Automatic Control Systems

Homework 1 Solution

Due date: 9/25/2017

* 總共五題，每題 20 分，滿分 100 分

1. **2-4.** Write the force equations of the linear translational systems shown in Fig. 2P-4.

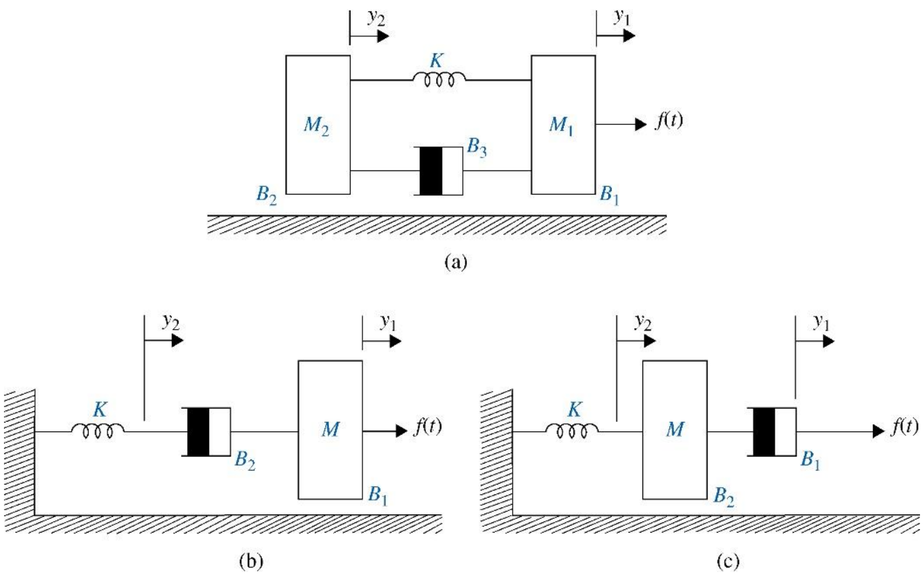
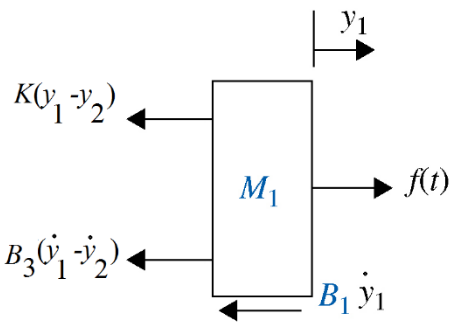
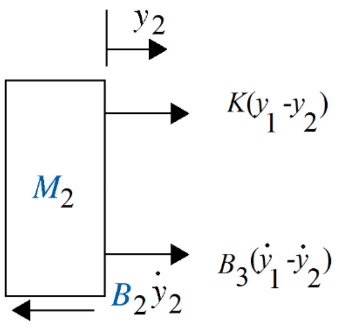


Figure 2P-4

(a) Force equations:



$$M_1\ddot{y}_1 = -B_1\dot{y}_1 - B_3\left(\dot{y}_1 - \dot{y}_2\right) - K\left(y_1 - y_2\right) + f\left(t\right)$$



$$M_2\ddot{y}_2 = -B_2\dot{y}_2 + B_3\left(\dot{y}_1 - \dot{y}_2\right) + K\left(y_1 - y_2\right)$$

$$f(t) = M_1\frac{d^2y_1}{dt^2} + B_1\frac{dy_1}{dt} + B_3\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) + K\left(y_1 - y_2\right)$$

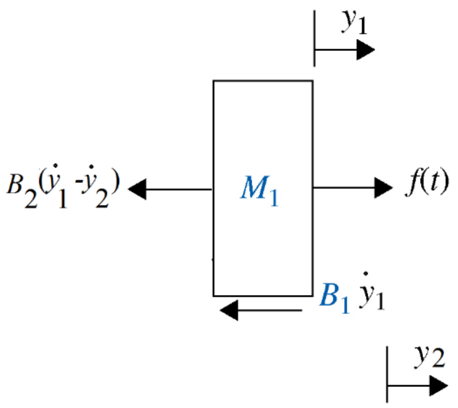
$$B_3\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) + K\left(y_1 - y_2\right) + M_2\frac{d^2y_2}{dt^2} + B_2\frac{dy_2}{dt}$$

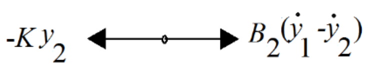
Rearrange the equations as follows:

$$\frac{d^2y_1}{dt^2} = -\frac{\left(B_1 + B_3\right)}{M_1}\frac{dy_1}{dt} + \frac{B_3}{M_1}\frac{dy_2}{dt} - \frac{K}{M_1}\left(y_1 - y_2\right) + \frac{f}{M_1}$$

$$\frac{d^2y_2}{dt^2} = \frac{B_3}{M_2}\frac{dy_1}{dt} - \frac{\left(B_2 + B_3\right)}{M_2}\frac{dy_2}{dt} + \frac{K}{M_2}\left(y_1 - y_2\right)$$

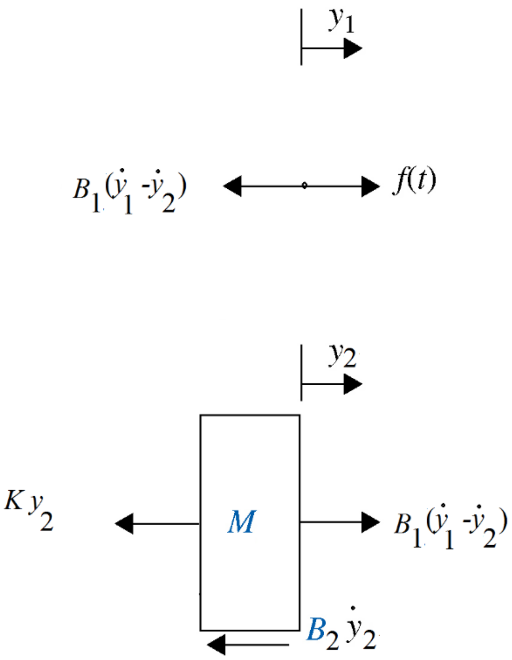
(b) Force equations:





$$\frac{d^2y_1}{dt^2} = -\frac{\left(B_1 + B_2\right)}{M}\frac{dy_1}{dt} + \frac{B_2}{M}\frac{dy_2}{dt} + \frac{1}{M}f \qquad \frac{dy_2}{dt} = \frac{dy_1}{dt} - \frac{K}{B_2}y_2$$

(c) Force equations:



$$\frac{dy_1}{dt} = \frac{dy_2}{dt} + \frac{1}{B_1} f \qquad \frac{d^2 y_2}{dt^2} = -\frac{(B_1 + B_2)}{M}\frac{dy_2}{dt} + \frac{B_1}{M}\frac{dy_1}{dt} - \frac{K}{M} y_2$$

2. **2-9.** Fig. 2P-9 shows an inverted pendulum on a cart.

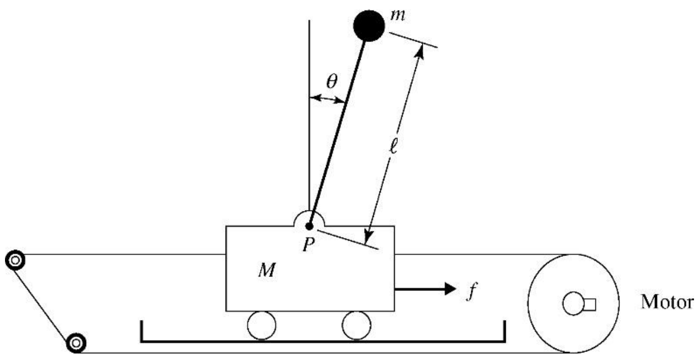


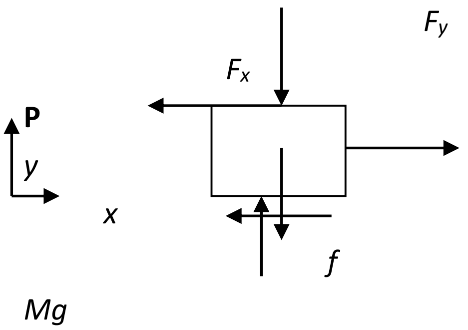
Figure 2P-9

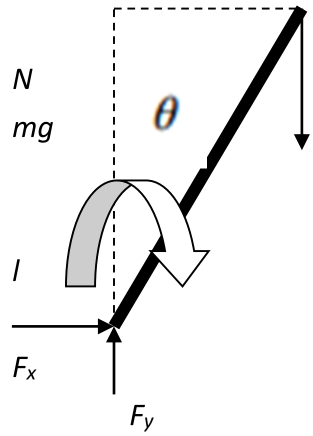
If the mass of the cart is represented by *M* and the force *f* is applied to hold the bar at the desired position, then

(a) Draw the free-body diagram.

(b) Determine the dynamic equation of the motion.

a)





b) **re of grav**

Then $x_g = x + l\sin\theta x_g = x + l\sin\theta x_g = x + l\sin\theta$ **and** $y_g = l\cos\theta$

From force balance, we have:

$$\begin{cases} F - f - F_x = M\ddot{x} \\ N - Mg - F_y = 0 \end{cases}$$

$$\begin{cases} F_x = m\ddot{x}_g \\ F_y - mg = m\ddot{y}_g \end{cases}$$

Combining the equations we have:

$$M\ddot{x} + m\ddot{x}_g = F - f$$

$$N - (M + m)g = m\ddot{y}_g$$

Assuming viscous damping for friction (rough assumption) $f = B\dot{x}$ **:**

$$M\ddot{x} + m\ddot{x}_g + B\dot{x} = F$$

Note: $\ddot{x}_g = \ddot{x} - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta$ **and**

$$\ddot{y}_g = -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta$$

Hence,

$$M\ddot{x} + m\ddot{x} + B\dot{x} + ml\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = F \qquad \textbf{(1)}$$

For the pendulum, if we take a moment about the point mass *mg*, we have:

$$ml^2\ddot{\theta} = -F_x l\cos\theta + F_y l\sin\theta$$

Where using:

$$F_x = m\ddot{x}_g = m\left(\ddot{x} - l\dot{\theta}^2 sin\theta + l\ddot{\theta}\cos\theta\right)$$

$$F_y - mg = m\ddot{y}_g = -m\left(l\dot{\theta}^2\cos\theta + l\ddot{\theta}\sin\theta\right)$$

We get:

$$ml^2\ddot{\theta} = -m\left(\ddot{x} - l\dot{\theta}^2 sin\theta + l\ddot{\theta}\cos\theta\right) l\cos\theta - m\left(l\dot{\theta}^2\cos\theta + l\ddot{\theta}\sin\theta\right)_y l\sin\theta \quad \textbf{(2)}$$

Simplifying equations (1) and (2) we arrive at the two equations of the system:

$$\Rightarrow \begin{array}{l} (M+m)\ddot{x} + B\dot{x} = F + ml\left(\dot{\theta}^2 \;\; \text{in}\,\theta - \ddot{\theta}\cos\theta\right) \\ ml^2\ddot{\theta} = mgl\sin\theta - ml\ddot{x}\cos\theta \end{array}$$

For small angles, linearized model of the system becomes

$$\begin{array}{l} (M+m)\ddot{x} + ml\ddot{\theta} + B\dot{x} = F \\ l\ddot{\theta} + \ddot{x} = g\theta \end{array}$$

3. 2-24. Fig. 2P-24 represents a vibration absorption system.

Assuming the harmonic force $F(t) = Asin(\omega t)$ is the disturbance applied to the mass *M*, derive the equations of motion of the system.

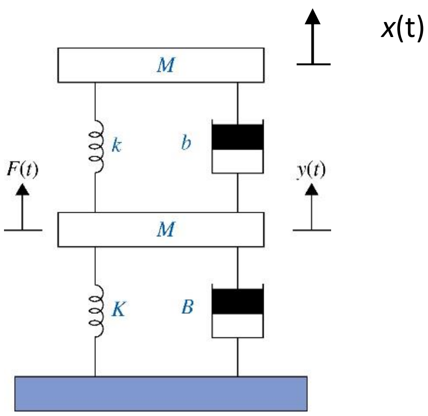


Figure 2P-24

Assume motion is such that both springs are in tension. Summation of vertical forces gives:

$$\begin{cases} M\ddot{y} + (B+b)\dot{y} - b\dot{x} + (K+k)y - kx = F \\ \qquad m\ddot{x} - b\dot{y} + b\dot{x} - ky - kx = 0 \end{cases}$$

Where $F(t) = Asin(\omega t)$

4. 2-26. Consider the electrical circuits shown in Figs. 4P-26(a) and (b).

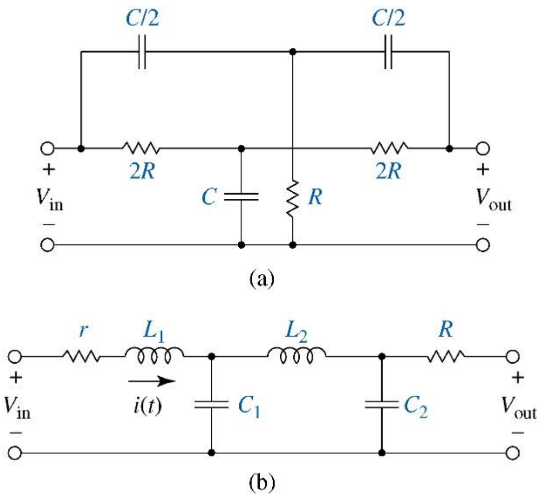


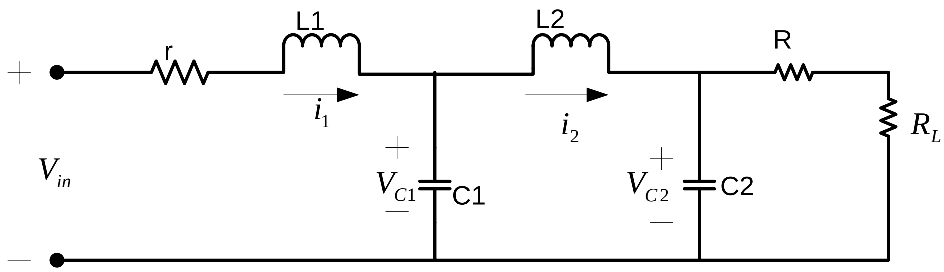
Figure 2P-26

(a) For each circuit find the dynamic equations.

According to the circuit:

$$\begin{cases} \dfrac{v_{in}-v_1}{2R}+C\dfrac{d}{dt}v_1+\dfrac{v_{out}-v_1}{2R}=0 \\ \dfrac{C}{2}\dfrac{d}{dt}(v_{in}-v_2)-\dfrac{v_2}{R}+\dfrac{C}{2}\dfrac{d}{dt}(v_{out}-v_2)=0 \\ \dfrac{C}{2}\dfrac{d}{dt}(v_2-v_{out})+\dfrac{v_1-v_{out}}{2R}=0 \end{cases}$$

Measuring V_{out} requires a load resistor, which means:



Then we have:

$$\begin{cases} L_1\dfrac{d}{dt}i_1 = v_{in} - ri_1 - v_{C1} \\ C_1\dfrac{d}{dt}v_{C1} = i_1 - i_2 \\ L_2\dfrac{d}{dt}i_2 = v_{C1} - v_{C2} \\ C_2\dfrac{d}{dt}v_{C2} = i_2 - \dfrac{v_{C2}}{R+R_L} \end{cases}$$

When

$$v_{out} = \frac{R_L}{R+R_L}v_{C2}$$

If R_L>>R, then $v_{out} = v_{C2}$

5. **2-28.** Fig. 2P-28 shows a circuit made up of two *RC* circuits. Find the dynamic equations of the system.

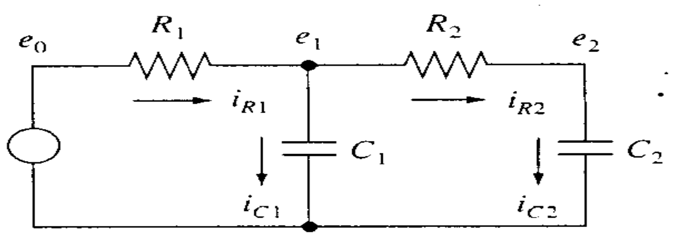


Figure 2P-28

Individual currents are:

$$i_{R1} = \frac{e_0 - e_1}{R_1}$$

$i_{C1} = C_1\dot{e}_1$ with $e_1(0)$

$$i_{R2} = \frac{e_1 - e_2}{R_2}$$

$i_{C2} = C_2\dot{e}_2$ with $e_2(0)$

The node equations are

$i_{R1} = i_{C1} + i_{R2}$

$i_{R2} = I_{C2}$

Substitute the current equations into the node equations and rearrange we get:

$R_1C_1\dot{e}_1 + (1 + R_1/R_2)e_1 = (R_1/R_2)e_2 + e_0$

$R_2C_2\dot{e}_2 + e_2 = e_1$

Since we are interested in e_2 as a function of e_0, we can substitute the second equation for e_1 into the first and rearrange to obtain

$$R_1C_1R_2C_2\ddot{e}_2+\left(R_1C_1+R_1C_2+R_2C_2\right)\dot{e}_2+\ e_2=e_0$$