x} - \frac{\partial\omega_x}{\partial z}\right\}, \tag{4.14}$$

$$\nabla^2 w = y\nabla^2\omega_x - x\nabla^2\omega_y + \nabla^2 w_0' + 2\left\{\frac{\partial\omega_x}{\partial y} - \frac{\partial\omega_y}{\partial x}\right\}, \tag{4.15}$$

and

$$\begin{aligned}\Delta = &\left\{y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right\}\omega_x + \left\{z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right\}\omega_y \\ &+ \left\{x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right\}\omega_z + \frac{\partial u_0'}{\partial x} + \frac{\partial v_0'}{\partial y} + \frac{\partial w_0'}{\partial z}.\end{aligned} \tag{4.16}$$

Before proceeding further, one feature of basic importance should be brought out which we by-passed without closer discussion at an earlier stage: namely, when by virtue of Equations (3.35)–(3.37) or (3.38)–(3.40) we replaced the *three* dependent variables u, v, w or u', v', w' on their left-hand sides by *six* new variables ω_x, ω_y, ω_z and u_0', v_0', w_0' or $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$ and u_0, v_0, w_0 on their right-hand sides. This deliberately created redundancy permits us to impose without the loss of generality additional constraints on these variables, not embodied in the fundamental equations of Section 2; and this we propose to do at the present time. We propose, in particular, to assume that the primed axes $x'y'z'$ obtained by a rotation of the inertial system xyz, about a fixed origin, in accordance with the transformation (3.1) remain rectangular – an assumption to which implies, in effect, that *the Eulerian angles θ, ϕ, ψ* involved in the direction cosines a_{ik} and, therefore, in the angular velocity components $\omega_{x,y,z}$ or $\omega_{x',y',z'}$ as defined by Equations (3.28)–(3.30) or (3.31)–(3.33) *are functions of the time t alone* (for should they depend, in addition, on the spatial coordinates x, y, z, a rotation as represented by Equations (3.1) would result in a curvilinear coordinate system).

This assumption will neatly separate the physical meaning of the two groups of variables: for while the angular velocity components $\omega_{x,y,z}$ will describe a *rigid-body rotation* of our dynamical system (during which the position of each particle remains unchanged in the primed coordinates), the remaining velocity components u_0', v_0', w_0' will represent *deformation* of our body, in the primed system, in the course of time. It is, therefore, the latter which will be of particular interest for the main problem which we have in mind; and in what follows, we propose to investigate the extent to which their occurrence may modify the structure of our equations.

In order to do so we notice first that, inasmuch as the angular velocity components are hereafter to be regarded as functions of t alone it follows from (4.13)–(4.15) that

$$\left.\begin{aligned}\nabla^2 u &= \nabla^2 u_0', \\ \nabla^2 v &= \nabla^2 v_0', \\ \nabla^2 w &= \nabla^2 w_0';\end{aligned}\right\} \tag{4.17}$$

and, similarly, the divergence (4.16) of the velocity vector will reduce to

$$\Delta_0' = \frac{\partial u_0'}{\partial x} + \frac{\partial v_0'}{\partial y} + \frac{\partial w_0'}{\partial z}. \tag{4.18}$$

In consequence, the corresponding expressions on the right-hand sides of Equations (4.10)–(4.12) are obtained if the velocity components u, v, w present there are replaced by u_0', v_0', w_0'; and Δ by Δ_0'.

Therefore,

$$\begin{aligned}\rho\{y\mathscr{H} - z\mathscr{G}\} = {}& \mu\{y\nabla^2 w_0' - z\nabla^2 v_0' + \tfrac{1}{3}D_1\Delta_0'\}\\ &+ \frac{\partial\mu}{\partial x}\{D_1 u_0' + \frac{\partial}{\partial x}(yw_0' - zv_0')\} +\\ &+ \frac{2}{3}\frac{\partial\mu}{\partial y}\{2D_1 v_0' + D_4 w_0'\}\\ &+ \frac{2}{3}\frac{\partial\mu}{\partial z}\{2D_1 w_0' - D_4 v_0'\}\\ &- \frac{2}{3}\frac{\partial u_0'}{\partial x}D_1\mu + \tfrac{1}{3}\xi D_4\mu,\end{aligned} \tag{4.19}$$

$$\begin{aligned}\rho\{z\mathscr{F} - x\mathscr{H}\} = {}& \mu\{z\nabla^2 u_0' - x\nabla^2 w_0' + \tfrac{1}{3}D_2\Delta_0'\}\\ &+ \frac{2}{3}\frac{\partial\mu}{\partial x}\{2D_2 u_0' - D_5 w_0'\}\\ &+ \frac{\partial\mu}{\partial y}\{D_2 v_0' + \frac{\partial}{\partial y}(zu_0' - xw_0')\}\\ &+ \frac{2}{3}\frac{\partial\mu}{\partial z}\{2D_2 w_0' + D_5 u_0'\}\\ &- \frac{2}{3}\frac{\partial v_0'}{\partial y}D_2\mu + \tfrac{1}{3}\eta D_5\mu,\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\rho\{x\mathscr{G} - y\mathscr{F}\} = {}& \mu\{x\nabla^2 v_0' - y\nabla^2 u_0' + \tfrac{1}{3}D_3\Delta_0'\}\\ &+ \frac{2}{3}\frac{\partial\mu}{\partial x}\{2D_3 u_0' + D_6 v_0'\}\\ &+ \frac{2}{3}\frac{\partial\mu}{\partial y}\{2D_3 v_0' - D_6 u_0'\}\\ &+ \frac{\partial\mu}{\partial z}\{D_3 w_0' + \frac{\partial}{\partial z}(xv_0' - yu_0')\}\\ &- \frac{2}{3}\frac{\partial w_0'}{\partial z}D_3\mu + \tfrac{1}{3}\zeta D_6\mu,\end{aligned} \tag{4.21}$$

where the symbols $D_j (j=1, \ldots, 6)$ stand for the following operators

$$D_1 \equiv y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \tag{4.22}$$

$$D_2 \equiv z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \tag{4.23}$$

$$D_3 \equiv x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \tag{4.24}$$

$$D_4 \equiv z \frac{\partial}{\partial z} + y \frac{\partial}{\partial y}, \tag{4.25}$$

$$D_5 \equiv x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z}, \tag{4.26}$$

$$D_6 \equiv x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}; \tag{4.27}$$

and where

$$\xi = \frac{\partial w_0'}{\partial y} - \frac{\partial v_0'}{\partial z}, \tag{4.28}$$

$$\eta = \frac{\partial u_0'}{\partial z} - \frac{\partial w_0'}{\partial x}, \tag{4.29}$$

$$\zeta = \frac{\partial v_0'}{\partial x} - \frac{\partial u_0'}{\partial y}, \tag{4.30}$$

denote the components of vorticity of the deformation vector.

As the next step of our analysis, let us integrate both sides of the Equations (4.4)–(4.6) over the entire mass of our configuration with respect to the mass element

$$\mathrm{d}m = \rho \, \mathrm{d}V = \rho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \,. \tag{4.31}$$

If, as usual,

$$A = \int (y^2 + z^2) \, \mathrm{d}m \,, \tag{4.32}$$

$$B = \int (x^2 + z^2) \, \mathrm{d}m \,, \tag{4.33}$$

$$C = \int (x^2 + y^2) \, \mathrm{d}m \tag{4.34}$$

denote the *moments of inertia* of our configuration with respect to the axes x, y, z; and

$$D = \int yz \, \mathrm{d}m \,, \tag{4.35}$$

$$E = \int xz \, \mathrm{d}m\,, \tag{4.36}$$

$$F = \int xy \, \mathrm{d}m \tag{4.37}$$

stand for the respective *products of inertia*, the mass integrals of the Equations (4.4)–(4.6) combined with (4.1)–(4.3) will assume the forms

$$\begin{aligned} A\dot{\omega}_x + (C - B)\,\omega_y\omega_z - D(\omega_y^2 - \omega_z^2) - E(\dot{\omega}_z + \omega_x\omega_y) \\ - F(\dot{\omega}_y - \omega_x\omega_z) \\ + 2\omega_x \int (yv_0' + zw_0')\,\mathrm{d}m - 2\omega_y \int yu_0'\,\mathrm{d}m - 2\omega_z \int zu_0'\,\mathrm{d}m \\ + \int D_1 P\,\mathrm{d}V - \int D_1\Omega\,\mathrm{d}m = \int \{z(\dot{v})_0' - y(\dot{w})_0'\}\,\mathrm{d}m + \int \rho\{y\mathscr{H} - z\mathscr{G}\}\,\mathrm{d}V\,, \end{aligned} \tag{4.38}$$

$$\begin{aligned} B\dot{\omega}_y + (A - C)\,\omega_x\omega_z - D(\dot{\omega}_z - \omega_x\omega_y) - E(\omega_z^2 - \omega_x^2) \\ - F(\dot{\omega}_x + \omega_y\omega_z) \\ + 2\omega_y \int (xu_0' + zw_0')\,\mathrm{d}m - 2\omega_z \int zv_0'\,\mathrm{d}m - 2\omega_x \int xv_0'\,\mathrm{d}m \\ + \int D_2 P\,\mathrm{d}V - \int D_2\Omega\,\mathrm{d}m = \int \{x(\dot{w})_0' - z(\dot{u})_0'\}\,\mathrm{d}m + \int \rho\{z\mathscr{F} - x\mathscr{H}\}\,\mathrm{d}V\,, \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} C\dot{\omega}_z + (B - A)\,\omega_x\omega_y - D(\dot{\omega}_y + \omega_x\omega_z) - E(\dot{\omega}_x - \omega_y\omega_z) \\ - F(\omega_x^2 - \omega_y^2) \\ + 2\omega_z \int (yv_0' + xu_0')\,\mathrm{d}m - 2\omega_x \int xw_0'\,\mathrm{d}m - 2\omega_y \int yw_0'\,\mathrm{d}m \\ + \int D_3 P\,\mathrm{d}V - \int D_3\Omega\,\mathrm{d}m = \int \{y(\dot{u})_0' - x(\dot{v})_0'\}\,\mathrm{d}m + \int \rho\{x\mathscr{G} - y\mathscr{F}\}\,\mathrm{d}V\,. \end{aligned} \tag{4.40}$$

The preceding three equations represent the exact form of the generalized Eulerian equations governing the precession and nutation of self-gravitating configurations which consist of a viscous fluid. They constitute a system of three ordinary differential equations for $\omega_{x,y,z}$ considered as functions of the time t alone. If the body in question were rigid (non-deformable) – or, if deformable, it were subject to no time-dependent deformation – all three velocity components u', v', w' relative to the rotating frame of reference (and thus, by (3.42), u_0', v_0', w_0') would be identically zero. In such a case, equations (4.38)–(4.40) would reduce to their more familiar form

$$\begin{aligned} A\dot{\omega}_x + (C - B)\,\omega_y\omega_z - D(\omega_y^2 - \omega_z^2) - E(\dot{\omega}_z + \omega_x\omega_y) \\ - F(\dot{\omega}_y - \omega_x\omega_z) + \int D_1 P\,\mathrm{d}V - \int D_1\Omega_0\,\mathrm{d}m = \int D_1\Omega_1\,\mathrm{d}m\,, \end{aligned} \tag{4.41}$$

$$
\begin{aligned}
&B\dot{\omega}_y + (A - C)\,\omega_x\omega_z - D(\dot{\omega}_z - \omega_x\omega_y) - E(\omega_z^2 - \omega_x^2) \\
&\quad - F(\dot{\omega}_x + \omega_y\omega_z) + \int D_2 P \,\mathrm{d}V - \int D_2\Omega_0 \,\mathrm{d}m = \int D_2\Omega_1 \,\mathrm{d}m\,,
\end{aligned}
\tag{4.42}
$$

and

$$
\begin{aligned}
&C\dot{\omega}_z + (B - A)\,\omega_x\omega_y - D(\dot{\omega}_y + \omega_x\omega_z) - E(\dot{\omega}_x - \omega_y\omega_z) \\
&\quad - F(\omega_x^2 - \omega_y^2) + \int D_3 P \,\mathrm{d}V - \int D_3\Omega_0 \,\mathrm{d}m = \int D_3\Omega_1 \,\mathrm{d}m\,,
\end{aligned}
\tag{4.43}
$$

where we have decomposed the total gravitational potential

$$\Omega = \Omega_0 + \Omega_1 \tag{4.44}$$

into its part arising from the mass of the respective body (Ω_0) and that arising from external disturbing forces (Ω_1) if any.

In the case of a rigid body, the existence of hydrostatic equilibrium requires that

$$\int D_i P \,\mathrm{d}V = \int D_i\Omega_0 \,\mathrm{d}m \tag{4.45}$$

exactly for $i=1, 2, 3$. If, moreover, we choose our system of inertial axes xyz to coincide with the principal axes of inertia of our configuration, it can be shown that all three moments of inertia (4.35)–(4.37) can be made to vanish; and for

$$D = E = F = 0 \tag{4.46}$$

our Equations (4.41)–(4.43) will reduce further to

$$
\left.
\begin{aligned}
A\dot{\omega}_x + (C - B)\,\omega_y\omega_z &= \int D_1\Omega_1 \,\mathrm{d}m\,, \\
B\dot{\omega}_y + (A - C)\,\omega_x\omega_z &= \int D_2\Omega_1 \,\mathrm{d}m\,, \\
C\dot{\omega}_z + (B - A)\,\omega_x\omega_y &= \int D_3\Omega_1 \,\mathrm{d}m\,,
\end{aligned}
\right\}
\tag{4.47}
$$

which is the familiar form of the Eulerian equations for the precession of rigid bodies.

If, however, the body in question were fluid and subject to distortion by external forces – though not necessarily (like equilibrium tides) fluctuating in time – Equations (4.47) would cease to be exact to the extent to which Equations (4.45) need no longer hold true. The reader may note that as long as the functions $P(r)$ and $\Omega_0(r)$ are purely radial (as they would be in the absence of any distortion) operation with $D_i (i=1, 2, 3)$ will annihilate them completely; so that Equations (4.45) continue to be fulfilled identically. The same argument discloses, however, that for fluid bodies, Equations (4.45) may become inequalities to the extent brought about by distortion; and – to this extent – the Eulerian differential equations for the precession and nutation of rigid and fluid bodies may be different even if the form of the fluid does not vary with the time.

If, however, this latter condition is not fulfilled – such as, for instance, in the case when the period of axial rotation of the fluid body differs from that of the revolution of an external attracting mass producing *dynamical tides* on the rotating fluid – the velocity components u_0', v_0', w_0' will emerge to give rise to supplementary terms in the Equations (4.38)–(4.40), which can be classified in two groups. Those on the left-hand sides of the respective equations are factored by the angular velocity components ω_x, ω_y, ω_z which play the role of dependent variables of our problem. However, their coefficients are not constants (like A, B, C; D, E, F), but functions of the time. The second group of new terms arising on the right-hand sides of the same equations are independent of $\omega_{x,y,z}$ and render our system non-homogeneous. The first mass-integral on the right-hand sides of Equations (4.38)–(4.40) arises from the accelerations $(\dot{u})_0'$, $(\dot{v})_0'$, $(\dot{w})_0'$ experienced by the body subject to deformation – irrespective of whether the flow due to this motion is inviscid or viscous; while the second group of volume integrals (the integrands of which are given by Equations 4.19–4.21) represent the effects of viscosity proper; and if the latter is large, these may be predominant.

In order to progress further, it is obviously necessary to ascertain the explicit forms of the time-dependent velocity components u', v', w' which arise from the deformation of the fluid body – be this through its non-uniform rotation, or due to possible proximity of external tide-generating bodies; for only then can the coefficients of all terms in Equations (4.38)–(4.40) factored by the dependent variables be evaluated as explicit functions of the time, and the equations themselves solved for the ω's. This task calls, however, for a prior complete specification of the internal structure of the rotating body as well as for that of the external disturbing function – which are very different in stellar or planetary problems; and such applications must, therefore, be postponed for subsequent investigations.

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