CLOSE BINARY SYSTEMS

THE INTERNATIONAL ASTROPHYSICS SERIES

VOLUME FIVE

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Close Binary Systems

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TO

VINCENT NECHVÍLE

Teacher and Friend

Editors' Note

The aim of the International Astrophysics Series is to provide a collection of authoritative volumes dealing with the main branches of Astrophysics and Radio-Astronomy. The need for such a series of books has arisen because of the great developments which have taken place in these fields of work during recent years.

The books will be suitable for both specialists and students. Some of the titles may have a wider and more popular appeal, but this will be secondary to their main purpose, which is to assist in the teaching of Astrophysics and Radio-Astronomy and in the advancement of these subjects themselves.

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Close Binary Systems

CHAPTER 1

Introduction

THE AIM of this volume will be to provide a general account of our knowledge of the theory of close binary systems, including such exposition of modern methods for the analysis of light changes of eclipsing variables as may be needed by students of the subject for independent work; and also to present some conclusions concerning the general physical characteristics and evolutionary trends in such binaries in so far as they have transpired from the available observed data.

The term *binary star* was apparently first used by Sir William Herschel in 1802, to designate . . . 'a real double star—the union of two stars that are formed together in one system, by the laws of attraction' [1]. The term *double star* is, on the other hand, of much earlier origin: at least its Greek equivalent was already used by Ptolemy to describe the appearance of ν Sagittarii, two fifth-magnitude stars whose angular separation is about 14′ (i.e., a little less than the apparent radius of the Moon); and it has been used ever since to describe close pairs of stars resolvable with the aid of a telescope. Not every 'double star' defined in this sense constitutes, to be sure, a 'binary system'; for a large majority of them may be optical pairs owing the accidental proximity of their projections on the celestial sphere to the laws of chance. The first double star which we know to form a binary system was ζ Ursae Maioris (Mizar), which was discovered around 1650 by Father Jean Baptista Riccioli in Bologna (and whose principal component was recognized by E. C. Pickering as the first spectroscopic binary in 1889). In 1656, Christiaan Huyghens saw θ Orionis resolved into the principal stars of the Trapezium; and in 1664 Robert Hooke noted that γ Arietis consisted of two stars. At least two additional pairs (one of which proved to be of more than ordinary interest) were discovered before the end of the seventeenth century—namely, α Crucis, discovered in 1685 by Father Fontenay, Jesuit missionary at the Cape of Good Hope; and α Centauri, discovered by his confrère, Father Richaud, while observing a comet at Pondicherry, India, in December 1689.

These discoveries were all accidental, and made in the course of observations taken for other purposes. No suspicion seems to have been entertained by all these observers or their contemporaries that the proximity of the two stars in such pairs was due to other reason than chance projection; but although they were therefore regarded as mere curiosities and no special effort made to increase their number, it grew from decade to decade until around 1750, several dozen of such pairs had been noted and recorded. As their number gradually became sufficiently large to lend itself to rudimentary statistical analysis, the cosmic significance of double stars appeared to conceal more than met the eye of their discoverers.

The first scientific argument in favour of the view that at least some, and probably many, double stars then known were the result of physical rather than optical association we owe to another faithful servant of Christ, Rev. John Michell. On May 7th and 14th of the year 1767, Michell read a remarkable paper before the Royal Society in London, published subsequently in the Society's *Transactions* [2], in which he pointed out that the frequency-distribution of the angular separations of double stars known in his time deviated grossly from one that could be expected for chance association of stars uniformly distributed in space—there appeared to be far too many close pairs among them—and according to Michell, 'the natural conclusion from hence is, that it is highly probable, and next to a certainty in general, that such double stars as appear to consist of two or more stars placed very near together, do really consist of stars placed nearly together, and under the influence of some general law . . . to whatever cause this may be owing, whether to their mutual gravitation, or to some other law or appointment of the Creator' [3].

The directness of Michell's expression left perhaps something to be desired, but the logic of his argument was unimpeachable and appears convincing to us to-day. Unfortunately, Michell's contemporaries did not see it in quite the same light. Consider, for example, the reaction of Michell's younger contemporary William Herschel. In a paper entitled 'On the Parallaxes of Fixed Stars' [4], Herschel is firmly convinced in 1781 that the components of double stars which are very unequal in brightness must be at very different distances from us and, therefore, particularly suitable for measurements of the relative parallax of their brighter (i.e., nearer) components. This prompted him to embark upon a systematic search for such pairs, and the results of his efforts were gathered in the *First* and *Second Catalogues of Double Stars* published in 1782 and 1784 [5]. Michell was, to be sure, quick to point out that Herschel's new discoveries greatly strengthened his earlier probabilistic argument [6], but Herschel still remained unimpressed. It was not till in 1803 that the ageing astronomer admitted, in a paper entitled 'Account of the Changes that have happened during the last Twenty-five Years, in the relative Situation of Double-stars; with an Investigation of the Cause to which they are owing' [7] that certain double stars observed by him must indeed be true binary systems. He did so, however, in terms which sound almost grudging for a person of Herschel's enthusiasm and temperament [8], and disclose that the weight of Michell's argument went clearly over his head.

Since Herschel's time, visual (and later photographic) double-star astronomy has made great strides and has been enriched by the discovery of thousands of new pairs. The *New General Catalogue of Double Stars* within 120° of the north pole, published by the late R. G. Aitken in 1932 [9], contains no less than 17,180 individual entries; and those binaries for which orbital elements as well as trigonometric parallax could be determined have become an important source of our knowledge of the masses of the stars. Except for this, however, wide pairs of stars actually resolvable through a telescope are individually of but limited interest for the astrophysicist; for apart from their mutual attraction which induces them to revolve in closed orbits around their common centre of gravity, the components of visual binary systems do not influence each other in any way and behave as mass-points—or single stars in space. If only the components could be moved sufficiently close together for their mutual interaction to make itself felt, it would at once be a very different and much more exciting story. In order to make this interaction appreciable, the distance between the components would, however, have to become comparable (or at least not too large) with their actual stellar dimensions. Such binaries will hereafter be referred to as *close*, and their study will constitute the main object of this book.

It goes without saying that binaries as close as those envisaged in the foregoing paragraph would not stand the ghost of a chance of optical resolution by conventional telescopes at a distance separating us from even the nearest stars; though other methods of identification come readily to our aid. Thus with increasing proximity of the components their velocity of orbital motion is bound to increase, and may become sufficient for its radial part to produce measurable Doppler shifts in the line spectra of the system. Although hundreds of spectroscopic binaries have been discovered in this way since 1889, and their observations have provided invaluable data for the study of such stars [10], the existence of close binary systems in the sky was first revealed to astronomers in another way—long before the advent of spectrographic techniques—and the story of it is of sufficient interest to be told in some detail.

The beginning of this story may take us back to the days of nomadic life in the ancient Near East [11] and, in particular, to the Arabian peninsula whose inhabitants, fond of giving each bright star a proper name, bestowed upon one of them—it happens to be the second brightest star in the constellation of Perseus—the name *Al Ghūl*, meaning literally 'changing spirit' [12]. Whether or not any specific significance should be attached to this name we do not know; the explanation, if any, may have perished in the flames of the Library of Alexandria. Nothing at least was heard of Al Ghūl—or Algol, as it is more commonly called—for more than ten centuries separating modern times from ancient civilization. It was not until toward the end of the seventeenth century, around 1670, that this star attracted again specific attention.

The Italian astronomer Montanari, then in Bologna, appears to be the first man on record to have noticed that the apparent brightness of Algol occasionally dropped appreciably below normal, and became so impressed by this curious phenomenon that he wrote a special pamphlet about it [13]. Twenty-five years later the variability of Algol was confirmed by Maraldi [14]; but neither he nor Montanari seem to have had any inkling of the *periodicity* of the phenomenon they happened to discover. Their observations passed, moreover, almost unnoticed; for Algol was not destined to make history until nearly a hundred years later.

The discovery of the periodicity of Algol's light changes was made far from sunny Italy, in the cloudy climate of northern England; and the following brief letter written by Sir Joseph Banks, then President of the Royal Society, to young William Herschel will introduce to us the discoverer:

> May 3, 1783.
>
> Dear Sir,
>
> 'I learnt at the Royal Society that the periodical occultation of the light of Algol happened last night at about 12 o'clock; the period is said to be 2 days 21 hours, and the discovery is now said to have been made by a deaf and dumb man, the grandson of Sir John Goodricke, who has for some years amused himself with astronomy. This is all I have yet made out.' [15]

In point of fact, John Goodricke, Junior, of York—then 18 years of age*—made the first observations of a minimum of Algol on November 12th, 1782, and with his friend Edward Pigott† assured himself shortly thereafter of the periodicity of the phenomenon. The two young men continued, however, their observations through the rest of the season, and it was not until May 12th, 1783, that Goodricke communicated (through the good offices of Rev. Anthony Shepherd, then Plumian Professor of Astronomy at Cambridge) the results to the Royal Society, in the form of a letter containing also some speculations as to the nature of the phenomenon.

This communication was read before the Society on May 15th, and created at once a considerable interest and excitement in astronomical circles‡—which prompted the Council of the Society to award to its youthful author the Copley medal for 1783. And well did Goodricke deserve it; for not only did he discover the first short-period variable ever known (all variables known in his day were long-periodic or irregular in nature) and establish a remarkably close estimate of its period;§ but at the end of his communication we find the following sentence which had truly made it historic: 'if it were not perhaps too early to hazard even a conjecture on the cause of this variation, I should imagine it could hardly be accounted for otherwise than . . . by the interposition of a large body revolving around Algol . . .' 'But' Goodricke went on, 'the intention of this paper is to communicate facts, not conjectures; and I flatter myself that the former are remarkable enough to deserve the attention and farther investigation of astronomers.' [16]

Nature had denied much to young Goodricke, but certainly not the gift of a splendid imagination. For it happens seldom indeed in the annals of science that the first conjecture of a discoverer strikes the nail on the head

* Born September 17th, 1764 at Groningen, as a deaf-mute grandson and heir of Sir John Goodricke of Ribston Hall, Yorkshire, and died on April 20th, 1786, at York.

† The future discoverer of the variability of η Aquilae.

‡ For an interesting echo of it as reflected in a correspondence between Rev. S. Vince and Wm. Sewell in 1791 *cf.* O. J. Eggen, *Observatory*, **77**, 191, 1957.

§ Goodricke's original value for Algol's period was 2 days, 20 hours, and 45 minutes, differing from its true period by only 4 minutes. A year later Goodricke revised his period to 2 days, 20 hours, 49 minutes and 9 seconds—a result on which modern observations had little to improve!

more accurately than this suggestion of a young man of eighteen, who within the short life allotted to him found time enough to discover, besides Algol, also the variability of β Lyrae and δ Cephei, and to be admitted to fellowship of the Royal Society—two weeks before his untimely death at the age of twenty-two.

Goodricke's bold suggestion that Algol was an *eclipsing binary* was, moreover, made too early to gain speedy acceptance. At least William Herschel, who at the request of Sir Joseph Banks had apparently acted as a referee of Goodricke's paper, was again plainly non-committal. 'The idea of a small Sun revolving around a large opake body has also been mentioned in the list of such conjectures' [17]. He had there were reasons for doubt, he thought; for prior to his report he had observed Algol repeatedly in the focus of his 7-foot telescope (of the discovery of Uranus fame) and found it 'distinctly single'. Twenty years later it became Herschel's destiny to demonstrate that many visual double stars observed by him form indeed physical binary systems conforming to his definition [7]; but whether or not the ageing astronomer ever made up his mind about Algol we do not know. It looks very much as if the uninhibited boldness which was so characteristic of most of his other astronomical speculations suddenly failed when confronted with this particular kind of stellar symbiosis—there is not a shred of evidence in all his prolific writings to suggest a rational cause of his apparent reluctance to accept its reality. At any rate, Goodricke's brilliant suggestion of 1783 was destined to remain in the realm of hypotheses for many more decades. It was not until 1889 when Vogel [18] recognized Algol as a spectroscopic binary whose conjunctions coincide with the minima of light that the binary nature of Algol and other similar eclipsing variables was at last established beyond any doubt. And it may be edifying for the reader—especially if he is a theoretician—to reflect on the historical fact that while the first representatives of wide (i.e., visual) as well as close (i.e., eclipsing) binary systems were discovered by Riccioli (1650) and Montanari (1670) who both happened to observe in Bologna—the site of the oldest of all Western Universities—under the clear Italian skies, the binary nature of each group was recognized by pure reasoning by two British amateurs—Michell (1767) and Goodricke (1783)—long before the observers like Herschel (1803) or Vogel (1889) provided the compelling observational proof.

Thus Algol became the first known eclipsing variable, with β Lyrae following shortly thereafter.* Further discoveries came slowly at first, but later in ever-increasing numbers—and never perhaps more rapidly than in our own lifetime. Thus Schneller's *Katalog und Ephemeriden der Veränderlichen Sterne* for 1940 [19] contained 1087 variables classified as eclipsing binaries for which at least the period and range of variation have been established, while Kukarkin and Parenago's *Obschij Katalog Peremennych Zvjozd* [20], published in 1948, listed over two thousand of such objects—and to-day, after the lapse of another decade, the number of known eclipsing

* In September 1784, discovered also by Goodricke.

variables appears to be between three and four thousands. These figures represent, moreover, little except a measure of tribute to the zeal of variable-star hunters in particular fields of the sky; for the real number of eclipsing systems in our own galaxy alone is probably quite beyond hope of individual discovery.

The percentage of double or multiple stars in the vicinity of the solar system is very high—with conservative estimates ranging from 30 to 50% of the total population [21]. To be sure, not all these binaries—or even a majority of them—are close binary systems [22]. But at least seven eclipsing variables are known within thirty parsecs from the Sun;* and as this volume contains some 3000 stars, eclipsing variables would seem about 0·2% of the total population; and the total number of close binaries for all values of orbital inclinations may be in the neighbourhood of 1%. If a similar ratio holds, however, for our galactic system as a whole, the total number of close binaries in it should be of the order of 10^9. Close binaries are, therefore, manifestly no exceptional or uncommon phenomena!

The significance of close binary systems is underlined by the fact that they have been found (with slight reservations) to occur among all types of stellar populations in all inhabited regions of the HR-diagram. Close binaries abound all along the Main Sequence, attaining conspicuous peaks in certain regions. Thus, according to Plaskett and Pearce [23], at least one-third of the O- and early B-type stars appear to be spectroscopic binaries; and since, according to all indications, their orbital planes are orientated at random in space, a considerable fraction of them is bound to exhibit eclipses. Among stars of later spectral types the frequency of close binaries appears to fall off rapidly—there are less than 5% of spectroscopic binaries among stars of the A-type [24]—until another high proportion of binaries is found among late F- and early G-stars (W UMa-type) [25]. Close binaries are known consisting of the pairs of giants (RX, SX Cas) or supergiants of late spectra (W Cru) as well as early spectral types (β Lyr), or of a symbiosis of a late-type supergiant with an early-type Main Sequence star (ζ Aur, VV Cep). Main-sequence stars are frequently found to be paired with subgiants (Algol-type stars); and close (eclipsing) binary systems have been discovered which consist of a pair of sub-dwarfs (UX UMa)—white dwarfs representing the only group of stellar populations among which no close binary has so far been detected;† though observational selection operates so strongly against the discovery of such pairs that their absence from known lists of close binaries may be wholly due to this cause [26].

But the variety of objects encountered among close binaries is thereby by no means exhausted. Most stars with composite spectra are now regarded

* β Aur, i Boo, R CMa, YY Gem (Castor C), VW Cep, α CrB, and β Per; to which we should probably add now also δ Cap (*cf.* O. J. Eggen, *P.A.S.P.*, **68**, 541, 1956), and HD 16157 (D. S. Evans, *Observatory*, **77**, 74, 1957).

† Unless the high-frequency (70-second) oscillation in brightness reported by M. Walker for DQ Her (*cf. Ap. J.*, **123**, 68, 1955) may be due to the fact that one component of this system consists of a pair of mutually-eclipsing white dwarfs.

as close binaries [27]; and it also appears that the majority—if not all—stars of the Wolf-Rayet type may constitute binaries [28]; though only some of them may be eclipsing like V444 Cyg, CQ Cep or CV Ser. On the heels of a realization that T CrB (Nova CrB 1866 and 1946) is probably a spectroscopic binary, there came Walker's remarkable discovery that DQ Her (Nova Her 1934) is an eclipsing variable consisting of a pair of subdwarfs which revolve around the common centre of gravity in a period of 4 hours and 39 minutes [29]; and Joy's discovery that SS Cyg, a prototype of subdwarf explosive variables, is a spectroscopic (though apparently not eclipsing [30]) binary as well. How many other post-Novae or explosive variables of SS Cyg or U Gem-type may prove to be close (and possibly eclipsing) binaries remains yet to be seen; but if we consider the effects of observational selection hampering their discovery, the possibility that many—and perhaps the majority—of such objects are close binary systems cannot be ruled out.

The significance of eclipsing variables among close binary systems is further emphasized by the fact that they represent the only kind of double stars which can so far be detected at great distances. In the neighbourhood of the solar system, up to distances of the order of a hundred parsecs, binary systems can be recognized either by visible orbital motion or (for wide pairs) by common proper motion of their components. At distances greater than a few hundred parsecs both types of motion become, in general, too small for a reliable recognition—so that no 'visual' or 'astrometric' binary can at present be detected beyond this limit. Spectroscopic binaries can be found, with the aid of large modern reflectors, up to distances of the order of a few thousand parsecs; but beyond this limit double stars can be detected if, and only if, they happen to be eclipsing variables.

A search for eclipsing variables in distant systems has led indeed to some thought-provoking results. It indicates, for instance, that eclipsing (i.e., close) binaries avoid, in general, regions in space in which the density of stars is high. They occur but rarely in galactic clusters,* and seem conspicuously absent from globular clusters [31]; their percentage in central parts of the galaxy is likewise well below the average [32]. And it should be pointed out that few eclipsing variables have so far been detected in neighbouring external galaxies. Approximately 50 such variables have recently been found in the large Magellanic Cloud by Shapley and McKibben [33]; while, still more recently, Baade and Swope reported the first crop of eclipsing variables in the Andromeda nebula (M 31) discovered on the 48-inch Palomar Schmidt plates [34]. The fact that their discovery had to await the largest telescopes now available indicates that the brighter eclipsing variables in M 31 must be of considerably fainter absolute magnitudes than the brightest eclipsing systems in our own galaxy. If systems like β Lyrae or Y Cygni, or a host of other

* Few visual binaries exhibiting a slow relative motion are known to be physical members of the Pleiades and Hyades. One eclipsing binary (TX Cnc) was discovered in Praesepe; possibly one (No. 1021) of Oosterhoff's catalogue in *Leiden Ann.*, **17**, No. 1, 1937 in χ and h Persei; while the membership of SZ Cam in N.G.C. 1502 is likewise still conjectural.

well-known O- and B-type galactic binaries belonged to the Andromeda or Triangulum nebula instead of our own Milky Way, they would have been well above the resolving limit of the 100-inch telescope thirty years ago; and yet no β Lyrae or Y Cygni—which shine like beacons through a major part of our galaxy—has ever been found beyond the boundaries of our Milky Way. Eclipsing binaries are conspicuously frequent among the absolutely brightest stars of our galactic system, apparently more so than in the neighbouring galaxies that are rich in high-luminosity stars.

Were this to be true, any progress of astronomy in these galaxies would be hampered considerably; for in our own galactic system, spectroscopic binaries which by virtue of their orbital inclination happen also to be eclipsing variables have been our principal source of information concerning the masses and absolute dimensions of the individual stars. A certain amount of information concerning stellar masses can be obtained, to be sure, also from accurate observations of visual binary systems which are near enough to reveal absolute orbital motions as well as the parallax. The number of such systems is, however, severely limited by their proximity to us in space, and the existing supply is not copious; moreover, it is limited to systems of moderate or small masses [35]. Eclipsing binaries, on the other hand, offer an inexhaustible source of most diverse observational material, of which only a minute fraction has so far been surveyed. Even that has provided the bulk of the data at the basis of all empirical mass-luminosity relations; while if also the parallax of such systems is known, the observed data lend themselves for a determination of effective temperature of the individual stars. The masses and absolute dimensions of the components of eclipsing binary systems, obtainable by a combination of their photometric and spectroscopic elements, have provided the basis for studies of the chemical composition of stellar interiors.

The astrophysical data which can be deduced from a study of eclipsing binary systems transcend, however, mere information concerning the dimensions or surface characteristics of the components, or the geometry of their orbits; for even their internal constitution is not wholly concealed from us. An insight into it is provided by the gravitational field emanating from invisible interiors, which the opaque outer layers cannot appreciably modify; and the radiant energy originating in deep interiors of distorted stars will experience obstacles in negotiating its way outwards which will depend on the direction, and cause the distribution of brightness over the visible surface to be non-uniform. As long as a star is single we have, of course, no way of gauging its external gravitational field, or learning anything about the distribution of its surface brightness. Place, however, another star in its proximity; and the properties of their combined gravitational field can be deduced from the characteristics of their motion (apsidal advance) or from their appropriate forms of equilibrium. The variation of light invoked by axial rotation of distorted components, or exhibited during their mutual eclipses, will permit their investigator to ascertain the distribution of surface brightness over the apparent disks of the constituent stars. Non-radial oscillations of tidal origin, invoked by varying radius-vector in close eccentric systems, can demonstrate whether stellar configurations behave like perfect fluids.

These and other related considerations make it evident that a study of detailed properties of close binary systems is bound to occupy a position of central importance in contemporary stellar astrophysics; and the aim of the present volume will be to outline its theoretical background on which all interpretation of their observations must necessarily be based. The entire contents of this monograph (with the exception of its first and last chapters) has accordingly been divided in parts covering the individual aspects of the underlying problem.

Chapter II following this introduction will contain an outline of the *dynamics* of close binary systems, with special emphasis on such phenomena, arising from mutual perturbations of the two components, as may be actually observable. The perturbations which we propose to consider will be due to axial rotation of the two components, to the tides raised by one component on the other, as well as to a possible presence of distant third companions accompanying close pairs. In doing so we shall not restrict the axes of rotation of the components to be perpendicular to the orbital plane, but shall investigate the motions (i.e., precession and nutation) of axes arbitrarily inclined in space, synchronized as they must be with the motion of the orbital plane itself. We shall, however, assume the mutual tidal distortion of both components to be governed by the equilibrium theory of tides, so that the tidal bulge raised by one star upon another will always remain directed towards the centre of gravity of the disturbing mass and will sweep around each star with the Keplerian angular velocity. Within the scheme of approximation of a linearized treatment, secular as well as periodic perturbations will be investigated of the elements of motion *in* the orbital plane as well as *of* this plane, with a special attention to the fluctuation of apparent period of the orbit.

Chapter III (The Roche Model) will contain a detailed geometrical analysis of a binary system whose components can be approximated, for gravitational purposes, by two mass-points and which should offer an ample approximation to the behaviour of components exhibiting a high degree of central condensation. Special attention will be paid to the geometry of the limiting model whose components occupy the largest closed volume capable of containing their whole mass and are in contact at a point on the line joining their centres.

The following Chapters IV and V contain a systematic development of theoretical light- and radial velocity-curves exhibited by distorted rotating components of close binary systems between minima as well as within eclipses, taking account of their limb- and gravity-darkening. An appropriate description of such phenomena will call for the introduction of a whole new class of special functions, whose properties will have to be studied in some detail. In addition to the light- and velocity-changes due to the rotational and tidal distortion of both components, supplementary effects of light reflected by each star will be considered in equal detail.

If Chapter IV was devoted in part to an investigation of the theoretical light changes due to mutual eclipses of two (spherical or distorted) stars, in Chapter VI we shall turn to confront the converse task: namely, a determination of such elements of eclipsing binary systems as may be deduced by an appropriate analysis of their observed light changes. This chapter should by itself be of primary interest to the astronomers interested mainly in applications—and this accounts for its size. It gives a self-contained systematic account of analytical procedures which are necessary and sufficient for a precise analysis of modern photoelectric observations of eclipsing binary systems exhibiting all types of eclipses, and can be read largely independently of other parts of the text.

Chapters II–VI have all been concerned with different aspects of the theory of close binary systems, or tasks of analysing their observations for different elements of such systems. Our presentation of this important branch of double-star astronomy would, however, be both disappointing and incomplete if we concluded the book at this stage and did not acquaint the reader with at least the main results of application of various methods described earlier in this volume to an interpretation of the actual observations of photometric or spectroscopic binaries. The principal physical characteristics of actual close binary systems, which have gradually emerged from such applications, will be described in the concluding Chapter VII, together with such discussion of their significance as can be attempted at the present time. The ultimate problem of double-star astronomy is to gain an understanding of the cosmic processes which led to their formation, or which may control their astronomical future. Needless to stress, we are still very far from this goal; but a critical examination of the present evidence revealed by close binaries existing at this time has furnished, in recent years, some highly important results and opened up new avenues of approach to the main problem which are both suggestive and intriguing. These results and trends of current research into the nature of close binary systems will be outlined in the concluding Chapter VII as far as can be done at the present time.

In conclusion of these introductory remarks a few additional considerations may be raised. In Chapters II–V theories of certain photometric and dynamical phenomena have been developed which should describe the external manifestation of our adopted model of close binary systems in considerable degree of detail. Should it follow that the observations—both photometric and spectroscopic—of actual binary systems should be accounted for by the existing theories to the degree of accuracy to which the latter are presented in this book? The answer is very probably in the negative; and the reasons for it call indeed for a few words of explanation.

In most problems of theoretical astronomy the principal objective of the investigator is to describe the observed facts in terms of a physically reasonable model of the respective phenomena. The relative success of this task must, in turn, depend on the inherent complexity of the phenomena confronting us. A dynamical model consisting of two or more mass-points is, for example, known to offer an exceedingly close approximation for studying the motions of celestial bodies in our solar system. On the other hand, in most astrophysical problems our working models are relatively as yet very insecure. Now we have every reason to feel confident that the model we used as a basis of our study of close binary systems in this book—with its assumed applicability of the equilibrium theory of tides—should offer a remarkably good approximation to reality; much better than is attainable in most other problems of contemporary stellar astronomy.

Nevertheless, we must recognize that a number of additional physical phenomena are likely to arise in close binary systems, of which our theory as developed in this book takes as yet but little or no account. Of such phenomena, we may mention possible effects due to the dynamical tides, which may be responsible for at least a part of the asymmetries of observed light changes of eclipsing binary systems. Dynamical consequences of the departures from rigid-body rotation of both components should be kept in mind, particularly in connection with the notorious fluctuations in periods of certain close binaries. We may, in this connection, quote possible photometric effects of gas streams circulating in many binary systems, or the effects of 'spots' or 'faculae' on the photospheres of early-type stars. Such phenomena may be, moreover, only partly amenable to treatment within the framework of any causal theory, and partly remain of random nature. When any individual binary system is observed in real detail, the picture emerging from actual observations is always likely to be more complicated and intriguing than any explanatory theory which we can so far construct.

Should, however, a possible presence of one or more additional phenomena likely to complicate the photometric or spectroscopic behaviour of our basic model be regarded as a legitimate excuse to lower the standards of accuracy of its analysis and abstain from considering its manifestations in all their aspects? The answer is again emphatically in the negative: for *the role of secondary physical phenomena, complicating our equilibrium model, cannot be satisfactorily elucidated until all major effects, which are bound to be present, have been fully understood and properly accounted for.* This sets the task and the challenge which we propose to face in this book; and constitutes our ground for the hope that the physical soundness and high degree of approximation attainable by our equilibrium model will earn its analysis a permanent place in the existing literature on double-star astronomy.

I. BIBLIOGRAPHICAL NOTES

[1] W. Herschel, 'On the Construction of the Universe', *Phil. Trans. Roy. Soc.* for 1802, p. 477.

[2] J. Michell, 'An Inquiry into the Probable Parallax and Magnitude of the Fixed Stars, from the Quantity of Light which they afford us, and the particular Circumstances of their Situation', *Phil. Trans. Roy. Soc.*, **57**, 234, 1767.

[3] *Ibidem*, p. 249.
[4] *Phil. Trans. Roy. Soc.*, **72**, 82, 1782.
[5] *Phil. Trans. Roy. Soc.*, **72**, 112, 1782; **75**, 40, 1785.
[6] In a paper entitled 'On the Means of discovering the Distance, Magnitude, etc. of the Fixed Stars . . .' and published in *Phil. Trans. Roy. Soc.*, **74**, 35, 1784, Michell pointed out that 'The very great number of stars that have been discovered to be double, triple, etc. particularly by Mr. Herschel, if we apply the doctrine of chances, as I have heretofore done . . . cannot leave a doubt with any one, who is properly aware of the force of those arguments, that by far the greatest part, if not all of them, are systems of stars so near to each other, as probably to be liable to be affected sensibly by their mutual gravitation; and it is therefore not unlikely, that the periods of the revolutions of some of these about their principals (the smaller ones being, upon this hypothesis, to be considered as satellites to the others) may some time or other be discovered' (op. cit., p. 36).
[7] *Phil. Trans. Roy. Soc.* for 1803, p. 339.
[8] 'We have already shewn the possibility' (*cf.* his note 'On the Construction of the Universe' in *Phil. Trans. Roy. Soc.* for 1802, p. 477) 'that two stars, whatsoever be their relative magnitudes, may revolve, either in circles or elipses, round their common centre of gravity; and that, among the multitude of the stars of the heavens, there should be many sufficiently near each other to occasion this mutual revolution, must also appear highly probable. But neither of these considerations can be admitted in proof of the actual existence of such binary combinations. I shall therefore now proceed to give an account of a series of observations on double stars, comprehending a period of about 25 years, which, if I am not mistaken, will go to prove, that many of them are not merely double in appearance, but must be allowed to be real binary combinations of two stars, intimately held together by the bond of mutual attraction' (op. cit. ante, pp. 339–340).
[9] R. G. Aitken, *New General Catalogue of Double Stars* in two volumes, Carnegie Institution of Washington, 1932.
10] For the latest compilation of such data *cf.* J. H. Moore and F. J. Neubauer, *Fifth Catalogue of the Orbital Elements of Spectroscopic Binary Stars*, Lick. Obs. Bull., No. 521, 1948.
[11] J. Schaumberger, in a paper entitled 'Haben die Babylonier Veränderliche Sterne gekannt?', and published on p. 350 of Ergänzungsheft to Kugler's monumental *Sternkunde und Sterndienst in Babel* (1935), unearthed certain archaeological evidence from which he concluded that . . . 'Wir haben also . . . die ältesten Berichte über Beobachtungen von veränderlichen Sternen vor uns, und zwar je des wichtigsten Vertreters der beiden Haupttypen der Stellae Variabiles, Algol und Mira. Die Entdeckung der Veränderlichkeit der Sterne is also um gut 2000 Jahre, vielleicht 3000 Jahre früher geschehen, also die Geschichte der Astronomie bisher angenommen hat.' This view has, however, not yet been generally accepted; for critical comments on it see, e.g., N. Bobrovnikoff, *Isis*, **33**, 687, 1942.
[12] The Arabs were not the only ancient people who gave this star a proper name. According to Allen's *Star Names and their Meaning* (New York, 1899) the Hebrews knew Algol as Rōsh-ha-Satan (Satan's Head); while the Chinese gave it the gruesome title Tseih She (the Piled-up Corpses)!
[13] 'Sopra la sparizione d'alcune stelle e altre novità celesti', in Prose di Signori Academici Gelati di Bologna, 1671.

The literal version of the relevant passage is, unfortunately, tantalizing by its incompleteness. 'The brightest star that shines in it,' (i.e., in the head of Medusa) 'affected by frequent mutations, attains but occasionally its greatest magnitude. I had observed it for several years to be of the 3rd magnitude. It faded in 1667 to the 4th magnitude; in 1669 it recovered again its previous lustre up to the 2nd magnitude, but in 1670 it exceeded only by little the 4th magnitude . . . '. This is all that Montanari had to say in his report, mentioning neither the month nor the day of his observations. Could any additional information about the times of the minima be extracted from the MSS of Montanari's observations?

A search in Bologna proved, however, fruitless; but Montanari left this city in 1678 for Padua in response to the invitation of the Venetian Republic to accept the chair once Galilei's at the Bo, and spent the rest of his unfortunately short life (died in 1687) in Padua. There one of his closest pupils became F. Bianchini, subsequently canon at the Liberian Basilica in Verona, who observed many years with his teacher, and left after his own death (in 1729) a voluminous collection of manuscripts which were published by Manfredi under the title *F. Bianchinis Astronomicae ac Geographicae Observationes Selectae, etc* at Verona in 1737. The material left by Bianchini was, however, so extensive that Manfredi's edition included only a selection of what their editor considered to be most important at that time. This left out, unfortunately, all observations of variable stars—but he did mention them; and this offered the clue to later investigators.

The most recent research on the Bianchini-Montanari MSS which contains their observations of variable stars was undertaken by A. Porro. In particular, one fascicle of 275 pages (No. 387 in the library of the Liberian Basilica at Verona) contains the magnitude differences of all stars of which its author had found between their own observations and those of Bayer. And in this survey (on p. 240 of the Verona MS) Porro discovered a statement revealing that one minimum of Algol (of 4th magn.) was apparently observed by Montanari on November 8th, 1670. This, then is the date of the first recorded minimum of Algol's light; though its hour or minute is, unfortunately, unknown.

It may be added that the relevant bibliographical evidence was published by Porro in this edition of the MSS under the title *Observationes circa fixas* (Stabilimento Fratelli Pagano, Genova 1902). A brief account of it was published by Porro in *A.N.*, **127**, 41, 1891. It is not impossible—as was stressed by Porro himself—that further bibliographical search in the Capitolar Library in Verona may reveal further evidence on the times of the early minima of Algol; but such evidence—if it is there—remains yet to be brought to light.

The present writer is indebted to Professor G. Horn d'Arturo from the University of Bologna for much of the information contained in this note.

[14] *Cf. Histoire de l'Acad. Roy. des Sciences*, tome II (Paris, 1733), pp. 139 and 223.

[15] Published by Lady C. A. Lubbock (granddaughter of Sir William Herschel) in the *Herschel Chronicle*, Cambr. Univ. Press, 1933, p. 184.

[16] *Phil. Trans. Roy. Soc.*, **73**, 474, 1783.

[17] In an informal report 'Observations upon Algol', read before the Royal Society on May 8th, 1783, but not printed till it appeared in *The Scientific Papers of Sir William Herschel*, London 1912, vol. I, p. cvii.

[18] H. C. Vogel, *A.N.*, **123**, 289, 1890.

[19] Kl. Veröff. Berlin-Babelsberg, No. 21, 1939.

[20] Publ. Sternberg State Astr. Inst., Moscow 1948.

[21] *Cf.* G. P. Kuiper, *P.A.S.P.*, **47**, 15, 121, 1935.

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[25] *Cf.* e.g., H. Shapley, *Harvard Centennial Symposia* (Harv. Obs. Mono., No. 7, Cambridge, Mass., 1948) pp. 249–260.

[26] For their discussion *cf.* e.g., Z. Kopal, *I.A.U. Trans.* **9**, 599, 1955.

[27] *Cf.* M. Johnson, *Vistas in Astronomy*, vol. II, London 1957, p. 1407; or Tcheng Mao-lin and M. Bloch, *ibidem*, p. 1412.

[28] As suggested, for instance, by O. Struve in *Sky and Telescope*, **15**, 209, 1956.

[29] M. Walker, *Ap. J.*, **123**, 68, 1956.

[30] *Cf.* G. Grant, *Ap. J.*, **122**, 566, 1955; A. H. Joy, *Ap. J.*, **124**, 317, 1956.

[31] Sawyer's latest *Second Catalogue of 1421 Variable Stars in Globular Clusters* (Toronto Publ., **2**, No.2, 1955) lists only 3 eclipsing variables in three clusters (N.G.C. 3201, 5139, 6838), which may quite likely be all foreground stars.

[32] *Cf.* S. I. Gaposchkin, *Per. Zvjozdy*, **10**, 337, 1955.

[33] H. Shapley and V. N. McKibben, *Proc. U.S. Nat. Acad. Sci.*, **41**, 1955.

[34] W. Baade and H. H. Swope, *A. J.*, **60**, 151, 1955.

[35] For an up-to-date list of visual binaries of known mass-ratios, *cf.*, e.g., K. Aa. Strand and R. G. Hall, *Ap. J.*, **120**, 322, 1954.

CHAPTER II

Dynamics of Close Binary Systems

THE AIM of the present chapter will be to provide a systematic introduction to the study of dynamical phenomena exhibited by close binary systems, with special regard to such effects as may be spectroscopically or photometrically observable. If the components of binary systems were sufficiently far apart to attract each other as a pair of mass-points, their orbits would be simple Keplerian ellipses—perturbed occasionally by encounters with neighbouring stars (or interstellar clouds) whose cumulative effects may affect appreciably the separation and other elements of wide pairs after sufficiently long intervals of time.* If, however—consistent with our definition of close binaries as set forth already in the introduction to this volume—proximity phenomena arising from mutual dynamical interaction of the two components are to be regarded as essential features of our problem, encounters with external celestial objects can be safely ignored; but significant perturbations are bound to arise from the *tides* raised by one component upon another, as well as by *axial rotation* responsible for polar flattening.

As long as the free periods of non-radial oscillations of the two components are sufficiently long in comparison with the period of their orbit (so that each star can adjust its instantaneous form to the prevailing field of force), the extent of their mutual tidal distortion will be governed by the equilibrium theory of tides; the tidal bulge raised by one star upon the other will always be directed towards the centre of mass of the system (i.e., will remain symmetrical with respect to the radius-vector), and rotate with the Keplerian angular velocity. On the other hand, the axis of rotation of each component may be orientated quite arbitrarily in space; and no force is known, moreover, which would impel their orientation to remain fixed. In point of fact, the axes of rotation are bound to move in the course of time (due to precession and nutation of fluid components)—and so will, in general, the orientation of the orbital plane itself relative to any fixed direction in space.

In order to study these phenomena, we shall have to depart from the general Lagrangian equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \frac{\partial W}{\partial q_i}, \tag{0-1}$$

where the q_i's stand for the respective degrees of freedom of our dynamical system, and to formulate its total potential energy W and the kinetic energy

* For the latest study of the cumulative effects of perturbations due to stellar encounters *cf.* S. Chandrasekhar, *Ap. J.*, **99**, 54, 1944; for the effects of encounters with interstellar gas-clouds, see B. Takase, *Tokyo Ann.*, **3**, 192, 1953.

T as functions of the q_i's before an explicit form of equations (0-1) can be established and their solution attempted.

The formulation of the potential and kinetic energy of close binary systems whose components are stars of arbitrary structure, rotating like rigid bodies, will occupy us in the introductory sections 1–3 of this chapter. In deriving the appropriate expression for the potential in sections 1–2 we can, in particular, follow a well-trodden path: the type of analysis which we shall employ goes back in principle to Clairaut, was developed further by Legendre and Laplace, and played an important part in the early development of geophysics and study of planetary structure. Any novelty which this part of our text may claim rests on its application to tidally-distorted configurations in close binary systems, which is indeed of relatively recent date and whose possibilities are still far from being fully exhausted. The procedure itself is extremely general and easy of application as long as superficial distortion remains small enough for its squares and higher powers to be negligible. The same is, moreover, true of the formulation of the kinetic energy T as given in section 3 of this chapter for components rotating like rigid bodies. The novelty of our treatment rests here on the fact that we shall not restrict our problem by assuming the position of the axis of rotation to be fixed in any arbitrary direction, or its angular velocity to be constant, but shall allow for the effects of its changes in the course of time, as well as for the response of polar flattening of a fluid spheroid to any variation of angular velocity. Due account will also be taken of the kinetic energy of travelling tidal waves, which will prove to be comparable with the effects, on kinetic energy, of rotational distortion, and without which the total kinetic energy of close binaries consisting of fluid components (in contrast to rigid components on which no tides could be raised) would be seriously incomplete. In view of the obvious nature of this comment, it may perhaps surprise the reader to learn that the present chapter represents the first treatment of its subject where this has been done.

In section 4 the specific form of the equations (0-1) of motion will then be formulated, subject only to such approximations as are inherent in the adopted expressions for W and T. These equations will, in general, be *non-linear* in nature, and offer but little encouragement to any attempt at establishing their exact solution. Therefore, in sections 5–7 a systematic effort will be made to construct their *approximate* solutions to a *linear* approximation. In section 5 we shall be concerned, within this scheme of approximation, with absolute motions of the equatorial planes of the two components (i.e., with the precession and nutation of fluid stars) as well as *of* the orbital plane in space; while in section 7 our aim will be to investigate the perturbations of different elements *in* the plane of the orbit; section 8 being devoted to a more detailed discussion of the behaviour of one particular element—namely, the *orbital period*—because of the possibility of sensitive observational checks of theoretical developments.

In all this work we propose to pay due attention to *secular* as well as *periodic* perturbations of all elements, and investigate them in equal detail. While certain types of secular perturbations (causing a motion of the apsidal line, or that of the nodes) already attracted some attention of previous investigators, a study of the corresponding periodic perturbations has so far been entirely neglected, and undeservedly so; for, quite apart from their theoretical significance, their practical importance for the student of eclipsing binary systems is considerable: namely, *their amplitudes represent the limits within which the geometrical elements of eclipsing binaries*—such as the fractional radii r_1, r_2, of the two components, orbital inclination i, etc.—*can be regarded as constant within each cycle.* With a gradually increasing precision of photometric observations, the time is bound to come when periodic variations of such elements due to perturbations of rotational or tidal origin will have to be taken into account in precise analysis of the observed light changes of close eclipsing systems. While we do not propose yet to incorporate such refinements in an outline of light curve analysis as given in Chapter VI of this book, we shall, by establishing the amplitudes of the dominant terms of periodic perturbations of principal elements of the orbit, ascertain at least the limit to which the conventional geometrical elements of eclipsing binary systems can legitimately be regarded as constant.

In sections 1–7 of this chapter we thus propose to consider the dynamical effects of perturbations, arising from rotation and tides, which *must* be operative in close binary systems whose components behave as fluid bodies. In the subsequent section 8 we shall consider the effects of a different type of perturbations which *may* be operative in such systems: namely, those arising from the possible presence of a *third body.* Triple systems consisting of a close pair accompanied by a relatively distant third component are indeed not rare—at least 10% of all known close binaries appear to possess such companions—and an investigation of the role of the perturbations due to such third bodies must, therefore, be given the place it deserves in any systematic survey of the dynamics of close binary systems.

The concluding section 9 (followed only by bibliographical notes and comments on subject matter discussed in the present chapter) will be devoted to a survey of principal results—many of them new—which have been established in this chapter—and a brief discussion of their significance in relation to the observed facts. This section alone may be of interest to the practical astronomer interested only in applications. The writer ventures, however, to hope that the apparent complexity of some of the earlier sections will not deter the more serious student of the subject from following us through the developments from which the results summarized in section 9 have eventually emerged—not only to grasp fully their accuracy or approximations involved in their derivation, but also because the dynamics of close binary systems constitutes a truly modern and fascinating branch of celestial mechanics which has not yet attained the degree of exactitude that has earned for its older branches their well-deserved pre-eminence in the whole field of exact science, and whose non-linear aspects represent still largely a closed book yet to be written by future investigators.

II.1. POTENTIAL ENERGY

In order to establish the potential energy of distorted components in close binary systems, of arbitrary internal structure, let us consider a fluid configuration in steady state whose density at any point be ρ and the pressure

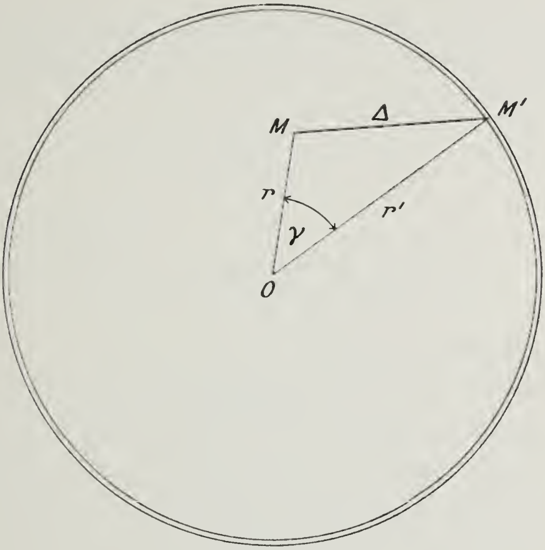

FIGURE 2-1.

P. The equations of its hydrostatic equilibrium are together equivalent to a single total differential equation

$$dP = \rho \, d\Psi, \tag{1-1}$$

where Ψ stands for the complete potential (gravitational plus disturbing) of forces acting upon our body. From (1-1) it follows that P is a function of Ψ, and that ρ is either constant, or another function of Ψ. In other words, over any surface of our body characterized by equal density and equal pressure,

$$\Psi = \text{constant}. \tag{1-2}$$

This contains, *in nuce*, the complete specification of our problem. In what follows our task will merely be to spell out the explicit form of this equation for any given type of disturbing force.

In order to do so, let us fix our attention on an arbitrary point $M(r, \theta, \phi)$ in the interior of our body (see Fig. 2-1) acted upon by the attraction of a stratum comprised between the radii $r = r_0$ and r_1, and let $M'(r', \theta', \phi')$ be an arbitrary point of this stratum. If so, the *interior potential* U at M will evidently be given by the equation

$$U = G \int \frac{dm'}{\Delta}, \tag{1-3}$$

where G stands for the constant of gravitation, and the mass element

$$dm' = \rho r'^2 \, dr' \sin\theta' \, d\theta' \, d\phi'; \tag{1-4}$$

while

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos\gamma, \tag{1-5}$$

where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \tag{1-6}$$

If, moreover, we expand Δ^{-1} in terms of Legendre polynomials $P_n(\cos\gamma)$, equation (1-3) may be rewritten as

$$U = \sum_{n=0}^{\infty} r^n U_n, \tag{1-7}$$

where

$$U_n = G \int_{r_0}^{r_1} \int_0^{\pi} \int_0^{2\pi} \rho r'^{1-n} P_n(\cos\gamma) \, dr' \sin\theta' \, d\theta' \, d\phi'. \tag{1-8}$$

Let, in the foregoing expressions,

$$r' = f(a, \theta', \phi') \tag{1-9}$$

denote symbolically the equation of an equipotential surface of constant density and pressure. By virtue of the uniqueness of the potential function, only one equipotential surface can pass through any point (i.e., for any value of a); and since the density must, by definition, remain constant over such a surface, it follows that ρ can hereafter be regarded as a function of a *single* variable a, introduced by the above equation (1-9), and denoting the *mean radius* of the corresponding equipotential; it is bounded so that $0 < a < a_1$, where a_1 represents the (smallest) root of the equation $\rho(a_1) = 0$. The fact that ρ can thus be regarded as the function of a single variable a suggests that it should be of advantage to change over from r' to a as the new variable of integration on the right-hand side of (1-8); and since during an integration with respect to r' both θ' and ϕ' can be regarded as constants, it follows that a transformation from r' to a can be effected by means of the equation

$$dr' = \frac{\partial r'}{\partial a} \, da \tag{1-10}$$

and that, furthermore, it should be legitimate without any loss of generality to assume for r' an expansion of the form

$$r' = a\left\{1 + \sum_{i,j} Y_j^i(a, \theta', \phi')\right\}, \tag{1-11}$$

where the Y's stand for tesseral harmonics taken with respect to the centre of mass of our configuration.

In order to proceed any further, let us simplify our tasks ahead by assuming that *the distortion of our configuration is so small that quantities of the order of squares and higher powers of the individual harmonics* Y_j^i *may be neglected.** If so, and if we make use of the well-known orthogonality conditions for tesseral harmonics requiring that

$$\int_0^{\pi}\int_0^{2\pi} P_n(\cos\gamma)\, Y_j^i(a,\theta',\phi')\sin\theta'\, d\vartheta'\, d\phi' = \frac{4\pi}{2j+1}\, Y_j^i(a,\theta,\phi) \tag{1-12}$$

if $j = n$ and zero when $j \neq n$, equation (1-7) after some operations may be reduced to the form

$$U = U_0 + \sum_{j=1}^{\infty} \frac{4\pi G r^j}{2j+1} \int_{a_0}^{a_1} \rho \frac{\partial}{\partial a}(a^{2-j} Y_j^i)\, da, \tag{1-13}$$

when we have abbreviated

$$U_0 = 4\pi G \int_{a_0}^{a_1} \rho a\, da. \tag{1-14}$$

The *exterior potential* V at $M(a, \theta, \phi)$ may be found and expressed in terms of a by closely analogous methods. In order to do so, let us start from the definition

$$V = \sum_{n=0}^{\infty} V_n r^{-(n+1)}, \tag{1-15}$$

where

$$V_n = G \int P_n(\cos\gamma) r'^n\, dm'. \tag{1-16}$$

For $n = 0$,

$$V_0 = G \int dm' = G m_0, \tag{1-17}$$

where

$$m_0 = 4\pi \int_0^{a_0} \rho a^2\, da \tag{1-18}$$

denotes the mass of our configuration interior to a_0. It can, furthermore, be shown† that if a is measured from the centre of gravity of our body. V_1 is bound to vanish. Therefore, on changing over from r' to a by means of

* Squares of second-harmonic rotational distortion were considered in connection with the theory of exact figure of the Earth by O. Callandreau, *Ann. Obs. de Paris*, **19**, E (1889), G. H. Darwin, *M.N.*, **60**, 82, 1900, and A. Véronnet, *Journ. de Math.* (6) **8**, 331, 1912.

† *Cf.*, e.g., F. Tisserand, *Mécanique Céleste*, Paris 1891, **2**, p. 295.

(1-10)–(1-11) and making use of (1-12), we can reduce equation (1-15) to

$$V = G\frac{m_0}{r} + \sum_{j=2}^{\infty} \frac{4\pi G}{(2j+1)r^{j+1}} \int_0^{a_0} \rho \frac{\partial}{\partial a}(a^{j+3} Y_j^i)\, da \tag{1-19}$$

again correctly to terms of the first order in the Y's.

The total potential arising from the mass of the distorted configuration is then equal to the sum

$$\begin{aligned} U + V = \Psi' = U_0 + \frac{V_0}{r} + \sum_{j=1}^{\infty} \frac{4\pi G r^j}{2j+1} \int_a^{a_1} \rho \frac{\partial}{\partial a}(a^{2-j} Y_j^i)\, da \\ + \sum_{j=2}^{\infty} \frac{4\pi G}{(2j+1)r^{j+1}} \int_0^{a} \rho \frac{\partial}{\partial a}(a^{j+3} Y_j^i)\, da, \end{aligned} \tag{1-20}$$

where we have replaced—there should be no danger of confusion—a_0 in the limits of our integrals by a. Lastly, let us suppose that the disturbing potential V'—whatever its origin—is also expansible in terms of surface harmonics $P_j^i(\theta, \phi)$ in a series of the form

$$V' = \sum_{i,j} c_{i,j} r^j P_j^i(\theta, \phi), \tag{1-21}$$

where the $c_{i,j}$'s are constants depending on the nature of distortion. If so, the *total potential* Ψ of forces acting on any arbitrary point will be represented by the sum

$$\Psi = U + V + V' = \Psi' + V', \tag{1-22}$$

where U and V are already known from equations (1-13) and (1-19), and their sum is given by (1-20). Now if, in conformity with (1-2), the function Ψ is to remain constant over any surface of equal pressure or density, over which $r = a\{1 + \Sigma Y_j^i\}$, *the coefficients corresponding to the individual values of j in Ψ must all be equal to zero*; and this will be true if (and only if),

$$\begin{aligned} Y_j^i \int_0^a \rho a^2\, da - \frac{1}{(2j+1)a^j} \int_0^a \rho \frac{\partial}{\partial a}(a^{j+3} Y_j^i)\, da \\ - \frac{a^{j+1}}{2j+1} \int_a^{a_1} \rho \frac{\partial}{\partial a}(a^{2-j} Y_j^i)\, da = \frac{c_{i,j}}{4\pi G} a^{j+1} P_j^i(\theta, \phi) \end{aligned} \tag{1-23}$$

for any i, j, correctly to the first order in small quantities. This well-known result is generally referred to in the literature as *Clairaut's Equation*, although its above form (for unrestricted j) was not actually derived until by Legendre.

Clairaut's equation (1-23) implicitly specifies the tesseral harmonics Y_j^i describing the actual form of equipotential level surfaces as distorted by an external force derived from the potential (1-21). In order to establish from it a more explicit form of the *boundary* of our configuration, let us multiply both sides of equation (1-23) by a^j, differentiate with respect to a, and divide by a^{2j}: we obtain

$$\left\{j\frac{Y_j^i}{a^{j+1}}+\frac{1}{a^j}\frac{\partial Y_j^i}{\partial a}\right\}\int_0^a \rho a^2\,da-\int_a^{a_1}\rho\frac{\partial}{\partial a}\left(\frac{Y_j^i}{a^{j-2}}\right)da=\frac{c_{i,j}}{4\pi G}(2j+1)P_j^i(\theta,\phi). \tag{1-24}$$

Since, at the boundary, $a = a_1$ and

$$4\pi\int_0^{a_1}\rho a^2\,da = m_1 \tag{1-25}$$

denotes the mass of the whole distorted configuration, the second integral on the left-hand side of (1-24) vanishes and the first can be replaced by $m_1/4\pi$. The remaining parts of this equation can, moreover, be symbolically solved for the surface value of $Y_j^i(a_1)$ in the form

$$Y_j^i(a_1)=c_{i,j}\Delta_j\frac{a_1^{j+1}}{Gm_1}P_j^i(\theta,\phi), \tag{1-26}$$

where we have abbreviated

$$\Delta_j=\frac{2j+1}{j+\eta_j(a_1)} \tag{1-27}$$

and

$$\eta_j(a)=\frac{a}{Y_j^i}\frac{\partial Y_j^i}{\partial a}. \tag{1-28}$$

The foregoing solution of (1-24) is, therefore, not yet explicit, as Y_j^i remains involved on the right-hand side of (1-26) through the surface value of its logarithmic derivative (1-28). A more detailed study of the behaviour of this derivative $\eta_j(a)$ is being postponed for the subsequent section II.2. Subject to a determination of its surface value at $a = a_1$ equation (1-26) reveals, however, the specific form of surface harmonics governing the external shape of a slightly distorted configuration of arbitrary structure, and enables us also to reduce the expressions (1-13) and (1-19) for the internal and external potentials U and V to more tractable forms.

In order to evaluate the integrals remaining on the right-hand sides of equations (1-13) and (1-19), let us divide (1-23) by a^{j+1}, differentiate with respect to a, and multiply by a^{2j+2}: we find that

$$\int_0^a \rho\frac{\partial}{\partial a}(a^{j+3}Y_j^i)\,da=a^jY_j^i\{j+1-\eta_j(a)\}\int_0^a\rho a^2\,da \tag{1-29}$$

so that, for $a = a_1$,

$$\int_0^{a_1}\rho\frac{\partial}{\partial a}(a^{j+3}Y_j^i)\,da=\frac{2j+1}{4\pi G}(\Delta_j-1)a_1^{2j+1}c_{i,j}P_j^i(\theta,\phi) \tag{1-30}$$

by virtue of (1-26). On the other hand, a multiplication of (1-23) by a^j, differentiation with respect to a, and subsequent division by a^{2j} yields

$$\int_a^{a_1} \rho \frac{\partial}{\partial a}(a^{2-j} Y_j^i)\, da = \frac{Y_j^i}{a^{j+1}}\{j + \eta_j(a)\} \int_0^a \rho a^2\, da - \frac{2j+1}{4\pi G} c_{i,j} P_j^i(\theta, \phi). \tag{1-31}$$

Inserting (1-31) in (1-13) we find that, for an arbitrary internal point of our configuration, the potential arising from the mass of a shell comprised between the mean radii a and a_1 assumes the more explicit form

$$U(a) = U_0 + G \sum_{j=1}^{\infty} \left\{ \left[\frac{j + \eta_j(a)}{2j+1} \right] \frac{m(a)}{a} Y_j^i(a; \theta, \phi) - \frac{c_{i,j} a^j}{G} P_j^i(\theta, \phi) \right\}, \tag{1-32}$$

where

$$m(a) = 4\pi \int_0^a \rho a^2\, da \tag{1-33}$$

denotes the mass of the core interior to a. Furthermore, by an insertion of (1-29) or (1-30) in (1-19) the potential arising from the mass of the core interior to a similarly becomes

$$V(a) = \frac{V_0}{a} + G \sum_{j=2}^{\infty} \left[\frac{j + 1 - \eta_j(a)}{2j+1} \right] \frac{m(a)}{a} Y_j^i(a; \theta, \phi) \tag{1-34}$$

for $r = a$; while for points exterior to our configuration ($r > a_1$),

$$V(r) = G\frac{m_1}{r} + \sum_{j=2}^{\infty} \frac{a_1^{2j+1}}{r^{j+1}} (\Delta_j - 1) c_{i,j} P_j^i(\theta, \phi), \tag{1-35}$$

where we substituted for $Y_j^i(a_1; \theta, \phi)$ from (1-26) and abbreviated $m_1 \equiv m(a_1)$. Moreover, the sum

$$\Psi'(a) = U(a) + V(a) \tag{1-36}$$

of potentials acting on any internal point, previously given by equation (1-20), now simplifies into

$$\Psi'(a) = U_0 + \frac{V_0}{a} + G \sum_{j=1}^{\infty} \left\{ \frac{m(a)}{a} Y_j^i(a; \theta, \phi) - \frac{c_{i,j} a^j P_j^i(\theta, \phi)}{G} \right\}. \tag{1-37}$$

The *total potential energy* W of a binary system consisting of two components of masses m_1 and m_2 and mean radii $a_{1,2}$ can generally be expressed as

$$W = W_1 + W_2 + W_{12}, \tag{1-38}$$

where

$$2W_i = \int_0^{m_i} \Psi_i'\, dm_i, \qquad i = 1, 2, \tag{1-39}$$

represents the contributions due to the attraction of each component on itself, while

$$2W_{12} = \int_0^{m_1} \Psi_2' \, dm_1 + \int_0^{m_2} \Psi_1' \, dm_2 \tag{1-40}$$

denotes a contribution of the mutual attraction exerted by each component on its mate. An evaluation of these latter expressions offers but little difficulty; for as the Ψ_i's are independent of dm_j for $i \neq j$ and, moreover, $U_1(a_1) = U_2(a_2) = 0$, it follows at once that

$$2W_{12} = m_2 V_1(r) + m_1 V_2(r), \tag{1-41}$$

where the external potentials $V_i(r)$ of the two components for $r > a_{1,2}$ are already known from equation (1–35) above.

An evaluation of the potential energy due to self-attraction of the two components becomes somewhat more involved. As all surface harmonics involved in $U(a)$ and $V(a)$ vanish when integrated over the whole sphere, it follows from (1-39) at once that

$$2W_i = \int_0^{m_1} \left(U_0 + \frac{V_0}{a} \right) dm_i = 16\pi^2 G \int_0^{a_1} \left\{ \int_a^{a_1} \rho a \, da + \frac{1}{a} \int_0^{a} \rho a^2 \, da \right\} \rho a^2 \, da. \tag{1-42}$$

In order to re-derive a more explicit form of this result, advantage can be taken of the fact that the functions $\Psi_i'(a)$ on the right-hand side of equation (1-36) is bound to satisfy Poisson's equation

$$\nabla^2 \Psi' = -4\pi G \rho, \tag{1-43}$$

by virtue of which it follows at once that

$$\int_0^{m_1} \Psi_i' \, dm_i = -\frac{1}{4\pi G} \int \Psi_i' \nabla^2 \Psi_i' \, d\tau_i \tag{1-44}$$

where

$$dm_i = \rho \, d\tau_i \tag{1-45}$$

and the integral on the right-hand side of (1-44) is to be evaluated over the whole volume of the respective configuration. Moreover, by Gauss's Theorem,

$$\int \Psi_i' \nabla^2 \Psi_i' \, d\tau_i = -\oint \left(\frac{d\Psi_i'}{dn} \right)^2 d\tau_i \tag{1-46}$$

where

$$\left(\frac{d}{dn} \right)^2 = \left(\frac{\partial}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right)^2 + \left(\frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right)^2, \tag{1-47}$$

and the circle over the integral sign on the right-hand side signifies that the limits of integration are to be extended over the entire space.

Now earlier in this section we have seen that both U and V constituting Ψ' depend on the angles θ and ϕ only through the tesseral harmonics $Y_j^i(a; \theta, \phi)$ whose squares and higher powers we agreed to ignore. Within this scheme of approximation, the operator d/dn evidently reduces to $\partial/\partial r$ and, as a result, equations (1-44) and (1-46) combined with (1-39) reveal that

$$8\pi G W_i = \int_0^{a_i} \left(\frac{\partial \Psi_i'}{\partial a}\right)^2 d\tau_i + \int_{a_i}^{\infty} \left(\frac{\partial V_i}{\partial r}\right)^2 d\tau_i \tag{1-48}$$

where $V_i(r)$ stands for the external potential, (1-35), and the limits of integration with respect to both angular variables in $d\tau_i$ are to be extended over the whole sphere. Inserting for Ψ_i' and V_i we find, moreover, that integrations with respect to θ and ϕ over these limits will annihilate again all surface harmonics of orders higher than zero, thus leaving us with the result

$$W_i = \frac{G}{2}\left\{\frac{m_i^2}{a_i} + \int_0^{a_i} \frac{m^2(a)}{a^2}\, da\right\} = G \int_0^{m_i} \frac{m(a)}{a}\, dm, \tag{1-49}$$

which is equivalent to, but more explicit than, that represented by equation (1-42).

As a result of the foregoing developments, it transpires that *the potential energy $W_{1,2}$ of both components, due to their self-attraction, remains unaffected by distortion to the order of accuracy to which squares and higher powers of the tesseral harmonics $Y_j^i(a; \theta, \phi)$ governing their distortion are ignorable.* On the other hand, *the potential energy W_{12} due to the gravitational interaction of both components departs from its equilibrium value by quantities of the first order*; and it is these terms that will bear the brunt of the responsibility for dynamical perturbations in close binary systems which we intend to study in subsequent sections of this chapter.

The results obtained in the foregoing paragraphs should enable us next to ascertain the *variation of gravity* over free surfaces of distorted components in close binary systems, which will be found later (cf. section IV.1) to control their surface brightness. This variation should be derivable from the total potential Ψ of forces acting on any point, as defined by equation (1-22), which on the surface ($a = a_1$) of the primary star reduces to

$$\Psi(a_1) = V(a_1) + V'(a_1) = G\frac{m_1}{r} + \sum_{j=2}^{\infty} c_{i,j} \left\{(\Delta_j - 1)\frac{a_1^{2j+1}}{r^{j+1}} + r^j\right\} P_j^i(\theta, \phi) \tag{1-50}$$

by use of (1-21) and (1-35). The surface gravity g is then defined by the relation

$$g = -\frac{d}{dn}\Psi(a_1) \tag{1-51}$$

where, consistent with equation (1-47), dn is measured in the direction of the outward normal. This normal makes, in general, a small angle with the radius-vector r; this angle is of the order of magnitude of Y, so that its cosine can differ from unity only by quantities of second order which we proposed to ignore. To the first order in such quantities we may, therefore, set

$$g = -\frac{d}{dr}\Psi(a_1) \tag{1-52}$$

to be evaluated at $r = a_1(1 + \Sigma Y)$. Differentiating (1-50) we find that a spherical harmonic distortion Y_j^i produces, on the surface, a variation of gravity given by the equation

$$\frac{g - g_0}{g_0} = -\{1 + \eta_j(a_1)\} Y_j^i, \tag{1-53}$$

where we have abbreviated $g_0 = Gm_1/r^2$; and if a number of distinct Y's are superposed,

$$\frac{g - g_0}{g_0} = -\sum_{i,j} \{1 + \eta_j(a_1)\} Y_j^i. \tag{1-54}$$

To each separate harmonic distortion there corresponds a variation of surface gravity such that the fractional variation of this gravity is proportional, over the star's surface, to fractional variation of the radius-vector.

As was already mentioned before, the effects of internal structure on the external form of distorted configurations enter equation (1-26) specifying the Y's only through the functions $\eta_j(a)$ as defined by equation (1-28), and their discussion is reserved for the next section. In order to render (1-26) otherwise uniquely determined, we are left with the specification of the constants $c_{i,j}$ from the nature of the disturbing forces. The potential V' arising from all sources other than the attraction of a star on itself has already been supposed to be expansible in a series of the form (1-21), and to each term on its right-hand side there should correspond a contribution Y_j^i to superficial distortion as defined by (1-26).

Suppose now that a gaseous (and therefore perfectly fluid) component of a close binary system is acted upon by *tidal forces* originating from its companion of mass m_2 at a distance R from the primary's centre of mass. Let, furthermore, this centre be taken as the origin of spherical polar coordinates (r, ϕ, θ), chosen so that the coordinates of the secondary's centre are $(R, 0, \pi/2)$. If R is sufficiently large in comparison with the primary's size, the potential V' arising from the secondary at any distance r' from the origin should be expansible in a well-known series of the form

$$V' = G\left\{\frac{m_2}{r'} + \frac{A + B + C - 3I}{2r'^3} + \dots\right\}, \tag{1-55}$$

where A, B, C denote the principal moments of inertia of the secondary about its centre of gravity, and I the moment of inertia about the line joining the centres of the two stars. The differences $(A-I)$, $(B-I)$ and $(C-I)$ are zero for spherical distribution of matter; and for a secondary distorted by the primary's tidal field they are of the order of $m_2a_2(a_2/R)^3$, where a_2 stands for the mean radius of the secondary component. Hence, the second term in braces on the right-hand side of (1-55) will clearly be of the order of magnitude of $G(m_2/a_2)(a_2/R)^6$. Through this and higher-order terms the potential V' should depend on the relative dimensions, mass and structure of the disturbing as well as distorted star. Its neglect, on the other hand, would be equivalent to considering the tidal effects of one star upon another as due to a mass-point.

To this latter order of accuracy, the potential (1-55) reduces simply to

$$V' = G\frac{m_2}{r'} = \frac{Gm_2}{\sqrt{R^2 - 2a_1R\lambda + a_1^2}}\,; \tag{1-56}$$

or, if r' in the denominator is expanded in terms of the Legendre polynomials, to

$$V' = \frac{Gm_2}{R}\sum_{j=1}^{\infty}\left(\frac{a_1}{R}\right)^j P_j(\lambda) + \text{a constant}, \tag{1-57}$$

where $\lambda \equiv \cos\phi \sin\theta$, ϕ being the azimuth reckoned from the line joining the centres of both components and θ, the polar distance. The constant term on the right-hand side of (1-57) gives rise to no forces acting on the primary star. The term corresponding to $j = 1$ gives rise to a uniform field of force of intensity m_2/R^2, which produces an acceleration of m_2/R^2 at the primary's centre. We can, however, neutralize it by supposing the axes of reference to move with this acceleration. Terms with $j > 1$ constitute finally the tide-generating potential; and a comparison of (1-21) with(1-57) discloses at once that the tide-generating potential (1-57) will produce upon the primary component of mass m_1 a superficial distortion specified by

$$c_{0,j} = G\frac{m_2}{R^{j+1}}, \qquad j = 2, 3, 4, \tag{1-58}$$

these being the only non-zero coefficients (to the first order in small quantities) consistent with the equilibrium theory of tides. In consequence, equation (1-26) reduces in this case to

$$Y_j(a_1) = \Delta_j\frac{m_2}{m_1}\left(\frac{a_1}{R}\right)^{j+1} P_j(\lambda); \tag{1-59}$$

zero superscripts on Y_j and P_j being hereafter omitted.

Earlier in this section we found it possible to describe the form of a distorted configuration in relatively simple terms to the order of accuracy in which quantities of the order of squares and higher powers of the lowest significant harmonic could be ignored. The extent of this approximation may now be tested quantitatively. Since the individual tidal Y_j's as given by the foregoing equation (1-59) turn out to be proportional to $(a_1/R)^{j+1}$, the fifth harmonic Y_5 is readily found to be as small as $(Y_2)^2$, and this we agreed to ignore. Therefore, *to the order of accuracy we have been working, a description of the external form of a tidally-distorted configuration should be limited to the second, third and fourth zonal harmonics alone.* It should, furthermore, be reiterated that *a neglect of quantities smaller than the fourth harmonic is precisely equivalent to considering the tidal action of one component upon another as being due to a mass-point.* Lastly, it should be added that the occurrence of other than these harmonics (i.e., of harmonics characterized by $i > 0$) would signify *tidal lag*; and *tides cannot lag in binary systems whose components behave as perfect fluids* and revolve in circular orbits.

Apart from the tides, the second physical cause of the departure of stellar configurations from spherical form is their *axial rotation*. Extensive spectroscopic evidence reveals that the components of close binary systems do rotate with an angular velocity ω which is generally equal to the Keplerian angular velocity of orbital motion* and occasionally very much larger—the sense of rotation being direct in every known case. If we assume, for the sake of simplicity, that the primary component of a close binary system rotates as a rigid body with a constant angular velocity ω_1 about a fixed axis which is perpendicular to the orbital plane, the disturbing centrifugal potential becomes

$$V'_{\mathrm{rot}} = -\tfrac{1}{3}\omega_1^2 a_1^2 P_2(\cos\theta) + \text{a constant}. \tag{1-60}$$

The constant term on the right-hand side may cause an expansion (or contraction) of the star as a whole, but cannot disturb the star from a sphere. Comparing (1-60) with (1-21), we see that a centrifugal force derived from the potential (1-60) will invoke a single second-harmonic distortion characterized by

$$c_{0,2} = -\tfrac{1}{3}\omega_1^2; \tag{1-61}$$

and, hence, the superficial form of a slowly rotating configuration should be specified by

$$Y_{\mathrm{rot}} = -\frac{\omega_1^2 a_1^2}{3Gm_1}\,\Delta_2 P_2(\cos\theta) = -\frac{\omega_1^2}{4\pi G\rho_m}\,\Delta_2 P_2(\cos\theta), \tag{1-62}$$

where ρ_m stands for the mean density of the distorted configuration, to the same degree of approximation as the effects of tidal distortion were by equation (1-40).

Next let us consider a somewhat more general case of a rigid-body rotation about a fixed axis which is *inclined* to the orbital plane. If so, the centrifugal potential (1-60) is merely to be replaced by

$$V'_{\mathrm{rot}} = -\tfrac{1}{3}\omega_1^2 a_1^2 P_2(\cos\Theta) + \text{a constant}, \tag{1-63}$$

* In systems exhibiting circular orbits synchronism between rotation and revolution may usually (though not always) be expected to exist; while components describing eccentric orbits rotate as a rule faster than they revolve.

where Θ, the angle between an arbitrary radius-vector specified by the direction cosines

$$\left.\begin{aligned} \lambda &= \cos\phi \sin\theta, \\ \mu &= \sin\phi \sin\theta, \\ \nu &= \cos\theta, \end{aligned}\right\} \tag{1-64}$$

and the axis of rotation whose orientation is described by the direction cosines

$$\left.\begin{aligned} \lambda' &= \cos\alpha \sin\beta, \\ \mu' &= \sin\alpha \sin\beta, \\ \nu' &= \cos\beta, \end{aligned}\right\} \tag{1-65}$$

is given by the equation

$$\cos\Theta = \lambda\lambda' + \mu\mu' + \nu\nu'. \tag{1-66}$$

Now by the addition theorem for Legendre polynomials

$$P_2(\cos\Theta) = P_2(\nu)P_2(\nu') + \tfrac{1}{3}P_2^1(\nu)P_2^1(\nu')\cos(\phi-\alpha) + \tfrac{1}{12}P_2^2(\nu)P_2^2(\nu')\cos 2(\phi-\alpha) \tag{1-67}$$

so that the corresponding superficial distortion assumes the form

$$Y_{\text{rot}} = -\frac{\omega_1^2\Delta_2}{4\pi G\rho_m}\{P_2(\nu)P_2(\nu') - \tfrac{1}{2}\sin 2\beta\cos(\phi-\alpha)P_2^1(\nu) + \tfrac{1}{4}\sin^2\beta\cos 2(\phi-\alpha)P_2^2(\nu)\}. \tag{1-68}$$

The centrifugal force alone would thus render the primary a rotational spheroid flattened at the poles; while tidal distortion will tend to elongate it in the direction of the secondary component. To the order of accuracy we have been working, both distortions are simply *additive*, so that the external surface of the primary should be given by

$$r = a_1\left\{1 + Y_{\text{rot}} + \sum_{j=2}^{4} Y_j\right\}, \tag{1-69}$$

where Y_j and Y_{rot} are given by equations (1-59) and (1-62) or (1-68), respectively, and the corresponding expressions for the secondary component can be obtained by an obvious interchange of indices.

II.2. EFFECTS OF INTERNAL STRUCTURE

In the preceding section we have found that the form of equipotential surfaces of a distorted configuration can be uniquely described in terms of the tesseral harmonics $Y_j^i(a, \theta, \phi)$ obeying the integral equation (1-23), whose surface form (at $a = a_1$) is given by equation (1-25). This form, as well as the external potential (1-30) of the distorted body was, moreover, found to depend on its internal structure only through the surface value of the logarithmic derivative of Y. In order to establish the values of such derivatives at $a = a_1$, and their dependence on the whole march of the density distribution $\rho(a)$ in the interior, it will be necessary to investigate the behaviour of $Y_j(a)$, not only at $a = a_1$, but throughout the whole interior $0 \leqslant a \leqslant a_1$. The aim of the present section will be to outline the course of such an analysis.

In order to do so, let us begin with differentiating (1-23) once more with respect to a: the result will be

$$a^2 \frac{\partial^2 Y_j}{\partial a^2} + 6 \frac{\rho}{\bar{\rho}} \left(a \frac{\partial Y_j}{\partial a} + Y_j \right) = j(j+1) Y_j, \tag{2-1}$$

where

$$\bar{\rho} = \frac{3}{a^3} \int_0^a \rho a^2 \, da \tag{2-2}$$

denotes the mean density interior to a. We have seen, however, before that the external form of a distorted configuration depends on its structure (i.e., the ratio $\rho/\bar{\rho}$), not through Y_j, but only through its logarithmic derivative η_j as defined by equation (1-28). If we insert it in (2-1), the latter reduces to

$$a \frac{d\eta_j}{da} + 6 \frac{\rho}{\bar{\rho}} (\eta_j + 1) + \eta_j(\eta_j - 1) = j(j+1), \tag{2-3}$$

which is of first order in the dependent variable. Equation (2-1) is usually also referred to as *Clairaut's equation* (being but a simple modification of 1-23), and (2-3) frequently carries the name of *Radau's equation*—in honour of two distinguished French mathematicians to whom much of the development of our subject is due. As usual, however, such simple terminology does but partial justice to historical truth; for Clairaut deduced equations (1-23) or (2-1) only in the particular case of $j = 2$ (equation 2-1 as it stands above did not make its appearance until in the works of Laplace), and the substitution (1-28) leading to equation (2-3) was used already before Radau's time. If, in what follows we persist, nevertheless, to associate the names of Clairaut and Radau with equations (2-1) and (2-3), the critical reader may regard such terminology at least as convenient labels which may contribute to clarity and save us some confusion.

For certain density distribution-functions $\rho(a)$ equations (2-1) or (2-3) admit of simple solutions in a closed form. Thus the reader may note that, at the centre of our configuration (i.e., $a = 0$) for *finite* values of $\rho(0)$ these equations reveal at once that

$$\left. \begin{aligned} Y_j(0) &= 0, \\ \eta_j(0) &= j - 2; \end{aligned} \right\} \tag{2-4}$$

and if the density remains constant throughout the interior (i.e., if $\rho = \bar{\rho}$ for any value of a),

$$\left. \begin{aligned} Y_j(a) &= k a^{j-2}, \\ \eta_j(a) &= j - 2, \end{aligned} \right\} \tag{2-5}$$

where k denotes an arbitrary constant. If, on the other hand, the whole mass of a distorted configuration is condensed at its centre (i.e., $\rho/\bar{\rho} = 1$ for $a = 0$, and zero for $a > 0$), it is seen that

$$\left.\begin{aligned} Y_j(a) &= ka^{j+1}, \\ \eta_j(a) &= j + 1. \end{aligned}\right\} \tag{2-6}$$

Lastly, if the whole mass of our configuration were confined to an infinitesimally thin surface shell (so that $\rho/\bar{\rho} = 0$ for $a < a_1$; and ∞ at $a = a_1$), the solution of equations (2-1) and (2-3) reduces to

$$\left.\begin{aligned} Y_j(a_1) &= ka^{-1}, \\ \eta_j(a_1) &= -1. \end{aligned}\right\} \tag{2-7}$$

Thus, if we presume nothing on the density distribution in the interior of our configuration, the absolute limits of the function $\eta_j(a_1)$ on the surface are

$$-1 \leqslant \eta_j(a_1) \leqslant j + 1 \tag{2-8}$$

and, hence, by (1-26) the constants Δ_j factoring the superficial distortion $Y_j(a_1)$ as given by equation (1-25) are found to be comprised between the limits

$$1 \leqslant \Delta_j \leqslant \frac{2j+1}{j-1} \tag{2-9}$$

for $j = 2, 3, 4$. If, moreover, the density is supposed not to increase outwards, the foregoing inequality (2-9) becomes restricted to

$$1 \leqslant \Delta_j \leqslant \frac{2j+1}{2(j-1)}. \tag{2-10}$$

It may also be noticed that, whatever the structure,

$$\Delta_j \geqslant \Delta_{j+1}. \tag{2-11}$$

The inequality (2-10) has first been proved (for $j = 2$) by Clairaut; and that of (2-9) by Kopal. The latter reveals that, in any given field of external force, *the superficial distortion of a configuration whose whole mass is confined to an infinitesimally thin surface shell would be twice as large as if the configuration were homogeneous, and five times as large as that appropriate for a mass-point model.*

Apart from these limiting cases, Clairaut's or Radau's equations admit of closed solutions also for certain intermediate density distributions in the interior, some of which deserve at least a brief mention. In order to exhibit them, we find it convenient to change over from Y_j in (2-1) to a new variable y defined by

$$y = \frac{a^3\bar{\rho}}{3} Y_j, \tag{2-12}$$

so that

$$\eta_j = \frac{a}{y}\frac{dy}{da} - 3\frac{\rho}{\bar{\rho}}. \tag{2-13}$$

In consequence, equation (2-1) will assume the form

$$\frac{d^2y}{da^2} = \left\{\frac{j(j+1)}{a^2} + g(a)\right\} y, \tag{2-14}$$

where we have abbreviated

$$g(a) = \frac{3}{a\bar{\rho}} \frac{d\rho}{da}. \tag{2-15}$$

Two cases arise immediately in which equation (2-14) becomes solvable in finite terms: namely, if

$$\rho(a) = ka^{-n}, \tag{2-16}$$

where both k and n (<3) are suitable constants—so that, in this particular case,

$$g(a) = \frac{n(n-3)}{a^2}; \tag{2-17}$$

or again if the internal density distribution is such that

$$g(a) = -m^2, \tag{2-18}$$

m being an arbitrary constant. The actual density distribution $\rho(a)$ consistent with this latter assumption follows as a solution of the differential equation

$$\frac{1}{a^2}\frac{d}{da}\left(a^2 \frac{d\rho}{da}\right) + m^2\rho = 0, \tag{2-19}$$

obtained by a differentiation of (2-18) with respect to a; and its particular solution rendering $\rho(0) \equiv \rho_c$ finite is easily found to be

$$\frac{\rho}{\rho_c} = \frac{\sin ma}{ma}. \tag{2-20}$$

If, moreover, we impose the condition that ρ is to vanish at $a_1 = \pi$ (say), it follows that $m = 1$ and, in consequence, $\rho/\rho_c = a^{-1} \sin a$.*

Let us return now to equation (2-14). Inserting (2-17) we find the particular solution of (2-14) vanishing at the origin to assume the form

$$y_j(a) = C_j a^{1/2 + \sqrt{(j+\frac{1}{2})^2 + n(n-3)}}, \tag{2-21}$$

where the C_j's are arbitrary constants; and, by (2-13), the corresponding functions $\eta_j(a)$ are given by

$$\eta_j(a) = \sqrt{(j+\tfrac{1}{2})^2 + n(n-3)} - \tfrac{5}{2} + n \tag{2-22}$$

for $a > 0$.

* It may be noted that the differential equation (2-19) and its solution (2-20) represent nothing but the internal density distribution of a polytropic gas sphere characterized by the polytropic index $n = 1$—a model which made its first appearance, in this context, in the writings of Laplace almost a century before Emden's work.

If, on the other hand, $g(a)$ is to be given by (2-18), the requisite particular solution of (2-14) can be expressed in terms of the Bessel functions $J_{j+\frac{1}{2}}(a)$ of fractional order in the form

$$y_j(a) = C_j a^{1/2} J_{j+\frac{1}{2}}(a). \tag{2-23}$$

As is well known, the Bessel functions of half-integral order are expressible in closed form in terms of trigonometric functions: in particular,

$$\left.\begin{aligned} J_{1/2}(a) &= \sqrt{\frac{2}{\pi a}}\sin a, \\ J_{-1/2}(a) &= \sqrt{\frac{2}{\pi a}}\cos a; \end{aligned}\right\} \tag{2-24}$$

and those of higher order can be successively built up by means of the recursion formula

$$J_{n+1}(a) = \frac{2n}{a} J_n(a) - J_{n-1}(a). \tag{2-25}$$

In consequence, all functions $y_j(a)$ should be expressible in terms of suitable combinations of $a^{-1}\sin a$ and $a^{-1}\cos a$ factored by polynomials in descending powers of a; and their logarithmic differentiation reveals that, on the surface $(a_1 = \pi)$,

$$\left.\begin{aligned} \eta_2(\pi) &= \frac{\pi^2}{3} - 2, \\ \eta_3(\pi) &= \frac{6\pi^2 - 45}{15 - \pi^2}, \\ \eta_4(\pi) &= \frac{\pi^4 - 55\pi^2 + 420}{10\pi^2 - 105}, \end{aligned}\right\} \tag{2-26}$$

etc., leading to

$$\left.\begin{aligned} \Delta_2 &= 1{\cdot}519\ 820\ldots, \\ \Delta_3 &= 1{\cdot}212\ 908\ldots, \\ \Delta_4 &= 1{\cdot}120\ 482\ldots, \end{aligned}\right\} \tag{2-27}$$

etc.

Another interesting model for which equation (2-1) becomes solvable in a closed form follows if the internal density distribution is such that

$$\bar{\rho} = \rho_c(1 - ka^\lambda)^\mu, \tag{2-28}$$

corresponding to

$$\rho = \rho_c(1 - ka^\lambda)^{\mu-1}\{1 - (1 + \tfrac{1}{3}\lambda\mu)ka^\lambda\}, \tag{2-29}$$

where $\rho_c \equiv \rho(0)$ and k, λ, μ are arbitrary constants. The mass of such a configuration is going to be given by

$$m(a_1) = \tfrac{4}{3}\pi\rho_c a_1^3(1 - ka_1^\lambda)^\mu, \tag{2-30}$$

and its finiteness makes it necessary that $\lambda\mu > 0$, in which case the external radius a_1 defined as the first zero of (2-29) follows from

$$ka_1^\lambda = \frac{1}{1 + \frac{1}{3}\lambda\mu} < 1. \tag{2-31}$$

In consequence, the density concentration of this model will be characterized by the ratio

$$\frac{\rho_c}{\rho_m} = \frac{\rho(0)}{\bar{\rho}(a_1)} = \left\{1 + \frac{3}{\lambda\mu}\right\}^\mu. \tag{2-32}$$

This ratio becomes equal to one (i.e., the model becomes homogeneous) if either $\lambda = \infty$ or $\mu = 0$; and infinite (centrally-condensed model) only if $\lambda = 0$; for the condition $\mu = 0$ leads to a finite density concentration specified by $\rho_c/\rho_m = \exp(3/\lambda)$ which tends to infinity only as $\lambda \to 0$.

If, consistent with equations (2-28) and (2-29), we set now

$$\frac{\rho}{\bar{\rho}} = 1 - (1 + \tfrac{1}{3}\lambda\mu)ka^\lambda \tag{2-33}$$

and introduce this ratio in (2-1), Clairaut's equation will assume the explicit form

$$\lambda^2x^2(1 - x)\frac{d^2Y_j}{dx^2} + \lambda x\{\lambda + 5 - (\lambda + 5 + 2\lambda\mu)x\}\frac{dY_j}{dx} + \{6 - j(j+1) - [6 + 2\lambda\mu - j(j+1)]x\}Y_j = 0, \tag{2-34}$$

where $x \equiv ka^\lambda$. This is a generalized hypergeometric equation, and its particular solution which is regular at the origin is known to assume the form

$$\begin{aligned} Y_j &= Ax^{(j-2)}\, x^{(j-2)/\lambda}\, F(\alpha, \beta, \gamma; x) \\ &= Aa^{j-2}F(\alpha, \beta, \gamma; ka^\lambda), \end{aligned} \tag{2-35}$$

where A is an integration constant,

$$\alpha, \beta = \mu + \frac{2j + 1 \pm \sqrt{(2j+1)^2 + 3\lambda\mu + \lambda^2\mu^2}}{2\lambda} \tag{2-36}$$

and

$$\gamma = 1 + \frac{2j+1}{\lambda}. \tag{2-37}$$

As the finiteness of the mass requires that, at the surface, $x_1 \equiv ka_1^\lambda < 1$ the series on the right-hand side of (2-35) is absolutely convergent for all values of α, β or γ; and the corresponding superficial distortion will be specified by

$$Y_j(a_1) = A\{3/k(3 + \lambda\mu)\}^{1/\lambda}F\{\alpha, \beta, \gamma; (1 + \tfrac{1}{3}\lambda\mu)^{-1}\} \tag{2-38}$$

and

$$\eta_j(a_1) = \left\{\frac{a}{Y_j}\frac{\partial Y_j}{\partial a}\right\}_{a_1} = j - 2 + \frac{3\alpha\beta\lambda}{(3 + \lambda\mu)}\,\frac{F\{\alpha + 1, \beta + 1, \gamma + 1; (1 + \frac{1}{3}\lambda\mu)^{-1}\}}{F\{\alpha, \beta, \gamma; (1 + \frac{1}{3}\lambda\mu)^{-1}\}}. \tag{2-39}$$

4

It may be added that whenever the combination of λ and μ happens to be such that either α or β as defined by equation (2-36) becomes a negative integer, the hypergeometric series in the foregoing equations will terminate and reduce to the appropriate Jacobi polynomials.

Apart from special cases of internal density distribution, which may be only of limited astrophysical interest, equations (2-1) or (2-3) are not generally solvable in a closed form; and in order to solve them, recourse must be had to approximate or purely numerical procedures. By far the most complete set of such integrations has so far been carried out for the polytropic family of models—i.e., for configurations in which $\rho = \rho_c \theta^n$, where the function θ is a solution of the differential equation

$$\frac{d}{da}\left(a^2 \frac{d\theta}{da}\right) + a^2\theta^n = 0, \tag{2-40}$$

subject to the initial conditions $\theta(0) = 1$ and $\theta'(0) = 0$, and n is a constant interior to the interval $0 < n < 5$. A survey of the values of $\eta_j(a_1)$ and $k_j = \frac{1}{2}(\Delta_j - 1)$ as obtained by Brooker and Olle* for this particular family of models is given in the accompanying Tables 2-1 and 2-2. A full list of references to numerical integrations of Clairaut's or Radau's equations for other models will be given in the Bibliographical Notes at the end of this chapter. Before reviewing such integrations we wish, however, to exhibit two further analytical procedures by which the solutions of equations (2-1) or (2-3) can be approximated, for *arbitrary* density distribution, if the density concentration inside our configuration is either weak or pronounced (i.e., is in the neighbourhood of the homogeneous or centrally-condensed model).

In order to consider the former case first, we rewrite (with Radau) equation (2-3) alternatively in the form

$$\frac{d}{da}\{\bar{\rho} a^5 \sqrt{1 + \eta_j}\} = 5 \bar{\rho} a^4 \psi_j(\eta), \tag{2-41}$$

where we have abbreviated

$$\psi_j(\eta) = \frac{4 + j(j+1) + 5\eta_j - \eta_j^2}{10\sqrt{1 + \eta_j}}. \tag{2-42}$$

An expansion of (2-42) in ascending powers of η_j yields

$$\psi_j(\eta) = \frac{4 + j(j+1)}{10} + \frac{6 - j(j+1)}{20}\eta_j + \frac{3j(j+1) - 16}{80}\eta_j^2 + \dots \tag{2-43}$$

Now Radau noticed that, for the second-harmonic distortion ($j = 2$), this expansion reduces to

$$\psi_2(\eta) = 1 + \tfrac{1}{40}\eta_2^2 + \dots; \tag{2-44}$$

and since (in accordance with 2-4) $\eta_2(a) = 0$ for the homogeneous model, it

* R. A. Brooker and T. W. Olle, *M.N.*, **115**, 101, 1955.

TABLE 2-1

Values of $\eta_j(a_1)$

$n \backslash j$	2	3	4	5	6	7
0	0·0	1·0	2·0	3·0	4·0	5·0
1	1·28987	2·77126	4·03229	5·19855	6·31464	7·40065
1·5	1·886337	3·330844	4·532536	5·647172	6·720567	7·771184
2	2·3558682	3·6743708	4·7975208	5·8605512	6·8976928	7·9215832
2·5	2·6741873	3·8601625	4·9225255	5·9513727	6·9670118	7·9763746
3	2·8596251	3·9485820	4·9746967	5·9885823	6·9909869	7·9939855
3·25	2·9145691	3·9711040	4·9866323	5·9927584	6·9956613	7·9972088
3·5	2·9512885	3·9848142	4·9934119	5·9966130	6·9980576	7·9987963
3·75	2·9744672	3·9926882	4·9970328	5·9985530	6·9992072	7·9995284
4	2·9880797	3·9968781	4·9988199	5·9994568	6·9997157	7·9998370
4·25	2·9953218	3·9988870	4·9996095	5·9998305	6·9999154	7·9999534
4·5	2·9986395	3·9997085	4·9999050	5·9999619	6·9999832	7·9999896
4·75	2·9998017	3·9999628	4·9999885	5·9999944	6·9999950	7·9999955
5	3·0	4·0	5·0	6·0	7·0	8·0

TABLE 2-2

Values of $k_j = \frac{1}{2}(\Delta_j - 1)$

$n \backslash j$	2	3	4	5	6	7
0	0·75	0·375	0·25	0·1875	0·15	0·125
1	0·259910	0·106454	0·060241	0·039292	0·027827	0·020810
1·5	0·1432792	0·0528489	0·0273930	0·0165691	0·0109835	0·0077454
2	0·07393839	0·02439400	0·01150774	0·00641997	0·00396610	0·00262763
2·5	0·03485234	0·01019200	0·00434151	0·00222015	0·00127200	0·00078876
3	0·01444298	0·00369989	0·00140970	0·00051953	0·00034690	0·00020056
3·25	0·00869160	0·00207256	0·00074375	0·00032938	0·00016693	0·00009306
3·5	0·00491907	0·00108706	0·00036627	0·00015400	0·00007472	0·00004013
3·75	0·00256639	0·00052282	0·00016490	0·00006578	0·00003049	0·00001572
4	0·00119488	0·00022309	0·00006557	0·00002469	0·00001093	0·00000543
4·25	0·00046826	0·00007951	0·00002170	0·00000770	0·00000325	0·00000155
4·5	0·00013609	0·00002082	0·00000528	0·00000173	0·00000065	0·00000035
4·75	0·00001983	0·00000266	0·00000064	0·00000025	0·00000019	0·00000015
5	0	0	0	0	0	0

follows that, for configurations whose density concentration is weak, the expansion of $\psi_2(\eta)$ reduces essentially to its first term and, to a high degree of approximation, $\psi_2(\eta)$ may be replaced by unity. If so, however, equation (2-41) reduces to

$$\frac{d}{da}\{\bar{\rho}a^5\sqrt{1+\eta_2}\} - 5\bar{\rho}a^4 = 0, \tag{2-45}$$

and can be integrated as it stands to yield

$$\sqrt{1+\eta_2(a)} = \frac{5}{\bar{\rho}a^5}\int_0^a \bar{\rho}a^4\,da. \tag{2-46}$$

This approximate solution agrees strictly with equations (2-5) valid for the homogeneous model. Furthermore, for the Laplace model (i.e., polytrope $n = 1$) characterized by the density distribution (2-20),

$$\bar{\rho}(a) = \frac{3}{a^3}\int_0^a \rho a^2\, da = \frac{3\rho_c}{a^3}\{\sin a - a\cos a\} \tag{2-47}$$

which, inserted in (2-46), yields on the surface ($a_1 = \pi$) an approximate solution

$$\eta_2(\pi) = \frac{225}{\pi^4} - 1 = 1{\cdot}3098458\ldots, \tag{2-48}$$

deviating from the exact solution, as given by the first one of equations (2-26), by 2% in excess. Thus, for a density concentration characterized by the ratio $\rho_c/\rho_m = \rho_c/\bar{\rho}(\pi) = \pi^2/3 = 3{\cdot}28987\ldots$, the approximation based on the simplified equation (2-46) is still tolerable; but fails completely in the neighbourhood of a centrally-condensed model, as the reader may verify by comparing its consequences with equations (2-6).

The significance of the particular metamorphosis (2-45) of Radau's equation rests not only on the circumstance that (2-45) is solvable by quadratures, but also on the fact that the density function $\rho(a)$ is involved in it only through the mean density $\bar{\rho}(a)$ interior to a. Therefore, within the scheme of approximation inherent in (2-45), *the boundary value of η_2 depends on the whole march of density distribution only through $\bar{\rho}(a)$.* This is an important result; though its validity is limited only to the second-harmonic distortion of a configuration exhibiting a weak degree of central condensation. It may be noted that, as far as the computation of external form of distorted configurations is concerned, the foregoing result goes a long way to meet all our needs; for (as will be proved below), with increasing degree of central condensation, Δ_2 (and still more Δ_3, Δ_4, etc.) approaches unity asymptotically with such speed as to make it possible to set $\Delta_2 = \Delta_3 = \Delta_4 = 1$ without appreciable error—whatever the detailed behaviour of the function $\rho(a)$ may be. The external form of a j-th harmonic distortion (which is proportional to Δ_j) is, however, only one of our interests; the other being the corresponding contributions of distortion to the external potential V which (by 1-35) are proportional to the differences $\Delta_j - 1$. Therefore, in order to investigate the values of this difference for centrally-condensed configurations for an arbitrary density-distribution $\rho(a)$ another approximation different from the above must obviously be sought.

In order to develop it, let us return to Radau's equation (2-3) and inquire as to the methods of constructing such solutions of it which, in addition to fulfilling the initial conditions (2-4), are required to assume at $a = a_1$ the values of $\eta_j(a_1) \equiv \lambda$ (say). If we revert from $\eta_j(a)$ to the variable y defined by the transformation (2-12), this problem is tantamount to a search for characteristic solutions of the linear boundary-value problem

$$L[y] = g(a)y, \qquad y(0) = 0, \qquad y'(a_1) = \lambda y(a_1), \tag{2-49}$$

where the operator

$$L \equiv \frac{d^2}{da^2} - \frac{j(j+1)}{a^2}, \tag{2-50}$$

and λ becomes a characteristic parameter.

The operator L is evidently self-adjoint. Its Green's function appropriate for the given boundary conditions will, therefore, be symmetrical and of the form

$$G(a, \xi) = -\frac{1}{2j+1}\{\xi^{-j} + C_j \xi^{j+1}\} a^{j+1}, \qquad (\xi \geqslant a), \tag{2-51}$$

where we have abbreviated

$$a_1^{2j+1} C_j = \frac{j + a_1\lambda}{j - a_1\lambda + 1} = \frac{1}{\Delta_j - 1} \tag{2-52}$$

by (1-27). With the aid of this Green's function our differential boundary-value problem (2-49) can be rewritten in the form of the integral equation

$$\begin{aligned}(2j+1)y(a) = &-a^{j+1}C_j \int_0^{a_1} g(\xi)y(\xi)\xi^{j+1}\,d\xi \\ &-a^{-j}\int_0^{a} g(\xi)y(\xi)\xi^{j+1}\,d\xi \\ &-a^{j+1}\int_a^{a_1} g(\xi)y(\xi)\xi^{-j}\,d\xi,\end{aligned} \tag{2-53}$$

which is equivalent in fact to Clairaut's equation (1-23).

We have seen earlier in this section (*cf.* equation 2-6) that, for a mass-point model, $\eta_j(a) = j+1$ and $\Delta_j = 1$, in consequence of which $C_j = \infty$. If, moreover, the density concentration in our configuration becomes finite but high, the C_j's will likewise become finite, though they are likely to remain numerically very large. Now the three integrals on the right-hand side of (2-53) are all quantities of the same order of magnitude, but only the first one is magnified by multiplication by C_j to such an extent as to render the other two terms very small in comparison with the first. Therefore, to a good approximation, the integral equation (2-53) may be replaced by

$$(2j+1)y(a) = -a^{j+1}C_j \int_0^{a_1} g(\xi)y(\xi)\xi^{j+1}\,d\xi; \tag{2-54}$$

but since the limits of integration on the right-hand side are now constant, the solution of this equation is clearly of the form

$$y(a) = Aa^{j+1}, \tag{2-55}$$

where A is a constant. To this degree of approximation, therefore,

$$Y_j(a) = \frac{Aa^{j+1}}{\int_0^a \rho a^2\, da} \tag{2-56}$$

and

$$\eta_j(a) = j + 1 - \frac{\rho a^3}{\int_0^a \rho a^2\, da} \tag{2-57}$$

represent generalizations of equations (2-6), valid if the internal density concentration of our configuration is finite though high—whatever $\rho(a)$ may be otherwise. Since, moreover, equation (2-54) is homogeneous in the dependent variable, its validity requires that the value of C_j involved in it be, not arbitrary, but specified by the equation

$$C_j^{-1} = a_1^{2j+1}(\Delta_j - 1) = -\frac{3}{2j+1}\int_0^{a_1} \frac{1}{\bar{\rho}}\frac{d\rho}{da}\, a^{2j+1}\, da, \tag{2-58}$$

which thus expresses the value of $\Delta_j - 1$ in terms of a simple quadrature of the ratio $\rho'/\bar{\rho}$.

It may be noted that the integral on the right-hand side of (2-58) admits of the following further simplification. A differentiation of the expression (2-2) defining $\bar{\rho}$ reveals that

$$\frac{a}{3}\frac{d\bar{\rho}}{da} = \rho - \bar{\rho} \tag{2-59}$$

which permits us to integrate the right-hand side of (2-58) by parts to obtain

$$\int_0^{a_1} \frac{1}{\bar{\rho}}\frac{d\rho}{da}\, a^{2j+1}\, da = 3\int_0^{a_1} D^2 a^{2j}\, da - 2(j+2)\int_0^{a_1} D a^{2j}\, da, \tag{2-60}$$

where we have abbreviated

$$D \equiv \frac{\rho}{\bar{\rho}}. \tag{2-61}$$

For centrally-condensed configurations $D \ll 1$ and, therefore, $D^2 \ll D$. In such cases, the first integral on the right-hand side of (2-60) is likely to be very small in comparison with the second. Neglecting it, we arrive at the approximate formula

$$a_1^{2j+1}(\Delta_j - 1) = \frac{6(j+2)}{2j+1}\int_0^{a_1} D a^{2j}\, da, \tag{2-62}$$

expressing the values of the constants Δ_j for centrally-condensed configurations in terms of a simple quadrature of the ratio $D \equiv \rho/\bar{\rho}$, weighted by a^{2j}.

It transpires, therefore, that for such configurations the values of Δ_j are determined by the behaviour of the ratio $\rho/\bar{\rho}$ (no longer, as in 2-46, of $\bar{\rho}$ alone) throughout the interval $0 \leqslant a \leqslant a_1$; with the relative contribution of the outer parts of the interior becoming the more important, the higher the value of j. If, lastly, we approximate $\bar{\rho}$ by an expression

$$\bar{\rho} = \frac{3m_1}{4\pi a^3} \tag{2-63}$$

appropriate for mass-point model, equation (2-62) simplifies further to

$$a_1^{2j+1}(\Delta_j - 1) = \frac{8\pi(j+2)}{(2j+1)m_1} \int_0^{a_1} \rho a^{2j+3}\, da, \tag{2-64}$$

revealing the approximate connection which exists between the constants Δ_j and the respective *moments of inertia* of centrally-condensed configurations.

In the general case of unrestricted $\rho(a)$ the boundary-value problem (2-49) does not admit of analytic solutions of even approximate validity, and recourse must be had to numerical integrations. As this represents a somewhat laborious process (facilitated as it may be by resort to automatic computing machines) whose directness leaves something to be desired, it may be pertinent to inquire into such general features common to all possible solutions as may be inferred from formal properties of the underlying differential equation (2-3). In considering their behaviour in the a-η plane, we may note that (2-3) defines the first derivative η_j' in terms of a and η_j uniquely everywhere except at the origin. Along the axis $a = 0$ the function $\eta_j'(a)$ ceases to be holomorphic, and (2-3) can be satisfied by η_j assuming the alternative values of $j - 2$ or $-j - 3$. At the points $(0, j - 2)$ and $(0, -j - 3)$ the integral curves obeying equation (2-3) are thus not uniquely defined; but it can be shown* that, of the total manifold of such curves passing through an arbitrary point of the a-η plane, only *one* will pass through $(0, j - 2)$; all others passing through $(0, -j - 3)$. Therefore, *the initial condition* $\eta_j(0) = j - 2$ *is sufficient to ensure the uniqueness of the particular solution of the non-linear equation* (2-3) which is of interest to us in this connection.

In further quest for general properties of such solutions, let us return to the equation (1-29) which, on integration by parts of its right-hand side, assumes the form

$$a^j Y_j^i \left\{ [j + 1 - \eta_j(a)] \int_0^a \rho a^2\, da - \rho a^3 \right\} = - \int_0^a \frac{d\rho}{da} a^{j+3} Y_j^i\, da. \tag{2-65}$$

Since, however,

$$\int_0^a \rho a^2\, da = \tfrac{1}{3} a^3 \bar{\rho} \tag{2-66}$$

* *Cf.* H. Poincaré, *Leçons sur les Figures d'Equilibre*, Paris 1903, pp. 67–81.

and

$$\rho = \bar{\rho}\left\{1 + \frac{1}{3}\frac{a}{\bar{\rho}}\frac{d\bar{\rho}}{da}\right\} = \bar{\rho}(1 - \tfrac{1}{3}f), \tag{2-67}$$

it follows from (2-65) that

$$a^{j+3}Y_j^i\{j - 2 + f - \eta_j(a)\}\bar{\rho} = 3\int_0^a \left(-\frac{d\rho}{da}\right) a^{j+3}Y_j^i\,da. \tag{2-68}$$

Now let us assume, in what follows, that $\rho(a)$ is a diminishing function of the radius. If so, however, both sides of (2-68) must be non-negative,* and this can obviously be true only if

$$\eta_j(a) \leqslant j - 2 + f. \tag{2-69}$$

The quantity f introduced by equation (2-67) above is, in turn, constrained to obey the inequality

$$0 \leqslant f \leqslant 3; \tag{2-70}$$

its left-hand side being necessarily true if $\bar{\rho}'$ in (2-67) is negative; the right-hand side, because ρ is positive. In consequence, any variation of $\eta_j(a)$ possible under these conditions will be bracketed by

$$j - 2 \leqslant \eta_j(a) \leqslant j - 2 + f \leqslant j + 1. \tag{2-71}$$

Let us inquire next as to the behaviour of the particular solutions of Radau's equation (2-3), consistent with the initial condition $\eta_j(0) = j - 2$, in the (η-η')-plane. For the limiting model of a homogeneous configuration $f = 0$ and $\rho/\bar{\rho} = 1$, so that (2-3) reduces to the parabola

$$a\eta' = j(j + 1) - (\eta + 2)(\eta + 3); \tag{2-72}$$

while for a centrally-condensed model ($f = 3$, $\rho/\bar{\rho} = 0$) equation (2-3) reduces likewise to

$$a\eta' = j(j + 1) - \eta(\eta - 1). \tag{2-73}$$

On the other hand, for the values of $f = \eta + 2 - j$ at which (2-69) becomes an equality,

$$a\eta' = (j + 1)(j - 2) - (2j - 1)\eta + \eta^2. \tag{2-74}$$

Any point of the solution of Radau's equation (2-3) for a configuration intermediate in structure between homogeneous and a mass-point model must lie, in the (η-η')-plane, in the region limited between the foregoing parabolae; in point of fact, for such models as render the integral on the right-hand side of equation (2-68) positive, the permitted region is comprised solely between the parabolae (2-72) and (2-74) intersecting at the points

* It may be noted that the positivity of both sides of (2-68) and, therefore, the validity of (2-69) does *not* necessarily require ρ'(a) to remain negative throughout the interior. This latter condition is sufficient, but not necessary, for this to be true. The validity of (2-69) would, for instance, not be impaired by an occurrence of positive density gradients in the interior, provided that these are not sufficient to alter the sign of the *integral* on the right-hand side of (2-68).

$j - 2$ and $j + 1$. A plot of these limiting curves is diagrammatically shown on the accompanying Fig. 2-2; and these delimit an area in which $\eta_j'(a)$ can be both positive and negative.

In order to investigate further the possible behaviour of $\eta_j(a)$ in this area, let us differentiate Radau's equation (2-3) with respect to a to obtain

$$a\eta'' + 2\eta'(\eta + 3) = 2(f\eta)' + 2f'. \tag{2-75}$$

If η'' is to vanish anywhere within this area, either both sides of this equation

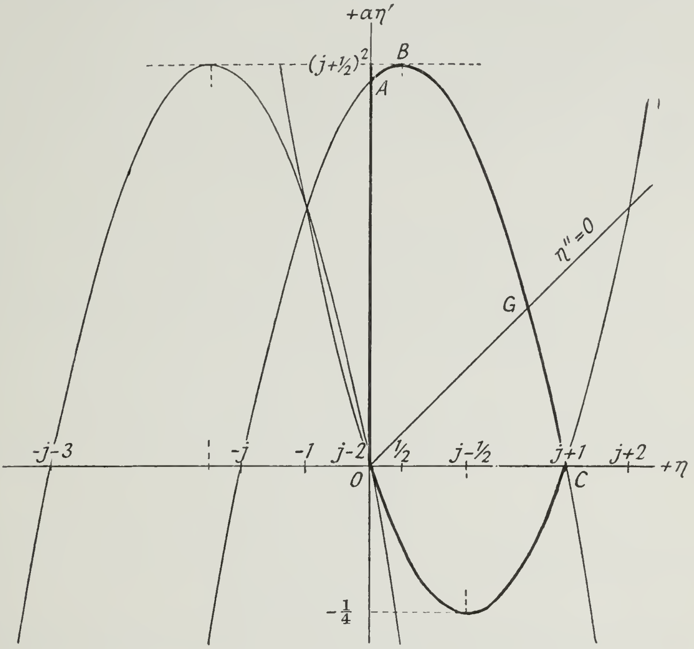

FIGURE 2-2.

must vanish identically (i.e., η must pass through an inflection point and increase thereafter), or we must have

$$\eta'(\eta + 3) = (f\eta)' + f'. \tag{2-76}$$

This latter equation can be readily integrated as it stands, and its particular solution satisfying the initial condition $\eta_j(0) = j - 2$ assumes the form

$$2f(\eta + 1) = \eta(\eta + 6) - (j - 2)(j + 4), \tag{2-77}$$

which inserted in (2-3) yields

$$a\eta' = \eta - j + 2. \tag{2-78}$$

Therefore, the locus $\eta'' = 0$ of inflection points will be represented on Fig. 2-2 by a straight line, passing through the point $\eta = j - 2$, and inclined by 45° to the locus $\eta'' = 0$ of the extrema which coincides with the η-axis.

At the centre of our configuration $\eta_j(0) = j - 2$, and in its immediate neighbourhood $\eta'(a) > 0$. Therefore, $\eta_j(a)$ will at first be an increasing function of a; but its subsequent behaviour may differ in different parts of the diagram. Thus, in the domain OABG of Fig. 2-2 we have $\eta'' > 0$, η' positive and increasing, and thus η increasing throughout. In the domain OGC, $\eta'' < 0$ and η' is therefore diminishing but still positive, so that η increases at a diminishing rate. Lastly, in the region OCD we find that both η'' and η' are negative and, therefore, η will be diminishing throughout. The general trend of variation of $\eta_j(a)$ in the interior of our configuration can, therefore, be clearly specified by the surface values of $\eta_j(a_1)$ and $\eta_j'(a_1)$ alone. If $\eta_j'(a_1) > 0$, $\eta_j(a)$ will be an increasing function throughout the interior; or a decreasing function if $\eta_j'(a_1) < 0$. Since, moreover, the line $\eta'' = 0$ lies above that of $\eta' = 0$ between $j - 2 < \eta < j + 1$, it is obvious that $\eta_j(a)$ can never have a minimum in the interior. Therefore, *if $\eta_j(a)$ is an increasing function at the surface of our configuration, it will remain so throughout its mass; but if it is diminishing at the surface, it will possess a maximum in the interior.*

The fact that the line $\eta'' = 0$ lies above that of $\eta' = 0$ for $\eta > j - 2$ in the (η-η')-coordinates (see again Fig. 2-2) entails one specific consequence restricting possible density distribution of our configurations. In order to demonstrate it, let us return to equation (2-75) and solve it for $\eta_j''(a)$ at a point when $\eta_j'(a) = 0$: we find that, at that point,

$$\eta'' = 2(\eta + 1)f. \tag{2-79}$$

Now the fact that the function $\eta_j''(a)$ cannot have a minimum for $0 < a < a_1$ reveals that f' must remain positive throughout that domain—i.e., that

$$\frac{d}{da}\left(\frac{\rho}{\bar{\rho}}\right) \leqslant 0 \tag{2-80}$$

or, in more specific terms, that

$$3\rho\bar{\rho} \leqslant 3\rho^2 - \bar{\rho}(a\rho'). \tag{2-81}$$

The equality sign holds good when $\rho = \bar{\rho} =$ constant, and corresponds to $f' = 0$ leading to an inflection point $\eta'' = \eta' = 0$ at $\eta = j - 2$. We found in fact before that the known solution (2-5) of Radau's equation (2-3) for the homogeneous model reduces indeed to the point $\eta_j(a) = j - 2$ in the (η-η')-plane. The conditions

$$\rho' < 0 \qquad \text{and} \qquad \rho'' < 0, \tag{2-82}$$

or

$$\bar{\rho}' < 0 \qquad \text{and} \qquad \bar{\rho}'' < 0, \tag{2-83}$$

stated by earlier writers* are sufficient, but not necessary, for the validity of the inequalities (2-80) or (2-81).

II.3. KINETIC ENERGY

In the preceding two sections the potential energy W of the components of close binary systems has been specified, and their mutual distortion

* *Cf.* A. Véronnet, *Journ. de Math.*, (6) **8**, 331, 1912.

described to the order of accuracy to which the disturbing effect of one component upon another can be regarded as that of a mass-point. The aim of the present section will be to ascertain, to the same degree of approximation, the *kinetic energy* T of our dynamical system, whose determination should enable us to formulate the corresponding Lagrangian or Hamiltonian functions $T \pm W$ and thus to open the road for a study of rotational and orbital motions of both components.

As is well known, the total kinetic energy of the system will consist of the sum of the kinetic energy T_0 of orbital motion of the two components, augmented by the kinetic energies $T_{1,2}$ of their rotation (and vibration). In order to formulate them, let us adopt a rectangular frame of reference XYZ whose axes are *fixed* in space. If, moreover, its origin is allowed to coincide with the centre of gravity of the binary system, the kinetic energy of the orbital motion assumes the simple form

$$T_0 = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2), \tag{3-1}$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) denote the coordinates of the centres of gravity of the two components of masses $m_{1,2}$. In what follows we find it, however, convenient to shift the origin of coordinates from the centre of gravity of the system to that of the primary component by setting

$$\left.\begin{aligned} x_2 - x_1 &= x, \\ y_2 - y_1 &= y, \\ z_2 - z_1 &= z, \end{aligned}\right\} \tag{3-2}$$

where, by the integrals of the mass-centre,

$$\left.\begin{aligned} m_1x_1 + m_2x_2 &= 0, \\ m_1y_1 + m_2y_2 &= 0, \\ m_1z_1 + m_2z_2 &= 0, \end{aligned}\right\} \tag{3-3}$$

so that

$$T_0 = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}\{\dot{x}^2 + \dot{y}^2 + \dot{z}^2\}, \tag{3-4}$$

where the coordinates x, y, z describe the instantaneous position of the secondary component in its relative orbit around the primary star.

The rotational energies $T_{1,2}$ of the two components are, quite generally, given by the well-known expression

$$T = \tfrac{1}{2}\{A\omega_x^2 + B\omega_y^2 + C\omega_z^2\} - D\omega_x\omega_y - E\omega_x\omega_z - F\omega_y\omega_z + G, \tag{3-5}$$

where A, B, C are the *moments of inertia*

$$A = \int (y^2 + z^2)\, dm, \tag{3-6}$$

$$B = \int (x^2 + z^2)\, dm, \tag{3-7}$$

$$C = \int (x^2 + y^2)\, dm, \tag{3-8}$$

of each component with respect to the fixed space-axes XYZ;

$$D = \int yz \, dm, \tag{3-9}$$

$$E = \int xz \, dm, \tag{3-10}$$

$$F = \int xy \, dm, \tag{3-11}$$

are the corresponding *products of inertia*; $\omega_{x,y,z}$ denote the angular velocity components relative to the respective space-axes; and, for *fluid* bodies,

$$G = \tfrac{1}{2} \int \dot{\mathbf{h}}^2 \, dm \tag{3-12}$$

represents the kinetic energy of possible *oscillations* of the respective star—whether free or forced (such as, for instance, non-radial oscillations bound to be invoked by varying radius-vector in close binary systems which exhibit eccentric orbits)—where **h** denotes the (vector) displacement of any particle. The limits of integration in equations (3-6)–(3-12) are to be extended over the whole sphere.

As is well-known, the full-dress expression (3-5) for rotational kinetic energy can be simplified by a suitable change of coordinates. If the axis of rotation were perpendicular to the XY-plane, all three products of inertia as given by equations (3-9)–(3-11) would vanish identically for each component; and the same feature can be retained if such products are referred to a new system of *body*-axes, having the same origin but *rotating* with the respective component and defined so that the $X'Y'$-plane coincides with the equatorial plane of the rotating star, Z' being in the direction of its axis of rotation. The transformation of coordinates between the fixed space-axes XYZ and rotating body-axes $X'Y'Z'$ is then governed by the following scheme of the direction cosines

	x'	y'	z'
x	a'_{11}	a'_{12}	a'_{13}
y	a'_{21}	a'_{22}	a'_{23}
z	a'_{31}	a'_{32}	a'_{33}

where

$$\left.\begin{aligned} a'_{11} &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi, \\ a'_{12} &= -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi, \\ a'_{13} &= \sin\theta\sin\phi; \end{aligned}\right\} \tag{3-13}$$

$$\left.\begin{aligned} a'_{21} &= \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi, \\ a'_{22} &= -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi, \\ a'_{23} &= -\sin\theta\cos\phi; \end{aligned}\right\} \tag{3-14}$$

$$\left.\begin{aligned} a'_{31} &= \sin\psi\sin\theta, \\ a'_{32} &= \cos\psi\sin\theta, \\ a'_{33} &= \cos\theta; \end{aligned}\right\} \tag{3-15}$$

the respective Eulerian angles ϕ, θ, ψ being defined as shown on the accompanying Fig. 2-3.

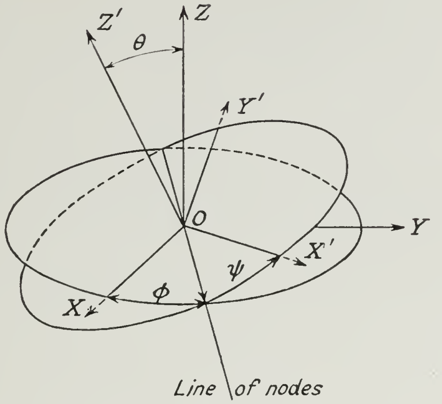

FIGURE 2-3. DEFINITIONS OF THE COORDINATE SYSTEMS AND OF EULERIAN ANGLES.

The virtue of the transformation of coordinates from space- to body-axes, as embodied in the equations

$$\left.\begin{aligned} x &= a'_{11}x' + a'_{12}y' + a'_{13}z', \\ y &= a'_{21}x' + a'_{22}y' + a'_{23}z', \\ z &= a'_{31}x' + a'_{32}y' + a'_{33}z', \end{aligned}\right\} \tag{3-16}$$

rests on the fact that while, as a result, $A = A'$, $B = B'$, $C = C'$, *all three products of inertia, D, E, F vanish when referred to the body-axes*; and, hence, equation (3-5) simplifies into

$$2T' = A'\omega_{x'}^2 + B'\omega_{y'}^2 + C'\omega_{z'}^2 + G, \tag{3-17}$$

where the new velocity components in the primed system are expressible in terms of $\omega_{x,y,z}$ by means of the equations

$$\left.\begin{aligned} \omega_{x'} &= a'_{11}\omega_x + a'_{21}\omega_y + a'_{31}\omega_z, \\ \omega_{y'} &= a'_{12}\omega_x + a'_{22}\omega_y + a'_{32}\omega_z, \\ \omega_{z'} &= a'_{13}\omega_x + a'_{23}\omega_y + a'_{33}\omega_z. \end{aligned}\right\} \tag{3-18}$$

As, moreover, the angular velocities of rotation $\omega_{x,y,z}$ relative to the space-axes XYZ are expressed in terms of our Eulerian angles θ, ϕ, ψ and their time derivatives by

$$\left.\begin{aligned} \omega_x &= \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi, \\ \omega_y &= \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi, \\ \omega_z &= \qquad\qquad \dot{\psi}\cos\theta \qquad + \dot{\phi}, \end{aligned}\right\} \tag{3-19}$$

it follows from (3-18) and (3-13)–(3-15) that

$$\left.\begin{aligned} \omega_{x'} &= \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi, \\ \omega_{y'} &= \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi, \\ \omega_{z'} &= \dot{\phi} \cos\theta \qquad\qquad\qquad + \dot{\psi}. \end{aligned}\right\} \tag{3-20}$$

The foregoing equation (3-17) specifies the kinetic energy of a star arising from free axial rotation and oscillation; and if this star were single (or rigid), our expression for kinetic energy would be complete. For fluid components of close binary systems we must, however, still augment it by the *kinetic energy of tidal bulge*, raised by the disturbing star, which is bound to remain pointed in the direction of the latter's centre of gravity and sweep around the distorted star in the plane, and with the period, of the binary orbit. It should, moreover, be stressed that this period need not be related in any way to the period of axial rotation; and the inclination of the equator (of axial rotation) to the orbital plane may likewise be arbitrary.

In order to account for the kinetic energy of tidal bulge, we find it convenient to introduce a third system of doubly-primed rectangular coordinates $X''Y''Z''$, with the same origin as our preceding coordinate systems, but defined so that its X''-axis coincides constantly with the radius-vector joining the centres of the two components, and the $X''Y''$-plane coincides with that of the binary orbit. Like the singly-primed system, the doubly-primed axes are, therefore, likewise in motion relative to the fixed space-axes XYZ, and rotate in a period equal to that of the binary orbit. Within the scheme of approximations adopted in section II.1, the tidal bulge is then symmetrical with respect to the X''-axis; and the transformation of coordinates from space- to doubly-primed axes is governed by the scheme of the direction cosines

	x''	y''	z''
x	a''_{11}	a''_{12}	a''_{13}
y	a''_{21}	a''_{22}	a''_{23}
z	a''_{31}	a''_{32}	a''_{33}

with

$$\left.\begin{aligned} a''_{11} &= \cos u \cos\Omega - \cos i \sin\Omega \sin u, \\ a''_{12} &= -\sin u \cos\Omega - \cos i \sin\Omega \cos u, \\ a''_{13} &= \sin i \sin\Omega; \end{aligned}\right\} \tag{3-21}$$

$$\left.\begin{aligned} a''_{21} &= \cos u \sin\Omega + \cos i \cos\Omega \sin u, \\ a''_{22} &= -\sin u \sin\Omega + \cos i \cos\Omega \sin u, \\ a''_{23} &= -\sin i \cos\Omega; \end{aligned}\right\} \tag{3-22}$$

$$\left.\begin{aligned} a''_{31} &= \sin u \sin i, \\ a''_{32} &= \cos u \sin i, \\ a''_{33} &= \cos i; \end{aligned}\right\} \tag{3-23}$$

where Ω, i, u denote, successively, the longitude of the node (from the X-axis), the inclination of the orbital and XY-planes, and the longitude of the disturbing component in its orbital plane as measured from the node. If, moreover, ω denotes, as usual, the longitude of the periastron (i.e., the angle between the nodal and apsidal lines) and v, the true anomaly of the disturbing component,

$$u = \omega + v. \tag{3-24}$$

The reader may note that should we equate $\phi = \Omega$, $\theta = i$, and $\psi = u$, the singly- and doubly-primed sets of direction cosines $\{a'_{ij}\}$ and $\{a''_{ij}\}$ would become identical.

If we adopt the doubly-primed system of axes $X''Y''Z''$ as our frame of reference, the kinetic energy of tidal bulge will be given by the equation

$$2T'' = A''\omega_{x''}^2 + B''\omega_{y''}^2 + C''\omega_{z''}^2 \tag{3-25}$$

with

$$\left.\begin{aligned} \omega_{x''} &= a''_{11}\omega_x + a''_{21}\omega_y + a''_{31}\omega_z, \\ \omega_{y''} &= a''_{12}\omega_x + a''_{22}\omega_y + a''_{32}\omega_z, \\ \omega_{z''} &= a''_{13}\omega_x + a''_{23}\omega_y + a''_{33}\omega_z, \end{aligned}\right\} \tag{3-26}$$

where $\omega_{x,y,z}$ continue to be given by equations (3-19) and A'', B'', C'' are the moments of inertia of the tidal bulge with respect to the respective doubly-primed axes.

In order to specify completely the kinetic energy due to the rotation of the components, it remains to evaluate their moments of inertia about the singly- and doubly-primed axes. Let us illustrate this process on one moment—say that about the x'-axis—and for others merely state the results. Thus in accordance with equation (3-6),

$$\left.\begin{aligned} A' &= \int_0^{r_1}\int_0^{2\pi}\int_0^{\pi} \rho r^4(\sin^2\alpha'\sin^2\beta' + \cos^2\beta')\sin\beta'\,dr\,d\alpha'\,d\beta' \\ &= 2\int_0^{r_1}\int_{-1}^{1}\int_{-\sqrt{1-\mu'^2}}^{\sqrt{1-\mu'^2}} \rho r^4\frac{(1-\lambda'^2)}{\lambda'}\,dr\,d\mu'\,d\nu', \end{aligned}\right\} \tag{3-27}$$

where

$$\left.\begin{aligned} x' &= r\cos\alpha'\sin\beta' = r\lambda', \\ y' &= r\sin\alpha'\sin\beta' = r\mu', \\ z' &= r\cos\beta' \qquad\quad\;\; = r\nu', \end{aligned}\right\} \tag{3-28}$$

and which by means of Clairaut's artifice (1-10) can be expressed as

$$A' = 2\int_0^{a_1}\int_{-1}^{1}\int_{-\sqrt{1-\mu'^2}}^{\sqrt{1-\mu'^2}} \rho r^4\frac{\partial r}{\partial a}\frac{1-\lambda'^2}{\lambda'}\,da\,d\mu'\,d\nu' \tag{3-29}$$

where, to a first approximation

$$r = a\{1 + Y_2 + \ldots\} \tag{3-30}$$

in accordance with (1-11). As, to the first order in Y_2,

$$\rho r^4 \frac{\partial r}{\partial a} = \rho a^4 + \rho \frac{\partial}{\partial a}(a^5 Y_2) \tag{3-31}$$

and, by (1-30)

$$\int_0^{a_1} \rho \frac{\partial}{\partial a}(a^5 Y_2)\, da = \frac{5c_2 a_1^5}{4\pi G}(\Delta_2 - 1)P_2(\theta, \phi), \tag{3-32}$$

it follows that

$$\begin{aligned} A' = 2 \int_0^{a_1}\int_{-1}^{1}\int_{-\sqrt{1-\mu'^2}}^{\sqrt{1-\mu'^2}} \rho a^4 \frac{1-\lambda'^2}{\lambda'}\, da\, d\mu'\, d\nu' \\ + \frac{5(\Delta_2 - 1)c_2 a_1^5}{2\pi G} \int_{-1}^{1}\int_{-\sqrt{1-\mu'^2}}^{\sqrt{1-\mu'^2}} \frac{1-\lambda'^2}{\lambda'} P_2\, d\mu'\, d\nu' \end{aligned} \tag{3-33}$$

where, of course, $\lambda'^2 = 1 - \mu'^2 - \nu'^2$.

The evaluation of the integrals on the right-hand side of (3-33) now offers no difficulties. Remembering that, for the distortion caused by rotation about an axis coincident with the Z'-axis, $P_2(\theta, \phi) \equiv P_2(\nu')$ and, by (1-61), $c_2 = -\frac{1}{3}\omega^2$, equation (3-33) readily yields

$$A' = \tfrac{8}{3}\pi \int_0^{a_1} \rho a^4\, da - \frac{(\Delta_2 - 1)\omega^2 a_1^5}{9G}. \tag{3-34}$$

Moreover, the corresponding expressions for B' and C' can be obtained, in accordance with (3-7), (3-8) and (3-28), by replacing merely

$$y^2 + z^2 \equiv r^2(1 - \lambda'^2) \tag{3-35}$$

in the integrand of (3-31) by

$$x^2 + z^2 \equiv r^2(1 - \mu'^2) \tag{3-36}$$

and

$$x^2 + y^2 \equiv r^2(1 - \nu'^2), \tag{3-37}$$

respectively. Doing so we find that

$$B' = \tfrac{8}{3}\pi \int_0^{a_1} \rho a^4\, da - \frac{(\Delta_2 - 1)\omega^2 a_1^5}{9G} \tag{3-38}$$

and

$$C' = \tfrac{8}{3}\pi \int_0^{a_1} \rho a^4\, da + \frac{2(\Delta_2 - 1)\omega^2 a_1^5}{9G}. \tag{3-39}$$

These results make it evident that, to the order of accuracy we have been working,

$$A' = B' \tag{3-40}$$

but

$$C' - A' = C' - B' = \tfrac{1}{3}(\Delta_2 - 1)\frac{\omega^2 a_1^5}{G}. \tag{3-41}$$

It should be added that, in the case of *rigid* bodies, the angular velocity ω of axial rotation of the respective component is an arbitrary constant (which may, or may not, be equal to the Keplerian angular velocity of orbital motion). However, for *fluid* bodies, which are capable of adjusting their form to the instantaneous centrifugal force,

$$\omega = \dot{\psi} + \dot{\phi}\cos\theta \tag{3-42}$$

and, as a result, ω can be regarded as constant only if $\dot{\psi} + \dot{\phi}\cos\theta =$ constant happens to be an integral of the Eulerian equations of motion (*cf.* section IV.4).

Lastly, the moments of inertia, with respect to the doubly-primed axes, associated with the tidal bulge readily follow from essentially the same expressions—i.e.,

$$A'' = \frac{5(\Delta_2 - 1)c_2 a_1^5}{2\pi G}\int_{-1}^{1}\int_{-\sqrt{1-\mu''^2}}^{\sqrt{1-\mu''^2}} \frac{1-\lambda''^2}{\lambda''} P_2\, d\mu''\, d\nu'' \tag{3-43}$$

etc., where doubly-primed direction cosines (of an arbitrary radius-vector in the doubly-primed frame of reference) have replaced the singly-primed ones, $P_2 \equiv P_2(\lambda'')$ and $c_2 = Gm_2/R^3$ in accordance with (1-58). Evaluating such expressions we establish that

$$A'' = -\tfrac{2}{3}(\Delta_2 - 1)m_2\frac{a_1^5}{R^3} + \ldots, \tag{3-44}$$

$$B'' = +\tfrac{1}{3}(\Delta_2 - 1)m_2\frac{a_1^5}{R^3} + \ldots, \tag{3-45}$$

and

$$C'' = +\tfrac{1}{3}(\Delta_2 - 1)m_2\frac{a_1^5}{R^3} + \ldots, \tag{3-46}$$

i.e., that, within the scheme of our approximation,

$$B'' = C'', \tag{3-47}$$

but

$$B'' - A'' = C'' - A'' = (\Delta_2 - 1)m_2\frac{a_1^5}{R^3}. \tag{3-48}$$

The foregoing expressions for the moments of inertia of the tidal bulge hold good for stars of arbitrary structure, but take account of only second-harmonic distortion. For third- and fourth-harmonic distortions stars of arbitrary structure the doubly-primed moments of inertia can no longer be evaluated in a closed form. If, however, such stars are centrally-condensed to the extent to which equation (2-56) represents a valid approximation to the solution of (2-1)—so that

$$Y_j(a_1) = c_j a_1^{j+1} \tag{3-49}$$

where, by (1-58) and (2-56),

$$c_j = \frac{m_2}{m_1} \frac{G}{R^{2j+1}}, \tag{3-50}$$

equation (3-29) can be integrated directly—without recourse to (3-31)—for any value of j to yield

$$A'' = -\frac{64\pi}{15R^3} \frac{m_2}{m_1} \int_0^{a_1} \rho a^7 \, da - \frac{6\pi}{R^4} \frac{m_2}{m_1} \int_0^{a_1} \rho a^8 \, da + \dots \tag{3-51}$$

and, similarly,

$$B'' = C'' = \frac{32\pi}{15R^3} \frac{m_2}{m_1} \int_0^{a_1} \rho a^7 \, da - \frac{3\pi}{2R^4} \frac{m_2}{m_1} \int_0^{a_1} \rho a^8 \, da + \dots; \tag{3-52}$$

the contributions of the fourth-harmonic distortion ($j = 4$) being identically zero. If, moreover, we take advantage of the fact that, in accordance with (2-64),

$$m_1(\Delta_2 - 1)a_1^5 = \frac{32\pi}{5} \int_0^{a_1} \rho a^7 \, da \tag{3-53}$$

to the same degree of approximation, the first terms on the right-hand sides of (3-51) and (3-52) prove indeed identical with those on the right-hand sides of the preceding equations (3-44)–(3-46); but terms due to third harmonics are new.

It may, furthermore, be added that if (in accordance with 3-40) $A' = B'$, the expression (3-17) for the kinetic energy of a rotationally-distorted configuration simplifies into

$$\begin{aligned} 2T' &= A'(\omega_{x'}^2 + \omega_{y'}^2) + C'\omega_{z'}^2 \\ &= A'(\dot\phi^2 \sin^2\theta + \dot\theta^2) + C'(\dot\phi\cos\theta + \dot\psi)^2 \end{aligned} \tag{3-54}$$

by virtue of (3-20); while if (in accordance with 3-47) $A'' = -2B'' = -2C''$, the expression (3-25) for the kinetic energy of the tidal bulge reduces to

$$\begin{aligned} 2T'' &= C''\{-2\omega_{x''}^2 + \omega_{y''}^2 + \omega_{z''}^2\} \\ &= -C''\{\dot\theta^2 P_2(l_1) + \dot\phi^2 P_2(l_2) + \dot\psi^2 P_2(l_3) \\ &\qquad + 3l_1 l_2 \dot\theta\dot\phi + 3l_1 l_3 \dot\theta\dot\psi + (3l_2 l_3 - \cos\theta)\dot\phi\dot\psi\}, \end{aligned} \tag{3-55}$$

where

$$\left.\begin{aligned} l_1 &= a''_{11}\cos\phi + a''_{21}\sin\phi, \\ l_2 &= \qquad\qquad\qquad\qquad + a''_{31}, \\ l_3 &= a'_{13}a''_{11} + a'_{23}a''_{21} + a'_{33}a''_{31}, \end{aligned}\right\} \tag{3-56}$$

denote cosines of the angles between the radius-vector (i.e., the X''-axis) and the nodal line in the XY-plane (of direction cosines $\cos\phi$, $\sin\phi$, 0), the

Z-axis (of direction cosines 0, 0, 1), and the Z'-axis (of direction cosines a'_{13}, a'_{23}, a'_{33}), respectively.

It may lastly be noted that, as long as squares and higher powers of stellar distortion are regarded to be negligible, the term (3-12) characterizing the oscillatory energy of the stars becomes a quantity of the second order even in an acoustic approximation, and can likewise be ignored. To this order of accuracy, the total kinetic energy $T_{1,2}$ of each component is represented by the sum

$$T_{1,2} = T'_{1,2} + T''_{1,2}, \tag{3-57}$$

where T' and T'' are defined by equations (3-54) and (3-55)* given earlier in this section. The total kinetic energy T of the binary system then becomes equal to

$$T = T_0 + T_1 + T_2, \tag{3-58}$$

where the kinetic energy T_0 of orbital motion has been specified by equation (3-4).

II.4. EQUATIONS OF MOTION

In the preceding sections of this chapter explicit expressions have been established approximating the potential and kinetic energy of close binary systems to the degree of accuracy to which the gravitational attraction of one component upon another can be regarded as that of a mass-point, and both components to rotate as rigid bodies. Reasons were, however, listed before why such approximations should be ample for application to close binary systems; and subject to their ultimate limitations, the Lagrangian equations of motion governing the dynamics of close binary systems may now be set forth in explicit form.

In order to do so we should remember that, of the complete expression (1-38) for the potential energy W of the system occurring on the right-hand side of equations (0-1), the terms $W_{1,2}$ due to the self-attraction of the two components remain constant to the first order in small quantities; and only the term W_{12} arising from mutual attraction of both bodies contains the independent variables q of our problem through its terms of zero and first order. Therefore, for dynamical purposes we may set $W = W_{12}$ as given by equation (1-41); and if we remember that the direction cosines of the centre of mass of either disturbing component as viewed from that of its mate are 1, 0, 0 (so that $P_j(\lambda) = 1$ for all values of j) W_{12} will assume (within the

* The singly- and doubly-primed moments of inertia as given earlier in this section refer explicitly to the primary component of mass m_1 and radius a_1. The corresponding expressions for the secondary are, however, obtained by a mere replacement of a_1 and m_2 by a_2 and m_1, respectively.

scheme of our approximation) the more explicit form

$$W_{12} = G\frac{m_1m_2}{r}\left\{1 + \sum_{i=1}^{2}\sum_{j=2}^{4}\frac{(\Delta_j)_i - 1}{2}\frac{m_{3-i}}{m_i}\left(\frac{a_i}{r}\right)^{2j+1} - \sum_{i=1}^{2}\frac{\omega_i^2}{4\pi G\bar{\rho}_i}\frac{(\Delta_2)_i - 1}{2}\left(\frac{a_i}{r}\right)^2 P_2(\cos\Theta_i)\right\}, \tag{4-1}$$

where r stands for the radius-vector of the relative orbit of the two components of our binary system, and $\Theta_{1,2}$ denote the angles between this radius-vector and the axis of rotation of the respective star. With kinetic energy T_0 of orbital motion as given by equation (3-1), and the rotational and tidal kinetic energies T_i' and T_i'' specified by equations (3-54) and (3-55), the formation of the corresponding Lagrangian equations (0-1) of motion calls for no further prerequisites.

The total number of the degrees of freedom characterizing our dynamical problem and involved explicitly in our expressions for the potential W_{12} and the kinetic energy T then becomes equal to *twelve*: six rectangular coordinates x_i, y_i, z_i ($i = 1, 2$) of the centres of gravity of the two components in space; and six Eulerian angles θ_i, ϕ_i, ψ_i specifying the positions of their axes of rotation. An identification of the centre of gravity of the primary component (of mass m_1) with the origin of our fixed set of space-axes, implied in the transformation (3-2), will reduce the degrees of freedom from twelve to *nine*, and render x, y, z the coordinates of the secondary's centre in its relative orbit around the primary star. If r denotes the radius-vector in this relative orbit, the position of the secondary in the doubly-primed system of axes of the preceding section II.3 simply becomes

$$x'' = r, \qquad y'' = 0, \qquad z'' = 0; \tag{4-2}$$

so that, relative to the space-axes,

$$\left.\begin{aligned} x &= ra_{11}'' = r(\cos\Omega\cos u - \sin\Omega\sin u\cos i),\\ y &= ra_{21}'' = r(\sin\Omega\cos u + \cos\Omega\sin u\cos i),\\ z &= ra_{31}'' = r(\sin i\sin u); \end{aligned}\right\} \tag{4-3}$$

or, changing over from rectangular to spherical polar coordinates,

$$\left.\begin{aligned} x &= r\cos P\sin Q,\\ y &= r\sin P\sin Q,\\ z &= r\cos Q, \end{aligned}\right\} \tag{4-4}$$

where

$$\left.\begin{aligned} P &= \Omega + \tan^{-1}(\cos i\tan u),\\ Q &= \qquad\cos^{-1}(\sin i\sin u). \end{aligned}\right\} \tag{4-5}$$

The kinetic energy (3-4) of the orbital motion in terms of these polar coordinates becomes

$$T_0 = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\{\dot{r}^2 + r^2(\dot{P}^2 \sin^2 Q + \dot{Q}^2)\}, \tag{4-6}$$

and the cosines of the angles Θ between the radius-vector and the axis of rotation of the respective components, figuring on the right-hand side of equation (4-1), should be given by

$$\cos\Theta = \cos\theta\cos Q + \sin\theta\sin Q\sin(\phi - P). \tag{4-7}$$

In order to arrive at the desired equations of motion, it is sufficient now to identify the variables q in the Lagrangian equations (0-1) successively with x, y, z or r, P, Q and θ_i, ϕ_i, ψ_i, and perform the requisite partial differentiation of T and W_{12}. The reader may note that the orbital part of the kinetic energy as given by equation (4-6) proves to depend explicitly on two coordinates r, Q and three velocities $\dot{r}$, $\dot{P}$ and $\dot{Q}$. The rotational kinetic energy T' as given by (3-54) depends only on θ and $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$;* while the kinetic energy T'' of tidal origin (equation 3-55) depends on r (through the respective moments of inertia), P, Q (through l_1, l_2, l_3), as well as on two Eulerian angles θ, ϕ and the time-derivatives $\dot{\theta}$, $\dot{\phi}$, of all three. Lastly, the potential energy W_{12} as given by equation (4-1) depends on r through rotational as well as tidal terms, and on P, Q as well as θ, ϕ through rotational terms involving Θ. Therefore, the three differential equations governing the motion of the secondary component in its relative orbit around the primary star will assume the form

$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{r}}\right) - \frac{\partial}{\partial r}(T_0 + T''_{12}) = \frac{\partial W_{12}}{\partial r}, \tag{4-8}$$

$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{P}}\right) - \frac{\partial T''_{12}}{\partial P} = \frac{\partial W_{12}}{\partial P}, \tag{4-9}$$

$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{Q}}\right) - \frac{\partial}{\partial Q}(T_0 + T''_{12}) = \frac{\partial W_{12}}{\partial Q}; \tag{4-10}$$

or, more explicitly,

$$\frac{d^2 r}{dt^2} - r\left\{\left(\frac{dP}{dt}\right)^2 \sin^2 Q + \left(\frac{dQ}{dt}\right)^2\right\} + \frac{G(m_1 + m_2)}{r} = \frac{\partial R_{12}}{\partial r}, \tag{4-11}$$

$$\frac{d}{dt}\left\{r^2 \sin Q\,\frac{dP}{dt}\right\} = \frac{\partial R_{12}}{\partial P}, \tag{4-12}$$

$$\frac{d}{dt}\left\{r^2\frac{dQ}{dt}\right\} - r^2\sin Q\cos Q\left(\frac{dP}{dt}\right)^2 = \frac{\partial R_{12}}{\partial Q}, \tag{4-13}$$

* The reader may note that T, is independent of ϕ for any distortion of a configuration rotating like a rigid body, but becomes independent of ψ only if $A' = B'$.

where the function R_{12} on the right-hand side is defined by the relation

$$W_{12} + T''_{12} = Gm_1m_2 \left\{ \frac{1}{r} + R_{12} \right\}, \tag{4-14}$$

and where we have abbreviated $T''_1 + T''_2 \equiv T''_{12}$.

The Eulerian differential equations governing the motion of each component about its respective centre of gravity follow likewise from (0-1) if we identify q in them successively with $\theta_{1,2}$, $\phi_{1,2}$ and $\psi_{1,2}$. Doing so and retaining only the non-zero terms we arrive at the following six fundamental equations

$$\frac{d}{dt}\left(\frac{\partial T'_i}{\partial \dot{\theta}_i} + \frac{\partial T''_i}{\partial \dot{\theta}_i}\right) - \frac{\partial T'_i}{\partial \theta_i} = Gm_1m_2 \frac{\partial R_{12}}{\partial \theta_i}, \tag{4-15}$$

$$\frac{d}{dt}\left(\frac{\partial T'_i}{\partial \dot{\phi}_i} + \frac{\partial T''_i}{\partial \dot{\phi}_i}\right) = Gm_1m_2 \frac{\partial R_{12}}{\partial \phi_i}, \tag{4-16}$$

$$\frac{d}{dt}\left(\frac{\partial T'_i}{\partial \dot{\psi}_i} + \frac{\partial T''_i}{\partial \dot{\psi}_i}\right) = 0. \tag{4-17}$$

The system (4-11)–(4-13) consists of three simultaneous differential equations, while the system (4-15)–(4-17) constitutes six simultaneous equations (three for each component), all of second order. In close binaries whose components behave like fluid bodies both these systems are, moreover, coupled through the derivatives of T'' on the right-hand sides and constitute, therefore, a simultaneous set of nine differential equations of 18-th order. The systems (4-11)–(4-13) and (4-15)–(4-17) would become independent only if the relative orbit of both components was circular (i.e., if $r =$ constant) *and* the axis of rotation with constant angular velocity (i.e., $\omega = \dot{\psi} + \dot{\phi} \cos \theta =$ constant) were constantly perpendicular to the orbital plane (i.e., if $\phi_1 = \phi_2 = \Omega$ and $\theta_1 = \theta_2 = i$). Under these conditions the distortion of both components would remain constant, and the two bodies would behave exactly as if they were rigid. An assumption of rigidity would, moreover, automatically fulfil the condition $\dot{\psi} + \dot{\phi} \cos \theta =$ constant (and thus render the amount of polar flattening independent of time).

However, even rigidity would not release the coupling between the systems (4-11)–(4-13) and (4-15)–(4-17) if the axes of rotation are inclined to the orbital plane; for the precession and nutation ensues which is bound to be coupled with the motion of the orbital plane. Moreover, a restoration of fluidity of both components will raise the tides travelling around each component in the orbital plane (which need not coincide with its equator) with the Keplerian angular velocity of orbital motion (which need not be related to the speed of axial rotation), and the height of such tides should vary with the radius-vector. Under these conditions, providing for still further bonds between equations (4-11)–(4-13) and (4-15)–(4-17), the two sets could be regarded as independent only if the necessary conditions

$$\left.\begin{aligned} r &= \text{constant}, \\ \phi_1 = \phi_2 = \Omega &= \text{constant}, \\ \theta_1 = \theta_2 = i &= \text{constant}, \\ \dot{\psi}_i + \dot{\phi}_i \cos\theta_i &= \text{constant}, \end{aligned}\right\} \tag{4-18}$$

happen to be the integrals of our dynamical system.

Later in this book we shall show that the first and the last conditions may indeed come close to such integrals and be fulfilled very nearly in a large majority of known close binary systems. On the other hand, no feature of our systems (or indeed no forces extraneous to them) are known to constrain the axes of rotation of the components in close binaries to remain fixed in space and perpendicular to the orbital plane: even if the conditions $\theta = i$ and $\phi = \Omega$ were approximately fulfilled at some particular moment, they cannot remain so constrained for all time. In consequence, the systems (4-11)–(4-13) and (4-15)–(4-17) must be regarded as simultaneous in any realistic treatment of the dynamics of close binary systems, and solved accordingly. A construction of such solutions represents the task which will be taken up in subsequent sections of this chapter.

Before embarking upon it we wish, however, to point out that the system of equations (4-11)–(4-13) admits also of alternative formulation. In particular we may assume—without any loss of generality—that the orbits governed by them are curves, in a plane specified by the angles Ω and i, of the form

$$r = A(1 - e\cos E), \tag{4-19}$$

where the eccentric anomaly E is related with the time t by means of the well-known equations

$$\left.\begin{aligned} E - e\sin E &= n(t - T), \\ n^2 &= \frac{G(m_1 + m_2)}{A^3} \end{aligned}\right\} \tag{4-20}$$

of the problems of two bodies. Such orbits are, in general, specified in space by the six elements

$$\Omega,\ i,\ A,\ e,\ \omega \text{ and } \varepsilon = \bar{\omega} - nT,$$

which, in the absence of perturbations, would represent the constant values of the longitude of the node Ω, orbital inclination i, semi-major axis A of the relative orbit, its eccentricity e, longitude of periastron $\bar{\omega}$ (referred, like Ω, to the fixed space-axis x),* and the difference ε between the true and mean anomaly of the periastron passage; moreover, n would stand for the mean daily motion.

* The reader should note that if, as usual, ω denotes the longitude of periastron in the orbital plane as measured from the direction of ascending node, $\bar{\omega} = \omega + \Omega$; but the angles ω and Ω are *not* measured in the same plane.

The effect of the *perturbations* due to rotation and tides will be to render all these elements, not constants, but *functions of the time* governed by the equations

$$\frac{1}{An}\frac{d\Omega}{dt} = \frac{1}{\sqrt{1-e^2}\sin i}\frac{\partial S_{12}}{\partial i}, \tag{4-21}$$

$$\frac{1}{An}\frac{di}{dt} = -\frac{1}{\sqrt{1-e^2}\sin i}\frac{\partial S_{12}}{\partial \Omega} - \frac{\tan\frac{1}{2}i}{\sqrt{1-e^2}}\left\{\frac{\partial S_{12}}{\partial \bar{\omega}} + \frac{\partial S_{12}}{\partial \varepsilon}\right\}, \tag{4-22}$$

$$\frac{1}{A^2n}\frac{dA}{dt} = 2\frac{\partial S_{12}}{\partial \varepsilon}, \tag{4-23}$$

$$\frac{1}{An}\frac{de}{dt} = -\sqrt{1-e^2}\left\{\frac{1-\sqrt{1-e^2}}{e}\frac{\partial S_{12}}{\partial \varepsilon} + \frac{1}{e}\frac{\partial S_{12}}{\partial \bar{\omega}}\right\}, \tag{4-24}$$

$$\frac{1}{An}\frac{d\bar{\omega}}{dt} = \frac{\tan\frac{1}{2}i}{\sqrt{1-e^2}}\frac{\partial S_{12}}{\partial i} + \frac{\sqrt{1-e^2}}{e}\frac{\partial S_{12}}{\partial e}, \tag{4-25}$$

$$\frac{1}{An}\frac{d\varepsilon}{dt} = \frac{\tan\frac{1}{2}i}{\sqrt{1-e^2}}\frac{\partial S_{12}}{\partial i} + \sqrt{1-e^2}\,\frac{1-\sqrt{1-e^2}}{e}\frac{\partial S_{12}}{\partial e} - 2A\frac{\partial S_{12}}{\partial A}, \tag{4-26}$$

where the *perturbing function* S_{12} is defined by the equation

$$W_{12} = Gm_1m_2\left\{\frac{1}{r} + S_{12}\right\} \tag{4-27}$$

and W_{12} continues to be given by (4-1).

The system of six first-order differential equations (4-21)–(4-26) is equivalent to the system (4-11)–(4-13) of three second-order equations governing the relative motion of the components in all aspects; but its adoption entails certain advantages. In order to point them out it is scarcely necessary to emphasize that our dynamical problem admits of no closed solution; and such solutions as we may seek will have to be obtained by successive approximations. Now as long as (consistent with our scheme of approximation adopted in section II.1) squares and higher powers of dominant perturbation terms are held ignorable, the orbits we wish to study should depart from closed conics of a classical two-body problem likewise only by quantities of the first order. In other words, as we are primarily interested in approximate solutions of our dynamical problem which are adjacent to closed conics, the adoption of equations (4-21)–(4-26) to solve in place of a frontal attack on the second order equations (4-11)–(4-13) possesses advantages which will become fully apparent in subsequent sections of this chapter.

After these preliminary remarks concerning the first half of the equations of motion of our problem, let us turn our attention to the Eulerian equations (4-15)–(4-17). A glance at (4-17) reveals that (within the scheme of our approximation) the angle ψ turns out to be of the nature of a cyclic coordinate; for integrating (4-17) as it stands with respect to t, we establish at once that

$$\frac{\partial T_i'}{\partial \dot{\psi}_i} + \frac{\partial T_i''}{\partial \dot{\psi}_i} = N_i \, , \tag{4-28}$$

where the N_i's are constants. Now a partial differentiation of (3-54) and (3-55) with respect to $\dot{\psi}$ yields

$$\frac{\partial T'}{\partial \dot{\psi}} = C'(\dot{\phi} \cos \theta + \dot{\psi}) \tag{4-29}$$

and

$$\frac{\partial T''}{\partial \dot{\psi}} = C''(\dot{\phi} \cos \theta + \dot{\psi}) - 3C'' l_3(l_1\dot{\theta} + l_2\dot{\phi} + l_3\dot{\psi}) \tag{4-30}$$

so that, according to (4-28),

$$(C' + C'')(\dot{\phi} \cos \theta + \dot{\psi}) - 3C'' l_3(l_1\dot{\theta} + l_2\dot{\phi} + l_3\dot{\psi}) = N \tag{4-31}$$

separately for each component. In this equation l_3 stands for the cosine of the angle between the respective axis of rotation and the radius vector; and unless the inclination between the orbital and equatorial planes of the two components is large, l_3 can be regarded as a small quantity whose product with C'' becomes ignorable. Moreover, in accordance with (3-39) and (3-46), the sum

$$C_i' + C_i'' = \tfrac{8}{3}\pi \int_0^{a_1} \rho a^4 \, da + \frac{\Delta_2 - 1}{3} \left\{ \frac{\omega_i^2 a_i^5}{3G} + \frac{m_j a_i^5}{r^3} \right\} , \tag{4-32}$$

for $j = 3 - i$ and $i = 1, 2$. The two terms in curly brackets on the right-hand side will be constant only to the extent to which ω and r remain so, but are much smaller (on account of the factor $\Delta_2 - 1$) than the first. If we ignore them and limit ourselves to an approximation

$$C_i' + C_i'' = \tfrac{8}{3}\pi \int_0^{a_1} \rho a^4 \, da = M_i \, , \tag{4-33}$$

the integral (4-31) can be interpreted to assert that

$$\dot{\phi} \cos \theta + \dot{\psi} = \frac{N}{M} = \omega, \tag{4-34}$$

the ratio N/M becoming identical with the *angular velocity* ω *of rotation* of the respective component, which (subject to our above approximations) proves to be *constant*.

The physical meaning of the argument which leads to (4-34) can be re-stated also in the following words. Our neglect of terms factored by C'' in equation (4-31) implies that each component is allowed to rotate freely and unaffected by tides. The tidal waves will merely superimpose upon the rotational spheroid and float passively on its surface following always the direction of radius-vector, but should not interact with the proper speed of axial rotation. As is well known, interaction between rotation and tides can occur only through tidal friction. Such friction is indeed bound to be operative in close binary systems, but its effects can make themselves felt only over long intervals of time. If our aim were to follow processes which may unroll in 10^7 or 10^8 years, it would be necessary to consider detailed coupling between T' and T'' in equation (4-31), consistent with the constancy of the total angular momentum of the system. Our principal aim in this monograph is much more modest: namely, to study dynamical phenomena which are likely to manifest observable consequences in time intervals of the order of 10 or 100 years; and if so, tidal friction can safely be ignored and our approximations inherent in (4-34) regarded as ample.

Accepting, therefore, the integral (4-34) of equation (4-17) as a basis for our subsequent work, we may take advantage of the fact that both T' and T'' depend on ψ only through $\dot{\psi}$, and eliminate this angular variable from (4-15) and (4-16) with the aid of the relation (4-34). In doing so we are left with θ_i and ϕ_i as the only dependent Eulerian variables, and the order of our remaining differential problem has thus been reduced by four. If bar over T' is used to denote the rotational kinetic energy subject to the constraint (4-34), we easily establish that

$$\begin{aligned} \frac{\partial \bar{T}'}{\partial \dot{\theta}} &= A'\dot{\theta}, \\ \frac{\partial \bar{T}'}{\partial \dot{\phi}} &= A'\dot{\phi} \sin^2 \theta + C'\omega \cos \theta, \end{aligned} \tag{4-35}$$

and, consequently,

$$\frac{d}{dt}\left(\frac{\partial \bar{T}'}{\partial \dot{\theta}}\right) = A'\ddot{\theta} \tag{4-36}$$

$$\frac{d}{dt}\left(\frac{\partial \bar{T}'}{\partial \dot{\phi}}\right) = A'(\ddot{\phi} \sin^2 \theta + \dot{\phi}\dot{\theta} \sin^2 \theta) - C'\omega\, \dot{\theta} \sin \theta; \tag{4-37}$$

moreover,

$$\frac{\partial \bar{T}'}{\partial \theta} = A'\dot{\phi}^2 \sin \theta \cos \theta - C'\omega\dot{\phi} \sin \theta, \tag{4-38}$$

and

$$\frac{\partial \bar{T}'}{\partial \phi} = 0. \tag{4-39}$$

The respective derivatives of the tidal kinetic energy $\bar{T}''$ so constrained can be formulated with equally little difficulty. Setting

$$2\frac{\partial \bar{T}''}{\partial \dot{\theta}} = -C''(f_{11}\dot{\theta} + f_{21}\dot{\phi} + f_{31}\omega), \tag{4-40}$$

$$2\frac{\partial \bar{T}''}{\partial \dot{\phi}} = -C''(f_{12}\dot{\theta} + f_{22}\dot{\phi} + f_{32}\omega), \tag{4-41}$$

we easily establish that

$$\left.\begin{aligned} f_{11} &= 2P_2(l_1), \\ f_{22} &= 2P_2(l_2 - l_3 \cos\theta) + \cos^2\theta, \\ f_{12} &= 3l_1(l_2 - l_3\cos\theta) = f_{21}, \\ f_{31} &= 3l_1 l_3, \\ f_{32} &= 3l_3(l_2 - l_3\cos\theta). \end{aligned}\right\} \tag{4-42}$$

Moreover, taking advantage of the fact that

$$\left.\begin{aligned} \sin\theta\,\frac{\partial l_1}{\partial\phi} &= l_2\cos\theta - l_3, \\ \sin\theta\,\frac{\partial l_3}{\partial\theta} &= l_3\cos\theta - l_2, \\ \frac{\partial l_3}{\partial\phi} &= l_1\sin\theta, \end{aligned}\right\} \tag{4-43}$$

while

$$\frac{\partial l_3}{\partial\theta} = \frac{\partial l_2}{\partial\theta} = \frac{\partial l_2}{\partial\phi} = 0, \tag{4-44}$$

it can be shown that

$$2\,\frac{\sin\theta}{C''}\,\frac{\partial\bar{T}''}{\partial\theta} = g_{11}\dot\theta^2 + g_{22}\dot\phi^2 + g_{33}\omega^2 + g_{12}\dot\theta\dot\phi + g_{13}\dot\theta\omega + g_{23}\dot\phi\omega, \tag{4-45}$$

and

$$2\,\frac{\sin\theta}{C''}\,\frac{\partial\bar{T}''}{\partial\phi} = h_{11}\dot\theta^2 + h_{22}\dot\phi^2 + h_{33}\omega^2 + h_{12}\dot\theta\dot\phi + h_{13}\dot\theta\omega + h_{23}\dot\phi\omega, \tag{4-46}$$

where

$$\left.\begin{aligned} g_{11} &= 0, \\ g_{22} &= 2\{3l_3(l_2 - l_3)P_2(\cos\theta) + \cos^3\theta P_2(l_3) - \cos\theta P_2(l_2)\}, \\ g_{33} &= 3l_3\{l_2 - l_3\cos\theta\}, \\ g_{12} &= 3l_1\{2l_3\cos^2\theta - l_2\cos\theta + l_3\}, \\ g_{13} &= 3l_1\{l_2 - l_3\cos\theta\}, \\ g_{23} &= 3\{3l_3^2\cos^2\theta - 3l_1l_2\cos\theta + l_2^2 - l_3^2\}, \end{aligned}\right\} \tag{4-47}$$

and

$$\left.\begin{aligned} h_{11} &= 3l_1(l_3 - l_2\cos\theta), \\ h_{22} &= -3l_1(l_3 - l_2\cos\theta)\sin^2\theta, \\ h_{33} &= -3l_1l_3\sin^2\theta, \\ h_{12} &= 3\{l_1^2\sin^2\theta + (l_2 - l_3)(l_3 - l_2\cos\theta)\}, \\ h_{13} &= -3\{l_1^2\sin^2\theta - l_3(l_3 - l_2\cos\theta)\}, \\ h_{23} &= -3(l_2 - 2l_3\cos\theta)l_1\sin^2\theta. \end{aligned}\right\} \tag{4-48}$$

The time-derivatives of equations (4-40) and (4-41) are, however, somewhat more complicated. Thus

$$\frac{2r}{C''}\frac{d}{dt}\left(\frac{\partial \bar{T}''}{\partial \dot{\theta}}\right) = -r\{f_{11}\ddot{\theta} + f_{21}\ddot{\phi} + \dot{f}_{11}\dot{\theta} + \dot{f}_{21}\dot{\phi} + \dot{f}_{31}\omega\} + 3\dot{r}\{f_{11}\dot{\theta} + f_{21}\dot{\phi} + f_{31}\omega\} \tag{4-49}$$

and

$$2\frac{r}{C''}\frac{d}{dt}\left(\frac{\partial \bar{T}''}{\partial \dot{\phi}}\right) = -r\{f_{12}\ddot{\theta} + f_{22}\ddot{\phi} + \dot{f}_{12}\dot{\theta} + \dot{f}_{22}\dot{\phi} + \dot{f}_{32}\omega\} + 3\dot{r}\{f_{12}\dot{\theta} + f_{22}\dot{\phi} + f_{32}\omega\} \tag{4-50}$$

where, for example,

$$\dot{f}_{11} = 6l_1\dot{l}_1 = 6l_1\{(\dot{a}''_{21} - \dot{\phi}a''_{11})\sin\phi + (\dot{a}''_{11} + \dot{\phi}a''_{21})\cos\phi\} \tag{4-51}$$

and

$$\left.\begin{aligned} \dot{a}''_{11} &= (-a''_{21})\dot{\Omega} + (a''_{12})\dot{u} + (a''_{13}\sin u)\dot{i}, \\ \dot{a}''_{21} &= (+a''_{11})\dot{\Omega} + (a''_{22})\dot{u} + (a''_{23}\sin u)\dot{i}. \end{aligned}\right\} \tag{4-52}$$

All other $\dot{f}_{ij}$'s can, moreover, be evaluated in exactly the same manner.

It is, however, perhaps unnecessary to proceed with this somewhat lengthy task to complete the explicit formulation of our Eulerian equations, as the foregoing developments should be sufficient to illustrate the process. The time-derivatives $\dot{a}''_{ij}$'s involved in the $\dot{f}_{ij}$'s are clearly expressible in terms of $\dot{\Omega}$, $\dot{i}$, and $\dot{u}$. Therefore, the left-hand sides of the equations (4-15) and (4-16) contain, in addition to the first and second time-derivatives of the Eulerian angles θ and ϕ, also $\dot{\Omega}$ and $\dot{i}$ (which themselves are defined by equations 4-21 and 4-22); and $\dot{r}$, $\dot{u}$ which, in turn, are expressible in terms of the four elements $\dot{a}$, $\dot{e}$, $\dot{\omega}$ and $\dot{\varepsilon}$ of the plane problem of two bodies. Through all these terms (in addition to those in R_{12}) the systems (4-15)–(4-16) and (4-21)–(4-26) are interdependent. The complexity of this full-dress system of differential equations at the basis of our dynamical problem is, therefore, considerable; and any attempt at its solution must inevitably start by introducing further simplifications to ease our task.

II.5. PRECESSION AND NUTATION OF FLUID COMPONENTS

The above remarks with which we concluded the preceding section, concerning the solvability of the dynamical problem outlined in it, did not perhaps fully reveal the magnitude of the problem and understated its difficulty. The reader who will, however, take the trouble to write out the explicit form of the corresponding equations of motion, with all their non-linear terms in full, will need little convincing to realize that a search for closed analytic solution of our problem by any methods known to mathematical analysis so far would constitute a task which would be more than difficult: it would be utterly hopeless. In order to learn anything at all on at least the essential features of our dynamical problem, the only approach which will enable us to bring known analytical methods to bear on its solution will be to *linearize* our equations of motion by suitable assumptions as to the range (or rate) of variation of the individual elements. As will be shown in this and the subsequent section, so linearized a system of equations does indeed admit of analytic solution in not too complicated terms; and such a solution will in fact prove adequate to disclose the principal types of motion likely to be exhibited in close binary systems, which are consistent with the general equations of our problem as set forth in the preceding section of this chapter.

In order to do so, let us return to the equations (4-15) and (4-16) which govern the behaviour of the remaining Eulerian angles θ and ϕ: by insertion from equations (4-36)–(4-39) the former can be made to assume the more explicit forms

$$A_i'\ddot{\theta}_i - A_i'\dot{\phi}_i^2 \sin\theta_i \cos\theta_i + C_i'\dot{\phi}_i\omega_i \sin\theta_i = Gm_1m_2 \frac{\partial R_{12}}{\partial \theta_i} - \frac{d}{dt}\left(\frac{\partial T_i''}{\partial \dot{\theta}_i}\right) \tag{5-1}$$

and

$$A_i'(\ddot{\phi}_i \sin^2\theta_i + \dot{\theta}_i\dot{\phi}_i \sin^2\phi_i) - C_i'\dot{\theta}_i\omega_i \sin\theta_i = Gm_1m_2 \frac{\partial R_{12}}{\partial \phi_i} - \frac{d}{dt}\left(\frac{\partial T_i''}{\partial \dot{\theta}_i}\right), \tag{5-2}$$

where, by (4-14) and (4-27),

$$\left.\begin{aligned} \frac{\partial R_{12}}{\partial \theta_i} &= \frac{\partial S_{12}}{\partial \theta_i} + \frac{1}{Gm_1m_2}\frac{\partial \bar{T}_{12}''}{\partial \theta_i}, \\ \frac{\partial R_{12}}{\partial \phi_i} &= \frac{\partial S_{12}}{\partial \phi_i} + \frac{1}{Gm_1m_2}\frac{\partial \bar{T}_{12}''}{\partial \phi_i}. \end{aligned}\right\} \tag{5-3}$$

To proceed further with our solution, let us hereafter assume that *the first derivatives $\dot{\theta}$ and $\dot{\phi}$ of the Eulerian angles are quantities so small that their squares and higher powers* (*as well as products with other small quantities*) *can be neglected*; and that *their second derivatives $\ddot{\theta}$ and $\ddot{\phi}$ are likewise ignorable.* Furthermore, let us consider the relative orbit of the two stars to be so nearly circular that the products of $\dot{r}$ with all other small quantities of first order can also be neglected. If so, however, the right-hand sides of the preceding equations (5-1) and (5-2) reduce to

$$C_i'\omega_i\dot{\phi}_i \sin\theta_i = Gm_1m_2 \frac{\partial S_{12}}{\partial \theta_i} + \frac{\partial \bar{T}_{12}}{\partial \theta_i} - \frac{d}{dt}\left(\frac{\partial \bar{T}_i''}{\partial \dot{\theta}_i}\right), \tag{5-4}$$

$$-C_i'\omega_i\dot{\theta}_i \sin\theta_i = Gm_1m_2 \frac{\partial S_{12}}{\partial \phi_i} + \frac{\partial \bar{T}_{12}''}{\partial \phi_i} - \frac{d}{dt}\left(\frac{\partial \bar{T}_i''}{\partial \dot{\phi}_i}\right), \tag{5-5}$$

where, it may be remembered,

$$S_{12} = S_1 + S_2 \tag{5-6}$$

and

$$\bar{T}_{12} = \bar{T}_1'' + \bar{T}_2''. \tag{5-7}$$

Now from equation (4-1) it follows that, to the first order in small quantities,

$$S_i = -\frac{\alpha_i}{r^3} P_2(\cos \Theta_i) + \sum_{j=2}^{4} \frac{\beta_{ij}}{r^{2(j+1)}}, \tag{5-8}$$

where

$$\alpha_i = \frac{a_i^2 \omega_i^2 (\Delta_{i2} - 1)}{8\pi G \bar{\rho}_i} \tag{5-9}$$

or, by (3-41),

$$\alpha_i = \frac{C_i' - A_i'}{2m_i} = \frac{\Delta_{i2} - 1}{6} \left\{1 + \frac{m_{3-i}}{m_i}\right\} \frac{\gamma_i a_i^5}{A^3} \tag{5-10}$$

in which A (without subscript) stands, as before, for the semi-major axis of the relative orbit. The quantity

$$\gamma_i = \frac{\omega_i}{n} \tag{5-11}$$

denotes the ratio of the actual angular velocity ω_i of axial rotation of the respective component to the Keplerian angular velocity (i.e., the mean daily motion) defined by

$$n^2 = G \frac{m_1 + m_2}{A^3}; \tag{5-12}$$

and

$$\beta_{ij} = \tfrac{1}{2}(\Delta_{ij} - 1) \frac{m_{3-i}}{m_i} a_i^{2j+1}. \tag{5-13}$$

Moreover, within the scheme of approximation adopted in this section, equation (3-55) for the kinetic energy of tidal bulge reduces likewise to

$$\bar{T}_i'' = -\tfrac{1}{2} C_i'' \omega_i^2 P_2(\cos \Theta_i), \tag{5-14}$$

where C'' continues to be given by equation (3-46).

In order to evaluate the remaining terms on the right-hand sides of equations (5-4) and (5-5) let us fall back on equations (4-49) and (4-50) which, under our conditions, reduce to

$$\frac{d}{dt}\left(\frac{\partial \bar{T}_i''}{\partial \dot{\theta}_i}\right) = -\tfrac{1}{2} C_i'' \omega_i f_{31}, \tag{5-15}$$

$$\frac{d}{dt}\left(\frac{\partial \bar{T}_i''}{\partial \dot{\phi}_i}\right) = -\tfrac{1}{2} C_i'' \omega_i f_{32}, \tag{5-16}$$

where, in accordance with equations (4-42),

$$f_{31} = 3l_1 l_3 \quad \text{and} \quad f_{32} = 3l_3(l_2 - l_3 \cos \theta), \tag{5-17}$$

the l_j's being defined by (3-56). Equations (4-43) reveal, however, that*

$$\left.\begin{aligned} f_{31} &= \frac{3}{2\sin\theta}\frac{\partial l_3^2}{\partial\phi}, \\ f_{32} &= -\tfrac{3}{2}\sin\theta\,\frac{\partial l_3^2}{\partial\theta}, \end{aligned}\right\} \tag{5-18}$$

where

$$l_3 = \cos\Theta \tag{5-19}$$

in accordance with (3-56).

In order to differentiate l_3 with respect to the time, observe that

$$\dot{a}''_{i1} = na''_{i2}, \qquad i = 1, 2, 3, \tag{5-20}$$

where n denotes, as before (*cf.* equation 5-12) the mean daily motion in the binary orbit and, therefore,

$$\dot{l}_3 = n(a'_{13}a''_{12} + a'_{23}a''_{22} + a'_{33}a''_{32}) = n\cos H, \tag{5-21}$$

H being the angle between the Y''- and Z''-axes. In consequence,

$$\left.\begin{aligned} \dot{f}_{31} &= +\frac{3n}{\sin\theta}\frac{\partial}{\partial\phi}(\cos\Theta\cos H), \\ \dot{f}_{32} &= -3n\sin\theta\,\frac{\partial}{\partial\theta}(\cos\Theta\cos H), \end{aligned}\right\} \tag{5-22}$$

and, if we insert all preceding results in the right-hand sides of equations (5-4)–(5-5), the latter assume the forms

$$\frac{1}{n\Gamma}\frac{d\theta}{dt} = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\cos^2\Theta) + \frac{1}{\gamma}\frac{\partial}{\partial\theta}(\cos\Theta\cos H), \tag{5-23}$$

$$\frac{1}{n\Gamma}\frac{d\phi}{dt} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\cos^2\Theta) + \frac{1}{\gamma\sin^2\theta}\frac{\partial}{\partial\phi}(\cos\Theta\cos H), \tag{5-24}$$

where we have abbreviated

$$\Gamma_i = \frac{3}{2}\frac{C''_i}{C'_i}\gamma_i = \frac{3m_{3-i}}{2(m_1+m_2)}\left\{\frac{C'-A'}{C'}\right\}_i\gamma_i \tag{5-25}$$

by (3-41) and (3-46). If our stars were distorted solely by centrifugal force (rigid spheroids), the coefficients of the first terms on the right-hand sides of the foregoing equations (5-23) and (5-24) would be halved, and the second terms would drop out altogether. When, however, due account is taken of the kinetic energy of revolving tidal bulge, the effects of polar flattening are doubled and new terms arise which had no counterpart in a purely rotational problem.

* The absence of subscripts 1, 2 on the Eulerian angles θ and ϕ hereafter signifies that the respective symbols may pertain to the primary as well as secondary component.

The reader may recall that, in deriving equations (5-23)–(5-24), we have regarded the first derivatives of the Eulerian angles θ, ϕ (as well as that of the radius-vector r) to be small quantities of the same order as Γ, with negligible squares and cross-products, but abstained so far from imposing any restriction on the magnitudes of the angles θ, ϕ, themselves. In order to proceed with the solution of equations (5-23)–(5-24) let us replace these angles by new variables μ, ν, defined by

$$\left.\begin{aligned} \mu &= 2 \sin \tfrac{1}{2}\theta \sin \phi, \\ \nu &= 2 \sin \tfrac{1}{2}\theta \cos \phi, \end{aligned}\right\} \tag{5-26}$$

and consider hereafter *the angle* θ *to be sufficiently small for* $\cos^2 \frac{1}{2}\theta$ *to be sensibly equal to one.* If so, the equations (5-23)–(5-24) may be rewritten more symmetrically as*

$$\frac{1}{n\Gamma}\frac{d\mu}{dt} = -\gamma \frac{\partial}{\partial \nu}(\cos^2 \Theta) + \frac{\partial}{\partial \mu}(\cos \Theta \cos H), \tag{5-27}$$

$$\frac{1}{n\Gamma}\frac{d\nu}{dt} = +\gamma \frac{\partial}{\partial \mu}(\cos^2 \Theta) + \frac{\partial}{\partial \nu}(\cos \Theta \cos H), \tag{5-28}$$

where, by (5-19) and (5-21), the cosines of the angles Θ and H can be rewritten as

$$\left.\begin{aligned} \cos \Theta &= \mathfrak{A} \sin u + \mathfrak{B} \cos u, \\ \cos H &= \mathfrak{A} \cos u - \mathfrak{B} \sin u, \end{aligned}\right\} \tag{5-29}$$

in which

$$\left.\begin{aligned} \mathfrak{A} &= \sin i \cos \theta - \cos i \sin \theta \cos (\phi - \Omega), \\ \mathfrak{B} &= \sin \theta \sin (\phi - \Omega). \end{aligned}\right\} \tag{5-30}$$

Through the coefficients $\mathfrak{A}$ and $\mathfrak{B}$ just defined, the equations (5-27)–(5-28) depend, not only on the angles θ and ϕ specifying the position of the equatorial planes of the respective components, but also on the angles i and Ω describing the orientation of the plane of their orbit. The variation of these latter angles with the time is, in turn, governed by the differential equations (4-21) and (4-22) of the preceding section. If, consistent with our present scheme of accuracy, we *regard the orbital inclination* i *as well as the eccentricity* e *sufficiently small for their squares and higher powers to be negligible*, equations (4-21) and (4-22) clearly reduce to

$$\frac{1}{n}\frac{d\Omega}{dt} = +\frac{A}{\sin i}\frac{\partial S_{12}}{\partial i} = -\frac{1}{\sin i}\frac{\partial}{\partial i}(\Pi_1 \cos^2 \Theta_1 + \Pi_2 \cos^2 \Theta_2), \tag{5-31}$$

$$\frac{1}{n}\frac{di}{dt} = -\frac{A}{\sin i}\frac{\partial S_{12}}{\partial \Omega} = \frac{1}{\sin i}\frac{\partial}{\partial \Omega}(\Pi_1 \cos^2 \Theta_1 + \Pi_2 \cos^2 \Theta_2), \tag{5-32}$$

* The reader may note that the neglect of θ^2 was necessary only to accomplish the reduction of the second (tidal) terms on the right-hand sides of (5-27) and (5-28) to their given forms; the reduction of the first terms to their desired forms was exact.

where we have abbreviated

$$\Pi_i = \frac{3\alpha_i}{2A^2} = \frac{3}{4A^2}\left\{\frac{C_i' - A_i'}{m_i}\right\}. \tag{5-33}$$

If, furthermore, in the manner of (5-26) we introduce the new variables

$$\left.\begin{aligned} p &= 2 \sin \tfrac{1}{2}i \sin \Omega, \\ q &= 2 \sin \tfrac{1}{2}i \cos \Omega, \end{aligned}\right\} \tag{5-34}$$

equations (5-31)–(5-32) can be rewritten as

$$\frac{1}{n}\frac{dp}{dt} = -\frac{\partial}{\partial q}\{\Pi_1 \cos^2 \Theta_1 + \Pi_2 \cos^2 \Theta_2\}, \tag{5-35}$$

$$\frac{1}{n}\frac{dq}{dt} = +\frac{\partial}{\partial p}\{\Pi_1 \cos^2 \Theta_1 + \Pi_2 \cos^2 \Theta_2\}. \tag{5-36}$$

The two sets of the equations (5-27)–(5-28) and (5-35)–(5-36) are evidently coupled through the coefficients $\mathfrak{A}$ and $\mathfrak{B}$ involved on their right-hand sides in $\cos \Theta$ and $\cos H$. Our former neglect of the squares of θ and i, inherent in the derivation of equations (5-23) and (5-24), entitles us now to ignore the squares and higher powers of p, q and μ, ν as well—in which case it follows readily that

$$\mathfrak{A} = \frac{p^2 + q^2 - (\mu p + \nu q)}{p^2 + q^2} \tag{5-37}$$

and

$$\mathfrak{B} = \frac{\mu q - \nu p}{p^2 + q^2}. \tag{5-38}$$

In order to proceed with the solution suppose that, consistent with our previous neglect of the second derivatives θ and ϕ of the Eulerian angles on the left-hand sides of equations (5-4) and (5-5), we confine at first our attention to the *secular perturbations* suffered by the angles θ and ϕ as well as i and Ω on account of the distortion of both components. With this limited objective in view it should, however, be sufficient to confine our attention to only the secular part of the disturbing functions $\cos^2 \Theta$ or $\cos \Theta \cos H$, which will hereafter be denoted by square brackets and obtained if the respective functions are replaced by their *average* values in the course of a cycle, as represented by the expressions

$$[\cos^2 \Theta] = \frac{1}{2\pi}\int_0^{2\pi} \cos^2 \Theta \, du = \tfrac{1}{2}(\mathfrak{A}^2 + \mathfrak{B}^2) \tag{5-39}$$

and

$$[\cos \Theta \cos H] = \frac{1}{2\pi}\int_0^{2\pi} \cos \Theta \cos H \, du = 0. \tag{5-40}$$

The vanishing of the secular part of the function $\cos \Theta \cos H$ reveals that (to the order of accuracy we have been working) *the sole effect of the revolving* *tidal bulge of each component on the secular motion of the elements θ, ϕ or i, Ω in close binary systems will be to double the effects caused by rotational distortion.*

For the sake of a gradual approach to the full complexity of the dynamical behaviour of close binary systems, and the desirability of unfolding its essential features in discrete stages, let us consider first the situation which arises if the primary component of finite size and density concentration is disturbed by a secondary which, for dynamical purposes, can be approximated by mass-point. If, in consequence, we set $\Gamma_2 = \Pi_2 = 0$ in equations (5-35) and (5-36), omit (for simplicity) the subscripts 1 referring to the primary, and differentiate

$$[\cos^2 \Theta] = \tfrac{1}{2}\{p^2 + q^2 - 2(p\mu + q\nu) + \mu^2 + \nu^2\} \tag{5-41}$$

with respect to p, q as well as μ, ν, the system (5-35)–(5-36) is found to reduce to

$$\frac{dp}{dt} = +n\Pi(\nu - q), \tag{5-42}$$

$$\frac{dq}{dt} = -n\Pi(\mu - p), \tag{5-43}$$

while the equations (5-27)–(5-28) reduce likewise to

$$\frac{d\mu}{dt} = -n\Gamma(\nu - q), \tag{5-44}$$

$$\frac{d\nu}{dt} = +n\Gamma(\mu - p), \tag{5-45}$$

respectively.

Equations (5-42)–(5-45) constitute a simultaneous system of linear differential equations of fourth order. In order to solve it, add (5-42) with (5-44) and (5-43) with (5-45): abbreviating

$$\frac{\Pi_i}{\Gamma_i} = k_i = \frac{m_1 + m_2}{2\gamma_i m_1 m_2} \frac{C_i'}{A^2} \tag{5-46}$$

by (5-25) and (5-33), we establish that, for $i = 1$,

$$\frac{d}{dt}(p + k\mu) = 0, \tag{5-47}$$

$$\frac{d}{dt}(q + k\nu) = 0, \tag{5-48}$$

which integrate readily into

$$\begin{aligned} p + k\mu &= c_1\,, \\ q + k\nu &= c_2\,, \end{aligned} \tag{5-49}$$

where $c_{1,2}$ are integration constants, whose values are related to the orientation of the coordinate system in which the variables p, q and μ, ν are defined.

The reader may recall that the orientation of our fixed system of space-axes XYZ, introduced in section II.3, has remained so far arbitrary and has not been restricted by any assumption (other than that both the angles θ and i are sufficiently small for their squares and higher powers to be negligible). In order to remove this arbitrariness we shall hereafter assume that *the XY-plane coincides with the invariable plane of our binary system*, and the Z-axis coincides in direction with the vector representing the total moment of momentum of the system. From this choice of orientation of our space axes it follows that the integration constants

$$c_1 = c_2 = 0, \tag{5-50}$$

by virtue of which

$$p = -k\mu \quad \text{and} \quad q = -k\nu. \tag{5-51}$$

If we insert these integrals in equations (5-42)–(5-45), these reduce to the simultaneous system

$$\left.\begin{aligned} \frac{dp}{dt} &= -\kappa q, \qquad & \frac{d\mu}{dt} &= -\kappa\nu, \\ \frac{dq}{dt} &= +\kappa p, \qquad & \frac{d\nu}{dt} &= +\kappa\mu, \end{aligned}\right\} \tag{5-52}$$

where

$$\kappa_i = n(\Gamma_i + \Pi_i), \qquad i = 1, 2, \tag{5-53}$$

and their solution reveals that

$$\left.\begin{aligned} p &= \tilde{A}\sin(\kappa t + \tilde{B}), \\ q &= -\tilde{A}\cos(\kappa t + \tilde{B}); \end{aligned}\right\} \tag{5-54}$$

$$\left.\begin{aligned} \mu &= -k^{-1}\tilde{A}\sin(\kappa t + \tilde{B}), \\ \nu &= k^{-1}\tilde{A}\cos(\kappa t + \tilde{B}); \end{aligned}\right\} \tag{5-55}$$

where $\tilde{A}$ and $\tilde{B}$ are integration constants.

The meaning of these constants is not difficult to interpret. By adding the squares of the foregoing equations (5-54) and (5-55) it transpires that

$$\tilde{A}^2 = p^2 + q^2 = k^2(\mu^2 + \nu^2) \tag{5-56}$$

which, combined with the definitions of p, q and μ, ν as given by equation (5-26) and (5-34), reveals at once that

$$\tilde{A} = \sin i = k \sin \theta \tag{5-57}$$

i.e., that *the angles θ and i remain secularly constant.* Moreover, as

$$\sin\Omega = \frac{p}{\sqrt{p^2 + q^2}} = -\sin(\tilde{B} - \kappa t), \tag{5-58}$$

$$\sin\phi = \frac{\mu}{\sqrt{\mu^2 + \nu^2}} = +\sin(\tilde{B} - \kappa t), \tag{5-59}$$

it follows that

$$\phi - \Omega = \pi, \tag{5-60}$$

and that

$$\left.\begin{aligned}\Omega &= \Omega_0 - \kappa t,\\ \phi &= \phi_0 - \kappa t,\end{aligned}\right\} \tag{5-61}$$

where $\phi_0 - \Omega_0 = \pi$. Therefore, *the nodal lines of the orbital plane as well as the equatorial plane of the primary component are found to be secularly receding at the same uniform rate, and to complete their regression in a period U_1 which is related with the orbital period P by the ratio*

$$\frac{P}{U_1} = \Gamma + \Pi. \tag{5-62}$$

In this équation, the quantity U_1 is obviously identical with the *period of precession* of the rotational axis of the primary component, which happens to be synchronized with the motion of the nodes: namely, the equatorial and orbital planes are inclined to the invariable plane of the system in such a way that the longitudes of their nodes differ constantly by 180°; and the absolute values of their inclinations are in the ratio $i/\theta = k$. As, for centrally-condensed stars, k is bound to be a rather small quantity, it follows that the inclination of the orbital plane to the invariable plane of the system will usually be quite small in comparison with the inclination of the equatorial plane.

All these results have so far been proved for the simplified case in which disturbing action of one (the secondary) component on its mate can be regarded as that of a mass-point. Should, in turn, the primary happen to be the disturbing component, all foregoing results would of course continue to hold good with appropriate interchange of indices. If, moreover, *both* components are regarded as stars of finite size and density concentration, the analysis of motion in this more general case remains closely analogous to that outlined above. Thus if subscripts 1, 2 refer hereafter to the primary and secondary components and

$$[\cos^2 \Theta_i] = \tfrac{1}{2}\{p^2 + q^2 - 2(p\mu_i + q\nu_i) + \mu_i^2 + \nu_i^2\}, \quad i = 1, 2, \tag{5-63}$$

equations (5-42) and (5-43) are to be replaced by

$$\frac{dp}{dt} = n \sum_{i=1}^{2} \Pi_i(\nu_i - q), \tag{5-64}$$

$$\frac{dq}{dt} = -n \sum_{i=1}^{2} \Pi_i(\mu_i - p); \tag{5-65}$$

while equations (5-44) and (5-45) become

$$\frac{d\mu_i}{dt} = -n\Gamma_i(\nu_i - q) \tag{5-66}$$

and

$$\frac{d\nu_i}{dt} = n\Gamma_i(\mu_i - p). \tag{5-67}$$

These equations possess again the integrals

$$\left.\begin{aligned} p + k_1\mu_1 + k_2\mu_2 &= c_1, \\ q + k_1\nu_1 + k_2\nu_2 &= c_2, \end{aligned}\right\} \tag{5-68}$$

where $k_i = \Pi_i/\Gamma_i$ as defined earlier by equation (5-46), and the constants $c_1 = c_2 = 0$ in accordance with (5-49) if XY represents the invariable plane of our system.

With the aid of the foregoing integrals (5-68) of motion, the order of our simultaneous system (5-64)–(5-67) can be suppressed from 6 to 4, and the remaining differential equations for $\mu_{1,2}$ and $\nu_{1,2}$ assume the explicit forms

$$\frac{d\mu_1}{dt} = -n\Gamma_1\{(1 + k_1)\nu_1 + k_2\nu_2\}, \tag{5-69}$$

$$\frac{d\mu_2}{dt} = -n\Gamma_2\{k_1\nu_1 + (1 + k_2)\nu_2\}, \tag{5-70}$$

$$\frac{d\nu_1}{dt} = +n\Gamma_1\{(1 + k_1)\mu_1 + k_2\mu_2\}, \tag{5-71}$$

$$\frac{d\nu_2}{dt} = +n\Gamma_2\{k_1\mu_1 + (1 + k_2)\mu_2\}. \tag{5-72}$$

Once these equations have been solved, the functions p and q are obtainable directly from (5-68).

The solution of the simultaneous system (5-69)–(5-72) can be expressed as

$$\left.\begin{aligned} \mu_i &= \tilde{A}_i \sin s_1 t + \tilde{B}_i \sin s_2 t, \\ \nu_i &= \tilde{A}_i \cos s_1 t + \tilde{B}_i \cos s_2 t, \end{aligned}\right\} \tag{5-73}$$

where the integration constants $\tilde{A}_i$, $\tilde{B}_i$, as well as $s_{1,2}$ are constrained to satisfy the relations

$$\left.\begin{aligned} \{s_1 + n(\Gamma_{1,2} + \Pi_{1,2})\}\tilde{A}_{1,2} + n\Gamma_{1,2}k_{2,1}\tilde{A}_{2,1} &= 0, \\ \{s_2 + n(\Gamma_{1,2} + \Pi_{1,2})\}\tilde{B}_{1,2} + n\Gamma_{1,2}k_{2,1}\tilde{B}_{2,1} &= 0. \end{aligned}\right\} \tag{5-74}$$

The requirement that the determinant of the coefficients of the $\tilde{A}_{ij}$'s and $\tilde{B}_i$'s$_j$ on the left-hand sides should vanish (if the solution for the $\tilde{A}$'s and $\tilde{B}$'s are to be non-trivial) renders $s_{1,2}$ the roots of the quadratic equation

$$s^2 + (\kappa_1 + \kappa_2)s + \kappa_1\kappa_2 = n^2\Pi_1\Pi_2\,, \tag{5-75}$$

where, consistent with (5-53), $\kappa_i = n(\Gamma_i + \Pi_i)$. Moreover, the constants $\tilde{A}_{1,2}$ and $\tilde{B}_{1,2}$ must satisfy the relations

$$\frac{\tilde{A}_2}{\tilde{A}_1} = -\frac{(\kappa_1 - \kappa_2) + (s_1 - s_2)}{2nk_2\Gamma_1} = -\frac{2nk_1\Gamma_2}{(\kappa_2 - \kappa_1) + (s_1 - s_2)} \tag{5-76}$$

and

$$\frac{\tilde{B}_2}{\tilde{B}_1} = \frac{(\kappa_2 - \kappa_1) + (s_1 - s_2)}{2nk_2\Gamma_1} = \frac{2nk_1\Gamma_2}{(\kappa_1 - \kappa_2) + (s_1 - s_2)}\,, \tag{5-77}$$

so that

$$\frac{\tilde{A}_2}{\tilde{A}_1} - \frac{\tilde{B}_2}{\tilde{B}_1} = \frac{s_2 - s_1}{nk_2\Gamma_1}\,. \tag{5-78}$$

The physical meaning of the constants $\tilde{A}_{1,2}$ and $\tilde{B}_{1,2}$ will become more apparent when we pass over from the quantities μ_i, ν_i to the angles θ_i and ϕ_i which they define. For squaring equations (5-73) and summing them up we find that (to the order of accuracy we have been working)

$$\mu_i^2 + \nu_i^2 = \sin^2 \theta_i = \tilde{A}_i^2 + \tilde{B}_i^2 + 2\tilde{A}_i\tilde{B}_i \cos (s_1 - s_2)t \tag{5-79}$$

and, similarly,

$$\left.\begin{aligned} p^2 + q^2 = \sin^2 i &= (k_1\mu_1 + k_2\mu_2)^2 + (k_1\nu_1 + k_2\nu_2)^2 \\ &= (k_1\tilde{A}_1 + k_2\tilde{A}_2)^2 + (k_1\tilde{B}_1 + k_2\tilde{B}_2)^2 \\ &\quad + 2(k_1\tilde{A}_1 + k_2\tilde{A}_2)(k_1\tilde{B}_1 + k_2\tilde{B}_2) \cos (s_1 - s_2)t. \end{aligned}\right\} \tag{5-80}$$

As the function $\cos (s_1 - s_2)t$ oscillates between ± 1, $\sin \theta_i$ oscillates between $\tilde{A}_i \pm \tilde{B}_i$. Therefore, the constants $\tilde{A}_{1,2}$ are seen to specify the *mean* values of the *inclinations* of equatorial planes of the respective components to the invariable plane of the system, while the $\tilde{B}_{1,2}$'s stand for the *amplitudes* of their *nutation*. For the equation (5-79) makes it evident that if *both* components of a close binary system are regarded to be of finite size (and internal density concentration), the inclination of their rotational axes will no longer be constant, but will exhibit an oscillation (analogous to nutation of the terrestrial axis) in a period U' bearing a ratio to the orbital period P which is given by

$$\frac{P}{U'} = \frac{|s_1 - s_2|}{n}. \tag{5-81}$$

Now it follows from (5-75) that the discriminant of this quadratic equation becomes

$$\left\{\frac{s_1 - s_2}{n}\right\}^2 = \{\Gamma_1 + \Pi_1 - \Gamma_2 - \Pi_2\}^2 + 4\Pi_1\Pi_2 . \tag{5-82}$$

The right-hand side of this equation represents the sum of two terms neither of which can be negative. The period of nutation will, therefore, always be real and can become infinite only if $\Gamma_{1,2} = \Pi_{1,2} = 0$ (corresponding to the case of undisturbed motion of two rigid spheres). In systems consisting of fluid (and therefore deformable) components equation (5-79) demonstrates that both stars constituting them are bound to nutate in the same period (though the amplitudes of such nutations are arbitrary). Moreover, the angle i between the orbital and invariable plane of our system will, in accordance with (5-80), oscillate about its mean value $k_1\tilde{A}_1 + k_2\tilde{A}_2$ with the *same* period U', and an amplitude bearing a definite relation to those of nutation of the two stars and (as $k_{1,2} \ll 1$) generally much smaller.

In order to investigate the simultaneous motion of *nodal lines* of the orbital and equatorial planes of the two components, consider the expressions

$$\sin \Omega = \frac{p}{\sqrt{p^2 + q^2}} = -\frac{\sin s_1 t + \varepsilon \sin s_2 t}{\sqrt{1 + 2\varepsilon \cos (s_1 - s_2)t + \varepsilon^2}}, \tag{5-83}$$

$$\cos \Omega = \frac{q}{\sqrt{p^2 + q^2}} = -\frac{\cos s_1 t + \varepsilon \cos s_2 t}{\sqrt{1 + 2\varepsilon \cos (s_1 - s_2)t + \varepsilon^2}}, \tag{5-84}$$

where we have abbreviated

$$\varepsilon = \frac{k_1\tilde{B}_1 + k_2\tilde{B}_2}{k_1\tilde{A}_1 + k_2\tilde{A}_2}. \tag{5-85}$$

As long as this latter constant can be regarded as small the line of the nodes will be *receding*, and will perform a complete revolution in retrograde direction in a period U given by the equation

$$\begin{aligned}\frac{P}{U} = -\frac{s_1}{n} &= \tfrac{1}{2}\{\Gamma_1 + \Pi_1 + \Gamma_2 + \Pi_2\} \\ &+ \tfrac{1}{2}\{(\Gamma_1 + \Pi_1 - \Gamma_2 - \Pi_2)^2 + 4\Pi_1\Pi_2\}^{1/2},\end{aligned} \tag{5-86}$$

which, for $\Gamma_2 = \Pi_2 = 0$, reduces indeed to (5-62). The reader may notice that, if $U_{1,2}$ denote the periods of nodal regression due to the separate action of each component (as given by equation 5-62); U, the period of regression under the combined field of force; and U', the period of the nutation as given by (5-81), it follows from (5-86) that

$$\frac{2}{U} = \frac{1}{U_1} + \frac{1}{U_2} + \frac{1}{U'} \tag{5-87}$$

i.e., the *period of nodal regression* in close binaries consisting of finite components *results as the harmonic mean of the respective contributions of each star, augmented by the reciprocal of the period of nutation.* In consequence, in every such system

$$U' > U, \tag{5-88}$$

i.e., the *period of nutation will always be longer than that of precession or of nodal revolution.* It is, however, also evident from (5-83) or (5-84) that if the ratio ε as defined by (5-85) becomes appreciable, the motion of the nodes ceases to be uniform and becomes a more complicated function of the time.

The motion of the equatorial planes of the two components becomes, in turn, defined by the equations

$$\sin\phi_i = \frac{\mu_i}{\sqrt{\mu_i^2 + \nu_i^2}} = \frac{\sin s_1 t + \eta_i \sin s_2 t}{\sqrt{1 + 2\eta_i \cos(s_1 - s_2)t + \eta_i^2}} \tag{5-89}$$

and

$$\cos\phi_i = \frac{\nu_i}{\sqrt{\mu_i^2 + \nu_i^2}} = \frac{\cos s_1 t + \eta_i \cos s_2 t}{\sqrt{1 + 2\eta_i \cos(s_1 - s_2)t + \eta_i^2}}, \tag{5-90}$$

where we have abbreviated

$$\eta_i = \frac{\tilde{B}_i}{\tilde{A}_i} \tag{5-91}$$

and $i = 1, 2$. These equations reveal that *the nodal lines of the equatorial planes of both components are parallel and recede at the same rate.* A combination of (5-83) and (5-84) with (5-89) and (5-90) reveals, moreover, that

$$\begin{aligned}\sin(\phi_i - \Omega) &= \sin\phi_i \cos\Omega - \cos\phi_i \sin\Omega \\ &= -\frac{(\varepsilon - \eta_i)\sin(s_1 - s_2)t}{\{[1 + 2\varepsilon\cos(s_1 - s_2)t + \varepsilon^2][1 + 2\eta_i\cos(s_1 - s_2)t + \eta_i^2]\}^{1/2}},\end{aligned} \tag{5-92}$$

and similarly for $\cos(\phi_i - \Omega)$, leading (for sufficiently small values of ε and η_i) to an oscillation of the difference

$$\phi_i - \Omega = \pi + (\eta_i - \varepsilon)\sin(s_1 - s_2)t + \dots \tag{5-93}$$

in the period of the nutation.

All foregoing developments have been concerned with the *secular* motion of equatorial planes of the components in close binary systems and of their orbital plane. The secular nature of even very slow motions of this type may render their effects observable after sufficiently long intervals of time; and for this reason they are of primary interest to the investigator. Our analysis would, however, be incomplete if we did not consider also the possibility of the occurrence of *periodic perturbations* of the angular variables θ, ϕ as well as Ω and i. The significance of such perturbations is due to the fact that *their amplitudes would represent the limits within which such elements can no longer be regarded as constant in the course of an orbital cycle.* In view of the tasks awaiting us in Chapter VI, the establishment of such limits becomes likewise a problem of considerable practical importance which will be dealt with below.

In order to investigate possible periodic oscillations of the Eulerian angles θ, ϕ specifying the position of the equatorial planes of each component in space, we must return to equations (5-1) and (5-2) at the beginning of this section and recognize the fact that, for short-periodic motions, the second derivatives θ and ϕ are no longer negligible in comparison with θ or ϕ, but may become quantities of the same order of magnitude. If so, however, the left-hand sides of equations (5-23) and (5-24) must be augmented to read

$$\frac{d^2\phi}{du^2} - \frac{\gamma}{\sin\theta}\frac{d\theta}{du} = -\frac{\gamma\Gamma}{\sin\theta}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\cos^2\Theta) + \frac{1}{\gamma}\frac{\partial}{\partial\theta}(\cos\Theta\cos H)\right\}, \tag{5-94}$$

$$\frac{d^2\theta}{du^2} + \gamma\sin\theta\frac{d\phi}{du} = \gamma\Gamma\left\{-\frac{\partial}{\partial\theta}(\cos^2\Theta) + \frac{1}{\gamma\sin\theta}\frac{\partial}{\partial\phi}(\cos\Theta\cos H)\right\}, \tag{5-95}$$

where we have abbreviated $nt \equiv u$. Ignoring (as before) the squares and cross-products of the angle θ and i involved in the expressions for $\cos\Theta$ and $\cos H$, we find it easy to show that

$$\frac{\gamma}{\sin\theta}\frac{\partial}{\partial\phi}(\cos^2\Theta) + \frac{\partial}{\partial\theta}(\cos\Theta\cos H) = (\gamma - 1)\sin\theta\sin 2(\phi - \Omega - u), \tag{5-96}$$

while

$$-\gamma\frac{\partial}{\partial\theta}(\cos^2\Theta) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(\cos\Theta\cos H)$$

$$= -\gamma\sin\theta + (\gamma - 1)\sin\theta\cos 2(\phi - \Omega - u). \tag{5-97}$$

A case of particular simplicity arises if $\gamma = 1$ (i.e., if the rotation of the respective component is synchronous with its revolution); for equations

(5-94) and (5-95) reduce then to

$$\frac{d^2\phi}{du^2} - \frac{1}{\sin\theta}\frac{d\theta}{du} = 0, \tag{5-98}$$

$$\frac{d^2\theta}{du^2} + \sin\theta\frac{d\phi}{du} = -\Gamma\sin\theta. \tag{5-99}$$

In order to obtain an approximate solution of this non-linear system for the case of *small oscillations*, let us replace $\sin\theta$ in the coefficients of the derivatives on the left-hand sides of the foregoing equations by its average constant value of $\sin\theta_0 = (\tilde{A}^2 + \tilde{B}^2)^{1/2}$ in accordance with equation (5-79): doing so we find the general solution of the system (5-96)–(5-97) to assume the form

$$\theta - \theta_0 = \tilde{C}\sin(u - u_0) \tag{5-100}$$

and

$$(\phi - \phi_0)\sin\theta_0 = -\tilde{C}\cos(u - u_0) - (u - u_0)\,\Gamma\sin\theta, \tag{5-101}$$

where θ_0, ϕ_0, $\tilde{C}$ and u_0 are arbitrary integration constants. The periodic terms on the right-hand sides of these equations reveal that, if the oscillations of θ and ϕ are sufficiently small for the equations defining them to be treated as linear, their amplitudes remain arbitrary,* but such that their ratio is fixed and equal to $\sin\theta_0$. Their period is, moreover, equal to that of the orbit, but their phases are displaced by 90°. The constant term on the right-hand side of (5-99) gives rise merely to a secular regression of nodes of the equatorial planes, of the amount established already before.

Periodic perturbations of the angular variables Ω and i, which specify the position of the orbital plane in space, can be dealt with with equal ease. In order to investigate them, let us fall back on equations (4-21) and (4-22) which (for small values of i) assume the more explicit forms

$$\frac{\sqrt{1-e^2}}{2}\frac{d\Omega}{du} = \frac{\sin u}{\sin i}\sum_{j=1}^{2}\{\Pi\mathfrak{A}_i(\mathfrak{A}\sin u + \mathfrak{B}\cos u)\}_j \tag{5-102}$$

and

$$\frac{\sqrt{1-e^2}}{2}\frac{di}{du} = \frac{\sin u}{\sin i}\sum_{j=1}^{2}\{\Pi\mathfrak{A}_\Omega(\mathfrak{A}\sin u + \mathfrak{B}\cos u)\}_j + \frac{\cos u}{\sin i}\sum_{j=1}^{2}\{\Pi\mathfrak{B}_\Omega(\mathfrak{A}\sin u + \mathfrak{B}\cos u)\}_j\,, \tag{5-103}$$

where the coefficients $\mathfrak{A}$ and $\mathfrak{B}$ continue to be given by equations (5-30) and, therefore,

$$\mathfrak{A}_i = \cos i\cos\theta + \sin i\sin\theta\cos(\phi - \Omega), \tag{5-104}$$

while

$$\left.\begin{aligned}\mathfrak{A}_\Omega &= -\cos i\sin\theta\sin(\phi - \Omega),\\ \mathfrak{B}_\Omega &= -\sin\theta\cos(\phi - \Omega).\end{aligned}\right\} \tag{5-105}$$

* A limit of such amplitudes is bound to emerge with the restoration of non-linear terms.

The right-hand sides of the foregoing equations (5-102) and (5-103) constitute small quantities of first order. To this degree of accuracy it is, therefore, permissible to treat $\mathfrak{A}$ and $\mathfrak{B}$ (as well as their partial derivatives with respect to Ω and i) as constants. Moreover, in eccentric orbits

$$\Pi_i = \frac{3}{2}\frac{\alpha_i A}{r^3} \tag{5-106}$$

as a generalization of (5-33). In order to perform the actual integration we find it, therefore, convenient to replace the mean anomaly u in its function of independent variable by the true anomaly v, which is related with u by means of Kepler's second law of elliptic motion asserting that

$$du = \frac{r^2}{A^2}\frac{dv}{\sqrt{1-e^2}} \tag{5-107}$$

to the zero order of accuracy. Doing so we easily establish that

$$\frac{d\Omega}{dv} = -\frac{3(1+e\cos v)\sin u}{(1-e^2)^2A^2\sin i}\sum_{j=1}^{2}\{\alpha\,\mathfrak{A}_i(\mathfrak{A}\sin u+\mathfrak{B}\cos u)\}_j \tag{5-108}$$

and

$$\begin{aligned}\frac{di}{dv} = {} & \frac{3(1+e\cos v)\sin u}{(1-e^2)^2A^2\sin i}\sum_{j=1}^{2}\{\alpha\,\mathfrak{A}_\Omega(\mathfrak{A}\sin u+\mathfrak{B}\cos u)\}_j \\ & + \frac{3(1+e\cos v)\cos u}{(1-e^2)^2A^2\sin i}\sum_{j=1}^{2}\{\alpha\,\mathfrak{B}_\Omega(\mathfrak{A}\sin u+\mathfrak{B}\cos u)\}_j\end{aligned} \tag{5-109}$$

for any value of orbital eccentricity. Moreover, as

$$u = \omega + v, \tag{5-110}$$

the integration of the preceding equations reduces to simple quadratures which can be performed in a closed form and reveals that

$$\begin{aligned}\tfrac{1}{3}\{(1-e^2)^2A^2\sin i\}(\Omega-\Omega_0) = {} & -\mathfrak{E}\int(1+e\cos v)\sin^2 u\,du \\ & - \mathfrak{F}\int(1+e\cos v)\sin u\cos u\,du\end{aligned} \tag{5-111}$$

and

$$\begin{aligned}\tfrac{1}{3}\{(1-e^2)^2A^2\sin i\}(i-i_0) = {} & \mathfrak{G}\int(1+e\cos v)\sin^2 u\,du \\ & + \mathfrak{H}\int(1+e\cos v)\cos^2 u\,du \\ & + \mathfrak{K}\int(1+e\cos v)\sin u\cos u\,du,\end{aligned} \tag{5-112}$$

where

$$2\int(1+e\cos v)\sin^2 u\,du = u - \sin u\cos u$$
$$+ \tfrac{2}{3}e\{\sin^2 u \sin(u-\omega) - 2\sin\omega\cos u\}, \qquad (5\text{-}113)$$
$$= u + e\sin(u+\omega) - \int(1+e\cos v)\cos^2 u\,du,$$

$$2\int(1+e\cos v)\sin u\cos u\,du = \sin^2 u + \tfrac{2}{3}e\{\sin^3 u\sin\omega - \cos^3 u\cos\omega\}, \qquad (5\text{-}114)$$

and where we have abbreviated

$$\mathfrak{E} = \alpha_1\mathfrak{A}_1\mathfrak{A}_{i1} + \alpha_2\mathfrak{A}_2\mathfrak{A}_{i2}, \qquad (5\text{-}115)$$

$$\mathfrak{F} = \alpha_1\mathfrak{B}_1\mathfrak{A}_{i1} + \alpha_2\mathfrak{B}_2\mathfrak{A}_{i2}; \qquad (5\text{-}116)$$

and

$$\mathfrak{G} = \alpha_1\mathfrak{A}_1\mathfrak{A}_{\Omega 1} + \alpha_2\mathfrak{A}_2\mathfrak{A}_{\Omega 2}, \qquad (5\text{-}117)$$

$$\mathfrak{H} = \alpha_1\mathfrak{B}_1\mathfrak{B}_{\Omega 1} + \alpha_2\mathfrak{B}_2\mathfrak{B}_{\Omega 2}, \qquad (5\text{-}118)$$

$$\mathfrak{K} = \alpha_1\mathfrak{B}_1\mathfrak{A}_{\Omega 1} + \alpha_2\mathfrak{B}_2\mathfrak{A}_{\Omega 2} + \alpha_1\mathfrak{A}_1\mathfrak{B}_{\Omega 1} + \alpha_2\mathfrak{A}_2\mathfrak{B}_{\Omega 2}. \qquad (5\text{-}119)$$

The reader may note that if, very approximately, $\phi - \Omega = \pi$, it follows from (5-30) that

$$\mathfrak{A} = \sin(\theta + i), \quad \mathfrak{B} = 0; \qquad (5\text{-}120)$$

and

$$\mathfrak{A}_i = \cos(\theta + i) \qquad (5\text{-}121)$$

while

$$\mathfrak{A}_\Omega = 0 \quad \text{and} \quad \mathfrak{B}_\Omega = \sin\theta. \qquad (5\text{-}122)$$

If so, and if the orbital eccentricity e becomes small enough to be ignored, the principal periodic parts of equations (5-111) and (5-112) will reduce to

$$(A^2\sin i)\,\delta\Omega = -\tfrac{3}{8}\{\alpha_1\sin 2(\theta_1 + i) + \alpha_2\sin 2(\theta_2 + i)\}\sin 2u \qquad (5\text{-}123)$$

and

$$(A^2\sin i)\,\delta i = -\tfrac{3}{4}\{\alpha_1\sin\theta_1\sin(\theta_1 + i) + \alpha_2\sin\theta_2\sin(\theta_2 + i)\}\cos 2u. \qquad (5\text{-}124)$$

We may, furthermore, note that in accordance with (5-57) we may expect, very approximately,

$$\sin i = k_j\sin\theta_j, \qquad (5\text{-}125)$$

where $k_{1,2}$ are constants defined by equation (5-46). If so, however, it follows from (5-123) and (5-124) that

$$\delta\Omega = -\tfrac{1}{2}\{\Gamma_1 + \Pi_1 + \Gamma_2 + \Pi_2\}\sin 2u + \ldots \qquad (5\text{-}126)$$

and

$$\delta i = -\tfrac{1}{2}\{(1 + k_1^{-1})\Gamma_1 + (1 + k_2^{-1})\Gamma_2\}\sin i\cos 2u + \ldots, \qquad (5\text{-}127)$$

where the constants $\Gamma_{1,2}$ and $\Pi_{1,2}$ continue to be given by equations (5-25) and (5-33).

Equations (5-126) and (5-127) reveal that the principal periodic perturbations of the angular elements Ω and i possess a period equal to one-half of that of the orbit, and (as the k_i's are likely to be small) their amplitudes should be essentially the same. It transpires, moreover, from the foregoing analysis that these amplitudes are not necessarily small: in point of fact, they turn out to be quantities of the same order of magnitude as the superficial distortion of the components; and although with increasing degree of central condensation the difference $C' - A'$ tends to zero, the corresponding amplitudes may not be too small for observational detection. This possibility should be kept in mind throughout Chapter VI, when we shall turn to the construction of practical methods by which the values of i can be determined from the observations of light-changes of eclipsing binary systems.

II.6. PERTURBATIONS OF THE ELEMENTS IN THE ORBITAL PLANE

Having investigated in the preceding section the perturbations of the elements defining the position *of* the orbital plane of close binary systems in space, we propose to turn now to an investigation of the perturbations of those elements which specify the position of the components *in* the orbital plane. In order to do so, let us revert to equations (4-21)–(4-26) and consider first the perturbations of the orbital eccentricity e and of the longitude ω of the apsidal line in space.

The relevant differential equations governing the variation of e and ω with the time have already been stated in section II.4 under numbers (4-24) and (4-25), and the perturbing function S_{12} on their right-hand sides continues to be given by equations (5-6) and (5-8). An inspection of these equations reveals that the only variable terms of this perturbing function are the radius-vector r and $\cos\Theta$. In order to formulate the right-hand sides of the relevant perturbation equations explicitly, the first prerequisite will obviously be to form the partial derivatives of r and $\cos\Theta$ with respect to the individual elements of the orbit. In order to accomplish this task in a systematic manner, let us depart from the standard expression for the radius-vector

$$r = \frac{A(1-e^2)}{1+e\cos v} = A(1-e\cos E), \tag{6-1}$$

in terms of the true and eccentric anomalies v and E, of which the latter is known to obey Kepler's equation

$$\left.\begin{aligned} E - e\sin E &= \sqrt{\frac{G(m_1+m_2)}{A^3}}\,(t-T) \\ &= nt + \varepsilon - \bar{\omega}. \end{aligned}\right\} \tag{6-2}$$

Let, furthermore, x stand hereafter for any one of the six elements Ω, i, A, e, $\bar{\omega}$, and ε of the orbit. Differentiating (6-1) with respect to it we readily find that

$$\frac{\partial r}{\partial x} = \frac{r}{A}\frac{\partial A}{\partial x} - A\cos E\frac{\partial e}{\partial x} + Ae\sin E\frac{\partial E}{\partial x}, \tag{6-3}$$

and a similar differentiation of (6-2) yields

$$\frac{\partial E}{\partial x} = \frac{A}{r}\left\{t\frac{\partial n}{\partial x} + \frac{\partial \varepsilon}{\partial x} - \frac{\partial \bar{\omega}}{\partial x} + \sin E\frac{\partial e}{\partial x}\right\}. \tag{6-4}$$

Eliminating $\partial E/\partial x$ from (6-3) with the aid of (6-4) and making use of the fact that

$$A\sqrt{1-e^2}\sin E = r\sin v, \tag{6-5}$$

we find that

$$\frac{\partial r}{\partial x} = \frac{r}{A}\frac{\partial A}{\partial x} - A\cos E\frac{\partial e}{\partial x} + \frac{Ae\sin v}{\sqrt{1-e^2}}\left\{t\frac{\partial n}{\partial x} + \frac{\partial \varepsilon}{\partial x} - \frac{\partial \bar{\omega}}{\partial x} + \sin E\frac{\partial e}{\partial x}\right\}, \tag{6-6}$$

where x may stand for any one of the six elements of the orbit.

In differentiating similarly the first one of equations (5-29) for $\cos\Theta$, the reader should note that its expression depends on the elements Ω and i explicitly through its coefficients $\mathfrak{A}$ and $\mathfrak{B}$, but implicitly it depends on all four remaining elements through the angle $u = \omega + v$. In order to establish the partial derivatives of $\cos\Theta$ with respect to x it is, therefore, necessary first to obtain those of the true anomaly v; and this can be accomplished if we remember that v and E are related by the equation

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}. \tag{6-7}$$

Differentiating the latter with respect to x we easily find that

$$\begin{aligned}\frac{1}{\sin v}\frac{\partial v}{\partial x} &= \frac{1}{\sin E}\frac{\partial E}{\partial x} + \frac{1}{1-e^2}\frac{\partial e}{\partial x} \\ &= \left\{\frac{1}{1-e^2} + \frac{A}{r}\right\}\frac{\partial e}{\partial x} + \frac{A}{r\sin E}\left\{t\frac{\partial n}{\partial x} + \frac{\partial \varepsilon}{\partial x} - \frac{\partial \bar{\omega}}{\partial x}\right\}\end{aligned} \tag{6-8}$$

by use of (6-4); and eliminating $\sin E$ from this latter equation with the aid of (6-5) we obtain

$$\frac{\partial v}{\partial x} = \frac{A^2\sqrt{1-e^2}}{r^2}\left\{t\frac{\partial n}{\partial x} + \frac{\partial \varepsilon}{\partial x} - \frac{\partial \bar{\omega}}{\partial x}\right\} + \sin v\left\{\frac{1}{1-e^2} + \frac{A}{r}\right\}\frac{\partial e}{\partial x} \tag{6-9}$$

as the final result, of which frequent use will be made later on. We may also note that, inasmuch as

$$u = v + \omega = v + \bar{\omega} - \Omega, \tag{6-10}$$

we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}$$

for $x = A$, e, ε, or i; but *not* for $\bar{\omega}$ or Ω, when in accordance with (6-10)

$$\frac{\partial u}{\partial \bar{\omega}} = \frac{\partial v}{\partial \bar{\omega}} + 1 = 1 - \frac{A^2\sqrt{1-e^2}}{r^2} \tag{6-11}$$

and

$$\frac{\partial u}{\partial \Omega} = \frac{\partial v}{\partial \Omega} - 1 = -1. \tag{6-12}$$

After these preliminaries let us turn to the perturbation equations (4-24) and (4-25) for e and $\bar{\omega}$. With the aid of the equations (6-6) and (6-9) just established we find it easy to prove that

$$\left\{\frac{1-\sqrt{1-e^2}}{e}\frac{\partial}{\partial \varepsilon} + \frac{1}{e}\frac{\partial}{\partial \bar{\omega}}\right\} r = -A \sin v \tag{6-13}$$

and

$$\left\{\frac{\tan \tfrac{1}{2}i}{\sqrt{1-e^2}}\frac{\partial}{\partial i} + \frac{\sqrt{1-e^2}}{e}\frac{\partial}{\partial e}\right\} r = -\frac{A\sqrt{1-e^2}}{e} \cos v \tag{6-14}$$

exactly; while setting, with ample approximation, $\phi - \Omega = \pi$ in the expressions (5-30) constituting $\cos \Theta$ we likewise find that

$$\begin{aligned}\left\{\frac{1-\sqrt{1-e^2}}{e}\frac{\partial}{\partial \varepsilon} + \frac{1}{e}\frac{\partial}{\partial \bar{\omega}}\right\} &\cos \Theta \\ &= -\frac{\sin(\theta + i)}{1-e^2}\{2 \cos v + e(1 + \cos^2 v)\} \cos u\end{aligned} \tag{6-15}$$

and

$$\begin{aligned}\left\{\frac{\tan \tfrac{1}{2}i}{\sqrt{1-e^2}}\frac{\partial}{\partial i} + \frac{\sqrt{1-e^2}}{e}\frac{\partial}{\partial e}\right\} \cos \Theta &= \left\{\frac{\cos(\theta + i) \tan \tfrac{1}{2}i}{\sqrt{1-e^2}}\right\} \sin u \\ &+ \left\{\frac{\sin(\theta+i)}{e\sqrt{1-e^2}}\right\}(2 + e \cos v) \sin v \cos u.\end{aligned} \tag{6-16}$$

Inserting these results in equations (4-24)–(4-25) and changing over from nt to v as our independent variable with the aid of the equation (5-107) expressing the 'law of areas' of the elliptic motion* we find that the variation of the orbital eccentricity should satisfy the equation

$$\begin{aligned}\frac{de}{dv} = -&\left\{\frac{3}{2}\frac{\alpha_1 + \alpha_2}{r^2} + \sum_{j=2}^{4} \frac{2(j+1)(\beta_{1j} + \beta_{2j})}{r^{2j+1}}\right\} \sin v \\ &+ \mathfrak{P}\left\{\frac{3 \sin^2 u \sin v}{r^2} + \frac{2 \cos v + e(1 + \cos^2 v)}{A(1-e^2)}\frac{\sin 2u}{r}\right\}\end{aligned} \tag{6-17}$$

* A resort to this law should again be legitimate if we limit ourselves to first-order quantities on the right-hand sides of our perturbation equations.

and the motion of the apsidal line should similarly be governed by

$$\frac{d\bar{\omega}}{dv} = \left\{ \frac{3}{2} \frac{\alpha_1 + \alpha_2}{r^2} + \sum_{j=2}^{4} \frac{2(j+1)(\beta_{1j} + \beta_{2j})}{r^{2j+1}} \right\} \frac{\cos v}{e} - \frac{\mathfrak{P}}{e} \left\{ \frac{3 \sin^2 u \cos v}{r^2} + \frac{2 + e \cos v}{A(1 - e^2)} \frac{\sin v \sin 2u}{r} \right\} - \frac{\mathfrak{Q} \sin^2 u}{A(1 - e^2) r}, \tag{6-18}$$

where we have abbreviated

$$\mathfrak{P} = \tfrac{3}{2}\{\alpha_1 \sin^2 (\theta_1 + i) + \alpha_2 \sin^2 (\theta_2 + i)\}, \tag{6-19}$$

$$\mathfrak{Q} = \tfrac{3}{2}\{\alpha_1 \sin 2 (\theta_1 + i) + \alpha_2 \sin 2(\theta_2 + i)\} \tan \tfrac{1}{2} i, \tag{6-20}$$

and α_i, β_{ij} continue to be given by equations (5-9) or (5-10) and (5-13) of the preceding section.

The *secular variation* of e or ω per cycle can now be obtained by integrating equations (6-17) and (6-18) with respect to v between 0 and 2π: doing so we find that

$$\frac{\Delta e}{2\pi} = \frac{1}{2\pi} \int_0^{2\pi} \frac{de}{dv} \, dv = 0, \tag{6-21}$$

while

$$\frac{\Delta \bar{\omega}}{2\pi} = \frac{3}{2} \frac{\alpha_1 + \alpha_2}{A^2(1 - e^2)^2} [F_2] + \sum_{j=2}^{4} \frac{2(j+1)(\beta_{1j} + \beta_{2j})}{[A(1 - e^2)]^{2j+1}} [F_{2j+1}] - \frac{3\mathfrak{P} + \mathfrak{Q}}{2A^2(1 - e^2)^2}, \tag{6-22}$$

where $[F_k]$ stands for the secular part of the integral

$$2\pi e F_k = \int (1 + e \cos v)^k \cos v \, dv. \tag{6-23}$$

Evidently

$$[F_2] = 1; \tag{6-24}$$

while

$$[F_5] = \tfrac{5}{2}(1 + \tfrac{3}{2}e^2 + \tfrac{1}{8}e^4), \tag{6-25}$$

$$[F_7] = \tfrac{7}{2}(1 + \tfrac{15}{4}e^2 + \tfrac{15}{8}e^4 + \tfrac{5}{64}e^6), \tag{6-26}$$

$$[F_9] = \tfrac{9}{2}(1 + 7e^2 + \tfrac{35}{4}e^4 + \tfrac{35}{16}e^6 + \tfrac{7}{128}e^8), \tag{6-27}$$

etc. Therefore, to the first order in small quantities, *the orbital eccentricity of close binary systems remains secularly constant*; whereas *the longitude of periastron secularly advances* at such a rate that the period U of its revolution bears a ratio to the orbital period P as given by the equation

$$\frac{P}{U} = \frac{\Delta \bar{\omega}}{2\pi}. \tag{6-28}$$

The rate of advance is governed primarily by the mutual tidal distortion of both components; their rotational distortion making but a moderate contribution. Moreover, as the terms involving $\mathfrak{P}$ and $\mathfrak{Q}$ on the right-hand side of (6-22) are clearly negative, this contribution should be maximum if the equatorial planes of both components coincide with that of the orbit, and diminish with an increasing angle between them.

Equations (6-21) or (6-22) summarize only the secular perturbations of the respective elements. Their *periodic perturbations* will be obtained if we integrate equations (6-17) and (6-18) between variable limits. Doing so and limiting ourselves to terms invoked by the second-harmonic distortion of both stars we find that, in the course of each revolution, the orbital *eccentricity* should *oscillate* in accordance with the equation

$$\begin{aligned}\delta e = {} & \frac{3}{2}\frac{\alpha_1 + \alpha_2}{A^2(1-e^2)^2}\{(1 + \tfrac{1}{4}e^2)\cos v + \tfrac{1}{2}e\cos 2v + \tfrac{1}{12}e^2\cos 3v\} \\ & + 6\frac{\beta_{12} + \beta_{22}}{A^5(1-e^2)^5}\{(1 + \tfrac{5}{2}e^2)\cos v + \tfrac{5}{4}e\cos 2v + \tfrac{5}{6}e^2\cos 3v + \ldots\},\end{aligned} \tag{6-29}$$

and the corresponding periodic *libration of the apsidal line* will be given by

$$\begin{aligned}\delta\bar{\omega} = {} & \frac{3}{2}\frac{\alpha_1 + \alpha_2}{A^2(1-e^2)^2}\left\{v + \left(\frac{1}{e} + \tfrac{3}{4}e\right)\sin v + \tfrac{1}{2}\sin 2v + \tfrac{1}{12}e\sin 3v\right\} \\ & + 6\frac{\beta_{12} + \beta_{22}}{A^5(1-e^2)^5}\left\{\tfrac{5}{2}v + \left(\frac{1}{e} + \tfrac{15}{2}e\right)\sin v + \tfrac{5}{4}\sin 2v + \tfrac{5}{6}e\sin 3v + \ldots\right\}.\end{aligned} \tag{6-30}$$

An inspection of these equations reveals that whereas the right-hand side of (6-29) tends to a finite limit as $e \to 0$, that of (6-30) tends actually to infinity as e^{-1}. This does not imply, of course, any anomaly in motion of the two stars since, for $e = 0$, the definition of the apsidal line loses any meaning. Moreover, as we shall explore more fully in section VI.9, the light (or radial velocity) changes of close binary systems are affected by e and $\bar{\omega}$, not separately, but only through the products

$$g = e\sin\bar{\omega} \qquad \text{and} \qquad h = e\cos\bar{\omega}. \tag{6-31}$$

As e and $\bar{\omega}$ can thus be determined only through g and h, it becomes of importance for our later work to investigate the manner in which these latter quantities can be affected by the secular as well as periodic perturbations of e and $\bar{\omega}$. This can be done simply if we observe that

$$\left.\begin{aligned}\delta g &= (\delta e)\sin\bar{\omega} + (\delta\bar{\omega})\,e\cos\bar{\omega} = g(\delta e/e) + h\,\delta\bar{\omega}, \\ \delta h &= (\delta e)\cos\bar{\omega} - (\delta\bar{\omega})\,e\sin\bar{\omega} = h(\delta e/e) - h\,\delta\bar{\omega},\end{aligned}\right\} \tag{6-32}$$

Inserting for δe and $\delta\bar{\omega}$ from (6-29) and (6-30) we easily establish that, as $e \to 0$,

$$\delta g = \left\{\frac{3}{2}\frac{\alpha_1 + \alpha_2}{A^2} + 6\frac{\beta_{12} + \beta_{22}}{A^5} + \dots\right\} \sin(\bar{\omega} + v) \tag{6-33}$$

and

$$\delta h = \left\{\frac{3}{2}\frac{\alpha_1 + \alpha_2}{A^2} + 6\frac{\beta_{12} + \beta_{22}}{A^5} + \dots\right\} \cos(\bar{\omega} + v). \tag{6-34}$$

The observational implications of the existence of these limits will be considered later in section VI.9.

The last element of the plane orbit whose perturbations we wish to investigate in this section is the semi-major axis A of the relative orbit. Its fluctuation caused by our perturbing function S_{12} is governed by the equation

$$\frac{1}{nA^2}\frac{dA}{dt} = 2\frac{\partial S_{12}}{\partial \varepsilon}; \tag{6-35}$$

and since, in accordance with (6-6) and (6-9),

$$\frac{\partial r}{\partial \varepsilon} = \frac{Ae \sin v}{\sqrt{1 - e^2}} \quad \text{and} \quad \frac{\partial v}{\partial \varepsilon} = \frac{A^2\sqrt{1 - e^2}}{r^2}, \tag{6-36}$$

it follows that

$$\begin{aligned}\frac{\partial S_{12}}{\partial \varepsilon} = &-\left\{\frac{3}{2}\frac{\alpha_1 + \alpha_2}{r^4} + \sum_{j=2}^{4}\frac{2(j+1)(\beta_{1j} + \beta_{2j})}{r^{2j+3}}\right\}\frac{Ae \sin v}{\sqrt{1 - e2}} \\ &+ A\sqrt{1 - e^2}\left\{\frac{3e \sin v \sin^2 u}{r^4(1 - e^2)} - \frac{A \sin 2u}{r^5}\right\}\mathfrak{P}.\end{aligned} \tag{6-37}$$

Moreover, if we change over from t to v as our independent variable by means of (5-107), equation (6-35) will assume the form

$$\begin{aligned}\frac{1}{A}\frac{dA}{dv} = &-\frac{2e \sin v}{1 - e^2}\left\{\frac{3}{2}\frac{\alpha_1 + \alpha_2}{r^2} + \sum_{j=2}^{4}\frac{2(j+1)(\beta_{1j} + \beta_{2j})}{r^{2j+1}}\right\} \\ &+ 2\mathfrak{P}\left\{\frac{3e \sin v}{1 - e^2}\frac{\sin^2 u}{r^2} - A\frac{\sin 2u}{r^3}\right\}.\end{aligned} \tag{6-38}$$

The integral of the right-hand side of the foregoing equation vanishes if taken over the whole cycle—which reveals again that, to the first order in small quantities, *the semi-major axis in close binary systems remains secularly constant*. Its principal *periodic perturbations* assume, however, the form

$$\delta A = A\{1 + (\mathfrak{P}/A^2)\cos 2(\omega + v) + \dots\} \tag{6-39}$$

if the eccentricity is negligible but the equatorial planes of the two components are inclined to the plane of the orbit, and

$$\delta A = A\left\{1 + 3\left[\frac{\alpha_1 + \alpha_2}{A^2} + 4\frac{\beta_{12} + \beta_{22}}{A^5} + \dots\right]e \sin v + \dots\right\} \tag{6-40}$$

if the equators and orbit are co-planar, but the eccentricity e is finite. *The coefficients of the periodic terms on the right-hand sides of* (6-39) *or* (6-40) *represent the quantities which must be negligible if* A *is to be regarded as constant in the course of a cycle.*

II.7. PERIOD VARIATION IN ECLIPSING BINARY SYSTEMS

In section 5 of this chapter we have investigated the first-order secular as well as periodic perturbations of the orbital elements Ω and i, due to the mutual attraction of rotating components of close binary systems; and in the preceding section II.6 we have done the same for e, $\bar{\omega}$, and A. The aim of the present section will be to investigate, in a similar manner, the perturbations of the last remaining orbital element: namely, of the *period* of the binary orbit. Before we embark on this task it is well to stress that, because of the cumulative effects of the light- or radial-velocity changes, the period of the orbit of a close (i.e., spectroscopic or eclipsing) binary can be determined usually with an accuracy far surpassing that with which any other elements of such systems can be ascertained from the available observational evidence. Observations have indeed revealed that the periods of many close binaries do exhibit fluctuations sometimes of very complex character—which have so far largely defied rational explanation. The object of the present section will not be to confront the problem of period variations in its most general form and to set out to account for all period changes observed in close binary systems; but rather *to investigate such fluctuations of orbital periods as are bound to be invoked by our perturbing function S_{12}, both directly and indirectly* (through its effect on other elements whose variation may influence the duration of the apparent periods). Lastly, we shall scrutinize also the methods by which the duration of the periods can actually be extracted from the observed data, and establish the exact form of the relation between instantaneous period and the observed epochs.

In order to embark on this task, our first step should be to establish the relation between the time t and the position (i.e., the true anomaly) of the two components in their relative orbit. For simple elliptic motion such a relation would be represented by Kepler's second law. A glance at the system of equations (4-11)–(4-13) reveals, however, that if their right-hand sides are not identically zero, the 'law of areas' (i.e., the integral of 4-12) will cease to be Keplerian, and its proper form must be investigated. In order to derive maximum advantage from the ground already covered in preceding sections of this chapter, let us set out to arrive at the desired result by differentiating equation (6-1) with respect to the time, obtaining

$$\frac{dv}{dt} = \frac{\sin v}{1 - e^2} \frac{de}{dt} + \frac{A\sqrt{1 - e^2}}{r} \frac{dE}{dt}, \tag{7-1}$$

while a differentiation of (6-2) yields

$$\frac{dE}{dt} = \frac{A}{r}\left\{\frac{dM}{dt} + \sin E \frac{de}{dt}\right\}, \tag{7-2}$$

where we have abbreviated

$$M = nt + \varepsilon - \bar{\omega}. \tag{7-3}$$

In consequence,

$$\frac{dv}{dt} = \frac{A^2\sqrt{1-e^2}}{r^2}\left\{n - \frac{3}{2}\frac{nt}{A}\frac{dA}{dt} + \frac{d\varepsilon}{dt} - \frac{d\bar{\omega}}{dt}\right\} + \sin v\left\{\frac{1}{1-e^2} + \frac{A}{r}\right\}\frac{de}{dt}, \tag{7-4}$$

where the time-derivatives of A, e, $\bar{\omega}$, and ε occurring on the right-hand side are, in turn, defined by equations (4-21)–(4-26).

The foregoing equation (7-4) represents the desired generalization of Kepler's 'law of areas,' to which it would indeed reduce if the orbital elements A, e, $\bar{\omega}$, and ε were all constant. Earlier in this chapter we have, however, seen that in close binary systems these elements are bound to vary, in accordance with the equations (4-21)–(4-26), as a result of the perturbations derivable from our disturbing function S_{12}. Moreover, as we have seen in the previous section, such perturbations will affect the direction ω of the apsidal line from which the true anomaly v is reckoned. In order to obviate complications arising from this source, let us replace v by another true anomaly w reckoned from the fixed X-axis in space (*cf.* section II.3), and related with the true anomalies u and v measured from the nodal and apsidal lines by means of the equation

$$w = u + \Omega = v + \bar{\omega}. \tag{7-5}$$

If we replace v on the left-hand side of (7-4) by $w - \bar{\omega}$ and insert for the time derivatives of A, e, $\bar{\omega}$, and ε on the right from (4-21)–(4-26), our generalized law of areas will assume the more explicit form

$$\begin{aligned}\frac{1}{An}\frac{dw}{dt} = {} & \frac{A\sqrt{1-e^2}}{r^2}\left\{1 - 3Ant\frac{\partial S}{\partial \varepsilon} - A\frac{1-e^2}{e}\frac{\partial S}{\partial e} - 2A^2\frac{\partial S}{\partial A}\right\} \\ & - \sin v\left\{\frac{1}{1-e^2} + \frac{A}{r}\right\}\left\{\frac{1-\sqrt{1-e^2}}{e}\frac{\partial S}{\partial \varepsilon} + \frac{1}{e}\frac{\partial S}{\partial \bar{\omega}}\right\}\sqrt{1-e^2} \\ & + \left\{\frac{\tan\frac{1}{2}i}{\sqrt{1-e^2}}\frac{\partial S}{\partial i} + \frac{\sqrt{1-e^2}}{e}\frac{\partial S}{\partial e}\right\}.\end{aligned} \tag{7-6}$$

Now to the first order in small quantities the disturbing function S_{12} depends on the elements of the orbit only through r and $\cos\Theta$. Making use of (6-6) and (6-9) we easily establish that

$$\left\{3nt\frac{\partial}{\partial \varepsilon} + \frac{1-e^2}{e}\frac{\partial}{\partial e} + 2A\frac{\partial}{\partial A}\right\}r = 2r - \frac{1-e^2}{e}A\cos v \tag{7-7}$$

and

$$\left\{ 3nt\frac{\partial}{\partial \varepsilon} + \frac{1-e^2}{e}\frac{\partial}{\partial e} + 2A\frac{\partial}{\partial A} \right\} \cos\Theta \tag{7-8}$$

$$= (2e^{-1} + \cos v)\sin(\theta + i)\sin v \cos u,$$

while the results of the operation on r or $\cos\Theta$ by the remaining terms on the right-hand side of (7-6) are already known from equations (6-13)–(6-16). With the aid of these results it readily transpires that

$$\frac{1}{n}\frac{dw}{dt} = \frac{A^2\sqrt{1-e^2}}{r^2} - \frac{A\mathfrak{Q}\sin^2 u}{r^3\sqrt{1-e^2}}, \tag{7-9}$$

where the constant $\mathfrak{Q}$ continues to be given by equation (6-20). As could have been surmised on general grounds, departures from Kepler's second law arise only if the equatorial plane of at least one component is inclined to the invariable plane of the system; and if so, the requisite correction term on the right-hand side of (7-9) is exact for any eccentricity. If, moreover, the product $e\mathfrak{Q}$ can be regarded as ignorable, the foregoing equation (7-9) reduces to

$$\frac{1}{n}\frac{dw}{dt} = \frac{A^2\sqrt{1-e^2}}{r^2} - \frac{\mathfrak{Q}}{A^2}\sin^2(w - \Omega), \tag{7-10}$$

representing an approximation to the 'law of areas' in close binary systems, which should be ample for all pairs of small orbital eccentricity, and will hereafter be adopted for all our subsequent work.

In order to proceed further with our principal object of inquiry in this section—which is to investigate the period variation in close binary systems—an exact definition of what we mean by the orbital period becomes prerequisite. It goes without saying that, as no close binary can be telescopically resolved, the adoption of a sideric period based on transit observations is out of the question. The orbital period of a spectroscopic binary could be defined as a time interval which elapses between two successive (superior or inferior) conjunctions. The timing of such conjunctions is, however, limited by the relatively low resolving power (i.e., exposure length) of radial-velocity determination; and also by the fact that, for spectroscopic binaries which happen to be eclipsing variables, the radial-velocity changes in the neighbourhood of conjunctions are largely complicated by the rotational effects (*cf.* section V.2). In eclipsing variables, on the other hand, *the orbital period can be defined as a time interval between two successive* (primary or secondary) *minima of light*; and since the photometric observations of such minima can be timed with an accuracy far surpassing that of radial-velocity measurements, our present knowledge of period variations in close binary systems is mainly based on eclipsing variables. In what follows we shall, therefore, limit ourselves to an analysis of the variation of period consistent with the foregoing definition, and shall set out to investigate the extent to which this period is bound to oscillate as a result of the perturbations of other elements which were established in the preceding sections of this chapter.

Before doing so we must, however, define first what is meant by the term 'moments of the minima'. *The light minima will,* by definition, *be expected to occur when the apparent separation of the centres of both components becomes a minimum.* The phase angles (i.e., true anomalies) of these minima can, in turn, be best expressed in terms of the geometry of our problem by a recourse to the fixed XYZ-system of space axes introduced in section II.3. In that section the orientation of all axes was left wholly arbitrary; but in the subsequent section II.5 we were led to identify the Z-axis with the vector representing the total moment of momentum of our binary system to render XY its invariable plane. In order to remove the last arbitrariness from its definition, let us hereafter *assume the X-axis to be oriented so as to lie in a plane defined by the line of sight and the Z-axis* (which is perpendicular to the celestial sphere). Let, furthermore, the angle between the invariable (XY-) plane and a plane perpendicular to the line of sight of a distant observer be denoted by I. If so, the direction cosines of this line of sight, referred to fixed space-axes, obviously are ($\sin I$, 0, $\cos I$) and, consequently, the cosine l_0 of the angle between the radius-vector (of direction cosines a''_{i1} as defined by equations 3-21 to 23) and the line of sight will obviously be given by

$$\left.\begin{aligned} l_0 &= a''_{11} \sin I + a''_{31} \cos I \\ &= L \sin u + M \cos u, \end{aligned}\right\} \tag{7-11}$$

where we have abbreviated

$$\left.\begin{aligned} L &= \sin i \cos I - \cos i \sin I \sin \Omega, \\ M &= \qquad\qquad\qquad\quad \sin I \cos \Omega. \end{aligned}\right\} \tag{7-12}$$

The apparent distance δ of the projected centres of the two components at any moment then follows from the equation

$$\delta^2 = r^2(1 - l_0^2) = \left\{\frac{A(1 - e^2)}{1 + e \cos (u - \omega)}\right\}^2 \{1 - (L \sin u + M \cos u)^2\}; \tag{7-13}$$

and our task becomes to establish the value of u for which δ^2 becomes a minimum. Differentiating (7-13) we find this to be a root of the equation

$$\left.\begin{aligned} (L \sin u + M \cos u)(L \cos u - M \sin u)[1 + e \cos (u - \omega)] \\ = e \sin (u - \omega)[1 - (L \sin u + M \cos u)^2] \end{aligned}\right. \tag{7-14}$$

which, for small values of e, readily assumes either one of the two forms

$$\begin{aligned} u_p^* &= u_0 + 2n\pi - e \sin (u_0 - \omega) \cot^2 j + \dots, \\ u_s^* &= u_0 + (2n + 1)\pi + e \sin (u_0 - \omega) \cot^2 j + \dots, \end{aligned} \tag{7-15}$$

appropriate for the primary or secondary minima (taking place in the neighbourhood of the superior or inferior conjunction). In these equations n stands for an arbitrary integer (including zero),

$$\sin u_0 = \frac{L}{\sqrt{L^2 + M^2}}, \qquad \cos u_0 = \frac{M}{\sqrt{L^2 + M^2}}, \tag{7-16}$$

and

$$\sin^2 j = L^2 + M^2 \tag{7-17}$$

defines the angle j between the orbital plane and the celestial sphere, such that

$$\cos j = \cos i \cos I + \sin i \sin I \sin \Omega \tag{7-18}$$

by (7-12). The reader should particularly note that, unlike the angle I between the celestial sphere and the invariable plane of our binary system which remains secularly constant,* the angle j between the actual position of the orbital plane and the celestial sphere must reflect all variations of Ω and i due to precession and nutation (*cf.* section II.5) and, in particular, is bound to oscillate between the limits $I \pm i$ in the period of nodal regression. Moreover, the angle u_0 as defined by equations (7-16) assumes on expansion the more explicit form

$$u_0 = -\Omega + \sin i \cos \Omega \cot I + \tfrac{1}{4} \sin^2 i \sin 2\Omega + \ldots \tag{7-19}$$

so that, consequently,

$$u_p^* + \Omega = 2n\pi + \sin i \cos \Omega \cot I + e \sin \bar{\omega} \cot^2 j + \ldots \tag{7-20}$$

and

$$u_s^* + \Omega = (2n + 1)\pi + \sin i \cos \Omega \cot I - e \sin \bar{\omega} \cot^2 j + \ldots, \tag{7-21}$$

where $n = 0, 1, 2, \ldots$ etc.

Suppose that, in what follows, we shall define the orbital period as the time interval between two successive primary minima (the choice of the secondary minima would lead to quite analogous results). The duration of this interval can now be obtained by integrating the equation

$$n dt = \frac{(1 - e^2)^{3/2}\, dw}{[1 + e \cos (w - \bar{\omega})]^2} + \frac{\mho}{A^2} \sin^2 (w - \Omega)\, dw. \tag{7-22}$$

resulting from (7-10), between the limits

$$\left.\begin{aligned} w_1 &= + \sin i_1 \cos \Omega_1 \cot I + e \sin \bar{\omega}_1 \cot^2 j_1 + \ldots \\ w_2 &= 2\pi + \sin i_2 \cos \Omega_2 \cot I + e \sin \bar{\omega}_2 \cot^2 j_2 + \ldots \end{aligned}\right\} \tag{7-23}$$

In considering these limits, it is of fundamental significance for us to realize that, unlike the elements A or e which exhibit (to the first order in small quantities) no secular perturbations, the angles Ω and $\bar{\omega}$ are subject to such perturbations and, moreover, i oscillates in a period which is very long in comparison with that of the orbit. In consequence, the values of $i_{1,2}$, $j_{1,2}$ or

* Or its value may change but slowly in time, as the Sun and the respective eclipsing system alter their relative position in space, due to their peculiar motions in the Galaxy.

$\Omega_{1,2}$ will not be the same in w_1 and w_2, but will differ by the amount of their displacement in the course of a cycle.

Integrating equation (7-22) between these limits and retaining, as before, only quantities of first order, we eventually establish that

$$\frac{P}{P_0} = 1 + \frac{1}{2\pi}\left\{w - 2e\sin(w - \bar{\omega}) + \dots\right\}_{w_1}^{w_2} + \frac{\Omega}{2A^2} + \dots, \qquad (7\text{-}24)$$

where P_0 denotes the period of the sideric (undisturbed) orbit. Now by the fundamental trigonometric formula

$$\begin{aligned}\sin(w_2 - \bar{\omega}_2) &- \sin(w_1 - \bar{\omega}_1)\\ &= 2\sin\tfrac{1}{2}\{w_2 - w_1 + \bar{\omega}_1 - \bar{\omega}_2\}\cos\tfrac{1}{2}\{w_2 + w_1 - \bar{\omega}_2 - \bar{\omega}_1\}\end{aligned} \qquad (7\text{-}25)$$

and, by (7-23),

$$\left.\begin{aligned}w_2 \pm w_1 = 2\pi + e\{&\sin\bar{\omega}_2\cot^2 j_2 \pm \sin\bar{\omega}_1\cot^2 j_1\}\\ &+ \{\sin i_2\cos\Omega_2 \pm \sin i_1\cos\Omega_1\}\cot I.\end{aligned}\right. \qquad (7\text{-}26)$$

In section II.6 we have, moreover, established that the longitude $\bar{\omega}$ of the apsidal line in close binary systems advances at a uniform rate ν in such a way that if

$$\left.\begin{aligned}\bar{\omega}_1 &= \bar{\omega}_0 + \nu t,\\ \bar{\omega}_2 &= \bar{\omega}_0 + \nu(t + P) = \bar{\omega}_1 + \nu P,\end{aligned}\right\} \qquad (7\text{-}27)$$

then

$$\left.\begin{aligned}\bar{\omega}_2 + \bar{\omega}_1 &= 2\bar{\omega}_1 + \nu P,\\ \bar{\omega}_2 - \bar{\omega}_1 &= \nu P,\end{aligned}\right\} \qquad (7\text{-}28)$$

where

$$\nu = \frac{2\pi}{U}, \qquad (7\text{-}29)$$

U being the period of revolution of the apsidal line. We may also note that the difference $\sin(w_2 - \bar{\omega}_2) - \sin(w_1 - \bar{\omega}_1)$ on the right-hand side of (7-21) is multiplied by e which we have been regarding as a small quantity of first order. In consequence, the exact equation (7-25) may be replaced, correctly to quantities of second order, by the following approximation

$$\sin(w_2 - \bar{\omega}_2) - \sin(w_1 - \bar{\omega}_1) = -2\cos\bar{\omega}\sin\tfrac{1}{2}(w_2 - w_1 - \nu P) \qquad (7\text{-}30)$$

revealing that, of the two combinations $w_2 \pm w_1$ represented by equation (7-26), only the difference $w_2 - w_1$ will occur explicitly in (7-24).

In order to evaluate it, let us (consistent with our present scheme of approximation) replace hereafter the cotangents of the angles I and j on the right-hand side of (7-26) by their cosines, and remember that

$$\sin\bar{\omega}_2 - \sin\bar{\omega}_1 = 2\sin\tfrac{1}{2}\nu P\cos(\bar{\omega}_1 + \tfrac{1}{2}\nu P). \qquad (7\text{-}31)$$

Moreover, $\cos j$ continues to be given by equation (7-18), while (5-68) and (5-73) reveal that, very approximately,

$$\left.\begin{aligned}\sin i\sin\Omega &= p = -\tilde{G}\sin s_1 t - \tilde{H}\sin s_2 t,\\ \sin i\cos\Omega &= q = -\tilde{G}\cos s_1 t - \tilde{H}\cos s_2 t,\end{aligned}\right\} \qquad (7\text{-}32)$$

and, therefore,

$$\sin^2 i = \tilde{G}^2 + 2\tilde{G}\tilde{H}\cos(s_1 - s_2)t + \tilde{H}^2, \tag{7-33}$$

where we have abbreviated

$$\left.\begin{aligned} \tilde{G} &= k_1\tilde{A}_1 + k_2\tilde{A}_2, \\ \tilde{H} &= k_1\tilde{B}_1 + k_2\tilde{B}_2 \end{aligned}\right\} \tag{7-34}$$

In consequence, equation (7-24) assumes the more explicit form

$$\begin{aligned} \frac{P}{P_0} = 1 &+ \frac{\cos I}{2\pi}\{(\tilde{G}s_1P_0)\sin s_1t + (\tilde{H}s_2P_0)\sin s_2t\} \\ &+ \frac{e}{2\pi}\Big\{\nu P_0\cos^2 I\cos(\nu t + \bar{\omega}_{0\cdot5}) \\ &\qquad + \tilde{G}^2s_1P_0\sin^2 I\sin(\nu t + \bar{\omega}_0)\sin 2s_1t \\ &\qquad - \tilde{G}s_1P_0\sin 2I\sin(\nu t + \bar{\omega}_0)\cos s_1t \\ &\qquad - 2\tilde{G}s_1P_0\cos I\cos(\nu t + \bar{\omega}_{0\cdot5})\sin s_1t \\ &\qquad - 2\tilde{H}s_2P_0\cos I\cos(\nu t + \bar{\omega}_{0\cdot5})\sin s_2t + \dots\Big\} \\ &+ \mathfrak{Q}^{(0)} + \mathfrak{Q}^{(1)}\cos(s_1 - s_2)t + \dots, \end{aligned} \tag{7-35}$$

where, to the order of accuracy we are working, the ratio $\mathfrak{Q}/2A^2$ as given by equation (6-20) has been decomposed into a time function with the coefficients

$$\mathfrak{Q}^{(0)} = \tfrac{1}{2}(\tilde{G}^2 + \tilde{H}^2)\sum_{j=1}^{2}(\Pi_j + \Gamma_j) \tag{7-36}$$

and

$$\mathfrak{Q}^{(1)} = \tilde{G}\tilde{H}\sum_{j=1}^{2}(\Pi_j + \Gamma_j), \tag{7-37}$$

if (as in section II.5) advantage is taken of the simplified relations (5-57) which are exact whenever (for arbitrary values of $\tilde{A}_1$ and $\tilde{A}_2$)

$$\varepsilon = \eta_1 = \eta_2 \tag{7-38}$$

and if, by (5-33) and (5-46), we verify that

$$\frac{3}{2}\frac{\alpha_j}{A_2}\left(1 + \frac{1}{k_j}\right) = \Pi_j + \Gamma_j. \tag{7-39}$$

Moreover,

$$\bar{\omega} - \Omega = \bar{\omega}_0 - \Omega_0 + (\nu + s_1)t, \tag{7-40}$$

while

$$\bar{\omega}_{0\cdot5} = \bar{\omega}_0 + \tfrac{1}{2}\nu P_0. \tag{7-41}$$

Within the scheme of approximation adhered to in this section, equation (7-35) represents the period variations of eclipsing binary systems caused, directly or indirectly, by mutual distortion of their components. It should be stressed that, with the exception of the terms factored by $\mathfrak{Q}$ going back to the additional terms on the right-hand side of the generalized 'law of areas' (7-10), all other terms in (7-35) represent *apparent* period changes, due to the particular way in which the lines of the minima of eclipsing variables are observed, and *not* any intrinsic changes of the period of sideric orbit (which could be observed if the two components were visually decomposed). Of such apparent period changes, only those caused by the revolution of the apsidal line and represented by a single term (factored by evP_0) on the right-hand side of equation (7-35) have so far been paid any attention;* while all other similar terms invoked by the recession of the nodes or oscillation of the orbital plane are new.

The period changes to be expected on the basis of equation (7-35) are not subject to direct observational verification. They are, however, bound to influence the observed *times of the minima* $M(E)$, as observed at different epochs $E \equiv t/P_0$ for integral values of E. As is well known, the relation between $M(E)$ and $P(E)$ can be approximated by the differential equation

$$\frac{dM}{dE} = P(E)\,, \tag{7-42}$$

which integrates to yield

$$M(E) = \int P(E)\,dE = \int (P/P_0)\,dt. \tag{7-43}$$

If we evaluate this last integral for P/P_0 as given by equation (7-35) and set $t = P_0E$, the result can eventually be expressed in the form

$$\begin{aligned} M(E) - M_0 = \{1 + \mathfrak{Q}^{(0)}\}P_0E + (P_0/2\pi)\{&-a_1 \cos r_1E - a_2 \cos r_2E \\ &+ a_3 \sin (r_3E + \bar{\omega}_{0\cdot5}) \\ &+ a_4 \cos (r_4E + \bar{\omega}_{0\cdot5}) - a_5 \cos (r_5E + \bar{\omega}_{0\cdot5}) \\ &+ a_6 \cos (r_6E + \bar{\omega}_{0\cdot5}) - a_7 \cos (r_7E + \bar{\omega}_{0\cdot5}) \\ &- \tfrac{1}{2}a_8 \sin (r_8E + \bar{\omega}_0) + \tfrac{1}{2}a_9 \sin (r_9E + \bar{\omega}_0) \\ &- a_{10} \sin r_{10}E + \dots\}\,, \end{aligned} \tag{7-44}$$

where the periods of the individual oscillations are given by

$$\left.\begin{aligned} r_1 &= s_1P_0\,, \\ r_2 &= s_2P_0\,, \\ r_3 &= \nu P_0\,, \\ r_4 &= (\nu + s_1)P_0 = r_3 + r_1\,, \\ r_5 &= (\nu - s_1)P_0 = r_3 - r_1\,, \\ r_6 &= (\nu + s_2)P_0 = r_3 + r_2\,, \\ r_7 &= (\nu - s_2)P_0 = r_3 - r_2\,, \\ r_8 &= (\nu + 2s_1)P_0 = r_4 + r_1\,, \\ r_9 &= (\nu - 2s_1)P_0 = r_5 - r_1\,, \\ r_{10} &= (s_1 - s_2)P_0 = r_1 - r_2\,, \end{aligned}\right\} \tag{7-45}$$

* F. Tisserand, *C. R. Acad. Paris*, **120**, 125, 1895.

and their amplitudes are

$$\left.\begin{aligned} a_1 &= \tilde{G}\cos I, \\ a_2 &= \tilde{H}\cos I, \\ a_3 &= e\cos^2 I, \\ a_4 &= (r_1/r_4)\, e\tilde{G}\,(1+\sin I)\cos I, \\ a_5 &= (r_1/r_5) e\tilde{G}\,(1-\sin I)\cos I, \\ a_6 &= (r_2/r_6) e\tilde{H}\cos I, \\ a_7 &= (r_2/r_7) e\tilde{H}\cos I, \\ a_8 &= (r_1/r_8) e\tilde{G}^2\sin^2 I, \\ a_9 &= (r_1/r_9) e\tilde{G}^2\sin^2 I, \\ a_{10} &= (2\pi\mathfrak{Q}^{(1)})/r_{10}\,. \end{aligned}\right\} \qquad (7\text{-}46)$$

The foregoing equation (7-44) with its coefficients as given by (7-45) and (7-46) constitutes the main outcome of the present section and reveals that, within the scheme of our approximation, the theoretical ephemeris $M(E)$ for the times of the light minima will contain a total of *ten* periodic teıms arising from different causes. Of these, the *first* and *second* term (characterized by the amplitudes $a_{1,2}$ and periods $r_{1,2}$ are due solely to the *regression of the nodes* and its periodic inequality, while the *third* (of amplitude a_3 and period r_3) arises from the *apsidal advance.* The *tenth* term (a_{10}, r_{10}) is caused again by the *nutation* of the rotational axes of both components and the accompanying motion of the orbital plane. The *six* terms with periods from r_4 to r_9 may be regarded as *apse-node terms,* which arise from an interaction between apsidal advance and nodal regression or nutation of rotational axes, and are by far the most interesting ones on account of the possibility of *commensurability* between these motions.

In order to explore the possibility of such an event, we should recall that the periods of nodal regression and nutational motion are uniquely determined by the values of the constants

$$s_{1,2}P_0 = -\pi\{\Gamma_1 + \Pi_1 + \Gamma_2 + \Pi_2 \pm [(\Gamma_1 + \Pi_1 - \Gamma_2 - \Pi_2)^2 + 4\Pi_1\Pi_2]^{1/2}\}, \qquad (7\text{-}47)$$

following as roots of the quadratic equation (5-75). If the distortion of only one (say the primary) component were appreciable, we should have

$$\left.\begin{aligned} s_1P_0 &= -2\pi(\Gamma_1 + \Pi_1), \\ s_2P_0 &= 0. \end{aligned}\right\} \qquad (7\text{-}48)$$

In general, however, both s_1 and s_2 will be distinct from zero and such that

$$s_1 < s_2 < 0, \qquad (7\text{-}49)$$

while

$$(s_1 - s_2)P_0 = -2\pi\{(\Gamma_1 + \Pi_1 - \Gamma_2 - \Pi_2)^2 + 4\Pi_1\Pi_2\}^{1/2}, \qquad (7\text{-}50)$$

the square-root to be taken positively.

On the other hand, the period of the apsidal advance will (within the scheme of our approximation in this section) be given by

$$\nu P_0 = 2\Delta\bar{\omega} = \pi\left\{3\,\frac{\alpha_1+\alpha_2}{A^2} + 30\,\frac{\beta_{12}+\beta_{22}}{A^5} + \dots\right\}, \tag{7-51}$$

in accordance with (6-22). The constants α_i and β_{i2} in equation (7-51) are defined by (5-9) or (5-10) and (5-13), respectively. Equations (5-33) and (5-46) disclose, however, that

$$\frac{\alpha_i}{A^2} = \tfrac{2}{3}\Pi_i = \tfrac{2}{3}k_i\Gamma_i \tag{7-52}$$

while a recourse to (5-13) and (3-41) reveals that

$$\frac{\beta_{i2}}{A^5} = \frac{2m_{3-i}}{m_1+m_2}\,\frac{\Pi_i}{\gamma_i^2} \tag{7-53}$$

for $i = 1, 2$—by virtue of which the foregoing equation (7-51) can be rewritten as

$$\nu P_0 = 2\pi \sum_{i=1}^{2}\left\{1 + \frac{30}{\gamma_i^2}\,\frac{m_{3-i}}{m_1+m_2}\right\}\Pi_i\,. \tag{7-54}$$

Now consider, in the light of the preceding results, the sum $\nu + js_1$ (j being an arbitrary number) which occurs in the periods r_4 (with $j = 1$), or in r_8 ($j = 2$). Equations (7-47) and (7-54) make it evident that

$$\begin{aligned}(\nu + js_1)P_0 = 2\pi \sum_{i=1}^{2}\Bigg\{&\left[\left(1 + \frac{30}{\gamma_i^2}\,\frac{m_{3-i}}{m_1+m_2}\right)\Pi_i - j(\Gamma_i + \Pi_i)\right] \\ &- j[(\Gamma_1 + \Pi_1 - \Gamma_2 - \Pi_2)^2 + 4\Pi_1\Pi_2]^{1/2}\Bigg\}.\end{aligned} \tag{7-55}$$

In order to present this result in a simpler form, consider a case in which the distortion of one (say the primary) component becomes negligible: if so, the foregoing equation (7-55) readily reduces to

$$(\nu + js_1)P_0 = 2\pi\left\{1 - j + \frac{30}{\gamma_2^2}\,\frac{m_1}{m_1+m_2} - \frac{j}{k_2}\right\}\Pi_2\,. \tag{7-56}$$

Of the quantities figuring between curly brackets, the ratio γ_2 as defined by equation (5-11) is likely to be in the neighbourhood of unity for a highly distorted star, and the mass-ratio m_2/m_1 is then likely to be small. The same is, however, apt to be true of the ratio $k_2 = \Pi_2/\Gamma_2$. Therefore, *it is possible for a combination of the values of m_2/m_1, γ_2, and k_2 in certain eclipsing systems to become such as to render the sum $\nu + js_1$ for either one of the values* *of* $j = \frac{1}{2}, 1, \frac{3}{2}$, or 2 *very small.** If so, the period of the respective term may become very long indeed, and its amplitude (containing the respective value of r as a divisor) appreciable.

Such a situation appears to obtain, for instance, in the well-known eclipsing system of Algol, whose ephemeris for the light minima contains a 'great inequality' term of the period of approximately 174 years and an amplitude of 0·127 days; and of many other eclipsing systems as well. A more detailed discussion of the individual cases is, of course, wholly outside the scope of this book. We may, however, note that unless some of the periodic terms on the right-hand side of equation (7-44) possess amplitudes sufficiently enhanced by the resonance (or other) effects, a harmonic analysis of the observed empirical function $M(E)$ into its individual proper constituents may prove to confront us with a task of some difficulty.†

In conclusion of the present section, one retrospective remark of theoretical interest concerning the limits of validity of equation (7-42) which we have used to relate the orbital period $P(E)$ with the times of the minima $M(E)$ may be added. If, as usual, we *define* the (instantaneous) period of an eclipsing variable as the difference between the moments of light minima taking place at two consecutive epochs E and $E + 1$, the functions $M(E)$ and $P(E)$ are bound to satisfy the linear *difference* equation

$$M(E + 1) - M(E) = P(E). \tag{7-57}$$

Both $M(E)$ and $P(E)$ so defined are, by their nature, *discontinuous* functions of their argument and exist only for *integral* values of E. It is, unfortunately, impossible to utilize equation (7-57) as it stands for a determination of the instantaneous periods in practice (as the consecutive minima can seldom—if ever—be observed with the requisite precision); instead, it is customary to rely on the *cumulative* effects of any period variation in as long a sequence of the minima as may be available for harmonic analysis. In order that we should be in a position to do so—or even to predict a single value of $M(E)$ at some future date from known values of $M(0)$ and $P(E)$—a knowledge of the explicit solution of the foregoing difference equation (7-57) is evidently prerequisite.

The exact solution of (7-57) can, to be sure, be written down at once in the form of a summation

$$M(E) = M(0) + \sum_{j=0}^{E-1} P(j); \tag{7-58}$$

but this is not only very unwieldy in use (calling, as it does, for an algebraic

* The fractional values of j are involved only in the apse-nutation terms, whose periods are likely to contain, to be sure, also s_2 besides $v + js_1$. The values of s_2 are, however, ordinarily so small in comparison with $v + js_1$ as not to influence the respective periods appreciably. It is only when $v + js_1$ itself becomes very small on account of a near-resonance between the apse and node terms that the values of s_2 may affect the respective periods by a significant amount.

† For practical methods of tackling it *cf.*, e.g., C. Lanczos, *Applied Analysis*, Prentice-Hall, New York 1956, Chapters III and IV.

sum of E discrete terms—a very laborious task when E happens to be a large number), but becomes also of no avail if our professed aim is to deduce the form of the function $P(E)$ from the observed sequence of $M(E)$.

In order to investigate the mutual relationship of these two functions, let us extend the intuitive definitions of $M(E)$ and $P(E)$, and replace them by $\bar{M}(\bar{E})$ and $\bar{P}(\bar{E})$ regarded as functions of a *continuous* variable $\bar{E} \geqslant 0$, defined so that $\bar{M}(E) = M(E)$ and $\bar{P}(E) = P(E)$ for each integral value of E. For non-integral values of $\bar{E}$, $\bar{P}(\bar{E})$ may be any continuous function which coincides with $P(E)$ for integral values of the independent variable; and $\bar{M}(\bar{E})$ is then defined as a solution of the generalized difference equation

$$\bar{M}(\bar{E} + 1) - \bar{M}(\bar{E}) = \bar{P}(\bar{E}) \tag{7-59}$$

for all $\bar{E} \geqslant 0$. Now let us assume that there exists an analytic solution of (7-59) with a radius of convergence greater than 1 for each $\bar{E} \geqslant 0$; if so, this solution should be expansible in a Taylor series of the form

$$\bar{M}(\bar{E} + 1) = \bar{M}(\bar{E}) + \sum_{j=1}^{\infty} \frac{1}{j!} \frac{d^j \bar{M}}{d\bar{E}^j}, \tag{7-60}$$

and its insertion in (7-59) will convert the latter *difference* equation into the equivalent *differential* equation

$$\bar{P}(\bar{E}) = \left(e^{\frac{d}{d\bar{E}}} - 1\right) \bar{M} \tag{7-61}$$

of infinite order.

A particular solution of this differential equation with constant coefficients $(j!)^{-1}$ can be constructed by standard methods to assume the form

$$\bar{M}(\bar{E}) = \bar{M}(0) + \int_0^{\bar{E}} \bar{P}(\bar{E})\, d\bar{E} + \sum_{j=1}^{n} A_j e^{m_j \bar{E}} \int_0^{\bar{E}} e^{-m_j \bar{E}} \bar{P}(\bar{E})\, d\bar{E}, \tag{7-62}$$

where the constants A_j and m_j are defined by the well-known relation

$$\left\{ \sum_{j=1}^{n} \frac{D^j}{j!} \right\}^{-1} = \sum_{j=1}^{n} \frac{A_j}{D - m_j} \tag{7-63}$$

and $n \rightarrow \infty$.

The general solution of (7-61) can now be represented as the sum of the above particular solution and any (analytic) periodic function of period one. Only one initial condition is given to us, which specifies $\bar{M}(0)$. In order that this condition be met, all additional periodic functions must vanish for $\bar{E} = 0$, but otherwise remain completely arbitrary. As we are interested only in what happens for integral values E of $\bar{E}$, no generality will clearly be lost if we choose all such functions to be identically zero. If so, however, all bars above E, M, or P may henceforward be dropped; and on evaluation of the A_j's and m_j's from (7-63) by repeated integration per partes equation

(7-62) may be reduced to

$$M(E) = M(0) + \int_0^E P(E)\, de - \tfrac{1}{2}\{P(E) - P(0)\}$$
$$+ \frac{1}{2!\,6}\{P'(E) - P'(0)\} - \frac{1}{4!\,30}\{P'''(E) - P'''(0)\} + \dots, \tag{7-64}$$

which represents the explicit expression for $M(E)$ in terms of $P(E)$, just as the differential equation (7-61) represented the converse relation. A comparison of the foregoing equation (7-64) with (7-43) reveals that the first two terms on the right-hand side of (7-64) represents the desired solution for $M(E)$ to the order of accuracy to which the difference equation (7-57) may be approximated by the differential equation (7-42) whose integral led to (7-63). The infinite series which follows the leading term on the right-hand side of (7-64) represents then the correction to (7-43) arising from the fact that the period $P(E)$ is defined in terms of $M(E)$ by a difference, and not differential, equation.

We may add that the expansion on the right-hand side of equation (7-64) can also be obtained, without recourse to (7-62), in the following alternative way. Let us assume that the solution of (7-57) is generally expressible in the form

$$M(E) = M(0) + \int_0^E P(E)\, dE + \Delta(E), \tag{7-65}$$

where $\Delta(E)$ stands for the error of approximating the solution of the difference equation (7-57) by that of the differential equation (7-42) of first order. By definition, therefore,

$$\Delta(E) = M(E) - M(0) - \int_0^E P(E)\, dE, \tag{7-66}$$

where the difference $M(E)$—$M(0)$ follows exactly from (7-58), and the integral of $P(E)$ can be approximated with the aid of the well-known Euler-Maclaurin quadrature formula by

$$\int_0^E P(E)\, dE = \tfrac{1}{2}P(0) + \sum_{j=1}^{E-1} P(j) + \tfrac{1}{2}P(E)$$
$$- \sum_{j=1}^{n} \frac{B_{2j}}{(2_j)!}\{P^{(2j-1)}(E) - P^{(2j-1)}(0)\}, \tag{7-67}$$

where the B_{2j}'s are the Bernoulli numbers, the first few of which are known to be:

j	1,	2,	3,	4,	5,	...
B_{2j}	$\frac{1}{6}$,	$-\frac{1}{30}$,	$\frac{1}{42}$,	$-\frac{1}{30}$,	$\frac{5}{66}$,	...

The foregoing formula is subject to a known truncation error R_n if terminated after n terms.* Therefore, by use of (7-58) and (7-67) equation (7-66) assumes the more explicit form

$$\Delta(E) = \tfrac{1}{2}\{P(0) - P(E)\} + \sum_{j=1}^{n} \frac{B_{2j}}{(2j)!}\{P^{(2j-1)}(E) - P^{(2j-1)}(0)\}, \qquad (7\text{-}68)$$

which, on insertion in (7-65), verifies (7-64).

What is the actual significance, in our present problem, of the correction $\Delta(E)$ which should be adjoined to the right-hand side of equation (7-44)? As long as the variable part in $P(E)$ consists solely of periodic terms oscillating in the periods $2\pi P_0/r_j$, the coefficients of the individual terms in the integral (7-44) of $P(E)$ will be of the order of r_j^{-1}, while those of the successive derivatives $P^{(2j-1)}(E)$ in $\Delta(E)$ will be of the order of r_j^{2j-1}. In consequence, for small values of r_j (corresponding to long-periodic oscillations—such as are represented, in our present problem, by equations (7-45), the rate of convergence of the expansion on the right-hand side of (7-68) is likely to be so rapid that the retention of the first term alone—as we have done in equation (7-43) and (7-44)—should guarantee an entirely sufficient accuracy. On the other hand, should our aim become to study the effects, on the orbital period, of short-periodic oscillations of the elements i, Ω, e, $\bar{\omega}$, or A (the existence of which has also been established in the preceding sections II·5 and 6), the correction term $\Delta(E)$ would become of a magnitude comparable with that of the integral of $P(E)$, and would have to be taken into account. The reason why we do not at present propose to study the short-periodic changes in the same detail as the long-periodic ones is merely the practical fact that such changes would be much more difficult to detect by such observations as are mostly available for eclipsing binary systems. As a result, the incentive of a comparison between theory and observations—which should be the crowning point of our analysis and which is bound to furnish important results for long-periodic oscillations—remains as yet largely unavailable as far as short-periodic oscillations are concerned.

II.8. PERTURBATIONS BY A THIRD BODY

In all foregoing sections of this chapter we have been concerned with the perturbations in close binary systems arising from the deformation of both components due to their axial rotation and mutual tidal action. Such perturbations *must* be present in binaries consisting of fluid components, and their magnitude is governed by the proximity of the components. The object of this penultimate section of a chapter concerned with the dynamics of close binary systems will be to investigate a wholly different class of perturbations

* For its explicit form *cf.*, *e.g.*, H. and B. Jeffreys, *Methods of Mathematical Physics*, Cambridge 1946, pp. 254–257.

which *may* be present if such a binary is attended by a *third body.* Triple systems consisting of a close pair accompanied by a third star are relatively frequent: some ten per cent of all known close binaries seem to possess such companions, whose contribution to the total light of the system is as a rule too feeble to be noticeable, and whose presence can be inferred only by the cumulative effects of dynamical perturbations which it exerts on the motion of the close pair in the course of time. The aim of this section will be to investigate the nature of such perturbations, and to predict the effects by which a third body can make its presence felt in the motion of a close or eclipsing binary.

Observations reveal that, in the large majority of known triple stars, the distance of the third component from the centre of gravity of the system is many times as large as the separation of the close pair; and its orbital period correspondingly longer. The third body will, therefore, move around the common centre of gravity in a closed elliptical orbit which is essentially undisturbed by the binary nature of its companion* and remains fixed in space. The angle of inclination between the orbital planes of the close and wide pairs may, however, be quite arbitrary; and whereas the orbital eccentricity of the close pair will as a rule be small, that of the wide orbit may be of cometary order.

In order to set up the corresponding equations of motion and to specify the *disturbing function* which generates the perturbations, let the plane of the third orbit be adopted as our plane of reference, and the motion of the third body of mass m_3 be referred to the centre of mass of the close pair as the origin of coordinates. Moreover, in following the motion in the close pair, let that of the secondary component of mass m_2 be referred to the centre of gravity of the star of mass m_1 as origin; this has the effect of making the perturbations of the third star by the close pair a minimum. If so, then (considering all three stars as mass-points) the disturbing function S arising from the presence of a third body can be shown† to assume the form

$$S = Gm_3 \frac{r^2}{r'^3} \sum_{j=1}^{\infty} \frac{(m_1)^j - (-m_2)^j}{(m_1 + m_2)^j} \left(\frac{r}{r'}\right)^{j-1} P_{j+1}(\sigma), \tag{8-1}$$

where r, r' denote the radii-vectors of the close and wide orbit; and σ, the cosine of the angle between them, will be given by

$$\left.\begin{aligned} \sigma &= \cos(u-\Omega)\cos(u'-\Omega) + \sin(u-\Omega)\sin(u'-\Omega)\cos i \\ &= \cos^2 \tfrac{1}{2}i \cos(u-u') + \sin^2 \tfrac{1}{2}i \cos(u+u'-2\Omega), \end{aligned}\right\} \tag{8-2}$$

where Ω stands for the longitude of the node (i.e., of the line of intersection of the two orbital planes); i, the angle of inclination between them; and

* Its perturbations—whatever they may be—are besides of little interest as they are (in most cases) not observable.

† For fuller details of its derivation *cf.*, e.g., E. W. Brown and C. A. Shook, *Planetary Theory*, Cambridge 1933, pp. 13–14.

u, u', the longitudes of m_2, m_3, reckoned from Ω in the planes of their respective orbits.

Moreover, the radii-vectors of the close and wide pairs will, in general, be given by

$$r = \frac{A(1 - e^2)}{1 + e \cos v}, \qquad r' = \frac{A'(1 - e'^2)}{1 + e' \cos v'}, \tag{8-3}$$

where v, v' denote the true anomalies of m_2 and m_3 in their respective orbits, both reckoned from the longitudes ω, ω' of the periastra. Consistent with our initial assumptions, the primed elements A', e', ω', and the time T' of the periastron passage of the orbit of m_3 in our plane of reference will be regarded as constant. The orbital elements A, e, ω, and T of the disturbed motion of the close pair, as well as the angles Ω and i, will then be governed by a set of six first-order variational equations similar to (4-21)–(4-26),* with the disturbing function S as given by equation (8-1); and our task will be to solve them.

In order to do so for unrestricted values of i or e' (which, unlike in sections II.4 or 5, can no longer be considered small), we find it expedient to change over from the elements of the orbit to a new set of areal velocities L, G, H and the corresponding angles l, g, h, introduced first by Delaunay in his celebrated theory of the motion of the moon, which are related with the elements of the orbit by means of the equations

$$\left.\begin{aligned} L &= nA^2 & l &= nt + \varepsilon - \omega, \\ G &= nA^2\sqrt{1 - e^2}, & g &= \omega - \Omega, \\ H &= nA^2\sqrt{1 - e^2}\cos i; & h &= \Omega; \end{aligned}\right\} \tag{8-4}$$

n standing again for the mean daily motion of the close pair, and consider them as the new dependent variables of our problem.† As is well known, the advantage of their use rests on the fact that their variational equations assume the canonical form

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial S}{\partial l}, & \frac{dl}{dt} &= -\frac{\partial S}{\partial L}, \\ \frac{dG}{dt} &= \frac{\partial S}{\partial g}, & \frac{dg}{dt} &= -\frac{\partial S}{\partial G}, \\ \frac{dH}{dt} &= \frac{\partial S}{\partial h}, & \frac{dh}{dt} &= -\frac{\partial S}{\partial H}, \end{aligned}\right\} \tag{8-5}$$

with S as given by (8-1); and once they have been solved, the respective

* Or identical with them if we set $\omega + \Omega = \bar{\omega}$.

† Their use in the earlier sections of this chapter was precluded by the coupling that existed between the variational equations for Ω and i and the Eulerian equations governing the motion of the rotational axes of the components.

orbital elements follow from

$$\left.\begin{aligned} A &= \sqrt{L/n}, & i &= \cos^{-1}(H/G), \\ e &= \sqrt{1-(G/L)^2}, & \Omega &= h; \\ \omega &= g + h, \end{aligned}\right\} \tag{8-6}$$

while l stands for the mean longitude of m_2 in its relative orbit.

Needless to stress, the canonical equations (8-5) of our stellar problem of three bodies do not, in general, admit of any closed integrals; and owing to their non-linearity, any construction of their solutions can proceed only by successive approximations. In what follows we propose to divide this task into a separate investigation of:

(1) *Short-range perturbations* of the close pair, secular or of periods comparable with that of its orbit, for which the position of m_3 in space (and, therefore, r') can be regarded as *constant*; and

(2) *long-range perturbations* of the close pair, caused by the *motion* of m_3, whose orbital period will be regarded sufficiently long to enable us to replace all elements of the binary system by their *time-averages* over one orbital cycle.

In order to do so, let us return to the disturbing function S and (as the ratio r/r' will hereafter be regarded small) break off the expansion on the right-hand side of equation (8-1) after the second term, retaining

$$S = \frac{Gm_3}{r'^3}\left\{r^2 P_2(\sigma) + \frac{m_1 - m_2}{m_1 + m_2}\frac{r^3}{r'}P_3(\sigma) + \dots\right\}. \tag{8-7}$$

Let us, moreover, form with its aid the time-integral

$$I = \int S\,dt, \tag{8-8}$$

whose evaluation will constitute the principal part of our problem; for once we have done so, a solution for the individual Delaunay elements can evidently be obtained by a partial differentiation of I with respect to the requisite element in accordance with equations (8-5).

Suppose that we set out to evaluate I first for the case of *short-range perturbations*. Consistent with the assumed constancy of r', the integral on the right-hand side of (8-8) can then be split up into

$$I = G\frac{m_3}{r'^3}\left\{\tfrac{3}{2}J_2 - \tfrac{1}{2}J_0 + \frac{m_1 - m_2}{m_1 + m_2}\frac{1}{r'}\left(\tfrac{5}{2}J_3 - \tfrac{3}{2}J_1\right) + \dots\right\}, \tag{8-9}$$

where we have abbreviated

$$\left.\begin{aligned} J_0 &= \int r^2\,dt, & J_1 &= \int r^3\sigma\,dt, \\ J_2 &= \int r^2\sigma^2\,dt, & J_3 &= \int r^3\sigma^3\,dt. \end{aligned}\right\} \tag{8-10}$$

In order to evaluate these integrals in a closed form, replace the time t behind the integral sign by the eccentric anomaly E which is related with t

by means of the equation (6-2) and, accordingly,

$$\left.\begin{aligned} r &= A(1 - e\cos E), \\ n\,dt &= (1 - e\cos E)\,dE. \end{aligned}\right\} \tag{8-11}$$

Moreover (remembering that $u = \omega + v$) the quantity σ as defined by equations (8-2) can evidently be rewritten as

$$\sigma = M\sin v + N\cos v \tag{8-12}$$

or

$$(1 - e\cos E)\sigma = M\sqrt{1 - e^2}\sin E + N(\cos E - e), \tag{8-13}$$

where

$$\left.\begin{aligned} M &= -\sin(\omega - \Omega)\cos(u' - \Omega) + \cos(\omega - \Omega)\sin(u' - \Omega)\cos i, \\ N &= +\cos(\omega - \Omega)\cos(u' - \Omega) + \sin(\omega - \Omega)\sin(u' - \Omega)\cos i. \end{aligned}\right\} \tag{8-14}$$

If, to a *linear* approximation, we now regard the elements A, e, ω, Ω, i as well as u' to be *constant* on the right-hand sides of equations (8-5), a literal evaluation of the integrals (8-10) yields

$$\frac{n}{A^2}J_0 = (1 + \tfrac{3}{2}e^2)E - 3e(1 + \tfrac{1}{4}e^2)\sin E + \tfrac{3}{4}e^2\sin 2E - \tfrac{1}{12}e^3\sin 3E, \tag{8-15}$$

$$\begin{aligned} \frac{n}{A^3}J_1 = M\sqrt{1 - e^2}\,\{&-(1 + \tfrac{3}{4}e^2)\cos E + \tfrac{3}{4}e(1 + \tfrac{1}{6}e^2)\cos 2E \\ &- \tfrac{1}{4}e^2\cos 3E + \tfrac{1}{32}e^3\cos 4E\} \\ &+ N\{-\tfrac{5}{2}e(1 + \tfrac{3}{4}e^2)E + (1 + \tfrac{21}{4}e^2 + \tfrac{3}{4}e^4)\sin E \\ &- (\tfrac{3}{4} + e^2)e\sin 2E + \tfrac{1}{4}e^2(1 + \tfrac{1}{3}e^2)\sin 3E - \tfrac{1}{32}e^3\sin 4E\}, \end{aligned} \tag{8-16}$$

$$\begin{aligned} \frac{n}{A^2}J_2 = \tfrac{1}{2}M^2(1 - e^2)\{&E - \tfrac{1}{2}e\sin E - \tfrac{1}{2}\sin 2E + \tfrac{1}{6}e\sin 3E\} \\ &+ \tfrac{1}{2}MN\sqrt{1 - e^2}\,\{5e\cos E - (1 + e^2)\cos 2E + \tfrac{1}{3}e\cos 3E\} \\ &+ \tfrac{1}{2}N^2\{(1 + 4e^2)E - \tfrac{1}{2}e(11 + 4e^2)\sin E \\ &+ \tfrac{1}{2}(1 + 2e^2)\sin 2E - \tfrac{1}{6}e\sin 3E\}, \end{aligned} \tag{8-17}$$

$$\begin{aligned} \frac{n}{A^3}J_3 = M^3(1 - e^2)^{3/2}\{&-\tfrac{3}{4}\cos E + \tfrac{1}{8}e\cos 2E + \tfrac{1}{12}\cos 3E - \tfrac{1}{32}e\cos 4E\} \\ &+ 3M^2N(1 - e^2)\{-\tfrac{5}{8}eE + \tfrac{1}{4}(1 + e^2)\sin E + \tfrac{1}{4}e\sin 2E \\ &\qquad - \tfrac{1}{12}(1 + e^2)\sin 3E + \tfrac{1}{32}e\sin 4E\} \\ &+ 3MN^2(1 - e^2)^{1/2}\{-\tfrac{1}{4}(1 + 6e^2)\cos E + \tfrac{5}{8}e(1 + \tfrac{2}{5}e^2)\cos 2E \\ &\qquad - \tfrac{1}{12}(1 + 2e^2)\cos 3E + \tfrac{1}{32}e\cos 4E\} \\ &+ N^3\{-\tfrac{15}{8}e(1 + \tfrac{4}{3}e^2)E + \tfrac{3}{4}(1 + 7e^2 + \tfrac{4}{3}e^4)\sin E \\ &\quad - e(1 + \tfrac{3}{4}e^2)\sin 2E + \tfrac{1}{12}(1 + 3e^2)\sin 3E - \tfrac{1}{32}e\sin 4E\}. \end{aligned} \tag{8-18}$$

By insertion of the foregoing results the integral (8-8) can, therefore, be eventually reduced to the form

$$I = p_0 E + \sum_{j=1}^{4} (P_j \sin jE + q_j \cos jE), \tag{8-19}$$

where

$$\begin{aligned} p_0 = {} & \tfrac{2}{3}k_1\{(1-e^2)P_2(M) + (1+4e^2)P_2(N)\}L \\ & - \tfrac{5}{6}ek_2\{(1-e^2)NP_3'(M) + (3+4e^2)P_3(N)\}L, \end{aligned} \tag{8-20}$$

$$\begin{aligned} p_1 = {} & -\tfrac{1}{3}ek_1\{(1-e^2)P_2(M) + (11+4e^2)P_2(N)\}L \\ & + \tfrac{1}{3}k_2\{(1-e^4)NP_3'(M) + (3+21e^2+4e^4)P_3(N)\}L, \end{aligned} \tag{8-21}$$

$$\begin{aligned} p_2 = {} & -\tfrac{1}{3}k_1\{(1-e^2)P_2(M) - (1+2e^2)P_2(N)\}L \\ & + \tfrac{1}{3}ek_2\{(1-e^2)NP_3'(M) - (4+3e^2)P_3(N)\}L, \end{aligned} \tag{8-22}$$

$$\begin{aligned} p_3 = {} & \tfrac{1}{9}ek_1\{(1-e^2)P_2(M) - P_2(N)\}L \\ & - \tfrac{1}{9}k_2\{(1-e^4)NP_3'(M) - (1+3e^2)P_3(N)\}L, \end{aligned} \tag{8-23}$$

$$p_4 = \tfrac{1}{24}ek_2\{(1-e^2)NP_3'(M) - P_3(N)\}L; \tag{8-24}$$

and

$$q_1 = 5ek_1MNG - \tfrac{1}{3}k_2\{3(1-e^2)P_3(M) + (1+6e^2)MP_3'(N)\}G, \tag{8-25}$$

$$\begin{aligned} q_2 = {} & -k_1(1+e^2)MNG \\ & + \tfrac{1}{6}ek_2\{(1-e^2)P_3(M) + (5+2e^2)MP_3'(N)\}G, \end{aligned} \tag{8-26}$$

$$q_3 = \tfrac{1}{3}ek_1MNG - \tfrac{1}{9}k_2\{(1-e^2)P_3(M) - (1+2e^2)MP_3'(N)\}G, \tag{8-27}$$

$$q_4 = -\tfrac{1}{24}ek_2\{(1-e^2)P_3(M) - MP_3'(N)\}G, \tag{8-28}$$

where the P_j's denote the respective Legendre polynomials of M or N (with primes indicating the derivatives), and where we have abbreviated

$$k_1 = \frac{3}{4}\frac{Gm_3}{n^2r'^3} = \frac{3}{4}\frac{m_3}{m_1+m_2+m_3}\left(\frac{P}{P'}\right)^2\left\{\frac{1+e'\cos v'}{1-e'^2}\right\}^3 \tag{8-29}$$

and

$$k_2 = k_1\left\{\frac{m_1-m_2}{m_1+m_2}\right\}\frac{A}{A'}\,\frac{1+e'\cos v'}{1-e'^2}, \tag{8-30}$$

P, P' denoting the orbital periods of the close and wide pair, respectively.

All that remains to be done now to specify the short-range perturbations of orbital elements of the close pair, caused by the attraction of an immovable distant third star of mass m_3, is to differentiate the foregoing expressions for p_j and q_j partially with respect to the individual Delaunay elements in accordance with equations (8-5), and differentiate similarly Kepler's equation (6-2) which, rewritten in terms of the Delaunay variables, assumes the form

$$LE - \sqrt{L^2 - G^2}\sin E = lL \tag{8-31}$$

and yields

$$\frac{\partial E}{\partial L} = \frac{el + \sin E}{enrA}, \tag{8-32}$$

$$\frac{\partial E}{\partial G} = -\frac{\sqrt{1-e^2}}{enrA}\sin E = -\frac{\sin v}{enA^2}, \tag{8-33}$$

and

$$\frac{\partial E}{\partial l} = \frac{A}{r}, \tag{8-34}$$

as the only non-vanishing derivatives. By insertion of all these results in equations (8-6) our task should then be complete. This task offers indeed no inherent difficulty; but the explicit forms of the results (which should include all short-periodic perturbations) becomes too lengthy to be given here in full. Particular interest attaches, however, to *secular* (short-range) *perturbations* which arise from the leading term p_0 on the right-hand side of equation (8-19); and these will be worked out here in full.

The partial derivatives of p_0 with respect to the conjugate elements G, H and g, h assume the explicit forms

$$\begin{aligned}\frac{1}{L}\frac{\partial p_0}{\partial g} &= 10e^2k_1MN \\ &\quad + \tfrac{5}{6}ek_2M\{3(1-e^2) - (1-e^2)P_3'(M) - (1+6e^2)P_3'(N)\},\end{aligned} \tag{8-35}$$

$$\begin{aligned}\frac{1}{L}\frac{\partial p_0}{\partial h} &= -2(1-e^2)M\frac{\partial M}{\partial h}\{k_1 - \tfrac{25}{4}ek_2N\} \\ &\quad - 2\frac{\partial N}{\partial h}\{(1+4e^2)k_1N - \tfrac{5}{12}ek_2[(1-e^2)P_3'(M) \\ &\qquad + (3+4e^2)P_3'(N)]\},\end{aligned} \tag{8-36}$$

while

$$\begin{aligned}\frac{\partial p_0}{\partial H} &= \frac{2k_1}{\sqrt{1-e^2}}\{(1-e^2)M\cos(\omega-\Omega) + (1+4e^2)N\sin(\omega-\Omega)\}\sin(u'-\Omega) \\ &\quad - \frac{5k_2}{6\sqrt{1-e^2}}\{[(1-e^2)P'(M) + (3+4e^2)P_3'(N)]\sin(\omega-\Omega) \\ &\qquad + 15MN(1-e^2)\cos(\omega-\Omega)\}\sin(u'-\Omega)\end{aligned} \tag{8-37}$$

and

$$\begin{aligned}\frac{\partial p_0}{\partial G} &= -\cos i\frac{\partial p_0}{\partial H} + \tfrac{4}{3}k_1\sqrt{1-e^2}\{P_2(M) - 4P_2(N)\} \\ &\quad + \frac{5}{6}\frac{\sqrt{1-e^2}}{e}k_2\{(1-3e^2)NP_3'(M) + (1+4e^2)P_3(N)\}.\end{aligned} \tag{8-38}$$

As to the partial derivatives of p_0 with respect to L and l, p_0 is clearly independent of the latter and the corresponding derivative vanishes. If so, however, the first one of equations (8-5) reveals at once that L is secularly constant; and the derivative with respect to it will be evaluated later by a different method. The secular motions of the orbital elements of the close pair per revolution should then follow from the equations

$$\frac{\delta A}{A} = -\frac{2}{L}\frac{\partial p_0}{\partial l}, \tag{8-39}$$

$$\frac{\delta e}{e} = -\frac{G}{L^2 - G^2}\frac{\partial p_0}{\partial g}, \tag{8-40}$$

$$\delta\omega = -\left\{\frac{\partial}{\partial G} + \frac{\partial}{\partial H}\right\} p_0, \tag{8-41}$$

$$\delta\Omega = -\frac{\partial p_0}{\partial H}, \tag{8-42}$$

$$\sin i\, \delta i = -\frac{1}{G^2}\left\{G\frac{\partial}{\partial h} - H\frac{\partial}{\partial g}\right\} p_0, \tag{8-43}$$

and

$$\delta(nt + \varepsilon) = \delta(l + g + h) = -\left\{\frac{\partial}{\partial L} + \frac{\partial}{\partial G} + \frac{\partial}{\partial H}\right\} p_0. \tag{8-44}$$

If, for simplicity, we limit ourselves to the leading terms of these expressions (factored by k_1) and ignore the eccentricity of the close pair on the right-hand sides, an insertion of the foregoing results in (8-39)–(8-44) reveals that

$$\delta A = 0, \tag{8-45}$$

$$e^{-1}\, \delta e = -10 k_1 M N, \tag{8-46}$$

$$\begin{aligned}\delta\omega &= -4k_1 \sin^2 \tfrac{1}{2}i\{M\cos(\omega - \Omega) + N\sin(\omega - \Omega)\}\sin(u' - \Omega)\\ &\quad - \tfrac{4}{3}k_1\{P_2(M) - 4P_2(N)\}\\ &= -4k_1 \sin^2 \tfrac{1}{2}i \cos i \sin^2(u' - \Omega) - 2k_1(2 - 5N^2),\end{aligned} \tag{8-47}$$

$$\begin{aligned}\delta\Omega &= -2k_1\{M\cos(\omega - \Omega) + N\sin(\omega - \Omega)\}\sin(u' - \Omega)\\ &= -2k_1 \sin^2(u' - \Omega)\cos i,\end{aligned} \tag{8-48}$$

and

$$\sin i\, \delta i = 2k_1\left\{M\frac{\partial M}{\partial h} + N\frac{\partial N}{\partial h}\right\} = 2k_1\{\tfrac{1}{2}\sin 2(u' - \Omega)\sin^2 i\}, \tag{8-49}$$

so that

$$\delta i = k_1 \sin 2(u' - \Omega)\sin i. \tag{8-50}$$

It should be stressed that none of these equations involve any restriction as to the magnitude of the angle i between the two orbital planes. The latter can be anywhere between 0° and 90° if the motion of both bodies m_2 and m_3 takes place in the same direction; and between 90° and 180° if the two bodies move in opposite directions.

The last element whose short-range variation remains yet to be specified is the mean longitude $nt + \varepsilon$ of the secondary component in its relative orbit, and its perturbations are governed by equation (8-44). In order to evaluate it, let us set

$$p_0 = k_1 \frac{\partial p_0}{\partial k_1} + k_2 \frac{\partial p_0}{\partial k_2} \tag{8-51}$$

and note that the constants $k_{1,2}$, as defined by equations (8-29) and (8-30), involve the element L through the period P or the mean daily motion n of the close pair. As*

$$n = \sqrt{\frac{G(m_1 + m_2)}{A^3}} = \frac{G^2(m_1 + m_2)^2}{L^3}, \tag{8-52}$$

the constants $k_{1,2}$ can be alternatively rewritten as

$$k_1 = \frac{3}{4} \frac{m_3 L^6}{r'^3 G^3 (m_1 + m_2)^4} \tag{8-53}$$

and

$$k_2 = \frac{3}{4} \frac{m_3(m_1 - m_2)L^8}{r'^4 G^4 (m_1 + m_2)^6}. \tag{8-54}$$

If so, however, the reader can easily verify that the function I as defined by equation (8-9) and, therefore, np_0 is homogeneous in the Delaunay variables L, G, H in such a manner that $n(\partial p_0/\partial k_1)$ is a homogeneous function of the 4th degree, while $n(\partial p_0/\partial k_2)$ is such a function of 6th degree. Now Euler's theorem on homogeneous functions discloses that

$$L \frac{\partial}{\partial L}\left(n \frac{\partial p_0}{\partial k_j}\right) + G \frac{\partial}{\partial G}\left(n \frac{\partial p_0}{\partial k_j}\right) + H \frac{\partial}{\partial G}\left(n \frac{\partial p_0}{\partial k_j}\right) = (2j + 2)n \frac{\partial p_0}{\partial k_j} \tag{8-55}$$

$j = 1, 2$; and thus, by (8-51),

$$L \frac{\partial p_0}{\partial L} + G \frac{\partial p_0}{\partial G} + H \frac{\partial p_0}{\partial H} = 4p_0 + 2k_2 \frac{\partial p_0}{\partial k_2}. \tag{8-56}$$

Now, in accordance with (8-44),

$$\left.\begin{aligned} L\,\delta(nt + \varepsilon) &= -L\left\{\frac{\partial}{\partial L} + \frac{\partial}{\partial G} + \frac{\partial}{\partial H}\right\} np_0 \\ &= -\left\{L \frac{\partial}{\partial L} + G \frac{\partial}{\partial G} + H \frac{\partial}{\partial H}\right\} p_0 \\ &\quad - (L - G)\frac{\partial p_0}{\partial G} - (L - H)\frac{\partial p_0}{\partial H}, \end{aligned}\right\} \tag{8-57}$$

* The reader should take care to note that, in equations (8-52)–(8-54), the symbol G stands for the constant of gravitation and *not* a Delaunay variable.

where

$$\left.\begin{aligned} L - G &= L(1 - \sqrt{1 - e^2}),\\ L - H &= L(1 - \sqrt{1 - e^2}\cos i),\end{aligned}\right\} \tag{8-58}$$

and, therefore,

$$\begin{aligned} \delta(nt + \varepsilon) = &-4\frac{p_0}{L} - 2\frac{k_2}{L}\frac{\partial p_0}{\partial k_2}\\ &- (1 - \sqrt{1 - e^2})\frac{\partial p_0}{\partial G} - (1 - \sqrt{1 - e^2}\cos i)\frac{\partial p_0}{\partial H}. \end{aligned} \tag{8-59}$$

If we retain again only the leading terms and ignore e on the right-hand side we eventually establish that, within the scheme of our approximation, the secular motion of the mean longitude should be given by

$$\begin{aligned} \delta(nt + \varepsilon) = &\tfrac{8}{3}k_1 P_2\{\sin(u' - \Omega)\sin i\}\\ &- 4k_1 \sin^2 \tfrac{1}{2}i \cos i \sin^2(u' - \Omega), \end{aligned} \tag{8-60}$$

and our solution is now complete.

In order to ensure whether the short-range secular perturbations of the orbital elements of the close pair—as given by equations (8-45)–(8-50) and (8-60)—are genuine secular perturbations, or whether they may prove to be long-periodic when the disturbing component of mass m_3 is allowed to move, we must proceed to investigate the *long-range perturbations* for which the longitude u' or v' of the third body becomes a function of the time. The orbital period of the third body is usually very long in comparison with that of the binary orbit;* hence, it should be sufficient in the studies of long-range perturbations to omit all short-periodic terms of our disturbing function and set

$$S' = Gm_3\frac{A^2}{r'^3}\{\tfrac{3}{2}\bar{J}_2 - \tfrac{1}{2}\bar{J}_0\} + Gm_3\frac{m_1 - m_2}{m_1 + m_3}\frac{A^3}{r'^4}\{\tfrac{5}{2}\bar{J}_3 - \tfrac{3}{2}\bar{J}_1\}, \tag{8-61}$$

where

$$\left.\begin{aligned} \bar{J}_0 &= (1 + \tfrac{3}{2}e^2),\\ \bar{J}_1 &= -\tfrac{5}{2}e(1 + \tfrac{3}{4}e^2)N,\\ \bar{J}_2 &= \tfrac{1}{2}\{(1 - e^2)M^2 + (1 + 4e^2)N^2\},\\ \bar{J}_3 &= -\tfrac{15}{8}e\{(1 - e^2)M^2 + (1 + \tfrac{4}{3}e^2)N^2\}N,\end{aligned}\right\} \tag{8-62}$$

are the *average* values of the J_j's, as given by equations (8-15)–(8-18) earlier in this section, over one cycle of the close pair. It should be observed that, unlike in the short-range case, the quantities M and N in the foregoing expressions (8-62) cannot be considered constant in the course of a long cycle on account of the presence of $n' = \omega' + v'$ among their factors.

* The shortest known period of such a third orbit occurs in the system of λ Tauri where, according to Struve and Ebbighausen (*Ap. J.*, **124**, 507, 1956), $P = 3{\cdot}953$ days and $P' = 33{\cdot}025$ days, rendering the ratio $P/P' = 0{\cdot}12$.

Similarly, by analogy with (8-8) we can now write

$$I' = \int S'\,dt = Gm_3\left\{(\tfrac{3}{2}J_2' - \tfrac{1}{2}J_0')A^2 + \frac{m_1 - m_2}{m_1 + m_2}(\tfrac{5}{2}J_3' - \tfrac{3}{2}J_1')A^3\right\}, \tag{8-63}$$

where

$$\left.\begin{aligned} J_0' &= \int \frac{\bar{J}_0}{r'^3}\,dt, \qquad & J_1' &= \int \frac{\bar{J}_1}{r'^4}\,dt, \\ J_2' &= \int \frac{\bar{J}_2}{r'^3}\,dt, \qquad & J_3' &= \int \frac{\bar{J}_3}{r'^4}\,dt. \end{aligned}\right\} \tag{8-64}$$

As in our previous work on the short-range case, in what follows we propose to construct a *linear* approximation to a long-range solution, in which the Delaunay elements L, G, H or l, g, h as well as A', e', and ω' behind the integral signs in (8-64) can again be considered constant. If so, then a change of variables from the time t to the true anomaly v' of the third body, which can be effected by means of the relations

$$\left.\begin{aligned} \frac{1}{r'} &= \frac{1 + e'\cos v'}{A'(1 - e'^2)}, \\ n'dt &= \frac{(1 - e'^2)^{3/2}\,dv'}{(1 + e'\cos v')^2}, \end{aligned}\right\} \tag{8-65}$$

n' denoting the mean daily motion of the third body; and

$$\begin{aligned} M = {} & \{\ \ G\sin g\sin(\omega' - h) + H\cos g\cos(\omega' - h)\}G^{-1}\sin v' \\ & + \{-G\sin g\cos(\omega' - h) + H\cos g\sin(\omega' - h)\}G^{-1}\cos v', \end{aligned} \tag{8-66}$$

$$\begin{aligned} N = {} & \{-G\cos g\sin(\omega' - h) + H\sin g\cos(\omega' - h)\}G^{-1}\cos v' \\ & + \{\ \ G\cos g\cos(\omega' - h) + H\sin g\sin(\omega' - h)\}G^{-1}\cos v' \end{aligned} \tag{8-67}$$

will enable us to evaluate the J_j''s—and, therefore, I'—in a closed form.

Moreover, once this has been done, the corresponding variations δL, δG, δH and δg, δh, δl of the Delaunay elements can be obtained by a partial differentiation of I' with respect to the appropriate conjugate element. Since the mean longitude l dropped out of our problem as a result of the averaging which led to (8-62), it follows once again from the first one of the equations (8-5) of motion that

$$\delta L = 0, \tag{8-68}$$

and thus L proves to be constant in the long-range case as well. The long-periodic variations of the other elements then follow from

$$\left.\begin{aligned} \delta G &= \frac{\partial I'}{\partial g}, \qquad & \delta g &= -\frac{\partial I'}{\partial G}, \\ \partial H &= \frac{\partial I'}{\partial h}, \qquad & \partial h &= -\frac{\partial I'}{\partial H}, \end{aligned}\right\} \tag{8-69}$$

and as

$$I' = k_1'\{[(\tfrac{2}{3} + e^2)P_2(q) + \tfrac{5}{2}e^2(1 - q^2)\cos 2(\omega - \Omega)](v' + e'\sin v') + \tfrac{1}{2}(1 + \tfrac{3}{2}e^2)(1 - q^2)S(2\omega' - 2\Omega) + \tfrac{5}{4}e^2(1 + q^2)S(2\omega' - 2\Omega)\cos 2(\omega - \Omega) - \tfrac{5}{2}e^2 qC(2\omega' - 2\Omega)\sin 2(\omega - \Omega)\}L, \tag{8-70}$$

their leading terms can be shown to assume the explicit forms

$$\delta G = -5k_1'e^2\{(v' + e'\sin v')(1 - q^2)\sin 2(\omega - \Omega) + \tfrac{1}{4}(1 + q)^2 C(2\omega' - 2\omega) - \tfrac{1}{4}(1 - q)^2 C(2\omega' + 2\omega - 4\Omega)\}L, \tag{8-71}$$

$$\delta H = -5k_1'\{\tfrac{1}{5}(1 + \tfrac{3}{2}e^2)(1 - q^2)C(2\omega' - 2\Omega) + \tfrac{1}{4}e^2(1 + q)^2 C(2\omega' - 2\omega) + \tfrac{1}{4}e^2(1 - q)^2 C(2\omega' + 2\omega - 4\Omega)\}L, \tag{8-72}$$

$$\sqrt{1 - e^2}\,\delta g = k_1'\{(v' + e'\sin v')[e^2 + 5q^2 - 1 + 5(1 - e^2 - q^2)\cos 2(\omega - \Omega)] + \tfrac{1}{2}(3 - 3e^2 - 5q^2)S(2\omega' - 2\omega) + \tfrac{5}{4}(1 + q)(1 + q - e^2)S(2\omega' - 2\omega) + \tfrac{5}{4}(1 - q)(1 - q - e^2)S(2\omega' + 2\omega - 4\Omega)\}, \tag{8-73}$$

$$\sqrt{1 - e^2}\,\delta h = -k_1'\{(v' + e'\sin v')q[2 + 3e^2 - 5e^2q^2\cos 2(\omega - \Omega)] - (1 + \tfrac{3}{2}e^2)qS(2\omega' - 2\Omega) + \tfrac{5}{4}(1 + q)e^2 S(2\omega' - 2\Omega) + \tfrac{5}{4}(1 - q)e^2 S(2\omega' + 2\omega - 4\Omega)\}, \tag{8-74}$$

where we have abbreviated

$$\left.\begin{aligned} C(\theta) &= -2\int (1 + e'\cos v')\sin(2v' + \theta)\,dv' \\ &= \cos(2v' + \theta) + e'\cos(v' + \theta) + \tfrac{1}{3}e'\cos(3v' + \theta), \end{aligned}\right\} \tag{8-75}$$

$$\left.\begin{aligned} S(\theta) &= 2\int (1 + e'\cos v')\cos(2v' + \theta)\,dv' \\ &= \sin(2v' + \theta) + e'\sin(v' + \theta) + \tfrac{1}{3}e'\sin(3v' + \theta), \end{aligned}\right\} \tag{8-76}$$

q (without subscript) $\equiv \cos i$, and

$$\left.\begin{aligned} k_1' &= \frac{3}{8}\frac{Gm_3}{nn'A'^3(1 - e'^2)} \\ &= \frac{3}{8}\frac{m_3}{m_1 + m_2 + m_3}\left(\frac{P}{P'}\right)\frac{1}{(1 - e'^2)^{3/2}}. \end{aligned}\right\} \tag{8-77}$$

The calculations of the actual perturbations are then carried out by expanding the true anomaly v' in terms of the mean anomaly, and the final results are obtained by inserting the instantaneous elements in the elliptic expressions for the coordinates. Since, in particular,

$$\left.\begin{aligned} \delta e &= -\frac{\sqrt{1-e^2}}{e}\frac{\delta G}{L}, & \delta i &= \frac{q\,\delta G - \delta H}{L\sqrt{(1-e^2)(1-q^2)}}, \\ \delta\omega &= \delta g + \delta h, & \delta\Omega &= \delta h, \end{aligned}\right\} \tag{8-78}$$

an insertion for δG, δH, δg, and δh from (8-70)–(8-73) then completes the results. If, moreover, consistent with our previous practice we ignore the eccentricity e of the close pair, the foregoing results are found to reduce to

$$\begin{aligned} \delta e/e = 5k_1'\{(v' + e'\sin v')\sin 2(\omega - \Omega)\sin^2 i \\ + C(2\omega' - 2\omega)\cos^4 \tfrac{1}{2}i \\ - C(2\omega' + 2\omega - 4\Omega)\sin^4 \tfrac{1}{2}i\}, \end{aligned} \tag{8-79}$$

$$\begin{aligned} \delta\omega = k_1'\{&2(v' + e'\sin v')[2 - \cos i - 5\sin^2(\omega - \Omega)\sin^2 i] \\ &+ (\tfrac{5}{2}\sin^2 i - 2\cos^2 \tfrac{1}{2}i)S(2\omega' - 2\Omega) \\ &+ 5\cos^4 \tfrac{1}{2}iS(2\omega' - 2\omega) \\ &+ 5\sin^4 \tfrac{1}{2}iS(2\omega' + 2\omega - 4\Omega)\}, \end{aligned} \tag{8-80}$$

$$\delta\Omega = -k_1'\{2(v' + e'\sin v') - S(2\omega' - 2\Omega)\}\cos i, \tag{8-81}$$

and

$$\delta i = k_1' C(2\omega' - 2\Omega)\sin i. \tag{8-82}$$

The perturbations of the last remaining element, the mean longitude $nt + \varepsilon$, can be expressed in terms of the quantities already obtained in the following manner. As the reader can easily verify, the expression (8-63) for I' again proves to be homogeneous in terms of the Delaunay variables L, G, H. As the expressions (8-77) for k_1' can be alternatively rewritten in the form

$$k_1' = \frac{3}{8}\frac{m_3 L^3}{G(m_1 + m_2)^2 n' A'^3 (1 - e'^2)}, \tag{8-83}$$

the leading term in I' turns out to be a homogeneous function of 4th degree. Consequently, the same argument which led us previously from equation (8-55) to (8-59) can now be invoked to prove that

$$\begin{aligned} \delta(nt + \varepsilon) &= -\left\{\frac{\partial}{\partial L} + \frac{\partial}{\partial G} + \frac{\partial}{\partial H}\right\} I' \\ &= -4L^{-1}I' + (1 - \sqrt{1 - e^2})\,\delta g + (1 - q\sqrt{1 - e^2})\,\delta h, \end{aligned} \tag{8-84}$$

where I', δg and δh are already known from equations (8-70), (8-73) and (8-74). Inserting them in (8-84) and ignoring terms factored by e we find that

$$\delta(nt + \varepsilon) = k_1'\{2(\tfrac{2}{3} - q - q^2)(v' + e'\sin v') - (1 - q)(2 + q)S(2\omega' - 2\Omega)\}. \tag{8-85}$$

Within the scheme of our approximations, the *secular motions* of the individual elements in the course of one revolution of the wide pair accordingly become

$$[\delta A] = [\delta e] = [\delta i] = 0, \tag{8-86}$$

but

$$[\delta\omega] = 2k_1'\{1 - q + q^2 - 5(1 - q^2)\sin^2(\omega - \Omega)\}, \tag{8-87}$$

$$[\delta\Omega] = -2k_1'q, \tag{8-88}$$

so that

$$[\delta(\omega + \Omega)] = 2k_1'\{(1 - q)^2 - 5(1 - q^2)\sin^2(\omega - \Omega)\} \tag{8-89}$$

and

$$[\delta(nt + \varepsilon)] = \tfrac{2}{3}k_1'\{2 - 3q - 3q^2\}. \tag{8-90}$$

A comparison of the foregoing equations with the 'secular' perturbations (8-45)–(8-50) and (8-60) of the short-range case reveals that the latter perturbations of the angle of inclination i, as given by equation (8-50), have now vanished. The reason is indeed not too hard to find; for *the long-range secular perturbations*, summarized by equations (8-86)–(8-90) above, *represent the time-averages of the respective short-range perturbations over one revolution of the third body.* If we integrate the right-hand sides of the equations (8-45)–(8-50) and (8-60) with respect to dt over an interval of 2π in u' or v', the identity of both sets of results is readily established; and the short-range 'secular' perturbations of i turn out to be long-periodic. *The only elements exhibiting genuine secular perturbations are* ω, Ω *and* ε; ω advancing, Ω receding at rates which, for $i = 0°$, become identical. If both orbits are co-planar, the longitude of the apsidal line as measured from the fixed equinox (and not from a moving line of the nodes) then becomes secularly immovable to the first order in small quantities. The periods U, V of revolution of the apsidal and nodal lines are then expressed in terms of the orbital period P' of the third body by means of the ratios

$$\frac{P'}{U} = \frac{[\delta\omega]}{2\pi} \quad \text{and} \quad \frac{P'}{V} = \frac{[\delta\Omega]}{2\pi}, \tag{8-91}$$

respectively, where the quantities $[\delta\omega]$ and $[\delta\Omega]$ are already known by (8-87) and (8-88).

As to the *periodic perturbations* of different elements, in the present section we have established the existence of *three* distinct families of such perturbations: namely,

(1) short-periodic perturbations, of periods P and its submultiples (argument: jE), arising from the harmonic terms on the right-hand side of equation (8-19);

(2) long-periodic perturbations, of periods P' and its submultiples (argument: jv'), arising from the harmonic function $C(\theta)$ and $S(\theta)$ as defined by equations (8-75) and (8-76); and

(3) apse-node terms, arising from the secular motion of the angles and Ω, and oscillating with the periods U, V (or their difference) as given by equations (8-91).

Not all orbital elements exhibit, to be sure, the perturbations of all these types. For instance, the semi-major axis A of the relative orbit of the close pair is (within the scheme of our approximations) found to be subject to neither secular nor long-periodic perturbations; and, as a result, the period P of the sideric orbit of the two bodies of masses m_1 and m_2 will likewise remain constant. Should this close binary happen to be an eclipsing variable, its apparent period defined as a time interval between two successive light minima will, of course, be subject to complicated fluctuations arising from the secular motion of ω and Ω, and long-periodic oscillation of i, of the type already investigated in the preceding section of this chapter. Even if it were not, however, for such oscillations (whose periods would be of the order of U or V, and therefore long in comparison with P'), the apparent (observed) period P of the light changes of an eclipsing binary which is accompanied by a third body would still not be constant on account of the *light equation* of our variable in its absolute orbit around the centre of gravity of the triple system; and the effects of such a light equation remain yet to be investigated.

In order to do so, consider a system of rectangular coordinates XYZ, with origin at the centre of mass of the triple system, oriented so that the Z-axis coincides with the line of sight and the XY-plane, is then tangent to the celestial sphere. Let, moreover, z denote the distance of the centre of mass of the eclipsing system from the XY-plane. If this system is accompanied through space by a third body, the ensuing orbital motion will cause z to vary as

$$z = r \sin i' \sin (v' + \omega'), \tag{8-92}$$

where r stands for the radius-vector of the absolute orbit of the mass centre of the close pair around that of the whole system; i', for the angle of inclination of this orbit to the celestial sphere; and (as before) v' denotes the true anomaly of the centre of mass of the system m_{12} and ω', the longitude of the periastron from the ascending node at which the orbit intersects the XY-plane. If, furthermore, the origin of coordinates itself is moving towards, or away from, the observer as a result of the space motion of the whole triple system with a velocity γ in the z-direction, the distance Z between this system and the observer at any time t becomes

$$Z = z_0 + \gamma(t - t_0) + r \sin i' \sin (v' + \omega'), \tag{8-93}$$

where z_0 denotes the initial value of Z at the time t_0 of a light minimum of the eclipsing system.

If, however, the distance Z between us and the eclipsing system ceases to be constant, *the time required for any light signal* (such as the occurrence of a minimum) *to reach us from our binary must obviously vary with z*, as a consequence of the 'light equation' in the absolute orbit of m_{12} (i.e., of the fact that the light takes a finite time to traverse this orbit). In more specific terms, even if the sideric period P_0 of the close pair were undisturbed by other sources, the ephemeris $M(E)$ of the light minima taking place at the epoch E

(not to be confused with the eccentric anomaly) should contain a term $(Z - z_0)/c$—where c denotes the velocity of light—and assume the form

$$M(E) = M(0) + \left(1 + \frac{\gamma}{c}\right) P_0 E + \frac{r}{c} \sin i' \sin (v' + \omega'), \tag{8-94}$$

where we have set

$$t - t_0 = P_0 E. \tag{8-95}$$

Now, according to the known expansions of the elliptic motion,*

$$r \sin v' = A_{12} \sum_{k=1}^{\infty} g_k(e') \sin n'k(t - T), \tag{8-96}$$

$$r \cos v' = A_{12} \sum_{k=1}^{\infty} h_k(e') \cos n'k(t - T) - \tfrac{3}{2} A_{12} e', \tag{8-97}$$

where A_{12} denotes the semi-major axis of the absolute orbit of the centre of mass of the eclipsing pair around that of the triple system; e', its eccentricity; n', the mean daily motion; T, the time of the periastron passage; and where we have abbreviated

$$g_k(e') = 2\sqrt{1 - e'^2}\, \frac{J_k(ke')}{ke'} \qquad \text{and} \qquad k_k(e') = \frac{2}{k^2} \frac{dJ_k(ke')}{de'}, \tag{8-98}$$

$J_k(ke')$ denoting the Bessel function of k-th order. Inserting (8-96) and (8-97) in (8-94) and making use of (8-95) we find that the ephemeris $M(E)$ for the times of the minima of the eclipsing pair, as influenced by the 'light equation' due to its orbital motion, will assume the more explicit form

$$\begin{aligned} M(E) &= M(0) + \left(1 + \frac{\gamma}{c}\right) P_0 E + K \left\{ \sum_{k=1}^{\infty} g_k(e') \cos \omega' \sin k(\nu E + \nu_0) \right. \\ &\qquad \left. + h_k(e') \sin \omega' \cos k(\nu E + \nu_0) - \tfrac{3}{2} e' \sin \omega' \right\} \\ &= M(0) + \left(1 + \frac{\gamma}{c}\right) P_0 E + K\{ \sin (\nu E + \nu_0 + \omega') \\ &\quad + \tfrac{1}{2} e' [2 \sin (2\nu E + 2\nu_0 + \omega') - 3 \sin \omega'] \\ &\quad + \tfrac{1}{8} e'^2 [18 \sin (3\nu E + 3\nu_0 + \omega') \\ &\quad - 4 \sin (\nu E + \nu_0 + \omega') + \sin \omega' \cos (\nu E + \nu_0) + \ldots\}, \end{aligned} \tag{8-99}$$

where we have abbreviated

$$\left.\begin{aligned} n'(t - t_0) &= 2\pi (P_0/P') E = \nu E, \\ n'(t_0 - T) &= 2\pi (t_0 - T)/P' = \nu_0, \end{aligned}\right\} \tag{8-100}$$

* *Cf.*, e.g., E. W. Brown and C. A. Shook, *Planetary Theory*, Cambridge 1933, p. 74.

and

$$K = \frac{A_{12} \sin i'}{c}; \tag{8-101}$$

P' denoting (as before) the orbital period of the wide pair.

Suppose now that a sufficiently long sequence of the times of the minima has been observed, and analysed harmonically to yield a Fourier representation of the form

$$M(E) = M(0) + a_0 E + \sum_{k=1}^{n} \{a_k \sin (k\nu E) + b_k \cos (k\nu E)\}, \tag{8-102}$$

where the a_k's and b_k's as well as ν are known empirical constants. If so, a comparison of the corresponding terms on the right-hand sides of equations (8-99) and (8-101) reveals that

$$a_0 = \left(1 + \frac{\gamma}{c}\right) P_0 \tag{8-103}$$

and, for $k \geqslant 1$,

$$a_k = K\{g_k(e') \cos \omega' \cos k\nu_0 - h_k(e') \sin \omega' \sin k\nu_0\}, \tag{8-104}$$

$$b_k = K\{g_k(e') \cos \omega' \sin k\nu_0 + h_k(e') \sin \omega' \cos k\nu_0\}, \tag{8-105}$$

and

$$\nu = 2\pi(P_0/P'). \tag{8-106}$$

If we square equations (8-103)–(8-104) and add them we obtain

$$(a_k^2 + b_k^2) = K^2\{g_k^2(e') \cos^2 \omega' + h_k^2(e') \sin^2 \omega'\}, \tag{8-107}$$

i.e., a relation from which ν_0 has been eliminated. Any pair of the ratios $(a_j^2 + b_j^2)/(a_k^2 + b_k^2)$ for two different values of $j = k$ thus represents a system of two simultaneous equations for the determination of e' and ω'; whereupon ν_0 can be determined from any available ratio a_j/a_k or b_j/b_k; and, with e', ω', and ν_0 thus known, the value of $A_{12} \sin i' = cK$ can be obtained from any a_k or b_k. Therefore, the minimum number n of terms on the right-hand side of equation (8-102) sufficient for the determination of a complete set of the elements $A_{12} \sin i'$, e', ω' and ν_0 (or T) of the orbit of the close pair around the mass centre of the triple system thus turns out to be *three*; and if more are available, the elements can be adjusted by least-squares or other suitable techniques.

A more detailed description of such techniques is outside the scope of this section. If, however, the orbital eccentricity e' happens to be small enough for its squares and higher powers to become ignorable, the procedure simplifies somewhat; and it can be shown that two terms of the summation on the right-hand side of (8-102) necessary for a determination of the orbital elements reduces them to *two*. These terms assume the forms

$$\left.\begin{aligned} a_1 &= K \cos (\nu_0 + \omega'), \\ b_1 &= K \sin (\nu_0 + \omega'); \end{aligned}\right\} \tag{8-108}$$

$$\left.\begin{aligned} a_2 &= \tfrac{1}{2} e' K \cos (2\nu_0 + \omega'), \\ b_2 &= \tfrac{1}{2} e' K \sin (2\nu_0 + \omega'); \end{aligned}\right\} \tag{8-109}$$

and their inversion yields

$$A_{12} \sin i' = c\sqrt{a_1^2 + b_1^2}, \tag{8-110}$$

$$e' = 2\sqrt{\frac{a_2^2 + b_2^2}{a_1^2 + b_1^2}}, \tag{8-111}$$

$$\omega' = \tan^{-1}\left\{\frac{(b_1^2 - a_1^2)b_2 + 2a_1a_2b_1}{(a_1^2 - b_1^2)a_2 + 2a_1b_1b_2}\right\}, \tag{8-112}$$

$$t_0 - T = \frac{P_0}{\nu}\tan^{-1}\left\{\frac{a_1b_2 - b_1a_2}{a_1a_2 + b_1b_2}\right\}, \tag{8-113}$$

$$P' = \frac{2\pi P_0}{\nu}, \tag{8-114}$$

as the solution of our problem.

If the orbital eccentricity e' becomes large (as revealed by an appreciable skewness of the function $M(E)$), the situation becomes more complicated. Fortunately, it is not necessary to go into it here, as it has been previously investigated exhaustively in connection with the determination of spectroscopic orbits of close binary systems from radial-velocity observations. Indeed, the radial velocity V of the centre of mass of the close pair revolving around that of the triple system should, by definition, be given by

$$V = \frac{dZ}{dt} = c\left\{\frac{dM}{dt} - 1\right\} = c\left\{\frac{1}{P_0}\frac{dM}{dE} - 1\right\}. \tag{8-115}$$

As, moreover, to a sufficient approximation

$$\frac{dM}{dE} = P(E) \tag{8-116}$$

if the ratio P_0/P' is small,* a combination of equations (8-115) and (8-116) reveals that

$$\frac{V}{c} = \frac{P - P_0}{P_0}, \tag{8-117}$$

i.e., that *the systemic radial velocity V of our eclipsing binary is directly expressible in terms of the apparent period changes due to the revolution of its centre of gravity around that of the triple system*, and vice versa. *The observed variation of $P(E)$ arising from this cause should, therefore, enable us to deduce from it all elements of such an orbit as are deducible from a single-spectrum*

* For the error of (8-116) arising if this is not the case, *cf.* the last part of the preceding section II.7.

radial velocity curve. In point of fact, the variation of the orbital period due to the 'light equation' specifies the corresponding radial-velocity changes just as well as the spectroscope—in place of the frequency-shifts of vibrating atoms we observe the period changes of the eclipsing pair!

In order to obtain a more explicit form of (8-117) for the puipose of orbit determination, let us return to our previous equation (8-93) for Z and differentiate with respect to the time: we obtain

$$\frac{V}{c} = \frac{P}{P_0} - 1 = \frac{\gamma}{c} + \frac{n'K}{\sqrt{1-e'^2}}\{\cos(v'+\omega') + e'\cos\omega'\}, \tag{8-118}$$

governing the systematic radial velocity (or period) changes of the eclipsing pair in terms of the elements of its orbit around the mass centre of the triple system. The methods—graphical or numerical—for extracting the elements of the spectroscopic orbit from such radial-velocity changes are too numerous and too well known to justify a review in this place.* If the orbital eccentricity e' happens to be small, a direct solution of our problem as represented by equations (8-110)–(8-114) should usually suffice. If, however, a plot of V against the time reveals appreciable skewness, recourse must be had to other methods (Lehmann-Filhés, Schwarzschild, Zurhellen, King, etc.) leading to elements which must, however, be regarded as preliminary in the first instance, and improved subsequently by least-squares corrections in the following manner.

Let a preliminary set of the elements

$$\left.\begin{array}{ccc} M_0 = M(0), & \gamma, & K = \dfrac{A_{12}\sin i'}{c}, \\ & e', \quad \omega', & \\ n' = \dfrac{2\pi}{P'}, & & \sigma' = n'T, \end{array}\right\} \tag{8-119}$$

obtained—graphically or otherwise—by preliminary methods be inserted in equation (8-99) and lead to the residuals ΔM in the times of the minima, observed at the different epochs E. If so, a differentiation of the equation (8-99) reveals that

$$\begin{aligned}\Delta M = \frac{\partial M}{\partial M_0}\Delta M_0 + \frac{\partial M}{\partial \gamma}\Delta\gamma + \frac{\partial M}{\partial K}\Delta K + \frac{\partial M}{\partial e'}\Delta e' \\ + \frac{\partial M}{\partial \omega'}\Delta\omega' + \frac{\partial M}{\partial n'}\Delta n' + \frac{\partial M}{\partial \sigma'}\Delta\sigma',\end{aligned} \tag{8-120}$$

* For their survey *cf.*, *e.g.*, R. G. Aitken, *Binary Stars*, New York 1935; Chapter VI.

where the Δ's denote the errors of the respective elements, multiplied by the coefficients

$$\frac{\partial M}{\partial M_0} = 1, \tag{8-121}$$

$$\frac{\partial M}{\partial \gamma} = t, \tag{8-122}$$

$$\frac{\partial M}{\partial K} = \frac{(1 - e'^2) \sin (v' + \omega')}{1 + e' \cos v'} \tag{8-123}$$

$$\frac{1}{K}\frac{\partial M}{\partial e'} = \frac{\sin v' \cos (v' + \omega')}{1 + e' \cos v'} - \sin \omega', \tag{8-124}$$

$$\frac{1}{K}\frac{\partial M}{\partial \omega'} = \frac{(1 - e'^2) \cos (v' + \omega')}{1 + e' \cos v'}, \tag{8-125}$$

$$\frac{1}{K}\frac{\partial M}{\partial n'} = \frac{t}{\sqrt{1 - e'^2}} \{\cos (v' + \omega') + e' \cos \omega' + e'^2 \sin v' \sin \omega'\}, \tag{8-126}$$

$$\frac{1}{K}\frac{\partial M}{\partial \sigma'} = \frac{1}{\sqrt{1 - e'^2}} \{\cos (v' + \omega') + e' \cos \omega' + e'^2 \sin v' \sin \omega'\}. \tag{8-127}$$

As many equations of conditions of the form (8-120) are available as there are observed times of the minima; and their least-squares solution should furnish the most probable values of the corrections to be applied to the preliminary values of the respective elements in order to obtain their definitive set. After this has been done, the values of K, e' and n' (or P') then specify, in particular, the value of the mass-function

$$\frac{m_3^3 \sin^3 i'}{(m_1 + m_2 + m_3)^3} = \frac{n'^2 (cK)^3}{G} \tag{8-128}$$

of the triple system, just as if its single-spectrum orbit were available.

The close eclipsing systems which are known to be accompanied by invisible third bodies, revealing their presence by their effects on the periods or systemic radial velocities of the close pairs, are too numerous to be all quoted here. Perhaps the best known example of such systems is Algol (β Persei), which consists of close eclipsing system of period $P_0 = 2{\cdot}867$ days, attended by a third star (at the limit of spectroscopic visibility) of orbital period $P' = 1{\cdot}873$ years. The presence of this third body and its period have been verified both spectroscopically by its effect on systemic radial velocity of the close pair,* and photometrically by the 'light equation' in its orbit as discussed in this section, from the observed times of the minima of the variable.† In addition, the astrometric orbit of the eclipsing system actually proved measurable on long-focus parallax plates;‡ and, quite

* *Cf.* D. B. McLaughlin, *Michigan Publ.*, **6**, 3, 1934.
† O. J. Eggen, *Ap. J.*, **108**, 1, 1948.
‡ P. v. de Kamp, S. M. Smith, and A. Thomas, *A. J.*, **55**, 251, 1951.

recently, indications of the presence of lines of the third body in the combined spectrum of Algol have been reported by several investigators.* The ensemble of these different aspects of observational evidence leaves no room for doubt about their internal consistency, and renders the existence of an unseen third body in the Algol system incontestable.

II.9. SURVEY OF THE RESULTS

The aim of the present chapter, as set forth in the introductory paragraphs of its text, has been to investigate the dynamics of close binary systems, isolated in space from external disturbances, with special regard to such effects as may be observable by photometric or spectroscopic means. This task, as far as it could be formulated and tackled in a book of this size, is now complete. In spite of our intention to limit ourselves to essentials and to develop, therefore, the requisite dynamical theory only to a first approximation (i.e., consistently to small quantities of first order), the unavoidable mathematical technicalities grew at times so unwieldy that the individual results are scattered over a large part of the text, and thus sometimes lost from view for the general reader. The aim of this concluding section will, therefore, be to take stock of our accomplishments and to place the results obtained in the individual sections in proper perspective. In doing so, we hope also to provide a guide to the principal points of interest established by our analysis for those astronomers who are concerned mainly with applications of the results to practical cases, rather than with the formalities of their derivation.

The point of departure of our investigation has been the Lagrangian equations of motion of the components of close binary systems about their common centre of mass, whose solution has been our principal concern throughout this chapter. In order to set them up, a specification of the potential and kinetic energy of the individual components of such systems constituted a necessary prerequisite. In section II.1, general expressions have been formulated which govern the interior potential U (equation 1-32) and exterior potential V (equation 1-35) as well as the total potential energy W (equation 1-38) of fluid configurations of arbitrary structure, distorted from their form of equilibrium by an arbitrary disturbing force in such a way that squares and higher powers of superficial distortion are ignorable. As a by-product of this analysis we have established the explicit form (1-69) of the free surfaces of distorted component by its definition as an equipotential, and deduced from it the variation of local gravity (defined as the gradient of surface potential).

We found, in particular, that the potential energy of both components

* A. Meltzer, *Ap. J.*, **125,** 359, 1957; O. Struve and J. Sahade, *Ap. J.*, **125,** 689, 1957; *P. A. S. P.*, **69,** 265, 1957; S. S. Huang, *Ap. J.*, **126,** 51, 1957.

due to their self-gravitation remains unaffected by distortion to the degree of accuracy to which squares of the superficial distortion are ignorable, but that the potential energy arising from gravitational interaction of two stars departs from its equilibrium values by quantities of first order. If, furthermore, the distortion of both components is caused by their rotation (about an arbitrarily oriented axis) and mutual tidal action, the neglect of the quantities of second order was shown to be tantamount to limiting the description of the components to the second, third and fourth spherical harmonic distortion; and equivalent to considering the tidal action of one component upon another as that of a mass-point.

An investigation of the effects of internal structure on the distortion of the component in close binary systems, subject to a given field of force, is so important in its implications that the whole section II.2 has been devoted to its discussion. All this discussion revolved around Clairaut's equation (2-1) or Radau's equation (2-3) and various ways of their analytical or approximate solution subject to prescribed initial conditions. The amount of information concerning the internal structure of a star contained in an empirical determination of the surface values of $\eta_j(a_1)$ or Δ_j has been precised in general terms.

In section II.3 we set out to specify the total kinetic energy of close binary systems due to the axial rotation and orbital motion of their components. In doing so, we have considered both stars to rotate about the axes inclined arbitrarily to the orbital plane, with arbitrary angular velocity which flattens them at the poles; and also took account of the kinetic energy of the tidal bulge raised by the disturbing component in the direction of the radius-vector, which must sweep around the distorted star with the Keplerian angular velocity in the plane of the orbit. This kinetic energy of the tidal bulge in the orbital plane is, in general, comparable with the kinetic energy of the equatorial bulge caused by axial rotation, and both must be equally considered in the formulation of a realistic expression for the total kinetic energy of binary systems consisting of fluid components. The moments of inertia of the distorted configurations about the respective axes have been established to the same order of accuracy to which their form was found in section II.1 by essentially the same methods.

With the aid of the results established in sections 1–3 we are now in a position to set up the complete Lagrangian function of our dynamical systems and deduce the explicit form of the corresponding equations of motion (section II.4). These consist of *three* simultaneous differential equations (4-11)–(4-13) of second order governing the motion of mass centres of both components about their common centre of gravity, plus *six* additional equations of the form (4-15)–(4-17)—*three* for *each* component—which describe the motion of each component (regarded as a fluid body of finite size) about its own centre of mass in terms of the customary Eulerian angles θ, ϕ, ψ.

The systems of equations (4-11)–(4-13) and (4-15)–(4-17) would be independent of each other only if the relative orbits of both components were circular, and their axes of rotation (with uniform angular velocity) constantly perpendicular to the orbital plane. Under these conditions, the distortion of both components would remain constant and the two stars would behave as if they were rigid. In binaries consisting of fluid components equations (4-11)–(4-13) and (4-15)–(4-17) are, however, *coupled* through those parts of their absolute terms which express the kinetic energy of rotating tidal wave; and constitute, therefore, a *simultaneous* set of 9 different equations of 18th order!

Certain integrals of this complete system can be obtained at once. Thus if the tides do not lag (i.e., if the periods of free non-radial oscillations of the components are short in comparison with the period of their orbit), equations (4-17) as they stand can be immediately integrated into (4-28). If, moreover, the rotational and tidal distortions do not interact and merely superpose on each other, a further integral (4-34) is admissible revealing that, under these conditions, the angular velocity of rotation about the instantaneous axis remains invariant. The total order of our simultaneous set of differential equations is thereby reduced from 18 to 14; but no further reductions can be effected. It is only for reasons of convenience (as we propose to search for solutions which are in the neighbourhood of conics) that we replaced the three second-order differential equations (4-11)–(4-13) by our equivalent set of six first-order equations (4-21)–(4-26) governing the variation of the elements of elliptic motion rather than the coordinates of the components themselves.

The differential equations (4-11)–(4-13) and (4-15)–(4-17) of our problems are not only simultaneous, but also *non-linear*; and it is precisely their non-linearity which presents the greatest stumbling block to any search for additional integrals in a closed form. In order to unravel at least the essential features which such integrals should possess, we had in section II.5 to *linearize* our equations of motion by assuming that the time derivatives $\dot{\theta}$ and $\dot{\phi}$ of the Eulerian angles specifying the position of the equatorial planes of the components are small quantities of first order—comparable in magnitude with the coefficients α_i and β_{ij} of distortion of both stars as defined by equations (5-9) and (5-13)—whose squares or cross-products with other first-order quantities can be ignored. Moreover, the same was assumed of the orbital eccentricity e and the inclination i between the orbital and invariable plane of the system. If, in addition, the same is true of the inclinations of the axes of rotation and the invariable plane, it becomes possible to split up the fundamental equations of our problem into two separate sets—one of which governs the behaviour of the elements $\theta_{1,2}$, $\phi_{1,2}$, Ω and i *of* the orbital (and equatorial) planes in space; and the other, the variations of the elements of motion *in* the (instantaneous) orbital plane (A, e, $\bar{\omega}$, ε).

In section II.5, the linearized equations controlling the time variation of the elements of the orbital and equatorial planes of both components (which are coupled inseparably) have been established and solved in a closed form for *secular* as well as *periodic* perturbations of the respective elements. The analytic distinction between the two types rests on the fact that, in treating the former, we are entitled to disregard in our equations of motion the second derivatives $\ddot{\theta}$ and $\ddot{\phi}$ of the Eulerian angles, and to average terms involving true anomaly over a cycle; while in solving for periodic perturbations of the individual elements, the second derivatives $\ddot{\theta}$ and $\ddot{\phi}$ should be regarded as comparable in magnitude with the first derivatives $\dot{\theta}$ and $\dot{\phi}$, and the true anomaly will become our independent variable.

The principal results of our search for secular perturbations of the elements θ, ϕ, Ω and i can be summarized as follows:—As long as only *one* component (say the primary) is distorted and its companion can be regarded as a mass-point, the position of the orbital plane remains fixed in space and makes a constant angle i with the invariable plane of the system. The inclination θ_1 of the primary's axis of rotation to the orbital plane remains likewise unaltered, but such that the ratio $i/\theta_1 = k_1$ is a specific constant depending on the mass-ratio of the system and the ratio of the primary's momentum C_1' to its mass. The angles θ_1 and i cannot, therefore, vanish except simultaneously—i.e., both the equatorial and orbital planes happen (at some particular time) to coincide with the invariable plane of the system. Moreover, as (for centrally-condensed configurations) the constant k_1 defined by equation (5-46) is apt to be a small quantity, the angle i will usually be *small* in comparison with θ. It further transpires that the orientations of the equatorial and orbital planes with respect to the invariable plane of the system are such that the longitudes ϕ_1 and Ω of their ascending nodes differ constantly by 180°, but both *recede* at a uniform rate, completing a whole cycle of their regression in a period as given by equation (5-62) in terms of the constants Γ and Π defined by equations (5-25) and (5-33), respectively. The free axis of rotation of the primary component performs, accordingly, a *precessional motion* whose period coincides with that of nodal recession.

If, more generally, *both* components of a close binary system must be regarded as stars of finite size and density concentration, the general type of motion in the system remains essentially the same. The inclinations $\theta_{1,2}$ of the axes of rotation of the individual components will, however, no longer be constant, but will *oscillate* in a period U' of *nutation*, as given by equation (5-81), where the quantities $s_{1,2}$ are expressible in terms of the constants $\Gamma_{1,2}$ and $\Pi_{1,2}$ as roots of the quadratic equation (5-75). The inclination i between the orbital and invariable planes will likewise oscillate, with the same period U', in accordance with equation (5-80); and so will the ratios $i/\theta_{1,2}$ (though still remaining small). The nodal lines of the orbital and equatorial planes continue to recede and perform a complete revolution in the period U, as given by equation (5-86), which represents a harmonic mean of the respective contributions of each of the two components and of the period of nutation. The latter is, therefore, always longer than that of precession or nodal regression. The rate of this regression ceases, however, to be uniform and exhibits a periodic inequality oscillating in a period of $2\pi/s_2$. The same is true of the role of precession of the equinoxes of each component; and the difference $\phi_{1,2} - \Omega$ likewise oscillates about its mean value of 180° in the period U' of nutation.

All this description characterizes the *secular* (or long-periodic) motion of the orbital plane of close binary systems and of equatorial planes of their components. The slow nature of such motions may render them discernible by their cumulative effects in long intervals of time; though within single orbital cycles their effects are likely to be minute. In addition to these there exists, however, a whole class of *periodic* perturbations (of periods commensurable with that of orbital revolution), whose principal point of interest rests on the fact that their amplitudes represent limits within which such elements can be regarded as constant in the course of a single cycle. The leading terms of periodic perturbations of the elements of θ and ϕ oscillate in the period of the orbit, while those of Ω and i, in half the orbital period—with phases displaced in both cases by 90°. The amplitudes of such perturbations of θ and ϕ are small quantities of first order, but arbitrary within the framework of our approximate linearized theory of motion; but those of Ω and i are uniquely defined (by equations 5-126 and 127) in terms of the distortion constants $\Gamma_{1,2}$ and $\Pi_{1,2}$. The role of short-periodic perturbations may be significant when comparing effects which may be produced by them, for instance, on the alternate primary and secondary minima of eclipsing binary systems.

Turning next to perturbations of the elements of motion *in* the orbital plane (section II.6), we find that the *only* such element which exhibits a *secular* perturbation is the longitude $\bar{\omega}$ of periastron, whose rate $\Delta\bar{\omega}$ of advance per cycle is given by equation (6-22). An empirical determination of this rate (by the method of section VI.9) offers then a royal road for determination of the constants α and β (or rather a certain weighted means of α_i and β_{ij}), as defined by equations (5-9) and (5-13), with all the information on internal structure of the respective components which can be deduced from them (section II.2). The period of revolution of the apsidal line, as expressed by equation (6-28), proves to be comparable in duration with those of nodal regression or nutational motion. In addition, all three elements e, $\bar{\omega}$ and A are subject to short-periodic oscillations described by equations (6-29)–(6-30) and (6-39)–(6-40), respectively, whose periods are equal to that of the orbit and its multiples, and whose amplitudes are again specified by the distortion of both components. The order of magnitude of such amplitudes indicates again the limits within which the respective elements can be regarded as constant in the course of a cycle.

Of all elements of close binary systems, the one whose perturbations can be deduced from the observed data with by far the greatest relative precision is the *orbital period*; and for this reason a whole section (II.7) has been devoted to its discussion. The perturbations of the orbital period arise from two distinct sources. One is the inclination of the equatorial planes of one (or both) components to the invariable plane of the system, which generates perturbing terms on the right-hand side of the appropriate 'law of areas' as represented by equation (7-9); while the other goes back to the way in which the periods of close (eclipsing) binaries can be deduced from the observational data. In eclipsing variables it is customary to define the orbital period as the interval of time which elapses between two successive minima of light; and the latter occur when the apparent (projected) separation of the centres of both components becomes a minimum. Now for any binary whose orbit is characterized by a finite eccentricity—no matter how small—the minimum projected distance between the components can be shown (cf. section VI.9) to depend on the elements e, $\bar{\omega}$, as well as Ω and i. Any perturbations affecting these elements (investigated earlier in this chapter) are then bound to be reflected in apparent variations of the orbital period as well.

Section 7 of this chapter has been devoted to an investigation of the period changes—both real and apparent—arising from these two sources to the first order in small quantities (the orbital eccentricity e being regarded as one of them). The final results are represented by equation (7-35), while the corresponding times of the minima then follow from (7-44). It transpires that, within the scheme of our approximation, the phenomena we have considered have given rise to not less than 10 distinct periodic terms in the equation (7-44) for the light minima, caused by the motion of the apse, nodes, and nutation—separately or interacting. The apse-node (or apse-nutation) terms, arising from interaction of the respective types of motion, are particularly interesting—because of the possibility that a *resonance* between these motions may (and probably does) enhance in certain systems the amplitudes of the corresponding terms of very long periods beyond the values they would otherwise possess, and thus render them observationally conspicuous. A determination of the frequency-spectrum of the function $M(E)$ from a harmonic analysis of the observed times of the minima thus opens up an unexpected, but most practicable, way for a determination of the periods of nodal regression or nutation* (with all their information on the structure of the constituent components) indirectly through its effects on orbital periods. Moreover, an interpretation of the amplitudes of the corresponding periodic terms in $M(E)$, on the basis of a theory first developed in this section, can throw light on the mean values of actual inclinations of the axes of rotation of both components, or of the plane of the orbit, to the invariable plane of the system (just as the amplitude of the periodic inequality in the times of the minima, caused by apsidal advance, can be utilized to specify the value of the orbital eccentricity).

In all preceding sections of this chapter we have discussed the perturbations (and their effects) arising from the axial rotation and mutual tidal distortion of both components in close binary systems. Such perturbations are inevitably operative in each system, to an extent determined by the proximity of its components, their masses and internal structure. In the penultimate section II.8 we have concluded our survey of the dynamics of close binaries by investigating perturbations of the orbital elements of such systems which may arise from the possible proximity of a *third body*. Not all close binaries, to be sure, possess such companions; but a known

* The period of the revolution of apsidal line can be determined alternatively from the varying relative displacement of the primary and secondary minima in eccentric eclipsing systems by the method of section VI.9.

approximate percentage of those which do is sufficiently significant (about 10%) to justify closer dynamical analysis.

In doing so we have, for simplicity, assumed the mean distance of a hypothetical third body from the close pair to be large in comparison with the binary's semi-major axis, but imposed no restrictions on the angle between the planes of the double and triple orbits, or the latter's eccentricity. Two types of perturbations of the elements of the binary (close) orbit have been investigated in some detail: the long-range perturbations produced by the motion of the third body (in which the relative positions of the components of the close pair in the course of a single cycle have been replaced by suitable time-averages), and the short-range perturbations (for which the relative position of the third body has been regarded as fixed).

The integration of the respective equations of motion reveals that orbital elements of the close pair exhibiting *secular* perturbations are ω, Ω and ε; ω advancing and Ω receding at the rates given by equations (8-87) and (8-88) which, for $i = 0°$, become identical. If the orbits of the close and the wide pair are co-planar, the longitude $\bar{\omega}$ of the apsidal line reckoned from a fixed equinox then becomes immovable (to the first order in small quantities) in space. As to the *periodic* perturbations of different elements, we have investigated the families of the (*a*) short-periodic perturbations (of periods P and its sub-multiples; (*b*) long-periodic perturbations (of periods P' and its sub-multiples); and (*c*) perturbations oscillating in the periods of apsidal and nodal revolutions (which are generally long in comparison with P'). Not all orbital elements of the close pair exhibit, to be sure, the perturbations of all these types. In particular, the semi-major axis A of its relative orbit is (within the scheme of our approximations) subject to neither secular nor long-periodic perturbations; and, as a result, the period P of the sideric orbit of the close pair will likewise remain constant. Should this pair happen to be an eclipsing variable, its apparent period would, of course, exhibit complicated fluctuations arising from the secular motions of ω and Ω, as well as from the long-periodic oscillation of i—of the type already investigated in section II.7—regardless of whether the cause of such motions is the mutual distortion of both components or the proximity of a third body.

Even if it were not, however, for such oscillations, the apparent (observed) period of the light changes of an eclipsing binary which is accompanied by a third body would still not be constant, but fluctuate on account of the 'light equation' in its absolute orbit around the centre of gravity of the triple system. In the concluding part of section II.8 the effects of this light equation on the apparent periods of revolving systems have been investigated in some detail. It was pointed out that the apparent period changes due to this cause are simply related (by means of equation 8-117) with the radial-velocity changes of the centre of mass of the close pair around that of the triple system; and that, consequently, the observed variation of P should enable us to deduce from it all elements of the absolute orbit of the mass-centre of the close pair as would be deducible from a single-spectrum radial velocity curve.

II. BIBLIOGRAPHICAL NOTES

II.1: The method underlying the discussion of this section and leading to the fundamental equation (1-23) is, in principle, contained in a book, by A. C. Clairaut, entitled *Théorie de la Figure de la Terre, tirée des Principes de l'Hydrostatique* (Paris, 1743); but its general form (for any value of j) does not appear to have been stated explicitly until fifty years later by A. M. Legendre in his *Recherches sur la Figure des Planètes* (Mémoires de Mathématique par divers Savants for 1789, but not actually published in Paris till 1793).

The subject is so severely classical that little of significance has been added to it since the days of Laplace. A comprehensive summary of the work by Clairaut, Legendre, and Laplace can be found in F. Tisserand, *Traité de la Mécanique Céleste* (Paris, 1891), Tome II, Chapitres XIII–XVIII; or (in a more abridged form) in H. Poincaré, *Leçons sur les Figures d'Equilibre* (Paris, 1903), Chapitre IV; H. Jeffreys, *The Earth* (Cambridge, 1924), Chapter XIII; or L. Lichtenstein, *Gleichgewichtsfiguren rotierender Flüssigkeiten* (Berlin, 1933). This latter book contains also a number of further references to purely mathematical investigations concerning Clairaut's equation, of which we may quote A. Liapounoff, *Mem. de l'Acad. des Sci. de St. Petersbourg* (8), **15**, No. 10, 1904; J. Lense, *Math. Zeit.*, **16**, 296, 1923, and others.

II.2: Equation (2-1) of this section has been known (for $j = 2$) to Clairaut, and for any j to Legendre and Laplace. Its logarithmic transformation (1-28) leading to (2-3) was likewise known (for $j = 2$) to Clairaut; but as it was the central point of Radau's investigations in *C. R. Acad. Paris*, **100**, 972, 1885; or *Bull. Astr.*, **2**, 157, 1885, it seems eminently appropriate to continue associating the first-order equation (2-3) with Radau's name.

The inequalities (2-8) or (2-9) were first proved in this form by Z. Kopal (*Proc. U.S. Nat. Acad. Sci.*, **27**, 359, 1941); while the more restricted form (2-10) was, of course, known long before: for $j = 2$ it was already implicit in Clairaut's work (*op. cit.*), although its rigorous proof was not supplied until the days of Poincaré. The transformation (2-12) was used by Legendre and Airy.

With regard to the closed solutions of the Clairaut-Radau equations for particular distributions of density in the interior, the density law (2-16) leading to a solution of the form (2-22) has already been considered by Clairaut in his *Théorie de la Figure de la Terre*; while the hypothesis (2-18) implying a distribution of the density as governed by (2-20) was introduced in Legendre's *Recherches sur la Figure des Planètes*, and Laplace discussed it (in a more general form) in the fifth volume of his *Mécanique Céleste* (Paris, 1825). The present form (2-23) of its solution in terms of the Bessel functions of half-integral orders is, however, due to Kopal (*M.N.*, **99**, 266, 1939).

The density law (2-29) and the solutions (2-38)–(2-39) based upon it represents a generalization of particular cases considered previously by E. Roche (*Mem. de l'Acad. des Sci. de Montpellier*, 1848) and R. Lipschitz (*Journ. de Crelle*, **62**, 1, 1863) for $k = \mu = 1$ and $j = \lambda = 2$. F. Tisserand (*C. R. Acad. Paris*, **99**, 579, 1884) and M. Levy (*C. R. Acad. Paris*, **106**, 1270, 1314, 1375, 1888) subsequently considered λ, μ as arbitrary constants, but retained the particular values $j = 2$ and $k = 1$.

Numerical integrations of the equation (2-3) for the polytropic family of models, whose results have been reproduced in Tables 2-1 and 2-2, are due to R. A. Brooker and T. W. Olle and have been published by them in *M.N.* **115**, 101, 1955, superseding some earlier integrations by S. Chandrasekhar (*M.N.*, **93**, 449, 1933). Of numerical integrations for other types of models we may refer to the investigations by Z. Kopal (*M.N.*, **98**, 414, 589, 1938), L. Motz (*Ap. J.*, **94**, 253, 1941; **112**, 434, 1950; **115**, 562, 1952; **118**, 147, 1953), G. Keller (*Ap. J.*, **108**, 347, 1947), I. Epstein and L. Motz (*Ap. J.*, **120**, 353, 1954), R. Härm and J. Rogerson (*Ap. J.*, **121**, 439, 1955), J. N. Pike (*Ap. J.*, **122**, 202, 1955), R. S. Kushwaha (*Ap. J.*, **125**, 242, 1957), etc.

The approximate solution (2-46) of Radau's equation for the case of weak central condensation of the distorted configuration is due to Radau (*C. R. Acad. Paris*, **100**, 972, 1885); and the approximate solutions (2-56) or (2-57) of the boundary-value problem (2-49) for high degrees of central condensation have been constructed by Kopal (*M.N.*, **113**, 769, 1953), to whom also equations (2-62) and (2-64) are due.

A general survey of more formal properties of admissible solutions of Radau's equation, which concludes the present section, represents a generalization of previous discussions of this subject by H. Poincaré (*Leçons sur les Figures d'Equilibre*, Paris, 1903, pp. 69–81) and A. Véronnet (*Journ. de Math.* (6), **8**, 331, 1912). Both Poincaré and Véronnet limited their discussions to the case of $j = 2$ only. The inequality (2-69) was, for $j = 2$, proved previously by O. Callandreau (*Bull. Astr.* **5,** 474, 1888) and A. Véronnet (*op. cit. ante*). The monotonically increasing nature of the function $\eta_2(a)$ under conditions (2-82) was first established by Callandreau (*C. R. Acad. Paris,* **100,** 1024, 1885).

II.3: The derivation of the kinetic energy of close binary systems, as given in this section, taking account of the revolution of tidal bulge in a plane which may be inclined to the equators of both components, is new.

II.4: The novelty of the equations of motion of the components of close binary systems, as set forth in the present section, rests on the inclusion of kinetic-energy terms arising from the revolution of the tidal bulge as well as from axial rotation of the two components.

II.5: A treatment of the precession and nutation of fluid components in close binary systems, followed in the present section, parallels closely a recent investigation of the same subject by D. Brouwer (*A.J.*, **52,** 57, 1946), but goes beyond his work in taking account of the kinetic-energy terms due to the revolution of the tidal bulge. In regarding the components as rigid spheroids flattened at poles by centrifugal force alone, Brouwer neglected in fact the major part of the perturbations (due to the tides) which must be present in close binary systems.

The actual method of integration of the canonical equations (5-27)–(5-28) and (5-35)–(5-36) of motion goes back to C. V. L. Charlier, *Die Mechanik des Himmels* (Leipzig, 1902), **1,** pp. 258 ff.

Of previous investigations of the precession of fluid spheroids *cf.* G. H. Darwin, *Phil. Trans. Roy. Soc.*, A, **170,** 447, 1879 (*Scientific Papers*, Cambridge, 1908, **2,** pp. 36 ff). With regard to the motion of the node, its secular regression was proved previously by G. Shajn (*Ap. J.*, **57,** 129, 1923); but his quantitative results are vitiated by the fact that he identified the equator of the precessing component with the invariable plane of the system (an error already pointed out by W. J. Luyten in *Zs. f. Ap.*, **15,** 97, 1938).

II.6: The results contained in this section are mostly new, as no perturbations of the elements in the orbital plane—secular or periodic—have so far been studied; with the exception of the longitude of the apsidal line whose secular advance has previously been established by T. G. Cowling (*M.N.*, **98,** 734, 1938) and T. E. Sterne (*M.N.*, **99,** 451, 1939). Cowling employed the method of small variation of the Lagrangian equations around steady (circular) motion and limited himself to the effects produced by second-harmonic rotational and tidal distortion. Sterne set out to solve our perturbation equation (6-18) for any value of (constant) orbital eccentricity and any spherical-harmonic distortion up to the fourth. Both Cowling and Sterne considered axial rotation of the components with any arbitrary (constant) speed; but identified both their equators with the orbital plane of the system. Our equation (6-22) represents a generalization of their results for the case of finite inclinations of the equatorial planes of the components to that of the orbit.

II.7: Complex changes of orbital periods, long known to be characteristic of many (mainly semi-detached) eclipsing systems, constitute a challenging phenomenon which has so far been held to defy convincing rational explanation. Partly as a result of this apparent impasse, various qualitative physical hypotheses have of late been proposed to account for the observed facts—the latest and most curious one being that by F. B. Wood (*Ap. J.*, **112,** 196, 1950), who would have gas jets of large mass squirting from the secondary component now in the direction of its motion, now in the opposite direction, to account for the observed increases and decreases of orbital periods—and in spite of a completely *ad hoc* nature of such an assumption, it seems to have found favour with at least some more credulous readers.

Our aim in this section has not been to augment the general confusion by adducing any further speculative hypotheses which could be neither proved nor disproved by independent arguments, but to point out that certain basic dynamical phenomena—such as the secular motions of the nodal and apsidal lines or long-periodic librations of the orbital plane, investigated previously in sections II.5 and 6—are *bound* to affect the apparent periods of eclipsing binary systems (defined as the time intervals between two successive minima of light) in a manner which has so far largely escaped the attention of previous investigators. The sole exception are the apparent period changes invoked by the secular advance of the apsidal line, whose existence was previously pointed out by F. Tisserand (*C. R. Acad. Paris,* **120,** 125, 1895).

There seems no doubt that a large, if not most, part of the period changes observed in eclipsing variables can be accounted for by dynamical considerations developed in this section, which are inadvertently operative in close binary systems and which should provide a rational basis for gaining further important insight into the internal structure of their constituent components. Whether or not all observed period changes can be explained in this way, or a part of them may yet require additional physical hypotheses to account for, constitutes a problem which cannot be settled until those changes which are necessarily produced by known phenomena have first been taken out of the observed data.

An analysis of the exact relationship between variable periods and times of the minima of eclipsing binaries, as outlined in the latter part of this section, was previously published by Z. Kopal and R. Kurth (*Zs. f. Ap.*, **42,** 90, 1957).

II.8: The perturbations of the orbital elements of an eclipsing binary, caused by the presence of a distant third body, have provided a rewarding ground for the application of different types of lunar theories to this stellar problem of three bodies. Thus P. Slavenas (*Yale Trans.*, **6,** 35, 1927) invoked the use of Hill's theory, and R. A. Lyttleton (*M.N.*, **95,** 42, 1934) of de Pontécoulant's theory, on the assumption that the close and wide orbits are circular and co-planar. Slavenas showed that, under these conditions, the period U of the secular advance of the apsidal line is given by the ratio

$$\frac{P'}{U} = 2k_1' + 50(k_1')^2 + 100(k_1')^3 + \dots,$$

where P' denotes the period of revolution of the third body, and the constant k_1' is given by our equation (8-77). Lyttleton proved that, correctly to the squares of k_1', the periods U, V of the apsidal advance and of nodal regression are given by

$$\frac{P'}{U} = 2k_1' + 50(k_1')^2 + \dots,$$

$$\frac{P'}{V} = 2k_1' - 2(k_1')^2 + \dots,$$

for co-planar orbits.

Later on, D. Y. Martynov (*Izvestia Engelhardt Obs. Kazan*, No. 25, 1948) employed Laplace's lunar theory to generalize these results for the case of finite eccentricities e, e' of the close and wide pair and inclination i between them; and retaining the first and second powers of these quantities he proved that

$$\frac{P'}{U} = 2k_1'(1 - \tfrac{1}{2}e^2 + \tfrac{3}{2}e'^2 - 2\tan^2 i + \dots) + 50(k_1')^2 + \dots$$

and

$$\frac{P'}{V} = 2k_1'(1 + 2e^2 + \tfrac{3}{2}e'^2 - \tfrac{1}{2}\tan^2 i + \dots) - 2(k_1')^2 + \dots$$

Lastly, E. W. Brown applied to the stellar problem of three bodies the methods of Delaunay's lunar theory in a series of papers published in *M.N.*, **97,** 56, 62, 1936; 116, 388, 1937. In particular, his second paper (*M.N.*, **97,** 62, 1936) contains explicit expressions for the periodic perturbations of the apse, node, and mean longitude of the close pair, correct (apart from a few algebraic slips) to terms of the first order in k_1', which still consider e to be a small quantity, but involve no restriction on the magnitude of either e' or i.

Our treatment of this subject in the present section is based on a recent work by M. Hodgkinson (as yet unpublished). Her work follows Brown in the use of Delaunay's canonical equations of motion; but has not been restricted to long-periodic perturbations arising from the motion of the third body. It includes, in addition, a full treatment of short-periodic perturbations (important for eclipsing systems) during which the position of the third body can be regarded as constant.

The variations of orbital periods caused by the 'light equation' in triple systems, and their use for a determination of elements of the single-spectrum orbit of the third star have been studied by J. Woltjer (*B.A.N.*, **1,** 93, 1922), R. S. Dugan (*Princ. Contr.*, No. 17, 1938), D. Y. Martynov (*Izvestia Engelhardt Obs. Kazan*, No. 25 (sec. 6), 1948), or J. B. Irwin (*Ap. J.*, **116,** 211, 1952). Several formulae deduced in this section are, however, new. For the least-squares adjustment of the preliminary elements *cf.* R. M. Scott (*Harv. Bull.* No. 912, 1940) or A. A. Vasilieva (*Astr. Circ. U.S.S.R. Acad. Sci.*, No. 75, 1948; and *Stalinabad Obs. Bull.*, No. 4, 1952)—both concerned with applications to the system of RT Persei.

CHAPTER III

The Roche Model

IN THE INTRODUCTORY SECTIONS of the preceding chapter it has been made sufficiently clear that the form of the individual components in close binary systems is determined by their axial rotation and tidal interaction, and is bound to depart from a sphere as soon as the components are brought close enough for these disturbing effects to become appreciable. Provided that the periods of free oscillation of the components are sufficiently short in comparison with the period of the orbit (so that their shape can adjust itself to the instantaneous field of force), the appropriate distortion of both stars will be governed by the equilibrium theory of tides. The level-surfaces of constant density then coincide with those of constant potential and the boundary of zero density becomes a particular case of surfaces over which the potential arising from all forces acting upon it remains constant. An outline of the theory of such equipotentials for stars of arbitrary structure has been given in section II.1 earlier in this volume, and developed to first-order terms in superficial distortion (i.e., to the order of accuracy to which squares and higher powers of the axial rotation are negligible, and to which the tidal pull of one component upon another can be regarded as that of a mass-point). In close binaries exhibiting conspicuous effects of photometric ellipticity (*cf.* section IV.2) this first-order distortion may, however, become quite large; and if so, the scheme of approximation adopted in section II.1 and stopping with first-order terms may become inadequate. It is true that the theory of the equipotentials as developed in section II.1 may, in principle, be extended to take account of higher-order terms. Such an extension would, however, lead to very complicated analysis and only first steps in this direction (concerned with second-order rotational distortion) have so far been made.

This last statement admits, however, of one exception: namely, the case of a *centrally-condensed model.* If the density concentration of the components forming our binary is so high that their actual gravitational potential can be approximated by that of a mass-point, the total potential of forces acting at any particular point can be formulated in a closed form. For let m, m' denote the masses of the two components of our system and R their mutual separation. Suppose, moreover, that the positions of the two stars are referred with respect to a rectangular system of coordinates, with origin at the centre of gravity of mass m, whose X-axis coincides with the line joining the centres of the two stars, while the Z-axis is perpendicular to the plane of the orbit.* If so, the coordinates of the centre of gravity of the

* This rectangular system is evidently identical with the doubly-primed system of axes $X''Y''Z''$ introduced in section II.3. At present we are merely dropping primes for reasons of simplicity.

system as a whole evidently are

$$\frac{m'r}{m+m'}, 0, 0,$$

and the total potential Ψ of combined forces acting at an arbitrary point $P(x, y, z)$ becomes expressible as

$$\Psi = G\frac{m}{r} + G\frac{m'}{r'} + \frac{\omega^2}{2}\left\{\left(x - \frac{m'r}{m+m'}\right)^2 + y^2\right\}, \tag{0-1}$$

where

$$\left.\begin{aligned} r^2 &= x^2 + y^2 + z^2, \\ r'^2 &= (r - x)^2 + y^2 + z^2, \end{aligned}\right\} \tag{0-2}$$

represent squares of the distance of P from the centres of gravity of the two components: and ω denotes the angular velocity of rotation of the system about an axis perpendicular to the XY-plane and passing through the centre of gravity of the system; while G stands, as before, for the gravitational constant. The first term on the right-hand side of (0-1) represents the potential arising from the mass of the distorted component of mass m; the second, the disturbing potential of its mate of mass m'; and the third, the potential arising from centrifugal force.

The significance of equation (0-1) is underlined by the fact that it represents not only an exact expression for equipotential surfaces of a binary whose components can be regarded as mass-points, but also a highly approximate expression for components whose degree of central condensation—though finite—happens to be high. This fact is clearly revealed by an inspection of the quantities Δ_j, as defined by equation (1-27) of Chapter I, which measure the amount of first-order superficial distortion of configurations of arbitrary structure. For the mass-point model we found (*cf.* equations 2-6 of Chapter II) that $\Delta_j = 1$; but a glance at Table 2-1 (or at numerical integrations reviewed in the Bibliographical Notes on p. 122) reveals the speed of asymptotic approach of the Δ_j's to 1 with increasing degree of central condensation for all models investigated so far. Subject to very small errors inherent in the adoption of $\Delta_j = 1$ (which the existing numerical integrations reveal to be of the order of 1%), the equipotentials W as defined by the equation (0-1) should, therefore, approximate very closely the actual form of centrally condensed components of close binary systems *irrespective of their proximity or mass-ratio.* The quantitative properties of such equipotentials are of eminent interest for students of close binary systems and invite a more detailed analysis, whose outline will constitute the object of the present chapter.

III.1. ROCHE EQUIPOTENTIALS

In order to approach our task, let us specify it one step further by assuming that the angular velocity ω on the right-hand side of equation (0-1) is identical with the Keplerian angular velocity

$$\omega^2 = G\frac{m + m'}{R^3}, \tag{1-1}$$

a representation which in very close systems (of principal interest to us in this connection) becomes almost inevitable. Suppose, therefore, that we introduce (1-1) in (0-1) and adopt the separation R of both components as

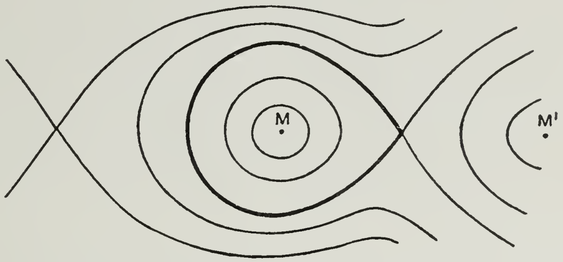

FIGURE 3-1. GEOMETRY OF THE ROCHE SURFACES
The Roche limit is marked by a heavy line

our unit of length. If, moreover, we change over from rectangular x, y, z coordinates to spherical polar coordinates defined by

$$\left.\begin{aligned} x &= r\cos\phi\sin\theta = r\lambda, \\ y &= r\sin\phi\sin\theta = r\mu, \\ z &= r\cos\theta \qquad\quad = r\nu, \end{aligned}\right\} \tag{1-2}$$

equation (0-1) may be expressed as

$$\Omega = \frac{1}{r} + q\left\{\frac{1}{\sqrt{1 - 2\lambda r + r^2}} - \lambda r\right\} + \frac{q+1}{2}r^2(1 - \nu^2), \tag{1-3}$$

where

$$\Omega = \frac{R\Psi}{Gm} - \frac{m'^2}{2m(m + m')} \tag{1-4}$$

and

$$q = \frac{m'}{m} \tag{1-5}$$

are non-dimensional parameters.

The surfaces generated by setting $\Omega =$ constant on the left-hand side of equation (1-3) will hereafter be referred to as the *Roche Equipotentials*—in honour of Edouard Albert Roche, a French mathematician (1820–1883), in whose writings such equipotentials made their first appearance by the middle of the nineteenth century. The form of such equipotentials depends evidently on the value of Ω. If Ω is large, the corresponding equipotentials will consist of two separate ovals (see Fig. 3-1) closed around each of the two mass-points; for the right-hand side of (1-3) can be large only if r (or $r' = \sqrt{1 - 2\lambda r + r^2}$) becomes small; and if the left-hand side of (1-3) is to be constant, so must be (very nearly) r or r'. Large values of Ω correspond, therefore, to equipotentials differing but little from spheres—the less so, the greater Ω becomes. With diminishing value of Ω the ovals defined by (1-3) become increasingly elongated in the direction of the centre of gravity of the system—until, for a certain critical value of Ω_1 characteristic of each mass-ratio, both ovals will unite in a single point on the x-axis to form a dumb-bell-like configuration (*cf.* again Fig. 3-1) which we propose to call the *Roche Limit*. For still smaller values of Ω the connecting part of the dumb-bell opens up and the corresponding equipotential surfaces would envelop both bodies. This latter case is, however, of no direct interest to us in this connection; as for $\Omega < \Omega_1$ the two initially distinct bodies would coalesce in one and we should no longer have the right to speak of a binary system. In what follows we shall, therefore, limit ourselves to a study of the geometry of surfaces characterized by $\Omega > \Omega_1$.

Before we embark upon this task we may, however, find it instructive to observe the effects of certain coordinate transformations designed to simplify the form of the Roche equipotentials and to render it more symmetrical. Thus introducing a complex transformation

$$\left.\begin{aligned} u^2 &= x + i\sqrt{y^2 + z^2}, \\ s^2 &= x - i\sqrt{y^2 + z^2}, \\ t^2 &= (z/u)^2, \end{aligned}\right\} \tag{1-6}$$

in place of (1-2) we find that

$$\left.\begin{aligned} r &= us, \\ r' &= \sqrt{(1 - u^2)(1 - s^2)}, \\ \lambda &= \frac{u^2 + s^2}{2us}, \qquad \nu = \frac{t}{s} \end{aligned}\right\} \tag{1-7}$$

and, consequently,

$$\Omega = \frac{1}{us} + q\left\{\frac{1}{\sqrt{(1 - u^2)(1 - s^2)}} - \frac{u^2 + s^2}{2}\right\} + \frac{1 + q}{2}\, u^2(s^2 - t^2). \tag{1-8}$$

A real transformation

$$\left.\begin{aligned} 1 - x &= p^2 - q^2 - s^2, \\ y &= 2pq, \\ z &= 2ps, \end{aligned}\right\} \tag{1-9}$$

permits us to write $r' = p^2 + q^2 + s^2$; but, unfortunately, this simplification is obtained only at the expense of more complicated expressions for r, λ and ν.

III.2. RADIUS AND VOLUME

Equation (1-3) of the Roche equipotentials (or its equivalents in other coordinate systems) represents an implicit equation defining, for given values of Ω and q, r as a function of λ and ν. When it has been rationalized and cleared of fractions, the result is an algebraic equation of *eighth* degree in r, whose analytical solution presents unsurmountable difficulties. In the case of pure rotational distortion (obtaining if $q = 0$), equation (1-3) can be reduced to a cubic solvable in terms of circular functions. In the case of a pure tidal distortion ($\Omega = 0$) equation (1-3) becomes a quartic, which could also be solved for r in a closed form (though its solution would be very much more involved). In the general case of rotational *and* tidal distortion interacting, however, any attempt at an *exact* solution of (1-3) for r becomes virtually hopeless, and approximate solutions must inevitably be sought.

In order to obtain them, let us begin by expanding the radical $(1 - 2\lambda r + r^2)^{-\frac{1}{2}}$ on the right-hand side of (1-3) in terms of the Legendre polynomials $P_j(\lambda)$. Doing so and removing fractions we find it possible to replace (1-3) by

$$(\Omega - q)r = 1 + q\sum_{j=2}^{\infty} r^{j+1}P_j(\lambda) + nr^3(1 - \nu^2), \tag{2-1}$$

where we have abbreviated

$$n = \frac{q+1}{2}. \tag{2-2}$$

Now if r is small in comparison with unity (i.e., if the linear dimensions of the equipotential surface are small in comparison with our unit of length R), the second and third terms on the right-hand side of (2-1) may be neglected in comparison with unity—in which case, to a first approximation,

$$r_0 = \frac{1}{\Omega - q}. \tag{2-3}$$

This result asserts that if Ω is large, the corresponding Roche equipotential will differ but little from a sphere of radius r_0.

Suppose now that

$$r_1 = r_0 + \Delta' r = r_0\left(1 + \frac{\Delta' r}{r_0}\right) \tag{2-4}$$

should represent our next approximation to r. Inserting it in (2-1) we find that

$$1 + \frac{\Delta' r}{r_0} = 1 + q\sum_{j=2}^{\infty} r_0^{j+1}P_j(\lambda) + nr_0^3(1 - \nu^2) \tag{2-5}$$

where, in small terms on the right-hand side, r was legitimately replaced by r_0. The foregoing equation then yields

$$\frac{\Delta' r}{r_0} = q\sum_{j=2}^{4} r_0^{j+1}P_j(\lambda) + nr_0^3(1 - \nu^2) \tag{2-6}$$

correctly to quantities of the order of r_0^5 (i.e., as far as squares and higher terms of first-order distortion remain negligible).

In order to improve upon this approximation let us set, successively,

$$r_2 = r_1 + \Delta'' r = r_0\left\{1 + \frac{\Delta' r}{r_0} + \frac{\Delta'' r}{r_0}\right\}, \tag{2-7}$$

$$r_3 = r_2 + \Delta''' r = r_0\left\{1 + \frac{\Delta' r}{r_0} + \frac{\Delta'' r}{r_0} + \frac{\Delta''' r}{r_0}\right\}, \tag{2-8}$$

$$\vdots \qquad \vdots$$

$$r_{j+1} = r_j + \Delta^{(j+1)} r = r_0\left\{1 + \sum_{i=0}^{j} \frac{\Delta^{(i+1)} r}{r_0}\right\}, \tag{2-9}$$

where

$$\frac{\Delta^{(i+1)} r}{r_0} = q \sum_{k=3}^{3(N-j)} (r_i^k - r_{i-1}^k) P_{k-1}^{(\lambda)} + n(r_i^3 - r_{i-1}^3)(1 - \nu^2), \tag{2-10}$$

$3N$ denoting the highest power of r_0 to which equation (2-9) represents a correct solution for r. We may note that, in general, the leading terms of the expression (2-10) for $\Delta^{(i+1)} r/r_0$ will be of $3(i+1)$st degree in r_0; and, similarly, the difference $r_i^k - r_{i-1}^k$ in higher terms on the right-hand side of (2-10) will be of the order of r_0^{3i+k}.

Suppose that, in what follows, we wish to construct the explicit form of an approximate solution of equation (2-1), in the form of (2-8), correctly to quantities of the order of $\Delta''' r/r_0$—which should, therefore, differ from the exact solution of (2-1) at most in quantities of the order of r_0^{12}. Then, within the scheme of our approximation,

$$\frac{\Delta' r}{r_0} = q \sum_{j=2}^{10} r_0^{j+1} P_j(\lambda) + n r_0^3 (1 - \nu^2), \tag{2-11}$$

$$\frac{\Delta'' r}{r_0} = q \sum_{j=2}^{7} (r_1^{j+1} - r_0^{j+1}) P_j(\lambda) + n(r_1^3 - r_0^3)(1 - \nu^2), \tag{2-12}$$

and

$$\frac{\Delta''' r}{r_0} = q \sum_{j=2}^{4} (r_2^{j+1} - r_1^{j+1}) P_j(\lambda) + n(r_2^3 - r_0^3)(1 - \nu^2). \tag{2-13}$$

But

$$\begin{aligned} r_1^3 - r_0^3 &= (r_1 - r_0)(r_1^2 + r_1 r_0 + r_0^2) \\ &= r_0^3 \frac{\Delta' r}{r_0}\left\{3 + 3\frac{\Delta' r}{r_0} + \ldots\right\} \end{aligned} \tag{2-14}$$

and, similarly,

$$r_1^4 - r_0^4 = r_0^4 \frac{\Delta' r}{r_0}\left\{4 + 6\frac{\Delta' r}{r_0} + \ldots\right\}, \tag{2-15}$$

$$r_1^5 - r_0^5 = r_0^5 \frac{\Delta' r}{r_0}\left\{5 + 10\frac{\Delta' r}{r_0} + \ldots\right\}; \tag{2-16}$$

the differences of higher powers of r_1 and r_0 being approximable by

$$r_1^k - r_0^k = kr_0^k\left(\frac{\Delta' r}{r_0}\right) + \ldots, \qquad 5 < k < 10, \tag{2-17}$$

and the differences of the k-th powers of r_2 and r_1 by

$$r_2^k - r_1^k = kr_0^k\left(\frac{\Delta'' r}{r_0}\right) + \ldots, \qquad 2 < k < 6. \tag{2-18}$$

With the aid of the foregoing formulae we establish by insertion in (2-12) and (2-13) that, to the order of accuracy we have been working,

$$\begin{aligned}\frac{\Delta'' r}{r_0} = \frac{\Delta' r}{r_0}\Big\{3\frac{\Delta' r}{r_0} &+ 3\frac{\Delta' r^2}{r_0} + \ldots \\ &+ qr_0^4(P_3 + 2r_0P_4 + 3r_0^2P_5 + 4r_0^3P_6 + 5r_0^4P_7) \\ &+ qr_0^4(3P_3 + 7r_0P_4 + \ldots)\frac{\Delta' r}{r_0} + \ldots\Big\}\end{aligned} \tag{2-19}$$

and

$$\frac{\Delta''' r}{r_0} = \frac{\Delta'' r}{r_0}\left\{3\frac{\Delta' r}{r_0} + qr_0^4(P_3 + 2r_0P_4) + \ldots\right\}, \tag{2-20}$$

where we have abbreviated $P_j \equiv P_j(\lambda)$. By use of the expression (2-11) already established for $\Delta' r/r_0$, the explicit forms of $\Delta'' r/r_0$ and $\Delta''' r/r_0$ can successively be found; and their insertion in (2-8) leads to the equation

$$\begin{aligned}\frac{r - r_0}{r_0} = r_0^3\{qP_2 &+ n(1 - \nu^2)\} \\ &+ r_0^4\{qP_3\} \\ &+ r_0^5\{qP_4\} \\ &+ r_0^6\{qP_5 + 3[qP_2 + n(1 - \nu^2)]^2\} \\ &+ r_0^7\{qP_6 + 7q[qP_2 + n(1 - \nu^2)]P_3\} \\ &+ r_0^8\{qP_7 + 8q[qP_2 + n(1 - \nu^2)]P_4 + 4q^2P_3^2\} \\ &+ r_0^9\{qP_8 + 9q[qP_2 + n(1 - \nu^2)]P_5 + 9q^2P_3P_4 \\ &\quad + 6[qP_2 + n(1 - \nu^2)]^3 + 6[q^3P_2^3 + n^3(1 - \nu^2)^3]\} \\ &+ r_0^{10}\{qP_9 + 10q[qP_2 + n(1 - \nu^2)]P_6 + 5q^2[P_4^2 + 2P_3P_5] \\ &\quad + 45q[qP_2 + n(1 - \nu^2)]^2P_3\} \\ &+ r_0^{11}\{qP_{10} + 11q[qP_2 + n(1 - \nu^2)]P_7 + 11q^2[P_3P_6 + P_4P_5] \\ &\quad + 55q[qP_2 + n(1 - \nu^2)]^2P_4 \\ &\quad + 55q^2[qP_2 + n(1 - \nu^2)]P_3^2\} + \ldots,\end{aligned} \tag{2-21}$$

which represents the desired approximate solution of equation (2-1) for r as a function of λ and ν in the form of an expansion in ascending powers of

r_0 (as defined by equation 2-3), which is correct as far as terms of the order of r_0^{11} are concerned. The reader may note that, to the first order in the superficial distortion (i.e., as far as terms of the order of r_0^5 are concerned), the foregoing equation agrees—as it should—with equation (1-69) of Chapter II if we set (for a mass-point model) $\Delta_j = 1$ in the corresponding tesseral harmonics Y_j. The present equation (2-21) should, therefore, be regarded as a generalization of the equation (1-69) of Chapter II up to the cubes of small quantities—a generalization possible only for a mass-point model, but offering without doubt an exceedingly good approximation to a parametric representation of the surfaces of centrally-condensed components of close binary systems as distorted by rotation and tides.

The *volume* V of a configuration whose radius-vector r is given by the foregoing equation (2-21) will be specified by

$$V = \tfrac{2}{3}\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \frac{r^3\,d\lambda\,d\nu}{\mu}, \tag{2-22}$$

where $\mu^2 = 1 - \lambda^2 - \nu^2$. By virtue of the algebraic identity

$$r^3 = r_0^3\left\{1 + \frac{r - r_0}{r_0}\right\}^3 \tag{2-23}$$

we find it convenient to express the integrand in (2-22) in terms of (2-21) as a function of λ and ν. This integrand will, in general, consist of a series of terms of the form $\lambda^m\nu^m/\mu$, factored by appropriate constant coefficients; therefore, the entire volume V will be given by an appropriate sum of partial expressions V_n^m of the form

$$V_n^m = \int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \frac{\lambda^m\nu^n}{\mu}\,d\lambda\,d\nu. \tag{2-24}$$

These expressions vanish (on grounds of symmetry) if either m or n is an odd integer. If, however, both happen to be even and such that $m = 2a$ and $n = 2b$, an evaluation of the foregoing integrals readily reveals that

$$V_{2b}^{2a} = \frac{\sqrt{\pi}\,\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}{\Gamma(a+b+\frac{3}{2})}, \tag{2-25}$$

where Γ denotes the ordinary gamma functions. As, in particular

$$\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \frac{P_j(\lambda)\,d\lambda\,d\nu}{\sqrt{1-\lambda^2-\nu^2}} \left.\begin{aligned} &= 2\pi \quad \text{if} \quad j = 0,\\ &= 0 \quad \text{if} \quad j > 0,\end{aligned}\right\} \tag{2-26}$$

and

$$\int_{-1}^{1}\int_{-\sqrt{1-\lambda^2}}^{\sqrt{1-\lambda^2}} \frac{\nu^{2j}\,d\lambda\,d\nu}{\sqrt{1-\lambda^2-\nu^2}} = \frac{2\pi}{j+1}, \tag{2-27}$$

we eventually find that the volume of a configuration whose surface is a Roche equipotential will be given by

$$
\begin{aligned}
V = \tfrac{4}{3}\pi r_0^3\{1 &+ \tfrac{12}{5}q^2r_0^6 + \tfrac{15}{7}q^2r_0^8 + \tfrac{18}{9}q^2r_0^{10} + \ldots \\
&+ \tfrac{22}{7}q^3r_0^9 + \tfrac{157}{7}q^3r_0^{11} + \ldots \\
&+ 2nr_0^3 + \tfrac{32}{5}n^2r_0^9 + \tfrac{176}{7}n^3r_0^9 + \ldots \\
&+ \tfrac{8}{5}nqr_0^6 + \tfrac{296}{35}nq(2q+n)r_0^9 \\
&+ \tfrac{26}{35}nq(q+3n)r_0^{11} + \ldots\},
\end{aligned}
\tag{2-28}
$$

correctly to quantities of the order up to and including r_0^{11}. With n and r_0 as given by equations (2-2) and (2-3) the volume V becomes an explicit function of Ω and q alone and can be tabulated in terms of these parameters.

III.3. CONTACT CONFIGURATIONS

It was pointed out earlier in this chapter (*cf.* section III.1) that a gradual diminution of the constant Ω on the left-hand side of equation (1-3) will cause the respective Roche equipotentials to expand from nearly spherical configurations to ovals of increased elongation in the direction of the attracting centre until, for a certain critical value of Ω characteristic of each mass-ratio, these ovals unite in a single point on the line joining their centres. Such configurations represent the largest *closed* equipotentials capable of containing the whole mass of the respective components, and will hereafter be referred to as their *Roche limits.* Any star filling its Roche limit will, moreover, be termed a *contact component*; and a binary system consisting of a pair of such components will be called a *contact system.* The fact that close binaries in which one, or both, components have attained their Roche limits actually exist in considerable numbers* adds importance to a study of the geometry of Roche limits in binary systems of different mass-ratios. The aim of the present section will be to investigate their relevant characteristics in some detail.

In order to do so, our first task should be to specify the values of Ω for which the two loops of the critical equipotential (*cf.* Fig. 3-2) develop a common point of contact at P_1; but its determination presupposes a knowledge of the position of P_1 on the x-axis. The location of this point is characterized by the vanishing of the gravity due to all forces—which means that, at that point,

$$\Omega_x = \Omega_y = 0. \tag{3-1}$$

* For their discussion *cf.* sections VII.4 to VII.7 later in this book.

Now a differentiation of (1-3), rewritten in terms of rectangular coordinates, with respect to x and y yields

$$\Omega_x = -xr^{-3} + q\{(1-x)(r')^{-3} - 1\} + 2nx \tag{3-2}$$

and

$$\Omega_y = -y\{r^{-3} + q(r')^{-3} - 2n\}, \tag{3-3}$$

where $r^2 = x^2 + y^2 + z^2$ and $r'^2 = (1-x)^2 + y^2 + z^2$ continue to be given by equations (0-2) and $2n = q + 1$ in accordance with (2-2).

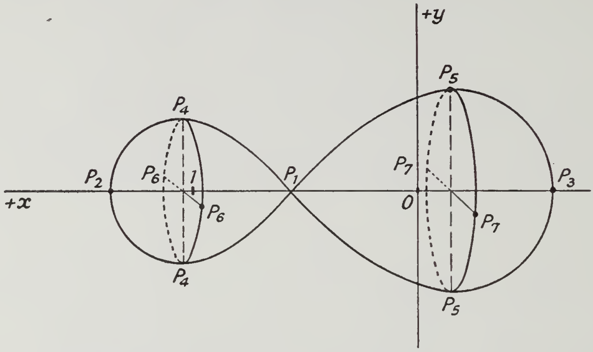

FIGURE 3-2. SCHEMATIC VIEW OF A CONTACT BINARY AT THE ROCHE LIMIT
In order to exhibit the geometry of this model, the diagram has not been drawn to scale for any specific mass-ratio, and certain features (such as the distance of the P_5P_7-plane from the origin) have been exaggerated.

The partial derivative Ω_y vanishes evidently everywhere along the x-axis; but the vanishing of the former renders the x-coordinate of P_1 to be a root of the equation

$$x^{-2} - x = q(1-x)^{-2} - (1-x) \tag{3-4}$$

which, after removal of the fractions, assumes the form

$$(1+q)x^5 - (2+3q)x^4 + (1+3q)x^3 - x^2 + 2x - 1 = 0. \tag{3-5}$$

For $q = 0$ the foregoing equation would evidently reduce to

$$(1-x)^3(x^2 + x + 1) = 0, \tag{3-6}$$

the value $x = 1$ becoming a triple root. Therefore, for small values of q, the root x_1 of equation (3-5) which is interior to the interval $0 < x < 1$ should be approximated by the expansion

$$x_1 = 1 - w + \tfrac{1}{3}w^2 + \tfrac{1}{9}w^3 + \ldots \tag{3-7}$$

in terms of the auxiliary parameter

$$w^3 = \frac{q}{3(1+q)}; \tag{3-8}$$

and more accurate values of x_1 can further be obtained by the method of differential corrections.

Once a sufficiently accurate value of x_1 has thus been obtained, the actual value of Ω_1 corresponding to our critical equipotential follows as

$$\Omega \equiv \Omega(x_1, 0, 0). \tag{3-9}$$

Moreover, the points $P_{4,5}$ in the xy-plane (see again Fig. 3-2) are evidently characterized by the vanishing of the derivative dy/dx at the Roche limit. Their coordinates $x_{4,5}$ and $y_{4,5}$ can, therefore, be evaluated by solving the simultaneous system

$$\left.\begin{aligned} \Omega(x, y, 0) &= \Omega_1, \\ \Omega_x(x, y, 0) &= 0; \end{aligned}\right\} \tag{3-10}$$

and once the values of $x_{4,5}$ have thus been found, the z-coordinates of points $P_{6,7}$ in the xz-plane (*cf.* Fig. 3 2) follow as roots of a single equation

$$\Omega(x_{4,5}, 0, z) = \Omega_1. \tag{3-11}$$

The accompanying Table 3-1 lists five-digit values of Ω_1, x_1; $x_{4,5}$, $y_{4,5}$; and $z_{6,7}$ for Roche limits appropriate for 15 discrete values of the mass-ratio.

TABLE 3-1

q	x_1	Ω_1	x_4	$\pm y_4$	x_5	$\pm y_5$	$\pm z_6$	$\pm z_7$
1·0	0·50000	3·75000	1·01134	0·37420	−0·01134	0·37420	0·35621	0·35621
0·8	0·52295	3·41697	1·01092	0·35388	−0·01168	0·39501	0·33770	0·37491
0·6	0·55234	3·06344	1·01029	0·32853	−0·01198	0·42244	0·31431	0·39909
0·4	0·59295	2·67810	1·00926	0·29465	−0·01213	0·46189	0·28260	0·43278
0·3	0·62087	2·46622	1·00847	0·27204	−0·01204	0·49015	0·26123	0·45599
0·2	0·65856	2·23273	1·00735	0·24233	−0·01163	0·52983	0·23294	0·48750
0·15	0·68392	2·10309	1·00656	0·22280	−0·01117	0·55774	0·21425	0·50781
0·1	0·71751	1·95910	1·00552	0·19746	−0·01034	0·59609	0·18991	0·53451
0·05	0·76875	1·78886	1·00397	0·15979	−0·00859	0·65804	0·15366	0·57291
0·02	0·82456	1·65702	1·00245	0·11992	−0·00618	0·73070	0·11522	0·61434
0·01	0·85853	1·59911	1·00165	0·09613	−0·00457	0·77779	0·09231	0·62867
0·005	0·88635	1·56256	1·00110	0·07689	−0·00327	0·81807	0·07379	0·64170
0·001	0·93231	1·52148	1·00041	0·04550	−0·00137	0·88816	0·04361	0·65762
0·0002	0·96001	1·50737	1·00015	0·02678	−0·00052	0·93264	0·02566	0·66348
0	1·00000	1·50000	1·00000	0·00000	0·00000	1·00000	0·00000	0·66667

It may further be noticed that if, in place of Ω_1, we introduce a new constant C_1 as defined by the equation

$$\left.\begin{aligned} C_1 &= \frac{2\Omega_1}{1+q} + \left(\frac{q}{1+q}\right)^2 \\ &= 2(1-\mu)\Omega_1 + \mu^2, \end{aligned}\right\} \tag{3-12}$$

where we have abbreviated

$$\mu = \frac{q}{1+q} = \frac{m'}{m+m'}, \tag{3-13}$$

the values of C_1 remain largely invariant with respect to the mass-ratio, and sensibly equal to 4 provided that q does not depart greatly from unity. This is demonstrated by an inspection of the tabulation of C_1 as given in column (2) of the following Table 3-2. In consequence, the corresponding simple expression

$$\Omega_1 = 2(1+q) - \frac{q^2}{2(1+q)} = \frac{2^2 - \mu^2}{2(1-\mu)} \tag{3-14}$$

is found to approximate the exact values of Ω_1 within 1% if $1 \geqslant q > 0{\cdot}5$, or within 10% for the wider range $1 \geqslant q > 0{\cdot}1$.

TABLE 3-2

q	C_1	$(r_0)_1$	$(r_0)_2$	V_1	V_2	$(r^*)_1$	$(r^*)_2$	v_1	v_2
1	4·00000	0·36363	0·36363	0·22704	0·22704	0·37845	0·37845	0·072267	0·072267
0·8	3·99417	0·38212	0·34528	0·26459	0·19374	0·39825	0·35896	0·075799	0·069377
0·6	3·96993	0·40594	0·32199	0·31974	0·15665	0·42420	0·33441	0·081422	0·066485
0·4	3·90749	0·43896	0·29025	0·40923	0·11444	0·46057	0·30115	0·091184	0·063726
0·3	3·84744	0·46163	0·26876	0·48148	0·09089	0·48622	0·27892	0·099619	0·062683
0·2	3·74900	0·49195	0·24018	0·59399	0·06492	0·52147	0·24933	0·113443	0·061996
0·15	3·67456	0·51201	0·22121	0·68002	0·05079	0·54552	0·22973	0·124462	0·061967
0·1	3·57027	0·53789	0·19642	0·80715	0·03564	0·57760	0·20414	0·141308	0·062385
0·05	3·40962	0·57509	0·15931	1·0289	0·01910	0·62626	0·16584	0·17193	0·063854
0·02	3·24945	0·61087	0·11974	1·2700	0·007961	0·67179	0·12387	0·2062	0·06462
0·01	3·16665	0·62928	0·09606	1·4656	0·004042	0·70465	0·09882	0·2356	0·06497
0·005	3·10959	0·64203	0·07686	1·5950	0·002038	0·7248	0·07865	0·2551	0·06520
0·001	3·03992	0·65769	0·04549	1·868	0·0004114	0·764	0·04614	0·298	0·06554
0·0002	3·01414	0·66350	0·02679	2·067	0·0000826	0·790	0·02702	0·329	0·06575
0	3·00000	0·66667	0·00000	2·26663	0·0000000	0·81488	0·00000	0·36075	0·065843

The mean radii $(r_0)_{1,2}$ of the two components of contact systems become (consistent with equations 2-3 and 3-12) equal to

$$(r_0)_{1,2} = \frac{2(1-\mu)}{C_1 - (1+\mu)^2 + 1} \tag{3-15}$$

where, for the primary component, $\mu = m_2/(m_1 + m_2)$ (so that $0 \leqslant \mu \leqslant 0{\cdot}5$); while, for the secondary, $\mu = m_1/(m_1 + m_2)$ (so that $0{\cdot}5 \leqslant \mu \leqslant 1$). Alternatively, we may fall back on equation (2-3) and, by inserting for Ω_1 from (3-9), write

$$(r_0)_1 = \frac{2x_1}{2 + 2qx_1^3(1-x_1)^{-1} + (q+1)x_1^3}\,; \tag{3-16}$$

while $(r_0)_2$ is obtainable from the same expression if we replace x_1 by $1 - x_1$ and q by its reciprocal. The values of $(r_0)_{1,2}$ so determined are listed as functions of the mass-ratio in columns (3) and (4) of Table 3-2. Having evaluated them, we are in a position to invoke equation (2-28) for expressing the volumes $V_{1,2}$ of contact components—the reader will find them tabulated in columns (5) and (6) of Table 3-2—while columns (7) and (8) list the equivalent radii $(r^*)_{1,2}$ of spheres having the same volume as the respective contact component. The penultimate and ultimate columns of Table 3-2 then contain the quantities

$$v_{1,2} = \frac{\omega_2}{2\pi G\bar{\rho}_{1,2}} = \frac{2}{3}\left\{1 + \frac{m_{2,1}}{m_{1,2}}\right\}(r^*)^3_{1,2}, \tag{3-17}$$

where ω denotes the (Keplerian) angular velocity of axial rotation of each component and $\bar{\rho}_{1,2}$, their respective mean densities.

The series on the right-hand side of the volume equation (2-28)—which are at the basis of our numerical data as given in columns (5)-(10)—converge with satisfactory rapidity if the masses of the two components are not too unequal, but fail to do so if the mass of one component becomes very much larger than the other. In order to attain adequate representation of the radii and volumes in such cases, asymptotic solutions of equation (2-1) must be sought as $\mu \rightarrow 0$ or 1.

In order to do so, we find it advantageous to rewrite (2-1) in the alternative form

$$(1 - 2\lambda r + r^2)\{(1 - \nu^2)r^3 - 2\lambda\mu r^2 + (\mu^2 - C_1)r + 2(1 - \mu)\}^2 = 4\mu^2 r^2, \tag{3-18}$$

where C_1 as well as μ are defined by equations (3-12) and (3-13) above. Consider first the case of negligible disturbing mass, when $\mu = 0$. As long as quantities of the order of μ^2 remain ignorable, equation (3-18) will admit of a real solution only if

$$(1 - \nu^2)r^3 - 2\lambda\mu r^3 - C_1 r + 2(1 - \mu) = 0. \tag{3-19}$$

For small values of μ, the solution of this latter equation can be sought in the form

$$r = S_{10} + S_{11}\mu + \ldots, \tag{3-20}$$

where S_{10}, S_{11}, . . . are defined by the equations

$$(1 - \nu^2)S_{10}^3 - C_1 S_{10} + 2 = 0, \tag{3-21}$$

$$3(1 - \nu^2)S_{10}^2 S_{11} - C_1 S_{11} - 2 = 2\lambda S_{10}^2, \tag{3-22}$$

etc., whose solutions become

$$S_{10} = 2\left\{\frac{C_1}{3(1 - \nu^2)}\right\}^{1/2} \sin\left\{\tfrac{1}{3}\sin^{-1}\frac{3}{C_1}\sqrt{\frac{3(1 - \nu^2)}{C_1}}\right\} \tag{3-23}$$

and

$$S_{11} = \frac{2(1 + \lambda S_{10}^2)}{3(1 - \nu^2)S_{10}^2 - C_1}, \tag{3-24}$$

respectively.

Equation (3-20) with its coefficients as given by (3-23) and (3-24) will closely approximate the form of the primary component of a contact system which is very much more massive than the secondary. Its first term S_{10}

defines obviously the form of a Roche equipotential distorted by centrifugal force alone. If $\mu \to 0$, $\Omega_1 \to 1{\cdot}5$ and $C_1 \to 3$, in which case the parametric equation of the corresponding critical equipotential assumes the neat form

$$r = \frac{2}{\sqrt{1-\nu^2}}\{\sin\tfrac{1}{3}\cos^{-1}\nu\}, \tag{3-25}$$

and its volume V_1, in accordance with equation (2-22), becomes

$$\left.\begin{aligned} V_1 &= \tfrac{32}{3}\pi\int_0^1 (1-\nu^2)^{-3/2}\sin^3(\tfrac{1}{3}\cos^{-1}\nu)\,d\nu \\ &= \tfrac{4}{3}\pi\left\{3\sqrt{3} - 4 + 3\log\frac{3(\sqrt{3}-1)}{\sqrt{3}+1}\right\} = 2{\cdot}26663\ldots \end{aligned}\right\}. \tag{3-26}$$

It is this foregoing value, rather than the one which would follow from a straightforward application of (2-28), which has been used to complete the last entry in column (5) of Table 3-2.

If the primary component accounts thus for most part of the total mass of our contact binary system, the volume of the secondary must clearly tend to zero. The form of its surface will, in turn, be given by an asymptotic solution of equation (3-18) as $\mu \to 1$. Let us, therefore, expand this solution in a series of the form

$$r = S_{20}(1-\mu) + S_{21}(1-\mu)^2 + \ldots; \tag{3-27}$$

inserting it in (3-18) we find the vanishing of the coefficients of equal powers of $(1-\mu)$ to require that

$$S_{20} = \frac{2}{C_1 - 3}, \tag{3-28}$$

$$S_{21} = -\left\{2 + \frac{\lambda}{C_1 - 3}\right\}S_{20}^2, \tag{3-29}$$

etc. An application of equation (2-22) reveals, moreover, that the volume V_2 of the respective configuration should be approximated by

$$V_2 = \tfrac{4}{3}\pi\{(1-\mu)^3 S_{20}^3 - 6(1-\mu)^4 S_{20}^4 - \ldots\}, \tag{3-30}$$

and the radius r_2^* of a sphere of equal volume becomes

$$r_2^* = (1-\mu)S_{20} - 2(1-\mu)^2 S_{20}^2 + \ldots \tag{3-31}$$

Now a glance at the second column of Table 3-2 reveals that, as $q \to 0$, $C_1 \to 3$ and, as a result, the product $(1-\mu)S_{20}$ tends to become indeterminate for $\mu = 1$. In order to ascertain its limiting value, let us depart from the equation (3-12) which, on insertion of Ω_1 from (3-9) assumes the form

$$C_1 = \frac{2(1-\mu)}{1-x_1} + \frac{2\mu}{x_1} + (1+\mu+x_1)^2, \tag{3-32}$$

with the root x_1 approximable by means of (3-7) where, by (3-13),

$$3w^3 = 1 - \mu. \tag{3-33}$$

Inserting (3-7) in (3-32) we find that, within the scheme of our approximation,

$$C_1 = 3\{1 + 3w^2 - 4w^3 + \ldots\}, \tag{3-34}$$

so that

$$(1 - \mu)S_{20} = \frac{2w}{3 - 4w} + \ldots \tag{3-35}$$

and, therefore,

$$r_2^* = \tfrac{2}{3}w - \tfrac{32}{27}w^3 + \ldots \tag{3-36}$$

In consequence, it follows from (3-17) that, for a secondary component of vanishing mass,

$$v_2 = \tfrac{1}{3}(\tfrac{2}{3})^4\{1 - \tfrac{16}{3}w^2 + \ldots\}. \tag{3-37}$$

An inspection of the last two columns of Table 3-2 reveals that, for the primary (more massive) component, the value of v_1 increases monotonously with diminishing mass-ratio m_2/m_1 from 0·07227 for the case of equality of masses to 0·36075 for $m_2 = 0$, at which point the primary component becomes rotationally unstable and matter would begin to be shed off along the equator if axial rotation were any faster. On the other band, for the secondary (less massive) component the values of v_2 diminish with decreasing mass-ratio from 0·07227 until, as $m_2 \to 0$, the value of $(2^4/3^5)$ has been attained.

III.4. GEOMETRY OF THE ECLIPSES

The data assembled in the foregoing section on the geometry of contact configurations lead to a number of specific conclusions regarding the eclipse phenomena to be exhibited by such systems. For suppose that a contact binary whose both components are at their Roche limits is viewed by a distant observer, whose line of sight does not deviate greatly from the x-axis of our model as shown on Fig. 3-2. If so, then in the neighbourhood of either conjunction one component is going to eclipse the other, and the system will exhibit a characteristic variation in brightness. If, in turn, the observed light variation is analyzed for the geometrical elements by methods which will be described in Chapter VI, the fractional 'radii' $r_{1,2}$ of the two components should (very approximately) be identical with the quantities $y_{4,5}$ as listed in columns (5) and (7) of Table 3-1. In the following Table 3-3 we have, accordingly, listed four-digit values of the sums $r_1 + r_2$ as well as the ratios r_2/r_1 of the 'radii' of such contact components as functions of their mass-ratio.

An inspection of this tabulation reveals that, within the scheme of our approximation, *the sum $r_1 + r_2$ of fractional radii of both components in contact binary systems is very nearly constant* and equal to $0{\cdot}75 \pm 0{\cdot}01$

for a very wide range of the mass-ratios q; whereas the *ratio* r_2/r_1 *decreases monotonically with diminishing value of* q. Therefore, a photometric determination of the sum $r_1 + r_2$—which, unfortunately, represents nearly all that can be deduced with any accuracy from an analysis of light curves due to shallow partial eclipses (*cf.* section VI.6)—cannot be expected to tell us anything new about contact systems, or in particular, about their mass-ratios. It is the ratio of the radii r_2/r_1 whose determination would provide

TABLE 3-3

q	$r_1 + r_2$	r_2/r_1
1	0·7484	1·0000
0·9	0·7486	0·9495
0·8	0·7489	0·8959
0·7	0·7496	0·8389
0·6	0·7510	0·7777
0·5	0·7529	0·7112
0·4	0·7565	0·6379
0·3	0·7622	0·5550
0·2	0·7722	0·4573
0·15	0·7805	0·3995
0·1	0·7935	0·3312
0·05	0·8178	0·2428
0	1·0000	0·0000

a sensitive photometric clue to the mass-ratio of a contact system. This underlines the importance of photometric determination of the ratios of the radii of contact binary systems; but owing to purely geometrical difficulties this important task of light curve analysis is, unfortunately, not yet well in hand.

Suppose next that a contact binary system, consisting of two components at their Roche limits, is viewed by a distant observer from an arbitrary direction. What will be the range of such directions from which this observer will see both bodies mutually eclipse each other during their revolution? In order to answer this question, let us replace the actual form of the corresponding Roche limit by an *osculating cone* which is tangent to it at the point of contact P_1. The equation of this cone may readily be obtained if we expand the function $\Omega(x, y, z)$ of Roche equipotentials in a Taylor series, in three variables, about P_1. The first partial derivatives Ω_x and Ω_y have already been given by equations (3-2) and (3-3) of the preceding section, and

$$\Omega_z = -zr^{-3} - qz(r')^{-3}. \tag{4-1}$$

Differentiating these equations further we find that

$$\Omega_{xx} = (3x^2 - r^2)r^{-5} + q\{3(1 - x)^2 - r'^2\}(r')^{-5} + q + 1, \tag{4-2}$$

$$\Omega_{yy} = (3y^2 - r^2)r^{-5} + q\{3y^2 - r'^2\}(r')^{-5} + q + 1, \tag{4-3}$$

$$\Omega_{zz} = (3z^2 - r^2)r^{-5} + q\{3z^2 - r'^2\}(r')^{-5}, \tag{4-4}$$

and

$$\Omega_{xy} = 3xyr^{-5} - 3q(1-x)y(r')^{-5}, \tag{4-5}$$

$$\Omega_{xz} = 3xzr^{-5} - 3q(1-x)z(r')^{-5}, \tag{4-6}$$

$$\Omega_{yz} = 3yzr^{-5} + 3qyz(r')^{-5}. \tag{4-7}$$

We note that all first (as well as mixed second) derivatives of Ω vanish at P_1. Hence, a requirement that the sum of nonvanishing second-order terms should add up to zero provides us with the desired equation of the osculating cone in the form

$$(x - x_1)^2 (\Omega_{xx})_1 + y^2(\Omega_{yy})_1 + z^2(\Omega_{zz})_1 = 0, \tag{4-8}$$

where

$$\left.\begin{aligned} (\Omega_{xx})_1 &= 2p + q + 1, \\ (\Omega_{yy})_1 &= -p + q + 1, \\ (\Omega_{zz})_1 &= -p, \end{aligned}\right\} \tag{4-9}$$

in which we have abbreviated

$$p = x_1^{-3} + q(1 - x_1)^{-3}. \tag{4-10}$$

The direction cosines l, m, n, of a line normal to the surface of this cone clearly are

$$l, m, n = \{f_\xi, f_y, f_z\} \div \{f_\xi^2 + f_y^2 + f_z^2\}^{1/2} \tag{4-11}$$

where $f(\xi, y, z)$ stands for the left-hand side of equation (4-8) and $\xi \equiv x - x_1$. Moreover, the direction cosines of the axis of this cone in the same coordinate system are (1, 0, 0). Consequently, the angle ε between any arbitrary line on the surface of the osculating cone and its axis will be defined by the equation

$$\cos\left(\frac{\pi}{2} - \varepsilon\right) = l \tag{4-12}$$

or, more explicitly,

$$\tan^2 \varepsilon = -\left\{\frac{(\Omega_{yy})_1\, y^2 + (\Omega_{zz})_1\, z^2}{(\Omega_{yy})_1^2\, y^2 + (\Omega_{zz})_1^2\, z^2}\right\} (\Omega_{xx})_1, \tag{4-13}$$

where the values of Ω_{xx}, Ω_{yy}, and Ω_{zz} at P_1 are given by equation (4-9) above.

Suppose now that the orbital xy-plane of the two components is inclined at an angle i to a plane perpendicular to the line of sight (i.e., one tangent to the celestial sphere at the origin of the coordinates), and that ψ_1 denotes the angle of the first contact of the eclipse (as measured from the moment of superior conjunction). If so, then obviously

$$\left.\begin{aligned} \cos \varepsilon = x &= \cos \psi_1 \sin i, \\ y &= \sin \psi_1 \sin i, \\ z &= \cos i, \end{aligned}\right\} \tag{4-14}$$

in equation (4-13), and the latter can be simplified to disclose that

$$\delta^2 = \frac{az^2}{a-1}\left\{\frac{a+2-4\delta^2}{a+2-3\delta^2}\right\}, \tag{4-15}$$

where

$$\delta^2 = y^2 + z^2 = \sin^2 \psi_1 \sin^2 i + \cos^2 i \tag{4-16}$$

denotes the apparent projected distance between the centres of both components at the moment of first contact of the eclipse, and

$$a = \frac{q+1}{p} = \frac{q+1}{x_1^{-3} + q(1-x_1)^{-3}}. \tag{4-17}$$

An inspection of equation (4-15) reveals several features deserving explicit statement. Thus the *maximum duration of eclipses* (i.e., the maximum value of ψ_1) will evidently obtain if the line of sight lies in the orbital plane and, therefore, $z = \cos i = 0$; and if so, equation (4-15) discloses that

$$\cos^2 \psi_{\max} = \tfrac{1}{3}(1-a). \tag{4-18}$$

With the aid of the x_1's as given in column (2) of Table 3-1 the values of a can be readily evaluated for any mass-ratio q; and the corresponding values of $\psi_{\max}$ are then listed in the second column of the following Table 3-4.

TABLE 3-4

q	$\pm(\psi_1)_{\max}$	$i_{\min}$
1	57°·3122	34°·4499
0·8	57·3192	34·4478
0·6	57·3474	34·4388
0·4	57·4280	34·4153
0·2	57·6446	34·3511
0·1	57·9296	34·2690

A glance at these data reveals that *the values of $\psi_{\max}$ are remarkably insensitive to the mass-ratio*—which constitutes a fact of considerable practical importance for the student of close binary systems. For the variations of light exhibited by such systems (*cf.* the forthcoming Chapter IV) in the course of each cycle are so smooth and continuous that it is often very difficult—if not impossible—to detect by an inspection of the observed light changes just where the eclipses may set in. Our present analysis now supplies a theoretical answer: namely, *no matter what the mass-ratio may be, the light changes of a close binary system are bound to be unaffected by eclipses for all phase angles outside the range of* $\pm 57\overset{\circ}{.}4$ *even if both components are in actual contact.* Therefore, the light changes exhibited at phases within $65\overset{\circ}{.}2$ around each quadrature must be due solely to the distorted form of both components, and may be subject to 'rectification' (*cf.* section VI.11) without fear of interference from eclipse phenomena.

The closed form of equation (4-15) invites another related question which can be asked in this connection: namely, what is the *minimum inclination* of the orbital plane to the celestial sphere below which no eclipses may occur for contact binary systems? This minimum inclination will evidently be attained if the eclipse becomes grazing just at the time of the conjunctions (i.e., if $\psi_1 = 0$, in which case $\delta = z$). Under these circumstances equation (4-15) can be shown to imply that

$$\cos^2 i_{\min} = \frac{2 + a}{3 - a}, \tag{4-19}$$

where a continues to be given by (4-17). A tabulation of the values of $i_{\min}$ as defined by the foregoing equation (4-19) for six discrete values of the mass-ratio can be found in column (3) of Table 3-4. An inspection of these data reveals that, *regardless of the mass-ratio, no binary system can exhibit eclipses if its orbit is inclined by less than* $34^\circ\!.4$ *to the celestial sphere—even if its components are so close as to be in actual contact.* For any value of i *above* this limit eclipses may occur (and *must* occur for contact binary systems) of durations requiring the angle ψ_1 of first contact to satisfy the inequality $0 \leqslant \psi_1 \leqslant 57^\circ$. A relation between ψ_1 and i for each particular value of the mass-ratio is then represented by the full-dress equation (4-15); and a compressed tabulation of the values ψ_1 as functions of i and q is given in the accompanying Table 3-5.

TABLE 3-5

	$\cos^2 i$							
q	0	0·10	0·20	0·30	0·40	0·50	0·60	$\cos^2 i_{\min}$
1	57°312	54°986	52°180	48°684	44°116	37°673	27°089	0°
0·8	57·319	54·995	52·188	48·693	44·128	37·688	27·112	0
0·6	57·347	55·018	52·211	48·714	44·147	37·706	27·129	0
0·4	57·428	55·093	52·282	48·782	44·212	37·775	27·213	0
0·3	57·509	55·167	52·352	48·849	44·277	37·843	27·297	0
0·2	57·645	55·295	52·471	48·963	44·391	37·959	27·437	0
0·1	57·930	55·555	52·716	49·197	44·619	38·195	27·725	0

III.5. EXTERNAL ENVELOPES

In the foregoing section III.2 we have been concerned with various properties of Roche equipotentials when $\Omega > \Omega_1$, and in section III.4 we investigated the geometry of limiting double-star configurations for which $\Omega = \Omega_1$. The aim of the present section will be to complete our analysis of formal properties of the Roche model by considering what happens when $\Omega < \Omega_1$. In introducing paragraphs of section III.1 we inferred on general grounds that if $\Omega < \Omega_1$, the dumb-bell figure which originally surrounded the two components will open up at P_1 (*cf.* again Fig. 3-1), and the corresponding equipotentials will enclose *both* bodies.

When will these latter equipotentials containing the total mass of our binary system cease to form a *closed* surface? A quest for the answer will take us back to equation (3-2) defining the partial derivative Ω_x. We may note that the right-hand side of this equation is positive when $x \to \infty$, but becomes negative when $x = 1 + \varepsilon$, where ε denotes a small positive quantity. It becomes positive again as $x \to 0$, and changes sign once more for $x \to -\infty$. Since Ω_x is finite and continuous everywhere except at $|x| = \infty$ and for $r = 0$ or $r' = 0$, it follows that it changes sign *three* times by passing through zero at points x_1, x_2, x_3, whose values are such that

$$\left.\begin{array}{ll} \text{(a)} & 0 < x_1 < 1, \\ \text{(b)} & \qquad x_2 > 1, \\ \text{(c)} & \qquad x_3 < 0; \end{array}\right\} \tag{5-1}$$

and of these, only the first one has been evaluated so far in section III.3 and its numerical values listed in column (2) of Table 3-1.

An evaluation of the remaining roots $x_{2,3}$ offers, however, no greater difficulty. In embarking upon it we should merely keep in mind that, regardless of the sign of x, the distances r and r' as defined by equations (0-2) are *positive* quantities. Thus, unlike in case (*a*)—when, by setting $r = x$ and $r' = 1 - x$, we were led to define x_1 as a root of equation (3-5),—in case (*b*), when $x_2 > 1$, we must set $r = x$ but $r' = x - 1$; and in caset (*c*) when $x_3 < 0$, $r = -x$ and $r' = 1 - x$. After doing so and clearing he, fractions we may verify that the equation $\Omega_x = 0$ in the case (*b*) and (*c*) assumes the explicit form

$$(1 + q)x^5 - (2 + 3q)x^4 + (1 + 3q)x^3 - (1 + 2q)x^2 + 2x - 1 = 0 \tag{5-2}$$

and

$$(1 + q)x^5 - (2 + 3q)x^4 + (1 + 3q)x^3 + x^2 - 2x + 1 = 0, \tag{5-3}$$

respectively.

For $q = 0$, the former equation (5-2) becomes identical with (3-5) and reduces to (3-6) admitting of $x = 1$ as a triple root. Hence, for small values of q, the root $x_2 > 1$ of the complete equation (5-2) should be expansible as

$$x_2 = 1 + \left(\frac{\mu}{3}\right)^{1/3} + \frac{1}{3}\left(\frac{\mu}{3}\right)^{2/3} + \frac{1}{9}\left(\frac{\mu}{3}\right) + \dots \tag{5-4}$$

in terms of fractional powers of $\mu \equiv q(q + 1)$. Similarly, equation (5-3) reduces for $q = 0$ to

$$(x - 1)^2 (x^3 + 1) = 0\,, \tag{5-5}$$

admitting of only one negative root (namely, -1). In consequence, the negative root x_3 of (5-3) should, for small values of μ, be approximable in terms of integral powers of μ by an expansion of the form

$$x_3 = -1 + \frac{7}{12}\mu - \frac{1127}{20736}\mu^3 + \dots \tag{5-6}$$

The approximate values of x_2 and x_3 as obtained from (5-4) or (5-6) may, moreover, be subsequently refined to any degree of accuracy by differential corrections or any other standard method.

Once sufficiently accurate values of $x_{2,3}$ have thus been established, the values of Ω corresponding to equipotentials which pass through these points can be ascertained from the equation

$$\Omega_{2,3} = \Omega(x_{2,3}, 0, 0); \tag{5-7}$$

while the corresponding values of $C_{2,3}$ then can be found from

$$C_{2,3} = 2(1-\mu)\Omega_{2,3} + \mu^2. \tag{5-8}$$

A tabulation of five-digit values of $x_{2,3}$ and $C_{2.3}$ is given in columns (2)-(5) of the accompanying Table 3-6. It may also be noticed that, to a high degree of approximation,

$$\Omega_3 \cong \tfrac{3}{2} + 2q - \frac{q^2}{2(1+q)} \tag{5-9}$$

or

$$C_3 \cong 3 + \mu\,; \tag{5-10}$$

while, somewhat less accurately,

$$(x_2 - 1)^2 = 1 - x_3^2. \tag{5-11}$$

A comparison of the values of $C_{2,3}$ as given in Table 3-6 with those of C_1 from Table 3-2 reveals that, for all values of $q > 0$,

$$C_1 > C_2 \geqslant C_3. \tag{5-12}$$

TABLE 3-6

q	x_2	C_2	$-x_3$	C_3	$C_{4,5}$
1·0	1·69841	3·45680	0·69841	3·45680	2·75
0·8	1·66148	3·49368	0·73412	3·41509	2·75309
0·6	1·61304	3·53108	0·77751	3·35791	2·76563
0·4	1·54538	3·55894	0·83180	3·27822	2·79592
0·3	1·49917	3·55965	0·86461	3·22675	2·82249
0·2	1·43808	3·53643	0·90250	3·16506	2·86111
0·15	1·39813	3·50618	0·92372	3·12959	2·88658
0·1	1·34700	3·45153	0·94683	3·09058	2·91735
0·05	1·27320	3·34671	0·97222	3·04755	2·95465
0·02	1·19869	3·22339	0·98854	3·01961	2·98077
0·01	1·15614	3·15344	0·99422	3·00990	2·99020
0·005	1·12294	3·10301	0·99710	3·00498	2·99504
0·001	1·07089	3·03838	0·99942	3·00099	2·99900
0·0002	1·04108	3·01387	0·99988	3·00020	2·99980
0	1·00000	3·00000	1·00000	3·00000	3·00000

For any value of C within the limits of the inequality $C_1 > C > C_2$ the corresponding equipotential will surround the whole mass of the system by a common *external envelope*, which may enclose the common atmosphere of the two stars. For $C = C_2$, this envelope will develop a conical point P_2 (at which $\Omega_x = \Omega_y = \Omega_z = 0$) at $x = x_2$—i.e., behind the centre of gravity of the less massive component (see Figs. 3-1 or 3-2); and if $C < C_2$, the respective equipotentials will open up at P_2. For $C = C_3$, a third conical point P_3 develops behind the centre of gravity of the more massive component at $x = x_3$; and if $C < C_3$, the equipotentials will become open at both ends. Their intersection with the xy-plane will then no longer represent a single closed curve, but will split up in two separate sections (symmetrical with respect to the x-axis), closing gradually around two points $Q_{1,2}$ (see again Figs. 3-1 or 3-2) which make equilateral triangles with the centres of mass of the two components. The coordinates of $Q_{1,2}$, specified by the requirement that $r = r' = 1$, consequently are $x = 0{\cdot}5$ and $y = \pm 1{\cdot}5$. These triangular points represent also the locus at which our equipotentials vanish eventually from the real plane—if (consistent with equations 1-3 and 3-12) their constants C reduce to

$$C_{4,5} = 3 - \mu + \mu^2. \tag{5-13}$$

The values of $C_{4,5}$'s as given by this equation are listed in column (6) of Table 3-6 for $1 > q > 0$, and represent the lower limits attainable by these constants; for if $C < C_{4,5}$, the equipotential curves $\Omega =$ *constant* in the xy-plane become imaginary, and thus devoid of any further physical significance.

III. BIBLIOGRAPHICAL NOTES

The concept of the Roche Model, as introduced in this chapter, owes its origin to an investigation by Edouard Roche, entitled 'La Figure d'une Masse Fluide, soumise à l'attraction d'un point éloigné', *Mémoires de l'Acad. des Sciences de Montpellier*, vol. **1**, pp 243*ff* and 333*ff*; **2**, pp 21*ff*, 1849–1951; *cf.* also Roche's memoir in *op. cit.*, **8**, 235, 1873.

For a formal discussion of the stability of such a model *cf.* L. Lichtenstein, *Berichte der Sächs. Akad. Wiss.*, **80**, 35, 1928, or K. Maruhn, *Math. Zeit.*, **33**, 300, 1931; summarized in Lichtenstein's book *Gleichgewichtsfiguren Rotierender Flüssigkeiten*, Berlin 1933, sections 24–28.

The complex transformation of Roche's equipotentials as given by equations (1-6) represents a generalization of the plane complex transformation used (for the Jacobian surfaces of zero velocity) by G. W. Hill (*cf.* his *Collected Works*, **1**, p. 290, 1905). For a formal mathematical discussion of such surfaces *cf.*, e.g., A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton Univ. Press, 1941. Chapter VI, and the references to it. For a further study of the Roche equipotentials *cf.* also P. ten Bruggencate *Zs. f. Ap.*, **8**, 344, 1934.

Most part of the discussion and results of sections 2–4 have been taken from Kopal's study of the Roche model which appeared in the *Jodrell Bank Annals*, **1**, 37, 1954.* Certain geometrical properties of this model have previously been derived (to a lower degree of accuracy) by G. P. Kuiper, in *Ap. J.* **93**, 133, 1941; and later by G. P. Kuiper and J. R. Johnson in *Ap. J.*, **123**, 90, 1956. A numerical study of the Jacobian surfaces of zero velocity by J. E. Rosenthal (*A.N.*, **244**, 169, 1931) contains a 4*D* tabulation of our values of k_1, k_2, and $-k_3$ of Table 7-8, and also of Ω_{xx} and Ω_{yy} at the Lagrangian collinear points.

* This paper contained, unfortunately, many slips and misprints, which have been corrected in its present version.

CHAPTER IV

Theoretical Light Changes of Close Binary Systems

In chapter II we have investigated the equilibrium of fluid components in close binary systems, and found their forms to be predictable in relatively simple terms as far as their axial rotation is slow enough for its squares and higher powers to be negligible, and the tidal pull of one component upon another can be regarded as that of a mass-point. The aim of the present chapter will be to investigate the photometric effects of such distortion, and the light changes which take place if our system happens to be an eclipsing variable. In order to formulate it in more specific terms, consider a close binary whose components revolve in a plane so inclined to the line of sight that, at the time of conjunctions, each star may alternately eclipse the other. Both components are distorted by their axial rotation and mutual tidal action. Moreover, the distribution of their apparent surface brightness is governed by the internal structure of distorted configurations. Our task will be to investigate the theoretical light changes exhibited by such systems—between minima as well as within eclipses—in any particular frequency of light, and to the same degree of accuracy to which the form of both components is known.

While the first few chapters of this book have been largely devoted to more general properties of close binary systems, not necessarily connected with any variation of their light, from now on we shall be increasingly concerned with the phenomena of *stellar eclipses.* If the students of eclipsing variables were to enter into conclave to elect the patron-saint of their subject, this honour would probably go by seniority to the Greek philosopher Anaxagoras (about 500–428 B.C.), who hailed from Clazomenae in Asia Minor and flourished in Athens of Pericles under the setting sun of the golden age of Greece. And a worthy patron he would be; for according to tradition, he neglected his earthly possessions—which were considerable—in order to devote himself to science; and when asked for the purpose of human life he responded: 'to study the Sun, the Moon, and the heaven'.* His end, in exile, as a victim of unscrupulous Athenian politicians who accused him of subversive activities (such as teaching that the Sun was a fiery stone, larger in size than the island of Peloponnesos), was prophetic of the tribulations awaiting his fellow-astronomers in subsequent centuries and millenia up to our own time. Our particular gratitude to Anaxagoras is, however, due to the fact that he appears to have been the first student

* According to the testimony of Diogenes Laertius (II 10, p. 294).

of science to discern the true nature of the solar and lunar eclipses.*

It is indeed truly inspiring to contemplate the astonishing range of the stimulus which the development of astronomy—and of physical sciences in general—received from the phenomena of eclipses. As early as in the third century B.C., the observed form of the terrestrial shadow during lunar eclipses enabled Aristotle to formulate the first scientific proof that the Earth was a sphere; and, somewhat later, the eclipses of the Sun provided Hipparchos with the geometric means to gauge the distance separating us from the Sun. In the seventeenth century A.D., the eclipses of the satellites of Jupiter afforded the first experimental demonstration of the fact that the velocity of light was finite. The total eclipses of the Sun have, in particular, proved a veritable godsend to many branches of science and their students—from the historians to whom they enabled to straighten out many an obscure chapter of ancient chronology, to the geophysicists who used the old eclipse records to detect secular irregularities in the length of the day, or to the chemists who learned of the existence of at least one new element (helium) from solar flash spectra.† In more recent days, astrophysicists have taken advantage of the fleeting minutes of total eclipses to measure the amount of light deflection in the gravitational field of the Sun, and thus to study the metric properties of space in the neighbourhood of large masses.

Facing this impressive array of facts, the reader may well ask whether a study of stellar eclipses—such as exhibited by countless eclipsing variables known in the sky—will prove rewarding to the same measure; and he will indeed not be disappointed. Before he reaps, however, his full reward in the concluding Chapter VII, he will be led through certain sections where progress may be impeded by sheer weight of mathematical analysis unequalled in complexity in any branch of astronomical science save for the theory of the motion of the Moon. If this complexity may at times seem exasperating, it should be kept in mind that the eclipses have never caused any astronomer to lose his head—at least not since the days of our somewhat legendary predecessors Hi and Ho in old China. Stellar eclipses will, to be sure, scarcely ever again have an opportunity to stop a battle as did the famous solar eclipse of 585 B.C. at the time of Thales; but the results of their study may well exert a more profound and lasting effect on science than that abortive skirmish between the Lydians and the Medes did on the history of the human race.

But mutual eclipses of two nearby stars do not represent the only phenomenon which may invoke observable light variation. A significant feature

* And, also, to have become father of the 'reflection effect'—together with Leonardo da Vinci, who centuries later was the first to recognize the true nature of lunar 'ashen light' (secondary reflection).

† It may be of interest to note, in this connection, that within our lifetime another new element was discovered in the Sun—namely, the negative hydrogen ion—not previously known from terrestrial laboratories. This last element did not, however, reveal its presence in the Sun by its line emission spectrum, but rather by its continuous absorption; and the instrument of discovery was not an eclipse of the Sun, but its limb-darkening.

of close binary systems is the fact that they, as a class, are bound to exhibit continuous light changes even if they do not happen to be eclipsing variables; moreover, their variability should persist throughout the whole period of the orbit, and may subside only under special circumstances. This is due to two principal reasons. First, since the components of such binaries are, in general, distorted ellipsoids with longest axes constantly in the direction of the radius-vector, their apparent area—and, therefore, the light—as seen by a distant observer should vary continuously in the course of a revolution (*ellipticity effect*). Secondly, it is inevitable in close binary systems that a part of the radiation of each component must fall on the surface of its mate where it will be absorbed and re-emitted (or scattered) in all directions—including the direction of the line of sight. The amount of such parasitic light 'reflected' by each component in the direction of the observer should again vary with the phase (*reflection effect*). Moreover, the magnitude of this effect should clearly depend on the target area exposed to incident light, and thus be only insensibly affected by distortion.

The changes of light due to the ellipticity and reflection are independent of, and supplementary to, the changes which may arise from eclipses. The magnitude of both effects should be governed by the relative proximity of both components as well as by the orientation of their orbital planes. The changes of light due to the ellipticity and reflection should vanish only if the plane of the orbit is perpendicular to the line of sight—and even then only if, in addition, this orbit happens to be circular. As long as this is so and the distance R between the centres of both stars remains constant, both the ellipticity and reflection effects should be symmetrical with respect to the conjunctions; but in eccentric orbits this will be so only if the apsidal line is parallel with the line of sight.

The changes of light exhibited by close binary systems between minima are, needless to stress, considerably simpler than those which accompany the eclipses. The occurrence of the latter will, in connection with distortion, give rise to some of the most intricate phenomena encountered in double-star astronomy. Their origin can be traced back to the particular form of the eclipsing and the eclipsed star. A distortion of the eclipsing component must produce a corresponding deformation of a shadow cylinder cast by it in the direction of the line of sight. On the other hand, a distortion of the eclipsed component will not only alter the portion of its apparent disk intercepted by the shadow cylinder at any moment, but will also influence the distribution of brightness over the eclipsed portion. The effects of distortion of the shadow cylinder are purely geometrical, and can be dealt with in relatively simple terms. On the other hand, those due to a distortion of the star undergoing eclipse involve also physical considerations (gravity-darkening), and their quantitative study will call for the introduction of whole new families of special functions.

Our strategy of approach to the problems to be discussed in this chapter is made evident by the headings of its sections. In section IV.1, which follows these introductory remarks, we shall set out to investigate the distribution of brightness over apparent disks of the components of close binary systems as governed by their limb- and gravity-darkening, as well as influenced by the light incident from their mates. Section IV.2 will be devoted to a discussion of photometric effects of distortion between minima; and section IV.3 will contain an analysis of light changes arising from eclipses of distorted components. Special functions which had to be introduced for this purpose will be discussed in sections IV.4 and IV.5; while section IV.6 will be concerned with the reflection effect (between minima as well as within eclipses). The concluding section IV.7 will then be devoted to a brief outline of the theory of atmospheric eclipses (i.e., eclipses by semi-transparent outer layers of stars possessing extended atmospheres).

IV.1. DISTRIBUTION OF BRIGHTNESS OVER APPARENT STELLAR DISKS

As is well known from the theory of stellar atmospheres, the distribution of brightness over the apparent disks of the stars is not, in general, likely to be constant, but rather a varying function of position on the surface; and as long as a star is solitary in space, the principal cause of such a variation will be the semi-transparency of stellar outer layers: if stars possessed no atmospheres and no continuous temperature gradient near the surface, their disks would indeed appear uniformly bright. If, however, the temperature in semi-transparent layers increases inwards, the radiation which emerges from such layers originates, on the average, at a greater depth—and, therefore, in regions of higher temperature—if viewed normally to the surface of a star than if viewed tangentially. As a consequence, the apparent surface brightness as seen by a distant observer should be expected to depend on the angle of foreshortening (i.e., one between the surface normal and the line of sight), being highest at the centre of the apparent disk of a star, and decreasing toward the edge.

In more specific terms, if $I(r, \theta, \phi)$ denotes the intensity of the radiation of a star at a distance r from its centre in a direction specified by the angles θ, ϕ, and $B(r, \theta, \phi)$ stands for the emissivity (source function) of the respective material at that particular point, the principle of the conservation of energy leads to the *equation of transfer* of the form

$$\frac{dI}{ds} = \kappa\rho(B - I), \tag{1-1}$$

where κ stands for the coefficient of opacity of stellar material; ρ, for its density; and s, for the distance which light has to pass through the atmosphere, so that

$$\left.\begin{aligned} \frac{d}{ds} &= \mu\frac{\partial}{\partial r} + \frac{1-\mu^2-\nu^2}{r}\frac{\partial}{\partial\mu} + \frac{\mu\nu}{r}\frac{\partial}{\partial\nu}, \\ \mu &= \cos\theta, \qquad \nu = \cos\phi\sin\theta. \end{aligned}\right\} \tag{1-2}$$

The second and third terms on the right-hand side of the foregoing operator contain the factor r^{-1} which will make them diminish with increasing distance from the centre (i.e., with the diminishing curvature of atmospheric layers); but as long as they remain perceptible, (1-1) constitutes a *partial* differential equation. In order to avoid difficulties arising from this fact let us hereafter assume—reasonably enough—that *the depth to which the boundary of a star is semi-transparent represents so small a fraction of the total radius of the star as a whole that the matter within the atmosphere can be regarded as stratified in plane-parallel layers.** If so, the operator d/ds reduces to $\mu(\partial/\partial r)$ and (1-1) becomes then an *ordinary* differential equation of the form

$$\mu\frac{dI}{d\tau} = I - B\,, \tag{1-3}$$

where the optical depth τ is defined, as usual, by

$$d\tau = -\kappa p\, dr\,, \tag{1-4}$$

and where we propose to orient our frame of reference in such a manner that

$$\mu = \cos\gamma; \tag{1-5}$$

γ denoting the angle between the radius-vector and the line of sight.

In order to establish a relation between the intensity I of emitted radiation and its source function B, appeal must be made to the theory of *radiative equilibrium.* The latter asserts that if there are no energy sources within the atmosphere,† so that the semi-transparent layers merely transmit without gain or loss the radiation arriving from the interior, the net flux

$$F = 2\int_0^\pi I\sin\gamma\cos\gamma\, d\gamma = 2\int_{-1}^{1} I\mu\, d\mu \tag{1-6}$$

* This assumption, which we shall consistently adhere to through all sections but IV.7 of this chapter, rules out of consideration stars with extended atmospheres. We should remember that, in eclipsing binary systems, we are dealing with close pairs of stars of comparable masses. Their mutual tidal action must necessarily tend to destroy their spherical symmetry. If the components are not too close, and the extent of their atmospheres is small, the material within such atmospheres may still be regarded with fair approximation as stratified in parallel layers. If, however, the atmosphere were extended—no matter what the dimensions of the parent star may be—the effects of tides raised in its outer layers by the disturbing component would be such that plane-parallel stratification, or even spherical symmetry would no longer provide a workable approximation. In order to obtain an equation of transfer relevant to such cases, the full-dress form of the operator in spherical polar coordinates would have to be employed. No attempt at solving equation (1-1) in such coordinates has so far been made; and as a result, the effects of darkening produced by extended atmospheres in distorted eclipsing systems appear so far to be unpredictable.

† Or if such sources are negligible in comparison with the energy generated in the interior, outflowing through the atmosphere.

must be constant and independent of τ; while the source function (representing the radiation emitted at any particular point) will be given by

$$B(\tau) = \tfrac{1}{2}\int_0^{\pi} I \sin\gamma\, d\gamma = \tfrac{1}{2}\int_{-1}^{1} I\, d\mu, \tag{1-7}$$

and consists of light incident upon it from all directions. Combining (1-3) and (1-7) we arrive thus at the following integro-differential equation

$$\mu\frac{dI}{d\tau} = I - \tfrac{1}{2}\int_{-1}^{1} I\, d\mu \tag{1-8}$$

of *radiative transfer* of energy which is absorbed and re-emitted (or scattered isotropically with unit albedo) in plane-parallel atmospheres, governing the intensity $I(\tau, \mu)$ of light at any optical depth τ in arbitrary direction μ. The boundary conditions of our problem, which the solution of (1-8) must satisfy, require that (if the radiation merely passes through the atmosphere without gain or loss) the net flux F as defined by equation (1-6) must be constant and independent of τ; and that, in addition,

$$I(0, \mu) = 0 \qquad \text{for} \qquad 0 \geqslant \mu \geqslant -1 \tag{1-9}$$

if no radiation is incident on the star from outside.

The integro-differential equation (1-8) with its associated boundary conditions defies formal solution in a closed form; but as was shown first by N. Wiener and E. Hopf,* its limiting form $I(0, \mu)$ for $\tau = 0$—describing the stellar distribution of brightness as seen by a distant outside observer, in which alone we are interested in this book—is expressible as

$$I(0, \mu) = \frac{\sqrt{3}}{4}\frac{F}{\sqrt{1+\mu}}\exp\left\{\frac{1}{\pi}\int_0^{\pi/2}\frac{\theta\tan^{-1}(\mu\tan\theta)}{1-\theta\cot\theta}\,d\theta\right\}. \tag{1-10}$$

The integral on the right-hand side can be evaluated by quadratures and, with its aid, the normalized variation $J(\mu)$ of surface brightness between the centre and the limb (i.e., the 'law of darkening') can be expressed as

$$I(0, \mu) = I(0, 1)\left\{\frac{I(0, \mu)}{I(0, 1)}\right\} = HJ(\mu), \tag{1-11}$$

where $H \equiv I(0, 1)$ denotes the intensity of radiation emerging normally to the surface; and $J(\mu)$ stands for the normalized law of darkening, varying from 1 at the centre ($\mu = 1$) to 0·3439 . . . at the limb. A four-digit tabulation of $J(\mu)$ for 11 discrete values of $\cos\gamma \equiv \mu$ is listed in column (2) of the accompanying Table 4-1. These data reveal that a semi-transparent

* *Berlin Berichte, Math.-Phys. Klasse*, (1931), p. 696. The form of the integral solution (1-10) as given by Wiener and Hopf in this paper was somewhat more complicated than the one given above, which is due to G. Placzek (*Phys. Rev.*, **72**, 556, 1947).

atmosphere stratified in plane-parallel layers has diminished the surface brightness $J(\mu)$ at the limb of the respective stellar disk to approximately 34% of its central value—the 'limb darkening' thus turns out to be appreciable.

TABLE 4-1

Approximations to Angular Distribution of Total Light Emerging from Stellar Atmospheres

cos γ	sin γ	$J(\mu)$ (exact)	$J(\mu)$ (linear)	O-C	$J(\mu)$ (quadratic)	O-C	$J(\mu)$ (quartic)	O-C
1·00	0·0000	1·0000	1·0000	0·0000	1·0000	0·0000	1·0000	0·0000
0·95	0·3123	0·9696	0·9700	−0·0004	0·9697	−0·0001	0·9697	−0·0001
0·90	0·4359	0·9391	0·9400	−0·0009	0·9393	−0·0002	0·9393	−0·0002
0·85	0·5268	0·9085	0·9100	−0·0015	0·9088	−0·0003	0·9088	−0·0003
0·80	0·6000	0·8779	0·8800	−0·0021	0·8781	−0·0002	0·8783	−0·0004
0·75	0·6614	0·8472	0·8500	−0·0028	0·8474	−0·0002	0·8477	−0·0005
0·70	0·7142	0·8164	0·8200	−0·0036	0·8165	−0·0001	0·8168	−0·0004
0·65	0·7599	0·7855	0·7900	−0·0045	0·7855	0·0000	0·7858	−0·0003
0·60	0·8000	0·7546	0·7600	−0·0054	0·7545	0·0001	0·7548	−0·0002
0·55	0·8352	0·7235	0·7300	−0·0065	0·7234	0·0001	0·7236	−0·0001
0·50	0·8660	0·6922	0·7000	−0·0078	0·6920	0·0002	0·6921	0·0001
0·45	0·8930	0·6608	0·6700	−0·0092	0·6607	0·0001	0·6605	0·0003
0·40	0·9165	0·6291	0·6400	−0·0109	0·6291	0·0000	0·6286	0·0005
0·35	0·9368	0·5972	0·6100	−0·0128	0·5973	−0·0001	0·5965	0·0007
0·30	0·9540	0·5649	0·5800	−0·0151	0·5656	−0·0007	0·5641	0·0008
0·25	0·9682	0·5321	0·5500	−0·0179	0·5337	−0·0016	0·5314	0·0007
0·20	0·9798	0·4988	0·5200	−0·0212	0·5017	−0·0029	0·4984	0·0004
0·15	0·9887	0·4646	0·4900	−0·0254	0·4696	−0·0050	0·4650	−0·0004
0·10	0·9950	0·4290	0·4600	−0·0310	0·4374	−0·0084	0·4312	−0·0022
0·05	0·99875	0·3909	0·4300	−0·0391	0·4050	−0·0141	0·3969	−0·0060
0·00	1·00000	0·3439	0·4000	−0·0561	0·3726	−0·0287	0·3622	−0·0183

In photometric investigations which will form the main subject of this chapter we cannot, of course, employ (1-10) directly to express this limb-darkening, but have to approximate it by more explicit expressions of semi-empirical nature. Thus if we wish to limit ourselves to an approximate 'law of limb-darkening' which is *linear* in μ, we obtain

$$J(\mu) = 0{\cdot}4 + 0{\cdot}6 \cos \gamma\,, \tag{1-12}$$

in accordance with Milne;* while if we expand (numerically) the integral on the right-hand side of (1-10) in a Taylor series around $\mu = 1$ then, to a *quadratic* approximation,

$$J(\mu) = 0{\cdot}3726 + 0{\cdot}6500 \cos \gamma - 0{\cdot}0226 \cos^2 \gamma\,. \tag{1-13}$$

These linear and quadratic approximate laws of darkening have been tabulated in columns (3) and (4) of Table 4-1, together with their O-C deviations from the exact four-digit solution of column (2). A glance at the respective residuals reveals that whereas Milne's linear approximation (1-12) errs

* E. A. Milne, *M.N.*, **81**, 361, 1921.

systematically by defect attaining almost 0·06 at the limb, the error of the quadratic approximation (1-13) becomes oscillatory and exceeds nowhere 0·02. This error can, moreover, be further diminished by the resort to a *quartic* approximation of the form

$$J(\mu) = 0{\cdot}3622 + 0{\cdot}6998 \cos\gamma - 0{\cdot}1053 \cos^2\gamma + 0{\cdot}0578 \cos^3\gamma - 0{\cdot}0145 \cos^4\gamma, \tag{1-14}$$

tabulated likewise in column (5) with its residuals which now amount to less than 0·02 even at the limb. A closer idea of the degree of approximation

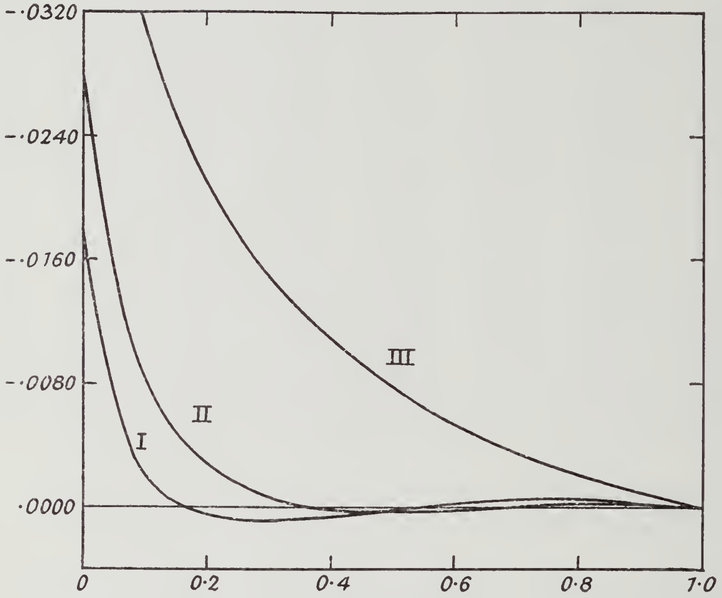

FIGURE 4-1

THE O-C DEVIATIONS OF THE APPROXIMATE LAWS OF DARKENING IN PLANE-PARALLEL ATMOSPHERES FROM THE EXACT LAW REPRESENTED BY EQUATION (1-10), PLOTTED AGAINST COS $\gamma \equiv \mu$.

The curves I, II, III indicate the deviations of the linear approximation (1–12), the quadratic approximation (1–13), and the quartic approximation (1–14) from the exact solution.

attained by equations (1-12), (1-13), and (1-14) can be obtained from the accompanying Fig. 4-1, on which the (O-C) residuals of the respective linear, quadratic, and quartic approximations to the exact law of darkening have been plotted against μ.

All results reviewed so far are relevant to the radiative transfer and angular distribution of the *total* flow of radiant energy (i.e., of light integrated over *all* frequencies) emerging from stellar atmospheres. In order to ascertain the distribution of brightness observable in the light of any effective wavelength λ, recourse must be had to the assumption of *local thermodynamic equilibrium.* For, as is well known, if the material of the atmosphere is in such a state of equilibrium, its source function should be uniquely specified by the Planck law

$$B_\lambda(\tau) = \frac{2hc/\lambda^3}{e^{hc/\lambda kT} - 1}, \tag{1-15}$$

where the quantities c, h, k denote the velocity of light, the Planck constant, and the Boltzmann constant, respectively; and the temperature T should vary with the optical depth τ in accordance with the Stefan's law

$$\sigma T^4 = \pi B(\tau); \tag{1-16}$$

σ being the Stefan–Boltzmann constant.

In actual stellar atmospheres the absorption coefficient κ in equation (1-4) will, moreover, be a certain function of λ and T to be determined from the atomic theory of absorption processes in ionized gas. In order to obviate at least a part of the complications arising from this source, let us introduce the concept of a mean absorption coefficient $\bar{\kappa}$, as defined by

$$\bar{\kappa} = \frac{1}{F}\int_0^\infty \kappa_\lambda F_\lambda^{(1)}\, d\lambda, \tag{1-17}$$

where κ_λ denotes the actual (frequency-dependent) absorption coefficient and $F_\lambda^{(1)}$, the monochromatic flux in a grey atmosphere (F being, as before, the net integrated flux). Next, let us express the optical depth τ as defined by equation (1-4) in terms of the mean absorption coefficient $\bar{\kappa}$ defined by (1-17). If, moreover, the ratio $\kappa_\lambda/\bar{\kappa}$ is assumed to be independent of τ, the relation (1-3)—regarded as a linear differential equation for $I(\tau, \mu)$ of first order—is then readily solved for the intensity $I_\lambda(0, \mu)$ of emergent radiation in the form of the particular integral

$$I_\lambda(0, \mu) = \int_0^\infty B_\lambda(\tau)\, e^{-(\kappa_\lambda/\bar{\kappa})\tau \sec\gamma}\, (\kappa_\lambda/\bar{\kappa}) \sec\gamma\, d\tau, \tag{1-18}$$

obeying the requisite boundary conditions, where $B_\lambda(\tau)$ behind the integral sign can be represented by the Planck function (1-15). The temperature T at any particular optical depth τ then follows, by virtue of (1-16), from the equation

$$T^4 = \tfrac{3}{4}T_e^4\{\tau + q(\tau)\}, \tag{1-19}$$

where T_e stands for the effective temperature of the respective star, and the auxiliary function $q(\tau)$, obtained from the transfer of integrated light, to a good approximation becomes*

$$\begin{aligned} q(\tau) = 0{\cdot}7069 &- 0{\cdot}0839\, e^{-4{\cdot}4581\tau} \\ &- 0{\cdot}0362\, e^{-1{\cdot}5918\tau} \\ &- 0{\cdot}0095\, e^{-1{\cdot}1032\tau} - \dots . \end{aligned} \tag{1-20}$$

* *Cf.* S. Chandrasekhar, *Ap. J.*, **100**, 76, 1944.

In order to evaluate the integral on the right-hand side of equation (1-18) analytically rather than by quadratures* it is necessary to express $B_\lambda(\tau)$ explicitly as a function of τ; and in order to do so we wish to expand the Planck function in a Taylor series in ascending powers of around any point at which this function remains analytic.† For reasons of convenience we choose to do so around a point $\tau = \tau_e$, at which the actual temperature T in the atmosphere is equal to the effective temperature T_e of the star as a whole. If we insert so resulting an expansion of the form

$$B_\lambda(T) = B_\lambda(T_e)\left\{1 + \sum_{j=1}^{\infty} \left(\frac{1}{B}\frac{d^jB}{d\tau^j}\right)_{\tau_e} \frac{(\tau - \tau_e)^j}{j!}\right\} \tag{1-21}$$

in (1-18) and make use of the fact that

$$\int_0^\infty e^{-x\tau}\, \tau^n\, d(x\tau) = \frac{n!}{x^n}, \tag{1-22}$$

equation (1-18) leads to

$$I_\lambda(0, \mu) = B_\lambda(T_e)\{A_0 + A_1\mu + A_2\mu^2 + A_3\mu^3 + \dots\}, \tag{1-23}$$

where

$$\left.\begin{aligned} A_0 &= 1 - Q_1\tau_e + \frac{1}{2!}\, Q_2\tau_e^2 - \frac{1}{3!}\, Q_3\tau_e^3, \\ A_1 &= (\bar{\kappa}/\kappa_\lambda)\left(Q_1 - Q_2\tau_e + \frac{1}{2!}\, Q_3\tau_e^2\right) \\ A_2 &= (\bar{\kappa}/\kappa_\lambda)^2\,(Q_2 - Q_3\tau_e), \\ A_3 &= (\bar{\kappa}/\kappa_\lambda)^3\, Q_3\,, \end{aligned}\right\} \tag{1-24}$$

etc., where we have abbreviated

$$Q_j \equiv \left(\frac{1}{B}\frac{d^jB}{d\tau^j}\right)_{\tau_e}. \tag{1-25}$$

The corresponding 'law of darkening' may then be expressed in the form

$$\begin{aligned} I(0, \mu) = H(1 - u_1 - u_2 - u_3 - \dots \\ + u_1 \cos\gamma + u_2\cos^2\gamma + u_3\cos^3\gamma + \dots), \end{aligned} \tag{1-26}$$

where $H \equiv I(0, 1)$ denotes the intensity of radiation emerging in the direction

* This integral has been repeatedly tabulated by means of numerical quadratures in terms of the arguments $\alpha = hc/\lambda kT_e$ and $\beta = (\kappa_\lambda/\bar{\kappa}) \sec\gamma$. Following the earlier work by Lindblad (Uppsala Univ. Årsskrift, **1**, 33, 1920) and Milne (*Phil. Trans. Roy. Soc.*, **223** A, 247, 1922) the most complete existing tables of $I_\lambda(\alpha, \beta)$ for $\alpha = 0(1)\,12$ and $\beta = 0(0, 1)\,2{\cdot}0$ have been prepared by Chandrasekhar and Breen, and published as Table XXIX on p. 306 of Chandrasekhar's book on *Radiative Transfer*, Oxford 1950.

† This requirement rules out, in particular, the possibility of an expansion around the outer boundary $\tau = 0$, as $B(\tau)$ is only piecewise-continuous at that point and its higher derivatives become infinite.

of the line of sight (about which we shall have more to say later in this section) and the *coefficients of limb-darkening* u_j are given by

$$u_j = \frac{A_j}{A_0 + A_1 + A_2 + \ldots + A_n}. \tag{1-27}$$

In order to evaluate these latter coefficients explicitly as functions of the temperature T, wave-length λ, and the ratio $\bar{\kappa}/\kappa_\lambda$, let us depart from the relations

$$\left.\begin{aligned} \frac{dB}{d\tau} &= \frac{dB}{dT}\frac{dT}{d\tau}, \\ \frac{d^2B}{d\tau^2} &= \frac{dB}{dT}\frac{d^2T}{d\tau^2} + \frac{d^2B}{dT^2}\left(\frac{dT}{d\tau}\right)^2, \\ \frac{d^3B}{d\tau^3} &= \frac{dB}{dT}\frac{d^3T}{d\tau^3} + \frac{d^2B}{dT^2}\left(3\frac{dT}{d\tau}\frac{d^2T}{d\tau^2}\right) + \frac{d^3B}{dT^3}\left(\frac{dT}{d\tau}\right)^3, \end{aligned}\right\} \tag{1-28}$$

etc., where it follows from (1-19) that

$$\begin{aligned} \frac{16}{3}\left(\frac{1}{T}\frac{dT}{d\tau}\right)_{\tau_e} &= q'(\tau_e) + 1, \\ \frac{16}{3T_e}\left(\frac{d^2T}{d\tau^2}\right)_{\tau_e} &= q''(\tau_e) - \tfrac{9}{16}\{1 + q'(\tau_e)\}^2, \\ \frac{16}{3T_e}\left(\frac{d^3T}{d\tau^3}\right)_{\tau_e} &= q'''(\tau_e) - \tfrac{27}{16}\{1 + q'(\tau_e)\}\, q''(\tau_e) + \tfrac{7}{16}\{1 + q'(\tau_e)\}^2, \end{aligned} \tag{1-29}$$

and τ_e is defined as a root of the equation

$$\tau + q(\tau) = \tfrac{4}{3}, \tag{1-30}$$

following likewise from (1-19) if $T = T_e$, which is equal to 0·649 . . . and can, with sufficient accuracy, be approximated by 0·65. Moreover, from equation (1-15) we find that, if we abbreviate

$$a = \frac{hc}{\lambda k T_e} \tag{1-31}$$

and

$$\left(\frac{T}{B}\frac{dB}{dT}\right)_{T_e} = \frac{a}{1 - e^{-a}} = b, \tag{1-32}$$

the higher requisite derivatives of the Planck function at $T = T_e$ assume the forms

$$\left.\begin{aligned} \left\{\frac{T^2}{B}\frac{d^2B}{dT^2}\right\}_{T_e} &= b\{2b - 2 - a\}, \\ \left\{\frac{T^3}{B}\frac{d^3B}{dT^3}\right\}_{T_e} &= b\{a^2 + 6(b^2 - ab + a - 2b + 1)\}, \end{aligned}\right\} \tag{1-33}$$

etc. With the aid of the foregoing relations (which can be easily continued) we are now in a position to express the coefficients Q_j, A_j, and thus the limb-darkening coefficients u_j, in terms of the parameters T_e and $(\kappa_\lambda/\bar{\kappa})$ characterizing each star in the effective wave-length of observation. Conversely, if the values of u_j could be deduced from the observations (by methods to be developed later in sections VI.7 and 8 of this book) of an eclipse of a star of known effective temperature, our foregoing theory should enable us to deduce from them the corresponding monochromatic values of the ratio $\kappa_\lambda/\bar{\kappa}$.

The expansions on the right-hand side of (1-23) or of the corresponding law of darkening (1-26) in ascending powers of μ converge, in general, but slowly and their use becomes thus rather unwieldy in detailed photometric analysis of the phenomena exhibited by close binary systems. The question, therefore, arises as to whether or not it is possible to *telescope** these expansions into other series with *reduced* number of terms, at the risk of an error which can be suitably bounded. To this end consider a Tchebycheff polynomial

$$T_3(\mu) = 32\mu^3 - 48\mu^2 + 18\mu - 1, \tag{1-34}$$

which is orthogonal with respect to a weight function $(1 - \mu^2)^{-1/2}$ in the interval (0, 1), and such that its value for $0 \leqslant \mu \leqslant 1$ does not exceed ± 1. If, therefore, we solve (1-34) for μ^3 and ignore the term $T_3(\mu)/32$ as small, we find that, within the interval of orthogonality,

$$\mu^3 = \tfrac{1}{32}(48\mu^2 - 18\mu + 1), \tag{1-35}$$

subject to an error which nowhere exceeds $\pm 0{\cdot}03125$.

The foregoing equation (1-35) expresses indeed a rather remarkable approximation which may call for a few words of comment. The cube of μ is, in general, algebraically independent of all lower powers and not expressible as their combination. But *in the range* (0, 1) of our immediate interest this independence becomes quite weak and, as a result, μ^3 is *almost* equal to a definite linear combination (1-35) of lower powers of μ, subject to a given maximum error. Moreover, the known properties of Tchebycheff polynomials reveal that the quality of the resulting approximation (1-35) to μ^3 in the interval (0, 1) cannot be improved any further with the given number of terms. However, by eliminating μ^3 from the right side of equation (1-23) by means of (1-35) we can replace (1-23) by a closely approximate equivalent expression

$$I_\lambda(0, \mu) = B_\lambda(T_e)\{B_0 + B_1\mu + B_2\mu^2\}, \tag{1-36}$$

where

$$\left.\begin{aligned} B_0 &= A_0 + \tfrac{1}{32}A_3\,, \\ B_1 &= A_1 - \tfrac{9}{16}A_3\,, \\ B_2 &= A_2 + \tfrac{3}{2}A_3\,; \end{aligned}\right\} \tag{1-37}$$

* *Cf.* C. Lanczos, *Journ. Math. Phys.*, **17**, 123, 1938; or his *Applied Analysis*, Prentice-Hall, New York 1956, Chapter VII.

and the coefficients $u_{1,2}$ in the law of limb-darkening (1-26) truncated after the first two terms will assume the form

$$u_{1,2} = \frac{B_{1,2}}{B_0 + B_1 + B_2} \tag{1-38}$$

as a particular case of (1-27).

The same procedure can, moreover, be carried out one step further by solving for the *square* of μ from the Tchebycheff polynomial

$$T_2(\mu) = 8\mu^2 - 8\mu + 1 \tag{1-39}$$

equated to zero, thus obtaining an approximation

$$\mu^2 = \tfrac{1}{8}(8\mu - 1) \tag{1-40}$$

subject to a maximum error of $\pm 0{\cdot}125$ in the interval (0, 1). By virtue of (1-40), the approximate equation (1-35) can be rewritten as

$$\mu^3 \cong \tfrac{5}{32}(6\mu - 1), \tag{1-41}$$

and thus enable us to telescope further the expansion on the right-hand side of (1-23) into a binomial

$$I_\lambda(0, \mu) = B_\lambda(T_e)\{C_0 + C_1\mu\}, \tag{1-42}$$

where

$$\left.\begin{aligned} C_0 &= A_0 - \tfrac{1}{8}A_2 - \tfrac{5}{32}A_3\,, \\ C_1 &= A_1 + A_2 + \tfrac{15}{16}A_3\,. \end{aligned}\right\} \tag{1-43}$$

The corresponding *linear* law of limb-darkening

$$J(\mu) = (1 - u + u\cos\gamma) \tag{1-44}$$

would then be characterized by the coefficient

$$u = \frac{C_1}{C_0 + C_1} = \frac{32(A_1 + A_2) + 30A_3}{32(A_0 + A_1) + 28A_2 + 25A_3}\,; \tag{1-45}$$

and it is *this* coefficient we deal with whenever the circumstances lead us to *force* a linear law of darkening through the observed photometric data in the course of this interpretation.

The coefficient u of the linearized law of limb-darkening is defined by the foregoing equation as a function of the parameters $a = hc/\lambda kT$ and $\kappa_\lambda/\bar{\kappa}$, and can be evaluated for any star for which the appropriate values of these parameters are known. The ratios $\kappa_\lambda/\bar{\kappa}$ of the absorption coefficients for main-sequence stars of different spectral types have been investigated by Chandrasekhar and Münch,* who have shown that the principal features of distribution of intensity in the continuous spectra of such stars with types between A0 and G2 can be accounted for in terms of the continuous absorption of neutral hydrogen atoms and of negative hydrogen ions; and in the same

* *Ap. J.*, **104**, 446, 1946.

paper its authors have also given the mean electron pressures in the atmospheres of such stars which will explain the observed discontinuities at the head of their Balmer and Paschen series. With these electron pressures and the corresponding values of $\kappa_\lambda/\bar{\kappa}$ we have evaluated the theoretical coefficients u of the *linearized* law of darkening, as given by equation (1-45), for various wave-lengths and spectral types and reproduced them in the accompanying Table 4-2.* The reader may gather for himself the extent of errors committed

TABLE 4-2
Monochromatic Coefficients u_1 *of the Linearized Law of Limb-darkening for Main-sequence Stars of Different Spectral Types*

Spectrum	Effective Wave-length							
	3647^-	3647^+	4000	4600	5300	6500	8203^-	8203^+
B0	0·65	0·65	0·65	0·65	0·65	0·65	0·65	0·65
B5	0·50	0·73	0·70	0·65	0·62	0·55	0·49	0·50
A0	0·38	0·79	0·75	0·67	0·60	0·49	0·37	0·39
A2	0·34	0·80	0·77	0·68	0·61	0·48	0·34	0·38
A5	0·41	0·82	0·79	0·70	0·62	0·50	0·35	0·38
A7	0·48	0·83	0·80	0·71	0·64	0·51	0·37	0·40
F0	0·58	0·85	0·82	0·73	0·65	0·53	0·39	0·43
F2	0·64	0·86	0·83	0·74	0·66	0·54	0·41	0·45
F4	0·71	0·87	0·84	0·76	0·68	0·55	0·42	0·47
F6	0·78	0·89	0·84	0·77	0·69	0·57	0·44	0·49
F8	0·84	0·90	0·85	0·78	0·70	0·58	0·46	0·51
G0	0·89	0·92	0·86	0·80	0·72	0·59	0·47	0·53
G2	0·92	0·93	0·87	0·81	0·74	0·61	0·49	0·55
G5	0·94	0·94	0·88	0·83	0·76	0·64	0·52	0·57
K0	0·96	0·95	0·91	0·86	0·79	0·67	0·56	0·62

by a linear theory of limb-darkening by a comparison of the results of our Table 4-2 (based on our third-order theory 'telescoped' into a linearized form) with those of Münch and Chandrasekhar† who employed a linear process (equivalent to the neglect of our coefficients A_2 and A_3) reducing to

$$u = \left\{1 + \frac{2}{3}\left(\frac{8}{b} - 1\right)\frac{\kappa_\lambda}{\bar{\kappa}}\right\}^{-1}. \tag{1-46}$$

The expansion (1-23) of $I_\lambda(0, \mu)$ in ascending powers of μ may also prove a source of weakness whenever the ratios $\kappa_\lambda/\bar{\kappa}$ become small (less than 0·5); for under such circumstances the contribution to emergent intensity $I(0, \mu)$ from great depth may become significant, and the variation of $\kappa_\lambda/\bar{\kappa}$ with τ may also become appreciable. From all this we conclude that, for stars of spectral types not later than F2, our values of u in the ultra-violet ($\lambda < 3747$ Å) and again in the near infrared (6500 Å $< \lambda <$ 8203 Å) where the absorption

* The values as given for early B-stars were extrapolated on the assumption that, for stars of class B0 and earlier spectral types, the atmospheric opacity becomes dominated by electron scattering.

† *Harv. Circ.*, No. 453, 1949.

by negative hydrogen ion becomes a maximum should provide a satisfactory approximation to reality. In the intermediate region (3747 Å $< \lambda <$ 6500 Å) our present theoretical values of u should still be viewed with some reserve. In comparing them with the observations we should bear in mind that the observed u's must also be affected by the 'blanketing effect' of line absorption spectrum* which, in general, tends to *lessen* the limb-darkening. A depletion of the continuous spectrum arising from this source should be serious in stars of earlier spectral types (B8–A2) above their Balmer limits; and in spectra later than F2 the effect should be pronounced throughout the whole blue and violet regions. In particular, the high computed values of u as given in Table 4-2 for λ 3547+ are unlikely to be observable on account of this phenomenon.

So far we have not considered the O- and early B-type stars, in whose atmospheres the principal source of opacity is not atomic absorption, but *electron scattering*; and a transfer of light scattered on free electrons constitutes a problem which remains yet to be considered. The fundamental equation (1-3) of transfer continues to hold good for the general case of a scattering atmosphere in radiative equilibrium as well, but the source-function B will be given by

$$B(\tau, \theta, \phi) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} I(\tau, \theta', \phi')\, p(\cos \Theta) \sin \theta'\, d\theta'\, d\phi' \tag{1-47}$$

in place of (1-7), where $p(\cos \Theta)$ denotes the phase-indicatrix of the respective scattering process as a function of the angle Θ between the incident and scattered rays, given by the equation

$$\cos \Theta = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \tag{1-48}$$

if θ', ϕ' and θ, ϕ describe the direction of the incident and scattered rays, respectively.

If the scattering were *isotropic*, and its indicatrix normalized so that $p(\cos \Theta) \equiv 1$, the respective transfer problem would become identical with that of light which is absorbed and re-emitted, and whose solution has already been given before. In the atmospheres of early-type stars, however, by far the most important scattering process is that caused by free electrons (i.e., the *Thomson scattering*) which is known to be *frequency-independent and to* obey the phase function

$$p(\cos \Theta) = \tfrac{3}{4}(1 + \cos^2 \Theta). \tag{1-49}$$

An insertion of (1-48) in (1-49) renders the latter an explicit function of our angular variables which, when substituted in (1-47), furnishes the respective source-function. Inserting it in (1-3) we find that the integro-differential transfer equation of radiation scattered in accordance with the indicatrix

* For its discussion *cf.*, e.g., E. A. Milne, *Phil. Trans. Roy. Soc.*, **223**A, 201, 1922.

(1-49) assumes the explicit form

$$\mu \frac{dI_s(\tau, \mu)}{d\tau} = I_s(\tau, \mu) - \frac{3}{16}\left\{(3 - \mu^2) \int_{-1}^{1} I_s(\tau, \mu')\, d\mu' + (3\mu^2 - 1) \int_{-1}^{1} I_s(\tau, \mu')\, d\mu'\right\}, \tag{1-50}$$

where we have abbreviated $\mu \equiv \cos\theta$, $\mu' \equiv \cos\theta'$; and if we identify the z-axis of our frame of reference with the line of sight, $\theta \equiv \gamma$ as used before. The boundary conditions of our problem, requiring that

$$F = 2 \int_{-1}^{1} I_s(\tau, \mu)\mu\, d\mu = \text{constant} \tag{1-51}$$

and (if no radiation is incident from outside)

$$I_s(0, \mu) = 0 \qquad \text{for} \qquad \mu < 0 \tag{1-52}$$

remain formally unaltered.

As was true of the absorption-re-emission case, the scattering transfer problem thus formulated does not admit of a closed analytic solution; but the emergent intensity $I_s(0, \mu)$ of scattered light can be expressed in the form

$$I_s(0, \mu) = \tfrac{3}{4} q F H_s(\mu), \tag{1-53}$$

analogous to (1-10), where the new function $H_s(\mu)$ satisfies an integral equation*

$$H(\mu) = 1 + \mu H(\mu) \int_{0}^{1} \frac{H(\mu')\Psi(\mu')}{\mu + \mu'}\, d\mu', \tag{1-54}$$

with

$$\Psi_s(\mu) = \tfrac{3}{16}(3 - \mu^2), \tag{1-55}$$

and where

$$q = \frac{2\sqrt{3}}{H(\sqrt{3}) - H(-\sqrt{3})} \tag{1-56}$$

is a constant.

The integral equation (1-54) can again be solved only by numerical means; and in doing so Chandrasekhar and Breen† were led to a law of darkening

* It is of interest to note that the same equation with $\Psi(\mu) = \frac{1}{2}$, is satisfied also by the emergent intensity $(2/\sqrt{3})I(0, \mu)$ of absorbed-re-emitted light, as defined by the Wiener-Hopf integral (1-10) given earlier in this section. A detail proof of this fact has been published by Busbridge in sec. 29 of Kourganoff's *Basic Methods in Transfer Problems*, Oxford Univ. Press, 1952.

† *Ap. J.*, **105**, 435, 1947. *Cf.* also Table XV on p. 135 of Chandrasekhar's *Radiative Transfer*, Oxford Univ. Press 1950.

$J(\mu) \equiv I_s(0, \mu)/I_s(0, 1)$ which is reproduced in column (2) of the accompanying Table 4-3. A comparison of the respective entries in columns (2) of Tables 4-1 and 4-3 reveals that the two laws of limb-darkening for light absorbed or scattered differ mostly in the third decimal place, and even at the limb ($\mu = 0$) their difference barely exceeds 0·01. Therefore, *the angular distribution of light scattered on free electrons turns out to be sensibly the same as that of light transferred by absorption-re-emission*; and, in particular, the same polynomial approximations to the law of darkening $J(\mu)$ in total radiation, deduced earlier in this section, should satisfactorily approximate the limb-darkening of scattered light as well.

TABLE 4-3

Angular Distribution of Scattered Light

μ	$J_s(\mu)$	$J_l(\mu)$	$J_r(\mu)$	δ
1·00	1·0000	1·0000	1·0000	0·0000
0·95	0·9691	0·9671	0·9700	0·0015
0·90	0·9381	0·9341	0·9400	0·0032
0·85	0·9070	0·9011	0·9100	0·0049
0·80	0·8759	0·8679	0·8799	0·0068
0·75	0·8447	0·8348	0·8497	0·0089
0·70	0·8135	0·8015	0·8195	0·0111
0·65	0·7821	0·7681	0·7892	0·0136
0·60	0·7506	0·7346	0·7589	0·0163
0·55	0·7189	0·7010	0·7284	0·0192
0·50	0·6879	0·6671	0·6979	0·0225
0·45	0·6551	0·6331	0·6672	0·0262
0·40	0·6229	0·5989	0·6363	0·0303
0·35	0·5904	0·5643	0·6053	0·0350
0·30	0·5575	0·5294	0·5740	0·0404
0·25	0·5242	0·4939	0·5423	0·0467
0·20	0·4901	0·4578	0·5102	0·0541
0·15	0·4552	0·4207	0·4774	0·0631
0·10	0·4188	0·3820	0·4435	0·0745
0·05	0·3798	0·3405	0·4077	0·0898
0·00	0·3312	0·2883	0·3647	0·1171

The transfer problem whose solution we have just outlined falls, however, short of describing the physical situation we have in mind adequately in one respect: namely, it fails to take account of the fact that the light scattered by free electrons is bound to be *plane-polarized*, with a ratio of intensities $\cos^2 \Theta : 1$ in the directions parallel with, and perpendicular to, a plane containing the directions of the incident and scattered light. Therefore, *diffuse radiation emerging from the atmosphere of any star in which electron scattering plays an appreciable part should be partially polarized*; and the intensities $I_l(\tau, \mu)$ and $I_r(\tau, \mu)$ referring to the state of polarization in which the electric vector vibrates in a plane parallel with, and perpendicular to, the line of sight should be governed by distinct transfer equations. These equations are found to be*

$$\mu \frac{d}{d\tau}\begin{pmatrix} I_l \\ I_r \end{pmatrix} = \begin{pmatrix} I_l \\ I_r \end{pmatrix} - \frac{3}{8}\int_{-1}^{1}\begin{pmatrix} 2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{pmatrix}\begin{pmatrix} I_l \\ I_r \end{pmatrix} d\mu', \tag{1-57}$$

subject to the boundary conditions

$$F = 2\int_{-1}^{1}\{I_l(\tau,\mu)+I_r(\tau,\mu)\}\mu\, d\mu = \text{constant} \tag{1-58}$$

and

$$I_l(0,\mu) = I_r(0,\mu) = 0 \qquad \text{for} \qquad 0 \geqslant \mu \geqslant -1. \tag{1-59}$$

The solution of these equations for the emergent intensities can be expressed in the form

$$\left.\begin{aligned} I_l(0,\mu) &= \tfrac{3}{8}F\sqrt{1-c^2}H_l(\mu), \\ I_r(0,\mu) &= \frac{3}{8}\frac{F}{\sqrt{2}}H_r(\mu)(\mu+c), \end{aligned}\right\} \tag{1-60}$$

where the auxiliary functions H_l and H_r satisfy again the integral equation (1-54) with

$$\Psi_l(\mu) = \tfrac{3}{4}(1-\mu^2) = 2\Psi_r(\mu) \tag{1-61}$$

and the constant c is given by

$$c = \frac{H_l(1)H_r(-1) + H_l(-1)H_r(1)}{H_l(1)H_r(-1) - H_l(-1)H_r(1)}. \tag{1-62}$$

The total intensity of emergent light is, moreover, given by

$$I_s(0,\mu) = I_l(0,\mu) + I_r(0,\mu). \tag{1-63}$$

The functions $H_l(\mu)$ and $H_r(\mu)$ have been evaluated numerically by Breen and Chandrasekhar,† and the corresponding laws of limb-darkening

$$J_l(\mu) = \frac{I_l(0,\mu)}{I_l(0,1)} \qquad \text{and} \qquad J_r(\mu) = \frac{I_r(0,\mu)}{I_r(0,1)}, \tag{1-64}$$

based upon them can be found in columns (3) and (4) of Table 4-3, together with the degree of polarization

$$\delta = \frac{I_r - I_l}{I_r + I_l} \tag{1-65}$$

given in column (5). According to these results, the emergent intensities of light of the two states of polarization, equal at the centre ($\mu = 1$), should differ by about 25% at the limb ($\mu = 0$); the degree of polarization

* *Cf.* S. Chandrasekhar, *Radiative Transfer* (Oxford 1950), sec. 17.3.
† F. Breen and S. Chandrasekhar, *Ap. J.*, **105**, 435, 1947.

δ increasing between the centre and limb from 0 to 11·7%. In the atmospheres of the early B- and O-stars, in which electron scattering may play a predominant role in radiation transfer, the variation of δ between centre and limb could, in principle, be detected from polarimetric measurements of the light changes exhibited by close binary systems whenever an early-type component of such systems undergoes eclipse by a late-type companion.

If a star were isolated in space, spherical in form and surrounded by a thin atmosphere, a description of the distribution of brightness on its apparent disk as seen by a distant observer would now be complete. If, however, the star under consideration is to be the component of a close binary system, two additional effects—both caused by the proximity of a companion and influencing apparent surface brightness—must yet be taken into account: namely, the radiation incident on the star from its companion and re-radiated (or scattered) in the direction of the line of sight (i.e., the *reflection effect*), and the variation of the star's internal energy flux passing through a surface distorted by rotational and (or) tidal forces (i.e., the *gravity effect*). As diffuse reflection of radiation incident from above occurs regardless of distortion and can, moreover, be treated by a simple extension of methods already employed to investigate the transfer of star's own light passing through the atmosphere from below, we propose to take it up next.

In doing so, we shall consider again the radius of the reflecting star to be so large in comparison with the extent of its atmosphere that all curvature effects can be ignored, and the atmosphere regarded as stratified in plane-parallel layers (irrespective of distortion). If, moreover, the illuminating component is sufficiently far removed, its radiation can be considered as consisting of a parallel beam, of flux πS per unit area normal to itself, falling on a plane surface of matter, in radiative equilibrium, in a direction making an angle α with the surface normal. The flux incident on the boundary per unit area is then $\pi S \cos \alpha$. If the actual mechanism of 'reflection' is absorption followed by re-emission, this beam will be attenuated exponentially as it penetrates the material; and at an optical depth τ, the flux contained in the beam will be reduced to $\pi S \exp(-\tau \sec \alpha)$. Let, further, $B^*(\tau)$ denote source function of the incident radiation (as superposed upon ordinary radiation). The fundamental equation of transfer for the intensity $I^*(\tau, \mu)$ of reflected radiation is then of the same form as (1-3): namely,

$$\mu \frac{dI^*}{d\tau} = I^* - B^*. \tag{1-66}$$

The absorption of the incident beam at a depth τ is

$$\pi S e^{-\tau \sec \alpha} \cos \alpha \sec \alpha \, d\tau = \pi S e^{-\tau \sec \alpha} \, d\tau. \tag{1-67}$$

If, moreover, this incident radiation is not to influence the equilibrium of the reflecting star as a whole, *all incident light must be wholly re-radiated* (i.e., the heat-albedo of a star in radiative equilibrium must necessarily be unity).

If so, however,

$$\int I^*(\tau, \mu)\, d\omega + \pi S e^{-\tau \sec \alpha} = 4B^*(\tau); \tag{1-68}$$

the integral on the left-hand side being extended over the entire sphere. Inserting (1-68) in (1-66) we find that the integro-differential equation governing the transfer of reflected radiation in plane-parallel atmospheres assumes the explicit form

$$\mu \frac{dI^*}{d\tau} = I^* - \tfrac{1}{2}\int_{-1}^{1} I^*\, d\mu - \tfrac{1}{4} S e^{-\tau/\mu_0}, \tag{1-69}$$

where we have abbreviated $\mu_0 \equiv \cos \alpha$; and its solution we seek should be constrained so that

$$2\int_{-1}^{1} I^*(0, \mu, \mu_0)\mu\, d\mu = S\mu_0 \tag{1-70}$$

and

$$I^*(0, \mu, \mu_0) = 0 \qquad \text{for} \qquad 0 > \mu \geqslant -1. \tag{1-71}$$

A glance at the equation (1-69) reveals that it differs from (1-8) governing radiative transfer of ordinary light re-emitted isotropically only by the presence of a term, factored by S, on the right-hand side which renders it non-homogeneous. Its linear character asserts, moreover, that the intensities $I(\tau, \mu)$ and $I^*(\tau, \mu, \mu_0)$ of ordinary and reflected light are simply additive—the net observable intensity being $I + I^*$. As has, furthermore, been shown first by Hopf,* the *emergent* intensity $I^*(0, \mu, \mu_0)$ of reflected radiation for positive values of μ can be expressed *exactly* in terms of a solution of the respective associated homogeneous equation (1-8). For, if the emergent intensity of ordinary radiation studied earlier in this section is given by

$$I(0, \mu) = \frac{\sqrt{3}}{4} FH(\mu), \tag{1-72}$$

where πF is the net flux of ordinary light, the intensity of reflected radiation is bound to be specified by

$$I^*(0, \mu, \mu_0) = \frac{\sqrt{3}}{4} S \frac{\mu_0}{\mu + \mu_0} H(\mu)H(\mu_0). \tag{1-73}$$

Should, in particular, the direction of the line of sight coincide with that of incident radiation (i.e., if $\mu = \mu_0$), the foregoing equation (1-73) would reduce to

$$I^*(0, \mu_0) = \frac{\sqrt{3}}{8} S\{H(\mu_0)\}^2. \tag{1-74}$$

This represents the angular distribution of reflected light over the apparent

* *Cf.* E. Hopf, *Mathematical Problems of Radiative Equilibrium*, (Cambridge Tract in Mathematics and Math. Physics, No. 31, Cambridge 1934), sections 17-18.

disk of an illuminated star at 'full' phase; and the corresponding 'law of darkening'

$$J^*(\mu_0) = \left\{\frac{H(\mu_0)}{H(1)}\right\}^2 \tag{1-75}$$

is tabulated in column (2) of the accompanying Table 4-4 as a function of μ_0. A glance at this tabulation or a comparison of the equations (1-72) and (1-73) readily discloses that $J^*(\mu_0)$ is bound to fall off from the centre to the limb very much more rapidly than the function $J(\mu)$ does for ordinary

TABLE 4-4

Angular Distribution of Light Reflected at 'Full' Phase

μ_0	$J^*(\mu_0)$	$J_s^*(\mu_0)$
1·00	1·0000	1·0000
0·95	0·9401	0·9140
0·90	0·8818	0·8375
0·85	0·8254	0·7695
0·80	0·7707	0·7092
0·75	0·7178	0·6553
0·70	0·6666	0·6073
0·65	0·6171	0·5642
0·60	0·5694	0·5253
0·55	0·5234	0·4898
0·50	0·4792	0·4572
0·45	0·4366	0·4269
0·40	0·3958	0·3982
0·35	0·3566	0·3708
0·30	0·3191	0·3441
0·25	0·2831	0·3177
0·20	0·2488	0·2913
0·15	0·2158	0·2644
0·10	0·1840	0·2366
0·05	0·1528	0·2070
0·00	0·1183	0·1702

radiation—i.e., that *the reflection will strongly exaggerate ordinary limb-darkening*—whatever that may be. A comparison of the values given in columns (1) and (2) of Table 4-4 discloses that, within a tolerable approximation, the function $J^*(\mu_0)$ varies linearly with μ_0 and that, in consequence, the actual distribution of reflected light can be approximated by the Lambert's law

$$J^*(\mu_0) = \mu_0 = \cos\alpha, \tag{1-76}$$

simulating a *complete* darkening at the limb, of which use will be made later on (section IV.6).

The reader may also observe that if the reflecting body were 'cold' and possessed no semi-transparent outer fringe, the function $H(\mu)$ would freeze into a constant, and equation (1-73) degenerate into another well-known law

$$I^*(0, \mu, \mu_0) = S\,\frac{\mu_0}{\mu + \mu_0}, \tag{1-77}$$

derived in connection with planetary photometry by Lommel and Seeliger* many years ago. Hopf's equation (1-73) may, therefore, be regarded as a generalization of Lommel-Seeliger's law (1-77) to the reflection by stars rather than planets.

Now, for integrated light, the function $H(\mu)$ can be evaluated to any degree of approximation by iterative solution of equation (1-54) with $\Psi(\mu') = \frac{1}{2}$ so that, by virtue of (1-73) the angular distribution of total reflected light which has been absorbed and re-emitted can likewise be regarded as known. We have, however, yet to consider the possibility that, in early-type stars, a part of the incident radiation may be 'reflected' by scattering on free electrons in their atmospheres. If so, the equation (1-66) of radiative transfer naturally continues to hold good; but the equation (1-68) of radiative equilibrium is to be replaced by

$$4\pi B^*(\tau) = \int I^* p(\cos \Theta)\, d\omega + \pi S e^{-\tau/\mu_0} p(\cos \Theta_0), \tag{1-78}$$

where the scattering indicatrix continues to be given by equation (1-49) and $\cos \Theta$ by (1-48), but

$$\cos \Theta_0 = -\mu\mu_0 + \sqrt{(1-\mu^2)(1-\mu'^2)} \cos \phi \tag{1-79}$$

As has been shown by Chandrasekhar,* the solution of this particular transfer problem for the total emergent intensity $I\dagger(0, \mu, \mu_0)$ of light reflected by scattering, as well as for its polarized components, can likewise be expressed in a closed form in terms of different solutions of the integral equation (1-54) with appropriate functions $\Psi(\mu)$. The explicit form of such solutions becomes, however, so complicated that we cannot reproduce them in this place. The reader wishing to consult them must be referred to Chandrasekhar's papers quoted above or to his subsequent summarizing presentation of his results.‡ The angular distribution $I_s^*(0, \mu, \mu_0)$ of radiation reflected by scattering on free electrons in the direction has been tabulated on the basis of Chandrasekhar's exact solution of the problem, and can be found in column (3) of Table 4-4 in terms of $I_s^*(0, 1, 1)$. A glance at these results is again interesting and instructive. Earlier in this section we had an opportunity to observe that the angular distribution of light transferred by electron scattering differs but little from that of light re-emitted isotropically. A comparison of the data given in columns (2) and (3) of Table 4-4 now reveals that very much the same situation holds good for reflected light as well.

In comparing the reflection of light re-emitted or scattered in stellar atmospheres, one feature should be particularly noted: namely, that although the forms of the corresponding equations of transfer are similar in both cases, the significance of $B^*(\tau)$ is not. In the former case the source function

* *Cf.* E. Lommel and H. v. Seeliger, *Sitzungsber. der Münchener Akad., II Klasse,* **17**, 95, 1887; **18**, 201, 1888.

† *Ap. J.*, **104**, 110, 1946; **106**, 152, 1947; **107**, 188, 1948.

‡ *Cf.* S. Chandrasekhar, *Bull. Amer. Math. Soc.*, **53**, 641, 1947; or his *Radiative Transfer*, Oxford 1950, sections 69-71.

is directly related with the temperature prevailing at the respective depth—in local thermodynamic equilibrium we have

$$\pi B^*(\tau) = \sigma T^{*4} \tag{1-80}$$

by analogy with (1-16). In the case of pure scattering, on the other hand, the function $B^*(\tau)$ ceases to have any relation with the temperature. The radiation will pass through the scattering medium without influencing its temperature; and the radiation temperature of the matter is taken to be zero.

If, however, the process of reflection is absorption followed by re-emission, the incident light will tend to *increase* the temperature within the atmosphere and thus produce a definite *heating effect*. In particular, the temperature increments T^* due to the incident light at $\tau = 0$ and ∞ are found to be*

$$B^*(0) = \frac{\sigma}{\pi} T_0^{*4} = \tfrac{1}{4} S H(\mu_0) \tag{1-81}$$

and

$$B^*(\infty) = \frac{\sigma}{\pi} T^{*4} = \frac{\sqrt{3}}{4} S \mu_0 H(\mu_0). \tag{1-82}$$

If, moreover, we set

$$\pi S = \sigma T_e^{*4}, \tag{18-3}$$

so that T_e^* represents the increase of effective temperature T_e caused by the incident radiation, then obviously

$$T_0^{*4} = \tfrac{1}{4} T_e^{*4} H(\mu_0) \tag{1-84}$$

and

$$T_\infty^{*4} = \tfrac{3}{4} T_e^{*4} \mu_0 H(\mu_0). \tag{1-85}$$

If, finally, $T_{(e)}$ denotes the true effective temperature of the reflecting star in the absence of external radiation (i.e., that of its hemisphere averted from the companion), the actual boundary temperature T_0 over the illuminated hemisphere should vary with the angle of incidence according to

$$T_0^4 = \frac{\sqrt{3}}{4} T_{(e)}^4 + \tfrac{1}{4} T_{(e)}^{*4} H(\mu_0). \tag{1-86}$$

Equations (1-81) to (1-86) are exact.

It should also be emphasized that all foregoing developments pertain to *reflection of integrated light*. An investigation of *spectral distribution* of re-emitted radiation could, in principle, be carried out along the same lines as we did for ordinary light earlier in this section on the assumption of local thermodynamic equilibrium. Adequate investigations of this kind are, however, still lacking completely; and in their absence we shall attempt later (section IV.6) to predict a spectral distribution of the reflected light in a simpler and more heuristic manner.

The transmission and reflection of light in plane-parallel stellar atmospheres, considered so far in this section, are relevant to the phenomena

* For their proof *cf.* again Hopf's *Tract*, or Chandrasekhar, *Ap. J.*, **101**, 348, 1945.

exhibited in close binary systems irrespective of whether their components are approximately spherical, or appreciably distorted. If, however, the latter is the case, an additional effect becomes operative which may influence profoundly the distribution of brightness over apparent disks of distorted components: namely, their *gravity-darkening.* For in 1924, at the conclusion of a series of remarkable papers, H. von Zeipel* has proved the following theorem: *The emergent flux of total radiation over the surface of a rotationally or tidally distorted star in radiative equilibrium varies proportionally to the local gravity.*

The following proof of this theorem is essentially due to Chandrasekhar.† Let F_n denote the flux of radiant energy across a level surface of constant potential at any point per unit area. Then, evidently,

$$F_n = -\frac{c}{\kappa\rho}\frac{dp}{dn}, \tag{1-87}$$

where, as before, c denotes the velocity of light; κ, the opacity coefficient; ρ, the density; p, the radiation pressure; and the differential dn corresponds to the normal distance between adjacent level surfaces. If, furthermore, P stands for the total pressure (gas plus radiation) and Ψ, for the combined potential (own plus disturbing) of the respective configuration, the equation of hydrostatic equilibrium requires that

$$\operatorname{grad} P = \rho \operatorname{grad} \Psi; \tag{1-88}$$

while the equation of radiative equilibrium implies that

$$\operatorname{div}\left(\frac{1}{\kappa\rho}\operatorname{grad} p\right) = -\frac{4\pi\varepsilon\rho}{c}, \tag{1-89}$$

where $4\pi\varepsilon$ stands for the rate of energy liberation. If, moreover, ω denotes the angular velocity of axial rotation of any particular layer,

$$\operatorname{div}\operatorname{grad}\Psi = -4\pi G\rho + 2\omega^2; \tag{1-90}$$

therefore,

$$\operatorname{div}\left(\frac{1}{\rho}\operatorname{grad} P\right) = -4\pi G\rho + 2\omega^2 \tag{1-91}$$

and the equation (1-89) of radiative equilibrium can be rewritten as

$$\frac{d}{dP}\left(\frac{1}{\kappa}\frac{dp}{dP}\right)\frac{1}{\rho}\left(\frac{dP}{dn}\right)^2 = \frac{2}{\kappa}\frac{dp}{dP}(2\pi G\rho - \omega^2) - \frac{4\pi\varepsilon\rho}{c}. \tag{1-92}$$

The right-hand side of this equation is evidently constant over a level surface. Hence, its left-hand side can be either constant, or identically zero. If it were constant, the differential element dn would have to be constant over a level surface—i.e., the equipotential surfaces should be

* *M.N.*, **84**, 702, 1924.
† *M.N.*, **93**, 539, 1933.

equidistant. This is impossible in a rotationally or tidally distorted configuration. Hence, the only admissible alternative is that

$$\frac{d}{dP}\left(\frac{1}{\kappa}\frac{dp}{dP}\right) = 0, \tag{1-93}$$

leading to

$$\frac{1}{\kappa}\frac{dp}{dP} = \text{constant.} \tag{1-94}$$

If we insert this in (1-87), we obtain

$$F_n \sim \frac{dP}{dn} \tag{1-95}$$

or, by equation (1-1) of Chapter II,

$$F_n \sim \frac{d\Psi}{dn}. \tag{1-96}$$

Since, moreover, $-(d\Psi/dn)$ is identical with the local gravity at that particular point (*cf.* equation 1-51 of Chapter II), the validity of von Zeipel's theorem is thus demonstrated.

The physical meaning of this proof can also be re-stated in the following terms. In accordance with our discussion of section II.1, the internal structure of a star in hydrostatic equilibrium under the influence of its own gravity and any disturbing potential must be such that the total pressure P and the density ρ are functions of Ψ alone. Hence, the temperature T must also be some function of Ψ, and so must be the opacity coefficient $\kappa(\rho, T)$. The outward flow of heat F_n is given by a product of the temperature gradient and a function of T, ρ, and κ. This function is constant over any level surface; and thus the outward flow of heat across such a surface is proportional to the temperature gradient. Since equipotential surfaces are isothermal, the temperature gradient over such a surface is proportional to the gradient of Ψ—and the latter is merely the value of local gravity, Thus the outward flow of heat over any equipotential surface and, therefore. also over the boundary of a star, should be proportional to gravity.

The preceding version of the proof embodied in equations (1-87)–(1-96) reveals also more clearly the limitations under which von Zeipel's theorem is valid. It need not, for instance, apply if there are convection currents beneath the photosphere; for a transfer of heat by material currents would make its flow no longer proportional to the temperature gradient. A system of meridional currents transporting matter from the poles to the equator would likewise accomplish the same effect. In the absence of a detailed knowledge of the structure of each star it should, therefore, be difficult to predict solely on a theoretical basis the extent to which von Zeipel's theorem should apply to actual stars. Its consequences upon changes of light exhibited by close binary systems have, however, been found in so close an agreement with the observed facts* that an application of the theorem seems justified, and accordingly we should expect that, at the boundary of a star, the intensity $H \equiv I(0, 1)$ of *total* radiation emerging normally from the atmosphere should vary as

$$\frac{H - H_0}{H_0} = \frac{g - g_0}{g_0}, \tag{1-97}$$

where g and g_0 denote the local and mean surface gravity, respectively, and H_0 is the value of H at points where $g = g_0$.

The intensity H of total radiation is evidently identical with the star's bolometric surface brightness viewed normally to the surface. If, moreover, the stars radiate like black bodies, equation (1-97) can alternatively be rewritten as

$$\frac{g - g_0}{g_0} = \frac{T^4 - T_0^4}{T_0^4}, \tag{1-98}$$

where T denotes the local effective temperature and T_0, the mean effective temperature averaged over the whole disk (not to be confused with the boundary temperature of a spherical star). *Equation* (1-98) *governs the variation of temperature over the surface of a distorted star radiating like a black body.* The surface brightness H_λ, observed at any particular wave-length λ, should then vary as

$$\frac{H_\lambda}{H_0} = \frac{e^{hc/\lambda k T_0} - 1}{e^{hc/\lambda k T} - 1}, \tag{1-99}$$

where $hc/k = 1{\cdot}438$ cm deg. If we substitute in (1-99) for T from (1-98) and expand the former in a Taylor series in the neighbourhood of $T = T_0$ then, to the first order in small quantities,

$$\frac{H_\lambda}{H_0} = 1 + \frac{b}{4}\left(\frac{g}{g_0} - 1\right) + \ldots, \tag{1-100}$$

where the ratio $(g - g_0)/g_0$ has already been defined in terms of first-order rotational and tidal distortion by equation (1-54) of Chapter II, and the constant b is given by equation (1-32) of the present section. If $b = 4$ (i.e., $\lambda T_0 = 0{\cdot}3652$ cm deg), equation (1-100) would give a distribution of brightness appropriate for total radiation. For observations carried out in more or less narrow spectral domains of effective wave-length λ,

* Strictly speaking, this is true of the photometric effects of *tidal* distortion (i.e., the ellipticity effect between minima) which will be studied in some detail in the next section of this chapter. Whether or not the *rotational* distortion gives rise to similar gravity-darkening remains yet unsettled by any empirical evidence; for polar flattening will be found to invoke light changes between minima—which remain our principal source of information—only if the axis of rotation is inclined to the orbital plane (*cf.* section IV.2) and these are likely to be minor. A gravity-darkening due to rotational distortion should, to be sure, impress observable effects on light changes of distorted eclipsing systems within minima (*cf.* the forthcoming section IV.3); but such effects are so thoroughly interlocked with those of tidal distortion that it is still too early to be sure of their presence even in the best observed light curves now available.

b as defined by (1-32) may be considerably greater or smaller than 4—as is shown by its 3D-tabulation, in the accompanying Table 4-5, as function of λ (in Å) and T_0 (in degrees). In particular, if $a = hc/\lambda k T_0$ is small (high temperatures or long wave lengths) $b \to 1$; while if a is large, b varies nearly proportionally to it.

TABLE 4-5

The Coefficient $\frac{1}{4}b = \tau_0$ of Gravity-darkening for Black-body Radiation

T_0 \ λ	3750	4000	4250	4500	4750	5000	5250	5500	6000	7000	8000	10000 Å
4000°	2·387	2·238	2·106	1·989	1·885	1·792	1·707	1·630	1·496	1·286	1·132	0·921
5000	1·908	1·792	1·687	1·594	1·511	1·437	1·370	1·309	1·203	1·040	0·921	0·759
6000	1·589	1·496	1·409	1·333	1·309	1·204	1·149	1·099	1·014	0·882	0·786	0·659
7000	1·370	1·333	1·213	1·149	1·092	1·040	0·995	0·953	0·882	0·772	0·713	0·588
8000	1·204	1·132	1·069	1·014	0·965	0·921	0·882	0·847	0·786	0·693	0·626	0·537
9000	1·076	1·014	0·959	0·911	0·871	0·830	0·796	0·766	0·713	0·634	0·576	0·500
10000	0·976	0·921	0·867	0·830	0·793	0·759	0·730	0·703	0·657	0·588	0·537	0·470
12000	0·830	0·786	0·747	0·713	0·683	0·657	0·634	0·613	0·576	0·521	0·481	0·428
14000	0·730	0·713	0·661	0·634	0·609	0·588	0·568	0·551	0·521	0·476	0·443	0·399
16000	0·657	0·626	0·600	0·576	0·556	0·537	0·521	0·506	0·481	0·443	0·416	0·378
18000	0·603	0·576	0·553	0·533	0·517	0·500	0·486	0·473	0·451	0·419	0·395	0·363
20000	0·560	0·537	0·517	0·499	0·484	0·470	0·458	0·447	0·428	0·399	0·378	0·350
24000	0·500	0·481	0·465	0·451	0·439	0·428	0·418	0·410	0·394	0·372	0·355	0·332
28000	0·458	0·443	0·430	0·419	0·408	0·399	0·391	0·384	0·372	0·353	0·338	0·319
35000	0·411	0·399	0·365	0·381	0·373	0·366	0·360	0·355	0·345	0·330	0·319	0·305
50000	0·357	0·350	0·344	0·338	0·333	0·329	0·325	0·321	0·314	0·305	0·298	0·288

Equation (1-100) with g/g_0 given by (1-54) of Chapter II describes the *gravity-darkening* to be expected over free surface of distorted components of close binary systems to the first order in small quantities; and a combination of equations (1-26) and (1-100) governing the limb- and gravity-darkening of such stars then describes the complete distribution of brightness, over their apparent disks, as seen by a distant observer at any phase of the orbital cycle. This resultant distribution of brightness is perhaps too complicated for its nature to be readily visualized. The limb-darkening tends to make brightest those parts of the visible surface which are nearest to the observer; the gravity-darkening, those which are nearest to the star's centre. The point, on a tidally-distorted star, which is closest to the observer will not in general be the geometric centre of its apparent disk; as a result, the corresponding system of isophotae will not only cease to be radially-symmetrical, but will vary with the phase. If, for instance, a tidally distorted star is viewed from the direction of its longest axis, its apparent distribution of brightness may in fact become tantamount to 'limb-brightening'. It is perhaps needless to stress that all such effects are likely to render the light changes to be expected in close binary systems—between eclipses as well as within minima—quite complicated, and their interpretation difficult. The reader who has followed us so far and has become hardened enough to be undeterred by such prospects, is now invited to accompany us through

Chapters IV and V in an investigation of theoretical light- and radial velocity-changes to be exhibited by distorted close binaries; and to find in Chapter VI an outline of the solution of a converse problem: namely, the determination of the elements of close eclipsing systems from an analysis of their light changes. In going through the complexity of even a first-order theory of such superlatively intriguing phenomena, many a reader may at times sigh in despair: if the Arabs named Algol a demon—what should they have called β Lyrae?

IV.2. LIGHT CHANGES OF ROTATING DISTORTED STARS

In order to investigate the light changes caused by the axial rotation of distorted components of close binary systems in the course of their orbital revolution, let us adopt a rectangular frame of reference whose X-axis coincides with the line joining the centres of both components and Z is perpendicular to the orbital plane;* their origin will be identified with the centre of gravity of the star under consideration (which, for the sake of convenience, will hereafter be referred to as the primary component). Let, further, the position of any point in this system be specified by the polar coordinates r, θ, ϕ, where ϕ denotes the angle, in the XY-plane, between a projected radius-vector and the X-axis, and θ is the polar distance. The light of the star as seen by an observer at a great distance can generally be expressed as

$$\mathfrak{L} = \int J \cos \gamma \, d\sigma, \tag{2-1}$$

where J denotes, as in the previous section, the intensity of light at any point of the visible surface; γ, the angle of foreshortening; and the surface element $d\sigma$ is evidently given by

$$\cos \beta \, d\sigma = r^2 \sin \theta \, d\theta \, d\phi, \tag{2-2}$$

where β stands for the angle between a radius-vector and surface normal. The range of integration in (2-1) is to be extended over the whole visible hemisphere.

The angles β and γ are clearly given by

$$\left.\begin{aligned} \cos \beta &= \lambda l + \mu m + \nu n, \\ \cos \gamma &= l l_0 + m m_0 + n n_0, \end{aligned}\right\} \tag{2-3}$$

* The reader may note that a system so defined is identical with the doubly-primed system of coordinates introduced previously in section II.3 to distinguish it from other coordinate systems then employed. As, in the present section, only one coordinate system will be used, its double primes can hereafter be dropped without risking any ambiguity.

where

$$\begin{array}{lll} \lambda, & \mu, & \nu, \\ l, & m, & n, \\ l_0, & m_0, & n_0, \end{array}$$

denote the direction cosines of a radius-vector, surface normal, and line of sight, respectively. The forms of λ, μ, ν, and l_0, m_0, n_0 depend upon the adopted frame of reference and the observer's position. In conformity with the definition of our XYZ-system,

$$\left.\begin{aligned} \lambda &= \cos\phi\sin\theta, \\ \mu &= \sin\phi\sin\theta, \\ \nu &= \cos\theta; \end{aligned}\right\} \tag{2-4}$$

while, if ψ denotes the true anomaly of the secondary component in the plane of the relative orbit, reckoned from the moment of superior conjunction (i.e., mid-primary minimum if the primary component is one of greater surface brightness); and i, the inclination of the orbital plane to a plane tangent to the celestial sphere at the origin of coordinates*

$$\left.\begin{aligned} l_0 &= \cos\psi\sin i, \\ m_0 &= \sin\psi\sin i, \\ n_0 &= \cos i. \end{aligned}\right\} \tag{2-5}$$

The direction cosines l, m, n of a surface normal at any point of the primary component are, in turn, defined by

$$(l, m, n) = -\frac{1}{g}\left\{\frac{\partial\Psi}{\partial x}, \frac{\partial\Psi}{\partial y}, \frac{\partial\Psi}{\partial z}\right\}, \tag{2-6}$$

where

$$\left.\begin{aligned} \frac{\partial\Psi}{\partial x} &= \lambda\left(\frac{\partial\Psi}{\partial r}\right) + \frac{\partial\lambda}{\partial\theta}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right) + \frac{\partial\lambda}{\partial\phi}\left(\frac{1}{r\sin^2\theta}\frac{\partial\Psi}{\partial\phi}\right), \\ \frac{\partial\Psi}{\partial y} &= \mu\left(\frac{\partial\Psi}{\partial r}\right) + \frac{\partial\mu}{\partial\theta}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right) + \frac{\partial\mu}{\partial\phi}\left(\frac{1}{r\sin^2\theta}\frac{\partial\Psi}{\partial\phi}\right), \\ \frac{\partial\Psi}{\partial z} &= \nu\left(\frac{\partial\Psi}{\partial r}\right) + \frac{\partial\nu}{\partial\theta}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right), \end{aligned}\right\} \tag{2-7}$$

and

$$g^2 = \left(\frac{\partial\Psi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\Psi}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial\Psi}{\partial\phi}\right)^2, \tag{2-8}$$

* It should be stressed that, in Chapter II (section II.5) we used the letter i to denote the inclination of the orbital plane to the invariable plane of the system; while the inclination of the orbital plane to the celestial sphere, in the sense defined above, was denoted in section II.7 by j.

where Ψ denotes the total potential of forces acting at that point which, in accordance with equation (1-50) of Chapter II can be expressed as

$$\Psi = G\frac{m_1}{r} + \sum_{i,j} c_{i,j}\left\{(\Delta_j - 1)\frac{a_1^{2j+1}}{r^{j+1}} + r^j\right\} T_j^i, \tag{2-9}$$

where the $c_{i,j}$'s are constants depending on the nature of distortion and

$$T_j^i = \frac{\cos i\phi}{\sin i\phi} P_j^i(\cos\theta) \tag{2-10}$$

stand for the tesseral harmonics of the respective order and index.

If we use the foregoing expression (2-9) for the potential in equations (2-7) and (2-8), perform the respective operations and so insert resulting direction cosines l, m, n in equations (2-3), it transpires that, as far as squares and higher powers of the $c_{i,j}$'s remain negligible, the cosines of the angles β and γ assume the more explicit forms

$$\cos\beta = 1 + \text{second-order terms,*} \tag{2-11}$$

while

$$\begin{aligned} \cos\gamma = l_0\left\{\lambda - \sum_{i,j} k_{i,j}\left[\frac{\partial\lambda}{\partial\theta}\frac{\partial T_j^i}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial\lambda}{\partial\phi}\frac{\partial T_j^i}{\partial\phi}\right]\right\} \\ + m_0\left\{\mu - \sum_{i,j} k_{i,j}\left[\frac{\partial\mu}{\partial\theta}\frac{\partial T_j^i}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial\mu}{\partial\phi}\frac{\partial T_j^i}{\partial\phi}\right]\right\} \\ + n_0\left\{\nu - \sum_{i,j} k_{i,j}\left[\frac{\partial\nu}{\partial\theta}\frac{\partial T_j^i}{\partial\theta}\right]\right\}, \end{aligned} \tag{2-12}$$

where we have abbreviated

$$k_{i,j} = \frac{\Delta_j a_1^{j+1}}{Gm_1} c_{i,j}\,. \tag{2-13}$$

It may be added (*cf.* section II.1) that, to the first order in the $k_{i,j}$'s, the square of the radius r connecting the origin with any surface point will be given by

$$r^2 = a_1^2\{1 + 2\sum_{i,j} k_{i,j} T_j^i\}, \tag{2-14}$$

by virtue of which the surface element $d\sigma$ as well as the foreshortening factor $\cos\gamma$ in the integral on the right-hand side of equation (2-1) can now be regarded as known functions of our polar coordinates θ and ϕ.

In order to be able to perform the integration indicated by (2-1), it remains to express also the apparent surface brightness J at any point

* That this must be so can be easily seen from the fact that the angle β between radius-vector and surface normal, zero for a sphere, is a quantity of the same order of magnitude as the superficial distortion. Hence, $\cos\beta$ can differ from unity at most by quantities of the order of *squares* of this distortion, which we consider negligible.

explicitly in terms of θ and ϕ. In the preceding section of this chapter we found that, over surfaces of rotationally or tidally distorted components, limb-darkening should cause the intensity J of light emerging from a stellar atmosphere in a direction making an angle γ with the line of sight to vary, in general, as

$$J = H(1 - u_1 - u_2 - \ldots + u_1 \cos\gamma + u_2 \cos^2\gamma + \ldots), \tag{2-15}$$

where H denotes the intensity of radiation emerging normally from the atmosphere which, for distorted stars, should vary in turn as

$$H = H_0\left\{1 + \tau_0\left(\frac{g - g_0}{g_0}\right)\right\}, \quad \tau_0 = \frac{h\nu}{4kT(1 - e^{-h\nu/kT})}, \tag{2-16}$$

on account of gravity-darkening. The values of the coefficients u_j or $\tau_0 = \frac{1}{4}b$ (*cf.* equation 1-32) appropriate for stars of different effective temperatures and observed in light of any particular frequency have already been discussed and tabulated in the preceding section IV.1; while the variation of the surface gravity g has been investigated already in section II.1 (*cf.* its equation 1-54) and found to obey the formula

$$\frac{g - g_0}{g_0} = -\sum_{i,j}\left\{\frac{2j+1}{\Delta_j} + 1 - j\right\} k_{i,j} T^i . \tag{2-17}$$

In consequence, if we collect the results represented by equations (2-12), (2-14) and (2-17) we find that the integrand on the right-hand side of (2-1) will consist of a series of terms of the form

$$\begin{aligned} r^2 H \cos^h\gamma = a_1^2 H_0 N^h \Bigg\{ 1 &- \sum_{i,j}\left[\left(\frac{2j+1}{\Delta_j} + 1 - j\right)\tau_0 - 2\right] k_{i,j} T_j^i \\ &- h\left[\frac{l_0}{N} - \lambda\right] \sum_{i,j} k_{i,j} \frac{\partial T_j^i}{\partial \lambda} \\ &- h\left[\frac{m_0}{N} - \mu\right] \sum_{i,j} k_{i,j} \frac{\partial T_j^i}{\partial \mu} \\ &- h\left[\frac{n_0}{N} - \nu\right] \sum_{i,j} k_{i,j} \frac{\partial T_j^i}{\partial \nu} \Bigg\}, \end{aligned} \tag{2-18}$$

where $h = 1, 2, 3, \ldots$, and where we have abbreviated

$$N = \lambda l_0 + \mu m_0 + \nu n_0 . \tag{2-19}$$

Moreover, by virtue of the fact that our surface harmonics T_j^i as homogeneous functions of λ, μ, ν of j-th degree are bound to satisfy the relation

$$\lambda \frac{\partial T_j^i}{\partial \lambda} + \mu \frac{\partial T_j^i}{\partial \mu} + \nu \frac{\partial T_j^i}{\partial \nu} = j T_j^i , \tag{2-20}$$

the above equation (2-18) for the integrand can be reduced to

$$r^2 H \cos^h \gamma = a_1^2 H_0 N^h \left\{ 1 - \sum_{i,j} \Omega_j^{(h)} k_{i,j} T_j^i \right.$$
$$\left. - \frac{h}{N} \sum_{i,j} k_{i,j} \left[l_0 \frac{\partial T_j^i}{\partial \lambda} + m_0 \frac{\partial T_j^i}{\partial \mu} + n_0 \frac{\partial T_j^i}{\partial \nu} \right] \right\}, \tag{2-21}$$

where we have abbreviated

$$\Omega_j^{(h)} = \left\{ \frac{2j+1}{\Delta_j} + 1 - j \right\} \tau_0 - jh - 2. \tag{2-22}$$

The light $\mathfrak{L}$ of the primary component, which we placed at the origin of our coordinate system, will now be obtained if the integration of (2-1) is carried out over the whole hemisphere visible to a distant observer at any particular phase. If its distortion could be ignored, the total light of the primary component would be constant and given by

$$\mathfrak{L}_0 = 2\pi \int_0^{a_1} J \sin^{-1} (r/a_1) r \, dr = \pi a_1^2 H_0 \left\{ 1 - \sum_{h=1}^{k} \frac{h-1}{h+1} u_h \right\}, \tag{2-23}$$

where k denotes the number of the terms in the law (2-15) of limb-darkening retained in our computation. Let us hereafter adopt $\mathfrak{L}_0$ so defined as our unit of light, and inquire as to the variation $\delta\mathfrak{L}$ of $\mathfrak{L}$, invoked by distortion, and defined by the equation

$$\mathfrak{L} = \mathfrak{L}_0(1 + \delta\mathfrak{L}). \tag{2-24}$$

In order to facilitate this task, suppose that we decompose $\delta\mathfrak{L}$ into a series of partial terms $\delta\mathfrak{L}^{(h)}$ denoting the individual contributions, to the resultant light variation $\delta\mathfrak{L}$, of photometric effects arising from the h-th term of the series on the right-hand side of (2-15). If so, it follows that

$$\delta\mathfrak{L} = \sum_{h=1}^{k} C^{(h)} \, \delta\mathfrak{L}^{(h)}, \tag{2-25}$$

where

$$C^{(1)} = \frac{1 - \sum_{h=1}^{k} u_h}{1 - \sum_{h=1}^{k} \frac{h-1}{h+1} u_h} \tag{2-26}$$

while, for $h > 1$,

$$C^{(h)} = \frac{2}{2+h} \frac{u_h}{1 - \sum_{h=1}^{k} \frac{h-1}{h+1} u_h}; \tag{2-27}$$

and

$$\frac{2\pi a_1^2}{2+h} H_0 \, \delta\mathfrak{L}^{(h)} = \int_0^{\pi} \int_{\varepsilon-\frac{\pi}{2}}^{\varepsilon+\frac{\pi}{2}} r^2 H \cos^h \gamma \sin\theta \, d\theta \, d\phi, \tag{2-28}$$

where ε denotes the angle between the radius-vector (joining the centres of the two components) and the line of sight. As the former has been identified with the X-axis of our coordinate system (and its direction cosines are, accordingly, 1, 0, 0) it follows from (2-5) at once that

$$\cos\varepsilon = l_0 = \cos\psi \sin i. \tag{2-29}$$

The integration of (2-28) causes no difficulty: performing it we find that

$$\begin{aligned} \delta\mathfrak{L}^{(h)} = & -\frac{h(4+\beta_2)}{h+3} k_{i,2} T_2^i(l_0, n_0) \\ & -\frac{(h-1)(h+1)}{(h+2)(h+4)}(10+\beta_3) k_{i,3} T_3^i(l_0, n_0) \\ & -\frac{h(h-2)}{(h+3)(h+5)}(18+\beta_4) k_{i,4} T_4^i(l_0, n_0) + \ldots, \end{aligned} \tag{2-30}$$

where the tesseral harmonics $T_j^i(l_0, n_0)$ are the same functions of l_0, n_0 as the $T_j^i(\lambda, \nu)$'s were of λ and ν in the integrand (2-21), and where we have abbreviated

$$\beta_j = \left\{\frac{2j+1}{\Delta_j} + 1 - j\right\}\tau_0 = \left\{1 + \eta_j(a_1)\right\}\tau_0 \,. \tag{2-31}$$

We may now also recall that, in section I.1, we have established, for *tidal* distortion, the absence of tidal lag to imply that the index $i = 0$ and, by equation (1-58) of Chapter II,

$$c_{0,j} = G\frac{m_2}{R^{j+1}}, \tag{2-32}$$

where m_2 denotes the mass of the secondary component and R, the distance between the two stars; so that, by (2-13),

$$k_{0,j} = \Delta_j \frac{m_2}{m_1}\left(\frac{a_1}{R}\right)^{j+1} = w_1^{(j)} \tag{2-33}$$

for the primary component (and with an appropriate interchange of indices for its mate). Moreover, the respective tesseral harmonics T_j^0 then reduce simply to the Legendre polynomials $P_j(l_0)$. In the case of *rotational* distortion, we have seen likewise in section II.1 that, if the primary rotates with a (constant) angular velocity ω_1 about an axis which is perpendicular to the orbital plane, the only non-zero $c_{i,j}$ turns out to be

$$c_{0,2} = -\tfrac{1}{3}\omega_1^2, \tag{2-34}$$

leading to

$$k_{0,2} = -\frac{\Delta_2 a_1^3 \omega_1^2}{3Gm} = -\frac{\Delta_2}{3}\left(1 + \frac{m_2}{m_1}\right)\left(\frac{a_1}{R}\right)^3 = -\frac{v_1^{(2)}}{3} \tag{2-35}$$

if the primary rotates with the Keplerian angular velocity

$$\omega_1^2 = G\frac{m_1 + m_2}{R^3}. \tag{2-36}$$

Moreover, the sole tesseral harmonic T_2^0 of rotational origin then reduces to $P_2(n_0)$.

Under these circumstances, an insertion of (2-30) and (2-26)–(2-27) in (2-25) leads to

$$\begin{aligned} \delta\mathfrak{L} = {} & X_2^{(k)}\{1 + \tfrac{1}{4}\beta_2\}\{\tfrac{1}{3}v_1^{(2)}P_2(n_0) - w_1^{(2)}P_2(l_0)\} \\ & - X_3^{(k)}\{1 + \tfrac{1}{10}\beta_3\}w_1^{(3)}P_3(l_0) \\ & - X_4^{(k)}\{1 + \tfrac{1}{18}\beta_4\}w_1^{(4)}P_4(l_0) + \dots, \end{aligned} \tag{2-37}$$

where for $k = 2$ (i.e., linear limb-darkening),

$$X_2^{(2)} = \frac{15 + u_1}{5(3 - u_1)}, \qquad X_3^{(2)} = \frac{5u_1}{2(3 - u_1)}, \qquad X_4^{(2)} = \frac{9(u_1 - 1)}{4(3 - u_1)}; \tag{2-38}$$

for $k = 3$ (quadratic limb-darkening),

$$\left.\begin{aligned} X_2^{(3)} &= \frac{2}{5}\,\frac{15 + u_1}{6 - 2u_1 - 3u_2}, \\ X_3^{(3)} &= \frac{1}{7}\,\frac{35u_1 + 48u_2}{6 - 2u_1 - 3u_2}, \\ X_4^{(3)} &= \frac{9}{8}\,\frac{4(u_1 - 1) + 7u_2}{6 - 2u_1 - 3u_2}; \end{aligned}\right\} \tag{2-39}$$

for $k = 4$,

$$\left.\begin{aligned} X_2^{(4)} &= \frac{2}{7}\,\frac{7(15 + u_1) - 9u_3}{30 - 10u_1 - 15u_2 - 18u_3}, \\ X_3^{(4)} &= \frac{5}{14}\,\frac{70u_1 + 96u_2 + 105u_3}{30 - 10u_1 - 15u_2 - 18u_3}, \\ X_4^{(4)} &= \frac{3}{56}\,\frac{420(u_1 - 1) + 735u_2 + 932u_3}{30 - 10u_1 - 15u_2 - 18u_3}; \end{aligned}\right\} \tag{2-40}$$

for $k = 5$,

$$\left.\begin{aligned} X_2^{(5)} &= \frac{1}{7}\,\frac{210 + 14u_1 - 18u_3 - 35u_4}{30 - 10u_1 - 15u_2 - 18u_3 - 20u_4}, \\ X_3^{(5)} &= \frac{5}{42}\,\frac{210u_1 + 288u_2 + 315u_3 + 320u_4}{30 - 10u_1 - 15u_2 - 18u_3 - 20u_4}, \\ X_4^{(5)} &= \frac{3}{56}\,\frac{420(u_1 - 1) + 735u_2 + 932u_3 + 1050u_4}{30 - 10u_1 - 15v_2 - 18u_3 - 20u_4} \end{aligned}\right\} \tag{2-41}$$

etc.

The changes of light exhibited by distorted components of close binary systems in the course of their revolution are now specified to the same degree of accuracy to which the external form of such configurations has been established in section II.1. In discussing the nature of such light changes we may note that, as $P_2(n_0)$ on the right-hand side of (2-37) is independent of the phase, *the rotational distortion of components whose equatorial planes are coplanar with that of their orbit does not influence the variation of light between eclipses*; all terms which cause it are due to the tides alone. Since, moreover, all such terms depend on the phase only through the direction cosine $l_0 = \cos \psi \sin i$, it follows that the *light variation due to terms of tidal origin should be symmetrical with respect to the phases $\psi = \pi$ of the secondary minimum*; as long as tides do not lag, the light changes arising from them cannot be asymmetric.

Let us consider next the effects, upon light changes, due to limb- and gravity-darkening. In the case of a linear approximation (i.e., $k = 2$) to the actual law of limb-darkening, equations (2-38) make it evident that *no light variation arises from the third-harmonic tidal distortion unless there is some limb-darkening, nor any from the fourth unless the limb-darkening is incomplete.* If the gravity-darkening were absent (i.e., if $\tau_0 = 0$) equation (2-37) would for $u_1 = 0$ reduce to

$$\delta\mathfrak{L}^{(1)} = \tfrac{1}{3}v_1^{(2)}P_2(n_0) - w_1^{(2)}P_2(l_0) + \tfrac{3}{4}w_1^{(4)}P_4(l_0) + \dots, \tag{2-42}$$

while for $u_1 = 1$ we should obtain

$$\delta\mathfrak{L}^{(2)} = \tfrac{8}{15}v_1^{(2)}P_2(n_0) - \tfrac{8}{5}w_1^{(2)}P_2(l_0) - \tfrac{5}{4}w_1^{(3)}P_3(l_0) + \dots . \tag{2-43}$$

The first of these equations would express the changes of light due to the distorted geometry alone.* A comparison of equations (2-42) and (2-43) reveals that, in particular, *the limb-darkening tends to exaggerate the variation arising from the second-harmonic distortion, to give rise to a significant third harmonic, and to damp the fourth-harmonic light variation.* On the other hand a full gravity-darkening (i.e., $\tau_0 = 1$) of a centrally-condensed ($\Delta_j = 1$) star would lead for $u_1 = 0$ to

$$\delta\mathfrak{L}^{(1)} = \tfrac{2}{3}v_1^{(2)}P_2(n_0) - 2w_1^{(2)}P_2(l_0) + w_1^{(4)}P_4(l_0) + \dots, \tag{2-44}$$

while for $u_1 = 1$

$$\delta\mathfrak{L}^{(2)} = \tfrac{16}{15}v_1^{(2)}P_2(n_0) - \tfrac{16}{5}w_1^{(2)}P_2(l_0) - \tfrac{15}{8}w_1^{(3)}P_3(l_0) + \dots . \tag{2-45}$$

The reader may notice that, in either case, *the effect of gravity-darkening upon the changes of total (integrated) light invoked by the second, third, and fourth tidal harmonic distortion of a centrally-condensed star is to multiply the variation due to distorted geometry alone by the factors of* 2, $\frac{3}{2}$, and $\frac{4}{3}$, *respectively.*

* It may be recalled that the coefficients $v^{(2)}$ and $w^{(j)}$ possess a simple geometrical meaning: namely, $v^{(2)}$ is identical with the oblateness of a slowly rotating configuration, while the $w^{(j)}$'s are proportional to the contributions to its equatorial ellipticity invoked by the j-th tidal harmonic distortion.

It may also be noted that, consistent with equation (1-46), the coefficient u_1 of linear law of limb-darkening is expressed as

$$u_1 = \left\{ 1 + \frac{2}{3}\left(\frac{2}{\tau_0} - 1\right)\frac{\kappa_\lambda}{\bar{\kappa}} \right\}^{-1}, \tag{2-46}$$

the functions $X_j^{(2)}$ as defined by equations (2-38) assume the particular forms

$$\left.\begin{aligned} X_2^{(2)} &= 1 + \frac{3}{5}\frac{\tau_0}{\tau_0 + (2-\tau_0)x}, \\ X_3^{(2)} &= \frac{5}{4}\frac{\tau_0}{\tau_0 + (2-\tau_0)x}, \\ X_4^{(2)} &= -\frac{3}{4}\frac{(2-\tau_0)x}{\tau_0 + (2-\tau_0)x}, \end{aligned}\right\} \tag{2-47}$$

where we have abbreviated $\kappa_\lambda/\bar{\kappa} \equiv x$. It transpires, therefore, that within the scheme of approximations on which a linear law of limb-darkening is based, the optical ellipticity of rotating distorted configurations is controlled by the magnitude of the two independent factors τ_0 and x. Of these, τ_0 represents a logarithmic derivative of the intensity-distribution in the continuous spectrum of the stars which, for black-body radiation, proves to depend on the temperature of the emitting body and the wave length of observation in a simple manner indicated by equation (2-16). On the other hand, as was pointed out in the preceding section, a purely theoretical determination of the ratio $\kappa_\lambda/\bar{\kappa} \equiv x$ would take us deep into the physics of radiation processes in stellar atmospheres as well as into the chemistry of their composition, which are still quite far from being well understood. Under these circumstances, the possibility suggests itself to accept theoretical (black-body) values (2-16) for τ_0, but to determine x empirically from observed photometric ellipticity of eclipsing binary systems of known fractional dimensions and mass-ratios, with the aid of the theoretical formulae which we have just established.

So far we have considered the photometric effects of limb-darkening which are linear in $\cos\lambda$. In order to investigate the effects which quadratic and higher terms in (2-15) are likely to exert on the light variation of distorted components of close binary systems in the course of their orbital revolution, let us fall back on the results established in the preceding section IV.1 for stellar limb-darkening in *total* radiation and recall that while, in the linear approximation, Milne's well-known solution (1-12) of the equation of radiative transfer led to

$$u_1 = 0{\cdot}6, \tag{2-48}$$

a quadratic approximation (1-13) to the exact law of darkening yielded

$$\left.\begin{aligned} u_1 &= 0{\cdot}6500, \\ u_2 &= -0{\cdot}0226; \end{aligned}\right\} \tag{2-49}$$

while the quartic approximation (1-14) furnished us with a set of four coefficients

$$\left.\begin{aligned} u_1 &= 0{\cdot}6998, \\ u_2 &= -0{\cdot}1053, \\ u_3 &= 0{\cdot}0576, \\ u_4 &= -0{\cdot}0145. \end{aligned}\right\} \tag{2-50}$$

The relative efficiency with which the formulae (1-12), (1-13), and (1-14) employing such coefficients approximates the exact solution (1-10) of the equation (1-8) of transfer in plane-parallel atmospheres has already been demonstrated graphically in Fig. 4-1. If we insert now the sets of coefficients (2-48), (2-49) and (2-50) successively in equations (2-38), (2-39) and (2-40), the following numerical results are obtained:

	$k = 2$	$k = 3$	$k = 4$
$X_2^{(k)}$	1·3	1·3130	1·3142
$X_3^{(k)}$	0·625	0·6492	0·65
$X_4^{(k)}$	—0·375	—0·3677	—0·3709

Although these data pertain, strictly speaking, only to effects that would be observable in integrated light, their inspection reveals what is probably the general situation for observations carried out in any reasonable spectral range: namely, that *the photometric effects of rotation of distorted stars do not appear to be very sensitive to the distribution of brightness near the limb* (where alone equations (1-12), (1-13) and (1-14) differ by appreciable amounts).

On the whole, the deviations of the actual distribution of brightness from a linear cosine law are found (in total radiation) to magnify the photometric effects of the second and third tidal harmonic by 1% and 4%, respectively, and to diminish the coefficient of the fourth harmonic by about 2%. Moreover, the differences between quadratic and quartic approximations to the actual law of limb-darkening, such as represented by equations (1-13) and (1-14), prove to have altogether negligible photometric consequences. This all entitles us to conclude that, for the time being and probably for many years to come, *an interpretation of even the most precise light curves of eclipsing binary systems between minima should not call for the retention of more than the quadratic term in the law* (2-15) *of limb-darkening*; while, in ordinary cases, *a failure to consider this quadratic term should vitiate the coefficients of harmonic light variation by not more than a few per cent.* It should be stressed, however, that these comments are meant to apply to the light changes observed between minima; when we come to deal with light changes exhibited within eclipses, a very different situation will turn out to be true.

In passing to equation (2-37) from (2-30) we have considered, for simplicity, the axis of rotation of the distorted component to be perpendicular to the orbital plane and found that, under these conditions, its rotational distortion will not (at least in systems characterized by circular orbits) give rise to any light variation in the course of a cycle. In Chapter II we pointed out, however, that no forces are known to be operative in close binary systems which should impel the axes of rotation of the individual components to remain constantly perpendicular to the orbital plane: even should they happen to be so at some particular time, the co-planarity of the equatorial planes with that of the orbit is bound to be destroyed by the nutation of the components as well as the oscillation of the plane of the orbit about the invariable plane of the system. Moreover, the orientation of the rotational axes in space is bound to vary in the course of time on account of the precession. All these phenomena, whose dynamical study was the principal object of Chapter II of this book, invite now an inquiry as to their photometric consequences. In particular, let the instantaneous inclination of the axis of rotation of our primary component be specified (as in section II.1) by the direction cosines

$$\left.\begin{aligned} \lambda' &= \cos\alpha \sin\beta, \\ \mu' &= \sin\alpha \sin\beta, \\ \nu' &= \cos\beta, \end{aligned}\right\} \tag{2-51}$$

where α denotes the longitude of the nodal line at which the orbital and equatorial planes intersect, and β is the angle between the axis of rotation and a direction normal to the orbital plane (i.e., our present Z-axis*); any variation of both α and β due to periodic perturbations of these elements in the course of a cycle (*cf.* section II.5) will hereafter be ignored. If so, and if the primary component rotates about this inclined axis as a rigid body with constant angular velocity ω_1, the single tesseral harmonic $T_j^i(\lambda, \nu)$ in the integrand (2-21) will become identical with $P_2(\cos\Theta)$, where (as in section II.1)

$$\cos\Theta = \lambda\lambda' + \mu\mu' + \nu\nu' \tag{2-52}$$

denotes the colatitude of any arbitrary surface point, measured from the pole of the rotating star. Since, by the addition theorem for Legendre polynomials.

$$\begin{aligned} P_2(\cos\Theta) = P_2(\nu)P_2(\nu') &+ \tfrac{1}{3} P_2^1(\nu)P_2^1(\nu')\cos(\phi - \alpha) \\ &+ \tfrac{1}{12}P_2^2(\nu)P_2^2(\nu')\cos 2(\phi - \alpha), \end{aligned} \tag{2-53}$$

it follows by integration that the light changes exhibited during each cycle by a star whose equator is inclined by an angle β to the orbital plane should be

* The reader should note that the direction cosines λ', μ', ν', so defined are *not* identical with $a'_{13}, a'_{23}, a'_{33}$ as given by equations (3-13)–(3-15) of Chapter II, since the former refer to the position of the axis of rotation in our present XYZ-frame of reference (identical with the doubly-primed system of section II.3), and the latter, to its position with respect to a fixed system of space axes (whose XY-plane is identical with the invariable plane of our binary system).

given by

$$\begin{aligned}\delta\mathfrak{L}_{\text{rot}} = X_2^{(k)}\{1 + \tfrac{1}{4}\beta_2\}\{\tfrac{1}{3}P_2(\nu')P_2(n_0) \\ + (\lambda'\mu')l_0m_0 + (\lambda'\nu')l_0n_0 + (\mu'\nu')m_0n_0 \\ + \tfrac{1}{4}(\lambda'^2 - \mu'^2)(l_0^2 - m_0^2)\}v_1^{(2)}.\end{aligned} \tag{2-54}$$

If the angle $\beta = 0$ (so that $\lambda' = \mu' = 0$ and $\nu' = 1$) this result reduces to

$$\delta\mathfrak{L}_{\text{rot}} = \tfrac{1}{3}X_2^{(k)}\{1 + \tfrac{1}{4}\beta_2\}v_1^{(2)}P_2(n_0) \tag{2-55}$$

in agreement with (2-37); but for $\beta > 0$ rotational distortion of components with inclined axes is seen to give rise to several additional terms on the right-hand side of (2-54) which depend on the phase, not only through cos ψ (in l_0), but also through sin ψ (i.e., m_0). In consequence, *the light variations* governed by them *will no longer be symmetrical with respect to the phase* $\psi = \pi$ *of the secondary minimum*, but will exhibit a distinct *asymmetry* of amplitude comparable with that of the photometric effects of tidal distortion.

This is an important result; for light changes exhibited between minima by many eclipsing systems are known to be asymmetric; and this asymmetry cannot be accounted for by phenomena within the framework of the equilibrium theory of tides. If it were, however, due to the inclination of the axes of rotation to the orbital plane, it would follow that the amplitude as well as sense of the asymmetry would have to vary continuously with the time. In section 5 of Chapter II we found that, to a first approximation, the angle β involved through the direction cosines λ', μ', ν' in the amplitudes of the asymmetry-generating terms on the right-hand side of (2-54) should remain secularly stable, and oscillate but slightly on account of nutation in a period given by equation (5-81) of Chapter II. On the other hand, the angle α involved in λ' and μ' should vary secularly with the time, on account of precession, and complete a whole revolution in a period given by equation (5-86) of Chapter II which, for typical eclipsing systems, proves to be of the order of a few years. Therefore, *the photometric asymmetry of light changes due to the rotational distortion of components whose axes are inclined to the orbital plane should oscillate in sign with the precessional period of the respective stars*; and this fact, if confirmed observationally, would identify the nature of asymmetric light variations beyond any reasonable doubt. Such a verification would, however, require a continuous (or at least intermittent) observational record of light changes which may not amount to more than 0·1 of a magnitude over several hundred orbital cycles; and, needless to stress, such long series of observations of requisite precision as well as homogeneity are not yet available for many eclipsing systems. For systems exhibiting asymmetries between minima which have been so observed (U Cephei, for instance), it would seem that our prediction is indeed borne out by the observed facts; but more detailed analysis will be necessary before such an assertion can be placed on a more solid footing.

Throughout all foregoing developments of this section we have confined our explicit attention to the light changes, invoked by distortion, of the *primary* component of mass m_1 which we placed at the origin of our coordinate system. It goes, however, without saying that the light of a binary *system* as seen by the observer at a great distance consists of the *sum* of luminosities of the two components; and the same is true of their light changes. The light changes of the *secondary* component arising from its own distortion by the primary follows obviously from the same equation (2-37) provided only that the subscripts of both $v^{(2)}$ and $w^{(j)}$ are changed appropriately, and the phase ψ in l_0 shifted by π. This difference in phase will change the sign of the only odd harmonic occurring in (2-37), but not those of even harmonics. As a result, *no odd harmonics can appear in the combined light of a binary system consisting of two identical components*; for the effects of odd harmxnics of one star would be neutralized by those due to its mate. On the other hand, *even-order harmonics will always tend to reinforce each other.*

So far we have also tacitly assumed the relative orbit of the two stars to be *circular*, and considered their distance R a constant. If, however, their orbit becomes eccentric, R naturally becomes a function of the time. The separation of the two components will then vary continuously and cause the distortion as well as distribution of brightness of both components to vary, in the course of a cycle. The dynamical theory of tides produced on each star by the tide-generating potential of the form given by equation (1-57) of Chapter II would call for the development of each term in (1-57) into a series of purely harmonic functions of the mean anomaly. To each such function there would correspond one or more partial tides; these tides would sweep around the primary each with a constant amplitude, speed, and phase; and their combination would represent the total distortion. Such a treatment would, however, be rather complicated and has not so far been given. In the limiting case, when the orbital period is long in comparison with the period of free non-radial oscillation of the components, their distortion should be essentially given by the *equilibrium theory of tides* as developed in section II.1. In the dynamical theory of tides the resulting distortion of either component must be periodic in the orbital period. In the equilibrium limit this must still be true; and since the viscosity in outer layers of a star (where tides attain appreciable height) is likely to be negligibly small, there should be no difference in phase, causing the tidal distortion to assume at each instant an equilibrium value appropriate for the instantaneous prevailing field of force.

The question naturally arises as to whether the free periods of non-radial oscillations of the stars are short enough to justify an application of the equilibrium theory of tides. All investigations of this subject carried out so far* reveal that this can indeed be expected to be the case. Quantitative investigations of this problem have, to be sure, been limited to the polytropic family of models; and their analysis reveals that at least the polytropes admit, in general, of an infinite discrete spectrum of characteristic frequencies of non-radial oscillation, with the fundamental mode oscillating in a period

* *Cf.*, e.g., T. G. Cowling, *M.N.*, **101**, 367, 1941; Z. Kopal, *Ap. J.*, **109**, 509, 1949; E. Sauvenier-Goffin, *Bull. Soc. R. Sci. Liège*, **20**, 20, 1951; J. Owen, *M.N.*, **117**, 384, 1957.

which is in general short in comparison with that of the disturbing force (i.e., the orbital period). The really arresting feature of their oscillatory properties is, however, the fact that, with increasing degree of central condensation, low modes of non-radial oscillations gradually disappear from the characteristic spectrum until, for very high degrees of central condensation, such configurations can oscillate freely only in periods which are either very short or very long.* The density of the low-frequency end of the characteristic spectrum is, moreover, such that the periods of some of its members may indeed come very close to that of the binary orbit and resonate with it. On the other hand, such modes will almost certainly be highly damped in the outer parts of a star, so that even a near approach to resonance may not give rise to observationally appreciable phenomena;† and if so, *the external form of the components in eccentric binary systems should sensibly be one of equilibrium under the instantaneous rotational and tidal forces.*

As long as this holds true, equations (2-37) or (2-54) continue to reproduce the changes of light exhibited by distorted components of close binary systems in the course of their revolution whether their relative orbit is circular or eccentric—provided only that, in this latter case, we allow for the respective variation of R and replace a_1/R in $v^{(2)}$ as well as $w^{(j)}$ by

$$\frac{a_1}{R} = \frac{a_1}{A(1-e^2)}\{1 + e\sin(\omega - \psi)\}, \tag{2-56}$$

where A denotes the semi-major axis of the relative orbit; e, its eccentricity; and ω, the longitude of periastron measured from the ascending node. The presence of the phase angle on the right-hand side of the foregoing equation (2-56) will obviously destroy the symmetry which characterized the light changes $\delta\mathfrak{L}$ of systems possessing circular orbits; and their theoretical light curves between minima will now consist of a number of terms of the form $e^m \sin^m \psi \cos^n \psi$ as well as $\cos^n \psi$.

The variability of a_1/R, in eccentric systems, leads to one additional interesting consequence. The Legendre polynomials $P_2(l_0)$ and $P_2(n_0)$ as well as $P_4(l_0)$ on the right-hand side of (2-37) contain (apart from different powers of l_0 or n_0) the constant terms $-\frac{1}{2}$ and $\frac{3}{8}$. As long as the orbit is circular, these terms do not give rise to any light changes; but the appearance of orbital eccentricity will evidently set them to vary with the phase in inverse proportion to the third and fifth powers of R. The amplitude of their variation should, moreover, be proportional to the orbital eccentricity. The variation of light $\delta\mathfrak{L}_e$ due to the most important term arising in this connection then readily takes the form

$$\delta\mathfrak{L}_e = \frac{1}{6}\left(\frac{a_1}{A}\right)^3 \frac{\Delta_2 X_2}{(1-e^2)^3}\{1 + \tfrac{1}{4}\beta_2\}\left\{2\frac{m_2}{m_1} - 1\right\}\{1 + e\sin(\omega - \psi)\}^3. \tag{2-57}$$

* *Cf.* J. Owen, *M.N.*, **117**, 384, 1957.

† For a fuller discussion of their implications *cf.* Z. Kopal, "Unsolved Problems in the Theory of Eclipsing Variables", *Harvard Centennial Symposia* (*Harv. Obs. Mono.*, No. 7, Cambridge, Mass., 1948) pp. 261-275.

Such terms do not depend on the orientations of the components with respect to the observer, but solely on the true anomaly. Since all are bound to vary inversely with the distance between the components, their effect will be to make a distorted star appear brighter at periastron than at apastron (*periastron effect*). The magnitude of the first-order periastron effect as represented by the foregoing term is evidently of the order $e(a_1/A)^3$. Its light effect should, therefore, be e times smaller than that due to the photometric ellipticity and should, therefore, become appreciable only in very eccentric or very close systems.

In conclusion, one more effect which may represent an additional cause of asymmetry of light variation of close binary systems should be mentioned in this place.* Observations of cepheid variables suggest that the variation of light emitted by radially oscillating stars lags behind the variation in radius. If the rotational and orbital velocities differ, the tidal distortion of components in eccentric binary systems becomes a forced non-radial oscillation. We might then expect that the variation in light which may accompany such non-radial oscillations will also lag behind vertical displacement; and this—if true—should render the light changes asymmetric. One might remark, of course, that it is not yet certain whether non-radial oscillations of components in eccentric binary systems are accompanied by any intrinsic light variation. Such variation of radially-oscillating stars originates presumably in deep interiors of the stars, while non-radial oscillations of tidal origin represent relatively a more superficial phenomenon. Whether or not this effect is actually operative in close binary systems remains to be clarified by future investigations.

IV.3. LIGHT CHANGES ARISING FROM THE ECLIPSES

In the preceding section we have investigated the changes of light in close binary systems which are due to the distortion of their components caused by rotation and tides. Provided only that the proximity of both components is sufficient to render distortion appreciable, such light changes are bound to arise irrespective of whether or not our binary system happens also to be an eclipsing variable. Should, however, the inclination of the orbital plane to the line of sight be such as to cause the components to eclipse each other at the time of the conjunctions, the results derived so far can represent only the variation of light taking place between eclipses. The changes of light exhibited by close eclipsing systems within minima—geometrically much more complicated phenomena—must, however, likewise be affected by the distortion of the constituent components. A distortion of the eclipsing component must produce a corresponding deformation of its shadow cylinder cast in the direction of the line of sight. A distortion of the component undergoing eclipse will, moreover, not only alter the proportion of its apparent

* *Cf.* T. G. Cowling, *M.N.*, **101**, 367, 1942.

disk intercepted at any phase by the shadow cylinder of its mate, but will also re-arrange the distribution of surface brightness over the eclipsed portion. The aim of the present section will be to analyze and describe such eclipse phenomena to the same degree of accuracy to which we dealt with the light changes between minima in the preceding section.

In order to do so, let us confine our attention to the star undergoing eclipse which, as before, will be consistently referred to as the primary component. Its loss of light $\Delta\mathfrak{L}$ as seen by a distant observer at any moment during eclipse can again be generally expressed as

$$\Delta\mathfrak{L} = \int_S J \cos\gamma \, d\sigma, \tag{3-1}$$

where as before in section IV.2, J denotes the surface brightness at any point of the primary star; γ, the angle of foreshortening; $d\sigma$, the surface element; and S, the range of integration which has now to be extended over the whole eclipsed area. The explicit form of the integrand $J \cos\gamma \, d\sigma$, including the effects of the second, third and fourth spherical harmonic distortion, has already been derived in section IV.2 and expressed in terms of polar coordinates with respect to a rectangular system XYZ, with origin at the centre of the primary star, whose X-axis coincides with the line joining the centres of the two components, and $Z = 0$ represents the plane of their orbit. This system rotates, therefore, with respect to the observer, with the radius-vector in the course of each cycle.

In studying the changes of light exhibited between minima we find it, however, convenient to change over to another rectangular system $X'Y'Z'$ stationary with respect to the observer, having the same origin but defined so that its Z'-axis coincides constantly with the line of sight, and its X'-axis is in the direction of the projected centre of the secondary component. A transformation of coordinates between the two systems is clearly governed by the scheme of the direction cosines

	Z'	Y'	X'
X	l_0	l_1	l_2
Y	m_0	m_1	m_2
Z	n_0	n_1	n_2

where, as before,

$$l_0 = \cos\psi \sin i, \qquad m_0 = \sin\psi \sin i, \qquad n_0 = \cos i; \tag{3-2}$$

but

$$l_1 = 0\,, \qquad m_1 = -\frac{n_0}{\sqrt{1 - l_0^2}}\,, \qquad n_1 = \frac{m_0}{\sqrt{1 - l_0^2}}\,; \tag{3-3}$$

and

$$l_2 = \sqrt{1 - l_0^2}\,, \quad m_2 = -\frac{l_0 n_0}{\sqrt{1 - l_0^2}}\,, \quad n_2 = -\frac{l_0 n_0}{\sqrt{1 - l_0^2}}\,. \tag{3-4}$$

The coordinates of the projected centre of the eclipsing component before maximum eclipse are assumed to be positive.

Let us, furthermore, adopt as before the radius R of the relative orbit of the two components as our unit of length and set

$$\frac{a_1}{R} = r_1 \quad \text{and} \quad \frac{a_2}{R} = r_2 , \tag{3-5}$$

where r_1 and r_2 denote the fractional radii of spheres having the same volume as the respective stars. The projected distance δ of their centres in the $X'Y'$-plane (which is tangent to the celestial sphere at the origin) is clearly given by

$$\delta^2 = \sin^2 \psi \sin^2 i + \cos^2 i = l_2^2 . \tag{3-6}$$

Moreover, the equation of the shadow cylinder cast by the secondary component in the direction of the line of sight in the primed coordinates then becomes

$$(\delta - X')^2 + Y'^2 = r_2^2 (1 + \Delta \bar{r}_2)^2, \tag{3-7}$$

where $\Delta \bar{r}_2$ stands for the deformation of the secondary's shadow cylinder in the plane $Z' = 0$. If, furthermore, Δr_1 stands for the corresponding distortion of the primary component, the surface of the latter admits evidently of the parametric representation

$$\left.\begin{aligned} X' &= r_1(1 + \Delta r_1)\, L, \\ Y' &= r_1(1 + \Delta r_1)\, M, \\ Z' &= r_1(1 + \Delta r_1)\, N, \end{aligned}\right\} \tag{3-8}$$

where L, M, N denote the direction cosines of any arbitrary radius in the primed system. These direction cosines satisfy the relation

$$L^2 + M^2 + N^2 = 1 \tag{3-9}$$

and can be regarded as rectangular coordinates, in the primed system, of unit radius.

Consistent with (3-5) we prefer, however, to keep the radius R of the relative orbit as our unit of length. In order to do so let us, therefore, put

$$L = \frac{x}{r_1}, \qquad M = \frac{y}{r_1}, \qquad N = \frac{z}{r_1} \tag{3-10}$$

and adopt x, y, z as our new independent variables which should replace the polar coordinates r, θ, ϕ used in section IV.2. These coordinates are obviously related with the direction cosines λ, μ, ν used formerly by means of the relations

$$\left.\begin{aligned} r_1\lambda &= l_0 z + l_1 y + l_2 x, \\ r_1\mu &= m_0 z + m_1 y + m_2 x, \\ r_1\nu &= n_0 z + n_1 y + n_2 x. \end{aligned}\right\} \tag{3-11}$$

By virtue of these equations the Legendre polynomials $P_j(\lambda)$ or $P_2(\nu)$ associated with the effects of tidal or rotational distortion may now be rewritten as polynomials of the j-th degree in terms of integral powers of x, y, z, with coefficients depending upon the amount of distortion and the relative position of the components in their orbit. Since, furthermore,

$$r_1 \sin\theta \, d\theta \, d\phi = \frac{dx\,dy}{z}, \tag{3-12}$$

the whole integrand in (3-1) is found to become a rational algebraic function of x, y and z; the general terms arising from the rotational and tidal distortion being of the form $x^m y z^n$ and $x^m z^n$, respectively, where $m \geqslant 0$ and $n \geqslant -1$.

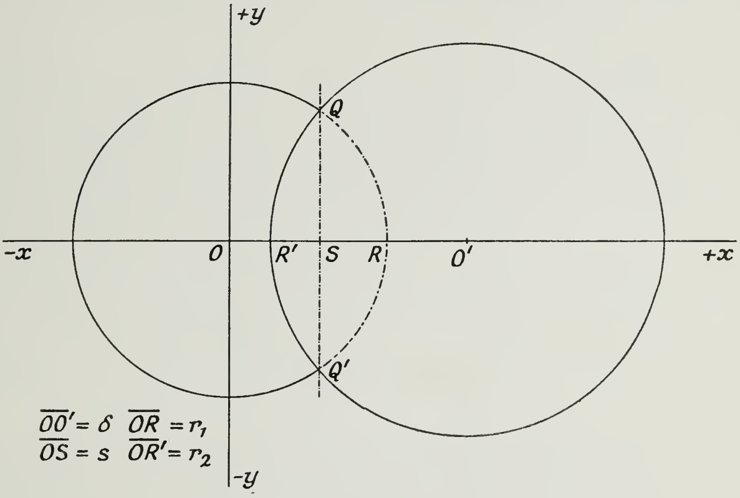

FIGURE 4-2. GEOMETRY OF THE ECLIPSES

In order to perform the integration on the right hand side of (3-1) we must still delimit the range S over which the operation is to be extended. Turning first to the parametric equations (3-8) we note that N, the cosine of the angle of foreshortening, vanishes at the limb of the primary star if the latter is spherical, and for distorted bodies becomes a small quantity whose squares we agreed to ignore. Hence, to the order of accuracy we are working, the intersection of the surface of the primary star with the xy-plane reduces to the circle

$$x^2 + y^2 = r_1^2, \tag{3-13}$$

which represents the arc QRQ' (see Fig. 4-2) limiting the eclipsed fraction of the primary star. In order to ascertain the form of the arc $QR'Q'$ limiting the opposite border of the eclipsed area, let us solve equation (3-7) of the shadow cylinder cast by the secondary (eclipsing) component with equations (3-8) defining the surface of the primary star. Inserting (3-8) in (3-7) we find that, to the first order in small quantities, the equation of the arc $QR'Q'$ becomes

$$(\delta - x)^2 + y^2 = r_2^2(1 + 2\Delta\bar{r}_2) - 2\Delta r_1\{r_2^2 - \delta(\delta - x)\} + \dots . \tag{3-14}$$

The deviation Δy of the arc $QR'Q'$ from the circle

$$(\delta - x)^2 + y^2 = r_2^2 \tag{3-15}$$

assumes the form

$$\Delta y_2 = \frac{r_2^2 \Delta\bar{r}_2}{r_2^2 - (\delta - x)^2} - \frac{r_2^2 - \delta(\delta - x)\,\Delta r_1}{r_2^2 - (\delta - x)^2} + \dots . \tag{3-16}$$

The first term on the right-hand side of this latter equation arises from the distortion of the eclipsing component, the second term from that of the eclipsed star.

The significance of a transformation of coordinates effected in this section rests not only on the fact that the whole integrand of (3 1) can be reduced to a rational algebraic function in the new coordinates x, y, z, but also that the limits of integration S now assume relatively simple forms. In more specific terms, the fraction of the primary's light lost at any moment during eclipse can be represented as consisting of two parts: the 'circular' one obtained by integrating (3-1) within the common region bounded by the circles (3-13) and (3-15), and the 'boundary corrections' arising from the distortion of the two arcs QRQ' and $QR'Q'$ (see Fig. 4-2). If

$$s = \frac{r_1^2 - r_2^2 + \delta^2}{2\delta} \tag{3-17}$$

denotes the x-coordinate of the common chord of the circles (3-15) and (3-15) at any phase of the eclipse, the 'circular' part of the resulting fractional loss of light can evidently be expressed in terms of the functions α_n^m as defined by the equation

$$\pi r_1^{m+n+2}\,\alpha_n^m = \int_s^{r_1}\int_{-\sqrt{r_1^2-x^2}}^{\sqrt{r_1^2-x^2}} + \int_{\delta-r_2}^{s}\int_{-\sqrt{r_2^2-(\delta-x)^2}}^{\sqrt{r_2^2-(\delta-x)^2}} x^m z^n\,dx\,dy \tag{3-18}$$

if the eclipse is partial, and by

$$\pi r_1^{m+n+2}\,\alpha_n^m = \int_{\delta-r_2}^{\delta+r_2}\int_{-\sqrt{r_2^2-(\delta-x)^2}}^{\sqrt{r_2^2-(\delta-x)^2}} x^m z^n\,dx\,dy \tag{3-19}$$

if it is annular. Such functions will hereafter be referred to as the *associated α-functions* of the (integral) order m and index n, and will play a central role in the theory of light changes of eclipsing binary systems.

The light changes arising from the 'boundary corrections' due to the distortion of the arcs QRQ' and $QR'Q'$ which limit the eclipsed area can, furthermore, be obtained by a *single* integration of $2\Delta y$ with respect to x between appropriate limits. These limits obviously range from $\delta - r_2$ to s as long as the eclipse remains partial, and from $\delta - r_2$ to $\delta + r_2$ when it becomes annular. If the disk of the primary component is uniformly bright, the integrand will consist solely of $2\Delta y$; if it were completely darkened at limb $2\Delta y$ is to be pre-multiplied by the foreshortening factor $2\delta(s - x)$. As, moreover, Δy is obviously a small quantity of first order, the terms arising from the gravity-darkening are clearly negligible in this connection.

In order to be able to perform the actual integration, we must express the quantity Δr_1 as well as $\Delta \bar{r}_2$ involved on the right-hand side of (3-14) in terms of x, y and z. If, for simplicity, we assume that the axis of rotation of the component undergoing eclipse is perpendicular to the orbital plane, it follows from the results established in section II.1 that

$$\Delta r = \sum_{j=2}^{4} w^{(j)} P_j(\lambda) - \tfrac{1}{3} v^{(2)} P_2(\nu), \tag{3-20}$$

where the constants $v^{(2)}$ and $w^{(j)}$ specifying the amount of distortion are given by equations (2-33) and (2-35), respectively. Now along the arc $QR'Q'$ at which the primary's surface is intersected by the shadow cylinder cast by the secondary component, $\Delta r_1 \equiv \Delta r(\lambda_1, \nu_1)$, where

$$\left.\begin{aligned} r_1\lambda_1 &= l_0\sqrt{2\delta(s - x)} + l_2 x\,, \\ r_1\nu_1 &= n_0\sqrt{2\delta(s - x)} + n_1\sqrt{r_2^2 - (\delta - x)^2} + n_2 x\,; \end{aligned}\right\} \tag{3-21}$$

while along the curve at which the shadow cylinder is tangent to the secondary component, $\Delta r_2 \equiv \Delta r_2(\bar{\lambda}_2, \bar{\nu}_2)$, where

$$\left.\begin{aligned} r_2\bar{\lambda}_2 &= + l_2(\delta - x)\,, \\ r_2\bar{\nu}_2 &= n_1\sqrt{r_2^2 - (\delta - x)^2} + n_2(\delta - x)\,. \end{aligned}\right\} \tag{3-22}$$

The 'boundary corrections' due to the deformation of the arc $QR'Q'$ which arises from the distortion of the eclipsed component should, therefore, be expressible in terms of the functions $\mathfrak{J}^m_{\beta,\gamma}$ as defined by the equation

$$\pi r_1^{\beta+\gamma+m+3}\, \mathfrak{J}^m_{\beta,\gamma} = (r_2^2 - \delta^2) I^m_{0,\beta,\gamma} + \delta I^{m+1}_{0,\beta,\gamma}, \tag{3-23}$$

where β, γ, m are arbitrary integers, and

$$I^m_{0,\beta,\gamma} = \int_{\delta - r_2}^{c_2} [r_2^2 - (\delta - x)^2]^{\beta/2} [2\delta(s - x)]^{\gamma/2} x^m \, dx; \tag{3-24}$$

where, in the limits, $c_2 = s$ or $\delta + r_2$ depending on whether the eclipse is partial or annular. Moreover, the remaining boundary corrections which arise from the distortion $\Delta \bar{r}_2$ of the secondary component should be expressible in terms of an additional family of integrals of the form

$$\pi r_2^{\beta+\gamma+m+1} I^m_{\beta,\gamma} = \int_{\delta - r_2}^{c_2} [r_2^2 - (\delta - x)^2]^{\beta/2}\, [2\delta(s - x)]^{\gamma/2}\, (\delta - x)^m \, dx. \tag{3-26}$$

The equations governing the changes of light due to eclipses in close binary systems can now be put down at once. Let, as in section IV.2, the loss of light $\Delta\mathfrak{L}$ during eclipse of an arbitrarily darkened star be generally expressed as

$$\Delta\mathfrak{L} = \sum_{h=1}^{k} C^{(h)}\alpha^0_{h-1} + \Delta\mathfrak{L}^{(h)}, \tag{3-27}$$

where the coefficients $C^{(h)}$ continue to be given by equations (2-26) and (2-27), and the α^0_n's are associated α-functions as defined by equations (3-18) and (3-19) above. The latter functions occurring in (3-27) arise from eclipses of spherical stars whose limb-darkening requires a retention of k terms in the law (2-15) for adequate description. Let, furthermore, the effects of distortion be represented by

$$\Delta\mathfrak{L}^{(h)} = f^{(h)}_* + f^{(h)}_1 + f^{(h)}_2, \tag{3-28}$$

where $f^{(h)}_*$ constitutes the contributions of geometrical distortion and gravity-darkening over the circular portion of the eclipsed disk and expressible, therefore, likewise in terms of the associated α-functions; $f^{(h)}_{1,2}$ are photometric contributions of the 'boundary corrections' along $QR'Q'$ arising from the distortion of the primary and secondary component.

In order to evaluate the explicit form of the function $f^{(h)}_*$, it is sufficient to re-write the integrand (2-21) in powers of x, y, z by means of equations (3-11) and (3-12) and remember that, in the rotational terms, odd powers of y vanish on account of symmetry while even powers can be expressed in terms of those x and z by means of the relation $y^2 = r_1^2 - x^2 - z^2$. A term-by-term integration over circular limits then yields

$$\begin{aligned} f^{(h)}_* = \tfrac{1}{3}\{&\tfrac{1}{2}\Omega^{(h)}_2[3(n_0^2 - n_1^2)\alpha^0_{h+1} + 3(n_2^2 - n_1^2)\alpha^2_{h-1} + 6n_0n_2\alpha^1_h + 2P_2(n_1)\alpha^0_{h-1}] \\ &+ h[2P_2(n_0)\alpha^0_{h-1} + 3n_0n_2\alpha^1_{h-1}]\}v^{(2)}_1 \\ &+ \{\tfrac{1}{2}\Omega^{(h)}_2[3l_0^2\alpha^0_{h+1} + 6l_0l_2\alpha^1_h + 3l_2^2\alpha^2_{h-1} - \alpha^0_{h-1}] + h[2P_2(l_0)\alpha^0_{h-1} \\ &+ 3l_0l_2\alpha^1_{h-2}]\}w^{(2)}_1 \\ &- \{\tfrac{1}{2}\Omega^{(h)}_3[5l_0^3\alpha^0_{h+2} + 15l_0^2l_2\alpha^1_{h+1} + 15l_0l_2^2\alpha^2_h - 5l_2^3\alpha^3_{h-1} - 3l_0\alpha^0_h \\ &- 3l_2\alpha^1_{h-1}] + h[(l_0\alpha^0_h + 2l_2\alpha^1_{h-1})P'_3(l_0) - \tfrac{3}{2}l_0(\alpha^0_h - 5l_2^2\alpha^2_{h-1} \\ &+ \alpha^0_{h-1})]\}w^{(3)}_1 \\ &- \{\tfrac{1}{8}\Omega^h_4[35l_0^4\alpha^0_{h+3} + 140l_0^3l_2\alpha^1_{h+2} + 210l_0^2l_2^2\alpha^2_{h+1} + 140l_0l_2^3\alpha^3_h \\ &+ 35l_2^4\alpha^4_{h-1} - 30l_0^2\alpha^0_{h+1} - 60l_0l_2\alpha^1_h - 30l_2^2\alpha^2_{h-1} + 3\alpha^0_{h-1}] \\ &+ \tfrac{1}{2}h[2l_0P'_4(l_0)\alpha^0_{h+1} + 15l_2^2(7l_0 - 1)\alpha^2_{h-1} - 2P'_3(l_0)\alpha^0_{h-1} \\ &+ 15l_0l_2(7l_0 - 2)\alpha^1_h + 35l_0l_2^3\alpha^3_{h-2} - 15l_0l_2\alpha^1_{h-2}]\}w^{(4)}_1 + \dots, \end{aligned} \tag{3-29}$$

where $\Omega^{(h)}_j$ continues to be given by equation (2-22).

A similar integration of (3-16) shows that

$$
\begin{aligned}
f_1^{(h)} = {} & \tfrac{1}{3}\{3n_0^2\,\mathfrak{I}^0_{-1,h+1} + 3n_1^2\,\mathfrak{I}^0_{1,h-1} + 3n_2^2\,\mathfrak{I}^2_{-1h-1,} + 6n_0n_2\,\mathfrak{I}^1_{-1,h} \\
& \quad - \mathfrak{I}^0_{-1,h-1}\}v_1^{(2)} \\
& - \{3l_0^2\,\mathfrak{I}^0_{-1,h+1} + 6l_0l_2\,\mathfrak{I}^1_{-1,h} + 3l_2^2\,\mathfrak{I}^2_{-1,h-1,} - \mathfrak{I}^0_{-1h,-1}\}\,w_1^{(2)} \\
& - \{5l_0^3\,\mathfrak{I}^0_{-1,h+2} + 15l_0^2l_2\,\mathfrak{I}^1_{-1,h+1} + 15l_0l_2^2\,\mathfrak{I}^2_{-1,h} + 5l_2^3\,\mathfrak{I}^3_{-1,h-1} \\
& \quad - 3l_0\,\mathfrak{I}^0_{-1,h} - 3l_2\,\mathfrak{I}^1_{-1,h-1}\}\,w_1^{(3)} \\
& - \tfrac{1}{4}\{35l_0^4\,\mathfrak{I}^0_{-1,h+3} + 140l_0^3l_2\,\mathfrak{I}^1_{-1,h+2} + 210l_0^2l_2^2\,\mathfrak{I}^2_{-1,h+1} \\
& \quad + 140l_0l_2^3\,\mathfrak{I}^3_{-1,h} + 35l_2^4\,\mathfrak{I}^4_{-1,h-1} - 30l_2^2\,\mathfrak{I}^0_{-1,h+1} - 60l_0l_2\,\mathfrak{I}^1_{-1,h} \\
& \quad - 30l_2^2\,\mathfrak{I}^2_{-1,h-1} + 3\mathfrak{I}^0_{-1,h-1}\}\,w_1^{(4)} + \dots,
\end{aligned}
\tag{3-30}
$$

and

$$
\begin{aligned}
(r_1/r_2)^2 f_2^{(h)} = {} & -\tfrac{1}{3}\{3n_1^2 I^0_{1,h-1} + 3n_2^2 I^2_{-1,h-1} - I^0_{-1,h-1}\}v_2^{(2)} \\
& + \{3l_2^2 I^2_{-1,h-1} - I^0_{-1,h-1}\}w_2^{(2)} \\
& + \{5l_2^3 I^3_{-1,h-1} - 3l_2 I^1_{-1,h-1}\}w_2^{(3)} \\
& + \tfrac{1}{4}\{35l_2^4 I^4_{-1,h-1} - 30l_2^2 I^2_{-1,h-1} + 3I^0_{-1,h-1}\}w_2^{(4)} + \dots
\end{aligned}
\tag{3-31}
$$

As all functions occurring on the right-hand sides of equations (3-27) and (32-8) have thus been explicitly formulated, the corresponding expressions for the theoretical light changes arising from the eclipses of mutually distorted components in close binary systems are now complete.

IV.4. ASSOCIATED ALPHA-FUNCTIONS AND RELATED INTEGRALS

In the preceding section of this chapter we were led to express the changes of light due to any type of the eclipse of a distorted star in terms of 'circular' integrals of the form (3-18) or (3-19) defining the *associated α-functions* α_n^m; while the supplementary light changes arising from the limb distortion of the eclipsed and eclipsing components were found to be expressible in terms of the additional 'boundary correction' integrals $\mathfrak{I}^m_{\beta,\gamma}$ and $I^m_{\beta,\gamma}$ as defined by equations (3-24) and (3-26), respectively. The aim of the present section will be to establish the explicit form of all such integrals in terms of the geometrical elements of the eclipse by performing their integration in a finite number of terms, and thus to prepare the ground for eventual tabulation of their numerical values for any phase of the eclipse.

In order to do so, we find it convenient to split up equation (3-18) in two parts by setting

$$\pi r_1{}^{m+n+2}\alpha_n^m = \mathfrak{A}_n^m + \mathfrak{B}_n^m, \tag{4-1}$$

where

$$\mathfrak{A}_n^m = \int_s^{r_1} \int_{-\sqrt{r_1{}^2-x^2}}^{+\sqrt{r_1{}^2-x^2}} x^m z^n \, dx \, dy, \tag{4-2}$$

$$\mathfrak{B}_n^m = \int_{\delta-r^2}^{s} \int_{-\sqrt{r_2{}^2-(\delta-x)^2}}^{+\sqrt{r_2{}^2-(\delta-x)^2}} x^m z^n \, dx \, dy, \tag{4-3}$$

and to deal with each in turn. For annular eclipses, equation (3-19) can likewise be written as

$$\pi r_1^{m+n+2}\alpha_n^m = \mathfrak{B}_n^m\,, \tag{4-4}$$

provided that s is replaced by $(\delta + r_2)$ in the limits of integration on the right-hand side.

As to $\mathfrak{A}$, integrating we obtain at once that

$$\begin{aligned} \mathfrak{A}_n^m &= B(\tfrac{1}{2}, 1 + \tfrac{1}{2}n) \int_s^{r_1} x^m (r_1^2 - x^2)^{(n+1)/2} \, dx \\ &= B(\tfrac{1}{2}, 1 + \tfrac{1}{2}n)\{D_{n+1}^m(r_1) - D_{n+1}^m(s)\}, \end{aligned} \tag{4-5}$$

where $B(m, n)$ denotes the complete beta-function (numerical factor), and

$$D_{2\nu}^m(x) = \int_0^x x^m (r_1^2 - x^2)^\nu \, dx, \tag{4-6}$$

which is a binomial integral tractable by elementary means. Its evaluation reveals in fact that, if $0 \leqslant s \leqslant r_1$,

$$(n+3)\mathfrak{A}_n^m = B(\tfrac{1}{2}, 1 + \tfrac{1}{2}n)\sigma^{n+2}\,{}_2F_1(\sigma^2), \tag{4-7}$$

while for $-r_1 \leqslant s \leqslant 0$

$$(n+3)\mathfrak{A}_n^m = B(\tfrac{1}{2}, 1 + \tfrac{1}{2}n)\{2\,{}_2F_1(1) - \sigma^{n+2}\,{}_2F_1(\sigma^2)\}, \tag{4-8}$$

where

$${}_2F_1(\sigma^2) \equiv {}_2F_1\{\tfrac{1}{2}(1-m), \tfrac{1}{2}(n+3), \tfrac{1}{2}(n+5), \sigma^2\} \tag{4-9}$$

stands for an ordinary hypergeometric series of an argument defined by

$$r_1^2\sigma^2 = r_1^2 - s^2 = r_2^2 - (\delta - s)^2. \tag{4-10}$$

The calculation of $\mathfrak{B}_n^m$ will prove to be more tedious. Integrating with respect to y we find that for n zero or an even integer,

$$\mathfrak{B}_{2\nu}^m = \frac{1}{\pi} B(\tfrac{1}{2}, 1 + \nu) \sum_{j=0}^{\nu} B(\tfrac{1}{2}, \tfrac{1}{2} + \nu - j) I_{2j,1,2(\nu-j)}^m\,; \quad \nu = 0, 1, 2\ldots; \tag{4-11}$$

while if n is odd,

$$\mathfrak{B}^m_{2\nu-1} = \frac{1}{\pi} B(\tfrac{1}{2}, \tfrac{1}{2} + \nu) \sum_{j=0}^{\nu} B(\tfrac{1}{2}, j) I^m_{2(\nu-j),1,j-1} + 2\Pi^m_{2\nu}, \tag{4-12}$$

where, as a generalization of (3-24),

$$I^m_{2\alpha,\beta,\gamma} = \int_{\delta-r_2}^{c_2} (r_1^2 - x^2)^\alpha [r_2^2 - (\delta - x)^2]^{\beta/2} [2\delta(s - x)]^{\gamma/2} x^m \, dx, \tag{4-13}$$

and

$$\Pi^m_{2\nu} = \int_{\delta-r_2}^{c_2} x^m (r_1^2 - x^2)^\nu \sin^{-1} \left\{ \frac{r_2^2 - (\delta - x)^2}{r_1^2 - x^2} \right\} dx, \tag{4-14}$$

where $c_2 = s$ or $(\delta + r_2)$ as to whether the eclipse is partial or annular. These are the two standard forms which we have to evaluate.

The methods of integration of (4-13) depend in principle upon the values of the three subscripts α, β, γ. The first can be easily suppressed by putting

$$I^m_{2\alpha,\beta,\gamma} = \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} r_1^{2(\alpha-j)} I^{m+2j}_{0,\beta,\gamma}. \tag{4-15}$$

Further, the nature of our problem is such that β can assume values of odd integers only.* Thus the character of (4-13) depends on whether γ is odd or even. If it is zero or an even integer, (4-13) can be solved in terms of circular and algebraic functions by elementary methods.

If, however, γ is odd, (4-13) becomes an elliptic integral. In order to evaluate it we change over to a new variable

$$t = x - h, \tag{4-16}$$

where h is a constant defined so as to render

$$[r_2^2 - (\delta - x)^2][s - x] = (t - e_1)(t - e_2)(t - e_3), \tag{4-17}$$

subject to conditions that

$$e_1 > e_2 > e_3 \qquad \text{and} \qquad e_1 + e_2 + e_3 = 0.$$

Evidently,

$$h = \tfrac{1}{3}(2\delta + s), \tag{4-18}$$

and

$$\left.\begin{aligned} e_1 &= +\tfrac{1}{3}(\delta - s) + r_2, \\ e_2 &= -\tfrac{2}{3}(\delta - s), \\ e_3 &= +\tfrac{1}{3}(\delta - s) - r_2, \end{aligned}\right\} \tag{4-19p}$$

if the eclipse is partial, and

$$\left.\begin{aligned} e_1 &= -\tfrac{2}{3}(\delta - s), \\ e_2 &= +\tfrac{1}{3}(\delta - s) + r_2, \\ e_3 &= +\tfrac{1}{3}(\delta - s) - r_2, \end{aligned}\right\} \tag{4-19a}$$

* Terms with β zero or an even integer do not occur in the light curve on account of symmetry.

if it is annular. In either case we are therefore entitled to put

$$t = \wp(u), \tag{4-20}$$

where $\wp$ denotes the Weierstrass π-function of an argument u which will replace t as our independent variable.

The integrals on the right-hand side of (4-15) in terms of this new variable become

$$I^m_{0,\beta,\gamma} = -2i^{\beta+}(2\delta)^{\gamma/2} \int_{\omega_2}^{\omega_1+\omega_2} \{[\wp(u) - e_1][\wp(u) - e_3]\}^{(\beta+1)/2} \times \{\wp(u) - e_2\}^{(\gamma+1)/2}\{\wp(u) + h\}^m \, du, \tag{4-21}$$

with limits defined by

$$\left.\begin{aligned} \wp(\omega_1) &= e_1 , \\ \wp(\omega_2) &= e_3 . \end{aligned}\right\} \tag{4-22}$$

If, as in equations (4-11) or (4-12), $\beta = 1$, then (4-22) can undergo further reduction. For, by definition, we have

$$2\sqrt{[\wp(u) - e_1][\wp(u) - e_2][\wp(u) - e_3]} = \wp'(u), \tag{4-23}$$

where accent denotes derivative with respect to u. Squaring (4-23) and inserting in (4-21) we obtain

$$I^m_{0,1,\gamma} = -\tfrac{1}{2}i^{\gamma+1}(2\delta)^{\gamma/2} \int_{\omega_2}^{\omega_1+\omega_2} [\wp(u) - e_2]^{(\gamma-1)/2}[\wp(u) + h]^m[\wp'(u)]^2 \, du. \tag{4-24}$$

But, if we abbreviate

$$\begin{aligned} g_2 &= -4(e_1e_2 + e_1e_3 + e_2e_3), \\ g_3 &= +4e_1e_2e_3 , \end{aligned} \tag{4-25}$$

it follows from (4-23) that*

$$\{\wp'(u)\}^2 = 4\wp^3(u) - g_2\wp(u) - g_3 \tag{4-26}$$

and hence, γ being odd, the whole integrand of (4-20) can be written out as a polynomial of the $\{\tfrac{1}{2}(\gamma - 1) + m + 3\}$th degree in powers of $\wp(u)$.

The last step in the evaluation of (4-17) or (4-20) consists in reducing integrals

$$\int_{\omega_2}^{\omega_1+\omega_2} \{\wp(u)\}^j \, du, \qquad j = 0, 1, 2 \ldots$$

to Legendre normal forms. This can proceed by expressing, by successive differentiation of (4-26), the powers of $\wp(u)$ in terms of its derivatives. A general expression for $\{\wp(u)\}^j$ contains, in addition to $\wp^{\mathrm{II}\,(j-1)}(u)$ and lower

* This is the differential equation defining $\wp(u)$; *cf.* E. T. Whittaker and G. N. Watson, *Modern Analysis*, Cambridge, 1920; section 20.22.

derivatives of even orders, also the first power of $\wp(u)$ (for $j > 2$) and a constant. If we put

$$\int_{\omega_2}^{\omega_1+\omega_2} du = \omega_1, \qquad \int_{\omega_2}^{\omega_1+\omega_2} \wp(u)\, du = -\eta_1 \tag{4-27}$$

and remember that odd derivatives of $\wp(u)$ with arguments $\omega_1 + \omega_2$ or ω_2 vanish, we readily see* that

$$\int_{\omega_2}^{\omega_1+\omega_2} \{\wp(u)\}^2\, du = \tfrac{1}{12}\, g_2\omega_1\,, \tag{4-28}$$

and for $j > 2$ all integrals of powers of $\wp(u)$ can be expressed as linear combinations of ω_1 and η_1, with coefficients involving powers and cross-products of the invariants g_2 and g_3. The functions ω_1 and η_1 rewritten in terms of Legendre normal integrals take finally the forms

$$\omega_1 = \frac{F\left(\dfrac{\pi}{2}, \kappa\right)}{\sqrt{e_1 - e_3}} \tag{4-29}$$

and

$$\eta_1 = \sqrt{e_1 - e_3}\,E\left(\frac{\pi}{2}, \kappa\right) - \frac{e_1}{\sqrt{e_1 - e_3}}\, F\left(\frac{\pi}{2}, \kappa\right), \tag{4-30}$$

where F and E denote the Legendre complete integrals of the first and second kind, with the modulus

$$\kappa^2 = \frac{e_2 - e_3}{e_1 - e_3}. \tag{4-31}$$

The reader should notice that the moduli appropriate for partial and annular eclipses are mutually reciprocal.

After having thus established the solution of (4-13) in a finite number of terms for any value of the subscripts and of m, let us return to (4-14). Integrating Π by parts we obtain

$$\Pi_{2\nu}^m = GD_{2\nu}^m(s) - \sqrt{\frac{\delta}{2}} \int_{c_3}^{c_2} \frac{XD_{2\nu}^m(x)\, dx}{\sqrt{(x - c_1)(x - c_2)(x - c_3)}}\,, \tag{4-32}$$

where $G = \dfrac{\pi}{2}$ or 0 according as the eclipse is partial or annular,

$$X = \frac{x^2 - 2sx + r_1^2}{r_1^2 - x^2}\,, \tag{4-33}$$

and

$$\left.\begin{aligned} c_1 &= \delta + r_2\,, \\ c_2 &= s, \\ c_3 &= \delta - r_2\,, \end{aligned}\right\} \tag{4-34p}$$

* *Cf.* Whittaker and Watson, *op. cit.*, section 20.52.

if the eclipse is partial, and

$$\left.\begin{aligned} c_1 &= s, \\ c_2 &= \delta + r_2, \\ c_3 &= \delta - r_2, \end{aligned}\right\} \tag{4-34a}$$

if it is annular. Substitute, as before,

$$x - \tfrac{1}{3}\sum_{j=1}^{3} c_j = \wp(u) \tag{4-35}$$

and expand

$$XD_{2\nu}^m(x) = \sum_{j=0}^{m+n+2} a_n^m(j)\{\wp(u)\}^j + b_n^m r_1^{m+n+2}\left\{\frac{r_1 - s}{\wp(u) + h - r_1} + (-1)^m \frac{r_1 + s}{\wp(u) + h + r_1}\right\}, \tag{4-36}$$

where the coefficients a_n^m are polynomials of the $(m + n + 2 - j)$th degree in r_1, δ, and s, and b_n^m is a positive fraction (numerical factor). Since, by definition,

$$dx = 2\{(x - c_1)(x - c_2)(x - c_3)\}^{\frac{1}{2}}\, du\,, \tag{4-37}$$

we see that the $\Pi_{2\nu}^m$ can be expressed in terms of integrals of powers of $\wp(u)$ which we have just solved, plus two integrals of the form

$$\int_{\omega_2}^{\omega_1+\omega_2} \frac{du}{\wp(u) + h \pm r_1},$$

which are new and remain to be evaluated.

In order to do so we introduce new arguments $v_{1,2}$ defined by

$$-(h \pm r_1) = \wp(v_{1,2}). \tag{4-38}$$

Then, by means of a well-known theorem* we have

$$\int_{\omega_2}^{\omega_1+\omega_2} \frac{du}{\wp(u) - \wp(v_j)} = \frac{2}{\wp'(v_j)}\{\omega_1\zeta(v_j) - \eta_1 v_j\}, \qquad j = 1, 2, \tag{4-39}$$

where prime denotes derivative with respect to v_j, and ζ is the Weierstrass zeta-function. As one can easily verify,

$$\wp'(v_{1,2}) = \mp 2i\sqrt{2\delta}(r_1 \pm s) \tag{4-40p}$$

if the eclipse is partial, and

$$\wp'(v_{1,2}) = -2i\sqrt{2\delta}(r_1 \pm s) \tag{4-40a}$$

if it is annular. In order to remove the imaginary unit, we put

$$\left.\begin{aligned} v_1 &= iw_1\,, \\ v_2 &= iw_2 + \omega_1\,. \end{aligned}\right\} \tag{4-41}$$

* *Cf.* Whittaker and Watson, *op. cit.*, section 20.53.

Remembering that $\zeta(\omega_1) = \eta_1$, $\wp(\omega_1) = e_1$, and $\wp'(\omega_1) = 0$, the addition theorem for Weierstrass zeta-functions yields

$$\zeta(\omega_1 + iw_j) = \eta_1 + \zeta(iw_j) + \frac{i}{\sqrt{2\delta}}\{r_1 - (\delta - r_2)\} \tag{4-42p}$$

if the eclipse is partial, and

$$\zeta(\omega_1 + iw_j) = \eta_1 + \zeta(iw_j) + i\sqrt{2\delta} \tag{4-42a}$$

if it is annular. If we further substitute

$$\zeta(iw) = -i\zeta^*(w), \tag{4-43}$$

where

$$\zeta^*(w; e_1, e_2, e_3) = \zeta(w; -e_1, -e_2, -e_3), \tag{4-44}$$

we eventually find that, for partial eclipses,

$$\sqrt{2\delta}(r_1 + s)\int_{\omega_2}^{\omega_1+\omega_2} \frac{du}{\wp(u) - \wp(v_1)} = \omega_1\zeta^*(w_1) + \eta_1 w_1 \tag{4-45}$$

and

$$\sqrt{2\delta}(r_1 - s)\int_{\omega_2}^{\omega_1+\omega_2} \frac{du}{\wp(u) - \wp(v_2)} = -\omega_1\zeta^*(w_2) - \eta_1 w_2 + \frac{\omega_1}{\sqrt{2\delta}}(r_1 + r_2 - \delta). \tag{4-46p}$$

If the eclipse is annular, equation (4-45) continues to hold good; but (4-46p) is to be replaced by

$$\sqrt{2\delta}(r_1 - s)\int_{\omega_2}^{\omega_1+\omega_2} \frac{du}{\wp(u) - \wp(v_2)} = \omega_1\zeta^*(w_2) + \eta_1 w_2 - \omega_1\sqrt{2\delta}. \tag{4-46a}$$

The functions $w_{1,2}$ and $\zeta^*(w_{1,2})$, expressed in terms of Legendre normal forms, can be shown to become,

$$w_{1,2} = \frac{F(\phi_{1,2}, \kappa')}{\sqrt{e_1 - e_3}} \tag{4-47}$$

and

$$\left.\begin{aligned} &(p){:}\ \ \zeta^*(w_{1,2}) = e_3 w_{1,2} + \sqrt{e_1 - e_3}\, E(\phi_{1,2}, \kappa') \\ &\qquad\qquad\qquad + \frac{1}{\sqrt{2\delta}}[r_1 \pm (\delta - r_2)], \\ &(a){:}\ \ \zeta^*(w_{1,2}) = e_3 w_{1,2} + \sqrt{e_1 - e_3}\, E(\phi_{1,2}, \kappa') + \sqrt{2\delta}, \end{aligned}\right\} \tag{4-48}$$

—according to whether the eclipse is partial (p) or annular (a)—where κ', the complementary modulus, is defined by

$$(\kappa')^2 = \frac{e_1 - e_2}{e_1 - e_3} = 1 - \kappa^2, \tag{4-49}$$

and the amplitudes for partial and annular eclipses take the respective forms

$$\text{(p)} \quad \phi_1 = \sin^{-1}\sqrt{\frac{2r_2}{r_1+r_2+\delta}}, \qquad \phi_2 = \sin^{-1}\sqrt{\frac{2\delta}{r_1+r_2+\delta}}, \tag{4-50p}$$

and

$$\text{(a)} \qquad \phi_1 = \phi_2 = \sin^{-1}\sqrt{\frac{r_1+r_2-\delta}{r_1+r_2+\delta}}. \tag{4-50a}$$

Let us put, for brevity's sake,

$$(r_1+s)\int_{\omega_2}^{\omega_1+\omega_2} \frac{du}{\wp(u)-\wp(v_1)} \mp (r_1-s)\int_{\omega_2}^{\omega_1+\omega_2} \frac{du}{\wp(u)-\wp(v_2)} = \frac{1}{\sqrt{2\delta}}\mathfrak{E}_{1,2}. \tag{4-51}$$

By combination of the above formulae it follows that, for partial eclipses,

$$\begin{aligned} \mathfrak{E}_{1,2} = &\left\{E\left(\frac{\pi}{2}, \kappa\right) - F\left(\frac{\pi}{2}, \kappa\right)\right\}\{F(\phi_1, \kappa') \pm F(\phi_2, \kappa')\} \\ &+ F\left(\frac{\pi}{2}, \kappa\right)\{E(\phi_1, \kappa') \pm E(\phi_2, \kappa') + \kappa\cos\phi_1 \sec\phi_2\}. \end{aligned} \tag{4-52}$$

If the upper sign is valid, this expression admits of a further simplification; for, by an obvious extension of a theorem due to Legendre,* the reader should have no difficulty to prove that

$$\mathfrak{E}_1 = \frac{\pi}{2} + \sqrt{\frac{\delta}{r_2}}\,F\left(\frac{\pi}{2}, \kappa\right); \tag{4-53p}$$

and, by subtraction of $\mathfrak{E}_1$ and $\mathfrak{E}_2$, the latter takes the form

$$\begin{aligned} \mathfrak{E}_2 = \frac{\pi}{2} &- 2\left\{E\left(\frac{\pi}{2}, \kappa\right) - F\left(\frac{\pi}{2}, \kappa\right)\right\}F(\phi_2, \kappa') \\ &- 2\left\{E(\phi_2, \kappa') - \tfrac{1}{2}\sqrt{\frac{\delta}{r_2}}\right\}F\left(\frac{\pi}{2}, \kappa\right), \end{aligned} \tag{4-54p}$$

in which both kinds of incomplete integrals possess a common amplitude.

If, finally, the eclipse is *annular* we similarly obtain

$$\mathfrak{E}_1 = \kappa\sqrt{\frac{\delta}{r_2}}\,F\left(\frac{\pi}{2}, \kappa\right), \tag{4-53a}$$

and

$$\begin{aligned} \mathfrak{E}_2 = 2F(\phi_{1,2}, \kappa')&\left\{E\left(\frac{\pi}{2}, \kappa\right) - F\left(\frac{\pi}{2}, \kappa\right)\right\} \\ &+ 2F\left(\frac{\pi}{2}, \kappa\right)\left\{E(\phi_{1,2}, \kappa') + \frac{\kappa}{2}\sqrt{\frac{\delta}{r_2}}\right\}. \end{aligned} \tag{4-54a}$$

* *Cf.* Whittaker and Watson, *op. cit.*, section 22.735.

The evaluation of circular integrals associated with effects of the tidal distortion has thus been completed. We found that the expressions for $\mathfrak{A}_n^m$ are all elementary, while those for $\mathfrak{B}_n^m$ are such only if n is zero or an even integer. If it is odd the $\mathfrak{B}$'s are found to involve elliptic integrals. Expressions of the form $I_{0,1,\gamma}^m$, where γ is an odd integer, or (if m is also odd) $\Pi_{2\nu}^m$, can all be solved in terms of Legendre complete integrals of the first and second kind. If, however, m is zero or even, the $\Pi_{2\nu}^m$'s involve also complete elliptic integrals of the third kind which belong to the 'circular' class and are therefore expressible in terms of incomplete integrals of the first and second kind with complementary moduli. By collecting the results established earlier in this section we find that, if n is zero or an even integer (i.e., $n = 2\nu$, $\nu = 0, 1, 2, \ldots$), the corresponding associated α-functions assume the forms

$$\pi^2 r_1^{m+2(\nu+1)} \alpha_{2\nu}^m = B(\tfrac{1}{2}, \nu+1)\{2G[D_{2\nu+1}^m(r_1) - D_{2\nu+1}^m(s)] + \sum_{j=0}^{\nu} B(\tfrac{1}{2}, \tfrac{1}{2}+\nu-j) I_{2j,1,2(\nu-j)}^m\}, \tag{4-55}$$

where, as before, $G = \frac{1}{2}\pi$ or 0 according as the eclipse is partial or annular; while if n is odd,

$$\pi^2 r_1^{m+2\nu+1} \alpha_{2\nu-1}^m = 2B(\tfrac{1}{2}, \tfrac{1}{2}+\nu)\left\{G D_{2\nu}^m(r_1) + \tfrac{1}{2}\sum_{j=1}^{\nu} B(\tfrac{1}{2}, j) I_{2(\nu-j),1,2j-1}^m - \sqrt{\frac{\delta}{2}} \int_{c_3}^{c_2} \frac{X D_{2\nu}^m(x)\, dx}{\sqrt{(x-c_1)(x-c_2)(x-c_3)}}\right\}. \tag{4-56}$$

Functions of the former kind can be evaluated in terms of circular and algebraic function; while those of the latter kind are expressible only by means of elliptic integrals.

As to the 'boundary corrections' integrals $\mathfrak{I}_{\beta,\gamma}^m$ or $I_{\beta,\gamma}^m$, little remains to be added on their evaluation except to note that, in accordance with equations (3-23) and (3-26),

$$\pi r_1^{\beta+\gamma+m+3} \mathfrak{I}_{\beta,\gamma}^m = (r_2^2 - \delta^2) I_{0,\beta,\gamma}^m + \delta I_{0,\beta,\gamma}^{m+1} \tag{4-57}$$

and

$$\pi r_2^{\beta+\gamma+m+1} I_{\beta,\gamma}^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \delta^{m-j} I_{0,\beta,\gamma}^j, \tag{4-58}$$

where the $I_{0,\beta,\gamma}^m$'s represent particular cases of integrals of the form (4-13) whose evaluation has already been dealt with before. Hence, all 'circular integrals' as well as 'boundary corrections' of section IV.3 can be expressed in terms of functions already treated, and the solution of our mathematical problem is thus complete.

An inspection of the equations defining the associated α-functions and the related boundary integrals reveals that all such functions are *homogeneous* in *three* geometrical elements r_1, r_2, δ (or s) and can, therefore, be made to depend only on *two* of the *ratios* which can be formed between them. The choice of such ratios remains, in principle, arbitrary; but in practice it proves most advantageous to adopt them as the *ratio of the radii k* and the *geometrical depth of the eclipse p*, as defined by the equations

$$k = \frac{r_a}{r_b} \qquad \text{and} \qquad p = \frac{\delta - r_b}{r_a}, \tag{4-59}$$

where r_a stands for the radius of the *smaller* component, and r_b for that of the *larger* component of the two. The reason for this particular choice (of many others that could be considered) rests on the simplicity of the limits within which k and p defined above are allowed to vary. First, it is immediately obvious from this definition that the range of k has been constrained to

$$0 \leqslant k \leqslant 1 \tag{4-60}$$

for any type of eclipse. Furthermore, it is equally easy to see that, during *partial* eclipse, the geometrical depth p will be bounded by

$$-1 \leqslant p \leqslant 1; \tag{4-61}$$

the upper limit of this inequality being attained at the moment of first contact of the eclipse (when $\delta = r_1 + r_2 = r_a + r_b$); and the lower limit, at the moment of internal tangency of the two disks (i.e., when $\delta = r_b - r_a$) which marks the beginning of totality if $r_a = r_1$, or of annular phase if $r_1 = r_b$.

During the *annular phase* itself, as the disk of the star of radius $r_a = r_2$ transits across that of its mate, the geometrical depth p as defined by (4-59) above would vary between

$$-1 \geqslant p \geqslant -k^{-1}; \tag{4-62}$$

the lower limit being attained at the moment of central eclipse ($\delta = 0$), leading to a range which for small values of k may become inconveniently large. In order to re-normalize it, it is convenient to replace p then by an auxiliary geometrical depth q of an annular eclipse, related with p by

$$q = \frac{k(1 + p)}{k - 1}, \tag{4-63}$$

which is constrained so that

$$0 \leqslant q \leqslant 1; \tag{4-64}$$

the lower limit being reached at the moment of internal tangency ($p = -1$); and the upper, at the moment of central eclipse ($p = -k^{-1}$).

As to the particular values which the associated α-functions and related integrals assume at the limiting phases of a partial eclipse,

$$\alpha_n^m(k, 1) = \mathfrak{J}_{\beta,\gamma}^m(k, 1) = I_{\beta,\gamma}^m(k, 1) = 0 \tag{4-65}$$

for every value of m, n, or β, γ. All such functions are bound to vanish at first contact of any eclipse. If $r_1 < r_2$, the respective boundary integrals vanish also at the onset of totality—so that, for eclipses of a smaller star by a larger one,

$$\mathfrak{I}^m_{\beta,\gamma}(k, \pm 1) = I^m_{\beta,\gamma}(k, \pm 1) = 0 \tag{4-66}$$

regardless of β, γ or m. On the other hand, the associated α-functions vanish at the beginning of totality only if their order m is odd, so that

$$\alpha_n^{2\mu+1}(k, \pm 1) = 0 \tag{4-67}$$

for $\mu = 0, 1, 2, \ldots$; while if m is zero or an even integer,

$$\alpha_{2\nu}^{2\mu}(k, -1) = \frac{\nu!\, \Gamma(\mu + \frac{1}{2})}{\sqrt{\pi}(\mu + \nu + 1)!} \tag{4-68}$$

and

$$\alpha_{2\nu-1}^{2\mu}(k, -1) = \frac{\Gamma(\mu + \frac{1}{2})\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\Gamma(\mu + \nu + \frac{3}{2})}\,, \tag{4-69}$$

depending on whether the index n is odd or even.

On the other hand, if $r_1 > r_2$, the associated α-functions as well as all related boundary integrals of even order remain non-vanishing functions of k at the moment of internal tangency and continue, moreover, to vary with δ throughout annular phase. Ultimately, at the moment of central eclipse ($\delta = 0$), the only functions whose coefficients in equations (3-29)–(3-31) of the theoretical light curves do not vanish are those of zero order ($m = 0$), which for $p = -k^{-1}$, can be shown to reduce to

$$\alpha_n^0(k, -k^{-1}) = \frac{2}{n+2}\{1 - (1 - k^2)^{1+\frac{1}{2}n}\}, \tag{4-70}$$

$$\pi\mathfrak{I}^0_{\beta,\gamma}(k, -k^{-1}) = B(\tfrac{1}{2}, \tfrac{1}{2}\beta + 1)k^{\beta+3}(1 - k^2)^{\gamma/2}, \tag{4-71}$$

and

$$\pi I^0_{\beta,\gamma}(k, -k^{-1}) = B(\tfrac{1}{2}, \tfrac{1}{2}\beta + 1)k^{-\gamma}(1 - k^2)^{\gamma/2}. \tag{4-72}$$

Of the associated α-functions figuring on the right-hand sides of equation (3-27) particular interest attaches to the two functions α_0^0 and α_1^0 of lowest orders: for α_0^0 represents nothing but the fractional loss of light due to the eclipse of a uniformly bright star (equal to its fractional area eclipsed); while α_1^0 stands for the fractional loss of light caused by the eclipse of a disk which is completely darkened at the limb. In the former case, equation (4-55) reveals that, during *partial* eclipse,

$$\pi r_1^2 \alpha_0^0 = r_1^2 \cos^{-1}\frac{s}{r_1} + r_2^2 \cos^{-1}\frac{\delta - s}{r_2} - \delta\sqrt{r_1^2 - s^2} \tag{4-73}$$

or, in terms of our normalized parameters k and p,

$$2\pi k^2\alpha_0^0(k, p) = 2\phi_1 - \sin 2\phi_1 + k^2(2\phi_2 - \sin 2\phi_2), \tag{4-74}$$

where

$$\sin \phi_1 = k \sin \phi_2 \tag{4-75}$$

and

$$\cos \phi_2 = \frac{kp^2 + 2p + k}{2(1 + kp)} \cdot \tag{4-76}$$

In the case of an *annular* eclipse of a uniformly bright disk, equation (4-55) reduces merely to

$$\alpha_0^0(k, p) = k^2. \tag{4-77}$$

Passing over to α_1^0 we find that, in this particular case, equation (4-56) reduces for *partial* eclipses to

$$\begin{aligned}\pi r_1^3\alpha_1^0 = {} & \tfrac{2}{3}\pi r_1^3 - \tfrac{4}{3}r_1^3\{[E(\tfrac{1}{2}\pi, \kappa) - F(\tfrac{1}{2}\pi, \kappa)]F(\phi_2\kappa') + F(\tfrac{1}{2}\pi, \kappa)E(\phi_2, \kappa')\} \\ & + \tfrac{2}{9}r_2\sqrt{\frac{\delta}{r_2}}\,(7r_2^2 - 4r_1^2 + \delta^2)\{2E(\tfrac{1}{2}\pi, \kappa) - F(\tfrac{1}{2}\pi, \kappa)\} \\ & - \frac{2}{9\delta}\sqrt{\frac{\delta}{r_2}}\,\{5\delta^2r_2^2 + 3(r_1^2 - r_2^2)^2 - 3\delta r_1^3\}F(\tfrac{1}{2}\pi, \kappa)\end{aligned} \tag{4-78}$$

where, consistent with equation (4-31) and (4-19p)

$$\kappa^2 = \tfrac{1}{2}\left\{1 - \frac{\delta - s}{r_2}\right\} = 1 - \kappa'^2 \tag{4-79}$$

and the amplitude ϕ_2 continues to be given by (4-50). If, on the other hand, the eclipse becomes *annular*

$$\begin{aligned}\pi r_1^3\alpha_1^0 = {} & \tfrac{4}{3}r_1^3\left\{\left[E\left(\frac{\pi}{2}, \frac{1}{\kappa}\right) - F\left(\frac{\pi}{2}, \frac{1}{\kappa}\right)\right]F(\sqrt{1 - \kappa^{-2}}, \phi)\right. \\ & \left. + E\left(\frac{\pi}{2}, \frac{1}{\kappa}\right)E(\sqrt{1 - \kappa^{-2}}, \phi)\right\} \qquad (4\text{-}80) \\ & + \tfrac{2}{9}(7r_2^2 - 4r_1^2 + \delta^2)\sqrt{2\delta(r_2 - \delta + s)}\,E\left(\frac{\pi}{2}, \frac{1}{\kappa}\right) \\ & + \tfrac{2}{9}\{(\delta^2 - r_2^2)^2 - r_1^2(2r_1^2 - r_2^2) + 6\delta r_1^3\}\,\frac{F\left(\frac{\pi}{2}, \frac{1}{\kappa}\right)}{\sqrt{2\delta(r_2 - \delta + s)}}\end{aligned}$$

where κ continues to be given by the equation (4-79) above, and ϕ by (4-50a).

The task of rewriting the foregoing expressions for α_1^0 in terms of k and p is somewhat tedious, and may be left as an exercise for the interested reader. We may note that, at the moment of internal tangency (i.e., for $\delta = |r_1 - r_2|$) if $r_2 > r_1$ equation (4-78) reduces to $\alpha_1^0 = \frac{2}{3}$ in accordance with (4-72); but if $r_2 < r_1$,

$$\alpha_1^0(k, -1) = \frac{4}{3\pi}\{\sin^{-1}\sqrt{k} + \tfrac{1}{3}(4k - 3)(2k + 1)\sqrt{k(1 - k)}\}, \tag{4-81}$$

where $k = r_2/r_1$. If $r_2/r_1 = 0$, the eclipsing component would reduce to a point and render both losses of light α_0^0 and α_1^0 identically zero. If, on the other hand, $r_2/r_1 = \infty$ and thus $k = r_1/r_2 = 0$ (i.e., if the limb of the eclipsing component acts as a straight occulting edge), equations (4-73) and (4-78) reveal that

$$\pi\alpha_0^0(0, p) = \cos^{-1}p - p\sqrt{1 - p^2} \tag{4-82}$$

and

$$\alpha_1^0(0, p) = \tfrac{1}{6}(2 - 3p + p^3), \tag{4-83}$$

respectively. For unrestricted values of k and p, the relationship between α_0^0, α_1^0 and these parameters becomes somewhat involved. This fact does not, however, constitute any serious impediment, because both functions $\alpha_0^0(k, p)$ as well as $\alpha_1^0(k, p)$ have been extensively tabulated* in terms of their arguments for any type of eclipse, and to a degree of accuracy which should be entirely adequate for illustrating the nature of these functions or any other practical purpose. This entitles us, therefore, to regard hereafter both associated α-functions of lowest orders and index $n = 0, 1$ as known, and use them for subsequent ends.

IV.5. ALGEBRA OF THE ASSOCIATED ALPHA-FUNCTIONS

In section three of this chapter we found it possible to express the light changes due to eclipses of distorted components of close binary systems in terms of the associated α-functions and certain related integrals depending on the phase; and in the preceding section we outlined the way in which all such functions can be evaluated. Explicit results have so far been obtained only for the functions α_0^0 and α_1^0 occurring as terms of zero order on the right-hand sides of equation (3-27). The first-order terms due to the second, third and fourth harmonic distortion of limb- and gravity-darkened components

* For an extensive review of such tables *cf.* Z. Kopal, *Journ. on Math. Tables and other Aids to Computation*, **3**, 191, 1948.

were found to invoke 22 additional associated α-functions, summarised in the following scheme,

$$
\begin{array}{ccccc}
\alpha_{-1}^0 & \alpha_{-1}^1 & \alpha_{-1}^2 & \alpha_{-1}^3 & \\
 & \alpha_0^1 & \alpha_0^2 & \alpha_0^3 & \alpha_0^4 \\
 & \alpha_1^1 & \alpha_1^2 & \alpha_1^3 & \alpha_1^4 \\
\alpha_2^0 & \alpha_2^1 & \alpha_2^2 & \alpha_2^3 & \\
\alpha_3^0 & \alpha_3^1 & \alpha_3^2 & & \\
\alpha_4^0 & \alpha_4^1 & & & \\
\alpha_5^0, & & & &
\end{array}
$$

plus an analogous number of functions of the type $\Im_{\beta,\gamma}^m$ and $I_{\beta,\gamma}^m$; and still higher functions will be needed in the next chapter to express the theoretical radial-velocity curves of close binary systems to the same accuracy. The methods of their integration have been investigated in section IV.4; and using them the writer has actually established the closed forms of them all* in a memoir which should be accessible† to every interested reader. The complexity of the results (which would occupy several pages of print) was, however, found to increase so rapidly with increasing values of m and n that a recourse to literal formulae for the evaluation of the α_n^m's of higher orders or indices—if inevitable—would prove excessively laborious. In order to circumvent this necessity, our aim in the present section will be to search for possible *recursion formulae* which connect associated α-functions of different orders and indices, and whose use may enable us to restrict literal evaluation to only a certain limited number of basic α-functions in terms of which all others may then be built up. As will be shown below, linear recursion formulae making this possible indeed exist; and the basic set of α-functions requisite for generating all others will in fact reduce to the pair of α_0^0 and α_1^0 nvestigated already in the preceding section, aided by integrals of the type $I_{\beta,\gamma}^m$ which are likewise already available in tabular form.

In order to prove it and to construct the requisite recursion formulae connecting associated α-functions of different orders and indices, we find it convenient to re-define such functions in terms of plane polar coordinates in the form

$$\pi r_1^{m+n+2}\alpha_n^m = 2\int_{\delta-r_2}^{r_1}\int_0^{\theta_0}(r_1^2-r^2)^{n/2}r^{m+1}\cos^m\theta\,d\theta\,dr, \tag{5-1}$$

where

$$\cos\theta_0 = \frac{\delta^2+r^2-r_2^2}{2\delta r}. \tag{5-2}$$

This definition remains valid only during partial eclipse as long as $\delta \geqslant r_2$.

* *Cf. Proc. Amer. Phil. Soc.*, **85**, 399, 1942.
† As Harvard Reprint, Series II, No. 1, 1942.

When the converse is true and the eclipsing limb covers the centre of the component of radius r_2, the above equation (5-1) must be replaced by

$$\pi r_1^{m+n+2}\alpha_n^m = \left\{\int_0^{r_1}\int_0^{\pi} - \int_{r_2-\delta}^{r_1}\int_0^{\pi-\theta_0}\right\}(r_1^2 - r^2)^{n/2} r^{m+1}\cos^m\theta\, d\theta\, dr \qquad (5\text{-}3)$$

while, for annular eclipses ($\delta < r_1 - r_2$), the upper limits r_1 and θ_0 in the foregoing integrals should be replaced by $\delta + r_2$ and π, respectively.

Integrating with respect to θ we find that

$$\int_0^{\theta_0} \cos^m\theta\, d\theta = \pi I^m_{-1,0}(\cos\theta_0), \qquad (5\text{-}4)$$

where $I^m_{\beta,\gamma}$ continues to be given by equation (3-26) revealing that, for $\theta_0 = \pi$,

$$\left.\begin{aligned} I^{2\mu}_{-1,0} &= \binom{\mu - \frac{1}{2}}{\mu}, \\ I^{2\mu+1}_{-1,0} &= 0. \end{aligned}\right\} \qquad (5\text{-}5)$$

In consequence, equations (5-1) and (5-3) assume the more explicit forms

$$r_1^{m+n+2}\alpha_n^m = 2\int_{\delta-r_2}^{r_1} (r_1^2 - r^2)^{n/2} I^m_{-1,0}(\cos\theta_0) r^{m+1}\, dr \qquad (5\text{-}6)$$

if $\delta \geqslant r_2$, and (if we remember that, for odd m, $I^m_{-1,0}(-x)$ becomes an odd function of x) for $\delta \leqslant r_2$

$$r_1^{2(\mu+1)+n+1}\alpha_n^{2\mu+1} = 2\int_{r_2-\delta}^{r_1} (r_1^2 - r^2)^{n+2} I^{2\mu+1}_{-1,0}(\cos\theta) r^{2(\mu+1)}\, dr \qquad (5\text{-}7)$$

if $m = 2\mu + 1$ is an odd integer, and

$$\begin{aligned} r_1^{2\mu+n+1}\alpha_n^{2\mu} = {} & \frac{\Gamma(\mu+\frac{1}{2})\Gamma(1+\frac{1}{2}n)}{\sqrt{\pi}\,\Gamma(\mu+\frac{1}{2}n+2)}\, r_1^{2\mu+n+1} \\ & - \int_{r_2-\delta}^{r_1} (r_1^2 - r^2) I^{2\mu}_{-1,0}(-\cos\theta_0) r^{2\mu+1}\, dr \end{aligned} \qquad (5\text{-}8)$$

if m is even.

Further evaluation of the α_n^m's encounters the same difficulties as we met already in the preceding section IV.4 while working with the rectangular coordinates. In the present approach we may note, however, that the integrals of the form $I^m_{-1,0}$ can be shown to obey the simple recursion formula.

$$mI^m_{-1,0}(x) = (m-1)I^{m-2}_{-1,0}(x) + x^{m-1}I^1_{-1,0}(x), \qquad (5\text{-}9)$$

where

$$\pi I^1_{-1,0}(x) = \sqrt{1 - x^2}. \qquad (5\text{-}10)$$

In consequence, an insertion of (5-9) in the foregoing formulae for the α_n^m's reveals at once that

$$\pi m\alpha_n^m + \pi(m-1)(\alpha_{n+2}^{m-2} - \alpha_n^{m-2}) = 2\delta r_1^{-m-n-2} I_{0,1,n}^{m-1}, \tag{5-11}$$

when $I_{0,1,n}^{m-1}$ stands for a particular member of the family of integrals of the form (4-13).

The foregoing equation (5-11) represents the first fundamental recursion formula between three distinct associated α-functions of different orders and indices, which follows as a direct consequence of the recursion property (5-9) of the I-integrals, and holds good for any integral values of m and n such that $m > 0$ and $n > -2$. The second such formula can be deduced if we integrate (5-7) by parts obtaining

$$r_1^{m+n+2}\alpha_n^m = \frac{2}{n+2}\int_{\delta-r_2}^{r_1} (r_1^2 - r^2)^{2+\frac{1}{2}n} \frac{d}{dr}\{r^m I_{-1,0}^m(\cos\theta)\}\, dr. \tag{5-12}$$

Replace, in this equation, n by $n-2$ and subtract from (5-12): we find that

$$r_1^{m+n+2}\{n\alpha_{n-2}^m - (n+2)\alpha_n^m\} = 2\int_{\delta-r_2}^{r_1} (r_1^2 - r^2)^{n/2} \frac{d}{dr}\{r^m I_{-1,0}^m(\cos\theta)\} r^2\, dr. \tag{5-13}$$

If, furthermore, we make use of the fact that

$$\frac{d}{dr}\{r^m I_{-1,0}^m\} = mr^{m-1} I_{-1,0}^m + r^m \frac{dI_{-1,0}^m}{dr} \tag{5-14}$$

and insert it on the right-hand side of (5-13), this latter equation assumes the form

$$r^{m+n+2}\{(m+n+2)\alpha_n^m - n\alpha_{n-2}^m\} = -2\int_{\delta-r_2}^{r_1} (r_1^2 - r^2)^{n/2} \frac{dI_{-1,0}^m}{dr} r^{m+2}\, dr, \tag{5-15}$$

which, by virtue of the fact that

$$\frac{dI_{-1,0}^m}{dr} = \cos^m\theta_0 \frac{d\theta_0}{dr}, \tag{5-16}$$

where θ_0 continues to be given by (5-2), can be rewritten as

$$(m+n+2)\alpha_n^m - n\alpha_{n-2}^m = 2\mathfrak{I}_{-1,n}^m \tag{5-17}$$

for $m > 0$ and $n > 0$ where, in accordance with (3-23),

$$\pi r^{m+n+2}\, \mathfrak{I}_{-1,n}^m = (r_2^2 - \delta^2) I_{0,-1,n}^m + \delta I_{0,-1,n}^{m+1}. \tag{5-18}$$

Equations (5-11) and (5-17) represent two fundamental and mutually independent recursion formulae valid for the associated α-functions, whose right-hand sides consist of integrals of the form $I_{0,-1,n}^m$. These are still rather complicated; but their essential features in this connection are the recursions which they obey. It can indeed be shown that

$$2\delta r_1 \mathfrak{I}_{-1,n}^m = 2\delta s \mathfrak{I}_{-1,n}^{m-1} - r_1^2 \mathfrak{I}_{-1,n+2}^{m-1}, \tag{5-19}$$

which combined with (5-17) reveals the existence of the recursion formula

$$\begin{aligned} 2\delta r_1\{(m+n+1)\alpha_{n-2}^{m+1} - (n-2)\alpha_{n-4}^{m+1}\} \\ - 2\delta s\{(m+n)\alpha_{n-2}^{m} - (n-2)\alpha_{n-4}^{m}\} \\ + r_1^2\{(m+n+2)\alpha_n^m - n\alpha_{n-2}^m\} = 0\,, \end{aligned} \tag{5-20}$$

relating five associated α-functions of different orders and indices. It is, furthermore, to be expected that the integrals $I_{0,1,n}^m$ occurring on the right-hand side of the equation (5-18) will obey the same recursion formula (5-19) as the $\mathfrak{I}$-integrals (which consist of them) and, as a result, (5-11) yields another recursion formula of the form

$$\begin{aligned} 2\delta r_1\{(m+1)\,[\alpha_n^m - \alpha_{n-2}^m] + (m+2)\alpha_{n-2}^{m+2}\} \\ - 2\delta s\{m[\alpha_n^{m-1} - \alpha_{2-n}^{m-1}] + (m+1)\alpha_{n-2}^{m+1}\} \\ + r_1^2\{m[\alpha_{n+2}^{m-1} - \alpha_n^{m-1}] + (m+1)\alpha_n^{m+1}\} = 0, \end{aligned} \tag{5-21}$$

relating eight different associate α-functions in a linear manner.

How many associated α-functions must be known before the equations (5-20) and (5-21) can be invoked to generate any number of them for a particular phase? Let us survey first what we know about such functions already. Apart from the two fundamental modes α_0^0 and α_1^0 which were adequately investigated in the preceding section, the easiest ones to obtain are the associated α-functions of order one; for equations (5-6)–(5-8) then reveal at once that, for $m = 1$,

$$r_1^{n+3}\alpha_n^1 = 2\delta r_2^{n+2} I_{1,n}^0 = \frac{2r_2^{n+3}}{n+2} I_{-1,n+2}^1\,, \tag{5-22}$$

where the $I_{\beta,\gamma}^m$'s on the right-hand side represent particular cases of the family of relatively simple integrals as defined by equation (3-26). Moreover, only the lowest two functions α_{-1}^1 and α_0^1 need to be evaluated on their own; all higher ones can be generated from them with the aid of the recursion formula

$$(n+4)r_1^2\alpha_n^1 + 2n\delta(\delta - s)\alpha_{n-2}^1 = 4\delta r_2(r_2/r_1)^{n+1} I_{-1,n}^0\,. \tag{5-23}$$

Suppose, therefore, that we consider the α_n^1's as known in addition to α_0^0 and α_1^0. If so, it is easily seen that the recursion relations (5-20) and (5-21) should permit us to generate any other associated α-function, of arbitrary order or index, provided that the following four additional functions

$$\begin{matrix} \alpha_{-1}^0, & \alpha_{-1}^2, \\ \alpha_2^0\;, & \alpha_0^2\;, \end{matrix}$$

are likewise given. And these, in turn, follow indeed by application of equation (5-17) as

$$\left.\begin{aligned} \alpha_{-1}^0 &= 3\alpha_1^0 - 2\mathfrak{I}_{-1,1}^0 \\ &= 3\alpha_1^0 - 2(r_2/r_1)^3\{I_{-1,1}^0 - (\delta/r_2)I_{-1,1}^1\}, \end{aligned}\right\} \tag{5-24}$$

and

$$\alpha_{-1}^2 = \alpha_{-1}^0 + 2(r_2/r_1)^3 I_{1,1}^0 \tag{5-25}$$

in terms of α_1^0 and the respective I-integrals; while

$$2r_1\alpha_2^0 = r_1\alpha_0^0 - (3\delta - 2s)\alpha_0^1 + 6\delta(r_2/r_1)^3 I_{1,0}^1 \tag{5-26}$$

and

$$4r_1\alpha_0^2 = r_1\alpha_0^0 + (5\delta - 2\delta)\alpha_0^1 - 10\delta(r_2/r_1)^3 I_{1,0}^1 \tag{5-27}$$

in terms of α_0^0, α_0^1 and $I_{1,0}^1$. Moreover, once this fundamental set of functions has thus been established, the five-term recursion relation (5-20) may now be employed to generate all subsequent members of the families α_n^0 and α_0^n; and after these have been found, the eight-term recursion relation (5-21) can be solved for α_{n-2}^{m+2} to yield successively all columns of associate α-functions of higher orders.

With regard to the *differential properties* of the associated α-functions, by direct differentiation of the equations (5-6)-(5-8) as they stand it is not difficult to establish the existence of the differential recursion formula

$$r_1 \frac{\partial \alpha_n^{m+1}}{\partial r_2} = \delta \frac{\partial \alpha_n^m}{\partial r_2} + r_2 \frac{\partial \alpha_n^m}{\partial \delta} = 2\mathfrak{J}_{-1,n}^m\,. \tag{5-28}$$

As, however, all associated α-functions depend on r_1, r_2 and δ only through the *ratios* of these parameters, Euler's theorem on homogeneous functions requires that, in addition,

$$r_1 \frac{\partial \alpha_n^m}{\partial r_1} + r_2 \frac{\partial \alpha_n^m}{\partial r_2} + \delta \frac{\partial \alpha_n^m}{\partial \delta} = 0\,. \tag{5-29}$$

A combination of (5-28) and (5-29) reveals then at once that

$$r_1 \frac{\partial \alpha_n^{m+1}}{\partial r_2} = -r_1 \frac{\partial \alpha_n^m}{\partial r_1} = 2\mathfrak{J}_{-1,n}^m \tag{5-30}$$

or, in more explicit form,

$$\pi r_1^{m+n+3} \frac{\partial \alpha_n^m}{\partial r_1} = 2\{(\delta^2 - r_2^2) I_{0,-1,n}^m - \delta I_{0,-1,n}^{m+1}\}\,, \tag{5-31}$$

$$\pi r_1^{m+n+2} \frac{\partial \alpha_n^m}{\partial r_2} = 2\, r_2 I_{0,-1,n}^m\,, \tag{5-32}$$

and, therefore,

$$\pi r_1^{m+n+2} \frac{\partial \alpha_n^m}{\partial \delta} = 2\{I_{0,-1,n}^{m+1} - \delta I_{0,-1,n}^m\}\,. \tag{5-33}$$

In particular, for $m = 0$,

$$\frac{\partial \alpha_n^0}{\partial r_2} = \frac{2}{r_2}\left(\frac{r_2}{r_1}\right)^{n+2} I_{-1,n}^0\,, \tag{5-34}$$

$$\frac{\partial \alpha_n^0}{\partial \delta} = -\frac{2}{r_2}\left(\frac{r_2}{r_1}\right)^{n+2} I_{-1,n}^1\,, \tag{5-35}$$

so that, by (5-29)

$$r_1 \frac{\partial \alpha_n^0}{\partial r_1} = \frac{2}{r_2}\left(\frac{r_2}{r_1}\right)^{n+2} \{\delta I_{-1,n}^1 - r_2 I_{-1,n}^0\}\,, \tag{5-36}$$

where the integrals of the form $I_{\beta,\gamma}^m$ and $I_{0,\beta,\gamma}^m$ are connected by equation (4-58).

In the foregoing developments of this section we have, therefore, made good our promise to express all associated α-functions, of any order or index, in terms of the two fundamental modes α_0^0 and α_1^0; but before such expressions can be regarded as explicit, the whole family of auxiliary integrals of the form $I^m_{\beta,\gamma}$ as defined by equation (3-26) remain yet to be evaluated. In point of fact, these integrals turn out to play so fundamental a role in the whole theory of light (and velocity) curves of eclipsing binary systems that their properties deserve closer study.

In order to investigate them, let us put

$$I^m_{\beta,\gamma} = (\delta/r_2)^{\gamma/2} J^m_{\beta,\gamma}\,, \tag{5-37}$$

and reduce the integral $J^m_{\beta,\gamma}$ so defined to a tractable form. If the eclipse is *partial* (i.e., $c_2 = s$ in equation 3-26), this can be done by introducing a new variable u as defined by

$$\delta - x = r_2(1 - 2\kappa^2 u), \tag{5-38p}$$

where*

$$\kappa^2 = \tfrac{1}{2}(1 - \mu) \tag{5-39p}$$

and

$$\mu = \frac{\delta - s}{r_2} = \frac{r_2^2 - r_1^2 + \delta^2}{2\delta r_2}\,. \tag{5-40}$$

If so, equations (3-26) and (5-37) reveal that

$$2\pi J^m_{\beta,\gamma} = (2\kappa)^{\beta+\gamma+2}\int_0^1 u^{\beta/2}(1 - \kappa^2 u)^{\beta/2}(1 - u)^{\gamma/2}(1 - 2\kappa^2 u)^m\, du, \tag{5-41p}$$

i.e., that $J^m_{\beta,\gamma}$ turns out to depend on the geometry of the eclipses only through the modulus κ. The aim of the representation (5-37) was obviously to express the function $I^m_{\beta,\gamma}$ of two variables as a product of two factors each depending on the single variable δ/r_2 or κ.

The integral on the right-hand side of the foregoing equation (5-41p) can be recognized as representing a *generalized hypergeometric series.* In fact, if we abbreviate

$$a = \tfrac{1}{2}(\beta + 2) \qquad \text{and} \qquad b = \tfrac{1}{2}(\beta + \gamma + 4), \tag{5-42}$$

equation (5-41p) can be rewritten as

$$2\pi J^m_{\beta,\gamma} = (2\kappa)^{2(b-1)} B(a, b - a) F^{(1)}(a; 1 - a, -m; b; \kappa^2, 2\kappa^2), \tag{5-43p}$$

where $F^{(1)}$ denotes the first one of Appell's generalized hypergeometric functions† in two variables $x = \kappa^2$ and $y = 2\kappa^2$. The function $F^{(1)}$ is known to satisfy, in general, a certain system of two simultaneous partial differential equations of the second order. If, however, as in the present case $x/y =$ constant, this system reduces to a single ordinary differential equation which was obtained by Burchnall‡. From his results we deduce that, if t denotes the operator

$$t \equiv \kappa^2 \frac{d}{d\kappa^2}\,, \tag{5-43}$$

* The reader may note that the modulus κ^2 as defined by equation (5-39) is identical with that defined by (4-31) before.

† *Cf.* P. Appell and J. Kampé de Fériet, *Fonctions Hypergéometriques et Hypersphériques*, Paris 1926; or W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge Tract in Mathematics and Math. Physics, No. 32), Cambridge 1935 (Chapter IX).

‡ J. L. Burchnall, *Quart. Journ. of Math.* (*Oxford*), **13**, 90, 1942.

the function $F^{(1)}(a; 1-a, -m; b; \kappa^2, 2\kappa^2)$ will satisfy the following ordinary differential equation

$$t(t+b-1)(t+b-2)F^{(1)} - \kappa^2(3t+1-a-2m)(t+b-1)(t+1)F^{(1)} + 2\kappa^4(t+1-a-m)(t+a)(t+a+1)F^{(1)} = 0 \tag{5-44}$$

of the third order. Since, moreover, the substitution of

$$J^m_{\beta,\gamma} = \kappa^{2(b-1)}F^{(1)} \tag{5-45}$$

in (5-44) leads to

$$t(t-1)(t+1-b)J^m_{\beta,\gamma} - \kappa^2(3t+4-a-3b-2m)t(t+a+1-b)J^m_{\beta,\gamma} + 2\kappa^4(t+2-a-b-m)(t+a+1-b)(t+a+2-b)J^m_{\beta,\gamma} = 0, \tag{5-46}$$

it follows that the function $J^m_{\beta,\gamma}(\kappa^2)$ satisfies an ordinary differential equation of the *third* order.

A solution of the foregoing equation (5-46) can obviously be sought in the form of a series in ascending powers of κ^2. Since the operator t occurs as a factor of the first as well as second term, the unit difference of the roots of the corresponding indicial equation will not give rise to a logarithmic singularity at the origin. Constructing the particular solution equivalent to the integral on the right-hand side of equation (5-41p) we find that

$$2\pi J^m_{\beta,\gamma} = 4^{b-1}B(a, b-a)\sum_{j=0}^{\infty}\frac{a_j(1-a)_j}{j!(b)_j}F(-j, -m, a-j, 2)\kappa^{2(b+j-1)}, \tag{5-47p}$$

where $(a)_j = a(a+1)(a+2)\ldots(a+j-1)$, $(a)_0 = 1$, and F stands for the ordinary hypergeometric function ${}_2F_1$. The foregoing series constitutes the most general representation of the integral $J^m_{\beta,\gamma}$ valid for all values (not necessarily integral) of β, γ, or m—provided only that their combination is such as to make the series convergent. Since, in our astronomical problem, m is restricted to assume the values of zero or a positive integer while $\beta + \gamma > -2$, the absolute and uniform convergence of the expansion on the right-hand side of (5-47p) for $\kappa^2 < 1$ is assured.

It may be noted that for three particular values of the parameters β, γ, m the differential equation (5-44) governing the function $F^{(1)}$ reduces to one of second order. This will evidently happen if $m = 0$; for then (5-47p) reduces to an ordinary hypergeometric series and, in consequence,

$$2\pi J^0_{\beta,\gamma} = 4^{b-1}B(a, b-a)\kappa^{2(b-1)}F(a, 1-a, b, \kappa^2). \tag{5-48p}$$

The second reducible case arises when $\gamma = 0$ and, therefore, $a = b - 1$. If so, equation (5-44) can (by removal of a factor) be reduced to

$$\{t - \kappa^2(3t+2-b-2m) + 2\kappa^4(t+2-b-m)\}(t+b-1)F^{(1)} = 0, \tag{5-49}$$

which is also one of second order*. The third reducible case, arising when

* *Cf.* K. Heun, *Math. Annalen*, **33**, 161, 1889.

$2(\beta + m + 2) + \gamma = 0$ has no relevance to astronomical applications and is being mentioned only for the sake of completeness.*

All equations given thus far hold good only for *partial* eclipses. If the eclipse becomes *annular*, the substitution required to reduce $J^m_{\beta,\gamma}$ to tractable form is

$$\delta - x = r_2(1 - 2v) \tag{5-38a}$$

and

$$\pi J^m_{\beta,\gamma} = 2^{\beta+\gamma+1}\kappa^{-\gamma}\int_0^1 v^{\beta/2}(1 - v)^{\beta/2}(1 - \kappa^2 v)^{\gamma/2}(1 - 2v)^m\, dv, \tag{5-41a}$$

where the modulus

$$\kappa^2 = \frac{2}{1 - \mu} \tag{5-39a}$$

turns out to be a reciprocal of the one appropriate for partial eclipses. The right-hand side of (5-41a) is once more an integral representation of the generalized hypergeometric series

$$2\pi J^m_{\beta,\gamma} = 4^{b-1}\kappa^{2(a-b+1)}B(a, a)F^{(1)}(a; -m, a - b + 1; 2a; 2, \kappa^2)\,. \tag{5-43a}$$

The differential equation governing this function is again of the form (5-46) and can be obtained from it by an appropriate permutation of parameters. Its solution is, in turn, expressible as

$$2\pi J^m_{\beta,\gamma} = 4^{b-1}B(a, a)\sum_{j=0}^{\infty}\frac{(a)_j(a - b + 1)_j}{j!(2a)_j}F(-m, a + j, 2a + j, 2)\kappa^{2(a-b+j+1)}; \tag{5-47a}$$

if $m = 0$, this series reduces to

$$2\pi J^0_{\beta,\gamma} = 4^{b-1}B(a, a)\kappa^{2(a-b+1)}F(a - b + 1, a, 2a, \kappa^2); \tag{5-48a}$$

while if $\gamma = 0$ (i.e., $a = b - 1$),

$$2\pi J^0_{\beta,0} = 4^a B(a, a)F(-m, a, 2a, 2). \tag{5-50}$$

This latter expression no longer involves κ and therefore does not vary during annular phase.

As it proved to be the case with associated α-functions, not all J-integrals of the foregoing forms need to be evaluated by a recourse to their literal expansions; but this task can be facilitated by the existence of several *recursion formulae* relating the $J^m_{\beta,\gamma}$'s for different values of β, γ and m in a linear manner. Thus we find (by virtue of simple algebraic identities) that

$$J^m_{\beta+2,\gamma} = J^m_{\beta,\gamma} - J^{m+2}_{\beta,\gamma} \tag{5-51}$$

or

$$J^m_{\beta,\gamma+2} = 2\{J^{m+1}_{\beta,\gamma} - \mu J^m_{\beta,\gamma}\}, \tag{5-52}$$

and (using known recursion properties of the hypergeometric series) we also establish that

$$(\beta + 2)J^1_{\beta,\gamma} = \gamma J^0_{\beta+2,\gamma-2}. \tag{5-53}$$

* *Cf.* T. W. Chaundy, *Quart. Journ. of Math.* (Oxford), **14**, 55, 1943.

If $m = 0$, we have two further recursion formulae

$$\left.\begin{aligned}(\beta+2)J^0_{\beta,\gamma+2} + 2(\beta+2)\mu J^0_{\beta,\gamma} &= 2\gamma J_{\beta+2,\gamma-2}\,,\\ \{\gamma+2(\beta+3)\}J^0_{\beta+2,\gamma} + 2(\beta+2)J^0_{\beta,\gamma} &= 2\gamma\mu J^0_{\beta+2,\gamma-2}\,,\end{aligned}\right\} \tag{5-54}$$

which can also be written as

$$\left.\begin{aligned}\{\gamma+2(\beta+3)\}\mu J^0_{\beta+2,\gamma} + (\beta+2)J^0_{\beta,\gamma+2} &= 2\gamma(1-\mu^2)J^0_{\beta+2,\gamma-2}\,,\\ \{\gamma+2(\beta+3)\}J^0_{\beta+2,\gamma} + (\beta+2)\mu J^0_{\beta,\gamma+2} &= 2(\beta+2)(1-\mu^2)J^0_{\beta,\gamma};\end{aligned}\right\} \tag{5-55}$$

and for $\gamma = 0$ we have the recursion formulae

$$\left.\begin{aligned}(m+\beta+1)J^m_{\beta,0} &= (\beta+2)\mu^{m-1}J^1_{\beta,0} + (m-1)J^{m-2}_{\beta,0}\\ &= -\beta\mu^{m+1}\,J^1_{\beta-2,0} + \beta J^m_{\beta-2,0}\,,\end{aligned}\right\} \tag{5-56}$$

of elementary character, of which equation (5-9) used earlier to deduce the recursion formula (5-11) for the associated α-functions happened to be a particular case. General recursion formulae relating the J-integrals for different values of m, β and γ depend, however, on recursion properties of Appell's generalized hypergeometric series which remain as yet largely unexplored.

A repeated use of the foregoing known recursion formulae for particular types of the J-integrals enable us eventually to express all members of their family in terms of functions of the form $J^m_{-1,0}$ and $J^m_{-1,1}$, constituting a basic set from which all higher requisite integrals can be generated. In their cases, the expansions found previously for functions $J^m_{\beta,\gamma}$ of unrestricted order reduce to a representation of certain more elementary functions whose identity can be established by integration in a closed form. Thus as regards $J^m_{-1,0}$, it is easy to see that

$$\pi J^0_{-1,0} = \cos^{-1}\mu \qquad \text{and} \qquad \pi J^1_{-1,0} = \sqrt{1-\mu^2}, \tag{5-57}$$

while the integrals corresponding to $m > 1$ can be generated by means of the recursion formula

$$mJ^m_{-1,0} = (m-1)\,J^{m-2}_{-1,0} + \mu^{m-1}J^1_{-1,0} \tag{5-58}$$

coincident with (5-9) to find that, during *partial* eclipses,

$$\left.\begin{aligned}2\pi J^2_{-1,0} &= \cos^{-1}\mu + \mu\sqrt{1-\mu^2},\\ 3\pi J^3_{-1,0} &= (\mu^2+2)\sqrt{1-\mu^2},\\ 8\pi J^4_{-10,} &= 3\cos^{-1}\mu + \mu(2\mu^2+3)\sqrt{1-\mu^2},\\ 15\pi J^5_{-1,0} &= (3\mu^4+4\mu^2+8)\sqrt{1-\mu^2},\end{aligned}\right\} \tag{5-59}$$

etc., while if the eclipse becomes *annular* then, in agreement with equations (5-5) and (5-37),

$$J^{2\mu}_{-1,0} = \frac{\Gamma(\mu+\frac{1}{2})}{\mu!\sqrt{\pi}} \qquad \text{and} \qquad J^{2\mu+1}_{-1,0} = 0, \tag{5-60}$$

depending on whether their superscript is even or odd.

When $\gamma = 1$, no simple recursion formula analogous to (5-58) above appears to exist to facilitate our task; but by direct integration we establish that

$$\left.\begin{aligned}
\pi J^0_{-1,1} &= 2\{2E - (1 + \mu)F\},\\
3\pi J^1_{-1,1} &= 2\{(1 + \mu)F - 2\mu E\},\\
15\pi J^2_{-1,1} &= 2\{2(9 - 2\mu^2)E - (1 + \mu)(9 - 2\mu)F\},\\
105\pi J^3_{-1,1} &= 2(25 - 6\mu + 8\mu^2)(1 + \mu)F - 4\mu(19 + 8\mu^2)E,\\
315\pi J^4_{-1,1} &= 4(147 - 24\mu^2 - 16\mu^4)E\\
&\qquad - 2(147 - 36\mu + 12\mu^2 - 16\mu^3)(1 + \mu)F,\\
3465\pi J^5_{-1,1} &= 2(675 - 204\mu + 252\mu^2 - 96\mu^3 + 128\mu^4)(1 + \mu)F\\
&\qquad - 4\mu(471 + 156\mu^2 + 128\mu^4)E,
\end{aligned}\right\} \tag{5-61}$$

etc. if the eclipse is *partial*; and

$$\left.\begin{aligned}
\pi k J^0_{-1,1} &= 4E,\\
3\pi k J^1_{-1,1} &= 4\{(1 + \mu)F - \mu E\},\\
15\pi k J^2_{-1,1} &= 4\{(9 - 2\mu^2)E + 2\mu(1 + \mu)F\},\\
105\pi k J^3_{-1,1} &= 4\{(25 + 8\mu^2)(1 + \mu)F - \mu(19 + 8\mu^2)E\},\\
315\pi k J^4_{-1,1} &= 4(147 - 24\mu^2 - 16\mu^4)E + 16\mu(9 + 4\mu^2)(1 + \mu)F,\\
3465\pi k J^5_{-1,1} &= 4(675 + 252\mu^2 + 128\mu^4)(1 + \mu)F\\
&\qquad - 4\mu(471 + 156\mu^2 + 128\mu^4)E,
\end{aligned}\right\} \tag{5-62}$$

etc. if it is *annular*, where μ continues to be given by (5-40) and F, E denote the complete elliptic integrals of the first and second kind, respectively, of modulus κ as defined by equations (5-39p) or (5-39a) according to whether the eclipse is partial or annular.

The functions just quoted represent a fundamental set from which all other functions involved directly or indirectly in the theory of light (and velocity) curves of close binary systems can be built up by suitable recursion formulae. As, moreover, all these fundamental functions (and many others) have been adequately tabulated* in terms of an angular variable $\alpha = 2\sin^{-1}\kappa$, the investigation of formal aspects of our photometric problem can now be regarded as complete.

IV.6. PHOTOMETRIC EFFECTS OF REFLECTION IN CLOSE BINARY SYSTEMS

It is inevitable in close binary systems that a part of the radiation of either component sent out in every direction will fall on the surface of its

* *Cf.* Z. Kopal, 'Theory and Tables of the Associated α-functions', *Harv. Circ.*, No. 450, 1947.

mate where it will be absorbed and re-emitted (or scattered). This should evidently cause the hemisphere of each star exposed to the radiation incident on it from outside to be brighter than that turned away; and as the components revolve the amount of light 'reflected' in the direction of the line of sight should vary with the phase. If the star is to remain in radiative equilibrium, it is obvious that all radiation incident upon it must be wholly returned—i.e., *the heat albedo of the star must necessarily be unity.* The incident radiation will exert an additional pressure on the boundary of the reflecting component. It may rearrange the atmospheric temperature-distribution over the illuminated hemisphere and, in particular, increase its surface temperature out of proportion to the effective temperature. But all these effects are too small to influence the interior of the reflecting star to any appreciable extent, and its equilibrium remains unchanged.

For the moment we propose to pass over the problem of translating a heat curve into light changes observable in a finite spectral range and to confine our attention to the amount of energy reflected at any particular phase. The latter can again be obtained by integrating an expression of the form (2-1) provided that we know the appropriate intensity-distribution of the reflected radiation and extend the limits of integration over the illuminated crescent visible at any particular phase. In discussing the angular intensity-distribution of reflected light in section IV.1 earlier in this chapter we noted that the intensity of reflected light falls off so rapidly between centre and limb that an adoption of Lambert's cosine law for its angular distribution should represent a tolerable approximation (*cf.* Table 4-4). This will constitute one of the approximations at the basis of our work in this section.

The other approximation, to be adopted likewise for the sake of simplicity, will concern the form of the illuminating and reflecting components: namely in what follows we propose to investigate the photometric effects of reflection to the order of accuracy to which both components can be regarded as spheres. The first-order reflection effect—proportional in magnitude to the area exposed to incident light—will be of the order of $(a/R)^2$ where a stands for the radius of the reflecting star and R, its distance from the illuminating source. On the other hand, in section II.1 we learned that the superficial distortion of the two components will, in general, be of the order of $(a/R)^3$. Hence, the photometric effects of reflection should not be influenced by distortion till through terms of the order of $(a/R)^5$ which we propose to ignore. The assumption that the components possess spherical form (implying the neglect of their rotational or tidal distortion) will, therefore, restrict the applicability of our results to actual binary systems to terms of the order of $(a/R)^2$, $(a/R)^3$ and $(a/R)^4$.

In order to proceed, under these assumptions, to establish the phase law for reflected light in close binary systems, let us adopt a rectangular frame of reference XYZ, with origin O at the centre of the reflecting star (which we shall hereafter refer to as the primary component), and oriented so that its X-axis coincides with the radius-vector joining the centres of the two stars and the Y-axis lies in the plane defined by the radius-vector and the line of sight.* Let, furthermore, r, ϕ, η be the spherical polar coordinates in this system, related with the rectangular coordinates x, y, z by

$$\left.\begin{aligned} x &= r\cos\phi\sin\eta, \\ y &= r\sin\phi\sin\eta, \\ z &= r\cos\eta, \end{aligned}\right\} \tag{6-1}$$

so that η denotes the polar distance of an arbitrary point P on the primary's surface (see Fig. 4-3); and ϕ, its longitude measured from the 'principal meridian' in the XZ-plane.

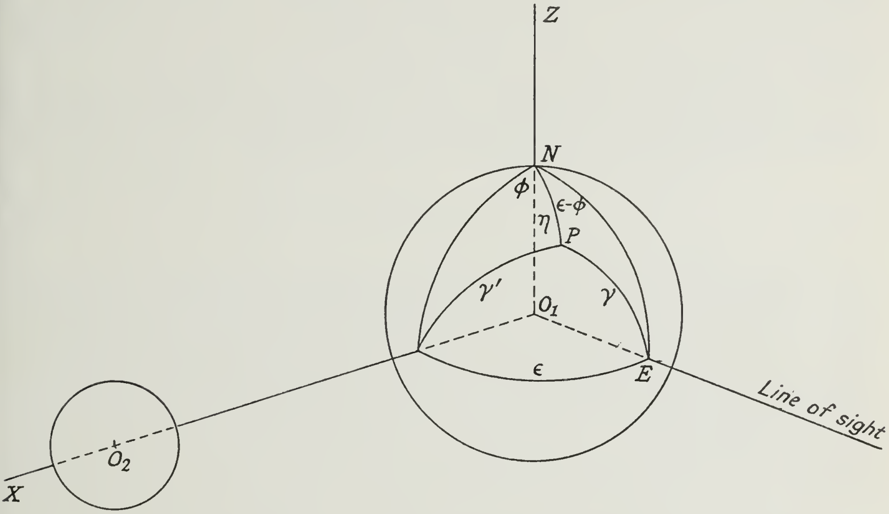

FIGURE 4-3. GEOMETRY OF THE REFLECTION EFFECT

As the components revolve, the XYZ system of axes will evidently rotate with respect to the line of sight whose direction cosines, in the XYZ system, are ($\cos\varepsilon$, $\sin\varepsilon$, 0) if ε denotes the angle between the radius-vector and the line of sight, such that (consistent with 2-29)

$$\cos\varepsilon = l_0 = \cos\psi\sin i\,, \tag{6-2}$$

where ψ denotes the orbital phase angle and i, the inclination of the orbital plane to the celestial sphere. Consistent with these definitions, the angle $\varepsilon = 0$ corresponds to 'full' phase of the illuminated primary component; and $\varepsilon = \pi$, to the moment when its phase as seen by the distant observer is 'new'. Moreover, the amount $\mathscr{L}$ of the secondary's light 'reflected' from the primary component in the direction of the line of sight should, at any

* The reader should take good care to note that this is *not* the plane of the binary orbit, unless the inclination i of the latter to the celestial sphere happens to be equal to 90°.

phase, be generally expressible as

$$\mathscr{L} = \int_{\mathbf{C}} \mathscr{I} \cos \gamma \, d\sigma_1 , \tag{6-3}$$

where $\mathscr{I}$ denotes the intensity distribution of reflected light; γ, the angle of foreshortening; and $d\sigma_1$, the corresponding surface element of the reflecting star; while the domain **C** of integration is to be extended over the crescent visible to a distant observer at any particular phase.

Consistent with the foregoing definitions, the angle γ of foreshortening can be solved for from the spherical triangle *NPE* (Fig. 4-3) in the form

$$\cos \gamma = \sin \eta \cos (\varepsilon - \phi), \tag{6-4}$$

while

$$d\sigma_1 = a_1^2 \sin \eta \, d\eta \, d\phi , \tag{6-5}$$

where a_1 denotes the (constant) radius of the primary component.

As to the limits of integration in both variables let us, for the sake of brevity, introduce the integral operator $K(x)$ defined by

$$\int_{\varepsilon-\pi/2}^{\cos^{-1} x} \int_{\sin^{-1}(x \sec \phi)}^{\pi-\sin^{-1}(x \sec \phi)} \ldots \cos \gamma \, d\sigma_1 = K(x) \{ \ldots \} \tag{6-6}$$

and consider the illumination of one finite sphere by another. The fully illuminated portion of the primary's surface (at which the whole apparent disk of the secondary component of radius a_2 will be visible to an observer situated at P) will evidently be subtended by the inner tangent cone to both spheres, within which

$$\eta_1 \leqslant \eta \leqslant \pi - \eta_1 , \tag{6-7}$$

where

$$\sin \eta_1 = \frac{a_1 + a_2}{R} , \tag{6-8}$$

$R \equiv \overline{O_1O_2}$ denoting the separation of centres of the two components. For $\eta = \eta_1$ (or $\pi - \eta_1$) the lower limb of the apparent disk of the secondary component will touch the horizon of the observer at P; and for η outside the inequality (6-7) this disk will gradually sink below the horizon of the observer until, for $\eta = \eta_2$ (or $\pi - \eta_2$) such that

$$\sin \eta_2 = \frac{a_1 - a_2}{R} , \tag{6-9}$$

it disappears completely. Therefore, as long as

$$\eta_1 \leqslant \varepsilon \leqslant \pi - \eta_1, \tag{6-10}$$

the fully illuminated zone will eventually be delimited by an application of the integral operator $K(\sin \eta_1)$; while the penumbral zone (illuminated by only a part of the apparent disk of the secondary) should be delimited by an application of the operator $K(\sin \eta_2) - K(\sin \eta_1)$.

The foregoing statements are, to be sure, exact only as long as the phase angle ε continues to lie between the limits of the foregoing inequality (6-10); for when

$$-\eta_1 \leqslant \varepsilon \leqslant \eta_1, \tag{6-11}$$

the fully illuminated zone becomes completely exposed to the external observer, and the lower limit $\varepsilon - 90^\circ$ of integration with respect to ϕ in the operator $K(\sin \eta_1)$ should be replaced by $-\cos^{-1}(\sin \eta_1)$.*

When, on the other hand,

$$\pi - \eta_1 \leqslant \varepsilon \leqslant \pi + \eta_1, \tag{6-12}$$

the whole fully illuminated zone becomes invisible and the integral $K(\sin \eta_1)$ $\{\dots\}$ then vanishes. Similarly, when

$$-\eta_2 \leqslant \varepsilon \leqslant \eta_2, \tag{6-13}$$

the limb of the penumbral zone becomes a circle, and the lower limit $\varepsilon - 90^\circ$ of integration with respect to ϕ in the operator $K(\sin \eta_2)$ should be replaced by $-\cos^{-1}(\sin \eta_2)$. When, ultimately,

$$\pi - \eta_2 \leqslant \varepsilon \leqslant \pi + \eta_2 \tag{6-14}$$

and $a_1 > a_2$, the whole penumbral zone will be lost out of sight of our external observer, and the integral $K(\sin \eta_2)$ $\{\dots\}$ will consequently vanish. Should, on the other hand, the illuminating component be the larger of the two the penumbral zone will not disappear for ε constrained by (6-14) but will continue to surround the primary component with an illuminated ring—obtained again by replacing the lower limit $\varepsilon - 90^\circ$ with respect to ϕ in $K(\sin \eta_2)$ by $-\cos^{-1}(\sin \eta_2)$ and remembering that, for $a_1 < a_2$, $\sin \eta_2$ becomes a negative quantity.

It may be added that the foregoing equations, as they stand, express the fraction of light of the secondary component reflected by the primary star. The fraction of the primary's light reflected from the secondary at any phase can likewise be deduced from the foregoing equations, provided only that the values of a_1 and a_2 are interchanged, and that the phase angle ε is consistently replaced by $\pi - \varepsilon$. The total amount of light reflected by a close binary system will then be equal to the sum $\mathscr{L}(a_1, a_2; \varepsilon) + \mathscr{L}(a_2, a_1; \pi - \varepsilon)$; and it is this sum which must be taken out of the observed light changes before the light curve of the system is analysed for photometric elements in the usual manner.

In order to be able to evaluate the integral (6-3) as a function of ε, we must specify the intensity $\mathscr{I}(\phi, \eta)$ of radiation reflected at any point P of the illuminated portion of the primary star. This intensity can indeed be established provided that we can ascertain (*i*) the flux of radiation of the secondary

* This case is, however, largely academic; for if $|\varepsilon| < |\eta_1|$, the secondary component is going to eclipse the primary star; and if $a_2 > 0$, a part of the reflected light is bound to be lost by eclipse.

component incident at P; and (*ii*) its transfer (by absorption-re-emission, or scattering) outwards consistent with the requirement that the heat-albedo of the stellar surface be unity. In what follows we shall by-pass the physical difficulty inherent in this proposition by assuming that the reflected radiation is uniformly distributed (which is always true to a first approximation), and that its intensity varies in accordance with Lambert's cosine law

$$\mathscr{I}(\phi, \eta) = S \cos \alpha\,, \tag{6-15}$$

where πS denotes the flux, per unit area normal to it, originating in the secondary component; and α is the angle of incidence (*cf.*, Fig. 4-4). In the present section our problem will, therefore, be limited to expressing both S and α in terms of our angular variables ϕ, η.

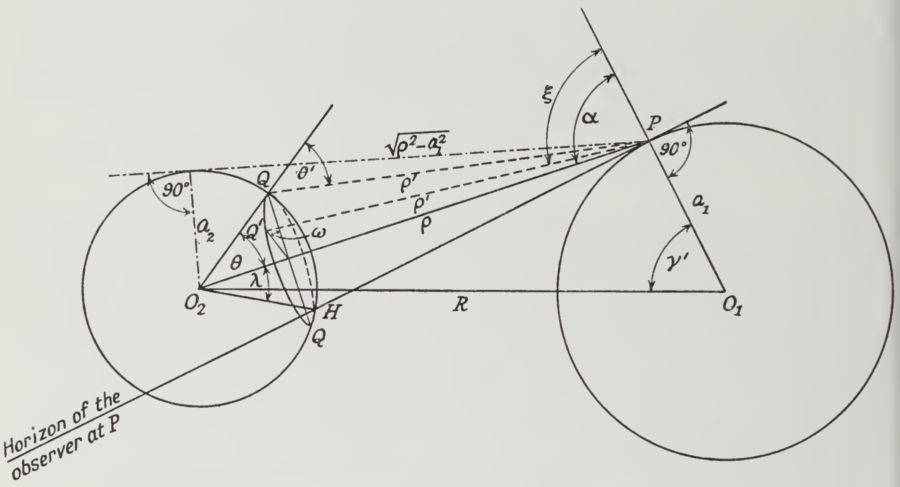

FIGURE 4-4. REFLECTION EFFECT: GEOMETRY OF THE PENUMBRA

In order to do so, consider a new rectangular system of axes, with origin at P (see Fig. 4-4) whose Z-axis coincides with $\overline{O_2P} \equiv \rho$, and the XZ-plane coincides with that of the triangle O_2PQ (i.e., with the plane of the diagram); the Y-axis being perpendicular to it. The direction cosines of an arbitrary radius-vector from O_1 through P then are

$$\sin \alpha,\ 0,\ \cos \alpha;$$

while the direction cosines of $PQ' \equiv \rho'$ in the same system become

$$\sin \beta \cos \omega,\ \sin \beta \sin \omega,\ \cos \beta,$$

where β denotes the angle O_2PQ between the vectors ρ and ρ', and $\omega = QP'Q'$ (see Fig. 4-4).

If so, the angle ξ between ρ' and the surface normal at P will evidently be specified by

$$\cos\xi = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\omega\,, \tag{6-16}$$

where, from $\triangle O_2PQ$,

$$\sin\beta = \frac{a_2}{\rho}\sin\theta' \qquad \text{and} \qquad \cos\beta = \frac{\rho - a_2\cos\theta}{\rho'}\,, \tag{6-17}$$

while (from the same triangle) it follows that

$$\cos\theta' = \frac{\rho^2 - a_2^2 - \rho'^2}{2\rho' a_2} = \frac{\rho\cos\theta - a_1}{\rho'}\,. \tag{6-18}$$

Therefore,

$$\rho'^2 = \rho^2 + a_2^2 - 2a_2\rho\cos\theta, \tag{6-19}$$

while (from $\triangle O_1O_2P$)

$$\rho^2 = R^2 + a_1^2 - 2a_1R\cos\gamma', \tag{6-20}$$

where $\gamma' = \measuredangle O_2O_1P$, defined by

$$\cos\gamma' = \sin\eta\cos\phi; \tag{6-21}$$

and, ultimately,

$$\cos\alpha = \frac{R\cos\gamma' - a_1}{\rho}. \tag{6-22}$$

The last three equations (6-20)–(6-22) define α as a function of ϕ and η; while the flux πS incident at P can be ascertained as follows. Let the distribution of brightness $I(\theta')$ over the apparent disk of the secondary component vary in accordance with the well-known cosine law of limb-darkening as

$$I(\theta') = I(0)\{1 - u + u\cos\theta'\}, \tag{6-23}$$

where $I(0)$ denotes the intensity at the apparent centre of the illuminating star and u, its coefficient of darkening. The intensity $\mathscr{I}(\phi, \eta)$ of light reflected at P can then be expressed as

$$\mathscr{I}(\phi, \eta) = I(0)\{(1-u)J_1 + uJ_2\}, \tag{6-24}$$

where

$$\pi J_n = \int \frac{\cos\xi\cos^n\theta'}{\rho'^2}\,d\sigma_2\,, \tag{6-25}$$

the surface element $d\sigma_2$ of a (spherical) secondary component being, in turn, expressible as

$$d\sigma_2 = a_2^2\sin\theta\,d\theta\,d\omega = a_2(\rho'/\rho)\,d\rho'\,d\omega; \tag{6-26}$$

while $\cos\xi$ and ρ' continue to be defined by (6-16) and (6-19).

As long as the whole apparent disk of the secondary component remains visible to the observer at P, the limits of integration in ω extend from 0 to 2π. If so, however, the integration with respect to ω clearly annihilates the second part of the right-hand side of (6-25) arising from the second term on the right-hand side of the equation (6-16) for cos ξ, leaving us with

$$\pi J_1 = \int_{\rho_1}^{\rho_2}\int_0^{2\pi} \frac{\cos\alpha\cos\beta}{\rho'^2}\,\frac{\rho^2 - a_1^2\rho'^2}{2\rho}\,d\rho'\,d\omega\,, \tag{6-27}$$

where

$$\left.\begin{aligned} \rho_1 &= \rho - a_2\,, \\ \rho_2 &= (\rho^2 - a_2^2)^{1/2}\,, \end{aligned}\right\} \tag{6-28}$$

and cos α as well as cos β have already been defined in terms of our variables of integration. Integrating, we thus establish that

$$J_2 = \frac{\cos\alpha}{2\rho^2}\int_{\rho_1}^{\rho_2} \frac{(\rho^2 - a_2^2)^2 - \rho'^4}{\rho'^3}\,d\rho' = \frac{a_2^2}{\rho^2}\cos\alpha\,; \tag{6-29}$$

while, similarly,

$$\left.\begin{aligned} J_2 &= \int_{\rho_1}^{\rho_2}\int_0^{2\pi} \frac{\cos\alpha\cos\beta}{\rho'^2}\left\{\frac{\rho^2 - a_1^2 - \rho'^2}{2a_2\rho'}\right\}^2 \frac{a_2\rho'}{\pi\rho}\,d\rho'\,d\omega \\ &= \frac{2}{3}\,\frac{a_2^2}{\rho^2}\cos\alpha\,. \end{aligned}\right\} \tag{6-30}$$

Inserting (6-29) and (6-30) in (6-24) we find, eventually, that

$$\mathscr{I}(\phi,\eta) = \frac{L_2\cos\alpha}{\pi\rho^2}\,, \tag{6-31}$$

where

$$L_2 = \pi a_2^2 I(0)\{1 - \tfrac{1}{3}u\} \tag{6-32}$$

represents the apparent luminosity of the secondary component. At any point of the *fully illuminated zone* the flux incident upon P proves to be proportional to the total light of the secondary component, and inversely proportional to the square of the distance of its centre from P; so that, by (6-31) and (6-20)–(6-23)

$$\mathscr{I}(\phi,\eta) = \frac{L_2}{\pi}\,\frac{R\lambda - a_1}{(R^2 + a_1^2 - 2a_1R\lambda)^{3/2}}\,, \tag{6-33}$$

where we have abbreviated

$$\lambda = \cos\gamma' = \cos\phi\sin\eta. \tag{6-34}$$

In the *penumbral zone* the situation becomes complicated by virtue of the fact that the secondary component now undergoes eclipse by the horizon of the observer (which acts as a straight occulting edge), and only the light of the uneclipsed portion will continue to illuminate the surface element at P. In order to evaluate the intensity of light reflected from any point of the penumbral zone, let us write again

$$\mathscr{I}'(\phi, \eta) = I(0)\{(1-u)J_1' + uJ_2'\}, \tag{6-35}$$

where now

$$\pi J_n' = 2a_2^2 \int_{-|\chi|}^{\cos^{-1}(a_2/\rho)} \int_0^{\cos^{-1}(\sin\chi \csc\theta)} (\rho')^{-2} \cos\xi \cos^n\theta' \sin\theta \, d\theta \, d\omega, \tag{6-36}$$

the limits of integration being now adjusted so as to extend over the visible portion of the secondary's disk visible to the observer at P. The (plane) angle χ, which will hereafter play the role of a 'geometrical depth of the eclipse' is, in turn, defined by the triangle HPO_2 (*cf.*, Fig. 4-4) as

$$\chi = \alpha - \cos^{-1}(\rho/a_2)\cos\alpha\,, \tag{6-37}$$

and normalized so as to vary between $\pm\cos^{-1}(a_2/\rho)$ as the apparent disk of the secondary component continues to set behind the horizon of the observer. We shall, moreover, reckon χ so that its upper limit corresponds to the beginning of the eclipse (i.e., to the observer situated at the boundary between the fully illuminated and penumbral zones).

The integration of (6-36) with respect to ω offers again but little difficulty; for inserting $\cos\xi$ from (6-35) we readily establish that

$$J_n' = M_n \cos\alpha + N_n \sin\alpha\,, \tag{6-38}$$

where

$$\pi M_n = 2a_2^2 \int_{-|\chi|}^{\cos^{-1}(a_2/\rho)} \frac{\cos\beta}{\rho'^2} \cos^{-1}\frac{\sin\chi}{\sin\theta} \cos^n\theta' \sin\theta \, d\theta \tag{6-39}$$

and

$$\pi N_n = 2a_2^2 \int_{-|\chi|}^{\cos^{-1}(a_2/\rho)} \frac{\sin\beta}{\rho'^2} (\sin^2\theta - \sin^2\chi)^{1/2} \cos^n\theta' \, d\theta, \tag{6-40}$$

respectively. Unlike the case of flux in the fully illuminated zone, however, the second integration of the foregoing expression with respect to θ can no longer be performed in a closed form without appeal to certain simplifications which we shall presently proceed to discuss.

In the foregoing paragraphs we have formulated rigorously the problem of mutual illumination of two spheres reflecting in accordance with Lambert's law. The expressions for the flux and limits of integration could, in fact, be used to study the reflection of light from two spheres in actual contact, or from a sphere illuminated by an infinite wall—in both of which cases there would be no fully illuminated zone, but only a penumbral zone (extending, in the latter case, over the entire surface of the reflecting sphere). Such extreme cases would, however, be of little or no astrophysical interest, since two stars brought so close together could not possibly retain spherical shape, and their mutual distortion would render the geometry of our preceding sections inexact. As was mentioned in the introductory paragraphs of this section, it is our intention to carry through the solution of our problem only to the order of accuracy to which both components of our binary system can be regarded as spherical (i.e., to quantities of the fourth order in their fractional radii). In doing so, we surmise, in particular, that the amount of light reflected from the penumbral zone will represent a small quantity whose square and higher powers become negligible within the scheme of our approximation.

If, however, only terms of the lowest order are to be retained, we may evidently set

$$\left.\begin{aligned} \sin\beta &\simeq (a_2/\rho)\sin\theta, \\ \cos\beta &\simeq 1, \\ \cos\theta' &\simeq \cos\theta, \\ \rho' &\simeq \rho, \end{aligned}\right\} \tag{6-41}$$

in virtue of which the flux of radiation incident on the penumbral zone may be approximated by

$$\left.\begin{aligned} M_1 &= \frac{2a_2^2}{\pi\rho^2}\int_{-|\chi|}^{\pi/2}\cos^{-1}\left(\frac{\sin\chi}{\sin\theta}\right)\sin\theta\cos\theta\,d\theta \\ &= \frac{a_2^2}{\rho^2}\left\{\frac{1}{2}+\frac{\chi+\sin\chi\cos\chi}{\pi}\right\}, \end{aligned}\right\} \tag{6-42}$$

and

$$\left.\begin{aligned} N_1 &= \frac{2a_2^2}{\rho^2}\int_{-|\chi|}^{\pi/2}(\sin^2\theta-\sin^2\chi)^{1/2}\sin\theta\cos\theta\,d\theta \\ &= \frac{2a_2^3}{3\pi\rho^3}\cos^3\chi \end{aligned}\right\} \tag{6-43}$$

for uniformly bright disks, while

$$\left.\begin{aligned} M_2 &= \frac{2a_2^2}{\pi\rho^2}\int_{-|\chi|}^{\pi/2}\cos^{-1}\left(\frac{\sin\chi}{\sin\theta}\right)\sin\theta\cos^2\theta\,d\theta \\ &= \frac{a_2^2}{3\rho^2}\left\{1+\sin\chi\left(1+\frac{\cos^2\chi}{2}\right)\right\} \end{aligned}\right\} \tag{6-44}$$

and

$$\left.\begin{aligned} N_2 &= \frac{2a_2^3}{\pi\rho^3} \int_{-|\chi|}^{\pi/2} (\sin^2\theta - \sin^2\chi)^{1/2} \sin\theta \cos^2\theta \, d\theta \\ &= \frac{a_2^3}{8\rho^3} \cos^4\chi \end{aligned}\right\} \tag{6-45}$$

for disks completely darkened at the limb.

Inserting the foregoing equations in (6-38) we are then in a position to assert that, within the scheme of our approximation,

$$J_1' = \frac{a_2^2}{\rho^2}\left\{\frac{1}{2} + \frac{\chi + \sin\chi\cos\chi}{\pi}\right\}\cos\alpha + \frac{2a_2^2}{3\pi\rho^3}\cos^3\chi\sin\alpha\,, \tag{6-46}$$

$$J_2' = \frac{a_2^2}{3\rho^2}\{1 + \sin\chi\,(1 + \tfrac{1}{2}\cos^2\chi)\}\cos\alpha + \frac{a_2^3}{8\rho^3}\cos^4\chi\sin\alpha\,, \tag{6-47}$$

where, to the order of accuracy we are working,

$$-\frac{\pi}{2} \leqslant \lambda \leqslant \frac{\pi}{2}\,\cdot \tag{6-48}$$

If $\chi = 90°$ (corresponding to the boundary between full-light and penumbral zone), the foregoing expressions reduce to

$$J_1'\left(\frac{\pi}{2}\right) = \frac{a_2^2}{\rho^2}\cos\alpha\,, \tag{6-49}$$

$$J_2'\left(\frac{\pi}{2}\right) = \frac{2}{3}\frac{a_2^3}{\rho^2}\cos\alpha\,, \tag{6-50}$$

in agreement with the results established previously by equations (6-29) and (6-30) for J_1 and J_2 in the fully illuminated zone; whereas if $\chi = -90°$,

$$J_1'\left(-\frac{\pi}{2}\right) = J_2'\left(-\frac{\pi}{2}\right) = 0\,, \tag{6-51}$$

as they should, since the eclipse of the secondary component has now become complete.

In order to simplify further the foregoing expressions for $J_{1,2}$ it should be remembered that within the penumbral zone α will be in the neighbourhood of 90°, thus rendering $\cos\alpha$ a small quantity of the order of a_2/ρ. If so, equation (6-37) defining χ can be replaced by

$$\chi = \sin^{-1}(R/a_2)\cos\alpha \tag{6-52}$$

and, in terms of the order of $(a_2/R)^2$, we may approximate $\sin\alpha \simeq 1$, while $\cos\alpha \simeq (a_2/R)x$. In consequence, equations (6-46) and (6-47) may be rewritten as

$$J_1' = \frac{a_2^3 x}{R^3}\left\{\frac{\sin^{-1} x + x(1-x^2)^{1/2}}{\pi} - \frac{1}{2}\right\} + \frac{2a_2^3}{3\pi R^3}(1-x^2)^{3/2}, \tag{6-53}$$

$$J_2' = \frac{a_2^3}{R^3}\left\{\frac{x}{6}(1+x)(-x^2+x+2) + \tfrac{1}{8}(1-x^2)^2\right\}, \tag{6-54}$$

and expressed in terms of the single variable $x \equiv \sin\chi$ which, consistent with (6-52), can be approximated by

$$x = \sin\chi = (R/a_2)\cos\alpha, \tag{6-55}$$

and is by (6-48) constrained to remain with $-1 \leqslant x \leqslant 1$.

Moreover, an appeal to equation (6-22) discloses that, to the order of accuracy we are working,

$$\frac{R}{a_2}\cos\alpha = \frac{R}{a_2}\lambda - \frac{a_1}{a_2}, \tag{6-56}$$

where, as before, $\lambda = \cos\gamma' = \cos\phi\sin\eta$. If, furthermore, L_2 stands again for the luminosity of the secondary component—as defined by equation (6-32) above—the equation (6-35) governing the intensity of light reflected by the penumbral zone will, with the aid of the foregoing expressions for $J_{1,2}'$, ultimately assume the form

$$\mathscr{I}'(\phi, \eta) = \frac{3(1-u)}{3-u}\mathscr{I}^{\mathsf{U}}(\phi, \eta) + \frac{2u}{3-u}\mathscr{I}^{\mathsf{D}}(\phi, \eta), \tag{6-57}$$

where, as before, u denotes the secondary's coefficient of limb-darkening, and

$$\mathscr{I}^{\mathsf{U}}(\phi, \eta) = \frac{a_2 L_2}{6\pi^2 R^3}\left\{x[\sin^{-1} x + x(1-x^2)^{1/2} - 3\pi] + 4(1-x^2)^{3/2}\right\}, \tag{6-58}$$

while

$$\mathscr{I}^{\mathsf{D}}(\phi, \eta) = -\frac{a_2 L_2}{16\pi R^3}(1+x)^2(-x^2+2x+3), \tag{6-59}$$

respectively. As x can, by means of (6-55) and (6-56), be readily expressed as a function of $\lambda = \cos\phi\sin\eta$, the requisite explicit forms of $\mathscr{I}^{\mathsf{U,D}}(\phi, \eta)$ —and, therefore, of $\mathscr{I}'(\phi, \eta)$—as functions of our angular variables are thereby established.

A possession of the results puts us at last in a position to ascertain the fraction of light of one component of a close binary system from the other at any phase; for this task calls merely for an application of the K-operator, as defined by equation (6-6), to the expressions for $\mathscr{I}(\phi, \eta)$ established in the preceding paragraphs. In the fully illuminated zone, $K \equiv K(\sin\eta_1)$, while the intensity $\mathscr{I}(\phi, \eta)$ of reflected radiation is known to be given by equation (6-33). Moreover, an expansion of the right-hand side of this latter equation in ascending powers of a_1/R in terms of the Legendre polynomials $P_j(\lambda)$ permits us to replace the closed form of (6-33) by

$$\mathscr{I}(\phi, \eta) = \frac{L_2}{\pi R^2}\left\{P_1(\lambda) + 2\frac{a_1}{R} P_2(\lambda) + 3\frac{a_1^2}{R^2}P_3(\lambda) + \dots\right\}, \tag{6-60}$$

correctly to the order of accuracy adopted at the outset. Hence, the corresponding amount of light of the secondary component reflected from the fully illuminated portion of the primary star in the direction of the line of sight should be given by

$$\mathscr{L}_1(a_1, a_2, \varepsilon) = K(\sin\eta_1)\left\{P_1(\lambda) + 2\frac{a_1}{R} P_2(\lambda) + 3\frac{a_1^2}{R^2} P_3(\lambda) + \dots\right\} \frac{L_2}{\pi R^2} \tag{6-61}$$

Performing the actual integration in a closed form we find that

$$K(\sin\eta_1)\,\{P_1(\lambda)\} = \tfrac{2}{3}a_1^2\,\{\Phi_2 - \Phi_3 \sin^3\eta_1\}\cos\varepsilon + \tfrac{2}{3}a_1^2\Psi\cos^2\eta_1\,, \tag{6-62}$$

$$\begin{aligned} K(\sin\eta_1)\,\{P_2(\lambda)\} = {} & \tfrac{1}{4}a_1^2\,\Phi_1 P_2(\cos\varepsilon) \\ & + \tfrac{1}{4}a_1^2\,\Phi_3(1 + 3\sin^2\eta_1)\cos^2\eta_1\cos\varepsilon \\ & + \tfrac{1}{4}a_1^2\,\Psi(\tfrac{1}{2} + 3\cos^2\eta_1)\sin\eta_1\,, \end{aligned} \tag{6-63}$$

$$\begin{aligned} K(\sin\eta_1)\,\{P_3(\lambda)\} = {} & a_1^2\,\Phi_3\sin^3\eta_1\cos^2\eta_1\cos\varepsilon \\ & + a_1^2\,\Psi\sin^2\eta_1\cos^2\eta_1 - \tfrac{1}{3}a_1\,\Psi^3, \end{aligned} \tag{6-64}$$

where we have abbreviated

$$\begin{aligned} \Phi_1 &= \cos^{-1}(\sin\eta_1\csc\varepsilon), \\ \Phi_2 &= \cos^{-1}(-\sec\eta_1\cos\varepsilon), \\ \Phi_3 &= -\cos^{-1}(\tan\eta_1\cot\varepsilon), \end{aligned} \tag{6-65}$$

and

$$\Psi = (\sin^2\varepsilon - \sin^2\eta_1)^{1/2}. \tag{6-66}$$

Since, moreover, these functions admit of expansion in ascending powers of $\sin\eta_1$ of the form

$$\begin{aligned} \Phi_1 &= \frac{\pi}{2} - \sin\eta_1\csc\varepsilon - \frac{1}{6}\sin^3\eta_1\csc^3\varepsilon - \dots, \\ \Phi_2 &= \pi - \varepsilon + \tfrac{1}{2}\sin^2\eta_1\cot\varepsilon + \tfrac{1}{8}\sin^4\eta_1(\cot^2\varepsilon + 3)\cot\varepsilon + \dots, \\ \Phi_3 &= -\frac{\pi}{2} + \sin\eta_1\cot\varepsilon + \frac{1}{6}\sin^3\eta_1(\cot^2\varepsilon + 3)\cot\varepsilon + \dots, \end{aligned} \tag{6-67}$$

and

$$\Psi = \sin\varepsilon - \tfrac{1}{2}\sin^2\eta_1\csc\varepsilon - \tfrac{1}{8}\sin^4\eta_1\csc^3\varepsilon - \dots, \tag{6-68}$$

correctly to quantities of the fourth order in $\sin \eta_1$, it follows that, within the scheme of our approximation,

$$\begin{aligned}\mathscr{L}_1(a_1, a_2; \varepsilon) = L_2 \Bigg\{ & \frac{2}{3}\left(\frac{a_1}{R}\right)^2 \frac{(\pi - \varepsilon)\cos\varepsilon + \sin\varepsilon}{\pi} \\ & + \frac{1}{8}\left(\frac{a_1}{R}\right)^3 (3\cos^2\varepsilon + 2\cos\varepsilon - 1) \\ & + \left(\frac{a_1}{R}\right)^4 \frac{\sin\varepsilon\cos^2\varepsilon}{\pi} - \frac{a_1^2 a_2^2}{R^4}\frac{\sin\varepsilon}{\pi} + \dots \Bigg\}\end{aligned} \tag{6-69}$$

for any phase angle ε within

$$\sin^{-1}\frac{a_1 + a_2}{R} \leqslant \varepsilon \leqslant \pi - \sin^{-1}\frac{a_1 + a_2}{R}. \tag{6-70}$$

In the immediate neighbourhood of the inferior conjunction, when

$$-\sin^{-1}\frac{a_1 + a_2}{R} \leqslant \varepsilon \leqslant \sin^{-1}\frac{a_1 + a_2}{R}, \tag{6-71}$$

the K-operator (6-6) must be modified as explained earlier, in order to account for the fact that the fully illuminated zone should become completely exposed to the external observer (though it is bound to be partially masked by eclipses); and the result of such integration reveals that, during this phase,

$$\mathscr{L}_1(a_1, a_2; \varepsilon) = L_2 \frac{a_1^2}{R^2}\left\{\frac{2}{3} + \frac{a_1 + a_2}{2R} + \dots\right\}\cos\varepsilon, \tag{6-72}$$

correctly to the same order of accuracy.* If, ultimately,

$$\pi - \sin^{-1}\frac{a_1 + a_2}{R} \leqslant \varepsilon \leqslant \pi + \sin^{-1}\frac{a_1 + a_2}{R} \tag{6-73}$$

and the fully illuminated zone becomes completely invisible to our obseiver, the foregoing expression for $\mathscr{L}_1(a_1, a_2; \varepsilon)$ is indeed seen to vanish foi $\varepsilon = \pi$, and to become an ignoiable quantity of the fifth order in a/R for all other ε's within the limits of the inequality (6-72).

The fraction $\mathscr{L}_2(a_1, a_2; \varepsilon)$ of the secondary's light reflected from the *penumbral zone* of the primary star may, in turn, be decomposed into

$$\mathscr{L}_2(a_1, a_2; \varepsilon) = \frac{3(1-u)}{3-u}\mathscr{L}_2^{\mathbf{U}} + \frac{2u}{3-u}\mathscr{L}_2^{\mathbf{D}}, \tag{6-74}$$

where u denotes, as before, the coefficient of limb-darkening of the illuminating star, and

$$\mathscr{L}_2^{\mathbf{U},\mathbf{D}} = \{K(\sin\eta_2) - K(\sin\eta_1)\}\{\mathscr{I}^{\mathbf{U},\mathbf{D}}(\phi, \eta)\}. \tag{6-75}$$

* A part of this light will, of course, be lost by the eclipse.

Now the expression (6-58) for $\mathscr{I}^{\mathbf{U}}$ can be expanded in ascending powers of x to yield

$$\mathscr{I}^{\mathbf{U}}(\phi, \eta) = \frac{a_2 L_2}{\pi R^3}\left\{\frac{2}{3\pi} + \frac{x}{2} + \frac{x^2}{\pi} - \frac{x^4}{12\pi} - \dots\right\} \tag{6-76}$$

within an error of less than 1% for the extreme values attainable by x; and equation (6-56) permits us to convert the foregoing expansion into a series of the form

$$\mathscr{I}^{\mathbf{U}}(\phi, \eta) = \frac{a_2 L_2}{\pi R^3} \sum_{n=0}^{\infty} C_n^{\mathbf{U}} \left(\frac{R\lambda}{a_2}\right)^n, \tag{6-77}$$

where

$$\left.\begin{aligned}
\pi C_0^{\mathbf{U}} &= \frac{2}{3} - \frac{\pi}{2}\left(\frac{a_1}{a_2}\right) + \left(\frac{a_1}{a_2}\right)^2 - \frac{1}{12}\left(\frac{a_1}{a_2}\right)^4 + \dots, \\
\pi C_1^{\mathbf{U}} &= \frac{\pi}{2} - 2\left(\frac{a_1}{a_2}\right) + \frac{1}{3}\left(\frac{a_1}{a_2}\right)^2 + \dots, \\
\pi C_2^{\mathbf{U}} &= 1 - \frac{1}{2}\left(\frac{a_1}{a_2}\right)^2 + \dots, \\
\pi C_3^{\mathbf{U}} &= \frac{1}{3}\left(\frac{a_1}{a_2}\right) + \dots, \\
\pi C_4^{\mathbf{U}} &= -\frac{1}{12} + \dots .
\end{aligned}\right\} \tag{6-78}$$

On the other hand, by virtue of equation (6-59),

$$\left.\begin{aligned}
\mathscr{I}^{\mathbf{D}}(\phi, \eta) &= \frac{a_2 L_2}{\pi R^3}\left\{\frac{3}{16} + \frac{x}{2} + \frac{3x^2}{8} - \frac{x^4}{16}\right\} \\
&= \frac{a_2 L_2}{\pi R^3} \sum_{n=0}^{4} C_n^{\mathbf{D}} \left(\frac{R\lambda}{a_2}\right)^n,
\end{aligned}\right\} \tag{6-79}$$

where

$$\left.\begin{aligned}
C_0^{\mathbf{D}} &= \frac{3}{16} - \frac{1}{2}\left(\frac{a_1}{a_2}\right) + \frac{3}{8}\left(\frac{a_1}{a_2}\right)^2 - \frac{1}{16}\left(\frac{a_1}{a_2}\right)^4, \\
C_1^{\mathbf{D}} &= \frac{1}{2} - \frac{3}{4}\left(\frac{a_1}{a_2}\right) + \frac{1}{4}\left(\frac{a_1}{a_2}\right)^3, \\
C_2^{\mathbf{D}} &= \frac{3}{8} - \frac{3}{8}\left(\frac{a_1}{a_2}\right)^2, \\
C_3^{\mathbf{D}} &= \frac{1}{4}\left(\frac{a_1}{a_2}\right), \\
C_4^{\mathbf{D}} &= -\frac{1}{16}.
\end{aligned}\right\} \tag{6-80}$$

Now as the reader can easily verify,

$$K(\zeta)\{\lambda^n\} = K(0)\{\lambda^n\} - \frac{2a_1^2}{n+1}\,\zeta^{n+1}\sin\varepsilon + \ldots, \tag{6-81}$$

by virtue of which

$$\{K(\sin\eta_2) - K(\sin\eta_1)\}\,\mathscr{I}^{\mathbf{U},\mathbf{D}}(\phi,\eta)$$

$$= 2\frac{L_2}{\pi}\frac{a_1^2 a_2}{R_2}\sum_{n=0}^{\infty}\frac{C_n^{\mathbf{U},\mathbf{D}}}{n+1}\left(\frac{R}{a_2}\right)^n(\sin^{n+1}\eta_1 - \sin^{n+1}\eta_2)\sin\varepsilon$$

$$= \frac{2}{\pi}\left(\frac{a_1a_2}{R^2}\right)^2 L_2\sin\varepsilon\sum_{n=0}^{\infty}\frac{C_n^{\mathbf{U},\mathbf{D}}}{n+1}\left\{\left(\frac{a_1+a_2}{a_2}\right)^{n+1} - \left(\frac{a_1-a_2}{a_2}\right)^{n+1}\right\}, \tag{6-82}$$

where all $C_n^{\mathbf{U}}$'s for $j > 4$ are identically zero. Inserting the appropriate coefficients C_n, as given by the foregoing equations (6-78) and (6-80) in (6-82) we find, moreover, that the whole summation on the right-hand side of (6-82) reduces to π^{-1} in the case of a uniformly bright secondary component ($u = 0$), and to 0·3 in the case of complete darkening ($u = 1$). In consequence, the requisite amount of light reflected by the penumbral zone of the primary component may, to the order of accuracy we have been working, be approximated by

$$\mathscr{L}_2^{\mathbf{U}}(a_1, a_2; \varepsilon) = \left\{\frac{2a_1a_2}{\pi R^2}\right\}^2 L_2\sin\varepsilon + \ldots, \tag{6-83}$$

$$\mathscr{L}_2^{\mathbf{D}}(a_1, a_2; \varepsilon) = \frac{6}{5\pi}\left\{\frac{a_1a_2}{R^2}\right\}^2 L_2\sin\varepsilon + \ldots, \tag{6-84}$$

so that the amount of light, reflected by the penumbral zone, incident from a secondary component arbitrarily darkened at limb becomes

$$\mathscr{L}_2(a_1, a_2; \varepsilon) = \frac{12}{5}\left\{\frac{a_1a_2}{\pi R^2}\right\}^2\left\{\frac{5 + (\pi - 5)\,u}{3 - u}\right\}L_2\sin\varepsilon + \ldots \tag{6-85}$$

for any phase angle ε within the limits of the inequality (6-70). The *total amount of light* reflected from the fully-illuminated as well as penumbral zones of the primary component is then represented by the sum

$$\mathscr{L}(a_1, a_2; \varepsilon) = \mathscr{L}_1(a_1, a_2; \varepsilon) + \mathscr{L}_2(a_1, a_2; \varepsilon), \tag{6-86}$$

where the functions $\mathscr{L}_{1,2}$ are (within the scheme of our approximation) given by the foregoing equations (6-69) or (6-72) and (6-85). To this extent, the geometrical solution of the problem set forth at the beginning of this section is now complete.

In order to facilitate the application of the phase-law (6-86), with its constituents as given by equations (6-69) or (6-72) and (6-85), to practical cases, we find it convenient to expand (6-86) in ascending powers of cos ε in a series of the form

$$\mathscr{L}(a_1, a_2; \varepsilon) = L_2 \sum_{n=0}^{\infty} C_n \cos^n \varepsilon, \tag{6-87}$$

where, within the scheme of our approximation,

$$\left.\begin{aligned}
C_0 &= \frac{2}{3\pi}\left(\frac{a_1}{R}\right)^2 - \frac{1}{8}\left(\frac{a_1}{R}\right)^3 - \frac{k}{\pi}\left(\frac{a_1a_2}{R^2}\right)^2 + \dots,\\
C_1 &= \frac{1}{3}\left(\frac{a_1}{R}\right)^2 + \frac{1}{4}\left(\frac{a_1}{R}\right)^3 + \dots,\\
C_2 &= \frac{1}{3\pi}\left(\frac{a_1}{R}\right)^2 + \frac{3}{8}\left(\frac{a_1}{R}\right)^3 + \frac{1}{\pi}\left(\frac{a_1}{R}\right)^4 + \frac{k}{2\pi}\left(\frac{a_1a_2}{R^2}\right)^2 + \dots,\\
C_3 &= 0,\\
C_4 &= \frac{1}{36\pi}\left(\frac{a_1}{R}\right)^2 - \frac{1}{2\pi}\left(\frac{a_1}{R}\right)^4 + \frac{k}{8\pi}\left(\frac{a_1a_2}{R^2}\right)^2 + \dots,
\end{aligned}\right\} \tag{6-88}$$

where we have abbreviated

$$k = 1 - \frac{12}{5\pi}\frac{5 + (\pi - 5)\,u}{3 - u}. \tag{6-89}$$

Alternatively, by expanding our phase-law (6-86) in a Fourier cosine series we get

$$\mathscr{L}(a_1, a_2; \varepsilon) = L_2 \sum_{n=0}^{\infty} C_n' \cos n\varepsilon, \tag{6-90}$$

where

$$\left.\begin{aligned}
C_0' &= \frac{8}{3\pi^2}\left(\frac{a_1}{R}\right)^2 + \frac{1}{16}\left(\frac{a_1}{R}\right)^3 + \frac{2}{3\pi^2}\left(\frac{a_1}{R}\right)^4 - \frac{2k}{\pi^2}\left(\frac{a_1a_2}{R^2}\right)^2 + \dots,\\
C_1' &= \frac{1}{3}\left(\frac{a_1}{R}\right)^2 + \frac{1}{4}\left(\frac{a_1}{R}\right)^3 + \dots,\\
C_2' &= \frac{16}{27\pi^2}\left(\frac{a_1}{R}\right)^2 + \frac{3}{16}\left(\frac{a_1}{R}\right)^3 + \frac{4}{15\pi^2}\left(\frac{a_1}{R}\right)^4 + \frac{4k}{3\pi^2}\left(\frac{a_1a_2}{R^2}\right)^2 + \dots,\\
C_3' &= 0,\\
C_4' &= \frac{16}{675\pi^2}\left(\frac{a_1}{R}\right)^2 - \frac{52}{105\pi^2}\left(\frac{a_1}{R}\right)^4 + \frac{4k}{15\pi^2}\left(\frac{a_1a_2}{R^2}\right)^2 + \dots,
\end{aligned}\right\} \tag{6-91}$$

etc.

The foregoing phase laws call, however, for one additional explanatory comment: namely, throughout all foregoing developments the symbol L_2 as introduced by equation (6-32) has been used to denote the fractional luminosity of the illuminating component *as seen from the reflecting star*—i.e., its intrinsic luminosity (due to the radiation of its own) augmented by that part of the primary's light which will be reflected from it back in the direction of the radius-vector (corresponding to $\varepsilon = 0$). Therefore, if $(L_{1,2})_0$ denotes the intrinsic luminosities of the respective components, it follows from (6-69) that

$$L_2 = (L_2)_0 + \left\{\frac{2}{3}\left(\frac{a_2}{R}\right)^2 + \frac{1}{3}\left(\frac{a_2}{R}\right)^3 + \ldots\right\}(L_1)_0 \tag{6-92}$$

and, in consequence, the phase-law (6-69) rewritten in terms of the $(L_{1,2})_0$'s should be augmented by the term

$$\mathscr{L}'(a_1, a_2; \varepsilon) = \left\{\frac{2}{3}\frac{a_1 a_2^2}{R^2}\right\}\left\{\frac{(\pi - \varepsilon)\cos\varepsilon + \sin\varepsilon}{\pi}\right\}(L_1)_0 + \ldots, \tag{6-93}$$

representing the phase effect of *secondary reflection*. The foregoing equation (6-92) taken alone would represent the light reflected by the primary component if its secondary had no light of its own; for the latter would still be bound to intercept a part of the radiation of the primary star and to illuminate it by back-reflection in the same way as lunar ashen light illuminates our Earth. Tertiary reflection (arising from reflection of the secondary's light on the primary, back to the secondary, and returned to the primary again) would clearly constitute a quantity of the order of $(a_1^2 a_2/R^3)^2 (L_2)_0$ and therefore negligible within the scope of our approximation.

The phase-law (6-86) with its constituents, augmented by (6-92), represents the final outcome of our analysis of the light changes due to the reflection effect in close binary systems. Its leading term, of the order of $(a_1/R)^2$, would alone represent such light changes if the incident radiation would constitute a parallel beam. The next term, of the order of $(a_1/R)^3$, arises from the convergence of incident beam emanating from a light point; while the finite angular size of the illuminating component makes itself felt, as we have seen, through terms of the order of $(a_1 a_2/R^2)^2$. Moreover, the effects of secondary reflection (i.e., the illumination by reflected light) prove to be of the same order of magnitude. The effects of distortion (both rotational or tidal) of the primary component would influence the amount of light reflected by it through terms of the order of $(a_1/R)^5$; while the distortion of the illuminating (secondary) components would not become appreciable till through terms of the order of $a_1^2 a_2^5/R^7$—i.e., concurrently with the appearance of terms due to the fourth-harmonic tidal distortion of the illuminating star (or with first-order effects of distortion on secondary reflection); and after the effects of tertiary reflection have been taken into account.

The phase-law (6-69) or its expansions (6-87) or (6-90) governs the amount of light of the secondary component intercepted by the primary star and reflected from it in the direction of the line of sight. The corresponding amount of the primary's light reflected from the secondary is, however, bound to be given by the *same* law as above, provided only that the radii $a_{1,2}$ and luminosities $L_{1,2}$ of both components are interchanged and the phase angle ε is replaced by $\pi - \varepsilon$. The total amount of light $\mathscr{L}(\varepsilon)$ thus *added* by reflection to the intrinsic light of a close binary system is, of course. represented by the *sum*

$$\mathscr{L}(\varepsilon) = \mathscr{L}(a_1, a_2; \varepsilon) + \mathscr{L}(a_2, a_1; \pi - \varepsilon); \tag{6-94}$$

and it is this sum which should be removed from the observed light changes of a close eclipsing system before its light curve can be properly analysed for its geometrical elements. The explicit form of this sum is so easily written down with the aid of the results already established that it need not be reproduced here in full. The reader may, however, note that the difference of 180° between the phases of the primary and secondary components will change the algebraic sign of all odd functions of ε in (6-69), but not of even functions. As a result, the terms arising from reflection of light of one star from another and varying as odd functions of the phase (i.e., odd powers of cos ε in 6-87, or cosines of odd multiples of ε on the right-hand side of 6-90) will tend to neutralize each other in the combined light of a close binary system, while those varying as even functions of ε will always reinforce. In the particular case of a system whose components are equal in size and brightness, all odd terms arising from reflection should cancel, and the remaining (additive) even terms should closely simulate the photometric effects of ellipticity as produced by tidal distortion.

It should also be added that all results derived earlier for the amount of reflected light and its variation with the phase hold good irrespective of whether the relative orbit of the two components is circular or eccentric. If it is circular, the phase-angle ε is simply identical with the mean anomaly reckoned from superior conjunction, and the radius-vector R is constant. Should, however, the orbit happen to be an ellipse of semi-major axis A and eccentricity e, it is easy to show that equation (6-2) should be replaced by

$$\cos \varepsilon = \sin (v + \omega) \sin i, \tag{6-94}$$

where v denotes the true anomaly of the secondary (illuminating) component in its relative orbit around the primary star, and ω stands for the longitude of periastron. Furthermore, the radius-vector R will vary in such a way that

$$\frac{a_{1,2}}{R} = \frac{a_{1,2}}{A} \frac{1 + e \cos v}{1 - e^2} \, . \tag{6-95}$$

In consequence, the reflection effect in eccentric binary systems will no longer be symmetrical with respect to conjunctions; and its observed asymmetry may even indicate on inspection an approximate position of the periastron.*

* For an illustrative example of such a case the reader is referred to a study of the reflection effect in the system of ζ Aurigae ($e = 0{\cdot}41$) by K. Walter in *Zs. f. Ap.*, **14**, 62, 1937.

Ultimately we should recall that the reflection effect—like the ellipticity—is bound to give rise to small but continuous variation of light in close binary systems regardless of whether or not the system happens to be an eclipsing variable; and this variation should theoretically disappear only if the relative orbit of the two stars is circular ($R =$ constant) and the line of sight happens to be perpendicular to the orbital plane (i.e., $i = 0°$). If so, then $\varepsilon = 90°$ all the time, and the reflected light continues to contribute a constant amount to the intrinsic luminosity of the system. Should, however, the relative orbit become eccentric, the amount of light added by reflection should vary (approximately) as R^{-2}—being greatest at the time of periastron passage and smallest at apastron—a fact which, if observed, could establish the time of the periastron passage.

Thus far in this section we have been concerned with an investigation of the light changes arising from reflection in close binary systems in the absence of eclipses. If, however, our binary system happens to be an eclipsing variable then, during eclipses, the system must suffer the loss of a varying amount of proper light of the eclipsed star as well as of the light reflected from it; for that portion of the surface of either component which undergoes eclipse is necessarily illuminated by its mate. The loss of proper light suffered at any phase of the eclipse was already investigated in section IV.3 of this chapter; and the loss of reflected light can be formulated with equal ease by much the same analytical means.

In order to do so, we propose to fall back on equation (6-3) and extend merely the limits of integration on the right-hand side over the eclipsed portion of the primary's disk rather than over the whole illuminated crescent. The integrand in (6-3) can again be expressed in terms of the $X'Y'$-coordinates of section IV.3 by a recourse to the transformation represented by equations (3-11), and integrated term-by-term. We may further recall that, throughout this section, we have been concerned with reflection on spherical stars only. This scheme of approximation disposes, in turn, at once of any 'boundary corrections' of section IV.3, leaving us only with 'circular integrals' which can be expressed in terms of the associated α-functions. If, lastly, the components are sufficiently well separated to justify setting, during eclipses, the direction cosines l_0 and l_2 as defined by equations (3-2) and (3-4) equal to one and zero, respectively,* the theoretical loss $\Delta\mathfrak{L}^*(k, p)$ of reflected light within eclipses becomes independent of the scale of each particular system and assumes the form

$$\Delta\mathfrak{L}^*(k,p) = \{\alpha_1^0 + (3\alpha_2^0 - \alpha_0^0)(a_1/R) + \tfrac{3}{2}(5\alpha_3^0 - 3\alpha_1^0)(a_1/R)^2 + \ldots\}(a_1/R)^2 L_2 \,. \tag{6-96}$$

The reader may recall that, when the eclipse becomes total (i.e., the geometrical depth $p = -1$), associated alpha-functions α_n^0 of zero order reduce (in

* This approximation being tantamount to considering the beam of light incident on the primary as parallel.

accordance with 4-68 and 4-69) to $2/(2+n)$ and, in consequence,

$$\Delta\mathfrak{L}^*(k, -1) = \left\{\frac{2}{3}\left(\frac{a_1}{R}\right)^2 + \frac{1}{2}\left(\frac{a_1}{R}\right)^3 + \ldots\right\} L_2 \tag{6-97}$$

in agreement with the phase-law (6-69) for $\varepsilon = 0$.

A glance at the foregoing equation (6-96) reveals that the expansion on its right-hand side is led by the function α_1^0, to express the fractional loss of light of a disk which is completely darkened at the limb. Therefore, it transpires that *if the star undergoing eclipse had no light of its own and the observed minimum were due entirely to the loss of reflected radiation, its light curve would closely approximate that of an eclipse of a completely darkened disk.* This could indeed have been anticipated; for when a_1 is small, the angles of incidence and reflection become sensibly equal near full phase, and equation (6-15) corresponds then to a total darkening at the limb.

All formulae derived so far for the amount of light reflected in close binary systems between eclipses as well as within minima, and its variation with the phase, have been based on the assumption that the albedo of the reflecting star is unity (i.e., that all incident light is re-emitted or scattered). The validity of this assumption is, however, evident only for integrated light; and hence all foregoing results should, in principle, be applicable only to *bolometric* light curves (corrected for atmospheric absorption). If, however, our theory is to be compared with observations carried out in more or less limited spectral ranges, the heat-curves have first to be translated into light curves, relevant to a given effective wave length. *When the illuminating and reflecting components differ in their spectra, the incident light will be returned to the outside radiation field by matter whose temperature is different from that of the emitting source.* If the reflected light were scattered in the atmosphere of the reflecting component (on, say, free electrons) this difference should not matter; for the spectral distribution of light so scattered remains unaltered. If, however, the scattering particles were atoms or ions, or if the actual mechanism of reflection were absorption followed by re-emission, the spectral distribution of light so reflected might differ drastically from that of incident radiation. Stars in radiative equilibrium must return all light received from outside; hence (for black bodies) the Planck curves specifying the incident and reflected radiation will delimit equal areas; but their maxima may be considerably displaced and their relative intensities of both radiations at any particular wave length may differ widely.* It is thus to be expected that, unless the temperatures of

* In order to illustrate such a situation on a drastic example, consider the case of sunlight reflected by the Earth. That part of it which is scattered from the surface will retain the spectral distribution of incident sunlight, characterized by $\lambda_{\max} = 0{\cdot}53\,\mu$ appropriate for the solar effective temperature of 5800°K. On the other hand, the sunlight absorbed by the terrestrial surface will be re-emitted at a (mean) temperature of some 300° K, for which $\lambda_{\max} = 10\,\mu$ happens to lie in deep infra-red.

both components are the same, *the observed efficiency of the incident and reflected light will generally be different.* In other words, the reflection actually observed in any particular frequency of light may be either greater or smaller than that appropriate for total radiation, depending on the temperature of the two stars.

In order to describe this process in more specific terms, let us confine our attention to the light emitted by the secondary (illuminating) component. Its luminosity $(L_2)_\lambda$ observable in an effective wave length λ will obviously be given by

$$(L_2)_\lambda = \left(\frac{J_\lambda}{J_b}\right)_2 L_2, \tag{6-98}$$

where L_2 stands, as before, for the secondary's integrated light and J_b, J_λ denote its surface brightness in total and discrete light, as defined by the equations $L_2 = \pi a_2^2 J_b$ and $(L_2)_\lambda = \pi a_2^2 J_\lambda$, respectively. A known fraction $\mathscr{L} = q(\varepsilon) L_2$ of the secondary's total light will be intercepted by the primary component and reflected in the direction of the line of sight. If now $\mathscr{L}_\lambda$ denotes the amount of reflected light observable in the light of an effective wave length λ, it follows by the same argument as above that

$$\mathscr{L}_\lambda = \left(\frac{J_\lambda}{J_b}\right)_1 \mathscr{L} = q(\varepsilon)\left(\frac{J_\lambda}{J_b}\right)_1 L_2. \tag{6-99}$$

Eliminating L_2 between (6-98) and (6-99) we find that

$$\mathscr{L}_\lambda = q(\varepsilon)(L_2)_\lambda f, \tag{6-100}$$

where

$$f = \left(\frac{J_2}{J_1}\right)_b \left(\frac{J_1}{J_2}\right)_\lambda \tag{6-101}$$

represents the *luminous-efficiency factor* to be applied—between minima as well as within eclipses—to the total (bolometric) reflection already established to reduce it to its observable amount. A similar factor for the light of the primary component reflected from the secondary follows likewise from (6-101) by a mere interchange of appropriate indices.

The value of $(J_1/J_2)_\lambda$ in equation (6-101) stands for the ratio of mean surface brightnesses of the illuminated hemispheres of both components in the light of effective wave length λ; and if the system happens to be an eclipsing variable, its value can be deduced directly from the depths of the primary and secondary minima in the manner of section VI.2. On the other hand, the ratio of the bolometric surface brightnesses $(J_1/J_2)_b$ does not follow directly from available observations; and in order to determine it recourse must be had to a specific law of radiation. Thus, if the stars radiate like black bodies, it follows that

$$\left(\frac{J_1}{J_2}\right)_b = \left(\frac{T_1}{T_2}\right)^4 \tag{6-102}$$

while, at the same time

$$\left(\frac{J_1}{J_2}\right)_\lambda = \frac{e^{c_2/\lambda T_2} - 1}{e^{c_2/\lambda T_1} - 1}, \tag{6-103}$$

where the constant $c_2 = 1{\cdot}438$ cm deg, and $T_{1,2}$ denotes the mean effective temperatures of the *illuminated hemispheres* of the two stars. It should be stressed that these will *not* be identical with the proper effective temperatures of their averted (dark) hemispheres, because of the heating effect of incident radiation which we discussed already in the introductory section IV.1 of this chapter. The effect of diluted radiation of a hot illuminating star in ionizing the outer atmospheric layers of its cool companion may, moreover, alter also the apparent spectral class of the illuminated hemisphere. In such cases it is the relative surface brightness of the dark hemisphere (obtainable from the rectified depth of the secondary minimum), rather than the spectrum, which can reveal to us the true effective temperature of the secondary component.

IV.7. ATMOSPHERIC ECLIPSES

It is a matter of direct experience to verify that the limb of the apparent disk of our sun appears to be perfectly sharp, and the physical reasons underlying this phenomenon are well understood: namely, the structure of the solar atmosphere is such that its optical depth increases inwards from 0·1 to 10 within the span of a few hundred kilometers—and thus a transition from practically complete transparency to complete opacity takes place within less than one part of ten thousand of its apparent diameter. The same reasons should apply to most other Main Sequence stars in similar measure; and on the strength of this evidence it is customarily assumed in the theory of light changes of eclipsing binary systems that their components, if actually seen, would appear in projection as sharp-edged disks. The recent two decades brought, however, the discovery of a number of binaries whose behaviour in this respect is likely to be anomalous: namely, of systems in which one (or both) components possess *extended atmospheres*, the optical depth of which varies in a more gradual manner. The classical examples of one type of such variables are the supergiant systems of ζ Aurigae or VV Cephei, in which the early-type component passes near superior conjunction behind the extended atmosphere of its mate and suffers an appreciable loss of light before being eventually eclipsed by its photosphere. In other words, the bodily eclipses in these and other similar systems are preceded by *atmospheric eclipses* of the early-type components, which may gradually merge one into another. A different type of atmospheric eclipse occurs again in Wolf-Rayet eclipsing variables (i.e., close binaries containing a Wolf-Rayet star as one of the components), of which V444 Cygni can be regarded as a representative example. Since the systems exhibiting evidence of atmospheric eclipses are likely to grow in numbers as well as importance at an accelerated pace, it seems desirable that a theory of atmospheric eclipses be developed along similar lines as was done in earlier sections of this chapter for bodily eclipses of opaque stellar disks. The aim of the present section will be to provide at least an outline of such a theory.

In more specific terms, our object will be to establish explicit relations connecting the observed loss of light caused by atmospheric eclipses with the structure and absorbing power of the atmosphere as well as with the geometrical circumstances of the eclipse. In order to do so let us consider, quite generally, an eclipsing system whose one (say the primary) component is surrounded by a semi-transparent radially-symmetrical envelope of varying opacity. When, in the neighbourhood of superior conjunction, the secondary will find itself shining through such an atmosphere, a loss of light occurs which we shall call an 'atmospheric eclipse'. Let $\mathfrak{L}(\delta)$ denote the instantaneous luminosity of the secondary component when its distance from the centre of the primary star is δ; and J be the true distribution of brightness over its apparent disk of fractional radius r_2. If this disk is divided into elements by circles concentric with the primary star, the light emitted by the secondary within an annulus comprised between the radii r and $r + dr$ will have its intensity reduced as a result of its passage through the primary's extended atmosphere by a factor of $e^{-\vec{\tau}}$, where $\vec{\tau}$ denotes the optical depth of this atmosphere along the line of sight.* In consequence, a passage of the light of the secondary component through semi-transparent layers of its mate will cause its apparent distribution of brightness as seen by a distant observer *through* such layers to be, not J, but $J \exp(-\vec{\tau})$.

Under such circumstances it follows from simple geometry that, during an atmospheric eclipse,

$$\pi r_2^2 \mathfrak{L}(\delta) = \int_{\delta - r_2}^{\delta + r_2} J e^{-\vec{\tau}(r)}\, r \cos^{-1} \frac{r^2 + \delta^2 - r_2^2}{2\delta r}\, dr. \tag{7-1}$$

If, moreover, the primary component happens to be so large in comparison with its mate that the curvature of its limb can be ignored over the angular diameter of the secondary—as is practically the case in the systems of ζ Aurigae, VV Cephei, and other similar variables—the foregoing equation (7-1) reduces to

$$\pi r_2^2 \mathfrak{L}(\delta) = 2 \int_{\delta - r_2}^{\delta + r_2} J e^{-\vec{\tau}(r)} \{r_2^2 - (\delta - r)^2\}^{1/2}\, dr \tag{7-2}$$

* And *not*, therefore, along the radius r as in section IV.1.

or, more symmetrically, to

$$\mathfrak{L}(\delta) = \int_{-1}^{1} e^{-\vec{\tau}(p,\delta)} \frac{\partial \alpha}{\partial p}\, dp, \tag{7-3}$$

where $\alpha \equiv \alpha(0, p)$ stands, as in the earlier sections of this chapter, for the fractional loss of light of the secondary component (whatever its own distribution of brightness J may be); and p denotes the customary geometrical depth of the eclipse. If, in particular, $J = H$ (uniformly bright disks), an insertion of the equation (4-82) for α_0^0 in (7-3) leads to

$$\pi\mathfrak{L}(\delta) = 2H \int_{-1}^{1} e^{-\vec{\tau}(p,\delta)} \sqrt{1-p^2}\, dp; \tag{7-4}$$

while in the case of a complete darkening at the limb an appeal to (4-83) discloses that

$$4\mathfrak{L}(\delta) = 3H \int_{-1}^{1} e^{-\vec{\tau}(p,\delta)}(1-p^2)\, dp. \tag{7-5}$$

The foregoing relation (7-1) or its particular case (7-2) represent different forms of the *first fundamental integral equation of our problem*, connecting the observed luminosity $\mathfrak{L}(\delta)$ of a star undergoing atmospheric eclipse with its unknown transform

$$\exp\{-\vec{\tau}(r)\} \equiv \Lambda(r), \tag{7-6}$$

which specifies the absorbing properties of the atmosphere. The physical meaning of this transform is simple: $\Lambda(r)$ represents nothing else but the remaining light of a point-source, of luminosity equal to that of the secondary component, at a distance r from the centre of the primary star. It is obvious that if $r_2 = 0$, then $r = \delta$ and the foregoing equations admit indeed of the trivial solution $\mathfrak{L}(\delta) = \Lambda(\delta)$. We should, however, bear well in mind that a mere numerical smallness of r_2 in comparison with the photospheric 'radius' of the opaque core of the primary component offers by itself still no guarantee that the $\mathfrak{L}$- and Λ-functions should behave similarly. What matters equally is the steepness of the density gradient in the semi-transparent outer layers of the primary component—i.e., *whether or not the function* $\Lambda(\mathrm{r})$ *would react appreciably to a change of its argument by* $\pm \mathrm{r}_2$. This, in turn, depends as much on the structure and absorbing properties of the primary's atmosphere as on the secondary's size; and since the former are usually not known beforehand, a solution of the integral equation connecting $\Lambda(r)$ with $\mathfrak{L}(\delta)$ becomes inevitable. Before we outline a method for solving such an equation, it may be stressed that *the instantaneous light* $\mathfrak{L}(\delta)$ *of the secondary component as seen through an extended atmosphere of its mate can be expressed by means of equations* (7-1) *or* (7-2) *irrespective of whether* *its eclipse is bodily or atmospheric* (or partly one and the other). Should the eclipsing component possess a sharp opaque limb at $r = r_1$, the solution of (7-1) would obviously split up in two branches $\Lambda(r) = 0$ and $\Lambda(r) > 0$, according to whether $r \lessgtr r_1$. In other words, *the Λ-transform of light changes due to a bodily eclipse is discontinuous at $r = r_1$.* Should, however, the primary's limb appear diffuse, $\Lambda(r)$ becomes a *continuous* function of its argument which will, in general, run its course more rapidly than $\mathfrak{L}(\delta)$—the more so, the smaller the range of r in which the atmospheric absorption runs from transparency to complete opacity.

The first task of an analysis of the light changes which accompany atmospheric eclipses consists clearly of devising a method by which equation (7-1) could be solved for $\Lambda(r)$ from the observed values of $\mathfrak{L}(\delta)$. Needless to say, a closed solution of this integral equation for an arbitrary form of $\mathfrak{L}(\delta)$ is scarcely obtainable. It is, however, also unnecessary; for our knowledge of $\mathfrak{L}(\delta)$ will, in general, be limited to a tabulation of its numerical values for discrete arguments δ corresponding to the times of observation. In consequence, a *numerical* solution of our fundamental integral equations (7-1) or (7-2) should be just as satisfactory—provided only that it makes full use of the information stored in the observed data.

In any process of constructing such a solution, our first step must be to replace the *integral* on the right-hand sides of (7-1) or (7-2) by an equivalent *summation* of a finite number of terms, in accordance with a quadrature formula of the general form

$$\int_{-1}^{1} w(x) f(x)\, dx = \sum_{j=1}^{n} H_j f(a_j), \tag{7-7}$$

which for suitable sets of the abscissae a_j and weight coefficients H_j can be made *exact* for all functions whose polynomial approximations in the domain concerned are not in excess of $2n - 1$.* Now the limits of the integrals on the right-hand sides of (7-1) or (7-2) involve δ; therefore, we must normalize them from $\delta \pm r_2$ to ± 1 by means of the transformation

$$\delta - r = r_2 x\,. \tag{7-8}$$

In further work, let us confine our attention to the integral on the right-hand side of (7-2) as an illustrative example. In changing over from r to x by means of (7-8), equation (7-2) will assume the form

$$\mathfrak{L}(\delta) = \frac{2}{\pi} \int_{-1}^{1} \Lambda(r) \sqrt{1 - x^2}\, dx, \tag{7-9}$$

* *Cf.*, e.g., Z. Kopal, *Numerical Analysis*, Chapman-Hall, London 1955, Chapter VII.

for which the associated Gaussian quadrature formula with a weight function $w(x) = \sqrt{1 - x^2}$, of the degree of precision $2n - 1$, assumes the explicit form*

$$\int_{-1}^{1} f(x)\sqrt{1-x^2}\,dx = \frac{\pi}{n+1}\sum_{j=1}^{n} \sin^2\frac{j\pi}{n+1} f\left(\cos\frac{j\pi}{n+1}\right); \qquad (7\text{-}10)$$

and with its aid the preceding equation (7-9) can be suitably approximated by a weighted sum

$$\mathfrak{L}(\delta) = \frac{2}{n+1}\sum_{j=1}^{n} \sin^2\frac{j\pi}{n+1}\,\Lambda\left(\delta - r_2\cos\frac{j\pi}{n+1}\right) \qquad (7\text{-}11)$$

of particular values of Λ corresponding to n discrete values of its argument.

As many relations of the form (7-11) can obviously be set up as there are observed measures of light changes; and they all employ the same set of the weight coefficients H_j. Their abscissae a_j are, however, different in each equation; for they involve δ and will respond to any change of it. Therefore, each equation of the form (7-11) will contain a set of n Λ's corresponding to *different* sets of the arguments. As, moreover, the local values of Λ are precisely the unknowns we seek to determine, it is clear that a set of m linear algebraic equations of the form (7-11) would constitute an *indeterminate* system containing $m \times n$ unknowns. Its indeterminacy can, however, be removed in the following manner. A representation of the integral on the right-hand side of equation (7-9) by a weighted sum of n unequally spaced ordinates of the form (7-10) tacitly presumes that the Λ-transform can be approximated by a polynomial of degree not in excess of $2n - 1$. Suppose, therefore, that out of the array of $m \times n$ ordinates of our indeterminate system *we single out 2n of them as unknowns, and express all others in terms of this selected pivotal set by means of an interpolation polynomial of* $(2n - 1)$th *degree*. This is indeed legitimate; for the Λ-transform is assumed to be continuous; and though the magnitude of each ordinate is unknown, its position with respect to any other one is specified as soon as δ and r_2 are fixed.

Let $\bar{\Lambda}(r)$ denote hereafter such an approximating polynomial of degree $2n - 1$, which for arbitrarily spaced values of $\bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots \bar{a}_{2n}$ of the argument r admits of the values $\Lambda_1, \Lambda_2, \ldots, \Lambda_{2n}$. For any other value of r, $\bar{\Lambda}(r)$ can accordingly be evaluated from the well-known Lagrangian interpolation formula

$$\bar{\Lambda}(r) = \sum_{k=1}^{2n} l_k(r)\Lambda_k, \qquad (7\text{-}12)$$

where the $l_k(r)$'s denote the Lagrangian interpolation coefficients (which are entire polynomials of degree $2n - 1$ in r).† Accordingly it should be

* For its proof *cf.* Z. Kopal, *op. cit.*, ante, section VII-K.

† For their definition *cf.* again the writer's *Numerical Analysis*, section II-B.

legitimate to set

$$\bar{\Lambda}\left(\delta - r_2 \cos \frac{j\pi}{n+1}\right) = \sum_{k=1}^{2n} l_k \left(\delta - r_2 \cos \frac{j\pi}{n+1}\right) \Lambda_k \tag{7-13}$$

and express thus the $m \times n$ unknown values of $\Lambda(r)$ as a set of m equations of the form (7-11) in terms of a selected set $2n$ ordinates can, in principle, be adopted at will (and need not, in particular, coincide with any one of the values of $r_j = \delta - r_2 \cos [j\pi/(n+1)]$. In doing so, we have at one stroke extricated ourselves from the indeterminate nature of our algebraic problem; for we are now in a position to rewrite equation (7-11) in the form

$$\mathfrak{L}(\delta_i) = \sum_{k=1}^{2n} A_k^{(n)} (\delta_i) \Lambda_k, \qquad i = 1, 2, \ldots, m, \tag{7-14}$$

where the coefficients

$$A_k^{(n)} (\delta_i) = \frac{2}{n+1} \sum_{j=1}^{n} l_k \left(\delta_i - r_2 \cos \frac{j\pi}{n+1}\right) \sin^2 \frac{j\pi}{n+1}, \tag{7-15}$$

and m denotes the number of available observations, while $2n - 1$ stands for the degree of the polynomial approximation $\bar{\Lambda}(r)$ to the exact solution $\Lambda(r)$ of our problem. If $m = 2n$, the system (7-14) will admit of a unique solution for the Λ_k's; if $m > 2n$, the system becomes over-determinate and may be solved by least-squares or any other appropriate method. When the values of $\Lambda_k (k = 1, 2, \ldots, 2n)$ thus found are inserted back in the interpolation polynomial (7-12), the latter represents a n-th approximation to $\bar{\Lambda}(r)$, whose accuracy may be arbitrarily increased by taking n sufficiently large.

The method suggested in the foregoing paragraphs for a numerical solution of the first fundamental equation of our problem consists, therefore, of replacing the original integral equation (7-9) by an equivalent system of linear algebraic equations of the form (7-11) by means of the Gaussian summation formula (7-10); and reducing subsequently the number of the unknowns by expressing $(m - 2)n$ of them as interpolates in terms of $2n$ pivotal points by means of a $2n$-point Lagrangian interpolation formula (7-12). The legitimacy of this procedure requires the unknown function $\Lambda(r)$ to be *continuous* within $r = \delta \pm r_2$; for if so, both approximate steps represented by equations (7-10) and (7-12) are easily justified (by a recourse to Weierstrass's theorem on polynomial approximation), and the process shown to lead to a rigorous solution of our problem when n is allowed to increase beyond any limit.* In practice, however, even a moderate number

* Our numerical process can, of course, be applied regardless of whether or not $\Lambda(r)$ is continuous across the disk of the secondary component. If it were not (bodily eclipse!), the failure of successive approximations to converge to a limit would give us a sufficient warning that we are trying to interpolate over a discontinuity.

of terms (i.e., a low value of n) should, as a rule, give a good approximation; for the use of a Gaussian quadrature in passing from (7-9) to (7-11) should guarantee the best possible fit with the smallest number of terms.

The physical advantages of the transformation we have just effected will be immediately apparent. In passing from $\mathfrak{L}(\delta)$ to $\Lambda(\delta)$ we have replaced a distended beam of light emitted by the secondary component and emerging from semi-transparent layers of its mate by an equivalent beam of a point-source at a distance $r = \delta$ from the primary's centre, which—unlike $\mathfrak{L}(\delta)$—can be related with the absorption coefficient κ_λ of the obscuring atmosphere by a series of purely analytical operations. For by definition of the optical depth it follows that

$$\Lambda \equiv e^{-\vec{\tau}} = \exp\left\{ - \int_{-\sqrt{r_A^2-\delta^2}}^{\sqrt{r_A^2-\delta^2}} \kappa_\lambda \, ds \right\}, \tag{7-16}$$

where r_A denotes the extent of the absorbing atmosphere and ds, the element of the line of sight which is related with r by means of the equation

$$s^2 = r^2 - \delta^2. \tag{7-17}$$

If we eliminate s for r by means of (7-17) and take logarithms of both sides, equation (7-16) readily assumes the form

$$\log \Lambda = - \int_{\delta}^{r_A} \frac{\kappa_\lambda \, dr^2}{\sqrt{r^2 - \delta^2}} . \tag{7-18}$$

If, furthermore, we abbreviate

$$r_A^2 - \delta^2 = \Delta, \qquad \text{and} \qquad r_A^2 - r^2 = x , \tag{7-19}$$

and introduce new auxiliary functions $\psi(x)$ and $\sigma(\Delta)$ related with κ_λ and Λ by

$$\kappa_\lambda = \frac{d\psi}{dx} \qquad \text{and} \qquad \log \Lambda = -\sigma(\Delta), \tag{7-20}$$

the foregoing equation (7-18) assumes the alternative form

$$\sigma(\Delta) = \int_0^\Delta \frac{d\psi}{dx} \frac{dx}{\sqrt{\Delta - x}} . \tag{7-21}$$

Equation (7-18) or its equivalent (7-21) represents the *second fundamental equation of our problem*, which relates the Λ-transform established earlier with the absorption coefficient κ_λ. In order to solve it, however, we no longer have to resort to any approximate procedure; for (7-21) is easily recognized as an integral equation of Abel's type, whose inversion yields

$$\psi(x) = \frac{1}{\pi} \int_0^x \frac{\sigma(\Delta) \, d\Delta}{\sqrt{x - \Delta}} . \tag{7-22}$$

Next integrate the right-hand side of (7-22) by parts. Since, by definition, the absorption ceases at $r = r_A$ and, hence, $\Lambda = 1$ for $\Lambda = 0$, the integrated part vanishes and we are left with

$$\psi(x) = \frac{2}{\pi} \int_0^x \frac{d\sigma}{d\Delta} \sqrt{x - \Delta}\, d\Delta\,. \tag{7-23}$$

Ultimately, differentiating equation (7-23) with respect to x, we find that

$$\frac{d\psi}{dx} = \frac{1}{\pi} \int_0^x \frac{d\sigma}{d\Delta} \frac{d\Delta}{\sqrt{x - \Delta}} \tag{7-24}$$

or, reverting to our original functions and variables,

$$\kappa_\lambda(r) = \frac{1}{\pi} \int_r^{r_A} \frac{d}{d\delta} \{\log \Lambda(\delta)\} \frac{d\delta}{\sqrt{\delta^2 - r^2}}, \tag{7-25}$$

where the function $\Lambda(\delta)$ can be approximated by $\overline{\Lambda}(\delta)$ as represented by our Lagrangian interpolation formula (7-12), and r_A denotes the limiting height beyond which atmospheric absorption becomes insensible. In the case of an atmosphere having no 'top', we can clearly replace r_A by ∞. This last equation (7-25) expresses $\kappa_\lambda(r)$ explicitly as a function of r in terms of definite integral whose evaluation calls at worst for an additional quadrature; and since the Λ-function has already been deduced from the observed light changes by a method developed earlier in this section, a solution of our problem is thus complete.

Having thus converted the light changes observed during an atmospheric eclipse into the variation of the absorption coefficient with height, we may wish to use this knowledge for determining the corresponding distribution of density, pressure, and temperature throughout the extended atmosphere; but in order to do so, an identification of the processes which cause this atmosphere to be semi-transparent becomes necessary. If the extinction of light within it were due to scattering on free electrons, it would follow at once from Thomson's formula that

$$\kappa(r) = \frac{8\pi}{3} \left(\frac{e^2}{mc^2}\right)^2 N_e(r) = 0{\cdot}66 \times 10^{-24}\, N_e(r) \tag{7-26}$$

in c.g.s. units, where e denotes the specific charge of an electron; m, its mass; c, the velocity of light; and N_e, the number of free electrons per cc. As is well known, the electron scattering represents a process which does not depend on the frequency of light, and is predominant in the extended atmospheres of Wolf-Rayet components of close binary systems and other hot stars. If, on the other hand, the principal cause of extinction were absorption by hydrogen (H or H^-) or metallic ions—as is true of the atmospheres surrounding late-type supergiants like the principal components of ζ Aurigae or VV Cephei—the situation becomes more complicated.

In what follows we propose to limit ourselves to explore the specific consequences aiising if the density of free electrons falls off exponentially with height (as it would in an electrically neutral atmosphere in hydrostatic equilibrium)—or, more generally, if

$$\kappa\rho = \kappa_0 e^{-\alpha r}, \tag{7-27}$$

where κ_0 and α are suitable constants. If so, however, then an integral of the equation

$$d\vec{\tau} = \kappa\rho\, ds \tag{7-28}$$

governing our optical depth $\vec{\tau}$ assumes the form

$$\vec{\tau} = 2\kappa_0 \int_0^\infty e^{-\alpha r}\, ds\,, \tag{7-29}$$

where, by (7-17), $s^2 = r^2 - \delta^2$. If, furthermore, we set $r = \delta x$ and replace ds by dx, the optical depth along the line of sight can be expressed as

$$\vec{\tau} = acK_1(a), \tag{7-30}$$

where

$$K_1(a) = \int_1^\infty \frac{x\, e^{-ax}}{\sqrt{x^2 - 1}}\, dx\,, \tag{7-31}$$

and

$$\left.\begin{aligned} a &= \alpha\delta, \\ c &= 2\kappa_0/\alpha\,, \end{aligned}\right\} \tag{7-32}$$

are non-dimensional constants.

As the reader may easily verify, the foregoing equation (7-31) constitutes an integral representation of the Bessel function $K_n(a)$ of the second kind, with imaginary argument, of index $n = 1$. As such it will, therefore, satisfy Bessel's differential equation

$$\frac{d^2K_n}{da^2} + \frac{1}{a}\frac{dK_n}{da} - \left(1 + \frac{n^2}{a^2}\right) K_n = 0, \tag{7-33}$$

subject to the boundary conditions requiring that

$$K_n(0) = \infty \qquad \text{and} \qquad K_n(\infty) = 0. \tag{7-34}$$

An actual construction of the respective particular solution of (7-33) in the form of a series in ascending powers of a is involved;* but its result discloses

* For its derivation *cf.*, e.g., Whittaker and Watson, *Modern Analysis*, Cambridge 1927, pp. 373–374.

that

$$K_n(a) = (-1)^{n+1} \log \frac{a}{2} \sum_{j=0}^{\infty} \frac{1}{j!(n+j)!} \left(\frac{a}{2}\right)^{2j+n}$$

$$+ (-1)^n \sum_{j=1}^{\infty} \frac{1}{2j(n+j)!} \left(\frac{a}{2}\right)^{2j+n} \{\psi(j) - \psi(j+n)\}$$

$$+ \sum_{j=0}^{n-1} \frac{(-1)^j (n-j-1)!}{2(j!)} \left(\frac{a}{2}\right)^{2j-n}, \tag{7-35}$$

where

$$\psi(j) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{j} - \gamma,$$

$\gamma = 0{\cdot}577215\ldots$ denoting the well-known Euler–Mascheroni constant. For *small* values of a, the series on the right-hand side of equation (7-36) for $n = 1$ reduces essentially to

$$K_1(a) = \frac{1}{a} + \frac{a}{2} \log \frac{a}{2} + \ldots; \tag{7-36}$$

while if a is *large*, equation (7-33) is known to admit of the asymptotic solution

$$K_1(a) = \sqrt{\frac{\pi}{2a}}\, e^{-a} \left\{1 + \sum_{j=1}^{\infty} \frac{(4-1^2)(4-3^2)\ldots(4-[2j-1]^2)}{j!\,(8a)^j}\right\}. \tag{7-37}$$

The instantaneous luminosity $\mathfrak{L}(\delta)$ of the secondary component as seen through the obscuring haze around its mate continues to be given by equation (7-1) where, for the sake of simplicity, we shall set $J = 1$ and hereafter use $\vec{\tau}(r) = 2\kappa_0 r K_1(\alpha r)$, in accordance with (7-30). If so, an evaluation of the integral on the right-hand side of (7-1) offers certain difficulties which can, however, be largely circumvented in the following manner. First, differentiate (7-1) with respect to δ, obtaining

$$\pi r_2 \frac{\partial \mathfrak{L}}{\partial \delta} = -2 \int_0^{\pi} e^{-\vec{\tau}(r)} \cos\theta \, d\theta, \tag{7-38}$$

where the new angular variable θ is related with r by the equation

$$r^2 = r_2^2 + \delta^2 - 2\delta r_2 \cos\theta. \tag{7-39}$$

In order to evaluate the integral on the right-hand side of (7-38), express next $\tau(r)$ as a function of θ by means of the addition theorem for modified

Bessel functions of the second kind* which asserts that

$$rK_1(\alpha r) = (2\alpha/\delta r_2)(r_2^2 + \delta^2 - 2\delta r_2 \cos\theta) \times \sum_{j=0}^{\infty} (j+1) U_j(\cos\theta) I_{j+1}(\alpha r_2) K_{j+1}(\alpha\delta), \tag{7-40}$$

where

$$U_j(\cos\theta) = \frac{\sin(j+1)\,\theta}{\sin\theta} \tag{7-41}$$

denotes a Tchebycheff polynomial of the second kind and j-th order, and I_j stands for the modified Bessel function of the first kind—i.e., the second independent solution of Bessel's differential equation (7-33), such that

$$I_n(0) = 0 \quad \text{and} \quad I_n(\infty) = \infty\,, \tag{7-42}$$

which is given by the series

$$I_n(a) = \sum_{j=0}^{\infty} \frac{1}{j!(j+n)!} \left(\frac{a}{2}\right)^{2j+n}. \tag{7-43}$$

The expansion on the right-hand side of (7-40) can be shown to converge absolutely when $\delta > r_2$; its j-th term being of the order of $(r_2/\delta)^{j+1}$. Should $\delta < r_2$, the convergence is again restored by a mere interchange of arguments of the two Bessel functions.

As the reader has undoubtedly noticed, an appeal to the addition theorem (7-40) enabled us to remove, in one stroke, the independent variable θ from the argument of the Bessel function K_1 and transfer it to the Tchebycheff polynomials U_j which contain only integral powers of $\cos\theta$. If we expand now $\exp(-\vec{\tau})$ in powers of $\vec{\tau}$ and make use of (7-40), the integral on the right-hand side of (7-38) can be evaluated term by term. If, in particular, $\vec{\tau}$ is sufficiently small for its squares and higher powers to be negligible, then by setting

$$\mathfrak{L}(\delta) = \mathfrak{L}_0(1 - \Delta\mathfrak{L}) \tag{7-44}$$

and integrating (7-38) we easily find that

$$\frac{r_2}{4}\frac{\partial}{\partial\delta}(\Delta\mathfrak{L}) = -I_1(\alpha r_2)K_1(\alpha\delta) - 2\sum_{j=1}^{\infty}(2j+1)I_{2j}(\alpha r_2)K_{2j}(\alpha\delta) + \frac{r_2^2+\delta^2}{\delta r_2}\sum_{j=0}^{\infty}(2j+1)I_{2j+1}(\alpha r_2)K_{2j+1}(\alpha\delta), \tag{7-45}$$

* *Cf.*, e.g., A. Gray and G. B. Matthews, *Treatise on Bessel Functions*, London 1922, p. 74.

while a subsequent integration with respect to δ yields

$$
\begin{aligned}
r_2\, \Delta\mathfrak{L}(\delta) = 4 \sum_{j=0}^{\infty} (2j+1) I_{2j+1}(\alpha r_2) \int \frac{r_2^2 + \delta^2}{\delta r} K_{2j+1}(\alpha\delta)\, d\delta \\
- 8 \sum_{j=1}^{\infty} (2j+1) I_{2j}(\alpha r_2) \int K_{2j}(\alpha\delta)\, d\delta - 2 I_1(\alpha\delta_2) \int K_1(\alpha\delta)\, d\delta,
\end{aligned} \tag{7-46}
$$

where the I_n's and K_n's are given by the series on the right-hand sides of equations (7-35) and (7-43).

Should, however, the rate of convergence of the expansion of $e^{-\vec{\tau}}$ become such that powers of $\vec{\tau}$ higher than the first would have to be retained,* our foregoing method of evaluating $\mathfrak{L}(\delta)$—while rigorous—may become too time-consuming for practical work. In such cases it may again be of advantage to resort to numerical quadratures. In point of fact, the Gaussian quadrature formula

$$
\int_0^{\pi} f(\cos\theta)\, d\theta = \frac{\pi}{n} \sum_{j=1}^{n} f\left(\cos \frac{2j-1}{2n} \pi\right), \tag{7-47}
$$

allied to (7-10), is exact for all polynomials $f(\cos\theta)$ of degree not in excess of $2n - 1$.† For instance, if $\vec{\tau}(r) \equiv \vec{\tau}(\delta, \cos\theta)$ could be approximated by a polynomial of fifth degree in $\cos\theta$, then

$$
\int_0^{\pi} e^{-\vec{\tau}} \cos\theta\, d\theta = \frac{\pi}{2\sqrt{3}} \{e^{-\vec{\tau}(\delta, \pi/6)} - e^{-\vec{\tau}(\delta, 5\pi/6)}\}, \tag{7-48}
$$

so that, by (7-38),

$$
\sqrt{3} r_2 \mathfrak{L}(\delta) = \int e^{-\vec{\tau}(\delta, 5\pi/6)}\, d\delta - \int e^{-\vec{\tau}(\delta, \pi/6)}\, d\delta; \tag{7-49}
$$

and similarly in other approximations.

It may also be mentioned that if a becomes a sufficiently small quantity for $K_1(a)$ to be approximable by (7-36)—i.e., if the extended atmosphere is *tenuous* and characterized by low density-gradient—then, to the first power of $\vec{\tau}$, the expression $\Delta\mathfrak{L}$ as defined by (7-44) will be given by the equation

$$
\pi r_2^2\, \Delta\mathfrak{L} = 2c \int_{\delta - r_2}^{\delta + r_2} \left\{1 + \frac{\alpha^2 r^2}{2} \log \frac{\alpha r}{2}\right\} r \cos^{-1} \frac{\delta^2 - r_2^2 + r^2}{2\delta r}\, dr\,; \tag{7-50}
$$

and, in spite of a somewhat uninviting aspect of the integral on the right,

* We may note that a simple quadratic approximation

$$
e^{-\tau} \simeq 1 - \frac{24}{25}\tau + \frac{8}{25}\tau^2
$$

would represent our exponential on the left-hand side within an error of less than two per cent throughout the interval (0, 1).

† For its proof *cf.* again the writer's *Numerical Analysis* (London 1955), section VII-K.

it can be evaluated in a closed form to reveal that, in this particular case,

$$\Delta\mathfrak{L} = c + \tfrac{1}{4}c(a\tan^2\lambda)^2\{1 + (1 + 2\cot^4\lambda)\cot^2\lambda\log\tfrac{1}{2}\alpha r_2\}, \tag{7-51}$$

where a and c continue to be given by equations (7-32) and where we have abbreviated

$$\frac{\delta}{r_2} = \cot^2\lambda\,. \tag{7-52}$$

If, on the other hand, the atmosphere is *thick* and its density-gradient sufficiently steep to justify an approximation of $K_1(a)$ by the leading term of the asymptotic series on the right-hand side of (7-37), the corresponding integral

$$\pi r_2^2\,\Delta\mathfrak{L} = (2\pi\alpha)^{1/2}\,c\int_{\delta-r_2}^{\delta+r_2} r^{3/2}\,e^{-\alpha r}\cos^{-1}\frac{\delta^2 - r_2^2 + r^2}{2\delta r}\,dr \tag{7-53}$$

can, after some transformations,* be evaluated in the form of an expansion

$$\Delta\mathfrak{L} = ac\cos^2\lambda\{K_1(a\sec^2\lambda) + a\sin^2\lambda K_2(a\sec^2\lambda) + \dots\} \tag{7-54}$$

in ascending powers of $\sin^2\lambda \equiv r_2/(\delta + r_2)$.

Throughout the foregoing paragraphs we have considered, in accordance with equation (7-27), our atmosphere to extend outwards to infinity. Should, however, this atmosphere possess a 'top' at a finite height—i.e., should it represent an absorbing *shell* which surrounds the primary component up to a radius r_A, the expression (7-30) for the optical depth along the line of sight must be generalized to read

$$\vec{\tau} = acK_1(a, b), \tag{7-55}$$

where

$$K_1(a, b) = \int_1^{\sqrt{1+b^2}} \frac{e^{-ax}x\,dx}{\sqrt{x^2 - 1}}\,, \tag{7-56}$$

and

$$b^2 = (r_A/\delta)^2 - 1. \tag{7-57}$$

By repeated differentiation behind the integral sign with respect to a, $K_1(a, b)$ can be shown to obey the differential equation

$$a^2\frac{\partial^2 K}{\partial a^2} + a\frac{\partial K}{\partial a} - (1 + a^2)\,K = f(a, b), \tag{7-58}$$

where

$$f(a, b) = -b(1 + a\sqrt{1 + b^2})\,e^{-a\sqrt{1+b^2}}\,; \tag{7-59}$$

and a similar differentiation with respect to b yields

$$(1 + b^2)\frac{\partial^2 K}{\partial b^2} - b\frac{\partial K}{\partial b} = f(a, b). \tag{7-60}$$

* For their details *cf.* Z. Kopal, *Proc. Amer. Phil. Soc.*, **89**, 590, 1945.

Eliminating $f(a, b)$ between (7-58) and (7-60) we find that the definite integral $K_1(a, b)$ actually satisfies the *partial* differential equation

$$a^2 \frac{\partial^2 K}{\partial a^2} - (1 + b^2) \frac{\partial^2 K}{\partial b^2} + a \frac{\partial K}{\partial a} + b \frac{\partial K}{\partial b} = (1 + a^2) K, \tag{7-61}$$

which is linear and homogeneous in the dependent variable. With regard to (7-58), its complete primitive can formally be expressed as

$$\begin{aligned} K_1(a, b) = I_1(a) \left\{ A(b) - \int^a f(x, b) K_1(x) d \log x \right\} \\ + K_1(a) \left\{ B(b) + \int^a f(x, b) I_1(x) d \log x \right\}, \end{aligned} \tag{7-62}$$

where $I_1(a)$, $K_1(a)$ denote the respective modified Bessel functions of index one, and the forms of $A(b)$, $B(b)$ are to be determined from given boundary conditions.* The complete primitive of (7-60) assumes likewise the form

$$\begin{aligned} K_1(a, b) = C(a) - \int^b \left\{ x + \frac{\sin \mathrm{h}^{-1} x}{\sqrt{1 + x^2}} \right\} f(x, a) \, dx \\ + \{C\sqrt{1 + b^2} + \sin \mathrm{h}^{-1} b\} \left\{ D(a) + \int^b \frac{f(x, a)}{\sqrt{1 + x^2}} dx \right\}, \end{aligned} \tag{7-63}$$

where again $C(a)$ and $D(a)$ are certain functions of a to be determined from the boundary conditions.

If a is small, the value of the integral (7-56) can be approximated by

$$K_1(a, b) = \frac{1}{a} \{1 - e^{-a\sqrt{1+b^2}}\} + \frac{a}{2} \log \frac{a}{2} + \dots ; \tag{7-64}$$

while if a is large, $K_1(a, b)$ admits of an asymptotic expansion

$$K_1(a, b) = \sqrt{\frac{\pi}{2a}} e^{-a} \left\{ \Phi(x) + \frac{3}{8a} \left[\Phi(x) + \Phi^{\mathrm{II}}(x) + \frac{1}{12} \Phi^{\mathrm{IV}}(x) \right] + \dots \right\}, \tag{7-65}$$

where

$$x = b\sqrt{a/2} \tag{7-66}$$

and

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \tag{7-67}$$

stands for the well-known error function (the primes denoting its respective derivatives), which for large values of its argument can, in turn, be approximated by the asymptotic series

$$\Phi(x) = 1 - \frac{e^{-x^2}}{x\sqrt{\pi}} \left\{ 1 - \frac{1}{2x^2} + \frac{3}{8x^4} - \dots \right\}. \tag{7-68}$$

* These imply that, obviously, $A(0) = B(0) = 0$, while $A(\infty) = 0$ and $B(\infty) = 1$.

Equations (7-64) or (7-65) should be regarded as generalizations of (7-36) or (7-37), and entail no restriction on b.

In conclusion of the present section on atmospheric eclipses, one retrospective consideration deserves explicit mention. Our assumption that the distribution of brightness impressed upon a stellar disk in the course of an atmospheric eclipse is of the form $\exp(-\vec{\tau})$, which underlies the whole discussion of this section, is consistent with the equations of radiative transfer of light through such atmospheres (as set forth in the introductory section IV.1 of this chapter) only *if the source-function of the secondary's radiation is disregarded*; for if so, the remaining part of the equation (1-1) of transfer for the emergent intensity $I(r_A, \mu)$ integrates indeed into an exponential of $\vec{\tau}$. In physical language, this neglect means that we consider the flux of the primary star's own radiation flowing radially outwards to be so much more intense than the secondary's flux passing along a secant through an extended atmosphere surrounding its mate, that the contribution of the secondary to the radiative equilibrium of the primary's atmosphere as a whole can be ignored.

Such an approximation is likely to be ample for atmospheric eclipses of the ζ Aurigae-type; but need not necessarily be so for atmospheric eclipses caused by extended envelopes of Wolf-Rayet stars, in which electron scattering plays a predominant role.* There a more orthodox procedure would be to consider a distribution of brightness $I(r_A, \mu)$ over the disk of a star undergoing atmospheric eclipse to be, not $\exp(-\vec{\tau})$, but (ignoring distortion) rather the solution of an equation of transfer of the form

$$\mu \frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} + \kappa\rho I = \tfrac{1}{2}\kappa\rho \int_{-1}^{1} I(r, \mu')\, p(\cos\Theta)\, d\mu' - \tfrac{1}{4} S e^{\int \kappa\rho ds} p(\cos\Theta_0), \qquad (7\text{-}69)$$

where the phase-function $p(\cos\Theta)$ of scattering on free electrons continues to be given by equation (1-49) and its arguments by (1-48) and (1-79), respectively.

The foregoing equation (7-68) is analogous to one governing the transfer of reflected light, of flux πS, coming from a direction which is parallel with the line of sight. The boundary conditions which its acceptable solutions must obey continue, accordingly, to be given by the flux integral (1-70), together with a requirement that $I(r_A, \mu)$ should be zero for all negative values of μ. A transfer of light through spherically-symmetrical (let alone distorted) atmospheres of finite height, as embodied in equation (7-69) and its boundary conditions, has received as yet no specific attention; and its solution represents another task awaiting the investigators of the future.

* For an empirical evidence of this fact in the WR-system of V444 Cygni *cf.* Z. Kopal and M. B. Shapley, *Ap. J.*, **104**, 160, 1946.

IV.8. SURVEY OF THE RESULTS

An analysis of the light changes of close binary systems within the limits set out in the introduction now being complete, the aim of this concluding section should be to summarize briefly its principal results.

As a prerequisite to our work, in section IV.1 we started with a description of brightness over the apparent disks of the stars, in radiative equilibrium, which can be deduced from the theory of stellar atmospheres. As long as we limit ourselves to grey atmospheres which are stratified in plane-parallel layers, the angular distribution of emergent radiation (i.e., its limb-darkening) admits of an exact solution in a closed form whether the actual transfer mechanism is isotropic, re-emission and scattering, or (in the atmospheres of hot stars) scattering on free electrons in accordance with Rayleigh's law. The latter is known to render the intensity of emergent radiation partly polarized; and the degree of this polarization should increase between the centre and the limb from zero to approximately 11%. The light re-emitted and scattered on free electrons is, moreover, found to emerge from stellar atmospheres with so similar an angular distribution that no practicable photometric tests—save that of polarization—can be involved to distinguish between them. This is true, of course, of grey atmospheres alone. The scattering of light on free electrons is known to be frequency-independent and so should, as a result, be its corresponding angular distribution. On the other hand, radiative transfer in non-grey atmospheres—a physically much more complicated problem—does lead to limb-darkening which depends on the wave length, and for which only approximate solutions could be outlined so far.

The angular distribution of the light of a component of a close binary system, which is intercepted by its mate and re-radiated (or scattered) by it in the direction of the line of sight, can be shown to be simply related with the angular distribution of its own emergent light, and is found to exhibit a nearly complete darkening at limb. If the actual mechanism of reflection is scattering on free electrons, the reflected light will again be partly polarized; and thanks to the efforts of Chandrasekhar its limb-darkening is now known irrespective of whether the incident beam itself was polarized or unpolarized.

The emission and reflection of light by stellar atmospheres will influence the distribution of light over the apparent disks of the components of close binary system regardless of whether the constituent stars are approximately spherical or appreciably distorted. If, however, the latter is the case, the apparent total brightness over stellar surfaces distorted by rotational or tidal forces can also be shown to vary proportionally to the local gravity. The gravity-darkening arising from this source is independent of, and supplementary to, the darkening at limb which is due to the finite transparency of the atmosphere. The limb-darkening tends to make brightest those parts of the visible surface which are nearest to the observer; the gravity-darkening, those which are nearest to the star's centre; and the two together give rise to a rather complicated system of isophotae which alters continuously with the phase.

The second section IV.2 of this chapter has been devoted to an investigation of the light changes of close binary systems invoked by the revolution of its components distorted by their axial rotation and mutual tidal action. Taking the geometry of such distorted configurations from section II.1 we prove that (within the framework of the equilibrium theory of tides and in the absence of orbital eccentricity) the light variation of tidal origin should be symmetrical with respect to the moments of conjunctions of the two components. A development of such light changes in zonal harmonics of the angle ε between the radius-vector and the line of sight reveals, moreover, that no light variation arises from the third harmonic unless there is some limb-darkening, nor any from the fourth unless the darkening is incomplete. The role of this darkening is to exaggerate the light variation arising from the second-harmonic distortion, to give rise to a significant third harmonic, and to damp the effects of the fourth harmonic. Such effects are, moreover, shown to be quite insensitive to the detailed distribution of surface intensity near the star's limb. On the other hand, the role of the gravity-darkening is always to enhance the light changes due to the distorted geometry. In particular, the gravity-darkening in total radiation ($\tau_0 = 1$) of centrally-condensed stars ($\Delta_j = 1$) is found to multiply the corresponding j-th harmonic variation of light by $j/(j-1)$. If, lastly, the relative orbit of the two stars were eccentric, each constant term in the zonal harmonics $P_j(\cos \varepsilon)$ of even orders would give rise to a term varying as inverse $(j+1)$st power of the radius-vector—becoming greatest at periastron and smallest at apastron—and thus giving rise to a 'periastron effect' which may be photometrically detectable.

When we turn to consider the photometric effects of the rotational distortion, we find that as long as the equatorial planes of the components are co-planar with that of their orbit, axial rotation gives rise to no observable light changes. Should, however, the axes of rotation be *inclined* to the orbital plane, the observed brightness of the respective stars would be bound to start varying with the phase; and for an arbitrary orientation of the nodes of the equatorial plane the corresponding light changes would be *asymmetric* with respect to the minima (i.e., would contain terms varying as odd powers of $\sin \varepsilon$ as well). Moreover, in section II.5 we learned that, if the axes of rotation are inclined to the orbital plane, the nodes of their equators must secularly recede and complete a whole revolution in the period U as given by equations (5-62) or (5-86) of Chapter II.

In consequence, the sense of the asymmetry of light changes due to inclined axes of rotation is bound to *oscillate* in the same period; and an observational determination of the period of such oscillations would thus open to us a photometric way for an empirical determination of the periods of precessional motion of fluid components of close binary systems—which is independent of, and supplementary to, its determination from an analysis of the fluctuations of the orbital period P arising from this source (*cf.* section II.7).* The amplitudes of the asymmetry-generating sine terms in a Fourier expansion of light changes between minima could then be used for an alternative specification of the inclination of the axes of rotation to the orbital plane. A consistency of the results obtained in two independent ways—photometric (section IV.2) and dynamical (section II.7)—should ensure that the nature of the processes giving rise to the observed phenomena has been correctly identified.

In the next section IV.3 of this chapter we have investigated the theoretical light changes of close eclipsing systems within the minima of light—i.e., when one distorted component eclipses another—to the same degree of accuracy to which the light changes between minima have been investigated in the preceding section. This particular task belongs among the most intricate problems encountered in the whole domain of the astronomy of double stars; and a theory of special functions which had to be introduced to render it at all tractable has been outlined in subsequent sections IV.4 and 5. These last two sections are primarily mathematical, and many a reader interested mainly in applications may be tempted to by-pass them. Yet it is certain that these functions represent the veritable entrance into higher parts of the theory of light- (or velocity-) changes of close binary systems; and it is through their fuller understanding and utilization that our subject may be expected to advance in the future. Besides, even now the associated α-functions and related integrals are indispensable, not only for a theoretical description of the theoretical light changes of distorted eclipsing system within minima in section IV.3 of this chapter, but also for a similar description of their radial-velocity changes in the forthcoming section V.2 of the next chapter; or again for proper understanding of the process of the 'rectification' of the observed light changes to be discussed in section VI.12. These diverse uses—both present and anticipated—should, therefore, make it eminently worth while for the reader to get acquainted with the special functions of sections IV.4 and 5, and their algebra, at this stage.

Section IV.6 is concerned with the photometric effects of reflection, and with the derivation of laws which govern their variation with the phase. In doing so we have used Lambert's approximation to the actual angular distribution of reflected light as discussed in section IV.1, and took account of the finite angular size of the illuminating as well as reflecting components to the order of accuracy to which both components can be regarded as spheres. The resulting phase-law, as embodied in equation (6-86) and its constituents, has been expanded in ascending powers of the fractional

* Strictly speaking, the method of section II.7 leads to a determination of the period of recession of the nodal line of the orbital plane (i.e., an intersection of the orbital and invariable planes of the system). As, however, we have seen in section II.5, the secular motions of the angles ϕ and Ω are synchronized and their periods of recession equal.

radii $a_{1,2}/R$ of the reflecting (primary) and illuminating (secondary) component. An inspection of it reveals that the leading term, of the order of $(a_1/R)^2$, would represent the actual phase-law if the incident light would constitute a parallel beam. The second term of the order of $(a_1/R)^3$ is necessary to account for the divergence of the incident beam if the illuminating component were a point-source of light; while the next terms—of the order of $(r_1/R)^4$ and $(r_1r_2/R^2)^2$—take account of the finite angular size of the secondary component (giving rise to a penumbral zone on the primary) as well as of the 'secondary reflection' (i.e., of light reflected by the primary to the secondary, and from it back to the primary star). This is where our present expansions have been terminated; for subsequent terms of the order beginning from $(a/R)^5$ would already be influenced by the distortion of the reflecting component; those of $(a/R)^6$ by 'tertiary reflection'; and those of $(a/R)^7$, by a distortion of the illuminating component as well. A treatment of none of these additional effects introduces any particular new difficulties; but it is not yet available and must be postponed for subsequent investigations.

The penultimate section IV.7 of this chapter has been devoted to a discussion of stellar occultations in close binary systems by extended atmospheres surrounding their mates, whose absorbing power may extinguish the light of a star before the ingress (or after the egress) of an actual bodily eclipse. In discussing this subject we have assumed that light of the star undergoing eclipse, transmitted through such extended atmospheres, does not influence their own radiative equilibrium to an appreciable extent. If so, a general method developed in this section should enable us to utilize the light changes, observed when a star of finite angular diameter gradually sets in behind an extended atmosphere, for extracting the variation of the coefficient of absorption in this atmosphere with the height, in a manner which is completely free from any physical assumptions or hypotheses. In the latter part of this section we considered, in some detail, the photometric effects of special atmosphere whose mass absorption coefficient decreases exponentially with height, and also the eclipses produced by discrete atmospheric shells of similar structure.

IV. BIBLIOGRAPHICAL NOTES

IV.1: The fact that the apparent disk of our nearest star—the Sun—is not uniformly bright, but darkened towards the limb, was first noted by Luca Valerio, of the Academy dei Lincei, in Rome 1612, and was given a prominent description in P. Johann Scheiner's *Rosa Ursina* (**4**, p. 618). Curiously enough, the reality of this phenomenon was vigorously contested by Galileo Galilei in a letter to Prince Cesi dated January 25th, 1613 (*cf. Opera di Galileo Galilei*, Edizio Nazionale, **6**, p. 198). J. Bouguer, in 1729, appears to have been the first to measure the amount of solar darkening quantitatively.

The physical theory of this limb-darkening invoked much speculation during the eighteenth and nineteenth centuries, and was not placed on a solid basis until the work of K. Schwarzschild (*Göttingen Nachr.*, Nr. 41, 1906) to whom the elements of the theory of radiative equilibrium are due. For a modern extensive survey of the theory of stellar limb-darkening in plane-parallel atmospheres *cf.*, e.g., S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, 1950; or V. Kourganoff, *Basic Methods in Transfer Problems*, Oxford Univ. Press, 1952. Of subsequent advances in this subject, *cf.*, J. B. Sykes, *M.N.*,

111, 377, 1951; M. Krook, *Ap. J.*, **122**, 488, 1955; or E. P. Gross and S. Zierling, *Ap. J.*, **123**, 343, 1956.

The exact solution (1-10) for limb-darkening in plane-parallel atmospheres was given first by N. Wiener and E. Hopf in *Berlin Berichte* (Math.-Phys. Klasse), 1931, p. 696, and subsequently tabulated by a number of writers (e.g., S. Chandrasekhar in *Ap. J.*, **99**, 180, 1944; or G. Placzek, *Phys. Rev.*, **72**, 556, 1947). For its expansion in powers of μ *cf.*, Z. Kopal, *Harv. Circ.*, No. 454, 1949. A realization that the emergent intensity $I(0, \mu)$ obeys an integral equation of the form (1-54) for $\Psi = \frac{1}{2}$ is due to V. A. Ambarzumian (*Journal of Physics*, U.S.S.R. Acad. Sci., **8**, 64, 1944), and an explicit use of the *H*-functions was introduced by S. Chandrasekhar in *Ap. J.*, **103**, 165, 1946. The limb-darkening in light scattered on free electrons was studied by S. Chandrasekhar by approximate means in *Ap. J.*, **103**, 351; and **104**, 110, 1946. For limb-darkening in non-grey atmospheres *cf.*, G. Münch and S. Chandrasekhar, *Harv. Circ.*, No. 453, 1949. The method of 'telescoping' non-linear laws of darkening into linearized approximations, using the minimum properties of Tchebycheff polynomials and developed in this section, is new.

The problem of the radiative transfer of *reflected* light was opened up by E. A. Milne in *Phil. Mag.*, **44**, 871, 1922, and subsequently in *M.N.*, **87**, 43, 1926. Milne attempted to solve the transfer of reflected light in plane-parallel atmospheres by Schwarzschild's method of anti-parallel streams, and was the first to point out the large limb-darkening of reflected radiation, as well as to formulate quantitatively the heating effect of incident light. Subsequently, E. Hopf in his *Mathematical Problems of Radiative Equilibrium* (Cambridge Tract in Mathematics and Math. Physics, No. 31; Cambridge Univ. Press, 1934) gave the exact solution (1-73) of Milne's equation of transfer for parallel incident light which is absorbed and re-emitted, or scattered isotropically. The exact equations (1-81) to (1-86) are also due to Hopf. Diffuse reflection of light scattered on free electrons has been investigated in terms of the appropriate *H*-functions by S. Chandrasekhar (*Ap. J.*, **103**, 165, 1946), to whom we also owe exact description of the polarization phenomena accompanying such scattering whether the incident light is unpolarized (*Ap. J.*, **104**, 110, 1946) or is itself polarized (*Ap. J.*, **105**, 151, 1947). Of other, approximate, solutions of the problem of radiative transfer of reflected light we may refer to S. Chandrasekhar, *Ap. J.*, **101**, 348, 1945 (solution by discrete-ordinates method, containing a re-derivation of Hopf's equation 1-73); D. H. Menzel and H. K. Sen, *Ap. J.*, **113**, 490, 1951; or S. S. Huang, *Ap. J.*, **117**, 221, 1953 (both by variational methods).

The existence of stellar gravity-darkening, and the variation of temperature over surfaces of rotationally and tidally distorted stars in radiative equilibrium was first predicted by H. von Zeipel in *M.N.*, **84**, 665, 684 and 702, 1924; and an independent proof of these phenomena for tidally-distorted configurations was later given by S. Chandrasekhar (*M.N.*, **93**, 539, 1933; Appendix I). Both von Zeipel and Chandrasekhar were concerned with gravity-darkening in total (integrated) light. Owing, perhaps, to the fact that their proofs followed in close relation with a physically improbable requirement that, in rotating stars, the rate of energy generation ε should depend on the velocity of their axial rotation and, in tidally-distorted stars, it should remain constant throughout the interior, astronomers were at first inclined to disbelieve also the existence of the gravity-darkening which followed as a corollary to the required improbable behaviour of ε—but the situation was by no means clear. Thus A. S. Eddington, in his discussion of the physical implications of von Zeipel's theorem on p. 288 of his *Internal Constitution of the Stars* (Cambridge, 1930) admitted that '. . . the approximation for it is used in a more specialized way than in a discussion of the total radiation of the star, and it would seem necessary to examine how closely the result is bound up with the accuracy of the approximation before we can be sure that it will apply to actual stars. I daresay '— and here Eddington revealed again his penetrating physical insight—'it will be found that the approximation still justifies itself, but it is not obvious that it is legitimate. The distribution of surface brightness over a tidally disturbed star is of considerable practical importance in the interpretation of the light curves of eclipsing variables'.

It is to be regretted indeed that Eddington's words of this last sentence were not heeded more promptly. The first investigators to consider the effects of gravity-darkening upon the light of close binary systems were S. Takeda (*Kyoto Mem. A*, **17**, 197, 1934) and W. A. Krat (*Zs. f. Ap.*, **9**, 319, 1935). Krat limited, however, his attention to the effects of gravity-darkening on the total light of a close binary system at the times of the conjunctions; but Takeda went far beyond it and his important contributions will be commented upon more fully in the notes to the next section. Both Krat and Takeda considered the effects of gravity-darkening in total light only. The law of gravity-darkening appropriate for any effective wave length for stars radiating like black bodies, as given by equation (1-100) and the coefficient b defined by (1-32), was first formulated by Z. Kopal in the *Annals of New York Acad. Sci.*, **41,** 13, 1941. For further general comments on stellar gravity-darkening *cf.* also T. E. Sterne, *Proc. U.S. Nat. Acad. Sci.*, **27,** 99, 1941; or Z. Kopal, *Harvard Centennial Symposia* (Harv. Obs. Mono., No. 7, Cambridge, Mass., 1948), pp. 261–275.

IV.2: A realization of the fact that the essential part of the light changes of close eclipsing systems exhibited between minima is due to their ellipticity of figure goes back to the closing decades of the last century. Its first defender in public appears to have been J. Plassmann in his book on *Die Veränderlichen Sterne*, Köln 1888. Plassmann stated, however, on p. 52 of this work that his explanation of the light changes of β Lyrae by a mutual tidal distortion of its components had occurred to him already in 1881; and had been published by him previously in the *Jahresbericht des Westfalischen Provinzial-Vereins der Görres-Gesellschaft für* 1885. This latter publication was, unfortunately, inaccessible to the present writer; but sufficient quotations from it have been reproduced by J. G. Stein on p. 308 of his work on *Die Veränderlichen Sterne*, Freiburg 1924, to establish Plassmann's priority in this matter beyond any doubt.

Of subsequent early investigations of the photometric consequences of stellar ellipticity (dealing with individual systems of this type) we may refer to G. W. Myers, *Untersuchung über den Lichtwechsel des Sterns β Lyrae*, Diss. München 1896, and *Ap. J.*, **7,** 1, 1898, *Illinois Obs. Bull.*, No. 1, 1898 (U Peg); A. W. Roberts, *Ap. J.*, **13,** 177, 1901 (V Pup), *M.N.*, **63,** 527, 1903 and **65,** 706, 1905 (RR Cen); or J. v. Hepperger, *Sitzungsber, der Akad. der Wiss. Wien* (*Math.-Naturwiss. Klasse*), **118,** *Abt.* IIa, 923, 1909. These investigations were systematized by H. N. Russell, *Ap. J.*, **36,** 54, 1912 for uniformly bright stars, and by H. N. Russell and H. Shapley, *Ap. J.*, **36,** 385, 1912 for completely limb-darkened stars—both on the assumption that the components of close binary systems are similar in form. For subsequent formal investigations of the light changes due to rotation of (uniformly bright) three-axial ellipsoids of arbitrary elongation *cf.* also B. Shchigolev, *Astr. Zhurnal*, **14,** 447, 1937, or K. Schütte, *A.N.*, **267,** 369, 1939.

The method of treatment of our subject as followed in the present section was initiated by S. Takeda in a paper of historical significance which appeared in *Kyoto Mem.*, *A*, **17,** 197, 1934. This investigation was relevant to centrally-condensed stars ($\Delta_j = 1$) exhibiting full gravity-darkening ($\tau_0 = 1$), and took explicit account of the second, third and fourth harmonic tidal distortion. Unfortunately, perhaps because Takeda's paper appeared in a non-astronomical publication and under a concealing title, his study failed to attract proper attention at the time of its publication; and so it happened that some of Takeda's results were later re-discovered by others. Thus H. N. Russell (*Ap. J.*, **90,** 641, 1939) investigated, by a different and more elementary method, the second-harmonic variation of light of distorted stars between minima, but went beyond Takeda in considering the effects of gravity-darkening other than full, and of finite central condensation of the distorted stars. Still later, T. E. Sterne (*Proc. U.S. Nat. Acad. Sci.*, **27,** 99, 1941) and Z. Kopal (*Proc. Amer. Phil. Soc.*, **85,** 399, 1942) extended Russell's investigation, by a method similar to Takeda's, to include the effects of the third and fourth harmonic distortion of arbitrarily limb- and gravity-darkened components of any structure.

An extension of this work to include the photometric effects of non-linear law of limb-darkening goes back to Kopal, *Harv. Circ.*, No. 454, 1949; and its extension to other than zonal harmonics (or to photometric effects of inclined axes of rotation) is new. A fluctuation of the photometric ellipticity in eccentric close binaries and the occurrence of a 'periastron effect' due to this cause were qualitatively predicted as early as in 1906 by A. W. Roberts (*M.N.*, **66,** 123, 1906); but an adequate treatment of these effects was not given until later by Z. Kopal (*Harv. Circ.*, No. 441, 1941) and T. E. Sterne (*Proc. U.S. Nat. Acad. Sci.*, **27,** 106, 1941). *Cf.* also D. J. K. O'Connell, *M.N.*, **111,** 642, 1942; *Riverview Publ.*, No. 10, 1951.

IV.3: An investigation of the theoretical light curves of close eclipsing systems within minima belongs among the most intricate problems of double-star analysis; and its inherent complexity appeared for a long time so discouraging as to make a list of the literature relevant to the subject of this section particularly brief.

In point of fact, a realistic treatment of its subject did not begin until with S. Takeda (*Kyoto Mem.*, *A*, **20,** 47, 1937) whose aim was to study the variation of integrated light of close eclipsing systems, with components built up according to Eddington's 'standard model' (in point of fact, Takeda's coefficients pertaining to the tidal distortion corresponded to a mass-point model). By its methodical originality as well as the width of treatment, this investigation by Takeda should rank with the most important theoretical papers ever devoted to the study of eclipsing variables. Its reading is, unfortunately, difficult because of uncommon notations as well as a great number of misprints (the latter being possibly due to the fact that, as this appears to be a posthumous publication, its author may not have had an opportunity to read the proofs). Moreover, Takeda did not set out to evaluate the explicit forms of his theoretical light curves, but expressed them in terms of certain integrals whose study and evaluation were largely left as an exercise for the reader.

Therefore, in a subsequent paper Z. Kopal (*Proc. Amer. Phil. Soc.*, **85**, 399, 1942) went further to evaluate in a finite number of terms the changes of light due to partial or annular eclipses of distorted components of any structure, and observable in any particular effective wave-length (i.e., for arbitrary limb- and gravity-darkening). His analysis included the effects of the second, third and fourth spherical harmonic distortion of both stars due to their axial rotation and mutual tidal action. It is this investigation on which the present version of section IV.3 is predominantly based. An abstract of this necessarily long and involved investigation can also be found in Z. Kopal, *Ap. J.*, **94**, 159, 1941, and **96**, 20, 1942.

IV.4: The present version of this section leans heavily on sections IV and VI of the writer's paper from the *Proc. Amer. Phil. Soc.*, **85**, 399, 1943.

IV.5: The subject matter of the present section is mostly new; such results of it as were known have been published by the writer in the introduction to his 'Theory and Tables of the Associated α-functions', *Harv. Circ.*, No. 450, 1947.

More limited tables of some of the *J*-integrals (5-37), introduced in this section have previously been published by A. B. Wyse and G. E. Kron (*Lick Bull.*, No. 496, 1939), and Z. Kopal (*Proc. Amer. Phil. Soc.*, **88**, 145, 1944).

IV.6: The existence of a phenomenon known as the 'reflection effect' in eclipsing binary systems was first noted and correctly explained by R. S. Dugan (*Publ. Amer. Astr. Soc.*, **1**, 311, 1908) in his study of RT Persei (*cf.* also Princ. Contr., No. 1, 1911). Two years later this effect was independently discovered and explained by J. Stebbins (*Ap. J.*, **32**, 213, 1910) and Ch. Nordmann (*Bull. Astr.*, **27**, 145, 1910) in Algol. It is interesting, however, to note that its existence was anticipated at least twenty years before actual discovery by J. Wilsing (*A.N.*, **124**, 121, 1890) in his investigation of the Algol system. Wilsing's formula for the amount of light reflected at 'full' phase proved essentially identical with the one deduced 36 years later by A. S. Eddington (*M.N.*, **86**, 320, 1926).

Eddington (*op. cit.*) was the first to give a closer consideration to the physical processes underlying the reflection of light by the stars, and to point out that their heat-albedo must necessarily be unity. His law for the variation of reflected light with the phase was based on the assumption that the incident light constitutes a parallel beam. He pointed out, moreover, that an expansion of this law is bound to contain a second-harmonic term which will tend to mask the ellipticity effect—a fact whose anticipation can be found already in Stebbins' study of Algol in the *Seeliger Festschrift* (Leipzig 1924, p. 422). Eddington also called attention to a difference which must exist between bolometric and observable reflection, and outlined the nature of the effects exerted by reflection upon radial-velocity curves of the components of close binary systems.

E. A. Milne (*M.N.*, **87**, 43, 1926) shortly thereafter formulated proper equations of transfer of the reflected radiation, and used their approximate solution to deduce a physically sounder law governing the variation of amount of reflected light with the phase—still on the assumption that the incident radiation constitutes a parallel beam. An alternative form of the result, based on Chandrasekhar's 'first approximation' to the solution of transfer of the reflected light (*cf.*, S. Chandrasekhar, *Ap. J.*, **101**, 348, 1945), was later given by Z. Kopal in his *Introduction to the Study of Eclipsing Variables* (Harvard Univ. Press, Cambridge, Mass., 1946), section 63; and a numerical tabulation of the phase-law based on Chandrasekhar's 'fourth approximation' was subsequently evaluated by J. Sahade and C. Cesco, and is likewise reproduced in section 63 of Kopal's book referred to above.

All these results just quoted have been based on the assumption that the incident light forms a parallel beam (which would correspond to an infinite distance of the illuminating source). A first attempt to break away from this limitation and take account of the convergence of incident beam was due to S. Takeda (*Kyoto Mem.*, *A*, **17**, 197, 1934), who regarded the secondary component as a non-dimensional light point illuminating the primary which (as seen from its mate) exposed to the incident light an apparent disk of finite angular size. Takeda carried through an approximate solution of this problem to quantities of the fourth order in the fractional radius of the primary (reflecting) component (i.e., to the order of accuracy to which a distortion of this star due to its own axial rotation as well as to the tidal action of its companion can be ignored). In doing so Takeda committed, however, the mistake of extending the cone of incident light beyond the horizon of visibility of the illuminating light course—an oversight which vitiated his fourth-order term and which was eventually corrected by H. K. Sen (*Proc. U.S. Nat. Acad. Sci.*, **34,** 311, 1948).

A reconnaissance of the effects of a finite angular size of the illuminating component on the amount of reflected light was first undertaken by H. N. Russell (*Harv. Circ.*, No. 452, 1949) by an elementary method which was essentially incapable of generalization, and which led him to underestimate the relative photometric importance of the penumbral zone. A year later, a discussion of the same problem was resumed by T. Matukuma (*Sendai Astr. Raportoj*, **2,** No. 10, 1950). Like Russell, Matukuma did not aim at a derivation of the phase-law of reflected light, but limited himself to ascertaining the proportion of incident light which gets reflected at full phase (and expressed by what Matukuma called the 'criterion formula'). However, in concluding his paper, Matukuma stated that '. . . recently, the writer had taken up this problem' (i.e., the derivation of phase-law) 'and has almost completed the necessary calculations. But some minute points, which are obscure in a physical sense, make him postpone publishing his results.' Unfortunately, shortly after writing these words Matukuma died (on January 14th, 1950)—apparently without being able to clear up his doubts; and an examination of his 'criterion formula' reveals its derivation to be in error invalidating the results.

These reasons led the present writer to work out an exact theory of the mutual illumination of two finite spheres reflecting light in accordance with Lambert's law. This theory was published in *M.N.*, **114,** 101, 1954, and has been used as a basis for the presentation of its subject in this section. It includes the treatment of the secondary (or tertiary) reflection, of reflection in eccentric systems, or of eclipse effects on reflected light. A transformation of the total (i.e., bolometric) reflection to that observable in any particular effective wavelength follows the treatment of this subject given earlier by the writer in *Ap. J.*, **89,** 323, 1939 or (more explicitly), in section 66 of his *Introduction to the Study of Eclipsing Variables* (Harvard Univ. Press, Cambridge, Mass., 1946).

Of other investigations of the photometric effects of reflection in close binary systems we may quote E. W. Pike, *Ap. J.*, **73,** 205, 1931; W. A. Krat, *M.N.*, **94,** 70, 1933, *Astr. Zhurnal*, **11,** 5, 1934; M. G. Odinzov, *Astr. Zhurnal*, **19,** 80, 1942; P. Pişmiş, *Ap. J.*, **104,** 141, 1946; and others.

IV.7: An outline of the theory of atmospheric eclipses by extended envelopes of the stars, as given in this section, is essentially taken from the earlier publications by Z. Kopal in *Proc. Amer. Phil. Soc.*, **89,** 590, 1945, and *Ap. J.*, **103,** 310, 1946. The method of solution of the integral equation (7-9) has been expounded in more detail in Z. Kopal, *Numerical Analysis*, London, 1955, section VIII-E.

Of other investigations dealing with specific systems *cf.*, e.g., D. H. Menzel, *Harv. Circ.*, No. 417, 1936, dedicated to an analysis of the eclipses of ζ Aurigae. (It should be noted, however, that all terms of Menzel's fundamental equation (7)—which should be identical with our equation (7-37)—of order higher than zero are in error, caused by the fact that Menzel expanded the radical in the exponent of his equation (5) in a binomial series converging only for $x < 1$, whereas in his investigation x ranged between 0 and ∞.) B. Strömgren (*Ap. J.*, **86,** 570, 1937) in his work on ε Aur again obtained a first approximation to our equation (7-65).

In addition, theroetical light changes arising from eclipses *of* stars with extended atmospheres by their mates—a subject not treated in this section—*cf.* C. P. and S. I. Gaposchkin, *Ap. J.*, **101,** 56, 1945; Z. Kopal and M. B. Shapley, *Ap. J.*, **104,** 160, 1946; A. M. Schulberg, *Odessa Obs. Bull.*, No. 3, 1947; G. E. Kron and K. C. Gordon, *Ap. J.*, **111,** 454, 1950; A. M. Schulberg, *Odessa Obs. Izvestia*, **3,** 249, 1953; also *Per. Zvjozdy*, **9,** 256, 1953 (all dealing with V444 Cygni); etc.

CHAPTER V

Theoretical Velocity Changes in Close Binary Systems

In the previous chapter we have investigated the photometric consequences of distortion of the components in close binary systems, and of their mutual irradiation, to the order of accuracy to which their axial rotation is slow enough for the effects of its squares and higher powers to be negligible and the tidal action of one component upon another can be regarded as that of a mass-point. The aim of the present chapter will be to investigate, in a similar manner and to the same order of accuracy, the effects produced by distortion and reflection on the second aspect of evidence on binary systems which is open to observational scrutiny: namely, the form of the spectral lines of their components and their radial velocities.

The nature of this subject may perhaps require a few words of introduction. It is a well-known fact that the observed periodic variations in radial velocities of the components of spectroscopic binaries can be regarded as wholly due to orbital motion only provided that their components are sufficiently far apart for their relative dimensions to be ignorable in comparison with their separation, and can thus be regarded as mere light points. Should, however, the dimensions of the components become—as they do in close binary systems—sizeable fractions of their separation, their observed radial velocities would continue to be purely orbital only if either component possessed no rotational momentum or (under certain circumstances) the distribution of brightness over their surfaces were uniform or symmetrical with respect to the projected centre of mass.

So far as the present observational evidence goes, however, the components of all close binaries are found to rotate—with a speed equal to the Keplerian angular velocity or exceeding it in many cases—around an axis which is perpendicular to the orbital plane or at least approximately so. Moreover, the average separations of components in most known spectroscopic binaries make it clear that such components cannot in general be approximated by spheres: centrifugal and tidal forces will cause such stars to deviate from spherical form and to possess non-uniform surface brightness. In point of fact, the intensity-distribution over apparent disks of even spherical components could not remain symmetrical with respect to the projected centre of mass if only because of the light mutually reflected between the two stars. The reflection, jointly with the darkening of distorted stars, will thus prevent the centre of light of the apparent disks of components in close binary systems from coinciding in projection with their respective centres of mass at any phase; and their varying displacement will allow the axial rotation of the components to influence distinctly their observed radial velocities in a characteristic manner. All these effects will necessarily render the observed radial velocities of rotating components of close binary systems different from velocities of their centres of mass; terms arising from axial rotation will superpose upon purely orbital velocity to yield a resultant which, if analyzed in a conventional manner, would furnish elements of spectroscopic orbits that are systematically in error. It is, therefore, evident that no detailed sound knowledge of the elements of spectroscopic binary systems (and, therefore, of their masses and absolute dimensions) can be expected until the effects, upon radial velocity, of axial rotation and orbital motion of their components are satisfactorily disentangled. The object of the present chapter will be to lay down a theoretical basis for the development of systematic methods which will make this possible.

In section V.1, following these introductory remarks, we shall set out to investigate the contributions, to radial velocity, arising from axial rotation of distorted components of close binary systems to the order of accuracy to which their form has been established in Chapter II—i.e., as far as effects of the second, third, and fourth spherical harmonic distortion of limb- and gravity-darkened stellar configurations of arbitrary structure are concerned. In doing so we shall not confine our attention to theoretical velocity changes in 'full light' alone. In the subsequent section V.2 we shall equally consider the more general and more difficult case of such phenomena within eclipses ('rotational effect') as well. This investigation will parallel, in its method, that of the light changes in eclipsing binary systems (sections IV.2 and IV.3) so closely that frequent appeals to our earlier work will materially shorten our present task. In particular, we shall find it possible to express the theoretical velocity changes of non-orbital origin in terms of the same types of special functions which describe the theoretical light changes of close eclipsing systems, and since all such requisite functions have already been exhaustively investigated in sections IV.4 and IV.5, we shall be able to express the desired results in an almost telegraphic style.

The next section V.3 will discuss the influences, upon radial velocity, of the reflection effect in close binary systems. Its treatment will again be closely parallel in method to the subject matter of section IV.6, and the requisite analysis will be carried out to the same order of accuracy as in our discussion of the light changes: account will be taken of the finite angular size of the illuminating as well as the reflecting component, but their distortion will be ignored. In section V.4, we shall consider the specific effects exerted by distortion and reflection on the interpretation of the observed radial-velocity changes of components in close binary systems, and point out the way in which contributions to radial velocity of non-orbital origin—if uncorrected—may vitiate determination of the individual elements of their spectroscopic orbits. In section V.5 we shall discuss the effects of rotation on line profiles of the stars undergoing eclipses; and in the concluding section V.6 we shall summarize the results.

V.1. EFFECTS OF DISTORTION

In order to investigate the effects of distortion of rotating components of close binary systems on their observed radial velocities, let us consider a binary whose components revolve in space around their common centre of gravity and rotate like rigid bodies—an assumption justified on grounds of simplicity—about fixed axes. Let, furthermore, the observed radial velocity of the component at any moment be decomposed into a sum $V + \delta V$, where V stands for velocity due to space and orbital motion, and δV for the contribution arising from axial rotation of the respective star. The amount of this contribution is obtained, by definition, if we multiply the radial velocity at any point on the surface of the rotating star by the corresponding element of light, integrate over the whole apparent disk (or a visible portion thereof) and divide by the light proper of the respective star.

In order to specify the non-orbital contribution δV to the observed radial velocity, let us identify the origin of coordinates with the centre of mass of the star under consideration—which will be hereafter consistently referred to as the primary component—and adopt (as in section IV.1) an orthogonal frame of reference XYZ whose X-axis coincides with the line joining the centres of the two components, and Z is perpendicular to the orbital plane. Let, furthermore, r, θ, ϕ denote the spherical polar coordinates in the XYZ-system. The light element dl is then evidently equal to

$$dl = r^2 J \cos\gamma \sin\theta \, d\theta \, d\phi \,, \tag{1-1}$$

where $r(\theta, \phi)$ specifies the boundary of the distorted star; γ, the angle of foreshortening; and J, the local surface brightness.

Now we have seen before that, to the first order in small quantities, the superficial distortion of a star should be governed by the equation

$$\frac{r - r_1}{r_1} = \sum_{j=2}^{4} w_1^{(j)} P_j(\lambda) - \tfrac{1}{3} v_1^{(2)} P_2(\nu) \,, \tag{1-2}$$

where

$$\left.\begin{aligned} \lambda &= \cos\phi \sin\theta \,, \\ \mu &= \sin\phi \sin\theta \,, \\ \nu &= \cos\theta \,; \end{aligned}\right\} \tag{1-3}$$

stand for the direction cosines of an arbitrary radius-vector; r_1 denotes the fractional radius of a sphere equal in volume to the distorted primary; and the constants $w_1^{(j)}$ and $v_1^{(2)}$ continue to be given by equations (2-33) and (2-35) of Chapter IV. The angle of foreshortening γ is, consistent with equations (2-3) of the same chapter, defined by

$$\cos\gamma = l l_0 + m m_0 + n n_0 \,, \tag{1-4}$$

where l_0, m_0, n_0, the direction cosines of the line of sight, continue to be given by equations (2-5) of Chapter IV; while the direction cosines l, m, n of a line normal to the surface (1-2) at an arbitrary point can be shown to become

$$l = \lambda\left\{1 - v_1^{(2)}\nu^2 + (\lambda - \lambda^{-1})\sum_{j=2}^{4} w_1^{(j)}P_j'(\lambda)\right\}, \tag{1-5}$$

$$m = \mu\left\{1 - v_1^{(2)}\nu^2 + \lambda\sum_{j=2}^{4} w_1^{(j)}P_j'(\lambda)\right\}, \tag{1-6}$$

$$n = \nu\left\{1 - v_1^{(2)}\nu^2 + v_1^{(2)} + \lambda\sum_{j=2}^{4} w_1^{(j)}P_j'(\lambda)\right\}, \tag{1-7}$$

as particular cases of equations (2-6) of Chapter IV.

Now as is well known, the distribution of brightness J over the surface of the distorted primary component will be governed by limb- and gravity-darkening which, in conformity with equations (2-15) and (2-16) of Chapter IV, leads us to expect that

$$J = H(1 - u_1 - u_2 - \ldots + u_1 \cos\gamma + u_2 \cos^2\gamma + \ldots) \tag{1-8}$$

where the u_i's are the respective coefficients of limb-darkening*, and

$$H = H_0\left\{1 - \tau_0 \frac{g - g_0}{g_0}\right\}, \tag{1-9}$$

in which τ_0 denotes the coefficient of gravity-darkening introduced in section IV.1; g and g_0 stand for the local and mean gravity over the distorted surface; and H_0, the mean intensity of radiation emerging normally to the star's surface, is regarded to be constant. Moreover, equation (2-17) of Chapter IV permits us to assert that, in the present case,

$$\frac{g - g_0}{g_0} = \frac{1}{3}\left\{\frac{5}{\Delta_2} - 1\right\} v_1^{(2)}P_2(\nu) - \sum_{j=2}^{4}\left\{\frac{2j+1}{\Delta_j} - j + 1\right\} w_1^{(j)}P_j(\lambda), \tag{1-10}$$

where the Δ_j's are constants depending on internal structure of the distorted configuration and defined already by equation (1-27) of Chapter II.

This completes the groundwork necessary for explicit formulation of the light element dl. The non-orbital velocity V', in the line of sight, at any surface point of a rotating star remains, however, yet to be specified.

* It should be pointed out that the coefficients u_j and τ_0 in Chapter IV and the present section need not necessarily be identical. The present values of all these coefficients refer to the darkening in the monochromatic light of spectral lines selected for measurement of radial velocities, while throughout Chapter IV these referred to the darkening in the light of the continuous spectrum—between absorption lines—at the effective wave length of photometric observations. As the line absorption and continuous radiation may, in general, originate at quite different optical depths of a stellar atmosphere, the respective monochromatic and continuous-light darkening coefficient may differ from each other.

In order to do so in the most general case, let ω_x, ω_y, ω_z denote the angular velocities of rotation about the respective axes: the individual components of the velocity vector in the xyz-direction are then expressible as

$$\left.\begin{aligned} V_x &= z\omega_y - y\omega_z\,, \\ V_y &= x\omega_z - z\omega_x\,, \\ V_z &= y\omega_x - x\omega_y\,. \end{aligned}\right\} \tag{1-11}$$

To determine the radial-velocity component in the direction of the line of sight, let us change over—as in section IV.3—to a primed rectangular system $X'Y'Z'$, with the same origin, which is stationary with respect to the observer and defined so that its Z'-axis coincides constantly with the line of sight, while the X'-axis points in the direction of the centre of the secondary component projected on the celestial sphere. A transformation of coordinates between the primed and unprimed axes is governed by the scheme of the direction cosines as given on p. 189. In particular, the radial-velocity component $V_{Z'} \equiv V'$ in the direction of the Z'-axis then assumes the form

$$\begin{aligned} V' = l_0 V_x + m_0 V_y + n_0 V_z = \omega_x(l_1 x' - l_2 y') \\ + \omega_y(m_1 x' - m_2 y') + \omega_z(n_1 x' - n_2 y')\,, \end{aligned} \tag{1-13}$$

where the direction cosines l_j, m_j, n_j $(j = 0, 1, 2)$ are given by equations (3-2)–(3-4) of Chapter IV, and

$$x' = l_2 x + m_2 y + n_2 z = r(\lambda l_2 + \mu m_2 + \nu n_2)\,, \tag{1-14}$$

$$y' = l_1 x + m_1 y + n_1 z = r(\mu m_1 + \nu n_1)\,. \tag{1-15}$$

By virtue of the foregoing relations, equation (1-13) can be rewritten also in the following alternative form

$$V'_{\text{rot}} = r\omega_x(\mu n_0 - \nu m_0) + r\omega_y(\nu l_0 - \lambda n_0) + r\omega_z(\lambda m_0 - \mu l_0)\,. \tag{1-12}$$

The behaviour of the angular velocity components ω_x, ω_y, ω_z was investigated in some detail in section II.3, to which the reader is referred back for relevant information.* If, however, the precession and nutation phenomena are ignored within a single cycle, it can be shown that

$$\left.\begin{aligned} \omega_x &= \omega_1 \sin\beta \sin u\,, \\ \omega_y &= \omega_1 \sin\beta \cos u\,, \\ \omega_z &= \omega_1 \cos\beta\,, \end{aligned}\right\} \tag{1-16}$$

where ω_1 denotes the (constant) angular velocity of rotation of the primary component about a fixed axis; β, the angle between this axis and a direction normal to the plane of the orbit; and u, the longitude (true anomaly)

* The reader should note that the present quantities ω_x, ω_y, ω_z are identical with those denoted by $\omega_{x''}$, $\omega_{y''}$, $\omega_{z''}$ in section II.3. *Cf.* also footnote on p. 174.

measured from the ascending node. If the axis of rotation were perpendicular to the orbital plane (i.e., if $\beta = 0$), it would follow that

$$\omega_x = \omega_y = 0 \qquad \text{and} \qquad \omega_2 = \omega_1 ; \tag{1-17}$$

but for $\beta \neq 0$ all three angular velocity components are bound to be distinct from zero and, moreover, ω_x as well as ω_y should oscillate between $\pm\omega_1 \sin \beta$, in the course of each cycle, with a phase difference of 90°.

The non-orbital contribution δV to the observed radial-velocity of a distorted star in rotation can be expressed as

$$\delta V = \frac{\iint V' \, dl}{\iint dl}, \tag{1-18}$$

where both V' and dl may now be written down explicitly in terms of the angular variables θ and ϕ (or λ and ν); the limits of integration being extended over the whole hemisphere visible at any particular phase. Suppose that, in what follows, we regard the angle β of deviation of the axis of rotation from perpendicularity to the orbital plane as a small quantity, whose squares and cross-products with other small quantities of first order can be ignored. Moreover, as the numerator on the right-hand side of the foregoing equation (1-18) becomes, on integration, such a small quantity, it follows that, within the scheme of our approximation, the denominator needs to be evaluated only to quantities of zero order (i.e., the distortion may legitimately be ignored). The result is then known from equation (2-23) of Chapter IV; and the integration of the numerator offers no greater difficulty.

Its outcome reveals that if the general law of limb-darkening (1-8) contains k terms of the form $u_{h-1} \cos^{h-1} \gamma$ $(h = 1, 2, \ldots, k)$ and if, by analogy with equation (2-25) of Chapter IV which expresses the corresponding light changes, we set

$$\delta V = \sum_{h=1}^{k} C^{(h)} \, \delta V^{(h)}, \tag{1-19}$$

the coefficients $C^{(h)}$ continue to be given by equations (2-26) and (2-27) of Chapter IV, and

$$\begin{aligned} \frac{\delta V^{(h)}}{\omega_1 R_1} = & -\frac{(h+1)(\beta_2+1-h)}{(h+2)(h+4)} \, w_1^{(2)} m_0 P_2'(l_0) \\ & -\frac{h(\beta_3+7-h)}{(h+3)(h+5)} \, w_1^{(3)} m_0 P_3'(l_0) \\ & -\frac{(h-1)(h+1)(\beta_4+15-h)}{(h+2)(h+4)(h+6)} \, w_1^{(4)} m_0 P_4'(l_0) - \ldots, \end{aligned} \tag{1-20}$$

where R_1 denotes the mean radius of the rotating star in absolute units; the constants β_j continue to be given by equations (2-31) of Chapter IV; and, it may be noted,

$$m_0 P_j'(l_0) = -n_1 P_j^1(l_0) \, . \tag{1-21}$$

It transpires, therefore, that *whereas the light changes arising from rotation of distorted stars are expansible in terms of the zonal harmonics* P_j^0 *of the phase, the corresponding radial-velocity changes are found to be expansible in terms of tesseral harmonics* P_j^1 *of the same argument.* A similar contribution to the radial-velocity of the *secondary* component is, moreover, easily obtained from (1-20) by an appropriate interchange of indices and a phase shift of 180°.

The foregoing result may, therefore, be also rewritten as

$$\delta V = \omega_1 R_1 n_1 \sum_{j=2}^{4} w_1^{(j)} f_j^{(k)} P_j^1(l_0) , \tag{1-22}$$

where

$$f_2^{(k)} = \sum_{h=1}^{k} \frac{(h+1)(\beta_2 + 1 - h)}{(h+2)(h+4)} C^{(h)} , \tag{1-23}$$

$$f_3^{(k)} = \sum_{h=1}^{k} \frac{h(\beta_3 + 7 - h)}{(h+3)(h+5)} C^{(h)} , \tag{1-24}$$

and

$$f_4^{(k)} = \sum_{h=1}^{k} \frac{(h-1)(h+1)(\beta_4 + 15 - h)}{(h+2)(h+4)(h+6)} C^{(h)} . \tag{1-25}$$

If $k = 2$ (corresponding to a linear law of limb-darkening),

$$f_2^{(2)} = \frac{8\beta_2 - 3u_1\beta_2 - 5u_1}{20(3 - u_1)} , \tag{1-26}$$

$$f_3^{(2)} = \frac{210 + 35\beta_3 - 3u_1\beta_3 - 50u_1}{280(3 - u_1)} , \tag{1-27}$$

$$f_4^{(2)} = \frac{u_1(\beta_4 + 13)}{32(3 - u_1)} ; \tag{1-28}$$

for $k = 3$ (quadratic limb-darkening)

$$f_2^{(3)} = \frac{56\beta_2 - 35u_1 - 48u_2 - 21u_1\beta_2 - 32u_2\beta_2}{70(6 - 2u_1 - 3u_2)} , \tag{1-29}$$

$$f_3^{(3)} = \frac{140(\beta_3 + 6) - 200u_1 - 420u_2 - 12u_1\beta_3 - 35u_2\beta_3}{560(6 - 2u_1 - 3u_2)} , \tag{1-30}$$

$$f_4^{(3)} = \frac{u_1(\beta_4 + 13)}{16(6 - 2u_1 - 3u_2)} + \frac{8u_2(\beta_4 + 12)}{105(6 - 2u_1 - 3u_2)} ; \tag{1-31}$$

etc.

An inspection of the foregoing results reveals, first, that *the non-orbital contributions to the radial velocities of rotating components of close binary systems are due to their tidal interaction alone.* This was indeed to be expected; for the aspect of a flattened spheroid rotating about an axis which is perpendicular to the orbital plane does not change, for a distant observer, in the course of a revolution. That, furthermore, the rotation of a uniformly bright prolate spheroid is likewise accompanied by no change of its radial velocity is also easily understood. If, however, there is some limb- or gravity-darkening (or both), equations (1-26) to (1-28) reveal that the contribution of the second harmonic to δV becomes distinct from zero. Proceeding further to consider the effects of tidal harmonic distortion of orders higher than the second, we notice that *the third harmonic will contribute to the radial velocity of a rotating distorted ellipsoid even if the latter were uniformly bright*—as a consequence of the fact that, as the star revolves, its centre of light would no longer project itself constantly on its centre of mass. The limb- and (or) gravity-darkening will merely reinforce this contribution. Ultimately, we notice that *the fourth-harmonic tidal distortion of a rotating star*, like the second, *will contribute nothing to its observed radial velocity unless this star exhibits some darkening at limb.*

A closer examination of the coefficients $f_j^{(k)}$ on the right-hand side of (1-22), as defined by equations (1-23)–(1-25), leads to certain interesting conclusions with regard to the character of isophotae on apparent disks of distorted stars. The coefficient f_2 may evidently be positive or negative —depending on whether the limb- or gravity-darkening preponderates—and vanishes when

$$\beta_2 = \frac{35u_1 + 48u_2 + \ldots}{56 - 21u_1 - 32u_2 - \ldots} \tag{1-32}$$

or, for linear law of limb-darkening, if

$$u_1 = \frac{8\beta_2}{5 + 3\beta_2}. \tag{1-33}$$

If this is the case, δV vanishes for ellipsoidal configurations; and we know on general grounds that this may happen if, and only if, the isophotae on the apparent disk of the star form a family of curves which remain symmetrical with respect to its projected centre of mass. We infer therefrom at once that, *to every degree of limb-darkening of a rotating ellipsoid there corresponds a certain amount of gravity-darkening*, as defined by equation (1-32), *which renders the isophotae on its apparent disk symmetrical at any phase with respect to its centre.*

For arbitrary values of u_j these curves of equal brightness need not necessarily be closed—in fact, they do freely cross the boundary of the apparent disk. A case of special interest arises, however, when $u_1 = 1$ and $u_2 = 0$. By clamping down full limb-darkening we compel the boundary to become an envelope of all isophotae of finite surface brightness. If in addition, $\beta_2 = 1$, the vanishing of the coefficient $f_2^{(2)}$ reveals that these isophotae must form a family of curves which are symmetrical with respect to the projected centre of mass of our ellipsoid; and since for $u_1 = 1$ the limb itself has become an isophote corresponding to $J = 0$, *all isophotae interior to it must represent a family of ellipses similar to the limb, and concentric* *with it.* When, however, $\beta_2 > 1$, equation (1-32) makes it evident that the symmetry of the isophotae is irrevocably lost and no amount of limb-darkening may help to restore it. We may add that, in real stars, there is every reason to expect that $\beta_2 \gg 1$. In accordance with equation (2-31) of Chapter IV, $\beta_2 \approx 4\tau_0$, where the coefficient τ_0 of gravity-darkening is, in turn, given by equation (2-16) of the same chapter and, for total radiation, becomes equal to one. Now the coefficients $u_{1,2}$ of a quadratic law of limb-darkening in total light are (*cf.* equations 2-49 of Chapter IV) closely approximated by the values $u_1 = 0{\cdot}65$ and $u_2 = -0{\cdot}02$, which inserted in (1-32) yield $\beta_2 = 0{\cdot}53$ as the value required to ensure the vanishing of $f_2^{(3)}$—while the physically probable value of β_2 is in the neighbourhood of 4. It is, therefore, to be expected that for all real stars the coefficient (1-23) of the second harmonic term on the right-hand side of (1-22) will be present in a significant amount.

Proceeding lastly to consider terms associated with tidal distortion of orders higher than the second we note that, unlike f_2, the coefficient f_4 vanishes only if the star is undarkened at limb; and f_3 would fail to vanish even then—which indicates that under no circumstances can the isophotae on apparent disks of distorted ellipsoids form a family of curves symmetrical with respect to their projected centre of mass—whatever their amount of limb- or gravity-darkening.

In conclusion of the present section one last remark may be appropriate. The explicit expressions established for δV throughout its text refer consistently to the non-orbital contributions to the radial velocity of the *primary* component. Those for the *secondary* can, however, be deduced from the same expressions at no greater modification than that entailed in interchanging appropriate indices 1 and 2, and shifting phase by 180°.

V.2. ROTATIONAL EFFECT DURING ECLIPSES

So far we have considered the effects of distortion upon the observed radial velocity of rotating components of close binary systems in 'full light' (i.e., at such phases when the apparent disks of both stars are fully exposed to view by a distant observer). If, however, our binary happens to be an eclipsing variable, the analysis of the foregoing section continues to furnish the non-orbital contributions to the radial velocity of a star undergoing eclipse as well, provided only that *the limits of integration in both the numerator and the denominator on the right-hand side of equation* (1-18) *for δV are extended over the visible fraction of the eclipsed star.* This task becomes, to be sure, geometrically much more complicated than our simple analysis carried out so far in this section. As, however, most part of these difficulties has already been faced and overcome in connection with theoretical light curves in section IV.3, the results relevant to our present problem can be written down in terms of special functions analyzed in sections IV.4 and 5 in an almost telegraphic style.

In doing so we may note that, since the denominator of the expression (1-18) for δV represents nothing but the instantaneous fractional light of the star undergoing eclipse, it is possible to set

$$\iint dl = \pi r_1^2 H_0\{\Delta\mathfrak{L}(k, -1) - \Delta\mathfrak{L}(k, p)\}, \tag{2-1}$$

where $\Delta\mathfrak{L}(k, p)$ represents the fractional loss of light at a geometrical depth p of the eclipse, which has already been given by equation (3-27) of the previous Chapter IV. Moreover, the numerator in (1-18) can be evaluated by essentially the same method. In embarking upon this task, it is useful to recall that all terms, in $V'dl$, containing odd powers of y' are bound to vanish when integrated over an area that is symmetrical with respect to the y'-axis. In consequence, the y'-terms in the expression (1-13) for V' will contribute nothing to the radial velocity except in connection with the rotational distortion of the eclipsed star.* Therefore, if

$$\iint V'dl = \pi\omega_1 r_1^2 R_1 H_0\{\mathfrak{V}(k, -1) - \mathfrak{V}(k, p)\} \tag{2-2}$$

where, by analogy with equation (3-27) of Chapter IV, we decompose the new function $\mathfrak{V}(k, p)$ into

$$\mathfrak{V}(k, p) = \sum_{h=1}^{k} C^{(h)}\{\alpha_{h-1}^1 + g_*^{(h)} + g_1^{(h)} + g_2^{(h)}\}, \tag{2-3}$$

the coefficients $C^{(h)}$ continue to be given by equations (2-26)–(2-27) of Chapter IV and, to the first order in small quantities,

$$\begin{aligned} g_*^{(h)} = {} & \{\tfrac{1}{2}[\Omega_2^{(h)} - 1][(n_0^2 - n_1^2)\alpha_{h+1}^1 + (n_2^2 - n_1^2)\alpha_{h-1}^3 + \tfrac{2}{3}P_2(n_1)\alpha_{h-1}^1 \\ & + h[\tfrac{2}{3}P_2(n_0)\alpha_{h-1}^1 + n_0 n_2\, \alpha_{h-2}^2]\}n_1 v_1^{(2)} \\ & - \{[\Omega_2^{(h)} - 1][n_0(\alpha_h^0 - \alpha_h^2 - \alpha_{h+2}^0) + n_2(\alpha_{h-1}^1 - \alpha_{h-1}^3 - \alpha_{h+1}^1)] \\ & + h\,[n_0(\alpha_{h-2}^0 - \alpha_{h-2}^2 - \alpha_h^0)]\}n_1 n_2 v_1^{(2)} \\ & - \{[\Omega_2^{(h)} - 1][3l_0^2\alpha_{h+1}^1 + 6l_0 l_2\alpha_h^2 + 3l_2^2\alpha_{h-1}^3 - \alpha_h^0] \\ & + h[2P_2(l_0)\alpha_{h-1}^1 + 3l_0 l_2\alpha_{h-2}^1]\}n_1 w_1^{(2)} \\ & - \{[\Omega_3^{(h)} - 1][5l_0^3\alpha_{h+2}^1 + 15l_0^2 l_2\alpha_{h+1}^2 + 15l_0 l_2^2\alpha_h^3 \\ & + 5l_2^3\alpha_{h-1}^4 - 3l_0\alpha_h^1 - 3l_2\alpha_{h-1}^2] + h[(l_0\alpha_h^1 + 2l_2\alpha_{h-1}^2)P_3'(l_0) \\ & - \tfrac{3}{2}l_0(\alpha_h^1 - 5l_2^2\alpha_{h-2}^3 + \alpha_{h-2}^1)]\}n_1 w_1^{(3)} \\ & - \{\tfrac{1}{8}[\Omega_4^{(h)} - 1][35l_0^4\alpha_{h+3}^1 + 140l_0^3 l_2\alpha_{h+2}^2 + 210l_0^2 l_2^2\alpha_{h+1}^3 \\ & + 140l_0 l_2^3\alpha_h^4 + 35l_2^4\alpha_{h-1}^5 - 30l_0^2\alpha_{h+1}^1 - 60l_0 l_2\alpha_h^2 \\ & - 30l_2^2\alpha_{h-1}^3 + 3\alpha_{h-1}^1] \\ & + \tfrac{1}{2}h[2l_0 P_4'(l_0)\alpha_{h+1}^1 + 15l_2^2(7l_0^2 - 1)\alpha_{h-1}^3 - 2P_3'(l_0)\alpha_{h-1}^1 \\ & + 5l_0 l_2(21l_0^2\alpha_h^2 - 6\alpha_h^2 + 7l_2^2\alpha_{h-2}^4 - 3\alpha_{h-2}^2)]\}w_1^{(4)} + \ldots, \end{aligned} \tag{2-4}$$

* The reason being that λ, unlike μ, does not contain y' when rewritten in terms of the primed rectangular coordinates as in equations (3-11) of Chapter IV.

where the constants $\Omega_j^{(h)}$ continue to be given by equation (2-22) of Chapter IV; and

$$\begin{aligned} g_1^{(h)} = \{ & n_0^2\mathfrak{I}^1_{-1,h+1} + n_1^2\mathfrak{I}^1_{1,h-1} + n_2^2\mathfrak{I}^3_{-1,h-1} \\ & + 2n_0n_2\mathfrak{I}^2_{-1,h} - \tfrac{1}{3}\mathfrak{I}^1_{-1,h-1}\}n_1v_1^{(2)} \\ & - 2\{n_0\mathfrak{I}^0_{1,h} + n_2\mathfrak{I}^1_{1,h-1}\}n_1n_2v_1^{(2)} \\ & - \{3l_0^2\mathfrak{I}^1_{-1,h+1} + 6l_0l_2\mathfrak{I}^1_{-1,h} + 3l_2^2\mathfrak{I}^2_{-1,h-1} \\ & - \mathfrak{I}^0_{-1,h-1}\}n_1w_1^{(2)} \\ & - \{5l_0^3\mathfrak{I}^1_{-1h+2} + 15l_0^2l_2\mathfrak{I}^2_{-1,h+1} + 15l_0l_2^2\mathfrak{I}^3_{-1,h} + 5l_2^3\mathfrak{I}^4_{-1,h-1} \\ & - 3l_0\mathfrak{I}^1_{-1,h} - 3l_2\mathfrak{I}^2_{-1,h-1}\}n_1w_1^{(3)} \\ & - \tfrac{1}{4}\{35l_0^4\mathfrak{I}^1_{-1,h+3} + 140l_0^3l_2\mathfrak{I}^2_{-1,h+2} + 210l_0^2l_2^2\mathfrak{I}^3_{-1,h+1} \\ & + 140l_0l_2^3\mathfrak{I}^4_{-1,h} + 35l_2^4\mathfrak{I}^5_{-1,h-1} - 30l_0^2\mathfrak{I}^1_{-1,h+1} - 60l_0l_2\mathfrak{I}^2_{-1,h} \\ & - 30l_2^2\mathfrak{I}^3_{-1,h-1} + 3\mathfrak{I}^1_{-1,h-1}\}n_1w_1^{(4)} + \dots \end{aligned} \tag{2-5}$$

and

$$\begin{aligned} g_2^{(h)} = & -\{n_1^2\mathfrak{R}^1_{1,h-1} + n_2^2\mathfrak{R}^3_{-1,h-1} - \tfrac{1}{3}\mathfrak{R}^1_{-1,h-1}\}n_1v_2^{(2)} \\ & + 2n_1n_2^2(r_2/r_1)^2I^1_{1,h-1}v_2^{(2)} \\ & + \{3l_2^2\mathfrak{R}^3_{-1,h-1} - \mathfrak{R}^1_{-1,h-1}\}n_1w_2^{(2)} \\ & + \{5l_2^3\mathfrak{R}^4_{-1,h-1} - 3l_2\mathfrak{R}^2_{-1,h-1}\}n_1w_2^{(3)} \\ & + \tfrac{1}{4}\{35l_2^4\mathfrak{R}^5_{-1,h-1} - 30l_2^2\mathfrak{R}^3_{-1,h-1} + 3\mathfrak{R}^1_{-1,h-1}\}n_1w_2^{(4)} + \dots, \end{aligned} \tag{2-6}$$

where the integrals $\mathfrak{I}^m_{\beta,\gamma}$ as well as $I^m_{\beta,\gamma}$ continue to be given by equations (3-23) and (3-26) of Chapter IV, and

$$\mathfrak{R}^m_{\beta,\gamma} = (\delta/r_2)I^{m-1}_{\beta,\gamma} - I^m_{\beta,\gamma}\,. \tag{2-7}$$

The most significant feature of equations (1-35)–(1-36) is the fact that, unlike the corresponding result for full light, the non-orbital part δV of the radial velocity does not vanish with the distortion but remains finite—within eclipses—even if the component undergoing eclipse becomes a sphere. In point of fact, for spherical stars linearly darkened at the limb ($k = 2$) we readily find from the foregoing results that

$$\frac{\delta V}{\omega_1 R_1} = -n_1\left\{\frac{(1-u)\alpha_0^1 + u\alpha_1^1}{(1-u)(1-\alpha_0^0) + u(\frac{2}{3} - \alpha_1^0)}\right\} \tag{2-8}$$

which, for uniformly bright disks ($u = 0$) would reduce further to

$$\frac{\delta V}{\omega_1 R_1} = -\frac{n_1\alpha_0^1}{1-\alpha_0^0}, \tag{2-9}$$

where the α_n^m's stand for the respective associated α-functions which have been explicitly evaluated in section IV.4. Moreover, if the axis of rotation of the star undergoing eclipse is *not* perpendicular to the plane of the orbit, but inclined to the direction of its normal by an angle β, an appeal to the full-dress expression (1-13) for V' reveals that, in this more general case,

$$\delta V = -R_1\{m_1\omega_y + n_1\omega_z\}\left\{\frac{(1-u)\alpha_0^1 + u\alpha_1^1}{(1-u)(1-\alpha_0^0) + u(\frac{2}{3}-\alpha_1^0)}\right\}, \quad (2\text{-}10)$$

where the direction cosines m_1 and n_1 are defined by equations (3-3) of Chapter IV, and where the angular velocities ω_y, ω_z are given by (1-16).

The quantity $\delta V/\omega_1 R_1$ on the left-hand sides of the foregoing equations is customarily referred to in the astronomical literature as the *rotation factor.* For a given orientation of the axis of rotation this factor represents a single-valued function of the phase which is readily expressible in terms of known quantities. It is symmetrical with respect to the moments of conjunctions if the axis is perpendicular to the orbital plane, vanishing at mid-minima due to partial eclipses, and becoming equal to $\pm\sin i$ at the inner contacts of a total eclipse; but any inclination of the axis of rotation to the orbital plane will cause this symmetry to disappear. Moreover—and this is significant—*the asymmetry of the rotational effect arising from this source should vanish and reappear periodically in the opposite direction as the axis of rotation precesses* in a period investigated in section II.5. Such variation of the asymmetry of the rotational effect within minima of close eclipsing systems should constitute a third independent proof of the precession of the axes of rotation, supplementary to the dynamical proof based on the oscillation of the apparent period of the orbit (sec. II.7), and a photometric proof based on the oscillating asymmetry of the light changes exhibited between minima (sec. IV.2).

If numerical values of the rotation factors are evaluated in terms of known elements of the eclipse at any phase with the aid of the equations established in this section, the observed departures δV of radial velocity from that resulting from the orbital motion should lend themselves for a determination of the velocity $\omega_1 R_1$ at the equator of the component undergoing eclipse, or—if ω_1 is identified with (say) the Keplerian angular velocity of revolution of the system—of its mean radius R_1 in absolute units. In applying such a procedure to practical cases we should, however, bear in mind that the star undergoing eclipse need not necessarily have to rotate like a rigid body (as was tacitly implied throughout all our analysis in this section); and if it does not, the equatorial velocities obtained from our present theory should represent only certain means of actual angular velocities of the primary component at different astrocentric latitudes, which may either exceed, or be smaller than, the actual velocity on the equator—depending on the circumstances of the eclipse. On the other hand, provided that the law of variation of the angular velocity with latitude were known from the theory of stellar structure, it could be readily incorporated in the derivation of our equation (1-13) for V'*; or, conversely, accurate spectroscopic observations of δV

* This has, for instance, been done by Y. Hosokawa (*Publ. Astr. Soc. Japan*, **5**, 88, 1953) for Faye's law making ω_1 diminish towards the poles with the square of the sine of the latitude.

might enable us to obtain at least some empirical information concerning the variation of angular velocity with latitude for stars other than the Sun.

V.3. EFFECTS OF REFLECTION ON RADIAL VELOCITY

Tidal distortion of rotating stars represents only one way in which the centre of light on apparent stellar disks can be made to deviate from the projected centre of mass by an amount varying with the phase, and thus give rise to a significant contribution to the observed radial velocity which is independent of, and supplementary to, that produced by orbital motion. The reflection effect, consisting of the light of each component incident on its mate and re-radiated (or scattered) by it in the direction of the line of sight, represents another way in which axial rotation of the components in close binary systems is bound to affect their observed radial velocity even if both components were spherical in form. The photometric consequences of this reflection effect have already been discussed in an earlier section IV.6 of this book; and the aim of the present section will be to explore its bearing on the observed radial velocity.

With the fundamentals of our method of investigation already laid down in the preceding sections, our present task will become relatively simple. The non-orbital contribution δV_r to the radial velocity arising from reflection will again be given by a general formula of the form

$$\delta V_r = \frac{\iint V' dl^*}{L_1 + \mathscr{L}_1}, \tag{3-1}$$

analogous to (1-18), in which dl^* represents, however, the light element of the reflected radiation, and the limits of integration in the numerator being now extended over the illuminated crescent visible at any particular phase. The denominator on the right-hand side of (2-1) consists of the total light of the primary (reflecting) component—proper (L_1) plus reflected (L_1^*).

The radial velocity V' at any point on the surface of the rotating star continues to be given by equation (1-12), while the form of the light element dl^* of the reflected radiation has already been investigated previously in section IV.6. In that section we found, in particular, that as long as terms of the fourth and higher degree in powers of the fractional radius r_1 of the reflecting component are ignorable, the illuminating component can be regarded as a light point, and the projection of the terminator of the illuminated hemisphere becomes a great circle. Under these conditions,

$$\iint V' dl^* = \left\{ \int_{-1}^{1} \int_{0}^{\sqrt{1-y'^2}} - \int_{-1}^{1} \int_{0}^{-l_0\sqrt{1-y'^2}} \right\} V' \mathscr{I} r_1^2 \, dx' \, dy' \tag{3-2}$$

where, in accordance with equation (6-60) of Chapter IV,

$$\pi\mathscr{I} = \{r_1^2 P_1(\lambda) + 2r_1^3 P_2(\lambda) + \ldots\}L_2 \tag{3-3}$$

and

$$\lambda = l_0 z' + l_2 x'; \tag{3-4}$$

the primed rectangular coordinates x', y', z' being identical with those used in the preceding sections. An evaluation of the foregoing integrals offers no difficulty. If, for the sake of simplicity, the axis of rotation of the reflecting component is regarded as perpendicular to the orbital plane*, it readily follows that

$$\frac{\delta V_r}{\omega_1 R_1} = \frac{L_2}{L_1 + \mathscr{L}_1}\left\{\tfrac{1}{8}r_1^2 m_0(1 + l_0) + \frac{2r_1^3}{15\pi} m_0[l_2 + 6l_0 \cos^{-1}(-l_0)] + \ldots\right\}, \tag{3-5}$$

where, in accordance with equation (6-69) of Chapter IV,

$$L_1 = \left\{\frac{2r_1^2}{3\pi}[(\pi - \cos^{-1} l_0)l_0 + l_2] + \frac{r_1^3}{8}[3l_0^2 + 2l_0 - 1] + \ldots\right\}L_2. \tag{3-6}$$

The non-orbital contribution to the radial velocity of the *secondary* component, due to light incident on it from the primary, can also be obtained from the foregoing equations if indices are interchanged and phase shifted by 180°.

If, however, the spectral lines of the reflecting component are to remain visible in the combined spectrum of the system, L_2 cannot be large in comparison with L_1. Hence, $\mathscr{L}_1$ must be small when compared with L_1; and as δV_r itself is a small quantity of the order of r_1^2, it follows that $\mathscr{L}_1$ should be negligible in the denominator as far as terms of the order of r_1^2 and r_1^3 in δV are concerned. If, on the other hand, L_2 were to become *large* in comparison with L_1, the light reflected from the primary star may become comparable with, or even greater than, its own light. The neglect of $\mathscr{L}_1$ in the denominator of δV_r would then be inadmissible: in fact, if the reflecting star had no light of its own, the reflected light might represent its entire luminosity. In such a case the denominator on the right-hand side of (3-5) would consist solely of $\mathscr{L}_1$; and an insertion for it from (3-6) would lead to

$$\frac{\delta V_r}{\omega_1 R_1} = \frac{3}{16}\frac{\pi m_0(1 + l_0)}{(\pi - \cos^{-1} l_0)l_0 + l_2} + \ldots, \tag{3-7}$$

correctly to a first approximation. The radial-velocity changes arising from axial rotation of a star shining only with reflected light are independent of intensity of the illuminating source, and represent the *maximum possible* effect which the reflection can exert. The spectrum of the reflected light should be characterized by its close resemblance to that of the bright star;

* If this were not the case, the term $\omega_1 m_0$ in equation (3-5) would have to be replaced by $\omega_z m_0 - \omega_y n_0$ for arbitrary inclination.

in fact, it should probably constitute its faithful replica in all observed features—except in intensity which should be small and should, moreover, undergo periodic fluctuation according to the phase of the reflecting star. The velocity curves derived from measured Doppler displacements of lines of the 'reflected' spectrum would evidently possess a peculiar character and, since δV_r as defined by equation (3-7) represents no longer a small quantity, would deviate widely from velocity curves controlled by orbital motion. A failure to consider the effects of reflection in a solution for orbital elements would, in such cases, be bound to lead to quite incorrect results.

V.4. EFFECTS OF DISTORTION AND REFLECTION ON THE ELEMENTS OF SPECTROSCOPIC BINARIES

In the foregoing two sections we have investigated the radial-velocity components of axial rotation caused by distortion of the components in close binary systems as well as mutual reflection of light between them, and established the magnitude of the principal terms of such velocities—between minima as well as within eclipses—in terms of known geometrical elements of the system. The radial-velocity changes δV or δV_r invoked by axial rotation will naturally superpose upon the velocities V of the respective stars due to their orbital motion around the centre of gravity of the system, and only their resultant sum $V + \delta V + \delta V_r$ becomes directly observable. The present section will, therefore, be concerned with a converse problem: namely, to investigate the significance of the rotational terms δV and δV_r, due to distortion and reflection, in observed radial velocities, and their effects on the determination of orbital elements of close spectroscopic binaries from a standard analysis of their observed changes of radial velocities.

In order to go about this task, the best way to proceed should be to compare the theoretical expressions δV and $\delta V'$, as established in the foregoing sections, with the radial velocity V arising from the orbital motion of the primary component around the centre of mass of the binary system. As is well known, this latter velocity (proportional to a change of the distance of the centre of mass of the star from a plane perpendicular to the line of sight) can generally be expressed as

$$V = \gamma + K_1(e \cos \omega + \cos u), \tag{4-1}$$

where γ denotes the radial velocity of the centre of mass of the system; K_1, the amplitude of the primary's orbital velocity curve; e, the eccentricity of the orbit; $u = \omega + v$, the true anomaly of the primary's centre of mass; and ω, the longitude of periastron—both measured from the direction of the ascending node. If—as is true of a large majority of close binary systems—the orbital eccentricity e is small, u may be conveniently expanded in terms of the mean anomaly M in a series whose first term, $\omega + M$, specifies the mean longitude which we shall hereafter denote by L. To the first order in e, equation (4-1) may then be expanded as

$$\left.\begin{aligned} V = \gamma + K_1 \cos L + K_1\, e \sin \omega \sin 2L \\ + K_1\, e \cos \omega \cos 2L + \dots \end{aligned}\right\} \tag{4-2}$$

and, to the same order of accuracy,

$$K_1 = a_1 n \sin i = \frac{m_2}{m_1 + m_2} An \sin i\,, \tag{4-3}$$

where $m_{1,2}$ denote the masses of the respective components; a_1, the semi-major axis of the absolute orbit of the primary star; A, the semi-major axis of the relative orbit of both components; and n, the mean daily motion.

Throughout sections 1–3 of this chapter the primary component was supposed to be at rest at the origin of our coordinate systems, and the angle ψ in the direction cosines l_0 and m_0 of the line of sight stood for the true anomaly of the secondary component reckoned from inferior conjunction. For the sake of comparison with spectroscopic orbits in which the primary is supposed to revolve, we now have to put $\psi = u - 90°$, in consequence of which $L = M + 90°$. Then

$$\left.\begin{aligned} l_0 &= \sin L \sin i\,, \\ m_0 &= -\cos L \sin i\,, \\ n_0 &= \cos i\,, \end{aligned}\right\} \tag{4-4}$$

and, as the reader may easily verify

$$m_0 P_2'(l_0) = -\tfrac{1}{2} P_2^2(n_0) \sin 2L\,, \tag{4-5}$$

$$m_0 P_3'(l_0) = -\tfrac{1}{4} P_3^1(n_0) \cos L - \tfrac{1}{8} P_3^3(n_0) \cos 3L\,, \tag{4-6}$$

$$m_0 P_4'(l_0) = +\tfrac{1}{12} P_4^2(n_0) \sin 2L + \tfrac{1}{48} P_4^4(n_0) \sin 4L\,, \tag{4-7}$$

and similarly

$$m_0(1 + l_0) = -\sin i \cos L - \tfrac{1}{2} \sin^2 i \sin 2L\,. \tag{4-8}$$

In slightly eccentric orbits we may also assume that the angular velocity ω_1 of axial rotation of the primary component bears a ratio q to the mean daily motion n, which is approximately constant and becomes equal to one if the star rotates with the Keplerian angular velocity. Since, furthermore, $R_1/A = r_1$ by definition, the non-orbital contribution δV, of tidal origin, to the observed radial velocity of the primary component, as represented by equation (1-22) and rewritten in terms of the mean longitude L with the aid of the foregoing equations (4-3) and (4-5)–(4-7), assumes eventually the form

$$\begin{aligned} \delta V = \tfrac{1}{48}\{&12 f_3^{(k)} w_1^{(3)} P_3^1(n_0) \cos L \\ &+ 4[6 f_2^{(k)} w_1^{(2)} P_2^2(n_0) - f_4^{(k)} w_1^{(4)} P_4^2(n_0)] \sin 2L \\ &+ 6 f_3^{(k)} w_1^{(3)} P_3^3(n_0) \cos 3L \\ &- f_4^{(k)} w_1^{(4)} P_4^4(n_0) \sin 4L + \dots\} \left(1 + \frac{m_1}{m_2}\right) \frac{q r_1}{\sin i} K_1\,, \end{aligned} \tag{4-9}$$

and equation (3-5) similarly yields

$$\delta V_r = -\{\cos L + \tfrac{1}{2}\sin i \sin 2L + \ldots\} \frac{L_2}{L_1}\left(1 + \frac{m_1}{m_2}\right) q r_1^3 K_1 . \qquad (4\text{-}10)$$

Equation (3-7) expanded in the same manner would turn out to be considerably more complex; but since it is valid only in a very special case we shall not discuss it in the same detail.

Now the observed radial velocity $\bar{V}$ of the primary component of our binary component will evidently be equal to the sum

$$\bar{V} = V + \delta V + \delta V_r , \qquad (4\text{-}11)$$

with the individual constituents as given by equations (4-2), (4-9), and (4-10). In order to exhibit the role of the additive terms δV and δV_r in a determination of the elements of spectroscopic orbits, let us assume—for the sake of simplicity—that the primary's orbit around the centre of gravity of the binary system is actually circular, so that $V = \gamma + K_1 \cos L$. The amplitude $\bar{K}_1$ of the observed velocity curve as deduced from a harmonic analysis of $\bar{V}$ will, however, be not K_1, but

$$\bar{K}_1 = \left\{1 + \tfrac{1}{4} f_3^{(k)} \frac{P_3'(n_0)}{\sqrt{1 - n_0^2}} \left(1 + \frac{m_1}{m_2}\right) q r_1 w_1^{(3)} - \frac{1}{8}\frac{L_2}{L_1}\left(1 + \frac{m_1}{m_2}\right) q r_1^3 + \ldots\right\} K_1 . \qquad (4\text{-}12)$$

The third-harmonic tidal distortion of components in close binary systems will accordingly tend to increase or diminish the true amplitudes of their radial-velocity variation (and thus their masses and absolute dimensions based upon them) depending on whether or not the term $f_3^{(k)} P_3^1(n_0)$ is positive or negative.

Equation (1-24) makes it evident that (barring exceptional conditions) the coefficient $f_3^{(k)}$ is essentially positive; and the same is true of $P_3'(n_0)$ if $n_0 < \frac{1}{5}$. Therefore, for high values of orbital inclination—as are common among ordinary eclipsing systems—the true amplitudes of radial-velocity changes should be *smaller* than those observed; while for relatively low inclinations the converse should be true. For the limiting value of $i = \cos^{-1}(\frac{1}{5}) = 63°435 \ldots$ the first-order effect of tidal distortion upon K_1 should disappear. On the other hand, the reflected radiation produces a non-vanishing term on the right-hand side of (4-12) which is always negative and thus tends to render $\bar{K}_1$ *less* than K_1—i.e., to offset the spurious increase of $\bar{K}_1$ caused by third-harmonic tidal distortion for highly inclined orbits. Which of the two effects may predominate depends on particular properties of each system and must be ascertained by individual analysis. It should, however, be clear that such an analysis must be undertaken, and the observed amplitudes $\bar{K}_1$ of radial-velocity changes reduced to their true values K_1, before it is legitimate to use them for a derivation of the masses and absolute dimensions of the respective components in the customary manner.

Turning now to the terms varying as $\sin 2L$ on the right-hand sides of equations (4-9) or (4-10) and comparing them with the series on the right-hand side of (4-2) it is immediately seen that the departures from simple-harmonic variation of radial velocity, caused by the distortion or reflection, would be interpreted by the orbit computer as indicating a slightly eccentric orbit, characterized by a spurious eccentricity

$$\bar{e} = \frac{1}{12}\left\{\left[f_2^{(k)}w_1^{(2)} - \frac{1}{4}\frac{L_2}{L_1}r_1^2\right]P_2^2(n_0) - f_4^{(k)}w_1^{(4)}P_4^2(n_0) + \ldots\right\}\left(1 + \frac{m_1}{m_2}\right)\frac{qr_1}{\sin i} \tag{4-13}$$

and a longitude of periastron

$$\bar{\omega} = 90^\circ \text{ or } 270^\circ, \tag{4-14}$$

depending on whether the foregoing expression (4-13) for $\bar{e}$ proves to be positive or negative. It should, however, be stressed that the spurious eccentricity of tidal (or reflection) origin may be regarded as descriptive of the distorted velocity curve only to a first approximation; for when squares and higher powers of orbital eccentricity are taken into account in the expansion on the right-hand side of (4-2), a correspondence between these and the higher harmonic terms in (4-9) or (4-10) ceases to hold good. The absence, from (4-9), of terms varying as cosines of even multiples of the mean longitude (or of sines of its odd multiples) proves that any deformation of radial-velocity curves of tidally distorted components is bound to remain symmetrical with respect to the line of sight. The spurious longitudes of periastra of such orbits would therefore remain 90° or 270°; but the amounts of spurious eccentricity as derived from higher individual terms of tidal distortion should generally differ. This state of affairs would, moreover, be further complicated by the reflection effect.

In dealing with two-spectra close binaries it should also be borne in mind that the spectroscopic values of $\bar{e}$ and $\bar{\omega}$ as deduced from the observed radial velocities of their components without regard to the ellipticity or reflection should not in general be identical. In particular, if the real orbits in space are circular and the perturbations of radial-velocity curves are due wholly to axial rotation of distorted components reflecting the light of each other, the spurious $\bar{\omega}$'s of their absolute orbits should, in general, differ by 180°.

If, on the other hand, the real orbits of the components cease to be circular, the spurious e's arising from ellipticity or reflection should be expected to combine with the real orbital e to yield a resultant which may be either larger or smaller than the real e. The general situation then becomes too complex to be analyzed in simple terms. In order to avoid such complications, *the theoretical contributions δV and δV_r, to radial velocities, arising from both the ellipticity and reflection should be taken out from measured velocities of the components of close binary systems before a conventional* *solution for orbital elements is actually carried out.* In systems which happen to be also eclipsing variables, all quantities constituting the δV's may be obtained from a prior analysis of their light curves; while in non-eclipsing systems the magnitude of certain factors must be estimated as best as we can. But it is essential that such estimations be made, and that the observed radial velocities $\bar{V}$ be freed from effects of non-orbital origin; for only then elements of spectroscopic orbits of close binaries may be obtained which are reasonably free from any obvious source of systematic errors.

V.5. LINE PROFILES OF ROTATING STARS WITHIN ECLIPSES

In the preceding sections 1–3 of this chapter we have investigated the radial-velocity changes produced by distortion and reflection of rotating components of close binary systems between minima as well as within eclipses—changes which superpose naturally upon those arising from orbital motion, and whose resultants are measurable as Doppler shifts of the respective spectral lines. The Doppler shifts corresponding to the non-orbital velocities δV investigated earlier in this chapter refer, however, by their definitions (1-18) or (3-1), to those of monochromatic lines of zero intrinsic width—or, in practical cases, to the *cores* of spectral lines which may be broadened by other physical causes (collisions, Stark effect, etc.) or for instrumental reasons (finite width of the slit). Even in the absence of such effects, however, the line profiles in the spectra of rotating stars are bound to be intrinsically broadened, because different surface points on their apparent disks are moving at different speeds relative to the observer on account of their axial rotation—the total width of a rotationally broadened profile of each line being determined by the maximum difference in radial velocity at opposite limbs of the star.

This phenomenon is bound to be characteristic of all stars—whether single or double—and the rotational broadening due to this cause should vanish only if the axis of rotation happens to be parallel with the line of sight (in which case the radial-velocity component at any surface point becomes identically zero). If stars rotating about an inclined axis are spherical, their rotation will broaden merely each line of their spectra into a profile which is symmetrical with respect to its position characterized by the velocity of its centre of mass. If, however, this star happens to be the component of a close binary system, such that the expressions δV investigated in sections V.1 to V.3 appropriate for it are not identically zero, the intrinsic profile of each line becomes not only *broadened*, but also *asymmetric*; and for stars undergoing eclipse this will be true even if they were spherical (rotational effect). If the results established in sections V.1 to V.3 have enabled us to ascertain the positions of the centre of gravity (maximum absorption or emission) of each line at any phase of the orbital cycle, the aim of the present section will be to provide a mathematical description of the distribution of intensity inside rotationally broadened lines in the spectra of the components of close binary systems.

In order to do so, consider a rectangular system of coordinates xyz (*cf.*, Fig. 5-1), with origin at the centre of the star undergoing eclipse (and referred to hereafter again as the primary component); defined so that the xy-plane

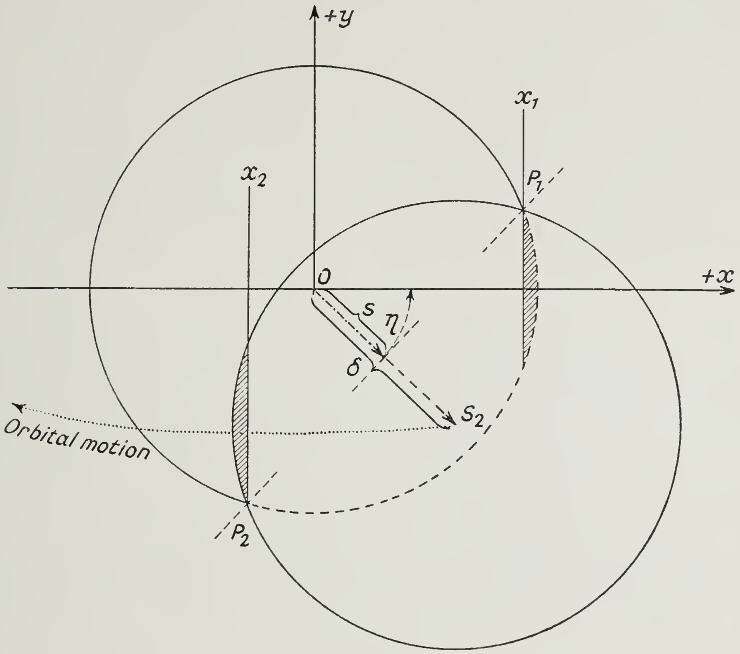

FIGURE 5-1

is tangent to the celestial sphere at the origin of coordinates, and the y-axis coincides with the projection of the axis of rotation of the primary star, while the z-axis coincides with the line of sight. Let, furthermore, the primary component—regarded as a sphere of radius R_1—rotate like a rigid body with a constant angular velocity ω_1; and let its axis of rotation make an angle of $90°-i$ with the y-axis in space. The distribution of surface brightness $J(x, y)$ at any point of the primary's apparent disk will be assumed to be radially-symmetrical and to obey the linear law of limb-darkening of the form

$$J(x, y) = H(1 - u + u \cos \gamma), \tag{5-1}$$

where H denotes the intensity of radiation (now regarded as constant) emerging normally to the surface; u, the coefficient of limb-darkening; and γ, the angle of foreshortening now defined by

$$\cos \gamma = \left\{1 - \frac{x^2 + y^2}{R_1^2}\right\}^{1/2}. \tag{5-2}$$

The radial velocity-component V', in the z-direction, of any surface point of the primary component will, for rigid-body rotation, be proportional to that point's distance from the projected axis of rotation and expressible, therefore, as

$$V' = \pm x\omega_1 \sin i\,; \tag{5-3}$$

the negative sign being true if the rotation is direct, and positive if it were retrograde. The corresponding Doppler shift $\Delta\lambda$ for a line of wave length λ will be given by

$$\Delta\lambda = (\lambda/c)V' = \pm(\lambda/c)x\omega_1 \sin i\,, \tag{5-4}$$

where c stands for the velocity of light. Since x in the foregoing equations ranges between $\pm R_1$, the maximum displacement of light of any wave length originating at opposite limbs of the primary stars will clearly be $\pm(\lambda/c)R_1\omega_1 \sin i$; and the total width $w(\lambda)$ of the intrinsically monochromatic spectral line of wave length λ should then be

$$w(\lambda) = 2(\lambda/c)R_1\omega_1 \sin i\,; \tag{5-5}$$

while the distribution of intensity *within* this width should be proportional to the relative brightness of each particular strip of the apparent stellar disk between x and $x + dx$, along which the radial velocity V' of all points is the same and given by equation (5-3).

The form $A(x)$ of the profile of an intrinsically monochromatic line broadened by stellar rotation will thus be found if we establish the fractional brightness of each such strip expressed in terms of the total brightness (in that wave length) of the primary's visible disk. In mathematical language, then,

$$A(x) = \frac{\int J(x, y)\, dy}{\iint J(x, y)\, dx\, dy}\,, \tag{5-6}$$

where the limits of integration in the numerator are to be extended over each x-strip of the entire disk or, within eclipses, of its visible portion; while the limits in the denominator should be extended over the entire visible disk (or crescent).

Outside eclipses—if the whole disk of the primary component remains visible to the observer, the requisite integrations become trivially simple: namely, the limits with respect to y become $\pm(R_1^2 - x^2)^{1/2}$ in both the numerator and the denominator on the right-hand side of the foregoing equation (5-6), while the x-limits in the denominator are $\pm R_1$.
As a result,

$$\iint J(x, y)\, dx\, dy = \pi R_1^2 H(1 - \tfrac{1}{3}u)\,, \tag{5-7}$$

while

$$\int J(x, y)\, dy = 2(1 - u)HR_1\sqrt{R_1^2 - x^2} + \tfrac{1}{2}\pi uH(R_1^2 - x^2); \tag{5-8}$$

so that, during full light,

$$\tilde{A}(x) = \frac{3(1-u)}{3-u}\frac{2}{\pi}\left(1-\frac{x^2}{R_1^2}\right)^{1/2} + \frac{3u}{3-u}\frac{1}{2}\left(1-\frac{x^2}{R_1^2}\right) \tag{5-9}$$

where, by (5-4),

$$\frac{x}{R_1} = \frac{c}{R_1\omega_1 \sin i}\left(\mp\frac{\Delta\lambda}{\lambda}\right). \tag{5-10}$$

Accordingly, the Doppler-broadened lines of rotating stars out of eclipse should be symmetrical with respect to the core of the line, and the form of their profiles should depend upon the coefficient of limb-darkening of the respective star in a manner indicated by equation (5-9).

In order to perform the requisite integrations when the primary component is partially eclipsed by its mate, our first task must be to specify the limits. For this purpose consider again Fig. 5-1 on which the secondary (eclipsing) component—hereafter regarded also as a sphere of radius R_2—appears in projection on the xy-plane as a circle with the centre at S_2. Let the radius-vector of the relative orbit (in what follows considered as circular) of the two stars be taken as our unit of length, and its projection on the xy-plane be denoted, as usual, by δ. Let, moreover, ψ stand for the mean anomaly (phase angle) of the secondary component in its relative orbit whose plane coincides with the equator of the rotating primary star, reckoned from the moment of inferior conjunction. If so, then

$$\delta^2 = \sin^2\psi \sin^2 i + \cos^2 i, \tag{5-11}$$

while the angle η between the projected radius-vector and the x-axis will be specified by

$$\left.\begin{aligned} \delta \sin\eta &= \cos\eta \sin i, \\ \delta\cos\eta &= -\sin\eta. \end{aligned}\right\} \tag{5-12}$$

In extending—as we should—the integration over the whole area of the primary's disk visible at any particular phase, we find it convenient to integrate first over the entire disk, then over the eclipsed area, and subtract the latter part from the former. The advantage of so doing rests with the fact that the results of the former integration are already known and represented by equation (5-9). Suppose, moreover, that we denote by

$$\tilde{A}'(x; k, p) = \frac{3(1-u)}{3-u} I^{\mathrm{U}}(x; k, p) + \frac{3u}{3-u} I^{\mathrm{D}}(x; k, p) \tag{5-13}$$

the right-hand side of equation (5-6), with the corresponding integrals evaluated over the eclipsed fraction of the primary's disk, which should depend on the ratio of the radii $k = R_1/R_2$ of the two components and on the geometrical depth $p = (\delta - r_1)/r_2$, where $r_{1,2}$ denote the fractional radii of the two stars expressed in terms of their separation. The effective profiles, during eclipses, will then be given by

$$A(x; k, p) = \tilde{A}(x) - \tilde{A}'(x; k, p), \tag{5-14}$$

and it is the $\tilde{A}'$-term which will be responsible for their asymmetry.

A proper formulation of the asymmetry-generating terms $I^{\mathbf{U},\mathbf{D}}(x;\ k,\ p)$ as defined by equation (5-13) requires, however, some care. In order to set it up, let us return to Fig. 5-1 and observe that the circles representing apparent disks of the two components intersect at points $P_{1,2}$ whose positions are specified by the coordinates

$$x_{1,2} = \quad s \cos \eta \pm \sqrt{r_1^2 - s^2} \sin \eta, \tag{5-15}$$

$$y_{1,2} = -s \sin \eta \pm \sqrt{r_1^2 - s^2} \cos \eta, \tag{5-16}$$

where (consistent with 3-17) we have abbreviated

$$2\delta s = r_1^2 - r_2^2 + \delta^2. \tag{5-17}$$

The lines $x_{1,2} =$ constant divide the eclipsed portion of the primary's disk in three parts characterized by the following limits:

Range in x	Limits in y
$r_1 \geqslant x \geqslant x_1$:	$\pm\sqrt{r_1^2 - x^2}$,
$x_1 \geqslant x \geqslant x_2$:	$y_0 + \sqrt{r_2^2 - (x - x_0)^2} - \sqrt{r_1^2 - x^2}$,
$x_2 \geqslant x \geqslant x_0 - r_2$:	$y_0 \pm \sqrt{r_2^2 - (x - x_0)^2}$,

where

$$x_0 = \delta \cos \eta, \qquad y_0 = -\delta \sin \eta, \tag{5-18}$$

are coordinates of projected centre S_2 of the secondary (eclipsing) component in the xy-plane.

Different limits in y as summarized in the foregoing tabulation will cause the functions $I^{\mathbf{U},\mathbf{D}}(x;\ k,\ p)$ to assume different explicit forms within each range. If we abbreviate them in accordance with the following scheme

Range	$I^{\mathbf{U},\mathbf{D}}(x; k, p)$
$r_1 \geqslant x \geqslant x_1$:	$I_1^{\mathbf{U},\mathbf{D}}(x)$
$x_1 \geqslant x \geqslant x_2$:	$I_2^{\mathbf{U},\mathbf{D}}(x)$
$x_2 \geqslant x \geqslant x_0 - r_2$	$I_3^{\mathbf{U},\mathbf{D}}(x)$

we find that

$$\pi r_1 I_1^{\mathbf{U}}(x) = 2\sqrt{r_1^2 - x^2}, \tag{5-19}$$

$$\pi r_1 I_2^{\mathbf{U}}(x) = y_0 + \sqrt{r_1^2 - x^2} + \sqrt{r_2^2 - (x - x_0)^2}, \tag{5-20}$$

$$\pi r_1 I_3^{\mathbf{U}}(x) = 2\sqrt{r_2^2 - (x - x_0)^2}; \tag{5-21}$$

and

$$\pi r_1^2 I_1^{\mathbf{D}}(x) = \frac{\pi}{2}(r_1^2 - x^2), \tag{5-22}$$

$$\begin{aligned}\pi r_1^2 I_2^{\mathbf{D}}(x) = {} & \tfrac{1}{2}a_1\{y_0 + \sqrt{r_2^2 - (x - x_0)^2}\} \\ & + \tfrac{1}{2}(r_1^2 - x^2)\cos^{-1}\frac{-y_0 - \sqrt{r_2^2 - (x - x_0)^2}}{r_1^2 - x^2},\end{aligned} \tag{5-23}$$

$$\begin{aligned}\pi r_1^2 I_3^{\mathbf{D}}(x) = {} & \tfrac{1}{2}(a_1 + a_2)\sqrt{r_2^2 - (x - x_0)^2} + \tfrac{1}{2}(a_1 - a_2)\, y_0 \\ & + \tfrac{1}{2}(r_1^2 - x^2)\sin^{-1}\frac{a_1 + a_2\sqrt{r_2^2 - (x - x_0)^2} - (a_1 - a_2)y_0}{r_1^2 - x^2},\end{aligned} \tag{5-24}$$

where we have abbreviated

$$a_{1,2} = \sqrt{2}\,\{x_0(x_0 - x) \mp y_0\sqrt{r_2^2 - (x - x_0)^2} - \delta(\delta - s)\}^{1/2}. \tag{5-25}$$

Should the y-coordinate of point P_1 become negative—i.e., should

$$\sqrt{r_1^2 - s^2}\cos\eta < s\sin\eta, \tag{5-26}$$

then

$$I_1^{\mathbf{U},\mathbf{D}}(x) = 0; \tag{5-27}$$

and similarly should the tangent at P_2 to the circle of radius R_2 circumscribed around S_2 make an acute angle with the positive direction of the x-axis—i.e., should

$$\sqrt{r_1^2 - s^2}\cos\eta < (\delta - s)\sin\eta, \tag{5-28}$$

then

$$I_3^{\mathbf{U},\mathbf{D}}(x) = 0; \tag{5-29}$$

$I_2^{\mathbf{U},\mathbf{D}}(x)$ remaining then the only asymmetry-generating terms which are not identically zero. It should also be added that the validity of the equations (5-17)–(5-25) extends only for *partial* phases of the eclipse. If this eclipse becomes *annular* $I_{1,2}^{\mathbf{U},\mathbf{D}}(x) = 0$ identically; while the x-limits within which $I_3^{\mathbf{U},\mathbf{D}}(x)$ as defined by equations (5-21) and (5-24) holds good become $x_0 + r_2 \geqslant x \geqslant x_0 - r_2$.

The entire foregoing analysis has been based on the assumption that the two components of our eclipsing system are so far apart that departures from spherical form due to their rotational distortion and mutual tidal action may be ignored. As long as this holds true, the loci of equal radial velocity on the apparent disks of the two stars coincide with the lines $x =$ constant, and the problem can be treated in relatively simple terms. If, however, the distortion becomes appreciable this ceases to be true: the loci of equal velocity become *curves* of high degree in x and y, and the integrations defining the relative intensity of strips characterized by $V'(x, y) =$ constant become correspondingly more complicated. This statement admits, however, of one exception: namely, when a tidally distorted star is seen at quadratures (i.e., when the radius-vector is at right angles to the line of sight), the loci of equal velocity become again straight lines $x =$ constant.

The corresponding expressions $A(x)$ defining the form of line profiles can be written down almost at once. For let, as in our earlier work, the tidal distortion of the primary component be expressible (to the first order in small quantities) as

$$\frac{R_1 - R_0}{R_0} = \sum_{j=2}^{4} w_1^{(j)} P_j(x), \tag{5-30}$$

where R_0 stands for the radius of a sphere of the same volume as the distorted star; $P_j(x)$, the respective Legendre polynomial; and the constants $w_1^{(j)}$ continue to be given by equation (2-33) of Chapter IV. The radial velocity $V'(x)$ at any point of the visible surface will then be given by

$$V'(x) = \omega_1 R_0 \left\{ 1 + \sum_{j=2}^{4} w^{(j)} P_j(x) \right\} x, \quad (-1 \leqslant x \leqslant 1), \tag{5-31}$$

attaining the extreme values of

$$V(\pm 1) = \pm \omega_1 R_0 \{ 1 + w_1^{(2)} + w_1^{(4)} \} \tag{5-32}$$

at opposite ends of a tidally-elongated star.

Within these limits, the theoretical line profiles of distorted stars will be governed by

$$A(x) = \frac{\int R_1^2 \, J(x, y) \, dy}{\int\int R_1^2 \, J(x, y) \, dx \, dy}, \tag{5-33}$$

with limits extended over the visible disk, where the product of the distribution of brightness $J(x, y)$—which, for distorted stars, is governed by both the limb- and gravity-darkening—and R_1 as given by equation (5-30) can be shown to assume the form

$$R_1^2 \, J(x, y) = R_0^2 H \{ (1 - u)(1 - Y^{\mathbf{U}}) + u(1 - Y^{\mathbf{D}}) \sqrt{1 - x^2 - y^2} \}, \tag{5-34}$$

where, to the order of accuracy we have been working,

$$Y^{\mathbf{U}}(x) = \sum_{j=2}^{4} \{ (\beta_j - 2) P_j(x) - x P_j'(x) \} w^{(j)} \tag{5-35}$$

and

$$Y^{\mathbf{D}}(x) = \sum_{j=2}^{4} \{ (\beta_j - 2) P_j(x) - 2x P_j'(x) \} w^{(j)}, \tag{5-36}$$

respectively, and the constants β_j continue to be given by equation (2-31) of Chapter IV.

An integral $P(x)$ of the product $R_1^2 J(x, y)$ with respect to y between $\pm\sqrt{1 - x^2}$ and expressed in terms of the brightness $\pi R_0^2 H(1 - \frac{1}{3}u)$ of the undistorted star then assumes the form

$$P(x) = \frac{2}{\pi}\frac{3(1-u)}{3-u}\{1 - Y^{\mathsf{U}}(x)\}\sqrt{1-x^2} + \frac{1}{2}\frac{3u}{3-u}\{1 - Y^{\mathsf{D}}(x)\}(1-x^2), \tag{5-37}$$

and represents a generalization of (5-9) to our particular case; while the luminosity of the entire disk of a tidally-distorted component exposed to us at quadratures (and obtained by integrating the foregoing expression with respect to x between ± 1) becomes

$$\begin{aligned} Q = 1 + \frac{3(1-u)}{3-u}\{\tfrac{1}{2}w^{(2)}(1+\tfrac{1}{4}\beta_2) + \tfrac{9}{32}w^{(4)}(1+\tfrac{1}{8}\beta_4)\} \\ + \frac{2u}{3-u}\{\tfrac{4}{5}w^{(2)}(1+\tfrac{1}{4}\beta_2)\} \end{aligned} \tag{5-38}$$

in terms of the same unit, and to the same order of accuracy. The ratio $P(x) \div Q$ then defined the line profile (5-33) of a tidally distorted star at quadratures—at a time when the asymmetry of such a profile due, not to eclipses, but to tidal elongation of the primary component in the direction of the radius-vector—attains its maximum.

In concluding the present section it should again be stressed that all functions $A(x)$ established above represent the rotational broadening of profiles of intrinsically monochromatic lines. Should the natural width $W(x)$ of the line (before it is affected by rotation) be comparable with its value (5-5) due to Doppler broadening, the effective profile $S(x)$ of such a line is known* to be given by the equation

$$S(x) = \int_{-\infty}^{\infty} W(x-\xi)A(\xi)\,d\xi, \tag{5-39}$$

where the form of the function $A(\xi)$ in the integrand—between minima as well as within eclipses—has been investigated in the present section, and that of $W(\xi)$ must, in general, be deduced from the theory of line formation in stellar atmospheres. The actual evaluation of the integral on the right-hand side of equation (5-39) can, however, invariably proceed only by graphical or numerical integration, the details of which are wholly outside the scope of the present section.†

V.6. SURVEY OF THE RESULTS

While the preceding Chapter IV has been devoted to an investigation of theoretical light curves of close binary systems between minima as well as within eclipses, taking account of all phenomena which arise from mutual

* *Cf.*, e.g., A. Unsöld, *Physik der Sternatmosphären*, Berlin 1938; sec. 77.

† For the requisite methods *cf.*, e.g., W. Gyllenberg, *A.N.*, **269**, 52, 1939; or H. C. v. d. Hulst, *B.A.N.*, **10**, 75, 1946.

distortion of the components to the first order in small quantities, the aim of the present chapter has been to prove that axial rotation of such components can also cause their observed radial velocity to differ significantly from that of their centres of mass in the course of orbital motion.

In the introductory section V.1, general expressions are set up for the radial component of the velocity vector of any surface point of a star whose axis of rotation is arbitrarily inclined to the line of sight (equation 1-12), and integrated over the whole hemisphere visible at any particular phase. If the isophotae on their apparent disks were symmetrical with respect to the geometrical centre of the respective configuration (i.e., if the centre of light were to project itself constantly on the centre of mass), such integrals would be bound to vanish. For actual components of close binary systems, of the form investigated in section II.1 and distribution of surface brightness as described in section IV.1, these integrals do not reduce to zero, but to small quantities of the same order as the superficial distortion of the components. These correspond to a finite radial velocity δV of each star which is not of orbital origin, and whose nature and interpretation have occupied us through most of this chapter.

In Chapter IV we found that the light changes arising from axial rotation of distorted stars are expansible in *zonal* harmonics $P_j(\cos\varepsilon)$ of the angle ε between the radius-vector and the line of sight. In the first section of the present chapter it transpired that the corresponding radial-velocity δV of non-orbital origin is expansible in terms of the *tesseral* harmonics $P_j^1(\cos\varepsilon)$ of the same argument. Moreover, all non-zero terms in δV as given by equation (1-22) are due to the tidal action alone: the second- and fourth-harmonic distortions of tidal origin give rise to significant contributions to δV if there is at least some limb- or gravity-darkening; while the third harmonic would contribute to the radial velocity of a rotating distorted ellipsoid even if the latter were uniformly bright.

We may also note from equation (1-22) that, as far as the second-harmonic effects are concerned, limb- and gravity-darkening prove to be *antagonistic* in their contributions to δV; and for a certain relation between u and τ_0, as represented by equations (1-32) or (1-33), this contribution can be made to vanish. This signifies a geometrically interesting fact that, to every degree of limb-darkening there corresponds a certain amount of gravity-darkening which renders the isophotae on the apparent disk of the respective star constantly symmetrical about its centre. The curves representing such isophotae are not necessarily closed; in point of fact, they freely intersect the boundary of the respective disk. It is only if we impose complete limb-darkening ($u = 1$) and, in the light of (1-33), $\beta_2 = 1$ corresponding (for centrally-condensed configurations) to the coefficient of gravity-darkening $\tau_0 = 0{\cdot}25$, that our system of isophotae become closed ellipses concentric with the limb.

Between minima—if no eclipses occur—the non-orbital contributions δV to the radial velocity of a rotating distorted star thus prove to be small quantities of first order. Within eclipses—when the limits of integrations furnishing δV are to be extended only over the crescent of the star undergoing eclipse which remains visible at any particular phase—the leading term of δV becomes, however, a quantity of zero-order which may be comparable, or even large, in comparison with the radial velocity of orbital motion. This *rotational effect*, as it is commonly called, can deform conspicuously the radial-velocity curves of close eclipsing systems within the minima and its geometry deserves, therefore, close study. Section V.2 of this chapter has been devoted to this end; and in it we found it possible to express the corresponding variation of δV with the phase in terms of the same families of associated α-functions and the boundary integrals introduced in section IV.3 (and discussed in some detail in sections IV.4 and 5) in connection with our study of the accompanying light changes. If the axis of rotation of the star undergoing partial eclipse is perpendicular to the orbital plane, the rotational effect should vanish at mid-primary minima; and be symmetrical but of opposite sign, before and after. Should, however, this axis be again inclined to the plane of the orbit, the effect should become *asymmetric* and its asymmetry should, moreover, vanish and reappear in opposite direction as the axis of rotation precesses under the gravitational influence of its mate (*cf.* section II.5). A study of the asymmetry of the rotational effect exhibited by the radial velocity of close eclipsing systems within minima thus offers a *third* independent way for a *spectroscopic* study of the precessional phenomena (i.e., the period U of the precessional motion and inclination of the axis of rotation) exhibited by the components of eclipsing binary systems—supplementary to the *dynamical* method of section II.7 or *photometric* method of section IV.2. It may, however, be stressed in this place that any and all of these three methods are applicable to our end only if the close binary happens to be an eclipsing variable.

A mutual tidal distortion of the components of close binary systems is not the only phenomenon which may cause the projected centre of light on the apparent disks of such stars to deviate from their centre of mass. Another phenomenon which is bound to produce the same effect is the reflection—whose photometric consequences have already been studied in section IV.6, and whose effects on the radial velocity are the subject of section V.3 of this chapter. As is easy to visualize, the mutual illumination of one star by another is bound to shift the apparent centre of light of each component in the direction of the centre of mass of the system, by amounts which vanish at the times of conjunctions and become maximum at quadratures.

The combined effects of distortion and reflection on a determination of the spectroscopic elements of the orbits of close binary systems and, through them, on their absolute masses and dimensions are then analyzed in the subsequent section V.4. It is pointed out that the principal effect of reflection is to diminish the true amplitudes K of the radial-velocity curves of the components of close binary systems by amounts defined by equation (4-12) which, if uncorrected for, would render the values of K as deduced directly from the observations spuriously too small—and would thus lead us to *underestimate* the true masses and absolute dimensions of the respective stars. Third-harmonic distortion of tidal origin influences K likewise; and will reinforce reflection if the inclination i of the orbital plane to the celestial sphere is less than approximately 63°.4, but tend in the opposite direction if $i > 63°.4$ (as is likely to be true in most eclipsing variables). In addition to influencing the observed *amplitudes* of the radial-velocity curves of close binary systems, the terms in δV due to both ellipticity and reflection tend to render them also *asymmetric*, in a way simulating a spurious orbital eccentricity, as given by equation (4-13), even if the true orbit were in reality circular. The corresponding longitudes of the periastra of such spuriously eccentric orbits should always be 90° or 270° (rendering the apsidal line parallel with the line of sight), depending on whether the expression on the right-hand side of (4-13) turns out to be positive or negative.

In an earlier section V.2 of this chapter we studied the rotational effect, within minima, which was found to produce systematic Doppler shifts of spectral lines of the component undergoing eclipse. Such shifts referred, by definition of δV, to the *cores* (i.e., centres of light, or absorption) of the individual spectral lines, and had no bearing on their profiles. Now the axial rotation of a star is, in general, bound to *widen* the profile of any line originating in its atmosphere; and if a rotating star undergoes eclipse, its line profiles should become also *asymmetric*. An investigation of the extent and form of line asymmetry arising from this source has been the subject of the preceding section V.5, in which closed analytic expressions have been set up which should permit us to predict the corresponding line asymmetry at any phase of the eclipse or, conversely, to utilize the observed asymmetric profiles for a determination of the velocity of rotation or of the geometry of eclipses.

V. BIBLIOGRAPHICAL NOTES

V.1: The contributions to the observed radial velocities of distorted components in close binary systems of non-orbital origin (due to the fact that the centre of light of a non-uniformly bright disk will not project itself constantly on the centre of mass of the respective body, but will oscillate around it in the course of an orbital cycle) have first been pointed out by T. E. Sterne (*Proc. U.S. Nat. Acad. Sci.*, **27**, 168, 1941) who limited himself, however to the specific effects of second-harmonic distortion of gravity-darkened stars. His investigation was subsequently extended in a systematic way to third-and fourth-harmonic distortion of arbitrarily limb- and gravity-darkened stars by Z. Kopal (*Proc. Amer. Phil. Soc.* **89**, 517, 1945), whose treatment of the subject has been followed in the present section.

The symmetry of the isophotae over apparent stellar disks for the coefficients of limb- and gravity-darkening related by equation (1-33) was proved by Kopal (op. cit. ante); though for the particular case of $u = 1$ and $\tau_0 = \frac{1}{4}$ this result was previously discovered by H. N. Russell (*Ap. J.*, **95**, 345, 1942). Russell's proof is, however, so replete with slips and misprints as to make it almost impossible to follow its meaning; and only a part of these mistakes were subsequently corrected by Russell in *Ap. J.*, **102**, 1, 1945. A correct reconstruction of Russell's argument was also given by Kopal in sec. 78 of his *Introduction to the Study of Eclipsing Variables* (Cambridge 1946).

V.2: The departures, within minima, from the Keplerian velocity-curves of close binary systems have first been noted by F. Schlesinger (*Allegh. Publ.*, **1**, 126, 1909) in δ Librae. The existence of such a 'rotational effect' was also predicted by J. Hellerich (*A.N.*, **216**, 276, 1922), and the effect was actually discovered by R. A. Rossiter (*Ap. J.*, **60**, 15, 1924) in β Lyrae, and by D. B. McLaughlin (*Ap. J.*, **60**, 22, 1924) in Algol.

A first analytical theory of this effect was given by R. M. Petrie in *Publ. D.A.O.*, **7**, 133, 1938, in which the stars were regarded as uniformly bright circular disks; and extended subsequently by Z. Kopal (*Proc. U.S. Nat. Acad. Sci.*, **28**, 133, 1942) to stellar disks of arbitrary limb-darkening. Still more recently, Z. Kopal (*Proc. Amer. Phil. Soc.*, **89**, 517, 1945; and Harv. Circ., No. 454, 1949) extended the whole theory of the rotational effect to distorted stars of arbitrary limb- and gravity-darkening; and Y. Hosokawa (*Publ. Astr. Soc. Japan*, **5**, 88, 1953) generalized it (for spherical stars) to such cases in which the axis of rotation of the star undergoing eclipse is not perpendicular to the orbital plane, and the angular velocity of rotation varies with the astrocentric latitude in accordance with Faye's law.

V.3: The effects of reflection on the observed radial velocities of the components of close binary systems, and their variation with the phase, have been foreseen by A. S. Eddington (*M.N.*, **86**, 320, 1926); and a rough estimate of their magnitude made by G. P. Kuiper in *Ap. J.*, **88**, 472, 1938 (Table 10). Their first analytical investigation is, however, due to Z. Kopal (*Proc. Amer. Phil. Soc.*, **86**, 351, 1943). In Kopal's treatment the illuminating component was regarded as a light point at a finite distance from the primary, which reflected light in accordance with Lambert's law. Subsequently, M. Kitamura (*Publ. Astr. Soc. Japan*, **5**, 114, **6**, 217, 1954) investigated the shift between the projected centres of light and mass of mutually illuminated stars at quadratures, by taking account of the finite angular size of both bodies; and A. H. Batten (*M.N.*, **117**, 521, 1957) investigated the corresponding phase-law by a method similar to that followed in this section.

V.4: The effects of the distortion of rotating components and of reflection in close binary systems on the determination of the elements of their spectroscopic orbits have been studied by T. E. Sterne (*Proc. U.S. Nat. Acad. Sci.*, **27**, 168, 1941), Z. Kopal (*Proc. Amer. Phil. Soc.*, **86**, 351, 1943; **89**, 517, 1945), M. Kitamura (*Publ. Astr. Soc. Japan*, **5**, 114, **6**, 217, 1954), and A. H. Batten (*M.N.*, **117**, 521, 1957).

V.5: The possibility of determining the velocity of axial rotation of the stars from an analysis of the Doppler broadening of their spectral line profiles appears to have first been pointed out by W. de W. Abney (*M.N.*, **37**, 278, 1877) and followed up, in more recent years, by many investigators. The corresponding theory which aims at predicting the line profiles for stars rotating with a given angular velocity was attempted by J. A. Carroll in *M.N.*, **93**, 478, 1933, and applied to Algol by J. A. Carroll and L. J. Ingram (*M.N.*, **93**, 508, 1933); while G. Shajn (*Pulkovo Circ.*, No. 7, 1933) tried to predict the theoretical line profiles of the distorted components of RR Centauri.

In all these investigations, their authors have been concerned with the forms of the spectral line profiles of the components of close binary systems in full light—when the entire apparent disk of each star remains exposed to the distant observer. A generalization of their results to stars undergoing eclipse, as given in this section, is new and has not been previously published.

CHAPTER VI

Determination of the Elements of Eclipsing Binary Systems

In Chapter IV of this book we investigated the theoretical light changes to be exhibited by close binary systems between minima as well as within eclipse, and succeeded in expressing them explicitly in terms of the elements of such systems to the degree of accuracy to which squares and higher powers of their mutual distortion can be ignored. The aim of the present chapter will be to consider systematically the converse problem: namely, a determination of the elements of eclipsing binary systems from an analysis of their observed light changes.

A study of the light changes of eclipsing binary systems—i.e., of close binaries which, by an accident of their space orientation with respect to our line of sight, happen to be eclipsing variables—occupies an important position in stellar astronomy for several reasons. First, because of the prodigious abundance of the objects of its study. Surveys of the stars in the neighbourhood of the Sun disclose that at least 0·1% of them form eclipsing systems; and if a similar ratio were to extend over the whole galaxy, the total number of eclipsing systems in it is enormous (of the order of 10^8) and quite beyond the hope of individual discovery. Eclipsing variables are, therefore, manifestly no exceptional or uncommon phenomena! And their significance is further emphasized by the fact that they represent the only class of double stars which can be discovered at great distances, and in other galaxies than our own. In the neighbourhood of the Sun—up to distances of the order of a few hundred parsecs—visual binaries can be recognized by their orbital motion, or common proper motion. Spectroscopic binaries can be discovered with modern reflectors of large apertures up to distances of a few hundred parsecs; but beyond this limit close double stars can be detected only if they happen to be eclipsing variables.

Spectroscopic binaries which are also found to be eclipsing variables have long been our principal source of information concerning the masses and absolute dimensions of the individual stars—our only source for massive stars—of which only a minute trickle has so far been tapped. Even this trickle has provided the bulk of all available data at the basis of all empirical mass-luminosity relations; and whenever also the parallax of such systems is known, the observed data can lend themselves to a determination of effective temperatures of the stars of different spectral types. The masses and absolute dimensions of the components of eclipsing systems have provided the basis for studies of the chemical composition of stellar interiors or, under certain conditions, even of their internal density distribution. At first sight stellar interiors would appear to be as completely concealed from external inspection as any region in nature. A gravitational field emanates, however, from the interiors, which the opaque outer layers cannot appreciably modify; and a radiant energy originating in the deep interior will govern, with the intervening layers, the distribution of brightness over the surface. As long as a star is single, there is no way of gauging its external gravitation field, or to learn anything about the distribution of light on its surface. Place, however, another star in its proximity; and various properties of the combined gravitational field of both components can be deduced from certain observable characteristics of their motion, which we have already discussed in Chapter II. The variation of light induced by the rotation of distorted components or exhibited during their mutual eclipses (Chapter IV) should permit us, in principle, to deduce many fundamental properties of such stars from an analysis of their light changes. The aim of the present chapter will be to outline such an analysis.

'The wonderful train of consequences that can be drawn from such regular occultations' (of Algol) 'should engage our utmost attention,' wrote William Herschel in a preface to this branch of double-star astronomy 174 years ago*; but owing to the difficulties inherent in such a proposition this train was rather slow to unroll. For little did Sir William realize that the task which he so shrewdly passed on to subsequent generations would confront us one day with some of the most complicated problems in double-star astronomy. To re-state their nature in the language of a modern communications engineer, the eclipsing binaries can also be regarded as individual television cameras sending out detailed information about distant stellar systems. The light by which such information is relayed carries a message which a long journey through space will not alter appreciably. Before we can, however, learn anything from such a message, it must be properly recorded and interpreted. Its reception is a problem of practical photometry which is currently well in hand, and its noise level (due to the fluctuation of atmospheric extinction and instrumental limitations) is at a satisfactorily low level. The reception is, unfortunately, only a part of the story; for the actual message turns out to be expressed in a code (in the form of the observed light changes)—quite akin to the working of a television camera which scans a scene and transforms it into a series of linear elements at a speed of hundreds per second. These elements are then reassembled by the television receiver into a picture resembling the original enough to be recognizable (sometimes) to the viewer at a glance. An eclipsing binary represents, in principle, an identical source of information: for as one component of a close binary system begins to eclipse its mate, a scanning takes place and the accompanying light variation should also contain information sufficient to describe the system—though manifestly not at a glance! In point of fact, as we shall

* In an informal report entitled 'Observations upon Algol', read before the Royal Society of London on May 8th, 1783, but not printed until it appeared in *The Scientific Papers of Sir William Herschel*, London 1912, vol. I, p. cvii.

show in this chapter, the light message can be properly deciphered only with the aid of many subtle tools borrowed from the workshop of the applied mathematician (including, of late, the electronic computers). That this must be so should be evident from the fact that eclipsing variables constitute an utterly simple one-line television system, and the simplicity of their emitting mechanism is bound to add to the complexity of interpreting their message.

The first period of development in methods of interpretation of the light curves of close binary systems did not begin properly until 1880, with Pickering's attempt to determine the elements of Algol. Between 1880 and 1912, he and other early pioneers of our subject had only gradually come to realize the full magnitude of their problem, and their efforts were concerned with a few specific systems known at that time, rather than with the general processes which should be applicable to any practical case. Although they laid thus the foundations for later efflorescence of the subject, their individual contributions are now only of historical significance.*

The second period in development of our subject opened up in 1912, when most of the earlier work was superseded by simple methods by Russell and Shapley,† which were applicable to any type of eclipse and aimed to provide rapid means for preliminary solutions for the elements of eclipsing variables on the basis of observations of moderate accuracy. In order to do so, several reasonable but over-simplified assumptions had to be made concerning the form of the two components (spheres, or similar ellipsoids), or the distribution of brightness over their apparent disks—assumptions which were prompted by the desire for simplicity rather than by their physical justification—and in order to facilitate the work, smooth curves drawn by free hand to follow the course of individual observations were substituted for the actual observed data.

Under these circumstances, Russell and Shapley succeeded in reducing a large part of light change analysis to semi-graphical procedures whose time-saving features have earned them well-deserved popularity, and which have retained their usefulness for preliminary work up to the present time. For this reason, as well as for the sake of methodical development of our subject, we shall render in section VI.3 an account of Russell's methods aiming at a *direct* solution for the geometrical elements; and in so doing shall point out their merits as well as drawbacks of partly practical, and partly fundamental nature. For as time went on, the gradually increasing accuracy of photometric observations on one hand, and a fuller understanding of stellar structure on the other, have provided a continuing incentive for further developments. By 1912, photoelectric photometry was still in its infancy; and the errors of normal points constituting the best light curves then available amounted to hundredths of a magnitude. In the course of time which has since elapsed, the accuracy of photometric measurements has

* For their account *cf.* the Bibliographical Notes to section VI.3.

† *Cf.* H. N. Russell, *Ap. J.*, **35**, 315; **36**, 54, 1912; H. N. Russell and H. Shapley, *Ap. J.*, **36**, 239, 385, 1912.

been increased more than ten times, and a limit has gradually been exceeded to any gain of fitting free-hand curves to oversimplified models. Schematic representations of average eclipsing systems, characteristic of earlier methods, have gradually given way to physically sounder models based on dynamical considerations.

A proper realization of all these facts has emerged only gradually under the pressure of increasing demand for rigour and deeper understanding, prompted by the observational feats of photoelectric photometry; and under their impact the need arose for a revision of the basic concepts of the methods for light curve analysis of eclipsing binary systems. This revision was carried out largely by the present writer and his associates in the decade between 1941–1950, which can perhaps be regarded as the third period of development of our subject, following previous achievements in 1880 and 1912–1914. Unlike this latter triennium which saw the development of *direct* methods of solution for photometric elements of eclipsing variables, the decade of 1941–1950 witnessed the birth of *iterative* methods to the same end; and these methods will be described in subsequent sections 4-6 of this chapter. This change of strategy from direct to iterative methods was, moreover, not optional, but necessary. Direct solutions could be attempted only on the basis of very simple models which, adequate as they may have seemed in 1912, were no longer so by 1940. Besides, even in the simplest cases to be described in section VI.3 the Russell–Shapley method led to no solution in the mathematical sense, as a part of it (i.e., the location of fixed points on free-hand curves) had to be anticipated beforehand. On the other hand, iterative methods leading to the result by successive approximations can be made very much more general. They utilize the observed data as they stand—obviating any recourse to free-hand curves or other inferences—and furnish the most probable values of the elements simultaneously with their uncertainty. They are of necessity more time-consuming in their execution than the earlier direct methods, but this additional time should be amply rewarded by increased 'resolving power' of the procedure and precision of the results. It should also be stressed that the time spent by a resort to them is still short in comparison with the time which the observer had previously to spend at the telescope, and in the reductions, to provide the basic photometric data. The observer should, therefore, have full right to demand that the analyst of his data (be he the same, or different, person) should not shrink from his obligations, and spare no time to ensure that all information stored in the original observations—not more and not less—has been extracted.

To lay before the reader the methods necessary to this end in sufficient detail to enable him to use them in practice will require more space than we have needed to expound any other topic in this book so far; and this accounts for the extent of the present chapter. The iterative process of solution described in sections 4–6 is not yet the end in itself; for the results of photometric observations may have to be suitably combined with the spectroscopic evidence (section VI.7); or may warrant generalization and improvement by way of differential corrections (section VI.8). If, moreover, the relative orbit of the two stars is markedly eccentric, special precautions must be observed (sections VI.9 and 10). Lastly, if the distortion of one (or both) components becomes appreciable or even conspicuous, the processes expounded earlier in this chapter will require a generalization developed in sections VI.11 and 12 which will bring us in contact with some of the most intricate developments encountered in light change analysis. The developments of section VI.12, concerned with an interpretation of the light curves of distorted eclipsing systems within minima is, in particular, of so recent a date that various possibilities of application to practical cases are still far from being exhausted. A demand for further work on subjects touched upon in that section is so pressing that its present version will largely have fulfilled its mission if it succeeds in attracting the attention of future investigators to this newly opened and still almost virgin field. The concluding section 13 of this long chapter will then contain a survey of all methods dealt with within its scope, and an outline of the strategy which may be followed by the investigator; while the Appendix will give some technical details concerning the least-squares processes, encountered at different stages of our analysis (sections VI.6 and 8), which may be of practical value to the investigator but which would have proved too discursive for the main text.

VI.1. OBSERVATIONAL DATA AND THEIR TREATMENT

Photometric observations of an eclipsing system furnish us, in general, with a series of measurements of the instantaneous brightness of the system performed each at a certain time. Of the two measurable quantities—i.e., light and time—the time measurements can usually be made with a relative accuracy far surpassing that of photometric measures, so that for all practical purposes the time readings of the moments at which the photometric measurements were made can be regarded as exactly known. It is obvious, however, that the accuracy with which the observations must be timed will have to be greater, the greater the precision of the photometric measures, and the steeper the light curve. Totally-eclipsing systems like RW Tauri or X Trianguli are known to diminish their light, in advanced stages of the eclipse, by as much as one magnitude in less than 15 minutes—which means that if the individual observations were accurate (say) to $0^{m}.001$, the time readings should be recorded to the nearest second; and the corrections needed to reduce the observed local time into the heliocentric Greenwich Mean Time should be applied with equal precision. Existing tables* for a conversion of the geocentric into heliocentric G.M.T. will permit us to do so expeditiously and with ample accuracy.

* *Cf.*, e.g., R. Prager, *Tafeln der Lichtgleichung*, Kl. Veröff. Berlin-Bab., Nr. 12, 1932.

When we pause to consider the uncertainty of the photometric measurements of stellar brightness, a comparison between eclipsing orbit work and celestial mechanics becomes once again illuminating. While single measurements of the position of a star or an asteroid can be made accurate to at least a millionth part of the quadrant, the most precise existing photometric measures of the brightness of a star are barely accurate to about one part in a thousand. It is possible (though by no means certain) that, in the decades to come, improved techniques will increase the accuracy of photometric measures to one part in ten thousand (corresponding to errors of the order of 0·0001 magnitude) but scarcely more; for beyond this limit, irregular anomalies in the extinction of light in the terrestrial atmosphere are likely to impose an impenetrable barrier to a further gain in precision—except possibly at very rare localities—just as the long-period refraction anomalies, ever present even at the zenith, imposed a similar limit to a further growth of accuracy obtainable by micrometric measurements.* In view of this great disparity in the ultimate precision that can be reached by photometric and astrometric observations, the need of a different approach to the problem of a determination of the elements of an eclipsing binary system from that of an asteroid will be immediately apparent.† In order to compensate for their low relative accuracy, the photometric observations underlying eclipsing orbit work must obviously be very many and will require a statistical, rather than individual, treatment.

The observed data which we shall have to analyze will consist, therefore, of a series of discrete measurements of the brightness of an eclipsing system performed at certain particular times; and it will be assumed that the timing of the observations can be so accurate as to permit us to regard the errors of photometric measures as practically the only source of imperfection inherent in our data. In ordinary cases, the total photometric evidence on the system under investigation may consist of several hundred to several thousand individual determinations of its brightness at strategic moments of time. In most methods which we are going to develop it would be utterly impracticable to treat such a mass of observational data simultaneously; we must, therefore, contract them into a more restricted number of *normal points*. This should be prompted by reasons of expediency rather than principle. In point of fact, it should be regarded as a necessary evil: an evil, because the normal points themselves are but a substitute for the original individual observations which we wish to interpret; and a necessary one, because large numbers of observations (which every investigator hopes to have at his disposal) could hardly be dealt with otherwise.

* *Cf.* F. Schlesinger, *M.N.*, **87**, 506, 1927.

† The difficulties characteristic to the eclipsing orbit work, based on even the best photometric observations available at present, should be comparable to those confronting the computer of asteroidal orbits if his individual observed positions were inaccurate by some 20 minutes of arc! In dynamical astronomy, this would correspond to quite a prehistoric degree of accuracy; for even Kidinnu and his Chaldean colleagues knew better twenty-five centuries ago.

The time interval within which the individual observations should be grouped into a single normal point is difficult to estimate with any generality. It can evidently be longer in the interval between minima—when the variation of light is slow, if any—than within minima when the light of the system may vary rapidly. Within eclipses, the number of available normal points should exceed the number of the quantities for which we wish to solve by a sufficiently wide margin to enable us to employ the well-known least-squares techniques in subsequent analysis, but should not be too large for the formation of the normal equations of each solution to become unnecessarily laborious. In practice, we may tentatively suggest that the number of normal points formed within each minimum should be not less than 10, and need not probably be greater than 20 or 25.

Once we have arrived at the formation of normal points, the question will naturally arise: what kind of a mean should we form? This is a point largely of theoretical interest, but its implications may be mentioned in this place. If the individual observations have been made visually, Seeliger* pointed out many years ago that, owing to the operation of Weber–Fechner psycho-physical law, the most probable value of discordant *visual* observations is their *geometric mean*; and the same should be true, by implication, of the *photographic* observations. On the other hand, *photoelectric* observations are performed in such a manner that the *arithmetic mean* should represent the most probable value of discordant readings. Another way of expressing these facts is the statement that *visual or photographic observations should be averaged in stellar magnitudes* (i.e., on a *logarithmic* scale) while the *photoelectric observations should be averaged in intensity units.* These facts are perhaps not very important in practice because, if the dispersion of the individual observations grouped in a single normal is small (as it should be), the arithmetic and geometric means will be closely the same; but they should be kept in mind.

When the normal points have thus been formed in an appropriate manner, it remains for us to ascertain their *weight.* It should be stressed that, in this connection, we are concerned only with their *empirical weight* specified by the quality and number of the individual observations included in each particular normal; the question of their *intrinsic weights* for the determination of the elements will be taken up at a later stage. Provided that the number of the individual observations constituting each normal is sufficiently large to make the laws of large numbers applicable, *the empirical weight of the normal point should be inversely proportional to the square of the standard deviation of individual observations from the mean.*

If the investigator of the elements happens also to be the observer, he will be intimately acquainted with the quality of his observations to form a good grasp of the empirical weights of his normal points. If, on the other hand, the observer wishes to leave the task of the determination of the elements of his eclipsing system to a future investigator, he should publish explicitly as

* H. v. Seeliger, *A.N.*, **132,** 209, 1893.

many details of his observations as may be needed to ascertain the weights of each normal *a posteriori*. This is provided best by a publication of all individual observations, with explicit indication of all circumstances which may affect their quality (such as moonlight, dew, fogged plates, galvanometer failures, etc.). Should this prove impossible on technical grounds, and should the observer be forced to limit himself to the publication of a list of normal points alone, he should by all means state explicitly the dispersion of the individual light measurements within each point from a mean curve, in order to enable the subsequent investigator to assess their relative weights. In the absence of any such information, the customary practice has so far been to assume that the *errors of visual photometric or photographic observations are equal on the magnitude scale* and that their *absolute amount* (in intensity units) *varies*, therefore, *in direct proportion to the instantaneous luminosity of the system*. *Photoelectric measures* of stellar brightness are, on the other hand, primarily subject to instrumental errors which are independent of the star's brightness, and their errors should, theoretically, be constant on the *intensity* scale. They would indeed be so if it were not for the disturbing effect of the atmosphere. The observational errors having their source in bad 'seeing' conditions are, however, proportional again to the star's brightness and, to this extent, the errors of even photoelectric observations should be constant on the logarithmic (i.e., magnitude) scale—just as for visual or photographic observations.

As long as the amplitude of light variations of our eclipsing system is small, the foregoing considerations do not make much difference; but for variables of large amplitudes they manifestly become important. Consider, for example, the eclipsing system RW Tauri whose primary minima attain an amplitude of 4·3 magnitudes in photographic light.* In consequence—if our above premises are right and the errors of observation are indeed constant on the magnitude scale—a photographic observation made near the inner contact should carry a weight 2291 times as large as any similar observation made at maximum light; and if the latter were of unit weight, an equation of condition based upon observations near the inner contact should be multiplied by $\sqrt{2291} = 47{\cdot}9$. On the other hand, if the same observations were made by means of a photoelectric photometer at the time of good seeing, no such difference in weights between the phases near the inner and outer contacts should be encountered. Inasmuch as so drastic a disparity in weights between various phases, as implied by the hypothesis of the errors of observation being equal on the magnitude scale, would significantly influence the outcome of the whole subsequent solution, it is a matter of real importance to be able to check up whether or not such a hypothesis corresponds to actual facts in each particular case; and to this end a knowledge of the dispersion of individual points in each observed normal is a necessary prerequisite.

One final remark relative to the weight of the normal points at different

* L. Binnendijk, *B.A.N.*, **9**, 173, 1941.

parts of the light curve may be added. Suppose, for the sake of argument, that the observations distributed uniformly with the phase have been averaged into normal points over *constant* intervals of time; would the standard deviations of the individual observations from their mean represent a true measure of their observational weight? The answer would be in the affirmative if, and only if, our light curve were a straight line in the coordinates characterizing the positions of our normals; for only then would the approximation of the appropriate mean (arithmetic or geometric) be independent of the *width* of the time interval within which the individual observations were grouped into a normal point. In reality, the light curve of an eclipsing system within minima is not straight, and its curvature varies with the phase. A strong curvature tends to diminish the weight of a normal point based upon observations made during the respective fraction of the cycle; and in order to avoid this loss we may have to shorten the time interval used for averaging—i.e., to break up the ordinary normals in the critical part of the light curve (such as in advanced stages of the eclipse, when the curve is rapidly changing direction) into smaller normals averaged over shorter intervals of time. This fact has been intuitively known and respected for a long time, though its explicit statement has rarely been given.

The measurements of stellar brightness—irrespective of the manner in which they are made—are customarily recorded in stellar magnitudes. From the intrinsic point of view such a scale is, of course, purely arbitrary; yet if we are dealing with visual or photographic observations, its logarithmic nature will permit us to average the individual observations by forming the arithmetic means of the respective stellar magnitudes—which is computationally more convenient than the formation of a geometric mean of their light intensities. After this has been done, however, and the normal points formed, the usefulness of stellar magnitude scale has come to an end and our next step should be to *convert the magnitudes into light intensities.* A number of convenient existing tables will enable us to perform this task with a minimum of difficulty. Furthermore, in order to *normalize* these intensities, *we shall express them in terms of a brightness of the system between minima* taken as our unit of light, and shall denote the light intensities normalized in such a manner by l. If the components constituting our system are distant, their combined light—our unit of light—will remain sensibly constant outside of eclipses and its value will be equal to an appropriate mean of all observations secured between minima. If the components are so close that their axial rotation alone gives rise to a significant light variation (which is independent of, and supplementary to, the variation due to the eclipses), the absolute value of our unit of light cannot be determined separately from all other parameters; but the proper process for doing so offers no difficulty and will be discussed later in this chapter.

Once we have thus normalized our *light* measurements, it remains for us to convert also the corresponding *time* readings into appropriate non-dimensional ratios. The unit of time which suggests itself for this purpose is the heliocentric orbital period P of revolution of the components of our eclipsing system. This quantity can as a rule be determined from the observations with an accuracy far surpassing that of all other elements. The methods for its derivation or for the determination of the time t_0 of the minima are, however, common to all other classes of periodic variable stars and have been described by previous writers* so thoroughly as to call for little elaboration in this place. Suffice it to mention that, since the orbital periods of most eclipsing systems are slightly variable, it is their *instantaneous* values (prevailing during the interval of observation) which are of direct interest for the orbit computer. It should, furthermore, be mentioned that such periods as are usually given by the observer and included in the catalogues are only *apparent* ones; for unless the respective system happens to be at rest relative to the sun, the observed period is *not* identical with the true period of revolution, but is equal to†

$$P_{\text{app}} = P_{\text{true}}[1 + (\gamma/c)], \tag{1-1}$$

where γ denotes the observed heliocentric radial velocity of the centre of mass of the eclipsing system and c is the velocity of light. For most eclipsing systems, the differences between the true and apparent (observed) periods of revolution may amount to several seconds. It should also be kept in mind that any variation of P (such, for instance, as that caused by a revolution of the eclipsing system around the centre of gravity of a triple system, or by a hyperbolic close encounter with a nearby star) is bound to invoke an apparent change in the observed orbital period which is due solely to the relative acceleration of the respective system along the line of sight.

When P has been determined, we are in a position to convert the observed times t of photometric measurements into the respective *phase angles* θ, defined by

$$\theta = (2\pi/P)(t - t_0) \text{ radians} = (360°/P)(t - t_0) \text{ degrees}, \tag{1-2}$$

where t_0 denotes the heliocentric time of the *primary* (deeper) minimum from which the phase is conventionally reckoned. The phase angle as given by the above equation represents the *mean anomaly* of the component of lesser surface brightness in its relative orbit around its mate; if the orbit is eccentric, the true anomaly v should be obtained from θ by means of the standard expansions of the two-body problem. The accuracy with which the photometric observations can as a rule be timed is such that the phase angles can be determined with a higher accuracy than is really needed for the subsequent analysis of the light changes. In actual practice, phase angles should be recorded, on the average, to the nearest hundredth of a degree—i.e., to *four* significant figures. If the accuracy of the underlying photometric measures is of the order of $0\overset{m}{.}01$ or even smaller, three significant figures in the phase angles are probably sufficient; while even the most precise observations now

* *Cf.*, in particular, J. G. Hagen, *Die Veränderlichen Sterne*, Freiburg 1921, pp. 635–671.
† *Cf.* section II.8 and, in particular, equation (8-103).

available do not call for the retention of more than five significant figures (i.e., of 0°001) in the corresponding phase angles.

The normalized fractional light l (expressed in terms of the total brightness of a system taken as unity) and the phase angle θ (or some function of it) will be the coordinates specifying the position of each normal point in a light–time diagram which we shall presently construct, and it is these two quantities alone which will be needed explicitly in our analysis of the observed light changes for the elements of our eclipsing system later on. Before proceeding to it, however, it may be advisable to plot l against $\sin^2 \theta$ on a suitable scale, and to inspect the general trend of the light changes which we are going to analyze. If the observations are numerous and well distributed, the form of the respective light changes will readily transpire and may disclose a good deal to inspection concerning the nature of the eclipses which give rise to it. Few investigators will, at this time, resist perhaps the temptation of drawing a continuous *light curve* by free hand to follow the course of observed normals, in order to obtain a more forceful representation of the anticipated light changes. *At no time, however, will there be need to use this curve for anything else but inspection,* or for an *estimate* of preliminary values of certain characteristics of the system (such as the depths of the minima, or the moments of outer or inner contacts) which will be improved by subsequent analysis. *No investigator should ever forget that his real task is to interpret the actual observations* (as represented by a discrete set of available normal points) rather than any inferences based upon them. For, irrespective of how closely a light curve could be drawn to follow the course of observed normals, it represents indeed merely a plausible inference which, in some parts, may approach the reality more closely than any individual normal point but which, in others, may be systematically off; and no matter how small the respective deviations may be, their systematic nature may entail serious cumulative effects.

The choice of $\sin^2 \theta$—rather than of some simpler functions of the time, such as θ or $\sin \theta$—as the abscissa for the plot of our light curve has been made for very good reasons. First, the choice of an even function of θ will permit us to *reflect* the ascending and descending branches of the light curve on one another. Unless complications of unknown physical nature are present, the ascending and descending branches of the light curve should be symmetrical and their normal points at opposite phases can be freely combined.* Secondly, the light curve plotted in the l-$\sin^2 \theta$ coordinates will be characterized by a steeper slope at the moments of the outer or inner contacts, than it would be if the time itself (or any odd function of it) were used as the abscissa. In consequence, *the time at which an eclipse begins or ends, is much easier to estimate from a plot of the observed normals in the* l-$\sin^2 \theta$ *coordinates* than it would be if the abscissa were θ or $\sin \theta$; such preliminary estimates may indeed be needed later (*cf.* section VI.4). Third,

* The orbital eccentricity would affect the symmetry of the light curves by scarcely perceptible amounts (*cf.* section VI.9).

the slope of the light curve in the l-$\sin^2\theta$ coordinates will turn out to be intimately connected with the *intrinsic weight* of the observations pertaining to various phases of the eclipse. Their connection will be fully discussed in section VI.4; and a knowledge—even approximate—of the derivative $dl/d\sin^2\theta$, which can be readily deduced from our plot, may then turn out to be quite an asset. All these independent reasons are mutually supplementary, and leave no room for doubt as to the choice of coordinates for plotting our light curve.

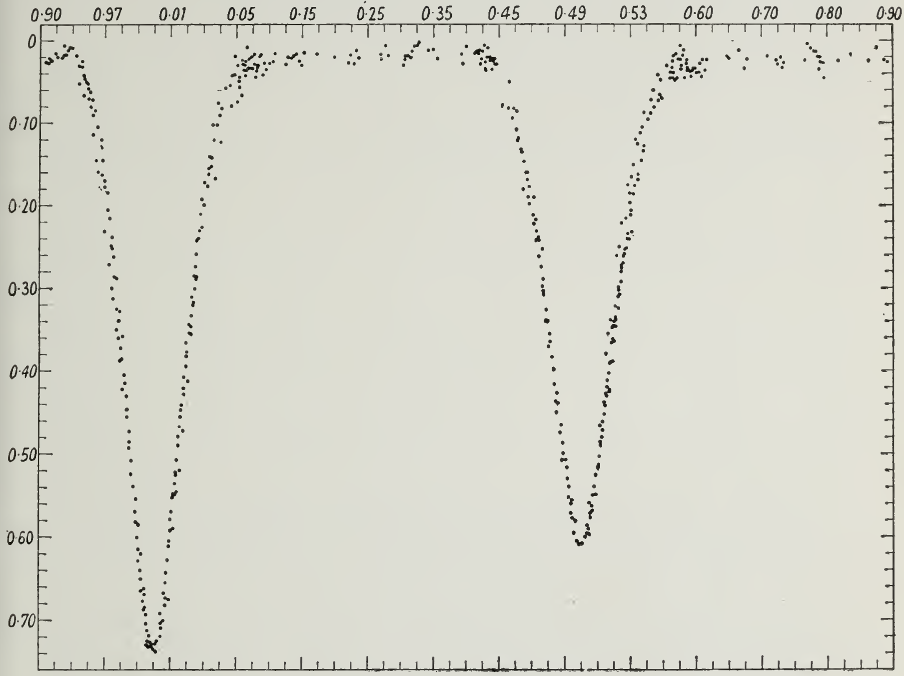

FIGURE 6-1. LIGHT CHANGES OF THE ECLIPSING SYSTEM OF WW AURIGAE according to the photoelectric observations by C. M. Huffer (*Ap. J.*, **114**, 297, 1951). Abscissae: the relative brightness of the system in stellar magnitudes; ordinates: the phase in fractions of the orbital cycle. The time-scale between minima has been contracted for convenience of presentation.

Once this has been done, we must inspect the available evidence with the aim of distinguishing between the following three possibilities:

(1) The observations bear out no evidence of any significant variation of light outside of eclipses, so that for all practical purposes the light between eclipses can be regarded as constant (*cf.* Fig. 6-1).

(2) The light of the system is found to vary noticeably between minima—due to the ellipticity and reflection effects—but the moments at which the eclipses set in can be clearly distinguished (Fig. 6-2).

(3) The changes of light exhibited between eclipses are comparable to those caused by the eclipses, and merge together to such an extent as to

render it effectively impossible to discern, even approximately, the times at which the eclipses begin and end (Fig. 6-3).

If the light curve of our eclipsing system falls in the first category, we shall proceed to analyse it for the elements by the methods outlined in sections 3–10 of this chapter, which are based on the assumption that both components are spherical in form and appear in projection as circular disks arbitrarily darkened at the limb. Some readers may wonder, perhaps, why so much attention is going to be devoted to a model which is of necessity somewhat artificial; for the components of all real eclipsing systems are bound to be

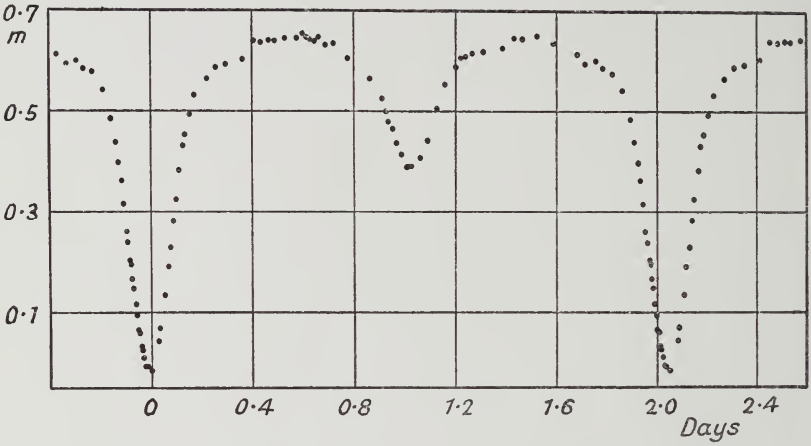

FIGURE 6-2. LIGHT CHANGES OF THE ECLIPSING VARIABLE U HERCULIS according to the photoelectric observations by W. A. Calder and H. Shapley (*Harv. Circ.*, No. 425, 1937). Abscissae: the relative brightness of the system in stellar magnitudes; ordinates: the phase in days.

distorted, depending on their proximity, and thus depart from spherical form to a smaller or greater degree. The answer to this is simple. In a great many known eclipsing systems the separation of both components is so wide, and, consequently, their distortion so small that, for practical purposes, they can be regarded as spheres—in which case the methods of sections 3–10 may furnish indeed the final solution.

On the other hand, there exists no royal road to a direct determination of elements of distorted eclipsing systems. Their elements are obtainable only by a process of successive approximations, starting from the preliminary elements deduced on the assumption of spherical stars. The details and practical aspects of such a process will be taken up later in sections 11 and 12. If the preliminary elements evaluated on the assumption that both components are spheres (or similar ellipsoids) can be modified by any practicable method to account for the effects of possible dissimilarity in form and of other perturbations, genuine properties of close eclipsing systems may still be obtained. Should, however, such a procedure fail to converge with a sufficient rapidity to make it practicable—as it almost invariably does for stars belonging to the above group (3)—the search for the elements will have to take an essentially different course. Such a course will have to be based on closed properties of the Roche model which have already been investigated in Chapter III. Their practical applications to the determination of the elements of contact eclipsing variables is as yet in its infancy and can be only briefly referred to at the end of section VI.12.

Returning, however, to the categories (1) and (2) for which the analysis is currently well in hand, we may observe that, irrespective of whether the elements derived on the assumption of spherical components are eventually

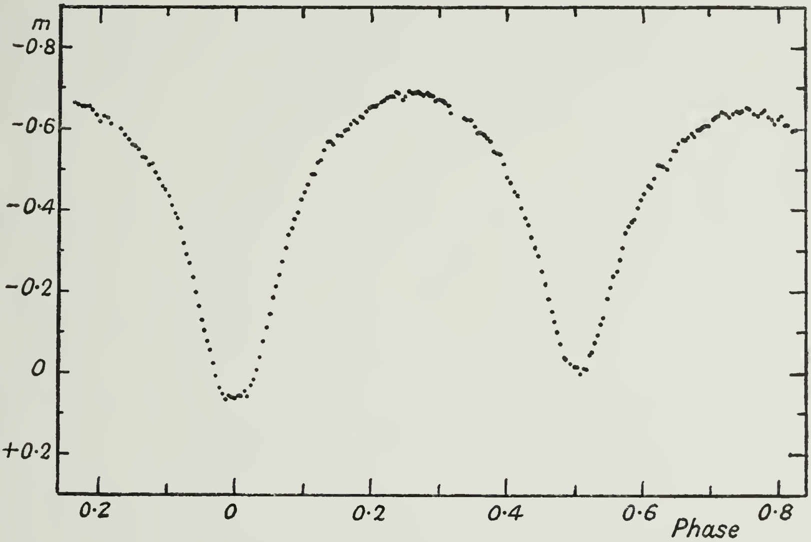

FIGURE 6-3. LIGHT CHANGES OF THE ECLIPSING VARIABLE W URSAE MAIORIS, according to the photoelectric observations by K. K. Kwee (*B.A.N.*, **12**, 330, 1956) on the night of January 23rd, 1954. Abscissae: the relative brightness of the system in stellar magnitudes; ordinates: the phase in fractions of the orbital period.

adopted as final (group 1) or are merely to serve as a basis for subsequent refinement to take account of the perturbations invoked by the proximity effects (group 2), the methods for obtaining them are evidently fundamental to the analysis of *any* light curve. This is why their discussion precedes that of the phenomena invoked by the proximity effects; and it is suggested that the reader should become acquainted with them in this order.

VI.2. GEOMETRY OF THE ECLIPSES

Having progressed thus far, we are ready now to embark upon the actual analysis. Suppose, in what follows, that the light changes of an eclipsing binary system, caused by mutual eclipses of two spherical stars appearing as uniformly bright or limb-darkened disks, have been measured quantitatively and plotted against the phase. For the sake of simplicity let us, furthermore, assume also that both components of our system revolve around their common centre of gravity in circular orbits; the effects invoked by orbital eccentricity will be postponed for a later discussion (section VI.9). If our eclipsing system conforms to this model, its light will remain constant between eclipses* and exhibit two minima alternating at each conjunction. These minima will be symmetrical, of equal duration but generally of unequal depth, and separated by exactly half the orbital period. At any moment during a minimum, one component will eclipse a certain area of the apparent disk of the other. Half a revolution later, at the corresponding phase during the other minimum, the geometrical relations of the two disks will be exactly the same, except that it is now the other star which is in front and eclipses an equal area—though not necessarily an equal proportion—of the disk of its mate.

If the disks of both components were uniformly bright, the light changes in both minima should be similar and differ only in their vertical scales (which would be in the ratio of the surface brightness of the two stars). If their disks were darkened at the limb to any arbitrary degree, this should continue to be true only if the darkening of both stars were the same and their radii were equal. Irrespective of darkening, however, *the deeper minimum always corresponds to the eclipse of a star of greater surface brightness*; whether this star is also the larger or the smaller of the two is not immediately apparent and remains to be ascertained by subsequent analysis. Since almost any combinations of spectral types can be encountered among the components of eclipsing binary systems, the disparity in depths of both minima may often be great. If—as it frequently happens—the surface brightness of one component is too low for its eclipse to cause any appreciable diminution of light, only one minimum can be seen in the light curve and used for subsequent analysis. In many other cases one minimum may be barely discernible, so that all one can use from it is its depth.

The problem of determining the elements of an eclipsing system from an analysis of such a light curve is one of considerable complexity and warrants a careful approach. The first problem confronting the investigator is to decide whether the observed minima are due to total (annular) or partial eclipses. A well-observed light curve will usually permit us to decide this by inspection; for if the variation of light during *both* minima is *continuous*, the eclipses giving rise to them are necessarily *partial*; while if there is a phase of *constant light* at the bottom of *one* (or *both*) minima, *total* and *annular* eclipses alternate. Difficulties in discriminating between these alternatives may arise if the eclipses are nearly grazing, or if only one minimum has been detected (since an annular eclipse of a limb-darkened star may closely simulate a partial eclipse). In such cases all we can do, at this stage,

* Apart from minor changes invoked by the reflection effect which have been discussed in section IV.6.

is to *conjecture* as to the type of eclipses under investigation, and to ascertain *a posteriori* whether or not our conjecture will be borne out by subsequent analysis. Should this not be the case, we may have to retrace our steps and attempt another solution on the basis of an alternative hypothesis. This may entail an added amount of work; but the investigator can content himself with the fact that if the first conjecture was wrong, the other will necessarily be correct; for *tertium non datur*.

In order to fix our terminology we shall, in all that follows, refer to the *deeper* minimum as the *primary*, and to the *shallower* minimum as the *secondary* one; and shall likewise denote the component of *greater surface brightness* as the *primary component*, and the one of *lesser* surface brightness as the *secondary* one. Thus the primary component will, by definition, be the one eclipsed at the time of the primary minimum, and vice versa. We shall, moreover, find it convenient to differentiate further in terminology between different types of eclipses encountered in close binary systems. If a minimum arises from an eclipse of the smaller star by the larger one, we shall call it an *occultation*; while if the smaller star is in front, we shall call it a *transit*. In the case of total or annular eclipses these terms are self-explanatory; for a total phase results necessarily in a complete occultation of the smaller component, while the annular phase represents a transit of the smaller star across the disk of its mate. We shall, however, find it convenient to extend the terms 'occultation' and 'transit' also to the partial phases of the total or annular eclipses, or to the partial eclipses in general.

If, in totally eclipsing systems, an inspection will permit us to identify the minimum due to a total eclipse, the question as to which one of the two minima is an occultation or a transit has automatically been settled. For partially eclipsing systems a mere inspection of the light curve, is, unfortunately, of very little avail. Whether or not, in such systems, the primary minimum is an occultation or a transit—i.e., whether the primary component exceeds the secondary, not only in surface brightness, but also in size and luminosity, or whether it is inferior to it in these respects—cannot, in general, be settled by inspection and must be established by subsequent analysis. This analysis will be discussed in detail in later parts of this chapter; for its first half will be devoted to a treatment of total or annular eclipses. We do not do this because the totally eclipsing systems are more important or numerous than those exhibiting partial eclipses. According to the laws of chance, a large majority of known eclipsing systems is bound to be characterized by partial eclipses (although the range of the light changes, usually greater for totally eclipsing systems, will favour the latter in the chances of discovery). On the other hand, the process of determination of the elements of totally eclipsing systems from an analysis of their light changes is so much simpler than if the eclipses are partial, that it seems advisable to take it up first foı purely didactic reasons. Before we do so, however, it is advisable to introduce explicitly certain quantities relative to the fractional loss of light which our system suffers during eclipses, as well as other abbreviations which will be frequently used, and consistently adhered to, throughout the present chapter.

In the preceding section, we found it expedient to convert the observed brightness of our eclipsing system into fractional intensities expressed in terms of the combined luminosities of its constituent components. In what follows, let L_a and L_b denote the fractional luminosities of the smaller and the larger component, respectively, expressed in terms of the same unit. Evidently, by definition,

$$L_a + L_b = 1, \tag{2-1}$$

and the value of l between eclipses is likewise equal to unity. During an occultation minimum (larger star in front), however,

$$l_a = 1 - f_a L_a , \tag{2-2}$$

while half a revolution later, during a transit (smaller star in front),

$$l_b = 1 - f_b L_b , \tag{2-3}$$

where f_a, f_b denote the respective fractional loss of light *expressed in terms of the total light of the component undergoing eclipse.* Let, moreover, the distribution of surface brightness J over the apparent disk of the eclipsed star be such as to obey the cosine law (2-15) of Chapter IV, which for our present purposes we propose to restrict to its linear form

$$J(\gamma) = H(1 - u + u \cos \gamma), \tag{2-4}$$

where u denotes the coefficient of limb-darkening and γ, the angle of foreshortening. If so, the fractional loss of light f appropriate for either type of the eclipse and any value of u can be found as a weighted mean of the two extreme cases by means of the equation

$$f = \frac{3(1-u)}{3-u} f^{\mathbf{U}} + \frac{2u}{3-u} f^{\mathbf{D}}, \tag{2-5}$$

where $f^{\mathbf{U},\mathbf{D}}$ denote the respective fractional loss of light if the disk undergoing eclipse is uniformly bright ($u = 0$) or completely darkened at the limb ($u = 1$). The symbol f in the foregoing equations may stand for either f_a or f_b provided that u occurring in them refers to the smaller or the larger star, respectively.

The functions $f^{\mathbf{U}}$ and $f^{\mathbf{D}}$ so defined can evidently be expressed in terms of the associated α-functions introduced in section IV.3 and subsequently investigated in sections 4 and 5 of the same chapter. In point of fact, it follows immediately that

$$f^{\mathbf{U}} \equiv \alpha_0^0(k, p) \qquad \text{and} \qquad f^{\mathbf{D}} = \tfrac{3}{2}\alpha_1^0(k, p), \tag{2-6}$$

where the two requisite associated α-functions have already been expressed in section IV.4 explicitly in terms of the ratio

$$k = \frac{r_a}{r_b} \tag{2-7}$$

of the fractional radii of the *smaller* (r_a) and the *larger* (r_b) component, respectively, and of the geometrical depth

$$p = \frac{\delta - r_b}{r_a} \tag{2-8}$$

of the eclipse, where δ denotes (as in Chapters IV and V) the apparent distance of the projected centres of the two stars on the celestial sphere.

Both k and p are suitably normalized; but the same is *not* true of the functions $f(k, p)$ because they do not possess a common upper bound. At the moment of first contact ($p = 1$)

$$f_{a,b}^{\mathbf{U},\mathbf{D}}(k, 1) = 0 \tag{2-9}$$

and thus, by (2-5),

$$f_{a,b}(k, 1) = 0, \tag{2-10}$$

irrespective of limb-darkening. At the moment of internal tangency ($p = -1$) two alternatives may, however, arise—depending on whether the star undergoing eclipse is the smaller or the larger of the two. If the eclipse is an *occultation* and the internal tangency marks, therefore, the beginning of *totality*, then evidently

$$f_a^{\mathbf{U}}(k, -1) = 1 = f_a^{\mathbf{D}}(k, -1), \tag{2-11}$$

so that

$$f_{a,b}(k, -1) = 1. \tag{2-12}$$

If, on the other hand, the eclipse has been a *transit* and internal tangency marks the onset of an *annular* phase, we have seen in section IV.4 that

$$f_b^{\mathbf{U}}(k, -1) = k^2, \tag{2-13}$$

while

$$f_b^{\mathbf{D}}(k, -1) = \frac{4}{3\pi k^2}\{\sin^{-1}\sqrt{k} + \tfrac{1}{3}(4k - 3)(2k + 1)\sqrt{k(1-k)}\} \equiv \Phi(k); \tag{2-14}$$

and thus, by (2-5),

$$f_b(k, -1) = \frac{3k^2}{3 - u_b}\{1 - u_b + u_b\Phi(k)\}, \tag{2-15}$$

where u_b denotes the coefficient of limb-darkening of the *larger* component (of fractional radius r_b) of the two.

As k may assume any value between 0 and 1, the amount of fractional light $f_b(k, -1)$ lost at the moment of internal tangency of a transit eclipse will vary within the same limits. Let us, therefore, normalize the fractional loss of light by expressing $f(k, p)$ in terms of the amount of light lost at the moment of internal tangency of the respective eclipse—i.e., by introducing the *normalized fractional loss of light* $\alpha(k, p)$, defined by*

$$\alpha_a(k, p) = \frac{f_a(k, p)}{f_a(k, -1)} \tag{2-16}$$

* The reader should note that functions so normalized are *not* identical with associated α-functions.

if the eclipse is an occultation, and

$$\alpha_b(k, p) = \frac{f_b(k, p)}{f_b(k, -1)} \tag{2-17}$$

if it is a transit. Since, however, $f_a(k, -1) = 1$ and $f_b(k, -1)$ is given by equation (2-15), the foregoing equations assume the explicit forms

$$\alpha_a^x(k, p) = (1 - x_a)\alpha^{\mathbf{U}}(k, p) + x_a\alpha_a^{\mathbf{D}}(k, p) \tag{2-18}$$

if the eclipse is an occultation, and

$$\left.\begin{aligned} \alpha_b^x(k, p) &= \frac{(1 - x_b)\alpha^{\mathbf{U}}(k, p) + \frac{3}{2}x_b\Phi(k)\alpha_b^{\mathbf{D}}(k, p)}{1 - x_b + \frac{3}{2}x_b\Phi(k)} \\ &= (1 - x_b')\alpha^{\mathbf{U}}(k, p) + x_b'\alpha_b^{\mathbf{D}}(k, p) \end{aligned}\right\} \tag{2-19}$$

if it is a transit. In these equations we have abbreviated

$$x_a = \frac{2u_a}{3 - u_a}, \quad x_b = \frac{2u_b}{3 - u_b}, \quad x_b' = \frac{u_b\Phi(k)}{1 - u_b + u_b\Phi(k)}, \tag{2-20}$$

where u_a, u_b denote the degrees of limb-darkening of the *smaller* and the *larger* component, respectively.

The new functions $\alpha(k, p)$ defined in this way are bound to vary between 0 and 1 as p diminishes from 1 to -1, irrespective of k or the degree of darkening of the star undergoing eclipse. The only time where $\alpha(k, p)$ can exceed unity is during the annular phase of a transit across the disk of a limb-darkened star. In deducing equations (2-18) and (2-19) advantage has been taken of the fact that $\alpha_a^{\mathbf{U}}(k, p) = k^2\alpha_b^{\mathbf{U}}(k, p)$, so that hereafter $\alpha^{\mathbf{U}} \equiv \alpha_a^{\mathbf{U}}$. No such simple relation exists unfortunately, between α_a^x and α_b^x for $x \neq 0$ unless $k = 1$; so that separate tables of $\alpha_a^{\mathbf{D}}(k, p)$ and $\alpha_b^{\mathbf{D}}(k, p)$ are generally required to obtain α_a^x or α_b^x for any intermediate degree of darkening.

After this digression, let us return to equations (2-2) and (2-3). Adding them we obtain

$$f_a = 1 - l_a + (1 - l_b)(f_a/f_b) \tag{2-21}$$

or, alternatively,

$$f_b = 1 - l_b + (1 - l_a)(f_b/f_a). \tag{2-22}$$

If we replace now, in these equations, f_a and f_b by the corresponding normalized fractional losses of light $\alpha(k, p)$ as defined by equations (2-18) and (2-19) above we find that, if the eclipse is an occultation

$$\alpha_a^x(k, p) = 1 - l_a + \frac{1 - l_b}{k^2 Y(k, p)}; \tag{2-23}$$

while if it is a transit

$$\alpha_b^x(k, p) = (1 - l_a)\frac{Y(k, p)}{Y(k, -1)} + \frac{1 - l_b}{k^2 Y(k, -1)}, \tag{2-24}$$

where

$$Y(k, p) = \frac{(1 - x_b)\alpha^{\mathbf{U}}(k, p) + \frac{3}{2}x_b\Phi(k)\alpha_b^{\mathbf{D}}(k, p)}{(1 - x_a)\alpha^{\mathbf{U}}(k, p) + x_a\alpha_a^{\mathbf{D}}(k, p)} \tag{2-25}$$

is a slowly varying function of k, p, x_a and x_b. The foregoing equations (2-23) and (2-24) make it evident that

$$\frac{\alpha_a^x(k,p)}{\alpha_b^x(k,p)} = \frac{Y(k,-1)}{Y(k,p)}. \tag{2-26}$$

If $x_a = x_b = 0$, or $x_a = x_b$ *and* $k = 1$, the ratio on the right-hand side reduces to unity; but for any other value of these parameters it will be different from one and will vary slowly with the phase.

The foregoing equation specifies the ratio of the normalized losses of light at the corresponding phases of the alternate minima. A ratio of the normalized losses of light at different phases of the same minimum follows, however, directly from the observations; for it is easy to see that

$$\frac{\alpha(k,p_1)}{\alpha(k,p_0)} = \frac{1-l_1}{1-l_0}. \tag{2-27}$$

In particular, let the subscript '1' refer to any arbitrary phase, and subscript '0' to that of the maximum eclipse. It is customary to denote l_0 as λ and write (2-27) as

$$\alpha(k,p) = \frac{1-l}{1-\lambda}\,\alpha(k,p_0) = n\alpha_0, \tag{2-28}$$

where

$$n = \frac{1-l}{1-\lambda} \qquad \text{and} \qquad \alpha_0 \equiv \alpha(k,p_0) \tag{2-29}$$

—equations of which frequent use will be made later on. By definition, $0 \leqslant n \leqslant 1$ during the partial phases of any type of the eclipse; the only case in which n may actually exceed unity is provided by an annular transit of a limb-darkened star. Should the eclipse happen to be total, $p_0 = -1$ and $\alpha(k,-1)$ is, by definition, unity. In such cases, the normalized fractional losses of light $\alpha(k,p)$ can be determined directly in terms of the fractional light l observed at the corresponding phase, and of the value λ which l would assume at the moment of internal tangency. Should, on the other hand, the eclipse happen to be partial, this is not the case and the maximum fractional loss of light α_0 or the maximum geometrical depth p_0 attained at the moment of deepest eclipse (when $l = \lambda$) become additional unknowns to be determined by subsequent analysis. The same is also true of the case of an annular transit of a limb-darkened star, when $\alpha > 1$ and $p < -1$.

VI.3. COMPUTATION OF THE ELEMENTS: DIRECT METHODS

In the preceding section of this chapter, the geometrical aspects of the eclipses of spherical stars have been investigated in some detail, and formulae have been established which express the loss of light, during eclipses, in terms of the elements of a system. The main aim of such an investigation was, however, to provide the basis for a solution of the *converse* problem which can be stated as follows: Suppose that the changes of light caused by mutual eclipses of two spherical stars, appearing as circular disks with radially-symmetrical distribution of brightness, have been measured and, as a result, we are in possession of an adequate record of the observed changes of light as a function of the time. The problem is to determine such elements of the respective eclipsing system as can be extracted from an analysis of its light changes.

Even within the scope of the foregoing restrictions the problem at issue is one of considerable complexity; for the number of the elements required for a complete specification of a system is considerable: namely,

Semi-major axis of the relative orbit	A
Orbital eccentricity	e
Longitude of periastron	ω
Longitude of ascending node	Ω
Inclination of the orbit to the celestial sphere	i
Orbital period	P
Epoch of principal conjunction	t_0

in addition to the following properites of the individual components:

	Star Smaller	Larger
Radius	a_a	a_b
Luminosity	L_a	L_b
Surface brightness	J_a	J_b
Degree of limb-darkening	u_a	u_b

Of the elements specifying the position of the orbital plane in space, the longitude Ω of the ascending node must in most cases remain unknown; for there is no hope at present, or in the near future, of telescopic separation of any eclipsing pair;* and only one or two known eclipsing binaries possess orbits which are large enough to be measured astrometrically.† Photometric (or spectroscopic) data alone leave Ω indeterminate. The light changes due to eclipses should enable us to determine the absolute value of the inclination i of the orbital plane to the celestial sphere, but its algebraic sign will likewise remain indeterminate. The orbital eccentricity e and the

* Unless it be by a new interferometric method for measuring extremely small angular separations, which is being developed by R. Hanbury Brown and R. Q. Twiss (*cf. Phil. Mag.* (7) **45,** 663, 1954, and subsequent publications).

† Attempts at determining the astrometric orbits of the supergiant eclipsing systems ε Aurigae and VV Cephei from long-focus parallax plates are in progress at the Sproul Observatory.

longitude of the periastron ω can, in principle, be established by either spectroscopic or photometric data. The semi-major axis A of the relative orbit of both components can be determined from the spectroscopic data in absolute units; but since the light changes of any eclipsing system are affected by A only through the ratios $a/A \equiv r$, in all that follows we shall adopt A as our *unit of length* and express all other linear dimensions in terms of it. The absolute values of L_a and L_b or J_a, J_b are likewise determinable only if the parallax of the system is known; but nothing should prevent us from adopting the combined luminosity of the system $L_a + L_b$ as our *unit of light* and expressing the apparent brightness of the system, at any moment, in terms of this unit. Moreover, the *ratio* J_a/J_b should then be expressible in terms of r_a/r_b and L_a/L_b by definition. The orbital period P can invariably be determined from an extended series of observations to a degree of accuracy far surpassing that of any other element, by methods which are sufficiently well known not to warrant a repetition in this place;* and the epoch t_0 of the primary minimum can likewise be determined directly from the observations and independently of other elements. Furthermore, binary systems with periods as short as those of the majority of known eclipsing variables possess orbits whose eccentricities are so small that, for the time being, we shall consider them as circular; complications arising from finite orbital eccentricity and its photometric determination will be discussed later in sections VI.9 and 10.

Under these conditions, our problem thus reduces to the determination of the following six elements (suitably normalized):

$$r_a, r_b, i; \qquad L_a, u_a, u_b \, .$$

Of these, the first three are usually referred to as the *geometrical elements* of an eclipsing system—in contrast to the ratio of the radii $k = r_a/r_b$ and the maximum obscuration α_0 (or the maximum geometrical depth p_0) which are frequently termed the *elements of the eclipse*. The six elements listed above do not result from the photometric data with equal ease. In particular, the coefficients u_a, u_b of limb-darkening of the two components will prove to be so difficult to determine simultaneously with all other elements that special methods will have to be developed later (sections VI.7 and 8) to deal effectively with this particular aspect of our problem. This fact (which we mention here ahead of its proof to be given later in this chapter) compels us, however, to introduce one further simplification and to restrict our present problem to the determination of the elements for *assumed* values of the darkening of one or both components. Whether or not such assumed values are indeed compatible with the reality can be verified only at a later stage of orbital analysis.

The fundamental equations of so restricted a problem can now be set up in the following manner. Let us adopt the centre of the component

* For their thorough discussion *cf.*, e.g., J. G. Hagen, *Die Veränderlichen Sterne*, Freiburg 1921, pp. 609–671.

undergoing eclipse as our origin of coordinates, and the (constant) radius of the relative orbit of the companion as the unity of length. The apparent separation δ of centres of the two stars as projected on the celestial sphere at any moment will evidently be given by the equation

$$\delta^2 = \sin^2 \theta \sin^2 i + \cos^2 i, \tag{3-1}$$

where θ stands for the phase-angle of the eclipsing star in its relative orbit, as defined by equation (1-1), and i the inclination of the orbital plane to the celestial sphere. On the other hand we have, by (2-7) and (2-8), the identity

$$\delta = r_b(1 + kp), \tag{3-2}$$

where $k = r_a/r_b$. Equating (3-1) and (3-2) we find that

$$\sin^2 \theta \sin^2 i + \cos^2 i = r_b^2(1 + kp)^2, \tag{3-3}$$

which constitutes the *first fundamental equation* of our problem. It represents, to be sure, merely the identity $\delta^2 \equiv \delta^2$; but its significance rests on the fact that its left- and right-hand sides are expressible in terms of different observable quantities: namely, the time (through θ) on the left, and the fractional light l of the system (through $p(k, \alpha)$) on the right. For if, as before in section VI.2, l denotes the fractional luminosity of a system during eclipse and λ, the value of l at minimum light, the fractional loss of light α of the star undergoing eclipse can be expressed as

$$\alpha = \frac{1 - l}{1 - \lambda} \alpha(k, p_0) = n\alpha_0, \tag{3-4}$$

which constitutes the *second fundamental equation* of our problem.

Equations (3-3) and (3-4) are basic for a determination of the geometrical elements of any eclipsing system from an analysis of its light changes; and all methods which have been devised to this end are rooted in them. Of the quantities involved in (3-3) and (3-4), *two* are supplied by the observations (i.e., θ and α) at any phase; while the number of the unknowns is *four* (i.e., r_a, r_b, i and α_0) if the eclipses are partial, and *three* for total eclipses (when $\alpha_0 = 1$). Therefore, in this latter case, a minimum of three observations during partial phases of the eclipse are necessary to render the problem determinate.*

The way in which we combine their use depends, in principle, on the characteristics of the eclipses (total or partial) as well as on the mathematical

* As in certain problems of celestial mechanics, three observations are thus found to be theoretically sufficient for specification of the geometrical elements of totally eclipsing systems; but the underlying photometric observations are so much less accurate than astrometric measurements (the former can be made accurate to at best one part in a thousand, while the latter to one part in several millions) that it is well-nigh impossible to accomplish this in practice. In the interpretation of light changes of eclipsing binary systems the relatively low accuracy of the underlying observational data must be compensated by a large number of them, and the weight of the determination of the elements must be distributed over the whole cycle rather than being based on a few isolated points.

strategy adopted for their solution. The aim of the present section will, in particular, be to outline the methods leading to a *direct* solution of our underlying photometric problem for any type of the eclipses; an expose of *iterative* methods and their comparison with direct solutions being postponed for subsequent sections. Attempts at direct solution of our photometric problem originated with Pickering's work on Algol in 1880, and assumed their well-known practical form at the hands of Russell and Shapley between 1912–1914; while iterative methods have been developed by the present writer and his collaborators in the decade between 1941–50.

Russell's Method

The method proposed by Russell in 1912 to solve directly for the elements of totally-eclipsing systems from an analysis of their light changes can be outlined as follows. Let $\theta_j, p_j\,(j = 1, 2, 3)$ denote the phase angles and geometrical depths at any three moments within partial phases of the eclipse. The equations of the form (3-3) written out for these phases will contain the following three quantities, k, r_b and i as unknowns. Now Russell conceived the idea of eliminating r_b and i between them in order to isolate k; in doing so we find that the latter should be a root of the determinantal equation

$$\begin{vmatrix} \sin^2\theta_1 & (1+kp_1)^2 & 1 \\ \sin^2\theta_2 & (1+kp_2)^2 & 1 \\ \sin^2\theta_3 & (1+kp_3)^2 & 1 \end{vmatrix} = 0. \tag{3-5}$$

In order to solve for it, Russell proposed to assign to two of the points (θ_1, p_1) and (θ_2, p_2) on which (3-5) is based a specific phase (such that, at these points, $\alpha = 0{\cdot}6$ and $0{\cdot}9$, respectively) while retaining the third point as arbitrary (and its subscript may, therefore, be dropped). By developing the determinant on the left-hand side of (3-5) he finds that

$$\sin^2\theta = A + B\psi(k, p), \tag{3-6}$$

where

$$\left.\begin{aligned} A &= \sin^2\theta_1, \\ B &= \sin^2\theta_1 - \sin^2\theta_2, \end{aligned}\right\} \tag{3-7}$$

and

$$\psi(k, \alpha) = \frac{2(p-p_1) + k(p^2 - p_1^2)}{2(p_1 - p_2) + k(p_1^2 - p_2^2)}. \tag{3-8}$$

The whole complexity of the problem is now evidently confined to this latter function $\psi(k, \alpha)$, but may be overcome by its tabulation in terms of k and α for specific degrees of limb-darkening of the star undergoing eclipse.* Such tables have indeed been constructed;† and with their aid any point on the

* It is at this stage that we have to commit ourselves to a definite value for the limb-darkening of the eclipsed star; the darkening of the eclipsing component is so far irrelevant.

† For their latest version *cf.* J. E. Merrill, *Princ. Contr.*, No. 23, 1950–53.

light curve within partial phases, combined with the two standard points (θ_1, p_1) and (θ_2, p_2) lends itself to an independent determination of the numerical values of $\psi(k, \alpha)$ and thus (for known α) of k, the mean of which may be adopted. Once, moreover, the value of k has thus been established, the remaining geometrical elements of our problem follow from

$$\cot^2 i = \frac{B}{\Phi_1(k)} - A \tag{3-9}$$

and

$$r_b^2 \csc^2 i = \frac{B}{\Phi_2(k)}, \tag{3-10}$$

where

$$\left.\begin{aligned} \Phi_1(k) &= 1 - \left\{\frac{1 + kp_2}{1 + kp_1}\right\}^2, \\ \Phi_2(k) &= (1 + kp_1)^2 \Phi_1(k), \end{aligned}\right\} \tag{3-11}$$

stand for two auxiliary functions* which have likewise been tabulated.

The foregoing equations (3-8), (3-9) and (3-10) represent (within the scheme of assumptions at the basis of the whole procedure) a direct solution of our photometric problem if the observed light minima are due to total eclipses. If, however, such eclipses become *partial* (as evidenced by a continuous variation of light throughout *both* minima) our problem becomes complicated by the fact that, although the fractional loss α of light of the eclipse continues to be given by equation (3-4), the *maximum obscuration* α_0 is no longer unity (as for total eclipses), but becomes an additional unknown to be determined simultaneously with r_a, r_b and i. In order to do so, let us depart again from the fundamental equation (3-3) which, at the moment of maximum eclipse ($\theta_0 = 0$ and $\alpha = \alpha_0$) reduces to

$$\cos^2 i = r_b^2(1 + kp_0)^2, \qquad p_0 \equiv p(k, \alpha_0). \tag{3-12}$$

Subtracting (3-12) from (3-3) we find that

$$\sin^2 \theta = \frac{r_a r_b}{\sin^2 i}\{k(p^2 - p_0^2) + 2(p - p_0)\}. \tag{3-13}$$

In order to eliminate the unknown multiplicative constant on the right-hand side of (3-13), Russell proposed to divide it by an analogous equation pertaining to a specific phase of the eclipse—such as, for instance, when $p_1 \equiv p(k, 0{\cdot}5)$. If so, it follows obviously that

$$\frac{\sin^2 \theta(n)}{\sin^2 \theta(\frac{1}{2})} = \frac{k(p^2 - p_0^2) + 2(p - p_0)}{k(p_1^2 - p_0^2) + 2(p_1 - p_0)} \equiv \chi(k, \alpha_0, n), \tag{3-14}$$

where the function $\chi(k, \alpha_0, n)$ thus introduced will play, for partial eclipses,

* Not to be confused with $\Phi(k)$ as defined by equation (2-14) of the preceding section.

a role similar to that played by $\psi(k, \alpha)$ if the eclipse is total: in point of fact, these two are related by the equation

$$\chi(k, \alpha_0, n) = \frac{\psi(k, n\alpha_0) - \psi(k, \alpha_0)}{\psi(k, \frac{1}{2}\alpha_0) - \psi(k, \alpha_0)}. \tag{3-15}$$

The function $\chi(k, \alpha_0, n)$ depends on three parameters, of which only one (i.e., n) is known from observations. Suppose, however, that the values of $\sin^2 \theta$ have been determined for three distinct phases at which $n = n_1, n_2$, and $\frac{1}{2}$. A solution of the relations

$$\left.\begin{aligned} \chi(k, \alpha_0, n_1) &= c_1, \\ \chi(k, \alpha_0, n_2) &= c_2, \end{aligned}\right\} \tag{3-16}$$

should then be sufficient to specify k and α_0—provided, of course, that the $\chi(k, \alpha_0, n)$'s are functionally independent. In practice, *the χ-functions have been found to simulate functional dependence so closely as to make a determination of the elements of partially-eclipsing systems from observations of one minimum effectively impossible.* In order to render the problem determinate, another independent relation between k and α_0 must be sought: and this is provided by the equation

$$\alpha_0 = 1 - \lambda_a + \frac{1 - \lambda_b}{k^2 Y(k, p_0)}, \tag{3-17}$$

deduced in the preceding section, where λ_a, λ_b stand for the fractional intensities of the system at the moments of internal tangency of the occultation (λ_a) or transit (λ_b) eclipses, and the auxiliary function $Y(k, p_0)$ has been defined by equation (2-25). The explicit form of (3-17) requires, therefore, a knowledge of not only the depths of both minima, but also of the degrees of darkening of both components (involved through $Y(k, p_0)$). Moreover, an application of (3-17) presumes a knowledge as to which one of the two minima is due to an occultation or transit eclipse. This may (in the absence of spectroscopic data) not be known *a priori*, and must in general be ascertained by trial and error; the necessary (though not sufficient) condition for the correctness of either alternative being a consistency of the equations

$$\left.\begin{aligned} \alpha_0 &= 1 - \lambda_a + (1 - \lambda_b)/k^2 Y, \\ \chi(k, \alpha_0, n) &= \text{constant}, \end{aligned}\right\} \tag{3-18}$$

which must be satisfied by a pair of k and α_0, such that $0 < k < 1$ and $0 < \alpha_0 < 1$. And once these have been found, the values of the remaining geometrical elements r_b and i may be obtained by a straightforward solution of two equations of the form (3-1) written down for $\theta = 0°$ and $\theta(\frac{1}{2})$; and the solution of our problem would appear to be complete.

If we pause, however, a little to reconsider the foundations on which the foregoing direct solution is based, our view of the situation confronting us will become considerably less optimistic. In point of fact, the mathematician might inquire by what right do we believe to have obtained any solution of our underlying problem at all—as certain essential parts of it (i.e., the location of the pivotal points corresponding to $\alpha = 0{\cdot}5$, $0{\cdot}6$, or $0{\cdot}9$ on the light curve) had to be guessed in advance. This is indeed a valid objection which strikes deep at the roots of the Russell–Shapley methods expounded in this section, or indeed of any other method aiming at a direct solution for the photometric elements of eclipsing binary systems. For it is a fact that the positions of the pivotal points of the light curves are seldom—if ever—directly observed for a lack of advance knowledge when the fractional loss of light α will assume the requisite round values; and even if it were perchance possible to observe them, their coordinates could not be determined with greater weight than those of any other point of the light curve. Moreover, any errors in their assigned positions would obviously go through the whole determination of the photometric elements, vitiating them to an extent to which Russell's method offers no clue.

In order to lessen the errors arising from this source, Russell and Shapley proposed to establish the location of the pivotal (or indeed any other) point on the light curve, not directly from the observations, but from a smooth curve drawn by free hand to follow the course of the individual observations or normal points. A graphical interpolation of this nature may have constituted a shrewd move, justifiable in the early days of astronomical photometry when the relative accuracy of the individual normals was between 1 and 10% of the measured light intensities. With the gradually increasing precision of photometric observations the time was, however, bound to come when an empirical interpolation and smoothing of the observed data, as represented by the drawing of free-hand curves, was bound to become a liability. For it is a fact that a substitution of such curves for the actual observations may amount to an unwarranted and possible risky interference with the basic data, which may vitiate all results based upon them in a systematic manner. Moreover, no matter what the scale of such curves, or the skill with which they could be drawn, their exclusive use would never permit us to learn the *uncertainty* with which the elements of eclipsing binary systems are defined by the available observational data.

As in all other branches of astronomy or of physical sciences in general, a determination of the uncertainty of any results based on photometric observations of finite accuracy should represent an integral part of each solution; for without it we should not know the extent to which such photometric elements can be trusted. The methods of Russell and Shapley, developed with the aim of directness and convenience rather than completeness or rigour, are ill-suited to cope with this all-important requirement. Any method relying on the use of fixed points read off a free-hand curve, and aiming primarily at a graphical representation of the observed data, assigns arbitrary weights to different parts of the light curve (in fact, infinite weights to the selected pivotal points) and, as a result, cannot enable us to ascertain any errors of the solution, or its stability. A mere inspection of the deviations of normal points from free-hand curves can ordinarily disclose very little about the way in which the observational errors can affect individual elements. In point of fact, the less determinate the solution the easier it is to obtain a seemingly good graphical representation of the observations by a right combination of wrong elements—as only too many investigators have learned in the past the hard way to their sorrow.

Kopal's Method

As was pointed out, however, by the present writer,* the drawbacks of the Russell–Shapley method inherent in the adoption of the pivotal points can be completely removed by resorting to a different strategy of solution. In order to outline it, let us return again to our fundamental equation (3-3) which, if we abbreviate

$$\sin^2\theta \equiv x \qquad \text{and} \qquad (1+kp)^2 = y(k, \alpha), \tag{3-19}$$

we may rewrite as

$$x = y(k, \alpha) r_b^2 \csc^2 i - \cot^2 i. \tag{3-20}$$

Since r_b and i are constants, equation (3-20) should, in the xy-coordinates, be represented by a straight line. The quantities x and y are, however, not independent; but are constrained by the condition that points specified by a given pair of x and y should lie on the light curve. This happens for total eclipses if, in accordance with equation (3-5),

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \tag{3-21}$$

—i.e., only for the value of the ratio of the radii k which is a solution of this equation. We may, therefore, set out to determine the ratio of the radii as such a value of k which, inserted in (3-20), renders this equation a straight line in the xy-coordinates. The remaining geometrical elements then follow at once from the intercepts subtended by this line on both axes.

The actual determination of k must proceed by trial and error, about which we shall have more to say later on (*cf.* section VI.4). Having adopted a provisional k, we should specify the values of θ and α for each individual observation or normal point appropriate for partial stage of the eclipse. With the aid of the provisional k and an appropriate table of $p(k, \alpha)$ we find the corresponding values of y and plot the observed points, in the xy-coordinates, thus obtaining a 'characteristic diagram' of each light curve. Unless we happen to adopt the right value of k to begin with, the points plotted out in the characteristic diagram will generally follow some curve. A small arbitrary change in the provisional value of k will either increase or diminish its curvature, indicating the direction in which k needs to be adjusted; and if we overcorrect it, the curvature will change its sign.† This procedure

* Z. Kopal, *Ap. J.*, **94**, 145, 1941, and subsequent publications.

† Time may be saved in these preliminary plots by using the round values of k which appear as arguments in available tables of the p-functions. Only when two such round values lead to curvatures in opposite sense will k-wise interpolation become necessary.

should, if necessary, be repeated until all observed normal points come as close as possible to a straight line in the xy-coordinates. The value of k which produces this is the true ratio of radii of the two components, while the intercept on the x-axis furnishes at once the value of $\cot^2 i$; and that on the y-axis, $r_b^{-2} \cos^2 i$.* A modification of the same technique to a treatment of partially-eclipsing systems is quite analogous, and may be left as an exercise for the reader. For two illustrative examples of such characteristic diagrams *cf.* the accompanying Fig. 6-4.

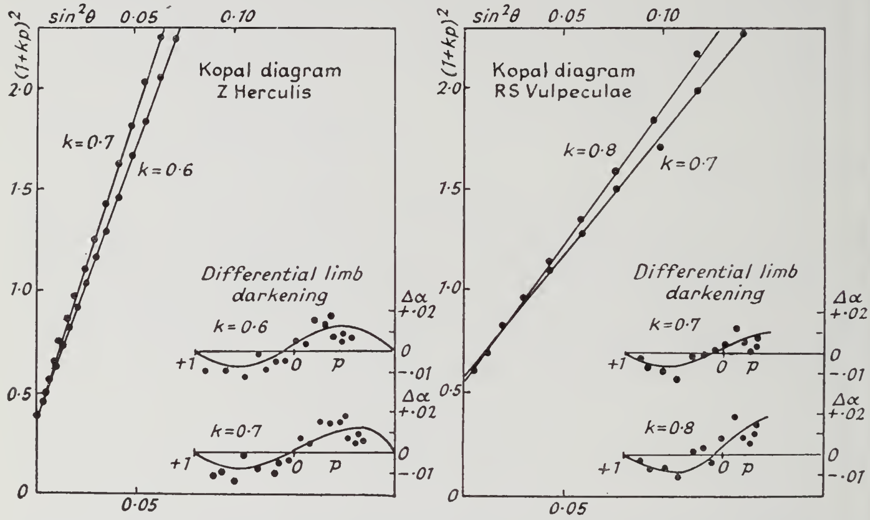

FIGURE 6-4. THE "CHARACTERISTIC DIAGRAM" OF Z HERCULIS AND RS VULPECULAE after R. L. Baglow, *Publ. David Dunlap Obs.* (Toronto), **2**, No. 1, 1952.

It may be added that the strategy outlined in the foregoing paragraph is also applicable to the fundamental equations (3-6) or (3-13) of Russell's method for total or partial eclipses. In the former case, by putting $\sin^2 \theta \equiv x$ and $\psi(k, \alpha) \equiv z$, a plot of equation (3-13) in xz-coordinates should also lead to a straight line for the correct value of k. The scale for z in such a diagram would, however, be much more open, and the z-function (as well as the intercepts on x- and z-axes) much more complicated functions of the geometrical elements than if the xy-coordinates (3-19) are used. The same comment applies, in general, to a treatment of partial eclipses in this manner.

Quite apart from questions of feasibility, the reader should take good care to note that, in their methodical aspects, Russell's and Kopal's methods as expounded in this section differ in fundamentals: namely, the latter method

* This makes it evident that, for any real value of the inclination, the intercept on the x-axis must be negative—just as, in Russell's method, we must have $A/B \leqslant \Phi_1(k)$. If, contrary to expectation, a line representing equation (3-20) happens to intersect the positive direction of the x-axis, the most plausible conclusion is that the degree of darkening at the basis of the adopted tables of $p(k, \alpha)$ is incompatible with reality (though the converse is not necessarily true).

gives up any attempt at a frontal solution of equation (3-21) for k, ahead of the determination of other elements, and seeks instead to locate k indirectly, as such value of the ratio of radii of the two components which can 'rectify' the fundamental equation (3-3) in suitable coordinates. The principal gain of such a strategy has been the ability to make use of the observed photometric data as they stand, without having to give undue weight to any point on the light curve, to which it may not be entitled on observational grounds. In doing so we had, however, to give up any pretence at a *direct* solution of our photometric problem. As soon as we decide, for reasons mentioned earlier in our discussion of the drawbacks of Russell's method, to base our solution solely on the original observed data, and wish to forego replacing them from the outset by the results of graphical smoothing as represented by free-hand curves, any hope of a direct solution must inadvertently be abandoned. Further research into the techniques of direct utilization of the observed data led the present writer and his associates to a systematic development of *iterative* methods to this end, whose exposition will be the subject of subsequent sections.

VI.4. ITERATIVE METHODS: TOTAL (ANNULAR) ECLIPSES

After covering the preliminary ground outlined in the preceding sections, the actual analysis may now begin. Suppose, for the sake of argument, that an inspection of the observational evidence leads us to conclude (or conjecture) that the light minima of the system under investigation are due to total and annular eclipses. The next step for us to take is to decide whether the eclipse giving rise to the primary (deeper) minimum is total or annular—i.e., whether it is the larger star or the smaller one which possesses greater surface brightness. This can also often be decided by inspection as already explained in section VI.1; for example, if there is a definite phase of constant light at the bottom of one minimum and a continuous light variation through the other, there is no room for doubt that the former minimum is due to a total eclipse and the latter to an annular one, and that the larger component is appreciably darkened at the limb. Unless this larger star is intrinsically variable, the light of the system during totality must necessarily remain constant.

Suppose, furthermore, that photometric observations made in the course of such an eclipse have been grouped in a suitable number of normal points, each characterized by a pair of observed values of θ and l. The problem which confronts us now is to determine the elements of the system by an appropriate analysis of such data. In more specific terms, the elements we seek to determine first simultaneously are r_a, r_b and i. We shall frequently refer to this trio as the *geometrical elements* of the system, in contrast to the *physical elements* (such as the fractional luminosities of the components, of their mean densities, or the ratio of their surface brightnesses, etc.) which can usually be determined separately once the geometrical elements are known.

The fundamental equation which relates the geometrical elements with the observed quantities results by eliminating the apparent fractional separation δ of the centres of the two components between equations (3-1) and (3-2): doing so we obtain

$$\sin^2 \theta \sin^2 i + \cos^2 i = r_b^2\{1 + kp(k, \alpha)\}^2, \tag{4-1}$$

which at the moment of the inner contact ($p = -1$) reduces to

$$\sin^2 \theta'' \sin^2 i + \cos^2 i = r_b^2(1 - k)^2, \tag{4-2}$$

where θ'' denotes the phase angle of the internal tangency. Subtracting (4-2) from (4-1) and solving for $\sin^2 \theta$ we obtain

$$(p^2 - 1)C_1 + 2(p + 1)C_2 + C_3 = \sin^2 \theta, \tag{4-3}$$

where

$$\left.\begin{aligned} C_1 &= r_a^2 \csc^2 i, \\ C_2 &= r_a r_b \csc^2 i, \\ C_3 &= \sin^2 \theta'', \end{aligned}\right\} \tag{4-4}$$

are constants to be determined. The reader may notice that, in principle, the choice of these auxiliary constants is not unambiguous. At the moment of the first contact ($\theta = \theta'$), for instance, equation (4-1) would reduce likewise to

$$\sin^2 \theta' \sin^2 i + \cos^2 i = r_b^2(1 + k)^2, \tag{4-5}$$

and subtracting it from (4-1) we would obtain

$$(p^2 - 1)C_1 + 2(p - 1)C_2 + C_3 = \sin^2 \theta \tag{4-6}$$

as an alternative to (4-3), in which C_1 and C_2 have the above meanings, but $C_3 = \sin^2 \theta'$. Theoretically, equations (4-3) and (4-6) are equivalent; in practice, the one should be chosen which will yield the values of the unknown constants with greater weight; and this may depend on the distribution of the observations and their weights along the light curve. In most cases, however, it is the value of θ'' which is defined by the observations much more precisely than θ', and equation (4-3) involving θ'' explicitly is, therefore, to be preferred.

Of the quantities involved in the fundamental equation (4-1) of the present problem, two are supplied by the observations (θ, α) and three are unknown (r_a, r_b, i). Therefore, at least three equations of the form (4-1) or of any of its derivatives are required to render our problem determinate. Thus, as in certain problems of classical celestial mechanics, *three* observations are found theoretically sufficient to specify uniquely the geometrical elements of a totally eclipsing system. Our problem differs, however, so drastically from celestial mechanics in the precision of the underlying observational data (*cf.* section VI.1) that it is impossible to do so in practice. In eclipsing orbit work a large number of photometric observations must compensate for their low relative accuracy; and the weight of a determination of the elements must be distributed over the whole light curve rather than based on a few isolated points.

The coefficients of C_1 and C_2 in equations (4-3) or (4-6) involve the geometrical depth of the eclipse p which is, in turn, an implicit function of k and δ. In section VI.2 we mentioned that the normalised fractional loss of light $\alpha(k, p)$ can be expressed (and tabulated) in terms of k and p for any type of eclipse and any degree of darkening of the eclipsed star. Conversely, the geometrical depth p can be regarded as a function of k and α and obtained by a numerical inversion of the respective α-tables. Since no formula analogous to (2-5) exists to yield the values of $p(k, \alpha)$ for intermediate degrees of darkening by linear interpolation between the two limiting cases, such inversion has to be performed independently for each degree of darkening and each type of the eclipse (i.e., occultation or transit). A construction of such tables represents an arduous piece of work which has taxed the skill and perseverance of several previous investigators.* The most extensive and accurate set of tables of the p-function in existence, appropriate for $u = 0$, 0·2, 0·4, 0·6, 0·8 and 1 and for occultation as well as transit eclipses, was recently completed by Tsesevich and published in the *Bulletin of the Astronomical Institute of the U.S.S.R. Academy of Sciences*, No. 45 and 50—two fundamental publications which will accompany us through most of the discussion in this chapter and will hereafter be referred to as Tsesevich 45 and Tsesevich 50, respectively. Their possession will permit us to obtain the numerical value of p appropriate for any pair of k and α with sufficient accuracy and a minimum of effort for any type of eclipse.

Preliminary Determination of the Ratio of the Radii

Of the two arguments of p, the value of α characteristic of any observed normal point can be obtained directly from equation (2-28) which, by virtue of the fact that, by definition, $\alpha(k, -1) = 1$, yields

$$\alpha(k, p) = \frac{1 - l}{1 - \lambda} = n, \tag{4-7}$$

where l is the instantaneous fractional luminosity of the system at the moment of observation and λ, the respective luminosity at the moment of internal tangency. Any measured value of l, combined with a value of λ estimated from the plot of our light changes, will, therefore, yield us directly the corresponding value of α. The value of the second parameter $k = r_a/r_b$ depends, however, by definition on the two quantities which are not known to us beforehand and which we seek to determine. An approximate value of k can, however, be inferred for totally eclipsing systems without difficulty.

* For appropriate references *cf.* the Bibliographical Notes to section VI.2 at the end of this chapter (and, in particular, Kopal's survey article in the *Journ. on Math. Tables and other Aids to Comp.*, **3**, 191, 1948).

Two ways for doing so are, in principle, open to us and their choice will depend on whether *one* or *both* minima of our eclipsing system have been adequately observed.

If the latter is the case, the possibility suggests itself to fall back on equation (2-23) and to remember that, when $p = -1$ and $l = \lambda$, $\alpha^x(k, -1) = 1$ for any type of eclipse. Equation (2-23) can then be solved for k to yield

$$k^2 = \frac{1 - \lambda_b}{\lambda_a Y(k, -1)} \tag{4-8}$$

where, by (2-25)

$$Y(k, -1) = 1 - x_b + \tfrac{3}{2}x_b\Phi(k), \tag{4-9}$$

and λ_a, λ_b denote the fractional intensities of the system at the moments of internal tangency of the *occultation* and *transit* eclipses, respectively; x_b being a quantity related to the degree of limb-darkening of the *larger* component by means of equation (2-20). The solution of equation (4-8) for k must obviously proceed by approximations; the use of a detailed table of $\Phi(k)$ in terms of k, as given, for instance, in Tsesevich 50 (Table 5, p. 307) will greatly expedite the numerical work. As many significant figures should be retained in this computation as can be guaranteed in the λ's; a knowledge of the approximate value of k to two decimals will be ample for all subsequent purposes. It should also be noted that, in order to approximate k by means of equation (4-8), an estimate of the degree of darkening of the larger component is prerequisite, irrespective of whether this component is the eclipsing or the eclipsed one.

Had it proved possible to identify the type of eclipse giving rise to the primary minimum at the outset of our analysis, equation (4-8) would permit us to approximate k uniquely. If, however, we do not yet know at this stage whether the primary minimum is an occultation or a transit, equation (4-8) will furnish us a pair of two possible values of k obtained by setting the fractional intensity λ_1 at the moment of internal tangency of the primary minimum successively equal to λ_a and λ_b. Fortunately, in many cases a discrimination between two such possible values of k can be readily effected. In more precise terms, if the orbital plane of our eclipsing system is parallel with the line of sight (i.e., if $i = 90°$)—an expectation which, for totally eclipsing systems, is not likely to be too far off—a simple geometry shows that

$$\frac{\sin^2 \theta''}{\sin^2 \theta'} = \frac{1 - k}{1 + k} \tag{4-10}$$

and, hence,

$$k = \frac{\tan \frac{1}{2}(\theta' - \theta'')}{\tan \frac{1}{2}(\theta' + \theta'')} \tag{4-11}$$

or, provided that θ' is not too large,

$$k = \frac{\theta' - \theta''}{\theta' + \theta''} \tag{4 12}$$

gives a somewhat larger but still sufficiently close value of k. For $i < 90°$, the foregoing equation yields a *lower limit* of a possible value of k which is consistent with the observed duration of the partial phase of the eclipse. If one of the k's deduced from equation (4-8) lies significantly below this limit, it becomes automatically eliminated, and the hypothesis on which it was based is disproved.

Considering the limited accuracy with which one can deduce θ'' and, in particular, θ' directly from the observed data, it is evident that the criterion offered by the above equation (4-12) will not be too effective unless the alternative values of k as obtained from equation (4-8) by setting $\lambda_1 = \lambda_a$ or λ_b differ widely. This will be true if the depths of the alternate minima of our eclipsing system are very unequal; and, in such cases, the chances for the deep minimum being due to a total eclipse are overwhelming. If, on the other hand, the depths of both minima are comparable and therefore, of necessity, rather shallow,* the two values of k consistent with equation (4-8) will not differ much and we may not be in a position to identify the type of the eclipse giving rise to the primary minimum before consulting the light changes which occur during partial phases. In the absence of any evidence to the contrary, it is in general recommended to embark upon subsequent analysis by assuming the primary (deep) minimum to be due to a total eclipse. An assumption of total eclipse at primary minimum always leads to a geometrically possible solution; while that of an annular eclipse does so only if $1 - \lambda_1 < k^2 Y(k, -1)$; otherwise the computed brightness of the smaller component would come out negative.

Summarizing the gist of the preceding paragraphs we may observe that if the depths of both minima can be estimated from our light curve with some accuracy, equation (4-8) should provide us with a relatively good provisional value (or a pair of values) of the ratio of the radii k; whereas if only one minimum is available for analysis, this can be estimated with a somewhat inferior accuracy by means of equations (4-11). Let us denote the provisional value of k—whatever its source—by K. Once the latter is fixed, the values of $p(k, \alpha)$ appropriate for each observed normal point can be found from Tsesevich's tables constructed for the respective type of eclipse (i.e., occultation or transit) and the adopted degree of darkening u of the star undergoing eclipse. The latter should be estimated to the best of our knowledge from the spectral type (or colour) of the component undergoing eclipse, and in accordance with the effective wave length of light (visual, photographic, photoelectric) in which the observations were carried out.† An opportunity to check upon the correctness of our estimate of u, and its compatibility with

* For equation (4-8) makes it evident that (since $k < 1$) in no eclipsing system can the fractional losses of light in both minima exceed 0·5 in intensity units, or 0·75 of a magnitude.

† For a discussion of the actual degrees of darkening appropriate for stars of different spectral types *cf.* the preceding section IV.1 of this book; while the available empirical information on this subject has been summarized by Z. Kopal on pp. 623–627 of the *I.A.U. Transactions*, **9**, 1957.

the ensemble of the photometric data secured within eclipses as well as between minima, will present itself at later stages of our analysis; but in order to start this analysis, *specific commitments concerning both the type of eclipse* giving rise to the minimum under investigation, *and the limb-darkening of the eclipsed star must be made now*, before entering the p-tables.* It goes without saying that if we have evaluated the provisional value of K from equation (4-8) assuming the primary minimum to be an occultation, we cannot enter with it the transit p-tables; or if we have assumed the primary minimum to be a transit, the degree of darkening of the star undergoing eclipse at the time of this minimum must have already been fixed when solving (4-8) for the respective value of k. If, on the other hand, the provisional value of K has been obtained from the estimated durations of the total (annular) and partial phases, the choice of the type of the eclipse giving rise to the primary minimum has thus far been left open.

Formation and Weighting of the Equations of Condition

Once these points have been settled with due care, the appropriate tables furnish readily the values of p (based on the adopted values of K and α) which permit us to evaluate the coefficients $p^2 - 1$ and $2(p - 1)$ of the quantities C_1 and C_2 in equations (4-3) or (4-6). Tsesevich's tables contain the values of $p(k, \alpha)$ correct to four decimal figures. Owing to the errors inherent even to the most precise photometric measure available at the present time, the fourth decimal in p would; however, carry as yet little significance. In all that follows we shall, therefore, limit ourselves to extracting from the tables the values of the geometrical depth p to three decimals, if the errors of the underlying photometric data are of the order of 0·1% (i.e., of the order of $0\overset{m}{.}001$), and to two decimals if the observations are of only a moderate precision (i.e., errors of the order of $0\overset{m}{.}01$ in the individual normals); in

* This is perhaps not the place to dwell too extensively on explaining the use of Tsesevich's tables; but it may be mentioned that Tsesevich uses the symbol $p(k, \alpha'^{x})$ to denote the tables appropriate for the occultation eclipses, and $p(k, \alpha''^{x})$ for the transits; his 'x' being identical with our degree of darkening u. The following concise key to Tsesevich's p-tables relevant to partial phases of the eclipse may perhaps be found useful.

u	Occultation		Transit		Source
	Table	page	Table	page	
0	VI	318	VI	318	Tsesevich 50
0·2	VII	322	XI	338	
0·4	VIII	326	XII	342	
0·6	IX	330	XIII	346	
0·8	X	334	XIV	350	
1·0	II	136	IV	148	Tsesevich 45

It should also be noted that Tsesevich's symbols α'^{x} and α''^{x} are identical with our α_a^x and α_b^x, respectively; our departure from Tsesevich's notations being prompted by the desire to avoid the use of double superscripts.

either case, the degree of subtabulation in Tsesevich's tables is such as to permit linear interpolation in both k and α over most of their range.*

The accuracy to which the coefficients of C_1 and C_2 in equation (4-3) are computed and recorded should, naturally, be commensurate with that of the basic p-values. In general, a three-digit accuracy is recommended for the coefficients of C_1 and C_2, and four digits in the absolute terms on the right-hand sides. *As many equations of this form can obviously be set up as there are observed normal points*, characterised by a known pair of l and θ, *and regarded as equations of condition, whose least-squares solution should yield the most probable values of* C_1, C_2 *and* C_3, corresponding to the adopted value of K. If the latter were correct, the ratio

$$\frac{C_1}{C_2} = k \tag{4-13}$$

should have verified it. If, however, this is not the case and this latter ratio differs significantly from K, the solution will have to be repeated—each time with the ratio C_1/C_2 resulting from the previous approximation taken as a basis of the next cycle—until the assumed and resulting values of the ratio of the radii agree.

This is, however, getting somewhat ahead of our subject; for before we can proceed to the actual formation of our first system of normal equations, which will permit us to iterate for k in the manner just described, the weights of the individual equations of condition will have to be duly considered. These weights are, in general, apt to be very different from equation to equation for two reasons. First, the normal points on which our equations are based may not consist of an equal number of individual observations, or these observations may be unequal in quality. Whether or not this is so must be made clear by the observer himself, and his information will have to guide us in ascribing to each normal a certain relative empirical weight.

The second reason which renders the weights of our equations of condition unequal is, however, intrinsic in our method of solution. Each normal point, on which such equations are based, is completely specified by a pair of measured quantities l and θ. In general, the measurements of the time (i.e., of θ) are so superior in relative accuracy to the measurements of light that practically the entire uncertainty of our solution will depend on the errors of the light measurements. This implies, in turn, that *it is the coefficients of the unknowns, rather than the absolute terms, of the equations of condition of the form* (4-3) *which are principally affected by observational errors.*

* In practice, this bi-variate interpolation can be avoided if we round off the provisional value K of the ratio of the radii to the nearest tabular argument. Since Tsesevich gives the p's for k = 0·00 (0·05) 1·00 if the eclipse is an occultation, and 0·20 (0·05) 1·00 if it is a transit, the rounding off of K to the nearest tabular argument could never entail an error in excess of 0·025 in K, which would be but seldom significant at the outset.

A small error Δl committed in measuring the instantaneous brightness of the system is, however, essentially equivalent to an error of

$$\frac{d \sin^2 \theta}{dl} \Delta l$$

on the right-hand side of equation (4-3); and if so, the weight $\sqrt{w}$ of the respective equation will be given by

$$\sqrt{w} = \frac{dl}{d \sin^2 \theta} \frac{\varepsilon}{|\Delta l|}, \tag{4-14}$$

where ε denotes the error of a light measurement of unit weight. Now, provided that there is no difference in empirical weight between the individual normals and that the errors of measurement are constant on the intensity scale, we may put $\varepsilon = |\Delta l|$—in which case

$$\sqrt{w} = \frac{dl}{d \sin^2 \theta}, \tag{4-15a}$$

whereas if the errors of observation are constant on the logarithmic (magnitude) scale, we may put $\varepsilon = |\Delta l|/l$, obtaining

$$\sqrt{w} = \frac{1}{l} \frac{dl}{d \sin^2 \theta}. \tag{4-15b}$$

The derivative $dl/d \sin^2 \theta$ in the preceding equations characterizes the slope of the light curve plotted in the (l-$\sin^2 \theta$) coordinates, and its value could be read off a smooth curve drawn freely to follow the course of observed normal points. It can, however, also be evaluated directly—without recourse to any free-hand curve—by a differentiation of the fundamental equation (4-3). Since

$$\frac{d \sin^2 \theta}{dl} = \frac{d \sin^2 \theta}{d\alpha} \frac{d\alpha}{dl} = -\frac{1}{1 - \lambda} \frac{d \sin^2 \theta}{d\alpha} \tag{4-16}$$

by (4-7), and by differentiating (4-3) we find that

$$\frac{d \sin^2 \theta}{d\alpha} = 2C_2(1 + kp) \frac{\partial p}{\partial \alpha}, \tag{4-17}$$

it follows that, if the errors of observation are constant on the *intensity* scale,

$$\sqrt{w} = -\frac{1 - \lambda}{2C_2(1 + kp)(\partial p/\partial \alpha)}; \tag{4-18a}$$

while if they are constant on the *magnitude* scale,

$$\sqrt{w} = -\frac{1 - \lambda}{2lC_2(1 + kp)(\partial p/\partial \alpha)}. \tag{4-18b}$$

It should be noted that the quantity C_2 in the denominator of the preceding expressions is not known at the stage when the individual equations of conditions are weighted; but since it is constant we can eliminate it by multiplying both sides of each equation by it. The term $1 - \lambda$ in the numerator, representing the fractional loss of light at the bottom of the minimum under investigation, is likewise a constant, and therefore immaterial as long as only *one* minimum is subject to our analysis. If, however, the equations of condition of *both* minima are being combined in a single set of normal equations, the factor $1 - \lambda$ is likely to be different for primary and secondary minima and must therefore be applied. It is *this* factor which will render the weight of the equations of condition pertaining to the secondary minimum small in comparison with those based on the primary minimum if the latter is deep and the former shallow.

The remainder of the terms in the denominator on the right-hand sides of equations (4-18) can be easily evaluated by setting K for k, $p(K, \alpha)$ for $p(k, \alpha)$, and replacing the partial derivative $\partial p/\partial\alpha$ by a ratio of first tabular differences taken from the appropriate table in the neighbourhood of the point in question. It may be mentioned that this derivative is negative in the entire domain (which reverses the minus sign of the right-hand sides of equations (4-18) and becomes infinite at the moment of either contact*—thus reducing the weight there to zero. A division by the instantaneous fractional luminosity l of the system will not be of much consequence if the amplitude of the light changes is small, but may become an important—and, in fact, controlling—element of the weight if the amplitude becomes large. An amplitude of 2·5 magnitudes can evidently cause $\sqrt{w}$ to vary from ten to one between the inner and outer contacts of the eclipse; an amplitude of five magnitudes will exaggerate this range from a hundred to one. This alone makes it obvious that whenever the errors of observation are constant on the magnitude scale (and, therefore, the absolute errors diminish with diminishing light), the weight of a determination of the elements of the eclipse will rest predominantly on observations made near the inner contact and in advanced partial phases, with the early stages of the eclipse contributing relatively little to the weight of the whole solution.

The expressions for $\sqrt{w}$ as defined by equations (4-18) represent—we repeat—the intrinsic weights of the individual equations of condition, arising solely from the way in which the observational errors affect their absolute terms. If, in addition, the individual equations are also observationally of unequal weights, the *total weight* of the respective equation should be a *product* of the *empirical times intrinsic weight*; its square-root should then be taken and the respective equations of condition multiplied by it before the normal equations are formed. In what follows, however, $\sqrt{w}$ will continue to stand for the intrinsic weight as defined by equations (4-18) alone; a multiplication of each equation by the square-root of its

* Except at the inner contact of a transit eclipse.

observational weight (whenever the latter differs from equation to equation) will be taken for granted. With regard to the numerical accuracy to which $\sqrt{w}$ should be evaluated, two significant figures are desirable and three figures are ample; while more than three figures in $\sqrt{w}$ would as yet be entirely superfluous.

Differential Corrections for λ and *U*

When the equations of condition of the form (4-8) have been formed for each observed normal point and multiplied by square-roots of their proper observational and intrinsic weights, we are almost ready to proceed with a solution for the elements of the eclipse, but not yet quite so. For the whole process of solution, as described in the preceding paragraphs, still makes use of two constants, the values of which had to be read off approximately from the observed data at the outset: namely, the brightness U of our system between minima (our unit of light), as well as the brightness λ of the system at the moment of internal tangency of the total or annular phase. The value of U is usually taken as the appropriate mean of all individual observations of the brightness of the system made in full light; and unless the minima are broad, or (what happens much more frequently) the number of observations made between minima is inadequate, its determination may carry considerable weight—much greater than that with which λ can as a rule be found. If the eclipse under consideration is an occultation which ultimately becomes total, λ_a remains constant during the whole interval of totality; and if its duration is appreciable, enough observations of minimum brightness can be secured to determine λ_a with fair accuracy. If, on the other hand, the eclipse happens to be a transit, the brightness of the system is apt to vary during annular phase, thus rendering a determination of λ_b at the moment of internal tangency inadvertently more difficult.

Suppose, however, that we have estimated both U and λ from the available observational data as well as we can, and have proceeded with iteration as outlined earlier in this section. If the solution of properly weighted equations of condition is performed by the method of least-squares, the most probable values of C_1, C_2 and C_3 will be obtained; but any error in the adopted values of U of λ will affect them all systematically. Moreover, if the errors within which the most probable values of C_1, C_2 and C_3 are defined by the available observational data were evaluated from the residuals of such a solution, they would faithfully reflect the dispersion of the observed normals between the outer and inner contacts, but would fail to take any cognizance of the uncertainty with which U as well as λ could be inferred from the observations and, in consequence, the real errors of all computed elements of the eclipse would be underestimated. This, if unremedied, would constitute indeed a serious defect of the method of solution which we propose to follow. It has, however, been shown by Piotrowski* that we can improve the provisionally

* *Cf.* S. L. Piotrowski, *Ap. J.*, **108**, 36, 1948.

adopted values of U and λ *simultaneously* with a determination of the most probable values of C_1, C_2 and C_3, and obtain the differential corrections ΔU and $\Delta\lambda$ which are required to ensure the best possible fit. Piotrowski's procedure which makes this possible is as simple as it is elegant and can be summarized as follows.

Let the fundamental equation (4-3) be rewritten as

$$\sin^2\theta = H(C_1, C_2, C_3;\ U, \lambda), \tag{4-19}$$

where

$$H \equiv (p^2 - 1)C_1 + 2(p+1)C_2 + C_3, \tag{4-20}$$

$p \equiv p(k, \alpha)$ and

$$\alpha = \frac{U - l}{U - \lambda}. \tag{4-21}$$

Let, furthermore,

$$\left.\begin{aligned} U &= 1 + \Delta U, \\ \lambda &= \lambda_0 + \Delta\lambda, \end{aligned}\right\} \tag{4-22}$$

where U, λ are the true values of the respective quantities; 1, λ_0, their estimated values; and ΔU, $\Delta\lambda$, the requisite corrections which we shall seek to determine. In order to do so, we shall expand $H(C_1, C_2, C_3, 1 + \Delta U, \lambda_0 + \Delta\lambda)$ in a Taylor series in ascending powers of ΔU and $\Delta\lambda$, and anticipate these quantities to be small enough for their squares and higher powers to be negligible. Our fundamental equation (4-19) generalized in this way then takes the form

$$H_0 + \left(\frac{\partial H}{\partial U}\right)_0 \Delta U + \left(\frac{\partial H}{\partial \lambda}\right) \Delta\lambda + \ldots = \sin^2\theta, \tag{4-23}$$

where we have abbreviated $H_0 \equiv H(C_1, C_2, C_3;\ 1, \lambda_0)$.

Now

$$\frac{\partial H}{\partial U} = \frac{\partial H}{\partial \alpha}\frac{\partial \alpha}{\partial U}, \qquad \frac{\partial H}{\partial \lambda} = \frac{\partial H}{\partial \alpha}\frac{\partial \alpha}{\partial \lambda}; \tag{4-24}$$

while from (4-21) it follows that

$$\frac{\partial \alpha}{\partial U} = \frac{1 - \alpha}{U - \lambda}, \qquad \frac{\partial \alpha}{\partial \lambda} = \frac{\alpha}{U - \lambda}. \tag{4-25}$$

Moreover, the square-root $\sqrt{w}$ of the intrinsic weight of our equations of condition can, with a sufficient accuracy, be approximated by

$$\sqrt{w} = -(1 - \lambda)\frac{d\alpha}{dH_0}. \tag{4-26}$$

Combining this with the preceding relations we easily establish that

$$\sqrt{w}\left(\frac{\partial H}{\partial U}\right)_0 = -(1 - \alpha) \tag{4-27}$$

and

$$\sqrt{w}\left(\frac{\partial H}{\partial \lambda}\right)_0 = -\alpha. \tag{4-28}$$

When we insert this in (4-23) properly weighted, our generalized equation of condition then takes the neat form:

$$\sqrt{w}(p^2 - 1)C_1 + 2\sqrt{w}(p + 1)C_2 + \sqrt{w}C_3 \\ - \alpha\Delta\lambda - (1 - \alpha)\Delta U = \sqrt{w}\sin^2\theta, \qquad (4\text{-}29)$$

and involves five unknowns to be simultaneously determined; the correction $\Delta\lambda$ pertaining always to the depth of the minimum under investigation—whether this be an occultation or a transit.

An inspection of the coefficients of the foregoing equation (4-29) discloses that $\alpha \equiv n$, as defined by equation (4-7), is a quantity which follows directly from the observations, while $p(K, \alpha)$ as well as $\partial p/\partial\alpha$ (in $\sqrt{w}$) can be excerpted from the appropriate tables for the adopted value of K and each one of the observed values of α. It should be added that, in equation (4-29) as it stands, the errors of the underlying observations are assumed to be constant on the intensity scale, and $\sqrt{w}$ to be given by equation (4-18a). Should, on the other hand, the photometric errors be equal on the magnitude scale and equation (4-18b) used to define $\sqrt{w}$, the coefficients of ΔU as well as of $\Delta\lambda$ in equation (4-29) would still have to be divided by l. It was also mentioned earlier that, of all factors involved in equation (4-18), C_2 in the denominator will usually not be known to us beforehand and we shall, therefore, have to get rid of this constant from $\sqrt{w}$ by multiplying the whole equation (4-29) by C_2. Since, however, the corrections ΔU and $\Delta\lambda$ in (4-29) are not multiplied by $\sqrt{w}$, it follows that, after a pre-multiplication of all equations of condition of the form (4-29) by C_2, *the last two unknowns will be $C_2\Delta U$ and $C_2\Delta\lambda$ rather than the corrections themselves*—a fact to be borne in mind when we shall proceed, in the next section, to combine such equations with other parts of the observational evidence.

The new equations of condition of the form (4-29) represent a very powerful tool for investigating the geometrical elements of the eclipse, and would permit us to determine the maximum or minimum brightness of our system even if no direct observations of these phases were available. One could simply estimate them—no matter how crudely—from the form of the light curve during partial phases, and refine them subsequently, by means of equation (4-29), to the degree of accuracy attainable from the given data.* In practice, however, a resort to such extreme measures will seldom be necessary; as a matter of fact, for well-covered light curves the corrections ΔU as well as $\Delta\lambda$—particularly the former—may come out insignificant. Even should it be so, however, the effort entailed in including ΔU and $\Delta\lambda$ in our solution of partial phases of the eclipse was not misspent, for it will enable us later to ascertain the extent to which the uncertainty of U and λ will contribute to the errors of the computed elements of the eclipse.

* Piotrowski (*Ap. J.*, **108**, 36, 1948) gives a convincing numerical example of such a process (*op. cit.*, p. 45).

In practical cases—even though the observations made during partial phases of the eclipse will contribute their share to our knowledge of U and λ—the main burden of their determination will naturally have to rest on the observations made during full light and during the total or annular phase. The question arises then as to the proper way in which this additional evidence should be combined with the observations of partial phases to compound the final result. If (as will usually be the case) our adopted unit of light represents an appropriate mean of all observations secured between minima and λ_a, the mean of all observations made during totality, the additional equations of condition bearing on their determination will be of the form

$$\sqrt{w_U}\Delta U = 0 \tag{4-30}$$

and

$$\sqrt{w_\lambda}\Delta\lambda_a = 0, \tag{4-31}$$

respectively. Any reader who may be inclined to regard them as trivial should remember that these are additional *equations of condition* which need not be fulfilled rigorously; how closely ΔU or $\Delta\lambda$ will actually approach zero will depend on the *weights* $\sqrt{w_U}$ and $\sqrt{w_\lambda}$ with which these equations will join the rest of our overdeterminate system. These weights are, in turn, specified by the relative uncertainty of the values adopted for U and λ_0—or, in more specific terms, by the standard deviations σ_U and σ_λ of the individual observations from their respective means—in accordance with the equations

$$\sqrt{w_U} = \varepsilon/\sigma_U \qquad \text{and} \qquad \sqrt{w_\lambda} = \varepsilon/\sigma_\lambda, \tag{4-32}$$

where ε denotes, as before, the corresponding error of a single normal point, within minima, of unit weight. If, in particular, the observations between minima are grouped into N normals equivalent in weight to those within minima, then obviously $w_U = N$; and similarly if N such normals are available during totality, $w_\lambda = N/\lambda_a^2$ if they are equal on the magnitude scale.

This completes a survey of the equations which will be fundamental to our analysis; for we are now in a position to take simultaneously into account the relative contribution of each part of the whole light curve to a solution for the elements of the eclipse. Every normal point formed from the observations made during partial phases of the primary or secondary minimum will furnish us with an equation of condition of the form (4-29) for five unknowns C_1, C_2, C_3, ΔU, and $\Delta\lambda_a$ if this minimum is an occultation, and C_1, C_2, C_3, ΔU, and $\Delta\lambda_b$ if it is a transit. The observations between minima will furnish a single equation of the form (4-30) which will participate in the determination of ΔU, while the total phase of an occultation eclipse furnishes an equation of the form (4-31) adding its weight to the determination of $\Delta\lambda_a$. In any attempt to solve equations (4-30) and (4-31) together with (4-29) for the most probable values of the unknown quantities which we are seeking to determine we should remember, however, that if we pre-multiplied all equations of condition of the form (4-29) by C_2 in order to cancel this constant from $\sqrt{w}$, the last two unknowns in (4-29) became $C_2\Delta U$ and $C_2\Delta\lambda$—in contrast with simple values of ΔU and $\Delta\lambda$ in equations (4-30) and (4-31). Before we shall, therefore, be able to combine all equations of condition of the form (4-29) with (4-30) and (4-31) into a single set of normal equations, we shall have to rewrite the latter two equations as

$$(\sqrt{w_U}/C_2)(C_2\Delta U) = 0 \tag{4-30a}$$

and

$$(\sqrt{w_\lambda}/C_2)(C_2\Delta\lambda_a) = 0. \tag{4-31a}$$

In order to evaluate the coefficients of these equations, a preliminary estimate of C_2 is clearly a prerequisite. Ultimately, we should mention that equations (4-30) or (4-31), as they stand, apply without qualification only to total eclipses and their depths. During the annular phase of a transit eclipse of a uniformly bright star, equations (4-30) or (4-31) continue to hold good, with the sole change that λ_a is to be replaced by λ_b. Should, however, the larger component be darkened at the limb to an arbitrary degree, equation (4-29) in the form appropriate for the partial phase of a transit continues to hold good during the annular phase as well, provided only that the requisite values of $p(K, \alpha)$ are taken from Tsesevich's 'annular' tables.*

Combination of Both Minima

Now we are in a position to construct a set of normal equations, appropriate for either minimum, and iterate their solutions for k. A combination of the normal equations based on both minima into a single set should, however, be postponed until the final stage of intermediary analysis; for before arriving at it we must make sure that

(*a*) the type of eclipse at either minimum has been correctly identified, and that

(*b*) both minima yield a consistent set of the geometrical elements.

The first point can be settled conclusively by the success (or failure) of the successive approximations for k inherent in our process. It is obvious that *our iterations will converge to the true value of k if, and only if, the type of the eclipse under investigation has been correctly identified*, and the correct set of p-tables used to evaluate the coefficients of C_1 and C_2 (as well as the

* In more specific terms, the geometrical depth p, during the annular phase of transit eclipses, must be found in terms of the auxiliary depth q as defined by equation (4-63) of Chapter IV. Tsesevich defines

$$\alpha_b^{\mathbf{D}}(k, q) = 1 + A^x(k)X(k, q),$$

where $X(k, q)$ is a function contained in Table XV (Tsesevich 50, p. 354), and the functions $A^x(k)$ for Tsesevich's x (i.e., our u) = 0·2 (0·2) 1·0 can be found in Tables XXI, XX, XIX, XVIII and XVII, (Tsesevich 50, pp. 364–366) respectively. The quantity $\alpha_b^{\mathbf{D}}(k, q)$ on the left-hand side of the preceding equation (4-33) is identical with 'α' of equation (4-7) and can thus be deduced directly from the observations. With the aid of the adopted values of K and u and of the respective tables we can use (4-33) to evaluate $X(K, q)$. A further auxiliary table of $q(k, \alpha)$ (Tsesevich 50, Table XVI, p. 358) will permit us to find the corresponding value of q, and solving for p from (4-63) of Chapter IV in terms of q we ultimately obtain

$$p = q - (q/K) - 1.$$

weights $\sqrt{w}$) in the equations of condition (4-29). If this has been done, the iterative process converges so rapidly that more than two cycles will but seldom be required. If, on the contrary, the difference between K and k after the second iteration fails to be substantially smaller than it was after the first one, the indications are that a wrong type of the eclipse has been used and that, instead of dealing (say) with an occultation, we are dealing with a transit. The alternative set of p-tables should then be used to set up a new system of our equations of condition, new normal equations formed, and their iterations repeated until k has at last been stabilized. In this way we shall always be able to find out *a posteriori* whether or not we were correct in our original assumption as to the type of the eclipse giving rise to the primary minimum, which we may have had to conjecture if no secondary minimum was observed (or if both minima were sensibly equal in depth).

If both minima have been adequately observed, we can go one step further. The foregoing discussion has made it clear that, in systems exhibiting total and annular eclipses, both minima can be subject to orbital analysis quite independently and on their own merits. It is, however, evident that not only should the solution of each minimum converge, but both should converge to the *same* set of geometrical elements. Let us fix our attention on the ratio k of the radii as the most sensitive one of such elements. If the values of the 'shape' k as deduced from the form of the light curve during partial phases of the occultation and transit eclipse fail to be the same within the limits of their uncertainty, while the value of the 'depth' k based on the depths of both minima and resulting from equation (4-8) comes out intermediate between the two, the evidence is unmistakable that the degrees of limb-darkening of both components have been estimated incorrectly and that, therefore, improper p-tables were used to calculate the coefficients of our equations of condition. If, on the other hand, the 'depth' k happens to agree sensibly with the 'shape' k, this indicates that the degree of darkening of the smaller star alone has been incorrectly estimated. In general, if the 'shape' k as deduced from an analysis of an *occultation* eclipse comes out *smaller* than the 'depth' k, this indicates that the degree of darkening of the *smaller* star has been *underestimated;* while if the 'shape' k as deduced from a *transit* is *less* than the 'depth' k, the darkening of the *larger* star was assumed to be *too large*. In such cases, the recalcitrant coefficients of darkening of one (or both) stars should be altered, by trial and error, until both 'shape' k's and the 'depth' k's are in agreement at least within the limits of their uncertainties. Thus, in totally eclipsing systems, an initially *separate* treatment of both minima should permit us, not only *to distinguish an occultation from a transit eclipse* (by a success or failure of the iterative process based on the respective type of tables), but also *to check upon the correctness of the assumed degree of limb-darkening of both stars* (by the requirement that both minima should yield the same set of geometrical elements). *It is not until all this has been done that the equations of condition of the form* (4-29), *based upon the observations of both the primary and the secondary minima,* *should be combined together with* (4-30) *and* (4-31) *into a single system of normal equations, which will yield the final set of the intermediary elements.*

The foregoing procedure tacitly assumes that the shape of both minima has been adequately observed. The same possibilities are, however, open to us in a diminished measure even if the secondary minimum is too shallow to enable us to make much use of its light record during partial phases, but if at least its amplitude can be safely established. It is true that if the secondary minimum is very shallow, the factor $1 - \lambda_2$ in (4-18) will reduce the weight of the equations of condition of the form (4-29) based upon it to a relative insignificance. Even though the *form* of a shallow secondary minimum by itself can disclose to us at this time very little, its *depth* λ_2, combined with λ_1 through the medium of equation (4-8) can furnish a relatively accurate value of the 'depth' k. This 'depth' k should now be compared with the 'shape' k as deduced from the primary minimum. If the type of eclipse giving rise to this minimum has been correctly identified, and the degree of darkening of the primary component correctly estimated, these two k's again should not differ significantly. If they do, the alternative type of eclipse should be adopted for the primary minimum, or the degree of darkening of the primary star adjusted until any significant discrepancy between the 'shape' and 'depth' k's has been removed.

The need of establishing a reliable value of the 'depth' k emphasizes the need of an accurate determination of the amplitude of the secondary minimum, even if the latter is too shallow to render the details of its shape to be of any avail. The difficulty of getting anything useful out of its shape arises from the fact that, for shallow eclipses, $1 - \lambda$ becomes a small divisor in equation (4-7), which will greatly magnify the errors in α due to the uncertainty of the observed values of $1 - l$. Suppose, however, that we reverse the problem and advance the following consideration. If we knew the shape of the secondary minimum, we could obviously evaluate $1 - \lambda$ from each normal point observed within partial phases of the secondary minimum by means of the equation $1 - \lambda = (1 - l)/\alpha$, which is a direct consequence of (4-7). In actual practice, the proper shape of the secondary minimum will, of course, not be known to us exactly. *It can*, however, *be approximated with an adequate precision from the shape of the primary* (*deep*) *minimum.* Barring the presence of complications of unknown origin, the minima should differ in shape only on account of possible unequal degree of darkening of the two components—a difference which is likely to be minor under any circumstances. Suppose that we ignore this difference in form between the primary and secondary minimum, and *assume the* α's *at corresponding moments of the alternate minima to be equal.* On the other hand, a differentiation of equation (4-21) yields

$$\alpha[U - \lambda + \Delta(1 - \lambda)] = 1 - l + l\Delta U, \tag{4-33}$$

from which, by substituting $\Delta(1 - \lambda) = \lambda\Delta U - \Delta\lambda$, we obtain

$$(n - 1)\Delta U - n\Delta\lambda = 1 - l - n(1 - \lambda)\,. \tag{4-34}$$

If the moment of the mid-secondary minimum and, therefore, the phase angle θ is known to us with a reasonable accuracy, we can replace, in the foregoing equation, the uncertain values of $\alpha(\pi + \theta)$ by those of $\alpha(\theta)$ which should be known much better from the well-observed light changes of the primary minimum. If, moreover, λ denotes the estimated fractional light of the system at the moment of internal tangency of the secondary minimum and l, that observed at any phase $\pi + \theta$, the only remaining unknown quantities in (4-34) are ΔU and $\Delta\lambda$. *As many equations of the form* (4-34) *can obviously be set up as there are normal points observed within the secondary minimum, and solved for* ΔU *and* $\Delta\lambda$ by the method of least squares; in many cases, we may be justified in putting $\Delta U = 0$ in advance, which will leave us with $\Delta\lambda$ as the only unknown. The quantity $\lambda + \Delta\lambda$ then represents the best approximation obtainable for the fractional light of the system at the moment of internal tangency of the secondary minimum, and should be used whenever the latter is needed.

It is not until this has been done that we are in a position to consider combining our knowledge of the depth of the secondary minimum, and the shape of the primary minimum in a single solution for the elements of the eclipse. The reader should recall that equation (4-13) permits us to write

$$C_1 - kC_2 = 0\,, \tag{4-35}$$

where k may be identified with our 'depth' k. Since a determination of the latter is subject to errors arising from the uncertainty of λ_a and λ_b, equation (4-35) does not represent an exact relation between C_1 and C_2, but merely an additional *equation of condition* to be solved simultaneously with equations (4-29), (4-30) and (4-31). The only point which still requires clarification is the relative weight $\sqrt{w_k}$ of equation (4-35) as compared with that of other equations of our over-determinate system; and this can be found from the following considerations. An error of $\pm\Delta k$ in the adopted value of the 'depth' k will evidently produce an error of $\pm C_2\Delta k$ on the right-hand side of equation (4-35). Now, differentiating (4-8), we find that

$$\Delta k = (2k\lambda_a)^{-1}\{(\Delta\lambda_b)^2 + (k^2\Delta\lambda_a)^2\}^{1/2}\,, \tag{4-36}$$

where the uncertainties $\Delta\lambda_a$ and $\Delta\lambda_b$ of depths of the two minima can be estimated from the available observational data. If, as before, ε denotes the respective error of a single normal point, within minima, of unit weight, the square-root of the relative weight $\sqrt{w_k}$ of the equation (4-35) will again be given by

$$\sqrt{w_k} = (\varepsilon/C_2)\Delta k \tag{4-37}$$

and should be evaluated (or estimated as well as we can; for an accurate value of C_2 may not be known in advance) to multiply equation (4-35) before the latter joins the rest of our overdeterminate system. The relative influence of this equation will depend on the ratio of uncertainties of the respective 'shape' and 'depth' k's. The reader should expect that, if the depths of both minima can be reliably estimated, the relative weight of the 'depth' k and, in consequence, of the additional equation (4-35) will always be considerable. If the primary minimum is deep, the 'shape' k based upon it may be of comparable weight; but *if*, on the other hand, *both minima are shallow, equation* (4-35) *properly weighted may exert an important*—and even controlling—*effect on the determination of all elements.*

Evaluation of the Elements

Once the value of the ratio of the radii of both components has been stabilized by repeated iterations and the most probable values of the auxiliary constants C_1, C_2 and C_3 obtained, we are in a position to complete our analysis by evaluating the geometrical elements of the system under investigation. Since, by definition,

$$\left.\begin{aligned} C_1 &= r_a^2 \csc^2 i\,, \\ C_2 &= r_a r_b \csc^2 i\,, \\ C_3 &= (r_b - r_a)^2 \csc^2 i - \cot^2 i\,, \end{aligned}\right\} \tag{4-38}$$

and the constants on the left-hand sides are now known, a solution of these equations yields

$$\left.\begin{aligned} r_a^2 &= C_1^2/G\,, \\ r_b^2 &= C_2^2/G\,, \\ \sin^2 i &= C_1/G\,, \end{aligned}\right\} \tag{4-39}$$

where we have abbreviated

$$G = (C_2 - C_1)^2 + (1 - C_3)C_1\,. \tag{4-40}$$

Since, during the interval of totality, the whole light of the system is due to that of the larger star, the fractional luminosities of the two components readily follow from

$$\left.\begin{aligned} L_a &= 1 - \lambda_a\,, \\ L_b &= \lambda_a\,. \end{aligned}\right\} \tag{4-41}$$

The ratio of the mean surface brightnesses J_a/J_b of the two components, defined by the relation

$$\frac{L_a}{L_b} = k^2 \frac{J_a}{J_b}\,, \tag{4-42}$$

follows from

$$\frac{J_a}{J_b} = \frac{1 - \lambda_a}{k^2 \lambda_a} = Y(k, -1)\frac{1 - \lambda_a}{1 - \lambda_b} \tag{4-43}$$

by (4-8) and (4-41); the ratio of the *central* brightnesses of the two apparent disks being

$$\frac{J_a}{J_{b_c}} = \frac{3 - u_b}{3 - u_a}\frac{J_a}{J_b}\,. \tag{4-44}$$

The theoretical angles θ' and θ'' of the outer and inner contacts of the eclipse (and thus the theoretical durations of the minima) can ultimately be obtained by inserting in equations (4-1) or (4-5) the appropriate values of r_a, r_b and i as deduced from (4-39).

It should be stressed that, in equations (4-41) and (4-43), λ_a and λ_b denote the fractional intensities corrected for $\Delta\lambda_a$ and $\Delta\lambda_b$, and pertaining to the corrected unit of light. It should also be mentioned that a least-squares solution for the most probable values of C_1, C_2, C_3, $\Delta\lambda_a$, $\Delta\lambda_b$ and ΔU should permit us to do more than to translate its results, by means of equations (4-41)–(4-43), into most probable values of the elements we seek to determine. The *uncertainty* of the auxiliary constants C_1, C_2, . . . etc., caused by the limited accuracy of the underlying observational data and implied in our least-squares solution, should permit us to determine also the corresponding *uncertainties of all elements*, which constitute an inseparable part of our solution. Before we come, however, to discuss the appropriate process by which this can be done, we propose first to outline the process of solution for the elements of partially eclipsing systems; for the methods of error analysis in either case will turn out to be so similar and to have so much in common that it is expedient to treat them both at the same time.

VI.5. ITERATIVE METHODS: PARTIAL ECLIPSES

If the light curve of an eclipsing system exhibits a continuous variation in both minima, the eclipses giving rise to them are necessarily partial and the problem of determining the geometrical elements of such eclipses from an analysis of their light curves becomes somewhat more complex. This is due to the fact that the fractional loss of light α of the eclipsed star, at any moment, will now be given by

$$\alpha = n\alpha_0 \tag{5-1}$$

where, in accordance with (4-21)

$$n = \frac{U - l}{U - \lambda}, \tag{5-2}$$

but the *maximum fractional loss of light*, α_0, is no longer unity as it was when the eclipses were total, or annular, and *becomes an additional unknown* to be determined by appropriate analysis. This analysis is, in general, analogous to the one described previously in connection with the treatment of total eclipses; but since the elements to be determined are now *four*—i.e., r_a, r_b, i *and* α_0—the problem confronting us will be found to possess certain particular features which will presently be discussed.

Let us start again from the fundamental equation of our problem

$$\sin^2\theta \sin^2 i + \cos^2 i = r_b^2(1 + kp)^2, \tag{5-3}$$

which at the moment of conjunction ($\theta = 0°$) now reduces to

$$\cos^2 i = r_b^2(1 + kp_0)^2. \tag{5-4}$$

Subtracting (5-4) from (5-3) we obtain

$$\left.\begin{aligned} (p^2 - p_0^2)C_1 + 2(p - p_0)C_2 &= \sin^2 \theta, \\ p \equiv p(k, n\alpha_0) \qquad p_0 &\equiv p(k, \alpha_0) \end{aligned}\right\} \tag{5-5}$$

where, as before,

and

$$\left.\begin{aligned} C_1 &= r_a^2 \csc^2 i, \\ C_2 &= r_a r_b \csc^2 i. \end{aligned}\right\} \tag{5-6}$$

Let us suppose, for the sake of argument, that only one minimum has been observed (and that its depth as well as our unit of light is free from observational errors, so that we may replace U by unity); is it possible, in such a case, to determine r_a, r_b, i as well as α_0 simultaneously by an appropriate analysis of the light changes? To begin with, we are usually in the dark as to the type of eclipse (i.e., occultation or a transit) giving rise to this minimum; moreover, the values of both k and α_0 must be estimated before equations of condition of the form (5-5) can be set up, on the assumption of an occultation or a transit, for every observed normal and solved for the unknown constants C_1 and C_2. Once such a solution has been performed, the initially assumed value of K can be checked by a comparison with the ratio C_1/C_2; but is there any way of checking upon the correctness of the assumed value of α_0? This should indeed be possible in principle if we again avail ourselves of the method of the preceding section—i.e., regard

$$(p^2 - p_0^2)C_1 + 2(p - p_0)C_2 \equiv H(C_1, C_2, \alpha_0 + \Delta\alpha_0), \tag{5-7}$$

and expand the H-function in a power series of the form

$$H_0 + \frac{\partial H_0}{\partial \alpha_0} \Delta\alpha_0 + \ldots = \sin^2 \theta, \tag{5-8}$$

where $H_0 \equiv H(C_1, C_2, \alpha_0)$. Before the equations of condition of the form (5-7) can be solved for the most probable values of the unknowns, they should (apart from the possible different observational weights) be multiplied by square-roots $\sqrt{w}$ of their intrinsic weights, which can again be derived by the method of the preceding section and, for partial eclipses, are found to take the explicit form*

$$\sqrt{w} = \frac{dl}{d \sin^2 \theta} = -\frac{1 - \lambda}{2\alpha_0 C_2(1 + kp)(\partial p/\partial \alpha)}, \tag{5-9}$$

which differs from equations (4-18) only by the presence of α_0 (which formerly

* Whenever, in what follows, the values of λ are used without subscripts, it is understood that they apply equally to either occultations or transits. The same will be true of α_0 when used without primes.

was equal to one) in the denominator. Differentiating H_0 with respect to α_0 and multiplying this derivative by $\sqrt{w}$ we readily establish that

$$\sqrt{w}\,\frac{\partial H_0}{\partial \alpha_0} = \frac{1-\lambda}{\alpha_0}\left\{\sqrt{\frac{w_0}{w}} - n\right\}, \tag{5-10}$$

where w_0 denotes the intrinsic weight at the moment of maximum eclipse.

When the expression (5-10) is inserted in (5-8), an equation results containing C_1, C_2 and $\Delta\alpha_0$ as unknowns. If this equation could actually be iterated for all the unknowns, this would imply that no matter how crude a guess at α_0 we started from, it could be improved by successive approximations until the correct value is obtained. This is indeed possible *in theory* but, unfortunately, *not in practice*, because the coefficient of $\Delta\alpha_0$ in equation (5-8) is not only numerically very small, but—to make matters worse—its variation simulates that of the coefficient of C_1 to such an extent as to render a simultaneous determination of C_1 and $\Delta\alpha_0$ from the same set of equations a virtual impossibility.* An inclusion of $\Delta\alpha_0$ as an additional unknown to be determined simultaneously with C_1 and C_2 by a least-squares solution of the equations of condition of the form (5-8) is found to entail so drastic a loss of weight of the whole solution that such a procedure appears to be futile even with the best light curves now available. This demonstrates, therefore, that *a determination of a complete set of geometrical elements of partially eclipsing systems from one minimum alone remains as yet impracticable.*

Combination of the Alternate Minima

When, however, at least the *depths* of *both* minima have been observed, the situation is thoroughly altered; for then the quantities k and α_0 can be connected by means of equations (2-23) and (2-24) which, for $l = \lambda$, assume the explicit forms

$$\alpha_a^x(k, p_0) \equiv \alpha_0' = 1 - \lambda_a + \frac{1-\lambda_b}{k^2 Y(k, p_0)} \tag{5-11}$$

if the eclipse is an occultation, and

$$\alpha_b^x(k, p_0) \equiv \alpha_0'' = (1-\lambda_a)\frac{Y(k, p_0)}{Y(k, -1)} + \frac{1-\lambda_b}{k^2 Y(k, -1)} \tag{5-12}$$

if it is a transit. In these equations, $Y(k, p_0)$ is obtained from (2-25) by putting $p = p_0$, and $Y(k, -1)$ is explicitly given by (4-9). For total eclipses ($\alpha_0' = \alpha_0'' = 1$), the foregoing equations led us to the determination of the 'depth' k; but now, with α_0 as an additional unknown, they furnish us with a relation between k and α_0 which is independent of the *form* of the light curve and which *will permit us to ascertain the value of* α_0 *corresponding to any assumed value of* k.

* It should be observed that the right-hand side of equation (5-10) vanishes, not only at the beginning of the eclipse (when $\sqrt{w} = n = 0$) as well as at the moment of maximum obscuration (when $\sqrt{w/w_0} = n = 1$), but also around $p = -p_0$ during the partial phases.

If, therefore, the depths of both minima of a partially eclipsing system are known, a determination of the elements of our system can take a different course.* As the first step toward such a solution, we must estimate a *provisional value* of k—let us denote it again by K—and settle the question of the *types of eclipse* giving rise to the two minima—i.e., decide whether $\lambda_1 = \lambda_a$ or λ_b. In contrast with the case of total eclipses, for partial eclipses there is—alas—no 'depth' or even 'duration' k available now to guide us at the beginning of our work; so that unless our system was already subject to previous investigations and some preliminary elements are available, our choice of k may have to be an almost outright guess. Spectroscopic evidence, if available, may sometimes enable us to specify k with some accuracy independently of the form of the light curve (*cf.* section VI.7), or may indicate whether or not the component of greater surface brightness is also likely to be greater in size. If, however, in the absence of any preliminary elements or other pertinent information we are completely in the dark as to the type of eclipses giving rise to the primary or secondary minima, the best strategy is to start the solution by assuming $K = 1$—in which case a distinction between occultations and transits loses any meaning.

Having estimated K and inferred from the observations the values of λ_a and λ_b as well as that of our unit of light to the best of our knowledge, *our next step should be to evaluate* α_0 *corresponding to the assumed value of* k, for either eclipse, *by means of equations* (5-11) *and* (5-12). This task is not as simple as it may seem at first, because α_0 occurs in these equations, not only explicitly on the left-hand side, but also implicitly through p_0 in $Y(k, p_0)$; and this latter feature will compel us to resort to successive approximations. In more specific terms, the quantity for which we wish to solve from (5-11) or (5-12) directly in terms of k is the maximum geometrical depth p_0 of the eclipse rather than α_0; for the former is an invariant characteristic of a given eclipsing system, while α_0' and α_0'' are numerically different at the alternate eclipses. Once, moreover, the proper value of p_0 has been established, the corresponding values of $\alpha_a^x(k, p_0)$ as well as $\alpha_b^x(k, p_0)$ can always be evaluated from equations (2-23) and (2-24) by means of the respective tables.

In order to ascertain p_0, we should recall that a combination of equations (2-23) or (2-24) with (5-11) and (5-12) yields, for an occultation eclipse,

$$(1 - x_a)\alpha^{\mathbf{U}}(k, p_0) + x_a\alpha_a^{\mathbf{D}}(k, p_0) = 1 - \lambda_a + \frac{1 - \lambda_b}{k^2 Y(k, p_0)} \qquad (5\text{-}13)$$

while, if the eclipse is a transit,

$$(1 - x_b)\alpha^{\mathbf{U}}(k, p_0) + \tfrac{3}{2}x_b\Phi(k)\alpha_b^{\mathbf{D}}(k, p_0) = (1 - \lambda_a)\,Y(k, p_0) + \frac{1 - \lambda_b}{k^2}. \qquad (5\text{-}14)$$

* In order to forestall any possible misunderstanding, let it be stressed that we do not require the secondary minimum to be deep. What matters is that an absolute (not proportional) error in its determination be small. A knowledge that the depth of the secondary minimum is quite negligible, with an upper limit of 0·01 magnitude, is just as valuable in fixing the value of α_0 as if its depth were (say) $0{\cdot}10 \pm 0{\cdot}01$ magnitude. It is only when we know *nothing whatever* of the secondary minimum—whether it is shallow or deep—that the determination of elements of partially eclipsing systems becomes well-nigh hopeless.

These equations constitute the desired relations between the geometrical invariants k and p_0 in terms of the observable quantities λ_a, λ_b and the estimated coefficients x_a and x_b. The reader should observe that, for setting up the explicit forms of such relations, an estimate of the degree of darkening of, not one, but of both components is now prerequisite. In order to solve these equations for the value of p_0 corresponding to an assumed value of k, advantage can be taken of the fact that their right-hand sides vary much more slowly with p_0 than the left-hand ones and that, furthermore, the numerical value of the function $Y(k, p_0)$ never exceeds unity by more than 19%.* Our successive approximation at a solution for p_0 can, therefore, start by putting $Y(k, p_0) = 1$, which permits us immediately to evaluate the right-hand sides of (5-13) or (5-14) in terms of an assumed value of K; this will be our first approximation to α_0.

Our next step should now be to find the value of p_0 which will permit us to equalize both sides of equations (5-13) and (5-14) for adopted values of x_a or x_b. If the assumed degree of darkening of the star undergoing eclipse corresponds to one of the tabular values, we simply enter the respective p-table with the arguments K and α equal to the preliminary value of α_0 just determined, to find the corresponding value of p_0. Once this has been done, we look up the corresponding values of $\alpha^{\mathbf{U}}(k, p_0)$ in Tsesevich 50 (Table V), $\alpha_a^{\mathbf{D}}(k, p_0)$ in Tsesevich 45 (Table I), $\alpha_b^{\mathbf{D}}(k, p_0)$ in Tsesevich 45 (Table III), as well as of $\Phi(k)$ (Tsesevich 50, Table V), and evaluate now $Y(k, p_0)$ by means of equation (2-25). The actual value of $Y(k, p_0)$ will as a rule turn out to be but slightly larger or smaller than unity; hence, it will bring about only a small change in the numerical value of the right-hand sides of equation (5-13) or (5-14). This change must, in turn, be offset by an appropriate change in p_0 on the left-hand sides of these equations, and the process continued until both sides of (5-13) or (5-14) have been stabilized to the requisite degree of accuracy. In practice, we shall again start as a rule with an estimated value of k rounded off to the nearest five hundredths to avoid bi-variate interpolation; and a stabilization of the corresponding value of p_0 (or α_0) to three decimals is all that will be needed in most practical cases. It may be mentioned that if, in the absence of any indication to the contrary, we initially assumed $K = 1$, equations (5-11) and (5-12) would have both reduced to

$$\alpha_0 = 2 - \lambda_a - \lambda_b \tag{5-15}$$

for either type of eclipse; α_0 could thus be determined directly—without approximations—from the observational evidence, and p_0 could be found from the appropriate tables as $p(k, \alpha_0)$.

Once the value of α_0 corresponding to an assumed value of k has been established, the normalized fractional loss of light at any other moment during eclipse follows from $\alpha = n\alpha_0$, and the corresponding value of $p(k, \alpha)$

* The maximum possible value of the Y-function is attained in the case of a grazing eclipse ($p_0 = -1$) and of complete limb-darkening ($x_b = 1$) at $Y(0{\cdot}67, -1) = 1{\cdot}193$; but in most practical cases we find that $0{\cdot}9 < Y(k, p) < 1{\cdot}1$.

can be extracted from Tsesevich's tables for the appropriate type of the eclipse and the degree of darkening adopted in solving the equation (5-11) or (5-12). An equation of condition of the form (5-5) can now be set up for every normal point based on observations secured within minima. With regard to the numerical accuracy to which the coefficients of these equations should be evaluated, the remarks made in connection with our treatment of total eclipses in section VI.4 continue to hold good—i.e., three decimals in the coefficients and four decimals in the absolute terms should be ample in all but very exceptional cases. A least-squares solution of the set of such equations, weighted in accordance with (5-9), should then yield the most probable values of C_1 and C_2 based on the assumed value of K. A check on the latter is again provided by the ratio $C_1/C_2 = k$. If K and k differ significantly, the iterative process should be repeated—each time with the previously improved value of k—until the assumed and resulting values of the ratio of the radii of both components are consistent.

Our ability to achieve this end presumes tacitly that our iterations represent a convergent process; and its convergence requires, in turn, that the types of eclipse giving rise to the primary and secondary minima are known *a priori* from the outset. If this was not so and if, for this reason, we started the iterative process by assuming $K = 1$, the ratio C_1/C_2 resulting from the first iteration will very likely differ significantly from unity, so that a decision as to which one of the two minima is an occultation or a transit can no longer be deferred. At this time we are, however, already in possession of a value of $k = C_1/C_2$ which should not be too far from the true value of this ratio; and with this value of k *two* new parallel solutions should be started: one on the assumption that the primary minimum is due to an occultation eclipse, the other assuming it to be a transit Only one of these alternatives can obviously be true and *only that one will converge to a definite answer*. Since the quantity K with which the two alternative solutions were started cannot be too far from reality, the outcome of one (the correct one) should closely confirm it; while, on the other hand, the ratio k resulting from the other solution should prove recalcitrant and refuse to stabilize in the course of repeated iterations. To iterate for k on the assumption of a wrong type of eclipse (i.e., using improper p-tables to calculate the coefficients of C_1 and C_2 in our equations of condition) is indeed as hopeless a proposition as chasing one's own shadow: for the ratio $C_1/C_2 = k$ will then be always different from K no matter how many times we put it through the mill. It is thus again the convergence (or divergence) of our iterations which will disclose to us *a posteriori* whether or not the types of our eclipses have been correctly identified.

Generalized Equations of Condition: Differential Corrections

Before we actually embark on the iterative process—or at least before the final iteration is performed—systematic errors produced by possible errors in the adopted values of minimum brightness of the system at the moments of either conjunction, or in the adopted maximum brightness of the system (i.e., in our unit of light) should again be considered. The whole process of solution, as outlined in the preceding sections, utilizes *three* constants whose values had to be approximately read off the observed data: namely, λ_a, λ_b and U; and their provisional values may, therefore, be subject to an error. The question then arises again as to the effect of these errors on the correctness of our solution for the elements of the eclipse, and the way in which the provisional values of all three constants might be improved simultaneously with a determination of the most probable values of the geometrical elements. This way is so closely parallel with that followed in the preceding section in the course of our treatment of total eclipses that a brief outline will now be sufficient.

Let, as before, equation (5-5) be written symbolically as

$$\sin^2\theta = H(C_1, C_2, 1 + \Delta U, \lambda_0 + \Delta\lambda, \alpha_0 + \Delta\alpha_0), \tag{5-16}$$

where λ_0 and α_0 pertain to the minimum under investigation and $\Delta\lambda$, $\Delta\alpha_0$ denote the corrections to the respective quantities which we seek to determine. If we expand (5-16) again in a Taylor series retaining only the first powers of the requisite corrections, we obtain

$$H_0 + \left(\frac{\partial H}{\partial U}\right)_0 \Delta U + \left(\frac{\partial H}{\partial \lambda}\right)_0 \Delta\lambda + \left(\frac{\partial H}{\partial \alpha}\right)_0 \Delta\alpha_0 + \ldots = \sin^2\theta, \tag{5-17}$$

where $H_0 \equiv H(C_1, C_2, 1, \lambda_0, \alpha_0)$. Before this equation can be used for a determination of the unknowns, it should again be multiplied by a square-root $\sqrt{w}$ of its intrinsic weight as defined by equation (5-9). Moreover, in evaluating its coefficients we find that, exactly as in the case of a total eclipse,

$$\sqrt{w}\left(\frac{\partial H}{\partial U}\right)_0 = n - 1, \tag{5-18}$$

$$\sqrt{w}\left(\frac{\partial H}{\partial \lambda}\right)_0 = -n, \tag{5-19}$$

$(\partial H/\partial\alpha_0)_0$ continues to be given by equation (5-10); but in virtue of equations (5-11)–(5-12) and of the fact that

$$\Delta(1 - \lambda) = \lambda\Delta U - \Delta\lambda \tag{5-20}$$

we may now express $\Delta\alpha_0$ in terms of the underlying errors of ΔU, $\Delta\lambda_a$ and $\Delta\lambda_b$ as

$$\Delta\alpha_0' = \lambda_a\Delta U - \Delta\lambda_a + \frac{\lambda_b\Delta U - \Delta\lambda_b}{k^2 Y(k, p_0)} \tag{5-21}$$

if the eclipse under investigation is an occultation, and

$$\Delta\alpha_0'' = \frac{Y(k, p_0)}{Y(k, -1)}\{\lambda_a\Delta U - \Delta\lambda_a\} + \frac{\lambda_b\Delta U - \Delta\lambda_b}{k^2 Y(k, -1)} \tag{5-22}$$

if it is a transit. If we insert all this in (5-17) and remember that, by equations

(5-11) and (5-12) $\alpha_0' Y(k, p_0) = \alpha_0'' Y(k, -1)$, our complete equation of condition for the determination of all unknown constants takes the explicit form

$$\sqrt{w}(p^2 - p_0^2)C_1 + 2\sqrt{w}(p - p_0)C_2 + \left\{n - 1 + \frac{1 - \lambda_a}{\alpha_0'}\left[\lambda_a + \frac{\lambda_b}{k^2 Y(k, p_0)}\right]\right.$$
$$\left.\times\left[\sqrt{\frac{w}{w_0}} - n\right]\right\}\Delta U - \left\{n + \frac{1 - \lambda_a}{\alpha_0'}\left[\sqrt{\frac{w}{w_0}} - n\right]\right\}\Delta\lambda_a$$
$$- \frac{1 - \lambda_a}{\alpha_0' k^2 Y(k, p_0)}\left\{\sqrt{\frac{w}{w_0}} - n\right\}\Delta\lambda_b = \sqrt{w}\sin^2\theta \tag{5-23}$$

if the eclipse under investigation is an occultation, and

$$\sqrt{w}(p^2 - p_0^2)C_1 + 2\sqrt{w}(p - p_0)C_2 + \left\{n - 1 + \frac{1 - \lambda_b}{\alpha_0'}\left[\lambda_a + \frac{\lambda_b}{k^2 Y(k. p_0)}\right]\right.$$
$$\left.\times\left[\sqrt{\frac{w}{w_0}} - n\right]\right\}\Delta U - \frac{1 - \lambda_b}{\alpha_0'}\left\{\sqrt{\frac{w}{w_0}} - n\right\}\Delta\lambda_a$$
$$- \left\{n + \frac{1 - \lambda_b}{\alpha_0' k^2 Y(k, p_0)}\left[\sqrt{\frac{w}{w_0}} - n\right]\right\}\Delta\lambda_b = \sqrt{w}\sin^2\theta \tag{5-24}$$

if it is a transit.* It should again be added that if, in order to remove the constant C_2 from the denominator of the equation (5-9) defining $\sqrt{w}$, we pre-multiplied all equations of condition of the above forms by C_2, the last three unknown quantities automatically become equal to $C_2\Delta U$, $C_2\Delta\lambda_a$ and $C_2\Delta\lambda_b$. If, in addition, we found it expedient to multiply the w's by any other arbitrary factor c (applied, for instance, to keep their numerical values within reasonable bounds), it goes without saying that the values of the last three unknowns resulting from our solution would be equal to c times the respective differential correction.

Whatever is the case, the most general form of the equations of condition of the form (5-23) or (5-24) is found to contain five unknowns to be determined simultaneously by an appropriate least-squares solution. This is the same number of unknowns as we encountered earlier in connection with the case of total eclipses; the only difference being that C_3 is now by definition zero and that the corrections to the depths of both minima occur explicitly, with different coefficients, in the equations of condition pertaining to each type of the eclipse. The reason is easy to understand; for whereas an error in the assumed depth of the minimum under investigation affects systematically all values of n as well as α_0, an error in depth of the other minimum will affect α_0 alone. An inspection of the numerical magnitude of the coefficients of ΔU, $\Delta\lambda_a$ and $\Delta\lambda_b$ in equations (5-23) and (5-24) reveals that,

* It may be of interest to note that, in the preceding equations $(1 - \lambda_a)/\alpha_0' = L_a$ and $(1 - \lambda_b)/\alpha_0' = k^2 Y(k, p_0)L_b$, where L_a, L_b are the fractional luminosities of the two components as defined by equations (5-29).

although the rigorous forms of these coefficients are rather complicated, that of the correction to the depth of the minimum under investigation is again essentially equal to n, and that of ΔU is sensibly equal to $n - 1$, as in the case of total eclipses; the numerical values of both coefficients will, therefore, range roughly from zero to one. The coefficient of the correction to the depth of the alternate minimum, proportional to the difference $(\sqrt{w/w_0} - n)$, is, on the other hand, apt to be numerically very small for reasons previously discussed. These facts disclose that the correction to the depth of the minimum under investigation will, in general, be a well-determined quantity, while the correction to the depth of the other minimum will remain largely indeterminate for the same reasons which made a determination of $\Delta\alpha_0$ well-nigh impossible.* If, however, we combine the equations of condition of the form (5-23) and (5-24) pertaining to both minima into a single set of normal equations—as we should always do at least in the final iteration—well-determined corrections to the depths of both minima should be obtained.

The light changes of a partially eclipsing system exhibit, by definition, no phase of constant light at the bottom of either minimum. In consequence, equation (4-31) of the case of total eclipses will have no analogy in the present problem; but equation (4-30) respecting the contribution of the observations made between minima to the determination of ΔU continues to hold good. Ultimately, we may add that if, in the foregoing equations, we set $p_0 = -1$ and thus let the eclipse become a grazing total (or annular) one, α_0' or α_0'' becomes exactly equal to unity (i.e., $\Delta\alpha_0' = \Delta\alpha_0'' = 0$) and, in consequence, equations (5-21) or (5-22) provide a closed relation between ΔU and $\Delta\lambda_a$ or $\Delta\lambda_b$ of the form

$$k^2 Y(k, -1)(\lambda_a \Delta U - \Delta\lambda_a) = \Delta\lambda_b - \lambda_b \Delta U. \tag{5-25}$$

If we insert (5-25) in (5-23) or (5-24), the latter equations reduce indeed to equation (4-29) appropriate for grazing eclipses (i.e., when $C_3 = 0$). In such a case, equations (5-23) and (5-24) would then become completely independent, and the corrections $\Delta\lambda_a$ and $\Delta\lambda_b$ could be solved separately from the data pertaining to either minimum. For partially eclipsing systems this is, however, not the case and should not be attempted.

Evaluation of the Elements

Once the final least-squares solution of a system of full-dress equations of condition of the form (5-23) and (5-24) has been obtained, and the value of C_1/C_2 found not to differ significantly from the adopted value K, we are in a position to reap the fruits of our investigation and evaluate at last the

* These reasons will be more readily apparent if we re-state them in the following terms. Since an error in the adopted depth of the minimum under investigation affects n as well as α_0, the corresponding error in n will affect all p's except p_0; its effect on the differences $p^2 - p_0^2$ or $p - p_0$ constituting the coefficients of C_1 and C_2 may, therefore, become appreciable. An error in the adopted depth of the other minimum, will, however, affect only α_0 and, through it, both p and p_0 alike—so that by forming their differences we can minimize its consequences to a large extent.

geometrical elements of our eclipsing system. In order to do so, we first establish the definitive values of the maximum obscuration α_0' and α_0'' at the bottom of each minimum from equations (5-11) and (5-12), by inserting in them the corrected values of λ_a and λ_b (expressed in terms of our corrected unit of light) and the final value of k. With the aid of the definitive values of k and α_0 enter next the Tsesevich table, appropriate for the accepted degree of darkening of the component undergoing eclipse at the respective minimum, to extract the corresponding value of $p(k, \alpha_0)$.

The equations defining the geometrical elements of partially eclipsing systems in terms of our auxiliary constants are

$$\left.\begin{aligned} C_1 &= r_a^2 \csc^2 i\,, \\ C_2 &= r_a r_b \csc^2 i\,, \\ p_0 &= (\cos i - r_b)/r_a\,; \end{aligned}\right\} \tag{5-26}$$

and solving them we obtain

$$\left.\begin{aligned} r_a^2 &= C_1^2/G', \\ r_b^2 &= C_2^2/G', \\ \sin^2 i &= C_1/G', \end{aligned}\right\} \tag{5-27}$$

where we have abbreviated

$$G' = C_1 + (p_0 C_1 + C_2)^2. \tag{5-28}$$

The fractional luminosities of the two components then follow from

$$\left.\begin{aligned} L_a + L_b &= 1, \\ \frac{L_a}{L_b} &= k^2 Y(k, p_0)\frac{1-\lambda_a}{1-\lambda_b}, \end{aligned}\right\} \tag{5-29}$$

where again corrected values of λ_a and λ_b, referred to the corrected unit of light, should be used. The ratio of the mean surface brightnesses of the two components is defined by

$$\frac{J_a}{J_b} = Y(k, p_0)\frac{1-\lambda_a}{1-\lambda_b}\,; \tag{5-30}$$

while the ratio of the central surface brightnesses of the two stars continues to be given by the equation

$$\frac{J_a}{J_{bc}} = \frac{3-u_b}{3-u_a}\frac{J_a}{J_b}. \tag{5-31}$$

The theoretical angle θ' of the first contact of a partial eclipse (and thus the computed duration of the minimum) can again be obtained from equation (4-5) by inserting in it the appropriate values of the elements r_a, r_b and i as deduced from (5-27).

Before concluding the present section, some retrospective considerations regarding the determination of elements of totally and partially eclipsing systems may be pointed out. The reader has undoubtedly noticed that the inclusion of α_0 among the unknown elements of partially eclipsing systems has not only complicated the procedure, but also diminished substantially the weight of the whole solution. Whereas, for total eclipses, three points of a light curve were in principle sufficient to specify the geometrical elements, all points within one minimum due to a partial eclipse were found inadequate to do so in practice, and in order to accomplish our aim we were compelled to draw further information from the secondary minimum as well. In solving for the elements of a totally eclipsing system we had to know the degree of darkening of one (the smaller) component only; whereas a knowledge (or estimate) of darkening of both components was found to be prerequisite for deducing the elements of partially eclipsing systems. All this is bound to render the determination of elements of partially eclipsing systems unavoidably less exact. Yet, according to the laws of chance, a great majority of known eclipsing variables is likely to exhibit partial eclipses. An interpretation of their light curves therefore represents an important, though perhaps less attractive and certainly more laborious, field of double-star astronomy; and as such it is bound to commend itself to the attention of the students of our subject.

VI.6. ERRORS OF THE ELEMENTS

Having evaluated the most probable values of the elements of a totally or partially eclipsing system as defined by equations (4-39)–(4-44) or (5-27)–(5-31), we have by no means come to the end of our investigation; for it remains for us still to determine the uncertainty with which such elements are defined by the available observational data. This constitutes a point whose importance cannot be overemphasized. Since the observations at the basis of our study are never infinitely numerous or precise, all results based upon them are bound to be inaccurate within certain limits; and the quantitative expressions of them are the mean or probable errors of the respective elements. *Such errors*—it cannot be repeated too often—*represent an integral part of our solution, which must be supplied by the investigator of any definitive set of the elements if his results are to merit serious consideration.*

The methods developed earlier for obtaining the most probable elements of an eclipsing system by successive iterations, as outlined in previous sections lend themselves readily to such a purpose; in fact, they were developed with the aim in view. They treat the whole ensemble of the given data rigorously and impartially; and while the $(O–C)$ residuals of our least-squares solutions specify the error ε of an observed normal point of unit weight, the elements of the inverse matrix of the coefficients of our normal equations specify the weights of all unknowns which we seek to determine. The aim of this section will be to establish the explicit form of the equations by which the uncertainty of all our elements can be evaluated. In doing so, we shall find ourselves confronted with a question which we tacitly by-passed earlier for reasons of convenience, but which is evidently basic to the justification of our whole process of solution. This process (the details of which were discussed in preceding sections) should enable us to arrive at the true value of k by successive iterations; but their *convergence* (for the right kind of the eclipse) has so far been taken for granted. Are we justified in doing so—the thoughtful reader may have inquired—and should the answer be in the affirmative, *how does the rate of convergence depend on the nature of the eclipses*? Or do cases exist in which the iteration may actually fail? Such questions, which obviously go deep to the roots of our method of approach to the problem, could so far only be asked; the specific answers to them are going to be given below.

In sections VI.4 and 5 we found it expedient to express the geometrical elements r_a, r_b and i of our eclipsing system in terms of the three auxiliary constants C_1, C_2 and C_3 or p_0. The errors of these elements, caused by the errors inherent to our determination of the respective auxiliary constants, are evidently given by

$$\Delta r_a = \frac{\partial r_a}{\partial C_1}\Delta C_1 + \frac{\partial r_a}{\partial C_2}\Delta C_2 + \frac{\partial r_a}{\partial C_3}\Delta C_3, \tag{6-1}$$

$$\Delta r_b = \frac{\partial r_b}{\partial C_1}\Delta C_1 + \frac{\partial r_b}{\partial C_2}\Delta C_2 + \frac{\partial r_b}{\partial C_3}\Delta C_3, \tag{6-2}$$

$$\Delta i = \frac{\partial i}{\partial C_1}\Delta C_1 + \frac{\partial i}{\partial C_2}\Delta C_2 + \frac{\partial i}{\partial C_3}\Delta C_3, \tag{6-3}$$

if the eclipses are total or annular, or from

$$\Delta r_a = \frac{\partial r_a}{\partial C_1}\Delta C_1 + \frac{\partial r_a}{\partial C_2}\Delta C_2 + \frac{\partial r_a}{\partial p_0}\Delta p_0, \tag{6-4}$$

$$\Delta r_b = \frac{\partial r_b}{\partial C_1}\Delta C_1 + \frac{\partial r_b}{\partial C_2}\Delta C_2 + \frac{\partial r_b}{\partial p_0}\Delta p_0, \tag{6-5}$$

$$\Delta i = \frac{\partial i}{\partial C_1}\Delta C_1 + \frac{\partial i}{\partial C_2}\Delta C_2 + \frac{\partial i}{\partial p_0}\Delta p_0, \tag{6-6}$$

if they are partial. The coefficient of the individual corrections on the right-hand sides of the preceding relations can, in turn, be obtained by an appropriate differentiation of equations (4-39) or (5-27) which yields

$$\left.\begin{aligned} \partial r_a/\partial C_1 &= \tfrac{1}{2}(1 + r_b^2 - r_a^2)r_a^{-1}\sin^2 i, \\ \partial r_a/\partial C_2 &= -(r_b - r_a)\sin^2 i, \\ \partial r_a/\partial C_3 &= \tfrac{1}{2}r_a \sin^2 i; \end{aligned}\right\} \tag{6-7}$$

$$\left.\begin{aligned} \partial r_b/\partial C_1 &= -\tfrac{1}{2}(1 - r_b^2 + r_a^2)r_a^{-2}r_b\sin^2 i, \\ \partial r_b/\partial C_2 &= (1 - r_b^2 + r_a r_b)r_a^{-1}\sin^2 i, \\ \partial r_b/\partial C_3 &= \tfrac{1}{2}r_b \sin^2 i; \end{aligned}\right\} \tag{6-8}$$

$$\left.\begin{aligned} \partial i/\partial C_1 &= \tfrac{1}{2}(r_b^2 - r_a^2)r_a^{-2}\sin^2 i\tan i, \\ \partial i/\partial C_2 &= -(r_b - r_a)r_a^{-1}\sin^2 i\tan i, \\ \partial i/\partial C_3 &= \tfrac{1}{2}\sin^2 i\tan i; \end{aligned}\right\} \tag{6-9}$$

if the eclipses are total or annular, and

$$\left.\begin{aligned}\partial r_a/\partial C_1 &= \tfrac{1}{2}r_a^{-1}\sin^4 i + r_a^{-1}r_b\sin^2 i\cos i\,,\\ \partial r_a/\partial C_2 &= -\sin^2 i\cos i\,,\\ \partial r_a/\partial p_0 &= -r_a^2\cos i;\end{aligned}\right\} \tag{6-10}$$

$$\left.\begin{aligned}\partial r_b/\partial C_1 &= -\tfrac{1}{2}(1-2r_b\cos i+\cos^2 i)r_a^{-2}r_b\sin^2 i\,,\\ \partial r_b/\partial C_2 &= (1-r_b\cos i)r_a^{-1}\sin^2 i\,,\\ \partial r_b/\partial p_0 &= -r_a r_b\cos i;\end{aligned}\right\} \tag{6-11}$$

$$\left.\begin{aligned}\partial i/\partial C_1 &= \tfrac{1}{2}(2r_b-\cos i)r_a^{-2}\sin^3 i\,,\\ \partial i/\partial C_2 &= -r_a^{-1}\sin^3 i\,,\\ \partial i/\partial p_0 &= -r_a\sin i;\end{aligned}\right\} \tag{6-12}$$

if they are partial.

Errors of the Auxiliary Constants

The errors of the auxiliary constants C_1, C_2 and C_3 or p_0 stem from two sources:

(a) the dispersion of the observed normal points and the uncertainty of the adopted depths of the minima, and

(b) the finite speed of convergence of our iterative process.

The first source of error is obvious enough, but the second is more subtle. It should, however, be remembered that (quite apart from the influence of any observational errors) *the elements computed from our set of auxiliary constants would be exact only if the assumed value K of the ratio of the radii were identical with the resulting value of k* (i.e., if the speed of convergence of our iterative process were *infinite*). If, as will always be the case in practice, the equality $K = k$ can be established, in the final iteration, only within a certain mean error, this error will constitute an *additional* source of uncertainty of all computed elements which is independent of, and supplementary to, that arising from the observational errors alone. This is another aspect of the problem, of basic importance, whose analysis we owe to Piotrowski; and his reasoning can be outlined as follows.

Let us consider, quite generally,

$$\left.\begin{aligned}C_1 &= F_1(K;\ L_1, L_2, \ldots, L_i),\\ C_2 &= F_2(K;\ L_1, L_2, \ldots, L_i),\end{aligned}\right\} \tag{6-13}$$

where $L_i \equiv \sqrt{w}\sin^2\theta_i$ stands for the absolute term of the i-th equation of condition relating C_1 and C_2; our aim being to determine them so that

$$K = k = C_1/C_2\,. \tag{6-14}$$

Insert (6-14) in the preceding relations and differentiate them with respect to L_i: we obtain

$$\frac{\partial C_1}{\partial L_i} = \frac{1}{C_2}\frac{\partial C_1}{\partial L_i}\frac{dC_1}{dK} - \frac{\partial C_2}{\partial L_i}C_1 + \frac{\partial F_1}{\partial L_i} \tag{6-15}$$

and

$$\frac{\partial C_2}{\partial L_i} = \frac{1}{C_2}\frac{\partial C_1}{\partial L_i}\frac{dC_2}{dK} - \frac{\partial C_2}{\partial L_i}C_1 + \frac{\partial F_2}{\partial L_i}, \tag{6-16}$$

respectively. If we remember that, in view of (6-14)

$$\frac{1}{C_2}\frac{dC_1}{dK} - \frac{k}{C_2}\frac{dC_2}{dK} = \frac{dk}{dK}, \tag{6-17}$$

a solution of equations (6-15)–(6-16) for $\partial C_1/\partial L_i$ and $\partial C_2/\partial L_i$ yields

$$\left.\begin{aligned} C_2\left(1-\frac{dk}{dK}\right)\frac{\partial C_1}{\partial L_i} &= \left(C_2 + k\frac{dC_2}{dK}\right)\frac{\partial F_1}{\partial L_i} - k\frac{dC_1}{dK}\frac{\partial F_2}{\partial L_i}, \\ C_2\left(1-\frac{dk}{dK}\right)\frac{\partial C_2}{\partial L_i} &= \left(C_2 - \frac{dC_1}{dK}\right)\frac{\partial F_2}{\partial L_i} + \frac{dC_2}{dK}\frac{\partial F_1}{\partial L_i}. \end{aligned}\right\} \tag{6-18}$$

Now the *total* errors of C_1 and C_2 are, by definition,

$$\Delta C_1 = \sum_i \frac{\partial C_1}{\partial L_i}\,\delta L_i \quad \text{and} \quad \Delta C_2 = \sum_i \frac{\partial C_2}{\partial L_i}\,\delta L_i, \tag{6-19}$$

where δL_i signifies the O–C residual of the i-th equation of condition and the summation is to be extended over all available normal points; while the *partial* errors, arising from the dispersion of the observations alone (with the assumed value of K considered as *fixed*), are

$$(\delta C_1)_K = \sum_i \frac{\partial F_1}{\partial L_i}\,\delta L_i \quad \text{and} \quad (\delta C_2)_K = \sum_i \frac{\partial F_2}{\partial L_i}\,\delta L_i, \tag{6-20}$$

respectively. *The latter are the errors whose root-mean-square values can be found from our least-squares solutions.* Our aim should, therefore, be to express the ΔC's in terms of the $(\delta C)_K$'s; and this can be done if we multiply both sides of equations (6-18) by δL_i and perform the summation over all i's: we obtain

$$\Delta C_1 = (\delta C_1)_K + X_1\{(\delta C_1)_K - k(\delta C_2)_K\}, \tag{6-21}$$

$$\Delta C_2 = (\delta C_2)_K + X_2\{(\delta C_1)_K - k(\delta C_2)_K\}, \tag{6-22}$$

and, if the eclipses are total,

$$\Delta C_3 = (\delta C_3)_K + X_3\{(\delta C_1)_K - k(\delta C_2)_K\}, \tag{6-23}$$

where we have abbreviated

$$X_j = \frac{\dfrac{1}{C_2}\dfrac{dC_j}{dK}}{1-\dfrac{dk}{dK}}, \qquad j = 1, 2, 3. \tag{6-24}$$

The contributions to the total uncertainty of C_1, C_2 and C_3 and, therefore, of the geometrical elements of the system, arising from the sources (a) and (b) mentioned above can now be distinctly localized. A dispersion of the individual normal points, in equations (6-21)–(6-22) or (6-23), while the finite speed of convergence of our iterative process will invoke additional terms multiplied by X_1, X_2 or X_3. If we pause to consider the relative magnitudes of (δC_1) and (δC_2) for partially eclipsing systems, we shall notice that their ratio will depend largely on the circumstances of the eclipse. In point of fact, in the simplest case—when C_1 and C_2 satisfy equation (5-5)—this ratio should be

$$\left(\frac{\partial C_2}{\partial C_1}\right)_K^2 = \frac{1}{4}\frac{[w(p^2 - p_0^2)^2]}{[w(p - p_0)^2]}, \tag{6-25}$$

where the square brackets on the right-hand side denote the customary sums taken over all equations of condition. If the eclipse were grazing or nearly so, this ratio should be in the neighbourhood of unity; the weights with which both C_1 and C_2 are specified by our solution can be substantial (unless the observations are of poor quality) and very nearly the same. As p_0 increases, however, the uncertainty of both C_1 and C_2 will gradually increase, but in such a way that *the ratio* $(\delta C_2)_K \div (\delta C_1)_K$ *diminishes*—which means that *the weight of* C_1 *diminishes, with increasing* p_0 *much more rapidly than that of* C_2. This situation persists until, for extremely shallow eclipses ($p_0 \sim 1$), the weights of C_1 and C_2 may become once more nearly the same—though this time both are effectively nil. These facts lead us to expect that, considering the observational errors alone, *the product* $r_a r_b$ *of the fractional radii* (i.e., C_2) *should be defined by the observations of partially eclipsing systems with considerably greater relative accuracy than their ratio* r_a/r_b (i.e., C_1/C_2).*

Irrespective of the value of the ratio (6-25), the numerical magnitude of the $(\delta C_j)_K$'s can be obviously diminished arbitrarily by increasing the number (or precision) of the underlying observational data. The magnitudes of the constants X_1, X_2 or X_3 in (6-21)–(6-22) or (6-23) depend, however, solely on the geometrical circumstances of the eclipse and cannot, therefore, be controlled by the observer. As long as the speed of convergence of the iterative process is high—which will, in general, be true if the eclipses under investigation are total or annular—the terms in equations (6-21)–(6-22) or (6-23) multiplied by X_1, X_2 or X_3 are likely to be unimportant (though not negligible). If, however, the eclipses happen to be partial, the *convergence of our process becomes in general the slower, the shallower the eclipses* and, as a result, the terms factored by X_1 or X_2 may become a large, or even dominant,

* It may be observed that C_2 specifies, in effect, the geometric mean of the two fractional radii r_a and r_b. Unless the disparity in radii is very large (i.e., unless k is very small), the geometric mean of the two radii is known to be very nearly equal to their arithmetic mean. Within the scheme of this approximation we are thus entitled to assert that, for shallow partial eclipses, the sum $r_a + r_b$ of the two fractional radii can be inferred from the observations with a much greater relative accuracy than their ratio.

part of the total error.* Hence, *the slower the convergence of the iterative process, the greater the inherent uncertainty with which the elements of the eclipse can be extracted by our method*—or, for that matter, by *any* method—from an analysis of the light changes.

A succinct measure of the rate of convergence of our iterative process is the quantity dk/dK related with the derivatives of C_1 and C_2 with respect to K by means of

$$\frac{dk}{dK} = \frac{1}{C_2}\frac{dC_1}{dK} - \frac{k}{C_2}\frac{dC_2}{dK}. \tag{6-26}$$

If the speed of convergence of our iterations were infinite this quantity would, by definition, be equal to zero; while if it were equal to (or greater than) unity, the iterations would manifestly fail. Therefore, for any eclipsing system whose elements can be deduced from its light curve,

$$0 \leqslant \frac{dk}{dK} < 1. \tag{6-27}$$

If the derivative is close to its lower limit, the denominator in the expressions for X and Y will be close to unity and the magnitude of these quantities will be controlled by dC_1/dK and dC_2/dK. If, on the other hand, dk/dK is close to unity, the denominator in the expressions (6-24) for X_1 and X_2 will tend to zero and, consequently, both ΔC_1 and ΔC_2 (and therefore also ΔC_3) will increase without limit; and this will make the errors of C_1, C_2 or C_3 grow beyond any limit regardless of the smallness of (δC_j)'s. This is an important revelation; for it shows that *if the eclipses under consideration possess such special features that dk/dK is close to unity, (shallow partial eclipses!) the elements of such a system are inherently indeterminate—irrespective of the amount or precision of the underlying observational data.*†

These considerations alone render dk/dK one of the most important auxiliary quantities of any solution for the elements, and a determination of its numerical value—or, what amounts to the same, of numerical values of the derivatives

$$\frac{dC_1}{dK}, \frac{dC_2}{dK}, \frac{dC_3}{dK}$$

is indispensable for establishing the genuine uncertainty of the geometrical elements of an eclipsing system as can be deduced from an analysis of its light changes. In order to ascertain these quantities, Piotrowski suggested carrying out two independent solutions based on two assumed values of

* In particular, it is in the nature of our problem that, in such cases, C_2 is very much less sensitive to a variation in the adopted value K than C_1 and that, in consequence, $X \gg Y$.

† It should, however, be clearly understood that this statement holds good whenever we are solving for the elements from an overdeterminate system of equations based on the actual observations, and not on the basis of fixed points equal in number to that of the unknowns. In this latter case, the uncertainty $(\delta C_j)_K$ remains, by definition, zero irrespective of a possible proximity of dk/dK to unity.

K_1 and K_2, and determining the derivatives of C_1, C_2 or C_3 with respect to K from the differences of the respective C's computed under the two assumptions divided by $K_1 - K_2$. Such a process would, however, not only just about double the computer's work, but the values of the derivatives approximated in this way would also pertain to the ratio of the radii half way between K_1 and K_2, which need not necessarily envelop k. Moreover, the approximate values of the derivatives based solely on the quotients of the first differences may be appreciably in error in regions where the variation of the C_j's with K is rapid. As it sometimes happens, however, a determination of actual derivatives dC_j/dK ($j = 1, 2, 3$) turns out to be easier than that of their finite-differences approximation and can be performed in the course of our intermediary solution with a minimum of repetitive work.

In order to do so, let us return again to our fundamental equation $H = \sin^2 \theta$ and consider what happens when we vary the value of K. It is obvious that the values of C_1 and C_2 or C_3 must be altered so as to counter the change of p with K because, if the light curve is to pass through the same points, the right-hand side of the fundamental equation must remain the same. In consequence, the function H on its left-hand side must be invariant with respect to an infinitesimal change in K, and this can be mathematically expressed as

$$\frac{dH}{dK} = 0, \tag{6-28}$$

which represents an implicit equation for determining dC_1/dK, dC_2/dK, or dC_3/dK. On performing the actual differentiation of equations (4-3) and (5-5) we find that, if the eclipse is total or annular,

$$(p^2 - 1)\frac{dC_1}{dK} + 2(p + 1)\frac{dC_2}{dK} + \frac{dC_3}{dK} = -2C_2(1 + kp)\frac{dp}{dK}; \tag{6-29t}$$

while if it is partial,

$$(p^2 - p_0^2)\frac{dC_1}{dK} + 2(p - p_0)\frac{dC_2}{dK} = 2C_2\left\{(1 + kp_0)\frac{dp_0}{dK} - (1 - kp)\frac{dp}{dK}\right\}. \tag{6-29p}$$

The computation of the ordinary derivatives of p with respect to K requires some care. In general,

$$\frac{dp}{dK} = \left(\frac{\partial p}{\partial K}\right)_\alpha + \left(\frac{\partial p}{\partial \alpha}\right)_K \frac{d\alpha}{dK}. \tag{6-30}$$

The reader may remember that, if the eclipse is *total* (or annular), $\alpha \equiv n$ is a quantity supplied directly by the observations and, therefore,

$$\frac{d\alpha}{dK} = 0. \tag{6-31t}$$

If, on the other hand, the eclipse happens to be *partial*, $\alpha = n\alpha_0$ and α_0 is, in turn, given by equations (5-11) or (5-12)—depending on whether the eclipse under investigation is an occultation or a transit—which both involve K. Differentiating them and ignoring, for simplicity, the minor variation of $Y(K, p_0)$ with K we find that

$$\frac{d\alpha}{dK} = -\frac{2n(1-\lambda_b)}{K^3 Y(k, p_0)} \tag{6-31p$'$}$$

if the eclipse is an occultation, and

$$\frac{d\alpha}{dK} = -\frac{2n(1-\lambda_b)}{K^3 Y(k, -1)} \tag{6-31p$''$}$$

if it is a transit. Moreover, differentiating partially the identity $p \equiv p[K, \alpha(K, p)]$ we have

$$\left(\frac{\partial p}{\partial K}\right)_\alpha = -\left(\frac{\partial p}{\partial \alpha}\right)_K \left(\frac{\partial \alpha}{\partial K}\right)_p. \tag{6-32}$$

Inserting equations (6-30), (6-31) and (6-32) in (6-33) and multiplying the latter by $\sqrt{w}$, we find that the explicit forms of equations (6-29) for determining the required derivatives of C_1, C_2 and C_3 with respect to K ultimately become

$$\sqrt{w}(p^2-1)\frac{dC_1}{dK} + 2\sqrt{w}(p+1)\frac{dC_2}{dK} + \sqrt{w}\frac{dC_3}{dK} = -(1-\lambda)\frac{\partial\alpha}{\partial K} \tag{6-33t}$$

if the eclipse is total (or annular), and

$$\sqrt{w}(p^2-p_0^2)\frac{dC_1}{dK} + 2\sqrt{w}(p-p_0)\frac{dC_2}{dK} = \frac{1-\lambda}{\alpha_0}\left\{\frac{2(1-\lambda_b)}{K^3Y(k,p_0)}\left[\sqrt{\frac{w}{w_0}} - n\right] + \left[\frac{\partial\alpha}{\partial K}\sqrt{\frac{w}{w_0}} - \frac{\partial\alpha}{\partial K}\right]\right\} \tag{6-33p}$$

if it is partial. It should be reiterated that, in this and other equations, α_0 and λ stand for either α_0' and λ_a or α_0'' and λ_b—depending on the type of the eclipse we are considering.

The reader should notice that $\sqrt{w}$ as defined by equation (4-18) (total eclipses) or (5-9) (partial eclipses) involves C_2^{-1}; hence, if we exclude this factor from $\sqrt{w}$ the actual unknowns in equations (6-33) are the ratios $C_2^{-1}(dC_1/dK)$, $j = 1, 2, 3$. Apart from this fact, however, the coefficients of the unknowns in equations (6-33) are identical with those of the fundamental equations (4-29) or (5-23)–(5-24) for the determination of C_1, C_2 or C_3, so that only the absolute terms on the right-hand sides of equations (6-33) need to be evaluated additionally before a solution for the derivatives can be made. Since, moreover, the corresponding numerical values of $\sqrt{w}$ as well as n are also available at this stage, the only new quantities needed are the derivatives $\partial\alpha/\partial K$ and their values can be extracted with the greatest ease from the tabular differences of the appropriate Tsesevich tables.* A determination of the derivatives dC_1/dK by means of the equations (6-33) is much more accurate and less time-consuming than the one based on a comparison of solutions started with two independent values of K. Moreover, if we set up equation (6-33t) for more than three values of p, or (6-33p) for more than two phases, we can ascertain not only the most probable value of the derivatives, but also the uncertainty with which these derivatives are defined by the available observational data. A computation of this uncertainty would, however, serve no specific purpose and may be dispensed with if necessary.

Once the values of dC_1/dK and dC_2/dK or dC_3/dK have thus been established, we shall insert them in (6-26) to evaluate dk/dK which, in turn, will permit us to specify the quantities X_1, X_2 and X_3 as defined by equation (6-24). When this has been done ΔC_1, ΔC_2 or ΔC_3 from equations (6-21)–(6-23) and, consequently, the expressions for Δr_1, Δr_2 and Δi as defined by (6-1)–(6-6) will become linear functions of $(\delta C_1)_k$, $(\delta C_2)_k$ or $(\delta C_3)_k$ with known numerical coefficients, and their uncertainty can be found from the elements of the inverse matrix of our least-squares solution for C_1, C_2 or C_3. The Appendix will contain a description of the requisite techniques as well as a number of illustrative examples of this process.

It should be pointed out, in this connection, that if the first derivative dk/dK can be evaluated by the method of this section while the second derivative d^2k/dK^2 may be considered to be negligible, our iterative process for a determination of the auxiliary constants characterizing an eclipsing binary system with known types of the minima can be *by-passed* by the following procedure:

1. Assume a plausible value of K and carry through a customary solution for C_1, C_2 (and C_3) leading to $C_1/C_2 = k$.
2. Solve equations (6-33t) or (6-33p) for the dC_j/dK's and evaluate dk/dK by means of equation (6-26).
3. The true value of k (i.e., the one for which the assumed K and the resulting k would be identical)—let us denote it by $\bar{k}$—follows from

$$\bar{k} = K - \frac{K - k}{1 - (dk/dK)}, \tag{6-34}$$

while the true (barred) values of the C_j's (such that $\bar{C}_1/\bar{C}_2 = \bar{k}$ are then given by

$$\bar{C}_j = C_j + (dC_j/dK)(\bar{k} - K) = C_j + C_2 X_j(k - K), \qquad j = 1, 2, 3; \tag{6-35}$$

and these values should be used to compute the true geometrical elements of our eclipsing system by the method of sections VI.4 or VI.5.

* This is due to the fact that the absolute terms of equations (6-33) involve only partial derivatives, as compared with ordinary derivatives on the right-hand sides of equations (6–29). The transformation of equations (6–29) into (6–33) was effected precisely for this purpose; for whereas partial derivatives $\partial\alpha/\partial k$ can be approximated from tabular differences in columns k = *constant*, ordinary derivatives would call for bi-variate differentiation.

VI.7. LIMB-DARKENING AND OTHER EFFECTS

As a corollary to the foregoing considerations, another generalization of the theory of intermediary orbits will presently be developed, which should enable us to gain insight into the dependence of the computed geometrical elements of a system on the assumed degree of darkening of the eclipsed star. As is well known (and will be demonstrated mathematically in the subsequent section), the light curves of partially eclipsing systems do not, in general, lend themselves to a determination of the limb darkening of the component undergoing eclipse; but a certain degree of it must be adopted on the basis of collateral physical evidence (spectrum, colour) before a solution for the elements can get under way. Since this adopted coefficient u of limb-darkening might often be subject to later revision, it may be desirable to know the extent to which the geometrical elements of a system may depend on our particular choice of u, and the manner in which these elements would individually respond to a given small change of u. One way of finding this out would be a repetition of a solution with several trial values of u and a comparison of the outcome; but such a way represents no method and would clearly be very laborious. In what follows we shall describe two methods, aiming at this end, which have been recently proposed by Piotrowski* and by Kopal,† and both of which require again only a minimum amount of additional work.

Piotrowski's approach follows closely the method outlined in sections VI.4 or VI.5 in connection with a determination of ΔU and $\Delta\lambda$. Consider again equation (4-19) or (5-16) and remember that the function H depends also (through p) on u. If an error of Δu has been committed in estimating u at the outset, we could in principle hope to determine it from an equation of the form

$$\sqrt{w}H_0 + \sqrt{w}\left(\frac{\partial H}{\partial U}\right)_0 \Delta U + \sqrt{w}\left(\frac{\partial H}{\partial \lambda}\right)_0 \Delta\lambda + \sqrt{w}\left(\frac{\partial H}{\partial u}\right)\Delta u + \dots = \sqrt{w}\sin^2\theta, \tag{7-1}$$

obtained by differentiating H with respect to u and retaining only first-order terms. Now

$$\frac{\partial H}{\partial u}\Delta u = \frac{\partial H}{\partial n}\frac{\partial n}{\partial u}\Delta u \tag{7-2}$$

and

$$\frac{\partial n}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\alpha}{\alpha_0}\right) = \frac{1}{\alpha_0}\left\{\frac{\partial \alpha}{\partial u} - n\frac{\partial \alpha_0}{\partial u}\right\} \tag{7-3}$$

where, for total (annular) eclipses, $\alpha_0 = 1$. On the other hand,

$$\frac{\partial H}{\partial n} = 2\alpha_0 C_2(1 + kp)\frac{\partial p}{\partial \alpha} = \frac{1-\lambda}{\sqrt{w}}, \tag{7-4}$$

* S. L. Piotrowski, *Ap. J.*, **108**, 36, 1948.
† Z. Kopal, *Ap. J.*, **108**, 46, 1948.

so that

$$\sqrt{w}\,\frac{\partial H}{\partial u}\,\Delta u = \frac{1-\lambda}{\alpha_0}\left\{\frac{\partial \alpha}{\partial u} - n\,\frac{\partial \alpha_0}{\partial u}\right\}\Delta u \tag{7-5p}$$

if the eclipse is partial, and

$$\sqrt{w}\,\frac{\partial H}{\partial u}\,\Delta u = (1-\lambda)\,\frac{\partial \alpha}{\partial u}\Delta u \tag{7-5t}$$

if it is total (in which case $\alpha_0 = 1$ and the derivative $\partial\alpha_0/\partial u$ then vanishes). The coefficient of Δu on the right-hand side of the preceding equation can be evaluated with great ease by means of the very convenient tables of $\partial\alpha/\partial u$ as a function of k and p which were recently published by Irwin;* with the aid of these tables, the computation of the coefficient of u appropriate for any particular phase should become a very simple matter.† The actual determination of Δu simultaneously with all other unknowns of equation (7-1) is, however, not so easy; in point of fact, in most cases it is virtually impossible on account of the numerical smallness of its coefficient, and of its strong correlation with that of the constant C_1. But—and this is what Piotrowski suggests—*we may transpose the term* (7-5) *on the right-hand side of our equations of condition of the form* (7-1), *and evaluate* C_1, C_2, *etc., in terms of* Δu retained as an arbitrary quantity. If we do so, our solution will take the form $C_j = a_j + b_j\Delta U$, $j = 1, 2, 3$, where a_j stands for the value of C_j corresponding to the adopted degree of darkening, and $b_j \equiv dC_j/du$.

Kopal set out to accomplish essentially the same purpose by a somewhat simpler and less laborious method, very similar to the one already described at the end of the last section. Let us start again from our fundamental

* J. B. Irwin, *Ap. J.*, **106**, 380, 1947.

† It should only be remembered that what Irwin actually gave in his Tables 4, 8, 12, 16, 20 and 24 are the partial derivatives of f, rather than of α, with respect to u. If the eclipse is an occultation, $f_a \equiv \alpha_a^x$ by virtue of equations (2-11) and (2-16) and the derivatives $\partial f/\partial u$ can, therefore, be extracted from Irwin's tables 4, 12 and 20 as they stand. If, on the other hand, the eclipse happens to be a transit, f_b and α_b^x are known to be related by equation (2-17) where the fractional loss of light $f_b(k, -1)$ at the moment of internal tangency depends itself on the degree of darkening of the larger component. Differentiating equation (2-17) with respect to u we find the partial derivatives of f and α to be related by

$$\frac{\partial f}{\partial u} = g\,\frac{\partial \alpha}{\partial u} + \alpha\,\frac{\partial g}{\partial u}, \tag{7-6}$$

where $\alpha \equiv \alpha(k, p), f \equiv f_b(k, p)$ and $g \equiv f_b(k, -1)$ as given by equation (2-15). Differentiating this latter equation logarithmically with respect to u we easily establish that

$$\frac{1}{g}\frac{\partial g}{\partial u} = \frac{3\Phi(k) - 2}{(3-u)\{1 - u + u\Phi(k)\}} \tag{7-7}$$

which, inserted in the foregoing equation (7-6), discloses that, for transit eclipses,

$$\frac{\partial \alpha}{\partial u} = \frac{1}{g}\left\{\frac{\partial f}{\partial u} - \frac{f}{g}\frac{\partial g}{\partial u}\right\}, \tag{7-8}$$

where $\partial f/\partial u$ can be obtained from Irwin's tables 8, 16 or 24 as a function of k and p, while g as well as its logarithmic derivative with respect to u are constants of a given solution.

equations (4-19) or (5-16) and consider the consequences of a small change of u on its left-hand side. All p's will change as a consequence; but so must C_1 and C_2 or C_3 if the resultant light curve is to remain the same. This is equivalent to an assertion that

$$\frac{dH}{du} = 0 \tag{7-9}$$

or, explicitly,

$$(p^2 - 1)\frac{dC_1}{du} + 2(p+1)\frac{dC_2}{du} + \frac{dC_3}{du} = -2C_2(1+kp)\frac{dp}{du} \tag{7-10t}$$

if the eclipse is total, and

$$(p^2 - p_0^2)\frac{dC_1}{du} + 2(p - p_0)\frac{dC_2}{du} = 2C_2\left\{(1+kp_0)\frac{dp_0}{du} - (1+kp)\frac{dp}{du}\right\} \tag{7-10p}$$

if it is partial. Differentiating the function $\alpha^x(k, p)$ we find that, in general,

$$d\alpha = \frac{\partial\alpha}{\partial k}\,dk + \frac{\partial\alpha}{\partial p}\,dp + \frac{\partial\alpha}{\partial u}\,du\,. \tag{7-11}$$

Now if α is supplied to us by the observations (i.e., if $d\alpha = 0$) and if we adopt a fixed value of K (thus rendering $dk = 0$), the foregoing equation discloses that

$$\frac{dp}{du} = -\frac{\partial p}{\partial\alpha}\frac{\partial\alpha}{\partial u}\,. \tag{7-12}$$

This equation specifies the rate at which p would change, for given K and α, if we vary the adopted degree of limb-darkening. If we insert this result in the preceding equations (7-10) and multiply them by the square-root of the intrinsic weight $\sqrt{w}$ we eventually find that

$$\sqrt{w}(p^2 - 1)\frac{dC_1}{du} + 2\sqrt{w}(p+1)\frac{dC_2}{du} + \sqrt{w}\,\frac{dC_3}{du} = -(1-\lambda)\frac{\partial\alpha}{\partial u} \tag{7-13t}$$

if the eclipse is total, and

$$\sqrt{w}(p^2 - p_0^2)\frac{dC_1}{du} + 2\sqrt{w}(p - p_0)\frac{dC_2}{du} = (1-\lambda)\left\{\frac{\partial\alpha_0}{\partial u}\sqrt{\frac{w}{w_0}} - \frac{\partial\alpha}{\partial u}\right\} \tag{7-13p}$$

if it is partial. It should again be remembered that, inasmuch as the expressions (4-18) or (5-9) for $\sqrt{w}$ involves C_2, the actual unknowns in these quantities are the ratios $C_2^{-1}(dC_j/du)$, $j = 1, 2, 3$; furthermore, since $\sqrt{w}$ as defined by (4-18) or (5-9) is itself multiplied by $1 - \lambda$, this latter factor can be cancelled on both sides of (7-13). If these equations are set up for more than two or three points of the light curve, we can again ascertain not only the most probable values of the requisite derivatives, but also their probable errors—if one is interested to know them.

Once equations of the form (7-13t) or (7-13p) have been solved and the requisite derivatives dC_1/du, dC_2/du or dC_3/du obtained, the derivatives of r_a, r_b and $\sin i$ with respect to u follow readily. The relations between the auxiliary constants and the geometrical elements are provided by equations (4-39) if the eclipses are total (or annular) or (5-27) if they are partial. Differentiating them we obtain

$$\frac{1}{C_2}\frac{dC_1}{du} = \frac{2}{r_b}\frac{dr_a}{du} - \frac{2k}{\sin i}\frac{d\sin i}{du} \tag{7-14}$$

and

$$\frac{1}{C_2}\frac{dC_2}{du} = \frac{1}{r_a}\frac{dr_a}{du} + \frac{1}{r_b}\frac{dr_b}{du} - \frac{2}{\sin i}\frac{d\sin i}{du}\,; \tag{7-15}$$

moreover, if the eclipses are total

$$\frac{1}{C_2}\frac{dC_3}{du} = -2\frac{r_b - r_a}{r_a r_b}\frac{dr_a}{du} + 2\frac{r_b - r_a}{r_a r_b}\frac{dr_b}{du} + 2\frac{\cos^2 \theta'' \sin i}{r_a r_b}\frac{d\sin i}{du}\,; \tag{7-16}$$

while if they are partial,

$$\frac{\partial p_0}{\partial \alpha}\frac{\partial \alpha_0}{\partial u} = \frac{p_0}{r_a}\frac{dr_a}{du} + \frac{1}{r_a}\frac{dr_b}{du} + \frac{\tan i}{r_a}\frac{d\sin i}{du}\,. \tag{7-17}$$

The left-hand sides of these equations being known, the rates of change of r_a, r_b and $\sin i$ corresponding to a given small change u in the adopted value of the coefficient of limb-darkening can be readily evaluated from (7-14), (7-15) and (7-16) if the eclipses are total (or annular), or from (7-14), (7-15) and (7-17) if they are partial, and the geometrical elements themselves expressed as

$$\left.\begin{aligned} r_a &= (r_a)_u + \left(\frac{dr_a}{du}\right)\Delta u + \ldots, \\ r_b &= (r_b)_u + \left(\frac{dr_b}{du}\right)\Delta u + \ldots, \\ \sin i &= (\sin i)_u + \left(\frac{d\sin i}{du}\right)\Delta u + \ldots, \end{aligned}\right\} \tag{7-18}$$

where $(r_a)_u$, $(r_b)_u$ and $(\sin i)_u$ stand for the values of the respective elements based on the initially assumed value of u.

The reader may notice that the methods by Kopal and Piotrowski differ only in the way in which the derivatives of the auxiliary constants C_1, C_2 and C_3 or p_0 are evaluated from the observed data; while the subsequent step of expressing the u-derivatives of the geometrical elements in terms of those of the auxiliary constants is common to both methods of approach.

Combination with Spectroscopic Evidence

The one admittedly unattractive feature of the determination of elements of partially eclipsing systems from the photometric evidence is, in general, our inability to embark on the iterative process with any close estimate of the expected ratio k of the radii of the two components. In contrast to the case of total eclipses, no 'depth' or even 'duration' k is available to start us off, and no other way suggests itself of inferring a provisional value of this ratio as long as the observations bearing on such a system are limited to the photometric data. If, however, spectroscopic observations are available to supplement the photometric evidence, the situation may be thoroughly altered. Suppose that the lines of both components are visible in the composite spectrum of an eclipsing system and that their relative intensities can be measured. The relative intensities of the lines belonging to each component are, in general, proportional to its luminosity; therefore, the ratio of line intensities of the two spectra yields, in principle, the ratio of luminosities of the two components. The techniques in which such determinations can be made were worked out in detail by Petrie,* but their full exposition is beyond the scope of this section. Let it suffice here to stress that, first, such determinations are possible only if the relative luminosities of the two components are not too unequal (since a difference of more than one magnitude in their brightness is usually sufficient to extinguish completely all traces of the secondary's lines from the composite spectrum); and, secondly, their ratios depend on the frequency of lines employed for their determination. Petrie has, however, indicated the way in which the individual determinations based on lines in different parts of the spectrum can be reduced to one standard wave length. If such determinations are to be used in combination with the photometric evidence to obtain the elements of the system, it is clear that *the standard wave length to which the spectrophotometric determination of the ratio of luminosities of the two components has been reduced must be identical with the effective wave length of photometric observations.*

Suppose that this is the case, and that we are in possession of such a determination of the ratio of luminosities of the components constituting our eclipsing system. It should be stressed that what the spectroscopic observations can yield directly is L_1/L_2 rather than L_a/L_b. Spectroscopic observations cannot readily discriminate between the relative dimensions of the components; all they can do is to disclose whether it is the primary or the secondary component (i.e., the star of earlier or later spectral type) which is the more luminous of the two. It is true that, in most cases, the decision as to which star is the larger or smaller of the two is already implied. If, for instance, the components differ greatly in surface brightness but their luminosities are found to be comparable, it is obvious that the primary component must be the smaller of the two and, therefore, the eclipse giving rise to the primary minimum must be an occultation; for if the converse were true, the primary component would be so bright that the light of its mate would be completely drowned in its glare and its spectrum could not be seen. If, on the other hand, the depths of both minima and, therefore,

* R. M. Petrie, *Publ. D.A.O.*, 7, 205, 1939.

the surface brightnesses of the two components are approximately equal, but their luminosities are spectroscopically found to be quite different, it follows that the eclipse taking place at a time when the less luminous component is in front must be an occultation, and the alternate minimum a transit; the velocity curves of the two components must disclose whichever conjunction coincides with the primary or secondary minima. It is only when both the photometric ratio of surface brightnesses and spectroscopic ratio of the luminosities are very nearly unity that the identification of the occultations and transits may remain ambiguous; but it is at least obvious that, in such cases, the ratio of radii k of the two components cannot be far from one.

Considerations of this nature will, therefore, usually permit us to decide whether or not the primary component is also the larger of the two—i.e., whether $L_1/L_2 = L_a/L_b$ or L_b/L_a—and the known value of this ratio will permit us, in turn, to ascertain the ratio of the radii k by means of the equation (5-30) rewritten as

$$k^2 = Y(k, p_0)\,\frac{1 - \lambda_b}{1 - \lambda_a}\,\frac{L_a}{L_b}, \tag{7-19}$$

where $Y(k, p_0)$ can, to a good approximation, be replaced by unity.* With the fractional losses of light $1 - \lambda_a$ and $1 - \lambda_b$ taken from the light curve, and the ratio L_a/L_b inferred from the spectroscopic observations, the 'spectroscopic' value of k can be readily evaluated from (7-19) and used as the initial value of K with which we can embark upon our iterative process.

Its knowledge provides us, moreover, with an additional equation of the form

$$C_1 - kC_2 = 0, \tag{7-20}$$

in which k can now be assigned its 'spectroscopic' value. This equation plays a similar role now as it did in our discussion of totally eclipsing systems (*cf.* section VI.4) when k occurring in it was the 'depth' k. As before, the main point of interest raised by this equation concerns, not its form—which is self-evident—but rather the relative weight to which such an equation should be entitled. For unless the 'spectroscopic' k occurring in (7-20) is subject to no error, the equation itself will not be exact, but rather one of condition for determining C_1 and C_2, which should be adjoined to all other equations of the form (4-29) or (5-23)–(5-24) *with the weight that is equal to a ratio of the expected errors of their right-hand sides.* We have already mentioned in section VI.4 that the error of the absolute terms of the equations of condition of the form (4-29) or (5-23)–(5-24) is equal to the uncertainty ε of the observed light intensity of unit weight, while the error of the right-hand side of (7-20), caused by an uncertainty ε_k in the 'spectroscopic' value of

* This we are compelled to do since, in the preliminary stages, we are ignorant of the true value of p_0. Irrespective of it, however, the Y-function remains always numerically so close to unity that an error in the computed value k, brought by this simplification, should never exceed a few %—which is usually less than the actual uncertainty of k due to the uncertainty of all other quantities on the right-hand side of (7-19).

k will evidently be $C_2\varepsilon_k$. The weight $\sqrt{w}$ by which equation (7-20) should be multiplied before it is solved together with all other available equations of condition will, therefore, continue to be given by equation (4-36), but ε_k, the uncertainty of the 'spectroscopic' k, will be a resultant of the uncertainties, not only (as for the 'depth' k) of λ_a and λ_b, but also of L_a/L_b, an estimate of which should always be provided by the spectroscopist. Since the errors of all these quantities are mutually independent, the uncertainty of the 'spectroscopic' k should be (very approximately) given by

$$\left(\frac{2\varepsilon_k}{k}\right)^2 = \left[\frac{\varepsilon(L_a/L_b)}{L_a/L_b}\right]^2 + \left[\frac{\varepsilon(1-\lambda_a)}{1-\lambda_a}\right]^2 + \left[\frac{\varepsilon(1-\lambda_b)}{1-\lambda_b}\right]^2, \qquad (7\text{-}21)$$

where $\varepsilon(L_a/L_b)$, $\varepsilon(1 - \lambda_a)$ and $\varepsilon(1 - \lambda_b)$ denote the errors of the estimated quantities. The reader will notice that, similarly as in section VI.4, an estimate of the value of C_2 is prerequisite for assessing the weight of equation (7-20).

In conclusion, it should be pointed out that the procedure we have just outlined is, of course, applicable whenever the spectroscopist has provided the underlying observational data—regardless of whether the eclipses of our system are total or partial. The usefulness of the process is, however, limited very largely to the latter case; for if the eclipses are total, the 'depth' or 'duration' k's which follow from the photometric observations alone will provide an ample approximation for starting the iterative process, and the 'depth' k in particular will as a rule be superior in accuracy to any determination of the 'spectroscopic' one that can be made with the present techniques. If, however, the system under investigation exhibits partial eclipses, no k other than the 'spectroscopic' one can guide us in the opening phases of our investigation. Its significance is, furthermore, emphasized by the fact that its spectrophotometric determination is favoured by a diminishing difference in light of the two components, and thus it can lend us a helpful hand precisely in such cases which photometrically border on indeterminacy.

The Influence of a Third Body

So far we have tacitly assumed that the whole light of our eclipsing system comes from the two components constituting the eclipsing pair. Many eclipsing systems are, however, known to be attended by third bodies which do not participate in the eclipses. Sometimes these companions can actually be seen—though if such a star is faint, or very close, a large telescope may be needed to reveal its presence—or in other cases, the companion may be too close to the eclipsing system to ever be seen, but its presence is revealed by the gravitational effect of its mass on the motion of the eclipsing system through space. Algol or λ Tauri are the best known examples of the systems containing such invisible companions. Whether or not such third bodies are physically related with the eclipsing pair or are mere optical companions is beside the point; the salient fact is that, even if their presence is known, the observer may find it impossible to eliminate their light from his measurements—in which case a third body necessarily contributes to the total luminosity of the system a certain fraction of light which does not vary with the time. The influence of this additional source of light on the determination of elements of totally or partially eclipsing systems remains, therefore, to be investigated.

If, as before, the total luminosity of the system out of eclipses is to remain our unit of light, this obviously means now that

$$L_a + L_b + L_c = 1, \tag{7-22}$$

where L_c denotes the fractional luminosity of the third body. If l_{obs} stands for the observed fractional light of the system, including the third body, expressed in terms of its total combined light taken as unity, while l_{corr} denotes the light corrected for the third body and expressed in terms of the combined light of the eclipsing system only, we easily find that

$$l_{\text{corr}} = \frac{l_{\text{obs}} - L_c}{1 - L_c} \,. \tag{7-23}$$

Now if L_c has been actually measured (or estimated) by the observer, the best course to pursue is to correct the observed luminosities l_{obs} for its presence by means of equation (7-23) and proceed hereafter as before. There are, however, many eclipsing systems in which the presence of a third body, while not actually known, is nevertheless suspected; or (for invisible companions), their brightness can be estimated within a fair margin of uncertainty only from their probable mass. This raises the question of possible influence L_c on the determination of elements of eclipsing binary systems. If its contribution to the combined light of the system were wholly neglected, which elements would be affected and to what extent?

The answer to this question turns out to be different for different types of eclipses. If total and annular eclipses alternate, we have seen earlier (section VI.4) that the observational evidence enters into the determination of elements only through θ and α characterizing each observation or each normal point. But equation (7-23) makes it evident that

$$n_{\text{corr}} = \frac{1 - l_{\text{corr}}}{1 - \lambda_{\text{corr}}} = \frac{1 - l_{\text{obs}}}{1 - \lambda_{\text{obs}}} \tag{7-24}$$

i.e., that regardless of whether or not we took the light of the third body out of the observed luminosities, the α's will remain exactly the same. This discloses that, for *totally-eclipsing systems, the determination of the geometrical elements r_a, r_b and i, as well as of the ratios of luminosities and surface brightnesses of the components of the eclipsing system is unaffected by the presence of any third body* or, in general, of any *additive* light. The only element which will be affected is the fractional luminosity L_b of the larger component, which is now specified by

$$L_b = \lambda_a - L_c \,. \tag{7-25}$$

If the eclipses are *partial*, however, the situation is different inasmuch as now $\alpha \equiv n\alpha_0$ and, in the presence of a third body, the equations (5-11) and (5-12) connecting k and α_0 are, as a consequence of (7-22), to be replaced by*

$$(1 - L_c)\alpha_0' = 1 - \lambda_a + \frac{1 - \lambda_b}{k^2 Y(k, p_0)} \tag{7-26}$$

if the eclipse is an occultation, and

$$(1 - L_c)\alpha_0'' = (1 - \lambda_a)\frac{Y(k, p_0)}{Y(k, -1)} + \frac{1 - \lambda_b}{k^2 Y(k, p_0)} \tag{7-27}$$

if it is a transit. In either case, *the presence of an unsuspected third body* of non-vanishing luminosity *would*, therefore, *vitiate* the whole solution for the elements *by rendering* α_0 *spuriously too small*; what we would ordinarily accept as α_0 would actually be $(1 - L_c)\alpha_0$. This error would naturally jeopardize the correctness of all geometrical as well as physical elements computed on the assumption that we are dealing with a binary rather than a triple system; and if L_c happened to be large, the consequences of its neglect might become serious.

The reader might inquire whether or not it is possible to determine L_c, in such cases, simultaneously with all other elements of the eclipse from an analysis of the observed light changes. The answer would be in the affirmative, provided only that $\Delta\alpha_0$ could be determined simultaneously with C_1, C_2 and $\Delta\lambda$ or ΔU from the light curves within each minimum by the method of section VI.5, without any recourse to equations (7-26)–(7-27); the latter relations together with the 'shape' k's could then be used to specify L_c. Reasons were, however, listed in section VI.5 why this is indeed possible in theory but not yet in practice; for the respective solution would be next to indeterminate even if based on the best observations now available. As long as this situation continues to be true and equations (7-26) or (7-27) represent relations which are indispensable, rather than supplementary, to the success of our method of approach to the problem, it will *not* be possible to determine L_c from the observed light changes *simultaneously* with all other elements of the eclipse, *no matter how large the relative luminosity of the third body may be.*

The implications of these facts are obvious. Before considering his observational work complete, the investigator should scrutinize the system carefully, on the nights of best seeing—particularly at the times of minima and with the largest instruments accessible to him—in search for any indication that his star may be more than a simple binary system. He should also scrutinize the observations of radial velocity—if available—in order to make sure that the radial velocity of the centre of mass of the eclipsing system remains constant within the limits of observational errors. In this way he will be able to assign at least an *upper limit* to the relative brightness of a hypothetical third body beyond which it would not have escaped detection.

* These equations are easily deduced by the method of section VI.2 if the underlying equation (2-1) is merely replaced by (7-22).

If this limit is low, the uncertainty of his elements arising from this source may be equal to, or possibly less than, the uncertainty caused by the accidental errors of observation. If, on the other hand, the possibility cannot be ruled out that the luminosity of a hypothetical third body may be considerable, the investigator should keep this in mind and caution his readers accordingly. This point is evidently more pressing if the eclipses are partial than if total and annular minima alternate.

VI.8. DIFFERENTIAL CORRECTIONS

In discussions of photometric measures of eclipsing variables, the ultimate aim of the investigator should be to derive a set of elements of the eclipsing system under investigation, and to demonstrate that the entire observational evidence can be represented by such elements within the limits of observational errors. The final residuals between theory and observations should enable him to ascertain the uncertainty within which his elements are defined by the underlying observational data. In the preceding chapter, systematic methods were developed for a determination, not of the elements themselves, but of certain auxiliary constants which specify the elements of our eclipsing system; and once the most probable values of these constants have been established, the rest of the solution follows readily and unambiguously. Of these auxiliary constants, C_1, C_2 or C_3 are arrived at essentially by successive iteration, combined with the process of differential corrections for establishing the most probable values of λ_a, λ_b and U at the same time.

The light residuals Δl between our intermediary theory and observation are likewise implied already in our solution for the intermediary elements; for as the reader can easily verify, we defined our weight $\sqrt{w}$ in section VI.4 in such a way that

$$\Delta l \equiv l_{\text{obs}} - l_{\text{comp}} = -\Delta(\sqrt{w}\sin^2\theta). \tag{8-1}$$

Hence, *the (O–C)-residuals of a least-squares solution of properly weighted equations of condition for* $C_1, C_2, \ldots$ etc. of the form (4-29) if the eclipses are total, and (5-23) or (5-24) if they are partial, *are, by definition, identical with the light residuals* $-\Delta l$; and if the average probable error of a single photometric observation l_{obs} is equal to ε, the theory of least-squares processes discloses that the average probable error of $l_{\text{comp}} = l_{\text{obs}} - \Delta l$ will be equal to $\varepsilon\sqrt{m/n}$, where m denotes the number of the unknown quantities of our solution, and n, the number of the available equations of condition.*

The reader should, however, clearly bear in mind that the foregoing identity (8-1) holds good only *provided that our equations of condition of the intermediary solution were multiplied by the full-dress expression for* $\sqrt{w}$ *as given by equations* (4-15). We mentioned already in section VI.4 that this

* *Cf.* A. Otrebski, *Acta Astr.* (*a*) **4,** 139, 1948.

would but seldom be feasible in practice. In particular, the value of C_2 which occurs in the denominator of (4-18) or (5-9) will rarely be known beforehand with a precision warranting its use in the formation of the weights; and if only one minimum has been observed (or if both minima are being treated separately), the term $1 - \lambda$ in the numerator on the right-hand sides of (4-18) or (5-9) can likewise be ignored as an irrelevant constant. If both C_2 and $1 - \lambda$ were disregarded in the formation of our weights, we should expect our light residuals to follow from

$$\Delta l = -[(1 - \lambda)/C_2](O-C), \tag{8-2}$$

where the value of the conversion factor $(1 - \lambda)/C_2$ can be determined accurately from the outcome of our solution. Furthermore, if (in treating the visual or photographic observations) we satisfied ourselves that the errors of observation were constant on the magnitude rather than intensity scale—i.e., if we multiplied our equations of condition by $\sqrt{w}$ as defined by equation (4-15b) rather than by (4-15a)—it would follow that

$$\Delta l = -l(O-C). \tag{8-3}$$

In general, if we multiplied the weight $\sqrt{w}$, as defined by equations (4-15), of any given equation of condition by any number N to account for unequal observational weight of the respective normal point, or simply in order to keep its numerical value within reasonable limits, the corresponding light residual should be specified by

$$\Delta l = -N^{-1}(O-C). \tag{8-4}$$

Once the conversion factor between Δl and $(O-C)$ has been appropriately determined, the task of converting the $(O-C)$-residuals of the intermediary least-squares solution into the corresponding light residuals Δl is completely straightforward and can be performed with a minimum of difficulty.

The light residuals Δl thus determined should now be scrutinized with care for their possible origin. If the timing of our observations was sufficiently accurate, and the correctness of the underlying photometric scale beyond any doubt, the remaining residuals should arise from

(*a*) accidental errors of observation
(*b*) systematic errors in the determination of any (or all) intermediary elements of the eclipse.

The accidental errors are characterized by a random distribution of signs and should be equal in magnitude to those exhibited by the observations of full light. If, however, the Δl's within minima are found to be noticeably larger and (or) their variation turns out to be systematic, we are led to suspect that our intermediary elements do not represent the observed light changes within the limits of observation errors and are in need of further refinement.

The aim of the present section will be to outline the way in which our elements may be further improved by the method of differential corrections.*

Equations of the Problem

Let, as in section VI.2, L_1 denote the fractional loss of light, at any moment, of the star undergoing eclipse. The corresponding fractional light l of the eclipsing system then follows from

$$l = 1 - fL_1 . \tag{8-5}$$

The fractional loss of light f is an implicit function of δ, r_1, r_2, and u; moreover, δ itself is a function of the orbital period P, of the time t_0 of the principal conjunction, and of the orbital inclination i. Therefore, symbolically, $f \equiv f(P, t_0, r_1, r_2, i, u)$, where r_1 and r_2 denote (throughout this section) the fractional radii of the eclipsed and eclipsing component respectively. Let us suppose, reasonably enough, that the orbital period P has been determined from the observations with a precision far surpassing that of all other arguments of the f-function; that t_0 was found by reflecting the two branches of the light curve upon one another before the solution was started; and that approximate values of r_1, r_2 and i were derived subsequently by preliminary or intermediary methods for an assumed value of u. Let, furthermore, Δl denote the difference between light observed at any particular phase and that computed on the basis of the approximate elements from the residuals of our intermediary solution by means of the formulae at the beginning of this section. If we find the light residuals Δl to be systematic in nature, arising presumably from the imperfection of our intermediary elements, then differentiating equation (8-5) partially with respect to all elements involved we find that each normal point observed within minima should supply us with an equation of the form

$$\Delta l = -f\Delta L_1 - L_1 \Delta f, \tag{8-6}$$

where

$$\Delta f = \frac{\partial f}{\partial t_0}\Delta t_0 + \frac{\partial f}{\partial r_1}\Delta r_1 + \frac{\partial f}{\partial r_2}\Delta r_2 + \frac{\partial f}{\partial i}\Delta i + \frac{\partial f}{\partial u}\Delta u, \tag{8-7}$$

and where the symbols ΔL_1, Δt_0, Δr_1, Δr_2, Δi and Δu stand for the values of the differential corrections which are to be added algebraically to the respective elements in order to obtain their improved values. As many equations of condition of the form (8-6)–(8-7) can obviously be set up as there are normal

* At this point, many a reader may be apt to ask: if proper least-squares techniques were employed at the intermediary stage, how could the residuals possibly turn out to be systematic? This is indeed a very appropriate question, and one to which we can give at present only a temporizing answer. It may be pointed out that, apart from its function as an independent check on the correctness of the intermediary elements and of their uncertainty evaluated by the methods of section VI.7, the method of differential correctness is more general than the intermediary iterative process and should, therefore, be regarded as a complementary rather than competitive stage of our analysis. Its full usefulness will, however, not become evident until in connection with the 'perturbations' of distorted eclipsing systems. The reader desirous to learn fuller details at once may turn to section VI.12 for a more complete answer.

points observed within minima. Provided that their number sufficiently exceeds the number of the unknowns, a least-squares solution of such a set of equations then yields the most probable values of the requisite corrections as well as of their errors.

In order to be able to set up such equations of condition, the explicit forms of partial derivatives of f with respect to all parameters involved remain to be investigated for every type of the eclipse. Of the five coefficients involved in (8-7), that of Δu presents the least difficulty; for as far as first-order (i.e., linear) effects of darkening are concerned,

$$f = \frac{3(1-u)}{3-u} f^{\mathsf{U}} + \frac{2u}{3-u} f^{\mathsf{D}} = f^{\mathsf{U}} + \frac{2u}{3-u}(f^{\mathsf{D}} - f^{\mathsf{U}}), \tag{8-8}$$

and a differentiation with respect to u readily yields

$$\frac{\partial f}{\partial u} = 6\frac{f^{\mathsf{D}} - f^{\mathsf{U}}}{(3-u)^2}. \tag{8-9}$$

Furthermore,

$$\frac{\partial f}{\partial t_0} = -\frac{2\pi}{P}\frac{\partial f}{\partial \theta}, \tag{8-10}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \delta}\frac{\partial \delta}{\partial \theta} = \frac{1}{\delta}\frac{\partial f}{\partial \delta}\sin^2 i \sin\theta\cos\theta, \tag{8-11}$$

and

$$\frac{\partial f}{\partial i} = \frac{\partial f}{\partial \delta}\frac{\partial \delta}{\partial i} = -\frac{1}{\delta}\frac{\partial f}{\partial \delta}\sin i\cos i\cos^2\theta. \tag{8-12}$$

It appears, therefore, that a complete specification of the coefficients of our equations of condition reduces to the evaluation of

$$\frac{\partial f}{\partial r_1}, \quad \frac{\partial f}{\partial r_2}, \quad \frac{\partial f}{\partial \delta}.$$

Since, however, the function f depends on r_1, r_2 and δ only through their ratios and is, therefore, a homogeneous function of δ/r_1 and r_2/r_1 of zero order, Euler's theorem on homogeneous functions reveals that

$$r_1\frac{\partial f}{\partial r_1} + r_2\frac{\partial f}{\partial r_2} + \delta\frac{\partial f}{\partial \delta} = 0, \tag{8-13}$$

in virtue of which any one of these derivatives can be expressed in terms of the two others. A direct evaluation of $\partial f/\partial r_2$ and $\partial f/\partial \delta$ offers little difficulty; for we have already found* that, in accordance with equations (5-34) and (5-35) of Chapter IV,

$$\left.\begin{aligned} \frac{\partial f^{\mathsf{U}}}{\partial r_2} &= 2\frac{r_2}{r_1^2} I^0_{-1,0}, & \frac{\partial f^{\mathsf{D}}}{\partial r_2} &= 3\frac{r_2^2}{r_1^3} I^0_{-1,1}, \\ \frac{\partial f^{\mathsf{U}}}{\partial \delta} &= -2\frac{r_2}{r_1^2} I^1_{-1,0}, & \frac{\partial f^{\mathsf{D}}}{\partial \delta} &= -3\frac{r_2^2}{r_1^3} I^1_{-1,1}, \end{aligned}\right\} \tag{8-14}$$

* *Cf.* Z. Kopal, *Proc. Amer. Phil. Soc.*, **86**, 342, 1943.

where the $I^m_{\beta,\gamma}$'s are integrals defined by equation (3-26) of Chapter IV, whose explicit evaluation in terms of the geometrical elements for every type of eclipse has already been accomplished in section IV.5.

The above expression (8-7) for Δf, as it stands, takes account only of the linear effects of limb-darkening. In section IV.1 we have already pointed out, however, that any linear approximation to the actual law of limb-darkening may offer a tolerable approximation to reality over central portions of apparent stellar disks, but is bound to become seriously deficient near the limb. On the other hand, in sections 4 and 5 of this chapter we found that the weight of a determination of the geometrical elements of eclipsing binary systems from an analysis of their light curves rests predominantly on the variation of light during advanced phases of the eclipse when only the limb regions of an apparent disk of the star undergoing eclipse remain still visible—i.e., *precisely those regions of the disk over which non-linear effects of darkening should be most pronounced.* This fortunate geometrical circumstance suggests that a simultaneous determination of the coefficients of linear as well as quadratic (or possibly even higher) terms of limb-darkening may indeed become possible on the basis of photometric observations of attainable precision; and if so, this task could best be accomplished by the method of differential corrections.

In order to do so, equation (8-8) of the present section should, in accordance with equations (3-27) and (2-26)–(2-27) of Chapter IV, be replaced by

$$f = f^{\mathbf{U}} + \sum_{j=1}^{n} x_j\{f^{\mathbf{D}}_{(j)} - f^{\mathbf{U}}\}, \tag{8-15}$$

where

$$f^{\mathbf{U}} \equiv \alpha_0^0, \qquad f^{\mathbf{D}}_{(j)} = \tfrac{1}{2}(j+2)\alpha_j^0, \tag{8-16}$$

and the coefficients x_j are related with the A_j's of the expansion on the right-hand side of equation (1-23) of Chapter IV by means of

$$(2+j)x_j = \frac{2A_j}{1 + \sum_{i=1}^{n} \frac{2A_i}{2+i}}\,. \tag{8-17}$$

If, in particular, $n = 1$ (corresponding to linear law of darkening)

$$x_1 = \frac{2A_1}{3+2A_1} = \frac{2u_1}{3-u_1} \quad \text{if} \quad u_1 = \frac{A_1}{1+A_1}\,; \tag{8-18}$$

while if $n = 2$ (quadratic limb-darkening),

$$x_1 = \frac{2A_1}{3+2A_1+\frac{3}{2}A_2} \qquad \text{and} \qquad x_2 = \frac{2A_2}{1+\frac{2}{3}A_1+\frac{1}{2}A_2}\,. \tag{8-19}$$

Suppose that, reasonably enough, $x_1 \gg x_2$. If so, the coefficient x_2 can be set equal to zero, and preliminary elements of the system determined on the basis of linear limb-darkening. In order to allow for the effects of quadratic (or higher) limb-darkening, the coefficients of the equation (8-6) should be evaluated, as before, in terms of the preliminary elements, but the right-hand side of (8-7) should be augmented by the term

$$\frac{\partial f}{\partial x_2}\Delta x_2 = \{2\alpha_2^0 - \alpha_0^0\}x_2 = 2\mathfrak{J}^0_{-1,2}x_2 \tag{8-20}$$

in accordance with (5-17) of Chapter IV in the case of quadratic limb-darkening; or, in general, by

$$\frac{\partial f}{\partial x_j}\Delta x_j = \left\{\frac{2+j}{2}\alpha_j^0 - \alpha_0^0\right\}x_j \tag{8-21}$$

for limb-darkening of higher orders. The coefficients of the x_j's on the right-hand sides of the foregoing equations (8-20) oı (8-21), whose preliminary values were adopted to be zero, should now be solved for simultaneously with the differential corrections of the adopted coefficient u of linear darkening as well as of all geometrical elements based upon it. It is then, and only then, that the correct values, and genuine uncertainty, of all elements of an eclipsing system can be established; for a truncation of the series on the right-hand side at $n = 1$ (i.e., adoption of a linear law of limb-darkening) represents an arbitrary act which can be justified only *a posteriori* (i.e., if x_2 and higher coefficients of limb-darkening actually prove to be insignificant); and even then its neglect (i.e., the fact that x_2 can be set equal to zero only within the limits of its uncertainty) would be reflected in a spurious diminution of the probable errors of all other elements (including that of Δu).

Apart from the effects of limb-darkening, however, equations (8-9)–(8-14) permit us to express all differential quotients of corrections to the geometrical elements in terms of known integrals of the form $I^m_{\beta,\gamma}$ which have been extensively tabulated.* The existence of such tables reduces the computation of the individual coefficients of equation (8-7) largely to one of interpolation. It may, however, be also pointed out in this connection that as long as we are satisfied with an accuracy of the order of $0{\cdot}1\%$ in the coefficients, all differential quotients involved in equation (8-7) may also be obtained by a numerical differentiation of existing tables of the α-function; and the process by which this can be accomplished is of sufficient interest to be outlined in some detail.

In section VI.2 we learned that the fractional loss of light f_a exhibited during an occultation eclipse can be expressed as

$$f_a(k,p) = (1 - x_a)\alpha^{\mathsf{U}}(k,p) + x_a\alpha_a^{\mathsf{D}}(k,p), \tag{8-22}$$

while during a transit

$$f_b(k,p) = (1 - x_b)q^{\mathsf{U}}\alpha(^{\mathsf{U}}k,p) + x_b q^{\mathsf{D}}\alpha_b^{\mathsf{D}}(k,p), \tag{8-23}$$

* *Cf.* Z. Kopal, *Harv. Circ.*, No. 450, 1947.

where

$$q^{\mathbf{U}} = k^2 \qquad \text{and} \qquad q^{\mathbf{D}} = \tfrac{3}{2}k^2\Phi(k), \tag{8-24}$$

so that

$$f_b(k, -1) = (1 - x_b)q^{\mathbf{U}} + x_b q^{\mathbf{D}} = k^2 Y(k, -1). \tag{8-25}$$

The evaluation of the derivatives of f with respect to u, which occurs in equations (8-22) and (8-23) explicitly, presents no difficulty and leads to equation (8-9); while the derivatives of f with respect to the geometrical elements r_a, r_b and i can be obtained as follows. Let, for brevity's sake, any one of these three geometrical elements be denoted by w. If so, then evidently

$$\frac{\partial f_a}{\partial w} = (1 - x_a)\frac{\partial \alpha^{\mathbf{U}}}{\partial w} + x_a \frac{\partial \alpha^{\mathbf{D}}}{\partial w} \tag{8-26}$$

and

$$\frac{\partial f_b}{\partial w} = (1 - x_b)\left\{\alpha^{\mathbf{U}}\frac{\partial q^{\mathbf{U}}}{\partial w} + q^{\mathbf{U}}\frac{\partial \alpha^{\mathbf{U}}}{\partial w}\right\} + x_b\left\{\alpha_b^{\mathbf{D}}\frac{\partial q^{\mathbf{D}}}{\partial w} + q^{\mathbf{D}}\frac{\partial \alpha_b^{\mathbf{D}}}{\partial w}\right\}. \tag{8-27}$$

Now in view of the equations (8-24), the only partial derivatives of $q^{\mathbf{U}}$ which are not identically zero are

$$\frac{\partial q^{\mathbf{U}}}{\partial r_a} = \frac{2k}{r_b} \qquad \text{and} \qquad \frac{\partial q^{\mathbf{U}}}{\partial r_b} = -\frac{2k^2}{r_b}; \tag{8-28}$$

whereas

$$\frac{\partial q^{\mathbf{D}}}{\partial w} = \frac{\partial q^{\mathbf{D}}}{\partial k}\frac{\partial k}{\partial w}, \qquad \frac{\partial q^{\mathbf{D}}}{\partial k} = (16k/\pi)\sqrt{k(1-k)}; \tag{8-29}$$

so that the only non-vanishing derivatives of $q^{\mathbf{D}}$ are again found to be

$$\frac{\partial q^{\mathbf{D}}}{\partial r_a} = (16k/\pi r_b)\sqrt{k(1-k)} \quad \text{and} \quad \frac{\partial q^{\mathbf{D}}}{\partial r_b} = -(16k^2/\pi r_b)\sqrt{k(1-k)}, \tag{8-30}$$

respectively.

The derivatives of the α-functions with respect to r_a, r_b or i are complicated functions of k and p; but, in practice, their analytical evaluation can be avoided by setting

$$\frac{\partial \alpha}{\partial w} = \frac{\partial \alpha}{\partial k}\frac{\partial k}{\partial w} + \frac{\partial \alpha}{\partial p}\frac{\partial p}{\partial w}. \tag{8-31}$$

In this form, the terms depending on the properties of each individual system have been separated from the purely geometrical ones; for it is only the derivatives of k and p with respect to r_a, r_b and i which will be different from star to star and need to be evaluated anew in each particular case. They are, moreover, simple: namely, as the reader can easily verify,

$$\frac{\partial k}{\partial r_a} = \frac{1}{r_b}, \qquad \frac{\partial k}{\partial r_b} = -\frac{k}{r_b}, \qquad \frac{\partial k}{\partial i} = 0, \tag{8-32}$$

and

$$\frac{\partial p}{\partial r_a} = -\frac{p}{r_a}, \quad \frac{\partial p}{\partial r_b} = -\frac{1}{r_a}, \quad \frac{\partial p}{\partial i} = \frac{1}{r_a}\frac{\partial \delta}{\partial i} = -\frac{\cos^2\theta \sin^2 i}{2\delta r_a}. \tag{8-33}$$

Of these derivatives, only $\partial p/\partial r_a$ and $\partial p/\partial i$ are found to vary with the phase; all others are constant for each particular system. Inserting (8-32) and (8-33) in (8·31),

$$\left.\begin{aligned} \frac{\partial\alpha}{\partial r_a} &= \frac{1}{r_b}\frac{\partial\alpha}{\partial k} - \frac{p}{r_a}\frac{\partial\alpha}{\partial p}, \\ \frac{\partial\alpha}{\partial r_b} &= -\frac{k}{r_b}\frac{\partial\alpha}{\partial k} - \frac{1}{r_a}\frac{\partial\alpha}{\partial p}, \\ \frac{\partial\alpha}{\partial i} &= -\frac{\cos\theta\sin 2i}{2\delta r_a}\frac{\partial\alpha}{\partial p}, \end{aligned}\right\} \tag{8-34}$$

as the final result of our transformations which, inserted in (8-26) or (8-27), specifies the required derivatives $\partial f/\partial w$.

The partial derivatives of α, in the foregoing equations, with respect to k and p continue to be complicated non-dimensional functions of k and p, but their numerical values can be obtained without difficulty by univariate numerical differentiation of columns $k = constant$ or $p = constant$ in Tsesevich's appropriate α-tables.* The reader interested primarily in applications need not, however, fall back on this process in every practical case, since this arduous piece of work was performed once and for all by Irwin,† to whose efforts we owe a complete set of tables of three- to four-decimal values of all requisite partial derivatives of f, evaluated in terms of k and p as arguments of tabulation.

With the numerical values of the coefficients of the most important differential corrections thus specified for any phase and any type of eclipse, let us proceed now to describe their general properties in somewhat greater detail. The coefficient of ΔL_1 calls for very little comment. In the preceding section we have recalled that f is expressible in terms of the normalized fractional losses of light α by means of equation (8-22) if the eclipse is an occultation, and by (8-23) if it is a transit; hence, the numerical values of f can be expressed with sufficient precision in terms of the α's of the intermediary solution without difficulty. With regard to the coefficient of Δt_0, equations (8-11) and (8-12) disclose that

$$\frac{\partial f}{\partial\theta} = \frac{\sin\theta\sin i}{\cos\theta\cos i}\frac{\partial f}{\partial i} \tag{8-35}$$

for any type of eclipse; hence, a tabular knowledge of $\partial f/\partial i$ supplies also that of $\partial f/\partial\theta$. It should, however, be borne in mind that Δt_0 can be included among the unknowns of our solution *only if the normal points of the ascending and descending branch of the minimum have been treated separately.* A 'reflection' of one branch of the light-curve upon another involves a tacit assumption that the times of maximum obscuration at either minimum can be estimated

* Table I (Tsesevich 45) gives five-decimal values of $\alpha_a^{\mathbf{D}}(k, p)$ for $k = 0{\cdot}00(0{\cdot}05)1{\cdot}00$ and $p = 1(-0{\cdot}01) - 1$, and Table III (*op. cit.*) contains $\alpha_b^{\mathbf{D}}(k, p)$ to the same accuracy and for the same range of the arguments; while Table V (Tsesevich 50) gives a corresponding tabulation of $\alpha^{\mathbf{U}}(k, p)$.

† *Ap. J.*, **106**, 380, 1947.

from the original obseivations, as they stand, so accurately as to require no subsequent coriection. Such a policy may, however, not always be legitimate in practice. If an inspection of the observed normals suggests that the moments of maximum eclipse are not defined by them with a convincing precision, we should abstain from combining the ascending and descending branches of either minimum (at the intermediary stage, it would have made no difference at all if the sign of θ were positive or negative), and include t_0 among the elements to be adjusted by the method of differential corrections. Equation (8-11) makes it evident that, at corresponding points of the two branches of either minimum, $\partial f/\partial \theta$ is equal in magnitude but opposite in sign, while the coefficients of all other unknowns are equal in both magnitude and sign; in consequence, Δt_0 should ordinarily be a well-determined quantity. It is true that if the observations are well distributed, the omission of Δt_0 in the general solution will have very little influence on the calculated elements of all other corrections. A knowledge of Δt_0 as well as of its probable error is, however, important for the determination of $e \cos \omega$ from the times of the minima (*cf.* section VI.9) and, in particular, for investigations of possible changes in $e \cos \omega$ due to apsidal motion, as well as in the investigation of possible variations of the orbital period and of their significance. It appears, therefore, to be good policy to include it in any precise investigation—even though this may entail some increase in numerical work.

The coefficients of Δr_1, Δr_2 and Δi again offer but little opportunity for comment—except for one concerning the behaviour of $\partial f/\partial i$ near $i = 90°$. As $i \rightarrow 90°$, equation (8-12) discloses that $\partial f/\partial i$ will approach zero with $\sin 2i$—a fact which is apt to render Δi nearly indeterminate for eclipsing systems whose orbits are nearly parallel to the line of sight. In such cases, it is reasonable to follow Irwin and other investigators in solving for, not Δi, but $\Delta \cos^2 i$ as the differential correction to be determined simultaneously with all others. Since

$$\frac{\partial f}{\partial \cos^2 i} = \frac{\cos^2 \theta}{2\delta} \frac{\partial f}{\partial \delta}, \tag{8-36}$$

the coefficient of $\Delta \cos^2 i$ remains finite for $i = 90°$; therefore, the quantity $\Delta \cos^2 i$ should be obtainable with a considerable weight even when that of Δi would become effectively zero.

The coefficient of Δu, (or x_2) and a simultaneous determination of this latter quantity with all other differential corrections, poses a special problem and represents one of the most difficult and exacting tasks encountered in the analysis of light curves of eclipsing binary systems. The reason will become evident as soon as we compare (by means of Irwin's tables or otherwise) the numerical magnitudes of $\partial f/\partial u$ with those of the coefficients of other differential corrections in equation (8-7). It will transpire that, whereas the partial derivatives of f with respect to r_1, r_2 and i (or $\cos^2 i$) are, in general, quantities of the order of magnitude zero (or one-tenth), that with respect to u (proportional to the difference $f^{\mathrm{D}} - f^{\mathrm{U}}$) is as a rule a quantity of the order of 0·001–0·01. This fact exposes at once a part of the difficulty which we are going to experience in trying to determine the degree of limb darkening simultaneously with other geometrical elements; but, unfortunately, even this is not yet the whole story. It appears that Nature has really conspired to thwart our efforts at determining the degree of darkening of the components of eclipsing binary systems in a very effective manner: for *not only is* $\partial f/\partial u$ *numerically so small* in comparison with other coefficients, but *its variation with* k and p *happens to simulate that of* $\partial f/\partial r_1$ *so closely as to render a significant separation of* Δr_1 *and* Δu *from a simultaneous solution a supremely difficult task.* This makes it evident that a determination of u (by *any* method) which does not include a simultaneous adjustment of all other geometrical elements would have very little meaning; for it transpires from the foregoing discussion that any arbitrary change in u can be countered to a large extent by an appropriate change in r_1 to restore an almost identical fit.

The extent to which the variation of $\partial f/\partial u$ with the phase simulates that of $\partial f/\partial r_1$ in any practical case can be readily ascertained by means of the numerical data compiled in Irwin's tables. An inspection of these data disclose that if $p_0 > -0{\cdot}6$—i.e., for most partially eclipsing systems—the ratio $(\partial f/\partial u) \div (\partial f/\partial r_1)$ remains so nearly constant throughout the eclipse as to defeat effectively any attempt at a separate determination of Δr_1 and Δu from a simultaneous solution. The only systems for which we expect to have any chance of success are obviously systems in which p_0 comes very close to -1, with k in the neighbourhood of 1 (i.e., if the eclipses are central and both components are nearly of the same size), or if p_0 actually becomes less than -1 and the eclipse becomes annular. The likelihood of eclipses of the former type is small; but annular eclipses of appreciable durations are not quite as rare. It is on such systems that the attention of observers aiming at a determination of limb-darkening *from observations of one minimum alone* should primarily be concentrated. For typical examples of the observed light changes within annular eclipses of limb-darkened stars *cf.* the accompanying Figs. 6-5 and 6-6.

In binaries which, on account of both the smallness of $\partial f/\partial u$ and its high degree of correlation with $\partial f/\partial r_1$, are unsuitable for a determination of the coefficient of limb-darkening—such as the partially eclipsing systems in general—a retention of Δu among the quantities to be determined by a simultaneous analysis, would simply result in a drastic loss of weight of the whole solution. This would indeed be unavoidable if we knew nothing whatever of the true value of u. Actually, we should nowadays be able to estimate u from extraneous evidence (spectrum, colour, and the available physical theory) within an uncertainty which may be close to 0·1 and should seldom exceed 0·2.* Under such circumstances, the

* For a discussion of the actual degrees of darkening for stars of different spectral types and observed in different colours *cf.*, again, section IV.1 of this book; while the available empirical evidence on this subject has been summarized by Z. Kopal on pp. 623–629 of the I.A.U. *Transactions*, **9**, 1957.

remaining elements of the system may then be adjusted by the method of differential corrections for an *assumed* value of u, which was at the basis of our intermediary (or preliminary) elements. Suppose, however, that we should like to know the extent to which all our elements would be changed should we decide to alter subsequently the accepted value of u by a small

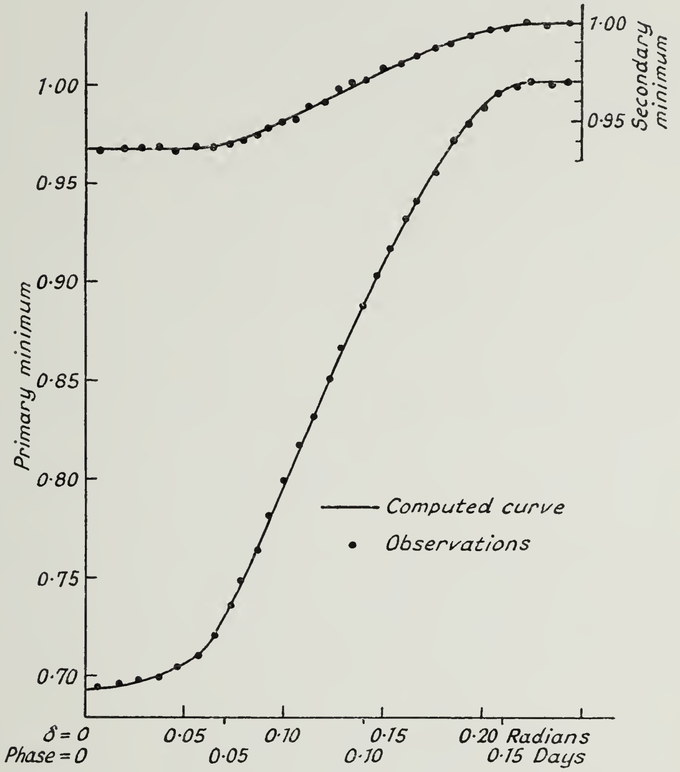

FIGURE 6-5. THE OBSERVED (PHOTOELECTRIC) NORMAL POINTS AND THEORETICAL LIGHT CURVE OF THE ECLIPSING VARIABLE YZ CASSIOPEIAE according to G. E. Kron (*Lick Bull.*, No. 499, 1939). Upper curve: secondary minimum (due to total eclipse); lower curve: primary minimum (annular eclipse). Abscissae: fractional intensity of the system (its intensity between minima being taken as unity of light); ordinates; the phase angle (in radians) and the phase (in days).

amount Δu. In order to ascertain the effect, upon other elements, of an arbitrary small change in u, all we need to do is to transpose the term $(\partial f/\partial u)\Delta u$ in equations (8-6)–(8-7) to its left-hand side and to treat it as a part of the absolute term. The solution of such a system will yield the values of all other corrections in the form $\Delta r_1 = a_1 + b_1\Delta u$, $\Delta r_2 = a_2 + b_2\Delta u$, . . . etc., where the a's and the b's are constants resulting from the solution; and these may have a considerable weight unless some other source of uncertainty is present.

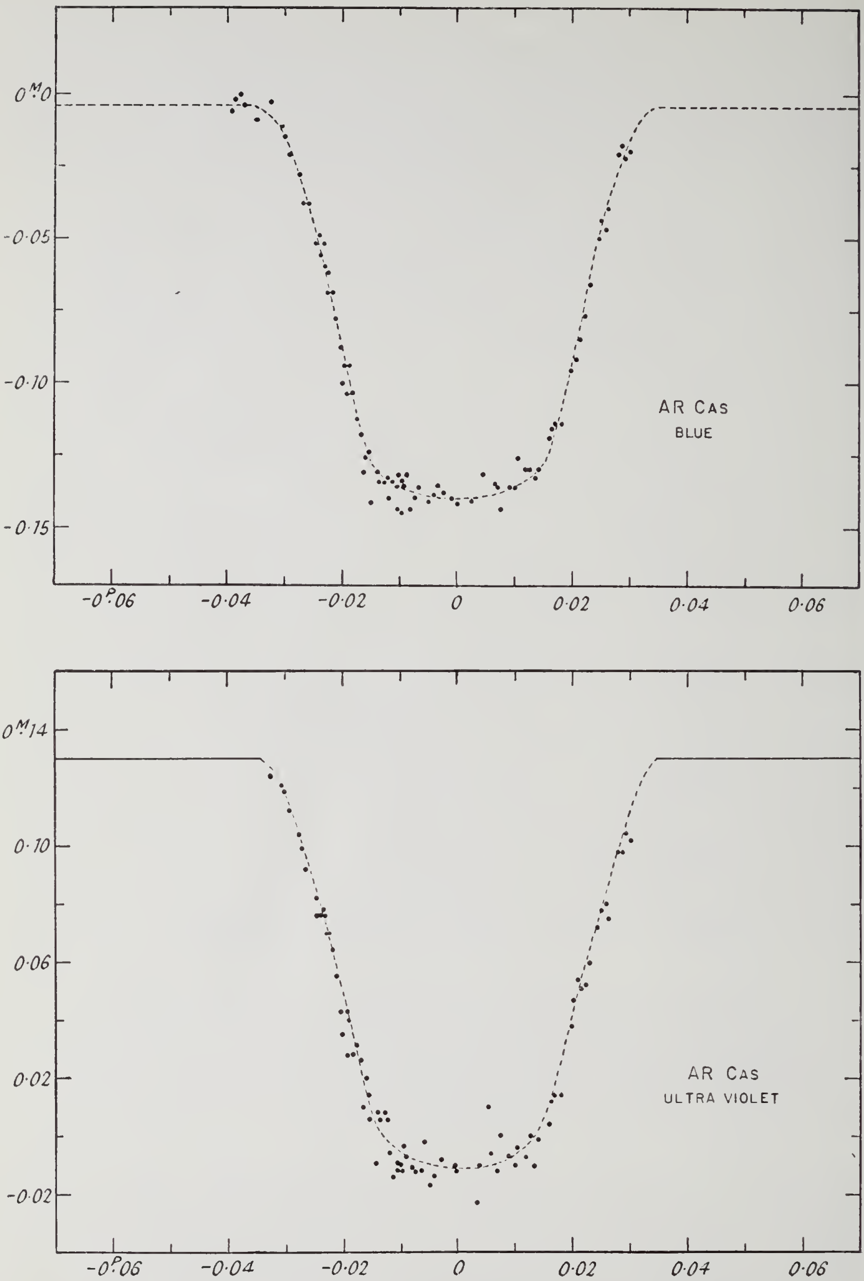

FIGURE 6-6. LIGHT CHANGES AND THEORETICAL LIGHT CURVES OF THE ANNULAR ECLIPSE IN THE SYSTEM OF AR CASSIOPEIAE

in the blue (λ 4200 A; above) and the ultra-violet (λ 3500 A; below), according to the photoelectric observations by C. M. Huffer (*A. J.*, 60, 164, 1955). Abscissae: the relative brightness of the system in stellar magnitudes; ordinates: the phase in fractions of the orbital cycle.

Solution of the Equations

The actual techniques by which the least-squares solutions of the equations defining our differential corrections may actually be performed need not detain us here, since a systematic exposition of such methods is being reserved for the Appendix. Let it suffice to point out a few safeguards which should be observed in combining the observational evidence pertaining to both minima into a single set of normal equations, or in combining the photometric and spectroscopic evidence in a single solution. It goes without saying that whenever two minima have been observed, all elements of the system should be adjusted so as to ensure the best possible fit to the light changes observed during both alternate eclipses. If the secondary minimum is shallow, the weight of the observational evidence based upon it may be too small to make its neglect of any consequence; but whenever this is not true, the equations of condition based upon both minima should be combined in a single set of normal equations before proceeding to the final solution. With respect to the weights of the individual equations of condition of the form (8-6)–(8-7), it should be borne in mind that *their intrinsic weights are already included in the numerical values of the coefficients of the respective differential corrections*, so that it is only the appropriate observational weights which should be applied to equations (8-6)–(8-7) if the underlying normal points themselves are of unequal quality.

In combining the equations of the form (8-6)–(8-7) into a single set of normal equations we should, furthermore, observe that our definition of the subscript 1, 2 to pertain to the eclipsed and eclipsing component, respectively, will render Δr_1 and Δr_2, as well as ΔL_1 and ΔL_2, different quantities at the alternate minima: namely, if the minimum under investigation is due to an occultation eclipse,

$$\Delta r_1 = \Delta r_a \qquad \Delta L_1 = \Delta L_a$$
$$\Delta r_2 = \Delta r_b \qquad \Delta L_2 = \Delta L_b$$

while, during a transit eclipse, the meaning of the respective quantities is interchanged and

$$\Delta r_1 = \Delta r_b\,, \qquad \Delta L_1 = \Delta L_b\,,$$
$$\Delta r_2 = \Delta r_a\,, \qquad \Delta L_2 = \Delta L_a\,.$$

In combining the evidence of the two minima, care should be taken to realize these facts. It is, furthermore, obvious that the correction Δu to the assumed degree of limb-darkening pertains likewise always to the darkening of the component undergoing eclipse; and hence, during an occultation eclipse $\Delta u \equiv \Delta u_a$; while during a transit $\Delta u \equiv \Delta u_b$.

The maximum total number of the unknowns to be encountered in a system of normal equations based on both alternate minima is, therefore, eight: namely,

$$\Delta t_0\,,\ \Delta r_a\,,\ \Delta r_b\,,\ \Delta i\,,$$
$$\Delta L_a\,,\ \Delta L_b\,,\ \Delta u_a\,,\ \Delta u_b\,.$$

Whether or not it is advisable to carry Δt_0 or Δu_a and Δu_b as unknowns to be determined by a simultaneous analysis, is a question which must be decided by the investigator in each particular case. Whether or not it is necessary to carry both ΔL_a and ΔL_b as independent unknowns of our impending least-squares solution constitutes, however, a question of principle which we may briefly consider. The reader may recall that, in section 2 of this chapter, we defined the sum $L_a + L_b$ to be our unit of light; and if the value of this unit were known exactly, we could obviously put $\Delta L_a = -\Delta L_b$. In actual practice, the value of our light unit can again be inferred from the observations with only a finite degree of accuracy, in which case the foregoing relation between ΔL_a and ΔL_b can no longer be regarded as exact and should be replaced by

$$\Delta L_a = \Delta U - \Delta L_b, \tag{8-37}$$

where ΔU represents, as before, a possible correction to a preliminary determination of our light unit. The foregoing equation makes it evident that if we retain both ΔL_a and ΔL_b as independent unknowns of our least-squares solution, their sum will furnish ΔU and the uncertainty of the linear function $\Delta L_a + \Delta L_b$ will be equal to that of ΔU. If so, equation (8-6) should be replaced by

$$\Delta l = \Delta L_1 + \Delta L_2 - f\Delta L_1 - L_1\Delta f. \tag{8-38}$$

Moreover, the observations between minima provide us with an additional equation of condition of the form

$$\Delta L_1 + \Delta L_2 = 0, \tag{8-39}$$

which is fulfilled within a certain probable error. If, on the other hand, we decide to eliminate ΔL_2 from our solution by putting it equal to $-\Delta L_1$, we should automatically include ΔU among the unknowns of our solution and replace (8-6) by

$$\Delta l = \Delta U - f\Delta L_1 - L_1\Delta f, \tag{8-40}$$

while the observations between minima assert that

$$\Delta U = 0, \tag{8-41}$$

again within a certain probable error.

The square-root $\sqrt{w}$ of the relative weight with which the foregoing equations (8-39) or (8-41) should join the rest of our overdeterminate system is—as in equations (4-32)—equal to the ratio ε/σ_U, where σ_U denotes again the probable error of our unit of light as determined from the observations between minima (if the light remains sensibly constant out of eclipses, σ_U becomes simply the probable error of the approximate mean of all observations secured during full light), and ε, the corresponding error of a single normal, within minima, of unit weight.

The foregoing considerations will permit us to bring the whole photometric evidence, between minima as well as within eclipses, to bear on the determination of differential corrections to all elements of our eclipsing system, which are required to establish the best possible fit. In order to complete our discussion let us, moreover, outline the way in which spectroscopic observations can be combined with the photometric evidence at this stage of our analysis. The significance of possible spectrophotometric contribution to the determination of elements of eclipsing binaries—particularly of partially-eclipsing systems—has already been discussed in the preceding section VI.7 and need not be repeated here. Our present aim will be confined to showing the way in which a spectrophotometric determination of L_1/L_2 can be combined with the purely photometric evidence before a final solution for the differential corrections is carried out.

Suppose that the lines of both components are visible in the composite spectrum of our eclipsing binary system, and that we are in possession of spectrophotometric measures of the relative intensities of both sets of lines which determine the ratio

$$\frac{L_1}{L_2} = A \pm \eta, \tag{8-42}$$

where η stands for the probable error of the spectrophotometric determination of the ratio A. In view of (2-1), the preceding equation gives

$$L_1 = \frac{A}{1 + A} \cdot \tag{8-43}$$

Moreover, let L_{1p} be the value of L_1 as determined from the photometric solution alone. A difference

$$\Delta L_1 = \frac{A}{1 + A} - L_{1p} \tag{8-44}$$

provides us now with an additional equation of condition to be solved together with the rest; all we need to do to make this possible is to ascertain its relative weight.

In order to do so we may observe that, as far as the uncertainty of spectroscopic measures is concerned, the error of the right-hand side of equation (8-42) is equal to

$$\frac{\partial \Delta L_1}{\partial A} \delta A = \pm \frac{\eta}{(1 + A)^2} \cdot \tag{8-45}$$

A square-root $\sqrt{w}$ of the relative weight by which both sides of equation (8-42) should be multiplied before being solved together with the rest of our equations of condition is then evidently given by

$$\sqrt{w} = (1 + A)^2(\varepsilon/\eta), \tag{8-46}$$

where ε denotes, as before, the probable error of a single normal point, observed within minima, of unit weight.

With all foregoing considerations duly in mind, the investigator may now proceed to complete his set of the equations of condition by including the gist of all pertinent extraneous evidence with an appropriate weight. The numerical accuracy requisite at this stage of our analysis is largely determined by the circumstances on hand. Irwin's tables furnish the coefficients of the individual equations of condition to three significant figures. This is somewhat more than is actually needed for an exhaustive interpretation of as accurate observational data as any that can be made at the present time; two significant figures in the coefficients of the equations of condition are all that are really necessary. Since, however, an additional decimal can be extracted from Irwin's tables with no more difficulty, most investigators may wish to avail themselves of the full facilities offered by Irwin's tables and evaluate all coefficients to as many decimal figures as these tables will permit. The coefficients of the *normal* equations should, in turn, be evaluated correctly to at least *twice* the number of significant figures retained in the equations of condition—or even more if the corrections Δu to the adopted degrees of limb darkening were included among the unknowns to be determined by a simultaneous analysis—in order to forestall a possible incipient degeneracy of the solution, and an undue accumulation of round-off errors in the process of solution which may arise from this source.

The number of digits which should be retained in the final values of the differential corrections, constituting the outcome of such a solution, should depend wholly on their significance. If the elements which we set out to improve by way of differential corrections represented a close approximation to reality (such as we should ordinarily expect of our intermediary elements), all corrections resulting from such a solution may be on the limit of significance. If, on the other hand, our underlying set of elements happened to be weak (as might easily happen if our preliminary elements were deduced by purely graphical methods), the amount of the resulting differential corrections might exceed their uncertainty by a wide margin. Should this turn out to be the case, we may find ourselves confronted with the unpleasant task of having to repeat the whole solution. For a significant feature of our problem has been the fact that *the coefficients of all differential corrections in equation* (8-7) *are not constants, but functions of the elements which we seek to improve*. Hence, if the corrections to one (or more) elements exceed their probable errors by a large factor, the original equations of condition may no longer be sufficiently accurate. The coefficients in (8-7) should then be recomputed with the use of the improved elements, a new set of equations should be solved—and this procedure should be repeated until the resulting corrections are no longer significant.

The labour involved in constructing and solving successive sets of the equations of condition by the method of least-squares may easily become taxing indeed. It is, however, by no means prohibitive—for Irwin's tables go a long way toward relieving us of a large part of the incumbent routine—and should be resorted to whenever the need becomes evident. But it should be plainly stated that this additional work would have been spent much more profitably in improving the quality of a deficient preliminary solution *before* we proceeded to calculate the differential corrections. *The further we progress in our analysis, the more laborious it becomes to correct any blunder committed in the earlier stages of our work.* Therefore, it is only after a really good set of elements has been obtained by any one of the more elementary methods, and the light residuals Δl reduced to amounts which are not far from the limits of observational errors, that a final adjustment of all elements by way of differential corrections becomes justifiable.

Suppose that, at last, a satisfactory set of simultaneous differential corrections has been established, and that their uncertainty has been found in terms of the appropriate elements of inverse matrix of our solution in the customary manner.* The most probable values of all elements of our eclipsing system are then clearly obtained by an algebraic addition of the respective differential correction to the previously adopted value of the element; but *will the uncertainty of this element be identical with that of its differential correction?* The answer is known to be unequivocally in the affirmative only if the equations of condition defining the differential corrections possess constant coefficients. In our present case, however, the coefficients in the equation (8-6)–(8-7) are functions of the variables we seek to determine. If so, the problem raised by our inquiry turns out to be by no means simple. This subtle point from the theory of linear equations has recently been investigated by Piotrowski,† and his answer turned out to be a conditional '*yes*.' The uncertainty of our differential corrections, as defined by our least-squares solution, remains sensibly (though not exactly) equal to that of the respective element, *provided* that the sums of products of all second partial derivatives of f multiplied by the respective (O–C) differences are negligible in comparison with the sums of cross-products of the first partial derivatives of f. If, for example, any one of the second derivatives of f could become numerically large at any phase of the eclipse, Piotrowski's condition might be violated regardless of the smallness of the (O–C)'s. The same situation could possibly occur if the signs of the (O–C) differences were not distributed at random, but were correlated with the sign of any one of the second derivatives of f. Fortunately for us, none of the second partial derivatives of f can become very large for any type of the eclipse. If, in addition, the algebraic signs of the (O–C)'s are found to oscillate approximately at random, we can derive confidence from Piotrowski's work for concluding that the probable errors of Δw and w will, for all practical purposes, be indeed the same—whatever element w may stand for.

VI.9. EFFECTS OF ORBITAL ECCENTRICITY

Throughout all preceding sections we have considered the relative orbit of components in an eclipsing system to be circular, and have adopted its

* For details of this process the reader is again referred to the Appendix.

† S. L. Piotrowski, *Proc. U.S. Nat. Acad. Sci.*, **34**, 23, 1948.

radius R as our unit of length. The real orbits of many eclipsing systems, however, may happen to be eccentric—in which case R ceases to be constant and comes to depend on the phase in accordance with the well-known integral of the problem of two bodies which states that

$$R = \frac{A(1 - e^2)}{1 + e \cos v}, \tag{9-1}$$

where A denotes the semi-major axis of the relative orbit; e, its eccentricity; and v, the true anomaly reckoned from the moment of the periastron passage in the direction of motion. Observations have shown that the orbital eccentricities encountered among close binary systems are as a rule so small* that the adoption of a circular orbit will invariably offer a safe approximation. Among well-separated systems of long periods, however, eccentric orbits turn out to be not too infrequent. The wider a binary system, the smaller, of course, is the probability that it will exhibit eclipses; hence, the percentage of eccentric orbits among eclipsing binaries may, in general, be expected to be small. Considering the frequency of eclipsing stars in space, however, even a small percentage may represent a great number of actual cases; therefore, the effect of orbital eccentricity upon the form of light curves and the determination of the elements invite close analysis. In the present section such an analysis will be given.

The effects of orbital eccentricity upon light curves of eclipsing variables are usually manifest at a glance. First, according to the laws of areas, *the times of the minima of light will generally not be separated by exactly half the orbital period*, but will be *displaced* by a relative amount depending on both the eccentricity and orientation of the apsidal line with respect to the line of sight. The same law will also render *both* minima to be *asymmetric*, and of *unequal durations*. From an analytical point of view, the treatment of these effects will differ in principle depending on whether or not the plane of the eclipsing orbit is parallel to the line of sight—i.e., *whether or not the moments of the minima of light happen to coincide with the conjunctions.* If this is the case (i.e., if $i = 90°$ and the ecilpses are central), the problem admits of a rigorous solution in a closed form; in the converse case, it can be solved only by approximations. In what follows, these two cases will be discussed in turn, and equations deduced which should permit us, not only to determine the orbital eccentricity e and the longitude of periastron ω from the observed characteristics of the respective light curves, but also to express the fractional radii r_a and r_b of the two components in terms of the semi-major axis A of the relative orbit.

Central Eclipses

For convenience in comparison with spectrographic orbits, let us consider the primary (brighter) component as moving about the secondary in a plane

* This is bound to be so, for an appreciable eccentricity might lead to collisions at the time of periastron passage.

parallel to the line of sight (i.e., $i = 90°$), and let the longitude of periastron ω be measured, as usual, from the ascending node in the direction of motion. The true anomaly v and the phase angle θ are then evidently related by

$$\theta = v + \omega - 90°, \tag{9-2}$$

and the primary and secondary minima occur when

and
$$\left.\begin{aligned} \theta_1 &= 0°, & v_1 &= 90° - \omega, \\ \theta_2 &= 180°, & v_2 &= 270° - \omega, \end{aligned}\right\} \tag{9-3}$$

respectively. According to Kepler's second law, the time interval $t_2 - t_1$ that elapses between the two minima, expressed in terms of the orbital period P, becomes

$$\frac{t_2 - t_1}{P} = \frac{1}{2\pi A^2\sqrt{1-e^2}} \int_{v_1}^{v_2} R^2\, dv = \frac{(1-e^2)^{3/2}}{2\pi} \int_{90°-\omega}^{270°-\omega} \frac{dv}{(1+e\cos v)^2}. \tag{9-4}$$

In order to evaluate this integral, we find it convenient to introduce as an auxiliary variable the eccentric anomaly E, defined by

$$\sqrt{\frac{1-e}{1+e}} \tan \tfrac{1}{2}v = \tan \tfrac{1}{2}E, \tag{9-5}$$

which discloses that

$$\left.\begin{aligned} (1-e^2)^{3/2} \int \frac{dv}{(1+e\cos v)^2} &= \int (1 - e\cos E)\, dE \\ &= E - e\sin E. \end{aligned}\right\} \tag{9-6}$$

In consequence, the integral on the right-hand side of (9-4) can be evaluated in a closed form: imposing the limits we find that

$$n(t_2 - t_1) = \pi + 2\tan^{-1}\frac{e\cos\omega}{\sqrt{1-e^2}} + \frac{2e\cos\omega\sqrt{1-e^2}}{1-e^2\sin^2\omega}, \tag{9-7}$$

where n denotes the mean daily motion $2\pi/P$. If, moreover, we abbreviate

$$\pi + 2\tan^{-1}\frac{e\cos\omega}{\sqrt{1-e^2}} = \Psi, \tag{9-8}$$

equation (9-7) can be made to assume the neat form

$$n(t_2 - t_1) = \Psi - \sin\Psi. \tag{9-9}$$

This is evidently a particular case of Kepler's celebrated equation appropriate for parabolic orbits. Thus, we find that, if the orbital inclination $i = 90°$ *the observed displacement of the minima specifies rigorously a quantity*

$$\frac{e\cos\omega}{\sqrt{1-e^2}}$$

whose function Ψ, *defined by equation* (9-9), *follows as the eccentric anomaly, in a parabolic orbit, corresponding to the mean anomaly* $n(t_2 - t_1)$.

The methods of solution of Kepler's equation are too numerous to be reviewed here; according to the late E. W. Brown, more than one hundred such procedures to this end have been offered in the past three centuries. If the displacement $n(t_2 - t_1)$ of the minima has been ascertained with only a moderate precision, graphical methods will usually be ample. Consider, for instance, a rectangular system of y-Ψ coordinates, and construct the sine curve and the straight line defined by the equations

$$\left.\begin{aligned} y &= \sin \Psi, \\ y &= \Psi - n(t_2 - t_1). \end{aligned}\right\} \tag{9-10}$$

The sine curve can evidently be drawn once and for all; a line inclined to the Ψ-axis by 45° and subtending an intercept of $n(t_2 - t_1)$ alone needs to be constructed for each individual case. By virtue of (9-9), *the abscissa of their point of intersection is the required value of* Ψ. Should a higher degree of accuracy be desired than is obtainable by a graphical construction, recourse may be had to tables expressing the numerical solution of Kepler's equation for parabolic orbits.* Whichever procedure is adopted, however, the reader should bear in mind that the observed displacement of the minima specifies, not e and ω separately, but only their combination $e(1 - e^2)^{-1/2} \cos \omega$ which, if quantities of the order of e^3 and higher are ignored, reduces to the *tangential component of the eccentricity* $e \cos \omega$.

The durations of the ascending and descending branches of either minimum can now be determined as follows. Let $t'_{1,2}$ and $t''_{1,2}$ denote the moments of the first and last contact of the primary and secondary eclipse, and let $\phi'_{1,2}$ be the corresponding phase angles reckoned regardless of sign from the respective conjunction at the time $t_{1,2}$. The differences

$$\left.\begin{aligned} t_{1,2} - t'_{1,2} &= \tau'_{1,2} , \\ t''_{1,2} - t_{1,2} &= \tau''_{1,2} , \end{aligned}\right\} \tag{9-11}$$

then give the durations of the descending and ascending branches of both minima. Now according to the law of areas,

$$n\tau'_{1,2} = \frac{1}{A^2\sqrt{1 - e^2}} \int_{v_{1,2} - \phi_{1,2}}^{v_{1,2}} R^2 \, dv \tag{9-12}$$

and

$$n\tau''_{1,2} = \frac{1}{A^2\sqrt{1 - e^2}} \int_{v_{1,2}}^{v_{1,2} - \phi''_{1,2}} R^2 \, dv, \tag{9-13}$$

where R denotes the radius-vector as given by equation (9-1), and the angles

* *Cf.*, for instance, R. T. Crawford's *Determination of Orbits of Comets and Asteroids*, New York 1930, Appendix C.

of the contacts ϕ' and ϕ'' are given by $\sin^{-1}(a_1 + a_2)/R$—where $a_{1,2}$ denote the radii of the two components—or, explicitly, by

$$\left.\begin{aligned} \sin \phi'_{1,2} &= \alpha\{1 \pm e \sin(\omega + \phi'_{1,2})\} \\ \text{and} \qquad \sin \phi''_{1,2} &= \alpha\{1 \pm e \sin(\omega - \phi''_{1,2})\}, \end{aligned}\right\} \tag{9-14}$$

where we have abbreviated

$$\alpha = \frac{a_1 + a_2}{A\sqrt{1 - e^2}}.$$

A solution of equations (9-14) yields

$$\left.\begin{aligned} \sin \phi'_{1,2} &= (\alpha/U^2)(1 \pm \alpha e \cos \omega \pm U^2 - \alpha^2 e \sin \omega), \\ \sin \phi''_{1,2} &= (\alpha/V^2)(1 \pm \alpha e \cos \omega \pm U^2 - \alpha^2 e \sin \omega), \end{aligned}\right\} \tag{9-15}$$

where

$$U^2 = 1 + \alpha^2 e^2 - 2\alpha e \cos \omega,$$
$$V^2 = 1 + \alpha^2 e^2 + 2\alpha e \cos \omega.$$

If we insert now the appropriate expressions for R and ϕ in equations (9-12)–(9-13) and integrate with respect to v (or, more conveniently, with respect to the eccentric anomaly E) between given limits, we obtain

$$\left.\begin{aligned} n\tau'_{1,2} = 2 \tan^{-1} &\sqrt{\frac{1-e}{1+e}} \tan \tfrac{1}{2}\phi'_{1,2} \\ &\pm \sqrt{1 - e^2}\left\{\frac{e \cos(\omega + \phi'_{1,2})}{1 \pm e \sin(\omega + \phi'_{1,2})} - \frac{e \cos \omega}{1 \pm e \sin \omega}\right\} \end{aligned}\right\} \tag{9-16}$$

and

$$\left.\begin{aligned} n\tau''_{1,2} = 2 \tan^{-1} &\sqrt{\frac{1-e}{1+e}} \tan \tfrac{1}{2}\phi'_{1,2} \\ &\pm \sqrt{1 - e^2}\left\{\frac{e \cos(\omega - \phi''_{1,2})}{1 \pm e \sin(\omega - \phi''_{1,2})} - \frac{e \cos \omega}{1 \pm e \sin \omega}\right\}, \end{aligned}\right\} \tag{9-17}$$

where the angles $\phi'_{1,2}$ and $\phi''_{1,2}$ are given by equations (9-15) above.

The sums

$$\tau'_1 + \tau''_1 \qquad \text{and} \qquad \tau'_2 + \tau''_2$$

specify the durations of both minima, while the differences

$$\tau'_1 - \tau''_1 \qquad \text{and} \qquad \tau'_2 - \tau''_2$$

furnish their asymmetry. An inspection of equations (9-16) and (9-17) discloses that, for $e \neq 0$, the differences $\tau' - \tau''$ vanish only if $\omega = 90°$ or 270°—i.e., if the apsidal line is parallel to the line of sight—and attain a maximum when $\omega = 0°$ or 180° (i.e., at the time when the displacement of the minima is greatest). If ω is between 270° and 90°, then

$$\tau'_1 > \tau''_1 \qquad \text{and} \qquad \tau''_2 < \tau'_2$$

while for $90° < \omega < 270°$ the opposite is true. The branch of each minimum which lies nearest to the neighbouring minimum is found to be the steeper of the two. On the other hand, the durations $(\tau_1' + \tau_1'')$ and $(\tau_2' + \tau_2'')$ of both minima turn out to be equal if $\omega = 0°$ or $180°$ and differ most when the apsidal line is parallel to the line of sight. Thus *the difference in durations of the minima and their asymmetry can never vanish both at the same time.*

Inclined Orbits

If the plane of the eclipsing orbit is not normal to the celestial sphere and the eclipses thus cease to be central, the geometry of our problem becomes considerably more complex. This is due to the fact that, if $e \neq 0$ and $i < 90°$, *the moments of the minima of light fail to coincide with the conjunctions,* and, as a consequence, *the areas of each component eclipsed at corresponding moments of either minimum will generally be unequal.** The minima will occur when the centres of the apparent disks of both components are closest—i.e., when their separation δ becomes a minimum. Now, for elliptic orbits, equation (2-6) is to be replaced by

$$\delta^2 = \frac{A^2(1 - e^2)^2(1 - \sin^2 i \cos^2 \theta)}{[1 - e \sin (\theta - \omega)]^2}. \tag{9-18}$$

A necessary condition that δ be a minimum is the requirement that $d\delta/d\theta$ vanishes. Differentiating (9-18) we find this to happen for values of θ which satisfy the equation

$$\{1 - e \sin (\theta - \omega)\} \sin^2 i \sin 2\theta + 2e \cos (\theta - \omega)(1 - \cos^2 \theta \sin^2 i) = 0. \tag{9-19}$$

If $i = 90°$, the foregoing equation possesses two roots at $\theta = 0°$ and $180°$, respectively, for which δ becomes a minimum. Let us, therefore, seek roots of equation (9-19) for $i < 90°$ which are in the neighbourhood of $\theta = 0°$ and $180°$. Their sines will obviously be quantities of the same order of magnitude as the orbital eccentricity e. If their third and higher powers are neglected, the roots of equation (9-19) which render δ a minimum are found to be

$$\left.\begin{aligned} \theta_1 &= \quad -e \cos \omega \cot^2 i \,(1 - e \sin \omega \csc^2 i) + \dots, \\ \theta_2 &= \pi + e \cos \omega \cot^2 i \,(1 - e \sin \omega \csc^2 i) + \dots, \end{aligned}\right\} \tag{9-20}$$

and the true anomalies, reckoned from the periastron passage, at the moment of deepest eclipses become

$$v_{1,2} = 90° - \omega + \theta_{1,2}\,. \tag{9-21}$$

Now, according to the law of areas, the relative displacement $(t_2 - t_1)/P$ of

* In particular, if one eclipse occurs in the neighbourhood of periastron and is (say) total or annular, the other may be partial. In extreme cases, there may be an eclipse near periastron and none near apastron.

the minima is given by equation (9-4) where, within the scheme of our approximation, the limits of integration v_1 and v_2 are now given by

$$\left.\begin{aligned} v_1 &= 90^\circ - \omega - e\cos\omega\cot^2 i\,(1 - e\sin\omega\csc^2 i) - \ldots, \\ v_2 &= 270^\circ - \omega + e\cos\omega\cot^2 i\,(1 - e\sin\omega\csc^2 i) - \ldots, \end{aligned}\right\} \tag{9-22}$$

correctly to quantities of the order of e^2. Integrating (9-4) between these limits to the same order of accuracy, we obtain

$$n(t_2 - t_1) = \pi + 2e\cos\omega\,(1 + \csc^2 i) + \ldots \tag{9-23}$$

which, for $i = 90^\circ$, reduces to

$$n(t_2 - t_1) = \pi + 4e\cos\omega + \ldots, \tag{9-24}$$

in agreement with equation (9-7); for the latter, when expanded in a Fourier series,

$$n(t_2 - t_1) = \pi + 4e\cos\omega - (\tfrac{2}{3}e^3 + \tfrac{1}{4}e^5)\cos 3\,\omega + \tfrac{3}{20}e^5\cos 5\omega + \ldots \tag{9-25}$$

The duration and asymmetry of both minima of eclipsing systems revolving in eccentric orbits inclined to the line of sight can now be found as follows. Equations (9-14) defining the angles of the first and last contacts for inclined orbits take the explicit forms

$$\left.\begin{aligned} \sqrt{1 - \sin^2 i\cos^2\phi'_{1,2}} &= \alpha\{1 \pm e\sin(\omega + \phi'_{1,2}), \\ \sqrt{1 - \sin^2 i\cos^2\phi''_{1,2}} &= \alpha\{1 \pm e\sin(\omega + \phi''_{1,2}). \end{aligned}\right\} \tag{9-26}$$

Introducing now an auxiliary angle ϕ_0 defined by the equation

$$\alpha = \frac{a_1 + a_2}{A\sqrt{1 - e^2}} = \sqrt{1 - \sin^2 i\cos^2\phi_0}\,, \tag{9-27}$$

we find that, correctly to the *first* order in e,

$$\left.\begin{aligned} \sin\phi'_{1,2} &= \sin\phi_0 \pm e\sin(\omega + \phi_0)(\sin^2\phi_0 + \cot^2 i)\csc\phi_0 + \ldots, \\ \sin\phi''_{1,2} &= \sin\phi_0 \pm e\sin(\omega - \phi_0)(\sin^2\phi_0 + \cot^2 i)\csc\phi_0 + \ldots, \end{aligned}\right\} \tag{9-28}$$

and, hence,

$$\left.\begin{aligned} \phi'_{1,2} &= \phi_0 \pm 2e\sin(\omega + \phi_0)(\sin^2\phi_0 + \cot^2 i)\csc 2\phi_0 + \ldots, \\ \phi''_{1,2} &= \phi_0 \pm 2e\sin(\omega - \phi_0)(\sin^2\phi_0 + \cot^2 i)\csc 2\phi_0 + \ldots. \end{aligned}\right\} \tag{9-29}$$

If we integrate now, to the same order of approximation, $R^2\,dv$ from the moment of deepest eclipse to that of either contact, the fractional *durations* of the two minima become

$$\left.\begin{aligned} d_1 &= n(\tau'_1 + \tau''_1) = 2\phi_0 - 2e\sin\omega\sin\phi_0(1 - \cot^2 i\csc^2\phi_0) + \ldots, \\ d_2 &= n(\tau'_2 + \tau''_2) = 2\phi_0 + 2e\sin\omega\sin\phi_0(1 - \cot^2 i\csc^2\phi_0) + \ldots; \end{aligned}\right\} \tag{9-30}$$

while their *asymmetries* are given by

$$\left.\begin{aligned} n(\tau_1' - \tau_1'') &= 2e\cos\omega\sec\phi_0(1-\cos\phi_0)(\csc^2 i - \cos\phi_0) + \dots, \\ n(\tau_2' - \tau_2'') &= -2e\cos\omega\sec\phi_0(1-\cos\phi_0)(\csc^2 i - \cos\phi_0) + \dots. \end{aligned}\right\} \quad (9\text{-}31)$$

The intervals between the time of maximum eclipse (i.e., the minimum of light) and the moment of conjunction are finally equal to

$$\tfrac{1}{2}(\tau_1' - \tau_1'') \qquad \text{and} \qquad \tfrac{1}{2}(\tau_2' - \tau_2''),$$

respectively.

In order to demonstrate the effects of orbital inclination on the duration of the alternate minima in eclipsing systems characterized by eccentric orbits, let us eliminate ϕ_0 from the foregoing equations by means of (9-27). In particular, since

$$1 - \cot^2\csc^2\phi_0 = \frac{\alpha^2 - 2\cos^2 i}{\alpha^2 - \cos^2 i}, \tag{9-32}$$

equations (9-27) make it evident that the effect of diminishing inclination upon durations of the alternate minima will essentially depend on whether $\cos i \lesseqgtr \alpha/\sqrt{2}$. *When* $\cos i < \alpha/\sqrt{2}$, *the minimum nearest apastron will be of longer duration, while for* $\alpha > \cos i > \alpha/\sqrt{2}$ *the converse will be true.* A decrease of orbital inclination therefore, will tend at first to diminish the disparity in durations of the two minima until $\cos i = \alpha/\sqrt{2}$; but beyond this limit the disparity will tend to increase again. When $\cos i = \alpha/\sqrt{2}$, both minima should be of equal duration—except for terms involving higher powers of e. The light curve should then simulate very closely one produced by a pair of stars of different radii and surface brightnesses, moving in an orbit for which $e \sin\omega = 0$; and the eclipses should be total or partial depending on whether the ratio of radii

$$k \lesseqgtr \frac{2 - \sqrt{2}}{2 + \sqrt{2}}. \tag{9-33}$$

If, ultimately, $\cos i$ approaches α, the denominator on the right-hand side of (9-26) will tend to zero and, consequently, $\cos\phi_0$ will increase beyond any limit. Should this happen, the expansions on the right-hand sides of equations (9-28)–(9-31) will cease to converge no matter how small e may be. This situation is explained by the obvious geometrical fact that, near this limit, there may no longer be any eclipse in the neighbourhood of the apastron, while one may remain near periastron.

The sum of durations of both minima follows from equations (9-30) as

$$d_1 + d_2 = 4\phi_0. \tag{9-34}$$

To the order of accuracy we are working, $d_1 + d_2$ turns out to be independent of e or ω, but diminishes steadily with i. On the other hand, their difference turns out to be

$$d_1 - d_2 = -4e\sin\omega\sin\phi_0(1 - \cot^2 i\csc^2\phi_0) + \dots. \tag{9-35}$$

Hence, provided that ϕ_0 is not too large,

$$\frac{d_1 - d_2}{d_1 + d_2} = -e \sin \omega \,(1 - \cot^2 i \csc^2 \phi_0) \tag{9-36}$$

which, by (9-32), yields

$$e \sin \omega = \frac{d_2 - d_1}{d_2 + d_1} \, \frac{\alpha^2 - \cos^2 i}{\alpha^2 - 2 \cos^2 i} \, . \tag{9-37}$$

A determination of the radial component $e \sin \omega$ of the eccentricity from the unequal durations of both minima is, therefore, possible in principle provided that the orbital inclination is not too far from 90°. In practice, such a determination will be much less precise than that of the tangential component, $e \cos \omega$, from the relative displacement of the minima. In the latter case, the error of the respective determination is proportional to the ratio of this displacement to the half period; in the former case, to the ratio of the difference to the sum of the durations of the minima—and this sum constitutes a much smaller divisor.

When both minima are fairly deep, the inclination is necessarily close to 90°, and a separation of e and ω from purely photometric data can therefore be effected. Shallow minima are, however, practically useless for this purpose, so that the theoretically interesting case when $\cos i > \alpha/\sqrt{2}$ is of no practical importance. All we can deduce, in such cases, from the photometric evidence is the tangential component of the eccentricity; and in order to obtain e and ω separately we have to wait either for a spectroscopic orbit, or for an unmistakable indication of apsidal motion.

Of all effects invoked by orbital eccentricity, the asymmetry of the minima proves to be by far the least conspicuous. We can demonstrate this fact by a comparison of the asymmetry with the displacement. Dividing (9-31) by $n(t_2 - t_1) - \pi$, we find the ratio of asymmetry A to displacement D of a minimum to be given by

$$\frac{A}{D} = \pm \frac{(\sec \phi_0 - 1)(\csc^2 i - \cos \phi_0)}{1 + \csc^2 i} \, , \tag{9-38}$$

which for $i = 90°$ reduces to

$$A/D = \pm 2 \sec \phi_0 \sin^4 \tfrac{1}{2}\phi_0 \, . \tag{9-39}$$

The former ratio (9-38) is always smaller than (9-39) which shows that, for a given e and ω, the asymmetry reaches a maximum when the eclipse is central and diminishes with decreasing inclination.

A ratio of displacement of the conjunctions with respect to the minima of light is equal to one-half of the above amounts. In either case, these ratios are always very small. If, for example, we set $\phi_0 = 45°$ and $i = 70°$—values well outside the range found ordinarily in practice—the ratio A/D as given by equation (9-38) becomes $\pm 0{\cdot}08$; if ϕ_0 is diminished to 30°, it becomes $\pm 0{\cdot}02$. *The asymmetry is, therefore, found to amount at best to a few per cent of the displacement*, which means that, *in most practical cases, it is likely to remain unobservably small.* This is a most fortunate fact; for, as will be shown below, it will open up the possibility of determining the true elements of the components of eclipsing systems revolving in circular oıbits in a fairly simple way.

VI.10. DERIVATION OF THE ELEMENTS OF ECCENTRIC ECLIPSING SYSTEMS

In the preceding paragraphs we have demonstrated that the light curves, within minima, of eclipsing systems whose components revolve in eccentric orbits may be regarded with a considerable approximation as symmetrical—even though the minima may be noticeably displaced and of unequal durations. The sensible symmetry of the minima implied that the variation of the radius-vector R or of dv/dt within each minimum may be ignored. If so, however, *the light curve, at either minimum, may evidently be treated as though the orbit were a circle of radius equal to the instantaneous radius-vector at the moment of each conjunction.* Consider, as in the preceding section, the eclipse of the brighter star by the fainter. At the moment of superior conjunction, $v = 90° - \omega$ and, therefore,

$$R = \frac{A(1-e^2)}{1+e\sin\omega}\,; \tag{10-1}$$

while according to the law of areas,

$$\frac{dv}{dt} = \frac{n\sqrt{1-e^2}}{R^2} = \frac{n(1+e\sin\omega\,)^2}{A^2(1-e^2)^{3/2}}\,, \tag{10-2}$$

where n stands again for the mean daily motion. If, moreover, $\theta = v + \omega - 90°$ denotes the true anomaly measured from superior conjunction, while the mean anomaly measured from mid-primary minimum is denoted by θ, then, ignoring the variation of R during the eclipse we may put

$$\frac{\theta'}{\theta} = \frac{(1-e^2)^{3/2}}{(1+e\sin\omega)^2}\,. \tag{10-3}$$

Now e is generally a small quantity unless the components are widely separated, and then θ remains small during eclipse. Within these precautions we may set $\theta = \sin\theta$, in which case

$$\sin^2\theta = \frac{(1+e\sin\omega)^4}{(1-e^2)^3}\sin^2\theta'. \tag{10-4}$$

In what follows *let the semi-major axis* A *of the relative orbit of the components be adopted as our unit of length.* If so, the fundamental equation (4-1) during primary minimum may, to the order of accuracy we are working, be rewritten as

$$\frac{(1 - e^2)^3}{(1 + e \sin \omega)^4} \cos^2 i + \sin^2 i \sin^2 \theta' = \frac{r_b^2(1 - e^2)}{(1 + e \sin \omega)^2} (1 + kp)^2. \quad (10\text{-}5)$$

Since, if there are to be any eclipses at all, the orbital inclination i must be fairly close to 90°, this latter equation is nearly equivalent to

$$\cos^2 i' + \sin^2 i' \sin^2 \theta' = r_b'^2(1 + kp)^2, \quad (10\text{-}6)$$

where

$$r_b' = \frac{\sqrt{1 - e^2}}{1 + e \sin \omega} r_b \quad (10\text{-}7)$$

and

$$\cot i' = \frac{(1 - e^2)^{3/2}}{(1 + e \sin \omega)^2} \cot i. \quad (10\text{-}8)$$

A glance at the foregoing equation (10-6) shows it to be identical with an exact equation that would hold good in a circular orbit of unit radius, with the same value of k, but with a fictitious radius r_b' of the larger component, and a fictitious inclination i'. It follows, therefore, that *if, in a practical case, we assume the orbit to be circular and determine the elements of the eclipse by the methods of section* VI.4, *the ratio of the radii* k *resulting from such an analysis will be correct to the order of accuracy we have been working; while the fictitious 'circular' elements* r_b' *and* i' *are related with the true 'elliptical' elements* r_b and i *by means of equations* (10-7) *and* (10-8). It should be added however, that these latter equations are valid only for the reduction of the 'circular' to the 'elliptical' elements deduced from the minimum which occurs at (or in the neighbourhood of) the superior conjunction. To obtain similar equations for the reduction of the fictitious 'circular' elements r_b'' and i'' derived from the alternate minimum, we have only to increase, in (10-7) and (10-8), ω by 180°; then

$$r_b'' = \frac{\sqrt{1 - e^2}}{1 - e \sin \omega} r_b \quad (10\text{-}9)$$

and

$$\cot i'' = \frac{(1 - e^2)^{3/2}}{(1 - e \sin \omega)^2} \cot i, \quad (10\text{-}10)$$

respectively. It should be stressed that a difference between 'circular' and 'elliptical' fractional radii of the two components is due solely to a difference in units in terms of which the respective quantities are evaluated. Whereas the 'circular' (fictitious) radii are expressed in terms of instantaneous values of the radius-vector at the moment of either conjunction, the 'elliptical' (true) radii are expressed in terms of the semi-major axis of the relative orbit adopted as our unit of length.

The reader should also notice that, within the scheme of our accuracy, equations (10-7)–(10-8) or (10-9)–(10-10) depend on e and ω only through $e \sin \omega$. In point of fact, the ratios

$$\frac{r_b'}{r_b''} = \frac{1 - e \sin \omega}{1 + e \sin \omega} \tag{10-11}$$

and

$$\frac{\cot i'}{\cot i''} = \left\{\frac{1 - e \sin \omega}{1 + e \sin \omega}\right\}^2 \tag{10-12}$$

of the respective pairs of fictitious elements following from both minima offer a way for the determination of the radial component of the eccentricity, which is alternative to equation (9-37). It is also evident that the singly- and doubly-primed fictitious 'circular' elements must be in the ratios

$$\left(\frac{r_a'}{r_a''}\right)^2 = \left(\frac{r_b'}{r_b''}\right)^2 = \frac{\cot i'}{\cot i''} \tag{10-13}$$

in order to be consistent.

It should be reiterated that all foregoing equations in this section have been obtained by adopting constant values of R and of dv/dt for each eclipse, and equal to their instantaneous values at the moments of the respective conjunction. If the eclipses are wide, however, it would be somewhat more accurate to use, instead, the values of R and of dv/dt averaged over the eclipses. The equations based on such averages are the same as equations given above, except that $e \sin \omega$, in equations (10-7)–(10-8) only, should be replaced by $\eta e \sin \omega$, where, to a sufficient approximation,

$$\eta = \tfrac{2}{3} + \tfrac{1}{3} \cos \tfrac{1}{2}(\theta_i + \theta_e), \tag{10-14}$$

$\theta_{i,e}$ being the mean anomaly of the first and last contact of the respective eclipse. When e is large, however, the eclipses will in general be narrow; η will then be effectively equal to unity regardless of asymmetry; and the equations (10-7)–(10-10) as they stand—without η—should be sufficiently accurate to yield good approximations to the true 'elliptical' elements of an eccentric eclipsing system.

The method discussed so far in this chapter for a determination of fictitious 'circular' sets of the elements from either minimum, and for their subsequent conversion into the true 'elliptical' elements, tacitly assumes that each minimum can be solved for the geometrical elements independently of the other. We learned earlier that this is indeed possible if the alternate minima are due to the total and annular eclipses; but if the eclipses were partial, we found it necessary to combine both minima in order to ascertain the maximum fractional loss of light α_0. As long as the relative orbit of both components was circular, the geometrical depth p_0 at the bottom of the alternate minima was the same. *If the relative orbit is eccentric*, however, *this will no longer be true*—as is easy to prove by means of the equations which we have already established.

In order to do so, let us recall that the geometrical depth p is defined by equation (2-8) as $(\delta - r_b)/r_a$ and, at the moment of deepest eclipse becomes equal to

$$p_0 = \frac{\delta_0 - r_b}{r_a}. \tag{10-15}$$

If the relative orbit of the two stars were circular, we should have $\delta_0 = \cos i$. If the orbit is eccentric, however, equations (9-20) inserted in (9-18) yield

$$\left.\begin{aligned} \delta_0' &= \cos i\,\{1 - e\sin\omega - \tfrac{1}{2}e^2\cos^2\omega(1 + \csc^2 i) + \ldots\}, \\ \delta_0'' &= \cos i\,\{1 - e\sin\omega - \tfrac{1}{2}e^2\cos^2\omega(1 + \csc^2 i) + \ldots\}, \end{aligned}\right\} \tag{10-16}$$

correctly to the second order in small quantities. Since, for high orbital inclinations, the functions $\cos i$ and $\cot i$ are very nearly equal, the foregoing equations may, with the aid of (10-8) and (10-10), be rewritten as

$$\left.\begin{aligned} \delta_0' &= \cos i'\{1 - \tfrac{1}{2}e^2\cos^2\omega(1 + \csc^2 i') + \ldots\}, \\ \delta_0'' &= \cos i''\{1 - \tfrac{1}{2}e^2\cos^2\omega(1 + \csc^2 i'') + \ldots\}, \end{aligned}\right\} \tag{10-17}$$

and

$$\left.\begin{aligned} p_0' &= \frac{\cos i'}{r_a'}\{1 - \tfrac{1}{2}e^2\cos^2\omega(1 + \csc^2 i') + \ldots\} - \frac{r_b'}{r_a'}, \\ p_0'' &= \frac{\cos i''}{r_a''}\{1 - \tfrac{1}{2}e^2\cos^2\omega(1 + \csc^2 i'') + \ldots\} - \frac{r_b''}{r_a''}. \end{aligned}\right\} \tag{10-18}$$

Now, in these equations, the terms r_b'/r_a' and r_b''/r_a'' are equal to each other and to the ratio of the true 'elliptical' fractional radii, expressed in terms of the semi-major axis of the relative orbit. The ratios $(\cos i')/r_a'$ and $(\cos i'')/r_a''$ are, however, not equal to each other; equations (10-13) disclose that they differ by an amount of the order of $e \sin \omega$. Hence, the maximum geometrical depths p_0' and p_0'' attained at the moments of deepest eclipse of the primary and secondary minima will, in eccentric orbits, be given in terms of the true (elliptical) elements by the equation

$$\left.\begin{aligned} p_0 &= \frac{(1 - e^2)(1 - e^2\cos^2\omega)}{1 \pm e\sin\omega}\,\frac{\cos i}{r_a} - \frac{r_b}{r_a} \\ &= \frac{\cos i - r_b}{r_a} \mp \frac{\cos i}{r_a}\,e\sin\omega + \ldots, \end{aligned}\right\} \tag{10-19}$$

where the signs $\pm$ refer to the primary and secondary minima. If, lastly, one of the two eclipses becomes annular, its maximum geometrical depth q_0 becomes, consistent with equation (4-63) of Chapter IV,

$$\left.\begin{aligned} q_0 &= 1 - \frac{\cos i}{r_b - r_a}\,\frac{1 \mp e\sin\omega}{1 - e^2} \\ &= 1 - \frac{\cos i}{r_b - r_a} \pm \frac{e\sin\omega\cos i}{r_b - r_a} + \ldots \end{aligned}\right\} \tag{10-20}$$

If the eclipses exhibited by a close eccentric binary are partial, this fact will exert important repercussions on the determination of the elements of the eclipse. In more specific terms, equations (2-23) and (2-24) of section VI.2 will continue to be valid, but equation (2-25) defining $Y(k, p)$ should be replaced by

$$Y'(k, p_0) = \frac{(1 - x_b)\alpha^{\mathbf{U}}(k, p_0'') + (3/2)x_b\Phi(k)\alpha_b^{\mathbf{D}}(k, p_0'')}{(1 - x_a)\alpha^{\mathbf{U}}(k, p_0') + x_a\alpha_a^{\mathbf{D}}(k, p_0')} \tag{10-21}$$

if the eclipse which occurs at the superior conjunction is an occultation, and by

$$Y''(k, p_0) = \frac{(1 - x_b)\alpha^{\mathbf{U}}(k, p_0') + (3/2)x_b\Phi(k)\alpha_b^{\mathbf{D}}(k, p_0')}{(1 - x_a)\alpha^{\mathbf{U}}(k, p_0'') + x_a\alpha_a^{\mathbf{D}}(k, p_0'')} \tag{10-22}$$

if it is a transit. When $p_0' \neq p_0''$, both $Y'(k, p_0)$ and $Y''(k, p_0)$ may be considerably different from unity and from each other.

A difference in the p_0's at the alternate minima will compel us to proceed by approximations, anticipating the ratio p_0'/p_0'' to begin with, and revising our original assumption whenever the outcome of the solution will lead us to do so. As long as the orbital eccentricity is small enough for terms of the order of e^3 and higher to remain negligible, the equations given in this section should enable us to perform this task without undue difficulty. If, however, the orbital eccentricity is large and, in addition, the orbital plane is sensibly inclined to the line of sight, partial eclipses are apt to give the computer of their elements a headache—and sometimes drive him to despair! For, under extreme conditions, the disparity in the two p_0's (i.e., in the area eclipsed by either component at the corresponding moments of the two minima) may be such that while the eclipse near periastron may be total (or annular), that near apastron may be partial—or non-existent! A solution for the elements in such extreme cases can evidently proceed only by successive approximations, based on the general properties of the problem of two bodies, and belongs without any doubt among the most tedious propositions encountered in eclipsing orbit work.

Suppose that the 'circular' elements of an eccentric eclipsing system have been determined by the methods preceding, and that although the representation of the observations within each minimum is tolerable, the two sets of singly and doubly primed elements fail to satisfy equations (10-13) within the limits of observational errors. Such a contingency would indicate unmistakably that the values of e and ω underlying our conversion of the 'circular' elements into 'elliptical' ones are in need of improvement. This improvement can be performed most accurately by the method of differential corrections discussed in section VI.8, by adding to the right-hand side of equation (8-7) new terms of the form

$$\frac{\partial f}{\partial e}\Delta e + \frac{\partial f}{\partial \omega}\Delta\omega ,$$

and solving for Δe and $\Delta\omega$ simultaneously with all other differential corrections required to establish the best possible fit to the observed data.

The coefficients of the corrections Δe and $\Delta\omega$ can evidently be expressed as

$$\left.\begin{aligned}\frac{\partial f}{\partial e} &= \frac{\partial f}{\partial r_1}\frac{\partial r_1}{\partial e} + \frac{\partial f}{\partial r_2}\frac{\partial r_2}{\partial e} + \frac{\partial f}{\partial i}\frac{\partial i}{\partial e},\\ \frac{\partial f}{\partial \omega} &= \frac{\partial f}{\partial r_1}\frac{\partial r_1}{\partial \omega} + \frac{\partial f}{\partial r_2}\frac{\partial r_2}{\partial \omega} + \frac{\partial f}{\partial i}\frac{\partial i}{\partial \omega},\end{aligned}\right\} \tag{10-23}$$

where r_1, r_2 and i stand for the singly or doubly primed values of the respective 'circular' elements, depending on whether we are considering the eclipse near the superior or inferior conjunction. If we remember that such 'circular' elements depend on e and ω through equations (10-7)–(10-10) and if we differentiate the latter we find that

$$\left.\begin{aligned}\frac{1}{r}\frac{\partial r}{\partial e} &= -\frac{e \pm \sin\omega}{(1-e^2)(1 \pm e\sin\omega)},\\ \frac{1}{r}\frac{\partial r}{\partial \omega} &= \pm\frac{e\cos\omega}{1 \pm e\sin\omega}\end{aligned}\right\} \tag{10-24}$$

and

$$\left.\begin{aligned}\frac{1}{\cot i}\frac{\partial \cot i}{\partial e} &= \frac{1}{r}\frac{\partial r}{\partial e} - \frac{e}{1-e^2},\\ \frac{1}{\cot i}\frac{\partial \cot i}{\partial \omega} &= \frac{2}{r}\frac{\partial r}{\partial \omega},\end{aligned}\right\} \tag{10-25}$$

where $r \equiv r_{1,2}$ stands for the fractional 'circular' radius of the respective component and i for the 'circular' (fictitious) inclination; the upper or lower sign in equations (10-24) and (10-25) holds good depending on whether these 'circular' elements are singly or doubly-primed.

If we insert equations (10-24)–(10-25) in (10-23) and express $\partial f/\partial i$ in terms of $\partial f/\partial\delta$ by means of equation (8-12), the coefficients of the differential corrections for Δe and $\Delta\omega$ readily take the forms

$$\begin{aligned}\frac{\partial f}{\partial e} = &-\frac{e \pm \sin\omega}{(1-e^2)(1 \pm e\sin\omega)}\left\{r_1\frac{\partial f}{\partial r_1} + r_2\frac{\partial f}{\partial r_2}\right\}\\ &-\frac{e(1 \pm e\sin\omega) + 2(e \pm \sin\omega)}{(1-d^2)(1 \pm e\sin\omega)}\,\frac{\sin^2 i\cos^2 i\cos^2\theta}{\delta}\,\frac{\partial f}{\partial\delta}\end{aligned} \tag{10-26}$$

and

$$\begin{aligned}\frac{\partial f}{\partial \omega} = &\frac{\pm e\cos\omega}{1 \pm e\sin\omega}\left\{r_1\frac{\partial f}{\partial r_1} + r_2\frac{\partial f}{\partial r_2}\right\}\\ &\pm\frac{2e\cos\omega}{1 \pm e\sin\omega}\,\frac{\sin^2 i\cos^2 i\cos^2\theta}{\delta}\,\frac{\partial f}{\partial\delta},\end{aligned} \tag{10-27}$$

where the upper or lower sign again holds good depending on whether the 'circular' (fictitious) elements r_1, r_2, and i are singly or doubly-primed, and where the partial derivatives of f with respect to r_1, r_2 and δ may be obtained by the methods of section VI.8. If the orbital eccentricity e is so small that, to a first approximation, we may replace it by zero on the right-hand sides of equations (10-26) and (10-27), the reader can easily verify that these equations will reduce to

$$\frac{\partial f}{\partial(e \sin \omega)} = \pm\left\{r_1 \frac{\partial f}{\partial r_1} + r_2 \frac{\partial f}{\partial r_2}\right\} \pm \frac{\cos^2 \theta \sin^2 2i}{2\delta} \frac{\partial f}{\partial \delta} \tag{10-28}$$

—a form in which the difference between the true ('elliptical') and fictitious ('circular') values of the elements on the right-hand side becomes immaterial. The preceding equation suggests itself for use whenever it is desirable to ascertain the most probable value of the radial component of the eccentricity $e \sin \omega$ for systems in which the preliminary orbit was assumed to be circular.

All foregoing considerations are based on a tacit assumption that the radius-vector joining the centres of the two components remains so nearly constant within each eclipse as to cause no appreciable departure from symmetry, so that the sole aim of an application of the differential corrections is to ensure the consistency of the two sets of 'circular' elements as deduced from the alternate minima. Suppose, however, that the orbital eccentricity happens to be large enough to give rise to a significant asymmetry of the two branches of each minimum. Can the method of differential corrections be employed to account for the effects of an asymmetry?

The answer is again in the affirmative; and in order to do so, it is advisable to adopt, as quantities to be adjusted by the differential corrections, a trio of e, T and t_0 (or ω), where T denotes the time of the periastron passage and t_0, that of the respective conjunction. A knowledge of T and t_0 is obviously tantamount to a knowledge of T and ω, and vice versa. Equations (10-23) should then be replaced by

$$\left.\begin{aligned}\frac{\partial f}{\partial e} &= \frac{\partial}{\partial e}\left(\frac{1}{R}\right)\left\{r_1 \frac{\partial f}{\partial r_1} + r_2 \frac{\partial f}{\partial r_2}\right\} + \frac{\partial f}{\partial \delta}\frac{\partial \delta}{\partial e},\\ \frac{\partial f}{\partial T} &= \frac{\partial}{\partial T}\left(\frac{1}{R}\right)\left\{r_1 \frac{\partial f}{\partial r_1} + r_2 \frac{\partial f}{\partial r_2}\right\} + \frac{\partial f}{\partial \delta}\frac{\partial \delta}{\partial T},\end{aligned}\right\} \tag{10-29}$$

where r_1, r_2 and i refer now to the true 'elliptical' elements of the system. In these equations, δ continues to be given by equation (9-18), while according to the well-known expansions of the problem of two bodies,*

$$\begin{aligned}R = A\{1 - e \cos M &- \tfrac{1}{2}e^2 (\cos 2M - 1)\\ &- \tfrac{3}{8}e^3 (\cos 3M - \cos M)\\ &- \tfrac{1}{3}e^4 (\cos 4M - \cos 2M) - \dots\},\end{aligned} \tag{10-30}$$

* *Cf.*, e.g., F. R. Moulton, *An Introduction to Celestial Mechanics* (second edition) New York 1914, p. 171.

where $M = n(t - T)$ denotes the mean anomaly and $n = 2\pi/P$ stands for the mean daily motion. With the aid of this equation, the partial derivatives of R^{-1} with respect to e or T may be obtained in the form of power series in e; the expansion on the right-hand side of (10-30) is known to converge for $e < 0{\cdot}6627$—an ample limit in eclipsing orbit work. The explicit form of the derivatives of δ with respect to e and T are rather involved;* but their evaluation offers no difficulty and may be left as an exercise to the interested reader.

VI.11. DERIVATION OF THE ELEMENTS OF DISTORTED ECLIPSING SYSTEMS

In all preceding sections of this chapter methods have been discussed by which the elements of eclipsing binary systems consisting of spherical components can be deduced from an analysis of their light curves. This task completed, the question is bound to arise as to the extent to which a model consisting of spherical components can be regarded as a satisfactoiy representation of actual eclipsing systems; for unless the constituent stars are rigid, tidal forces coupled with axial rotation must cause them to depart from spherical form. As we have seen in Chapters II–IV, the actual shape of each component should, in principle, depend upon its size, the distribution and the rate of rotation of matter within it, as well as upon the separation and the ratio of masses of both stars. If the components are widely separated and their rotation is sufficiently slow, the distortion may be so small that, for all practical purposes, they can be regarded as spheres. Even within the limits of accuracy attainable by modern photoelectric photometers this is, for instance, true of most eclipsing systems in which the fractional radii of the components are less than one-tenth. When, however, the values of such fractional radii approach, or exceed; say 0·2, the approximation regarding both components as spherical is certain to become inadequate, and a neglect of the effects of distortion would open the way to significant—or even serious—systematic errors; for distorted stars are not likely to appear in projection as circular disks exhibiting a uniform or radially-symmetrical distribution of brightness tacitly postulated so far throughout this chapter.

If so, many a reader may perhaps wonder why so much attention and space has been devoted to an analysis of a purely geometrical model consisting of spherical stars, when the actual components of all eclipsing binaries are bound to be distorted to a smaller or greater degree. The answer to this question is simple. In many eclipsing systems the components are too widely separated and their distortion is consequently so small that, within the limits of observational errors, they can be regarded as spheres—in which case the methods of solution as developed so far can indeed furnish the final elements. On the other hand—and this should be emphasized at the outset—there

* For fuller details *cf.* J. Stein, *Die Veränderlichen Sterne*, Freiburg 1924, p. 224.

exists no royal road to the direct determination of elements of distorted eclipsing systems. Their elements will be eventually obtainable by a process of successive approximations starting from a set of preliminary elements deduced on the assumption of spherical stars. If such preliminary elements can be modified, by any practicable method, to account for the effects of distortion, a genuine picture of the respective system may still be obtained; but if any such procedure fails, the problem of a determination of the elements becomes well-nigh indeterminate.

For any precise interpretation of light curves of close binary systems, a knowledge of the form of their components, and the distribution of brightness over their apparent disks, is an obvious prerequisite. The theory of equilibrium of fluid components of close binary systems has been investigated to a fair degree of completeness as far as the effects of first order are concerned, and the form of the components can be predicted in relatively simple terms provided that their axial rotation is slow and the tidal pull of one component upon another can be regarded as that of a mass-point. Spectroscopic observations suggest that the photospheric layers of components revolving in circular orbits rotate with the corresponding Keplerian angular velocity or approximately so, which in most cases is indeed relatively slow; whether or not their interiors rotate in this way is largely irrelevant as far as the external form of centrally-condensed stars is concerned. Furthermore, it has been proved that the tidal action of each component on its mate can be regarded as that of a mass-point as long as quantities of the order of sixth and higher powers of their fractional radii are negligible, and that the apparent brightness over the visible surfaces of distorted stars should, in addition to the ordinary limb-darkening, vary proportionally as the local gravity (section IV.1). The task which will confront us in the present section will be to develop the methods by which the elements of a moderately distorted (i.e., moderately close) eclipsing system can be deduced from an analysis of its light curve.

In its full scope, the problem just formulated is one of considerable complexity, which warrants a careful approach. To begin with, a significant feature of 'close' binary systems consists in the fact that *the observable light changes are no longer confined to the times of the minima, but extend over the whole cycle*. This is due to two fundamental reasons. First, since the components of close binaries are, in general, distorted ellipsoids with longest axes constantly in the direction of the line joining their centres, their apparent area—and, therefore, the light—exposed to the observer at a great distance should vary continuously in the course of a revolution (*ellipticity effect*). Secondly, it is inevitable in close binary systems that a fraction of the radiation of each component will be intercepted by its mate, where it will be absorbed and re-emitted (or scattered) in all directions—including the direction of the line of sight. The amount of such parasitic light 'reflected' by each component towards the observer should again clearly vary with the phase (*reflection effect*).

The changes of light arising from the ellipticity and reflection effects are independent of, and supplementary to, the light changes due to the eclipses. Both effects are indeed present in the light curves of close binary systems, and their magnitude increases with increasing proximity of the components. The variation of light invoked by both effects should, furthermore, be stimulated by increasing inclination of the orbital plane to the celestial sphere, and be absent only if the plane of the orbit happened to be perpendicular to the line of sight.* As long as the orbits of both components are circular and the distance between their centres remains constant, both the ellipticity and reflection effects should be symmetrical with respect to the times of the minima of light; though for eccentric orbits they will remain so only if the apsidal line is parallel with the line of sight.

The changes of light exhibited by close binary systems between minima are—needless to say—considerably simpler than those which accompany the eclipses; and for this reason it will be our strategy to deal with both in turn. In all that follows *we shall assume that*, although the light variation persists over the whole cycle, *the phases at which the eclipses begin and end can be estimated from the observations before a solution has been started.* This will enable us to divide our task by confining first our attention to the light changes exhibited *between minima*, and utilize the knowledge which can be extracted from such parts of the light curve to *simplify* the observed changes of light within eclipses by a process commonly but rather vaguely referred to as the *rectification.* Next, we shall treat such a rectified light curve as if it were (very nearly) produced by the eclipses of two spherical stars of radii equal to the semi-major axes of the distorted ellipsoids, and by an application of methods which will be essentially identical with those of sections VI.4–VI.8 we shall arrive at a corresponding set of intermediary elements. These elements are, however, but an *intermediary* product in our solution; since before accepting them we must *correct* them for the admitted shortcomings and simplifications inherent in our process of rectification and subsequent treatment of the rectified curve. This will be accomplished by means of a process usually referred to as the '*perturbations*'; and it is only after such perturbations have been duly applied that genuine elements of distorted eclipsing systems can be obtained. These three steps—rectification, derivation of the intermediary elements, and perturbations—will thus be found to constitute distinct and successive stages of our solution; and in what follows we shall discuss them in this order.

Rectification of the Light Curves

Let us ignore, for the time being, the changes of light invoked by the eclipses and confine our attention to the less conspicuous and more gradual

* This statement holds true for systems exhibiting circular orbits. Should, however, the orbit happen to be eccentric, some variation of light between minima, due both to the ellipticity and reflection, would persist on account of the variable distance between the centres of both components.

light variation exhibited between minima, which is due to the ellipticity and reflection. The theory as developed in Chapter IV discloses that as long as the form of both components is governed by the equilibrium theory of tides, the changes of light invoked by the ellipticity should be expansible as a Fourier cosine series of the phase angle θ; and the same is true, for geometrical reasons, of the reflection effect as well. Let us, therefore, assume, quite generally, that the variation arising from the ellipticity of the components in an eclipsing system may be expressed as

$$l_c = 1 - b\cos\theta - c\cos^2\theta - \ldots, \tag{11-1}$$

while the amount of reflected light varies as

$$l^* = \alpha - \beta\cos\theta - \gamma\cos^2\theta - \ldots, \tag{11-2}$$

where $b, c, \ldots$; $\alpha, \beta, \gamma, \ldots$ are quantities which can be predicted by the theory and whose specification can be deferred until later—save for the fact that *they all are*, by definition, *small enough for their squares and cross-products to be negligible.* Now it should be clearly borne in mind that the *effects of ellipticity are proportional to the luminosity of the distorted stars* and can, therefore, be removed from the observed light changes of a rotating ellipsoid by *division*; whereas the *reflection* represents an *additive* source of light (proportional to the size of the illuminated star and to the luminosity of its mate) and can, if desirable, be removed by *subtraction.*

Suppose now that the changes of light of an eclipsing system, observed between eclipses can be represented by a series of the form

$$l_{\text{obs}} = 1 - B\cos\theta - C\cos^2\theta - \ldots, \tag{11-3}$$

where the values of the small coefficients $B, C, \ldots$ can be determined by a harmonic analysis of the observed data.* As many equations of condition are available for their determination as there are observations (or normal points) secured between minima, and their least-squares solution should yield the most probable values of B and C. A comparison of (11-3) with (11-1) and (11-2) discloses that

$$B = b + \beta \qquad \text{and} \qquad C = c + \gamma. \tag{11-4}$$

Let us define now the '*rectified luminosity*' l_{rec} of the system as one which coincides with the observed luminosity at quadratures, but remains constant and equal to one everywhere outside eclipses. It can evidently be obtained if we *divide* the observed luminosity by its expected variation due to the ellipticity and consistent with equation (11-1) and *add* to it, in accordance

* For the purpose of normalization it is convenient to adopt the luminosity of the system at the quadrature ($\theta = 90°$), when the maximum area of the surface becomes exposed to us, as our unit of light. If, as will usually be the case, the harmonic analysis of the observed data will proceed in the form

$$l_{\text{obs}} = c_0 - c_1\cos\theta - c_2\cos^2\theta - \ldots, \tag{11-3$'$}$$

it goes without saying that $B = c_1/c_0$ and $C = c_2/c_0$.

with (11-2), the amount of light necessary to complete the phase of each component, reflecting the light of its mate, to 'full.' Doing so, we find that

$$l_{\mathrm{rec}} = \frac{l_{\mathrm{obs}}(1 - b\cos\theta - c\cos^2\theta)^{-1} + \alpha - \beta\cos\theta - \gamma\cos^2\theta}{1 + \alpha}. \qquad (11\text{-}5)$$

The values of the coefficients b, c, α, β, γ are, however, not known to us beforehand. What we can extract directly from the observations are the values of B and C; our aim should be, therefore, to rewrite equation (11-5) as completely as possible in terms of the coefficients of equation (11-3). If, in doing so, we remember that all these coefficients are expected to be small quantities whose squares and cross-products can be ignored, equation (11-5) can be easily transformed into

$$l_{\mathrm{rec}} = \frac{l_{\mathrm{obs}}(1 + C\cos^2\theta) + B\cos\theta + \alpha - b\cos\theta - \gamma\cos^2\theta}{1 + \alpha - b\cos\theta - \gamma\cos^2\theta} \qquad (11\text{-}6)$$

which, consistently with our degree of accuracy, may be rewritten as

$$l_{\mathrm{rec}} = l_{\mathrm{obs}}(1 + C\cos^2\theta) + B\cos\theta + (1 - l_{\mathrm{obs}})(\alpha - b\cos\theta - \gamma\cos^2\theta) - \ldots \qquad (11\text{-}7)$$

Of the constants figuring on the right-hand side of equation (11-7), the values of B and C follow directly from the observational evidence; but those of α, b or γ will as a rule not be known to us beforehand. Their explicit forms to be expected on theoretical grounds will be given later in section VI.12, when their use will become indispensable. If a good set of preliminary elements is available at this stage, the investigator may utilize these theoretical expressions to estimate the numerical values of α, b and γ as best he can, and use the complete equation (11-7) for rectification. In the absence of any reliable estimate of α, b or γ the investigator cannot, however, do otherwise than ignore the whole term

$$(1 - l_{\mathrm{obs}})(\alpha - b\cos\theta - \gamma\cos^2\theta - \ldots) \qquad (11\text{-}8)$$

and base the rectification on the approximate formula

$$l_{\mathrm{rec}} = l_{\mathrm{obs}}(1 + C\cos^2\theta) + B\cos\theta. \qquad (11\text{-}9)$$

A glance at the term (11-8) discloses that, significantly, it is multiplied by the factor $(1 - l_{\mathrm{obs}})$. Between eclipses, $1 - l_{\mathrm{obs}}$ is obviously a small quantity of the same order of magnitude as α, b or γ; to the first order in small quantities the whole term (11-8) between eclipses becomes, therefore, negligible. Moreover, as long as the amplitude of light changes invoked by the eclipses (and consequently the factor $(1 - l_{\mathrm{obs}})$ is not too large, the term (11-8) will continue to remain small though not strictly negligible. If, however, the amplitude of light variation during the eclipses does become large and l_{obs} grows small, the term (11-8) may become appreciable, or even large, in comparison with the terms on the right-hand side of equation (11-9).

When the minima are deep (typical Algol systems) the neglect of (11-8) in the process of rectification would constitute no approximation at all, and the magnitude of this term must be estimated on the basis of the preliminary elements before the rectification is performed.

Intermediary Elements of Distorted Eclipsing Systems

Suppose now that equation (11-7) or its simplified version (11-9) has been used to convert the observed values l_{obs} of the measured fractional light intensities of a distorted eclipsing system into the corresponding values of l_{rec}. This operation will not only reduce the light changes between minima to unity, but will also modify the form of the light curve within minima as well. This naturally gives rise to a question as to the significance of this transformation. Exactly what did we accomplish by the process of rectification? To what kind of eclipses does the rectified curve now correspond? These questions have been considered by several investigators who pointed out that *if both components can be regarded as three-axial ellipsoids which are similar in form, and if the distribution of brightness upon their apparent disks is such that the isophotae form at any moment a family of curves similar to the limb of the disk and concentric with it, the rectified light curve will become equivalent to one produced by mutual eclipses of two spherical disks of radii equal to the semi-major axes of the actual ellipsoids.* Now, weighty physical reasons (*cf.* the following section VI.12) lead us to expect that neither one of the foregoing assumptions as to the form and distribution of brightness of components is likely to be fulfilled in the majority of close binary systems with a sufficient accuracy to enable us to adopt the ellipsoidal model as final. On the other hand, there is no doubt that such a model, inadequate as it may be for the final analysis, represents a closer approximation to reality than one consisting of a pair of spherical stars; and for this reason we shall, in what follows, develop a procedure for the computation of intermediary elements of an ellipsoidal model, which will be later modified (section VI.12) to yield the final elements of a distorted eclipsing system.

Within the scheme of approximation afforded by the ellipsoidal model, a derivation of the intermediary elements by an analysis of the rectified light curve presents no difficulty and deviates, in fact, but little from a corresponding treatment of the spherical model discussed in sections VI.4–VI.8. The only essential difference consists of the fact that, for a model consisting of two similar ellipsoids, the fundamental equation (4-1) of our problem is to be replaced by

$$\delta^2 = \sin^2\theta\sin^2 i + \cos^2 i = r_b^2(1 - z\cos^2\theta)(1 + kp)^2, \tag{11-10}$$

where all symbols have the same meaning as in section VI.4 and before, except that r_b denotes now the semi-major axis of the larger component and that, as usual,

$$z = \varepsilon_e^2\sin^2 i, \tag{11-11}$$

ε_e denoting the eccentricity of the equatorial cross-section (supposed to be the same for both stars). On going through exactly the same procedure as in sections VI.4 or VI.5 we easily establish that, for total eclipses, equation (4-3) generalized to ellipsoidal stars and properly weighted becomes

$$\sqrt{w}(Ep^2 - E'')D_1 + 2\sqrt{w}(Ep + E'')D_2 + \sqrt{w}D_3 = \sqrt{w}\sin^2\theta; \qquad (11\text{-}12)$$

while if the eclipses are partial, equation (5-5) should similarly be replaced by

$$\sqrt{w}(Ep^2 - E_0p_0^2)D_1 + 2\sqrt{w}(Ep - E_0p_0)D_2 = \sqrt{w}\sin^2\theta, \qquad (11\text{-}13)$$

where

$$\left.\begin{aligned} D_1 &= \frac{C_1^2}{C_1 - zC_2^2}, \\ D_2 &= \frac{C_1C_2}{C_1 - zC_2^2}, \\ D_3 &= C_3, \end{aligned}\right\} \qquad (11\text{-}14)$$

and where we have abbreviated

$$\left.\begin{aligned} E &= 1 - z\cos^2\theta, \\ E'' &= 1 - z\cos^2\theta'', \\ E_0 &= 1 - z. \end{aligned}\right\} \qquad (11\text{-}15)$$

As the reader can easily verify,

$$k = \frac{C_1}{C_2} = \frac{D_1}{D_2}. \qquad (11\text{-}16)$$

In order to be able to utilize equations (11-12) or (11-13) for a determination of the unknown constants D_1, D_2 and D_3 by the iterative process described in sections VI.4 or VI.5, we must commit ourselves to a definite choice of z before these unknowns can be evaluated. To a sufficient approximation we may put here

$$z \simeq C, \qquad (11\text{-}17)$$

where C denotes the coefficient of $\cos^2\theta$ in equation (11-3), which has been one of our rectification constants. In more precise terms (*cf.* section IV.2),

$$z = \frac{10(3-u)c}{(15+u)(1+\tau_0)}, \qquad (11\text{-}18)$$

where u denotes the coefficient of limb-darkening, τ_0, that of gravity darkening, and c, the coefficient of $\cos^2\theta$ in equation (11-1). In typical cases we may adopt $u = 0{\cdot}6$ and $\tau_0 = 1$ as values closely representative of a large class of eclipsing variables, which inserted in (11-19) yields $z = 0{\cdot}8c$. The actual value of c is not, however, obtainable directly from the observations but is, by (11-4), equal to $C - \gamma$. Since, however, γ is as a rule a *positive* quantity (though small in comparison with C), we should expect that $c > C$ and, therefore, $z > 0{\cdot}8C$. Since the average value of the ratio γ/C is of the order of one-tenth, equation (11-16) is apt to represent as plausible a schematic approximation to reality as can be expressed in simple terms. The error committed by this approximation will be determined and corrected at a later stage of our analysis.

With the values of $\sqrt{w}$ as evaluated by means of equations (4-18) or (5-9), and those of p taken from the appropriate Tsesevich tables, as many equations of condition of the form (11-12) or (11-13) can be set up as there are normal points observed within the minima and iterated (within precautions already discussed in section VI.6) for the most probable values of D_1, D_2 or D_3.* These equations could, if desirable, be generalized to the forms (4-29) or (5-23)–(5-24) suitable for a simultaneous determination of the differential corrections to the adopted (rectified) values of λ_a, or λ_b,† but the need of such a generalization in the present case is much lessened by the fact that the outcome of our intermediary solution will, under any circumstances, represent but an intermediate stage which must be subject to a general least-squares adjustment later on (section VI.12). Therefore, a determination of the differential corrections with the unknown quantities D_1, D_2 (and D_3) may well be dispensed with at the intermediary stage; and once the most probable values of the two (or three) principal unknowns have been determined by iteration, the constants C_1 and C_2 as defined by equations (5-6) follow in terms of the D's from

$$\left.\begin{aligned} C_1 &= \frac{D_1^2}{D_1 + zD_2^2}, \\ C_2 &= \frac{D_1 D_2}{D_1 + zD_2^2}. \end{aligned}\right\} \tag{11-19}$$

Solving these for the actual geometrical elements of the system we obtain

$$r_a^2 = \frac{C_1^2}{G}, \qquad r_b^2 = \frac{C_2^2}{G}, \qquad \sin^2 i = \frac{C_1}{G}, \tag{11-20}$$

where we have abbreviated

$$G = (C_2 - C_1)^2 E'' + C_1(1 - C_3) \tag{11-21t}$$

if the eclipses are total (or annular), or

$$G = (C_2 + p_0 C_1)^2 E_0 + C_1 \tag{11-21p}$$

if they are partial.

It should be reiterated that *the fractional 'radii' r_a and r_b of the two components as resulting from equations* (11-20) *are, in effect, the semi-major axes a*

* The reader may notice that the coefficients of D_1 and D_2 in equation (11-12) involve the quantity $E'' = 1 - z\cos^2\theta'' = 1 - z + zD_3$ and, in consequence, the equation itself is not strictly linear in the D's. Since, however, D_3 occurs in the coefficients of D_1 and D_2 only multiplied by the small quantity z and, moreover, a good provisional value of θ'' (*i.e.*, of D_3) can be inferred from an inspection of a tentative light curve, no serious error is likely to result from treating E'' in (11-12) as an observable quantity, and the equation (11-12) as linear in the D's.

† *Cf.* S. L. Piotrowski, *Ap. J.*, **108**, 36, 1948.

of the respective components. Their polar axes c should then be equal to $a\sqrt{E_0}$; and it is these quantities which are borne out by the observational evidence more directly than the a's because they are largely independent of any error in our preliminary estimate of z. Furthermore, *the angle i as defined by the last one of equations* (11-20) *is identical with the actual inclination of the orbital plane only if both components can be regarded as prolate spheroids* (with semi-axes $a > b = c$). Should, however, $b > c$, *the actual value of the orbital inclination is, not i, but* $\tan^{-1}[(b/c)\tan i]$.* The ratio b/c (i.e., the amount of polar flattening of our components) cannot, unfortunately, be inferred directly from the observations, but must be approximated by the theory (*cf.* section VI.12) before the true values of the orbital inclination can be obtained.

VI.12. PHOTOMETRIC PERTURBATIONS

In the preceding section we learned that if the components of a close eclipsing system could be regarded as similar three-axial ellipsoids (characterized by a symmetrical distribution of brightness around the centre of the apparent disk) it should be possible to determine the elements of such systems by methods closely analogous to those developed earlier in sections 4–6 of this chapter (with modifications outlined in section VI.11). Needless to stress, however, these special assumptions will but seldom—if ever—be realized in practical cases; for if the components of a binary differ in mass and (or) relative dimensions, they must generally differ in form; and the special nature of the conditions under which a distribution of brightness over the apparent disks of distorted stars can be symmetrical around their centres has already been pointed out in section V.1. If these are not fulfilled, the process of rectification as described in the preceding section VI.11 will, in general, *fail* of its primary purpose to render a rectified light curve equivalent to one that can be produced by eclipses of two spherical stars; and its use will be subject to *errors* the extent of which remains to be ascertained. These errors stem from two sources:

(*a*) imperfect rectification between minima;

(*b*) errors of rectification within minima;

and in what follows we shall investigate them in turn.

The errors of the former kind arise from:

(1) A neglect of the superposition of the ellipticity and reflection effect varying as $\cos\theta$ or $\cos^2\theta$;

(2) a neglect of the effects of spherical harmonic distortion of order higher than the second; and

(3) the errors inherent in our empirical determination of the rectification constants B and C, due to a neglect of the effects mentioned under (2) above.

* For a proof of this fact *cf.* equation (12-68) and its sequel in the next section.

The amount of the error arising from (1) has already been determined in section VI.11, and is, in conformity with (11-8) equal to

$$\mathfrak{E}_1 = (1 - l_{\text{obs}})(b \cos \theta + c \cos^2 \theta + \ldots); \tag{12-1}$$

while the error arising from (2) should be given by

$$\mathfrak{E}_2 = l_{\text{obs}}(d \cos^3 \theta + e \cos^4 \theta + \ldots), \tag{12-2}$$

where d and e denote the coefficients of the first two terms neglected on the right-hand side of equation (11-1). With their magnitude approximated on the basis of our theory developed in section IV.2 in terms of the intermediary elements, suppose that we define a new pair of constants B' and C' as coefficients of the equation

$$l_{\text{obs}} - \mathfrak{E}_2 = 1 - B' \cos \theta - C' \cos^2 \theta, \tag{12-2$'$}$$

which subtracted from (11-3) yields

$$(B' - B) \cos \theta + (C' - C) \cos^2 \theta = \mathfrak{E}_2 . \tag{12-3}$$

As many equations of condition of this form can obviously be set up as there are normal points available between minima, and solved for the most probable values of the differences $B' - B$ and $C' - C$. Once these have been found, the error of rectification committed by using the straight instead of the primed values of the constants B and C will be

$$\mathfrak{E}_3 = (B' - B) \cos \theta + l_{\text{obs}}(C' - C) \cos^2 \theta. \tag{12-4}$$

The *total error* Δl_{rec} entailed in the process of approximate rectification as outlined in section VI.11 should then be represented by the sum

$$\Delta l_{\text{rec}} = \mathfrak{E}_1 + \mathfrak{E}_2 + \mathfrak{E}_3, \tag{12-5}$$

and should be evaluated to the same precision (i.e., the same number of significant figures) to which the observed light measurements l_{obs} have been recorded.

Before all this can be done, however, a knowledge of the magnitude of the coefficients b, d, e and γ is a necessary prerequisite, and should be based on our theory of sections IV.2 and IV.6. The reader may perhaps question in his mind the necessity of such an apparently retrogressive step. Why is it not possible—he may ask—to generalize equation (11-3) to the form

$$l_{\text{obs}} = 1 - b \cos \theta - c \cos^2 \theta - d \cos^3 \theta - e \cos^4 \theta - \ldots, \tag{12-6}$$

and to regard it as an equation of condition for a simultaneous determination of the constants d and e with b and c? There is nothing indeed in principle which should prevent us from doing so but, in practice, any such attempt would result in a drastic loss of weight of the whole solution because, in the phase range of practical interest, the functions $\cos^3\theta$ and $\cos^4\theta$ are closely correlated with $\cos\theta$ and $\cos^2\theta$.

In order to demonstrate the extent to which this is true, let us consider the phase range within which the light changes of close eclipsing systems can be regarded as free from the eclipse phenomena; for observations in this range alone could obviously be used for a determination of the constants b, c, d, e from equation (12-6). In section III.4 we learned that, for contact systems, seen in the direction of their orbital planes the eclipse effects may extend within $\theta = \pm 57\overset{\circ}{.}4$; and even in moderately close detached close binaries the duration of eclipses is likely to extend over 30° of the phase angle around the moments of conjunctions. Therefore, the range in θ available for a determination of the constants B and C, from (12-6) may be limited to 90° (or 270°) $\pm$ 60°; and may be as small as $\pm 45°$ or even $\pm 30°$.* In order to demonstrate the degree of dependence of the higher powers of $\cos\theta$ on the lower ones in these ranges, consider the family of Tchebysheff polynomials

$$\left.\begin{aligned} T_j(\cos\theta) &= \cos\{j\cos^{-1}(\sqrt{2}\cos\theta)\}, && 45° \leqslant \theta \leqslant 135°;\\ &= \cos\left\{j\cos^{-1}\left(\frac{2}{\sqrt{3}}\cos\theta\right)\right\}, && 30° \leqslant \theta \leqslant 150°;\end{aligned}\right\} \tag{12-7}$$

normalized within the respective ranges. In particular, for $j = 3$

$$\left.\begin{aligned} T_3(\cos\theta) &= \sqrt{2}\,(8\cos^3\theta - 3\cos\theta), && 45° \leqslant \theta \leqslant 135°;\\ &= \frac{32}{3\sqrt{3}}\cos^3\theta - 2\sqrt{3}\cos\theta, && 30° \leqslant \theta \leqslant 150°;\end{aligned}\right\} \tag{12-8}$$

and for $j = 4$

$$\left.\begin{aligned} T_4(\cos\theta) &= 32\cos^4\theta - 16\cos^2\theta + 1, && 45° \leqslant \theta \leqslant 135°;\\ &= \tfrac{128}{9}\cos^4\theta - \tfrac{32}{3}\cos^2\theta + 1, && 30° \leqslant \theta \leqslant 150°.\end{aligned}\right\} \tag{12-9}$$

A solution of these equations for the highest powers occurring in these polynomials reveals that

$$\left.\begin{aligned} \cos^3\theta &= \frac{3}{8\sqrt{2}}\cos\theta + \frac{1}{8\sqrt{2}}T_3(\cos\theta), && 45° \leqslant \theta \leqslant 135°;\\ &= \tfrac{9}{16}\cos\theta + \frac{3\sqrt{3}}{32}T_3(\cos\theta), && 30° \leqslant \theta \leqslant 150°;\end{aligned}\right\} \tag{12-10}$$

* For actual values of the angles of first contacts of the eclipses in typical systems *cf.*, e.g., Z. Kopal and M. B. Shapley, *Catalogue of the Elements of Eclipsing Binary Systems* (*Jodrell Bank Ann.*, **1**, 141, 1956).

and

$$\left.\begin{aligned}\cos^4\theta &= \tfrac{1}{2}\cos^2\theta - \tfrac{1}{32} + \tfrac{1}{32}T_4(\cos\theta), &\quad 45^\circ \leqslant \theta \leqslant 135, \\ &= \tfrac{3}{4}\cos^2\theta - \tfrac{9}{128} + \tfrac{9}{128}T_4(\cos\theta), &\quad 30^\circ \leqslant \theta \leqslant 150^\circ.\end{aligned}\right\} \tag{12-11}$$

If we insert them in (12-6), the latter equation can clearly be rewritten in the alternative form

$$\begin{aligned} l_{\text{obs}} = 1 - \frac{e}{32} &- \left(b + \frac{3d}{8\sqrt{2}}\right)\cos\theta \\ &- \left(c + \frac{e}{2}\right)\cos^2\theta \\ &- \frac{d}{8\sqrt{2}}\,T_3(\cos\theta) - \frac{e}{32}\,T_4(\cos\theta)\end{aligned} \tag{12-12}$$

for $45^\circ \leqslant \theta \leqslant 135^\circ$, and

$$\begin{aligned} l_{\text{obs}} = 1 - \frac{9c}{128} &- (b + \tfrac{9}{16}d)\cos\theta \\ &- (c + \tfrac{3}{4}e)\cos^2\theta \\ &- \frac{3\sqrt{3}}{32}\,dT_3(\cos\theta) - \tfrac{9}{128}eT_4(\cos\theta)\end{aligned} \tag{12-13}$$

for $30^\circ \leqslant \theta \leqslant 150^\circ$, which are algebraically identical with (12-6). Now the Tchebysheff polynomials $T_j(\cos\theta)$ as defined by equation (12-7) are obviously constrained by the inequality

$$-1 \leqslant T_j(\cos\theta) \leqslant 1 \tag{12-14}$$

within the range of their respective normalization. Therefore, the last two terms on the right-hand side of equation (12-12) are necessarily smaller than $0{\cdot}08d$ and $0{\cdot}03e$; and those on the right-hand side of (12-13) necessarily less than $0{\cdot}16d$ and $0{\cdot}07e$—a circumstance which greatly aggravates the task of their simultaneous determination with the coefficients of $\cos\theta$ and $\cos^2\theta$—in fact, it renders it well-nigh impossible.

What is the bearing of this fact on the magnitude of the errors of rectification $\mathfrak{E}_2$ and $\mathfrak{E}_3$ as defined earlier in this section? Equations (12-10) and (12-11) can obviously be combined to yield

$$\begin{aligned} d\cos^3\theta + e\cos^4\theta = \frac{e}{32} &+ \frac{3d}{8\sqrt{2}}\cos\theta + \frac{e}{2}\cos^2\theta \\ &+ \frac{d}{8\sqrt{2}}\,T_3(\cos\theta) + \frac{e}{32}\,T_4(\cos\theta)\end{aligned} \tag{12-15}$$

if $45° \leqslant \theta \leqslant 135°$, and

$$d \cos^3 \theta + e \cos^4 \theta = \frac{9e}{128} + \frac{9d}{16} \cos \theta + \frac{3e}{4} \cos^2 \theta + \frac{3\sqrt{3}}{32} dT_3(\cos \theta) + \frac{9e}{128} T_4(\cos \theta) \tag{12-16}$$

for $30° \leqslant \theta \leqslant 150°$, where the T_j's normalized for the respective ranges (and different, of course, in each) are given by (12-7). Therefore, if the empirical constants B and C from equation (11-3) are identified—as a generalization of equations (11-4)—with

$$\left.\begin{aligned} B &= b + \frac{3}{8\sqrt{2}} d + \ldots + \beta + \frac{3}{8\sqrt{2}} \delta + \ldots, \\ C &= c + \tfrac{1}{2} e + \ldots + \gamma + \tfrac{1}{2} \varepsilon + \ldots, \end{aligned}\right\} \tag{12-17}$$

if the values of B and C have been determined by a harmonic analysis of the observed light changes within the range $45° \leqslant \theta \leqslant 135°$ (or $225° \leqslant \theta \leqslant 315°$), and with

$$\left.\begin{aligned} B &= b + \tfrac{9}{16} d + \ldots + \beta + \tfrac{9}{16} \delta \ldots, \\ C &= c + \tfrac{3}{4} e + \ldots + \gamma + \tfrac{3}{4} + \ldots, \end{aligned}\right\} \tag{12-18}$$

if $30° \leqslant \theta \leqslant 150°$ (or $210° \leqslant \theta \leqslant 330°$), it follows from equations (12-15) and (12-16) that the corresponding errors of rectification

$$\left.\begin{aligned} \mathfrak{E}_2 &= \frac{d}{8\sqrt{2}} T_3(\cos \theta) + \frac{e}{32} T_4(\cos \theta) + \ldots, \qquad 45° \leqslant \theta \leqslant 135° \\ &= \frac{3\sqrt{3}}{32} dT_3(\cos \theta) + \frac{9e}{128} T_4(\cos \theta) + \ldots, \; 30° \leqslant \theta \leqslant 150° \end{aligned}\right\} \tag{12-19}$$

reduce actually to a small fraction of the actual values of d and e, and may often be ignored with impunity.

Suppose that we have completed a harmonic analysis of those parts of the observed light changes of a close binary system which are unaffected by eclipses, and determined the values of the coefficients B and C (or, if the errors $\mathfrak{E}_2$ and $\mathfrak{E}_3$ turn out to be appreciable, B' and C') in equation (11-3) satisfying best the observed data. What is the exact information which a knowledge of such values contains? In order to interpret these constants, let us decompose the coefficients of different powers of $\cos \theta$ in equation (11-1), which governs the light variation due to the ellipticity of figure, in accordance with the scheme

$$\left.\begin{aligned} b &= L_1 b_1 - L_2 b_2, \\ c &= L_1 c_1 + L_2 c_2, \\ d &= L_1 d_1 - L_2 d_2, \\ e &= L_1 e_1 + L_2 e_2, \end{aligned}\right\} \tag{12-20}$$

where $L_{1,2}$ denote, as before, the fractional luminosities of the primary and secondary component (normalized so that $L_1 + L_2 = 1$). If so then, in conformity with equation (2-37) of Chapter IV,

$$b_j = -\frac{15u(2+\tau_0)}{8(3-u)}\, w_j^{(3)} \sin i + \dots, \tag{12-21}$$

$$\begin{aligned} c_j = &\frac{3(15+u)}{10(3-u)}(1+\tau_0)w_j^{(2)} \sin^2 i \\ &+ \frac{45(1-u)}{16(3-u)}(3+\tau_0)w_j^{(4)} \sin^2 i + \dots, \end{aligned} \tag{12-22}$$

$$d_j = \frac{25u(2+\tau_0)}{8(3-u)}\, w_j^{(3)} \sin^3 i + \dots, \tag{12-23}$$

$$e_j = -\frac{105(1-u)}{32(3-u)}(3+\tau_0)w_j^{(4)} \sin^4 i + \dots, \tag{12-24}$$

($j = 1, 2$) which are exact (for linear law of limb-darkening and centrally-condensed components) as far as first-order terms are concerned, and where all symbols have otherwise the same meaning as in section IV.2.

If we decompose similarly the coefficients on the right-hand side of equation (11-2), representing the light variation due to the reflection of light on both components, in accordance with

$$\left.\begin{aligned} \beta &= L_2\beta_1 - L_1\beta_2, \\ \gamma &= L_2\gamma_1 + L_1\gamma_2, \\ \delta &= L_2\delta_1 - L_1\delta_2, \\ \varepsilon &= L_2\varepsilon_1 + L_1\varepsilon_2, \end{aligned}\right\} \tag{12-25}$$

equations (6-88) and (6-93) of Chapter IV reveal that

$$\beta_j = \{\tfrac{1}{3}r_j^2 + \tfrac{1}{4}r_j^3 \qquad \tfrac{2}{9}(r_i r_j)^2 + \dots\} f_j \sin i, \tag{12-26}$$

$$\gamma_j = \left\{\frac{1}{3\pi}r_j^2 + \frac{3}{8}r_j^3 + \frac{1}{\pi}r_j^4 + \frac{1}{\pi}\left[\frac{13}{18} - \frac{6}{5\pi}\frac{5+(\pi-5)u_i}{3-u_i}\right](r_i r_j)^2 + \dots\right\} f_j \sin^2 i, \tag{12-27}$$

$$\delta_j = 0,$$

$$\varepsilon_j = \left\{\frac{1}{36\pi}r_j^2 - \frac{1}{2\pi}r_j^4 + \frac{1}{4\pi}\left[\frac{31}{54} - \frac{6}{5\pi}\frac{5+(\pi-5)u_i}{3-u_i}\right](r_i r_j)^2 + \dots\right\} f_j \sin^4 i, \tag{12-28}$$

where r_j denotes the (mean) fractional radius of the respective component, and f_j stands for the 'luminous-efficiency' coefficient as defined by equation (6-101) of Chapter IV. If the physical process of reflection is absorption-re-emission (and if both stars radiate like black bodies or approximately so) equations (6-102) and (6-103) of Chapter IV reveal that

$$f_j = (J_j/J_i)(T_i/T_j)^4, \tag{12-29}$$

where J_j/J_i denotes the ratio of mean surface brightnesses of the respective components (obtainable from the *rectified* depths of both minima by means of the equations 4-43 or 5-30); and T_i/T_j, the ratio of effective temperatures of both stars (which must be estimated from their spectra). If, on the other hand, the 'reflected' light is scattered on free electrons, it was pointed out already in section IV.1 that its spectral distribution remains unaltered and, accordingly,

$$f_1 = f_2 = 1. \tag{12-30}$$

The available observational evidence discloses that, for stars of spectral types *A* and later, equation (12-29) should be applicable; while for *O* and early *B* stars (12-30) should prevail, with late *B* stars representing intermediate cases.

With the constants b_j, c_j, d_j and e_j as defined by equations (12-21)–(12-24), and $\beta_j, \gamma_j, \delta_j$ and ε_j as given by (12-26)–(12-28) above, we are now in a position to set up the relations between the observed rectification constants B, C (or B', C') and other elements of the system in their explicit form. If the range in phase from which these constants have been determined is limited to $45° \leqslant \theta \leqslant 135°$ and $225° \leqslant \theta \leqslant 315°$ to avoid the effects of eclipses, these equations assume the explicit forms

$$\begin{aligned} &\left(b_1 + \frac{3}{8\sqrt{2}}\, d_1 + \ldots + \beta_2 + \frac{3}{8\sqrt{2}}\, \delta_2 + \ldots\right) L_1 \\ &- \left(b_2 + \frac{3}{8\sqrt{2}}\, d_2 + \ldots + \beta_1 + \frac{3}{8\sqrt{2}}\, \delta_1 + \ldots\right) L_2 = B, \end{aligned} \tag{12-31}$$

$$\begin{aligned} &(c_1 + \tfrac{1}{2}e_1 + \ldots + \gamma_1 + \tfrac{1}{2}\varepsilon_1 + \ldots)L_1 \\ &+ (c_2 + \tfrac{1}{2}e_2 + \ldots + \gamma_2 + \tfrac{1}{2}\varepsilon_2 + \ldots)L_2 = C; \end{aligned} \tag{12-32}$$

while if the eclipses are narrower, and the interval between them extends to $30° \leqslant \theta \leqslant 150°$ and $210° \leqslant \theta \leqslant 330°$, the corresponding equations become

$$\begin{aligned} &(b_1 + \tfrac{9}{16}d_1 + \ldots + \beta_2 + \tfrac{9}{16}\delta_2 + \ldots)L_1 \\ &- (b_2 + \tfrac{9}{16}d_2 + \ldots + \beta_1 + \tfrac{9}{16}\delta_1 + \ldots)L_2 = B, \end{aligned} \tag{12-33}$$

$$\begin{aligned} &(c_1 + \tfrac{3}{4}e_1 + \ldots + \gamma_1 + \tfrac{3}{4}\varepsilon_1 + \ldots)L_1 \\ &+ (c_2 + \tfrac{3}{4}e_2 + \ldots + \gamma_2 + \tfrac{3}{4}\varepsilon_2 + \ldots)L_2 = C. \end{aligned} \tag{12-34}$$

The absolute terms B and C (or B$'$ and C') of these equations are known from an analysis of photometric observations made between the minima of light; while sufficient approximations to the elements $r_{1,2}$, $L_{1,2}$ and i have previously been established from an analysis of the light changes within minima by the methods of sections 4–7 of this chapter and can thus likewise be regarded as known.

Equations (12-32) or (12-34) are dominated by the photometric ellipticity, and (12-31) or (12-33) by the reflection effect. The former equations, based on our theoretical interpretation of the ellipticity effect in section IV.2, can now be employed for a variety of purposes. First, it should be used to ensure that the assumed coefficient u of limb-darkening, which has been at the basis of our determination of the intermediary geometrical elements in sections 4–6 of this chapter, as well as the theoretical coefficient of gravity-darkening as given by equation (2-16) of Chapter IV, are indeed compatible with the observed ellipticity effect within the limits of its observational errors. It is true that the coefficient u can, under certain favourable circumstances discussed in section VI.8, be determined from an analysis of the observations within light minima with a greater weight than from the ellipticity effect; and even in less favourable cases its value can be fairly well estimated on the basis of the physical theory developed in section IV.1. The same is, unfortunately, not yet true of the coefficient of gravity-darkening; and the investigator who may not wish to place his whole confidence in the black-body approximation as given by equation (2-16) of Chapter IV, may employ the foregoing equations (12-32) or (12-34) for an empirical determination of the mean value of τ_0 of both components. In point of fact, our equations (12-32) or (12-34) offer then *the only way in which the amount of stellar gravity-darkening can be inferred from the observations* with any hope of success.

If, however, the investigator is ready to accept the theoretical values of u and τ_0, equations (12-32) or (12-34) can be solved for the mass-ratio m_2/m_1 (involved through the $w_{1,2}^{(j)}$'s) of the two components, required to produce their tidal elongation which is observed as the ellipticity effect. The significance of such a possibility is underlined by the fact that *it offers a clue for the determination of masses and absolute dimensions of single-spectrum eclipsing binaries*—i.e., of such systems in which a great disparity in masses of their constituent components will render the less massive star to be spectroscopically invisible. As is well known, the spectroscopic method for establishing mass-ratios of close binary systems favours (by virtue of the mass-luminosity relation) the discovery of pairs with mass-ratios in the neighbourhood of unity. The photometric method, based on the observed ellipticity effect, is largely free from this limitation and may yield valuable results. In more specific terms, both equations (12-32) or (12-34) are of the general form

$$M_1 \frac{m_2}{m_1} + M_2 \frac{m_1}{m_2} = M_0 , \tag{12-35}$$

where

$$M_0 = C - \gamma - \tfrac{1}{2}\varepsilon - \dots , \tag{12-36}$$

and

$$M_j = \frac{3L_j}{3 - u_j}\left\{\frac{1}{10}(15 + u_j)(1 + \tau_{0j})\Delta_{j2} + \frac{5}{64}(1 - u_j) \right.$$

$$\left. \times (3 + \tau_{0j})\Delta_{j4}(12 - 7\sin^2 i)r_j^2 + \dots\right\} r_j^3 \sin^2 i \tag{12-37}$$

for the phase range of $\pm 45°$ around the quadratures, and

$$M_0 = C - \gamma - \tfrac{3}{4}\varepsilon - \ldots, \tag{12-38}$$

$$M_j = \frac{3L_j}{3-u_j}\left\{\frac{1}{10}(15+u_j)(1+\tau_{0j})\Delta_{j2} + \frac{15}{256}(1-u_j)\right.$$

$$\left.\times (3+\tau_{0j})\Delta_{j4}(16 - 7\sin^2 i)r_j^2 + \ldots\right\} r_j^3 \sin^2 i \tag{12-39}$$

for the phase-range of $\pm 60°$. Equation (12-25) is obviously *quadratic* in the mass-ratio and will, in general, yield a *pair* of the roots which are consistent with the observed ellipticity (each ascribing the bulk of the distortion to one of the two stars). If the mass-ratio is not far from unity, extraneous evidence may have to be called for to discriminate between them; but if the two roots are very unequal (as is likely to be the case with the majority of single-spectrum systems), one of them will usually be ruled out on grounds of stability (i.e., it may make the size of the less massive star exceed its Roche limit).

Equations (12-31) or (12-33), based on our theory of the reflection effect of section IV.6, should be used in a similar manner to ensure the consistency of the light changes exhibited between minima and within eclipses. They may, in particular, be invoked to determine empirically the values of the luminous-efficiency coefficients f_j which may, in reality, be intermediate between those given by equations (12-39) and (12-30). These equations make it evident, however, that

$$f_1 f_2 = 1; \tag{12-40}$$

and that, hence, equations (12-31) or (12-33) can be regarded as quadratic in f_j, of the general form

$$N_1 f_1 + N_2 f_1^{-1} = N_0, \tag{12-41}$$

where

$$N_{1,2} = \mp L_{2,1}\{\tfrac{1}{3}r_{1,2}^2 + \tfrac{1}{4}r_{1,2}^3 + \tfrac{2}{9}r_1^2 r_2^2 + \ldots\}\sin i \tag{12-42}$$

and

$$N_0 = B - b - \frac{3}{8\sqrt{2}}d - \ldots \tag{12-43}$$

for the phase-range of $\pm 45°$ around the quadratures, and

$$N_0 = B - b - \tfrac{9}{16}d - \ldots \tag{12-44}$$

for the phase-range of $\pm 60°$. A determination of the actual values of $f_{1,2}$ may disclose the actual mechanism of reflection—i.e., whether it is pure absorption followed by re-emission, or scattering, or both—in the atmospheres of the components of close binary systems, and thus furnish us with another fact of considerable astrophysical interest. If, moreover, the reflection does take place by absorption-re-emission, the values of f thus found offer—by equation (12-29)—an indirect but sensitive way for a determination of the ratios T_1/T_2 of effective temperatures of the two stars.

Perturbations within Minima

Even if the process of rectification of the light changes exhibited by close eclipsing systems between minima, as described in the preceding section VI.11, had been subject to no error, a determination of the intermediary elements by the method of sections 4-7 and 11 of this chapter would lead to exact results only if it could be proved that this rectification has removed also all photometric effects of distortion from the light changes within eclipses, and rendered so rectified a light curve equivalent to one that can be produced by the eclipses of spherical stars. In order to find out whether or not this is indeed the case, let us return to the general expressions for the theoretical light changes of distorted eclipsing systems within minima—as given in section IV.4—and segregate in them the terms due exclusively to eclipses from those which also produce the light changes between minima. As the reader can easily verify, the only terms in equation (3-28) of Chapter IV which do not vanish out of eclipses are those in $f_*^{(h)}$ involving the associated α-functions of *even* orders. The limiting values attained by such functions at inner contacts of occultation eclipses are given by equations (4-68) and (4-69) of Chapter IV.

Let us generalize now these latter equations for *any* value of the geometrical depth p in the interval (—1, 1) by putting

$$\alpha_{2\nu}^{2\mu}(k, p) = \frac{\nu!\Gamma(\mu + \frac{1}{2})}{\sqrt{\pi}(\mu + \nu + 1)!}\, \alpha_0^0(k, p) + A_{2\nu}^{2\mu}(k, p), \tag{12-45}$$

and

$$\alpha_{2\nu-1}^{2\mu}(k, p) = \frac{3\Gamma(\mu + \frac{1}{2})\Gamma(\nu + \frac{1}{2})}{2\sqrt{\pi}\Gamma(\mu + \nu + \frac{3}{2})}\, \alpha_1^0(k, p) + A_{2\nu-1}^{2\mu}(k, p), \tag{12-46}$$

where μ and ν are zero or a positive integer. The foregoing equations evidently define a new class of *modified α-functions* $A_n^m(k, p)$, such that $A_n^m(k, \pm 1) = 0$ for any m or n if the eclipse happens to be an occultation. The fractional loss of light during eclipse of an arbitrarily darkened star may then be expressed as

$$\Delta\mathfrak{L} = \{1 + \delta\mathfrak{L}\}\left\{\sum_{h=1}^{k} C^{(h)}(\alpha_{h-1}^0 + \Delta\mathfrak{L}^{(h)})\right\}, \tag{12-47}$$

where $\Delta\mathfrak{L}$ continues to be given by equation (3-27) of Chapter IV, and all other symbols possess the same meaning as in that chapter. In particular, $\Delta\mathfrak{L}^{(h)}$ has been defined by equation (3-28) of Chapter IV, and its constituents continue to be given by equations (3-29)–(3-31) of the same chapter—*except that all associated α-functions of even orders in $f_*^{(h)}$ are to be replaced by the A-functions of corresponding order and index*, as defined by equations (12-45) and (12-46) above.

A glance at the foregoing equation (12-47) reveals that the terms in the first pair of curly brackets on the right-hand side are identical (to the first order in small quantities) with those invoking the variation of light between eclipses, and can presumably be removed by rectification. The remaining terms in the second set of curly brackets then represent the rectified theoretical light changes within minima. These, apart from the fundamental modes $C^{(h)}\alpha^0_{h-1}$ of zero order contain a considerable number of *additional* terms which—however small—are of the *same* order of magnitude as those responsible for the variation of light between minima. Thus, in general, *a rectification can remove only a part of the terms associated with the distortion.* It renders the light changes simpler; but is *not* capable of rendering a rectified light curve equivalent to one produced by eclipses of two spherical disks even to the *first* approximation. The remaining terms constitute then the second part of the general perturbations of our problem and will hereafter be referred to as the 'perturbations within minima' Δl_{per}.

Their explicit expressions may be further simplified by taking advantage of the fact that the associated α-functions and the integrals expressing the boundary corrections are closely interrelated. Their algebra has already been studied in section IV.5, in anticipation of our present needs, where we established that

$$2\mathfrak{I}^m_{-1,n} = (m+n+2)\alpha^m_n - n\alpha^m_{n-2} \tag{12-48}$$

if m is an odd integer; and we find now likewise from equations (12-45) and (12-46) that

$$2\mathfrak{I}^m_{-1,n} = (m+n+2)A^m_n - nA^m_{n-2} \tag{12-49}$$

if m is zero or even. In particular, the reader should experience no difficulty to verify with the aid of different identities established in section IV.5 that, if $n = 0$ or 1 and $m = 0$,

$$\mathfrak{I}^m_{-1,n} - \tfrac{1}{2}(m+n+2)A^m_n = (r_2/r_1)^2 I^m_{-1,n} - (\delta r_2/r_1)^2 I^1_{-1,n}\,, \tag{12-50}$$

while

$$\mathfrak{I}^1_{-1,n} - \tfrac{1}{2}(n+3)\alpha^1_n = -(r_2/r_1)^{n+1} I^1_{-1,n}\,, \tag{12-51}$$

$$\mathfrak{I}^3_{-1,n} - \tfrac{1}{5}(n+5)\alpha^3_n = -(r_2/r_1)^{n+3}(I^3_{-1,n} - I^1_{-1,n}), \tag{12-52}$$

and

$$(r_2/r_1)^2 I^0_{1,n} = \mathfrak{I}^0_{1,n} + \tfrac{1}{4}(n+4)(A^0_{n+2} + A^2_n). \tag{12-53}$$

By virtue of the foregoing identities, the expressions in the second curly brackets on the right-hand side of (12-47), for *linear* limb-darkening (i.e., $k = 2$) of the star undergoing eclipse which is characterized by the darkening coefficient u, lead to

$$\Delta l_{\text{per}} = -\left\{\frac{3(1-u)}{3-u}\,h^{(1)} + \frac{2u}{3-u}\,h^{(2)}\right\} L_1\,, \tag{12-54}$$

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where

$$
\begin{aligned}
h^{(1)} = {} & \tfrac{1}{2}\beta_2[(n_0^2 - n_1^2)A_2^0 + 2n_0n_2\alpha_1^1 + (n_2^2 - n_1^2)A_0^2]v_1^{(2)} \\
& - \tfrac{3}{2}\beta_2\,[l_0^2A_2^0 + 2l_0l_2\alpha_1^1 + l_2^2A_0^2]w_1^{(2)} \\
& - \tfrac{1}{2}\beta_3\,[5l_0^3A_3^0 + 15l_0^2l_2\alpha_2^1 + 15l_0l_2^2A_1^2 + 5l_2^3\alpha_0^3 - 3l_2\alpha_0^1]w_1^{(3)} \\
& - \tfrac{1}{8}\beta_4\,[35l_0^4A_4^0 + 140l_0^3l_2\alpha_3^1 + 210l_0^2l_2^2A_2^2 \\
& \qquad + 140l_0\alpha_2^3 + 35l_2^4A_0^4 - 30l_0^2A_2^0 \\
& \qquad - 60l_0l_2\alpha_1^1 - 30l_2^2A_0^2]w_1^{(4)} + \dots \\
& + (r_2/r_1)^2\{\tfrac{1}{3}[v_1^{(2)} - v_2^{(2)}][3n_1^2I_{1,0}^0 + 3n_2^2I_{-1,0}^2 - I_{-1,0}^0] \qquad (12\text{-}55) \\
& \qquad - [w_1^{(2)} - w_2^{(2)}][3l_2^2I_{-1,0}^2 - I_{-1,0}^0] \\
& \qquad - \tfrac{1}{4}[w_1^{(4)} - w_2^{(4)}][35l_2^4I_{-1,0}^4 - 30l_2^2I_{-1,0}^2 + 3I_{-1,0}^0] \\
& \qquad - g(\delta/r_2)I_{-1,0}^1\} \\
& + (r_2/r_1)^2\{5l_2^3[(r_2/r_1)w_1^{(3)} + w_2^{(3)}]I_{-1,0}^3 \\
& \qquad - 3l_2[(r_1/r_2)w_1^{(3)} + w_2^{(3)}]I_{-1,0}^1 \\
& + 5l_2^3w_1^{(3)}(r_2/r_1)(1 - r_2^2/r_1^2)I_{-1,0}^1 + \dots\},
\end{aligned}
$$

and

$$
\begin{aligned}
\tfrac{2}{3}h^{(2)} = {} & \tfrac{1}{2}(\beta_2 - 1)[(n_0^2 - n_1^2)A_3^0 + 2n_0n_2\alpha_2^1 + (n_2^2 - n_1^2)A_1^2]v_1^{(2)} \\
& - \tfrac{3}{2}(\beta_2 - 1)[l_0^2A_3^0 + 2l_0l_2\alpha_2^1 + l_2^2A_1^2]w_1^{(2)} \\
& - \tfrac{1}{2}[(\beta_3 - 2)(5l_0^3A_4^0 + 15l_0^2l_2\alpha_3^1 + 15l_0l_2^2A_2^2 \\
& \qquad + 5l_2^3\alpha_1^3 - 3l_0A_2^0 - 3l_2\alpha_1^1) + 5l_0^2(2l_0A_2^0 + 3l_2\alpha_1^1) \\
& \qquad - 6(l_0A_2^0 + l_2\alpha_1^1)]w_1^{(3)} \\
& - \tfrac{1}{8}\,[(\beta_4 - 3)(35l_0^4A_5^0 + 140l_0^3l_2\alpha_4^1 \\
& \qquad + 210l_0^2l_2^2A_3^2 + 140l_0l_2^3\alpha_2^3 + 35l_2^4A_1^4 \\
& \qquad - 30l_0^2A_3^0 - 60l_0l_2\alpha_1^1 - 30l_2^2A_1^2) \\
& \qquad + 35l_0^2(3l_0^2A_3^0 + 8l_0l_2\alpha_2^1 + 6l_2^2A_1^2)]w_1^{(4)} + \dots \qquad (12\text{-}56) \\
& + (r_2/r_1)^2\{\tfrac{1}{3}[v_1^{(2)} - v_2^{(2)}][3n_1^2I_{1,1}^0 + 3n_2^2I_{-1,1}^2 - I_{-1,1}^0] \\
& \qquad - [w_1^{(2)} - w_2^{(2)}][3l_2^2I_{-1,1}^2 - I_{-1,1}^0] \\
& \qquad - \tfrac{1}{4}[w_1^{(4)} - w_2^{(4)}][35l_2^4I_{-1,1}^4 - 30l_2^2I_{-1,1}^2 + 3I_{-1,1}^0] \\
& \qquad - g(\delta/r_2)I_{-1,1}^1\} \\
& + (r_2/r_1)^2\{5l_2^3[(r_2/r_1)w_1^{(3)} + w_2^{(3)}]I_{-1,1}^3 \\
& \qquad - 3l_2[(r_1/r_2)w_1^{(3)} + w_2^{(3)}]I_{-1,1}^1\} \\
& + 5l_2^3w_1^{(3)}(r_2/r_1)(1 - r_2^2/r_1^2)I_{-1,1}^1 \\
& - (\tfrac{5}{4} + \tfrac{1}{8}\beta_3)(\tfrac{2}{3}\alpha_0^0 - \alpha_1^0)w_1^{(3)}P_3(l_0) + \dots,
\end{aligned}
$$

respectively, where

$$g = \tfrac{2}{3}v_1^{(2)}P_2(n_2) - 2w_1^{(2)}P_2(l_2) - 2w_1^{(4)}P_4(l_2) - \dots \qquad (12\text{-}57)$$

and all other symbols possess the same meanings as in sections 2-5 of Chapter IV. In particular, the constants β_j continue to be given by equation (2-31), and $v^{(2)}$, $w^{(j)}$, by equations (2-33) and (2-35) of that chapter.

An inspection of the foregoing equations now clearly reveals the nature of the individual terms representing the theoretical light changes of close eclipsing systems within minima. First, it should be noted in passing that the terms factored by g are of purely formal character, as their presence arises solely from our definition of the 'standard radius' of distorted stars. Thus far the latter has been taken as the radius of a sphere having the same volume as the distorted configuration. Should we, however, adopt the diametral semi-axis as the 'standard radius' of a distorted ellipsoid, the terms in (12-56) and (12-57) involving g disappear,* and we are left with terms characteristic of the eclipse alone.

These terms can, in turn, be divided into *two groups*: one arising directly from the geometry of the distorted configurations, the other indirectly (through limb- and gravity-darkening). We notice that *even* harmonic distortions of uniformly bright as well as limb-darkened stars give rise to terms which are proportional to $[v_1^{(2)} - v_2^{(2)}]$ and $[w_1^{(j)} - w_2^{(j)}]$, where $j = 2$ or 4. Such terms express the effects of the *difference of both components in form*, and would vanish if the two stars were similar. The effect of limb-darkening in this connection is merely to raise the second subscripts of the respective 'boundary integrals' by unity. The occurrence of the third harmonic, unfortunately, rather spoils this symmetric picture; for even if both components were equal in size the respective terms would reduce no more than to $[w_1^{(3)} + w_2^{(3)}][5l_2^3 I_{-1,n}^3 - 3_2 I_{-1,n}^1]$, where $n = 0$ or 1 according as to whether the star undergoing eclipse appears uniformly bright, or completely darkened at the limb. We infer therefrom that *the effects, upon light changes within minima, of second- and fourth-harmonic distortion in form of both components tend to neutralize each other, while those of the third-harmonic tend to reinforce*—i.e., the *converse* of what we found to hold good for light changes exhibited *between* minima (*cf.* section IV.2).

The remaining terms in (12-55) are then due to the gravity-darkening; and those in (12-56) to the combined effect of limb- and gravity-darkening. If both components were similar in form, and their apparent disks uniformly bright (i.e., $u = \beta_2 = 0$), then $\Delta l_{\text{per}} = 0$ correctly to terms arising from second-harmonic distortion; and the same will, moreover, be true if $u = \beta_2 = 1$ (for the effects of limb- and gravity-darkening invoked by the second-harmonic rotational and tidal distortion then cancel). A closer inspection of equations (12-55) and (12-56) discloses, however, that this noteworthy

* This becomes obvious when we recall that the g-terms stem from *constant* terms in Legendre polynomials of even orders, invoked by the rotational and tidal distortion, and these terms can clearly be removed by an appropriate change in the definition of standard radius. Since any preliminary solution for the geometrical elements is bound to furnish the 'radii' which are much closer to the diametral semi-axis than to the radii of spheres of equal volume, the adoption of the former as the standard radius of distorted stars appears as natural as it is expedient.

fact—which we predicted already in section V.1—has no analogy in the realm of harmonics higher than the second. For them, no rectification can render $\Delta l_{per} = 0$—whatever the limb- or gravity-darkening.

The relative magnitude of the individual terms on the right-hand sides of equations (12-55) and (12-56) is a matter of important concern from the practical point of view. The relative contribution of the terms invoked by the dissimilarity of both components in form depends predominantly on the magnitude of the coefficients $[v_1^{(2)} - v_2^{(2)}]$ and $[w_1^{(j)} - w_2^{(j)}]$, and these may be large or small depending on the relative dimensions and mass-ratio of the two stars. The majority of the terms invoked by the distortion arises, however, in connection with the gravity-darkening. Is the relative magnitude of such terms governed exclusively by the coefficients $v^{(j)}$ or $w^{(j)}$? The tabulations* of the respective functions have revealed a noteworthy fact that the '*modified associated alpha-functions*' $A_n^m(k, p)$ *of all orders and indices are approximately ten times smaller than the corresponding* '*ordinary*' *alpha-functions* $\alpha_n^m(k, p)$, *and can be frequently ignored in comparison with the latter.* If we do so, and limit ourselves to the second-harmonic distortion as the most conspicuous one, then the approximate relations

$$h^{(1)} = \beta_2\{n_0 n_2 v_1^{(2)} - 3 l_0 l_2 w_1^{(2)}\}\alpha_1^1 + \ldots \tag{12-58}$$

and

$$\tfrac{2}{3}h^{(2)} = (\beta_2 - 1)\{n_0 n_2 v_1^{(2)} - 3 l_0 l_2 w_1^{(2)}\}\alpha_2^1 + \ldots \tag{12-59}$$

represent, under ordinary circumstances, the bulk of the photometric effects of distortion arising in connection with the limb- and gravity-darkening. These, together with the dissimilarity effects (if significant) constitute the 'special perturbations' of our problem—in contrast with the 'general perturbations' as defined by the full-dress equations (12-55) and (12-56). If, in particular, the special perturbations turn out to be small (or negligible) in comparison with the accuracy of the available photometric measurements, there is obviously no need of having a recourse to the general ones.

Another form of special perturbations becomes frequently useful in typical Algol systems, exhibiting total eclipses, for which the angle of orbital inclination i must evidently be close to 90°. For such systems the direction cosine $n_0 \equiv \cos i$ must become a small quantity which approaches zero as $i \to 90°$. It is, furthermore, equally evident that l_2, though larger than n_0, is bound to become within minima a quantity of the same order of magnitude as the fractional radii of both components—diminishing steadily with decreasing geometrical depth until it becomes equal to n_0 at the moment of maximum eclipse. Suppose that the orbital inclination is sufficiently close to 90°, and that the relative dimensions of both components are not too large, for quantities of the order of $n_{0,2}^2 v^{(2)}$ or $l_2 w^{(2)}$ to be negligible. If so, then

* *Cf.* Z. Kopal, 'Theory and Tables of the Associated α-functions' *Harv. Circ.*, No. 450, 1947.

to the second harmonic distortion equations (12-55) and (12-56) reduce to

$$h^{(1)} = -\tfrac{1}{2}\beta_2[(A_2^0 + A_0^2)v_1^{(2)} + 3w_1^{(2)}A_2^0] + (r_2/r_1)^2[v_1^{(2)} - v_2^{(2)}]I_{1,0}^0 + \dots \tag{12-60}$$

and

$$\begin{aligned}\tfrac{2}{3}h^{(2)} = -\tfrac{1}{2}(\beta_2 - 1)[(A_3^0 + A_1^2)v_1^{(2)} + 3w_1^{(2)}A_3^0]\\ + (r_2/r_1)^2[v_1^{(2)} - v_2^{(2)}]I_{1,1}^0 \dots,\end{aligned} \tag{12-61}$$

respectively. The reader may notice that to the order of accuracy we are working, the magnitude of the perturbations is independent of the *scale* (i.e., of the actual values of the direction cosines) of each particular system.

Of the two groups of terms on the right-hand sides of the preceding equations, those varying as $I_{1,n}^0$ account for the difference in polar flattening of both components, while the remaining ones represent the effects of limb- and gravity-darkening. The former consist of a *difference* of small quantities and, unless the dissimilarity in form is appreciable, they are likely to be relatively minor. The latter consist, on the other hand, of a *sum* of small quantities and (unless the A_n^m's turn out to be too small in comparison with $I_{1,n}^0$) are therefore likely to be dominant. To this order of approximation the rectified light changes do not depend on the tidal distortion of the secondary (eclipsing) component,* while the effects of tidal distortion of the primary star manifest themselves only indirectly (through limb- and gravity-darkening).

The outcome of all foregoing discussion renders it a very remote possibility that any kind of frontal attack on the determination of elements of distorted eclipsing systems could meet with success. Any attempts to allow for the difference in form of both components or for their gravity-darkening by an extension of the intermediary methods of section VI.11 would be bound to impose upon the latter such an elaborate superstructure of refinements that their original attractive simplicity would be completely lost and the practical outcome uncertain. There seems, in brief, to exist no *royal road* for a direct determination of the elements of distorted eclipsing systems—the only equitable avenue being again one of successive approximations. Our actual strategy may be briefly described as follows. After having performed an empirical rectification (section VI.11) we may neglect, at first, the general effects of distortion and solve such a rectified light curve by the intermediary methods of sections 4–7 of this chapter, with modifications outlined in section 11. With the aid of the intermediary elements as defined by equations (11-20), evaluate now the errors of the rectification (or, more aptly, 'perturbations between minima') Δl_{rec} as well as the 'perturbations within minima' Δl_{per} as defined by equations (12-5) and (12-54), and with their aid construct a 'theoretical light curve' (or 'theoretical normal points') which—with all effects included—would correspond to the provisionally adopted set of our

* This is easy to understand, because the tides do not affect the form of the diametral cross-sections—exposed to us during minima—of the distorted stars. The effect of tidal distortion of the eclipsing component, small as it is, tends to simulate slightly increased limb-darkening of the star undergoing eclipse, whereas of the effects of rotational distortion the opposite is true.

intermediary elements. Let O be the observed curve (or set of normals) and C, the computed one. *Construct now a curve new which is everywhere as far below O as C is above it and vice versa.* On the basis of this curve (or normals) derive now a new set of intermediary elements by the method of section VI.11, and from these a new computed curve C'. The latter should be much closer to O than C was. *Provided that the differences* $O—C$, $C—C'$, $C'—C''$, . . . etc., *represent a converging sequence*—and the speed with which it may converge depends, in general, on the proximity of the two components—a repetition of this process (if necessary) should eventually result in a good fit.*

The foregoing paragraph summarizes the basic idea which we propose to follow (and which is nothing else but a recourse to the well-known 'method of false position'); but how to translate our plan into practical terms? This task will turn out to be quite simple and we shall, in fact, find ourselves well prepared to do it. Having obtained the intermediary elements, let us now set up a system of equations of condition of the form (8-6)–(8-7) for improving our intermediary set of the elements by the method of differential corrections as developed in section VI.8. The coefficients of the individual differential corrections Δr_1, Δr_2, Δi, . . . , etc., on the right-hand side of (8-7) can be obtained by exactly the same way as in section VI.8—care being taken merely to make sure that, in dealing with the ellipsoidal model, the apparent separation δ of the centres of both stars is given by equation (11-10), where the numerical value of $\varepsilon_e^2 \sin^2 i$ can be approximated from the observations by means of equation (11-17). Therefore, the *whole process of an improvement of the intermediary elements of spherical components by way of least-squares corrections continues to be literally applicable to the ellipsoidal model and rectified light curves as well, provided that the quantity* 'δ' *throughout section* VI.8 *is consistently replaced by* '$\delta/\sqrt{E}$', *with E defined by equations* (11-15).

The goodness of fit of our ellipsoidal model (i.e., the agreement between computation and the rectified light intensities) provides the light residuals Δl on the right-hand sides of our equations of condition. These would, of course, be genuine residuals arising from the observational errors only if the process of rectification and our ellipsoidal model were exact. In reality, the actual (O–C)-residuals based on the intermediary elements would be, not Δl, but

$$\sum \Delta = \Delta l + \Delta l_{\rm rec} + \Delta l_{\rm per}\,, \tag{12-62}$$

where Δl represents the *random* errors arising from the imperfection of photometric measures, and $\Delta l_{\rm rec}$, $\Delta l_{\rm per}$ are *systematic* errors due to the

* It should, however, be kept well in mind that a mere agreement between the observed and computed light curve is a necessary, but not sufficient condition, for the correctness of the elements on which the comparison is based. It is probable that the ellipsoidal model underlying our intermediary elements will in many cases enable us to obtain a fairly good fit—perhaps no worse than the one which we shall eventually obtain. The reasons which prompt us to apply the perturbations are, in general, not due to any conspicuous discrepancy between the intermediary theory and the observations, but rather to the realization that the ellipsoidal model is physically inadequate and must be gradually replaced by one more closely representative of the actual facts.

perturbations between as well as within minima. Suppose, however, that we *evaluate* the amount of these latter corrections by the method of this section to the same precision (i.e., to the same number of decimal places) to which the light intensities have been recorded by the observer, *add* these corrections to the left-hand sides of the equation *with the opposite algebraic sign*, and *then* solve for the differential corrections to the individual elements. The (ellipsoidal) elements so corrected would obviously reproduce back the systematic residuals of the form $\Delta l - \Delta l_{\text{rec}} - \Delta l_{\text{per}}$; so that when the proper amounts of both kinds of perturbations are *added* to them, the residuals at last become (or should become) accidental.

The correctness of the process just described involves a tacit assumption that the matrix of the coefficients on the left-hand side of our equations of condition of the form (8-6)–(8-7) remains sensibly unaffected by the perturbations—which is equivalent to an assertion that the perturbations are small enough not to alter the resulting values of the differential corrections by large amounts. Should the corrections invoked by the perturbations turn out to be large, the whole process must obviously be *repeated*—each time with the *corrected* values of the intermediary elements—until the resulting corrections are no longer significant. The number of iterations required for an exhaustive analysis of the light changes of a given eclipsing system depends primarily on the relative magnitude of the perturbations—which, in turn, depends essentially on the proximity of both components: the closer the pair, the more important the photometric effects of distortion are bound to become. No general guarantee can be given that the iterations involved in the above process will converge in every practical case; yet if they fail, no other method is known to restore convergence and the analysis becomes indeterminate.

Let us suppose, however, that our process of solution turned out to converge and that, after one or more iterations, the definitive set of differential corrections has been obtained from a system of the equations of condition whose absolute terms were diminished by $\Delta l_{\text{rec}} + \Delta l_{\text{per}}$. When such corrections have been added algebraically to the intermediary value of the corresponding element, a set of *definitive elements* has been obtained which (subject to the approximations at the basis of our process) represent the best approach to the actual properties of the respective eclipsing system we can hope to attain. Moreover, the uncertainty with which such elements are defined by the available observational data (within precautions discussed in section VI.8) can be identified with that of the respective differential correction. The last question which we must answer before concluding the present chapter concerns the exact *meaning* of the individual elements which we have obtained.

To the order of accuracy we are working (i.e., up to and including fourth-order harmonic distortion), the form of both components has been approximated by distorted ellipsoids, with axes $a > b > c$, the meridional ellipticity ε_m of which is given by

$$\varepsilon_m^2 = 1 - (c/a)^2 = v^{(2)} + 3w^{(2)} + \tfrac{5}{4}w^{(4)} + \dots; \tag{12-63}$$

while their equatorial ellipticity ε_e becomes

$$\varepsilon_e^2 = 1 - (b/a)^2 = 3w^{(2)} + \tfrac{5}{4}w^{(4)} + \dots, \qquad (12\text{-}64)$$

where $v^{(2)}$ and $w^{(n)}$'s continue to be defined by equations (2-33) and (2-35) of Chapter IV. Now we emphasized already before that the radii 'r' as resulting from our analysis in this section are in fact identical with the semi-major axes of the actual ellipsoids. The three principal semi-axes of such ellipsoids are, therefore, given by

$$\left.\begin{aligned} a &= r, \\ b &= r\sqrt{1 - \varepsilon_e^2}, \\ c &= r\sqrt{1 - \varepsilon_m^2}; \end{aligned}\right\} \qquad (12\text{-}65)$$

or, to the order of accuracy we are working,

$$b = r\{1 - \tfrac{3}{2}w^{(2)} - \tfrac{5}{8}w^{(4)} + \dots\} \qquad (12\text{-}66)$$

and

$$c = r\{1 - \tfrac{1}{2}v^{(2)} - \tfrac{3}{2}w^{(2)} - \tfrac{5}{8}w^{(4)} - \dots\}, \qquad (12\text{-}67)$$

respectively.

Furthermore, it can be shown that *the true value of the orbital inclination of a distorted eclipsing system is, not* i *as resulting directly from our process of solution, but rather*

$$\mathrm{i} = \tan^{-1}(b/c)\tan i, \qquad (12\text{-}68)$$

where the ratio b/c *is the reciprocal of polar flattening.* In order to prove it, consider a three-axial ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (12\text{-}69)$$

referred to a rectangular system of coordinates whose axes coincide with those of our ellipsoid. If we change over—as in section IV.3—to a primed frame of reference $X'Y'Z'$ such that Z' becomes identical with the line of sight while X' is constantly in the direction of the secondary component, the orthogonal projection of our ellipsoid (12-69) on the plane $Z' = 0$ tangent to the celestial sphere will be the ellipse

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1, \qquad (12\text{-}70)$$

with the semi-axes

$$\left.\begin{aligned} a'^2 &= a^2l_2^2 + b^2m_2^2 + c^2n_2^2, \\ b'^2 &= \phantom{a^2l_2^2 +{}} b^2m_1^2 + c^2n_1^2, \end{aligned}\right\} \qquad (12\text{-}71)$$

where the direction cosines l_2, $m_{1,2}$, and $n_{1,2}$ continue to be given by equations (3-3) and (3-4) of Chapter IV.

Suppose that, for simplicity, the surface brightness over the apparent disk of our ellipsoid is uniform and equal to H. Its light $\mathfrak{L}$ as seen at a great distance by an observer whose line of sight is specified by the direction cosines l_0, m_0, n_0 will then be given by

$$\mathfrak{L} = \pi a'b'H = \pi abcH\{(l_0/a)^2 + (m_0/b)^2 + (n_0/c)^2\}^{1/2} \tag{12-72}$$

or, on insertion for l_0, m_0, n_0 from (3-2) of Chapter IV,

$$\mathfrak{L} = \pi a^2 H \cos \mathrm{i}\{1 - \varepsilon_e^2 + (1 - \varepsilon_m^2)(1 - \varepsilon_e^2 \cos^2 \psi) \tan^2 \mathrm{i}\}^{1/2}. \tag{12-73}$$

The variation of light between minima thus turns out to depend on *two* constants ε_e^2 and $(1 - \varepsilon_m^2) \tan^2 \mathrm{i}$, involving *three* unknown quantities ε_e^2, ε_m^2, i. This demonstrates that it is impossible in principle to determine them all from photometric observations between minima alone. In particular, *the meridional eccentricity can under no circumstances be separated from orbital inclination*—a fact first pointed out by Roberts.* If, however, we introduce a fictitious inclination i related with the true inclination i by means of

$$(1 - \varepsilon_m^2) \tan^2 \mathrm{i} = (1 - \varepsilon_e^2) \tan^2 i, \tag{12-74}$$

equation (12-74) readily assumes the form

$$\mathfrak{L} = \pi a^2 H \cos \mathrm{i} \sec \mathrm{i}\ \{(1 - \varepsilon_e^2)(1 - z \cos^2 \psi)\}^{1/2}, \tag{12-75}$$

where we have abbreviated

$$z = \varepsilon_e^2 \sin^2 i. \tag{12-76}$$

If our components were prolate spheroids (i.e., if $b = c$ and thus $\varepsilon_m = \varepsilon_e$), then obviously $i = \mathrm{i}$. Otherwise, however, this will not be true; and the orbital inclination obtained after rectification from our analysis will be, not i, but i as defined by equation (12-74) which, by virtue of (12-65), can be rewritten as

$$\tan \mathrm{i} = (b/c) \tan i. \tag{12-77}$$

in conformity with (12-68). It may be noted that, by (12-66) and (12-67),

$$\frac{b}{c} = 1 + \tfrac{1}{2}v^{(2)} + \dots, \tag{12-78}$$

so that

$$\mathrm{i} = i + \tfrac{1}{2}v^{(2)} \cot i + \dots. \tag{12-79}$$

It thus transpires that one of the effects of polar flattening of the components of eclipsing binary systems is to render the *computed* values i of orbital inclination *smaller* than their true values by appreciable amounts. Their difference cannot, moreover, be deduced directly from the observations and must be evaluated from equation (12-68) *after* the whole solution has been carried out.

Lastly, in determining the fractional luminosities of the two components and the ratio of their mean surface brightnesses from the rectified losses of light at the time of both minima, proper attention should be paid to the effects of reflected light. A discussion contained in section VI.11 has made it

* *M.N.*, **63**, 535, 1903.

evident that the rectification for reflection amounts in effect to adding, to each star and at any moment, the amount of light required to complete its phase to 'full'. Hence, if the fractional luminosities L_1 and L_2 are evaluated from equations (4-41) if the eclipses are total, and from (5-29) if they are partial, by substituting merely the rectified for the observed losses of light, such luminosities will represent a sum of the proper light of each component *augmented* by the amount of light reflected from it, and will be expressed in terms of the maximum brightness of the respective system (attained at the time of quadratures) taken as our unit of light. If L_1^*, L_2^* denote now the values of the fractional luminosities *proper* (i.e., freed from the effects of reflection), it follows from equations (6-69) and (6-93) of Chapter IV that (for the phase $\varepsilon = 0$) we have

$$L_{1,2}^* = L_{1,2} - L_{2,1}\{\tfrac{2}{3}r_{1,2}^2 + \tfrac{1}{2}r_{1,2}^3 + \tfrac{4}{9}r_1^2r_2^2 + \ldots\}f_{1,2}\,, \tag{12-80}$$

where $f_{1,2}$ denotes again the luminous-efficiency factor (12-29).

Now if the ratio J_1/J_2 of *mean* surface brightnesses of both components continues, as before, to be defined by

$$J_1/J_2 = (r_2/r_1)^2 L_1/L_2\,, \tag{12-81}$$

where the ratio L_1/L_2 has been evaluated from equations (4-41) or (5-29) by using the *rectified*, instead of the observed, values of the λ's, it follows again that the ratio J_1/J_2 will refer to the surface brightnesses which are those of their *following* (unilluminated) hemispheres. If the latter surface brightnesses are denoted again by the asterisks, their ratio J_1^*/J_2^* follows from

$$J_1^*/J_2^* = (r_2/r_1)^2 L_1^*/L_2^*\,, \tag{12-82}$$

which is of the same form as (12-81), but in which the proper luminosities of both components, as obtained from equation (12-80), have replaced the apparent luminosities augmented by reflection. *The ratios J_1^*/J_2^* of mean surface brightnesses of the following hemispheres*, unexposed to any incident light, *are genuine indicators of proper effective temperatures of both components*; the values of J_1/J_2 are *not*, because they may be seriously affected by reflection.

Ultimately, the ratio of mean surface brightnesses of the advancing and following hemispheres of either component is clearly given by

$$J_j/J_j^* = L_j/L_j^*\,, \tag{12-83}$$

where $j = 1$ or 2. The ratios $(J/J^*)_j$ may be used to investigate the 'heating effect' of the incident radiation, which may increase the mean effective temperature of the advancing ('daylight') hemisphere by several hundred degrees in excess over that of the following ('night') hemisphere of the component of a close eclipsing system. In typical Algol systems, consisting of intensely bright primaries powerfully illuminating large but feebly luminous secondary components, the ratios J/J^* may occasionally become very large indeed. In extreme cases, the investigator may discover that the whole apparent luminosity L_2 of the secondary component may be due to the reflected light, the proper value of L_2^* being effectively zero. In such cases, a shallow secondary minimum may be due to an eclipse, by the primary component, of its own light reflected from the secondary. Since the reflected radiation obeys its own characteristic law of limb-darkening, which is rather different from that represented by equation (2-4), the light changes arising from an eclipse of reflected radiation would not be completely describable in terms of the functions introduced in section VI.2. Such hypothetical minima would, however, be necessarily very shallow and their contribution to a determination of the elements of the respective eclipsing system could not be but very meagre—a fact which perhaps explains why a theory of such eclipses has not so far been elaborated.

VI.13. SURVEY OF THE METHODS

The preceding sections of this chapter contain the main outline of analytical methods for a determination of the elements of eclipsing binary systems; and the Appendix will give additional details of the necessary computational techniques. In spite of our consistent desire to limit to the essentials the scope of the topics treated in this chapter, the size to which its text has grown may have caused the reader, at times, to lose perspective in the midst of numerous but unavoidable technical details. In order to minimize the dangers arising from this source, the aim of this concluding section should be to remedy this situation by providing the perspective and summarizing briefly the essential steps which lead to a determination of the elements of an eclipsing binary system from an analysis of its light curve, for the details of which the reader can now be conveniently referred to the preceding appropriate sections.

The empirical data which the observer furnishes to their analyst and which will constitute the basis of his investigation consists of a (usually) large number of discrete measurements of the instantaneous brightness of an eclipsing system performed each at a certain time. The first move of the computer will be to arrange such observations according to the phase and to group them in a more limited number of normal points, representing the appropriate mean of the individual observations averaged over a certain small fraction of the cycle. The interval over which it may be legitimate to average the observations depends on the rapidity of the observed light changes and need not be the same all over the cycle. It should be chosen so that, on the one hand, the number of normal points is not too large to make the subsequent computations unnecessarily cumbersome but, on the other hand, the resulting normals should be sufficiently numerous and well-distributed to represent faithfully the actual light changes.

Once the normal points have been formed, their coordinates should be appropriately normalized, The measurements of stellar brightness—irrespective of the manner in which they have been made—are customarily recorded in stellar magnitudes. From the intrinsic point of view such a scale is, however, purely arbitrary; therefore, after the normals have been formed, their magnitudes should be converted into light intensities which will hereafter be used exclusively throughout our work. Furthermore, in order to normalize these intensities, we shall express them in terms of the maximum brightness of the system between minima adopted as our unit of light. The time readings of each normal should likewise be converted into non-dimensional phase angles, reckoned from the primary (deeper) minimum. The accuracy to which these two coordinates characterizing each normal should be recorded depends on the precision of the underlying observational data. When dealing with the photoelectric measurements, the fractional light

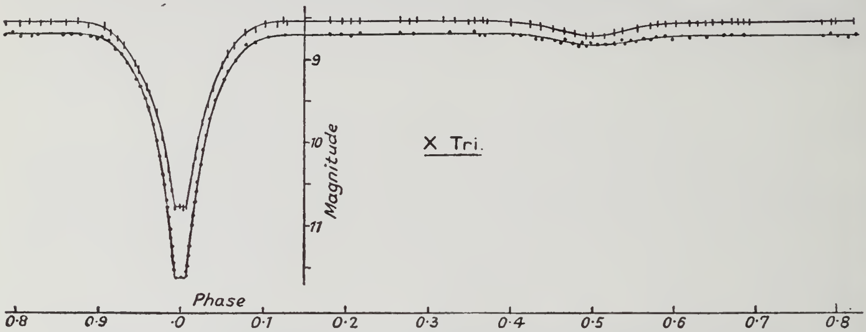

FIGURE 6-7. THE OBSERVED (PHOTOELECTRIC) NORMAL POINTS AND THEORETICAL LIGHT CURVES OF THE ECLIPSING SYSTEM OF X TRIANGULI (type *a*-1) according to F. Lenouvel (*Thèse*, Paris 1953). Upper curve: observations in the yellow; lower curve: observations in the blue.

Abscissae: the actual brightness of the system in stellar magnitudes; ordinates: the phase in fractions of the orbital period.

intensity l characterizing each normal should be recorded to at least three (and possibly four) decimal figures, and the same should be true of the corresponding phase angles, while less precise observations may permit us to carry a proportionally smaller number of digits throughout all subsequent computations.

The normal points thus formed should now be plotted, on a convenient scale, against $\sin^2 \theta$. This process effectively superimposes the two halves of the minima upon one another. A smooth curve may then be drawn by freehand to follow the course of the observed normals, and the standard deviation of the individual observations grouped in each normal point from such a curve should be used as a measure of the empirical weight of the respective normal. Furthermore, the slope of a light curve in the l-$\sin^2 \theta$ coordinates defines the intrinsic weight of each normal which, together with its empirical weight, will specify the total weight to which each normal point will be entitled in the course of our subsequent analytical work.

As the next step of our analysis, we should inspect our plot of the observations, with the aim of distinguishing between the following alternatives:

(*a*) The observations show no evidence of any significant variation of light outside of eclipses, so that for all practical purposes the brightness of the system between minima can be regarded as constant (*cf.* Figs. 6-7 and 6-8).

(*b*) The light of the system is found to vary noticeably between minima—due to the proximity effects (ellipticity and reflection)—which shows that the distortion of the components from the spherical form can no longer be ignored. (Figs. 6-9 or 6-10).

If the light curve of our eclipsing system falls in category (*a*), we shall proceed to analyse it for the elements by the methods of sections VI.4–VI.8 which are based on the assumption that both components are spherical in

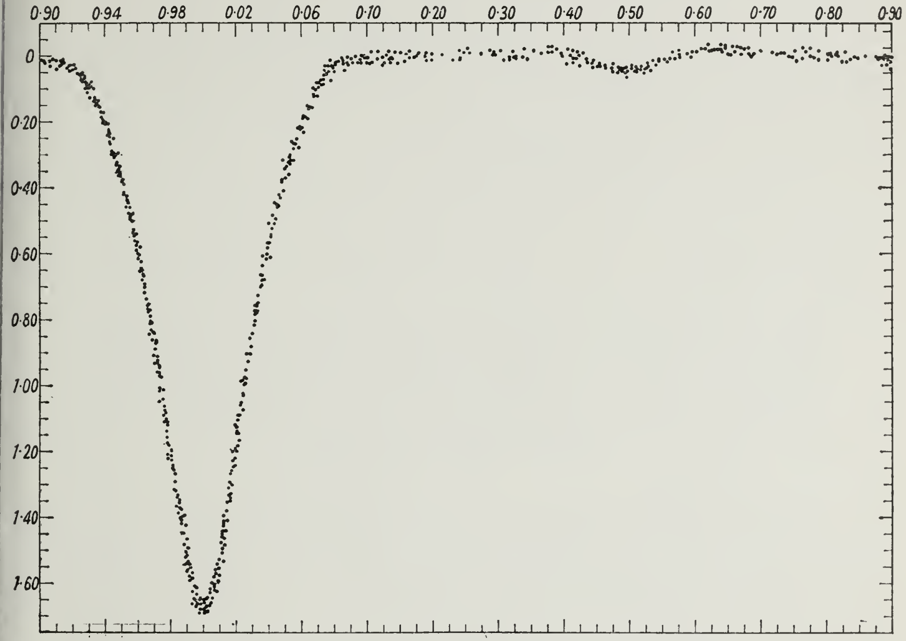

FIGURE 6-8. LIGHT CHANGES OF THE ECLIPSING SYSTEM OF RZ CASSIOPEIAE (type *a*-2) according to the photoelectric observations by C. M. Huffer (*Ap. J.*, **114,** 297, 1951). Abscissae: the relative brightness of the system in stellar magnitudes; ordinates: the phase in fractions of the orbital cycle. The time-scale outside primary minimum has been contracted for convenience of presentation.

form and appear in projection as circular disks arbitrarily darkened at the limb. Two alternatives may again arise, which should be distinguished by inspection:

(1) the light curve exhibits a phase of constant light at the bottom of one (or both) minima, which indicates that total and annular minima alternate (Fig. 6-7); or

(2) a continuous variation of light persists during both minima, disclosing that the eclipses are necessarily partial (Fig. 6-8).

Difficulties in discriminating between these alternatives may arise if the eclipses are nearly grazing, or if only one minimum has been observed (since the annular eclipse of a limb-darkened star may closely simulate a partial eclipse; see Figs. 6-5 or 6-6). Should a clear-cut decision turn out to be

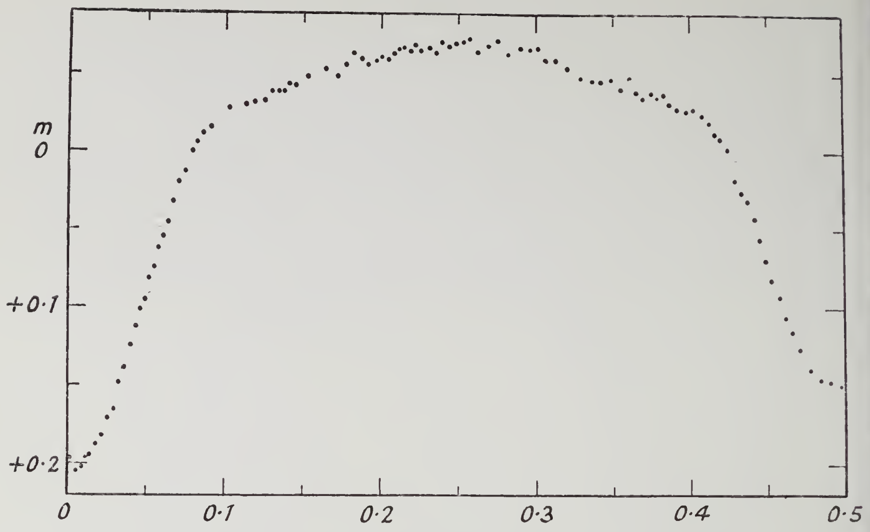

FIGURE 6-9. LIGHT CHANGES OF THE ECLIPSING SYSTEM OF SZ CAMELOPARDALIS (type *b*-1), according to the photographic observations by A. J. Wesselink (*Leiden Ann.*, **17**, Part III, 1941). Abscissae: the relative brightness of the system in stellar magnitudes; ordinates: the phase in fractions of the orbital cycle. Observations corresponding to the second half of the cycle have been reflected on the first.

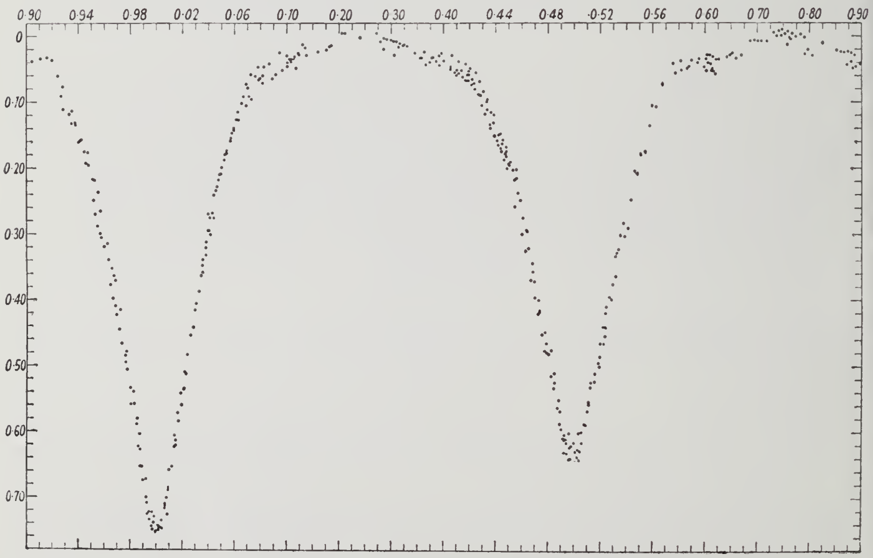

FIGURE 6-10. LIGHT CHANGES OF THE ECLIPSING SYSTEM OF U OPHIUCHI (type *b*-2) according to the photoelectric observations by C. M. Huffer (*Ap. J.*, **114**, 297, 1951). Abscissae: the relative brightness of the system in stellar magnitudes; ordinates: the phase in fractions of the orbital cycle. The time-scale between minima has been contracted for convenience of presentation.

impossible, the investigator may have to conjecture as to the proper type of the eclipses giving rise to the observed light changes, and to test his hypothesis *a posteriori* by trial and error. It may be mentioned that if the original hypothesis has perchance been wrong, our subsequent analysis will necessarily point out the correct alternative; so that the only risk entailed in our original estimate is an added amount of computation.

Total (Annular) Eclipses of Spherical Stars

Suppose, for the sake of argument, that an inspection of the observational evidence indicates the case (a-1) to be true. If so, the next step confronting us will be to decide which one of the two minima is due to a total and annular eclipse. This question may also sometimes be decided by inspection, since an annular eclipse may reveal its identity by a slow but continuous variation of light taking place as the track of the smaller component passes in front of the limb-darkened disk of its mate (during totality, the light of the system must necessarily remain constant unless the larger star is intrinsically variable); but the range of possible light changes during the annular phase is always apt to be small and an unfavourable distribution of observations may easily conceal it. If the two minima differ greatly in depth, the case for the deep minimum being due to a total eclipse may become overwhelming (*cf.* section VI.4); but if both minima are shallow and of comparable depths, the inspection by itself becomes of little avail. Spectroscopic observations often prove, in such cases, of considerable assistance—indicating as they may whether the component of greater surface brightness is or is not one of greater fractional luminosity. If the binary under investigation happens to be a single-spectrum system (which indicates a considerable disparity in luminosities of its components) and if, at the same time, both minima are nearly equal in depth (which indicates a comparable surface brightness), there is no doubt that the brighter star is the larger one, and its radial velocity curve discloses readily whether this star is the eclipsing or the eclipsed component at the time of either minimum.

In the absence of any spectroscopic observations, the whole identification of the types of the eclipses must be based on the light curve; and in order to do so we should proceed as follows. As the first move, let us evaluate the 'depth' k by means of equation (4-8). The use of this equation calls for the adoption of a definite degree of limb-darkening of the larger component. As long as we do not know the right type of the eclipse, equation (4-8) will furnish us with a pair of the 'depth' k's depending on whether $\lambda_a = \lambda_1$ or λ_2. If the two k's evaluated on the basis of the alternative assumptions are very different, and if the duration of the partial phases of the eclipse can be inferred from the plot of the observations without too much uncertainty, equation (4-12) may permit us to distinguish between the two admissible 'depth' k's. If, however, these two k's happen to differ but little, the criterion represented by equation (4-12) is apt to become ineffective, and recourse must be had to the actual shape of the light changes exhibited during partial phases of the eclipse.

In order to do this, we shall adopt one of the admissible values of the 'depth' k's—let us denote it by K—corresponding to a definite type of the eclipse at the primary minimum, and look up for each normal point the value of the geometrical depth $p(K, \alpha)$ corresponding to the adopted degree of limb-darkening of the star undergoing eclipse. With regard to the numerical accuracy, as many figures should be retained in the p's as are significant in the observed values of α; for photoelectric observations, this will usually imply a three-digit accuracy, which should permit linear interpolation over most part of Tsesevich's tables. Having found the p's, we now set us as many equations of condition of the form (4-3) or (4-6) as there are observed normals, weigh them in accordance with the reasoning developed in section VI.4, and iterate for the ratio $C_1/C_2 = k$ to which we shall hereafter refer as the 'shape' k. For total or annular eclipses the rate of convergence of our iterative process should be very rapid; but the process will, of course, converge only if the type of the eclipse giving rise to the respective minimum has been correctly identified. A failure of the difference $K - k$ to diminish drastically after the first iteration will, therefore, disclose usually at once improper initial identification.

If the right kind of eclipse has been chosen, however, our iterative process must not only converge, but converge to a value which does not differ significantly from the 'depth' k. This, in turn, can be true only if the proper degree of darkening of the star undergoing eclipse has been adopted. Its value can thus be established in this way by trial and error, and the precision of such a determination will obviously be the greater, the greater the accuracy with which the 'shape' and 'depth' k's can be inferred from the observed data. Moreover, it goes without saying that an agreement between the 'shape' and 'depth' k must be true of both minima. In particular, if the 'shape' k as deduced from the partial phases of an occultation eclipse comes out smaller than the 'depth' k, this indicates that the degree of darkening of the smaller star has been underestimated; while if the 'shape' k as deduced from the partial phases of a transit is less than the 'depth' k, the darkening of the larger component must have been overestimated. In general, if both alternate minima have been observed, the proper degrees of darkening of both components can be adjusted separately by invoking the necessity to harmonize the 'shape' and 'depth' k's. Should the secondary minimum be too shallow for its shape to be of any avail, though its depth may be discernible, the proper limb-darkening can be inferred at least for the component undergoing eclipse at the time of the primary minimum. If, ultimately, only one minimum has been observed, its analysis is still capable of furnishing a complete set of the geometrical elements, but in the absence of any 'depth' k we have no way of checking upon the correctness of the assumed degree of darkening of the eclipsed star.

When the types of the eclipses giving rise to the alternate minima have been correctly identified (which often may be true from the very beginning of our analysis), the time has come to fall back on the full-dress equation (4-29) containing the differential corrections $\Delta\lambda$ and ΔU, and to put our best previous approximation to k for the last time through the mill of our iterative process, which should reproduce it without appreciable change. As long as the convergence itself of our iteration was in doubt, the presence of the small corrections $\Delta\lambda$ or ΔU in our equations of condition would have constituted a rather superfluous impediment; but now, when we are approaching the final solution for the intermediary elements, these corrections should be carried through even if the chances are that they will come out insignificant; for their uncertainty will affect that of the individual geometrical elements, and the extent to which it may do so will be of interest to us. In carrying out a least-squares solution of our system of equations of the form (4-29), care should be taken to adjoin to it equations (4-30) and (4-31) based on the observations of full light or of the totality. Similarly, if only one minimum lends itself for an analysis of its partial phases, we should not fail to add to our system the properly weighted equation (4-35) based on our knowledge of the 'depth' k. If, in particular, the minimum under analysis is shallow, equation (4-35) may have to bear the brunt of the weight of the whole solution.

Once recourse to the full-dress equation (4-29) has been made, it is not recommended to repeat its least-squares solution, since the iteration of a system with five unknowns would be bound to become rather laborious. Instead, the computer should next turn his attention to equation (6-33t) and solve it for the k-derivatives of the auxiliary constants C_1, C_2 and C_3. In doing so, he will notice that the left-hand side of this equation is (apart from the absence of terms involving ΔU and λ) identical with the left-hand side of (4-3). In consequence, the matrix of the coefficients of C_1, C_2 and C_3 of our former system of normal equations remains unchanged (except that the columns and rows corresponding to the unknowns ΔU and $\Delta\lambda$ can now be crossed out), so that it is only the right-hand sides of the equations of condition, and of the normal equations, which remain to be recomputed.

As soon as this has been done, we are in a position to determine, by means of equation (6-26), the quantity dk/dK which is intimately connected with the rate of convergence of our iterative process and which, in turn, will permit us to evaluate the definitive (barred) values of the C_j's from equations (6-35) without any further iteration. The definitive values of C_1, C_2 and C_3, together with the corrected values of the depth of the minimum under investigation and of our unit of light, open the way for completing our analysis by the evaluation of the most probable values of the individual elements of our eclipsing system by means of the formulae of section VI.4. Such elements are all based on an assumed degree of limb-darkening of the component undergoing eclipse. If this degree could not be estimated with a satisfactory accuracy to begin with, and may be subject to a later revision, the investigator may wish to evaluate the auxiliary C_j's, and all elements based upon them, in terms of an arbitrary small correction Δu. If the methods of section

VI.7 are used for this purpose, it is only the absolute terms of our equations which will have to be recomputed, the matrix of the coefficients on the left-hand side remaining again the same.

When the intermediary elements of our eclipsing system have thus been established, the investigator may wish to evaluate the light residuals of each normal (i.e., the differences of observed light, minus that computed on the basis of our intermediary elements), in order to ascertain the 'goodness of fit' of his elements and the extent to which such elements actually satisfy the observational data. This task can be performed most expeditiously by converting the $O-C$ residuals of our intermediary solution for the auxiliary constants into the corresponding Δl-residuals by the method of section VI.8. Such residuals should be carefully inspected for any indication of a possible systematic trend with the phase which, if present, should become a subject of further examination. Moreover, a sum of the squares of individual residuals thus obtained should be compared with that following directly from the least-squares solution and any discrepancy exceeding the limits of round-off errors traced to a specific error in the formation of our system of the normal equations.

The uncertainty with which the individual elements are defined by the available observational data can now be evaluated as the error of the corresponding linear function, which can be constructed by the method of section VI.6, and whose uncertainty can be expressed in terms of the appropriate elements of the inverse matrix of our intermediary solution by the techniques in the Appendix. Another, and in certain respects more powerful, way of ascertaining the errors of the individual elements is the method of the differential corrections discussed in section VI.8. It calls for the setting up, and a least-squares solution, of a new set of as many equations of condition as there are available normal points; and its results should be essentially (though not exactly) equal to those obtainable from an intermediary solution. An experienced investigator will probably find the former method requiring less additional work. The novice should, however, gain much by determining the errors of his elements in both ways—if only for the feeling of confidence in the correctness of his results which an independent evaluation by two different methods alone can provide.

All foregoing remarks tacitly presume that the relative orbit of the components of our eclipsing system is circular. Should, however, the orbit happen to be eccentric—as evidenced by the displacement and (or) unequal duration of both minima—but the eccentricity being not large enough to render the minima asymmetric, the foregoing methods of analysis continue to apply almost literally equally well—except for the fact that, in general, the solutions of both minima can no longer be freely combined. Instead, each minimum treated on its own merits will furnish a corresponding set of 'circular' (fictitious) elements, which should be subsequently converted into the true 'elliptical' elements with the aid of the relations established in section VI.10. If the two independent sets of 'circular' elements fail to lead to essentially the same set of 'elliptical' elements, the adopted values of the radial component $e \sin \omega$ of the eccentricity should be adjusted by way of the differential corrections discussed in section VI.10. Recourse to the individual differential corrections for e and ω becomes indispensable when only the orbial eccentricity happens to be so large that the asymmetry of the light changes observed within minima can no longer be ignored.

Partial Eclipses of Spherical Stars

Having outlined the tentative strategy recommended for the determination of the elements of totally eclipsing systems, let us return now to the alternative (a-2) which we shall have to follow if the light of an eclipsing system varies continuously in both minima, indicating that the eclipses giving rise to the observed light variations are necessarily partial (*cf.* Fig. 6-7). In this case, the problem of determining the elements of such eclipses from an analysis of their light curves becomes somewhat more complex—mainly because the fractional loss of light α of the eclipsed star is defined by the observations only as a fraction of the maximum fractional loss of light α_0, which increases the number of the unknowns to be determined by a simultaneous analysis to four. This fact will not only complicate the orbital solution, but will also diminish substantially its weight. Whereas, for total eclipses, three points of a light curve within a minimum are sufficient in principle to specify the geometrical elements of the corresponding eclipse (and each minimum can be solved independently of the other), all points within one minimum due to a partial eclipse are inadequate to do so in practice, and a combination of both minima into a single solution—which was optional for totally eclipsing systems—now becomes a necessity. Furthermore, in solving for the elements from one minimum of a totally eclipsing system, we had to know the degree of darkening of one (the eclipsed) component only; and if both alternate minima of such a system were available for analysis, the degrees of limb-darkening of both components could be determined by the requirement that the geometrical elements defined by both minima must be the same. If, on the other hand, the eclipses are partial, a knowledge (or estimate) of the darkening of both components becomes prerequisite and will admit of no subsequent verification. These facts are bound to render the determination of elements of partially eclipsing systems inadvertently less exact than it was for systems exhibiting total or annular eclipses, and the investigator will find them reflected in increased errors of the individual elements which he desires to obtain.

Since a determination of the elements of a partially eclipsing system from one (the primary) minimum alone remains as yet a practical impossibility (*cf.* section VI.5), at least the depth of the secondary minimum must be known to the investigator before he can embark upon his work. As the first step toward such a solution, he must estimate a provisional value of k—let us denote it again by K—and settle the question of whether the primary (deeper) minimum is due to an occultation or a transit eclipse (i.e., whether $\lambda_1 = \lambda_a$ or λ_b). In contrast with the case of total eclipses, there is—alas—no 'depth' or even 'duration' k available to lend us a hand at the beginning of our work; so that unless our eclipsing system was already subject to some previous investigation and a set of preliminary elements is available to us (which may, however, not necessarily be always a safe guide, since a discrimination between occultations and transits in systems consisting of components that are nearly equal in size is a matter of some delicacy, for which graphical or semi-graphical methods are not particularly well suited), our initial choice of K may have to be an almost outright guess. Spectroscopic evidence—if available—may sometimes enable us to specify K with some accuracy independently of the form of the light curve (*cf.* section VI.7), or may at least indicate whether or not the component of greater surface brightness is also likely to be greater in size. If, however, in the absence of any preliminary elements or other pertinent information we are completely in the dark as to the type of eclipses giving rise to the alternate minima, the best strategy appears to be to start the solution by assuming $K = 1$—in which case a distinction between occultations and transits becomes immaterial.

Having estimated K and inferred from the observations the values of λ_a and λ_b as well as that of our unit of light to the best of our ability, our next step will be to evaluate, by successive approximations, the values of p_0 and α_0, for either eclipse, by means of equations (5-11) and (5-12) and with the aid of the requisite auxiliary tables, by the process described in section VI.5. As soon as this has been done, the normalized fractional loss of light $\alpha = n\alpha_0$ can be readily evaluated for every normal point within minima, and the corresponding value of $p(K, \alpha)$ can be extracted from Tsesevich's tables for the appropriate type of the eclipse and the degree of darkening which underlies our solution of (5-11) or (5-12) for p_0. An equation of condition of the form (5-5) can now be set up for every observed normal (with regard to the numerical accuracy to which the coefficients of such equations should be evaluated, the remarks made in connection with total eclipses continue to hold good), weighted in accordance with equation (5-9), and solved by least-squares for the most probable values of C_1 and C_2. If the initially assumed value of K has been correct, the ratio $C_1/C_2 = k$ should verify it. If, however, the values of K and k differ significantly, the iterative process should be repeated —each time on the basis of the latest approximation to k—until the assumed and resulting values of the ratio of the radii of both components are significantly the same.

Our ability to achieve this end presumes tacitly that our iterations converge; and this can be true only if the type of the eclipse under investigation has been correctly identified to begin with. If, however, in order to avoid having to make such a decision we started our iterative process by assuming $K = 1$, the ratio C_1/C_2 resulting from the first iteration will very likely differ significantly from unity, so that a hypothesis as to the type of the eclipses at the primary and secondary minima can no longer be deferred. At this time we are, however, already in possession of a value of C_1/C_2 which should not be too far from the true value of K; and with this value two new solutions should be started: one on the assumption of the primary minimum to be an occultation eclipse, the other assuming it to be a transit. The alternative for which the iterations happen to converge is obviously the correct one and should hereafter be adopted.

When the types of the eclipses giving rise to the alternate minima have thus been identified (which, in some cases, may have been true at the outset of our analysis), the time has again come to invoke the full-dress equation (5-23) for the occultation eclipses, or (5-24) for transits, and to put our best previous approximation to k for the last time through the mill of these equations containing the differential corrections $\Delta\lambda_a$, $\Delta\lambda_b$ as well as ΔU. In carrying out a least-squares solution of a properly weighted system of such equations, care should be taken, not only to adjoin to it the additional equation of condition of the form (4-30) based on the observations of full light (equation 4-31 encountered in connection with total eclipses has no counterpart in the present problem), but also to include the equations of condition based on observations of both minima into the same normal system. A simultaneous solution of both minima, which was optional for totally eclipsing systems, becomes mandatory if the eclipses are partial and of comparable weights; for a single minimum by itself could not disclose anything definite about the elements of the system. It is only if the secondary minimum is relatively shallow, and its relative weight small, that its depth alone may be used for the determination of the elements; but, in such cases, the adopted depth of the secondary minimum is not susceptible of any significant improvement (since the weight of $\Delta\lambda_2$ as deduced from the primary minimum alone is then apt to be vanishingly small). It may be added that, if both eclipses are shallow and of comparable depths, the solution for the geometrical elements is known to be notoriously indeterminate unless a 'spectroscopic' k can be secured (*cf.* section VI.7) to add weight to our solution and restore its determinacy.

Once recourse to the full-dress equations (5-23) or (5-24) has been made, it is not advisible to continue its iteration by the method of least-squares because, with five unknowns, such a task would represent quite a laborious proposition. Instead, the computor should next fall back on equation (6-33p) and solve it for the k-derivatives of C_1 and C_2. As in the case of total eclipses, he will find that it is only the absolute terms in the equations of condition of the form (6-33p) (and the absolute terms in the corresponding normal equations) which must be computed anew for this purpose; the matrix of the coefficients on the left-hand side being the same as in the solution for C_1 and C_2. When the derivatives dC_1/dk and dC_2/dk are thus available, we are in a position to evaluate dk/dK from (6-26) and, with the aid of this latter quantity, to obtain the definitive (barred) values of C_1 and C_2 by means of equations (6-35). Next, we establish the definitive values of the maximum obscurations $\alpha'(k, p_0)$ and $\alpha''(k, p_0)$ at the bottom of each minimum by inserting in equations (5-11) or (5-12) the definitive value of k and the corrected values of λ_a and λ_b (expressed in terms of the corrected unit of light). With the aid of the definitive values of k and α_0 we enter then the Tsesevich p-table appropriate for the proper type of the eclipse and the accepted degree of darkening of the eclipsed star, to obtain the corresponding value of $p(k, \alpha_0)$.

A knowledge of the definitive values of C_1, C_2 and p_0 puts us, at last, in a position to evaluate the most probable values of the individual elements of our eclipsing system by means of the formulae collected in section VI.5. Any adjustment of such elements for a small change Δu in the degree of limb-darkening underlying our solution can be done by the methods of section VI.6 which are closely analogous to those discussed previously in connection with the total eclipses; and the same is true of the computation of light residuals for each particular normal, as well as of the uncertainty with which the individual elements are defined by the available observational data—except perhaps for the increased importance of the derivative dk/dK, which becomes one of the most revealing auxiliary quantities of our solution. A discussion of section VI.6 has made it evident that as long as this derivative remains numerically small (which will, in general, be true if the eclipses are total or annular), the errors of the individual elements of our system can be diminished almost arbitrarily by increasing the number (or precision) of the underlying observational data. If, however, dk/dK becomes large enough to approach unity (shallow partial eclipses!), the elements of the respective system become inherently indeterminate—regardless of the amount or precision of the observations. In such cases, the only useful information we can obtain (apart from the ratio of surface brightnesses of the two components) is stored in the auxiliary constant C_2, which may retain its significance long after C_1 has become effectively insignificant. This implies (*cf.* section VI.6) that, in general, the product of the fractional radii of both components can be obtained from observed light changes of partially eclipsing systems with considerably greater accuracy than their ratio, and may remain significant long after k has ceased to be so.

If the relative orbit of the components happens to be eccentric—as evidenced by the displacement and (or) unequal duration of both minima—the foregoing methods of analysis are complicated by the fact that the maximum geometrical depth p_0 at the alternate minima ceases to be the same. The solution must then proceed by way of successive approximation as described in section VI.10, and recourse to differential corrections becomes indispensable whenever the orbital eccentricity happens to be so large that the asymmetry of the light changes observed within minima can no longer be ignored.

Total or Partial Eclipses of Distorted Stars

Suppose now that an inspection of the observed data has revealed that the light of the system between minima does not remain constant, but exhibits a continuous variation due to the proximity effects (ellipticity and reflection)—as on Figs. 6-9 or 6-10. If this is so, a determination of the elements from an analysis of the light curve takes a different course.

As the first step of such an analysis, we shall proceed to fit the light changes exhibited between minima to an equation of the form (11-3) (or rather 11-3′) and use all available normal points pertaining to the phase θ to determine the values of c_0, c_1, c_2, and from them B and C by the method of least-squares. Next we shall 'rectify' the observed light changes throughout the whole cycle by means of equation (11-9)—or, if a good set of preliminary elements has been derived by some previous investigator, which should enable us to estimate the values of a, b and γ, we can rectify by means of equation (11-7). As is well known, the aim of the rectification is to eliminate the light variation between minima by dividing the observed light by an amount required to offset the varying cross-section of distorted components exposed to the observer at different phases, and by adding to it the amount of light required to complete the phase of each component as illuminated by its mate to 'full'. The rectified light l_{rec} of the system between minima will remain constant and equal to the actual brightness of the system at quadratures, while within minima the rectified light curve should closely simulate one produced by the eclipses of spherical disks of radii equal to the semi-major axes of the actual distorted ellipsoids. If so, the intermediary methods of solution can be invoked almost literally to yield the corresponding set of intermediary elements of a distorted eclipsing system, which (for moderately distorted systems) will require but minor subsequent modification to yield the definitive set of the elements.

In order to do so, we set up as many equations of condition of the form (11-12) if the eclipses are total (annular), or (11-13) if they are partial, as there are normals observed within minima and solve them for the auxiliary constants D_1, D_2 or D_3. In evaluating the E's in the individual equations of condition we should tentatively set $z = C$ (equation 11-17), and estimate the value of θ'' involved in E'' as well as we can from a plot of our observations. The intrinsic part of the weights $\sqrt{w}$ to which the individual equations of condition of the form (11-12) or (11-13) are entitled is, furthermore, the same as in section VI.4. When a least-squares solution of a properly weighted set of the equations of condition has been performed, the corresponding values of the geometrical elements should be determined by means of equations (11-20). These elements, together with the adopted values of the mass-ratio and the coefficients of limb- and gravity-darkening, should then be harmonized with the observed values of the rectification constants by the method of section VI.12. Moreover, the $O-C$ residuals of the absolute terms of each one of our equations of condition should be evaluated and converted into the corresponding light residuals Δl by the method of section VI.8.

Unlike the case (*a*), the intermediary elements of distorted eclipsing systems should never be identified with the definitive elements, but regarded as a mere intermediary stage of our analysis. Once we have obtained them, we should proceed to evaluate, with their aid.

(*a*) The 'errors of the rectification' Δl_{rec},

(*b*) the 'perturbations within minima' Δl_{per},

as defined in section VI.12, add their absolute amounts and apply them, with the opposite sign, to the intermediary O–C light residuals. As long as the sum $\Delta l_{rec} + \Delta l_{per}$ remains small (say one-tenth of the observed light range or less), their effect upon a determination of the geometrical elements can be ascertained by

(1) setting up a system of the equations of condition of the form (8-6)–(8-7) for an evaluation of the differential corrections to the individual elements, as described in section VI.8 (with the sole exception of the formation of δ, for which *cf.* equation 11-10 of section VI.11);

(2) replacing the absolute terms of (8-6) by the intermediary O–C residuals from which we have subtracted $\Delta l_{rec} + \Delta l_{per}$;

(3) solving a properly weighted set of such equations of condition by the method of least-squares; and

(4) applying the resulting corrections to the respective values of the intermediary elements. The relation between such results and the definitive values of the individual elements is ultimately provided by equations (12-65)–(12-68) of section VI.12.

For moderately distorted systems, the sum of the perturbations $\Delta l_{rec} + \Delta l_{per}$ will consist perhaps of two significant figures throughout most of the eclipse. If so, a good deal of time and labour in the application of such perturbations may be saved if we evaluate first the total amount of the perturbations for a few pivotal points at regular intervals of the phase, plot them against the phase, and then obtain their particular values for each observed normal to the desired accuracy from a smooth curve drawn through the pivotal points. The labour-saving feature of this device becomes obviously the greater, the greater the number of the observed normal points; and its application is facilitated by the fact that, barring exceptional cases, the perturbations are found to vary smoothly with the phase and exhibit no singular points in the domain with which the investigator will be usually concerned.

If, however, the sum $\Delta l_{rec} + \Delta l_{per}$ becomes appreciable in comparison with the observed depths of the respective minima (very close eclipsing systems), the foregoing process (1)–(4) may have to be repeated (each time on the basis of the previously corrected set of the elements), the matrix of coefficients of the differential corrections recomputed (which constitutes a rather laborious step of our process), or the iterations may even refuse to converge to a definite answer. If so, the limit has been reached of what can be accomplished by perturbation methods as they have been developed so far, and the problem of determining the elements of such eclipsing systems becomes one whose solution will call for a different method of approach. One such possible method is based on the closed properties of the Roche model (Chapter III); but its elaboration represents a task and a challenge which remains yet to be taken up by future investigators.

Having progressed thus far, we now must part company with the reader and wish him God-speed in the pursuit of his own investigations. In concluding the present chapter on the computation of photometric elements of eclipsing variables, the writer wishes to emphasize that its text cannot claim to offer more than an introduction to the study of the subject rather than any exhaustive treatment. It should be sufficient to accompany the reader to the front-lines of further research, where a great deal of activity is currently in progress. In particular, an interpretation of the light curves of close eclipsing systems is still at the very beginning and various possibilities of application to practical cases are far from being exhausted. It is needless to stress, however, that a study of the book itself cannot replace the actual experience which can be gained only by independent investigation.

The apparent complexity of some parts of our analysis may have perhaps acted as a deterrent to the less interested readers, but for this fact the writer is prepared to offer no apology. For it is a fact that a solution for the photometric elements of eclipsing binary systems poses some of the most difficult and delicate problems of double-star astronomy. The number of elements to be determined by a simultaneous analysis of the light changes is large, and their relation to the observed quantities is intricate. Moreover, the underlying photometric observations cannot be made indefinitely accurate, and their number must compensate for their relatively low proportional precision. Such data must be treated by well-known methods of the combination of observations as developed in older branches of theoretical astronomy. The specific difficulties of our problem are, as we have seen, met in two ways: first, by the introduction of certain sets of auxiliary constants in terms of which the actual elements can later be expressed; and second, by the introduction of a class of special functions which simplify the relations between the observed quantities and the auxiliary constants. Even with the help of such functions, the task of solving for the elements or the auxiliary constants from the observed data turns out to be by no means simple. Throughout preceding sections, the complexity of this task has perhaps been exaggerated by the fact that, in many instances, more than one method was given which should lead to essentially the same results. In this last section we have attempted to place all such methods and devices in proper proportion; and while an experienced investigator may be able to short-cut certain steps of our process without impairing the quality of the results, the writer suggests that this be done with caution and that certain crucial results should be evaluated, whenever possible, by different methods—even if only as a safeguard against accidental errors and for the satisfaction of knowing that an involved astronomical problem is computationally well in hand.

Suppose, however, that the investigator took all necessary precautions to treat his observational evidence impartially by the methods discussed earlier in this book and succeeded in iepresenting the observations by a consistent set of the elements within a certain margin of errors. Would he be justified in concluding that the remaining light residuals are due wholly to the errors of observation? In most cases, the answer should probably be in the negative; and the reasons which lead us to assert it should be set forth in some detail. In most problems of theoretical astronomy it is the principal objective of the investigator to fit the observed facts to certain physically reasonable models.

The well-known dynamical model consisting of two or more mass-points, for example, offers an exceedingly close approximation to the actual motions of celestial bodies of our solar system. In many astrophysical problems our working models may, on the other hand, be as yet very insecure. Now we have every reason to believe that the model which we used as a basis of our investigation of the elements of eclipsing binary systems in this book (with refinements discussed in section VI.12) should offer a remarkably good approximation to reality—much better than we can hope for in most other problems of stellar astronomy. Nevertheless, it should be recognized that a number of additional physical phenomena are likely to be present in close binary systems of which our present theory of the light curves takes as yet no account. Of such phenomena, we may mention possible effects invoked by the dynamical tides, which may share responsibility for the asymmetric light-curves of many eclipsing systems. We may quote possible photometric effects of complicated gas streams enveloping one or both components of many eclipsing systems, which have been investigated, in recent years, by Struve and his associates at the Yerkes and McDonald Observatories with such success. We may add the effects of 'spots' on the surfaces of early-type components of several eclipsing systems discovered in recent years by Kron and his collaborators at Lick.

Should we, in view of a possible presence of additional physical phenomena complicating the normal state of affairs, compromise the standards of accuracy of our analysis and abstain from the application of various refinements discussed earlier in this book? Unless the amplitude of the light changes invoked by physical complications is really large, the answer is definitely in the negative; for *the secondary physical phenomena, complicating our model, cannot be brought into the open until the major effects, which are bound to be present, have first been removed from the observed light changes.* A neglect of small secondary phenomena in our solution may, however, affect the photometric elements to some extent and increase their uncertainties as calculated by the methods of this chapter. In systems in which the presence of such effects is at least suspected, the investigator should bear this in mind.

From the methodical point of view, the processes discussed in sections 4–7 of this chapter may represent, in a certain sense, both the culmination and the end of an epoch in the development of our branch of double-star analysis. Similarly as the work by Russell and Shapley between 1912 and 1914 represented probably the culminating point in the development of graphical or semi-graphical methods for the determination of elements of eclipsing binary systems, based on the use of free-hand curves—so the investigations of the past decade, summarized in the present chapter, may mark a similar high point in the development of analytical methods adapted for the use of desk-type computing machines. It is, however, likely that even this represents but a temporary stage in the further development of our subject. For with the increasing accuracy and amount of the observational data—such as provided by modern recording photoelectric photometers—the burden of the numerical work involved in the present methods is apt to become prohibitive to the observer, and may overtax the facilities at his disposal. The situation in which the investigator of the elements of eclipsing binary systems will soon find himself may be compared with that which confronted the computer of the orbits of comets or asteroids a quarter of a century ago. The example set by these older branches of classical astronomy indicates the direction in which the future development of eclipsing orbit work will probably be heading: namely, towards an increasing automatization of the underlying numerical processes and their adaptation to large-scale digital machines—particularly to such types of automatic machines as may be commercially available. The logic of such processes and their coding for any given type of machine opens up a new field of research which may occupy an important place in future monographs on the computation of elements of eclipsing binary systems. On the other hand, the observer of the future is likely to become primarily an expert in intricate photoelectric techniques of light measurement, whose complexity may grow in proportion to the increasing accuracy of the results. As the pioneer days of the development of our subject are gradually receding into the past, the sheer weight of technicalities thus threatens to convert a growing separation between the observer and the computer into a permanent divorce—whether for better or worse, only the future can tell.

VI. BIBLIOGRAPHICAL NOTES

VI.1: Four-digit tables giving the light intensity l in terms of the magnitude differences $m = 0{\cdot}00(0{\cdot}01)2{\cdot}59$ can be found, for example, in K. Schiller's *Einführung in das Studium der Veränderlichen Sterne*, Leipzig 1923, p. 371; or J. Stein's *Die Veränderlichen Sterne*, Freiburg 1924, p. 347; or R. G. Aitken's *The Binary Stars*, New York 1935, p. 191. The same table was also included among other auxiliary tables by H. N. Russell in *Ap. J.*, **35**, 315, 1912 (p. 330); and was reprinted by F. C. Henroteau in his contribution to the *Handbuch der Astrophysik*, Bd. VI (Zweiter Teil), Berlin 1928, p. 246. The most extensive and convenient conversion table of this kind which has appeared so far is that by E. Schoenberg, and can be found in the Appendix to his article on theoretical photometry in *Handbuch der Astrophysik*, Bd. II (Erste Hälfe, Zweiter Teil), Berlin 1929 (Tafel 1a, pp. 235–239). This table gives 4D values of l for $m = 0{\cdot}000(0{\cdot}001)2{\cdot}090$ and $2{\cdot}0(0{\cdot}01)5{\cdot}09$, and requires scarcely more than linear interpolation in its use to four decimals. It has been reproduced (in complement form) by J. E. Merrill on pp. 340–344 of his *Princeton Contr.*, No. 23, 1953.

For a most complete treatment of the determination of the times of the minima of eclipsing (and other) variables *cf.*, e.g., J. G. Hagen, *Die Veränderlichen Sterne*, Freiburg 1921, pp. 561–670. A shorter treatment of the same subject can be found in K. Schiller's *Einführung*, *op. cit.*, pp. 209–227.

VI.2: Formulae expressing the fractional loss of light due to eclipses of uniformly bright stars as a function of the phase were first given by E. C. Pickering in *Proc. Amer. Acad. Sci.*, **16**, 1, 1880; while the fractional loss of light due to eclipses of limb-darkened stars was first rigorously evaluated by P. Harzer (*Kiel Publ.*, No. 16, 1927) and subsequently (for partial as well as annular eclipses of any type) by V. P. Tsesevich (*Publ. Univ. Obs. Leningrad*, **6**, 48, 1936). The concept of the geometrical depth $p(k, \alpha)$ of the eclipse as an inverse function of the fractional loss of light $\alpha(k, p)$ is due to H. N. Russell (*Ap. J.*, **35**, 315, 1912). This p-function has proved singularly useful in all practical applications, and has remained a permanent feature of all subsequent work.

The first table of the α-functions for uniformly bright disks was published by M. Wend ('Eine Tafel zur Theorie der Bedeckungsveränderlichen', Diss. Leipzig 1931) to 4D as function of the parameters k and δ/r_b; and its inverse $\sqrt{w} = 1 + kp$ in terms of k and α by E. Hetzer (Diss. Leipzig, 1931). K. Ferrari (*Sitzungsber. d. Akad. Wiss. Wien, Abt.*

IIa, **147**, 497, 1938; and **148**, 217, 1939) published 4D tables of $\alpha(k, p)$ relevant to completely darkened disks for both occultation and transit eclipses, and partial as well as annular phase. In addition, he gave a 4D table of the function $\frac{3}{2}k^2\Phi(k)$ as defined by our equation (2-14). The most extensive and accurate existing tables of the α- as well as p-functions have been published by V. P. Tsesevich in *Bull. Astr. Inst. U.S.S.R. Acad. Sci.*, No. 45, 1939, and No. 50, 1940. Tsesevich's work included 5D tables of $\alpha(k, p)$ for uniform and completely darkened disks—in the latter case for both occultations and transits, and partial as well as annular eclipses. In addition, he gave 5D tables of $p(k, \alpha)$ for uniform and completely darkened disks, and 4D p-tables for partial darkening (corresponding to $u = 0{\cdot}2$ $(0{\cdot}2)0{\cdot}8$). Darkened p-tables are again provided for both occultations and transits, as well as partial and annular eclipses. The intervals of tabulation are the same as in Ferrari's work (namely, 0·01 in p and 0·05 in k). Introductions to both sets of tables (in Russian) contain much valuable information concerning the formal properties of the respective functions. Tsesevich also included within the scope of his tabular work some auxiliary functions, such as $\Phi(k)$ which he tabulated to 6D for $k = 0{\cdot}20(0{\cdot}01)1{\cdot}00$.

Unlike for the p-functions, Tsesevich did not give explicit tables of the α-functions for intermediate degrees of limb-darkening (as these can be obtained by simple linear interpolation between the two limiting 'uniform' and completely darkened cases). Such tables (accurate to 4D) were, however, subsequently provided by J. E. Merrill in *Princ. Contr.*, No. 23, 1950.

For an exhaustive review and critical survey of all existing tables discussed in this section and published up to 1947 *cf.* Z. Kopal, *Journ. on Math. Tables and other Aids to Computation*, **3**, 191, 1948.

The equations relating the fractional losses of light at the alternate minima with the geometry of eclipses were derived by H. N. Russell (*Ap. J.*, **35**, 315, 1912) for uniformly bright disks, and by H. N. Russell and H. Shapley (*Ap. J.*, **36**, 239 and 385, 1912) for disks completely darkened at the limb. Their generalization to partially darkened stars presented in this section is due to Z. Kopal, *Ap. J.*, **94**, 145, 1941.

VI.3: Early attempts at determining the photometric elements of eclipsing systems from an analysis of their light curves by direct methods were concerned with specific variables—ordinarily Algol—rather than with laying down the general principles applicable to any particular system.

Historically the first attempt at interpreting the light changes of Algol on eclipse hypothesis* should be credited to E. C. Pickering, who in a paper published in *Proc. Amer. Acad. Sci.*, **16**, 1, 1880, derived the relative dimensions of the components of the Algol system and inclination of their orbit to the celestial sphere assuming that (1) both stars were spherical, (2) the primary component appeared in projection as a uniformly bright disk, (3) its companion was completely dark, and (4) the eclipses were grazing. The details of Pickering's procedure and his numerical results are to-day only of historical interest. He was, however, the first to point out that (under certain restricting conditions) the problem of light change analysis admits indeed of a solution; and his paper contains implicitly so many features common to all subsequent investigations that it should rightly mark the beginning of development of the theory of our subject.

Subsequently, J. Harting, in his 'Untersuchung über den Lichtwechsel des Sterns β Persei' (Diss. München, 1889) repeated Pickering's investigations abandoning his hypothesis of grazing eclipse; while E. Hartwig (*A.N.*, **152**, 309, 1900), in an investigation of the eclipsing system Z Herculis, was led to consider also its secondary component as a luminous body. G. W. Myers, in his 'Untersuchungen über den Lichtwechsel des Sterns β Lyrae' (Diss. München, 1896) and in *Ap. J.*, **7**, 1, 1898, appears to have been the first to consider a determination of the elements of distorted eclipsing systems of the β Lyrae-type by assuming that their components are uniformly bright and similar prolate spheroids. He was soon joined in this endeavour by A. W. Roberts (*M.N.*, **63**, 527, 1903) who in this and subsequent papers carried out applications to several practical cases. Further advances were made in A. Pannekoek's 'Untersuchungen über den Lichtwechsel Algols' (Diss. Leiden, 1902) and C. Rödiger's 'Untersuchungen über das Doppelsternsystem Algol' (Königsberg, 1902)

* For the latter had remained a hypothesis until H. C. Vogel (*A.N.*, **123**, 289, 1890) recognized Algol as a spectroscopic binary whose conjunctions coincided with the minima of light.

who both simultaneously and independently considered the effects, upon light changes, of stellar limb-darkening. These and other effects were later studied by S. Blažko (*Ann. de l'Obs. Astr. de Moscou* (*2ème Serie*) **5**, 76, 1911; and by R. S. Dugan (*Princ. Contr.*, No. 1, 1911). A comprehensive review of these early investigations of the first period of development of our subject can be found, for instance, in Ch. André's *Traité d'Astronomie Stellaire*, Paris 1900, vol. 2; or in J. Stein's treatise on *Die Veränderlichen Sterne*, Freiburg 1924.

The methods by Russell and Shapley described in this section have been published by their authors in the following papers: H. N. Russell, *Ap. J.*, **35**, 315; **36**, 54, 1912; H. N. Russell and H. Shapley, *Ap. J.*, **36**, 239, 385, 1912. In addition, an extensive application to practical cases has been carried out by H. Shapley in *Princ. Contr.*, No. 3, 1915 (Diss. Princeton, 1914). More recently these methods have been written up, with many modifications, by H. N. Russell and J. E. Merrill in *Princ. Contr.*, No. 26, 1952; and new sets of the auxiliary ψ- and χ-functions for different degrees of darkening have been published by J. E. Merrill in *Princ. Contr.*, No. 23, 1950–53; nomographs to facilitate their use have been prepared, also by Merrill, in *Princ. Contr.*, No. 24, 1953.

Of investigations subsequent to the 1912 papers by Russell and Shapley and developing or modifying their methods in different respects we may quote J. Weber, 'Zur Theorie der Algolveränderlichen' (Diss. Göttingen, 1913); H. Vogt (*Heidelberg Veröff.*, **7**, 183, 1919); J. Fetlaar (*Utrecht Recherches*, **9**, pt. 1, 1923; or Diss. Utrecht, 1923); J. Stein, (*B.A.N.*, **2**, 123, 1924); B. W. Sitterly (*Princ. Contr.*, No. 11, 1930; Diss. Princeton 1922; S. Scharbe (*Pulkovo Bull.*, **10**, No. 94, 1925); W. A. Krat (*Astr. Zhurnal.* **11**, 407, 1934; **12**, 21, 1935; **13**, 521, 1936); S. L. Piotrowski (*Acta. Astr.*, (*a*) **4**, 1, 1937); J. Ellsworth (Lyon Publ., (1) **2**, Fasc. 1, 1936) or O. E. Brown (*A.J.*, **47**, 93, 1938), and others.

Kopal's method described in the second part of this section has been published in *Ap. J.*, **94**, 145, 1941; *cf.* also Kopal's *Introduction to the Study of Eclipsing Variables*, Cambridge 1946, sections 15 and 26.

VI.4: The iterative methods for a determination of the elements of eclipsing binary systems, developed in this section, were initiated by Z. Kopal in *Ap. J.*, **94**, 145, 1941, and described later in sections 17 and 23 of his *Introduction to the Study of Eclipsing Variables* (Harvard Univ. Press, Cambridge 1946), or in sections 3.5–3.13 of his *Computation of the Elements of Eclipsing Binary Systems* (*Harv. Obs. Mono.*, No. 8, 1950). Of other basic investigations which materially contributed to the outgrowth of the present subject *cf.* S. L. Piotrowski, *Ap. J.*, **106**, 472, 1947; **108**, 36, 510, 1948; or Z. Kopal, *Ap. J.*, **108**, 46, 1948.

VI.5: *Cf.* Z. Kopal, *Computation of the Elements of Eclipsing Binary Systems* (*Harv. Obs. Mono.*, No. 8, 1950), sections 3.14–3.18, and other references quoted in the preceding section.

VI.6: *Cf.* Z. Kopal, *Computation of the Elements of Eclipsing Binary Systems* (*Harv. Obs. Mono.*, No. 8, 1950), sections 3.19–3.21; or S. L. Piotrowski, *Ap. J.*, **108**, 36, 510, 1948; also Z. Kopal, *Ap. J.*, **108**, 46, 1948.

VI.7: *Cf.* Z. Kopal, *Computation of the Elements of Eclipsing Binary Systems* (*Harv. Obs. Mono.* No. 8, 1950), sections 3.22–3.24; and S. L. Piotrowski, *Ap. J.*, **108**, 36, 510, 1948; or Z. Kopal, *Ap. J.*, **108**, 46, 1948; also R. M. Petrie, *Harvard Centennial Symposia* (*Harv. Obs. Mono.*, No. 7, 1948) pp. 231*ff*, or *Publ. D.A.O.*, **7**, 205, 1939, and **8**, 319, 1950.

VI.8: An application of least-squares techniques to an adjustment of photometric elements of eclipsing binaries is of an early date, and goes back in principle to G. W. Myers (Illinois Obs. Bull., No. 1, 1898), A. W. Roberts (*M.N.*, **68**, 490, 1908), or H. N. Russell and H. Shapley (*Ap. J.*, **39**, 405, 1914).

A. Pannekoek and E. van Dien (*B.A.N.*, **8**, 141, 1937) worked out an application of least-squares to the determination of stellar limb-darkening by the variation of constants. They did not, however, include the depths of the minima among the quantities to be simultaneously adjusted. The first adequate treatment of the problem was given by A. B. Wyse in *Lick Bull.*, No. 494, 1939; but in this study the geometry of the problem was solved in a finite number of terms only for uniformly bright disks; while the case of limb-darkened stars, not tractable in terms of elementary functions, was dealt with only by approximate methods.

The determination of the coefficients of equation (8-7) by an appropriate differentiation of the α-tables has been developed by Z. Kopal in *Proc. Amer. Phil. Soc.*, **86**, 342, 1943. Of subsequent investigations, an exhaustive study of J. B. Irwin (*Ap. J.*, **106**, 380, 1947) deserves particular notice. It contains a set of 29 auxiliary bi-variate tables of the requisite coefficients, with arguments of tabulation $k = 0(0{\cdot}1)0{\cdot}8(0{\cdot}05)1$, $p = -1(0{\cdot}05) - 0{\cdot}8(0{\cdot}1)$ $0{\cdot}8(0{\cdot}05)1$ if the eclipse is an occultation ($r_1 = r_a$); and $k = 0{\cdot}2(0{\cdot}1)0{\cdot}8(0{\cdot}05)1$, $p = -1$ $(0{\cdot}05) - 0{\cdot}8(0{\cdot}1)0{\cdot}8(0{\cdot}05)1$ during a transit while, during annular phase, $q = 0{\cdot}1(0{\cdot}1)1$. The intervals of tabulation in both k and p or q were chosen small enough to justify linear interpolation over most part of the tables; except for tables of $\partial f/\partial u$, additional columns for $k = 0{\cdot}975$ and $p = -0{\cdot}975$ were inserted to facilitate this task. The following key will permit us to identify the appropriate number of Irwin's table containing the desired quantity (expressed in terms of our present notations; Irwin's notations differ to some extent from ours).

I. *Occultation Tables* (smaller star eclipsed; $r_1 = r_a$)

Tabulated quantity	$u = 0$	0·4	0·6	1
$r_a \dfrac{\partial f}{\partial r_b}$	1A	9	17	1B
$r_a \dfrac{\partial f}{\partial r_a}$	2A	10	18	2B
$\dfrac{r_a r_b}{\cos^2\theta \sin 2i}\dfrac{\partial f}{\partial i}$	3A	11	19	3B
$\dfrac{\partial f}{\partial u}$		12	20	

II. *Transit Tables* (larger star eclipsed: $r_1 = r_b$)

Tabulated quantity	$u = 0$	0·4	0·6	1
$\dfrac{r_b}{k}\dfrac{\partial f}{\partial r_b}$	5A	13	21	5B
$\dfrac{r_b}{k}\dfrac{\partial f}{\partial r_a}$	6A	14	22	6B
$\dfrac{r_b}{2k\cos^2\theta \sin 2i}\dfrac{\partial f}{\partial i}$	3A	15	23	7
$\dfrac{\partial f}{\partial u}$		16	24	

In addition, Tables 4 and 8 list four-decimal values of

$$\frac{(3-u)}{6}\frac{\partial f}{\partial u}$$

for the case of an occultation and transit, respectively, which can be used to evaluate the coefficient of Δu in equation (8-7) for any arbitrary degree of darkening.

The present version of the subject as given in this section leans heavily on Z. Kopal, *Proc. Amer. Phil. Soc.*, **86**, 342, 1943, and subsequent developments presented in the writer's *Introduction to the Study of Eclipsing Variables* (Harv. Univ. Press, Cambridge 1946), sections 29–32; or his *Computation of the Elements of Eclipsing Binary Systems* (*Harv. Obs. Mono.*, No. 8, 1950), sections 4.1–4.9.

Of other investigations dealing with the subject of the present section we may mention W. A. Krat, *Zs. f. Ap.*, **5**, 60, 1932 and **6**, 96, 1933; A. Hnatek, *A.N.*, **261**, 361, 1936 and **272**, 159, 1942 (*cf.*, however, the comments by A. Pannekoek and E. van Dien in *B.A.N.*, **8**, 141, 1937) or by S. L. Piotrowski, *Acta Astr.* (*c*) **3**, 29, 1937; or Y. Hosokawa, *Publ. Astr. Soc. Japan*, **1**, 73, 1949.

VI.9: The effects, upon light changes, of the orbital eccentricity have received attention of early investigators of our subject—mainly in connection with the classical variable

Y Cygni (discovered in 1886) exhibiting appreciable eccentricity. Thus our formulae for relative displacement and unequal durations of the alternate minima, based on the theory of elliptic motion, were given (though not in their present form) already by N. C. Dunér in *Ap. J.*, **11,** 175, 1900; or by Ch. André in the second volume of his *Traité d'Astronomie Stellaire*, Paris 1900, pp. 240-250. A comprehensive account of most of this early work can be found in J. Stein's *Die Veränderlichen Sterne*, Freiburg 1924, pp. 212–217.

The present version of our subject as given in this section leans heavily on sections 35–40 of the writer's *Introduction to the Study of Eclipsing Variables* (Harvard Univ. Press, 1946), and sections 5.1–5.5 of his *Computation of the Elements of Eclipsing Binary Systems* (*Harv. Obs. Mono.*, No. 8, 1950).

Of other investigations dealing with our subject we may mention J. Uitterdijk, *B.A.N.*, **6,** 241, 1931; W. A. Krat, *Per. Zvjozdy*, **4,** 97, 1933, and *Astr. Zhurnal*, **12,** 21, 1935; or M. P. Savedoff, *A.J.*, **56,** 1, 1951; J. de Kort, *Ric. Astr. Vatican*, **3,** 109, 1954; or *Vistas in Astronomy*, **2,** 1187, London 1956; and others.

VI.10: The problem of conversion between the 'circular' and 'elliptical' elements of eclipsing binary systems was opened up by H. N. Russell (*Ap. J.*, **36,** 54, 1912) who set up the requisite conversion formulae correctly to first powers in orbital eccentricity. Russell's formulae were subsequently superseded by those of T. E. Sterne (*Proc. U.S. Nat. Acad. Sci.*, **26,** 37, 1940) which are correct to squares of the orbital eccentricity and which have been reproduced in this section.

The present version of our subject follows closely that given by the writer in sections 41–42 of his *Introduction* (1946), and sections 5.6–5.8 of his *Computation* (1950). A simplified treatment of the differential correction (correct to first powers in orbital eccentricity) may be found in A. Pannekoek and E. van Dien, *B.A.N.*, **8,** 141, 1937, or in A. B. Wyse, *Lick Bull.*, No. 496, 1939.

VI.11: The technique of 'rectification' of the light changes of close eclipsing systems between minima was introduced in astronomical practice apparently by H. N. Russell and H. Shapley (*Ap. J.*, **36,** 54, 385, 1912); but its version as given in this section follows Z. Kopal's *Computation of the Elements of Eclipsing Binaries* (Cambridge, Mass., 1950), sections 6.1–6.3.

The legitimacy of basic assumptions on which rectification rests was not investigated by H. N. Russell until thirty years later in a paper published in *Ap.J.*, **95,** 345, 1942, whose text contains, unfortunately, many slips and misprints. A corrected version of Russell's work can be found in section 78 of Kopal's *Introduction to the Study of Eclipsing Variables*. In a subsequent investigation Z. Kopal (*Proc. Amer. Phil. Soc.*, **89,** 517, 1945) proved that the most general sufficient condition for the legitimacy of the rectification—namely, the symmetry of the isophotae on apparent disks of ellipsoidal configurations with respect to their centre—will be fulfilled if the coefficients of limb- and gravity-darkening are related by

$$u = \frac{8\beta_2}{5 + 3\beta_2},$$

where the constant β_2 is defined by equation (2-31) of Chapter IV. The cases $u = \tau_0 = 0$ (uniformly bright disks) or $u = 1$, $\tau_0 = (5\Delta_2^{-1} - 1)^{-1} \approx 0{\cdot}25$ (full limb- and partial gravity-darkening) represent particular examples of such a situation. However, even the foregoing condition is sufficient to ensure the symmetry of the isophotae only for second-harmonic rotational or tidal distortion; and with the emergence of the third harmonic this symmetry will be irrevocably lost whatever the amount of limb- or gravity-darkening.

VI.12: An application of the 'errors of rectification' between minima, or of 'perturbations' within eclipses due to the impossibility of complete rectification, represents a recent chapter in the development of our subject. Its present version as given in this section is largely new; for previous investigations *cf.* S. Takeda, *Kyoto Mem.*, *A*, **20,** 47, 1937; Z. Kopal, *Proc. Amer. Phil. Soc.*, **85,** 399, 1942 (section VII); or the latter's *Introduction to the Study of Eclipsing Variables*, sections 79–81, and *Computation of the Elements of Eclipsing Binary Systems*, sections 6.4–6.11. For auxiliary tables to facilitate the application of photometric 'perturbations' *cf.* S. Takeda and R. Kamiya, *Kyoto Mem.*, *A*, **20,** 47, 1937 (pp. 67–86); and Y. Hosokawa, *Science Reports of the Tohoku Univ., Sendai,* (1) **39,** 110, 1955.

Appendix

SOLUTION OF LEAST-SQUARES SYSTEMS AND COMPUTATION OF THE ERRORS

IN THE discussion of the determination of elements of eclipsing binary systems from an analysis of their light curves, as outlined in the preceding chapters, one computational operation was met repeatedly: namely, the solution of simultaneous systems of linear algebraic equations, with varying number of the unknowns, by the method of least-squares. Least-squares solutions of such systems were encountered at an early stage of computation of the intermediary elements (sections VI.4, 5, 6) and were again later found essential to an improvement of intermediary elements by way of differential corrections (section VI.8), and to the application of the perturbations by the method of section VI.12. Such solutions will, therefore, always constitute a large part of the computational routine involved in an analytical determination of the elements of eclipsing binary systems, and the amount of time required to carry out a complete solution will, consequently, depend to a large extent on the choice of techniques by which the solutions of the requisite systems of linear equations are performed. In view of the fact that the treatment of this subject such as can be found in most existing textbooks is still largely antiquated, we shall, in the first part of this Appendix, summarize briefly some of the modern methods which appear, at present, to be the most suitable to this end. For an expert in the theory of linear algebraic equations this part should hold little that was not known to him before; for it was written primarily for the benefit of the general astronomical reader who is not a specialist in applied mathematics. The second part of this Appendix will be devoted to a discussion of the methods by which the uncertainty of the elements, or of combinations of the elements, can be evaluated from known properties of our least-squares systems. The way in which this can be done has already been indicated in sections VI.6 and 8; but a discussion of its specific details had to be postponed for the present Appendix to follow the exposition of our least-squares techniques.

The solution of systems of linear equations characterized by a symmetry about the principal diagonal has been of particular interest to astronomers ever since the introduction of the method of least-squares into astronomical practice at the beginning of the nineteenth century; and the scheme most widely used to solve the normal equations continued for many decades to be that of Gauss [1]. This, in retrospect after more than one hundred years, appears to have been a major setback. There is no doubt that, in the derivation of his elimination method, Gauss's genius must have been momentarily dormant and rather deeply so; for the process was later recast in a much more logical form by much lesser mathematicians. Yet such was the weight of Gauss's authority throughout the greater part of the nineteenth century that efforts to improve upon his scheme were not taken up until 1878 by Doolittle [2], several generations after Gauss's method had reigned supreme. When one pauses to realize that this was all before the advent of computing machines —when all tedious operations had to be performed by logarithms or long-hand—the waste of time and effort entailed in the use of inferior techniques by at least three generations of astronomers and geodesists is grievous indeed to contemplate.

There is no doubt nowadays that the school-book methods of Cramer's rule or of Gaussian elimination in its original form are only of historical interest for the modern computer. Thanks to the efforts of Doolittle [2], de Sitter [3], Cholesky [4] and, in particular, of Banachiewicz [5], the Gaussian elimination method was gradually simplified to such an extent as to entail only a small fraction of the work originally needed. The most feasible existing scheme for the solution of normal equations of least-squares is that by Banachiewicz [5], and was discovered by him by the use of cracovian calculus. Three years later, the main features of Banachiewicz's algorithm were rediscovered independently by Crout [6] using the methods of ordinary matrix calculus; and seven years later the exact form of Banachiewicz's method was discovered by Dwyer [7]. It can be easily shown that, in so far as the determination of numerical values of the unknowns themselves is concerned, both Banachiewicz's and Crout's methods entail essentially the same number of arithmetical operations; though Crout's method calls for fewer divisions, no extraction of square-roots, and is in general more amenable to mechanization. When it comes, however, to the evaluation of the weights with which these unknowns (or any linear combination of them) are defined by the available data—and these will be of particular interest for us—Banachiewicz's method appears to be superior to all alternative schemes in so far as it yields, in effect, *all elements of the customary inverse matrix, which are required for the computation of weights of any linear combination of the unknowns, as simple by-products of the actual evaluation of these unknowns.* The advantage of this feature (eliminating completely the need of computation of any additional minors) is obviously the greater, the greater the number of the unknowns. Since in the problems encountered in eclipsing orbit work this number is usually in the neighbourhood of five, the choice of Banachiewicz's method for a solution of our systems of normal equations becomes, therefore, obvious.* As this method is still relatively new and not so well known as it deserves, we shall, in what follows, summarize briefly the arithmetical operations required for carrying out a complete solution; for their

* This assertion was recently questioned by Russell and Merrill (*Princ. Contr.*, No. 26, 1952; pp. 77*ff*), but their criticisms appear to be ill-founded. For these investigators limited their discussion to linear systems with only 3 unknowns, and in so simple a case *all* methods of their solution are essentially indistinguishable. It is only when one turns to larger systems (with 5 or more unknowns) that the differences between various methods will make themselves felt. The points raised by Russell and Merrill have since been answered by Banachiewicz (*Rocznyk Astr.*, No. 25, p. 7, 1954; *Acta Astr.*, (*a*) **5**, 151, 1954), and the author of the present book cannot but agree with his position.

proof the interested reader must be referred to Banachiewicz's original memoirs already quoted. In translating Banachiewicz's algorithms from the elegant formalism of cracovian calculus into the language of ordinary algebra we are bound to lose some generality, but hope to make them more intelligible for the non-mathematical reader.

Solution of the Normal Equations

The process of solution of the normal equations of a least-squares system which we propose to outline can, in effect, be summarized in two sentences: *if we regard the diagonally-symmetrical set of given coefficients of the system of normal equations as a matrix-cracovian, we first evaluate its triangular square-root, and find the reciprocal of this root. The last row of the inverse square-root of the given matrix-cracovian contains the values of all unknowns, while the elements of the preceding rows specify completely their weights.* Let the given matrix of coefficients of our normal equations be abbreviated by

$$\begin{array}{cccccccc|c}
R_{11} & R_{21} & R_{31} & R_{41} & R_{51} & \dots & R_{n1} & & S_1 \\
 & R_{22} & R_{32} & R_{42} & R_{52} & \dots & R_{n2} & & S_2 \\
 & & R_{33} & R_{43} & R_{53} & \dots & R_{n3} & & S_3 \\
 & & & R_{44} & R_{54} & \dots & R_{n4} & & S_4 \\
 & & & & & & \cdot & & \\
 & & & & & & \cdot & & \\
 & & & & & & \cdot & & \\
 & & & & & & R_{n,n-1} & & S_{n-1}
\end{array} \qquad \{\mathbf{R}\}$$

etc., where the S_j's to the right of the solid line indicate the respective elements of an auxiliary check column. Since a matrix of coefficients of any set of normal equations is characterized by a symmetry about the principal diagonal, the elements below this diagonal (omitted in the preceding scheme) are equal to the corresponding element above it (i.e., $R_{ij} = R_{ji}$).

Of the actual process of formation of the coefficients of normal equations from a given system of the equations of condition, little needs to be said in this place—except perhaps for the formation of the check column. The latter is an important feature of our process; for since the formation of the coefficients R_{ij} involves a fair amount of routine computation, safeguards should always be taken against the possibility of numerical errors which—if allowed to slip through—would vitiate all subsequent computations and completely escape detection until the very last stage of our analysis (i.e., in the formation of the $O-C$'s), thus forcing us eventually to repeat a large part of the work already performed. In order to forestall any such contingency, a single additional check column S_j should be evaluated in accordance with the following instructions. Suppose that there are at our disposal N equations of condition for $n - 1$ unknown quantities x_j of the form

$$a_{1k}x_1 + a_{2k}x_2 + a_{3k}x_3 + \dots + a_{n-1,k}x_{n-1} + a_{n,k} = 0, \tag{1}$$

where $k = 1, 2, 3 \ldots N \, (N > n)$. One way of evaluating the element S_j of the j-th row of the auxiliary column is to do so from

$$S_j = [a_{jk}\sigma_k] \qquad k = 1, 2, 3, \ldots, N, \tag{2}$$

where

$$\sigma_k = [a_{i,k}] \qquad i = 1, 2, 3, \ldots, N, \tag{3}$$

denotes the sum of all coefficients a_i in the k-th equation of the form (1). On the other hand, we should also expect that

$$S_j = [R_{ij}], \qquad i = 1, 2, 3, \ldots, n, \tag{4}$$

i.e., that S_j should be equal to the sum of all elements of the j-th row to the left of the solid line in matrix $\{\mathbf{R}\}$. A failure of any single value of S_j as evaluated by means of equations (2) and (4) independently to be identical indicates a numerical slip in the corresponding row, which should be promptly traced and corrected.

It is not until all such possible mistakes have been eliminated and agreement established for every single element of the auxiliary column that we are ready to proceed to the next step of our process, which will be the evaluation of a triangular square-root of $\{\mathbf{R}\}$. The elements r_{ij}

$$\begin{array}{ccccccc|c}
r_{11} & r_{21} & r_{31} & r_{41} & r_{51} & \cdots & r_{n1} & s_1 \\
 & r_{22} & r_{32} & r_{42} & r_{52} & \cdots & r_{n2} & s_2 \\
 & & r_{33} & r_{43} & r_{53} & \cdots & r_{n3} & s_3 \\
 & & & r_{44} & r_{54} & \cdots & r_{n4} & s_4 \\
 & & & & & & \cdot & \cdot \\
 & & & & & & \cdot & \cdot \\
 & & & & & & \cdot & \cdot \\
 & & & & & & r_{nn} & s_n
\end{array} \qquad \{\mathbf{r}\}$$

of this matrix-cracovian are obtained by the following schemes of algebraic operations:

$$\begin{array}{ll}
r_{11} = R_{11} \div r_{11} & r_{22} = (R_{22} - r_{21}r_{21}) \div r_{22} \\
r_{21} = R_{21} \div r_{11} & r_{32} = (R_{32} - r_{21}r_{31}) \div r_{22} \\
r_{31} = R_{31} \div r_{11} & r_{42} = (R_{42} - r_{21}r_{41}) \div r_{22} \\
r_{41} = R_{41} \div r_{11} & r_{52} = (R_{52} - r_{21}r_{51}) \div r_{22} \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
r_{n1} = R_{n1} \div r_{11} & r_{n2} = (R_{n2} - r_{21}r_{n1}) \div r_{22} \\
s_1 = S_1 \div r_{11} & s_2 = (S_2 - r_{21}s_1) \div r_{22} \\
\hline
\end{array}$$

$$r_{33} = (R_{33} - r_{32}r_{32} - r_{31}r_{31}) \div r_{33}$$
$$r_{43} = (R_{43} - r_{32}r_{42} - r_{31}r_{41}) \div r_{33}$$
$$r_{53} = (R_{53} - r_{32}r_{52} - r_{31}r_{51}) \div r_{33}$$
$$r_{63} = (R_{63} - r_{32}r_{62} - r_{31}r_{61}) \div r_{33}$$
$$\cdot$$
$$\cdot$$
$$r_{n3} = (R_{n3} - r_{32}r_{n2} - r_{31}r_{n1}) \div r_{33}$$
$$s_3 = (S_3 - r_{32}s_2 - r_{31}s_1) \div r_{33}$$

$$r_{44} = (R_{44} - r_{43}r_{43} - r_{42}r_{42} - r_{41}r_{41}) \div r_{44}$$
$$r_{54} = (R_{54} - r_{43}r_{53} - r_{42}r_{52} - r_{41}r_{51}) \div r_{44}$$
$$r_{64} = (R_{64} - r_{43}r_{63} - r_{42}r_{62} - r_{41}r_{61}) \div r_{44}$$
$$\cdot$$
$$\cdot$$
$$r_{n4} = (R_{n4} - r_{43}r_{n3} - r_{42}r_{n2} - r_{41}r_{n1}) \div r_{44}$$
$$s_4 = (S_4 - r_{43}s_3 - r_{42}s_2 - r_{41}s_1) \div r_{44}$$

$$r_{55} = (R_{55} - r_{54}r_{54} - r_{53}r_{53} - r_{52}r_{52} - r_{51}r_{51}) \div r_{55}$$
$$r_{65} = (R_{65} - r_{54}r_{64} - r_{53}r_{63} - r_{52}r_{62} - r_{51}r_{61}) \div r_{55}$$

etc.,

and $r_{nn} = \pm 1$, depending on whether the absolute terms of the equations of condition were transposed to the left—as in equation (1)—or were kept on the right-hand side, It should be stressed that, inasmuch as the above matrix $\{\mathbf{r}\}$ is *triangular*, all its elements *below* the principal diagonal (i.e., all r_{ij}'s for which $i < j$) are identically *zero.* The computation of all elements r_{ii} on the principal diagonal by the above rules evidently requires extraction of $n - 1$ square-roots of the quantities in parentheses on the right-hand sides of the foregoing relations. In the case of normal equations of a least-squares system all these quantities are, by definition, positive and their square-roots are consequently real.*

The elements $s_j \equiv r_{n+1,j}$ of the check column on the extreme right of the matrix $\{\mathbf{r}\}$ are obtained by exactly the same rules as all other elements r_{ij}, and should satisfy a system of n equations exactly analogous to (4)—i.e.,

$$s_j = [r_{ij}], \qquad i = 1, 2, 3, \ldots, n, \tag{5}$$

* It should be noted, however, that a mere diagonal symmetry of a given matrix $\{\mathbf{r}\}$ is no guarantee in itself that the r_{ii}'s will be all real. In point of fact, cases can easily be constructed in which some of the diagonal elements are imaginary—though the final solution may happen to be real.

the fact that all elements of the **r**-matrix below the principal diagonal are now zero greatly simplifies the computation. A failure of any one condition of the form (5) to be fulfilled within the limits of rounding-off errors indicates again a numerical slip on the corresponding line. There exists an additional simple condition which must be fulfilled by our check columns if our computations are correct: namely,

$$[(S_j - R_{nj})] = [(s_j - r_{nj})^2], \qquad j = 1, 2, 3, \ldots, n, \tag{6}$$

In practice this should be verified before we proceed to compute the reciprocal of the above **r**-matrix, of the form

$$\begin{matrix} q_{11} \\ q_{12} & q_{22} \\ q_{13} & q_{23} & q_{33} \\ q_{14} & q_{24} & q_{34} & q_{44} \\ q_{15} & q_{25} & q_{35} & q_{45} & q_{55} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{1n} & q_{2n} & q_{3n} & q_{4n} & q_{5n} & \cdots & q_{nn} \end{matrix} \qquad \{\mathbf{q}\}$$

the coefficients of which can be obtained by the following rules:

$$\begin{aligned} q_{ii} &= (r_{ii})^{-1}, \\ q_{12} &= -(r_{21}q_{11}) \div r_{22} \\ q_{13} &= -(r_{31}q_{11} + r_{32}q_{12}) \div r_{33} \\ q_{14} &= -(r_{41}q_{11} + r_{42}q_{12} + r_{43}q_{13}) \div r_{44} \\ q_{15} &= -(r_{51}q_{11} + r_{52}q_{12} + r_{53}q_{13} + r_{54}q_{14}) \div r_{55} \\ &\text{etc.,} \\ q_{23} &= -(r_{32}q_{22}) \div r_{33} \\ q_{24} &= -(r_{42}q_{22} + r_{43}q_{23}) \div r_{44} \\ q_{25} &= -(r_{52}q_{22} + r_{53}q_{23} + r_{54}q_{24}) \div r_{55} \\ &\text{etc.,} \\ q_{34} &= -(r_{43}q_{33}) \div r_{44} \\ q_{35} &= -(r_{53}q_{33} + r_{54}q_{34}) \div r_{55} \\ &\text{etc.,} \\ q_{45} &= -(r_{54}q_{44}) \div r_{55} \\ &\text{etc.,} \end{aligned}$$

while all elements *above* the principal diagonal (i.e., all q_{ij}'s for which $i > j$) are identically equal to *zero*.

The Unknowns and their Errors

The above matrix $\{\mathbf{q}\}$ represents the last step of our solution, since *the elements of its last row are the numerical values of our unknowns, while all* *elements lying above them specify their weights.* If $x_1, x_2, x_3, \ldots, x_{n-1}$ denote these unknowns, our complete solution takes the form

$$\begin{aligned} x_1 &= q_{1n} \pm \varepsilon\{q_{11}^2 + q_{12}^2 + q_{13}^2 + \ldots + q_{1,n-1}^2\}^{1/2}, \\ x_2 &= q_{2n} \pm \varepsilon\{q_{22}^2 + q_{23}^2 + q_{24}^2 + \ldots + q_{2,n-1}^2\}^{1/2}, \\ x_3 &= q_{3n} \pm \varepsilon\{q_{33}^2 + q_{34}^2 + q_{35}^2 + \ldots + q_{3,n-1}^2\}^{1/2}, \\ &\cdot \\ &\cdot \\ &\cdot \\ x_{n-1} &= q_{n-1,n} \pm \varepsilon q_{n-1,n-1}, \end{aligned} \tag{7}$$

where ε, the probable error of a single equation of unit weight, is equal to

$$\varepsilon = 0{\cdot}6745 \left\{\frac{[(O-C)^2]}{N-n+1}\right\}^{1/2}, \tag{8}$$

N being the total number of our equations of condition, and $[(O-C)^2]$, the sum of the squares of the individual residuals.

In certain cases, it may happen that the absolute terms in the given system of normal equations are not pure numbers, but polynomials involving arbitrary terms. Such a case arises, for instance, if the expressions involving Δu in equations of the form (7-1) or (8-7) of Chapter VI are transposed to the absolute terms, and all other unknowns are to be evaluated in terms of arbitrary Δu-corrections. In the face of such a task, the cracovian method continues to be unrivalled in performance by virtue of the fact—which does not seem to have been noticed before—that *in the elements $r_{n,j}$ of the last row of the auxiliary* **r**-*matrix, the coefficients of the absolute terms $R_{n,j}$* ($j = 0, 1, 2, \ldots n - 1$) *of the given* **R**-*matrix are identical with the elements of the first n rows of the final* **q**-*matrix.* In more specific terms, an evaluation of the elements of the last column of the auxiliary **r**-matrix by the rules summarized on pp. 451–452 yields

$$\begin{aligned} r_{n1} &= q_{11}R_{n1} \\ r_{n2} &= q_{12}R_{n1} + q_{22}R_{n2} \\ r_{n3} &= q_{13}R_{n1} + q_{23}R_{n2} + q_{33}R_{n3} \\ &\cdot \\ &\cdot \\ &\cdot \\ r_{n,n-1} &= q_{1,n-1}R_{n1} + q_{2,n-1}R_{n2} + q_{3,n-1}R_{n3} \\ &\qquad + \ldots + q_{n-1,n-1}R_{n,n-1}. \end{aligned}$$

Once the computation of the **r**-matrix has been performed, therefore, all elements of the corresponding **q**-matrix are already known—except for its last row containing the values of the unknowns which can, however, be obtained by the usual rules. The numerical coefficients in the expressions for q_{in} (which are now polynomials in R_{ij}) are *the only additional terms* which must be evaluated if the absolute terms in our normal equations are not pure numbers.

The reader may notice, moreover, that if we express the unknown quantities of our solution as

$$\left.\begin{aligned}
x_1 &= q_{1n} = a_{11}R_{n1} + a_{21}R_{n2} + a_{31}R_{n3} + \ldots + a_{n-1,1}R_{n,n-1} \\
x_2 &= q_{2n} = a_{12}R_{n1} + a_{22}R_{n2} + a_{32}R_{n3} + \ldots + a_{n-1,2}R_{n,n-1} \\
x_3 &= q_{3n} = a_{13}R_{n1} + a_{23}R_{n2} + a_{33}R_{n3} + \ldots + a_{n-1,3}R_{n,n-1} \\
&\cdot \\
&\cdot \\
&\cdot \\
x_{n-1} &= q_{n-1,n} = a_{1,n-1}R_{n1} + a_{2,n-1}R_{n2} + a_{3,n-1}R_{n3} \\
&\qquad + \ldots + a_{n-1,n-1}R_{n,n-1},
\end{aligned}\right\} \tag{9}$$

only $\frac{1}{2}n(n - 1)$ out of the whole array of $(n - 1)^2$ coefficients a_{ij} need to be evaluated, since their matrix again turns out to be diagonally symmetrical (i.e., $a_{ij} = a_{ji}$). It is, furthermore, evident from the definition of the q's that all elements a_{ii} $(i = 1, 2, 3, \ldots n - 1)$ on the principal diagonal of this matrix must be negative, because each one is equal to a negatively taken sum of squares of certain preceding elements of the **q**-matrix.

Two checks remain to be performed, at this stage, to verify the correctness of our solution. The *first* and most obvious one consists in inserting the values of our unknowns in our original system of normal equations to ensure that our solution satisfies them. The second, and more powerful check, is based on the fact that the sum of the squares $[(O–C)^2]$ of the residuals can be evaluated in two different ways. First, the individual $(O–C)$'s can obviously be obtained by inserting $x_1, x_2, \ldots x_{n-1}$ in each equation of condition, and their squares summed. *Secondly*, advantage can be taken of the algebraic identity asserting that

$$[(O–C)^2] = [mm] - r_{n1}^2 - r_{n2}^2 - r_{n3}^2 - \ldots - r_{n,n-1}^2, \tag{10}$$

where $[mm]$ stands for the sum of squares of the absolute terms $m \equiv a_{n,k}$ in all equations of condition of the form (1). The first check, based on the insertion of our unknowns in the basic system of normal equations, discloses merely whether or not these latter equations have been solved correctly. The second check, based on the equality of $[(O–C)^2]$ computed in two different ways, verifies the *formation* of the normal equations as well as their *solution.**

* If the computer has carried the check column S_j in matrix $\{\mathbf{R}\}$ and verified that equations (2) and (4) both yield the same result, this latter check may seem superfluous. In such cases, equation (10) can be used to compute the quantity $[(O–C)^2]$ directly, without going into the laborious process of evaluation of the individual residuals of all equations of condition. In practice, however, the computer may as a rule wish to form the individual residuals just the same, in order to make sure that their behaviour does not exhibit any systematic trend. This is a point of prime importance; for an obviously systematic behaviour of the residuals would mean that the errors causing the uncertainty of the unknowns are only partly accidental, and partly systematic. The presence of systematic errors would, in turn, indicate that the relation between the observed and the unknown quantities as represented by the general form of our equations of condition is deficient. In view of a possible presence of physical peculiarities in various eclipsing systems, complicating the normal state of affairs, such a possibility should always be kept in mind.

If both checks are fulfilled within the limits of numerical round-off errors, the solution of our system can be accepted as correct. The most probable values of the individual unknowns are then given by equations (7), while any *linear* function F of $x_1, x_2, x_3, \ldots, x_{n-1}$ of the form

$$F = f_1x_1 + f_2x_2 + f_3x_3 + \ldots + f_{n-1}x_{n-1}; \tag{11}$$

where $f_1, f_2, f_3, \ldots, f_{n-1}$ are known coefficients, will be numerically equal to

$$F = (f_1q_{1n} + f_2q_{2n} + f_3q_{3n} + \ldots + f_{n-1}q_{n-1,n}) \pm \varepsilon_F(p.e.), \tag{11a}$$

where ε_F, the probable error within which F is defined by the available data, can be evaluated from

$$\begin{aligned}(\varepsilon_F/\varepsilon)^2 = (&f_1q_{11})^2 \\ &+ (f_1q_{12} + f_2q_{22})^2 \\ &+ (f_1q_{13} + f_2q_{23} + f_3q_{33})^2 \\ &\cdot \\ &\cdot \\ &\cdot \\ &+ (f_1q_{1,n-1} + f_2q_{2,n-1} + f_3q_{3,n-1} + \ldots + f_{n-1}q_{n-1,n-1})^2.\end{aligned} \tag{12}$$

If, in the more general case, the function $F(x_1, x_2, \ldots, x_{n-1})$ depends on its arguments in a *non-linear* manner, but is continuous and differentiable in these parameters, we can consider, in its place, a differential linear function of the form

$$\Delta F = \left\{\frac{\partial F}{\partial x_1}\Delta x_1 + \frac{\partial F}{\partial x_2}\Delta x_2 + \frac{\partial F}{\partial x_3}\Delta x_3 + \ldots + \frac{\partial F}{\partial x_{n-1}}\Delta x_{n-1}\right\}. \tag{13}$$

The error $\varepsilon_{\Delta F}$ of this new function, arising from the uncertainty of Δx_1, $\Delta x_2, \ldots$ etc., will again be given by the right-hand side of the foregoing equation (12), where $f_j \equiv \partial F/\partial x_j$ ($j = 1, 2, 3, \ldots n - 1$); moreover, under precautions discussed already at the end of section VI.8, *the errors ε_F and $\varepsilon_{\Delta F}$ can be regarded to be identical* [8].

Uncertainty of the Geometrical Intermediary Elements

The properties of the least-squares solutions as summarized in the preceding section of this Appendix can now be utilized to evaluate the uncertainty, not only of the actual unknown quantities of our solution, but also of any of their linear or non-linear combinations. A decomposition of this uncertainty into partial constituents as represented by the individual rows of the q's on the right-hand side of equation (12) will, moreover, permit us to study conveniently the 'anatomy' of the errors of the individual elements or combination of elements, and to locate precisely the source of its major contributors. Consider, for instance, a least-squares solution for the intermediary elements of a totally or partially eclipsing system, with unknowns denoted according to the following scheme:

Eclipses	q_{16}	q_{26}	q_{36}	q_{46}	q_{56}
Total	C_1	C_2	C_3	$\Delta\lambda$	ΔU
Partial	C_1	C_2	$\Delta\lambda_a$	$\Delta\lambda_b$	ΔU

where the quantity $\Delta\lambda$ in the first row refers to the depth of the eclipse under investigation (i.e., to $\Delta\lambda_a$ if this eclipse is total, and to $\Delta\lambda_b$ if it is annular). With the expressions for the errors of any one of the preceding five quantities alone already contained in equations (7) we shall, in what follows, confine our attention to establishing explicit expressions for the errors of certain commonly used *functions* which are expressible in terms of the auxiliary constants of our intermediary solution. Since all such expressions are bound to be of the form (12), our actual task will be reduced to establishing the explicit forms of the f's appropriate for each particular function.

Of all possible functions which can be expressed in terms of our auxiliary constants, the most important ones are doubtless those defining the geometrical elements r_a, r_b and i themselves. If the eclipses are total (or annular), these geometrical elements are expressible in terms of the auxiliary trio C_1, C_2 and C_3 by means of the equations (4-39) of Chapter VI; while if the eclipses are partial, these elements can be deduced from C_1, C_2 and p_0 by means of the equations (5-27) of the same chapter. In both cases, the geometrical elements proved to be non-linear functions of the auxiliary constants resulting from our least-squares solution; hence, the errors of such elements will have to be identified with the errors of differential linear functions of the form (13), where F stands for the respective element. The explicit forms of Δr_a, Δr_b and Δi in terms of the uncertainties of our auxiliary constants have already been given in section 6 of Chapter VI by equations (6-1)–(6-3) if the eclipses are total (annular), and by (6-4)–(6-6) if the eclipses are partial.

In applying these equations to the actual evaluation of the errors we should, however, keep in mind that the values ΔF and, consequently, ΔC_1, ΔC_2 and ΔC_3 or Δp_0 represent, by definition, the *total* errors of the respective quantities, arising from the dispersion of the individual observations *and* from the finite rate of convergence of the iterative process by which the values of our auxiliary constants have been established. The relative contributions of these two independent sources of error to the total uncertainty of our results have already been discussed in some detail in section VI.6. Now a *single* least-squares solution for the auxiliary constants knows nothing about the rate of convergence of our iterations; the uncertainty of the constants resulting from such a solution for a fixed value of K reflects, therefore, solely the dispersion of the individual normal points. But if, as in section VI.6,

ΔC_1, ΔC_2, . . . etc., denote the total uncertainties of the respective quantities, and $(\delta C_1)_k$, $(\delta C_2)_k$, . . . etc., those arising from the dispersion of the observations alone and therefore obtainable from our least-squares solutions with a *fixed* value of K, equations (6-21)–(6-23) of Chapter VI permit us to write

$$\Delta C_1 = (1 + X_1)(\delta C_1)_K - kX_1(\delta C_2)_K , \tag{14}$$

$$\Delta C_2 = (1 - kX_1)(\delta C_2)_K - X_2(\delta C_1)_K , \tag{15}$$

$$\Delta C_3 = (\delta C_3)_K + X_3(\delta C_1)_K - kX_3(\delta C_2)_K , \tag{16}$$

where the quantities X_1, X_2, X_3 are defined by equations (6-24) of Chapter VI. Their magnitude depends on the speed of convergence of the iterative process, and must be ascertained by the methods of section VI.6 prior to the evaluation of the errors.

If we insert the foregoing equations (14)–(16) in (6-1)–(6-3) of Chapter VI, the latter can evidently be rewritten as

$$\Delta w = f_1(\delta C_1)_K + f_2(\delta C_2)_K + f_3(\delta C_3)_K , \tag{17}$$

where w stands for any one of the geometrical elements r_a, r_b or i and

$$f_1 = \frac{\partial w}{\partial C_1} + \left\{X_1 \frac{\partial w}{\partial C_1} + X_2 \frac{\partial w}{\partial C_2} + X_3 \frac{\partial w}{\partial C_3}\right\}, \tag{18}$$

$$f_2 = \frac{\partial w}{\partial C_2} - k \left\{X_1 \frac{\omega}{C_1} + X_2 \frac{\omega}{C_2} + X_3 \frac{\partial w}{\partial C_3}\right\}, \tag{19}$$

$$f_3 = -kx_3 \frac{\partial w}{\partial C_3} . \tag{20}$$

With the derivatives $\partial w/\partial C_j$ $(j = 1, 2, 3,)$ explicitly given by equations (6-7)–(6-12) of Chapter VI and with the values of X_1, X_2, X_3, ascertained by the method of section VI.6, the specification of the coefficients f_j defined by the above equations presents no difficulty; and once they have been evaluated, the total error ε_w of any one of the three geometrical elements readily follows from

$$\begin{aligned}(\varepsilon_w^2/\varepsilon^2) = {} & (f_1 q_{11})^2 \\ & + (f_1 q_{12} + f_2 q_{22})^2 \\ & + (f_1 q_{13} + f_2 q_{23} + f_3 q_{33})^2 \\ & + (f_1 q_{14} + f_2 q_{24} + f_3 q_{34})^2 \\ & + (f_1 q_{15} + f_2 q_{25} + f_3 q_{35})^2 . \end{aligned} \tag{21}$$

The first three lines on the right-hand side of this equation represent the error caused by the observational uncertainty of C_1, C_2 and C_3. Of the terms constituting these lines, only those involving the squares of the f's would be present if all three constants C_1, C_2, C_3 could be determined independently of each other, while the cross-products $f_i f_j$ multiply the terms which account for their mutual inter-dependence; the latter terms may increase or diminish the sum of the purely quadratic terms depending on the algebraic signs of the respective cross-products $f_i f_j$. If both the depth of the minimum under investigation and our unit of light were fixed and subject to no error, the sum of all terms of the first three lines on the right-hand side of (21) would constitute the total error of the element w. In reality, however, this will never be the case, and we have already recognized this fact by including the corrections $\Delta\lambda$ as well as ΔU among the unknowns to be determined simultaneously with C_1, C_2 and C_3. Accordingly, the terms of the fourth line on the right-hand side of (21) including the squares and cross-products of q_{14}, q_{24} and q_{34} arise from the observational uncertainty of λ; and the terms of the last line, involving the squares and cross-products of q_{15}, q_{25} and q_{35} are due to the uncertainty in our knowledge of the proper unit of light. *All these terms, summed up by the square-root of the sum of their squares and cross-products, constitute the total error of* w, and its major contributors can be easily recognized by their numerical magnitude in the course of computation.

Having established explicit expressions for the errors of the individual geometrical elements, let us outline the way in which errors of *non-linear combinations* of such elements can be evaluated. As an illustrative example of such a process, consider the ratio of the radii $k = C_1/C_2$. Differentiating the latter we easily find that

$$\Delta k = \frac{\Delta C_1}{C_2} - k\frac{\Delta C_2}{C_2} = \frac{(\delta C_1)_K - k(\delta C_2)_K}{C_2\left(1 - \dfrac{dk}{dK}\right)} \tag{22}$$

by use of equations (14) and (15). If we assume the uncertainty of this linear function of $(\delta C_1)_K$ and $(\delta C_2)_K$ to be equal to that of k, an appeal to equation (12) discloses that

$$\begin{aligned}(\varepsilon_k/\varepsilon)^2 = {} & (f_1 q_{11})^2 \\ & + (f_1 q_{12} + f_2 q_{22})^2 \\ & + (f_1 q_{13} + f_2 q_{23})^2 \\ & + (f_1 q_{14} + f_2 q_{24})^2 \\ & + (f_1 q_{15} + f_2 q_{25})^2 ,\end{aligned} \tag{23}$$

where we have abbreviated

$$f_1 = \frac{1}{C_2\left(1 - \dfrac{dk}{dK}\right)} \quad \text{and} \quad f_2 = -\frac{k}{C_2\left(1 - \dfrac{dk}{dK}\right)}. \tag{24}$$

As in the case of equation (21) valid for the errors of the individual elements, the last two lines of equation (23) contain terms arising from the uncertainty of our knowledge of the maximum and minimum light. If the latter were known exactly (or if $\Delta\lambda$ and (or) ΔU were omitted from our simultaneous least-squares solution), these terms would be absent. Under ordinary circumstances, their contribution to the total uncertainty of k should as a rule be small; the main source of error being the uncertainty with which the constants C_1, C_2 and C_3 can be deduced from the observed data. In order to elucidate the meaning of equation (23) from a different angle, let us rewrite this equation as

$$\begin{aligned}(\varepsilon_k/\varepsilon)^2 = {} & f_1^2(q_{11}^2 + q_{12}^2 + q_{13}^2 + q_{14}^2 + q_{15}^2) \\ & + f_2^2(q_{22}^2 + q_{23}^2 + q_{24}^2 + q_{25}^2) \\ & + 2f_1 f_2(q_{12}q_{22} + q_{13}q_{23} + q_{14}q_{24} + q_{15}q_{25}).\end{aligned} \tag{23a}$$

The terms in the first two parentheses on the right-hand side of this equation alone would constitute the total uncertainty of k if the errors of C_1 and C_2 were independent. Since, however, both these constants result from the same simultaneous solution their errors are interrelated; and the magnitude of the last term multiplied by the cross-product $2f_1f_2$ represents a measure of their interdependence. Whether this term will increase or diminish the total error of k depends again on its algebraic sign.

The foregoing equations for computation of total errors of the individual geometrical elements are, as they stand, directly applicable only to the case of total or annular eclipses; for if the eclipse happens to be partial, the situation becomes complicated by the fact that the maximum geometrical depth p_0—which is one of the auxiliary constants defining the geometrical elements by means of equations (5-27)–(5-28) of Chapter VI—does not occur among the unknowns of our least-squares solution and must, therefore, be expressed in terms of the latter before the errors of r_a, r_b or i can be found. This task can be accomplished by a differentiation of our fundamental equations (5-11) or (5-12) also of Chapter VI relating k and α_0. On one hand, the functional relationship between α, k and p permits us to expect that

$$\Delta\alpha_0 = \left(\frac{\partial\alpha}{\partial k}\right)_0 \Delta k + \left(\frac{\partial\alpha}{\partial p}\right)_0 \Delta p_0 . \tag{25}$$

On the other hand, by differentiating equations (5-11) and (5-12) (and disregarding, as usual, the slow variation of $Y(k, p_0)$ with k or p_0) we find that

$$\Delta\alpha_0' = \lambda_a \Delta U - \Delta\lambda_a + \frac{\lambda_b \Delta U - \Delta\lambda_b}{k^2 Y(k, p_0)} - \frac{2(1 - \lambda_b)}{k^3 Y(k, p_0)} \Delta k \tag{26}$$

if the eclipse is an occultation, and

$$\Delta\alpha_0'' = \frac{Y(k, p_0)}{Y(k, -1)} \Delta\alpha_0' \tag{27}$$

if it is a transit. Solving equations (26) and (27) together with (25) for Δp_0, inserting for Δk from (22), and remembering that

$$\left(\frac{\partial \alpha}{\partial k}\right)_p = -\left(\frac{\partial p}{\partial k}\right)_\alpha \div \left(\frac{\partial p}{\partial \alpha}\right)_k,$$

while

$$\left(\frac{\partial \alpha}{\partial p}\right)_k \left(\frac{\partial p}{\partial \alpha}\right)_k = 1,$$

we establish that

$$\Delta p_0 = f_1(\delta C_1)_k + f_2(\delta C_2)_k + f_3 \Delta\lambda_a + f_4 \Delta\lambda_b + f_5 \Delta U, \tag{28}$$

where

$$f_1 = \frac{\dfrac{1}{C_2}\dfrac{dp_0}{dk}}{1 - \dfrac{dk}{dK}}, \qquad f_2 = -\frac{\dfrac{k}{C_2}\dfrac{dp_0}{dk}}{1 - \dfrac{dk}{dK}}, \tag{29}$$

and, if the eclipse is an occultation,

$$\frac{dp_0}{dk} = \left(\frac{\partial p}{\partial k}\right)_0 - \frac{2(1-\lambda_b)}{k^3 Y(k, p_0)}\left(\frac{\partial p}{\partial \alpha}\right)_0, \tag{30}$$

$$\left.\begin{aligned} f_3 &= -\left(\frac{\partial p}{\partial \alpha}\right)_0 \\ f_4 &= -\frac{1}{k^2 Y(k, p_0)}\left(\frac{\partial p}{\partial \alpha}\right)_0 \\ f_5 &= \left\{\lambda_a + \frac{\lambda_b}{k^2 Y(k, p_0)}\right\}\left(\frac{\partial p}{\partial \alpha}\right)_0; \end{aligned}\right\} \tag{31}$$

while if it is a transit

$$\frac{\partial p_0}{\partial k} = \left(\frac{\partial p}{\partial k}\right)_0 - \frac{2(1-\lambda_b)}{k^3 Y(k, -1)}\left(\frac{\partial p}{\partial \alpha}\right)_0, \tag{32}$$

and the 'transit' values of f_3, f_4 and f_5 are equal to their corresponding 'occultation' values multiplied by $Y(k, p_0) \div Y(k, -1)$. With coefficients $f_1, f_2, \ldots f_5$ thus susceptible to numerical evaluation in terms of known quantities for either type of the eclipse, equation (28) can be used to evaluate by known methods the uncertainty of p_0 itself—or, by inserting (28) together with (14) and (15) in equations (6-4)–(6-6) of Chapter VI and formulating the appropriate linear functions, to evaluate the uncertainty of any other geometrical element (or a combination of elements) of a partially eclipsing system.

Uncertainty of the Physical Intermediary Elements

Having outlined the way in which the errors of the geometrical elements can be extracted from known properties of our intermediary solution, let us complete our survey by establishing explicit forms of the formulae by which the uncertainty of the fractional luminosities of the two components and of the ratio of their mean surface brightnesses can likewise be obtained. If the eclipses are *total*, equations (4-41) together with (5-20) of Chapter VI disclose at once that

$$\Delta L_a = \Delta(1-\lambda_a) = \lambda_a \Delta U - \Delta\lambda_a , \tag{33}$$

while

$$\Delta L_b = \Delta\lambda_a . \tag{34}$$

Hence, the úncertainty of L_a should be identical with that of the foregoing linear function (33) which, by (12), is numerically equal to ε_{L_a} as given by the equation

$$(\varepsilon_{L_a}/\varepsilon)^2 = (f_3 q_{33})^2 + (f_3 q_{34})^2 + (f_3 q_{35} + f_5 q_{55})^2, \tag{35}$$

where $f_3 = -1$ and $f_5 = \lambda_3$. Similarly, the error of L_b should result from

$$(\varepsilon_{L_b}/\varepsilon)^2 = q_{33}^2 + q_{34}^2 + q_{35}^2 . \tag{36}$$

On the other hand, the uncertainty of the ratio J_a/J_b of the mean surface brightnesses of the two components, as defined by equation (4-43) of Chapter VI, should be equal to that of the linear function

$$\Delta(J_a/J_b) = Y(k,-1)\frac{\Delta(1-\lambda_a)}{1-\lambda_b} - \frac{J_a}{J_b}\frac{\Delta(1-\lambda_b)}{1-\lambda_b}, \tag{37}$$

in which the slow variation of $Y(k, -1)$ with k has again been ignored and $Y(k, -1)$ treated as a known constant. Equation (5-20) of Chapter VI permits us to rewrite (37) in terms of the actual unknowns of our solution as

$$\Delta(J_a/J_b) = f_3\Delta\lambda_a + f_4\Delta\lambda_b + f_5\Delta U, \tag{38}$$

where

$$\left.\begin{aligned} f_3 &= -\frac{Y(k,-1)}{1-\lambda_b}, \\ f_4 &= \frac{J_a/J_b}{1-\lambda_b}, \\ f_5 &= \frac{\lambda_a Y(k,-1) - \lambda_b(J_a/J_b)}{1-\lambda_b}. \end{aligned}\right\} \tag{39}$$

In consequence, the numerical value of the error ε_{J_a/J_b} of J_a/J_b, in so far as it is caused by the observational errors of λ_a, λ_b and U, should follow from the equation

$$(\varepsilon_{J_a/J_b}/\varepsilon)^2 = (f_3 q_{33})^2 + (f_3 q_{34} + f_4 q_{44})^2 + (f_3 q_{35} + f_4 q_{45} + f_5 q_{55})^2. \tag{40}$$

If the eclipses happen to be *partial*, the uncertainty of the ratio J_a/J_b as defined by equation (5-30) of Chapter VI continues to be equal to that of the preceding linear function (37), where $Y(k, -1)$ has been replaced by $Y(k, p_0)$ in the coefficients f_3 and f_5. The uncertainties of the fractional luminosities

L_a and L_b are, however, no longer equal to those of the simple linear functions (33) or (34), since L_a and L_b are now defined by the simultaneous system (5-29) of Chapter VI. Differentiating the latter and solving for ΔL_a and ΔL_b we find that

$$\left.\begin{aligned} \Delta L_a &= L_a\Delta U + 2kL_b^2(J_a/J_b)\Delta k + (kL_b)^2\Delta(J_a/J_b), \\ \Delta L_b &= L_b\Delta U - 2kL_b^2(J_a/J_b)\Delta k - (kL_b)^2\Delta(J_a/J_b), \end{aligned}\right\} \tag{41}$$

where Δk and $\Delta(J_a/J_b)$ continue to be defined by the linear functions (22) and (38) already given.* Inserting these expressions in the preceding equations (41) we ultimately obtain

$$\Delta L_{a,b} = f_1(\delta C_1)_k + f_2(\delta C_2)_k + f_3\Delta\lambda_a + f_4\Delta\lambda_b + f_5\Delta U, \tag{42}$$

with

$$\left.\begin{aligned} f_1 &= \pm\frac{2L_aL_b}{C_2\left(1-\dfrac{dk}{dK}\right)}, \qquad f_2 = \pm\frac{2kL_aL_b}{C_2\left(1-\dfrac{dk}{dK}\right)}, \\ f_3 &= \mp\frac{L_aL_b}{1-\lambda_a}, \qquad f_4 = \pm\frac{L_aL_b}{1-\lambda_b}, \\ f_5 &= L_{a,b} \pm \frac{L_aL_b}{1-\lambda_a}\,\frac{\lambda_a-\lambda_b}{1-\lambda_b}, \end{aligned}\right\} \tag{43}$$

where the upper sign pertains to ΔL_a and the lower one to ΔL_b. The coefficients $f_1, f_2, \ldots, f_5$ of equations (42), expressed in terms of known quantities by the preceding relations, can now be regarded as known; and the uncertainties of the linear functions $\Delta L_{a,b}$, which should be identical with the errors of L_a and L_b themselves, can be evaluated in terms of the appropriate elements of the **q**-matrix by means of equation (12) in exactly the same way as in the preceding examples.

Uncertainty of the Elements by the Method of Differential Corrections

If a set of the intermediary elements of an eclipsing system (whether spherical or distorted) has subsequently been adjusted by the least-squares process described in section VI.8, the task of evaluating the uncertainty of some combinations of the elements becomes easier than at the intermediary stage, and for others it becomes more involved. Since the corrections Δr_a, Δr_b and Δi to the geometrical elements result directly from such a solution, their errors are obtainable from equations (7) as they stand, without recourse to any linear functions F and, within the precautions discussed in section VI.8, such errors can be identified with those of the respective elements themselves. With regard to such elements let us, therefore, limit ourselves to establishing explicit expressions by means of which the errors of the *sum* $r_a + r_b$ and of the *ratio* $k = r_a/r_b$ of the fractional radii can be evaluated.

* Care should merely be taken to replace $Y(k, -1)$ by $Y(k, p_0)$ in the coefficients of (38).

Suppose that the following five differential corrections have been determined by a simultaneous solution in the following order:

$$\left.\begin{aligned} \Delta L_a &= q_{16}\,, \\ \Delta L_b &= q_{26}\,, \\ \Delta r_a &= q_{36}\,, \\ \Delta r_b &= q_{46}\,, \\ \Delta \cos i &= q_{56}\,. \end{aligned}\right\} \tag{44}$$

Furthermore, let the uncertainty of $r_a + r_b$ be again identified with that of the sum $\Delta r_a + \Delta r_b$, which constitutes a linear function F of the form (11) such that $f_1 = f_2 = f_5 = 0$ and $f_3 = f_4 = 1$. If so, equation (12) reduces to

$$(\varepsilon_F/\varepsilon)^2 = q_{33}^2 + (q_{34} + q_{44})^2 + (q_{35} + q_{45})^2, \tag{45}$$

where ε_F is numerically equal to the uncertainty of $r_a + r_2$. The uncertainty ε_k of $(r_a/r_b) \equiv k$ may, in turn, be identified with that of a linear function

$$\Delta k = \Delta r_a/r_b - (r_a/r_b^2)\Delta r_b\,, \tag{46}$$

and its magnitude should be given by

$$(\varepsilon_k/\varepsilon)^2 = (f_3 q_{33})^2 + (f_3 q_{34} + f_4 q_{44})^2 + (f_3 q_{35} + f_4 q_{45})^2, \tag{47}$$

where

$$f_3 = r_b^{-1} \qquad \text{and} \qquad f_4 = -k r_b^{-1}, \tag{48}$$

all other f's being equal to zero.

Let us consider next the uncertainty of other elements—such as p_0 and α_0 for partially eclipsing systems—which were not adjusted explicitly by our least-squares solution. Such elements must evidently be first expressed in terms of the unknowns (44) of our solution before an evaluation of the errors can be undertaken. Since, by definition (*cf.* equations 5-26 of Chapter VI) $p_0 = (\cos i - r_b)/r_a$, differentiating this latter relation we obtain

$$\Delta p_0 = f_3 \Delta r_a + f_4 \Delta r_b + f_5 \Delta \cos i \tag{49}$$

and the error ε_{p_0} of p_0 is, therefore, given by

$$(\varepsilon_{p_0}/\varepsilon)^2 = (f_3 q_{33})^2 + (f_3 q_{34} + f_4 q_{44})^2 + (f_3 q_{35} + f_4 q_{45} + f_5 q_{55})^2\,, \tag{50}$$

where

$$\left.\begin{aligned} f_3 &= -r_a^{-2} \cos i\,, \\ f_4 &= -r_a^{-1}, \\ f_5 &= r_a^{-1}. \end{aligned}\right\} \tag{51}$$

The uncertainty of the maximum fractional loss of light α_0 of a partial eclipse can be evaluated as that of the linear function (25), where Δk and Δp_0 are given by the foregoing equations (46) and (49), respectively. If we insert them in (25), the outcome can be rewritten as

$$\Delta \alpha_0 = f_3 \Delta r_a + f_4 \Delta r_b + f_5 \Delta \cos i\,, \tag{52}$$

which represents a linear function of the same form as equation (49) for Δp_0 above. Hence, the error ε_{α_0} of α_0 will be given by an equation of exactly the same form as (50) which specifies the error of p_0—except that, in the present case, equations (51) defining the f's should be replaced by

$$\left.\begin{aligned} f_3 &= \left\{\frac{1}{r_b}\left(\frac{\partial\alpha}{\partial k}\right)_0 - \frac{\cos i}{r_a^2}\left(\frac{\partial\alpha}{\partial p}\right)_0\right\}, \\ f_4 &= -\left\{\frac{k}{r_b}\left(\frac{\partial\alpha}{\partial k}\right)_0 + \frac{1}{r_a}\left(\frac{\partial\alpha}{\partial p}\right)_0\right\}, \\ f_5 &= \frac{1}{r_a}\left(\frac{\partial\alpha}{\partial p}\right)_0 . \end{aligned}\right\} \tag{53}$$

As the concluding example, we wish to derive the formulae which should permit us to evaluate the uncertainty of the ratio of mean surface brightnesses J_a/J_b of the two components of our eclipsing system in terms of the errors of the differential corrections (44) of our least-squares solution. In order to do so we may recall that, by definition,

$$\frac{L_a}{L_b} = k^2 \frac{J_a}{J_b} \tag{54}$$

for any type of the eclipse; hence, by differentiating this equation, inserting Δk from (46) and solving for $\Delta(J_a/J_b)$ we obtain

$$\Delta(J_a/J_b) = f_1\Delta L_a + f_2\Delta L_b + f_3\Delta r_a + f_4\Delta r_b, \tag{55}$$

where

$$\left.\begin{aligned} f_1 &= \frac{1}{L_a}\frac{J_a}{J_b}, \qquad & f_2 &= -\frac{1}{L_b}\frac{J_a}{J_b}, \\ f_3 &= -\frac{2}{r_a}\frac{J_a}{J_b}, \qquad & f_4 &= \frac{2}{r_b}\frac{J_a}{J_b}. \end{aligned}\right\} \tag{56}$$

Once the numerical values of these coefficients have been determined, an appeal to equation (12) discloses that the total error of J_a/J_b caused by the uncertainty of the constituent elements ΔL_a, ΔL_b, Δr_a and Δr_b of our least-squares solution can be evaluated from

$$\begin{aligned} (\varepsilon_{J_a/J_b}/\varepsilon)^2 = (f_1 q_{11})^2 &+ (f_1 q_{12} + f_2 q_{22})^2 \\ &+ (f_1 q_{13} + f_2 q_{23} + f_3 q_{33})^2 \\ &+ (f_1 q_{14} + f_2 q_{24} + f_3 q_{34} + f_4 q_{44})^2 \\ &+ (f_1 q_{15} + f_2 q_{25} + f_3 q_{35} + f_4 q_{45})^2 , \end{aligned} \tag{57}$$

where the f's assume the values given by equations (56) above and the q's are the customary respective elements of the inverse matrix-cracovian of our least-squares solution.

In concluding the present Appendix, we are well aware that the illustrative examples of the preceding three sections—limited, of necessity, to a few basic requirements encountered most frequently in practice—do not by any means exhaust the survey of all combinations of various elements of eclipsing binary systems, whose uncertainty may be of interest to the investigator. Yet the writer ventures to hope that any reader, who has followed us through the foregoing discussion limited to a few basic examples, will find no difficulty in applying the general principles of our method to any particular situation which he may encounter in pursuit of his own investigations.

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[8] S. L. Piotrowski, *Proc. U.S. Nat. Acad. Sci.*, **34**, 23, 1948.

CHAPTER VII

Physical Properties of Close Binary Systems

THE GREATER part of this book (Chapters II–VI) has been devoted to stating the principal *methods of analysis* of the diverse phenomena exhibited by close binary systems; and a mastery of such methods is indeed indispensible for a correct and meaningful interpretation of the observed data. This book would, however, remain incomplete if analysis would not be followed by synthesis, and if an examination—sometimes microscopic—of the anatomy of individual trees would make us to forget the wood in which we happen to find ourselves equipped with all our analytical tools. Time has now come to put them to tasks for which they were designed. This has indeed been done; but to describe this part of our story in sufficient detail would require several volumes of the size of the present one—an impracticable task—and an ungrateful one, too. For the observational evidence—both photometric and spectroscopic—is currently increasing at so rapid a rate that this part of the work would be the first one to become obsolete in the future. In order to avoid such a prospect, we shall forego detailed analysis of the observations on examples which could be outdated virtually before the printer's ink has dried over these pages. Instead, we shall attempt to survey the whole field from a more detached view—with the hope that its salient features, emerging slowly in front of our eyes from the accumulating evidence, may throw already now some light on such fundamental problems of double-star astronomy as to why do the close binaries look as we see them, what is their past and the future, and their particular significance in the general framework of stellar evolution. The aim of this concluding chapter of the present book will be to outline such a survey, and to attempt its synthesis as far as it appears possible at the present time.

In order to do so, our first goal should be to set up methods—individual or statistical—which should enable us to establish the *absolute properties* of close binaries (such as the absolute masses and radii of their component, and their absolute luminosities) for as many systems as may lend themselves to such a determination. The requisite methods—some of them new—will be outlined in the opening section VII.1 of this chapter. Our next step will be to *classify* the abundant data so obtained, with the aim of distinguishing among them the existence—if any—of natural grouping. The immediate object of any classification is, to be sure, a segregation of the systems into groups with certain common characteristics. But if such a classification is to possess more than heuristic value, its distinguishing characteristics must be such as to endow each group with the properties of a definite physical association.

Let us examine, from this point of view, the schemes of classification used so far for close binary systems; and in doing so we shall be led almost from the outset to limit our attention to eclipsing variables. This is not only because the binaries which reveal their identity by variation of light outnumber those for which the radial velocity alone is known to vary at least four-to-one in our catalogues; but also because an analysis of the light changes can reveal much greater wealth of information than the radial-velocity changes. It goes, however, without saying that both spectroscopic and eclipsing binaries constitute the same physical group and differ in their observable mañifestations only by an accident of orientation of their orbits in space.

It has been customary, for many years, to divide the eclipsing variables into three principal groups—with Algol, β Lyrae, and W Ursae Maioris as their respective prototypes. It is, however, easy to show now that any such classification lacks any real physical basis and can be maintained only on historical grounds. Its principal distinguishing characteristic—the presence or absence of photometric ellipticity effect between minima—is but a natural resultant of the fractional dimensions, mass-ratio, and relative luminosities of the constituent components, and can be predicted in their terms from a reliable theory (section IV.2). Statistical investigations reveal, moreover, no apparent break in frequency—distribution of the fractional radii, or the mass- and luminosity-ratios, of eclipsing variables. Hence, with increasing separation of the components, stars of the β Lyr-type are bound to merge gradually with the Algol systems; and any dividing line between them becomes essentially only a matter of precision with which the photometric ellipticity can be ascertained from the observational data.

This fact renders, however, any such classification not only inexact, but ambiguous as well; for the dependence of the mean photometric ellipticity of a system on the relative luminosities of the constituent components (which are, in general, of different spectral types) makes the ellipticity dependent on the effective wave-length of observation. In order to illustrate this point consider, for example, the well-known eclipsing system of Algol and u Herculis. The former—a prototype of its group—exhibits a light curve which (in visible light) is virtually free from any ellipticity effect; while that of u Her is conspicuously convex between minima—thus rendering this system one of the β Lyrae-type according to the conventional classification. Geometrically, however, both these systems are very similar, and so are their mass-ratios: in both, the primary component of earlier spectral type is several times as massive as its mate and its form departs but little from a sphere. In each system the secondary component appears to be slightly larger of the two (the ratios of the radii being $k = 0{\cdot}95 \pm 0{\cdot}02$ for Algol, and $0{\cdot}99 \pm 0{\cdot}04$ for u Her), and the inequality in masses renders its tidal distortion conspicuous.

The principal difference between these two systems reduces, indeed, to a difference in the ratio J_1/J_2 of surface brightnesses of their components:

for whereas, in blue light (λ 4500 Å), J_1/J_2 in Algol is equal to 18, in u Her it happens to be 2·5. Since the ratios k of radii in both systems are very nearly the same, the inequality of the J_1/J_2's reveals, therefore, a corresponding disparity in fractional luminosities of the two components. The secondaries in both systems are highly distorted; but owing to its low fractional luminosity, the secondary of Algol influences the combined light of the system very much less than does the secondary of u Her in ordinary frequencies—and hence a conspicuous difference in the appearance of their light curves.

A disparity between surface brightnesses of the two stars is, however, bound to diminish with increasing wave-length of observation. A recourse to Planck's law discloses that the value of $J_1/J_2 = 18$ for Algol at $\lambda = 4500$ Å should decrease to 2·5 at approximately $\lambda = 2{\cdot}7\mu$—i.e., at an effective wave-length accessible to modern infrared detectors. In point of fact, an infrared light curve of Algol observed through the atmospheric 'window' between $\lambda = 2{\cdot}0$–$2{\cdot}5\mu$ should markedly resemble that of u Her, and thus earn for Algol a claim to be included in the β Lyr-group. Moreover, a generalization of this point makes it clear that whereas eclipsing systems whose components are of similar spectral types should exhibit similar changes in the light of any frequency, those consisting of dissimilar components may exhibit very different light curves in different effective wave-lengths if their components are also different in form. These facts show that a simple division of the eclipsing systems into Algol- and β Lyr-type stars is not only *inaccurate* for the lack of a suitable dividing line between the two types, but also *ambiguous*; for it may refer at best only to observations carried out in a certain frequency range.

In view of this situation, in section VII.3 we shall re-open the problem of classification *de novo*, with the aim of setting up a system of greater physical meaning. This we shall indeed do, and the subsequent sections VII.3 to VII.6 will be devoted to a more specific discussion of various properties of the individual groups introduced in section VII.3—until, in the concluding section VII.7 of this chapter and of the whole book, we shall attempt a grand synthesis of the whole material and of its evolutionary implications as far as it can be outlined at the present time.

VII.1. DETERMINATION OF ABSOLUTE DIMENSIONS

If our knowledge of close binary systems is to become sufficiently intimate to enable us to appreciate their real significance in proper perspective, it is essential to get acquainted—not only with the relative dimensions and other geometrical properties of such systems as can be obtained from an analysis of their light changes (and which we learned to obtain in the preceding Chapter VI)—but also the *absolute* masses and dimensions of the components and of their orbit; and the object of the present section will be to survey the principal methods which can be employed to this end. As is well known, if we are to accomplish this for individual systems by methods not involving any statistical considerations, spectrographic observations of their radial-velocity changes become a necessary prerequisite.

The methods of procurement of such observational data and their analysis have already been written up in so many other books and so exhaustively as to scarcely warrant a repetition in this place. Suffice it to state that if K_1 denotes the amplitude (in km/sec) of the radial-velocity curve of the primary (more luminous) component due to its motion in an absolute orbit of semi-major axis A_1, eccentricity e, and sideric period P,* Kepler's third law reveals that

$$A_1 \sin i = 13751\sqrt{1-e^2}K_1P \text{ kms} \tag{1-1}$$

if P is expressed in mean solar days; while the 'mass-function' of the system will be given by

$$f(m) \equiv \frac{m_2^3 \sin^3 i}{(m_1+m_2)^2} = 1{\cdot}0385 \times 10^{-7}(1-e^2)^{3/2}K_1^3P \odot, \tag{1-2}$$

where $m_{1,2}$ denote the masses of the two components (in solar units) and i, the inclination of the orbital plane to the celestial sphere.

This latter angle can, in principle, assume any value between 0° and 90°, and remains indeterminate from spectroscopic observations alone. If, however, our binary system happens to be an eclipsing variable, methods have been developed in Chapter VI which should enable us to specify i from an analysis of the observed light changes with considerable precision;† and once this has been accomplished, a knowledge of the mass-ratio m_2/m_1 alone stands between us and a complete description of all absolute properties of the system. The actual value of this ratio can, in turn, be determined in a variety of ways. If the luminosities of the two components are not too unequal—so that the lines of both stars can be measured in their combined spectrum‡—a spectroscopic determination of the amplitudes $K_{1,2}$ of radial-velocity curves of *both* components yields directly the mass-ratio in the form

$$\frac{m_1}{m_2} = \frac{K_2}{K_1}, \tag{1-3}$$

where the values of $K_{1,2}$ observed in close binary systems must, of course, be first corrected for the systematic effects of reflection (as well as of gravity-darkening invoked by odd-harmonic distortion of tidal origin) in a manner discussed previously in section V.4. A combination of (1-1) and (1-2) with

* The observed period of the system should, strictly speaking, be corrected for the effect of the systemic radial velocity γ in accordance with equation (8-103) of Chapter II.

† From good photoelectric observations of reasonably deep eclipses it is nowadays possible to specify the value of i with an uncertainty of the order of 0°.1.

‡ Barring exceptional cases, a difference in brightness of the order of one magnitude is usually sufficient to extinguish any trace of the secondary's lines from the combined spectrum of the system.

(1-3) then yields the absolute dimensions of the system and the masses of its components as

$$\left.\begin{aligned} A = A_1 + A_2 &= 13751(1-e^2)^{1/2}(K_1+K_2)P \csc i \text{ km}, \\ m_{1,2} &= 1{\cdot}0385 \times 10^{-7}(1-e^2)^{3/2}(K_1+K_2)^2 K_{2,1} P \csc^3 i \odot, \end{aligned}\right\} \tag{1-4}$$

within the limits of observational errors with which the quantities occurring on the right-hand sides of these equations can be deduced from the photometric and spectroscopic data.

The requirement that the amplitudes $K_{1,2}$ of radial-velocity changes of both components be measurable restricts, unfortunately, the applicability of the foregoing method to systems whose components do not differ too much in luminosity (and, therefore, in mass). For, in practice, a difference of about one magnitude is usually sufficient to make the less luminous component spectroscopically invisible, and thus to deprive us of the possibility of measuring K_2. It is indeed possible to lessen the disparity in the fractional luminosities of the two components in systems where one star is much hotter than the other by going sufficiently far to the red in their combined spectrum;* or to attempt establishing at least the slope of the secondary's velocity-curve at a time when the primary's luminosity may be sufficiently dimmed by eclipse.† But barring such exceptional cases, the spectroscopic method will, in general, furnish the masses and absolute dimensions of the components of only such systems whose mass-ratios are not very different from unity—which is bound to introduce a severe observational selection into any material based exclusively on it.

Indirect individual methods may, to be sure, be invoked to lessen the degree of this selection. Thus it is possible, in principle, to determine the mass-ratios of close binary systems from purely photometric data—namely, from the extent of tidal elongation mutually produced by both components, as evidenced by the observed ellipticity effect‡—by a method whose details have already been outlined in section VI.12. Or, within minima, the absolute dimensions of the component undergoing eclipse may be ascertained from measurements of the 'rotational effect' produced by its axial rotation (*cf.* section V.2), and its radius evaluated in absolute units from the observed equatorial velocity on the assumption that the star rotates in the same period as it revolves.§ With the absolute radius R_1 thus determined and the fractional radius r_1 known from the photometric solution, the absolute dimensions of the system and its mass-ratio follows from the equations

$$A = \frac{R_1}{r_1} = A_1\left(1 + \frac{m_1}{m_2}\right), \tag{1-5}$$

* *Cf.*, e.g., A. Beer and Z. Kopal (*Ann. d'Astroph.*, **17**, 443, 1954) in their work on Algol.

† *Cf.*, e.g., A. H. Joy (*Ap. J.*, **71**, 336, 1936) in the case of U Sge.

‡ *Cf.* Z. Kopal, *Ap. J.*, **93**, 92, 1941, for its application to β Lyrae, or *Harv. Circ.*, No. 439, 1941, for an application to V 380 Cygni.

§ D. B. McLaughlin, *Michigan Publ.*, **6**, 3, 1934; and elsewhere.

where A_1, the semi-major axis of the absolute orbit of the primary (visible) component is already known from (1-1). Lastly, in the forthcoming sections VII.3 and VII.4 we shall learn of two additional 'photometric' methods for individual determination of the mass-ratios of close binary systems, based on the near-equality of surface potentials in detached Main-Sequence systems (section VII.3), or on the contact properties of the secondary components in semi-detached systems of section VII.4, which are both completely free from any bias for pairs of comparable masses or luminosities.

If, however, our knowledge of the absolute properties of close binary systems were confined to couples to which any one of the foregoing methods is individually applicable, the total material at our disposal would still be rather seriously limited. In order to extend our knowledge to a greater number of such systems, we propose next to resort to certain *statistical* methods, which should enable us to extend our quest to all eclipsing binaries with known photometric elements, for which single-spectrum orbits may (or may not) be available. The only assumption which is required to make this possible is an expectation that one (or both) component of such systems conforms statistically to an empirical mass-luminosity relation.

The principle of such methods is exceedingly simple. Let the symbolical mass-luminosity law

$$m = F(M), \tag{1-6}$$

in solar units, be approximable by a linear expression* of the form

$$\log m = \alpha - \beta M, \tag{1-7}$$

where α and β are appropriate constants (whose values will be determined in the forthcoming section VII.3), and the absolute bolometric magnitude M is related with the absolute radius R and effective temperature T of the star in question by the Stefan's law

$$M = M_\odot - 5 \log (R/R_\odot) - 10 \log (T/T_\odot), \tag{1-8}$$

where the solar radius $R_\odot = 6{\cdot}96 \times 10^5$ km, temperature $T_\odot = 5730°$, and $M_\odot = 4^m\!.7$.

Suppose next that we are in possession of the elements $A_1 \sin i$ and $f(m)$ of the single-spectrum orbit of the primary component of our system, as defined by equations (1-1) and (1-2) above, If so, the absolute properties of the primary component will evidently be given by

$$m_1 = q(1 + q)^2 f(m) \csc i \tag{1-9}$$

and

$$R_1 = (1 + q)(A_1 \sin i)(r_1 \csc i), \tag{1-10}$$

where we have abbreviated

$$q = \frac{m_1}{m_2}. \tag{1-11}$$

* The use of a quadratic (or higher) approximation at this stage would lead to no difficulty except for more complicated algebra.

A combination of (1-7) and (1-9) leads to

$$\beta M_1 = \alpha - \log \{q(1+q)^2 f(m) \csc^3 i\}, \tag{1-12}$$

while an elimination of R_1 between (1-8) and (1-10) yields

$$M_1 = M_\odot - 5 \log \{(1+q)[(A_1/R_\odot) \sin i](r_1 \csc i)\} - 10 \log (T_1/T_\odot). \tag{1-13}$$

Ultimately, the elimination of M_1 between (1-12) and (1-13) leaves us with an equation reducible to the form

$$\log q + B \log (1+q) = Q, \tag{1-14}$$

where the coefficients

$$B = 2 - 5\beta \tag{1-15}$$

and

$$\begin{aligned} Q = \alpha - \beta\{M_\odot &- 5 \log [(A_1/R_\odot) \sin i] - 5 \log (r_1 \csc i) \\ &- 10 \log (T_1/T_\odot)\} - \log \{f(m) \csc^3 i\} \end{aligned} \tag{1-16}$$

are known functions of the photometric elements r_1, i; of the spectroscopic elements $A_1 \sin i$, $f(m)$; and of the effective temperature T_1 of the primary (visible) component which can be estimated from its spectrum. Once this has been done, equation (1–14) contains the mass-ratio q as the only unknown which can be solved for graphically or otherwise, and its solution contains the key to the masses and absolute dimensions of the whole system. The reader should note, in particular, that the whole determination of q has been based on the properties of the primary component alone; those of the secondary being wholly irrelevant in this connection. Therefore, provided only that the primary component (of known single-spectrum orbit) can be expected to behave like a normal star obeying a mass-luminosity relation of the form (1-7), the masses and absolute dimensions of the whole system can be determined by the present method no matter how anomalous the secondary component may turn out to be.

In order to illustrate this method on a practical example, let us consider the eclipsing system of Y Camelopardalis, for which Dugan's observations* rediscussed by Kopal and Shapley† established the fractional radius r_1 of the primary component to be $0{\cdot}233 \pm 0{\cdot}007$, while the orbital inclination i proved to be $85° \pm 0^\circ\!.4$. Y Cam is a single-spectrum binary, whose radial-velocity changes as observed by Struve and his associates‡ gave $A_1 \sin i = 1{\cdot}6 \times 10^6$ km, and $f(m) = 0{\cdot}015$ ⊙. Adopting, moreover, $T_1 = 10700°$ as the effective temperature appropriate for Y Cam A of spectral class A0, and anticipating that the total mass of the system will turn out to be moderate (in which case

* R. S. Dugan, *Princ. Contr.*, No. 6, 1924.

† Z. Kopal and M. B. Shapley, *Catalogue of the Elements of Eclipsing Binary Systems*, *Jodrell Bank Ann.*, **1**, 143, 1956.

‡ O. Struve, H. Horak, R. Cavanaggia, V. Kourganoff, and A. Colacevich, *Ap. J.*, **111**, 658, 1950.

the mass-luminosity relation 3-2 of section VII.3 will be appropriate), we have $\alpha = 0{\cdot}42$ and $\beta = 0{\cdot}086$. An insertion of all these data in equations (1-15) and (1-16) leads to

$$B = 1{\cdot}57 \qquad \text{and} \qquad Q = 1{\cdot}85;$$

and these constants, introduced in (1-14), render the mass-ratio q of the system of Y Cam to be a root of the equation

$$\log q + 1{\cdot}57 \log (1 + q) = 1{\cdot}85,$$

which is found to be

$$q = \frac{m_1}{m_2} = 4{\cdot}6.$$

Inserting it in (1-9) we find that the masses of the individual components of Y Cam then become

$$m_1 = 2{\cdot}2\ \odot, \qquad m_2 = 0{\cdot}48\ \odot;$$

the absolute radii result from (1-7) and (1-8) as

$$R_1 = 3{\cdot}0\ \odot, \qquad R_2 = 3{\cdot}0\ \odot;$$

and the absolute bolometric magnitudes

$$M_1 = 0^m{\cdot}7\,, \qquad M_2 = 3^m{\cdot}4.$$

In accordance with our expectations, the primary component of Y Cam conforms indeed by its properties satisfactorily to the Main Sequence; but the secondary turns out to be anomalous—one of the 'undersize' subgiants (smaller than their Roche limits) which we shall discuss in more detail in the forthcoming section VII.4.

To give another example, consider the well-known system of TX Ursae Maioris, consisting of a B8 principal component which at the time of the primary minima is deeply eclipsed by a late-type mate. The latest set of the elements of a single-spectrum orbit of its B8-star was established by Hiltner,* according to whom $A_1 \sin i = 2{\cdot}18 \times 10^6$ km and $f(m) = 0{\cdot}0442\ \odot$. On the other hand, Piotrowski's accurate photometric solution† revealed that $r_1 = 0{\cdot}158 \pm 0{\cdot}001$ and $i = 81^\circ{\cdot}0 \pm 0^\circ{\cdot}1$. If, moreover, we adopt $T_1 = 12300^\circ$ as the effective temperature of the B8-star, and anticipate that its mass is likely well to exceed $2\odot$, we have (by 3-2) $\alpha = 0{\cdot}45$ and $\beta = 0{\cdot}143$. An insertion of these data in equations (1-15) and (1-16) then leads to $B = 1{\cdot}29$, $Q = 1{\cdot}38$, rendering the mass-ratio of TX UMa to be a root of the equation

$$\log q + 1{\cdot}29 \log (1 + q) = 1{\cdot}38,$$

which is

$$q = \frac{m_1}{m_2} = 3{\cdot}4,$$

* W. A. Hiltner, *Ap. J.*, **101**, 108, 1945.
† S. L. Piotrowski, *Ap. J.*, **106**, 472, 1947.

so that $m_2/m_1 = 0{\cdot}29$. This value can be confronted with the mass-ratio $m_2/m_1 = 0{\cdot}30 \pm 0{\cdot}02$, as established spectroscopically by Pearce* from observations made during the brief visibility of the secondary spectrum around mid-primary minima. Moreover, an assumption that the secondary component of fractional radius $r_2 = 0{\cdot}277 \pm 0{\cdot}001$ fills completely its Roche limit (*cf.* section VII.4) leads to $m_2/m_1 = 0{\cdot}315 \pm 0{\cdot}005$, which is identical with the above determinations of the mass-ratio within the limits of observational errors. Either one of our indirect methods leads, therefore, to as good a determination of the mass-ratio of TX UMa as can be obtained from direct spectroscopic observations of the secondary component.

The foregoing method for a determination of the absolute properties of close binary systems consists, in brief, of combining the elements of their photometric and single-spectrum orbits on the assumption that the spectroscopically visible component conforms to the mass-luminosity relation. Suppose, however, that no spectroscopic observations are available at all; is it still practicable to attempt a determination of the absolute properties of such systems from a knowledge of their photometric elements alone? At the first sight, such a task might indeed appear to be impossible; but a second thought should dispel this impression completely: for in spite of a lack of spectrographic data we are still in possession of at least one element which bears directly on the absolute properties of the respective system: namely, its period P—by far the best determined element of the whole photometric orbit.

Let us, therefore, set out to make the best use of it by falling back on Kepler's third law, which asserts that

$$\frac{4\pi^2}{P^2} = \frac{G(m_1 + m_2)}{A^3} \tag{1-17}$$

and that, hence,

$$R_1 = Ar_1 = \{(G/4\pi^2)(m_1 + m_2)P^2\}^{1/2} r_1 . \tag{1-18}$$

On the other hand,

$$m_1 = \tfrac{4}{3}\pi R_1^3 \bar{\rho}_1 , \tag{1-19}$$

where $\bar{\rho}_1$ denotes the mean density of the primary component. Therefore, by use of (1-18),

$$\bar{\rho}_1 = \frac{3m_1}{4\pi R_1^3} = \frac{3\pi/G}{P^2 r_1^3} \frac{m_1}{m_1 + m_2} . \tag{1-20}$$

Inserting (1-20) back in (1-19) and taking logarithms we find that

$$\log m_1 = \log D + \log \frac{m_1}{m_1 + m_2} + 3 \log \left(\frac{R_1}{R_\odot}\right) , \tag{1-21}$$

where we have abbreviated

$$D = \frac{1}{Gm_\odot} \left(\frac{2\pi}{P}\right)^2 \left(\frac{R_\odot}{r_1}\right)^3 = \frac{0{\cdot}01344}{P^2 r_1^3} ; \tag{1-22}$$

* J. A. Pearce, *Publ. A.A.S.*, **8**, 251, 1936.

the masses $m_{1,2}$ now being expressed likewise in solar units, and the period P in mean solar days. An elimination of R_1 between (1-8) and (1-21) leads, in turn, to

$$\log(m_1 + m_2) = \log D - \tfrac{3}{5}(M_1 - M_\odot) - 6\log(T_1/T_\odot), \tag{1-23}$$

from which

$$m_1 + m_2 = D(T_\odot/T_1)^6 10^{-\frac{3}{5}(m_1 - m_\odot)}. \tag{1-24}$$

In order to proceed further, let us assume now that *both* components of our binary system obey the mass-luminosity law

$$\left.\begin{aligned} m_1 &= F(M_1), \\ m_2 &= F(M_2) = F(M_1 + \Delta M), \end{aligned}\right\} \tag{1-25}$$

where the difference ΔM of bolometric magnitudes of the two components will be given by

$$\Delta M = \tfrac{5}{2}\log\frac{L_1}{L_2}, \tag{1-26}$$

$L_{1,2}$ denoting (as in Chapters IV–VI) the fractional luminosities of both stars obtainable from the photometric solution.* If so it follows that, symbolically

$$F(M_1) + F(M_1 + \Delta M) = D(T_\odot/T_1)^6 10^{-\frac{3}{5}(M_1 - M_\odot)} \tag{1-27}$$

or, by use of (1-7),

$$10^{\alpha - \beta M_1} + 10^{\alpha - \beta M_1 - \beta\Delta M} = D(T_\odot/T_1)^6 10^{-\frac{3}{5}(M_1 - M_\odot)}. \tag{1-28}$$

This latter equation contains M_1 as the only unknown, and can be solved for it to yield

$$\left.\begin{aligned} (\tfrac{3}{5} - \beta)M_1 &= \tfrac{3}{5}M_\odot - \alpha + \log\frac{D(T_\odot/T_1)^6}{1 + 10^{-\beta\Delta M}} \\ &= 2{\cdot}82 - \alpha + \log\left\{\frac{0{\cdot}01344}{P^2 r_1^3(1 + 10^{-\beta\Delta M})}\left(\frac{5730}{T_1}\right)^6\right\} \end{aligned}\right\} \tag{1-29}$$

in terms of the quantities P, r_1 and ΔM obtainable from an analysis of the light changes, and of T_1 estimated from the visible spectrum.

Once the values of M_1 and $M_2 = M_1 + \Delta M$ have thus been established, the solution of our problem is virtually complete; for the corresponding masses $m_{1,2}$ then follow directly from the mass-luminosity law (1-7), and the

* If, however, the ratio $J_1/J_2 > 1$ suggests that the two components are of different effective temperatures, the photometric value of ΔM as obtained from the above equation (1-26) must still be corrected for a difference in bolometric corrections appropriate for the two stars.

absolute radii $R_{1,2}$ from the Stefan's law (1-8); the semi-major axis $A = A_1 + A_2$ of the relative orbit becomes, in absolute units, equal to $R_1/r_1 = R_2/r_2$. In order to obtain these results without recourse to any spectrographic elements we have, however, been forced to assume that *both* components obey a statistical mass-luminosity relation and behave, therefore, as normal stars in this respect. Moreover, the results so obtained will be affected, not only by the uncertainty of the photometric elements P, $r_{1,2}$, $T_{1,2}$ and ΔM, but also by the dispersion inherent in the adopted mass-luminosity relation.

In order to illustrate also this whole procedure on a practical example, let us consider the well-known eclipsing system of U Ophiuchi, for which a first-class light curve was secured by Huffer (see Fig. 6-10) and analyzed for the photometric elements by Huffer and Kopal.* The spectrum of U Oph A is B5; and the ratio $J_1/J_2 = 1{\cdot}12 \pm 0{\cdot}02$ of surface brightnesses (at $\lambda = 4500$ Å) leads us to expect that the spectrum of U Oph B should be approximately B6. The fractional luminosity $L_1 = 0{\cdot}56 \pm 0{\cdot}01$ of the primary component corresponds, by (1-26), to a difference $\Delta M = 0{\cdot}26$ in blue light or (as the difference in bolometric corrections between the B5 and B6 spectral types is 0·13) to $\Delta M_{\rm bol} = 0{\cdot}39$. The effective temperature of the B5 component of U Oph should be, very approximately, $T_1 = 14800° = 2{\cdot}58\ T_{\odot}$.

The orbital period of the system is known to be $P = 1{\cdot}667$ days, and the fractional radius of its primary component $r_1 = 0{\cdot}263 \pm 0{\cdot}002$. In consequence, the constant D as defined by equation (1-22) turns out to be 0·266. Moreover, anticipating that the masses of both the B5 and B6 component of U Oph are likely to exceed $2\odot$, we shall adopt the mass-luminosity relation as represented by equation (3-1) later in this chapter to give $\alpha = 0{\cdot}45$ and $\beta = 0{\cdot}143$. When all these quantities are inserted in (1-29), the computed absolute bolometric magnitude M_1 of the B5 component of U Oph proves to be $-2\overset{m}{.}1$; and that of the B6 component, $-2\overset{m}{.}1 + 0\overset{m}{.}4 = -1\overset{m}{.}7$. The mass-luminosity relation (1-7) then yields $m_1 = 5{\cdot}6\odot$, $m_2 = 4{\cdot}9\odot$; and the Stefan's law (1-8) can be solved for R to yield $R_1 = 3{\cdot}5\odot$, $R_2 = 3{\cdot}2\odot$—making both components to conform satisfactorily to the Main Sequence.

The smallness of the difference $\Delta M = 0{\cdot}26$ in the blue magnitudes of the components of U Oph should lead one to expect that the lines of both stars should be visible in the combined spectrum of the system; and this expectation is indeed not disappointed. The spectrographic orbits of both components have in fact been established by Plaskett† long ago to provide a royal road for a determination of the absolute properties of the system of U Oph; and their latest values are $m_1 = 5{\cdot}3\odot$, $m_2 = 4{\cdot}7\odot$, and $R_1 = 3{\cdot}4\odot$, $R_2 = 3{\cdot}1\odot$, respectively.‡ A comparison of these values with those obtained above by our statistical procedure reveals a close agreement—which becomes

* C. M. Huffer and Z. Kopal, *Ap. J.*, **114**, 297, 1951.
† J. S. Plaskett, *Publ. D.A.O.*, **1**, 138, 1919.
‡ *Cf.* Z. Kopal and M. B. Shapley, *Catalogue of the Elements of Eclipsing Binary Systems* (*Jodrell Bank Ann.*, **1**, 141, 1956), Table I.

all the more striking when we consider that our procedure knew nothing whatever of the spectroscopic orbit except for its period.

Of all the absolute properties of the components of close binary systems, one stands apart from others in the minimum amount of information required for its specification: namely, the *mean density* of the two stars. As has already transpired from equation (1-20),

$$\bar{\rho}_{1,2} = \frac{0{\cdot}01344}{P^2 r_{1,2}^3} \frac{m_{2,1}}{m_1 + m_2} \odot \tag{1-30}$$

in solar units ($1{\cdot}408$ g/cm^3) if P is expressed in mean solar days. The reader may observe that, of all quantities on the right-hand side of (1-30), only the constant $0{\cdot}01344\ldots$ possesses a physical dimension; P, $r_{1,2}$ and m_2/m_1 being non-dimensional ratios. The latter may, moreover, be conveniently approximated by

$$\frac{m_2}{m_1} = 10^{-\beta\Delta M} = \left(\frac{L_2}{L_1}\right)^{0,4\beta}, \tag{1-31}$$

in accordance with (1-7) and (1-26).

An equation of the form (1-30) continues, moreover, to hold good irrespective of whether the components are spherical or distorted. In the latter case, $r_{1,2}$ should merely be understood to denote the fractional radii of spheres having the same volume as the respective distorted configurations. In the limiting case of contact configurations (*cf.* section III.3) whose geometry itself depends solely on the mass-ratio, equation (1-30) may be replaced by

$$\bar{\rho}_{1,2} = \frac{0{\cdot}008963}{P^2 v_{1,2}} \odot, \tag{1-32}$$

where the quantities $v_{1,2}$ can be found tabulated in columns (9) and (10) of Table 3-2 as functions of the mass-ratio. This latter formula gives, incidentally, the *minimum* value of $\bar{\rho}$ which the components in any binary system—not necessarily eclipsing—of known period and mass-ratio may actually attain. Moreover, as the maximum value of v becomes $0{\cdot}36075$ (attained if we allow the volume of the secondary component to become infinitesimal), the *absolute minimum* of the mean density $\bar{\rho}$ which any component of a close binary revolving in a period P may attain is given by

$$(\bar{\rho})_{\min} = \frac{0{\cdot}02485}{P^2} \odot. \tag{1-33}$$

Thus we find that both components of a system whose period P is equal to $\sqrt{0{\cdot}02485}$ d $= 3$ h 47 min must possess a mean density at least as high as the Sun ($1{\cdot}4$ g/cm^3); for those with $P = 5{\cdot}17$ d, $\bar{\rho}$ must be at least as high as air at atmospheric pressure ($0{\cdot}00093\ \odot$); and for those with $P = 19{\cdot}6$ d, must exceed that of hydrogen gas at one atmosphere ($0{\cdot}00065\ \odot$).

The *geometric mean* of the mean densities of both components in a binary system will, in accordance with equation (1-30), be given by

$$(\bar{\rho}_1\bar{\rho}_2)^{1/2} = \frac{0{\cdot}01344}{P^2(r_1r_2)^{3/2}} \frac{m_1m_2}{(m_1+m_2)^2} \odot, \tag{1-34}$$

where

$$\frac{m_1m_2}{(m_1+m_2)^2} \leqslant \tfrac{1}{4}. \tag{1-35}$$

Hence, whatever the mass-ratio,

$$(\bar{\rho}_1\bar{\rho}_2)^{1/2} \geqslant \frac{0{\cdot}00336}{P^2(r_1r_2)^{3/2}} \odot; \tag{1-36}$$

the equality sign being valid when $m_1 = m_2$. Moreover, the reader may recall from section VI.5 that the product r_1r_2 of fractional radii of the two components, occurring on the right-hand side of the preceding equation (1-36), is also contained in the constant

$$C_2 = \frac{r_1r_2}{\sin^2 i}, \tag{1-37}$$

which can be determined significantly from an analysis of the light changes of even such shallow partial eclipses which are insufficient to define r_1 and r_2 separately. In such cases, the formulae

$$\left.\begin{aligned}(\bar{\rho}_1\bar{\rho}_2)^{1/2} &= \frac{0{\cdot}01344}{P^2C_2^{3/2}} \frac{m_1m_2}{(m_1+m_2)^2} \csc^3 i \quad \odot\\ &\geqslant \frac{0{\cdot}00336}{P^2C_2^{3/2}} \csc^3 i \quad \odot\end{aligned}\right\} \tag{1-38}$$

may retain significance even when the individual mean densities $\bar{\rho}_{1,2}$ of the two components become well-nigh indeterminate.

The *harmonic mean* density of the system, defined by the equation

$$\bar{\rho}_{12} = \frac{\bar{\rho}_1\bar{\rho}_2}{\dfrac{m_2}{m_1+m_2}\bar{\rho}_1 + \dfrac{m_1}{m_1+m_2}\bar{\rho}_2} \tag{1-39}$$

becomes equal to

$$\bar{\rho}_{12} = \frac{m_1+m_2}{\frac{4}{3}\pi(r_1^3+r_2^3)A^3} = \frac{3\pi/G}{P^2(r_1^3+r_2^3)} \tag{1-40}$$

by virtue of Kepler's third law. As, however,

$$r_1^3 + r_2^3 \geqslant \tfrac{1}{4}(r_1+r_2)^3 \tag{1-41}$$

and

$$r_1 + r_2 \geqslant \sin \pi(d/P), \tag{1-42}$$

where d denotes the total duration of the eclipses, a combination of (1-41) and (1-42) with (1-40) leads to the following *upper limit*

$$\bar{\rho}_{12} \leqslant \frac{0{\cdot}0537}{P^2 \sin^3 \pi(d/P)} \odot \tag{1-43}$$

for the harmonic mean density of the system in terms of its period (in mean solar days) and the duration of the eclipses (in the same time unit).

The equality sign in (1-43) becomes valid for central eclipses ($i = 90°$) of equally large stars; but when $i < 90°$, it may be further refined in the following manner. As

$$r_1 + r_2 \geqslant 2\sqrt{r_1 r_2}, \tag{1-44}$$

a recourse to (1-37) reveals that

$$r_1 + r_2 \geqslant 2\sqrt{C_2} \sin i, \tag{1-45}$$

which can be used in place of (1-42) to assert that

$$\bar{\rho}_{12} \leqslant \frac{0{\cdot}00672}{P^2 C_2^{3/2}} \csc^3 i \odot; \tag{1-46}$$

the equality sign being valid if $r_1 = r_2$. Should, in addition, the masses of the two components also become equal, equations (1-38) and (1-46) become identical, as then $\bar{\rho}_1 = \bar{\rho}_2$ and their geometric and harmonic means are necessarily the same.

VII.2. CLASSIFICATION OF CLOSE BINARY SYSTEMS

In the foregoing section of this chapter the principal direct as well as indirect methods have been described for a determination of the absolute properties of close binary systems from different aspects of the available photometric and spectroscopic data. With the aid of such methods, the masses and absolute dimensions of the components of about a hundred close binaries have been individually established up to this time; and this number may be augmented by some fifty additional systems whose absolute properties have been approximated by a statistical approach. Confronted as we are with this considerable material, our aim should be to search it for indications of any latent groups of physical significance, whose existence may enable us to specify more fully the role of the double stars in the general framework of stellar evolution; and a quest for leads can scarcely start more profitably if we pause to ask ourselves the following question: *how many parameters are necessary and sufficient for a complete specification of the form of both components in a close binary system*?

This form should, in principle, be specified by the nature of the forces (of centrifugal and tidal origin) acting on the surface. Within the framework of the equilibrium theory of tides, the surfaces of constant density within a star should coincide with those of constant potential; and the boundary of zero density becomes a particular case of surfaces over which the potential Ψ arising from all forces which act upon it remains constant. If, moreover, the density concentration inside the constituent components is sufficiently high for their attraction to be approximable by that of central mass-points, the total potential of forces acting on any other point has already been investigated in Chapter III under the name of the *Roche equipotentials*, as defined by equation (0-1). Now if, in that equation, we identify the angular velocity ω of rotation about an axis perpendicular to the orbital plane with the Keplerian angular velocity of revolution, the outcome assumes the more explicit form

$$(1+q)C = 2r^{-1} + 2q(r'^{-1} - \lambda r) + n^2r^2(1+q)(1-\nu^2)n^2q^2(1+q)^{-1}, \tag{2-1}$$

where (as in Chapter III), λ, ν stand for the direction cosines of the radius-vector r of the distorted configuration (expressed in terms of the separation A of the two components taken as unity), $r'^2 = 1 - 2\lambda r + r^2$,

$$n^2 = \frac{\omega^2 A^3}{G(m_1 + m_2)} \tag{2-2}$$

denotes the ratio of the actual velocity ω of axial rotation of the respective star to the Keplerian angular velocity of orbital motion,

$$q = \frac{m_{2,1}}{m_{1,2}} \tag{2-3}$$

is the mass-ratio, and

$$C = \frac{2A\Psi_{1,2}}{G(m_1 + m_2)}, \tag{2-4}$$

the normalized value of the surface potential.

The form of the Roche equipotentials generated by setting $C =$ constant on the left-hand side of equation (2-1) will evidently depend on the adopted value of C as well as on n and q. We have already seen in section III.1 that if C is numerically large, the corresponding equipotentials represent two separate ovals enclosing each one of the two mass-centres and differ but little from spheres. With diminishing values of C the oval defined by (2-1) becomes increasingly elongated in the direction of the centre of gravity of the system until, for a certain critical value C_0 characteristic of each mass-ratio, both ovals will come in contact at a single point on the axis joining the centres of the two components to form a dumb-bell-like configuration (Fig. 3-2) which we called the *Roche limit*. For still smaller values of C the connecting part of the dumb-bell would open up and a single equipotential enclose both bodies, thus depriving us of the possibility to regard equation (2-1) as the representation of a binary system. For each $C > C_0$ equation (2-1) will, however, define two detached equipotentials which approximate the forms of

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centrally-condensed components in close binary system to a remarkable extent. It is, furthermore, clear from the analytic character of (2-1) that, for a given pair of n and q, the fractional size and form of each component will be specified by a single value of C.* Therefore, the question raised at the outset of this section turns out to admit of the following answer: *the minimum number of parameters sufficient for a complete geometrical description of a close binary system whose components rotate with Keplerian angular velocity is three, and consists of the values of* C_1, C_2 and q.

A determination of the ratios n and q from the spectroscopic and photometric data has been discussed in the preceding section of this chapter. On the other hand, the values of $C_{1,2}$ are not accessible to observation, nor do they follow directly from an analysis of the light curves of eclipsing binary systems. A knowledge of the coordinates of any arbitrary point $P(r, \lambda, \nu)$ on the surface of a Roche equipotential would, however, be sufficient to specify C uniquely. In actual practice, it is expedient to confine our attention to a pair of points at the intersection of the y-axis with the respective equipotential (*cf.* Fig. 3-1); for the absolute value of the y-coordinates at these points is (essentially) equal to the fractional radii $r_{1,2}$ of the two components, obtainable from an analysis of the light changes of the respective eclipsing system by the methods of Chapter VI. Since, then $r = r_{1,2}$ if $\lambda = \nu = 0$, it follows from (2-1) that

$$C_j = \frac{2(1+\mu_j)}{r_j} + \frac{2\mu_j}{\sqrt{1+r_j^2}} + r_j^2 + \mu_j^2, \tag{2-5}$$

where

$$\mu_j = \frac{m_{3-j}}{m_1 + m_2} \tag{2-6}$$

and $j = 1, 2$. Moreover, the uncertainty δC_j of the constants C_j evaluated with the aid of the preceding equations are (to a good approximation) equally distributed between the uncertainties δr_j and $\delta\mu_j$ of the basic data, and follow from the equation

$$\left(\frac{\delta C}{C}\right)^2 = \left(\frac{\delta\mu}{\mu}\right)^2 + \left(\frac{\delta r}{r}\right)^2; \tag{2-7}$$

while the uncertainties δC_0 of the constants specifying the Roche limits depend, of course, on those of the mass-ratios alone.

Suppose then that, after these preliminaries, we set out to determine the size and form of the Roche limits of all available eclipsing systems with known mass-ratios, in order to find out *how do their components actually fit in these limits*—or, in other words, what is the ratio of the Roche constants C_j characterizing free surfaces of the individual components to their limiting value C_0 in each particular system? The outcome of this quest reveals that

* Such a value, when introduced in (2-1), defines, to be sure, a pair of the equipotentials, each enveloping one of the two mass-centres, of which the one surrounding the respective (primary or secondary) component is relevant for our purpose.

this criterion permits us to divide the vast majority of all known eclipsing systems in the following three groups of similar characteristics:

I. *Detached Systems.* The volumes of both components are significantly *smaller* than their Roche limits (i.e., $C_1 > C_0$, and $C_2 > C_0$). Prototypes: β Aurigae or U Ophiuchi.

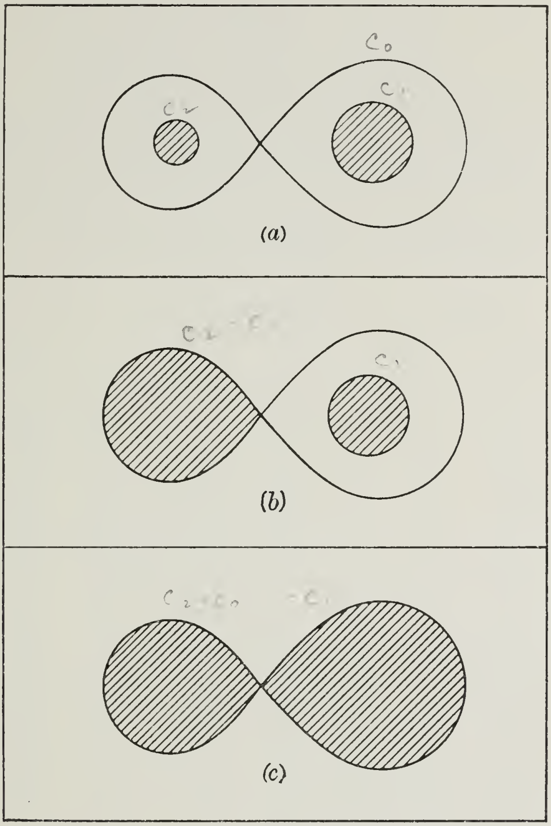

FIGURE 7-1. A SCHEMATIC VIEW OF THE THREE PRINCIPAL TYPES OF CLOSE BINARY SYSTEMS: (*a*) main sequence systems, (*b*) semi-detached systems (*c*) contact binaries.

II. *Semi-Detached Systems.* The primary (more massive) components are distinctly smaller than their Roche limits, but their secondaries appear to fill *exactly* the largest closed equipotentials capable of containing their whole mass (i.e., $C_1 > C_0$, but $C_2 = C_0$ within the limits of observational errors). Prototype: Algol.

III. *Contact Systems.* Both components appear to fill completely the respective loops of their Roche limits (i.e., $C_1 = C_2 = C_0$) and are therefore, probably in actual contact. Prototype: W Ursae Maioris.

The geometry of these types is illustrated on the accompanying Fig. 7-1 (drawn to scale for a mass-ratio of $m_2/m_1 = 0{\cdot}6$).

In what follows, this proposed system of classification will be further elaborated and more detailed properties of each group described in succession. The deeper significance of these groups and their respective places in the general framework of stellar evolution constitutes a supreme problem of double-star astronomy, which we shall discuss in the concluding section VII.7 of this book.

VII.3. DETACHED SYSTEMS

As the name suggests, a close binary system will hereafter be regarded as belonging to this group if the free surfaces of *both* of its components are *detached* from (i.e., smaller than) their respective Roche limits; and a list of the best known systems of this type with spectroscopically-established mass-ratios has been compiled in the following Tables 7-1 and 7-2. The headings

TABLE 7-1

Absolute Properties of Detached Eclipsing Systems

Star	Period	Spectra	m_1	m_2	R_1	R_2	M_1	M_2
	d						m	m
V 805 Aql	2·408	A2 + A6	1·85$\odot$	1·50$\odot$	2·16$\odot$	1·84$\odot$	0·7	1·6
σ Aql	1·950	B8 + B9	6·8	5·4	4·2	3·3	−1·9	−0·9
TT Aur	1·333	B3 + B7	6·7	5·3	3·3	3·2	−3·1	−1·8
WW Aur	2·525	A7 + F0	1·92	1·90	1·92	1·90	1·7	2·0
AR Aur	4·135	B9 + A0	2·55	2·30	1·82	1·82	0·3	0·6
β Aur	3·960	A2 + A2	2·33	2·25	2·48	2·27	−0·1	0·2
SZ Cam	2·698	09·5 + B2	21	6	10·1	4·5	−6·1	−3·7
AH Cep	1·775	B0 + B0·5	16·5	14·2	6·06	5·50	−5·6	−5·2
Y Cyg	2·996	09·5 + 09·5	17·4	17·2	5·9	5·9	−6·0	−6·0
V 477 Cyg	2·347	A3 + F6	2·4	1·6	1·5	1·2	1·7	3·7
YY Gem	0·814	M1 + M1	0·64	0·64	0·62	0·62	7·7	7·7
RX Her	1·779	A0 + A1	2·1	1·9	2·1	1·8	0·3	0·7
TX Her	2·060	A5 + F1	2·1	1·8	1·65	1·6	1·7	2·5
UV Leo	0·600	G0 + G1	1·36	1·25	1·21	1·20	4·0	4·1
U Oph	1·677	B5 + B6	5·30	4·65	3·4	3·1	−2·4	−1·9
V 451 Oph	2·197	A0 + A1·5	2·3	1·9	2·4	1·9	0·0	0·8
AG Per	2·029	B5 + B7	5·1	4·5	2·98	2·74	−2·3	−1·7

of the individual columns of these tables are self-explanatory; and most of their material has been taken from the recent *Catalogue of the Elements of Eclipsing Binary Systems* by Kopal and Shapley,* to which the reader is referred for the sources of the underlying observational data. Table 7-1 contains the absolute properties of the individual components of such systems, while Table 7-2 summarizes their geometrical characteristics.

An analysis of these data reveals that the stars found well inside their Roche limits cluster closely around the Main Sequence† (*cf.* Fig. 7-2) and

* Jodrell Bank *Annals*, **1**, 141, 1956.

† Exceptions to this fact are exhibited only by long-period systems of ζ Aurigae-type, whose primary components are late-type supergiants detached from their Roche limits (about which more will be said in section VII.7); and "undersize" subgiants of section VII.4, which are similarly detached. Detached supergiants of late spectral types continue, however, to conform to the mass-luminosity relation valid for detached stars; but the 'undersize' subgiants (being much nearer to their Roche limits) do not.

TABLE 7-2

Geometrical Properties of Detached Eclipsing Systems

Star	r_1	r_2	r_2/r_1	C_1	C_2	$(C_1/C_2)_0$	$(C_1/C_2)_C$	$O-C$	$(m_2/m_1)_0$	$(m_2/m_1)_C$	$O-C$
V 805 Aql	0·191 ± 0·006	0·163 ± 0·006	0·85 ± 0·04	6·9 ± 0·3	6·9 ± 0·3	1·00 ± 0·04	1·00 ± 0·01	0·00	0·81 ± 0·02	0·80 ± 0·04	+0·01
σ Aql	0·278 ± 0·011	0·218 ± 0·009	0·78 ± 0·04	5·1 ± 0·2	5·5 ± 0·3	0·93 ± 0·03	1·15 ± 0·01	−0·22	0·79 ± 0·01	0·68 ± 0·04	+0·11
TT Aur	0·283 ± 0·014	0·275 ± 0·028	0·97 ± 0·13	5·1 ± 0·3	4·7 ± 0·5	1·08 ± 0·07	1·02 ± 0·03	+0·06	0·85 ±	0·95 ± 0·14	−0·10
WW Aur	0·161 ± 0·001	0·160 ± 0·001	1·00 ± 0·01	7·6 ± 0·2	7·4 ± 0·2	1·03 ± 0·02	1·00 ± 0·00	+0·03	0·96 ± 0·02	1·00 ± 0·02	−0·04
AR Aur	0·098 ± 0·001	0·098 ± 0·001	1·00 ± 0·02	11·9 ± 0·5	11·0 ± 0·4	1·08 ± 0·05	1·00 ± 0·00	+0·08	0·90 ± 0·02	1·00 + 0·03	−0·10
β Aur	0·142 ± 0·005	0·130 ± 0·004	0·92 ± 0·03	8·4 ± 0·3	8·9 ± 0·3	0·94 ± 0·04	1·05 ± 0·01	−0·11	0·97 ± 0·01	0·90 ± 0·04	+0·07
SZ Cam	0·425 ± 0·003	0·188 ± 0·001	0·44 ± 0·01	4·3 ± 0·3	4·6 ± 0·3	0·93 ± 0·07	1·60 ± 0·06	−0·67	0·29 ± 0·02	0·25 ± 0·02	+0·04
AH Cep	0·324 ± 0·005	0·294 ± 0·008	0·91 ± 0·02	4·5 ± 0·1	4·6 ± 0·1	0·98 ± 0·02	1·06 ± 0·00	−0·08	0·86 ±	0·84 ± 0·02	+0·02
Y Cyg	0·208 ± 0·008	0·202 ± 0·008	0·97 ± 0·03	6·7 ± 0·5	6·8 ± 0·5	0·99 ± 0·06	1·02 ± 0·01	−0·03	0·99 ± 0·01	0·96 ± 0·03	+0·03
V 477 Cyg	0·124 ± 0·002	0·099 ± 0·002	0·80 ± 0·01	10·6 ± 0·2	9·7 ± 0·2	1·09 ± 0·04	1·14 ± 0·01	−0·05	0·67 ± 0·01	0·76 ± 0·02	−0·09
YY Gem	0·156 ± 0·001	0·156 ± 0·001	1·00 ± 0·02	7·7 ± 0·2	7·7 ± 0·2	1·00 ± 0·03	1·00 ± 0·00	0·00	1·00 ± 0·03	1·00 ± 0·02	0·00
RX Her	0·217 ± 0·004	0·190 ± 0·004	0·88 ± 0·05	6·1 ± 0·3	6·3 ± 0·3	0·97 ± 0·04	1·07 ± 0·02	−0·10	0·89 ± 0·06	0·83 ± 0·06	+0·06
TX Her	0·156 ± 0·003	0·154 ± 0·003	0·99 ± 0·04	8·1 ± 0·4	7·3 ± 0·5	1·11 ± 0·05	1·01 ± 0·01	+0·10	0·85 ± 0·03	0·99 ± 0·05	−0·14
UV Leo	0·295 ± 0·006	0·293 ± 0·005	1·00 ± 0·03	4·8 ± 0·6	4·6 ± 0·4	1·04 ± 0·13	1·00 ± 0·00	+0·04	0·93 ± 0·13	0·99 ± 0·03	−0·06
U Oph	0·263 ± 0·002	0·247 ± 0·002	0·94 ± 0·01	5·2 ± 0·2	5·2 ± 0·2	1·00 ± 0·04	1·04 ± 0·00	−0·04	0·88 ± 0·03	0·90 ± 0·02	−0·02
V 451 Oph	0·209 ± 0·004	0·167 ± 0·003	0·80 ± 0·02	6·4 ± 0·3	6·8 ± 0·3	0·94 ± 0·05	1·14 ± 0·00	−0·20	0·83 ± 0·02	0·73 ± 0·03	+0·10
AG Per	0·211 ± 0·007	0·194 ± 0·007	0·92 ± 0·03	6·2 ± 0·2	6·2 ± 0·2	1·00 ± 0·03	1·01 ± 0·01	−0·01	0·88 ± 0·03	0·89 ± 0·03	−0·01

conform statistically to an empirical mass-luminosity (m-M) relation (Fig. 7-3) as well as a corresponding mass-radius (m-R) relation (Fig. 7-4). Both these relations appear to be *linear* within the limits of observational errors, but their slope indicates a distinct change in statistical behaviour of massive and less massive stars (with a transition between $m \sim 1{\cdot}5$ and 2 ⊙).*

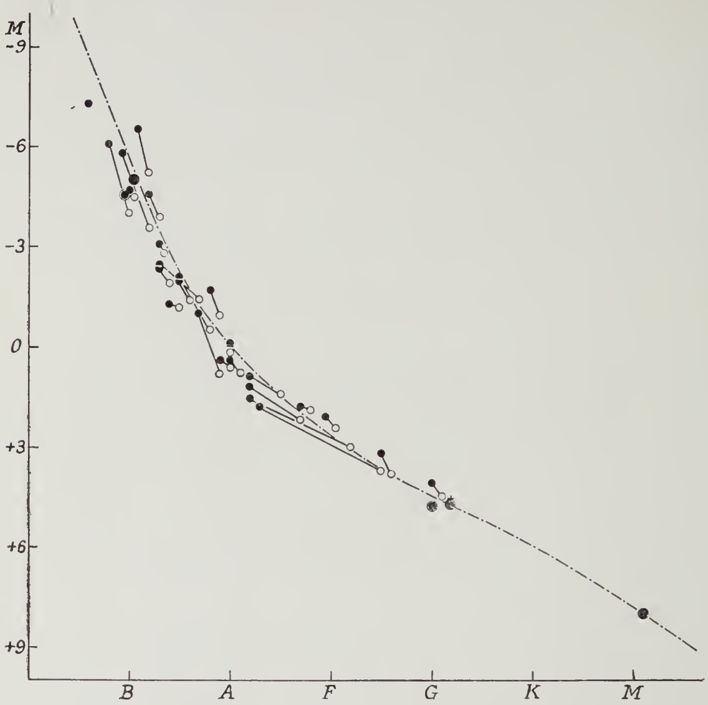

FIGURE 7-2. HERTZSPRUNG-RUSSELL DIAGRAM OF DETACHED BINARY SYSTEMS Full circles: primary (more massive) components; open circles, secondary components. Broken line represents the standard Main Sequence for single stars according to Morgan and Keenan.

For stars with $m > 2$ ⊙ we have, to a sufficient approximation,

$$\left.\begin{aligned} \log m &= 0{\cdot}45 - 0{\cdot}143M, \\ &= 1{\cdot}57 \log R - 0{\cdot}15; \end{aligned}\right\} \tag{3-1}$$

while for less massive stars ($m \ll 2\odot$)

$$\left.\begin{aligned} \log m &= 0{\cdot}42 - 0{\cdot}086M, \\ &= 1{\cdot}02 \log R; \end{aligned}\right\} \tag{3-2}$$

irrespective if they happen to be primary or secondary components. The lines defined by the foregoing equations have been graphically represented on Fig. 7-4 by broken dashes.

As a result of the properties just enumerated, the primary (more massive)

* Connected, no doubt, with a transition from the proton-proton reaction to the carbon-nitrogen cycle as the principal source of energy of the respective stars.

components in binary systems of this type are bound to be the *larger* of the two, and of *earlier* spectral type. If, therefore, such systems happen to be eclipsing variables, their *primary* (deeper) minima will represent eclipses of the *transit* type. The orbits of well-separated eclipsing systems of this type are frequently quite eccentric; but the orbital periods are seldom—if ever—subject to appreciable fluctuations or changes.

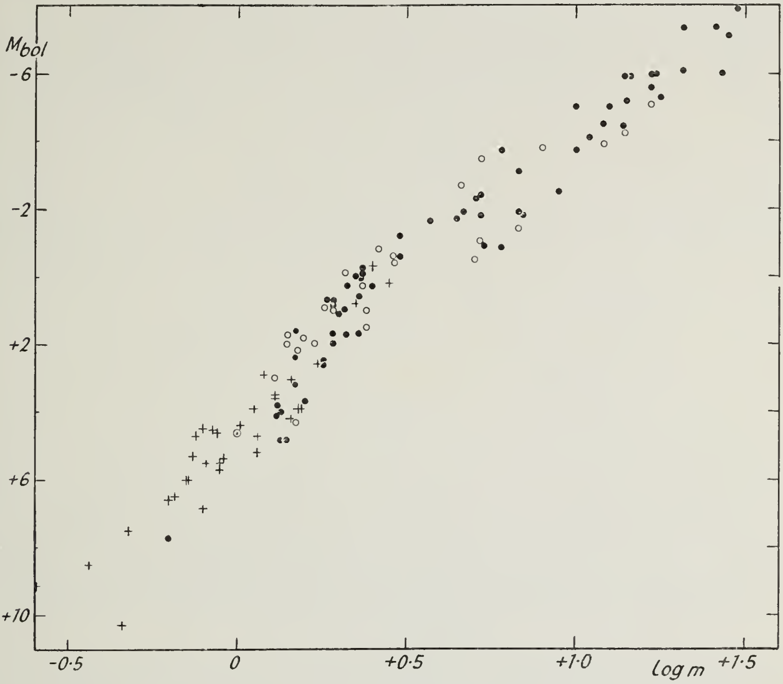

FIGURE 7-3. MASS-LUMINOSITY RELATION FOR DETACHED COMPONENTS OF BINARY SYSTEMS

Full circles (●) denote the detached components (primary or secondary) of close binary systems; open circles (○), the primary components of semi-detached systems. The crosses (+) indicate the components of visual binaries of known masses and luminosities; and the symbol ⊙ represents the position of our Sun.

One additional characteristic of the detached close binaries deserves explicit mention. A glance at columns (6)–(8) of Table 7-2 reveals that, *in systems consisting of Main-Sequence stars, the Roche constants $C_{1,2}$ characterizing the surface potentials of both components are sensibly equal.* Although the absolute value of surface potential of typical Main-Sequence stars varies by a factor of more than 10 from one end of the Main Sequence to another, the ratios C_1/C_2 do not seem to deviate from unity by more than $\pm 10\%$ and their mean value turns out to be $\overline{C_1/C_2} = 1{\cdot}007 \pm 0{\cdot}013$; the standard deviation of an individual value of C_1/C_2 from this mean being $\pm 0{\cdot}055$. In more general terms, the masses and absolute dimensions of the individual components in Main-Sequence detached systems appear to be apportioned in such a way as to render the resultant potentials over their free surfaces approximately the same—whatever the disparity in their masses and radii may be.

This phenomenon gives rise at once to some speculation as to its cause or possible evolutionary significance. Is, in particular, the prevailing near-equality of surface potentials of the components in close binary system a mere

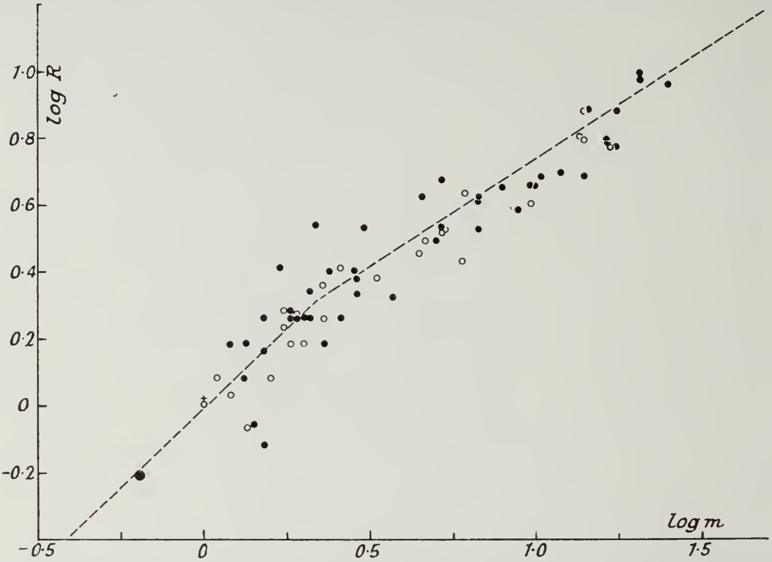

FIGURE 7-4. MASS-RADIUS RELATION FOR DETACHED COMPONENTS OF CLOSE BINARY SYSTEMS
Full circles: primary (more massive) components; open circles, secondary components. Broken lines represent the statistical relations as given by equations (3-1) and (3-2).

consequence of their present evolutionary stage, or is it a lingering reminiscence of the past from the ways of their origin? If, moreover, the mechanism of their formation was such that the surface potentials of both components were indeed exactly the same at the time of their origin,* how long could this

* Of all hypotheses concerning the origin of close binary systems which have so far been proposed, only the fission theory would seem to account logically for this fact. It is true that the original theory, as advanced by Darwin and Jeans several decades ago, has more recently become the target of well-founded grave criticisms. The original Darwin–Jeans theory involved, however, many simplifying assumptions (the stars to consist of a homogeneous liquid possessing no internal energy sources, etc.) which were made solely for the sake of mathematical tractability and which would be quite unacceptable today as a basis for any realistic discussion of the subject. So extreme a case of stellar hydrodynamics as a fission, under stress, of an originally single star in two components of comparable masses has, however, never yet been properly formulated—let alone solved! In particular, the oft-repeated argument that fission resulting in the formation of two bodies of comparable masses is impossible for stars exhibiting high degree of central condensation is largely irrelevant, because even though it is known that stars *in their near-equilibrium state* are indeed highly centrally-condensed, nobody knows as yet to what extent this central condensation may be weakened by increasing rotational disturbance, and what it may be *when the conditions ripe for fissions may have been attained.* As long as this is true, it seems impossible to rule fission out of the theories contesting for the explanation of the origin of close binary systems by any final verdict.

initial condition be remembered by two stars in the course of their subsequent separate evolution? None of these questions can yet be answered conclusively; but the problem raised by them admits of the following empirical test.

Suppose, for the sake of argument, that any vestige of initial conditions in the present characteristics of the individual components has long been obliterated by the passage of time, and that their present properties are the same as those of any pair of typical Main-Sequence stars, of the same masses, picked up at random in space. If so, the ratio of their Roche constants should (ignoring the distortion) be given by

$$\frac{C_1}{C_2} = \frac{m_1}{m_2}\frac{R_2}{R_1}, \tag{3-3}$$

which by use of the statistical relations (3-1) and (3-2) for the empirical mass-radius laws of Main-Sequence stars reveals that

$$\left.\begin{aligned} \log\frac{C_1}{C_2} &= 0{\cdot}57\log\frac{R_1}{R_2}, \qquad m > 2\,\odot; \\ &= 0{\cdot}02\log\frac{R_1}{R_2}, \qquad m \leqslant 2\,\odot. \end{aligned}\right\} \tag{3-4}$$

From these equations it transpires that, for stars of small masses, the ratios C_1/C_2 should indeed come very close to unity for any arbitrary combination of Main-Sequence stars, of average characteristics, regardless of their origin or age. However, for more massive stars ($m > 2\,\odot$) this should cease to be statistically true and a more specific test of the underlying assumption becomes possible.

In order to carry it out, we have evaluated the theoretical ratios of C_1/C_2 from the above equations (3-4), based on the assumed random association of average Main-Sequence stars, and listed them in column (8) of Table 7-2 after the observed values of this ratio; and the next column (9) of the same table then contains the corresponding (O–C)-differences. With the exception of a somewhat anomalous case of SZ Cam, the mean value of the theoretical C_1/C_2's turns out to be $1{\cdot}044 \pm 0{\cdot}014$, as compared with the 'observed' mean of $1{\cdot}007 \pm 0{\cdot}013$. The respective ($O$–$C$)-differences of column (9) appear accordingly to be of systematic character (i.e., mostly negative). The observational material at present at our disposal is, unfortunately, not extensive enough to enable us to follow this argument further in greater detail; but future investigations along this line, based on larger material, may well lead to valuable new information on the past history of detached binary systems.

In conclusion, it may be added that the near-equality of the potentials over free surfaces of the components in detached binary systems entails one interesting consequence. If, accordingly, we set

$$C_1 = C_2, \tag{3-5}$$

an appeal to equation (2-1) reveals that (3-5) represents, in fact, *a relation between the fractional radii* $r_{1,2}$ *and the mass-ratio* m_2/m_1 *of the two components*; for inserting (2-1) in (3-5) we can solve the outcome for m_2/m_1 to find that, for systems obeying (3-5),

$$\left.\begin{aligned} \frac{m_2}{m_1} &= \frac{r_2}{r_1}\left\{\frac{2 - r_1 + r_1(r_1 - r_2) - 2r_1(1 + r_2)^{-1/2}}{2 - r_2 + r_2(r_2 - r_1) - 2r_2(1 + r_1)^{-1/2}}\right\} \\ &= \frac{r_2}{r_1}\left\{\frac{2 - 3r_1 + r_1^3 - \dots}{2 - 3r_2 + r_2^3 - \dots}\right\}, \end{aligned}\right\} \tag{3-6}$$

the error of which should be sensibly equal to that of the linear function

$$\delta\left(\frac{m_2}{m_1}\right) = 2\left\{\frac{\delta r_2}{r_2(2 - 3r_2)} - \frac{\delta r_1}{r_1(2 - 3r_1)}\right\}, \tag{3-7}$$

whose value can be ascertained by the methods of the Appendix to the previous Chapter VI. In particular, it should be observed that the errors $\delta r_{1,2}$ of the respective fractional radii (as deducible from an analysis of the light changes) are not mutually independent. If, moreover, we are willing to ignore the terms $3r_{1,2}$ in comparison with 2, the uncertainty of a mass-ratio based on (3-6) becomes identical with that of the ratio k of the radii in the respective eclipsing system.

In view of the second one of equations (3-4), the condition (3-5) at the basis of (3-6) should certainly be fulfilled, for detached Main-Sequence systems of moderate or small masses, to enable us to determine their mass-ratios by way of (3-6) from the fractional radii of their components alone; and the same is true, to a lesser extent, of all detached Main-Sequence systems of any mass. The penultimate column (11) of Table 7-2 gives such theoretical mass-ratios evaluated from equation (3-6) for the 17 Main-Sequence systems compiled in it; and the ultimate column (12) contains the respective *O–C* differences obtained by subtracting the computed mass-ratios of column (11) from their spectroscopically observed values as given in the preceding column (10). As the reader can easily verify, the r.m.s. value of all the *O–C* residuals becomes ±0·07—i.e., of the same magnitude as the observational uncertainty of the spectroscopic values of these ratios.

VII.4. SEMI-DETACHED SYSTEMS

Having surveyed some properties of close binary systems whose *both* components are smaller than their Roche limit and belong to the Main Sequence, let us turn our attention next to the second group of our classification outlined in section VII.2, whose primary components are (and continue to behave as) detached stars, but *the secondaries happen to be in contact with their Roche limits.* A list containing the best known representatives of this group with spectroscopically established mass-ratios is given in the following Tables 7-3 and 7-4. The headings of their individual columns are self-explanatory; and their material has been taken again from the Kopal and Shapley's *Catalogue.**

TABLE 7-3

Absolute Properties of Semi-detached Eclipsing Systems

Star	Period	Spectra	m_1	m_2	R_1	R_2	M_1	M_2
	d						m	m
U Cep	2·493	B8 + G8	2·9 ⊙	1·4 ⊙	2·4 ⊙	3·9 ⊙	−0·6	2·3
U CrB	3·452	B5 + (A5)	6·5	2·5	3·5	5·5	−2·4	−0·9
u Her	2·051	B3 + B7·5	7·9	2·8	4·5	4·3	−3·8	−2·1
V Pup	1·454	B1 + B3·5	16·6	9·8	6·0	5·3	−5·1	−3·9
U Sge	3·381	B9 + G2	6·7	2·0	4·1	5·4	−1·4	1·2
V 356 Sgr	8·896	B3 + A2	12	4·7	5·0	13	−3·9	−3·2
μ^1 Sco	1·446	B3 + B7	14·0	9·2	4·8	5·3	−4·2	−3·3
TX UMa	3·063	B8 + G3	2·8	0·85	2·16	3·79	−0·4	2·1
Z Vul	2·455	B3 + A2·5	5·4	2·3	4·7	4·7	−3·5	−1·0
RS Vul	4·478	B5 + (F9)	4·6	1·4	3·9	5·3	−2·6	0·9

An examination of these data reveals that whereas the primary (more massive) components of these systems continue to behave statistically as normal Main-Sequence stars of their mass (see again Fig. 7-2), conforming to the empirical mass-luminosity and mass-radius relations as given by equations (3-1) and (3-2), their secondary (less massive) components lie systematically *above* the Main-Sequence—being too luminous for their mass—and exhibit other characteristics (density, spectrum) generally associated with *subgiants.* This is diagrammatically demonstrated on the accompanying figures showing the HR-diagram of semi-detached systems (Fig. 7-5) and the empirical mass-luminosity relation of contact stars (Fig. 7-6) to which frequent references will be made later on. As a result of the properties summarized on these diagrams, it transpires that if binary systems of this type happen to be eclipsing variables, their *primary* components will, as a rule, be the *smaller* of the two, and of *earlier* spectral type. The primary (deeper) minima of such systems are, therefore, likely to be due to *occultation* eclipses. The orbits of all known semi-detached systems of this group are sensibly circular; but the orbital periods often exhibit complicated fluctuations.

A glance at column (8) of Table 7-4 reveals the most important characteristic of semi-detached binary systems: namely, *the ratios C_2/C_0 of the Roche constants for subgiant secondaries are sensibly equal to unity.* Their mean is $1{\cdot}005 \pm 0{\cdot}007$, and the standard deviation of a single value of C_2/C_0 from this mean becomes $\pm 0{\cdot}021$ which is well within the limits of observational errors. A more detailed inspection of the data on hand discloses, moreover, no evidence of real dispersion: the fractional dimensions of the secondary components appear to coincide with those of the corresponding Roche limits (as given in the ultimate column 9 of Table 7-4) the more closely, the greater the precision of the underlying observational data. This fact suggests that

*Jodrell Bank *Ann.*, **1**, 14, 1956; with the exception of Z Vul for which the elements as published subsequently by D. H. Popper (*Ap. J.*, **126**, 53, 1957) have been adopted.

TABLE 7-4

Geometrical Properties of Semi-detached Eclipsing Systems

Star	r_1	r_2	m_2/m_1	C_0	C_1	C_2	C_2/C_0	$(r_2)_c$
U Cep	0·19 ± 0·01	0·31 ± 0·01	0·49 ± 0·02	3·941 ± 0·006	7·9 ± 0·4	3·9 ± 0·1	0·99 ± 0·03	0·311 ± 0·005
U CrB	0·175 ± 0·010	0·274 ± 0·008	0·38 ± 0·02	3·898 ± 0·009	8·9 ± 0·5	4·0 ± 0·3	1·03 ± 0·06	0·291 ± 0·004
u Her	0·301 ± 0·003	0·287 ± 0·003	0·35 ± 0·02	3·881 ± 0·012	5·6 ± 0·2	3·9 ± 0·2	1·00 ± 0·05	0·285 ± 0·004
V Pup	0·368 ± 0·006	0·327 ± 0·004	0·58 ± 0·02	3·966 ± 0·004	4·4 ± 0·1	4·0 ± 0·2	1·01 ± 0·04	0·324 ± 0·004
U Sge	0·210 ± 0·002	0·278 ± 0·003	0·30 ± 0·02	3·847 ± 0·014	7·9 ± 0·5	3·8 ± 0·3	0·99 ± 0·06	0·272 ± 0·005
V 356 Sgr	0·11 ± 0·01	0·28 ± 0·01	0·38 ± 0·03	3·898 ± 0·013	14 ± 1	4·0 ± 0·2	1·03 ± 0·05	0·292 ± 0·007
μ^1 Sco	0·313 ± 0·003	0·347 ± 0·004	0·66 ± 0·02	3·980 ± 0·003	4·9 ± 0·1	3·9 ± 0·1	0·98 ± 0·03	0·337 ± 0·004
TX UMa	0·158 ± 0·001	0·277 ± 0·001	0·30 ± 0·02	3·847 ± 0·014	10·3 ± 0·6	3·8 ± 0·3	0·99 ± 0·06	0·272 ± 0·004
Z Vul	0·31 ± 0·01	0·31 ± 0·02	0·43 ± 0·03	3·920 ± 0·011	5·3 ± 0·3	3·9 ± 0·3	0·99 ± 0·04	0·301 ± 0·005
RS Vul	0·186 ± 0·005	0·26 ± 0·01	0·31 ± 0·03	3·85 ± 0·02	8·8 ± 0·8	4·0 ± 0·5	1·04 ± 0·08	0·275 ± 0·007

this coincidence is probably *exact* in these systems, and possibly in many others as well.

It should be stressed that semi-detached eclipsing systems with subgiant secondaries are in reality much more numerous than would appear from their sample listed in Tables 7-3 and 7-4. Photometrically, such systems are

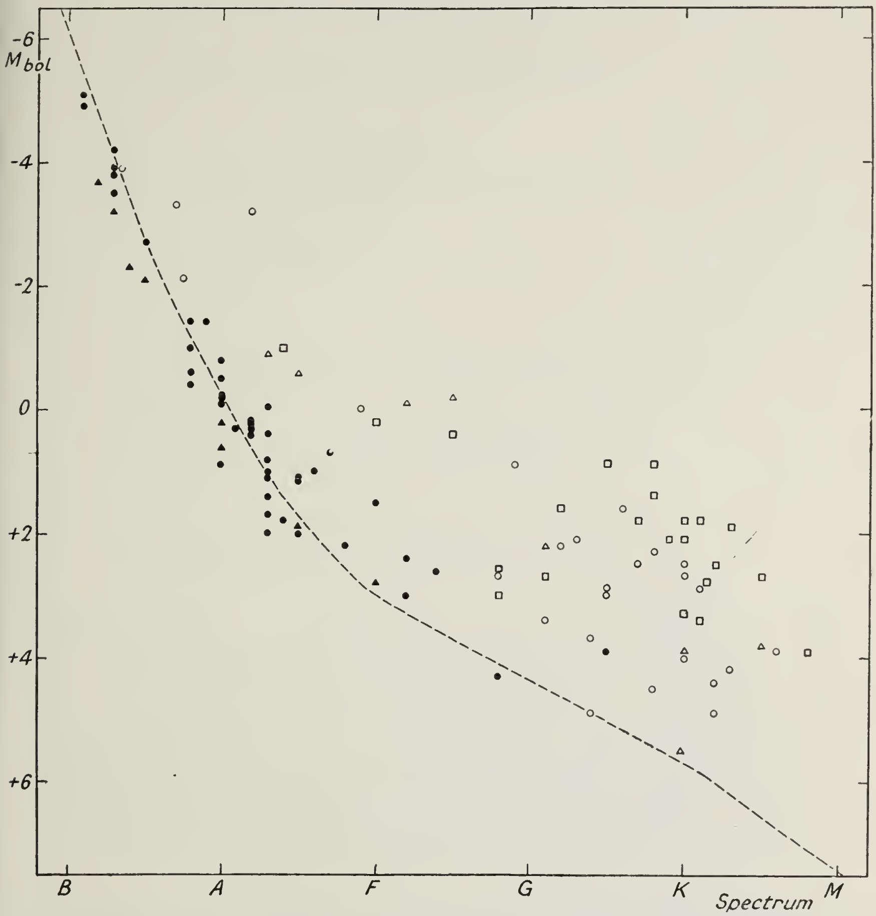

FIGURE 7-5. HR-DIAGRAM OF ECLIPSING SYSTEMS WHICH POSSESS SUBGIANT SECONDARY COMPONENTS

Full circles (●) denote the positions of the primary (detached) components of subgiant-possessing systems; open circles (○) indicate those of the contact secondaries; and open squares (□), those of "undersize" secondaries which are smaller than their Roche limits. Triangles represent the positions of the primary (▲) and secondary (△) components of R CMa-type systems. The dashed line on the diagram schematizes the run of the Keenan-Morgan standard Main Sequence for stars of luminosity class V.

easily recognizable by a great disparity in depths of the alternate minima due to total (or advanced partial) eclipses. A considerable difference in luminosities of the two components (due to their very different surface brightness) is, however, bound to render a large majority of such pairs single-spectra systems—with the secondary's spectrum becoming observable only around totality at the bottom of the primary minima (which seldom last long enough to enable us to establish the slope of the secondary's velocity-curve from which spectroscopic mass-ratio could be deduced). This accounts for the smallness of the sample collected in Table 7-3, which has been limited to systems with known spectroscopic mass-ratios, We can, however, extend its

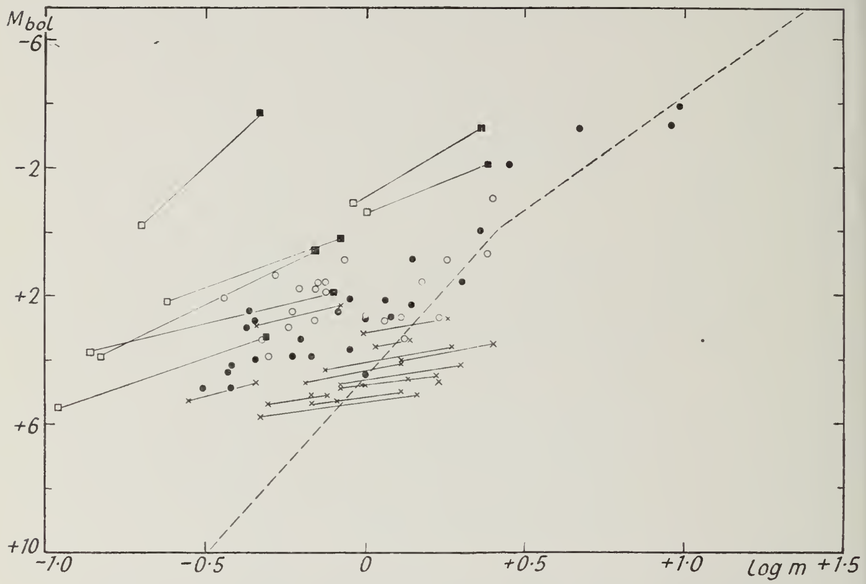

FIGURE 7-6. MASS-LUMINOSITY RELATION FOR CONTACT COMPONENTS OF CLOSE BINARY SYSTEMS

Full circles (●) denote the secondary components at their Roche limits; open circles (○) indicate the "undersize" subgiants; squares represent the primary (■) and secondary (□) components of the systems of R CMa-type. Contact components of the systems of W UMa-type are marked on the diagram by crosses (×). The dashed line indicates the empirical mass-luminosity relation valid for detached components of Main-Sequence systems, as represented by equations (3-1) and (3-2).

data considerably *if we set out to determine the mass-ratios of all known single-spectrum systems with subgiant secondaries on the assumption that these secondaries*—like those of Table 7-4—*fill completely their respective Roche limits.* The correspondence between the fractional size of such limits and the mass-ratio has already been investigated quantitatively in section III.3 of this book, and the requisite information can be found in Table 3-1. These data are, to be sure, exact only for contact configurations rotating with the Keplerian angular velocity about an axis perpendicular to the orbital plane; but are rather insensitive to moderate departures from these conditions.

The following Table 7-5 then contains a list of additional 18 semi-detached systems with known single-spectrum orbits, whose mass-ratios and absolute

TABLE 7-5

Single-Spectrum Systems with Contact Secondary Components

Star	Period	Spectra	$A_1 \sin i$	$f(m)$	r_1	r_2	i	m_2/m_1	m_1	m_2	R_1	R_2	M_1	M_2
	d												m	m
RT And	0·629	F8 + (gG8)	1·14	0·152 $\odot$	0·233	0·334	85°8	0·65	1·5 $\odot$	0·98 $\odot$	0·97 $\odot$	1·4 $\odot$	4·3 $\odot$	4·5
TW And	4·123	F0 + (gK1)	1·70	0·0115	0·156	0·238	86·7	0·19	2·4	0·46	2·4	3·7	1·5	2·8
RZ Cas	1·195	A0 + (gG1)	1·13	0·041	0·241	0·284	82·1	0·35	1·8	0·63	1·5	1·8	0·9	3·4
W Del	4·806	A0 + (gK0)	1·94	0·013	0·153	0·245	85·1	0·21	2·1	0·44	2·5	4·0	−0·1	2·5
Z Dra	1·357	A5 + (gK2)	0·92	0·017	0·234	0·263	86·3	0·27	1·4	0·38	1·5	1·6	2·0	4·9
TW Dra	2·807	A6 + (gG7)	2·40	0·071	0·212	0·306	86·6	0·43	1·9	0·82	2·4	3·5	1·0	2·5
RX Hya	2·282	A8 + (gK3)	1·25	0·015	0·180	0·258	86·5	0·25	1·5	0·38	1·6	2·3	2·2	4·2
SX Hya	2·896	A3 + (gK6)	2·00	0·038	0·118	0·284	80·8	0·35	1·7	0·59	1·3	3·2	2·0	3·9
Y Leo	1·686	A3 + (gK0)	1·4	0·038	0·251	0·295	88·1	0·41	1·1	0·46	1·8	2·1	1·4	4·0
δ Lib	2·327	A0 + (gG2)	2·42	0·104	0·301	0·302	79·5	0·44	2·6	1·1	3·5	3·5	−0·8	2·2
TY Peg	3·092	A2 + (gG1)	0·98	0·0039	0·217	0·220	87·9	0·15	1·5	0·23	2·3	2·4	0·6	2·9
RT Per	0·849	F2 + (gG4)	0·60	0·0117	0·310	0·255	86·9	0·24	1·3	0·31	1·4	1·1	3·0	4·9
RY Per	6·864	B4 + (F2)	2·6*	0·015*	0·12	0·24	83	0·20	2·5	0·49	2·7	5·4	−2·3	−0·1
ST Per	2·648	A3 + (gG5)	1·20	0·0099	0·190	0·234	90	0·18	2·4	0·43	2·2	2·6	1·0	3·0
β Per	2·867	B8 + (gK0)	1·73	0·0253	0·227	0·239	82·0	0·19	5·2	1·0	3·6	3·8	−1·0	2·7
V 505 Sgr	1·183	A1 + (gF8)	1·69	0·137	0·316	0·314	80·8	0·52	2·3	0·2	2·3	2·3	0·3	2·7
X Tri	0·972	A3 + (gG4)	1·47	0·134	0·283	0·330	88·5	0·63	1·4	0·90	1·6	1·8	1·7	3·7
W UMi	1·701	A3 + (gG5)	2·03	0·115	0·388	0·308	83·0	0·48	2·4	1·1	3·5	2·8	−0·1	2·9
UW Vir	1·811	A2 + (gK2)	1·05	0·0139	0·187	0·256	86	0·24	1·6	0·37	1·5	2·0	1·8	4·4

References.—For references to basic data (both photometric and spectroscopic) the reader is referred to Kopal and Shapley's *Catalogue of the Elements of Eclipsing Binary Systems* (*Jodrell Bank Annals*, **1**, 141, 1957), for all but four systems to be listed below. Of the pairs of following references, the first indicates the source of photometric elements and the second, the spectroscopic data:

SX Hya : M. B. Shapley, *Harv. Bull.*, No. 797, 1924.
Sanford, *Ap. J.*, **86**, 153, 1937.

Y Leo : Shapley, *P. A. S. P.*, **30**, 343, 1918.
Struve, *Ap. J.*, **102**, 1945, 74.

ST Per : Wood, *Princ. Contr.*, No. 21, 1946.
Struve, *Ap. J.*, **104**, 253, 1946. The values of $a_1 \sin i$ and $f(m)$ as given by Struve in this paper (and quoted in Table VII of our Paper I), are, however, numerically incorrect and inconsistent with Struve's semi-amplitude $K_1 = 33$ km/sec. Corrected values are given above.

UW Vir : M. B. Shapley, *Harv. Bull.*, No. 848, 1927.
Struve, *Ap. J.*, **106**, 2, 1947.

* The adopted values of $a_1 \sin i$ and $f(m)$ for RY Per refer to the velocity curve based on the hydrogen lines. Helium lines lead to substantially larger values of $a_1 \sin i = 3{\cdot}4 \times 10^6$ km and $f(m) = 0{\cdot}033\ \odot$.

dimensions have been obtained by an identification of the fractional size of their secondary components with their Roche limit. Columns (4) and (5) indicate the spectroscopic values of $A_1 \sin i$ (in units of 10^6 km) and of the mass-function $f(m)$ (in solar masses); while column (9) contains the mass-ratios corresponding, on contact hypothesis, to the fractional radii r_2 of the secondary components as listed in the preceding column (7). As many decimals are retained in all columns as are regarded to be significant. Their entries may be uncertain by as much as a few units of the last place; for a more specific discussion of the errors the reader should consult the references listed at the foot of the table.

How do we know, however, that the masses and absolute dimensions of these systems, based on the hypothetical 'contact' mass-ratios are genuine? The test is provided for us by the properties of the *primary* components obtained on this basis. It goes without saying that the 'contact' mass-ratio furnishes a clue to the absolute masses and luminosities of *both* components; and if the primary star is well detached from its Roche limit, it should lie on the Main Sequence and conform to the empirical mass-luminosity and mass-radius relations established in the preceding section VII.3. In order to verify whether or not this is indeed the case, we have plotted the primary components of Table 7-5 together with all other detached stars on Figs. 7-2 and 7-3, and found that they do not show any systematic difference from the behaviour of components in purely detached systems. The semi-detached nature of the 18 systems compiled in Table 7-5 is thereby confirmed; and they have been added to those of the Tables 7-3 and 7-4 on the HR-diagram of Fig. 7-4 and the mass-luminosity relation of Fig. 7-5.

It should, however, be also mentioned that other eclipsing systems—similar to those listed in Table 7-5—exist for which an assumed contact nature of their secondary components would displace the primary seriously off the Main Sequence and render its mass too large for its luminosity. If so, the only way of diminishing it (for a given mass-function) is to *increase* the mass-ratio m_2/m_1 above that appropriate for contact hypothesis, and thus render the secondary component somewhat *smaller* than its Roche limit. As the mass of the primary reacts very sensitively to any change of the mass-ratio, 'undersize' subgiant secondaries—which are inferior in size to their Roche limits—can be satisfactorily segregated in this way from genuine contact configurations. The absolute dimensions of single-spectrum systems possessing them can then be best determined by fitting the primary component to the empirical mass-luminosity relation in the manner described in section VII.1. This has been done with 13 systems listed in the following Table 7-6, to which five additional two-spectra systems of similar characteristics have been added. The arrangement and contents of the individual columns is exactly the same as in the preceding Table 7-5. The mass-ratios of column (9) are the values necessary to render the primary components genuine Main-Sequence stars; while the values given alongside in parentheses represent the ratios that would follow on contact hypothesis. The locations

TABLE 7-6

Eclipsing Systems with 'Undersize' Subgiant Secondaries

Star	Period	Spectra	$A_1 \sin i$	$f(m)$	r_1	r_2	i	m_2/m_1	m_1	m_2	R_1	R_2	M_1	M_2
	d												m	m
KO Aql	2·864	A0 + (gF8)	1·49	0·0161 ⊙	0·20	0·16	90°	0·20 (0·05)	2·9 ⊙	0·58 ⊙	2·6 ⊙	2·1 ⊙	−0·1	3·0
QY Aql	7·230	F0 + (gG9)	3·57	0·035	0·185	0·240	88	0·28 (0·195)	2·7	0·75	4·4	5·7	0·2	1·6
Y Cam	3·305	A7 + (gK1)	1·6	0·015	0·233	0·228	85	0·21 (0·165)	2·2	0·48	3·0	3·0	0·7	3·4
S Cnc	9·485	A0 + gG5	6·52	0·123	0·086	0·193	84·3	0·36*(0·095)	6·8	2·4	3·4	7·6	−0·8	0·7
RS CVn	4·798	F4 + gK5	6·04	0·383	0·092	0·28	80·9	0·93*(0·34)	1·8	1·7	1·7	5·1	2·8	2·9
TV Cas	1·813	A0 + gF8	2·19	0·128	0·307	0·283	80·8	0·56*(0·34)	1·7	1·0	2·6	2·4	−0·2	2·6
RS Cep	12·420	A5 + (gK3)	6·06	0·058	0·068	0·213	85·4	0·38 (0·13)	2·0	0·75	2·2	6·8	1·1	1·9
SW Cyg	4·573	A2 + (gK1.5)	2·58	0·033	0·149	0·257	85·5	0·28 (0·245)	2·5	0·70	2·5	4·4	0·3	2·8
VW Cyg	8·430	A3 + (gK0)	3·88	0·033	0·111	0·220	89	0·28 (0·145)	2·5	0·70	2·8	5·7	0·4	1·8
WW Cyg	3·318	B8 + (gG2)	3·10	0·108	0·207	0·261	87·3	0·36 (0·26)	4·3	1·5	3·5	4·4	−1·4	1·6
RX Gem	12·209	A4 + (gG8)	5·07	0·018	0·070	0·190	84	0·22 (0·09)	2·4	0·53	2·4	6·0	0·7	1·4
RY Gem	9·301	A2 + (gK1)	4·00	0·022	0·080	0·208	85	0·23 (0·12)	2·7	0·62	2·6	6·3	0·2	1·8
Z Her	3·993	F2 + gG1	4·80	0·284	0·12	0·185	83·5	0·87*(0·08)	1·5	1·3	1·8	2·8	2·4	2·7
TT Hya	6·953	A3 + (gG7)	3·90	0·049	0·100	0·227	84·2	0·36 (0·16)	2·0	0·71	2·1	4·8	1·1	1·8
AR Lac	1·983	G5 + gK0	3·26	0·356	0·169	0·313	86	0·99*(0·51)	1·3	1·3	1·5	2·9	3·8	3·4
AQ Peg	5·548	A2 + (gK2)	2·59	0·023	0·134	0·242	85	0·24 (0·20)	2·5	0·59	2·6	4·7	0·3	2·5
RW Per	13·198	A5 + (gK0)	3·36	0·0087	0·071	0·163	90	0·18 (0·053)	2·0	0·36	2·2	5·1	1·1	2·1
Y Psc	3·766	A3 + (gK8)	1·90	0·019	0·161	0·236	84·8	0·24 (0·185)	2·1	0·51	2·3	3·3	0·8	3·9

* Spectroscopic mass-ratios.

REFERENCES.—Of the pairs of references whenever given, the first indicates the source of photometric data, and the second, of spectroscopic data:

KO Aql: PLAUT, *Groningen Publ.*, No. 54, 1950. SAHADE, *Ap. J.*, **102**, 470, 1945.
QY Aql: WHITNEY, *Ap. J.*, **108**, 519, 1948. STRUVE, *Ap. J.*, **103**, 76, 1946.
Y Cam: KOPAL and SHAPLEY, *Catalogue of the Elements of Eclipsing Binary Systems*, Jodrell Bank Ann., **1**, 141, 1957.
S Cnc: KOPAL and SHAPLEY, *op cit.*
RS CVn: KOPAL and SHAPLEY, *op. cit.*
TV Cas: KOPAL and SHAPLEY, *op. cit.*
RS Cep: SHAPLEY, *Princ. Contr.*, No. 3, 1915. STRUVE, *Ap. J.*, **104**, 253, 1946.
SW Cyg: SHAPLEY, *Princ. Contr.*, No. 3, 1915. STRUVE, *Ap. J.*, **104**, 253, 1946.
VW Cyg: SHAPLEY, *Princ. Contr.*, No. 3, 1915. STRUVE, *Ap. J.*, **103**, 76, 1946.
WW Cyg: SHAPLEY, *Princ. Contr.*, No. 3, 1915. STRUVE, *Ap. J.*, **104**, 253, 1946.
RX Gem: GAPOSCHKIN, *Ap. J.*, **104**, 376, 1946.
RY Gem: GAPOSCHKIN, *Ap. J.*, **104**, 383, 1946. MCKELLAR, *Publ. D. A. O.*, **8**, 235, 1950
Z Her: KOPAL and SHAPLEY, *op. cit.*, supra.
TT Hya: REILLY, *Pop. Astr.*, **55**, 332, 1947. SAHADE and CESCO, *Ap. J.*, **103**, 71, 1946.
AR Lac: KOPAL and SHAPLEY, *op cit.*, supra.
AQ Peg: GAPOSCHKIN, *Berl. Bab. Veröff.*, **9**, No. 5, 1932. STRUVE, *Ap. J.*, **103**, 76, 1946.
RW Per: SHAPLEY, *Princ. Contr.*, No. 3, 1915. STRUVE, *Ap. J.*, **102**, 74, 1945.
Y Psc: SHAPLEY, *Princ. Contr.*, No. 3, 1915. STRUVE, *Ap. J.*, **104**, 253, 1946.

TABLE 7-7

Eclipsing Systems of the R CMa-type

Star	Period	Spectra	$A_1 \sin i$ (in 10^6 km)	$f(m)$	r_1	r_2	i	m_2/m_1	m_1	m_2	R_1	R_2	M_1	M_2
	d						°						m	m
R CMa	1·136	F0 + (gG9)	0·491	0·0037 ⊙	0·275	0·254	76·5	0·24	0·49 ⊙	0·11 ⊙	1·1 ⊙	1·0 ⊙	3·3	5·5
RW Gem	2·865	B5 + F5	2·56	0·082	0·233	0·303	89	0·45	1·9	0·85	2·8	3·6	−1·9	1·2
T LMi	3·020	A0 + (gK0)	1·04	0·0049	0·217	0·248	86·6	0·22	0·69	0·15	1·8	2·1	0·6	3·9
TU Mon	5·049	B5 + A5	3·8	0·085	0·18	0·30	80	0·44	2·3	1·0	3·0	4·9	−2·1	−0·6
UU Oph	4·397	A0 + (gG1)	1·81	0·0123	0·19	0·27	85	0·29	0·84	0·24	2·2	3·1	0·2	2·2
XZ Sgr	3·276	A3 + (gK1)	1·03	0·0040	0·098	0·197	85·5	0·10	4·8	0·48	1·6	3·2	1·6	3·2
RZ Sct	15·190	B2 + (F5)	4·91	0·020	0·150	0·306	76·8	0·47	0·47	0·20	3·4	6·9	−3·7	−0·2
S Vel	5·934	A5 + K5	1·4	0·0031	0·112	0·233	90	0·175	0·80	0·14	1·5	3·1	1·9	3·8

REFERENCES.—For references to basic data (both photometric and spectroscopic) relative to R CMa, RW Gem, and T LMi, cf. KOPAL and SHAPLEY'S *Catalogue of the Elements of Eclipsing Binary Systems* (Jodrell Bank Annals, **1**, 101, 1957). For all other stars such references are given below (of the pairs of references given in each case, the first again pertains to the source of photometric elements, and the second to that of the spectrographic data):

TU Mon : GAPOSCHKIN, *Berl. Bab. Veröff.*, **9**, No. 5, 1932.
DEUTSCH, *Ap. J.*, **102**, 433, 1945.

UU Oph : GAPOSCHKIN, *Berl. Bab. Veröff.*, **9**, No. 5. 1932.
STRUVE, *Ap. J.*, **103**, 76, 1945.

XZ Sgr : GAPOSCHKIN, *Berl. Bab. Veröff.*, **9**, No. 5, 1932.
SAHADE, *Ap. J.*, **109**, 439, 1949.

RZ Sct : SHAPLEY, *Princ. Contr.*, No. 3, 1915.
NEUBAUER and STRUVE, *Ap. J.*, **101**, 240, 1945.
STRUVE, *M. N.*, **109**, 487, 1949.

S Vel : SHAPLEY, *Princ. Contr.*, No. 3, 1915.
SAHADE, *Ap. J.*, **116**, 35, 1952.

of the 'undersize' subgiants in the HR-diagram of Fig. 7-4 are marked by open squares; and their relative positions in the mass-luminosity diagram of contact stars (Fig. 7-5) are shown as open circles.

The significance of the results contained in Tables 7-3 to 7-6 will be discussed more fully in section VII.7 later on. Here we merely wish to point out that, out of a total of 55 eclipsing systems with subgiant secondary components examined in this section, 37 of them (i.e., 67%) turn out to possess secondaries that are in contact with (or at least indistinguishably near to) their Roche limits; while in the remaining 18 systems (33% of the total) the secondaries appear to be 'undersize' subgiants which are appreciably smaller than their Roche limits.

These data do not, however, exhaust yet the variety of objects encountered under the heading of this section. The following Table 7-7 will introduce to us an even more remarkable—though small—group of close binary systems which differ from those compiled in Tables 7-3–7-6 in so far as their *both* components appear to be grossly over-luminous for *any* permissible value of the mass-ratio. Earlier in this section we pointed out that if an assumption of contact nature of the secondary component renders the primary much too massive for its luminosity, the adoption of a larger value of m_2/m_1 leading (for a given mass-function) to a smaller value of m_1 may restore the primary component to normalcy (and detach the secondary from its Roche limit). Now there exists a small but highly significant group of eclipsing systems—listed in the following Table 7-7—for which even the *minimum* mass-ratios m_2/m_1, obtained on the assumption of contact nature of the secondary component, lead to the values of m_1 which are, not too large, but too *small* for the observed luminosity of the primary component; and any increase of m_2/m_1 would exaggerate this disparity still further. In point of fact, the masses of both components of the eight systems listed in Table 7-7 represent the *maximum* masses which these stars can possess if their secondaries' sizes are not to exceed their Roche limits; and a comparison of the data given in columns (9) and (10) with those in (14) and (15) reveals that even these maximum masses render *both* components grossly over-luminous. The extent of this anomaly can be gathered by a glance at Figure 7-5, on which the components of such systems have been marked as squares; while Fig. 7-4 reveals that at least their primary components continue to conform fairly satisfactorily to the Main Sequence.

The best-known example of this remarkable group of stars is undoubtedly R Canis Maioris; and the smallness of its mass has previously been commented upon by several investigators.* Our present results are in accord with estimates of the masses and luminosities of its components by the writers just quoted; but they also reveal that R CMa, while typical of other systems of this group, represents by no means the most extreme case. This latter distinction undoubtedly belongs at present to RZ Scuti—a most peculiar eclipsing system of relatively long orbital period of 15·19 days, whose principal

* K. Walter, *Zs. f. Ap.*, **19**, 157, 1940; F. B. Wood, *Princ. Contr.*, No. 21, 1946.

component (the only one visible spectroscopically) exhibits a spectrum of class B2. The only available set of photometric elements of RZ Sct is due to Shapley.* According to him the fractional radius of the secondary component is $r_2 = 0{\cdot}306$ which, on contact hypothesis, points to a mass-ratio $m_2/m_1 = 0{\cdot}47$. If, however, we combine this ratio (and Shapley's value of $i = 77°$ for orbital inclination) with the mass-function $f(m) = 0{\cdot}0204 \odot$ of the single-spectrum orbit of RZ Sct as established by Neubauer and Struve,† the mass of the primary B2 component results as $m_1 = 0{\cdot}42 \odot$; and that of the secondary (which is spectroscopically invisible, but whose surface brightness corresponds to an F5 type), $m_2 = 0{\cdot}20 \odot$!

The absolute radii of both stars are $R_1 = 3{\cdot}4 \odot$ and $R_2 = 6{\cdot}9 \odot$, corresponding to absolute bolometric magnitudes of the components $M_1 = -3^{m}_{\cdot}7$ and $M_2 = -0^{m}_{\cdot}2$. The values of R_1 and M_1 are quite consistent with the B2 spectrum of RZ Sct A, and would locate this star fairly well on the Main Sequence—the striking peculiarity is only the fact that the mass of this early B-star of $-3{\cdot}7$ absolute bolometric magnitude appears to be less than one-half of that of the sun! In cther words, RZ Sct A would seem to be a B2 Main-Sequence star of average size and luminosity—except that it is largely empty inside, containing scarcely one-tenth of the mass which a Main-Sequence star of its absolute magnitude would be expected to possess and maintaining, therefore, its extravagant external appearances on manifestly shaky grounds. And the same is true of the secondary component of RZ Sct in an even more exaggerated manner. The abnormally small mass of both stars is made manifest by the weak gravitational bond between them (as revealed by long duration of the orbital period, and low orbital velocity); but if both components were travelling singly through space, no obvious sign would warn us that such outwardly normal stars are in reality but over-distended 'empty windbags' posing on the Main Sequence (or even above it) under false pretences! And RZ Sct is by no means the only system which we have caught in this act; seven other eclipsing binaries (i.e., full 15% of systems considered in this section), listed in Table 7-7, are found to differ from RZ Sct in degree rather than in kind. How can the stars whose masses are so small maintain—even temporarily—their observed sizes and energy output remains one of the most provocative puzzles of the current theories of stellar structures, and its solution is nowhere yet in sight.

In conclusion of this survey of available data on close binary systems of semi-detached type, one additional but highly significant piece of *negative* observational evidence should be pointed out: namely, *the non-existence of semi-detached systems* of the converse type, *in which the more massive component fills completely its Roche limit*, while its companion is significantly *smaller* than this limit. This complementary type of semi-detached systems seems indeed conspicuous by its absence from all lists of known eclipsing systems and—let it be stressed—its absence *cannot* be explained on grounds of

* H. Shapley, *Princ. Contr.*, No. 3, 1915.

† F. Neubauer and O. Struve, *Ap. J.*, **101**, 240, 1945.

observational selection; if anything the converse should be true.* For the larger fractional size of primary contact components would cause eclipses to occur for a wider range of orbital inclinations. The type of such eclipses would depend on the ratio of surface brightnesses of the two stars. The primary (more massive) component would necessarily possess greater luminosity; but being so much larger than its mate, its surface brightness would be likely to be less. In consequence, the primary (deeper) minima in hypothetical systems of this type would as a rule be caused by occultation eclipses of long duration—which constitute the most probable type of eclipses for photometric discovery. As none have, however, been detected so far, the reasons of their apparent non-existence must be of fundamental nature; but their discussion must now be postponed till we return to it in the concluding section VII.7.

VII.5. DYNAMICS OF MATTER EJECTED FROM UNSTABLE COMPONENTS

In the foregoing section VII.4 we have established the existence of a large group of close binary systems in which secondary components appear to fill completely the largest closed equipotentials capable of containing their whole mass. Whatever the cause of this striking phenomenon may be, the observed clustering of the subgiant secondaries around their Roche limits must evidently be the result of some *non-equilibrium process* (which we shall attempt to specify later in section VII.7). It cannot represent a random distribution of fractional dimensions of the stars in static equilibrium, which could assume any value inside their Roche limits; for the probability of a distinction exhibiting so peculiar a peak just at the Roche limit is negligible in so large a sample.

Suppose then, for the sake of argument, that the stars at their Roche limits do not represent equilibrium configurations, but are secularly contracting or expanding. If they were contracting, the distribution of the ratios C_2/C_0 should tend in the course of time to exhibit peak for *large* values; and there would be no reason why any of them should linger around the Roche limit. If, on the other hand, these stars are secularly expanding, there is a compelling reason why their fractional sizes should cluster around this limit; for the growth of a slowly expanding star is bound *arrested* there—thus accounting for the observed bottleneck—and once this maximum distension permissible on dynamical grounds has been attained, a continuing tendency to expand should bring about a secular *loss of mass* from the contact star; and the manner of its ejection should completely specify all its subsequent motion. The aim of the present section will be to investigate more closely the conditions under which matter can be expelled from rotating stars

* *Cf.* Z. Kopal, *Proc. Nat. Acad. Sci.* (India), 26, 462, 1957.

in expansion, and its subsequent motion under the gravitational influence of the two components of close binary systems.

If we disregard minor perturbations arising from the finite degree of central condensation of both stars in such systems and regard them acting, for dynamical purposes, as two mass-points, the motion of a gas particle (or of a cloud of mass negligible in comparison with those of the two finite mass centres) will be governed by the well-known differential equations of the restricted problem of three bodies. In setting up such equations, we shall hereafter regard the orbit of the two finite bodies of masses $m_{1,2}$ as circular and adopt their mutual separation A as our unit of length, while the combined mass $m_1 + m_2$ of the system will be taken as our unit of mass; the unit of time being, moreover, chosen so that the value of the gravitation constant G shall become unity. The cartesian coordinates x, y, z will hereafter refer to the instantaneous position of the infinitesimal mass-particle in a *moving* rectangular system, rotating with the Keplerian angular velocity of the orbit of the two finite masses in such a way that the x-axis coincides constantly with the line joining m_1 and m_2, while the z-axis is perpendicular to the plane of their orbit; the centre of gravity of the system being taken as the origin of coordinates. If, moreover, we restrict at first our attention to motions in the equatorial xy- plane of the two components, the differential equations of so restricted a dynamical problem will assume the well-known form

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2\frac{dy}{dt} &= \frac{\partial U}{\partial x}, \\ \frac{d^2y}{dt^2} + 2\frac{dx}{dt} &= \frac{\partial U}{\partial y}, \end{aligned}\right\} \tag{5-1}$$

where the potential U of forces acting upon our particle assumes the form

$$U = \tfrac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \tag{5-2}$$

and where

$$\left.\begin{aligned} r_1^2 &= (x - \mu)^2 + y^2, \\ r_2^2 &= (x + \mu - 1)^2 + y^2, \end{aligned}\right\} \tag{5-3}$$

denote the squares of the distances of any arbitrary point in the xy-plane from the two finite mass-centres. Lastly,

$$\mu = \frac{m_2}{m_1 + m_2} \leqslant \tfrac{1}{2} \tag{5-4}$$

will hereafter stand for the (normalized) mass of the secondary component.

The dynamical system represented by equations (5-1)–(5-3) is one of fourth order; but on account of its non-linearity caused by (5-3) it cannot be solved completely in a closed form. Particular closed integrals are, however, known to exist. Thus we know since the days of Lagrange that by setting

$$\left.\begin{aligned} x &= x_{1,2,3} - \mu, \\ y &= 0, \end{aligned}\right\} \tag{5-5}$$

where $x_{1,2,3}$ are roots of the equations (3-5), (5-2) and (5-3) of Chapter III,* or again

$$\left.\begin{aligned} x &= \mu - \tfrac{1}{2}, \\ y &= \pm\tfrac{1}{2}\sqrt{3}, \end{aligned}\right\} \tag{5-6}$$

both sides of equations (5-1) can be made to vanish identically and, hence, the *constant* values (5-5) or (5-6) represent particular solutions of our problem. Equations (5-5) define the positions of the well-known *Lagrangian collinear points* L_1, L_2, L_3 on the x-axis; while (5-6) define the positions of additional such points L_4, L_5 as vertices of two *equilateral triangles* on either side of the x-axis. The fact that both velocities *and* accelerations vanish simultaneously at these points imply that a mass particle placed there would remain permanently at rest with respect to the rotating xy-frame (i.e., would describe circular orbits in space with the same period as that of the two finite masses) unless disturbed by forces extraneous to our system.

What is the physical significance of the Lagrangian points in the orbital planes of close binary systems? Any matter which may happen to find itself at such a point will evidently be in neutral equilibrium, and a minimum energy will be required for its infinitesimal displacement. Moreover, any matter which will approach it may be 'trapped' in its neighbourhood for an indefinite time—and, as a result of this bottleneck, *gas condensations could gradually develop around such points*, with the possibility that they may actually become spectroscopically observable.

Suppose that this were indeed the case: what would be the amplitudes of radial-velocity changes which we should expect? Such variations are known to be proportional to the distance of the moving gas from the centre of gravity of the system; and this centre is located on the x-axis at a distance μ from the centre of mass of the primary component. If, therefore, $K_{1,2}$ denote the (normalized) amplitudes of the radial-velocity changes of the two stars of masses $m_{1,2}$, we evidently have

$$\left.\begin{aligned} K_1 &= \mu, \\ K_2 &= 1 - \mu, \end{aligned}\right\} \tag{5-7}$$

and the similar amplitudes of the motion of any gas agglomerating around the collinear points L_1, L_2, L_3 should be

$$\left.\begin{aligned} k_1 &= x_1 - \mu, \\ k_2 &= x_2 - \mu, \\ k_3 &= -x_3 + \mu, \end{aligned}\right\} \tag{5-8}$$

* In which $q = \mu/(1 - \mu)$.

while any matter agglomerating at the triangular points L_4, L_5 should likewise be characterized by the amplitudes

$$k_{4,5} = (1 - \mu + \mu^2)^{1/2} \tag{5-9}$$

and phases displaced by half a period. A tabulation of the normalized velocities $K_{1,2}$ and k_j ($j = 1, 2, 3, 4, 5$) expressed in terms of the unity employed in this section can be found in the accompanying Table 7-8. In order to

TABLE 7-8

q	K_1	K_2	k_1	k_2	k_3	$k_{4,5}$
1·0	0·50000	0·50000	0·00000	1·19841	1·19841	0·86603
0·8	0·44444	0·55556	0·07851	1·21704	1·17856	0·86781
0·6	0·37500	0·62500	0·17734	1·23804	1·15251	0·87500
0·4	0·28571	0·71429	0·30724	1·25967	1·11751	0·89214
0·3	0·23077	0·76923	0·39010	1·26840	1·09538	0·90691
0·2	0·16667	0·83333	0·49189	1·27141	1·06917	0·92796
0·15	0·13043	0·86957	0·55349	1·26770	1·05415	0·94158
0·1	0·09091	0·90909	0·62660	1·25609	1·03774	0·95778
0·05	0·04762	0·95238	0·72113	1·22558	1·01984	0·97706
0·02	0·01961	0·98039	0·80495	1·17908	1·00815	0·99034
0·01	0·00990	0·99010	0·84863	1·14624	1·00412	0·99509
0·005	0·00498	0·99502	0·88137	1·11796	1·00208	0·99752
0·001	0·00099	0·99901	0·93132	1·06990	1·00041	0·99950
0·0002	0·00020	0·99980	0·95981	1·04088	1·00008	0·99990
0	0·00000	1·00000	1·00000	1·00000	1·00000	1·00000

convert them to absolute units, they should be multiplied by the value of our velocity unit

$$v_0 = \{G(m_1 + m_2)/A\}^{1/2}, \tag{5-10}$$

appropriate for each particular system.

One additional closed integral of equations (5-1)–(5-3) exists which holds good for particles describing any trajectories: namely, as has first been shown by Jacobi, if the first one of equations (5-1) is multiplied by $2(dx/dt)$ and the second by $2(dy/dt)$, their sum can be integrated to yield

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2U - C, \tag{5-11}$$

where U continues to be given by equation (5-3) and C is a constant.* If, moreover, the derivatives x and y on the left-hand side of (5-11) are allowed

* Should the orbit of the two finite bodies become eccentric, so that U becomes an explicit function of the time, the vis-viva integral (5-11) can be shown (*cf.* F. R. Moulton, *Periodic Orbits*, Washington 1920, p. 253) to assume the form

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2U - 2\int \frac{\partial U}{\partial t}\,dt - C, \tag{5-12}$$

where C continues to be a constant.

to vanish identically, the resulting equation then defines the *surfaces of zero velocity* as

$$C = \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} + x^2 + y^2, \tag{5-13}$$

and becomes in fact identical with equation (1-3) of Chapter III defining the Roche equipotentials; the constant C is the same as that in equation (3-12) of the same chapter. This identity should cause no surprise; for if the Roche equipotentials are to represent surfaces of equilibrium under the gravitational influence of the two finite masses, any particle placed on them must indeed possess zero velocity (for otherwise such surfaces would have to expand or shrink).

The Roche equipotentials, whose geometry we discussed in some detail in Chapter III, are thus seen to assume an added significance: for not only do they describe the external form of the individual components in close binary systems, but also *impose a barrier which no mass particle moving inside their folds under the gravitational influence of the two finite bodies can ever penetrate*; since if its velocity is real on one side of such folds, it becomes imaginary on the other. If, in particular, the value of C as specified by the initial conditions of any trajectory is such that the corresponding surfaces of zero velocity are *closed* around the binary system, the particle following such a trajectory can never escape the system and remains a part of it for all time.

In quest of conditions under which this may or may not be true, it is important to realize that the Roche equipotentials investigated in Chapter III represent only one branch of the family of surfaces of zero velocity, as defined by equation (5-13). In order to demonstrate it suppose that the value of C on the left-hand side of equation (5-13) is large. In order that it be so, it is necessary that either r_1 or r_2 be small (which would lead to closed ovals surrounding the two finite mass centres, whose geometry we investigated already in Chapter III), or that $x^2 + y^2$ be large—in which case equation (5-13) defines a 'curtain' in the direction of the z-axis, whose intersection with the xy-plane becomes very nearly a circle. As $C \to \infty$, the two ovals containing m_1 and m_2 shrink to points, and the radius of the 'curtain' becomes infinite. With diminishing value of C the ovals expand and the curtain shrinks—until, for $C = C_2$ as tabulated in column (3) of Table 3-6, the curtain and the largest common envelope (i.e., the outer Roche limit) which originated by coalescence of the two originally distinct ovals and which surrounds now both stars come into contact at L_2. When this happens, the outer Roche limit has developed a comical point at L_2—similar to the one developed previously at L_1 when $C = C_1$—and for $C < C_2$ the sole remaining surface of zero velocity intersecting the xy-plane opens up at L_1 while the 'curtain' has disappeared from it in the direction of increasing $\pm z$-coordinate; and the opening continues to grow until, for still smaller value of $C = C_3$ as tabulated in column (5) of Table 3-6 a third conical point develops at L_3. For $C < C_3$, the surviving surfaces of zero velocity will become open at both ends: their intersection with the xy-plane splits up in two separate sections (symmetrical with respect to the x-axis) closing down gradually on the Lagrangian triangular points $L_{4,5}$ and vanishing from the xy-plane for values of $C_{4,5}$ as given by equation (5-13) of Chapter III and tabulated in column (6) of Table 3-6. A gradual development and transformation of the surfaces of zero velocity for four different values of the mass-ratios and a sufficiently wide range of C (summarized in Table 7-9) has recently been studied quantitatively

TABLE 7-9

The Values of C for the Computed Surfaces of Zero Velocity

$q = 1$	$q = 0{\cdot}8$	$q = 0{\cdot}6$	$q = 0{\cdot}4$
5·00000	5·00000	5·00000	5·00000
4·00000	3·99417	3·96993	3·90749
3·99000*	3·50417	3·53108	3·55894
3·96000	3·49368	3·47993	3·41749
3·84000*	3·41509	3·35791	3·27822
3·51000	3·35417	3·32993	3·26749
3·45680	3·18417	3·15993	2·90749
3·36000	2·99417	2·96993	2·90000
3·00000	2·78417	2·80000	
2·76000	2·76000	2·76000	
2·75500	2·75500		
2·75005			

* To prevent overcrowding of the diagram only a part of the curve has been plotted.

by V. Hewison using the University of Manchester's electronic computers, and her results are diagrammatically shown on the accompanying Fig. 7-7 to 7-10.

What do these data bear on the possibility of mass escape from close binary systems? The foregoing argument makes it obvious that *no particle moving in the xy-plane can escape the system if the value of C characterizing its trajectory is greater than* C_2 (and no escape in *any* direction in space is possible if $C > C_1$). If $C < C_2$, escape becomes possible if the particle in motion manages to navigate out through the ends of the surface of zero velocity opening up at L_2 or L_3 (and not to collide with it, rebounding in normal direction). Lastly, for $C < C_{4,5}$, the particle may travel anywhere it wishes and will no longer meet with any obstacle in the xy-plane. Now, if consistent with the views expressed earlier in this section, the mass particle is likely to escape from the inner Lagrangian point L_1 on the surface of a contact component, its value of C will be *less* than C_1 and such that, consistent with the vis-viva integral (5-11),

$$C_1 - C = (v/v_0)^2\,, \tag{5-14}$$

where v denotes the initial velocity of ejection and v_0 our velocity unit as given by equation (5-10). If

$$(v/v_0)^2 = C_1 - C_2 \tag{5-15}$$

the particle ejected from L_1 and moving henceforward freely under the

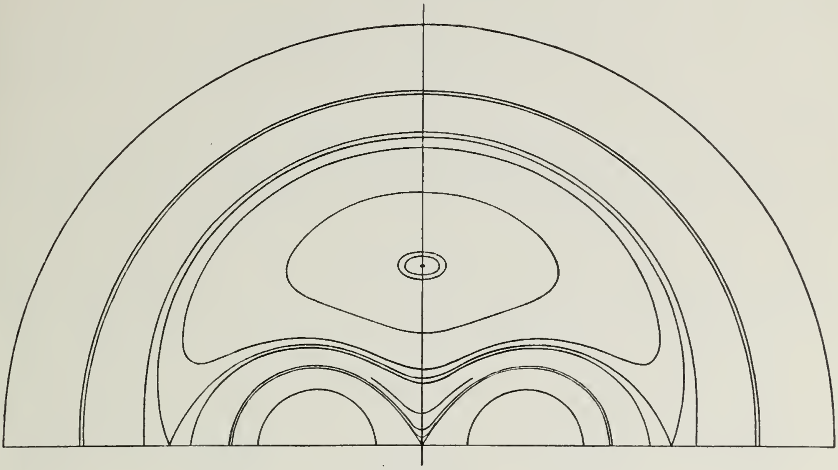

FIGURE 7-7.
CROSS-SECTIONS OF THE JACOBIAN SURFACES OF ZERO VELOCITY WITH THE *xy*-PLANE for the mass-ratio $m_2/m_1 = 1$ and the values of the constants C as listed in Table 7-9.

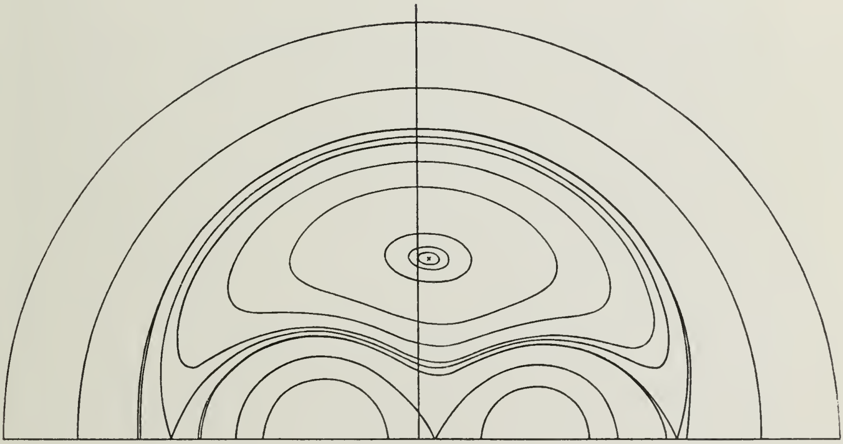

FIGURE 7-8.
CROSS-SECTIONS OF THE JACOBIAN SURFACES OF ZERO VELOCITY WITH THE *xy*-PLANE for the mass-ratio $m_2/m_1 = 0{\cdot}8$ and the values of the constants C as listed in Table 7-9.

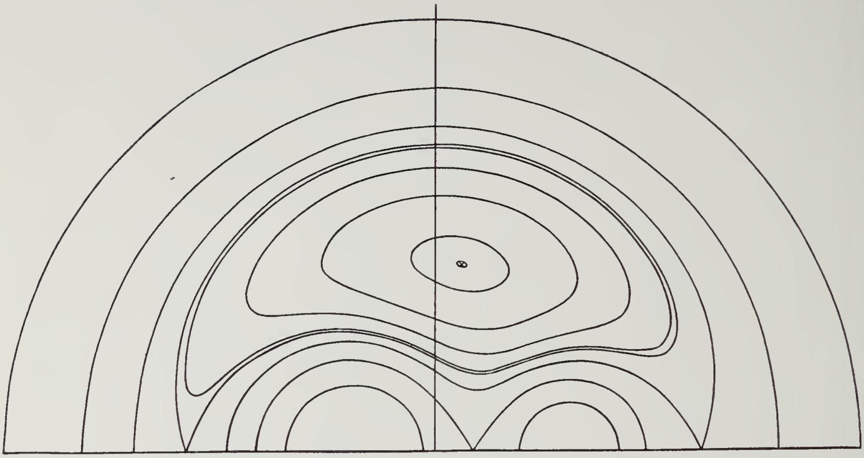

FIGURE 7-9.
CROSS-SECTIONS OF THE JACOBIAN SURFACES OF ZERO VELOCITY WITH THE xy-PLANE for the mass-ratio $m_2/m_1 = 0{\cdot}6$ and the values of the constants C as listed in Table 7-9.

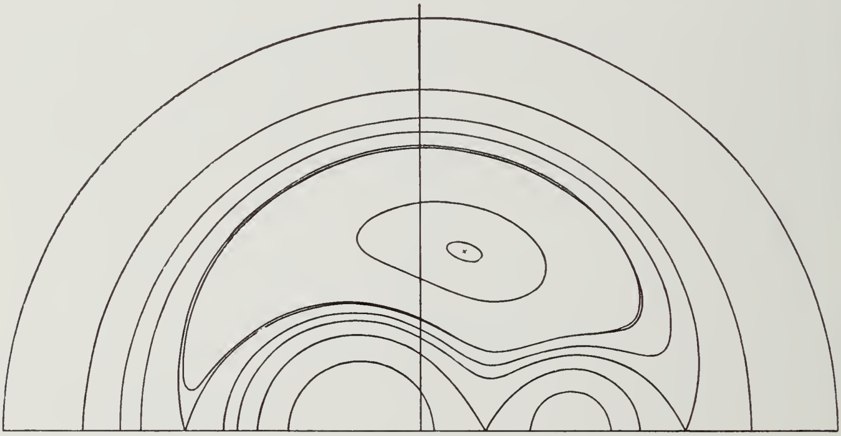

FIGURE 7-10.
CROSS-SECTIONS OF THE JACOBIAN SURFACES OF ZERO VELOCITY WITH THE xy-PLANE for the mass-ratio $m_2/m_1 = 0{\cdot}4$ and the values of the constants C as listed in Table 7-9.

attraction of the two finite masses cannot under any conditions leave the system (being trapped by the 'curtain'). It *may* do so (depending on the direction of launching) if its initial velocity obeys the inequality

$$C_1 - C_{4,5} < (v/v_0)^2 - C_1 < C_2\,; \tag{5-16}$$

and is free to escape in any direction whenever

$$(v/v_0)^2 > C_1 - C_{4,5}\,. \tag{5-17}$$

A tabulation of the actual values of these critical velocities of ejection at the five different Lagrangian points can be found in the accompanying Table 7-10.

TABLE 7-10

Normalized Velocities $v_E/v_0 = \sqrt{C_1 - C_j}$ *of Ejection from Contact Secondary Components in Close Binary Systems, for which the Corresponding Surfaces of Zero Velocity Pass Through the Lagrangian Point* L_j

q	$j = 1$	$j = 2$	$j = 3$	$j = 4, 5$
1	0·00000	0·73702	0·73702	1·11803
0·8	0·00000	0·70745	0·76097	1·11404
0·6	0·00000	0·66246	0·78232	1·09741
0·4	0·00000	0·59038	0·79326	1·05431
0·3	0·00000	0·53712	0·78784	1·01240
0·2	0·00000	0·46105	0·76416	0·94228
0·1	0·00000	0·34459	0·69259	0·80804
0	0·00000	0·00000	0·00000	0·00000

The actual form of the trajectories governed by the equations (5-1)–(5-2) in the *xy*-plane depends, of course, on the initial conditions of escape, and these remain yet to be specified. Let us assume hereafter that the locus of ejection is the inner Lagrangian point L_1; and as the immediate cause of escape let us consider the phenomenon of *thermal evaporation*—i.e., the velocities acquired by gas particles as a result of thermal agitation corresponding to a finite temperature T. This temperature is, to be sure, difficult to describe with any accuracy; for the gravity-darkening (section IV.1) should, in principle, reduce the radiant flux at L_1 to zero. In reality this may not be literally true, as the exact conditions under which gravity-darkening is operative may break down to some extent;* but $T(L_1)$ will no doubt be considerably less than the mean temperature over the distorted surface corresponding to the observed spectral type of the respective star. Whatever it may be, however, the mean velocity v_T of gas particles of molecular weight μ and at a temperature T is going to be given by

$$v_T = \left(\frac{\Re T}{\pi\mu}\right)^{1/2}. \tag{5-18}$$

* *Cf.* Z. Kopal in the Harvard Observatory's *Centennial Symposia*, part III-4, pp. 261–275.

where $\mathfrak{R}$ denotes the universal gas constant; or, when expressed in terms of our velocity unit v_0 inherent in our normalization of the variables,

$$\frac{v_T}{v_0} = \left\{\frac{8\mathfrak{R}AT}{\pi G\mu(m_1 + m_2)}\right\}^{1/2}. \tag{5-19}$$

For stellar matter we can put, to a sufficient approximation, $\mu \sim 1$. If, furthermore, we adopt $m_1 = 2m_2 = 2 \odot$ and $A = 10 \odot$ for the sake of an illustrative example, the ratio v_T/v_0 as defined by the foregoing equation (5-19) assumes for different temperatures the following numerical values:

T	v_T/v_0
2000°	0·0273
4000°	0·0386
6000°	0·0472
8000°	0·0546

The initial conditions of ejection at L_1 accordingly become

$$\left.\begin{aligned} x_0 &= x_1, & \dot{x}_0 &= (v_T/v_0)\cos\psi, \\ y_0 &= 0, & \dot{y}_0 &= (v_T/v_0)\sin\psi, \end{aligned}\right\} \tag{15-20}$$

where ψ, the angle of ejection of an evaporating particle, may be distributed at random outside the apex angle of the cone tangent to the surface of our configuration at L_1.

An inspection of the foregoing tabulation of the values of v_T/v_0 versus T reveals that, for most reasonable temperatures, the ratios v_T/v_0 are of the order of 0·01. Consequently, an ejection of gas from L_1 (or any other Lagrangian point) by thermal evaporation can be regarded as a 'small perturbation' and treated as such by analytic methods. In order to do so, let x', y' denote rectangular coordinates of the mass particle with respect to any one of the three Lagrangian collinear points $L_i(i = 1, 2, 3)$ taken as origin, so that

$$x = x_i + x', \qquad y = y', \tag{5-21}$$

where x_i is the x-coordinate of the respective Lagrangian point. If so, the equations of motion of our mass-particle in the neighbourhood of L_1 assume the forms*

$$\left.\begin{aligned} \frac{d^2x'}{dt^2} - 2\frac{dy'}{dt} &= (1 + 2A_i)x', \\ \frac{d^2y'}{dt^2} + 2\frac{dx'}{dt} &= (1 - A_i)y', \end{aligned}\right\} \tag{5-22}$$

* For their derivation *cf.*, e.g., F. R. Moulton, *Periodic Orbits*, Washington 1920, Chapter V.

where

$$A_i = \frac{1-\mu}{[(x_i+\mu)^2]^{3/2}} + \frac{\mu}{[(x_i-1+\mu)^2]^{3/2}}. \tag{5-23}$$

Now let us seek the solutions of the equations (5-22) in the form

$$x' = He^{\lambda t}, \qquad y' = Ke^{\lambda t}, \tag{5-24}$$

where H, K and λ are suitable constants. If these expressions are to satisfy (5-22)–(5-23) identically, it follows that

$$\left.\begin{aligned} [\lambda^2-(1+2A_i)]H - 2\lambda K = 0\,, \\ 2\lambda H + [\lambda^2-(1-A_i)]K = 0\,, \end{aligned}\right\} \tag{5-25}$$

which will possess a non-trivial solution for H and K provided that the determinant of this homogeneous system vanishes—and this renders λ a root of the biquadratic equation

$$\lambda^4 + (2-A_i)\lambda^2 + (1-A_i)(1+2A_i) = 0\,. \tag{5-26}$$

As was first shown by Plummer,* the discriminant of this equation is positive for any mass-ratio—and, consequently, of the four roots of (5-26), two are real and equal numerically but of opposite sign, and the other two are conjugate pure imaginaries.

Let the real roots be denoted by $\pm\lambda_1$ and the imaginary ones by $\pm i\lambda_2$. For each of these there exists a particular solution of our equations (5-22)–(5-23) of the form (5-24), and their linear combination leads to the general solutions

$$\left.\begin{aligned} x' &= H_1e^{\lambda_1 t} + H_2e^{-\lambda_1 t} + H_3e^{i\lambda_2 t} + H_4e^{-i\lambda_2 t}, \\ y' &= K_1e^{\lambda_1 t} + K_2e^{-\lambda_1 t} + K_3e^{i\lambda_2 t} + K_4e^{-i\lambda_2 t}, \end{aligned}\right\} \tag{5-27}$$

in which (by virtue of 5-25) the constants H and K are related

$$K_{1,2} = \pm mH_{1,2}\,, \qquad K_{3,4} = \pm inH_{3,4}\,, \tag{5-28}$$

where

$$m = \frac{\lambda_1^2-(1+2A_i)}{2\lambda_1} \quad \text{and} \quad n = \frac{\lambda_2^2+(1+2A_i)}{2\lambda_2}\,. \tag{5-29}$$

In consequence, the general solutions (5-27) assume the more explicit form

$$\left.\begin{aligned} x' &= H_1e^{\lambda_1 t} + H_2e^{-\lambda_1 t} + H_3e^{i\lambda_2 t} + H_4e^{-i\lambda_2 t}\,, \\ y' &= mH_1e^{\lambda_1 t} - mH_2e^{-\lambda_1 t} - inH_3e^{i\lambda_2 t} + inH_4e^{-i\lambda_2 t}\,, \end{aligned}\right\} \tag{5-30}$$

which, for $t = 0$, reduce to

$$\left.\begin{aligned} x_0' &= H_1 + H_2 + H_3 + H_4\,, \\ y_0' &= m(H_1 - H_2) - inH(_3 - H_4)\,, \\ x_0' &= \lambda_1(H_1 - H_2) + i\lambda_2(H_3 - H_4)\,, \\ y_0' &= m\lambda_1(H_1 + H_2) + n\lambda_2(H_3 + H_4)\,. \end{aligned}\right\} \tag{5-31}$$

* H. C. Plummer, *M.N.*, **62**, 6, 1901.

Now in the dynamical problem under consideration $x_0' = y_0' = 0$ while $\dot{x}_0'$ and $\dot{y}_0'$ are as given by the second pair of the equations (5-20). Solving for the H's from the foregoing system (5-31) in terms of these initial conditions we readily find that

$$H_{1,2} = \frac{\dot{y}_0'}{m\lambda_1 - n\lambda_2} \pm \frac{n\dot{x}_0'}{n\lambda_1 + m\lambda_2} \tag{5-32}$$

and

$$H_{3,4} = \frac{\dot{y}_0'}{n\lambda_2 - m\lambda_1} \mp \frac{im\dot{x}_0'}{n\lambda_1 + m\lambda_2} . \tag{5-33}$$

A glance at the solution (5-30) reveals that while the terms factored by $H_{3,4}$ are purely periodic, those multiplied by $H_{1,2}$ are exponential functions of the time and their presence renders the solution *unstable*. Moreover, as the foregoing equation (5-33) makes it evident that both coefficients $H_{1,2}$ cannot be made to vanish for any (non-zero) choice of initial conditions,* it follows that *any gas particle displaced but little from the Lagrangian collinear point as a result of thermal evaporation will eventually depart to a great distance*, and its actual trajectory must be followed by the process of numerical integration. Such integrations (*cf.* Figs. 7-11 to 7-18) reveal conclusively that, for the initial velocities as small as those produced by thermal evaporation, the gravitational force of the primary component is sufficient to attract particles ejected in random directions towards it through a relatively narrow 'gravitational pipeline' enveloping the x-axis. The net result of this process should be a gradual transfer of mass from the contact component to its mate along trajectories which (after a certain amount of initial dispersion) deviate but little from those of free fall (i.e., straight lines in the rotating frame of reference and spirals in fixed space-axes).

The secular and irreversible change in the mass-ratio which is likely to be brought about by thermal evaporation will, however, operate but very slowly and changes produced by it are probably altogether small. Further inquiry into what may cause an expanding star at its Roche limit to lose mass leads, however, to the detection of another process which is likely to be very much more important and which will emerge from the following considerations. One of the quantities which must be conserved in the course of any evolutionary history of an isolated binary in space is its total *angular momentum*

$$M = m_1\tilde{R}_1^2\omega_1 + m_2\tilde{R}_2^2\omega_2 + \frac{m_1m_2}{m_1 + m_2} A^2\omega_K , \tag{5-34}$$

* It is, to be sure, possible to annihilate the coefficient of the positive exponentials in (5-30) by choosing the direction of escape in such a way that

$$\tan\psi = \left(\frac{\dot{x}'}{\dot{y}'}\right)_0 = n\left\{\frac{n\lambda_2 - m\lambda_1}{n\lambda_1 + m\lambda_2}\right\},$$

but no evaporation process favouring such a direction can be suggested.

where $\tilde{R}_{1,2}$ denotes the radii of gyration of the two components; $\omega_{1,2}$ their angular velocities of rotation; and ω_K, the Keplerian angular velocity of orbital revolution. The first two terms on the right-hand side of (5-33) stand for the rotational momenta of the components, and the third for the orbital momentum of the system.

The conservation of angular momentum requires that the sum of all three terms constituting M should be invariant in time. However, any interaction between them is afforded only by tidal friction; and this operates so slowly that time intervals of the order of 10^8 to 10^9 years are required to make its effects felt.* Now if an expansion of the components in close binary systems to their Roche limits were to proceed on a time scale of the order of 10^6 years (and in section VII.7 we shall adduce arguments in favour of this view), tidal friction would be entirely powerless to effect any exchange between the rotational and orbital momenta—with the consequence that, as the expansion proceeds, the three terms on the right-hand side of equation (5-34) must be conserved *identically* and (at least before the Roche limit has been attained and any appreciable loss of mass occurs) *the angular velocity of rotation of the expanding components must vary inversely to the square of their radii of gyration.* It is, moreover, reasonable to assume that the pre-expansion state may have lasted sufficiently long for the equalization of $\omega_{1,2} = \omega_K$ to have taken place. If so, however, a conservation of the angular momentum should *slow down* the axial rotation of the expanding stars so that, at the Roche limit, we should expect $\omega_{1,2} \ll \omega_k$.

Now the inequality between $\omega_{1,2}$ and ω_K, if present, would imply certain important consequences. If $\omega_2 = \omega_K$, the position of every particle on the surface of the secondary component should remain fixed with respect to the (rotating) *xy*-frame of reference; while the inequality $\omega_2 \neq \omega_K$ would imply a differential *rotation*—i.e., a clockwise or counterclockwise motion in the *xy*-plane depending on whether $\omega_2 \lessgtr \omega_K$. The inequality $\omega_2 \ll \omega_K$, which we are led to expect on grounds of the conservation of angular momentum, would be interpreted by an observer in the *xy*-plane as a clockwise differential rotation; and, as a result, every particle at the equator is bound to pass through the point x_1 of the secondary component in the course of time. If the mass of such a particle were zero, it should momentarily come to rest there, and depart again in an altered direction along the respective equipotential. However, a particle of *finite* mass—no matter how small—would *not* be brought to a standstill at x_1 on account of its inertia, but would be *ejected* in the direction of a tangent to the branch of the equipotential along which it has been moving towards x_1 at an angular speed equal to the difference $\omega_2 - \omega_k$.

In order to explore the kinematic properties of gas streams which may be ejected by expanding components in this manner, the corresponding initial conditions remain yet to be specified. In doing so let us suppose that the actual angular velocity ω_2 of the secondary component is related to the

* *Cf.*, e.g., J. H. Jeans, *Astronomy and Cosmogony*, Cambridge, 1929, pp. 293–298.

Keplerian angular velocity by means of the equation

$$\omega_2 = f\omega_K, \tag{5-35}$$

where f denotes an arbitrary numerical factor. If $f = 1$, synchronism prevails between rotation and revolution; while for $f = 0$ the star will be non-rotating. Relative to our moving frame of reference the rotation of any

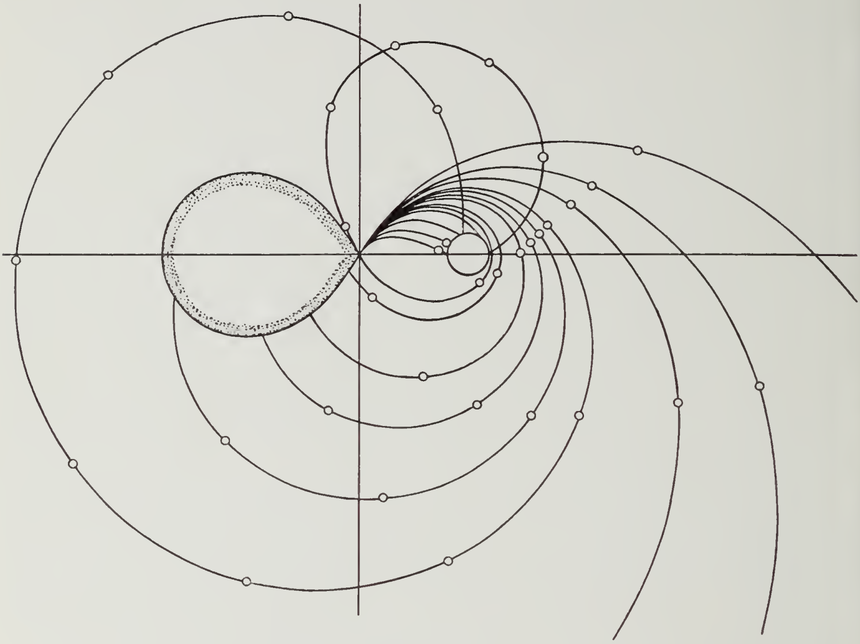

FIGURE 7-11.
Trajectories of direct orbits of mass particles ejected from the conical point of a contact secondary's surface in tangent direction, with different values of the initial velocity as listed in Table 7-12 for a mass-ratio $m_2/m_1 = 1$.

point of the secondary's surface will be direct or retrograde depending on whether $f \lessgtr 1$, and the differential velocity of angular rotation.

$$\omega_2 - \omega_K = (f - 1)\omega_K \tag{5-36}$$

will correspond to an absolute velocity of ejection

$$v_E = (f - 1)\omega_K A(1 - x_1), \tag{5-37}$$

where $A(1 - x_1)$ denotes the distance of the conical point on the secondary's surface from its centre of gravity.

Extensive numerical integrations of the equations (5-1)–(5-2) subject to the initial velocity (5-37) have been undertaken at Manchester in 1954–1955 with the aid of the University's electronic computers* for several mass-ratios of the two finite bodies and a sufficient range of the velocities of ejection to reveal the nature of such orbits. The principal results are shown graphically on the accompanying Figs. 7-11 to 7-18. On each diagram the centre of gravity of the system has been taken as the origin of coordinates, and the outline of the secondary's equator is drawn to scale for the respective mass-ratio; but the circle of radius 0·1 enclosing the centre of the primary (more

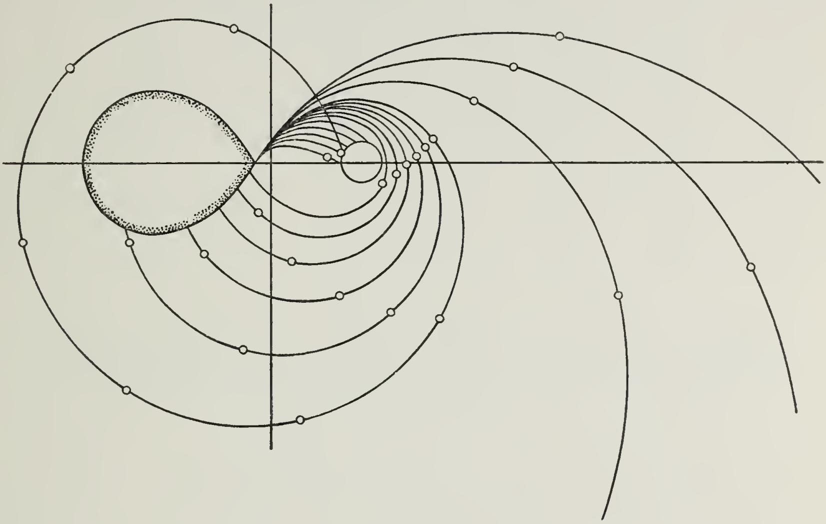

FIGURE 7-12.
Trajectories of direct orbits of mass particles ejected from the conical point of a contact secondary's surface in tangent direction, with different values of the initial velocity as listed in Table 7-12 for a mass-ratio $m_2/m_1 = 0{\cdot}8$.

massive) component represents merely a limit inside which difficulties have been encountered with the scale factors of numerical integrations rather than any anticipated size or shape of the primary star (which may be arbitrary inside its own Roche limit). A list of the initial conditions of all trajectories plotted on Figs. 7-11 to 7-14 ($\dot{y}_0$ positive; *direct* orbits) for four different mass-ratios is given in the following Table 7-11; while Table 7-12 contains the same information for *retrograde* orbits ($\dot{y}_0$ negative). Small circles on each trajectory separate the distance traversed by moving particles in constant time interval of $P/4\pi$, where P denotes the orbital period of the close pair.

Of the two families of orbits reproduced on Figs. 7-11–7-14 and 7-15–7-18, the latter group representing *retrograde* orbits is of particular interest, as it is such orbits which should be expected if (consistent with the conservation of its angular momentum) the secondary component attains the Roche limit with a velocity of rotation reduced below ω_K. An inspection of the orbits

* The actual machine work was programmed and carried out by R. A. Brooker, with the assistance of V. Hewison.

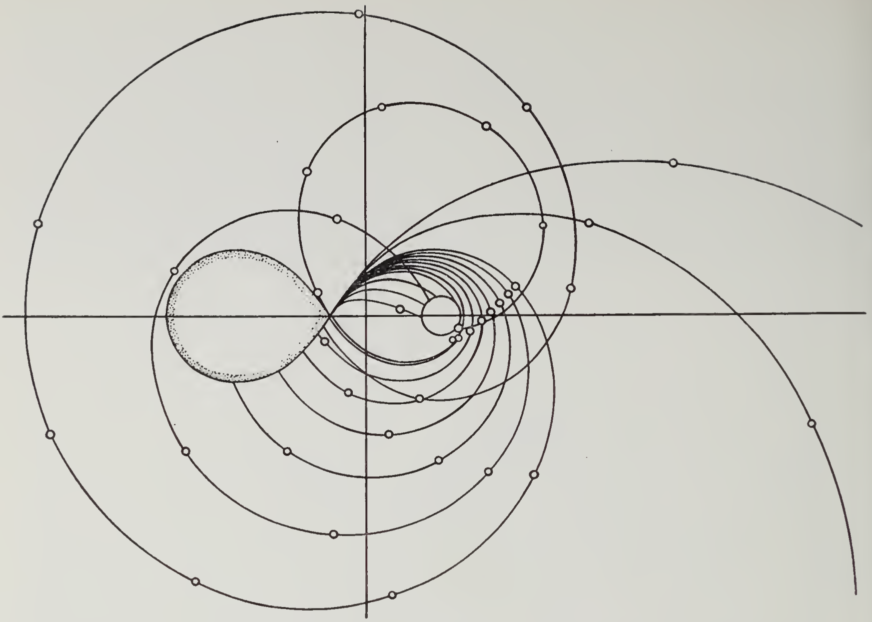

FIGURE 7-13.
Trajectories of direct orbits of mass particles ejected from the conical point of a contact secondary's surface in tangent direction, with different values of the initial velocity as listed in Table 7-12 for a mass-ratio $m_2/m_1 = 0{\cdot}6$.

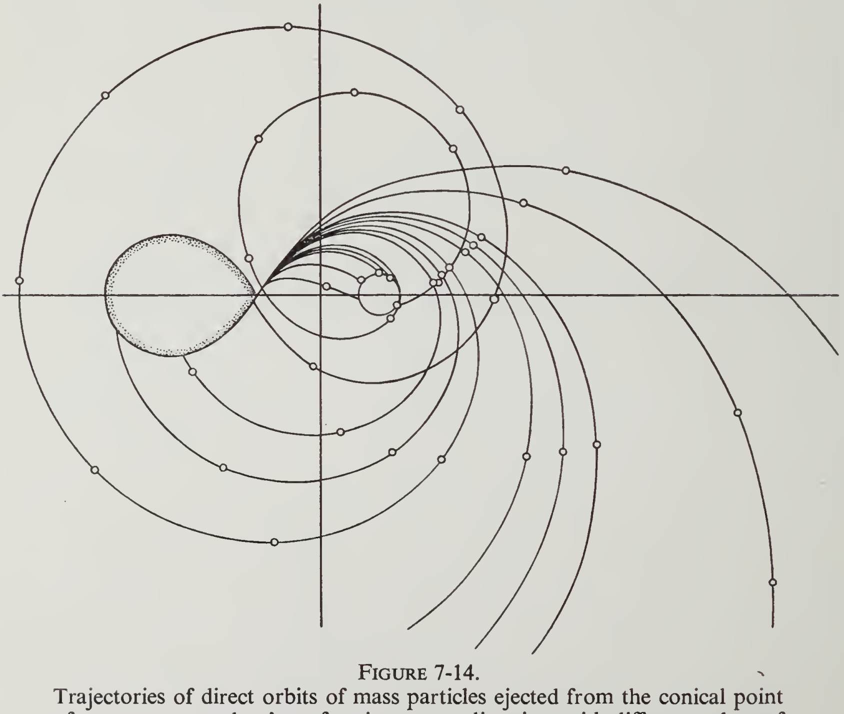

FIGURE 7-14.
Trajectories of direct orbits of mass particles ejected from the conical point of a contact secondary's surface in tangent direction, with different values of the initial velocity as listed in Table 7-12 for a mass-ratio $m_2/m_1 = 0{\cdot}4$.

TABLE 7-11

Initial Velocities of Ejection (Direct Orbits) and the Corresponding Values of the Jacobi Constants

$q = 1$		$q = 0·8$		$q = 0·6$		$q = 0·4$	
v_E/v_0	C	v_E/v_0	C	v_E/v_0	C	v_E/v_0	C
0·5	3·75	0·5	3·74	0·5	3·72	0·5	3·66
0·7	3·51	0·7	3·50	0·8	3·33	0·9	3·10
1·0	3·00	0·9	3·18	1·1	2·76	1·3	2·22
1·3	2·31	1·1	2·78	1·4	2·01	1·4	1·95
1·4	2·04	1·3	2·30	1·49	1·75	1·5	1·66
1·5	1·75	1·5	1·74	1·5	1·72	1·9	0·30
1·7	1·11	1·6	1·43	1·6	1·41	2·0	−0·09
1·8	0·76	1·7	1·10	1·7	1·08	2·1	−0·50
1·9	0·39	1·8	0·75	1·8	0·73	2·3	−1·38
2·0	0·00	1·9	0·38	1·9	0·36	2·4	−1·85
2·25	−1·06	2·0	−0·01	2·0	−0·03	2·5	−2·34
2·5	−2·25	2·5	−2·26	2·5	−2·28	3·0	−5·09
3·0	−5·00	3·0	−5·01	3·0	−5·03	3·5	−8·34
		3·5	−8·26	4·0	−12·03		

TABLE 7-12

Initial Velocities of Ejection (Retrograde Orbits) and the Corresponding Values of the Jacobi Constants

$q = 1$		$q = 0·8$		$q = 0·6$		$q = 0·4$	
v_E/v_0	C	v_E/v_0	C	v_E/v_0	C	v_E/v_0	C
0·5	3·75	0·5	3·74	0·5	3·72		
0·6	3·64	0·6	3·63				
0·7	3·51	0·7	3·50				
0·8	3·36	0·8	3·35	0·8	3·33		
0·9	3·19	0·9	3·18	0·9	3·16		
1·0	3·00	1·0	2·99	1·0	2·97	1·0	2·91
1·05	2·90						
		1·1	2·78	1·1	2·76	1·1	2·70
1·15	2·68						
1·2	2·56	1·2	2·55	1·2	2·53	1·2	2·47
1·3	2·31	1·3	2·30	1·3	2·28	1·3	2·22
1·4	2·04	1·4	2·03	1·4	2·01	1·4	1·95
1·5	1·75	1·5	1·74	1·5	1·72		
1·6	1·44	1·6	1·43				

shown diagrammatically on Figs. 7-15 to 7-18 reveals that all these trajectories can be divided into the following essential types:

(1) Particle escaping from the secondary on to the primary (either directly, or after describing a cusp caused by a close approach to the corresponding surface of zero velocity).

(2) Particle returns to the secondary along a path enclosing the primary by a simple or multiple loop.

(3) Particle returns to the secondary by a path not enclosing the primary (as the loop developed in Class 1 orbits opens up still further, its path will cross the secondary's surface).

(4) Particle falls on the primary along a path enclosing the secondary.

(5) Particle describes an orbit around the whole system before falling back on the secondary.

(6) Particle escapes from the system along a clockwise spiral.

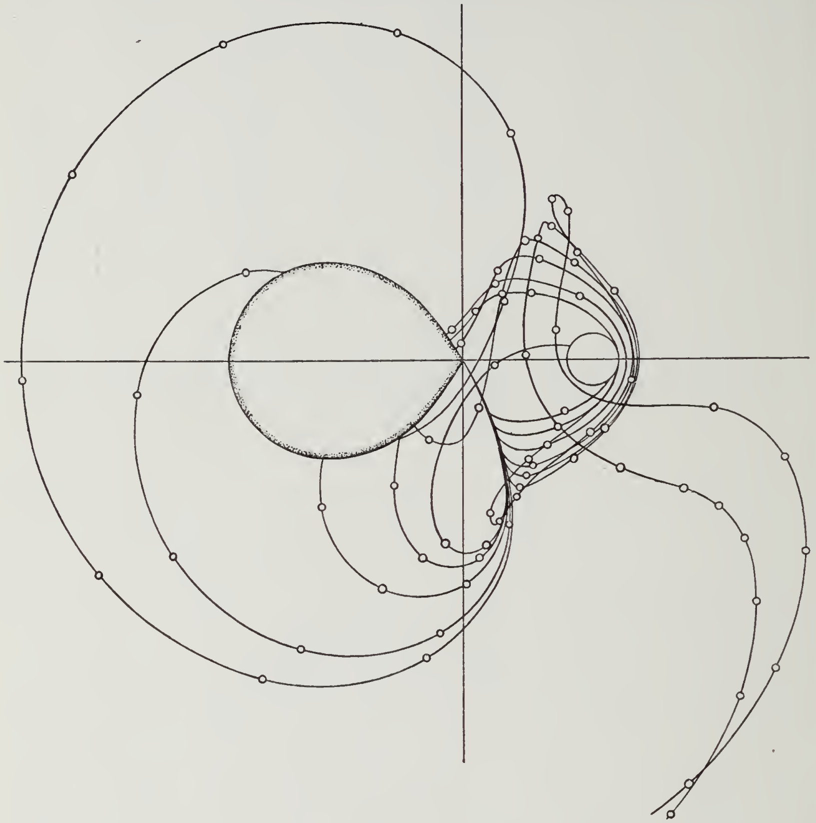

FIGURE 7-15.
Trajectories of retrograde orbits of mass particles ejected from the conical point of a contact secondary's surface in tangent direction, with different values of the initial velocity as listed in Table 7-13 for a mass-ratio $m_2/m_1 = 1$.

These classes of orbits are arranged in order of the increasing initial velocity necessary to give rise to them; for the actual velocity limits bracketing each particular class, *cf.* the individual Figs. 7-15 to 7-18.

The kinematic history of particles ejected from the conical end of the secondary's Roche limit can be briefly summarized as follows. Unless the velocity is sufficiently high to enable the particle to escape from the gravitational field of the system by direct route (Class 6), the particle is transferred on to the primary component (Classes 1, 4), or is intercepted on its way by the surface of the secondary (Classes 2, 3 and 5). But if this latter component already fills completely its Roche limit, it cannot accommodate any incoming matter indefinitely: its infall may merely accelerate the existing equatorial currents—leading to renewed ejection with higher speed—until the particles either fall on the primary component, or escape from the system altogether.

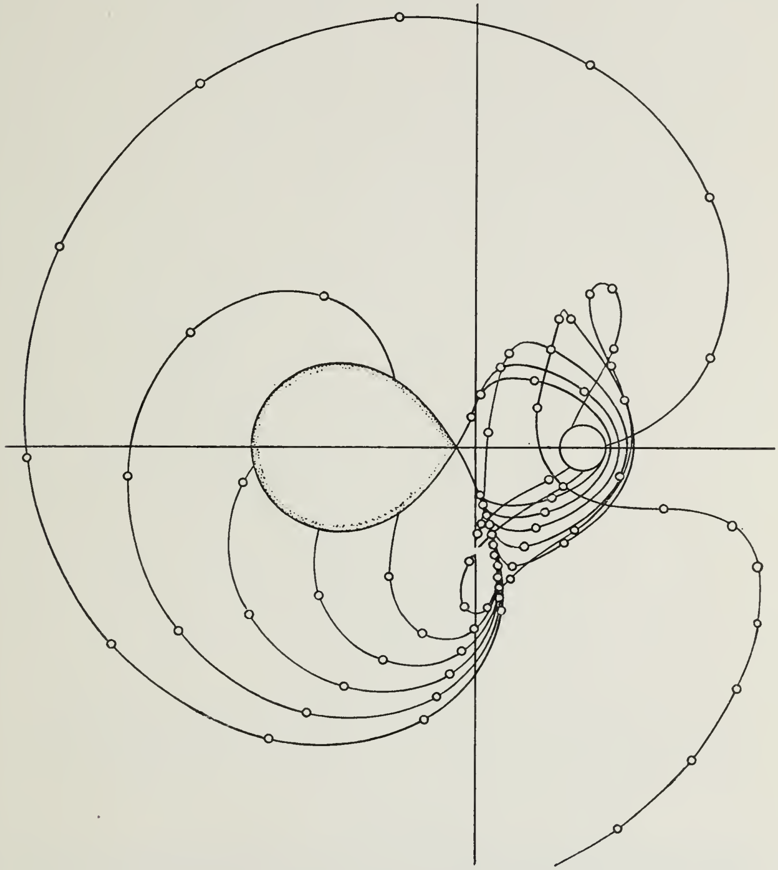

FIGURE 7-16.
Trajectories of retrograde orbits of mass particles ejected from the conical point of a contact secondary's surface in tangent direction, with different values of the initial velocity as listed in Table 7-13 for a mass-ratio $m_2/m_1 = 0{\cdot}8$.

The net effect of such a process is bound to be a secular transfer of mass from m_2 to m_1, leading to a gradual diminution of the mass-ratio m_2/m_1; but unless the velocity of ejection becomes sufficiently high, the total mass $m_1 + m_2$ of the system remains secularly constant.

Earlier in this section we have established the necessary conditions for an escape of mass from the system in the form of equations (5-16) or (5-17). Now it should be stressed here that these conditions are *necessary*, but *not* *sufficient*, for actual escape to occur. In order that this should happen, it is also necessary that the trajectory of a moving particle should not intersect the surface of the two components of finite size (or else the particle will end up its journey right there). An inspection of our data reveals that an initial velocity greater than $1{\cdot}8v_0$ is required for the ejection of a particle from the secondary's conical point to infinity for retrograde orbits ($\dot{y}_0 < 0$), and

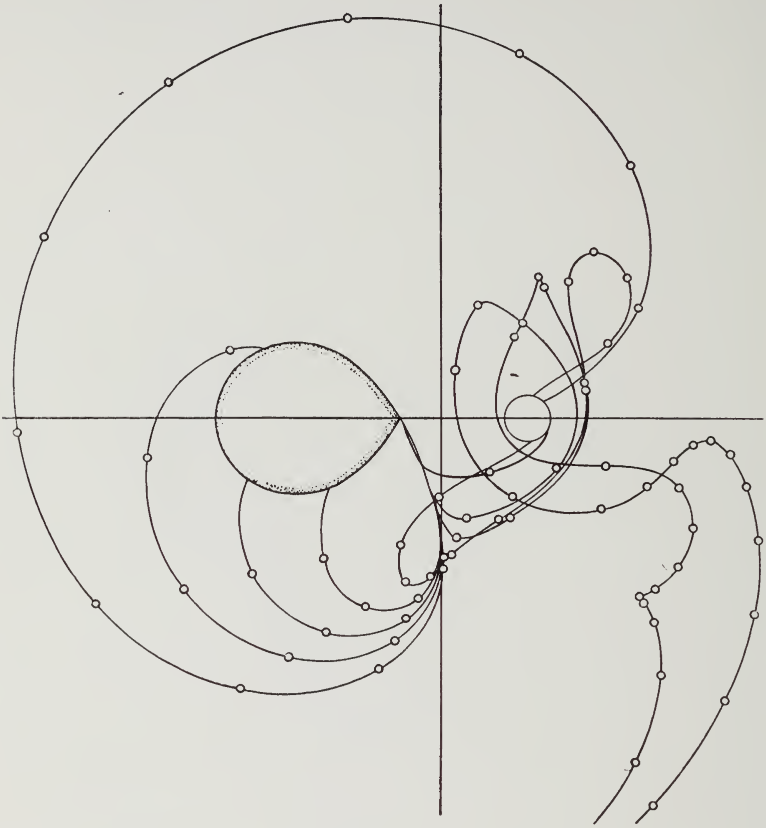

FIGURE 7-17.
Trajectories of retrograde orbits of mass particles ejected from the conical point of a contact secondary's surface in tangent direction, with different values of the initial velocity as listed in Table 7-13 for a mass-ratio $m_2/m_1 = 0{\cdot}6$.

greater than $2{\cdot}3v_0$ in the case of direct orbits ($\dot{y}_0 > 0$), largely irrespective of the mass-ratio. These velocities evidently *exceed* the parabolic velocity $v_0\sqrt{2} \simeq 1{\cdot}4v_0$ of escape which would obtain if the two components were to act as one in attracting the particle from a single centre; the excess of the factors of 1·8 and 2·3 over 1·4 being due to the fact that the binary system acts like a gravitational dipole. Moreover, the reason why the actual velocity of escape turns out to be greater for direct than for retrograde orbits becomes manifest when we stop to realize that, in the case of a retrograde orbit, the primary component moves away from the path of ejection whereas, for direct orbits, the ejected particle and the primary component are moving towards each other. If, for the sake of an illustrative example, we adopt the values of $m_1 + m_2 = 3 \odot$ and $A = 10 \odot$, equation (5-10) yields the circular velocity $v_0 = 240$ km/sec, and the approximate threshold velocities of escape

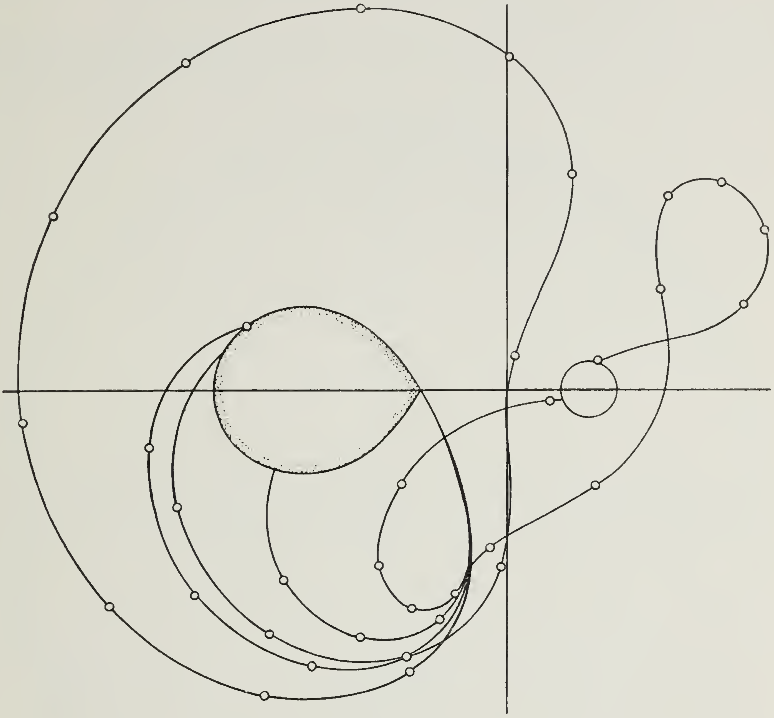

FIGURE 7-18.
Trajectories of retrograde orbits of mass particles ejected from the conical point of a contact secondary's surface in tangent direction, with different values of the initial velocity as listed in Table 7-13 for a mass-ratio $m_2/m_1 = 0{\cdot}4$.

in direct and retrograde orbits become 550 km/sec and 430 km/sec, respectively. For more massive systems such velocities would prove to be proportionally higher, and exceed velocities associated with the moving gas streams observed in all close binary systems—with the possible exception of those containing the Wolf-Rayet stars as a component.

It should still, however, be stressed that the velocities just referred to are relative to the point of ejection in the xy-coordinates rotating in space with the Keplerian angular velocity. If X, Y denote the rectangular axes in the orbital plane whose orientation in space is fixed, it follows from simple geometry that

$$\left.\begin{aligned} X &= x \cos t - y \sin t\,, \\ Y &= x \sin t + y \cos t\,, \end{aligned}\right\} \tag{5-38}$$

and, hence, the space velocity $V^2 = \dot{X}^2 + \dot{Y}^2$ becomes expressible as

$$V^2 = \dot{x}^2 + \dot{y}^2 + 2(x\dot{y} - y\dot{x}) + x^2 + y^2 . \tag{5-39}$$

Therefore, at the secondary's conical point, the space velocity of ejection will be given by

$$V^2 = v^2 + 2v(\mu - x_1)\cos\psi + (\mu - x_1)^2 , \tag{5-40}$$

where $v^2 = \dot{x}_0^2 + \dot{y}_0^2 = (v_E/v_0)^2$.

While matter ejected from a contact secondary component at its point of instability can thus escape from a system if its initial velocity is sufficiently

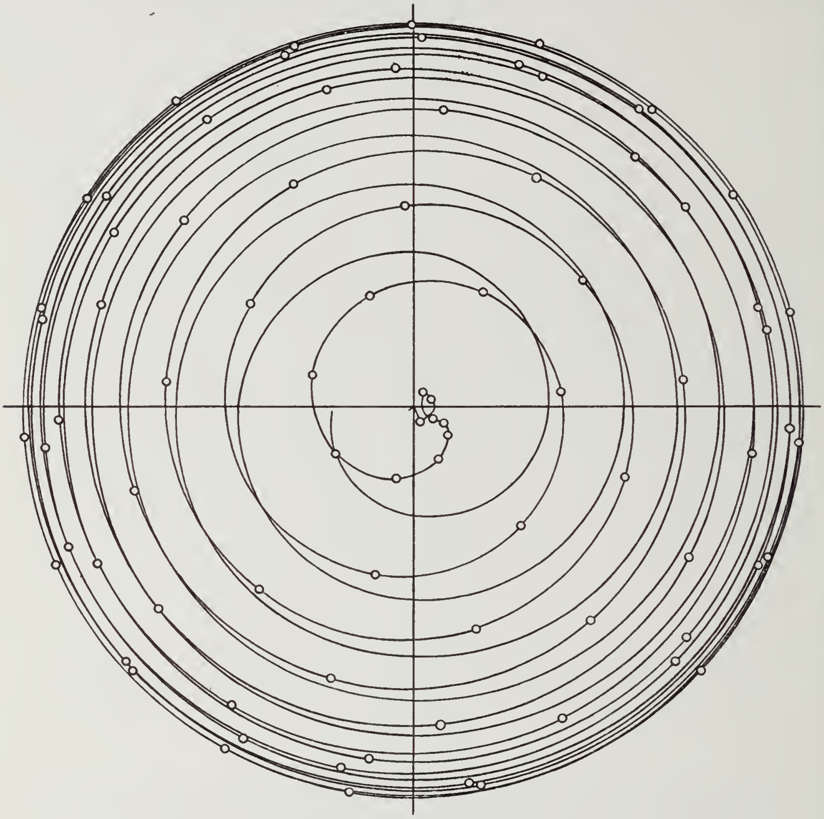

FIGURE 7-19.
The retrograde orbit of a particle ejected from the inner Lagrangian point L_1, corresponding to an initial velocity $v_E/v_0 = 0{\cdot}9$ and a mass-ratio $m_2/m_1 = 1$. Figure 7-15 should be consulted for details of the inner part of the trajectory.

high (generally of the order of several hundred km/sec), it is of interest to note that, in the case of retrograde motion, a particle leaving the secondary with (approximately) *one-half* the requisite velocity of escape can spiral out to a considerable distance from the binary system before eventual return. The corresponding numerical integrations reveal, however, that such unwinding spirals possess a definite envelope (which has nothing to do with the outer 'curtain' of the Jacobian surfaces of zero velocity, by now completely lifted

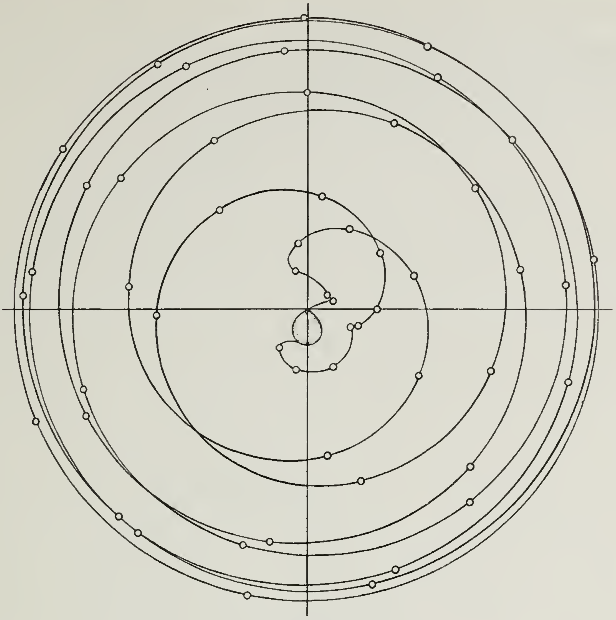

FIGURE 7-20.
The retrograde orbit of a particle ejected from the inner Lagrangian point L_1, corresponding to an initial velocity $v_E/v_0 = 1$ and a mass-ratio $m_2/m_1 = 1$. Figure 7-15 should be consulted for details of the inner part of the trajectory.

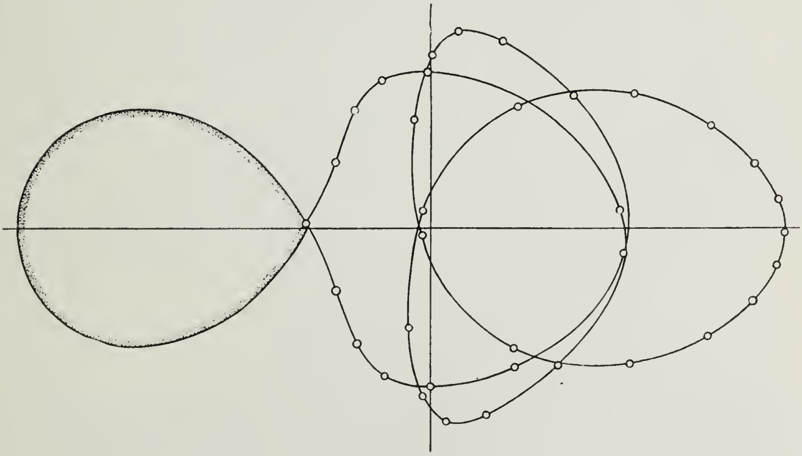

FIGURE 7-21.
The retrograde orbit of a particle ejected from the inner Lagrangian point L_1, corresponding to an initial velocity $v_E/v_0 = 0{\cdot}7$ and a mass-ratio $m_2/m_1 = 0{\cdot}4$.

from the *xy*-plane) which our particle will approach asymptotically before it will eventually spiral inwards again to end its motion in a collision with one of the two components. The asymptotic nature of such orbits (an example of which is exhibited on Figs. 7-19 and 7-20 for the mass-ratio

$(m_2/m_1) = 1$, and the initial velocities $v_E/v_0 = 0{\cdot}9$ and $1{\cdot}0$) suggests that a continuous stream of gas particles moving along them may lead to the establishment and maintenance of a *gas ring* encircling the whole binary system at a considerable distance (of the order of ten times the separation

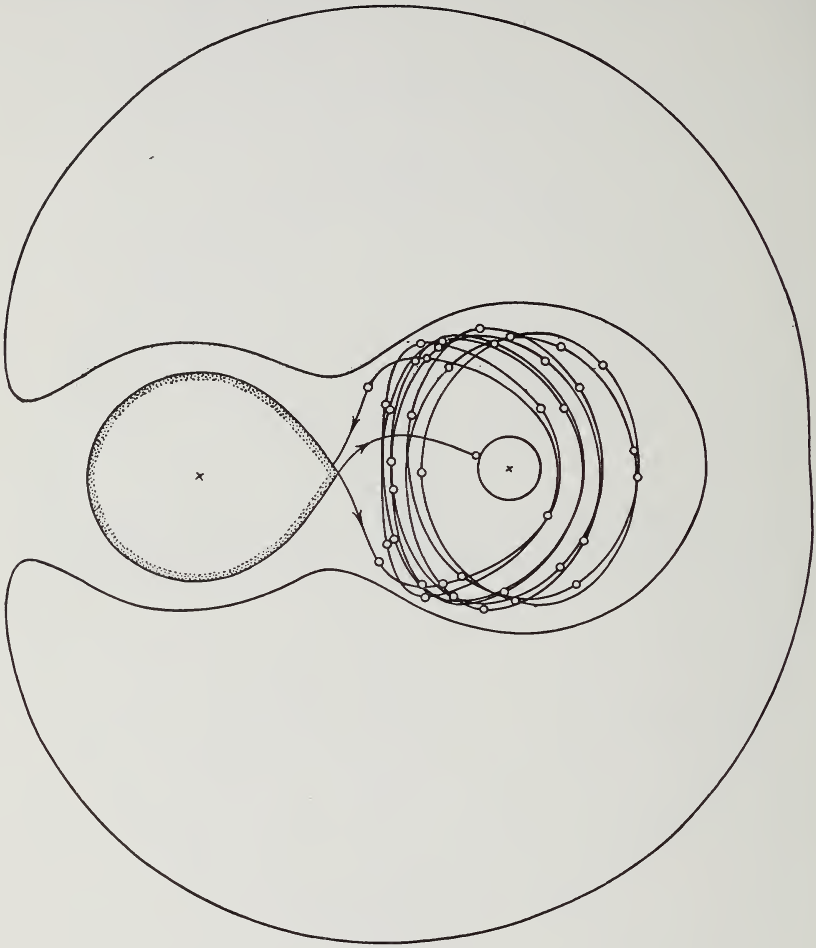

FIGURE 7-22.
The trajectories (direct and retrograde) of a particle ejected from the inner Lagrangian point L_1, corresponding to an initial velocity $v_E/v_0 = 0{\cdot}7$ and a mass-ratio $m_2/m_1 = 0{\cdot}6$. The respective Jacobian zero-velocity curve is shown for comparison.

of its components) and rotating around it in the plane of its orbit with an (almost) constant angular velocity (relative to the moving coordinates). Another example of a trajectory of a multiply-periodic orbit in the equatorial plane of the binary systems, emanating from the secondary and encircling the primary by a number of loops before eventual return is shown on Figs. 7-21 and 7-22 for $(v_E/v_0) = 0{\cdot}7$ and the mass-ratios $0{\cdot}4$ and $0{\cdot}6$, respectively.

The *direct* orbit ejected from x_1 with this velocity (also shown on Fig. 7-22) leads to the primary component along but a moderately curved path. The *retrograde* orbit will, on the other hand, take the particle describing it not less than eight times around the primary before its return to the secondary component; and only a slight adjustment of the initial velocity is needed to render the orbit exactly re-entrant.

It should, however, be stressed in this connection that gas streams whose motions may follow such trajectories cannot have anything in common with the rings reported to exist around primary components of certain eclipsing systems by several spectroscopists* in recent years, because the observed motions of such rings are invariably *direct*, while all such orbits as shown on Figs. 7-19 or 7-20 are necessarily *retrograde*. In point of fact, the results of all our numerical integrations render it a very remote possibility that matter escaping from a contact secondary component at x_1 could ever be prevented from falling on the primary star after not more than a few revolutions—unless excessively high velocities of ejection can actually be attained. For the conditions of escape envisaged earlier in this section, such a possibility can be ruled out almost categorically; and whether or not *any* conditions of escape from x_1 could give rise to direct orbits encircling the primary component for an astronomically long time remains problematic to the highest degree.

This argument should not be construed to imply that the mechanism of mass ejection from the contact components in close binary systems, discussed earlier in this section, cannot account for the maintenance of gas streams circulating in the orbital planes; but asserts that such streams are most unlikely to assume the form of closed rings rotating in a direct sense around the primary component alone.† It should, perhaps, be mentioned that the available spectroscopic evidence does not by any means provide a sufficient proof that the actual gas streams must necessarily be simple rings encircling the primary components; but if they are, their origin is most probably to be sought elsewhere than in the instability of the secondary components at their Roche limits.

VII.6. CONTACT BINARIES

The two groups of eclipsing systems introduced in sections VII.3 and VII.4 under the names of 'detached' and 'semi-detached' systems do not by any means exhaust all types of known close binaries; for they leave out by far the most numerous kind of such variables encountered in stellar population in the

* A. H. Joy, *P.A.S.P.*, **54**, 21, 1942; **59**, 171, 1947; O. Struve, *Stellar Evolution*, pp. 189–192; S. S. Huang and O. Struve, *Ap. J.*, **61**, 277, 1956.

† Another group of quasi-periodic orbits of a mass particle around the primary component, which resemble simple rings, was discovered numerically by F. R. Moulton (*cf.* his *Periodic Orbits*, Chapter XVI). It should, however, be stressed that all motions in such rings are again retrograde.

neighbourhood of the Sun: namely, the dwarf pairs of the W Ursae Maioris type which we shall hereafter refer to—by the geometry of their surfaces—as *contact binary systems.* Such binaries constitute a remarkably compact group, possessing a number of well-defined characteristics which can be briefly summarized as follows:

(1) The orbital period P is without exception shorter than one day; and for a large majority of such systems it is comprised between 7 and 12 hours. The periods of many systems fail, moreover, to remain constant and fluctuate by appreciable amounts in an apparently irregular manner. The orbital eccentricities e are as a rule vanishingly small.

(2) The variation of light ranges from a few tenths to just over a full magnitude (largely regardless of the colour); and the light changes due to the mutual distortion of both components merge smoothly with those due to eclipses. The alteınate minima in most systems are almost equally deep, indicating but small differences in surface brightnesses of their constituent components. Asymmetric light changes are frequently encountered, and the sense of asymmetry is changing with the time.

(3) An interpretation of the observed light changes—as far as it can be carried out by the methods of Chapter VI—reveals that *both* components of W UMa-type systems appear to fill completely their respective Roche limits—a property which has earned them the designation of *contact* systems. As a consequence, therefore, the *primary* (more massive) component is bound to be the *larger* of the two.

(4) The mass-ratios in dwarf contact binaries of W UMa-type are frequently quite different from unity. Moreover, in at least a large majority of known cases, the time of the primary (deeper) minima coincide with the middle of the ascending branch of radial-velocity changes of the more luminous component—a fact signifying that the more massive components are characterized by lesser surface brightness (i.e., later spectrum). The primary (deeper) minima are, therefore, due to *occultation* eclipses. Only two ambiguous cases (SW Lac and RZ Tau) and one exception (ER Ori) to this rule are known.

(5) The spectra of the components in all contact dwarf systems are very much alike and belong to the classes F and G. Very few spectra later than G9, and none earlier than F0 have so far been encountered among typical systems belonging to this group.

(6) The absolute magnitudes of the components of contact dwarf systems range mostly between +4 and +6 magn. Such stars cluster loosely around the Main Sequence and the mass-luminosity relation (*cf.* Fig. 7-6). In contrast to the behaviour of contact components in semi-detached systems, the contact primaries in W UMa-type pairs are decidedly *not* overluminous for their mass; and appear in fact to lie systematically *below* the statistical mass-luminosity law valid for detached stars, as their secondaries are *above* it.

A list of all contact dwarf systems of the W UMa-type with spectroscopically known mass-ratios—14 in number—is contained in the following

TABLE 7-14

Geometrical Properties and Absolute Dimensions of the Contact Eclipsing Systems of W UMa-type

Star	Period	Spectra	m_2/m_1	C_0	r_1	r_2	m_1	m_2	R_1	R_2	ρ_1	ρ_2	M_1	M_2	W
AB And	$0^{d}.332$	G5+G4	0·62±0·04	3·97 ±0·01	0·42 ±0·01	0·33 ±0·01	1·65⊙	1·03⊙	1·2⊙	0·95⊙	1·0⊙	1·2⊙	$4^{m}.5$	$4^{m}.8$	1·90
i Boo	0·268	G2+F9	0·50±0·01	3·944±0·003	0·441±0·002	0·312±0·002	1·35	0·68	0·98	0·70	1·4	2·0	4·6	5·1	1·82
TX Cnc	0·383	F8+F7	0·52±0·05	3·95 ±0·01	0·44 ±0·01	0·32 ±0·01	2·5	1·3	1·5	1·1	0·72	0·94	3·5	4·2	2·21
VW Cep	0·278	K1+G6	0·32±0·04	3·86 ±0·03	0·48 ±0·02	0·28 ±0·01	1·44	0·47	1·1	0·62	1·2	1·9	5·1	5·8	1·65
TW Cet	0·317	G5+G4	0·53±0·04	3·95 ±0·01	0·43 ±0·01	0·32 ±0·01	1·3	0·69	1·05	0·77	1·1	1·4	5·0	5·4	1·64
RZ Com	0·339	K0+G9	0·48±0·05	3·94 ±0·02	0·44 ±0·01	0·31 ±0·01	1·7	0·82	1·2	0·85	0·91	1·2	4·7	5·3	1·81
YY Eri	0·321	G5+G4	0·65±0·06	3·98 ±0·01	0·42 ±0·01	0·33 ±0·01	0·76	0·50	0·88	0·71	1·1	1·3	5·1	5·4	1·18
SW Lac	0·321	G3+G2	0·85±0·09	4·00 ±0·00	0·39 ±0·01	0·36 ±0·01	0·97	0·83	0·93	0·86	1·1	1·3	4·8	4·9	1·50
V 502 Oph	0·453	G2+F9	0·40±0·03	3·91 ±0·01	0·46 ±0·01	0·30 ±0·01	1·85	0·74	1·6	1·01	0·48	0·69	3·6	4·3	1·49
ER Ori	0·423	G1+G2	0·61±0·11	3·97 ±0·02	0·42 ±0·02	0·33 ±0·02	0·46	0·28	0·90	0·71	0·62	0·76	4·7	5·3	0·70
U Peg	0·375	F3+F3	0·80±0·07	3·99 ±0·00	0·40 ±0·01	0·35 ±0·01	1·35	1·1	1·2	1·1	0·84	0·91	3·4	3·6	1·63
RZ Tau	0·416	F0+F0	0·54±0·05	3·96 ±0·01	0·43 ±0·02	0·32 ±0·01	1·8	0·97	1·4	1·1	0·63	0·80	2·7	3·2	1·71
W UMa	0·334	F8+F7	0·50±0·02	3·94 ±0·01	0·441±0·004	0·312±0·03	1·30	0·65	1·11	0·79	0·94	1·24	4·1	4·7	1·52
AH Vir	0·408	K0+G6	0·42±0·05	3·92 ±0·01	0·46 ±0·01	0·30 ±0·01	2·0	0·84	1·5	0·98	0·60	0·86	4·2	4·8	1·68

Table 7-14, which summarizes the geometrical as well as physical characteristics of their constituent components. The data contained in it have been taken again from the recent Kopal and Shapley's *Catalogue* (Table IV). The headings of the individual columns of our present Table 7-14 are self-explanatory. Its ultimate column (15) lists the values W of the potential

$$W = \frac{m_1 + m_2}{A} \frac{C_0}{2}, \tag{6-1}$$

prevailing over free surfaces of both components, in solar units ($1{\cdot}905 \times 10^{15}$ g/cm²/sec²). It is interesting to note that (apart again from the recalcitrant case of ER Ori), the values of W appear to be very much the same for all known systems of this type, and their mean $1{\cdot}67 \pm 0{\cdot}07$ possesses a remarkably small standard deviation.*

The intrinsic significance of this fact is still rather debatable, but its practical value should not for a minute be in doubt. In order to demonstrate it, let us return to equations (1-4) and (1-5) which combined with (6-1) reveal that, for circular orbits,

$$3{\cdot}81 \times 10^5 W = \{(K_1 + K_2) \csc i\}^2 C_0, \tag{6-2}$$

where $K_{1,2}$ denote (as in section VII.1) the amplitudes of the radial-velocity changes, in km/sec, of the two components in their absolute orbits around the centre of gravity of the system. On the other hand, the contact nature of the binaries of W UMa-type permits us to ascertain the ratio $K_1/K_2 = m_2/m_1$ from the ratio k of the fractional radii of the two components by means of the relation between them as summarized numerically in Table 3-3; and once the mass-ratio has thus been established, the corresponding value of C_0 can be found from column (2) of Table 3-2.†

With the values of k and i taken from the photometric solution, the equation

$$\bar{W} = 1{\cdot}67 \pm 0{\cdot}07, \tag{6-3}$$

combined with (6-2) can now be used to evaluate the *sum* $K_1 + K_2$ of the radial-velocity changes of both components, while the equation

$$\frac{K_1}{K_2} = \frac{m_2}{m_1} = f(k) \tag{6-4}$$

furnishes their *ratio*. A solution of the foregoing equations should then enable us to establish the values of K_1 *and* K_2—and thus of all absolute properties of the respective contact system—*without recourse to any actual measurement of the radial velocity by spectroscopic means.*‡

* For ER Ori the value of W so expressed turns out to be $0{\cdot}70$—i.e., less than a half of the mean for all others.

† In point of fact, any determination of the geometrical elements of contact binary systems is inseparably connected with a determination of their mass-ratios (and *vice versa*); for the fractional dimensions of *both* components are obviously specified in all details by the geometry of the Roche model as soon as the mass-ratio is known.

‡ If a single-spectrum orbit were available, the sum $K_1 + K_2$ deduced from it should replace the one obtainable from the statistical relation (6-3).

The external characteristics of the contact dwarf systems of W UMa-type do not render such objects very conspicuous in the stellar population at large. None of them is visible to the naked eye; and only 18 are known to be brighter than the 10th apparent magnitude. Yet when their low intrinsic luminosity is duly taken into account it transpires that, surprisingly enough, *the contact dwarf binaries outnumber all other types of close binary systems per unit volume at least* 20 *times*—and possibly even more. They represent, therefore, without doubt by far the most common type of stellar symbiosis known to astronomy.

This astonishing abundance alone lifts the existence of W UMa-type stars into the front ranks of the unsolved problems of our science; and the explanation of their origin as well as of their present evolutionary stage constitutes an unsurpassed challenge; for the observed frequency of their occurrence precludes a resort to any unusual or unlikely process. Notwithstanding a considerable amount of speculation in this field,* an acceptable solution of this problem does not yet appear to be anywhere within sight. The origin of their most conspicuous attribute—namely, their contact nature—is itself uncertain; for although the possibility that their components have filled their Roche limits by expansion cannot be dismissed, the alternative possibility that such binaries originated *in their present form* in a relatively recent past—and thus represent a group of *very young* stars—should likewise be kept in mind. If contact binaries are indeed young stars, their astonishingly high abundance would demand that their formation must represent a very common process. If, on the other hand, they represent stars which are well on in life, their present stage must represent a veritable 'bottleneck' in stellar evolution if, at any particular time, so many of them are to be seen around. Which one of these alternatives may, however, come closer to truth only the future can tell.

VII.7. ORIGIN AND EVOLUTION OF CLOSE BINARY SYSTEMS

Our long narrative on various formal aspects and physical properties of close binary systems, to which this book has been largely dedicated, has brought us at last to grips with the central problem of their study, which towers over a large section of modern astrophysics and for which most technicalities discussed throughout chapters II–VI have been but prerequisites: namely, what are the underlying causes of the observed appearances of close binary systems? Any book on this subject—and let alone one of this size—would be seriously incomplete if it did not contain at least an attempt to explain the observed phenomena in terms of the natural physical causes. Our preparations being now complete, this task can no longer be deferred. The aim of the present section will, therefore, be to take stock of the known essential facts in our field in an attempt to learn what they may disclose

* *Cf.*, in particular, O. Struve, *Stellar Evolution*, Princeton 1950, p. 232ff.

concerning the origin and evolution of close binary systems as we see them to-day.

As is well known, the evolution of ordinary stars constitutes a process so slow when measured in terms of our human time-scales that (barring unstable objects) no changes arising from it can become perceptible within the brief span of a few generations during which this subject has been studied. All evolutionary theories constructed so far can, therefore, be tested only by confronting their consequences with the observed *statistical* properties of different types of stellar populations; and the discriminating power of such tests will, in turn, depend on the range of information offered by such objects. Single stars which move alone in space are, unfortunately, not very revealing in this respect; for their basic physical characteristics—such as their masses and absolute dimensions—are not individually obtainable and can be surmised only by statistical methods.

It has been generally recognized for some time that the principal factors governing stellar evolution are the initial mass and chemical composition of the stars—the former remaining essentially unaltered (at least well into the post-Main Sequence stage), while the latter keeps changing irreversibly as a result of gradual hydrogen burning. A sample of stars selected at random in any arbitrary volume of space would no doubt contain objects of quite different ages as well as of initial masses and chemical compositions; and, in consequence, an identification of their respective evolutionary stages may become quite complex. A more favourable situation would obtain if we could single out a group of stars for which at least some of the initial conditions are the same—such as stars of the same age and initial composition, differing only in mass. Such groups are indeed known to exist: for instance, the star clusters of different types, whose HR-diagrams reflect nothing but the evolutionary dispersion of equally old stars of different masses after a certain lapse of time.

The individual properties of stars in these clusters are, however, not known to us any more than if such stars were solitary travellers through space. In order to learn more about stars as individuals, recourse must be had to *close binary systems* (and, in particular, to *eclipsing variables*), whose manifest significance for the studies of stellar evolution is two-fold. Not only does a combination of the photometric and spectroscopic elements of such systems furnish practically the entire store of the data on *individual* masses, dimensions and other absolute properties of the stars which we possess. An intimate symbiosis of their components provides, in addition, a unique opportunity for tracing the effects of *differential evolution* of stars of exactly the same age and initial composition, but differing in mass by a known factor. 'The study of Algol variables', wrote Alexander Roberts in the pioneer days of double-star astronomy, 'should bring us to the very threshold of the question of stellar evolution, and to the heart of not a few of the greatest cosmical problems'.* The aim of the present section will be to describe the distance we have

* A. W. Roberts, *Proc. Roy. Soc. Edinburgh*, **24,** 73, 1902.

traversed in this direction since Roberts's time, and to outline the extent to which his prediction of 1902 may since have come true.

To commence our story at its beginning, the *origin* of close binary systems is no doubt to be sought in the same general processes which lead to the formation of the stars in general—be they single or double. It has been recognized in recent decades that the stars originate by a gravitational collapse of cosmic gas-clouds containing enough mass to give birth to, not one or a few, but hundreds or thousands of individual stars at the same time. If the total energy of such a group is positive (i.e., of the sum of the kinetic energies of the individual members exceeds the potential energy of the group as a whole), we deal with a fleeting 'association' of stars, bound to dissolve in a relatively short time; while if the total energy of the new-born group is negative, we have a 'cluster' with a life expectancy of a different order of magnitude.

On account of their short life-times, the known associations of stars are relatively few in number; but two at least are known to contain eclipsing variables among their members: namely, AG Persei in the ξ-Persei association,* and μ^1 Scorpii in the Scorpio-Centaurus stream.† The former group is quite young—its observed rate of expansion indicates that its age cannot exceed $1{\cdot}3 \times 10^6$ years—and the geometrical characteristics of the components of AG Per render it a 'detached' system of a B5 and B7 star, conforming to the Main Sequence. The Scorpio-Centaurus stream represents a much older group of stars (Blaauw† estimates it to be 70×10^6 years); and it is of interest to note that the secondary (B7) component of μ^1 Sco, of mass $9{\cdot}2\ \odot$, has already developed into a 'contact' star.

Close binary systems are, therefore, represented among the members of young association, and they occur also (though infrequently) in open clusters (*cf.* p. 7). Their undoubted absence from globular clusters, as well as reported scarcity among the stars of the central bulge of our galaxy, suggest, however, that among the very much older stars of Population II close binaries are likely to be very much less frequent. Whether this is due to the fact that the conditions obtaining at the time of the formation of Population II stars were less conducive to the formation of close pairs, or whether such pairs gradually fell prey to the disrupting forces acting over an interval of 10^9 or 10^{10} years, remains still uncertain. But as far as Population I stars are concerned, the relative frequency of close binaries among the diffuse star field around us—estimated in Chapter I to $0{\cdot}1\%$—appears to be comparable with the frequency of such objects in open clusters and stellar associations. This, together with the established presence of at least one close binary (AG Per) in a very young association, lends support to a tentative view that close binaries were formed simultaneously with single stars, as by-products of essentially the same formative process.

* A. Blaauw, *B.A.N.*, **11**, 405, 1952; *cf.* also J. Delhaye and A. Blaauw, *B.A.N.*, **12**, 72, 1953.

† A. Blaauw, *Groningen Publ.*, No. 52, 1946.

Whether this process was simultaneous condensation of matter around a pair of mass centres so close that the two stars already entered the stage of stellar evolution as components of a binary system; or whether such binaries originated as a result of a capture of two stars at a time when the density of stars per unit volume in the new-born cluster or association was sufficiently high (for a capture requires the interaction of a minimum of three bodies), remains likewise uncertain. The fact that the relatively high density of stars per unit volume maintained in certain clusters did not apparently lead to the formation of more binaries even over long intervals of time would seem to lend support to the view that, in the process of double-star formation, captures have played a subordinate (if not insignificant) role. But whatever the case may be, there seems no escape from the conclusions that

(*a*) single and double stars have been formed at the same time by similar processes; and

(*b*) the chemical composition of the two components in each pair was initially the same.

Their masses may have been different as a result of the accident of their formation; and if so, this has made all the difference in subsequent evolution.

Let us try to follow the course of events which have unrolled when the two components of a close binary, after their initial contraction, have reached the Main Sequence. If both their masses and chemical compositions were initially the same, their whole subsequent evolution should follow a parallel course and the two stars should remain alike in their external characteristics (radius, spectrum) for all time. If, however, their masses were unequal at the time of the formation of the system, the more massive component was bound to develop a higher internal temperature and, therefore, a higher rate of hydrogen consumption* than its less massive and less luminous mate. As a result, the chemical composition of both stars—if initially the same—will gradually alter and the primary (more massive) component will consume its original hydrogen supply the more rapidly, the greater the disparity in the mass-ratio. As, moreover, this ratio is quite unlikely to change by any appreciable amount as long as both stars remain on the Main Sequence, it follows that it is the more massive (and luminous) component which is bound to exhaust its hydrogen supply, and suffer whatever consequences this may entail, *ahead* of its less luminous mate.

Now a number of investigations undertaken in recent years have converged to the view that the most conspicuous consequence of incipient hydrogen shortage, caused by its depletion through helium-building thermonuclear reactions, should be the *expansion* of the star as a whole and its

* For moderately massive stars ($m < 2 \odot$) deriving their energy from a transmutation of hydrogen into helium by way of the proton-proton cycle of reactions, the rate of hydrogen consumption for given energy output should vary approximately with the square of the central temperature; while for massive stars which burn their hydrogen by way of the carbon-nitrogen cycle the rate of hydrogen consumption could be as much as ten times higher.

evolution away from the Main Sequence. Whether the hydrogen combustion at the centre leads (in massive stars) to the shrinkage of the convective core, or (in stars of smaller mass) to an outright formation of isothermal core of increasing degree of degeneracy, either process is bound to lead to a diminution of potential energy in central regions, the excess of which is spent in the expansion of the outer layer. Theoretical computations indicate, moreover, that the rate at which this expansion occurs should depend vitally on the star's mass: a high energy output associated with large masses is bound to make the onset of hydrogen deficiency more abrupt—and subsequent expansion triggered by it more rapid—than if the mass were small.

How do these theoretical expectations fit in with the phenomena observed in close binary systems? It needs hardly any further emphasis to point out that the existence of such systems offers us the best opportunity to probe the nature of stellar expansion in post-Main Sequence stage that we could hope for. In single stars, expansion could go on unhampered by external influences until the excess of potential energy has been expended. In close binaries, on the other hand, a mere proximity of the companion has surrounded the expanding star by a dynamical barrier—namely, the Roche limit—which the volume of the star cannot exceed. If, therefore, any component of a close binary attains the stage at which incipient internal hydrogen shortage brings about expansion, the gradual growth of the star's volume would be bound to be *arrested* at the Roche limit. As, moreover, the relative dimensions of this limit depend only on the mass-ratio of each respective system (which can be independently ascertained), the identification of stars which have attained their Roche limit in binary systems of known mass-ratios should offer no observational difficulty, and should enable us to confront the theoretical expectations with the actual facts.

Now we pointed out already in section VII.4 that there exists indeed a distinct group of eclipsing systems in which one component has indubitably attained its Roche limit—but, unfortunately for our expectations, it is the wrong star! For the most striking feature of the semi-detached systems discussed in section VII.4 is the fact that, in every single case known so far, *it is the secondary* (*less massive*) *component which appears to be at its Roche limit, while its more massive mate remains well interior to the limit.* In order to illustrate this point of cardinal importance for our subject, attention is invited to the accompanying Fig. 7-23, exhibiting schematically two possible types of semi-detached systems (drawn to scale for a mass-ratio $q = 0{\cdot}4$). The upper model (a) represents such a system in which the secondary has attained its Roche limit, while in the lower model (b) the primary has done so. Now we wish to stress the fact that *semi-detached systems of type* (a) *alone have so far been detected*; *type* (b) *being conspicuous by its absence.* At least among the 55 eclipsing systems with subgiant components examined in this chapter, 37 (i.e., 67%) turned out to be semi-detached binaries of type (a), while not a single case of type (b) has so far been found. The relative abundance of systems of this latter type would, therefore, seem to be less than

2% of the total; and a disproportion in occurrence of types (*a*) and (*b*) should be at least 10 : 1 or greater.

In order to reconcile this situation with the requirements of the hypothesis that the primary cause of expansion is the incipient hydrogen shortage, Crawford* proposed an ingenious hypothesis which was elaborated by

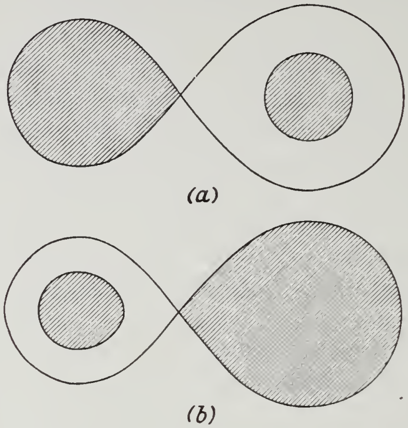

FIGURE 7-23. A SCHEMATIC VIEW OF TWO ALTERNATIVE TYPES OF POSSIBLE SEMI-DETACHED SYSTEMS

The diagrams (drawn to scale for a mass-ratio $m_2/m_1 = 0{\cdot}4$) represent semi-detached systems in which (*a*) the secondary (less massive), or (b) the primary (more massive) component is in contact with its Roche limit.

Hoyle† under the playful name of a 'dog-eats-dog' scheme, and can be summarized (largely in Hoyle's own words) about as follows:—From a theoretical point of view it seems inevitable that the primary (more massive) component A must reach first the stage of incipient expansion; and . . . 'as the evolving star swells, its companion B takes up more and more of its material. Thus B, initially the more sedate of the two stars, becomes increasingly predatory as the process gets under way. Where does it stop? Only after B has swallowed so much of its companion that the evolution of the latter is pushed to the evolutionary stage where shrinkage begins to occur. Eventually the shrinkage of the evolving star becomes sufficient for it to escape at last from its marauding companion.' And 'what will happen when the now brighter component begins its own evolution away from the Main Sequence? It may be expected to swell and to engulf the companion that it robbed so unfeelingly in the past. There is the possibility that the predatory star will be forced to make amends for its former behaviour by returning material to the (at present) fainter star. In the interest of cosmic justice it is to be hoped that this happens; but whether it does or not is unsure.'‡

This entertaining picture of a predatory struggle for mass between ageing

* J. A. Crawford, *Ap. J.*, **121**, 71, 1955.

† F. Hoyle, *Frontiers of Astronomy*, London 1955.

‡ *Cf.* F. Hoyle, *op. cit.*, p. 200.

components of close binary systems does not, however, fare any too well when confronted with the observed facts. For according to this picture, the present subgiant secondary components of close binary systems should be erstwhile primaries which have lost so much mass to their mates that their original mass-ratio, was actually reversed. If, however, this were indeed so, nothing should prevent the same process from going back and forth several times. It is true that we have since probably identified the policeman—namely, the Roche limit—who sees to it that 'cosmic justice' is done and the mass once gobbled up is duly returned on subsequent expansion. Moreover, we have seen in the preceding section VII.5 that most part of the mass lost by components expanding at the Roche limit should indeed be captured by their mates. Yet, in spite of these facts, at least two grave obstacles remain in the way of the 'dog-eats-dog' hypothesis which are bound to render its claims rather doubtful, and which can be summarized as follows:

(*a*) The complete lack of subgiant components, at their Roche limits, with masses larger than those of their mates, which would represent the transient evolutionary stage when the exchange of the roles of both components is in progress. It may, to be sure, be argued that the absence of specific examples of this type is due to the fact that this transition stage occurs at cosmically so rapid rate (of the order of 10^5–10^6 years) that very few systems may be actually caught in doing so at any particular time. Now several recent theoretical investigations* make it indeed plausible that incipient hydrogen deficiency may set a massive star expanding at the requisite rate, but *not* a star whose mass is comparable with (or less than) that of the sun. For such stars, time intervals of the order of 10^8–10^9 years have been found necessary to allow a hydrogen exhaustion to increase their radii to their maximum limits. Besides, even if the requisite rate of expansion could be attained for sufficiently massive stars, it would still follow that subsequent repetitions of this act would be bound to recur with corresponding frequency and that, as a result, the transitory stage could not escape statistical detection.

(*b*) The absence of the internal energy requisite to effect an exchange of the roles of the two components. If a rapid expansion followed by ejection of mass at the Roche limit were to be effective enough to convert an erstwhile primary into a secondary (less massive) component, then (for average semi-detached systems) a transfer of some 80% of the mass of one star on to the other would be called for in the course of each such metamorphosis. Now, in their pre-expansion stage, such stars are bound to exhibit a rather high degree of central condensation; and in order to remove some 80% of their mass it would be necessary to despoil them virtually down to their isothermal cores. But such cores no longer contain internal energy sufficient to lift so much mass up to the Roche limit (which would be prerequisite for its subsequent escape)—certainly not without resort to rapid helium burning

* S. Chandrasekhar and M. Schoenberg, *Ap. J.*, **96**, 161, 1942; A. R. Sandage and M. Schwarzschild, *Ap. J.*, **116**, 463, 1952; M. Schwarzschild, I. Rabinovitz and R. Härm, *Ap. J.*, **118**, 326, 1953; M. Schwarzschild, and F. Hoyle, *Ap. J. Suppl.* No. 13, 1955.

which, in turn, would almost certainly give rise to luminous phenomena sufficient for attracting attention to the object. In order to meet the requirements of the 'dog-eats-dog' hypothesis, the expanding star would not only have to manage to despoil itself quickly of some 80% of its original mass—i.e., of an amount expended in supernova explosions—but also to do it inconspicuously enough to avoid being seen in the act!

Needless to stress, these requirements are almost certainly impossible to fulfil; for even if (*a*) could be met by assuming a sufficiently rapid expansion and ejection, (*b*) cannot be circumvented. We are, instead, facing a tantalizing problem which does not seem to admit of any obvious solution. As the known contact components in close binary systems appear to be restricted exclusively to the secondaries, perhaps the most straightforward way of interpreting the known data would be to assume that the secondary components—for reasons best known to themselves—begin to expand at a certain stage of their evolution before primaries get around to do so, irrespective of their smaller masses. For why otherwise do the massive secondary components (of masses 10–15 ⊙) in systems like V Puppis or μ^1 Scorpii exhibit contact properties (overluminosity, overextension) in just the same way as the contact secondaries of (say) Algol ($m_2 = 1{\cdot}0$ ⊙) or TX UMa ($m_2 = 0{\cdot}85$ ⊙), while the primary components of intermediate masses ($m_1 = 5{\cdot}2$ ⊙ for Algol, or 2·8 ⊙ for TX UMa) appear to be normal and well detached from their Roche limits? If the secondaries are thus in the process of secular expansion, the 'undersize' subgiants of section VII.4 constitute a transitory stage between Main-Sequence stars and contact subgiants.

Such an interpretation seems indeed quite consistent with all available empirical evidence, but encounters theoretical difficulties. For in order to reconcile it with the view that the underlying cause of expansion is incipient hydrogen shortage, it would be necessary to argue that the internal structure of the secondary components possesses such special features as to render their central temperatures higher than those of the primary components, in spite of their smaller masses. This certainly does not seem to be likely; and even if we were willing, in the face of this difficulty, to give up the incipient hydrogen shortage as the motive of expansion and look for some other cause, the question would still be with us as to what else are then the actual manifestations of the hydrogen shortage which is eventually *bound* to develop in all ageing Main-Sequence stars?

While few of these questions can be satisfactorily answered so far, and fewer possibilities can as yet be categorically ruled out, an alternative to the 'dog-eats-dog' hypothesis can be put forward which seems to satisfy the observed data, but is not at variance with the theoretical expectations. If, consistent with the present theories of stellar structure, the cause of expansion in post-Main Sequence stage is an incipient hydrogen shortage affecting the primary (more massive) component first on account of its higher internal temperature, let us assume that its rate is such as to extend the primary to its Roche limit in a time of the order of 10^5–10^6 years. If so, less than one star in a hundred would be performing the expansion at any particular time, and their apparent absence in our catalogues could be accounted for. Moreover, when the Roche limit has been reached, a continued tendency to expand will bring about a loss of mass to the secondary component. This loss is, however, likely to be limited only to the outer layers of the primary component. A relatively small transfer of mass skimmed from its surface should be sufficient to fill the secondary's Roche limit (or, if more were ejected, to provide a gas pool circulating between, or around, the two stars); but the mass-ratio of the system would not be altered appreciably and the secondary's luminosity would be only moderately increased; for an influx of rarefied gas (no matter how hydrogen-rich) on the surface would alter the secondary's energy output only in so far as the additional mass would alter the hydrostatic pressure in central regions of the star. Thus the secondary component would still remain the less massive and luminous of the two; but its volume has increased and its Roche limit may be filled to the brim.

If a process outlined in the foregoing paragraph is to result in the formation of semi-detached systems of the observed type (*a*) on Fig. 7-21, it is necessary that, following its initial expansion, the primary component should collapse back now to its original state with the same rapidity as it formerly expanded leaving the contact secondary enlarged by its newly acquired matter to settle down at a much slower pace. If this were possible, the 'undersize' subgiant components of systems listed in Table 7-6 would then represent a later evolutionary stage than those still clinging to their Roche limits, as stars which are contracting from their Roche limits rather than expanding towards them. And it is perhaps not too much to expect that this contraction may proceed at such a rate that, if and when the primary component gets ready to repeat its rapid act of mass ejection, the secondary component may again possess some spare volume inside its Roche limit to receive it.

If the hypothesis outlined tentatively in the preceding paragraphs will withstand more detailed scrutiny, several notions held previously will have to be modified. Consider, for instance, the well-known conspicuous disparity in masses of the components of semi-detached systems—exceeding considerably that observed in detached Main-Sequence binaries. This fact has previously been interpreted by several investigators (including the present writer) as an indication that, in the past, a transfer of matter from the secondary to the primary component tended to exaggerate the inequality of their masses. Now it is at least possible that this argument might also be reversed, and an assertion made that systems with initially very unequal mass-ratios should also be the first to develop contact secondaries, because their primary components will run through the course of their Main-Sequence life much more rapidly than their mates. As time progresses, and more than one transfer of mass from the primary to the secondary has taken place, their mass-ratios may gradually tend to equality. If these views were correct, the subgiant-possessing systems like RS CVn ($m_2/m_1 = 0{\cdot}93$) should then represent a more advanced evolutionary stage than say TX UMa ($m_2/m_1 = 0{\cdot}30$)

or Algol ($m_2/m_1 = 0{\cdot}19$). Lastly, is it perhaps not possible to surmise that the rather enigmatic group of R CMa-type systems of section VII.4 owe the present smallness of their masses to the fact that, in the course of repeated mass transfers, a large fraction of the initial masses of their primary components may have escaped from the system rather than be captured by the secondary?

Another relevant consideration which should be mentioned in this connection concerns the rate of axial rotation of components in contact binary systems. In the preceding section VII.5 we stressed that, on account of the conservation of the angular momentum, a rapidly expanding component of a close binary system should reach its Roche limit with a velocity of axial rotation of its surface layers reduced well below the Keplerian angular velocity of the system. Earlier in this section we emphasized further the fact that no systems consisting of primary contact components accompanied by detached secondaries are known. This statement should now be qualified: although for Main-Sequence stars it holds good without reservation, at least one or two supergiant systems are known whose primary components appear to be contact stars; the best-known case being that of β Lyrae. This system is remarkable indeed in many respects; not the least being the fact that the axial rotation of its principal cB9 component appears to be abnormally slow. If this star were rotating with the same period as it revolves, its equatorial velocity could not be less than some 230 km/sec.* The narrowness of observed line profiles in the spectrum of this star rules out, however, any such velocity. Struve† and, more recently, Böhm-Vitense‡ and Mitchell§ have shown that the maximum velocity of axial rotation of the cB9 component of β Lyrae, consistent with its observed line-profiles, is between 40–50 km/sec —i.e., barely one-fifth of that corresponding to synchronism between rotation and revolution. This confronts us with a serious and significant discrepancy, whose probable cause is a rapid expansion of the cB9 component in relatively recent past. As this expansion was probably too rapid to give the tidal friction a chance to exert any appreciable effect, the conservation of the angular momentum inadvertently slowed down the rate of rotation of the outer surface layers in inverse proportion to the square of their increase in size. In point of fact, the existing disparity between the velocities of rotation and revolution offers a direct clue to the original size of the principal component of β Lyr (though not to its luminosity) in the pre-expansion stage, and implies that its mean radius on the Main Sequence must have been $\sqrt{5} = 2{\cdot}24$ times smaller (or approximately 20 $\odot$) than it is at the present time.

Complex gas-streams evidenced by the spectrum of β Lyr¶ have long

* Z. Kopal, *Ap. J.*, **93**, 92, 1941.
† O. Struve, *P.A.S.P.*, **64**, 180, 1952.
‡ E. Böhm-Vitense, *Ap. J.*, **120**, 271, 1954.
§ R. I. Mitchell, *Ap. J.*, **120**, 274, 1954.
¶ O. Struve, *Ap. J.*, **93**, 104, 1941; J. R. Gill, *Ap. J.*, **93**, 118, 1941; J. L. Greenstein and T. L. Page, *Ap. J.*, **93**, 128, 1941.

suggested that its cB9 component is likely to be at its limit of dynamical instability. We may add here that such a surmise is indeed borne out by the fractional dimensions of the primary component, whose diametral semi-axis is very approximately equal to $r_1 = 0{\cdot}39$.* The latter value corresponds, on contact hypothesis, to a mass-ratio $m_2/m_1 = 0{\cdot}85$—in close agreement with the ratio of 0·82 as deduced earlier by the present writer* from the observed photometric ellipticity of the system by the method of section VI.12. Thus it appears very probable that, in case of β Lyr A, we have caught—off Main Sequence—at least one example of the elusive type (*b*) of semi-detached binary system, and no wonder; for with its absolute visual brightness of -6 magn, β Lyrae shines like a beacon through most part of our galaxy, and is easily spotted from afar. The contact nature of its primary component is suggested by its fractional dimensions and overluminosity,† while the observed slowness of the axial rotation of its surface layers (whatever it may be inside) attests its rapid expansion in a relatively recent past. The activity of the star at this stage of its evolution, as manifested by the available spectroscopic evidence, makes one regret all the more the severely transient nature of this elusive phenomenon—a regret which should be perhaps mingled with gratitude that just at the present, we have the privilege of witnessing it in one of the naked-eye stars.

Next consider, from this point of view, the existence of binary systems like ζ Aurigae or VV Cephei, which consist of a late-type supergiant attended by a *B* star. As far as we know, this *B* component does not seem to deviate much from the Main Sequence; but the principal (more massive) components are conspicuously to the right of it in the HR-diagram. In spite of their large absolute dimensions (205 $\odot$ and 1200 $\odot$), the principal components of both ζ Aur and VV Cep are well interior to their respective Roche limits (because the dimensions of the systems are so large) and, therefore, both systems should be classified as 'detached.' However, the extreme disparity in sizes of two components in each pair ($k = 0{\cdot}018$ for ζ Aur, and 0·011 for VV Cep) strongly suggests that while the secondary components—on account of their smaller mass—have not yet exhausted their hydrogen and are, therefore, still on the Main Sequence, the primary components have already fallen victims to the hydrogen shortage and started expanding as a consequence. Moreover, at least in the case of VV Cep we again possess some evidence that the expansion of its primary component must have been relatively rapid;

* Z. Kopal, *Ap. J.*, **93**, 92, 1941.

† If the mass-ratio of the system is 0·82–0·85, and if both components possessed luminosities appropriate for detached stars of their masses, the lines of the secondary component should be clearly seen in the combined spectrum of β Lyrae. The fact that they are not reveals that the magnitude-difference of the two stars must be enhanced by an overluminosity of the primary component, or underluminosity of the secondary. Now the diametral fractional radius $r_2 = 0{\cdot}23$ of the secondary component renders the latter distinctly smaller than its Roche limit. On the other hand, the known overluminosity of the contact secondary components of semi-detached systems makes it very plausible that the cB9 component of β Lyrae is similarly overluminous and thus extinguishes spectroscopically its mate (whose lines should otherwise be discernible).

for spectroscopic observations* indicate that the primary's angular velocity of rotation is once more considerably below the Keplerian. Thus it appears reasonable to surmise that close binary systems whose primary component is a late-type supergiant, accompanied by an early-type Main-Sequence star, can be regarded as representing an evolutionary stage in which the primary (more massive) star did already exhaust its hydrogen and was impelled thereby to expand, while the secondary component has not yet reached the same predicament on account of its smaller mass—though it will no doubt also have to face it in the future. Thus, while the principal components of ζ Aur or VV Cep (and of systems like Antares, or Mira Ceti) may already possess helium cores which are completely devoid of hydrogen, their less massive mates probably represent Main-Sequence stars of less than average hydrogen contents; the difference in the external characteristics of the primary and secondary components being due to the existing disparity in their masses, which caused both components to evolve at a different rate.

It may be mentioned, in this connection, that many—if not all—of the 'symbiotic' stars exhibiting simultaneous early- and late-type features in their composite spectra† are nothing but close binaries, in which the disparity in masses of both components caused them to reach a stage at which the less massive early-type star remains still on the Main Sequence, while its more massive mate has already developed into a late-type giant or supergiant. The fact that, in most composite spectra of known symbiotic stars, the late-type features appear to predominate over the early-type characteristics lends support to this hypothesis. The absence of radial-velocity changes which should arise from orbital motion is easily explained if the planes of the orbit are inclined but little to the celestial sphere.

If the mass-ratio of an ageing close binary were closer to unity, or if the masses are smaller than in systems like ζ Aur or VV Cep, both components should approach the stage of hydrogen-exhaustion and start expanding at more nearly the same time, to develop temporary characteristics of giants or supergiants—depending on their mass and internal energy supply. Such systems appear also to be known; WW Draconis or RT Lacertae can be quoted as examples of moderately massive stars ($m_1 = 3{\cdot}9 \odot$, $m_2 = 2{\cdot}3 \odot$ for WW Dra; and $m_1 = 1{\cdot}9 \odot$, $m_2 = 1{\cdot}0 \odot$ for RT Lac) which have attained giant dimensions of $R_1 = 3{\cdot}0 \odot$, $R_2 = 5{\cdot}0 \odot$ for the gG2 + gK0 components of WW Dra, and $R_1 = 4{\cdot}8 \odot$, $R_2 = 3{\cdot}8 \odot$ for the gG9 + gK1 components of RT Lac.

Moreover, a truly breath-taking example of two supergiants near their maximum distension appears to be provided by the southern eclipsing system of W Crucis. This unique binary of the period of 198·5 days consists of a close pair of G-supergiants, whose fractional dimensions and superficial

* *Cf.* V. Goedicke, *Michigan Publ.*, **8**, No. 1, 1939.

† *Cf.*, e.g., M. Johnson, *Vistas in Astronomy*, (London 1957), vol. II, p. 1407; or Tcheng Mao-Lin and M. Bloch, *op. cit.*, p. 1412.

distortion indicate* that they may not be far from their Roche limit—if they have not already attained it. Unfortunately, no spectrographic orbit of this system has been secured so far (because its apparent photographic magnitude of $8^{m}_{\cdot}9$–$9^{m}_{\cdot}5$ would call for a rather large instrument); but the photometric evidence alone leaves but little doubt that W Cru is likely to prove a southern counterpart of β Lyr, and may well exceed it in total mass (as it certainly does in the absolute dimensions of its components)—thus having a chance to become the most massive known star in the whole sky. Its complicated emission spectrum reported by early observers from objective-prism plates gives a preview to what may transpire from a more detailed study with high-dispersion; and the first astronomer who will undertake it with the aid of a large reflector in the southern hemisphere will certainly collect a rich prize.

Summarizing, we may say that the existence of primary (more massive) components exhibiting unmistakable evidence of expansion ahead of their secondaries in supergiant systems like β Lyr, ζ Aur, or VV Cep (which are favoured in observational detection on account of their great absolute brightness) lends support to our view, expressed earlier in this section, that it is the more massive star, and *not* the secondary, which expands first in every binary system; and that the observed contact secondaries discussed in sections VII.4 and 5 are the result of acquisition of matter which filled their Roche limits to the brim rather than of an expansion prompted by internal reasons. Provided that—as seems highly probable—the real reason of stellar expansion in post-Main Sequence stage, the secondaries—any star, for that matter—are bound to undergo expansion on their own one day; but before this happens, their primaries must have been through this experience possibly more than once.

A closer inquiry into fuller details of such a process gives, unfortunately, rise to many more questions than can be answered so far. Will, in particular, the internal energy made available by the formation of an isothermal core be released gradually, and spent in a continuous expansion; or can the expansion be arrested at times by the growth of the core getting out of step with the structure of its surrounding layers—to be resumed again after balance has been re-established? If the star were single and the secular growth of the core sufficiently slow, a continued expansion would seem to be indicated. However, the evidence available for close binary systems indicates that the expansion on their primaries, once it starts, must represent a relatively rapid process and, moreover, each expansion to the Roche limit may lead to a spasmodic ejection and loss of mass—sufficient perhaps to arrest the expansion for a time.

But what happens when the available internal energy store gives in at last, and the maximum limit of distension can no longer be maintained? The theory of the internal structure of the stars cannot, unfortunately, offer as yet any secure guidance. It seems, however, more than probable that the

* *Cf.* H. N. Russell, *Ap. J.*, **36**, 148, 1912; D. J. K. O'Connell, *Riverview Publ.*, No. 3, 1936. Z. Kopal, *Harv. Circ.*, No. 439, 1941.

star will have to contract—and eventually traverse the Main Sequence on its way to become a subdwarf and, ultimately, a white dwarf. Whether the bulk of the mass of the star will be conserved in this process—or, if lost, at what stage—remains uncertain. One fact is, however, clear: any process involving rapid shrinkage to subdwarf stage is bound (on account of the conservation of angular momentum) to step up again the rate of axial rotation of the respective star—possibly to the limit of rotational instability. Do we know of any binaries which may fall in this category? It may perhaps be noted, in this connection, that the principal component of RZ Sct—the extreme case of a system belonging to the peculiar group of R CMa-stars introduced in section VII.4—exhibits conspicuous evidence of abnormally rapid axial rotation;* as does, to a lesser extent, RW Gem of the same group.† Whether or not their present rapid axial rotation indicates past shrinkage remains, however, conjectural.

And last but not least, how can we reconcile our present views on stellar evolution with the existence of such binary systems as represented, for instance, by Sirius—in which a Main-Sequence star of spectral class A2V is attended by a white dwarf? On account of their high degeneracy and low potential energy, white dwarfs are generally regarded as representing the last stage of stellar evolution, which can be attained only after all hydrogen has been completely exhausted in the interior. Now it is very probable that both components of the Sirius system originated at the same time, and from very much the same primordial matter. Therefore, if Sirius B attained now the white-dwarf stage while its mate is still on the Main Sequence, it must have passed through its evolution at very much more rapid pace. The present mass of Sirius B is, however, *less* than that of its mate ($1{\cdot}1\ \odot$ as against $2{\cdot}3\ \odot$). Its more advanced evolutionary stage is, therefore, compatible with the view that the motive power of stellar evolution is gradual hydrogen burning only if we assume that, initially and throughout most part of its past, *the mass of Sirius B was actually larger than that of its mate*—probably much larger judging from their present disparity in evolutionary stage—and that their present mass-ratio is of relatively recent date (as the collapse of Sirius B into white-dwarf stage may have entailed a considerable loss of mass). If so, however, then the conclusion is inevitable that, in its post-Main Sequence stage, the *erstwhile primary* (and now secondary) *component of Sirius must have temporarily been a red giant*—dominating the light of the system—which must have been by a few magnitudes brighter than now and distinctly reddish in colour.

Such a possibility recalls to mind that, in the old days when scholars knew their Latin and Greek better than astronomy, considerable attention was attracted by the fact that the majority—if not all—astronomical authorities

* According to F. Neubauer and O. Struve (*Ap. J.*, **101**, 240, 1945) the equatorial velocity of rotation of the B2 component is some 50 km/sec based on helium lines—as compared with the Keplerian angular velocity of only 11 km/sec.

† O. Struve, *Ap. J.*, **104**, 253, 1946.

of the ancient world described Sirius as a 'red' star—a fact quite at variance with its blue-white colour as we know it today. This apparent discrepancy gave rise to a lively discussion reflected in an extensive literature on the subject,* in which our intrepid astronomical ancestors eventually routed the philologists by the weight of the argument that so profound a change in the colour of a star in a time-span of some 2000–3000 years was a sheer physical impossibility.† In the time which elapsed since this discussion died down, our knowledge of the probable trends of stellar evolution progressed considerably—and, behold, arguments can now be advanced (unknown to our predecessors) which make it at least plausible that, some time in the past, Sirius may indeed have appeared as a red star and very much brighter than now. To be sure, the difficulty of the time-scale remains unsurmounted; for few astronomers would venture to concede even today that a transition from red giant to a white dwarf could be accomplished within a few thousand years; but in 10^5 or 10^6 years—who knows? It may thus have been the incomprehending eye of a Pithecanthropus or of the Neanderthal man, rather than that of an Egyptian priest or Greek sage, which actually beheld this heavenly wonder; but the phenomenon could scarcely have been even a memory at the time when writing was invented. Or are we still mistaken about the time-scale of stellar evolution?

And by this we have come to the end of our narrative, and of this book. Some reader may perhaps find this conclusion rather abrupt and the whole last chapter—which should have been the culminating point of the book—disappointingly brief. Its brevity is, however, certainly not due to any improper haste to part company with the reader, but rather to an incomplete state of research of the subject itself. Most of the preceding Chapters II–VI, dealing with methodological aspects of the study of close binary systems, have been concerned with topics which can at present be regarded as reasonably well understood. On the other hand, an interpretation of the observational evidence emerging from such studies is still relatively near to its beginnings, and further investigations of our heavenly twins may reveal many a surprise throwing new light on age-old problems. A demand for further research on subjects dealt with in this chapter is particularly pressing; and the writer ventures to hope that this book may help to attract the attention of many future investigators to this newly opening and still almost virgin field.

If we prefaced this section with a quotation from Alexander Roberts—and the reader should judge for himself the extent to which the developments of the past half-century have borne out his expectations—let us conclude it by recalling the words of another great pioneer of bygone days, Sir George Howard Darwin. At the close of a chapter entitled 'Speculations as to the

* For its summary *cf.*, e.g., T.J.J., See, *A.N.*, **229**, 245, 1926.

† Thus, according to A. Dittrich (*A.N.*, **231**, 385, 1928) 'Es ist mehr als waghalsig, wenn man wegen zweier Worte der antiken Literatur die Existenz roter Mittelsterne postulierte.'

Origin of Double Stars' in his little classic book on *The Tides*,* in the last year of his long life, Darwin recorded the following thoughts which ring as true to-day as when they were written in 1911, and which the present writer with due modesty now ventures to invoke for closing a book of his own:—'In the sketch which I have endeavoured to give of this fascinating subject, I have led my reader to the very confines of our present knowledge. It is not much more than a quarter of a century' (in Darwin's time; some eighty years now) 'since this subject has claimed the close attention of astronomers; something considerable has been discovered already, and there seems scarcely a discernible limit to what will be known in this field a century from now. Many of the speculations which I have set forth may then be shown to be false, but it seems profoundly improbable that we are being led entirely astray by a will-of-the-wisp.'

VII. BIBLIOGRAPHICAL NOTES

VII.1: A determination of the masses and absolute dimensions of two-spectra eclipsing variables by a combination of their photometric and spectroscopic elements is classical. The first successful use of this method occurred in 1911, when a combination of his own photometric elements with the spectroscopic orbit by R. H. Baker (*Allegh. Publ.*, **1**, 163, 1910) enabled Joel Stebbins (*Ap. J.*, **34**, 112, 1911) to determine the absolute properties of the eclipsing system β Aurigae, whose components thus became the first stars (other than the Sun) whose absolute radii were obtained without recourse to any law of radiation.

The corrections to the observed amplitudes of radial-velocity changes of the components of close binary systems, necessitated by the reflection effect, have been considered by A. S. Eddington (*M.N.*, **86**, 320, 1926), G. P. Kuiper (*Ap. J.*, **88**, 472, 1938), and first worked out by Z. Kopal (*Proc. Amer. Phil. Soc.*, **86**, 351, 1943). Of their subsequent investigations we should refer to M. Kitamura (*Publ. Astro. Soc. Japan*, **5**, 114; **6**, 217, 1954) and, in particular, A. H. Batten (*M.N.*, **117**, 521, 1957). The corrections arising from the gravity-darkening of third-harmonic tidal distortion are due to Z. Kopal (*Proc. Amer. Phil. Soc.*, **89**, 517, 1945).

A method of photometric determination of the mass-ratios from an interpretation of photometric ellipticity effect between minima is also due to Kopal, and was expounded by him (with applications) in *Ap. J.*, **93**, 92, 1941, and *Harv. Circ.*, No. 339, 1941. A determination of the absolute dimensions of eclipsing binary systems from the spectroscopically observed rotational effect within minima appears to have first been made by D. B. McLaughlin in *Ap. J.*, **60**, 22, 1924 (*cf.*, also *Michigan Publ.*, **6**, 3, 1934).

A determination of the masses and absolute dimensions of eclipsing variables with known single-spectrum orbits by fitting their primary components to an empirical mass-luminosity relation, as developed in this section, was published by Z. Kopal in *Ann. d'Astroph.*, **19**, 298, 1956; though its underlying idea goes back to P. P. Parenago (*Astr. Zhurnal*, **27**, 41, 1950). In his approach to the problem, Parenago fitted the primary component to the Main Sequence rather than a mass-luminosity relation; but for stars radiating like black bodies the two methods are essentially identical; and we prefer the latter because of the greater linearity of the mass-luminosity relation. For applications of these methods to practical cases, *cf.* P. P. Parenago and A. G. Massevich (*Astr. Zhurnal*, **27**, 137, 1950; *Trudy Sternberg State Astr. Inst.*, **20**, 81, 1950); I. A. Kurzemniece (*Astr. Zhurnal*, **31**, 36, 1954; **32**, 578, 1955); or Z. Kopal (*op. cit. ante*).

A determination of the absolute dimensions of eclipsing binaries with no spectrographic orbits, by fitting *both* components to an empirical mass-luminosity relation, originated with Z. Kopal (*Zs. f. Ap.*, **9**, 239, 1934) and was followed up later by G. Durand (*Toulouse Ann.*, **11**, 209, 1935; **14**, 5, 1938. *Cf.* also *C. R. Acad. Paris.*, **202**, 1762, 1936; **206**, 490, 1093, 1938); E. A. Kreiken (*A.N.*, **259**, 349, 1936); K. Pilowski (*A.N.*, **261**, 18, 1936); V. A. Krat (*Engelhardt Obs. Bull.*, No. 19, 1937); A. Colacevich (*Mem. Soc. Astr. Ital.*, **11**, 115, 1938); J. Ellsworth (*Journ. des Observ.*, **21**, 1, 1938); J. Gabovitš (*Publ. Obs. Tartu.* **30**,

* G. H. Darwin, *The Tides and Kindred Phenomena* (third edition), London 1911, p. 402.

11, 1938); S. Gaposchkin (*Proc. Amer. Phil. Soc.*, **79,** 327, 1938; **82,** 291, 1940) and others. A useful (though incomplete) list of bibliographical references to later investigations in this field was recently compiled by A. Beer (*Vistas in Astronomy*, vol. II, p. 1387, London 1957).

For a determination of mean densities of the components of close binary systems *cf.*, A. W. Roberts (*Ap. J.*, **10,** 308, 1899), H. N. Russell (*Ap. J.*, **10,** 315, 1899), J. H. Jeans (*Ap. J.*, **22,** 93, 1905), etc. For a derivation of the equations (1-38) or (1-46) *cf.* Z. Kopal, *Ap. J.*, **94,** 145, 1941; the absolute lower bound (1-33) of mean densities is new.

VII.2: The system of classification of close binary systems as outlined in this section has been proposed by Z. Kopal in *Ann. d'Astroph.*, **18,** 379, 1955 (*cf.* also his report on behalf of the Commission 42 in *I.A.U. Trans.*, **9,** 599, 1957; or his paper in the *I.A.U. Symposium on Non-Stable Stars*, Cambridge, 1957, pp. 123–137). For additional details of the proposed classification *cf.* also Z. Kopal and M. B. Shapley, *Jodrell Bank Ann.*, **1,** 141, 1956.

Of earlier work on the classification of close binary systems *cf.* E. W. Pike (*Ap. J.*, **41,** 76, 1931); V. A. Krat (*Astr. Zhurnal*, **21,** 20, 1944); O. Struve (*The Observatory*, **71,** 197, 1951); A. Kranjc (*Mem. Soc. Astr. Ital.*, **22,** 131, 1951); L. Plaut (*Groningen Publ.*, No. 55, 1953); etc.

VII.3: The observational material discussed in this section has been taken from Z. Kopal and M. B. Shapley's *Catalogue of the Elements of Eclipsing Binary Systems*, (*Jodrell Bank Ann.*, **1,** 141, 1956, Table II). The empirical mass-luminosity and mass-radius relations, as represented by equations (3-1) and (3-2), were previously published by Z. Kopal in the *Ann. d'Astroph.*, **18,** 379, 1955; and so was the method for a determination of the mass-ratios based on equation (3-5).

VII.4: The problem of the subgiant components in eclipsing binary systems belongs among the most exciting (and revealing) stories in double-star astronomy. The first investigator who became aware of its existence appears to be K. Walter. In a paper published in the *Königsberg Veröff.*, No. 2, 1931—which proved in retrospect one of the investigations inaugurating a modern era in the study of eclipsing binaries—Walter clearly pointed out the existence of 'spezielle Algolsterne' in which a nearly spherical primary component is attended by a highly distorted companion (and which are, in effect, identical with our present semi-detached systems); and on p. 26 of his paper just quoted he went on to say that . . . 'Man geht wohl nicht fehl, wenn man annimmt, dass nicht weit entfernt von dieser höchsten vorkommenden Elliptizität sich die Stabilitätsgrenze dieser deformierten Sterne befindet. Es ist nun interessant zu sehen, dass nicht nur die β Lyrae- und W Ursae Maioris-Sterne nahe der Stabilitätsgrenze stehen, wie bisher allgemein angenommen wurde, sondern dass ein grosser Teil der Algolsterne, wenigstens in bezug auf eine Komponente, sich in ähnlicher Lage befindet.'

The relation of the fractional dimensions of the secondary components of these 'special Algol stars' to their stability limits was followed by Walter in his subsequent investigations; and in *Zs. f. Ap.*, **19,** 157, 1940, Walter was the first to point out that the components of R CMa must be at the limit of stability—a conclusion later re-iterated by F. B. Wood (*Princ. Contr.*, No. 21, 1946). The first quantitative application of the geometrical properties of the Roche model to close binary systems appears, however, to have been made before Walter by A. J. Wesselink (*Diss. Leiden*, 1938) and G. P. Kuiper (*Ap. J.*, **88,** 497, 1938).

On the other hand, the peculiarly small masses of the secondary components in 'special Algol systems' attracted likewise the attention of the spectroscopists. Thus in a paper published in *Ann. d'Astroph.*, **11,** 117, 1948, O. Struve summed up the situation in the following trenchant words: . . . 'At the present time we have no reason to doubt the very simple conclusions based upon spectrographic determination of the mass-functions. No matter what we do, we encounter an amazing discrepancy. Rather than to conceal it by camouflaging the spectroscopic results or by artificially distributing the discrepancy among several factors, we should probably recognize that at least the fainter components of the Algol type binaries are stars not previously recognized in astronomy' (*op. cit.*, p. 122).

Of subsequent literature on the observational aspects of the problem of subgiants in eclipsing binary systems *cf.* P. P. Parenago and A. G. Massevich (*Trudy Sternberg State Astr. Inst.*, **20,** 81, 1950); O. Struve (*Communications presentées au cinquième Colloque International d'Astrophysique*, Liège 1953, pp. 236–253); O. Struve and N. Gould (*P.A.S.P.*, **66,** 28, 1954); O. J. Eggen, *P.A.S.P.*, **67,** 315, 1955; etc.

A demonstration that the fractional dimensions of the secondary components in semi-detached eclipsing systems coincide with those of their Roche limit within the limits of observational errors was not given until by Z. Kopal in *Ann. d'Astroph.*, **18**, 379, 1955; while the existence of a considerable population of 'undersize' subgiant secondaries, as well as of a whole group of systems resembling R CMa in their absolute physical properties was pointed out by the same writer in *Ann. d'Astroph.*, **19**, 298, 1956.

VII.5: An interpretation of the observed clustering of the secondary components of semi-detached eclipsing systems at their Roche limits as evidence of their instability was proposed by Z. Kopal (*Mem. Soc. Roy. des Sci. de Liège*, **15**, 684, 1954), and independently by J. A. Crawford (*Ap. J.*, **121**, 71, 1955). For other papers dealing with this subject *cf.*, e.g., D. Y. Martynov (*I.A.U. Symposium on Non-Stable Stars*, Cambridge 1957, pp. 138–143); V. A. Krat (*op. cit.*, pp. 159–164); Z. Kopal, *Proc. Indian Nat. Acad. Sci.*, **26**, 462, 1957; etc.

The present version of the subject of this section follows largely the investigations by Z. Kopal published in *Ann. d'Astroph.*, **18**, 379, 1955; **19**, 298, 1956; and *I.A.U. Symp. on Non-Stable Stars*, pp. 123–137.

The relevant aspects of the restricted problem of three bodies are, of course, classical. The five point-solutions were discovered by J. L. Lagrange in his 'Essai sur le problème des trois corps', submitted to the Paris Academy in 1772 (*cf.* his *Collected Works*, **6**, p. 229). Its vis-viva integral defining the surfaces of zero velocity was published by C. G. J. Jacobi in *C.R. Acad. Paris.* **3**, 59, 1836. An adequate summary of the principal features of the whole problem can be found, e.g., in F. R. Moulton's *Introduction to Celestial Mechanics*, New York 1914, Chapter VIII; or, more rigorously, in A. Wintner's *Analytical Foundations of Celestial Mechanics*, Princeton 1941, Chapter VI. For a generalization of the restricted problem of three bodies to eccentric orbits of the two finite masses *cf.* F. R. Moulton, *Periodic Orbits*, Washington 1920, Chapter VII.

The escape of mass from contact components was previously invoked by F. B. Wood (*Ap. J.*, **112**, 196, 1950) to explain the period changes frequently observed in semi-detached binary systems. The relevance of Wood's argument to the problem has already been commented upon on p. 123 of this book; here we wish to add that, from the dynamical point of view, Wood's thesis was based on completely fallacious grounds. Thus, according to Wood, . . . 'Computation of the Jacobian surface shows that the star is nearest instability at those portions of its surface near the ends of its shortest equatorial axes' (*op. cit.*, p. 204), but this is quite wrong; for what matters for an escaping particle is, not the linear distance along which it travels, but the *gradient* of the potential which it must overcome on its way. In actual fact, the star will be nearest instability at that point of its surface where this gradient becomes a minimum—and this happens at the inner Lagrangian point L_1 on the line joining the centres of the two components and, therefore, in a direction perpendicular to Wood's 'shortest equatorial axis'.

The escape of mass from the outer Lagrangian point L_2 at which the largest common equipotential surrounding *both* components opens up behind the less massive star, has previously been discussed by G. P. Kuiper (*Ap. J.*, **93**, 133, 1941) in connection with his work on β Lyrae. As Kuiper's results have been rather loosely interpreted by many subsequent writers, let it be stressed here that the filling up of the outer Roche limit with matter reaching to L_2 does *not* by itself render the common envelope of both stars unstable. Its equilibrium is merely neutral at L_2; and instability—if any—must be produced by extraneous forces whose magnitude and direction then controls the respective ejection trajectories.

The numerical results presented in this section have been taken partly from Kopal's study in *Jodrell Bank Annals*, **1**, 37, 1954, *Ann. d'Astroph.*, **19**, 298, 1956, and partly from an unpublished M.Sc. Thesis (Manchester, 1956) by V. Hewison. For other numerical integrations leading to quasi-circular retrograde orbits around one of the two finite masses *cf.* Chapter XVI of Moulton's *Periodic Orbits*; or an unpublished M.Sc. Thesis (Manchester, 1954) by C. T. Britten.

VII.6: The term 'contact binary' appears to have been introduced in astronomical literature by G. P. Kuiper in his paper 'On the Interpretation of β Lyrae and other Close Binaries' in *Ap. J.*, **93**, 133, 1941. It should, however, be stressed here that Kuiper used this term in a sense essentially different from ours: for whereas we propose to regard as contact binary (or component) a star whose surface coincides with its Roche limit, Kuiper's definition . . . 'does not mean that mere contact exists, but a common envelope as well' (*op. cit.*, p. 137). In other words, Kuiper's 'contact binaries' were systems in which the Roche limit was *exceeded* by a common envelope surrounding *both* stars. Now (with the exception of eclipsing systems containing a Wolf-Rayet star as one component) no known binary appears to possess such a common envelope which is sufficiently dense to produce an appreciable *continuous* (as distinct from discrete line) absorption in optical frequencies; and for this reason such hypothetical models have been left out of discussion in the present book.

The literature dealing with specific problems of the contact dwarf binary systems is so far rather limited. The data presented in Table 7-14 of this section have been taken from the *Catalogue of the Elements of Eclipsing Binary Systems by* Z. Kopal and M. B. Shapley (*Jodrell Bank Ann.*, **1**, 141, 1956). The contact nature of such systems, long conjectured by previous investigators, was demonstrated on the basis of the photometric evidence by Z. Kopal in *Ann. d'Astroph.*, **18**, 379, 1955. Most part of our knowledge of the spectroscopic aspects of these stars is due to O. Struve and his associates, and has been summarized by Struve in his book on *Stellar Evolution*, Princeton 1950, p. 175*ff*. *Cf.* also a note by G. P. Kuiper in *Ap. J.*, **108**, 541, 1948.

The astonishingly high frequency of W UMa-type systems per unit volume of galactic space was demonstrated by H. Shapley in the *Harvard Centennial Symposia*, (*Harv. Obs. Mono. No.* 7, Cambridge, Mass., 1948), pp. 249–260. For recent studies of the kinematic properties of such systems, based on their observed proper motions and radial velocities, *cf.* E. Schatzman and J. L. Rigal, *C. R. Acad. Paris*, **238**, 2392, 1954; J. L. Rigal, *C. R. Acad. Paris.* **240**, 50, 1955; or V. M. Bradley, M.Sc. Thesis (Manchester 1955; unpublished).

VII.7: The outline of the evolution of close binary systems as given in this section is largely new, and represents a development of the ideas initiated by the writer in *Ann. d'Astroph.*, **19**, 298, 1956.

Of the older literature on this subject (since 1930) *cf.* Z. Kopal, *Zs. f. Ap.*, **9**, 239, 1934; *M.N.*, **96**, 854, 1936; **97**, 646, 1937; *Zs. f. Ap.*, **13**, 302, 311, 1937; *M.N.*, **98**, 651, 1938; *The Observatory*, **61**, 201, 1938; *Harv. Centennial Symposia*, III-5 (pp. 261–275), Cambridge, 1948; *Ann. d'Astroph.*, **18**, 379, 1955. G. P. Kuiper, *P.A.S.P.*, **47**, 15, 121, 1935; **67**, 387, 1955. R. A. Lyttleton, *M.N.*, **98**, 646, 1938. O. Struve, *Harv. Centen. Symposia*, III-2 (pp. 211–230), Cambridge 1948; *M.N.*, **109**, 487, 1949; *Stellar Evolution*, Princeton 1950, pp. 232–259. S. S. Huang and O. Struve, *A. J.*, **61**, 300, 1956. K. Walter, *Königsberg Veröff.*, Nr. 2, 1931; *Zs. f. Ap.*, **13**, 294, 309, 1937; **15**, 315, 1938; **19**, 157, 1940; *VJS. d. A.G.* **72**, 334, 1937; **74**, 261, 1939; etc.

In looking over this literature to-day, we cannot fail to notice that some of our current views on the evolution of close binary systems were anticipated long before sufficient grounds could be adduced in their support. We mentioned already that the existence of what we call 'semi-detached' systems in this book was foreshadowed by K. Walter as far back as in 1931. The large distortions of their secondary components misled, however, Walter into belief that these were young systems contracting towards the Main Sequence. In a discussion of this subject between Walter and Kopal twenty years ago (*Zs. f. Ap.*, **13**, 294, 302, 309, 311, 1937), the present writer defended the opposite view—namely, that the Algol systems represented a relatively late stage of stellar evolution—a view which seems to have been vindicated by more recent developments.

Needless to stress, only inadequate support could have been adduced in favour of this view before the decisive role of stellar hydrogen content and the consequences of its depletion have become more fully understood through the work of S. Chandrasekhar and M, Schoenberg, *Ap. J.*, **96**, 161, 1942; A. R. Sandage and M. Schwarzschild, *Ap. J.*, **116**, 463. 1952; M. Schwarzschild, I. Rabinovitz and R. Härm, *Ap. J.*, **118**, 326, 1953; M. Schwarzschild and F. Hoyle, *Ap. J. Suppl.*, No. 13, 1955; and others. It is of interest to note, however, that the question of the relative hydrogen contents of the components of close binary systems was also raised as far back as 20 years ago by K. Walter (*Zs. f. Ap.*, **15**, 315, 1938), who noted that the secondary components of his 'special Algol systems' appeared to contain much less hydrogen than their more massive mates. As such secondaries are highly distorted and their primaries very nearly spherical, Walter claimed that the disparity in the hydrogen contents of the two stars was a consequence of their different degree of distortion. This was, however, like putting a cart before the horse; for we know now that it is the diminishing hydrogen content which causes a star to expand and thus become distorted on account of its increasing dimensions.

But do the contact components in close binary systems actually possess smaller hydrogen contents than their detached mates? Walter's work referred to above is now only of historical interest, as it was carried out under so restricted assumptions (chemically homogeneous stars, built on Eddington's standard model) that its results may well have no bearing whatever on the actual situation; and our continuing ignorance of the true model of actual subgiants still prevents us from ascertaining their actual hydrogen contents on a more realistic basis. If and when future theoretical advances will eventually make this possible, the identity of the component expanding on account of its developing hydrogen shortage will be automatically resolved.

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