Chapter 16: Frequency Response

Part I

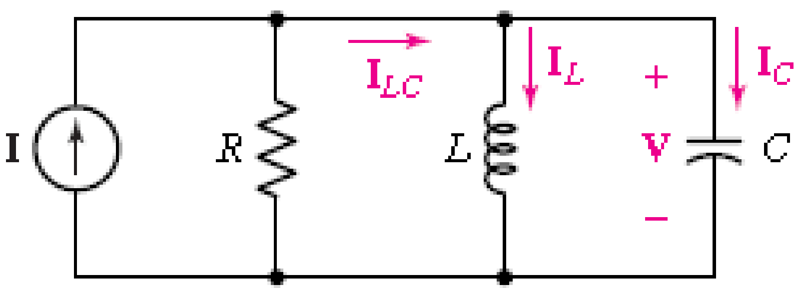
Outline

- Parallel Resonance
- Bandwidth and High-Q circuits
- Series Resonance
- Other Resonant Forms
- Scaling
- Bode Diagrams
- Filters

16.1 Parallel Resonance

Resonance

- In a two-terminal electrical network containing at least one inductor and one capacitor, we define resonance as the condition which exists when the input impedance of the network is **purely resistive**.
- A network is in resonance (or resonant) when the voltage and current at the network input terminals are in phase.
- In Fig. 16.1, the steady-state admittance offered to the ideal current source is



$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \qquad [1]$$

Resonance occurs when the voltage and current at the input terminals are in phase.

$$\omega C - \frac{1}{\omega L} = 0$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \text{rad/s}$$

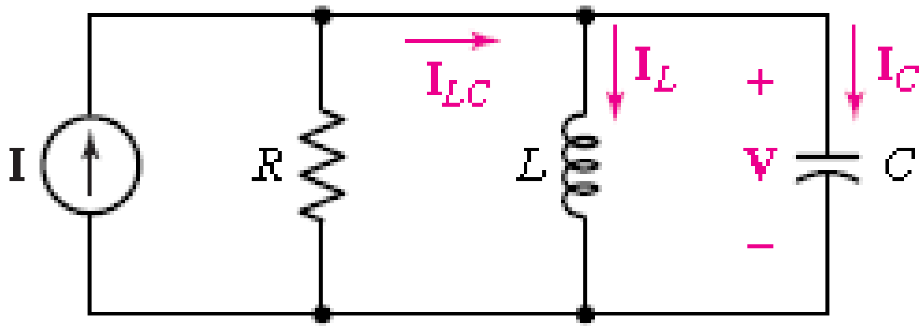
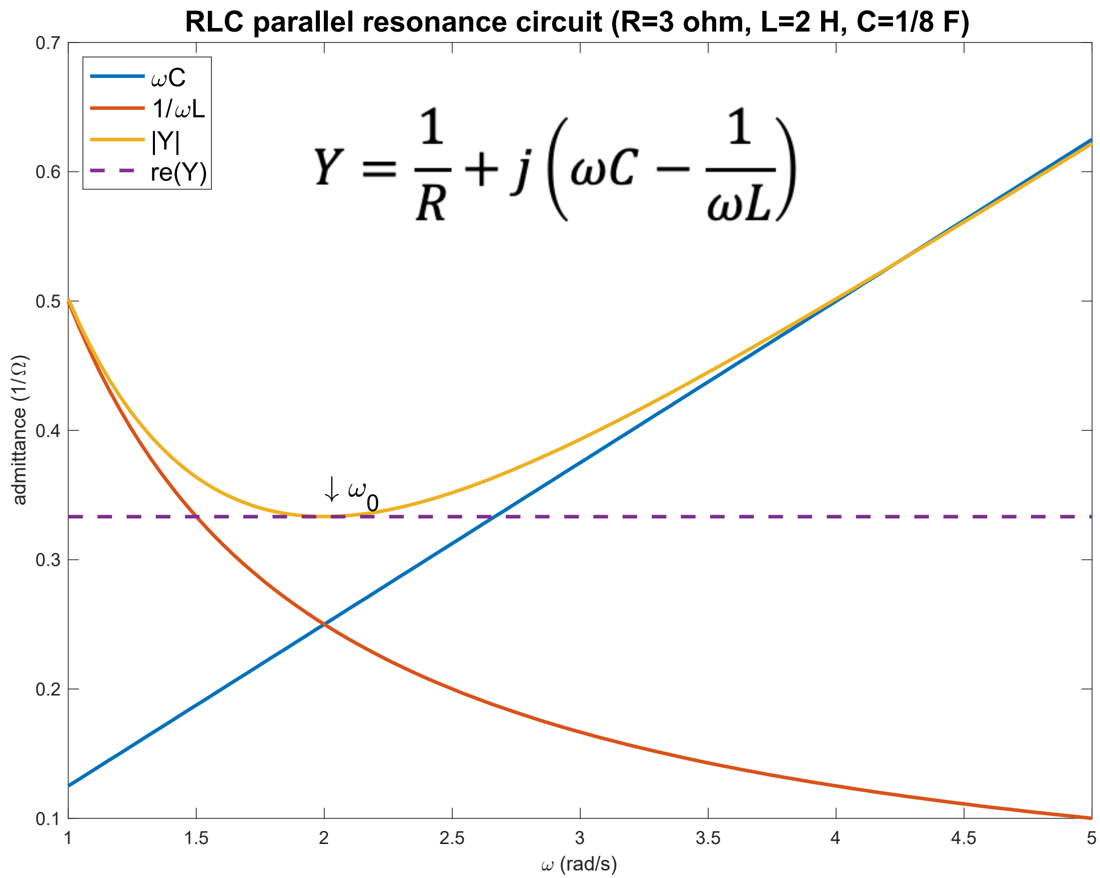


FIGURE 16.1 The parallel combination of a resistor, an inductor, and a capacitor, often referred to as a *parallel resonant circuit.*

Simulation


RLC parallel resonance circuit (R=3 ohm, L=2 H, C=1/8 F)
ωC
1/ωL
|Y|
re(Y)
$$Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$
↓ ω_0
admittance (1/Ω)
ω (rad/s)
0.1
0.2
0.3
0.4
0.5
0.6
0.7
1
1.5
2
2.5
3
3.5
4
4.5
5

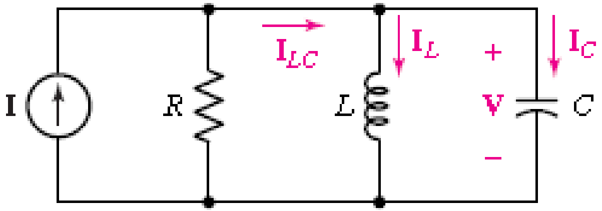
Hence, the resonant frequency ω_0 is

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \text{rad/s}$$

or

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \qquad \text{Hz}$$

The pole-zero configuration of the admittance function can also be used to considerable advantage here. Given $\mathbf{Y(s)}$,



■ **FIGURE 16.1** The parallel combination of a resistor, an inductor, and a capacitor, often referred to as a *parallel resonant circuit.*

$$\mathbf{Y}(\mathbf{s}) = \frac{1}{R} + \frac{1}{\mathbf{s}L} + \mathbf{s}C$$

$$\mathbf{Y}(\mathbf{s}) = C\frac{\mathbf{s}^2 + \mathbf{s}/RC + 1/LC}{\mathbf{s}} \qquad [4]$$

$$\mathbf{Y}(\mathbf{s}) = C\frac{\mathbf{s}^2 + \mathbf{s}/RC + 1/LC}{\mathbf{s}} = C\frac{(\mathbf{s}+\alpha - j\omega_d)(\mathbf{s}+\alpha + j\omega_d)}{\mathbf{s}}$$

α is the exponential damping coefficient,

$$\alpha = \frac{1}{2RC}$$

ω_d is the damped natural resonant frequency

$$\omega_d = \frac{1}{2}\sqrt{4\left(\frac{1}{\sqrt{LC}}\right)^2 - \left(\frac{1}{RC}\right)^2}$$

$$= \sqrt{\omega_0^2 - \alpha^2}$$

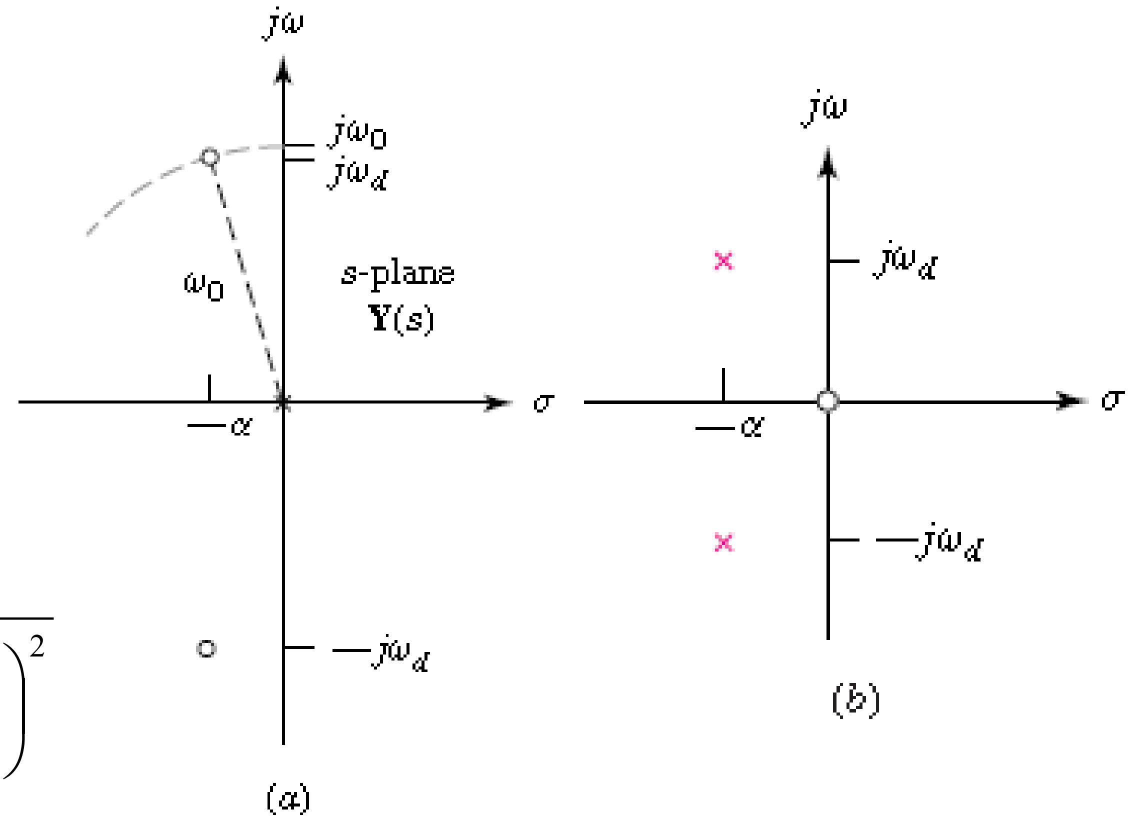


FIGURE 16.2 (a) The pole-zero constellation of the input admittance of a parallel resonant circuit is shown on the s-plane; $\omega_0^2 = \alpha^2 + \omega_d^2$. (b) The pole-zero constellation of the input impedance.

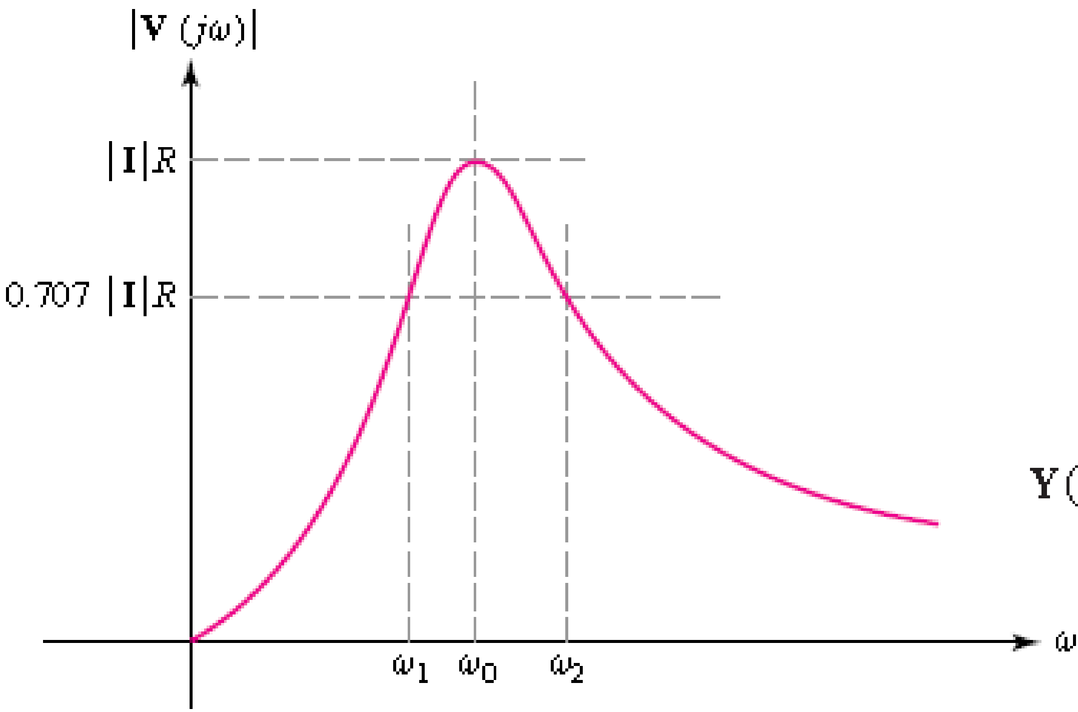
Resonance and the Voltage Response

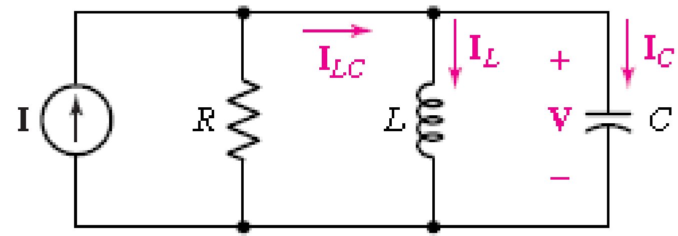
- Next let us examine the magnitude of the response, the voltage $\mathbf{V(s)}$ indicated in Fig. 16.3, as the frequency ω of the forcing function is varied.
- If we assume a constant-amplitude sinusoidal current source, the voltage response is proportional to the input impedance.

$$\mathbf{Z}(\mathbf{s}) = \frac{\mathbf{s}/C}{(\mathbf{s}+\alpha - j\omega_d)(\mathbf{s}+\alpha + j\omega_d)}$$

- The frequency response is sketched in Fig. 16.3 by using input admittance.

$$\mathbf{Y}(\mathbf{s}) = C\frac{(\mathbf{s}+\alpha - j\omega_d)(\mathbf{s}+\alpha + j\omega_d)}{\mathbf{s}}$$





$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

$$\mathbf{Y}(\mathbf{s}) = C\frac{(\mathbf{s} + \alpha - j\,\omega_d)(\mathbf{s} + \alpha + j\,\omega_d)}{\mathbf{s}}$$

$$\mathbf{Z}(\mathbf{s}) = \frac{\mathbf{s}/C}{(\mathbf{s} + \alpha - j\omega_d)(\mathbf{s} + \alpha + j\omega_d)}$$

■ **FIGURE 16.3** The magnitude of the voltage response of a parallel resonant circuit is shown as a function of frequency.

$$\mathbf{V}(j\omega) = \frac{\mathbf{I}(j\omega)}{\mathbf{Y}(j\omega)} = \frac{\mathbf{I}(j\omega)}{1/R + j(\omega C - 1/(\omega L))}$$

$$\left|\mathbf{V}(j\omega)\right| = \frac{\left|\mathbf{I}(j\omega)\right|}{\left|\mathbf{Y}(j\omega)\right|} = \frac{\left|\mathbf{I}(j\omega)\right|}{\sqrt{1/R^2 + (\omega C - 1/(\omega L))^2}}$$

$$\left|\mathbf{V}(j\omega_0)\right| = \left|\mathbf{I}(j\omega_0)\right| R$$

Resonance and the Voltage Response (cont.)

- The response maximum is shown to occur exactly at the resonant frequency ω_0.
- The admittance, as specified by Eq. [1], possesses a constant conductance and a susceptance which has a minimum magnitude (zero) at resonance.
- The minimum admittance magnitude therefore occurs at resonance, and it is $1/R$. Hence, the maximum impedance magnitude is R, and it occurs at resonance.
- At the resonant frequency, therefore, the voltage across the parallel resonant circuit of Fig. 16.1 is simply $\mathbf{I}R$, and the entire source current $\mathbf{I}$ flows through the resistor.
- The current is also present in L and C.

Since $1/(\omega_0 C) = \omega_0 L$ at resonance,

$$\mathbf{Y}(j\omega)\big|_{\omega=\omega_0} = \frac{1}{R} + j[\omega C - \frac{1}{\omega L}]_{\omega=\omega_0} = \frac{1}{R}$$

or

$$\mathbf{I}_{C,0} = j\omega_0 C\mathbf{V} \qquad = j\omega_0 C(\mathbf{I}/\mathbf{Y}(j\omega_0)) = j\omega_0 CR\mathbf{I}$$

$$\mathbf{I}_{L,0} = \frac{\mathbf{V}(j\omega_0)}{j\omega_0 L} \qquad = \frac{\mathbf{I}/\mathbf{Y}(j\omega_0)}{j\omega_0 L} = \frac{R\mathbf{I}}{j/(\omega_0 C)} = -j\omega_0 CR\mathbf{I}$$

$$\mathbf{I}_{C,0} = -\mathbf{I}_{L,0} = j\omega_0 CR\mathbf{I} \qquad [5]$$

$$\mathbf{I}_{C,0} + \mathbf{I}_{L,0} = \mathbf{I}_{LC} = 0$$

Thus, the net current flowing into the LC combination is zero.

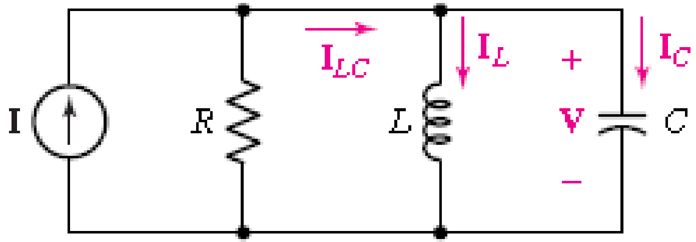


FIGURE 16.1 The parallel combination of a resistor, an inductor, and a capacitor, often referred to as a *parallel resonant circuit.*

Quality Factor

- Although the height of the response curve of Fig. 16.3 depends only upon the value of R for constant-amplitude excitation, the width of the curve or the steepness of the sides depends upon the other two element values also.
- The response curve of any resonant circuit is determined by the maximum amount of energy that can be stored in the circuit, compared with the energy that is lost during one complete period of the response.

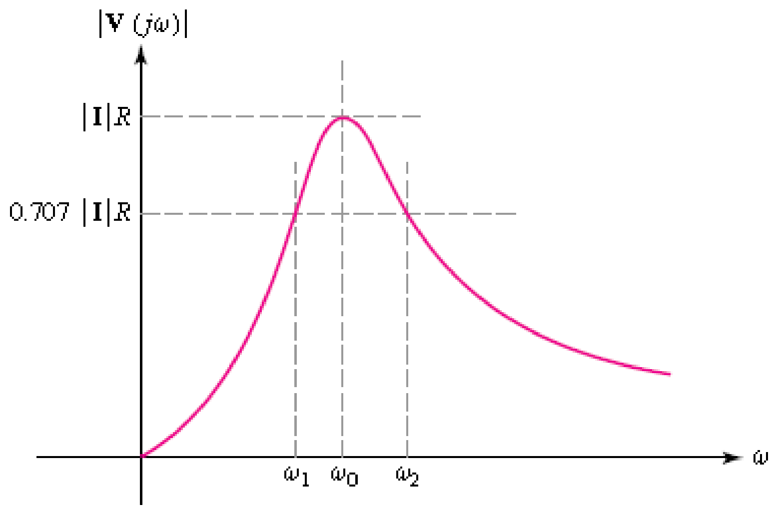
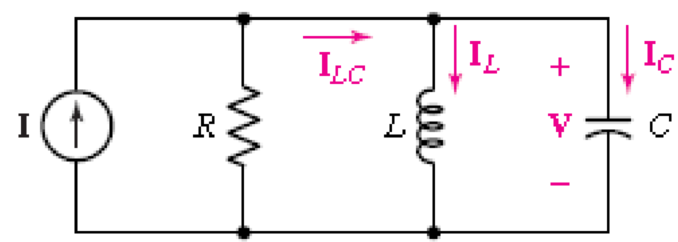


FIGURE 16.3 The magnitude of the voltage response of a parallel resonant circuit is shown as a function of frequency.



$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

We define Q as

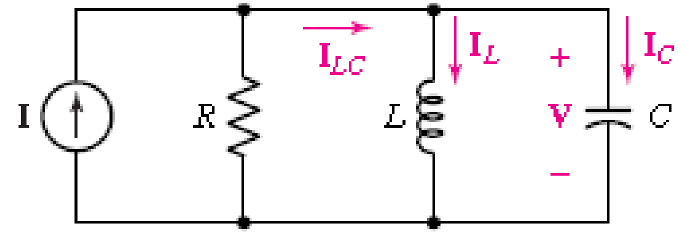
$$Q = \text{quality factor} = 2\pi \frac{\text{maximum energy stored}}{\text{total energy lost per period}}$$

Since energy can be stored only in the inductor and the capacitor, and can be lost only in the resistor, we may express Q in terms of the instantaneous energy associated with each of the reactive elements and the average power P_R dissipated in the resistor:

$$Q = 2\pi \frac{[w_L(t) + w_C(t)]_{\max}}{P_R T}$$

where T is the period of the sinusoidal frequency at which Q is evaluated.

To the parallel *RLC* circuit of Fig. 16.1, the value of *Q* at the resonant frequency is determined as follows.



$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

$$i(t) = \mathbf{I}_m \cos \omega_0 t$$

$$v(t) = Ri(t) = R\mathbf{I}_m \cos \omega_0 t$$

$$w_C(t) = \frac{1}{2}Cv^2 = \frac{\mathbf{I}_m^2 R^2 C}{2} \cos^2 \omega_0 t$$

$$w_L(t) = \frac{1}{2}Li_L^2 = \frac{1}{2}L\left(\frac{1}{L}\int v\,dt\right)^2 = \frac{1}{2L}\left[\frac{R\mathbf{I}_m}{\omega_0}\sin \omega_0 t\right]^2$$

$$= \frac{\mathbf{I}_m^2 R^2 C}{2} \sin^2 \omega_0 t$$

The total instantaneous stored energy is

$$w(t) = w_L(t) + w_C(t) = \frac{\mathbf{I}_m^2 R^2 C}{2}$$

In order to find the energy lost in the resistor in one period, we find the power lost in the resistor:

$$P_R = \tfrac{1}{2}\mathbf{I}_m^2 R$$

and multiply by one period, obtaining

$$P_R T = \frac{1}{2 f_0}\mathbf{I}_m^2 R$$

Substituting all we have in the definition of Q, obtaining

$$Q_0 = 2\pi \frac{\mathbf{I}_m^2 R^2 C/2}{\mathbf{I}_m^2 R/2 f_0}$$

$$= 2\pi f_0 RC = \omega_0 RC = R/(\omega_0 L)$$

$$Q_0 = 2\pi f_0 RC = \omega_0 RC = R/(\omega_0 L)$$

Equivalent expressions for Q_0 which are often quite useful may be obtained by simple substitution:

$$Q_0 = R\sqrt{\frac{C}{L}} = \frac{R}{|X_{C,0}|} = \frac{R}{|X_{L,0}|}$$

for this specific circuit, decreasing the resistance decreases Q_0;

Other Interpretation of Q

- Another useful interpretation of Q is obtained when we inspect the inductor and capacitor currents at resonance, as given by,

$$\mathbf{I}_{C,0} = -\mathbf{I}_{L,0} = j\omega_0 CR\mathbf{I} = jQ_0\mathbf{I}$$

- Note that each is Q_0 times the source current in amplitude and that each is 180° out of phase with the other.
- Thus, if we apply 2 mA at the resonant frequency to a parallel resonant circuit with a Q_0 of 50, we find 2 mA in the resistor, and 100 mA in both the inductor and the capacitor.

Other Interpretation of Q (cont.)

- A parallel resonant circuit can therefore act as a current amplifier, but not, of course, as a power amplifier, since it is a passive network.
- Resonance, by definition, is fundamentally associated with the forced response, since it is defined in terms of a (purely resistive) input impedance, a sinusoidal steady-state concept.
- The two most important parameters of a resonant circuit are perhaps the resonant frequency ω_0 and the quality factor Q_0.
- Both the exponential damping coefficient and the natural resonant frequency may be expressed in terms of ω_0 and Q_0:

$$\alpha = \frac{1}{2RC} = \frac{1}{2(Q_0/\omega_0 C)C} = \frac{\omega_0}{2Q_0} \quad \because Q_0 = \omega_0 RC$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega_0\sqrt{1 - \left(\frac{1}{2Q_0}\right)^2}$$

$$\mathbf{Y}(\mathbf{s}) = C\frac{\mathbf{s}^2 + \mathbf{s}/RC + 1/LC}{\mathbf{s}}$$

Damping Factor

- The quadratic factor appearing in the numerator of

$$\mathbf{s}^2 + \frac{1}{RC}\mathbf{s} + \frac{1}{LC}$$

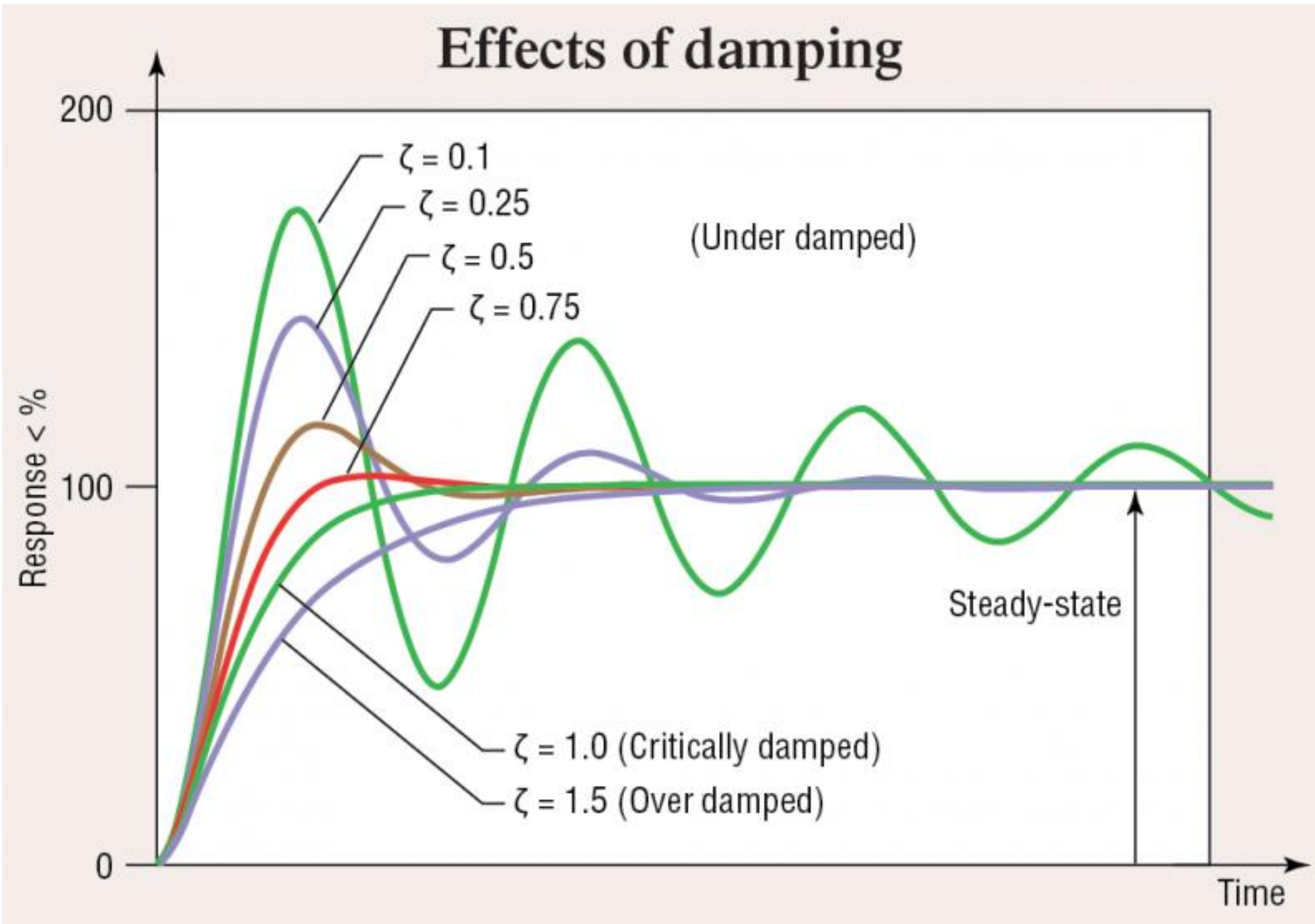
may be written in terms of α and ω_0:

$$\mathbf{s}^2 + 2\alpha\mathbf{s} + \omega_0^2$$

- In the field of system theory or automatic control theory, it is traditional to write this factor that utilizes the dimensionless parameter ζ (zeta), called the damping factor:

$$\boxed{\zeta = \frac{\alpha}{\omega_0} = \frac{1}{2Q_0}}$$

Effects of Damping Factor



Example 3.38 ***MEMS Accelerometer: Frequency Response and Resonance***

$$\frac{d^2}{dt^2}y(t)+\frac{\omega_n}{Q}\frac{d}{dt}y(t)+\omega_n^2 y(t)=x(t)$$

$$H(j\omega)=\frac{1}{(j\omega)^2+\frac{\omega_n}{Q}(j\omega)+\omega_n^2}$$

$$\zeta=\frac{\alpha}{\omega_0}=\frac{1}{2Q_0}$$

$$\mathbf{s}^2+2\alpha\mathbf{s}+\omega_0^2 \quad =\mathbf{s}^2+2\zeta\omega_0\mathbf{s}+\omega_0^2$$

$$=\mathbf{s}^2+\frac{\omega_0}{Q_0}\mathbf{s}+\omega_0^2$$

$$\omega_d=\sqrt{\omega_0^2-\alpha^2} \quad =\omega_0\sqrt{1-\left(\frac{1}{2Q_0}\right)^2}$$

$$=\omega_0\sqrt{1-\zeta^2}$$

Example

Calculate numerical values of ω_0, α, ω_d, and R for a parallel resonant circuit having $L = 2.5$ mH, $Q_0 = 5$, and $C = 0.01\ \mu$F.

From Eq. [2], we see that $\omega_0 = 1/\sqrt{LC} = 200$ krad/s, while $f_0 = \omega_0/2\pi = 31.8$ kHz.

The value of α may be obtained quickly by using Eq. [10],

$$\alpha = \frac{\omega_0}{2Q_0} = \frac{2 \times 10^5}{(2 \times 5)} = 2 \times 10^4 \text{ Np/s}$$

Now we may make use of our old friend from Chap. 9,

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Example (cont.)

to find that

$$\omega_d = \sqrt{(2 \times 10^5)^2 - (2 \times 10^4)^2} = 199.0 \text{ krad/s}$$

Finally, we need a value for the parallel resistance, and Eq. [7] gives us the answer:

$$Q_0 = \omega_0 RC$$

so

$$R = \frac{Q_0}{\omega_0 C} = \frac{5}{(2 \times 10^5 \times 10^{-8})} = 2.50 \text{ k}\Omega$$

Example 16.1

Consider a parallel *RLC* circuit such that $L = 2$ mH, $Q_0 = 5$, and $C = 10$ nF. Determine the value of *R* and the magnitude of the steady-state admittance at $0.1\omega_0$, ω_0, and $1.1\omega_0$.

We derived several expressions for Q_0, a parameter directly related to energy loss, and hence the resistance in our circuit. Rearranging the expression in Eq. [8], we calculate

$$R = Q_0\sqrt{\frac{L}{C}} = 2.236\ \text{k}\Omega$$

Next, we compute ω_0, a term we may recall from Chap. 9,

$$\omega_0 = \frac{1}{\sqrt{LC}} = 223.6\ \text{krad/s}$$

or, alternatively, we may exploit Eq. [7] and obtain the same answer,

$$\omega_0 = \frac{Q_0}{RC} = 223.6\ \text{krad/s}$$

Example 16.1 (cont.)

The admittance of any parallel *RLC* network is simply

$$\mathbf{Y} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

and hence

$$|\mathbf{Y}| = \left|\frac{1}{R} + j\omega C + \frac{1}{j\omega L}\right|$$

evaluated at the three designated frequencies is equal to

$$|\mathbf{Y}(0.9\omega_0)| = 6.504 \times 10^{-4}\ \text{S} \qquad |\mathbf{Y}(\omega_0)| = 4.472 \times 10^{-4}\ \text{S}$$

$$|\mathbf{Y}(1.1\omega_0)| = 6.182 \times 10^{-4}\ \text{S}$$

admittance

We thus obtain a minimum ~~impedance~~ at the resonant frequency, or a *maximum voltage response* to a particular input current. If we quickly compute the reactance at these three frequencies, we find

$$X(0.9\omega_0) = -4.72 \times 10^{-4}\ \text{S} \qquad X(1.1\omega_0) = 4.72 \times 10^{-4}\ \text{S}$$

$$X(\omega_0) = -1.36 \times 10^{-7}$$

We leave it to the reader to show that our value for $X(\omega_0)$ is nonzero only as a result of rounding error.

Practice 16.1

A parallel resonant circuit is composed of the elements $R = 8\ \text{k}\Omega$, $L = 50\ \text{mH}$, and $C = 80\ \text{nF}$. Find

(a) ω_0

(b) Q_0

(c) ω_d

(d) α

(e) ζ.

Ans: 15.811 krad/s; 10.12; 15.792 krad/s; 781 Np/s; 0.0494.

Practice 16.2

Find the values of R, L, and C in a parallel resonant circuit for which $\omega_0 = 1000$ rad /s, $\omega_d = 998$ rad/s, and $\mathbf{Y}_{\text{in}} = 1$mS at resonance.

Ans: $1000\ \Omega$; 126.4 mH; $7.91\ \mu$F.

$$\mathbf{Y}(\mathbf{s}) = C\frac{\mathbf{s}^2 + \mathbf{s}/RC + 1/LC}{\mathbf{s}}$$

Damping Factor (cont.)

- Now let us interpret Q_0 in terms of the pole-zero locations of the admittance $\mathbf{Y}(\mathbf{s})$ of the parallel RLC circuit.
- The two zeros are indicated in Fig. 16.4, and the arrows show the path they take as R increases. When R is infinite, Q_0 is also infinite, and the two zeros are found at $\mathbf{s} = \pm j\omega_0$ on the $j\omega$ axis.
- As R decreases, the zeros move toward the σ axis along the circular locus, joining to form a double zero on the σ axis at $\mathbf{s} = -\omega_0$ when $R = 1/2\sqrt{L/C}$ or $Q_0 = 1/2$.
- This condition may be recalled as that for critical damping, so that $\omega_d = 0$ and $\alpha = \omega_0$.

Now let us interpret Q_0 in terms of the pole-zero locations of the admittance $\mathbf{Y}(\mathbf{s})$ of the parallel RLC circuit. We will keep ω_0 constant; this may be done, for example, by changing R while holding L and C constant. As Q_0 is increased, the relationships relating α, Q_0, and ω_0 indicate that the two zeros must move closer to the $j\omega$ axis. These relationships also show that the zeros must simultaneously move away from the σ axis. The exact nature of the movement becomes clearer when we remember that the point at which $\mathbf{s} = j\omega_0$ could be located on the $j\omega$ axis by swinging an arc, centered at the origin, through one of the zeros and over to the positive $j\omega$ axis; since ω_0 is to be held constant, the radius must be constant, and the zeros must therefore move along this arc toward the positive $j\omega$ axis as Q_0 increases.

The two zeros are indicated in Fig. 16.4, and the arrows show the path they take as R increases. When R is infinite, Q_0 is also infinite, and the two zeros are found at $\mathbf{s} = \pm j\omega_0$ on the $j\omega$ axis. As R decreases, the zeros move toward the σ axis along the circular locus, joining to form a double zero on the σ axis at $\mathbf{s} = -\omega_0$ when $R = \frac{1}{2}\sqrt{L/C}$ or $Q_0 = \frac{1}{2}$. This condition may be recalled as that for critical damping, so that $\omega_d = 0$ and $\alpha = \omega_0$. Lower values of R and lower values of Q_0 cause the zeros to separate and move in opposite directions on the negative σ axis, but these low values of Q_0 are not really typical of resonant circuits and we need not track them any further.

Later, we will use the criterion $Q_0 \geq 5$ to describe a high-Q circuit. When $Q_0 = 5$, the zeros are located at $\mathbf{s} = -0.1\omega_0 \pm j0.995\omega_0$, and thus ω_0 and ω_d differ by only one-half of 1 percent.

$$\mathbf{Y}(\mathbf{s}) = C\frac{\mathbf{s}^2 + \mathbf{s}/RC + 1/LC}{\mathbf{s}}$$

$$\mathbf{Y}(\mathbf{s}) = C\frac{(\mathbf{s} + \alpha - j\omega_d)(\mathbf{s} + \alpha + j\omega_d)}{\mathbf{s}}$$

$$\mathbf{Z}(\mathbf{s}) = \frac{\mathbf{s}/C}{(\mathbf{s} + \alpha - j\omega_d)(\mathbf{s} + \alpha + j\omega_d)}$$

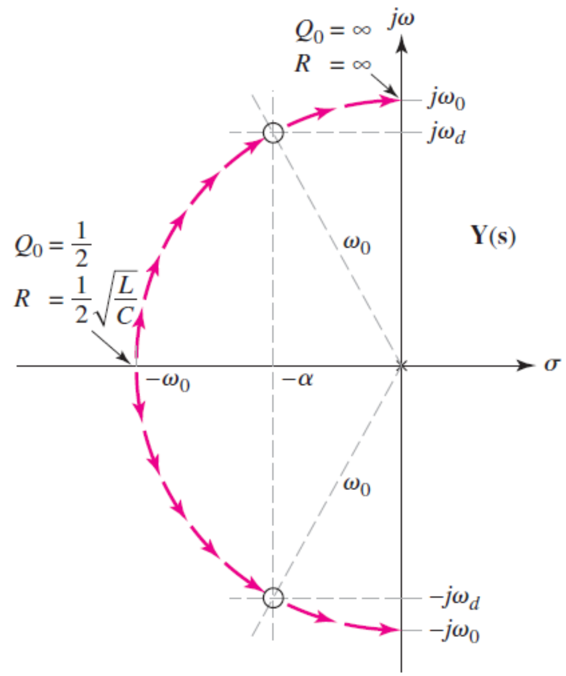
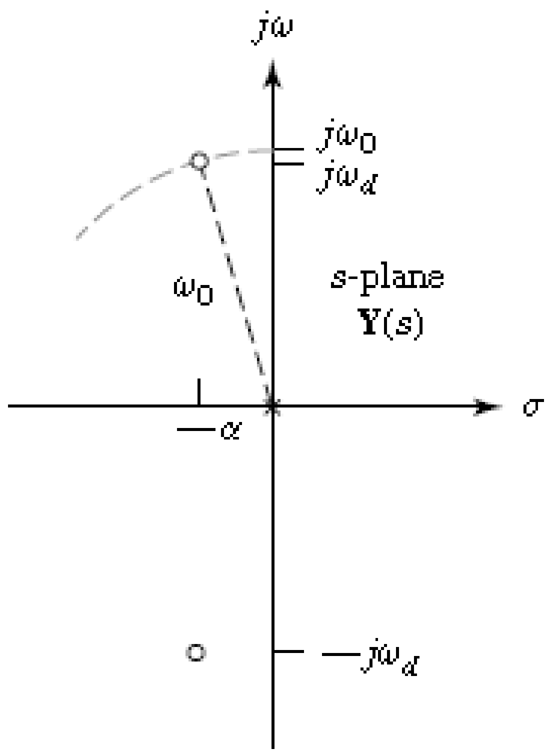


FIGURE 16.4 The two zeros of the admittance $\mathbf{Y}(\mathbf{s})$, located at $\mathbf{s} = -\alpha \pm j\omega_d$, provide a semicircular locus as R increases from $\frac{1}{2}\sqrt{L/C}$ to ∞.



(a)

$$\mathbf{Y}(\mathbf{s}) = C\frac{\mathbf{s}^2 + \mathbf{s}/RC + 1/LC}{\mathbf{s}}$$

$$\mathbf{Y}(\mathbf{s}) = C\frac{(\mathbf{s} + \alpha - j\,\omega_d)(\mathbf{s} + \alpha + j\,\omega_d)}{\mathbf{s}}$$

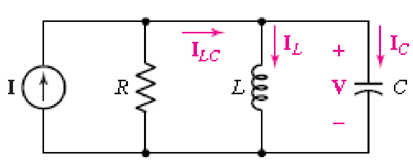
$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$\alpha = \frac{1}{2RC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \text{rad/s}$$

16.2 Bandwidth and High-Q Circuit

- As the one shown in Fig. 16.3, Let us first define the two half-power frequencies ω_1 and ω_2 as those frequencies at which the magnitude of the input admittance of a parallel resonant circuit is greater than the magnitude at resonance by a factor of $\sqrt{2}$.
- Since the response curve of Fig. 16.3 displays the voltage produced across the parallel circuit by a sinusoidal current source as a function of frequency, the half-power frequencies also locate those points at which the voltage response is $1/\sqrt{2}$ or 0.707 times its maximum value.
- A similar relationship holds for the impedance magnitude. We will designate ω_1 as the lower half-power frequency and ω_2 as the upper half-power frequency.



$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

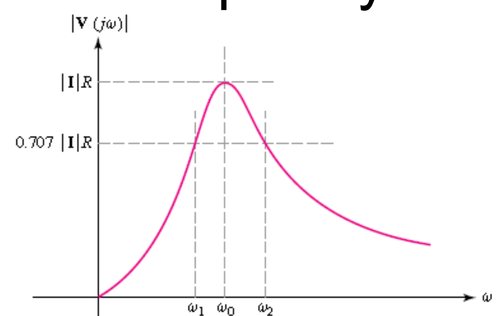


FIGURE 16.3 The magnitude of the voltage response of a parallel resonant circuit is shown as a function of frequency.

Bandwidth

- The (half-power) bandwidth of a resonant circuit is defined as

$$\mathcal{B} \equiv \omega_2 - \omega_1$$

- The half-power bandwidth is measured by that portion of the response curve which is equal to or greater than 70.7 percent of the maximum value, as depicted in Fig. 16.5.

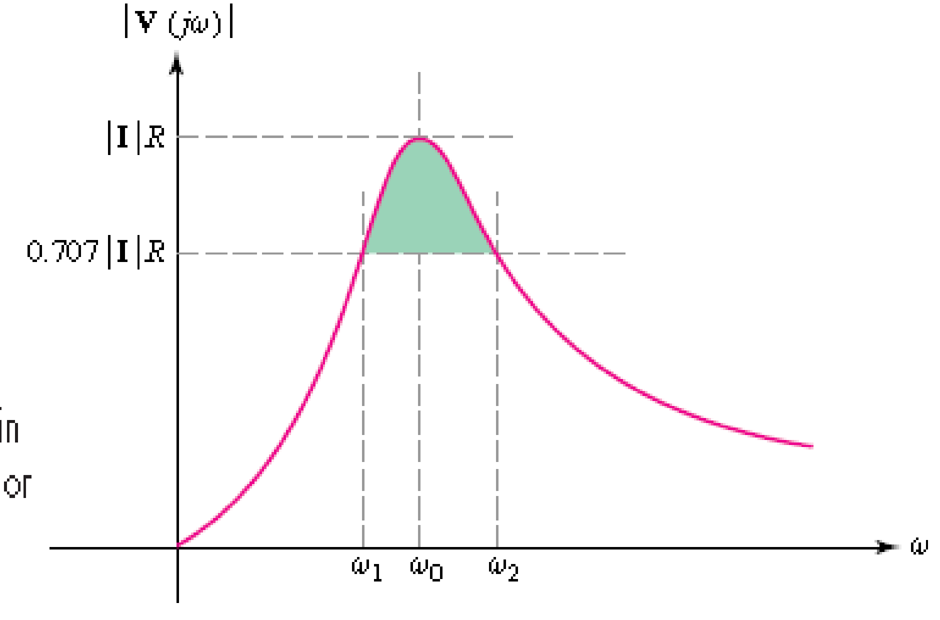


FIGURE 16.5 The bandwidth of the circuit response is highlighted in green; it corresponds to the portion of the response curve greater than or equal to 70.7% of the maximum value.

Now let us express the bandwidth in terms of Q_0 and the resonant frequency.

$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

$$\mathbf{Y} = \frac{1}{R} + j\frac{1}{R}\left(\frac{\omega\omega_0 CR}{\omega_0} - \frac{\omega_0 R}{\omega\omega_0 L}\right)$$

$$\mathbf{Y} = \frac{1}{R}\left[1 + jQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

an admittance magnitude of $\sqrt{2}/R$ can occur only when a frequency is selected such that the imaginary part of the bracketed quantity has a magnitude of unity.

Thus,

$$Q_0\left(\frac{\omega_2}{\omega_0}-\frac{\omega_0}{\omega_2}\right)=1 \qquad \text{and} \qquad Q_0\left(\frac{\omega_1}{\omega_0}-\frac{\omega_0}{\omega_1}\right)=-1$$

$$\omega_1=\omega_0\left[\sqrt{1+\left(\frac{1}{2Q_0}\right)^2}-\frac{1}{2Q_0}\right] \quad =\omega_0\left(\sqrt{1+\zeta^2}-\zeta\right) \qquad [15]$$

$$\omega_2=\omega_0\left[\sqrt{1+\left(\frac{1}{2Q_0}\right)^2}+\frac{1}{2Q_0}\right] \quad =\omega_0\left(\sqrt{1+\zeta^2}+\zeta\right) \qquad [16]$$

$$\mathcal{B}=\omega_2-\omega_1=\frac{\omega_0}{Q_0}$$

Equations [15] and [16] may be multiplied by each other to show that ω_0 is exactly equal to the geometric mean of the half-power frequencies:

Thus,

$$\omega_0^2 = \omega_1 \omega_2$$

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

Circuits possessing a higher Q_0 have a narrower bandwidth, or a sharper response curve; they have greater frequency selectivity, or higher quality (factor).

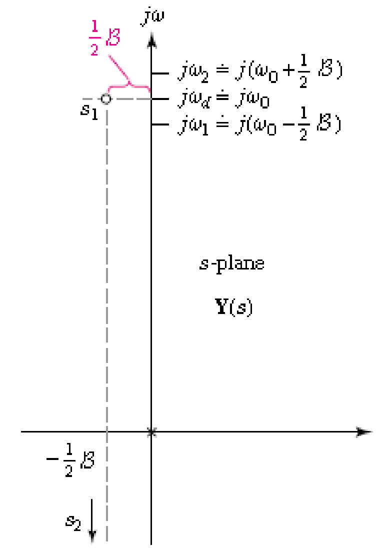
Approximations for High-Q Circuits

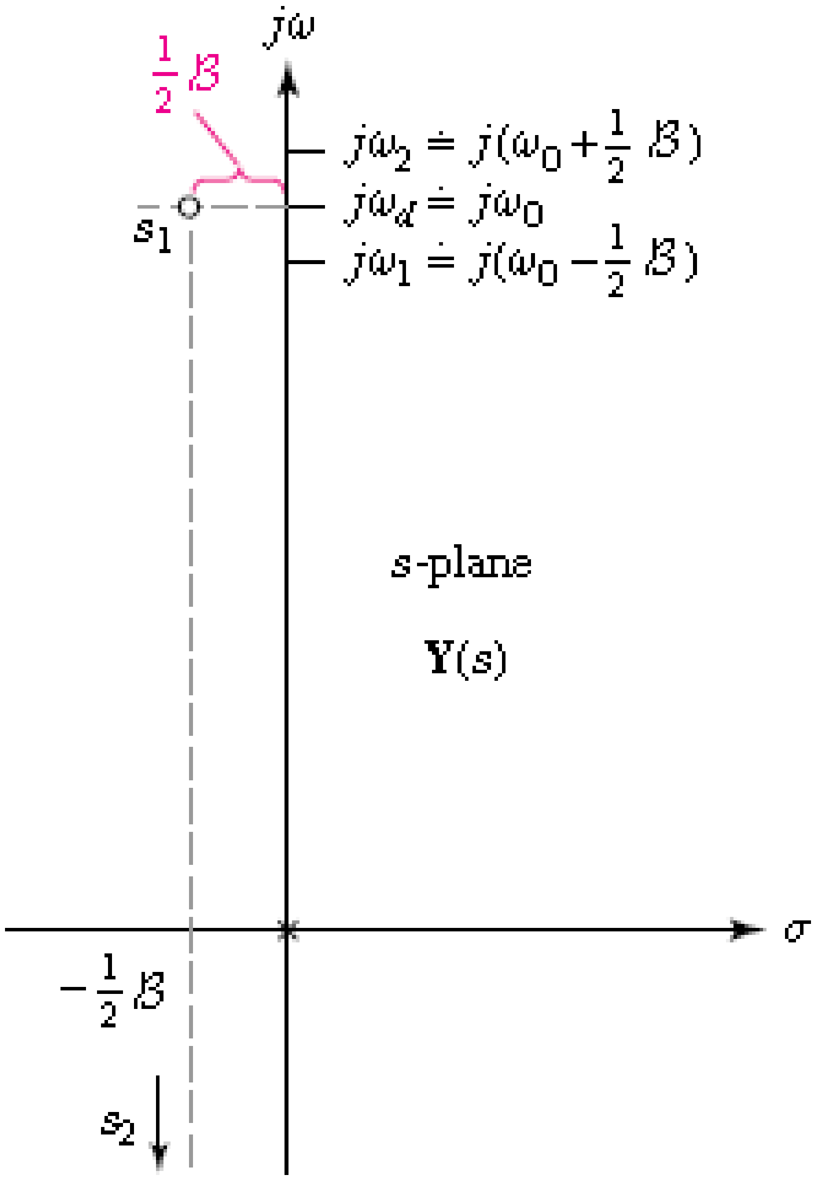
- Many resonant circuits are deliberately designed to have a large Q_0 in order to take advantage of the narrow bandwidth and high frequency selectivity associated with such circuits.
- When Q_0 is larger than about 5, it is possible to make some very useful approximations in the expressions for the upper and lower half-power frequencies and in the general expressions for the response in the neighborhood of resonance.
- The pole-zero configuration of $\mathbf{Y}(\mathbf{s})$ for a parallel RLC circuit having a Q_0 of about 5 is shown in Fig. 16.6.

$$\alpha = \frac{\omega_0}{2Q_0} \doteq \frac{1}{2}\mathcal{B}$$

$$\mathbf{s}_{1,2} = -\alpha \pm j\omega_d$$

$$\approx -\frac{1}{2}\mathcal{B} \pm j\omega_0$$





$$\mathbf{s}_{1,2} = -\alpha \pm j\omega_d$$
$$\approx -\frac{1}{2}\mathcal{B} \pm j\omega_0$$

FIGURE 16.6 The pole-zero constellation of $Y(s)$ for a parallel *RLC* circuit. The two zeros are exactly $\frac{1}{2}\mathcal{B}$ Np/s (or rad/s) to the left of the $j\omega$ axis and approximately $j\omega_0$ rad/s (or Np/s) from the σ axis. The upper and lower half-power frequencies are separated exactly $\mathcal{B}$ rad/s, and each is approximately $\frac{1}{2}\mathcal{B}$ rad/s away from the resonant frequency and the natural resonant frequency.

Approximations for High-Q Circuits (cont.)

- Thus

$$\omega_{1,2} = \omega_0 \left[\sqrt{1 + \left(\frac{1}{2Q_0} \right)^2} \mp \frac{1}{2Q_0} \right] \approx \omega_0 \left(1 \mp \frac{1}{2Q_0} \right)$$

$$\omega_{1,2} \approx \omega_0 \mp \tfrac{1}{2} \mathcal{B}$$

- In a high-Q circuit, therefore, each half-power frequency is located approximately one-half bandwidth from the resonant frequency; this is indicated in Fig. 16.6.

$$\omega_0 \approx \tfrac{1}{2}(\omega_1 + \omega_2)$$

- It can be shown that the error in magnitude or phase is less than 5 percent if $Q_0 \geq 5$ and $0.9\omega_0 \leq \omega \leq 1.1\omega_0$.

Approximations for High-Q Circuits (cont.)

- Although this narrow band of frequencies may seem to be prohibitively small, it is usually more than sufficient to contain the range of frequencies in which we are most interested.
- The parallel resonant key conclusions:
  - For this circuit, $\omega_0 = 1/\sqrt{LC}$
  - For this circuit, $Q_0 = \omega_0 RC$.
  - Two half-power frequencies, ω_1 and ω_2, as the frequencies at which the admittance magnitude is $\sqrt{2}$ times the minimum admittance magnitude.
  - The exact expressions for ω_1 and ω_2 are

$$\omega_{1,2} = \omega_0 \left[\sqrt{1 + \left(\frac{1}{2Q_0} \right)^2} \mp \frac{1}{2Q_0} \right]$$

Approximations for High-Q Circuits (cont.)

- The parallel resonant key conclusions:
  - The approximate (high-Q_0) expressions for ω_1 and ω_2 are

$$\omega_{1,2} \approx \omega_0 \mp \frac{1}{2}B \qquad [17]$$

  - The half-power bandwidth B is given by

$$B = \omega_2 - \omega_1 = \frac{\omega_0}{Q_0}$$

  - The input admittance in approximate form for high-Q circuits:

$$\mathbf{Y} = \frac{1}{R}\left[1 + jQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right] \longrightarrow \boxed{\mathbf{Y} \approx \frac{1}{R}(1 + jN) = \frac{1}{R}\sqrt{1+N^2}\,\underline{/\tan^{-1} N} \qquad N = \frac{\omega - \omega_0}{\frac{1}{2}B}}$$

$$= \frac{1}{R}\left[1 + jQ_0\frac{(\omega^2 - \omega_0^2)}{\omega\omega_0}\right]$$

$$= \frac{1}{R}\left[1 + jQ_0\frac{(\omega - \omega_0)(\omega + \omega_0)}{\omega\omega_0}\right] \approx \frac{1}{R}\left[1 + jQ_0\frac{2(\omega - \omega_0)}{\omega_0}\right] = \frac{1}{R}\left[1 + j\frac{(\omega - \omega_0)}{B/2}\right]$$

This approximation is valid for $0.9\omega_0 \le \omega \le 1.1\omega_0$.

Example 16.2

Determine the approximate value of the admittance of a parallel RLC network for which $R = 40\ \text{k}\Omega$, $L = 1\text{H}$, and $C = 1/64\ \mu\text{F}$ if the operating frequency is $\omega = 8.2$ krad/s.

▶ ***Identify the goal of the problem.***

We seek the lower and upper half-power frequencies of the voltage response as well as $\mathbf{Y}(\omega_0)$. Since we are asked to "estimate" and "approximate," the implication is that this is a high-Q circuit, an assumption we should verify.

▶ ***Collect the known information.***

Given R, L, and C, we are able to compute ω_0 and Q_0. If $Q_0 \geq 5$, we may invoke approximate expressions for half-power frequencies and admittance near resonance, but regardless could compute these quantities exactly if required.

Example 16.2 (cont.)

Determine the approximate value of the admittance of a parallel RLC network for which $R = 40\ \text{k}\Omega$, $L = 1\text{H}$, and $C = 1/64\ \mu\text{F}$ if the operating frequency is $\omega = 8.2\ \text{krad/s}$.

▶ ***Devise a plan.***
To use approximate expressions, we must first determine Q_0, the quality factor at resonance, as well as the bandwidth.

$$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{1\times 1/64\times 10^{-6}} = 8\times 10^3\ \text{rad/s} = 8\ \text{krad/s}$$

$$Q_0 = \omega_0 RC = 8\times 10^3 \times 40\times 10^3 \times (1/64)\times 10^{-6} = 5$$

The quality factor is sufficiently large to employ the high-Q approximation.

Construct an appropriate set of equations.

The bandwidth is simply

$$\mathcal{B} = \frac{\omega_0}{Q_0} = 1600 \text{ rad/s}$$

and so

$$\omega_1 \approx \omega_0 - \frac{\mathcal{B}}{2} = 7200 \text{ rad/s} \qquad \omega_2 \approx \omega_0 + \frac{\mathcal{B}}{2} = 8800 \text{ rad/s}$$

Equation [19] states that

$$\mathbf{Y}(\mathrm{s}) \approx \frac{1}{R}(1 + jN)$$

so

$$|\mathbf{Y}(j\omega)| \approx \frac{1}{R}\sqrt{1 + N^2} \qquad \text{and} \qquad \text{ang } \mathbf{Y}(j\omega) \approx \tan^{-1} N$$

Determine if additional information is required.

We still require N, which tells us how many half-bandwidths ω is from the resonant frequency ω_0:

$$N = (8.2 - 8)/0.8 = 0.25$$

$$N = \frac{\omega - \omega_0}{\frac{1}{2}\mathcal{B}}$$

Attempt a solution.

Now we are ready to employ our approximate relationships for the magnitude and angle of the network admittance,

$$\text{ang } \mathbf{Y} \approx \tan^{-1} 0.25 = 14.04^\circ$$

and

$$|\mathbf{Y}| \approx 25\sqrt{1 + (0.25)^2} = 25.77\ \mu\text{S}$$

Verify the solution. Is it reasonable or expected?

An exact calculation of the admittance using Eq. [1] shows that

$$\mathbf{Y}(j8200) = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) = 25.75 \underline{/13.87^\circ}\ \mu\text{S}$$

The approximate method therefore leads to values of admittance magnitude and angle that are reasonably accurate (better than 2 percent) for this frequency. We leave it to the reader to judge the accuracy of our prediction for ω_1 and ω_2.

Practice 16.3

A marginally high-Q parallel resonant circuit has $f_0 = 440$ Hz with $Q_0 = 6$. Use Eqs. [15] and [16] to obtain accurate values for

(a) f_1;

(b) f_2;

Now use Eq. [17] to calculate approximate values for

(c) f_1;

(d) f_2.

Ans: 404.9 Hz; 478.2 Hz; 403.3 Hz; 476.7 Hz.

16.3 Series Resonance

Consider the circuit shown in Fig. 16.8.

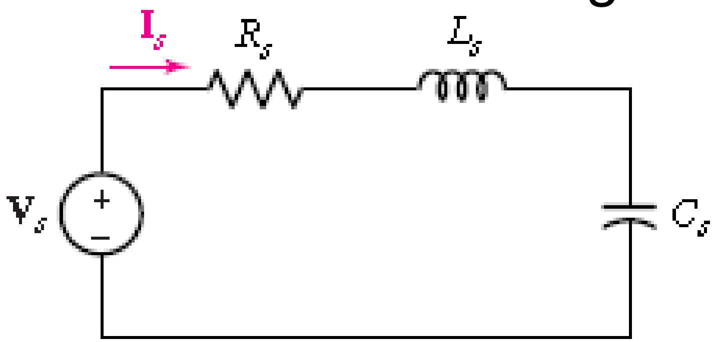


FIGURE 16.8 A series resonant circuit.

$$Z(s) = R + sL + 1/(sC)$$

$$j\omega_0 L = -1/(j\omega_0 C)$$

$$\omega_0 = 1/\sqrt{LC}$$

$$Z(s) = R + sL + 1/(sC)$$

$$= L\frac{s^2 + sR/L + 1/(LC)}{s}$$

$$= L\frac{(s+\alpha - j\omega_d)(s+\alpha + j\omega_d)}{s}$$

$$\alpha = R/(2L) \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$Q_0 = \omega_0 L/R$$

$$\longrightarrow$$

$$Z(s) = L\frac{s^2 + 2\alpha s + \omega_0^2}{s}$$

$$= L\frac{s^2 + 2\zeta\omega_0 s + \omega_0^2}{s}$$

$$= L\frac{s^2 + \dfrac{\omega_0}{Q_0}s + \omega_0^2}{s}$$

The series resonant circuit key conclusions:

- For this circuit, $\omega_0 = 1/\sqrt{CL}$.
- The circuit's figure of merit Q_0 is defined as 2π times the ratio of the maximum energy stored in the circuit to the energy lost each period in the circuit. For this circuit, $Q_0 = \omega_0 L/R$.
- Two half-power frequencies, ω_1 and ω_2, as the frequencies at which the impedance magnitude is 2 times the minimum impedance magnitude.
- The exact expressions for ω_1 and ω_2 are

$$\omega_{1,2} = \omega_0 \left[\sqrt{1 + \left(\frac{1}{2Q_0} \right)^2} \mp \frac{1}{2Q_0} \right]$$

- The approximate (high-Q_0) expressions for ω_1 and ω_2 are

$$\omega_{1,2} \approx \omega_0 \mp \frac{1}{2} B$$

The series resonant circuit key conclusions (cont.):

- The half-power bandwidth is given by

$$\mathcal{B} = \omega_2 - \omega_1 = \frac{\omega_0}{Q_0}$$

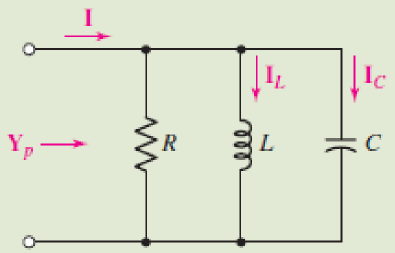
- The input impedance may also be expressed in approximate form for high-Q circuits:

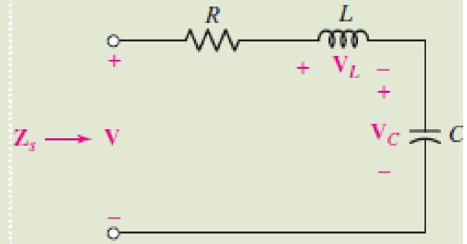
$$\mathbf{Z}_{\text{eq}} \approx R\sqrt{1 + N^2}\,\underline{/\tan^{-1} N}$$

$$N = \frac{\omega - \omega_0}{\frac{1}{2}\mathcal{B}}$$

If $Q_0 \geq 5$, this approximation is valid for $0.9\omega_0 \leq \omega \leq 1.1\omega_0$.

TABLE 16.1 A Short Summary of Resonance





$Q_0 = \omega_0 RC \qquad \alpha = \dfrac{1}{2RC}$	$Q_0 = \dfrac{\omega_0 L}{R} \qquad \alpha = \dfrac{R}{2L}$
$\lvert \mathbf{I}_L(j\omega_0)\rvert = \lvert \mathbf{I}_C(j\omega_0)\rvert = Q_0\lvert \mathbf{I}(j\omega_0)\rvert$	$\lvert \mathbf{V}_L(j\omega_0)\rvert = \lvert \mathbf{V}_C(j\omega_0)\rvert = Q_0\lvert \mathbf{V}(j\omega_0)\rvert$
$\mathbf{Y}_p = \dfrac{1}{R}\left[1 + jQ_0\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)\right]$	$\mathbf{Z}_s = R\left[1 + jQ_0\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)\right]$

Exact expressions

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\omega_1\omega_2}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega_0\sqrt{1 - \left(\frac{1}{2Q_0}\right)^2}$$

$$\omega_{1,2} = \omega_0\left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} \mp \frac{1}{2Q_0}\right]$$

$$N = \frac{\omega - \omega_0}{\frac{1}{2}\mathcal{B}}$$

$$\mathcal{B} = \omega_2 - \omega_1 = \frac{\omega_0}{Q_0} = 2\alpha$$

Approximate expressions

$$(Q_0 \geq 5 \qquad 0.9\omega_0 \leq \omega \leq 1.1\omega_0)$$

$$\omega_d \approx \omega_0$$

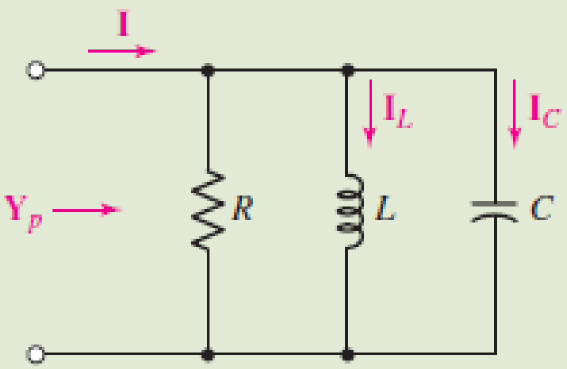
$$\omega_{1,2} \approx \omega_0 \mp \tfrac{1}{2}\mathcal{B}$$

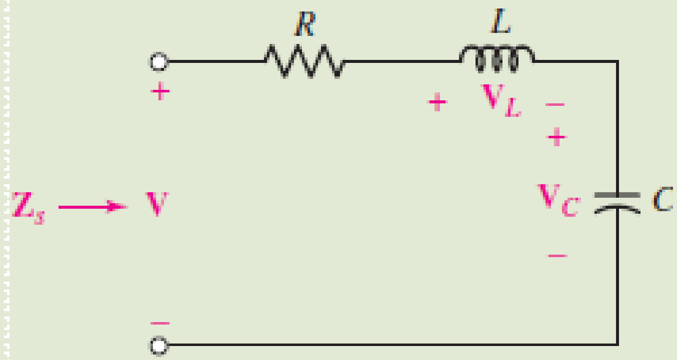
$$\omega_0 \approx \tfrac{1}{2}(\omega_1 + \omega_2)$$

$$\mathbf{Y}_p \approx \frac{\sqrt{1+N^2}}{R}\underline{/\tan^{-1} N}$$

$$\mathbf{Z}_s \approx R\sqrt{1+N^2}\underline{/\tan^{-1} N}$$

TABLE 16.1 A Short Summary of Resonance





$Q_0 = \omega_0 RC \qquad \alpha = \dfrac{1}{2RC}$	$Q_0 = \dfrac{\omega_0 L}{R} \qquad \alpha = \dfrac{R}{2L}$
$\lvert\mathbf{I}_L(j\omega_0)\rvert = \lvert\mathbf{I}_C(j\omega_0)\rvert = Q_0\lvert\mathbf{I}(j\omega_0)\rvert$	$\lvert\mathbf{V}_L(j\omega_0)\rvert = \lvert\mathbf{V}_C(j\omega_0)\rvert = Q_0\lvert\mathbf{V}(j\omega_0)\rvert$
$\mathbf{Y}_p = \dfrac{1}{R}\left[1 + jQ_0\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)\right]$	$\mathbf{Z}_s = R\left[1 + jQ_0\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)\right]$

Approximate expressions

$$(Q_0 \geq 5 \qquad 0.9\omega_0 \leq \omega \leq 1.1\omega_0)$$

$$\omega_d \approx \omega_0$$

$$\omega_{1,2} \approx \omega_0 \mp \tfrac{1}{2}\mathcal{B}$$

$$\omega_0 \approx \tfrac{1}{2}(\omega_1 + \omega_2)$$

$$\mathbf{Y}_p \approx \frac{\sqrt{1+N^2}}{R}\underline{/\tan^{-1} N} \qquad N = \frac{\omega - \omega_0}{\frac{1}{2}\mathcal{B}}$$

$$\mathbf{Z}_s \approx R\sqrt{1+N^2}\underline{/\tan^{-1} N} \qquad \mathcal{B} = \omega_2 - \omega_1 = \frac{\omega_0}{Q_0} = 2\alpha$$

Example 16.3

The voltage $v_s = 100\cos(\omega t)$ mV is applied to a series resonant circuit composed of a $10\ \Omega$ resistance, a 200 nF capacitance and a 2 mH inductance. Use both exact and approximate methods to calculate the current amplitude if $\omega = 48$ krad/s.

The resonant frequency of the circuit is given by

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(2 \times 10^{-3})(200 \times 10^{-9})}} = 50 \text{ krad/s}$$

Since we are operating at $\omega = 48$ krad/s, which is within 10 percent of the resonant frequency, it is reasonable to apply our approximate relationships to estimate the equivalent impedance of the network provided that we find that we are working with a high-Q circuit:

$$\mathbf{Z}_{\text{eq}} \approx R\sqrt{1 + N^2}\,\underline{/\tan^{-1} N}$$

where N is computed once we determine Q_0. This is a series circuit, so

$$Q_0 = \frac{\omega_0 L}{R} = \frac{(50 \times 10^3)(2 \times 10^{-3})}{10} = 10$$

which qualifies as a high-Q circuit. Thus, which qualifies as a high-Q circuit. Thus,

$$\mathcal{B} = \frac{\omega_0}{Q_0} = \frac{50 \times 10^3}{10} = 5 \text{ krad/s}$$

The number of half-bandwidths off resonance (N) is therefore

$$N = \frac{\omega - \omega_0}{\mathcal{B}/2} = \frac{48 - 50}{2.5} = -0.8$$

Thus,

$$\mathbf{Z}_{\text{eq}} \approx R\sqrt{1 + N^2}\underline{/\tan^{-1}\ N} = 12.81\underline{/-38.66^\circ}\ \Omega$$

The approximate current magnitude is then

$$\frac{|\mathbf{V}_s|}{|\mathbf{Z}_{\text{eq}}|} = \frac{100}{12.81} = 7.806 \text{ mA}$$

Using the exact expressions, we find that $\mathbf{I} = 7.746\underline{/39.24^\circ}$ mA and thus

$$|\mathbf{I}| = 7.746 \text{ mA}$$

Practice 16.4

A series resonant circuit has a bandwidth of 100 Hz and contains a 20 mH inductance and 2 μF capacitance. Determine

(a) f_0

(b) Q_0

(c) $\mathbf{Z}_{\text{in}}$ at resonance

(d) f_2.

Ans: 796 Hz; 7.96; $12.57 + j0\ \Omega$; 846 Hz (approx.).

16.4 Other Resonant Forms

- The degree of accuracy with which the idealized model fits the actual circuit depends on the operating frequency range, the Q of the circuit, the materials present in the physical elements, the element sizes, and many other factors.
- The network shown in Fig. 16.9a is a reasonably accurate model for the parallel combination of a physical inductor, capacitor, and resistor.
- The resistor labeled R_1 is a hypothetical resistor that is included to account for the ohmic, core, and radiation losses of the physical coil.
- The losses in the dielectric within the physical capacitor, as well as the resistance of the physical resistor in the given RLC circuit, are accounted for by the resistor labeled R_2.

A simpler equivalent may be constructed which is valid over a frequency band that is usually large enough to include all frequencies of interest. The equivalent will take the form of the network shown in Fig. 16.9b.

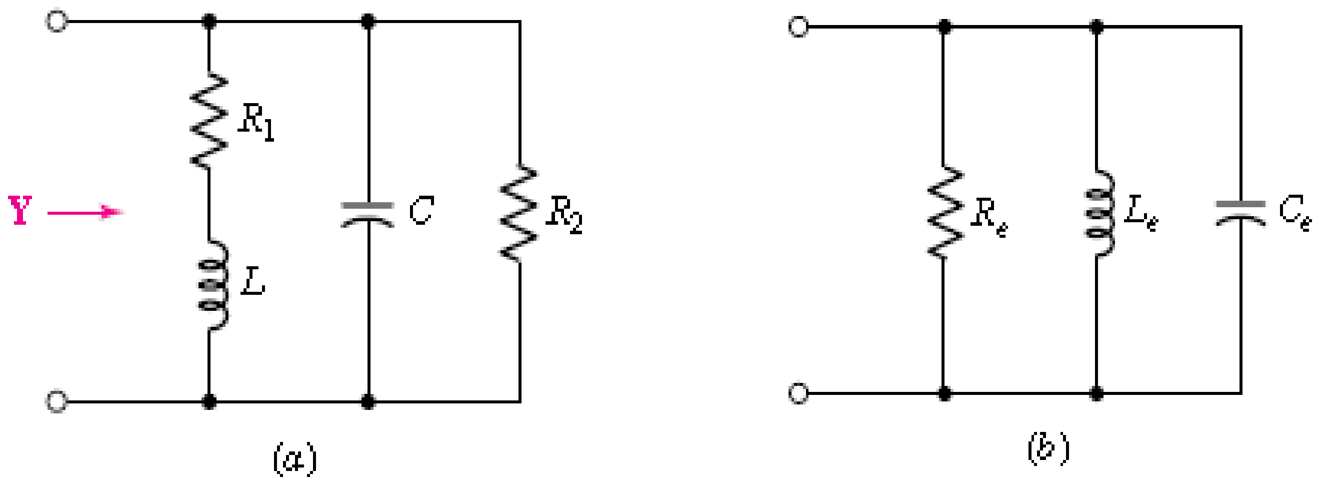


FIGURE 16.9 (*a*) A useful model of a physical network which consists of a physical inductor, capacitor, and resistor in parallel. (*b*) A network which can be equivalent to part *a* over a narrow frequency band.

In Fig. 16.9a, determine the resonant frequency by setting the imaginary part of the input admittance equal to zero:

$$\mathrm{Im}\{\mathbf{Y}(j\omega)\} = \mathrm{Im}\left\{\frac{1}{R_2} + j\omega C + \frac{1}{R_1 + j\omega L}\right\} = 0$$

or

$$\mathrm{Im}\left\{\frac{1}{R_2} + j\omega C + \frac{1}{R_1 + j\omega L}\,\frac{R_1 - j\omega L}{R_1 - j\omega L}\right\}$$

$$= \mathrm{Im}\left\{\frac{1}{R_2} + j\omega C + \frac{R_1 - j\omega L}{R_1^2 + \omega^2 L^2}\right\} = 0$$

$$C = \frac{L}{R_1^2 + \omega^2 L^2}$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R_1}{L}\right)^2}$$

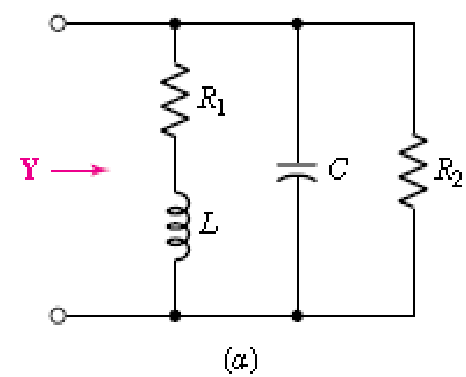


FIGURE 16.9

Example 16.4

Using the values $R_1 = 2\ \Omega$, $L = 1$ H, $C = 125$ mF, and $R_2 = 3\ \Omega$ for 16.9a, determine the resonant frequency and the impedance at resonance.

Substituting the appropriate values in Eq. [21], we find

$$\omega_0 = \sqrt{8 - 2^2} = 2 \text{ rad/s}$$

and this enables us to calculate the input admittance,

$$\mathbf{Y} = \frac{1}{3} + j2\left(\frac{1}{8}\right) + \frac{1}{2 + j(2)(1)} = \frac{1}{3} + \frac{1}{4} = 0.583 \text{ S}$$

and then the input impedance at resonance:

$$\mathbf{Z}(j2) = \frac{1}{0.583} = 1.714\ \Omega$$

At the frequency which would be the resonant frequency if R_1 were zero,

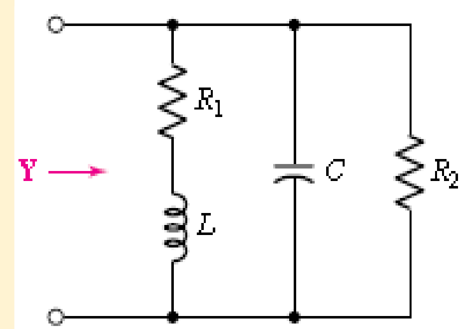
$$\frac{1}{\sqrt{LC}} = 2.83 \text{ rad/s}$$

the input impedance would be

$$\mathbf{Z}(j2.83) = 1.947\underline{/-13.26^\circ}\ \Omega$$

$$\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R_1}{L}\right)^2}$$

$$Y(s) = \frac{1}{R_2} + sC + \frac{1}{R_1 + sL}$$



(a)

FIGURE 16.9

As can be seen in Fig. 16.10, however, the frequency at which the *maximum* impedance magnitude occurs, indicated by ω_m, can be determined to be $\omega_m = 3.26$ rad/s, and the *maximum* impedance magnitude is

$$\mathbf{Z}(j3.26) = 1.980\underline{/-21.4^\circ}\ \Omega$$

The impedance magnitude at resonance and the maximum magnitude differ by about 16 percent. Although it is true that such an error may be neglected occasionally in practice, it is too large to neglect on an exam. (The later work in this section will show that the Q of the inductor-resistor combination at 2 rad/s is unity; this low value accounts for the 16 percent discrepancy.)

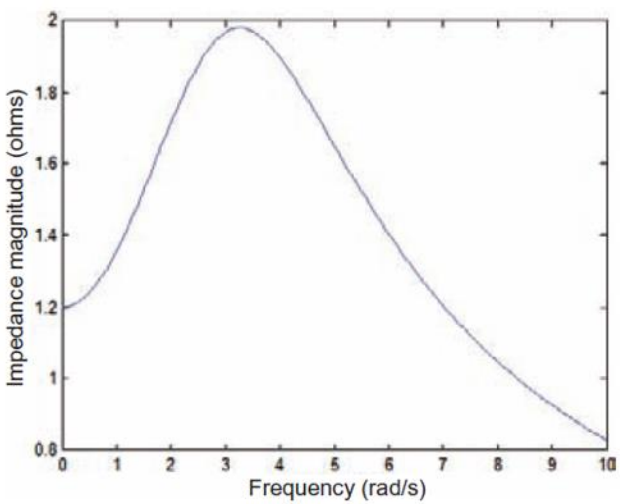


FIGURE 16.10 Plot of |Z| vs. ω, generated using the following MATLAB script:

```
EDU» omega = linspace(0,10,100);
EDU» for i = 1:100
Y(i) = 1/3 + j*omega(i)/8 + 1/(2 + j*omega(i));
Z(i) = 1/Y(i);
end
EDU» plot(omega,abs(Z));
EDU» xlabel('frequency (rad/s)');
EDU» ylabel('impedance magnitude (ohms)');
```

Practice 16.5

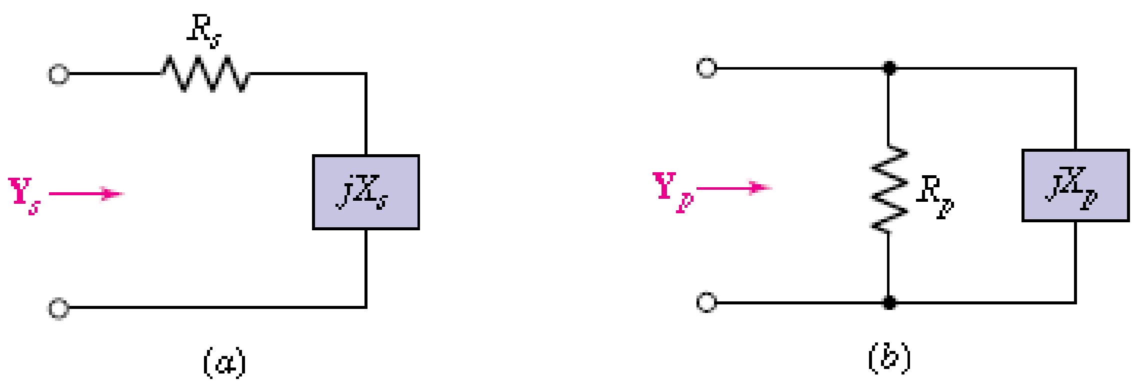
Referring to the circuit of Fig. 16.9a, let $R_1 = 1\ \mathrm{k}\Omega$ and $C = 2.533$ pF. Determine the inductance necessary to select a resonant frequency of 1 MHz. (Hint: Recall that $\omega = 2\pi f$.)

Ans: 10 mH.

Equivalent Series and Parallel Combinations

Consider the series circuit shown in Fig. 16.11a.
The Q may be evaluated at any frequency we choose. In other words, Q is a function of ω.
The Q of this series arm is $|X_s|/R_s$,
whereas the Q of the parallel network of Fig. 16.11b is $R_p/|X_p|$.

$$Q_s = \frac{|X_s|}{R_s}$$



$$Q_p = \frac{R_p}{|X_p|}$$

FIGURE 16.11 (a) A series network which consists of a resistance R_s and an inductive or capacitive reactance X_s may be transformed into (b) a parallel network such that $\mathbf{Y}_s = \mathbf{Y}_p$ at one specific frequency. The reverse transformation is equally possible.

Let us now carry out the details necessary to find values for R_p and X_p so that the parallel network of Fig. 16.11b is equivalent to the series network of Fig. 16.11a at some single specific frequency.

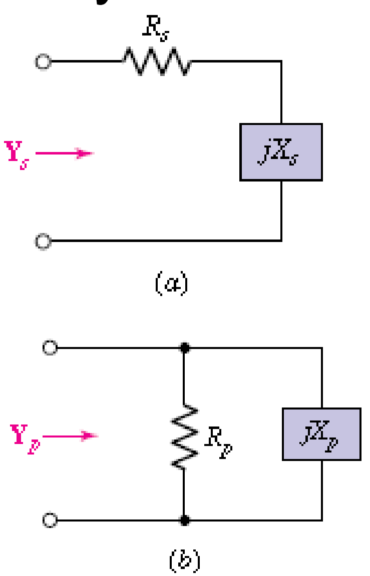


FIGURE 16.11

$$\mathbf{Y}_s = \frac{1}{R_s + jX_s} = \frac{R_s - jX_s}{R_s^2 + X_s^2}$$

$$= \mathbf{Y}_p = \frac{1}{R_p} - j\frac{1}{X_p}$$

$$R_p = \frac{R_s^2 + X_s^2}{R_s}$$

$$X_p = \frac{R_s^2 + X_s^2}{X_s}$$

$$\frac{R_p}{X_p} = \frac{X_s}{R_s}$$

It follows that the Q's of the series and parallel networks must be equal:

$$Q_p = Q_s = Q$$

$$R_p = \frac{R_s^2 + X_s^2}{R_s} = R_s + \frac{X_s^2}{R_s} = R_s\left(1 + \frac{X_s^2}{R_s^2}\right)$$

$$X_p = \frac{R_s^2 + X_s^2}{X_s} = \frac{R_s^2}{X_s} + X_s = X_s\left(\frac{R_s^2}{X_s^2} + 1\right)$$

$$\boxed{\begin{aligned} R_p &= R_s(1 + Q^2) \\ X_p &= X_s\left(1 + \frac{1}{Q^2}\right) \end{aligned}}$$

R_s and X_s may also be found if R_p and X_p are the given values; the transformation in either direction may be performed.

If $Q \geq 5$,

$$R_p \approx Q^2 R_s$$

$$X_p \approx X_s \qquad (C_p \approx C_s \qquad \text{or} \qquad L_p \approx L_s)$$

$$R_p \approx Q^2 R_s$$

Example 16.5

$$X_p \approx X_s \qquad (C_p \approx C_s \qquad \text{or} \qquad L_p \approx L_s)$$

Find the parallel equivalent of the series combination of a 100 mH inductor and a 5 Ω resistor at a frequency of 1000 rad/s. Details of the network to which this series combination is connected are unavailable.

At $\omega = 1000$ rad/s, $X_s = 1000(100 \times 10^{-3}) = 100\ \Omega$. The Q of this series combination is

$$Q = \frac{X_s}{R_s} = \frac{100}{5} = 20$$

Since the Q is sufficiently high (20 is much greater than 5), we use Eqs. [24] and [25] to obtain

$$R_p \approx Q^2 R_s = 2000\ \Omega \qquad \text{and} \qquad L_p \approx L_s = 100 \text{ mH}$$

Our assertion here is that a 100 mH inductor in series with a 5 Ω resistor provides *essentially the same* input impedance as does a 100 mH inductor in parallel with a 2000 Ω resistor at the frequency 1000 rad/s.

Example 16.5 (cont.)

To check the accuracy of the equivalence, let us evaluate the input impedance for each network at 1000 rad/s. We find

$$\mathbf{Z}_s(j1000) = 5 + j100 = 100.1\underline{/87.1^\circ}\ \Omega$$

$$\mathbf{Z}_p(j1000) = \frac{2000(j100)}{2000 + j100} = 99.9\underline{/87.1^\circ}\ \Omega$$

and conclude that the accuracy of our approximation at the transformation frequency is pretty impressive. The accuracy at 900 rad/s is also reasonably good, because

$$\mathbf{Z}_s(j900) = 90.1\underline{/86.8^\circ}\ \Omega$$

$$\mathbf{Z}_p(j900) = 89.9\underline{/87.4^\circ}\ \Omega$$

Practice 16.6

At $\omega = 1000$ rad/s, find a parallel network that is equivalent to the series combination in Fig. 16.12 a.

Ans: 8 H, 640 kΩ.

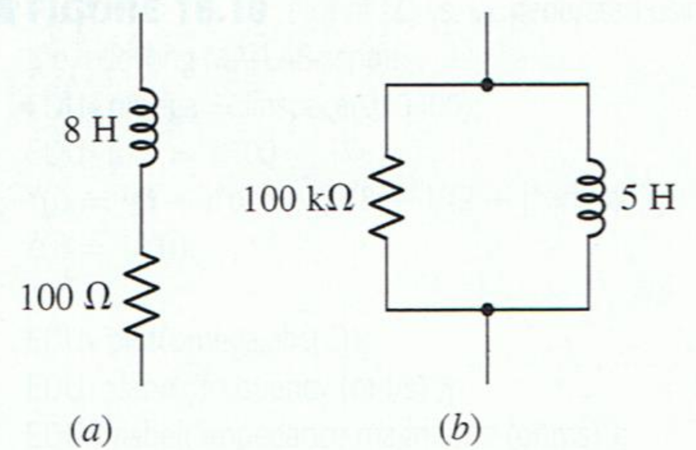


FIGURE 16.12 (*a*) A series network for which an equivalent parallel network (at $\omega = 1000$ rad/s) is needed. (*b*) A parallel network for which an equivalent series network (at $\omega = 1000$ rad/s) is needed.

Practice 16.7

Find a series equivalent for the parallel network shown in Fig. 16.12 b, assuming $\omega = 1000$ rad/s.

Ans: 5 H, 250 kΩ.

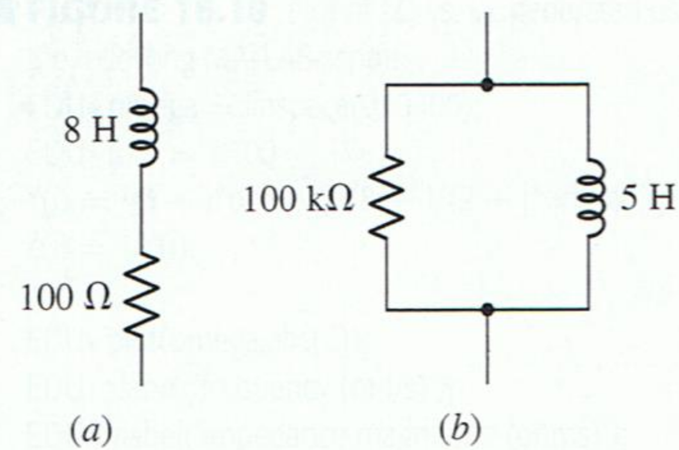


FIGURE 16.12 (*a*) A series network for which an equivalent parallel network (at $\omega = 1000$ rad/s) is needed. (*b*) A parallel network for which an equivalent series network (at $\omega = 1000$ rad/s) is needed.