Dynamic Predator-Prey Interactions with Modified Lotka-Volterra Models

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Abstract

The study of predator-prey interactions is essential for understanding ecological dynamics and is traditionally modeled by the Lotka-Volterra equations. These models have been instrumental in identifying the oscillatory behavior of ecological populations but are limited in their ability to fully represent the complexity of natural ecosystems. This research advances the classic equations by incorporating a Holling type II functional response and a logistic growth term, with the aim of developing a more realistic depiction of biological interactions. The primary focus of this research is to enhance mathematical models to better simulate the complex dynamics of predator-prey relationships. While prior research has expanded upon the Lotka-Volterra model to include additional ecological factors, there remains an incomplete understanding of how these factors affect system stability and long-term population dynamics. Our study addresses this by applying a numerical analysis using Python's `scipy.integrate.solve_ivp` function, which was selected for its effectiveness in solving non-linear differential equations. The results from our numerical simulations indicate a deviation from traditional model behaviors, with findings suggesting more varied stability conditions and dynamic patterns that are more consistent with real-world ecological data. These outcomes highlight the critical role of more detailed biological considerations in modeling efforts. The implications of this work are multifaceted, offering contributions to ecological management and biodiversity conservation strategies. Additionally, the enhanced model provides an advanced framework for academic purposes, enriching the educational resources available to those studying ecological systems and contributing to the advancement of mathematical ecology.
Keywords: predator-prey model, numerical simulation, Lotka-Volterra equations, ecological dynamics, stability analysis

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1 Introduction

Predator-prey interactions are a central component of ecological systems, characterized by complex dynamics that have engaged scientists for many years. The foundational framework in mathematical ecology is formed by the Lotka-Volterra equations, which were developed in the early 20th century [4][6]. These equations provided the first systematic approach to model the interactions between predators and their prey. Although important in identifying the oscillatory patterns of predator-prey populations, the original Lotka-Volterra model is often criticized for its simplicity and its limited capacity to accurately reflect the complexities of natural ecosystems.

This research seeks to bridge the gap between theoretical simplicity and ecological complexity by undertaking a numerical analysis of modified Lotka-Volterra dynamics. Specifically, our study aims to:

- Incorporate nonlinear terms into the Lotka-Volterra equations to more accurately model predator-prey interactions.
- Analyze the implications of these modifications for understanding the dynamics of ecological systems through numerical methods.

Understanding predator-prey interactions through mathematical models has evolved significantly since Lotka and Volterra's time. Subsequent models introduced by researchers with the Holling type II functional response and further dynamic models and bifurcation analyses[2] have expanded our understanding of these interactions. Our study acknowledges this historical progression and seeks to add to the continuum of knowledge by addressing unanswered questions regarding the impact of more complex predator-prey dynamics that have not been fully explored in previous models.

By enhancing the Lotka-Volterra model with nonlinear terms, this study introduces a more nuanced approach to examining predator-prey dynamics. This modification allows for a more detailed exploration of factors such as resource limitations, predator efficiency at varying prey population densities, and the natural regulatory mechanisms that shape ecological communities. Our use of numerical analysis, particularly through Euler's method, enables the simulation of these modified dynamics, providing new insights into the cyclic patterns and stability of predator-prey systems[3].

The implications of this research extend far beyond theoretical interest. A deeper understanding of predator-prey dynamics through our modified model has significant applications in ecological management and conservation. By providing a more accurate representation of natural interactions, our findings can inform strategies for the sustainable management of wildlife populations and the conservation of biodiversity. Additionally, this study serves as an educational resource, offering a more comprehensive framework for students and researchers to grasp the complexities of ecological systems.

2 Method

To address the dynamics between predators and their prey, we extended the classic Lotka-Volterra equations to incorporate elements that more accurately reflect biological processes. These modifications include the integration of a Holling type II functional response to account for predator satiation and the inclusion of a logistic growth term to model the carrying capacity for the prey population. We offer a detailed derivation of the modified equations and provide an analysis of the role of the nonlinear terms.

The classic Lotka-Volterra equations are represented as:

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{1}$$

$$\frac{dy}{dt} = \delta xy - \gamma y \tag{2}$$

Here, we denote x as the prey population, y as the predator cohort, α as the natural growth rate of prey, β the predation coefficient, δ the rate at which prey consumption is converted into predator offspring, and γ as the mortality rate of the predators.

To better mirror the complexities observed in nature, the equations are modified to include:

$$\frac{dx}{dt} = x(B - x) - \frac{xy}{1+x} \tag{3}$$

$$\frac{dy}{dt} = y\left(\frac{x}{1+x} - ay\right) \tag{4}$$

Here, the logistic term $x(B - x)$ is introduced, where B represents the environment's carrying capacity, to model the limitation on the exponential growth of the prey population. This term ensures that the population growth slows as it approaches B, reflecting the limiting resources available in their habitat.

Furthermore, the term $\frac{xy}{1+x}$ models the predation rate adjusted by a functional response to account for the effect of predator satiation. As the prey population increases, the predation rate approaches a maximum value due to factors like limited predator feeding capacity and the increasing challenge of capturing more prey.

Similarly, the predator equation includes the functional response term, $\frac{x}{1+x}$, which moderates the growth rate of predators in response to an increase in the prey population, accounting for the reduced efficiency in converting prey to predator biomass as prey abundance increases. The term $-ay$ represents the mortality rate of the predators, with a as the mortality rate constant.

The introduction of these elements into the model allows for a more accurate representation of phenomena such as predator satiation and the logistic growth of prey populations, commonly observed in ecological studies. The modified equations provide a more accurate description of the dynamic behavior of these populations, capturing stable states and oscillatory behavior that classical Lotka-Volterra equations cannot.

2.1 Mathematical Analysis

2.1.1 Finding the Fixed Points

We seek to find the fixed points where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. Setting these derivatives to zero, we solve for x and y.

2.1.2 Stability Analysis

The stability of each fixed point is determined by evaluating the Jacobian matrix J at each point. The Jacobian matrix is defined as:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix},$$

where $f(x, y) = x(B - x - \frac{y}{1+x})$ and $g(x, y) = y(\frac{x}{1+x} - ay)$. The eigenvalues of this matrix will help us understand the stability of the fixed points.

2.2 Simulation

For our simulations, we defined the following parameters:

The initial populations were:

- x_0, the initial prey population size, was set to 10.
- y_0, the initial predator population size, was set to 5.

These parameters were selected to illustrate the model behavior under typical ecological conditions.

The differential equations were solved numerically over a time span from $t = 0$ to $t = 50$ using 400 time points to ensure a detailed resolution of the dynamics. This discretization provides a sufficient number of data points for accurately capturing the oscillatory nature of the predator-prey population sizes over time. We implemented the numerical solution using Python's `scipy.integrate.solve_ivp` function, providing an efficient and reliable method for integrating ordinary differential equations. Initial conditions were specified as $z_0 = [x_0, y_0]$, corresponding to the initial sizes of the prey and predator populations.

3 Results

3.1 Mathematical Analysis Results

3.1.1 Fixed Point 1 (0, 0):

- Type: Unstable Node.
- Eigenvalues: [1.5, 0].
  Since one of the eigenvalues is 0 and the other is 1.5, we sketched a phase portrait to identify the stability of this fixed point. The portrait below shows that arrows move away from the equilibrium point, suggesting that it is an unstable node.

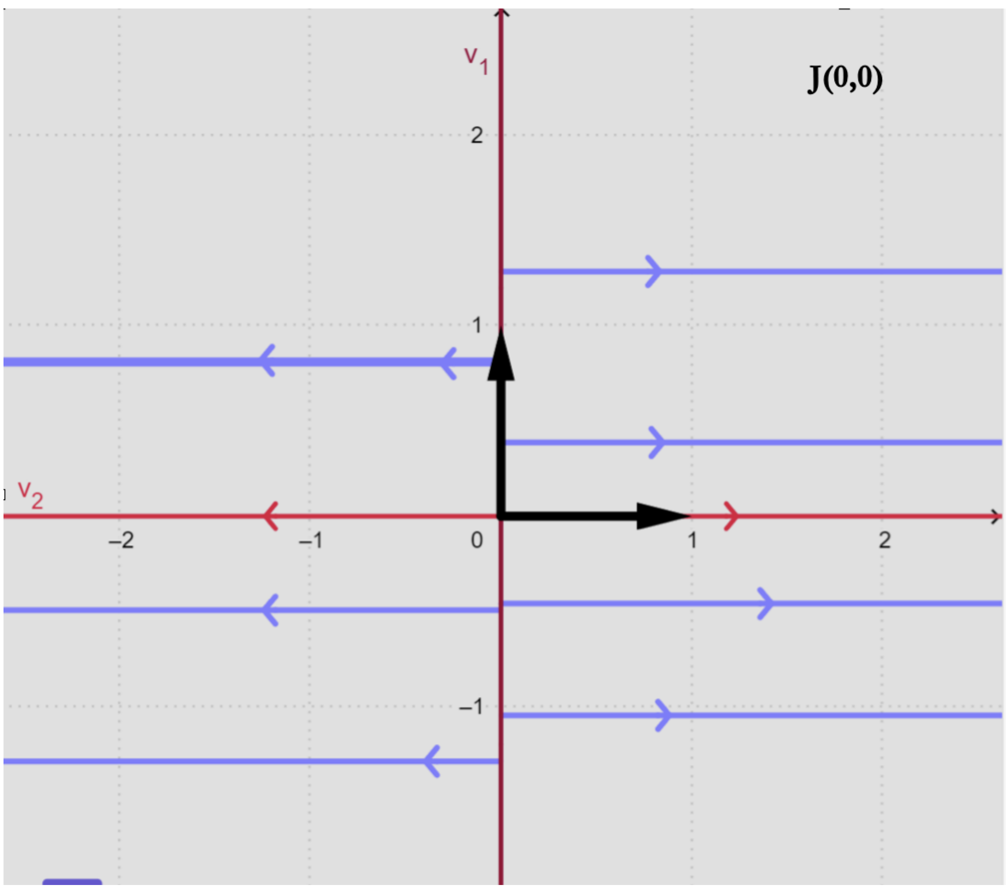


Figure 1: Phase Portrait at E.P. (0,0)

3.1.2 Fixed Point 2 (1, 1):

- Type: Stable Spiral.
- Eigenvalues: $[-0.625 \pm 0.33071891j]$.
  This fixed point is a stable spiral, indicated by complex eigenvalues with negative real parts, suggesting oscillatory behavior that dampens over time and eventually settles back to this equilibrium.

3.1.3 Fixed Point 3 (1.5, 0):

- Type: Unstable Node.
- Eigenvalues: $[3, \frac{3}{5}]$.
  This fixed point is an unstable node, indicated by real and positive eigenvalues. It suggests that the system, when perturbed from this point, will diverge from the equilibrium, making it an unstable state in the system dynamics.

3.1.4 Phase Space Analysis

The phase portrait, or phase space analysis, provides a deeper understanding of the system's stability and the interactions between the predator and prey populations. Figure 2 offers a visual representation of the

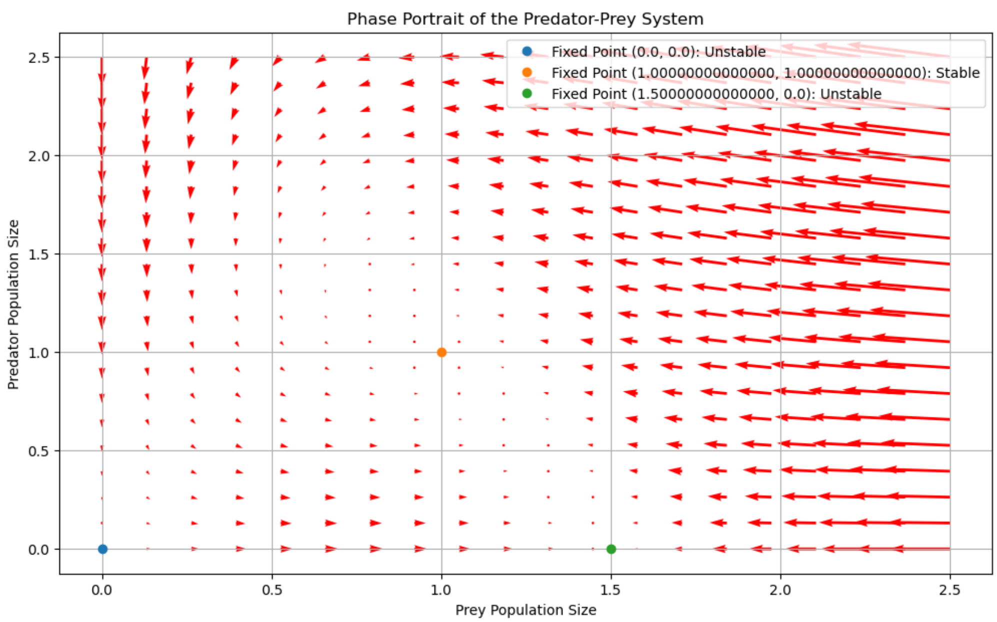


Figure 2: Predator-Prey Phase Space indicating the system's evolution.

trajectories that the predator and prey populations follow in the phase space. In real life, the origin $(0, 0)$ represents an unstable node where both species are extinct, the coexistence point $(1, 1)$ is a stable equilibrium where both species coexist with stable populations. Finally, the additional point $(1.5, 0)$ represents an unstable node in the system, suggesting that both the species will expand without limit.

3.2 Simulation results

The simulation results showcase the dynamic and interdependent nature of predator-prey relationships as described by the modified Lotka-Volterra model. The system's evolution through various phases demonstrates the complex interplay between the species, influenced significantly by the model's parameters.

3.2.1 Two-Dimensional Population Dynamics

The initial phase of the simulation is captured in a 2D plot, providing a straightforward visualization of population changes over time. In Figure 3, the prey population, depicted in blue, initially experiences exponential growth, driven by the birth rate parameter B. However, the introduction of the predator population, shown in red, leads to an immediate and steep decline in the prey population, signifying the effect of predation. As the simulation progresses, both populations reach lower densities, stabilizing as the model reaches a dynamic equilibrium. This plot highlights the initial conditions' critical role and the rapid adjustments that the populations undergo in response to each other's presence.

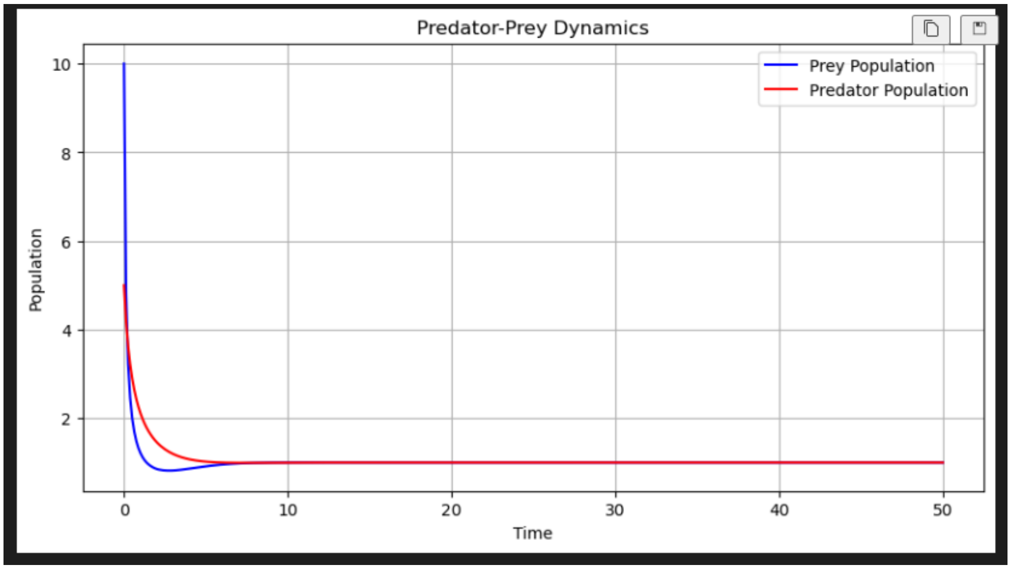


Figure 3: Predator-Prey Dynamics over Time.

3.2.2 Detailed Iterative Analysis Over Time

An iterative analysis provided snapshots of the predator-prey dynamics at key intervals. This section presents the population changes observed at steps 10, 20, 50, 60, and 90, offering a window into the immediate and subsequent responses of the system to ecological pressures.

3.2.3 Initial Impact at Step 10

At step 10, a sharp decline in the prey population is observed, indicative of an initial over-predation scenario. The corresponding predator population does not immediately reflect this decline, demonstrating a delayed response due to the time required for changes in prey abundance to affect predator numbers.

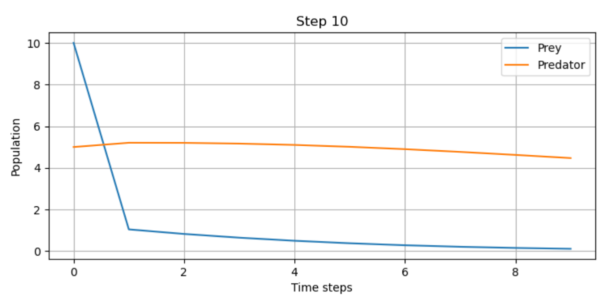


Figure 4: Prey and predator populations at step 10, showing the initial impact of predation on the prey population.

Response and Adjustment by Step 20

By step 20, the data shows the beginning of the prey population's stabilization as the immediate effects of predation subside. The predators now start to exhibit a decline as the reduced prey availability impacts their numbers.

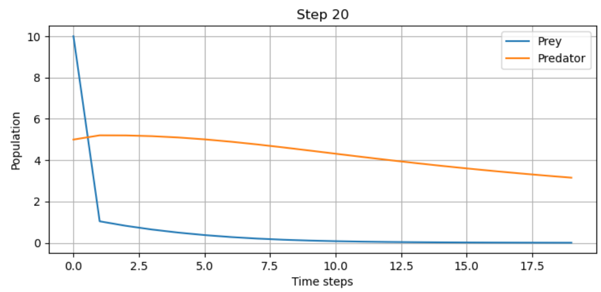
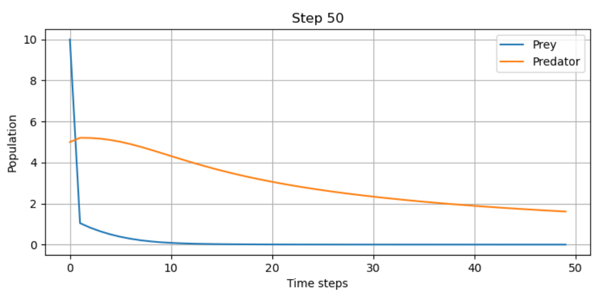


Figure 5: Population dynamics at step 20, highlighting the prey's stabilization and the predator's consequent decline.

Intermediate Dynamics at Steps 50 and 60

The intermediate steps, 50 and 60, illustrate the populations as they navigate through their respective recoveries and declines. A pattern of stabilization for the prey suggests a balance is being found, while a steady decrease in the predator population may offer an opportunity for prey recovery.



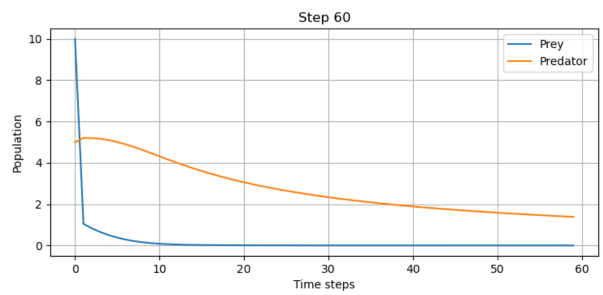


Figure 6: The progression of population dynamics at steps 50 and 60, showing the push and pull of the predator-prey relationship.

Approaching Equilibrium at Step 90

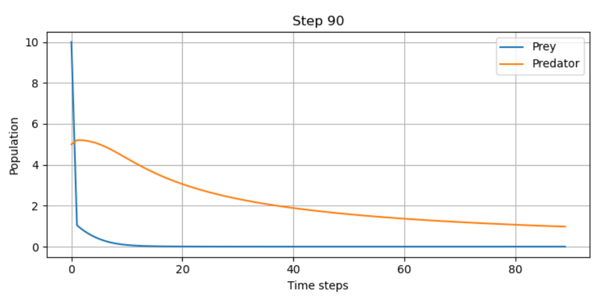


Figure 7: Indications of an approaching equilibrium in predator and prey populations at step 90.

By step 90, the predator and prey populations appear to be approaching a state of equilibrium. The prey population is low but stable, and the predator population is on a slow decline. This step may represent the system's move towards a long-term equilibrium point.

The iterative snapshots underscore the non-linear and complex nature of ecological dynamics, where timing and feedback mechanisms are crucial. The changing trajectories of populations at these selected steps provide insight into the balancing forces within natural ecosystems.

4 Discussion

In analyzing predator-prey relationships with an adapted Lotka-Volterra model, our work has added depth to the understanding of these ecological processes. Our approach, integrating a Holling type II functional response, captures the essence of predator saturation. Despite this advancement, we must consider the simplifications and limitations present in our model.

Our model is built on the assumption of consistent predator responses and steady prey growth rates[4] [6]. Such simplifications do not take into account the variable strategies that real-world populations exhibit as they respond to environmental changes and pressure from other species[1]. The model's parameters are static, potentially not reflecting the dynamic nature of real ecosystems where interactions change with environmental shifts[2]. It also leaves out random events that can significantly affect ecosystem dynamics, limiting its ability to predict the randomness seen in nature[5].

The ecosystem in our model is considered closed, ignoring important factors like immigration and emigration that influence population numbers. The use of Euler's method for numerical integration, while straightforward, introduces the risk of accumulating errors, especially when using larger time steps. These could undermine the simulation's accuracy over time.

Overcoming these limitations sets the stage for future research. Improving the model to include variations in space through reaction-diffusion equations can offer a better look at how predators and prey move and interact over an area. Adding time-dependent ecological factors would make the model more dynamic and true to the adaptability seen in actual ecosystems. These enhancements are steps toward stronger, more relevant models that can better support ecological forecasting and management.

A Appendices

A Python Code for Generating 2D Plot and Phase Portrait

The Python code provided here was used for generating the 2D plot and phase portrait in our analysis of the predator-prey model. The code utilizes the SciPy library for numerical integration and Matplotlib for visualization.

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

# Define the predator-prey model
def predator_prey_model(t, z, B, a):
    x, y = z
    dxdt = x * (B - x - y / (1 + x))
    dydt = y * (x / (1 + x) - a*y)
    return [dxdt, dydt]

# Parameters and Initial Conditions
B = 1.5  # Intrinsic growth rate of the prey
a = 0.5  # Death rate parameter of predators
x0 = 10  # Initial prey population
y0 = 5   # Initial predator population
z0 = [x0, y0]

# Time vector for solution
t = np.linspace(0, 50, 400)

# Solve the differential equations
sol = solve_ivp(predator_prey_model, [t[0], t[-1]], z0, args=(B, a), t_eval=t)

# 2D Plot of Populations vs Time
plt.figure(figsize=(10, 5))
plt.plot(sol.t, sol.y[0], label='Prey Population', color='b')
plt.plot(sol.t, sol.y[1], label='Predator Population', color='r')
plt.title('Predator-Prey Dynamics')
plt.xlabel('Time')
plt.ylabel('Population')
plt.legend()
plt.grid(True)
plt.show()

# Generate a grid for the phase portrait
x_values = np.linspace(0.1, 15, 400)
y_values = np.linspace(0.1, 15, 400)
X, Y = np.meshgrid(x_values, y_values)

# Compute growth rates on the grid
dx, dy = predator_prey_model(0, [X, Y], B, a)

# Normalize the vector field
M = np.sqrt(dx**2 + dy**2)
dx /= M
dy /= M
```

```
# Phase Portrait Plot
plt.figure(figsize=(10, 7))
plt.streamplot(X, Y, dx, dy, color=M, linewidth=1, cmap='autumn')
plt.xlabel('Prey Population')
plt.ylabel('Predator Population')
plt.title('Predator-Prey Phase Space')
plt.colorbar(label='Speed')
plt.grid(True)
plt.show()
```

B Python Code for Detailed Iterative Analysis

The code presented here was used for the detailed iterative analysis of the predator-prey system, where we observed the dynamics at every 10th step over a course of 100 steps.

```
import numpy as np
import matplotlib.pyplot as plt

# Adjust the number of iterations and interval
N = 100
interval = 10  # Observe populations every 10th step

# Initialize arrays for populations
x = np.zeros(N)
y = np.zeros(N)
x[0] = 10  # Initial prey population
y[0] = 5   # Initial predator population

def dxdt(x, y, B):
    return x * (B - x - y / (1 + x))
def dydt(x, y, a):
    return y * (x / (1 + x) - a * y)

# Iteratively calculate population changes
for t in range(N - 1):
    x[t + 1] = x[t] + dxdt(x[t], y[t], B) * dt
    y[t + 1] = y[t] + dydt(x[t], y[t], a) * dt

# Plotting populations at every 10th step
fig, axes = plt.subplots(nrows=5, ncols=2, figsize=(15, 20))
fig.suptitle('Prey and Predator Populations at Every 10th Step', fontsize=16)
for i, ax in enumerate(axes.flatten(), start=1):
    step = i * interval
    if step >= N:
        break
    ax.plot(x[:step], label='Prey')
    ax.plot(y[:step], label='Predator')
    ax.set_title(f'Step {step}')
    ax.set_xlabel('Time steps')
    ax.set_ylabel('Population')
    ax.legend()
    ax.grid(True)
plt.tight_layout(rect=[0, 0.03, 1, 0.95])
plt.show()
```

References

[1] Peter A. Abrams. Adaptive changes in prey vulnerability shape the response of predator populations to mortality. *Journal of Theoretical Biology*, 261(2):294–304, Nov 2009.

[2] C. S. Holling. The components of predation as revealed by a study of small-mammal predation of the european pine sawfly. *The Canadian Entomologist*, 91(5):293–320, 1959.

[3] R. Lavanya, S. Vinoth, K. Sathiyanathan, Zeric Njitacke Tabekoueng, P. Hammachukiattikul, and R. Vadivel. Dynamical behavior of a delayed holling type-ii predator-prey model with predator cannibalism. *Journal of Mathematics*, 2022:15, 2022.

[4] A. J. Lotka. Elements of physical biology. *Nature*, 116:461, 1925.

[5] Peter Turchin. *Complex Population Dynamics: A Theoretical/Empirical Synthesis*. Princeton University Press, 2003. Course Book.

[6] V. Volterra. Fluctuations in the abundance of a species considered mathematically. *Nature*, 118:558–560, 1926.