DYNAMICAL TIDES IN CLOSE BINARY SYSTEMS, I

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Abstract. The aim of the present paper will be to develop from the fundamental equations of hydrodynamics a theory of dynamical tides in close binary systems, the components of which are regarded to consist of heterogeneous viscous fluid, and to revolve around their common centre of gravity in eccentric orbits; moreover, the equatorial planes of their axial rotation and the orbital plane need not be co-planar, but all may be inclined to the invariable plane of the system of arbitrary amounts. The changes in the pressure or density invoked by time-dependent deformation will be regarded as adiabatic; but, in the equilibrium state, both the density and viscosity of the material of our components may be arbitrary functions of the radial distance.

Following a brief exposition in Section 2 of the fundamental equations linearized to small oscillations – be these free or forced – in Section 3 we shall particularize them to describe spheroidal deformations; with due regard to all terms arising from viscosity. Section 4 will contain a specification of the boundary conditions to be imposed upon such oscillations; and in Section 5 we shall solve the problem of non-radial oscillations of self-gravitating inviscid configurations in terms of hypergeometric series. The remaining Sections 6–8 will be devoted to a discussion of the phenomena arising from viscosity: in particular, we shall solve in a closed form the problem of non-radial oscillations of incompressible viscous globes in the terms of Bessel functions. It will be shown that the effect of viscosity – like those of compressibility – tend to de-stabilize all non-radial oscillations of homogeneous configurations.

At the other extreme, a similar treatment of a mass-point model – as well as of one exhibiting high but finite degree of central condensation – is being postponed for a subsequent communication.

1. Introduction

As is well known, the proximity effects in close binary systems produced by gravitational interaction of their components represent the *royal road* for studies of the physical properties of the constituent stars; and of these the most important (and observationally most conspicuous) are the *tides* raised on each star by its mate, causing characteristic deformations in shape. If the relative orbit of the two stars is circular, and their rotation synchronized with revolution, the tidal waves raised on each component by its mate will remain stationary, and give rise to no motion relative to a system of coordinates rotating with each star (equilibrium tides). However, if the relative orbit of the two stars becomes eccentric (so that the mutual distance between two stars varies in the course of each cycle), or if synchronism between rotation and revolution breaks down, the tidal waves cease to be stationary, and will move relative to a rotating system of coordinates in astrocentric longitude as well as latitude if the equatorial planes of the stars cease to coincide with that of the orbit (dynamical tides).

As long as the material constituting such stars can be regarded as inviscid, the direction of a tidal bulge raised on each star by any spherical-harmonic term in the disturbing function will coincide in direction with that of the disturbing force. Inviscid fluid represents, however, only a mathematical abstraction which can be only approached (but never actually fulfilled) in any real physical system. In the case of stellar matter, viscosity is bound to arise from several sources – plasma, radiative, turbulent – and their joint action will give rise to a new class of phenomena having no counterpart in inviscid treatment, which are of importance for an interpretation of the phenomena observed in close binary systems.

First, in the presence of viscosity, the individual tidal bulges raised by each harmonic term of the distributing function can no larger be orientated exactly towards the disturbing body – but will advance, or lag behind it, depending on whether the angular velocity of axial rotation of the respective component is greater or smaller than the instantaneous angular motion of its mate*. This phenomenon – generally referred to as *tidal lag* – can render the light changes due to the 'ellipticity effect' in close binary systems *asymmetric* with respect to the moments of conjunction; and the extent to which this may be the case should disclose the actual magnitude of the viscosity of stellar matter (in particular, in the outer layers of a star).

Secondly, the relative motion of stellar matter of finite viscosity, invoked by dynamical tides, is bound to produce a dissipation of kinetic energy into heat through the medium of *viscous friction*. This phenomenon is, in turn, not only bound to bring about a secular decrease in angular momentum of axial rotation of the individual components (and thus of the system as a whole) – the effects of which can be reflected in characteristic changes of the orbital period. It will also provide an additional source of heat to the interior, and through it may affect also the internal structure of the constituent stars.

In point of fact, if stellar material were inviscid, the components of close binary systems would possess no means of interacting with each other in any way except through their gravitational attraction; their evolution would be quite independent of each other; and the angular momenta of their axial rotation would be separately conserved. It is the viscosity and the phenomena arising from it that provides other than purely gravitational bond between them; and thus opens up a possibility that their evolution may follow a different course from that which the components of a binary would follow separately if they were single.

In view of the obvious importance of these phenomena and of their bearing on the interpretation of many phenomena observable in close binary systems, it may perhaps be surprising that their mathematical theory has not yet been adequately explored; but in actual fact this is far from being the case. The main reason is no doubt the mathematical complexity which we encounter in any attempt to describe the underlying phenomena with a satisfactory degree of generality; for it must be remembered that the effects of viscosity affect not only the equations of motion, but also that of the

* The reader may note that, in eccentric orbits, this will be true even if the angular velocity ω of axial rotation happens to be identical with the Keplerian angular velocity ω_K of orbital revolution; for if $e > 0$, the actual angular velocity is bound to vary between periastron and apastron in accordance with Kepler's second law of elliptic motion; so that $\omega \gtrless \omega_K$ in the course of each orbit.

energy. The first treatment of tidal lag and viscous friction phenomena within the confines of the solar system are due to DARWIN (1879a,b) who in his great memoirs laid down the foundations of the subject more than eighty years ago. However, in his work Darwin limited himself to incompressible bodies of constant density and viscosity; and his mathematical analysis focussed on this task proves incapable of generalization to problems encountered in astronomy of double-star systems. Such problems were considered more recently by ZAHN (1966) but were treated by him so far only in general outline rather than in a quantative manner.

It will be the aim of the present investigation (and subsequent papers) to make good of this lack of quantative treatment, and to embark on a systematic development of the theory of dynamical tides in close binary systems taking full account of the phenomena due to viscosity. In doing so we shall depart from the fundamental equations of hydrodynamics of viscous flow; but shall linearize them by assuming that the three spatial velocity components invoked by the tides are small enough for their squares and higher powers to be negligible. We shall, moreover, assume the changes in the pressure or density invoked by such deformation to be adiabatic; but apart from this the generality of our treatment will not be restricted in any manner. In particular, the equilibrium density or viscosity in the interior of the constituent stars may be arbitrary functions of the distance from their centres.

Not the entire problem outlined in the foregoing paragraphs can be solved within the scope of a single paper. Thus the present communication will be limited to a derivation of the fundamental equations with their associated boundary conditions (Sections 2–4), and to their solution for the limiting case of homogeneous models (Sections 5–8), in order to demonstrate our methods on a case which is solvable in a closed form. A similar treatment of centrally-condensed models (which should approximate much more closely to the actual behaviour of real stars) will be taken up in Part II of the present communication, which should follow shortly.

2. Deformation of Self-Gravitating Viscous Fluids in External Field of Force: Fundamental Equations

The fundamental equations in viscous fluid motion in Cartesian coordinates have already been stated in Section 2 of our previous investigation of the precession and nutation of deformable bodies recently published in this journal (KOPAL, 1968; hereafter referred to as Paper I). In order to use them for studies of the motions invoked by *tides* in self-gravitating systems – i.e., deformations of configurations which, in the absence of external force would be spherical in form – we find it expedient to change over from rectangular coordinates x, y, z to spherical polar coordinates r, θ, ϕ, related with the former by

$$\left.\begin{aligned} x &= r\cos\phi\sin\theta\,, \\ y &= r\sin\phi\sin\theta\,, \\ z &= r\cos\theta\,. \end{aligned}\right\} \tag{1}$$

If so, then – at a price of some loss of symmetry – Equations (1)–(11) of Paper I can be rewritten as

$$\rho\frac{DU}{Dt}-\rho\frac{V^2+W^2}{r}=\rho\frac{\partial\Omega}{\partial r}-\frac{\partial P}{\partial r}+\mu\left[\nabla^2U+\frac{1}{3}\frac{\partial\Delta}{\partial r}-\frac{2}{r^2}\frac{\partial V}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial W}{\partial\phi}\right.$$
$$\left.-\frac{2U}{r^2}-\frac{2V\cot\theta}{r^2}\right]+2\frac{\partial\mu}{\partial r}\left[\frac{\partial U}{\partial r}-\frac{\Delta}{3}\right]$$
$$+\frac{1}{r}\frac{\partial\mu}{\partial\theta}\left[\frac{1}{r}\frac{\partial U}{\partial\theta}+\frac{\partial V}{\partial r}-\frac{V}{r}\right]+\frac{1}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\left[\frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi}+\frac{\partial W}{\partial r}-\frac{W}{r}\right],\tag{2}$$

$$\rho\frac{DV}{Dt}+\rho\frac{UV}{r}-\rho\frac{W^2\cot\theta}{r}=\frac{1}{r}\left[\rho\frac{\partial\Omega}{\partial\theta}-\frac{\partial P}{\partial\theta}\right]+\mu\left[\nabla^2V+\frac{1}{3r}\frac{\partial\Delta}{\partial\theta}\right.$$
$$\left.+\frac{2}{r^2}\frac{\partial U}{\partial\theta}-\frac{2\cos\theta}{\sin^2\theta}\frac{\partial W}{\partial\phi}-\frac{V}{r^2\sin^2\theta}\right]$$
$$+\frac{\partial\mu}{\partial r}\left[\frac{\partial V}{\partial r}+\frac{1}{r}\frac{\partial U}{\partial\theta}-\frac{V}{r}\right]+\frac{2}{r}\frac{\partial\mu}{\partial\theta}\left[\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{U}{r}-\frac{\Delta}{3}\right]$$
$$+\frac{1}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\left[\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}+\frac{1}{r}\frac{\partial W}{\partial\theta}-\frac{W\cot\theta}{r}\right],\tag{3}$$

$$\rho\frac{DW}{Dt}+\rho\frac{W(U+V\cot\theta)}{r}=\frac{1}{r\sin\theta}\left[\rho\frac{\partial\Omega}{\partial\phi}-\frac{\partial P}{\partial\phi}\right]+\mu\left[\nabla^2W+\frac{1}{3r\sin\theta}\frac{\partial\Delta}{\partial\phi}\right.$$
$$\left.+\frac{2}{r^2\sin\theta}\frac{\partial U}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V}{\partial\phi}-\frac{W}{r^2\sin^2\theta}\right]$$
$$+\frac{\partial\mu}{\partial r}\left[\frac{\partial W}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi}-\frac{W}{r}\right]+\frac{1}{r}\frac{\partial\mu}{\partial\theta}\left[\frac{1}{r}\frac{\partial W}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}-\frac{W\cot\theta}{r}\right]$$
$$+\frac{2}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\left[\frac{1}{r\sin\theta}\frac{\partial W}{\partial\phi}+\frac{U+V\cot\theta}{r}-\frac{\Delta}{3}\right],\tag{4}$$

where the polar velocity-components U, V, W are related with the rectangular velocities u, v, w of Paper I by

$$U=\dot{r}=u\cos\phi\,\sin\theta+v\sin\phi\,\sin\theta+w\cos\theta,\tag{5}$$

$$V=r\dot{\theta}=u\cos\phi\cos\theta+v\sin\phi\cos\theta-w\sin\theta,\tag{6}$$

$$W=(r\sin\theta)\,\dot{\phi}=-u\sin\phi\qquad+v\cos\phi;\tag{7}$$

the operators

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2},\tag{8}$$

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2U)+\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(V\sin\theta)+\frac{\partial W}{\partial\phi}\right\},\tag{9}$$

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+U\frac{\partial}{\partial r}+\frac{V}{r}\frac{\partial}{\partial\theta}+\frac{W}{r\sin\theta}\frac{\partial}{\partial\phi}; \tag{10}$$

and all other symbols possess the same meaning as in Section 2 of Paper I.

In order to study the displacements governed by the foregoing system of equations, we shall hereafter assume that all three velocity components U, V, W of viscous motion are small enough for their squares and cross-products to be negligible. Let, moreover, the pressure P, density ρ, and gravitational potential Ω characterizing the internal structure of our configuration be expressible as

$$P=P_0(r)+P'(r,\theta,\phi;t), \tag{11}$$

$$\rho=\rho_0(r)+\rho'(r,\theta,\phi;t), \tag{12}$$

$$\Omega=\Omega_0(r)+\Omega'(r,\theta,\phi;t), \tag{13}$$

where P_0, ρ_0, and Ω_0 refer to the respective properties of our configuration in its stationary (equilibrium) state; and P', ρ', Ω' stand for their changes brought about by motion with the velocity components U, V, W.

In the state of equilibrium (when $U=V=W=0$), Equation (2) reveals at once that

$$\frac{\partial P_0}{\partial r}-\rho_0\frac{\partial\Omega_0}{\partial r}=-g\rho_0, \tag{14}$$

where the gravitational acceleration

$$g=G\frac{m(r)}{r^2}=\frac{4\pi G}{r^2}\int_0^r\rho_0 r^2\,\mathrm{d}r. \tag{15}$$

If, moreover, we regard the coefficient μ of viscosity to be a function of r only, and assume that the primed functions P', ρ', Ω' are – like the velocity components U, V, W – small enough for their squares and cross-products to be negligible, the fundamental Equations (2)–(4) of motion reduce to their linearized forms

$$\rho\frac{\partial U}{\partial t}=\rho\frac{\partial\Omega'}{\partial r}-\frac{\partial P'}{\partial r}-g\rho'+\frac{\mu}{r}\left[\nabla^2(rU)-2\Delta+\frac{r}{3}\frac{\partial\Delta}{\partial r}\right]+2\frac{\partial\mu}{\partial r}\left[\frac{\partial U}{\partial r}-\frac{\Delta}{3}\right], \tag{16}$$

$$\begin{aligned}\rho\frac{\partial V}{\partial t}=\frac{1}{r}\left[\rho\frac{\partial\Omega'}{\partial\theta}-\frac{\partial P'}{\partial\theta}\right]+\frac{\mu}{r^2}\Bigg[r^2\nabla^2(V)+2\frac{\partial U}{\partial\theta}-\frac{V}{\sin^2\theta}\\-\frac{2\cos\theta}{\sin^2\theta}\frac{\partial W}{\partial\phi}+\frac{r}{3}\frac{\partial\Delta}{\partial\theta}\Bigg]+\frac{\partial\mu}{\partial r}\left[\frac{\partial V}{\partial r}+\frac{1}{r}\frac{\partial V}{\partial\theta}-\frac{V}{r}\right],\end{aligned} \tag{17}$$

$$\begin{aligned}\rho\frac{\partial W}{\partial t}=\frac{1}{r\sin\theta}\left[\rho\frac{\partial\Omega'}{\partial\phi}-\frac{\partial P'}{\partial\phi}\right]+\frac{\mu}{r^2\sin^2\theta}\Bigg[(r^2\sin^2\theta)\nabla^2W+2\frac{\partial U}{\partial\phi}\\+2\cos\theta\frac{\partial V}{\partial\phi}-W+\tfrac{1}{3}(r\sin\theta)\frac{\partial\Delta}{\partial\phi}\Bigg]+\frac{\partial\mu}{\partial r}\left[\frac{\partial W}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi}-\frac{W}{r}\right],\end{aligned} \tag{18}$$

where we have dropped for simplicity – there should be no danger of confusion – the zero subscript of ρ_0.

Moreover, the Equation (12) of continuity and the Poisson equation (13) of Paper I can be similarly linearized to yield

$$\frac{\partial \rho'}{\partial t} + U \frac{\partial \rho}{\partial r} + \rho \Delta = 0 \tag{19}$$

and

$$\nabla^2 \Omega' = - 4\pi G \rho' , \tag{20}$$

respectively.

The foregoing Equations (16)–(20) constitute a simultaneous linear system of five relations between six dependent variables : U, V, W; P', ρ' and Ω'. In order to render this system determinate, an additional relation between them must be sought; and this can be deduced from the principle of the conservation of energy. If, for this purpose, we assume the changes in the state variables to be adiabatic, the respective equation can be shown to assume the form

$$\frac{DP}{Dt} = a^2 \frac{D\rho}{Dt}, \tag{21}$$

where

$$a^2 = \gamma \frac{P}{\rho} \tag{22}$$

denotes the square of the velocity of sound in the material characterized by a ratio γ of specific heats; and which for small changes in state can be replaced by its linearized version

$$\frac{\partial P'}{\partial t} + U \frac{\partial\ {}_0}{\partial r} = a_0^2 \left\{ \frac{\partial \rho'}{\partial t} + U \frac{\partial \rho_0}{\partial r} \right\}, \tag{23}$$

which by use of (14) and (19) can be rewritten to assert that

$$\frac{\partial P'}{\partial t} = \rho (gU - a^2 \Delta), \tag{24}$$

where zero subscripts of ρ and a^2 have likewise been omitted.

3. Spheroidal Deformations

In order to proceed further, let us hereafter assume that the anticipated deformation of our configuration is spheroidal – which implies that the velocity components U, V, W are constrained to be of the form

$$U = u(r, t)\ Y_j^i(\theta, \phi), \tag{25}$$

$$V = v(r, t) \frac{\partial Y_j^i}{\partial \theta}, \tag{26}$$

$$W = \frac{v(r, t)}{\sin\theta} \frac{\partial Y_j^i}{\partial\phi}, \tag{27}$$

where $u(r, t)$ as well as $v(r, t)$ are functions of r and t only, while the $Y_j^i(\theta, \phi)$'s are surface harmonics of index i and order j, obeying the differential equation

$$\frac{1}{\sin^2\theta} \frac{\partial^2 Y}{\partial\phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + j(j+1)\, Y = 0. \tag{28}$$

If, furthermore, we abbreviate

$$\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 u) - j(j+1) \frac{v}{r} = y \tag{29}$$

and

$$\frac{1}{r} \frac{\partial}{\partial r}(rv) - \frac{u}{r} = z, \tag{30}$$

it follows by insertion of (25)–(27) in (9) that

$$\Delta = y Y_j^i; \tag{31}$$

while the linearized Equation (19) of continuity will assume the form

$$\frac{\partial\rho'}{\partial t} = -\left\{\frac{1}{r^2} \frac{\partial}{\partial r}(\rho r^2 u) - j(j+1) \frac{\rho v}{r}\right\} Y_j^i = -\left\{\rho y + u \frac{\partial\rho}{\partial r}\right\} Y_j^i = -f Y_j^i; \tag{32}$$

and the energy Equation (24) transforms into

$$\frac{\partial P'}{\partial t} = \rho(gu - a^2 y)\, Y_j^i = -a^2 h Y_j^i. \tag{33}$$

As a result

$$\frac{\partial}{\partial t}\left\{\frac{\partial P'}{\partial r} + g\rho'\right\} = -\left\{gf + \frac{\partial}{\partial r}(a^2 h)\right\} Y_j^i \tag{34}$$

and

$$\frac{\partial}{\partial t}\left(\frac{\partial P'}{\partial\theta}\right) = -a^2 h \frac{\partial Y}{\partial\theta}, \quad \frac{\partial}{\partial t}\left(\frac{\partial P'}{\partial\phi}\right) = -a^2 h \frac{\partial Y}{\partial\phi}. \tag{35}$$

On the other hand, an insertion of (25)–(27) on the right-hand sides of Equations (16)–(18) reveals that the viscous terms transform into

$$\begin{aligned} &\frac{\mu}{r}\left\{\nabla^2(rU) - 2\Delta + \frac{r}{3} \frac{\partial\Delta}{\partial r}\right\} + 2 \frac{\partial\mu}{\partial r}\left\{\frac{\partial U}{\partial r} - \frac{\Delta}{3}\right\} \\ &\quad = \left\{2\left[\mu \frac{\partial y}{\partial r} + \frac{\partial\mu}{\partial r} \frac{\partial u}{\partial r}\right] + \frac{j(j+1)}{r} \mu z - \frac{2}{3} \frac{\partial}{\partial r}(\mu y)\right\} Y_j^i \equiv F(r, t)\, Y_j^i(\theta, \phi), \end{aligned} \tag{36}$$

while

$$\mu\left\{\nabla^2 V + \frac{2}{r^2}\frac{\partial U}{\partial \theta} - \frac{V}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial W}{\partial \phi} + \frac{1}{3r}\frac{\partial \Delta}{\partial \theta}\right\}$$
$$+ \frac{\partial \mu}{\partial r}\left\{\frac{\partial V}{\partial r} + \frac{1}{r}\frac{\partial U}{\partial \theta} - \frac{V}{r}\right\} = \left\{\frac{1}{r}\frac{\partial}{\partial r}(r\mu z) + \frac{4\mu}{3r}y + \frac{2}{r}\frac{\partial \mu}{\partial r}(u - v)\right\}\frac{\partial Y}{\partial \theta} \equiv \frac{G(r,t)}{r}\frac{\partial Y}{\partial \theta} \tag{37}$$

and

$$\mu\left\{\nabla^2 W + \frac{2}{r^2\sin^2\theta}\frac{\partial U}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V}{\partial \phi} - \frac{W}{r^2\sin^2\theta} + \frac{1}{3r\sin\theta}\frac{\partial \Delta}{\partial \phi}\right\}$$
$$+ \frac{\partial \mu}{\partial r}\left\{\frac{\partial W}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi} - \frac{W}{r}\right\} = \frac{G(r,t)}{r\sin\theta}\frac{\partial Y}{\partial \phi}. \tag{38}$$

If, lastly, we set

$$\frac{\partial \Omega'}{\partial t} = R(r,t)\, Y_j^i(\theta, \phi), \tag{39}$$

a differentiation of the first fundamental Equation (16) of motion with respect to the time renders the latter to assume, for spheroidal deformations, the explicit form

$$\rho\frac{\partial^2 u}{\partial t^2} = \rho\frac{\partial R}{\partial r} + \frac{\partial}{\partial r}(a^2 h) + gf + \frac{\partial F}{\partial t}; \tag{40}$$

while Equations (17) and (18) similarly treated can both be reduced to

$$\rho r\frac{\partial^2 v}{\partial t^2} = \rho R + a^2 h + \frac{\partial G}{\partial t}, \tag{41}$$

where the quantities f and h are defined by Equations (32) and (33); the viscous terms F and G by (36) and (37) and R, by (39).

The foregoing Equations (40) and (41) constitute a simultaneous set of partial differential equations for the unknown functions $u(r, t)$ and $v(r, t)$ introduced by (25)–(27). Before, however, we solve for them it is desirable to eliminate the disturbed potential function $R(r, t)$ between them; and this can be done in the following manner.

First, divide both sides of Equation (41) by ρ, differentiate with respect to r, and then eliminate $\partial R/\partial r$ between it and Equation (40): the outcome will assume the form

$$\frac{\partial^2}{\partial t^2}(rz) + \frac{\partial}{\partial t}\left\{\frac{F}{\rho} - \frac{\partial}{\partial r}\left(\frac{G}{\rho}\right)\right\} + a^2 A y = 0, \tag{42}$$

where y and z continue to be defined by (29) and (30), and where we have abbreviated

$$A = \frac{1}{\rho}\frac{\partial \rho}{\partial r} - \frac{1}{\gamma P}\frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\log \rho P^{-1/\gamma}, \tag{43}$$

so that, by (14) and (22),

$$a^2 A = g + \frac{a^2}{\rho}\frac{\partial \rho}{\partial r}. \tag{44}$$

In order to obtain a second independent relation between u and v which does not involve R, recourse must be had to the Poisson Equation (20). Differentiating the latter with respect to the time and inserting from (39) we find that the radial part of $\partial \Omega'/\partial t$ should satisfy the differential equation

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r} - j(j+1)\frac{R}{r^2} = -4\pi G\frac{\partial \rho'}{\partial t} = 4\pi G f \tag{45}$$

by (32). Now multiply (40) by r^2/ρ and differentiate with respect to r: the result will be

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \frac{\partial^2}{\partial t^2}\left\{\frac{\partial}{\partial r}(r^2 u)\right\} - \frac{\partial}{\partial r}\left\{\frac{r^2}{\rho}\left[\frac{\partial}{\partial r}(a^2 h + gf)\right]\right\} - \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial r}\left(\frac{r^2 F}{\rho}\right)\right\}, \tag{46}$$

which inserted in (45) together with (41) reveals that

$$\frac{\partial^2 y}{\partial t^2} = 4\pi G f + \frac{1}{r^2}\frac{\partial}{\partial r}\left\{\frac{r^2}{\rho}\frac{\partial}{\partial r}(a^2 h)\right\} + \frac{1}{r^2}\frac{\partial}{\partial r}\left\{\frac{r^2 g f}{\rho}\right\} - \frac{j(j+1)}{r^2}\frac{a^2 h}{\rho} + \frac{1}{r^2}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial r}\left(\frac{r^2 F}{\rho}\right) - j(j+1)\frac{G}{\rho}\right\} \tag{47}$$

where, it may be remembered,

$$4\pi G \rho = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 g) \tag{48}$$

by (15).

If we eventually insert for F and G from (36) and (37), our fundamental simultaneous Equations (42) and (47) for u and v will assume the more explicit forms

$$\frac{\partial^2}{\partial t^2}(rz) + a^2 A y = \frac{1}{\rho}\frac{\partial}{\partial t}\left\{\left[\rho\frac{\partial}{\partial r}\left(\frac{1}{\rho}\frac{\partial}{\partial r}\right) - \frac{j(j+1)}{r^2}\right](\mu r z) + 2\left(y - \frac{\partial u}{\partial r}\right)\frac{\partial \mu}{\partial r} + 2\rho\frac{\partial}{\partial r}\left(\frac{u-v}{\rho}\frac{\partial \mu}{\partial r}\right) - \frac{4}{3}\left(\frac{\mu}{\rho}\frac{\partial \rho}{\partial r}\right) y\right\} \tag{49}$$

and

$$\frac{\partial^2}{\partial t^2}(r^2 y) - 4\pi G r^2 f - \frac{\partial}{\partial r}\left\{\frac{r^2}{\rho}\left[\frac{\partial}{\partial r}(a^2 h) + gf\right]\right\} + j(j+1)\frac{a^2 h}{\rho} = \frac{\partial}{\partial t}\left\{\frac{4}{3}\left[\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial}{\partial r}\right) - \frac{j(j+1)}{\rho}\right](\mu y) - 2\frac{\partial}{\partial r}\left[\frac{r^2}{\rho}\left(y - \frac{\partial u}{\partial r}\right)\frac{\partial \mu}{\partial r}\right] - \frac{j(j+1)}{\rho}\left[2(u-v)\frac{\partial \mu}{\partial r} + \frac{1}{\rho}\frac{\partial \rho}{\partial r}(\mu r z)\right]\right\}. \tag{50}$$

The foregoing system (49)–(50) of two simultaneous linear differential equations for u and v is evidently one of fourth order in t, and of sixth order in r. If viscosity μ were absent, the right-hand sides of both (49) and (50) would vanish identically; and their left-hand sides equated to zero would constitute equations each of which would be of second order with respect to the time. However, (49) so truncated would reduce to a first-order equation with respect to r; while (50) would remain one of third order in the spatial variable. The appearance of even a constant viscosity μ would raise the order of (49) from one to three; but the order of (50) would remain unaltered.

4. Boundary Conditions

Before we can proceed with the construction of particular solutions of the equations established in the preceding section, we must specify the boundary conditions which the solutions of interest for our double-star problem will be called upon to satisfy. The fundamental equations of our problem are represented by the simultaneous system (40), (41) and (45) if u, v, and R are retained as dependent variables, or (49) and (50) in terms of u and v only. Since, in either case, the respective system of differential equations proves to be one of sixth order, six boundary conditions will be necessary for a complete specification of the desired particular solutions; and in what follows we shall proceed to establish their explicit form.

First, the obvious requirement that there be no displacement at the centre necessitates that

$$u(0, t) = v(0, t) = 0. \tag{51}$$

At the boundary $r = r_*$ of a self-gravitating configuration of viscous fluid we may require the vanishing of the radial viscous stress-components

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\phi} = 0. \tag{52}$$

The explicit forms of the six components σ_{ij} of viscous stresses in Cartesian coordinates have already been given by Equations (5)–(10) of Paper I. Rewriting these in spherical polar coordinates we find that

$$\sigma_{rr} = \mu \frac{\partial U}{\partial r} = \mu \left(\frac{\partial u}{\partial r}\right) Y_j^i, \tag{53}$$

$$\sigma_{r\theta} = \mu \left\{\frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r}\right\} = \mu \left\{\frac{\partial v}{\partial r} + \frac{u - v}{r}\right\} \frac{\partial Y}{\partial \theta}, \tag{54}$$

$$\sigma_{r\phi} = \mu \left\{\frac{1}{r \sin\theta}\frac{\partial U}{\partial \phi} + \frac{\partial W}{\partial r} - \frac{W}{r}\right\} = \mu \left\{\frac{\partial v}{\partial r} + \frac{u - v}{r}\right\} \frac{1}{\sin\theta}\frac{\partial Y}{\partial \phi} \tag{55}$$

by (25)–(27); so that all three radial components of the viscous stress tensor will vanish on the surface provided that

$$\left(\frac{\partial u}{\partial r}\right) = 0 \tag{56}$$

and

$$\frac{\partial v}{\partial r} = \frac{v - u}{r} \tag{57}$$

for $r = r_*$.

The remaining type of the boundary conditions which we must investigate consists of the requirement that the total gravitational potential and its normal derivative (i.e., gravitational acceleration) must be continuous across the free boundary of the distorted configuration; and this can be enforced in the following manner.

Let the total potential W of all forces acting upon any point of our configuration be expressed as the sum

$$W = \Omega + V_T, \tag{58}$$

where Ω denotes (as before) the potential arising from the mass of the respective body, and

$$V_T = \sum_{i,j} C_{i,j}(t)\, r^j Y_j^i(\theta, \phi) \tag{59}$$

stands for a potential of the forces causing the distortion (such as the attraction of external masses, for instance), specified by a set of the functions $C_{i,j}(t)$.

The total potential W must obviously satisfy the Poisson Equation

$$\nabla^2 W = -\,4\pi G\rho\,. \tag{60}$$

If, moreover, the potential Ω arising from the mass and the internal distribution of density ρ are likewise expansible in the form

$$\Omega = \sum_{i,j} \mathscr{R}_{i,j}(r, t)\; Y_j^i(\theta, \phi) \tag{61}$$

and

$$\rho = \sum_{i,j} \mathscr{F}_{i,j}(r, t)\; Y_j^i(\theta, \phi), \tag{62}$$

where the Y_j^i's are surface harmonics satisfying Equation (28) and $\mathscr{R}_{i,j}(r, t)$, $\mathscr{F}_{i,j}(r, t)$ are functions as yet unspecified, it follows from (61) and (62) that Equation (60) can be expressed more explicitly in the form

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2}\right\} \mathscr{R}_{i,j} = -\,4\pi G \mathscr{F}_{i,j}\,. \tag{63}$$

This linear non-homogeneous equation for $\mathscr{R}$ can, in turn, be shown (by standard methods) to admit of the solution

$$\mathscr{R}_{i,j}(r, t) = C_{i,j}(t)\, r^j + \frac{4\pi G}{2j+1} \times$$

$$\times \left\{\frac{1}{r^{j+1}} \int_0^r \mathscr{F}_{i,j}(r, t)\, r^{j+2}\, \mathrm{d}r + r^j \int_r^\infty \mathscr{F}_{i,j}(r, t)\, r^{1-j}\, \mathrm{d}r\right\}, \tag{64}$$

the particular integral of which consists of a sum of the interior and exterior potential arising from the mass itself, and the complementary function represents the disturbing potential (59). The reader may note that, for $j=0$ (i.e., in the case of spherical symmetry), the foregoing Equation (64) reduces to

$$\mathscr{R}_0 = \frac{4\pi G}{r} \int_0^r \rho r^2 \, \mathrm{d}r + 4\pi G \int_r^\infty \rho r \, \mathrm{d}r + a \text{ constant}, \tag{65}$$

which makes the nature of this solution obvious. Moreover, the infinite upper limit in the second interval on the right-hand side of (64) or (65) can be replaced by the mean radius r_* of the respective configuration without altering the value of the respective integral; since $\rho(r \leq r_*)=0$.

Let us differentiate now the Equation (64) with respect to r: in doing so we find that

$$\frac{\partial \mathscr{R}}{\partial r} = jC_j r^{j-1} + \frac{4\pi G}{2j+1} \left\{ -\frac{j+1}{r^{j+2}} \int_0^r \mathscr{F} r^{j+2} \, \mathrm{d}r + j r^{j-1} \int_r^{r_*} \mathscr{F} r^{1-j} \, \mathrm{d}r \right\}, \tag{66}$$

which combined with (64) yields

$$\frac{\partial \mathscr{R}}{\partial r} + \frac{j+1}{r} \mathscr{R} = (2j+1) \, C_j r^{j-1} + 4\pi G r^j \int_r^{r_*} \mathscr{F} r^{1-j} \, \mathrm{d}r \tag{67}$$

for any value of i and j; and at the surface of our configuration (i.e., for $r=r_*$) Equation (67) reduces further to

$$\left\{ \frac{\partial \mathscr{R}}{\partial r} + \frac{j+1}{r} \mathscr{R} \right\}_{r=r_*} = (2j+1) \, C_j r_*^{j-1}. \tag{68}$$

Let us next differentiate the foregoing relation with respect to the time. Since, by a comparison of (61) with (39) it follows that

$$\frac{\partial}{\partial t} \mathscr{R}(r, t) = R(r, t), \tag{69}$$

the boundary condition (68) can at once be rewritten to assert that

$$\left\{ \frac{\partial}{\partial r} R(r, t) \right\}_{r=r_*} + \frac{j+1}{r_*} R(r_*, t) = (2j+1) \frac{\partial C_j}{\partial t} r_*^{j-1}, \tag{70}$$

where the surface values of R and $\partial R / \partial r$ can be inserted from (40) and (41) in terms of the local values of u and v which, in turn, are constrained by the condition (52) to satisfy the Equations (56) and (57).

In order to ascertain the form which the function $R(r, t)$ must satisfy at the centre $r=0$ of our configuration, let us return to Equation (64), divide its both sides by r^j, and differentiate with respect to r: the outcome discloses that

$$\frac{\partial}{\partial r}\left(\frac{\mathscr{R}}{r^j}\right) = -\frac{4\pi G}{r^{2j+2}} \int_0^r \mathscr{F} r^{j+2}\, \mathrm{d}r\,. \tag{71}$$

Differentiate now again both sides of this equation with respect to t: inasmuch as

$$\frac{\partial \mathscr{F}}{\partial t} = -f(r, t), \tag{72}$$

where the function f has already been defined in terms of u and v by Equation (32), it follows from (69 and (71) that

$$r\frac{\partial R}{\partial r} - jR = \frac{4\pi G}{r^{j+1}} \int_0^r f(r, t)\, r^{j+2}\, \mathrm{d}r\,. \tag{73}$$

Since

$$\lim_{r\to 0} \frac{1}{r^{j+1}} \int_0^r f(r, t)\, r^{j+2}\, \mathrm{d}r = \lim_{r\to 0} \frac{fr^2}{j+1} = 0\,, \tag{74}$$

it follows from (73) that

$$R(0, t) = 0\,, \tag{75}$$

which together with (70) represents the boundary conditions imposed on the acceptable solutions of Equation (45).

Therefore, the particular solutions of interest to us of the system of Equations (40)–(41) and (45) must obey the conditions

$$u(0, t) = v(0, t) = R(0, t) = 0 \tag{76}$$

at the centre of our configuration, and

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} + \frac{u - v}{r} = \frac{\partial R}{\partial r} + \frac{j+1}{r} R - (2j+1)\frac{\partial C_j}{\partial t} r^{j-1} = 0 \tag{77}$$

on the surface ($r = r_*$). Of these, all but (70) are homogeneous in the dependent variables; and even the latter becomes so if no time-dependent external forces are present (and, consequently, all C_j's are constants).

As regards the boundary conditions at the centre, inserting (72) in (41) and particularizing the latter for $r = 0$ we find it to reduce to

$$(a^2 h)_0 + \left(\frac{\partial G}{\partial t}\right)_0 = 0\,. \tag{78}$$

Since, however, by (33) and in view of the fact that $g(0)=0$,

$$(a^2 h)_0 = (\rho a^2)_0 \left\{3\frac{\partial u}{\partial r} - j(j+1)\frac{\partial v}{\partial r}\right\}_0 \tag{79}$$

by d'Hospital rule; and since, moreover, it follows from (37) that

$$G(0, t) = \mu_0 \left\{3\frac{\partial u}{\partial r} - (j-1)(j+2)\frac{\partial v}{\partial r}\right\}_0, \tag{80}$$

and an insertion of (79) and (80) reveals (by 29) that, at the centre,

$$(\rho a^2 y)_0 + \mu_0 \frac{\partial}{\partial t}\left\{y + 2\frac{\partial v}{\partial r}\right\}_0 = 0. \tag{81}$$

In the inviscid case ($\mu_0=0$) this equation reduces to

$$y(0, t) = 0 \tag{82}$$

and represents (by 33) a sufficient condition that there be no variation of pressure at the centre of our configuration; but for $\mu_0 \neq 0$ this will continue to be true only if, in addition, $(\partial v/\partial r)_0=0$; for purely radial oscillations – characterized by $v(r)=0$ throughout the configuration – this will always be the case.

Should we, on the other hand, prefer to eliminate the potential disturbance R from our solution and regard Equations (49) and (50) as our fundamental system, the boundary conditions (70) and (75) must be rewritten in terms of the local values of u and v. This can be readily effected by means of the Equations (40) and (41); for an insertion for R from the latter in (75) yields

$$(\rho a^2)_0 \left\{3\frac{\partial u}{\partial r} - j(j+1)\frac{\partial v}{\partial r}\right\}_0 + \mu_0 \frac{\partial}{\partial t}\left\{3\frac{\partial \mu}{\partial r} - (j-1)(j+2)\frac{\partial v}{\partial r}\right\}_0 = 0 \tag{83}$$

in terms of u_0 and v_0.

A similar transformation of the boundary condition (70) at $r=r_*$ requires a little more work. In order to render its form more explicit, let us insert in (70) for $\partial R/\partial r$ and R from (40) and (41); in doing so and particularizing the outcome for $r=r_*$ we find that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\{u + (j+1)v\} - a^2 A y - \left\{\frac{\partial}{\partial r} + \frac{j+1}{r}\right\}(a^2 y - gu)\\ = -\frac{1}{\rho}\frac{\partial}{\partial t}\left\{F + \frac{j+1}{r}G\right\} + (2j+1)\frac{\partial C_j}{\partial t} r_*^{j-1}\end{aligned} \tag{84}$$

on the outer boundary.

Since, moreover, by (56) and (57) Equations (29)–(30) and their radial derivatives reduce to

$$y(r_*) = \frac{2u(r_*) - j(j+1)\,v(r_*)}{r_*}, \tag{85}$$

$$\left(\frac{\partial y}{\partial r}\right)_* = \left(\frac{\partial^2 u}{\partial r^2}\right)_* + \frac{(j-1)(j+2)}{r_*^2}\,u(r_*), \tag{86}$$

while

$$z(r_*) = \frac{2}{r_*}\{v(r_*) - u(r_*)\} \tag{87}$$

and

$$\left(\frac{\partial z}{\partial r}\right)_* = \left(\frac{\partial^2 v}{\partial r^2}\right)_*, \tag{88}$$

it follows, on the outer boundary ($r = r_*$),

$$\begin{aligned}\left\{\frac{\partial}{\partial r} + \frac{j+1}{r}\right\}(a^2 y - gu) = a^2\left\{\frac{\partial^2 u}{\partial r^2} + \left[j(j+3) + \frac{2r}{a^2}\frac{\partial a^2}{\partial r} - \frac{(j-1)\,gr}{a^2}\right]\frac{u}{r^2}\right.\\ \left. - j(j+1)\left[j+1+\frac{r}{a^2}\frac{\partial a^2}{\partial r}\right]\frac{v}{r^2}\right\},\end{aligned} \tag{89}$$

while

$$\begin{aligned}F + \frac{j+1}{r}\,G = \frac{\mu}{3r^2}\left\{4r^2\frac{\partial^2 u}{\partial r^2} + 3(j+1)\,r^2\frac{\partial^2 v}{\partial r^2}\right.\\ \left. - 2\left[(j^2+3)\,u + (2j-3)(j+1)^2\,v\right]\right\} - \frac{2}{3r}\frac{\partial \mu}{\partial r}\{2u - j(j+1)\,v\}.\end{aligned} \tag{90}$$

Inserting (89) and (90) in (74) we eventually establish that the boundary condition (70) rewritten in terms of the velocity components $u(r_*)$ and $v(r_*)$ in place of $R(r_*)$ assumes the alternative form

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}\{u + (j+1)\,v\} - a^2\left\{\frac{\partial^2 u}{\partial r^2} + \left[j(j+3) + \right.\right.\\ &\quad\left. + 2r\left(A + \frac{1}{a^2}\frac{\partial a^2}{\partial r}\right) - \frac{(j-1)\,gr}{a^2}\right]\frac{u}{r^2}\\ &\quad\left. - j(j+1)\left[j+1+r\left(A + \frac{1}{a^2}\frac{\partial a^2}{\partial r}\right)\right]\frac{v}{r^2}\right\} =\\ &= \frac{\mu}{3\rho}\frac{\partial}{\partial t}\left\{2\left[2\frac{\partial^2 u}{\partial r^2} - (j^2+3)\frac{u}{r^2}\right] + (j+1)\left[3\frac{\partial^2 v}{\partial r^2} - 2(j+1)(2j-3)\frac{V}{r^2}\right]\right.\\ &\quad\left. - \frac{2}{3r\rho}\left(\frac{\partial \mu}{\partial r}\right)\frac{\partial}{\partial t}\{2u - j(j+1)\,v\} + (2j+1)\frac{\partial C_j}{\partial t}\,r^{j-1}\right\},\end{aligned} \tag{91}$$

where, it may be noted from (22) and (43),

$$A + \frac{1}{a^2}\frac{\partial a^2}{\partial r} = \frac{\partial}{\partial r}\log(\gamma P^{(\gamma-1)/\gamma}); \tag{92}$$

while, in hydrostatic equilibrium,

$$\frac{(j-1)\,g}{a^2} = \frac{\partial}{\partial r}\log P^{(1-j)/\gamma}. \tag{93}$$

The reader may note that – unlike the boundary conditions (76)–(77) of Equations (40)–(41) and (45) – the conditions (83) and (91) to be adjoined to Equations (49)–(50) involve local values of the time-derivatives of the dependent variables u and v; or – in the case that the motion governed by (49)–(50) proves to be periodic – the frequencies of the respective motion. The latter will, therefore, occur not only in the actual Equations (49)–(50) of motion, but also in their associated boundary conditions.

The reader may also note that – in the absence of any external disturbing forces which depend on the time – all boundary conditions of our problem are homogeneous in their dependent variables. Should this be the case, the solutions of our equations would correspond to *free* spheroidal oscillations of compressible self-gravitating configurations of viscous fluid, with arbitrary amplitudes (as long as these are small enough for their squares and higher powers to be negligible).

Dynamical tides in close binary systems represent, to be sure, *forced* oscillations, with amplitudes governed by disturbing forces which depend on the time. The actual determination of the constants C_j in terms of the characteristics of the disturbing forces will be carried out later on. Before we do so, however, we wish to investigate briefly the frequency-spectra of free oscillations of certain limiting types of our configurations – both as a preparatory step for the study of forced oscillations, as well as to be able to consider later a possibility of *resonance* between free and forced oscillations in close binary systems; and to this task we shall address ourselves in the next four sections of our present paper.

5. Free Oscillations: Homogeneous Bodies

In the preceding Sections 2–4 of this paper equations have been set up which govern spheroidal oscillations – free or forced – of self-gravitating configurations of arbitrary structure, consisting of viscous gas. The aim of the present section will be to particularize our problem to an idealized case in which the density distribution $\rho_0(r)$ as well as the viscosity $\mu(r)$ throughout the interior can be regarded as constant; for their adoption will prove to simplify the solution of our equations of Section 3 to such an extent that the underlying problem can be solved in a *closed* form, in terms of certain hypergeometric series which we shall now proceed to construct.

To begin with, let us change over from the physical independent variables r, t of our problem to the corresponding non-dimensional variables x, τ defined by the equations

$$\left.\begin{aligned} r &= r_* x\,, \\ t &= \frac{\tau}{\sqrt{2\pi G\rho_0}}\,, \end{aligned}\right\} \tag{94}$$

normalized so that

$$0 \leq x \leq 1 \tag{95}$$

between the centre and the surface of our configuration. If so, then for constant ρ_0 Equation (14) can be readily integrated to

$$P_0 = \tfrac{2}{3}\pi G\rho_0^2 r_*^2 (1 - x^2)\,, \tag{96}$$

while (15) yields

$$g = \tfrac{4}{3}\pi G\rho_0 r_* x \tag{97}$$

and from (22) combined with (96),

$$a^2 = \tfrac{2}{3}\pi G\rho_0 r_* \gamma (1 - x^2)\,. \tag{98}$$

Moreover, in accordance with (43),

$$A = \frac{2x}{\gamma r_* (1 - x^2)}\,, \tag{99}$$

so that (44) assumes the form

$$a^2 A = \tfrac{4}{3}\pi G\rho_0 r_* x\,; \tag{100}$$

and, lastly, by (32) and (33)

$$gf = \tfrac{4}{3}\pi G\rho_0^2 x\tilde{y} \tag{101}$$

and

$$a^2 h = \tfrac{2}{3}\pi G\rho_0^2 r_* \{\gamma (1 - x^2)\,\tilde{y} - 2xu\}\,, \tag{102}$$

where we have abbreviated

$$\tilde{y} = r_* y \quad \text{and} \quad \tilde{z} = r_* z\,. \tag{103}$$

If so, the system (49)–(50) of our fundamental equations of motion can, for constant μ, be rewritten as

$$\frac{3}{2}\frac{\partial^2 \tilde{y}}{\partial \tau^2} = 3\tilde{y} + \frac{1}{x^2}\frac{\partial}{\partial x}(x^3 \tilde{y}) + \left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\left[\frac{\gamma}{2}(1 - x^2)\,\tilde{y} - xu\right] + \tilde{\mu}\frac{\partial}{\partial \tau}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\tilde{y}\,, \tag{104}$$

and

$$\frac{3}{2}\frac{\partial^2 \tilde{z}}{\partial \tau^2} + \tilde{y} = \frac{3}{4}\tilde{\mu}\frac{\partial}{\partial \tau}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\tilde{z}\,, \tag{105}$$

where

$$\tilde{\mu} = \frac{2\mu}{\rho_0 r_*^2 \sqrt{2\pi G\rho_0}} \tag{106}$$

stands for a non-dimensional constant parameter proportional to the viscosity.

Let us disregard at first the terms factored by $\tilde{\mu}$ in the foregoing equations (104)–(105) and – anticipating the motion governed by them to be harmonic – set

$$\frac{\partial^2}{\partial\tau^2} = -\tfrac{3}{2}\tilde{v}^2, \tag{107}$$

where $\tilde{v}$ stands for the (normalized) frequency of the respective motion. If so, the system(104)–(105) obviously reduces to

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right\}\left[\frac{\gamma}{2}(1-x^2)\,\tilde{y} - xu\right] + x\frac{\partial\tilde{y}}{\partial x} + (\tilde{v}^2+6)\,\tilde{y} = 0 \tag{108}$$

and

$$\tilde{y} = \tilde{v}^2\tilde{z}, \tag{109}$$

respectively.

The foregoing Equations (108) and (109) constitute a fourth-order simultaneous system for the velocity components u and v. However, in this particular case of a homogeneous configuration, their integration can be split up in two stages. First, let us note that, by virtue of Equations (29)–(30) and (109),

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{\partial x^2}\right\}(xu) = \frac{1}{x}\frac{\partial}{\partial x}(x^2\tilde{y}) + \frac{j(j+1)}{\tilde{v}^2}\,\tilde{y}, \tag{110}$$

which on insertion in (108) yields

$$x^2(1-x^2)\frac{\partial^2\tilde{y}}{\partial x^2} + 2x(1-3x^2)\frac{\partial\tilde{y}}{\partial x} + [Kx^2 - j(j+1)]\,\tilde{y} = 0 \tag{111}$$

where we have abbreviated

$$K = \frac{2}{\gamma}\left\{\tilde{v}^2 + 4 - \frac{j(j+1)}{\tilde{v}^2}\right\} + (j-2)(j+3). \tag{112}$$

The foregoing relation (110) constitutes a second-order differential equation for $\tilde{y}$ which can be integrated as it stands; and once its solution has been obtained, the velocity components u and v can be solved for from the equations

$$\tilde{y} = \frac{\partial u}{\partial x} + \frac{2u}{x} - j(j+1)\frac{v}{x} = \tilde{v}^2\left\{\frac{\partial v}{\partial x} + \frac{v-u}{x}\right\} \tag{113}$$

resulting from (103) and (109).

Let us, however, return now to Equation (111). If we substitute $x^2 = \xi$, the latter can be rewritten as

$$\xi^2(\xi-1)\frac{\partial^2\tilde{y}}{\partial\xi^2} + \frac{7\xi-3}{2}\,\xi\frac{\partial\tilde{y}}{\partial\xi} + \tfrac{1}{4}\{j(j+1) - K\xi\}\,\tilde{y} = 0, \tag{114}$$

and its complete primitive can be expressed as a linear combination of two hypergeometric series of the form

$$\tilde{y} = Ax^jF(a,b,c,x^2) + Bx^{-j-1}F(a-c+1,\, b-c+1,\, 2-c,\, x^2), \tag{115}$$

where A, B are arbitrary integration constants, and

$$\left.\begin{aligned} a &= \tfrac{1}{4}\left[2j + 5 \pm \sqrt{25 + 4K}\right], \\ b &= \tfrac{1}{4}\left[2j + 5 \mp \sqrt{25 + 4K}\right], \\ c &= j + \tfrac{3}{2}. \end{aligned}\right\} \tag{116}$$

The finiteness of $\tilde{y}$ as given by Equation (115) at the origin obviously requires that $B=0$. Moreover, inasmuch as, by (116),

$$a + b - c = 1, \tag{117}$$

an application of standard tests for the convergence of hypergeometric series discloses that both series on the right-hand side of (115) diverge for $x=1$. The solution (115) for $\tilde{y}$ expressed in their terms can, therefore, remain finite at the boundary only if the respective hypergeometric series are made to terminate by setting (say)

$$\tfrac{1}{4}\{2j + 5 \mp \sqrt{25 + 4K}\} = - k, \tag{118}$$

where k stands for an arbitrary positive integer. Since, however, then

$$a = j + k + \tfrac{5}{2} \quad \text{and} \quad b = - k, \tag{119}$$

the particular solution of (115) which remain finite for $0 \leq x \leq 1$ assumes the form

$$\tilde{y} = A_{j,k} x^j F(j + k + \tfrac{5}{2}, - k, j + \tfrac{3}{2}, x^2) = A_{j,k} x^j G_k(j + \tfrac{5}{2}, j + \tfrac{3}{2}, x^2) \tag{120}$$

where G_k denotes the corresponding Jacobi polynomial of degree k.

Moreover, Equation (118) implies that

$$(2j + 4k + 5)^2 = 25 + 4K, \tag{121}$$

which on insertion for K from (112) and after some rearrangement of terms assumes the form

$$\tilde{v}^2 + 4 - \frac{j(j + 1)}{\tilde{v}^2} = \gamma(k + 1)(2j + 2k + 3); \tag{122}$$

and the frequency of the corresponding oscillation will be given by

$$\tilde{v}^2 = \omega \pm \sqrt{\omega^2 + j(j + 1)}, \tag{123}$$

where we have abbreviated

$$\omega = \gamma(k + 1)(j + k + \tfrac{3}{2}) - 2. \tag{124}$$

For $j>0$, one of the conjugate roots of (122) is bound to be negative – implying dynamical instability. If ω (i.e., k) is large, the conjugate roots (123) will be led by the terms

$$\tilde{v}^2 = 2\omega + \frac{j(j + 1)}{2\omega} + \cdots \quad \text{or} \quad - \frac{j(j + 1)}{2\omega} + \cdots. \tag{125}$$

Thus, for any given j, the requirement that $\tilde{y}$ be finite throughout the interval $0 \leq x \leq 1$ leads to two types of the spectra: one consisting of positive eigenvalues tending towards infinity as k increases; the other of negative eigenvalues tending towards zero.

With the explicit form of $\tilde{y}$ as given by (120), the Equations (113) for u and v assume the explicit form

$$\frac{1}{x^2}\left\{\frac{\partial}{\partial x}(x^2 u) - j(j+1)\, xv\right\} = \frac{\tilde{\nu}^2}{x}\left\{\frac{\partial}{\partial x}(xv) - u\right\} = A_{j,k} x^j F(a, b, c, x^2), \tag{126}$$

where, as before,

$$\left.\begin{aligned} a &= j + \tfrac{5}{2} + k, \\ b &= \qquad\;\; - k, \\ c &= j + \tfrac{3}{2}; \end{aligned}\right\} \tag{127}$$

and can be integrated to furnish polynomial solutions for u and v in the following manner.

First, we note that an elimination of xv between the first two parts of the Equation (126) yields a relation of the form

$$\frac{\partial^2}{\partial x^2}(x^2 u) - j(j+1)\, u = A_{j,k}\left\{\frac{\partial}{\partial x}(x^{j+2}F) + \frac{j(j+1)}{\tilde{\nu}^2} x^{j+1} F\right\}, \tag{128}$$

with $F \equiv F(a, b, c, x^2)$; and its particular integral which remains finite at the origin becomes

$$u_{j,k}(x) = \frac{A_{j,k}}{2j+1}\left\{ j\left[1 + \frac{j+1}{\tilde{\nu}^2}\right] x^{j-1}\int_0^x xF\mathrm{d}x \right.$$
$$\left. + (j+1)\left[1 - \frac{j}{\tilde{\nu}^2}\right] x^{-j-2}\int_0^x x^{2(j+1)}F\mathrm{d}x\right\}. \tag{129}$$

Since, moreover,

$$xF(a, b, c, x^2) = \frac{c-1}{2(a-1)(b-1)}\frac{\partial}{\partial x} F(a-1, b-1, c-1, x^2) \tag{130}$$

and

$$(2j+3)\, x^{2(j+1)} F(a, b, c, x^2) = \frac{\partial}{\partial x}\{x^{2j+3} F(a, b, c+1, x^2)\}, \tag{131}$$

the integrals on the right-hand side of (129) can be evaluated in a closed form to yield

$$u_{j,k}(x) = \frac{A_{j,k} x^{j-1}}{4(a-1)(b-1)\,\tilde{\nu}^2}\{(j+1)(j-\tilde{\nu}^2)\, F(a-1, b-1, c, x^2)$$
$$+ 2(c-1)\,\tilde{\nu}^2 F(a-1, b-1, c-1, x^2) - j(j+\tilde{\nu}^2+1)\}, \tag{132}$$

where advantage has been taken of the identity

$$x^2 F(a, b, c+1, x^2) = \frac{c(c-1)}{(a-1)(b-1)} \times$$
$$\times \{F(a-1, b-1, c-1, x^2) - F(a-1, b-1, c, x^2)\}. \tag{133}$$

The arguments $b-1=-(k+1)$ in both hypergeometric series on the right-hand side of (132) are negative integers; hence, both series are terminating and can be expressed in terms of the Jacobi polynomials

$$F(a-1, b-1, c, x^2) \equiv G_{k+1}(j+\tfrac{1}{2}, j+\tfrac{3}{2}, x^2) \tag{134}$$

and

$$F(a-1, b-1, c-1, x^2) \equiv G_{k+1}(j+\tfrac{1}{2}, j+\tfrac{1}{2}, x^2) \tag{135}$$

of degrees $2(k+1)$ in x. Moreover, inasmuch as

$$(j+1)(j-\tilde{v}^2)+2(c-1)=j(j+\tilde{v}^2+1), \tag{136}$$

the leading term of the polynomial expression (132) will be

$$u_{j,0}(x)=A_{j,0}\left\{\frac{j(j+1)+(j+2)\tilde{v}^2}{2(2j+3)\tilde{v}^2}\right\}x^{j+1}. \tag{137}$$

With a polynomial solution for u thus established, that for v follows algebraically from the equality of the first and third term of the equation (126), disclosing that

$$j(j+1)\,v=\frac{1}{x}\frac{\partial}{\partial x}(x^2u)-x^{j+1}F(a, b, c, x^2). \tag{138}$$

Inserting in (138) for u from (132) we find that

$$\begin{aligned} v_{j,k}(x)=\frac{A_{j,k}x^{j-1}}{4(a-1)(b-1)\tilde{v}^2}\{&2(c-1)F(a-1, b-1, c-1, x^2) \\ &+(\tilde{v}^2-j)F(a-1, b-1, c, x^2)-(j+\tilde{v}^2+1)\} \end{aligned} \tag{139}$$

which for $k=0$ reduces again to

$$v_{j,0}(x)=A_{j,0}\left\{\frac{j+\tilde{v}^2+3}{2(2j+3)\tilde{v}^2}\right\}x^{j+1}. \tag{140}$$

The foregoing Equations (120) with (132) and (139) represent closed polynomial solutions for free non-radial oscillations of order j and mode k of a homogeneous configuration of compressible inviscid fluid, with characteristic frequencies $\tilde{v}$ as given by Equations (122) or (123).

Non-radial oscillations of compressible homogeneous configurations considered in this section were previously investigated by PEKERIS (1938) who noted their instability, and was the first to express the divergence of motion in terms of a second-order differential equation. He failed, however, to notice, and take advantage of, the hypergeometric character of this equation; nor did he carry out the explicit solution for the actual velocity components u and v which are presented here for the first time.

6. The Effects of Viscosity

Throughout most part of the preceding section we have regarded our configuration to consist of inviscid fluid – a simplifying assumption which enabled us temporarily to ignore the effects of the operator $\tilde{\mu}(\partial/\partial t)$ on the right-hand sides of Equations (104) and (105). The aim of the present section will be now to restore these terms and proceed to construct such solutions of Equations (104) and (105) which take the effects of constant viscosity duly into account.

The principal new feature of this problem which distinguishes it from the one treated in the preceding section is the fact that, for harmonic motion,

$$\frac{\partial}{\partial t} = i\tilde{v}; \tag{141}$$

in consequence of which the right-hand sides of Equations (104) and (105) become complex; and their solutions must be sought in terms of complex velocity components $u(x, \tau)$ and $v(x, \tau)$ of the form

$$u(x, \tau) = \{u_1(x) + i\tilde{\mu}u_2(x)\}\, e^{(\kappa + i\lambda)\tau} \tag{142}$$

and

$$v(x, \tau) = \{v_1(x) + i\tilde{\mu}v_2(x)\}\, e^{(\kappa + i\lambda)\tau}, \tag{143}$$

where $u_{1,2}(x)$, $v_{1,2}(x)$ are real functions of x and κ, λ are real constants. As $\tilde{\mu}\rightarrow 0$, the functions $u_1(x)$ and $v_1(x)$ should, moreover, tend to the limits represented by Equations (127) and (128); but the new functions $u_2(x)$ and $v_2(x)$ arising when $\tilde{\mu} \neq 0$ remain yet to be defined and solved for. This can be done by a method initiated by the present writer (cf. KOPAL, 1964) for the case of purely radial oscillations ($j=0$), and extended by MOUTSOULAS (1967) to non-radial oscillations, which are of primary interest to us in the present paper, in a manner to be developed in this section.

In embarking on this task, consider first the complex frequency $\kappa + i\lambda$ of harmonic oscillations in (142) and (143), the real part of which represents the rate of damping of the oscillatory motion, due to a viscous dissipation of kinetic energy into heat. As, however, the viscous dissipation function is known to be homogeneous and quadratic in the velocity components u and v, its magnitude becomes ignorable within the framework of our linearized theory initiated in Section 2; and, hence, consistent with our scheme of approximation, κ can hereafter be neglected.

Next, as a consequence of (142) and (143), the normalized quantities $\tilde{y}$ and $\tilde{z}$ should likewise be expressible as complex functions

$$\tilde{y}(x, \tau) = \{y_1(x) + i\tilde{\mu}y_2(x)\}\, e^{i\lambda\tau}, \tag{144}$$

$$\tilde{z}(x, \tau) = \{z_1(x) + i\tilde{\mu}z_2(x)\}\, e^{i\lambda\tau}, \tag{145}$$

where, in accordance with (29)–(30) and (103),

$$xy_{1,2} = x\frac{\partial u_{1,2}}{\partial x} + 2u_{1,2} - j(j+1)\,v_{1,2} \tag{146}$$

and

$$xz_{1,2} = x\frac{\partial v_{1,2}}{\partial x} + v_{1,2} - u_{1,2}\,. \tag{147}$$

If we insert the foregoing complex expressions for u, v, and $\tilde{y}$ in (104), and separate the real and imaginary parts we find that (104) splits up into the following symmetrical pair of simultaneous differential equations

$$3(4+\lambda^2)\, y_{1,2} + 2x\frac{\partial y_{1,2}}{\partial x} + \left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right]\left[\gamma(1-x^2)\, y_{1,2} - 2xu_{1,2}\right] \pm 2\lambda\tilde{\mu}^{2,0}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right] y_{2,1} = 0. \quad (148)$$

Equations (146) and (147) reveal that

$$\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right](xu_{1,2}) = \frac{1}{x}\frac{\partial}{\partial x}(x^2 y_{1,2}) + j(j+1)\, z_{1,2} \quad (149)$$

and

$$\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right](1-x^2)\, y_{1,2} = \left[(1-x^2)\frac{\partial^2}{\partial x^2} + 2\left(\frac{1-3x^2}{x}\right)\frac{\partial}{\partial x} - \frac{j(j+1)(1-x^2)}{x^2} - 6\right] y_{1,2}, \quad (150)$$

which inserted in (148) permit us to rewrite the latter as

$$(1-x^2)\frac{\partial^2 y_{1,2}}{\partial x^2} + 2\left(\frac{1-3x^2}{x}\right)\frac{\partial y_{1,2}}{\partial x} + \left\{\frac{2}{\gamma}(4+\tfrac{3}{2}\lambda^2) + (j-2)(j+3) - \frac{j(j+1)}{x^2}\right\} y_{1,2} = \frac{2}{\gamma}\, j(j+1)\, z_{1,2} \pm 2\frac{\lambda}{\gamma}\tilde{\mu}^{2,0}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right] y_{2,1} \quad (151)$$

constituting two relations between the unknown functions $y_{1,2}$ and $z_{1,2}$. The remaining two can, in turn, be obtained by an insertion of (145) in (105): in doing so and separating the real and imaginary parts we find them to assume the forms

$$\tfrac{2}{3}y_{1,2} = \lambda^2 z_{1,2} \mp \tfrac{1}{2}\lambda\tilde{\mu}^{2,0}\left[\frac{\partial^2}{\partial x^2} + \frac{2}{x}\frac{\partial}{\partial x} - \frac{j(j+1)}{x^2}\right] z_{2,1}, \quad (152)$$

which together with (151) represent the fundamental system of differential equations governing the oscillations of self-gravitating configurations of viscous fluid; for once these have been solved for $y_{1,2}$ and $z_{1,2}$, the corresponding velocity components $u_{1,2}$ and $v_{1,2}$ can be determined from (146)–(147).

It is of interest to note that these equations are quadratic in $\tilde{\mu}$ – a fact which underlines the importance of the terms introduced by viscosity. If $\tilde{\mu}\to 0$, only the functions with subscript 1 remain relevant to the problem; and a combination of Equations (151) and (152) reduces then to $y_1 \equiv \tilde{y}$ and $z_1 \equiv \tilde{v}^{-2}y$, with y as previously given by (111), and $\lambda^2 = 1.5\,\tilde{v}^2$. If, on the other hand, $\tilde{\mu}\to\infty$, the relevant solutions for y_2

and z_2 which remain finite at the origin will both vary as x^j and can differ only in their multiplicative constants; the same being true of u_2 and v_2 which vary as x^{j+1}.

For finite values of $\tilde{\mu}$, Equations (151)–(152) must be treated as a simultaneous system governing four unknown functions $y_{1,2}(x)$ and $z_{1,2}(x)$; a solution of which may also be approached by successive approximations. For insert for $z_{1,2}$ from (152) in (151): the outcome discloses that

$$(1-x^2)\frac{\partial^2 y_{1,2}}{\partial x^2}+2\left(\frac{1-3x^2}{x}\right)\frac{\partial y_{1,2}}{\partial x}+\left\{K-\frac{j(j+1)}{x^2}\right\}y_{1,2}=$$
$$=\pm 2\frac{\lambda}{\gamma}\tilde{\mu}^{2,0}\left[\frac{\partial^2}{\partial x^2}+\frac{2}{x}\frac{\partial}{\partial x}-\frac{j(j+1)}{x^2}\right]\left[\frac{1}{\lambda^2}j(j+1)z_{2,1}+y_{2,1}\right] \tag{153}$$

where the constant K continues to be given by (112). In consequence, the relation

$$(1-x^2)\frac{\partial^2 y_2}{\partial x^2}+2\left(\frac{1-3x^2}{x}\right)\frac{\partial y_2}{\partial x}+\left\{K-\frac{j(j+1)}{x^2}\right\}y_2=$$
$$=-2\frac{\lambda}{\gamma}\left[\frac{\partial^2}{\partial x^2}+\frac{2}{x}\frac{\partial}{\partial x}-\frac{j(j+1)}{x^2}\right]\left[\frac{1}{\lambda^2}j(j+1)z_1+y_1\right] \tag{154}$$

becomes independent of viscosity; and the operator on its left-hand side is identical with that of Equation (111). If the right-hand side of the foregoing Equation (154) we set equal to zero, the complete primitive of the homogeneous equation for y_2 would indeed be of the form (111). If, moreover, we approximate the functions on the right-hand side of (154) by polynomial expressions of the form (120), the particular integral of the complete nonhomogeneous Equation (154) for $(y_2)_0$ can be obtained by standard method; and the corresponding expression for $(z_2)_0$ then follows from (152) algebraically as

$$(z_2)_0=\frac{2}{3\lambda^2}(y_2)_0-\frac{1}{2\lambda}\left[\frac{\partial^2}{\partial x^2}+\frac{2}{x}\frac{\partial}{\partial x}-\frac{j(j+1)}{x^2}\right](z_1)_0, \tag{155}$$

where, by (152), $(z_1)_0=\frac{2}{3}\lambda^{-2}(y_1)_0$.

As the next step of our approximation procedure, we revert to the second one of the Equations (151) in the form

$$(1-x^2)\frac{\partial^2 y_1}{\partial x^2}+2\left(\frac{1-3x^2}{x}\right)\frac{\partial y_1}{\partial x}+\left\{K-\frac{j(j+1)}{x^2}\right\}y_1=$$
$$=2\frac{\lambda}{\gamma}\tilde{\mu}^2\left[\frac{\partial^2}{\partial x^2}+\frac{2}{x}\frac{\partial}{\partial x}-\frac{j(j+1)}{x^2}\right]\left[\frac{1}{\lambda^2}j(j+1)z_2+y_2\right], \tag{156}$$

where y_2 and z_2 on the right-hand side can be inserted from the solutions of (154) and (155); and (156) regarded as a non-homogeneous equation for a second approximation to $(y_1)_1$; and

$$(z_1)_1=\frac{2}{3\lambda^2}(y_1)_1+\frac{\tilde{\mu}^2}{2\lambda}\left[\frac{\partial^2}{\partial x^2}+\frac{2}{x}\frac{\partial}{\partial x}-\frac{j(j+1)}{x^2}\right](z_2)_0. \tag{157}$$

The same procedure can obviously be continued until the expressions for $(y_{1,2})_{i+1}$ or $(z_{1,2})_{i+1}$ cease to differ from $(y_{1,2})_i$ or $(z_{1,2})_i$ by significant amounts.

Suppose that such stabilized solutions for $y_{1,2}$ and $z_{1,2}$ have been established; and from these the velocity components $u_{1,2}$ and $v_{1,2}$ with the aid of (142) and (143). The complex velocity components $u(x, \tau)$ and $v(x, \tau)$ then follow by insertion of $u_{1,2}$ and $v_{1,2}$ in (142)–(143); and their real parts – which are of interest in connection with our physical problem – should be of the form

$$\mathrm{Re}\,[u(x, \tau)] = e^{\kappa\tau}\sqrt{u_1^2 + u_2^2}\cos\{\lambda\tau + \tan^{-1}\tilde{\mu}(u_2/u_1)\} \tag{158}$$

and, similarly,

$$\mathrm{Re}\,[v(x, \tau)] = e^{\kappa\tau}\sqrt{v_1^2 + v_2^2}\cos\{\lambda\tau + \tan^{-1}\tilde{\mu}(v_2/v_1)\}, \tag{159}$$

whose amplitudes

$$\sqrt{u_1^2(x) + u_2^2(x)}, \qquad \sqrt{v_1^2(x) + v_2^2(x)}$$

are identical with the moduli of the respective complex velocity components, while the angles

$$\varepsilon_u = \tan^{-1}(\tilde{\mu}u_2/u_1) \tag{160}$$

and

$$\varepsilon_v = \tan^{-1}\tilde{\mu}(v_2/v_1) \tag{161}$$

represent their phase lags.

It is evident from the expressions (158) and (159) that the phase with which a self-gravitating configuration of a viscous fluid can oscillate – freely or in response to an external field of force – is bound to vary between the center and the surface of our configuration. The respective motion represents, therefore, a travelling rather than a standing wave. Only inviscid configurations can oscillate with constant phase; and the latter is bound to become a function of r as soon as $\tilde{\mu}>0$. Moreover, our results make it evident that the amount by which the oscillations – free or forced – will lag in phase may be different for different velocity components. In particular, in the case of spheroidal symmetry represented by Equations (25)–(27), the phase lag in the radial velocity component U will be different from that in the angular velocity components V and W.

7. Incompressible Viscous Configurations

The aim of the present section will be to consider the consequences of an assumption that our homogeneous configuration becomes incompressible (or its compressibility becomes so small that it can be neglected). If so, the divergence Δ of the velocity vector as defined by Equation (9) will obviously vanish; and so will, by (31) and (103), the function y – implying, by (144), that $y_1(x)=y_2(x)=0$. On the other hand, incompressibility implies that the ratio of specific heats γ of the material become infinite – but in such a way that the product $\gamma\tilde{y}$ or $\gamma\tilde{y}_{1,2}$ should be regarded as a finite quantity.

If so, however, Equations (151) will reduce to

$$\left\{(1-x^2)\frac{\partial^2}{\partial x^2}+2\,\frac{1-3x^2}{x}\,\frac{\partial}{\partial x}+(j-2)(j+3)-\right.$$
$$\left.-\frac{j(j+1)}{x^2}\right\}(\gamma y_{1,2})=2j(j+1)\,z_{1,2}\,, \tag{162}$$

while Equations (152) for $z_{1,2}$ similarly become

$$\lambda^2 z_{1,2}=\pm\tfrac{1}{2}\lambda\tilde{\mu}^{2,0}\left\{\frac{\partial^2}{\partial x^2}+\frac{2}{x}\frac{\partial}{\partial x}-\frac{j(j+1)}{x^2}\right\}z_{2,1}\,. \tag{163}$$

Equations (162) and (163) constitute a simultaneous system for the determination of the functions $\gamma y_{1,2}$ and $z_{1,2}$. The reader may note that (163) is independent of (162) and contains alone an explicit reference to viscosity. In what follows we shall, therefore, take up its solution first for subsequent insertion in (162). In order to do so, let us substitute

$$z_{1,2}=\frac{\xi_{1,2}}{\sqrt{x}}, \tag{164}$$

which on insertion in (163) transforms the latter in the following pair of simultaneous equations

$$\lambda\xi_1=\frac{\tilde{\mu}^2}{2}\left\{\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}-\frac{\nu^2}{x^2}\right\}\xi_2\,, \tag{165}$$

$$\lambda\xi_2=-\frac{1}{2}\left\{\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}-\frac{\nu^2}{x^2}\right\}\xi_1\,, \tag{166}$$

where we have abbreviated

$$\nu^2=(j+\tfrac{1}{2})^2\,. \tag{167}$$

The fact that the operators in curly brackets on the right-hand sides of Equations (165) and (166) are Besselian leads us to expect the solutions of (165) and (166) to be of the form

$$\xi_{k,l}=A_{k,l}J_\nu(\alpha_{k,l}x)\,, \tag{168}$$

where J_ν denotes a general Bessel function of order $\pm(j+\frac{1}{2})$, and $A_{k,l}$, $\alpha_{k,l}$ are constants not necessarily real, but such as to render the solution real and finite at both ends of the closed interval $0\leq x\leq 1$.

The boundedness of $\xi_{k,l}(0)$ is sufficient to rule out the negative sign of ν; and the compatibility conditions for (168) to represent solutions of (165)–(166) will (on insertion) assume the form of the determinantal equation

$$\begin{vmatrix}\lambda & \frac{1}{2}\alpha_l^2\\ -\frac{1}{2}\alpha_k^2 & \lambda\end{vmatrix}=0\,, \tag{169}$$

leading to the condition

$$\lambda^2 + \left(\frac{\alpha_k \alpha_l}{2} \tilde{\mu}\right)^2 = 0 \tag{170}$$

according to which, for arbitrary α_k,

$$\alpha_l = \pm \frac{i}{\alpha_k} \left(\frac{2\lambda}{\tilde{\mu}}\right) \tag{171}$$

where i stands for the imaginary unit.

Thus the two linearly independent solutions of the system (165)–(166) which remain finite at the origin will assume the forms

$$\xi_{k,l} = A_k J_\nu(\alpha_k x) + A_l J_\nu\left(\pm \frac{i}{\alpha_k} \frac{2\lambda}{\tilde{\mu}} x\right), \tag{172}$$

where α_k remains so far completely arbitrary. As is, however, well known, for any real variable $z > 0$,

$$J_\nu(iz) = i^\nu I_\nu(z) \tag{173}$$

and

$$J_\nu(-iz) = J_\nu(e^{3i\pi/2} z) = (-1)^\nu J_\nu(e^{i\pi/2} z) = i^{3\nu} I_\nu(z), \tag{174}$$

where $I_\nu(z)$ stands for the modified Bessel function of the first kind, the complete primitive (172) can be rewritten in the real form

$$\xi_{k,l}(x) = A_k J_\nu(\alpha x) = A'_l I_\nu\left(\frac{2\lambda}{\alpha\tilde{\mu}} x\right), \tag{175}$$

where $\alpha \equiv \alpha_k$ and $A'_l \equiv A_l i^{3\nu}$ should represent real constants.

Having solved thus Equations (163), let us turn to (162). If $z_{1,2}$ were zero, the complete primitive of the homogeneous equation would be

$$\gamma y_{1,2}(x) = \frac{B_1 x^j + B_2 x^{-j-1}}{1 - x^2}, \tag{176}$$

where $B_{1,2}$ are arbitrary integration constants. Therefore, the complete primitive of the nonhomogeneous Equation (162), obtained by the addition to the right-hand side of (176) of the appropriate particular integral, will in turn assume the form

$$\begin{aligned}(1 - x^2)\gamma y_{1,2} &= B_1 x^j + B_2 x^{-j-1} \\ &+ \frac{2j(j+1)}{2j+1}\left\{x^j \int_0^x x^{-j+1} z_{1,2}(x)\,\mathrm{d}x - x^{-j-1}\int_0^x x^{j+2} z_{1,2}(x)\,\mathrm{d}x\right\}.\end{aligned} \tag{177}$$

In order to keep $y_{1,2}$ finite at the origin, it is obviously necessary that $B_2 = 0$; but B_1 can assume arbitrary value. Since, moreover,

$$\lim_{x \to 0} x^j \int_0^x x^{-j+1} z_{1,2}(x)\, \mathrm{d}x = -\lim_{x \to 0} \frac{x^2}{j} z_{1,2}(x) \tag{178}$$

and

$$\lim_{x \to 0} x^{-j-1} \int_0^x x^{j+2} z_{1,2}(x)\, \mathrm{d}x = \lim_{x \to 0} \frac{x^2 z_{1,2}(x)}{j+1}, \tag{179}$$

the particular integral on the right-hand side of (177) will clearly vanish at the origin provided that $z_{1,2}(0)$ is bounded.

Having thus established the explicit form of the real and imaginary parts of $\gamma\tilde{y}$ and $\tilde{z}$ in (144) and (145) as defined by Equations (162) and (163) of this section, let us proceed to establish the corresponding expressions for the real and imaginary parts of the velocity-components $u_{1,2}(x)$ and $v_{1,2}(x)$ in (142) and (143). Since, in the incompressible case, $y_{1,2}=0$ (though $\gamma y_{1,2}\neq 0$) and – in the presence of viscosity – $z_{1,2}\neq 0$, a combination of Equations (29) and (30) reveals that

$$\frac{\partial^2}{\partial x^2}(x^2 u_{1,2}) - j(j+1)\, u_{1,2} = j(j+1)\, x z_{1,2}; \tag{180}$$

and once this has been solved, $v_{1,2}$ follows from (29) as

$$v_{1,2}(x) = \frac{1}{j(j+1)\, x} \frac{\partial}{\partial x}(x^2 u_{1,2}). \tag{181}$$

The homogeneous part of (180) would possess a complete primitive of the form $a_1 x^{j-1} + a_2 x^{-j-2}$, where $a_{1,2}$ are integration constants; therefore, the complete primitive of the full-dress Equation (180) will be

$$\begin{aligned} u_{1,2}(x) = x^{j-1} &\left\{a_1 + \frac{j(j+1)}{y+1} \int_0^x x^{-j+1} z_{1,2}(x)\, \mathrm{d}x\right\} \\ &+ x^{-j-2}\left\{a_2 - \frac{j(j+1)}{y+1} \int_0^x x^{j+2} z_{1,2}(x)\, \mathrm{d}x\right\}, \end{aligned} \tag{182}$$

which on insertion in (181) yields

$$\begin{aligned} v_{1,2}(x) = \frac{x^{j-1}}{j} &\left\{a_1 + \frac{j(j+1)}{y+1} \int_0^x x^{-j+1} z_{1,2}(x)\, \mathrm{d}x\right\} \\ &- \frac{x^{-j-2}}{j+1}\left\{a_2 - \frac{j(j+1)}{y+1} \int_0^x x^{j+2} z_{1,2}(x)\, \mathrm{d}x\right\}. \end{aligned} \tag{183}$$

The finiteness of $u_{1,2}(x)$ and $v_{1,2}(x)$ at the origin obviously requires that $a_2=0$; but (for $z_{1,2}(x)$ bounded and $j>0$) there is so far no restriction on a_1.

The last remaining step for the completion of our task is to evaluate explicitly the particular integrals on the right-hand sides of Equations (177), (182) and (183), with $z_{1,2}(x)$ as defined previously by Equations (163), with solutions of the form (164) and (175) yielding

$$z_k(x) = A_k x^{-1/2} J_{j+1/2}(\alpha x) \tag{184}$$

and

$$z_l(x) = A_l' x^{-1/2} I_{j+1/2}(\beta x), \tag{185}$$

where we have abbreviated

$$\beta = \frac{2\lambda}{\alpha\tilde{\mu}}. \tag{186}$$

Obviously,

$$\int_0^x x^{j+2} z_k(x)\, \mathrm{d}x = A_k \int_0^x x^{j+3/2} J_{j+1/2}(\alpha x)\, \mathrm{d}x \tag{187}$$

and

$$\int_0^x x^{j+2} z_l(x)\, \mathrm{d}x = A_l' \int_0^x x^{j+3/2} I_{j+1/2}(\beta x)\, \mathrm{d}x. \tag{188}$$

Since, however,

$$x^{j+3/2} J_{j+1/2}(\alpha x) = \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\{x^{j+3/2} J_{j+3/2}(\alpha x)\} \tag{189}$$

and

$$x^{j+3/2} I_{j+1/2}(\beta x) = \frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}x}\{x^{j+3/2} I_{j+3/2}(\beta x)\}, \tag{190}$$

it follows at once that the first pair of the integrals (187) and (188) can be expressed in terms of the respective Bessel functions of the form

$$\int_0^x x^{j+2} z_k(x)\, \mathrm{d}x = (A_k/\alpha)\, x^{j+3/2} J_{j+3/2}(\alpha x), \tag{191}$$

$$\int_0^x x^{j+2} z_l(x)\, \mathrm{d}x = (A'/\beta)\, x^{j+3/2} I_{j+3/2}(\beta x); \tag{192}$$

and, in a similar fashion,

$$\int_0^x x^{-j+1} z_k(x)\, \mathrm{d}x = -(A_k/\alpha)\, x^{-j+1/2} J_{j-1/2}(\alpha x), \tag{193}$$

$$\int_0^x x^{-j+1} z_l(x)\,\mathrm{d}x = -\,(A_l'/\beta)\,x^{-j+1/2} I_{j-1/2}(\beta x). \tag{194}$$

In consequence, on insertion from (191)–(194) to (177) for $k=1$, $l=2$, the latter equation assumes the form

$$(1-x^2)\,\gamma y_1(x) = B_1 x^j - \frac{2j(j+1)\,A_1}{\alpha^2\sqrt{x}}\,J_{j+1/2}(\alpha x), \tag{195}$$

$$(1-x^2)\,\gamma y_2(x) = B_1' x^j - \frac{2j(j+1)\,A_1'}{\beta^2\sqrt{x}}\,I_{j+1/2}(\beta x), \tag{196}$$

if advantage is taken of the identity

$$xZ_{\nu-1}(x) + xZ_{\nu+1}(x) = 2\nu Z_\nu(x), \tag{197}$$

which holds good for any type of Bessel function $Z_\nu(x)$ of argument x. The Equations (182) for $u_{1,2}(x)$ similarly yield

$$u_1(x) = a_1 x^{j-1} - \frac{j(j+1)\,A_1}{\alpha^2\sqrt{x^3}}\,J_{j+1/2}(\alpha x), \tag{198}$$

$$u_2(x) = a_1' x^{j-1} - \frac{j(j+1)\,A_1'}{\beta^2\sqrt{x^3}}\,I_{j+1/2}(\alpha x); \tag{199}$$

while Equations (183) for $v_{1,2}(x)$ reduce to

$$v_1(x) = \frac{a_1}{j}\,x^{j-1} + \frac{A_1}{\alpha^2\sqrt{x^3}}\left\{jJ_{j+1/2}(\alpha x) - \alpha x J_{j-1/2}(\alpha x)\right\}, \tag{200}$$

$$v_2(x) = \frac{a_1'}{j}\,x^{j-1} + \frac{A_1'}{\beta^2\sqrt{x^3}}\left\{jI_{j+1/2}(\beta x) - \beta x I_{j-1/2}(\beta x)\right\}, \tag{201}$$

where we took advantage of the fact that, for a Bessel function $Z_\nu(x)$ of any type,

$$2\,\frac{\mathrm{d}Z_\nu}{\mathrm{d}x} = Z_{\nu-1}(x) - Z_{\nu+1}(x). \tag{202}$$

8. Frequency of Oscillations

Having determined in the preceding section the form of the characteristic amplitudes $u_{1,2}(x)$ and $v_{1,2}(x)$ which describe the gravitational vibrations of incompressible viscous globes, let us turn now to the task of determining the nature of their characteristic frequency spectrum; and to this end we must fall back on the boundary conditions of our problem set forth previously in Section 4.

At the center, the conditions (51) have – in view of (178) and (179) – already been satisfied by setting $a_2=0$ on the right-hand sides of Equations (182) and (183); while, on the boundary of a viscous globe, the conditions (56) and (57) remain yet to be enforced. In order to do so, let us begin by differentiating equations (198) and (199) with respect to x; in doing so and making use of (197) and (202) we easily establish that

$$\frac{\partial u_1}{\partial x} = (j-1)\, a_1 x^{j-2} + \frac{j(j+1)}{\alpha^2\sqrt{x^5}} \{(j+2)\, J_{j+1/2}(\alpha x) - (\alpha x) J_{j-1/2}(\alpha x)\}\, A_1\,, \tag{203}$$

$$\frac{\partial u_2}{\partial x} = (j-1)\, a_1' x^{j-2} + \frac{j(j+1)}{\beta^2\sqrt{x^5}} \{(j+2)\, I_{j+1/2}(\beta x) - (\beta x)\, I_{j-1/2}(\beta x)\}\, A_1'\,; \tag{204}$$

while a similar differentiation of (200) and (201) yields

$$\frac{\partial v_1}{\partial x} = \frac{j-1}{j}\, a_1 x^{j-2} + \frac{A_1}{\alpha^2\sqrt{x^5}} \times \{-\, j(j+2)\, J_{j+1/2}(\alpha x) + (\alpha x)\, J_{j-1/2}(\alpha x) + (\alpha x)^2\, J_{j+1/2}(\alpha x)\} \tag{205}$$

and

$$\frac{\partial v_2}{\partial x} = \frac{j-1}{j}\, a_1' x^{j-2} + \frac{A'}{\beta^2\sqrt{x^5}} \times \{-\, j(j+2)\, I_{j+1/2}(\beta x) + (\beta x)\, I_{j-1/2}(\beta x) + (\beta x)^2\, I_{j+1/2}(\beta x)\}\,. \tag{206}$$

Since, moreover, again by (198)–(201)

$$v_1 - u_1 = \frac{1-j}{j}\, a_1 x^{j-1} + \frac{A_1}{\alpha^2\sqrt{x^3}} \{j(j+2)\, J_{j+1/2}(\alpha x) - (\alpha x)\, J_{j-1/2}(\alpha x)\} \tag{207}$$

and

$$v_2 - u_2 = \frac{1-j}{j}\, a_1' x^{j-1} + \frac{A_1'}{\beta^2\sqrt{x^3}} \{j(j+2)\, I_{j+1/2}(\beta x) - (\beta x)\, I_{j-1/2}(\beta x)\}\,, \tag{208}$$

equation (56) on insertion from (203) and (204) on the surface ($x=1$) assumes the forms

$$\{(j+2)\, J_{j+1/2}(\alpha) - \alpha J_{j-1/2}(\alpha)\}\, A_1 + \frac{(j-1)\,\alpha^2}{j(j+1)}\, a_1 = 0 \tag{209}$$

and

$$\{(j+2)\, I_{j+1/2}(\beta) - \beta I_{j-1/2}(\beta)\}\, A_1' + \frac{(j-1)\,\beta^2}{j(j+1)}\, a_1' = 0; \tag{210}$$

while equation (57) combined with (205)–(208) for $x=1$ similarly yields

$$\{[\tfrac{1}{2}\alpha^2 - j(j+2)]\, J_{j+1/2}(\alpha) + \alpha J_{j-1/2}(\alpha)\}\, A_1 + \frac{(j-1)\,\alpha^2}{j}\, a_1 = 0 \tag{211}$$

and

$$\{[\tfrac{1}{2}\beta^2 - j(j+2)]\, I_{j+1/2}(\beta) + \beta I_{j-1/2}(\beta)\}\, A_1' + \frac{j-1}{j}\, \beta^2 a_1' = 0 \tag{212}$$

The boundary conditions (209) and (211) are homogeneous in a_1 and A_1. If they are to hold good for any arbitrary values of these constants, it is necessary that

$$\begin{vmatrix} (j+2)\, J_{j+1/2}(\alpha) - \alpha J_{j-1/2}(\alpha) & \dfrac{(j-1)\,\alpha^2}{j(j+1)} \\ [\tfrac{1}{2}\alpha^2 - j(j+2)]\, J_{j+1/2}(\alpha) + \alpha J_{j-1/2}(\alpha) & \dfrac{(j-1)\,\alpha^2}{j} \end{vmatrix} = 0, \tag{213}$$

which by repeated use of (197) can be reduced to

$$\alpha J_{j+5/2}(\alpha) + J_{j+3/2}(\alpha) = 0 \tag{214}$$

an equation which specifies the unknown constants α. Moreover, the compatibility of the homogeneous equations (210) and (212) for arbitrary values of a_1' and A_1' requires that

$$\begin{vmatrix} (j+2)\, I_{j+1/2}(\beta) - \beta I_{j-1/2}(\beta) & \dfrac{(j-1)^2}{j(j+1)} \\ [\tfrac{1}{2}\beta^2 - j(j+2)]\, I_{j+1/2}(\beta) + \beta I_{j-1/2}(\beta) & \dfrac{j-1}{j}\, \beta^2 \end{vmatrix} = 0, \tag{215}$$

which similarly reduces to

$$\beta I_{j+5/2}(\beta) + I_{j+3/2}(\beta) = 0. \tag{216}$$

The foregoing Equations (214) and (216) represent the solution of the problem set forth in this section; for once they have been solved for α and β, the corresponding characteristic frequencies follow from (170) as

$$\lambda = \tfrac{1}{2}\bar{\mu}\alpha\beta. \tag{217}$$

It is evident from this equation that these characteristic frequencies λ will be real if both α and β are either real, or purely imaginary. Should only one of them prove real and the other imaginary – so that λ itself proves to be imaginary – the motion would become anharmonic: and the displacements would vary exponentially with the time – increasing or decreasing according as to whether the imaginary part of λ is positive or negative. This latter case would be indicative of damping; the former, of instability. Lastly, if some roots α and (or β) of equations (214) or (216) would prove to be complex, a complex frequency λ would lead to harmonic oscillations with periods determined by the real part of λ, and amplitudes increasing or decreasing exponentially with the time.

In order to investigate the actual nature of the roots of Equations (214) and (216), let us make use of the well-known expansions

$$J_\nu(\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}\alpha)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} \tag{218}$$

and

$$I_\nu(\beta) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}\beta)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} \tag{219}$$

for Bessel functions with the real and imaginary argument.* An insertion of the latter in (216) leads to an algebraic equation of the form

$$(\tfrac{1}{2}\beta)^{j+3/2} \sum_{n=0}^{\infty} \left\{1 + \frac{\frac{1}{2}\beta^2}{j+n+\frac{5}{2}}\right\} \frac{(\frac{1}{2}\beta)^{2n}}{n!\,\Gamma(j+n+\frac{5}{2})} = 0\,, \tag{220}$$

which clearly does not possess any real root different from zero. Moreover, by setting

$$\beta = \pm\, i\alpha'\,, \tag{221}$$

Equation (216) can be rewritten as

$$i^{-\nu}\{\pm\,\alpha' J_{j+5/2}(\mp\,\alpha') + J_{j+3/2}(\mp\,\alpha')\} = 0\,, \tag{222}$$

which is closely analogous to (214).

In point of fact, it transpires therefrom that equation (217) must now be rewritten as

$$\lambda = \tfrac{1}{2} i\tilde{\mu}\alpha\alpha'\,, \tag{223}$$

where the two α's are real roots of the equations

$$\pm\,\alpha J_{j+5/2}(\alpha) + J_{j+3/2}(\alpha) = 0 \tag{224}$$

or, on insertion from (218),

$$(\tfrac{1}{2}\alpha)^{j+3/2} \sum_{n=0}^{\infty} \left\{1 \pm \frac{\frac{1}{2}\alpha^2}{j+n+\frac{5}{2}}\right\} \frac{(-1)^n (\frac{1}{2}\alpha)^{2n}}{n!\,\Gamma(j+n+\frac{5}{2})} = 0\,. \tag{225}$$

Numerical solutions of this equation, carried out by Dr. M. D. Moutsoulas with the aid of the ATLAS electronic computer of the University of Manchester, disclosed that (225) admits – in addition to the trivial roots $\alpha = \alpha' = 0$ – an infinity of discrete roots for every value of j; the first few of which (smaller than 20 in absolute value) are listed in the accompanying tabulation.

* For $\alpha > 1$, it is advantageous to replace (218) by an equivalent expansion in descending powers of α.

TABLE I

Numerical Values of the First Roots of Equations (225)

α		α'
	$j=2$	
		2.8646728
8.0661337		8.3088120
11.6215001		11.7920328
14.9741389		15.1069475
18.2471531		18.3563351
	$j=3$	
		3.1894320
9.2539416		9.4673704
12.8913408		13.0453713
16.2945382		16.4167040
19.6028335		19.7045222
	$j=4$	
		3.4853848
10.4220615		10.6121650
14.1388112		14.2794473
17.5922626		17.7055016
20.9363758		
	$j=5$	
		3.7589004
11.5750363		11.7465710
15.3681567		15.4976772
18.8710689		18.9766964
	$j=6$	
		4.0143329
12.7159254		12.8723069
16.5824525		16.7025836
20.1337959		20.2328453
	$j=7$	
		4.2548015
13.8469003		13.9906682
17.7840077		17.8960915
21.3826487		
	$j=8$	
		4.4827291
14.9695652		15.1026617
18.9746069		19.0797093
	$j=9$	
		4.701135
16.085143		16.209088
20.155662		20.254644

With the aid of the values of α and α' listed in Table I the frequency λ of the corresponding oscillations can be ascertained from Equation (223). The fact that this frequency proves to be *imaginary* (or, at most, zero corresponding to a state of neutral equilibrium) demonstrates the *instability* of our configuration consisting of incompressible viscous matter, which is no more capable of purely harmonic oscillations than if it were compressible though inviscid.

In conclusion of this section we wish to demonstrate the way in which our present results, valid for gravitational displacements of incompressible viscous globes, reduce to Kelvin's oscillations for the limiting case of zero viscosity. In order to obtain the latter in terms of our present analysis, it is sufficient to return to Equations (40) and (41) of Section 3, from which equations (162) and (163) were derived.

In case of incompressibility, a requirement that the divergence Δ of the velocity vector be zero entails, by (31), the necessity that both the real and imaginary part of $\tilde{y}$ in (144) vanish identically. If so, however, it follows from (101) that

$$gf = 0; \tag{226}$$

while from (102) and (142) with (144) we find that

$$a^2h = \tfrac{2}{3}\pi G\rho^2 r_* \{[(1-x^2)(\gamma y_1) - 2xu_1] + i[(1-x^2)(\gamma y_2) - 2u_2]\}, \tag{227}$$

where by insertion from (195)–(196) and (198)–(199) it follows that, regardless of viscosity,

$$(1-x^2)(\gamma y_{1,2}) - 2xu_{1,2} = k_{1,2}x^j, \tag{228}$$

where

$$\left.\begin{aligned} k_1 &= B_1 - 2a_1 \\ k_2 &= B_1' - 2a_1' \end{aligned}\right\} \tag{229}$$

denote real arbitrary constants. This result could indeed have been anticipated directly from (104) for $\tilde{y}=0$, and discloses that the pressure oscillations a^2h as defined by equations (33) for homogeneous incompressible configurations are independent of viscosity. If, moreover, such oscillations are made to vanish on the surface by setting $B_1=2a$, or $B_1'=2a_1'$ in (229), they would be annihilated throughout the configuration.

Furthermore, in the case of constant density and viscosity, Equation (36) and (37) combined with (145) reveal that

$$F = \frac{j(j+1)}{x}\tilde{\mu}\{z_1(x) + i\tilde{\mu}z_2(x)\}, \tag{230}$$

$$G = \tilde{\mu}\frac{\partial}{\partial x}\{xz_1(x) + i\tilde{\mu}xz_2(x)\}; \tag{231}$$

while in order to obtain the appropriate expression for R we must proceed as follows. In the absence of any external forces, let us differentiate Equation (64) with respect to the time, make use of (69) and (72), and insert

$$f = \rho y + u\frac{\partial \rho}{\partial r} \tag{232}$$

from (32); the outcome reveals that, quite generally,

$$R = \frac{4\pi G r_*}{2j+1}\left\{\frac{1}{x^{j+1}}\int_0^x \left(\rho\tilde{y} + u\frac{\partial\rho}{\partial x}\right)x^{j+2}\,\mathrm{d}x + x^j\int_x^1 \left(\rho\tilde{y} + u\frac{\partial\rho}{\partial x}\right)x^{1-j}\,\mathrm{d}x\right\}, \tag{233}$$

which on partial integration yields, for $\rho(1)=0$,

$$R = \frac{4\pi G r_*}{2j+1}\left\{\frac{1}{x^{j+1}}\int_0^x \rho\left[\frac{\partial}{\partial x}(ux^{j+2}) - \tilde{y}x^{j+2}\right]\mathrm{d}x + x^j\int_x^1 \rho\left[\frac{\partial}{\partial x}(ux^{1-j}) - \tilde{y}x^{j+2}\right]\mathrm{d}x\right\}, \tag{234}$$

which in the incompressible case ($\tilde{y}=0$) of constant density ρ reduces at once to

$$R = \frac{4\pi G\rho r_*}{2j+1}\left\{u_1(1) + i\tilde{\mu}u_2(1)\right\} x^j. \tag{235}$$

Let us now proceed to the inviscid case $\tilde{\mu}=0$. If so, the incompressibility condition $\tilde{y}=0$ entails, by (105), the vanishing of $\tilde{z}$ as well; and the only way to render $z_1(x)=z_2(x)=0$ is to set $A_1'=A_1=0$; in which case the constants B_1 and B_1' in (195) and (196) must also vanish, leaving us with

$$a^2 h = -\tfrac{2}{3}\pi G\rho^2 r_* a_1 x^j. \tag{236}$$

Moreover, Equations (198) and (201) will, for $A_1=A_1'=0$, clearly reduce to

$$u_1(x) = a_1 x^{j-1} \quad \text{and} \quad jv_1(x) = a_1 x^{j-1}, \tag{237}$$

respectively; the first of which reduces (236) further to

$$R = \frac{4\pi G\rho r_*}{2j+1} a_1 x^j. \tag{238}$$

Since, moreover, both functions F and G vanish in the inviscid case, an insertion from (226) and (236)–(238) together with (107) in (41) reveals that, in the incompressible inviscid case,

$$a_1 r_* v^2 x^j = 2\pi G\rho r_*\left\{\frac{2}{2j+1} - \frac{1}{3}\right\} a_1 x^j; \tag{239}$$

while a similar insertion in (40) yields

$$ja_1 r_* v^2 x^{j-1} = 2\pi G\rho r_*\left\{\frac{2j}{2j+1} - \frac{j}{3}\right\} a_1 x^{j-1}. \tag{240}$$

Equations (240) merely proves to be a radial derivative of (239); and both are satisfied for any x by

$$\frac{v^2}{2\pi G\rho} = \tilde{v}^2 = \frac{4j(j-1)}{3(2j+1)}, \tag{241}$$

a well-known result deduced first by KELVIN (1863).

Summarizing the results of Sections 5–7 we may observe that the effects of viscosity – like those of compressibility – tend to *de-stabilize* free non-radial oscillations of self-gravitating homogeneous configurations; the latter being able to perform such oscillations only in the incompressible inviscid case.

The reader may, of course, justly remark that the particular cases of our general problem treated in Sections 5–7 are not too closely relevant to the physical problems encountered in close binary systems, the components of which – whatever their detailed structure – will be characterized by generally high degree of central condensation. This is indeed true; and the homogeneous case was treated in the second part of this paper in such detail mainly to illustrate our general procedure on a case which can be solved in close form in terms of known functions. In order to approach more closely the problems actually encountered in real binary systems, we propose to apply the general equations of sections 2–4 to the other extreme case of a mass-point model, as well as to one exhibiting high but finite degree of central condensation; but their more detailed treatment must be postponed for a subsequent communication.

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