DYNAMICAL TIDES IN CLOSE BINARY SYSTEMS

III: *Dissipation of Energy*

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Abstract. The aim of the present paper will be to investigate the circumstances under which an irreversible dissipation of the kinetic energy into heat is generated by the dynamical tides in close binary systems if (a) their orbit is eccentric; (b) the axial rotation of the components is not synchronized with the revolution; or (c) the equatorial planes are inclined to that of the orbit.

In Section 2 the explicit form of the viscous dissipation function will be set up in terms of the velocity-components of spheroidal deformation arising from the tides; in Section 3, the principal partial tides contributing to the dissipation will be detailed; Section 4 will be devoted to a determination of the extent of stellar viscosity – both gas and radiative; while in the concluding Section 5 quantitative estimates will be given of the actual rate at which the kinetic energy of dynamical tides gets dissipated into heat by viscous friction in stellar plasma.

The results disclose that the amount of heat produced per unit time by tidal interaction between components of actual close binaries equals only about 10^{-10}th part of their nuclear energy production; and cannot, therefore, affect the internal structure of evolution of the constituent stars to any appreciable extent. Moreover, it is shown that the kinetic energy of their axial rotation can be influenced by tidal friction only on a nuclear, rather than gravitational (Kelvin) time-scale – as long as plasma or radiative viscosity constitute the sole sources of dissipation. However, the emergence of turbulent viscosity in secondary components of late spectral types, which have evolved away from the Main Sequence, can accelerate the dissipation 10^5–10^6 times, and thus give rise to appreciable changes in the elements of the system (particularly, in the orbital periods) over time intervals of the order of 10^5–10^6 years. Lastly, it is pointed out that, in close binary systems consisting of a pair of white dwarfs, a dissipation of the kinetic energy through viscous tides in degenerate fermion-gas could produce enough heat to account, by itself, for the observed luminosity of such objects.

1. Introduction

In two papers previously published in this journal (KOPAL, 1968a, b; hereafter referred to as Papers I and II, respectively), differential equations together with their associated boundary conditions have been set up, which govern the dynamical tides in close binary systems consisting of components of any structure; and their properties have been related explicitly with the disturbing function which arises from mutual attraction and axial rotation with arbitrary angular velocity. The resulting deformation has been found to consist of a considerable number of discrete partial tides, specified by certain spherical-harmonic symmetry, each of which sweeps around the respective component with a constant amplitude, velocity, and phase. Their motion relative to the centre of gravity of a mass of gas of finite viscosity is, however, bound to bring about also a *dissipation of the kinetic energy into heat* through the medium of viscous friction, the irreversible effects of which can cause *secular changes* to take place in the internal structure of the components and in the elements of their orbit.

The aim of the present paper will be to investigate in a quantitative manner the extent of the phenomena which can be produced in close binary systems by the dissipation of dynamical tides through the medium of viscosity. In Section 2, which follows these introductory remarks, the explicit form of the viscous dissipation function will be set up in terms of the velocity-components of spheroidal deformation investigated in Papers I and II. In Section 3 we shall detail the principal partial tides which can contribute to energy dissipation; Section 4 will be devoted to a determination of the viscosity (both gas and radiative) in stellar models approximating the properties of the components in Main-Sequence binary systems; while in the concluding Section 4 we shall attempt quantitative estimates of the actual rate at which the kinetic energy of dynamical tides can get dissipated into heat through viscous friction of stellar plasma. The results obtained have already been summarized in the Abstract; so that, in what follows, our task will be to substantiate them in detail.

2. Dissipation Function

As is well known (cf. e.g., Lamb, 1932) the viscous dissipation function Φ, which measures the degradation of the kinetic energy of motion into heat through the action of viscosity, can be generally expressed as a quadratic function of the components $\sigma_{i,j}$ of the respective stress-tensor, which in the spherical polar coordinates r, θ, ϕ assumes the form

$$\Phi = \tfrac{1}{2}\mu \{\sigma_{rr}^2 + \sigma_{\theta\theta}^2 + \sigma_{\phi\phi}^2 + 2(\sigma_{r\theta}^2 + \sigma_{\theta\phi}^2 + \sigma_{\phi r}^2)\} - \tfrac{2}{3}\mu \Delta^2 \tag{1}$$

in erg/cm^3 sec, where μ denotes (as in Papers I and II) the viscosity of the fluid medium; Δ, the divergence of the velocity vector; and

$$\sigma_{rr} = 2\frac{\partial U}{\delta r}, \tag{2}$$

$$\sigma_{\theta\theta} = 2\left(\frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r}\right), \tag{3}$$

$$\sigma_{\phi\phi} = 2\left(\frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi} + \frac{U}{r} + \frac{V\cot\theta}{r}\right), \tag{4}$$

$$\sigma_{r\theta} = \frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r}, \tag{5}$$

$$\sigma_{r\phi} = \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi} + \frac{\partial W}{\partial r} - \frac{W}{r}, \tag{6}$$

$$\sigma_{\theta\phi} = \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} + \frac{1}{r}\frac{\partial W}{\partial \theta} - \frac{W\cot\theta}{r}; \tag{7}$$

U, V, W being the polar velocity components in the direction of increasing r, θ, and ϕ, respectively; which for spheroidal oscillations (cf. Section 2 of Paper I) can be expressed as

$$U(r, \theta, \phi; t) = u(r, t)\, Y_j^i(\theta, \phi) \tag{8}$$

$$V(r, \theta, \phi; t) = v(r, t)\, \frac{\partial Y_j^i}{\partial \theta}, \tag{9}$$

$$W(r, \theta, \phi; t) = \frac{v(r, t)}{\sin\theta}\, \frac{\partial Y_j^i}{\partial \phi}, \tag{10}$$

where u, v are functions of r and t only; and the Y_j^i's are surface harmonics which satisfy the differential equation

$$\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \phi^2} + \frac{\partial^2 Y}{\partial \theta^2} + \cot\theta\, \frac{\partial Y}{\partial \theta} + j(j+1)\, Y = 0. \tag{11}$$

A determination of the radial functions $u(r, t)$, $v(r, t)$ for dynamical tides has already been outlined in Papers I and II; so that hereafter these can be regarded as known.

If equations (8)–(10) are inserted in (2)–(7) and use is made of (11), the stress components $\sigma_{i,j}$ assume the particular forms

$$\sigma_{rr} = 2\frac{\partial u}{\partial r}\, Y, \tag{12}$$

$$\sigma_{\theta\theta} = \frac{2}{r}\left(v\, \frac{\partial^2 Y}{\partial \theta^2} + uY\right), \tag{13}$$

$$\sigma_{\phi\phi} = \frac{2}{r}\left(uY - j(j+1)vY - v\, \frac{\partial^2 Y}{\partial \theta^2}\right), \tag{14}$$

$$\sigma_{r\theta} = \left(\frac{\partial v}{\partial r} + \frac{u - v}{r}\right)\frac{\partial Y}{\partial \theta}, \tag{15}$$

$$\sigma_{r\phi} = \left(\frac{\partial v}{\partial r} + \frac{u - v}{r}\right)\frac{1}{\sin\theta}\frac{\partial Y}{\partial \phi}, \tag{16}$$

$$\sigma_{\theta\phi} = \frac{2v}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{\partial Y}{\partial \theta} - Y\cot\theta\right), \tag{17}$$

and

$$\Delta = \left\{\frac{\partial u}{\partial r} + \frac{2u}{r} - j(j+1)\frac{v}{r}\right\} Y \equiv yY. \tag{18}$$

The total rate of energy dissipation E inside the oscillating configurations is then given by the volume integral

$$\frac{\mathrm{d}E}{\mathrm{d}t} \equiv \dot{E} = \int \Phi \mathrm{d}V = \int_0^{a_*}\int_0^{\pi}\int_0^{2\pi} \Phi r^2\, \mathrm{d}r \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\,, \tag{19}$$

where a_* denotes the mean radius of the oscillating configuration. On insertion from

(12)–(18) in (1) and (19) the integral on the right-hand side of (19) can be evaluated term-by-term. This task is one of some complexity, as only one of the partial integrals involving squares of the surface harmonics Y or of their derivatives with respect to the angular variables is classical. After some rather arduous piece of analysis we have found it, however, possible to decompose the foregoing expression (19) for E into a summation

$$\dot{E} = \sum_{i,j} \dot{E}_{i,j}, \tag{20}$$

where, for $i=0$,

$$\begin{aligned}\dot{E}_{0,j} = \frac{4\pi}{2j+1} \int_0^{a_*} \mu \left\{ 2\left(\frac{\partial u}{\partial r}\right)^2 + \left[\frac{2u}{r} - j(j+1)\frac{v}{r}\right]^2 \right. & \\ + j(j+1)\left[\frac{\partial v}{\partial r} + \frac{u-v}{r}\right]^2 & \\ \left. + (j-1)j(j+1)(j+2)\frac{v^2}{r^2} - \frac{2}{3}y^2 \right\} r^2\,\mathrm{d}r\,; & \end{aligned} \tag{21}$$

while, for $i>0$,

$$\dot{E}_{i,j} = \frac{(j+i)!}{(j-i)!}\frac{\dot{E}_{0,j}}{2}. \tag{22}$$

It should be kept in mind that the functions $u(r,t)$ as well as $v(r,t)$ in the integrand on the right-hand side of (21) are different for each value of j; moreover, that the expressions $\dot{E}_{i,j}(t)$ represent the energy dissipation caused by viscous friction of a partial tide characterized by the i, j-th spherical-harmonic symmetry per unit time t. The total amount of energy dissipated by each tide per cycle should then be given by

$$\bar{E}_{i,j} = \int_0^P \dot{E}_{i,j}(t)\,\mathrm{d}t, \tag{23}$$

where P denotes the period of the respective binary orbit.

3. Principal Tides

In order to estimate the importance of the dissipation of the kinetic energy by tidal friction of viscous gas in actual binary systems, let us set out to evaluate the volume integral (21) of the dissipation function Φ in so far as it can be done in general terms. The exact evaluation of this integral requires, to be sure, a prior determination of the functions $u(r,t)$ and $v(r,t)$ for the adopted models of the stars by numerical integration of the equations set up in Papers I and II. The only closed solutions which we were able to construct pertain to a homogeneous incompressible model of Paper I, Section 7; and these lead us to expect that the leading terms of both u and v will vary with the radius as r^{j+1}, with appropriate coefficients.

In what follows, two limiting cases will be considered: one corresponding to

$$\text{(A)} \quad \left\{ \begin{aligned} u &= j(j+1)\, c_{i,j} r^{j+1}, \\ v &= (j+3)\, c_{i,j} r^{j+1}, \end{aligned} \right.$$

where $c_{i,j}$ are appropriate constants (or functions of the time) which specify the nature of the oscillation; the other to

$$\text{(B)} \quad \left\{ \begin{aligned} u &= (j+2)\, c_{i,j} r^{j+1}, \\ v &= c_{i,j} r^{j+1}. \end{aligned} \right.$$

The reader may note that these forms constitute the limiting cases of Equations (137) or (140) of Paper I if, in these equations, the frequency $\tilde{\nu}$ is allowed to tend to (A) zero, or (B) infinity; moreover, an insertion of the above-adopted forms for u and v in Equations (29) or (30) of Paper I reveals that, in these particular cases,

$$\text{(A)}: \; y = 0, \tag{24}$$

or

$$\text{(B)}: \; z = 0, \tag{25}$$

respectively. The former condition implies *poloidal* oscillations, and will always be fulfilled in the case of incompressibility; while the latter implies u, v to be *gradients* of the same function.

Inserting successively (A) and (B) on the right-hand side of our present Equation (21) and performing the requisite integrations we find that, in the case (A),

$$\dot{E}_{0,j} = \frac{8\pi j(j+1)(2j+3)}{2j+1} \left[j(j+2)(2j+1) - 3 \right] c_{0,j}^2 \int_0^{a_*} \mu r^{2j+2}\, \mathrm{d}r\,; \tag{26}$$

while in the (B) case,

$$\dot{E}_{0,j} = \frac{8\pi j(2j+3)}{3(2j+1)} \left[6j^2 + 9j + 1 \right] c_{0,j}^2 \int_0^{a_*} \mu r^{2j+2}\, \mathrm{d}r\,; \tag{27}$$

and the corresponding $E_{i,j}$'s can then be obtained from (22).

In order to estimate the rate of the anticipated dissipation of energy on this basis, let us confine our attention to the effects produced by the principal second-harmonic tides (corresponding to $j=2$). From the discussion of Section 4 of our preceding Paper II we deduce that the most important zonal-harmonic tide arises from the eccentricity of the relative orbit, with the height of the tide varying ('breathing') in inverse proportion of the cube of the instantaneous radius-vector. If A denotes the semi-major axis of the relative orbit; e, its eccentricity; m'/m, the ratio of the masses of the disturbing (m') and the disturbed (m) component; n, their mean daily motion; and a_*, as before, the mean radius of the distorted star, the most important partial tide corresponding to $i=0, j=2$ will be factored by the coefficient

$$c_{0,2} = \frac{1}{2}\left(\frac{m'}{m}\right) \frac{n a_*}{A^3}\, e \sin nt\,, \tag{28}$$

where t denotes the time.

Partial tides of the type $i=1, j=2$ arise in connection with a finite inclination of the plane of the equator of a rotating distorted star to that of the orbit. If ω denotes the angular velocity of axial rotation of the component of mass m, and I, the angle between the equator and the orbit, this latter inclination will give rise to a tesseral-harmonic tide led by the coefficient

$$c_{1,2} = \frac{1}{2}\left(\frac{m'}{m}\right)\frac{\sin I}{A^3}\left\{[\omega a_*]\sin\omega t + [(2n \mp \omega)\,a_*]\sin(2u \mp \omega t)\right\}, \tag{29}$$

where u stands for the true longitude of the component of mass m' in its relative orbit, measured from the line of the nodes; and the $\mp$ sign of ω corresponds to the case of direct $(-)$ or retrograde $(+)$ rotation. The first time-dependent term in curly brackets on the right-hand side of Equation (29) represents the periodic motion of the crest of the respective tide in (astrocentric) latitude; the second, in longitude. Should the axial rotation be synchronized with the revolution (i.e. $\omega=n$) and $u=nt$, the right-hand side of (29) becomes independent of the longitude and both terms in curly brackets are identical.

The most important tide associated with the sectorial harmonic $i=j=2$ is characterized by the coefficient

$$c_{2,2} = \frac{1}{6}\left(\frac{m'}{m}\right)\frac{(n \mp \omega)\,a_*}{A^3}\cos 2(n \mp \omega)\,t\,, \tag{30}$$

and represents a wave sweeping around the equator of the star of mass m with an angular velocity $n \mp \omega$, equal to a difference between those of rotation and revolution.

Partial tides of the type $i=0, j=2$ vanish whenever $e=0$ and $I=0$; but persist if either one of these conditions fails to be met. Tides characterized by $i=1, j=2$, arise only if $I>0$ regardless of the eccentricity; while the sectorial tides $(i=j=2)$ vanish in the case of synchronism between rotation and revolution only if $I=0$; otherwise other tides of this type arise with non-vanishing coefficients. A full account of such tides can be found in Section 4 of Paper II. In the most general case (i.e. $e>0$, $I>0$, $\omega \neq n$) altogether 14 partial tides of different frequencies are associated with the zonal harmonics $(i=0, j=2)$ of the disturbing function; 19 with the tesseral harmonics $i=1$, $j=2$; and 21 with the sectorial harmonics $i=j=2$. The damping of each in a viscous medium will contribute to the total dissipation of kinetic energy into heat; and their individual contributions would have to be summed up in exact work to obtain the total effect. This we do not propose to carry out in the present section, since we are concerned merely with estimates of the expected magnitude of the total effect; and to this end the results obtained so far should be sufficient.

4. Viscous Friction

In order to complete our task, it remains for us to evaluate the integrals on the right-hand sides of Equations (26) or (27); and to this end we must specify the viscosity of stellar matter which gives rise to friction. Apart from turbulent viscosity (which we shall postpone for subsequent discussion) its principal constituents in stellar interiors are the gas (i.e. plasma) viscosity μ_G and the radiative viscosity μ_R, the sum of which will hereafter be identified with our μ. As is well known (cf. e.g., CHAPMAN, 1954; OSTER, 1957), the coefficient of viscosity μ_G of stellar plasma (consisting essentially of hydrogen) is sensibly equal to

$$\mu_G = 0.96\,\mu_i\,, \qquad \mu_i = \frac{5\sqrt{m_H}(kT)^{5/2}}{4\sqrt{\pi}\varepsilon^4 A_2(\xi)}\,, \tag{31}$$

where T denotes the local temperature; k, the Boltzmann constant; m_H, the mass of a proton; and where

$$A_2(\xi) = \log(1 + \xi^2) - \frac{\xi^2}{1 + \xi^2}\,, \tag{32}$$

with

$$\xi = \frac{4kT}{\varepsilon^2}\left(\frac{m_H}{\rho}\right)^{1/3}; \tag{33}$$

ρ denoting the density and ε, the electronic charge.

If in the preceding formulae we insert $k = 1.379 \times 10^{-16}$ erg/deg, $m_H = 1.672 \times 10^{-24}$ g, and $\varepsilon = 4.802 \times 10^{-10}$ e.s.u., we find that

$$\mu_G = \frac{3.68 \times 10^{-15} T^{5/2}}{\log(1 + \xi^2) - \dfrac{\xi^2}{1 + \xi^2}} \quad \frac{\text{g}}{\text{cm sec}}\,, \tag{34}$$

where

$$\xi = 2.84 \times 10^{-5} T\rho^{-1/3}\,. \tag{35}$$

On the other hand, the radiative viscosity μ_R is known (cf. e.g., HAZLEHURST and SARGENT, 1959) to be given by

$$\mu_R = \frac{4aT^4}{15\,c\kappa\rho} \quad \frac{\text{g}}{\text{cm sec}}\,, \tag{36}$$

where $a = 7.55 \times 10^{-15}$ erg/cm^3 deg^4 denotes the Stefan constant; $c = 2.998 \times 10^{10}$ cm/sec, the velocity of light; and κ, the absorption coefficient of stellar matter per unit volume.

If we combine Equations (34) and (36) for the plasma and radiative viscosity, the integrals on the right-hand sides of Equations (19) and (21) for $j = 2$ can be expressed as

$$\int_0^{a_*} (\mu_G + \mu_R)\, r^6\, \mathrm{d}r = a_*^7 \left\{(\mu_G)_c I_G + (\mu_R)_c I_R\right\}, \tag{37}$$

where $(\mu_G)_c$, $(\mu_R)_c$ stand for the central values of gas or radiative viscosity, and

$$I_G = \int_0^1 \left(\frac{T}{T_c}\right)^{5/2} \frac{A_2(\xi_c)}{A_2(\xi)} x^6 \, dx, \tag{38}$$

$$I_R = \int_0^1 \frac{(\kappa\rho)_c}{\kappa\rho} \left(\frac{T}{T_c}\right)^4 x^6 \, dx, \tag{39}$$

are non-dimensional quantities, the values of which can be ascertained by numerical integration for any desired model of a star.

In what follows, we wish to evaluate these parameters for six models of typical stars, published by SCHWARZSCHILD (1958) on pp. 254–259 of his book *Structure and Evolution of the Stars.* All these models pertain to Main-Sequence stars of different masses and evolutionary stages; and their principal characteristics have been compiled in Table I, the columns of which indicate, successively, the star's mass m (in units of

TABLE I

Fundamental Properties of Stellar Models

Model No.	m (in ☉)	a^* (in ☉)	L (in ☉)	Spectrum	$\log T_c$ (in deg)	$\log \rho_c$ (in g/cm^3)	ρ_c/ρ_m	h
I	10	3.65	3000	O8	7.442	0.892	26.9	0.29
II	2.5	1.59	21.2	A2	7.297	1.684	55.0	1.24
III	1.0	1.021	0.578	K1	6.906	1.813	51.3	0.25
IV	0.603	0.644	0.565	F8	6.906	1.813	20.5	0.32
V	10	6.09	5220	B0	7.545	1.075	191	0.18
VI	1	1	1	G2	7.165	2.128	95.2	0.22

☉ $= 1.985 \times 10^{33}$ g); radius a_* (expressed in solar units ☉ $= 0.695 \times 10^{11}$ cm); the luminosity L (in terms of ☉ $= 3.78 \times 10^{33}$ erg/sec); the corresponding spectral class; decimal logarithm of the central temperature T_c (in degrees K); logarithm of the central density ρ_c (in g/cm^3); the ratio ρ_c/ρ_m of the central to the mean density of the respective configuration; and h, its fractional radius of gyration.

Models I–IV represent initial Main-Sequence stars of different masses and luminosities (Model III approximating the properties of the initial sun; and Model IV, the initial state of the components of the eclipsing variable Castor C); while Models V–VI correspond to evolved Main-Sequence stars (Model VI to that of the present sun).

The viscous properties of these models are listed in the accompanying Table II, the contents of the columns of which are self-explanatory. The central values of μ_G and μ_R (in g/cm.sec) have been computed from Equations (34) and (36) with the aid of the data compiled in Table I; and those for I_G and I_R obtained from (38) and (39)

TABLE II

Viscous Properties of Stellar Models

Model No.	$\log(\mu_G)_c$ (in g/cm.sec)	$\log(\mu_R)_c$ (in g/cm.sec)	$10^3 I_G$	$10^3 I_R$	$(\mu_G)_c I_G + (\mu_R)_c I_R$ (in g/cm.sec)	
I	3.47	3.97	1.70	3.87	5.0	36.3
II	3.20	2.35	0.91	2.58	1.45	0.58
III	2.77	1.16	1.41	2.31	0.83	0.03
IV	2.35	0.22	4.86	3.26	1.10	0.01
V	3.72	4.31	0.23	2.03	1.2	41.7
VII	2.95	1.33	1.14	2.78	1.02	0.06

by numerical integration; the last column then gives the effective viscosity (in g/cm. sec) of the respective configuration as a whole, as it occurs in curly brackets on the right-hand side of Equation (37).

An inspection of the individual columns of the foregoing tables discloses several noteworthy facts. First, it reveals that the central viscosity of all stellar models under consideration proves to be remarkably high – of the order of 10^2–10^4 g/cm.sec – thus bearing out an earlier surmise by EDDINGTON (1926) that "...For hydrodynamical purposes, one must think of the star as a thick oily liquid. This applies even to the regions of low density. ... I suppose that even the photosphere will be rather sticky" (*op. cit.*, p. 281).

Secondly, our present computations disclose that, whereas in stars of masses comparable with (or smaller than) that of the sun radiative viscosity remains unimportant in comparison with plasma viscosity, in massive stars ($m > 5\odot$) radiative viscosity becomes dominant not only near the centre, but throughout the interior (as is borne out by the fact that $I_R \gg I_G$), because the ratio $T^4/\kappa\rho$ diminishes outwards less rapidly than $T^{5/2}$.

5. Application to Binary Systems

With the aid of the numerical results listed in Tables I and II we are now in a position to evaluate the total rate of the energy dissipation, through viscous tides, in close binary systems consisting of the components for which our Models I–VI of the preceding section can be regarded as representative.

In order to do so, let us return to Equations (19) or (21) of Section 2, which for the dominant second-harmonic tides (i.e., with $j=2$) assume the forms

$$\left.\begin{aligned} \dot{E}_{0,2} &= \frac{12432\pi}{5} c_{0,2}^2 \int_0^{a*} \mu r^6 \, \mathrm{d}r \ldots \text{(A)} \\ &= \frac{4816\pi}{15} c_{0,2}^2 \int_0^{a*} \mu r^6 \, \mathrm{d}r \ldots \text{(B)} \end{aligned}\right\} \qquad (40)$$

and, in either case,

$$\dot{E}_{i,2} = \frac{(2+i)!}{(2-i)!} \frac{\dot{E}^{*}_{0,2}}{2} \tag{41}$$

by (22).

Now, for $i=0$ (i.e. the zonal-harmonic tides), a combination of Equations (28) and (37) discloses that

$$c_{0,2}^2 \int_0^{a_*} \mu r^6 \,\mathrm{d}r = a_* \left\{ \frac{1}{2} \frac{m'}{m} \left(\frac{a_*}{A} \right)^3 n a_* e \sin nt \right\}^2 \{(\mu_G)_c I_G + (\mu_R)_c I_R\}, \tag{42}$$

where the second expression in curly brackets on the right-hand side has been tabulated for our models in the ultimate column of Table II. Inserting (42) in (40) and adopting, typically,

$$\frac{m'}{m} \sim 1 \quad \text{and} \quad \frac{a_*}{A} \sim 0.3 \tag{43}$$

we find that, in this case,

$$\dot{E} \sim a_* (n a_* e)^2 \{(\mu_G)_c I_G + (\mu_R)_c I_R\}, \tag{44}$$

where the constant of proportionality – 1.424 in Case (A) and 0.092 in Case (B) – will generally be of the order of 0.1. If, lastly, we adopt for a_* and na_* the values of 10^{11} cm (i.e., 1.4$\odot$) and 10^6 cm/sec, respectively (which should well represent typical cases), we find that, ultimately,

$$\dot{E} = (10^{22} - 10^{23}) e^2 \sin^2 nt \text{ erg/sec}, \tag{45}$$

corresponding to a secular energy loss of

$$\bar{E} = (10^{27} - 10^{28}) e^2 \text{ ergs per cycle}. \tag{46}$$

In the case of partial tides corresponding to $i=1$, the result proves to be identical with the preceding one, provided that the angular velocity ω of axial rotation is made to replace the mean daily motion n of the binary pair, and that $\sin I$ replaces e as the factor whose non-zero value is responsible for the tide. As regards the sectorial tide corresponding to $i=2$, the same continues to be true provided that n is replaced by $n \mp \omega$ and the result multiplied by $\frac{1}{3}(n \mp \omega)$; but it contains no other small factor. A combination $n \mp \omega$ (the sign $\mp$ corresponding to direct and retrograde rotation, respectively) need not, moreover, be small; but can be of the same order of magnitude as n or even larger if $\omega \gg n$. In such a case, a contribution to $\dot{E}$ arising from sectorial second-harmonic tide due to non-synchronism between rotation and revolution can, in effect, be 10–100 times as large as that arising from the orbital eccentricity or equatorial inclination, and attain the order of magnitude of 10^{23}–10^{24} ergs/sec or 10^{28}–10^{29} ergs per cycle.

Let us compare these rates with the total amount of kinetic energy possessed by typical components of close binary systems, and with their energy loss due to radiation. As the latter are mostly of the order of 10^{33} ergs/sec, it is immediately obvious that a production of heat by viscous friction at a rate of 10^{23} or 10^{24} ergs/sec is utterly negligible in comparison with the rate of nuclear energy production in stellar interiors – or even with the rate of gravitational energy release during the stages of contraction. *The heat produced by tidal interaction between components in close binary systems cannot*, therefore, *affect the internal structure or evolution of such stars to any appreciable extent* – at least as far as the effects of plasma or radiative viscosity are concerned – and the corresponding terms can be safely ignored in the equations for the energy-balance. This does not necessarily mean yet that the evolution of the individual components of close binary systems will be unaffected by their symbiosis, and proceed as if they were single; for the effects of their mutual attraction on their mechanical equilibrium are very much larger; but the interaction effects on the energy balance are clearly negligible.

Let us compare next the kinetic energy – both orbital and rotational – of the components in typical binary systems with the rate of its loss by tidal interaction. If possible orbital eccentricity is ignored, the kinetic energy $\mathfrak{T}_0$ of orbital motion of the system can be expressed by

$$\mathfrak{T}_0 = \frac{Gmm'}{2A}, \tag{47}$$

where $G = 6.68 \times 10^{-8}$ cm^3/g sec^2 denotes the constant of gravitation; m, m', the masses of the constituent components; and A, their mutual separation. For typical values of $m \sim m' \sim 10^{33}$ g and $A \sim 10^{12}$ cm, the orbital kinetic energy $\mathfrak{T}_0$ of the corresponding pair proves to be a quantity of the order of 10^{46} ergs. On the other hand, the kinetic energy $\mathfrak{T}_\omega$ of axial rotation of the component of mass m, radius a_*, and angular velocity ω, is known to be given by

$$\mathfrak{T}_\omega = \tfrac{4}{3}\pi\omega^2 \int_0^{a_*} \rho r^4 \,\mathrm{d}r = \tfrac{1}{2} m (a_* h\omega)^2, \tag{48}$$

where h, the fractional radius of gyration of the respective star, has been tabulated for our six models in the ultimate column of Table I; and its average value proves to be close to 0.2. Adopting again $m \sim 10^{33}$ g, $a_* \sim 10^{11}$ cm, and $\omega \sim 10^{-5}$ sec^{-1}, we find that $\mathfrak{T}_\omega \sim 10^{44}$ ergs – i.e., a quantity some 100 times smaller than the orbital kinetic energy of the pair.

If, now – in accordance with the results stated earlier in this section – each component is to lose 10^{22}–10^{23} ergs of kinetic energy by dissipative action of viscous tides (mainly those corresponding to the case of $i = j = 2$), it follows that *the kinetic energy of axial rotation would be affected by them appreciably* – say, within 10% of the actual value – *only after time-intervals of the order of* 10^{20} sec *or* 10^{13} years – *i.e., on a slow nuclear, rather than gravitational* (Kelvin) *time-scale*; while during time-span of the order of 10^6 or 10^7 years the axial rotation would be thoroughly uninfluenced by tides.

This appears incontrovertibly so as long as the dissipative action is due solely to plasma or radiative viscosity. However, should *turbulent viscosity* appear, it may be a very different story. Proper quantitative treatment of turbulent viscosity in stellar interiors still encounters difficulties which force us to postpone its more specific discussion for the future. However, it is known that turbulent zones of stellar interiors are characterized by macroscopic viscosity of the order of $\mathfrak{R}\mu_G$, where $\mathfrak{R}$ – the Reynolds number – must be a quantity of the order of 10^5 or 10^6 for turbulence to occur at all. In other words, in turbulent regimes the macroscopic viscosity becomes at least 10^5–10^6 times as large as the corresponding plasma viscosity μ_G; and so will be the corresponding rate of energy dissipation.

On the Main Sequence, turbulent zones are known to develop only in central parts of massive stars, where tidal effects are minor. However, as soon as the star begins to evolve away from the Main Sequence, turbulent zones begin to develop on the outside – where tides are relatively of greatest importance – and extend rapidly into the interior for stars which frequently occur as *secondary* components in close binary systems. Such stars exhibit spectroscopic characteristics of subgiants; and it is in systems possessing such components where dissipative effects of viscosity should be primarily anticipated; for *turbulent viscosity of the order of* 10^6 μ_G *could make the respective component lose (say)* 10% *of its rotational kinetic energy in* 10^6 years *or even shorter intervals of time – and thus produce dissipative effects on the contractional rather than nuclear time-scale.*

As is well known, the most sensitive detector of such effects are the period changes which should occur in binary systems as a result of the degradation of kinetic energy into heat. It may indeed be recalled (for the underlying basic data cf., e.g., KOPAL and SHAPLEY, 1956) that orbital periods of eclipsing binaries with both components on the Main Sequence are generally stable; while systems with one (or both) components evolved away from the Main Sequence exhibit as a rule complicated period fluctuation. This fact has in recent years been mostly attributed to mass loss, or exchange of mass between evolved components, stimulated by the coincidence of the surfaces with their 'Roche limits'. However, this latter characteristic alone would not explain why similar fluctuations in period are observed also in systems containing 'undersize subgiants'. In the light of the results of the present investigation, dissipative phenomena of viscous tides should be likewise considered as possible causes of the observed period fluctuations (the magnitude of which could, in fact, disclose to us the extent – and the Reynolds numbers – of the turbulent zones in the respective components); but a closer analysis must be postponed for future investigations.

One last remark may be added in this connection, and this concerns the role of viscous tides in close binaries which may consist of a pair of white dwarfs. At present no such system is known with certainty to exist. Since, however, their discovery may be only a matter of time, it may be of interest to consider the role which the tides raised in such systems would play in the conversion of mechanical energy into heat.

In doing so we should first recall that the viscosity of a degenerate fermion gas (cf., e.g., NISHIMURA and MORI, 1961) is by several orders of magnitude larger than that of a non-degenerate plasma; and should exceed that expected in turbulent zones of subgiant stars. Secondly, the typical values of a_* or na_* to be expected in close binaries consisting of white dwarfs (cf. KOPAL, 1957) should be of the order 10^9 cm and 10^8 cm/sec, respectively – which together with the fermion-gas viscosity should lead to an energy dissipation at a rate of 10^{30} ergs/sec. Since, moreover, the total luminosity output of such objects is of the order of 10^{30}–10^{31} ergs/sec, it follows that *a dissipation of the kinetic energy into heat through viscous tides in fermion-gas systems should, by itself, be able to provide a major part of the source of the luminosity of such objects.*

Secondly, since the kinetic energy of a rotating white dwarf (of the radius of gyration $h=0.453$, corresponding to that of a polytrope $n=1.5$ (MOTZ, 1952)) is about 10^{49} ergs, it follows that a depletion of so large a store even at a rate of 10^{30} ergs/sec could maintain a steady source of heat arising from the dissipation of viscous tides in white-dwarf systems for astronomically long intervals of time.

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