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Stability of Tidal Equilibrium

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Summary. It is proved rigorously that a binary system can be in tidal equilibrium only if coplanarity, circularity and corotation have been established. A complete analysis is given of the stability of this equilibrium against general perturbations. Stability occurs only if the orbital angular momentum exceeds the sum of the spin angular momenta of the stars by more than a factor of three. This condition is identical to the criterium for stability against perturbations of corotation only, as given by Counselman (1973).

Key words: binary stars – celestial mechanics – tidal evolution – X-ray binaries

1. Introduction

In detached close binary systems, where both stars are well inside their Roche lobes, tidal evolution is the main mechanism for inducing secular changes in orbital parameters. The details of the tidal evolution are highly dependent on the mechanism of energy dissipation. A review of possible dissipation mechanisms is given by Zahn (1977), showing the complications caused by the forced oscillations of the stars in a variety of eigen modes.

Fortunately a few general results can be obtained, using energy considerations only, without specifying the exact dissipation mechanism. Thus equilibrium states can be found by minimizing the total energy under the constraint of conservation of total angular momentum. General considerations already suggest that only one type of equilibrium is possible (e.g. Darwin, 1879); characterized by coplanarity (the equatorial planes of the two stars coincide with the orbital plane), circularity (of the orbit) and corotation (the rotation periods of the stars are equal to the revolution period). If these conditions are not simultaneously fulfilled, dissipative effects cause the tidal bulges to be misaligned. This produces a torque, resulting in an exchange between orbital and spin angular momenta. In Sect. 2, the multiplicator method of Lagrange is used to prove rigorously that this is the only possible type of equilibrium, and that, if the total angular momentum L exceeds a critical value L_c, there are two such equilibrium states, which degenerate to one if $L=L_c$.

Several authors have analyzed the stability of these equilibrium states against restricted perturbations; preserving coplanarity and circularity (Counselman, 1973), or only coplanarity (van Hamme, 1979). In Sect. 3 an analysis is presented of stability against general perturbations, thereby extending the analysis of the previous authors to all orbital degrees of freedom.

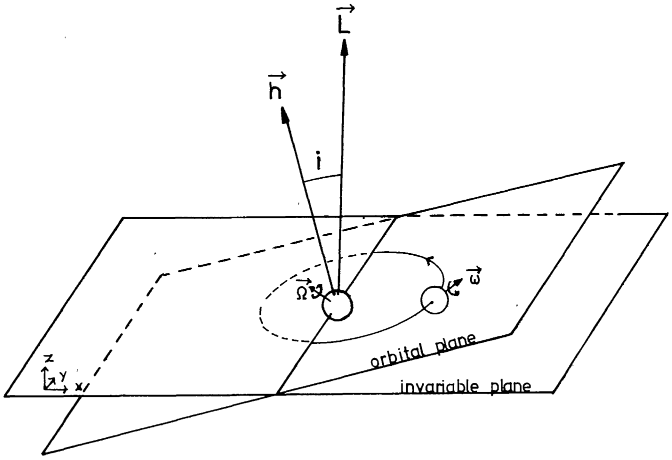

Fig. 1. Dynamical variables relevant for tidal evolution

2. Equilibrium States

The dynamical state of a binary system can be specified by 12 parameters: here we choose to take the 6 classical orbital elements together with the spin vectors of the two stars. To investigate the exchange and dissipation of energy and the exchange of angular momentum only the following three orbital elements are relevant: the semimajor axis a, the eccentricity e, and the inclination i with respect to the invariant plane, which is the angle between the orbital angular momentum $\boldsymbol{h}$ and the total angular momentum $\boldsymbol{L}$. The longitude of the ascending node and the argument of periastron, as well as the time of periastron passage, are of no interest here. The reason is that rotational effects such as spin precession and apsidal motion take place on a much shorter timescale than tidal effects (e.g. Alexander, 1973); making it very plausible that they can be averaged over before considering tidal evolution.

Thus there remain 9 relevant parameters, *viz.*: a, e, i and the angular velocities of rotation $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$. It is convenient to choose the z-axis along $\boldsymbol{L}=(o, o, L)$ and the x-axis such that $\boldsymbol{h}=(h \sin i, o, h \cos i)$ is in the (x, z) plane (Fig. 1). The dynamical evolution of the binary system is restricted to a six-dimensional subspace determined by the constraint of conserved total angular momentum

$$\boldsymbol{L}(a, e, i, \boldsymbol{\Omega}, \boldsymbol{\omega})=\boldsymbol{h}+I_1\boldsymbol{\Omega}+I_2\boldsymbol{\omega} \tag{2.1}$$

where

$$h^2=G\frac{M^2m^2}{M+m}a(1-e^2). \tag{2.2}$$

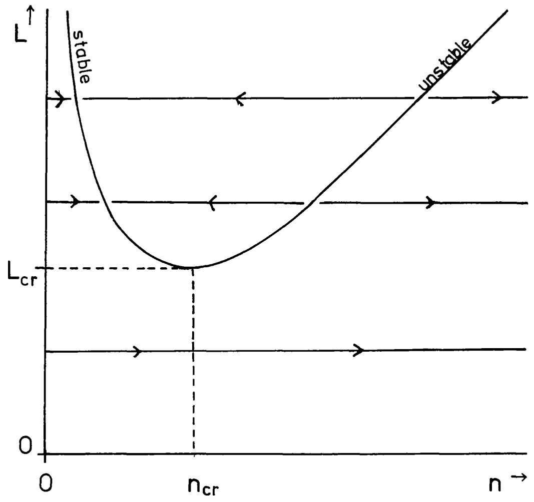

Fig. 2. The curve depicts the L-values for which tidal equilibrium is possible, given by Eq. (2.8). Arrows indicate the direction of decreasing total energy for the restricted case of a circular coplanar orbit

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G is the gravitational constant, and I_1, I_2 are the moments of inertia of the stars with masses M and m, respectively.

To find the stationary points of the total energy E in this six-dimensional subspace we apply the multiplicator method of Lagrange, i.e.:

$$E=-G\frac{Mm}{2a}+\frac{1}{2}I_1|\boldsymbol{\Omega}|^2+\frac{1}{2}I_2|\boldsymbol{\omega}|^2 \tag{2.3}$$

$$\frac{\partial}{\partial x_i}E+\boldsymbol{\lambda}\cdot\frac{\partial}{\partial x_i}\boldsymbol{L}=0 \tag{2.4}$$

where $x_i(i=1\ldots9)$ is any of the nine parameters a, e, i, $\boldsymbol{\Omega}$, $\boldsymbol{\omega}$, and $\boldsymbol{\lambda}$ is a constant vector (the multiplicator). Equations (2.1–4) constitute 13 equations for the set of unknowns (a_0, e_0, i_0, $\boldsymbol{\Omega}_0$, $\boldsymbol{\omega}_0$) at the equilibrium configuration, the corresponding energy E_0 and the multiplicator $\boldsymbol{\lambda}$. [Equation (2.4) can be visualized as demanding grad E to be perpendicular to the subspace of constant $\boldsymbol{L}$.]

It is convenient to introduce units such that $M=I_1=G=1$ (the units of length, time and mass then become $I_1^{1/2}M^{-1/2}$; $G^{-1/2}M^{-5/4}I_1^{3/4}$; and M respectively).

Equations (2.1–2.3) now read

$$E=-\frac{m}{2a}+\frac{1}{2}(\Omega_x^2+\Omega_y^2+\Omega_z^2)+\frac{1}{2}\gamma(\omega_x^2+\omega_y^2+\omega_z^2) \tag{2.5a}$$

$$\begin{aligned} &L=h\cos i+\Omega_z+\gamma\omega_z\\ &0=h\sin i+\Omega_x+\gamma\omega_x=\Omega_y+\gamma\omega_y. \end{aligned} \tag{2.5b}$$

The variational Eq. (2.4) become

$$\begin{aligned} &\frac{m}{a}+\lambda_x h\sin i+\lambda_z h\cos i=0\\ &\frac{e}{1-e^2}h\{\lambda_x\sin i+\lambda_z\cos i\}=0\\ &\lambda_x\cos i-\lambda_z\sin i=0\\ &\Omega_i+\lambda_i=\omega_i+\lambda_i=0\ (i=x,y,z) \end{aligned} \tag{2.5c}$$

where h is given by Eq. (2.2) and m and γ are the ratios of the masses and moments of inertia of the two stars, respectively.

Solution of Eqs. (2.5a–c) yields

$$\begin{aligned} &\Omega_z=\omega_z=n\\ &e=i=\Omega_x=\Omega_y=\omega_x=\omega_y=0 \end{aligned} \tag{2.6}$$

where n is the orbital angular velocity, given by Kepler's third law:

$$n^2=\frac{1+m}{a^3}. \tag{2.7}$$

Equation (2.6) shows that all equilibrium states are characterized by coplanarity ($i=\Omega_x=\Omega_y=\omega_x=\omega_y=0$), circularity ($e=0$) and corotation ($\Omega_z=\omega_z=n$).

The number of equilibrium states depends on the amount of total angular momentum L. The condition for corotation follows from Eqs. (2.1, 2.7) to be

$$L=\frac{m}{(1+m)^{1/3}}n^{-1/3}+(1+\gamma)n. \tag{2.8}$$

If the total angular momentum is smaller than the critical value

$$L_{cr}=4\left\{\frac{1}{27}(1+\gamma)\frac{m^3}{1+m}\right\}^{1/4} \tag{2.9}$$

Eq. (2.8) has no solutions for positive n, and therefore no equilibrium state exists.

For $L>L_{cr}$ Eq. (2.8) has two solutions which degenerate into one at $L=L_{cr}$ (see Fig. 2). At $L=L_{cr}$ one has

$$\begin{aligned} &h=3(1+\gamma)n_{cr}\\ &\Omega+\gamma\omega=(1+\gamma)n_{cr} \end{aligned} \tag{2.10}$$

which implies that the orbital angular momentum constitutes three quarters of the total angular momentum.

For a further discussion we refer the reader to the article by Counselman (1973), where evolution towards (or away from) corotation is investigated under the restrictions of coplanarity and circularity.

3. Stability

In the previous section the equilibrium states of a binary system were found as the stationary points of the total energy under the constraint of conserved total angular momentum. To investigate their stability, we have to go one step further. For an equilibrium to be stable, the total energy has to reach a local minimum. Thus we have to compute the second order variations of the energy at the equilibrium points.

Using conservation of total angular momentum, $\boldsymbol{\Omega}$ can be eliminated by (2.1) leaving a, e, i, $\boldsymbol{\omega}$ as local coordinates in the six-dimensional subspace in which the system can evolve. The total energy can then be expressed as

$$\begin{aligned} E=&-\frac{m}{2a}+\frac{1}{2}\gamma|\omega|^2+\frac{1}{2}(L-h\cos i-\gamma\omega_z)^2\\ &+\frac{1}{2}(h\sin i+\gamma\omega_x)^2+\frac{1}{2}\gamma^2\omega_y^2 \end{aligned} \tag{3.1}$$

in which the same dimensionless notation is used as in Sect. 2. The second order variation of E is given by the Hessian, defined as $\frac{\partial^2 E}{\partial x_i\partial x_j}$, where $x_i(i=1\ldots.6)$ are local coordinates.

Denoting $x_1=a$, $x_2=e$, $x_3=i$, $x_4=\omega_x$, $x_5=\omega_y$, $x_6=\omega_z$, the Hessian takes the following form at an equilibrium configuration:

$$\frac{\partial^2 E}{\partial x_i \partial x_j}=\begin{pmatrix} \frac{m}{4a^3}(\alpha-3) & 0 & 0 & 0 & 0 & \frac{\gamma}{2\sqrt{a}}\frac{m}{(1+m)^{1/2}} \\ 0 & \frac{m}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m}{a}(1+\alpha) & \gamma\sqrt{a}\frac{m}{(1+m)^{1/2}} & 0 & 0 \\ 0 & 0 & \gamma\sqrt{a}\frac{m}{(1+m)^{1/2}} & \gamma(1+\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma(1+\gamma) & 0 \\ \frac{\gamma}{2\sqrt{a}}\frac{m}{(1+m)^{1/2}} & 0 & 0 & 0 & 0 & \gamma(1+\gamma) \end{pmatrix} \tag{3.2}$$

where

$$\alpha=\frac{m}{1+m}a^2$$

is the ratio of the orbital moment of inertia, and the moment of inertia of the first star (keeping in mind that $I_1 \equiv 1$).

The total energy reaches a local minimum if all eigenvalues of the Hessian are positive at the equilibrium configuration (as can be illustrated by a Taylor's expansion around this point). Fortunately, the Hessian has a simple block structure: Only the pairs of coordinates (a, ω_z) and (i, ω_x) are connected by off diagonal elements; while e and ω_y are isolated. Since $\frac{\partial^2 E}{\partial e^2}=\frac{m}{a}>0$ and $\frac{\partial^2 E}{\partial \omega_y^2}=\gamma(1+\gamma)>0$, stability occurs if the following two equations have only positive eigenvalues λ and $\tilde{\lambda}$:

$$\begin{vmatrix} \frac{m}{4a^3}(\alpha-3)-\lambda & \frac{\gamma}{2\sqrt{a}}\frac{m}{(1+m)^{1/2}} \\ \frac{\gamma}{2\sqrt{a}}\frac{m}{(1+m)^{1/2}} & \gamma(1+\gamma)-\lambda \end{vmatrix}=0 \tag{3.3}$$

for the (a, ω_z) coordinates, and

$$\begin{vmatrix} \frac{m}{a}(1+\alpha)-\tilde{\lambda} & \gamma\sqrt{a}\frac{m}{(1+m)^{1/2}} \\ \gamma\sqrt{a}\frac{m}{(1+m)^{1/2}} & \gamma(1+\gamma)-\tilde{\lambda} \end{vmatrix}=0 \tag{3.4}$$

for the (i, ω_x) coordinates.

The solutions of the first eigenvalue equation are

$$\lambda_\pm=\frac{1}{2}\left\{\frac{m}{4a^3}(\alpha-3)+\gamma(1+\gamma)\right\}+ \pm\frac{1}{2}\left[\left\{\frac{m}{4a^3}(\alpha-3)+\gamma(1+\gamma)\right\}^2-\frac{m}{a^3}\gamma\{\alpha-3(1+\gamma)\}\right]^{1/2} \tag{3.5}$$

Both roots are positive only for $\alpha>3(1+\gamma)$. The second equation has only positive roots:

$$\tilde{\lambda}_\pm=\frac{1}{2}\left\{\frac{m}{a}(1+\alpha)+\gamma(1+\gamma)\right\}+ \pm\frac{1}{2}\left[\left\{\frac{m}{a}(1+\alpha)+\gamma(1+\gamma)\right\}^2-4\frac{m}{a}\gamma\{\alpha+1+\gamma\}\right]^{1/2}. \tag{3.6}$$

Thus three cases are possible, i.e.: for $\alpha>3(1+\gamma)$ all eigenvalues of the Hessian are positive; for $\alpha=3(1+\gamma)$ one eigenvalue is zero; whereas for $\alpha<3(1+\gamma)$ one negative eigenvalue occurs. In the degenerate case $\alpha=3(1+\gamma)$, the energy is not minimized, as can be seen explicitly from the third order derivatives of the energy. Thus stability only occurs if $\alpha>3(1+\gamma)$. For a given $L>L_{cr}$, the equilibrium configuration with the wider orbit (smaller n) is stable, the other one is unstable (cf. Fig. 2). This follows directly from Eqs. (2.8, 9) since at the critical point $(L=L_{cr})$, $\alpha=3(1+\gamma)$. Figure 1 illustrates this for the limited case in which coplanarity and circularization are always realized, and can be generalized to include more dimensions.

4. Conclusion

Without making any assumptions about the specific mechanism of tidal dissipation, still some important general results can be obtained:

1. Equilibrium with respect to tidal evolution is possible only if the total angular momentum L equals or exceeds a critical value

$$L_{cr}=4\left\{\frac{1}{27}G\frac{M^3m^3}{M+m}(I_1+I_2)\right\}^{1/4}. \tag{4.1}$$

For $L>L_{cr}$ two equilibrium states exist, involving coplanarity, circularity and corotation. For $L=L_{cr}$ the two solutions degenerate.

2. An equilibrium is stable only if the orbital angular momentum h exceeds a critical value

$$h_{cr}=3(I_1+I_2)n \tag{4.2}$$

where n is the angular velocity of the synchronous rotation and revolution. Thus more than three quarters of the total angular momentum must be in the form of orbital angular momentum. For $h \leqq h_{cr}$ the equilibrium is unstable. As is illustrated in Fig. 2, for $L>L_{cr}$ always one stable and one unstable equilibrium exists.

The stability analysis presented here includes all dominant rotational and orbital parameters, thereby generalizing the result of previous authors (Counselman, 1973; v. Hamme, 1979). It is possible to include also higher order terms in the energy Eq. (2.3), taking into account the energy deviations caused by the tidal bulges. This would include higher order terms in the expansion of the potential energy, and slight variations in the moments of inertia. However, these corrections vary only slowly in the neighbourhood of the equilibrium states. Therefore they will not invalidate the present stability analysis, unless the total angular momentum is exceedingly close to the critical value (2.9), where the equilibrium is already unstable. For all other values of the total angular momentum the present analysis is conclusive.

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