Q.24 Let $X_1, X_2, ..., X_n$ be a random sample of size n from a population having uniform distribution over the interval $\left(\frac{1}{3}, \theta\right)$, where $\theta > \frac{1}{3}$ is an unknown parameter. If $Y = \max X_1, X_2, ..., X_n$, then which one of the following statements is true?

1. $\left(\frac{n+1}{n}\right)\left(Y - \frac{1}{3}\right) + \frac{1}{3}$ is an unbiased estimator of θ
2. $\left(\frac{n}{n+1}\right)\left(Y - \frac{1}{3}\right) + \frac{1}{3}$ is an unbiased estimator of θ
3. $\left(\frac{n+1}{n}\right)\left(Y + \frac{1}{3}\right) - \frac{1}{3}$ is an unbiased estimator of θ
4. Y is an unbiased estimator of θ

Solution:

Any one of the above expressions is an unbiased estimator of θ if

$$E(\text{estimator}) = \theta$$

We will first calculate expected value of Y

The PDF of X_i can be expressed as

$$f_{X_i}(x) = \frac{1}{\theta - \frac{1}{3}} \; , \; \frac{1}{3} < \theta < 3 \tag{1}$$

$$\begin{aligned} F_{X_i}(y) &= \int_{\frac{1}{3}}^{y} f_{X_i}(x)dx && (2)\\ &= \int_{\frac{1}{3}}^{y} \frac{1}{\theta - \frac{1}{3}} dx && (3)\\ &= \frac{y - \frac{1}{3}}{\theta - \frac{1}{3}} && (4) \end{aligned}$$

$$\begin{aligned} F_X(y) &= \Pr(X \leq y) && (5)\\ &= \Pr(X_1 \leq y, X_2 \leq y, ..., X_n \leq y) && (6)\\ &= \Pr(X_1 \leq y)\Pr(X_2 \leq y) ... \Pr(X_n \leq y) && (7)\\ &= \left(\frac{y - \frac{1}{3}}{\theta - \frac{1}{3}}\right)^n && (8) \end{aligned}$$

$$\begin{aligned} f_X(y) &= \frac{d}{dy} F_X(y) && (9)\\ &= \frac{n}{\left(\theta - \frac{1}{3}\right)^n} \left(y - \frac{1}{3}\right)^n && (10) \end{aligned}$$

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\begin{aligned}
E(Y) &= \int_{\frac{1}{3}}^{\theta} y f_X(y) dy && (11)\\
&= \int_{\frac{1}{3}}^{\theta} y \frac{n}{\left(\theta - \frac{1}{3}\right)^n} \left(y - \frac{1}{3}\right)^n dy && (12)\\
&= \frac{n}{\left(\theta - \frac{1}{3}\right)^n} \int_{\frac{1}{3}}^{\theta} y \left(y - \frac{1}{3}\right)^{n-1} dy && (13)\\
\text{Let } y - \frac{1}{3} &= t && (14)\\
\implies y = t + \frac{1}{3} \implies dy &= dt && (15)\\
\text{Therefore,} & && (16)\\
E(Y) &= \frac{n}{\left(\theta - \frac{1}{3}\right)^n} \int_{0}^{\theta - \frac{1}{3}} \left(t^n + \frac{t^{n-1}}{3}\right) dt && (17)\\
&= \frac{n}{\left(\theta - \frac{1}{3}\right)^n} \left(\frac{\left(\theta - \frac{1}{3}\right)^{n+1}}{n+1} + \frac{\left(\theta - \frac{1}{3}\right)^{n}}{3n}\right) && (18)\\
&= \frac{n}{n+1} \left(\theta - \frac{1}{3}\right) + \frac{1}{3} && (19)\\
&= \frac{3n\theta + 1}{3(n+1)} \neq \theta && (20)
\end{aligned}
$$

Therefore, fourth option is incorrect.

1.

$$
\begin{aligned}
E(\left(\tfrac{n+1}{n}\right)\left(Y - \tfrac{1}{3}\right) + \frac{1}{3}) &= \frac{n+1}{n} E(Y) - \frac{n+1}{3n} + \frac{1}{3} && (21)\\
&= \frac{3n\theta + 1 - (n+1) + n}{3n} && (22)\\
&= \theta && (23)
\end{aligned}
$$

It is the unbiased estimator of θ.

2. Similarly,

$$
\begin{aligned}
E(\left(\tfrac{n}{n+1}\right)\left(Y - \tfrac{1}{3}\right) + \frac{1}{3}) &= \frac{n}{n+1} E(Y) - \frac{n}{3(n+1)} + \frac{1}{3} && (24)\\
&= \frac{n(3n\theta + 1) - n(n+1) + (n+1)^2}{3(n+1)^2} && (25)\\
&= \frac{3n^2 + n}{3(n+1)^2} \neq \theta && (26)
\end{aligned}
$$

It is not an unbiased estimator of θ.

3. In the same way,

$$E(\left(\tfrac{n+1}{n}\right)\left(Y + \tfrac{1}{3}\right) - \frac{1}{3}) = \frac{n+1}{n}E(Y) + \frac{n+1}{3n} - \frac{1}{3} \tag{27}$$

$$= \frac{3n\theta + 1 + n + 1 - n}{3n} \tag{28}$$

$$= \frac{3n\theta + 2}{3n} \neq \theta \tag{29}$$

It is not an unbiased estimator of θ.

Hence, the first option is the correct option.