# Automatic Control Systems
# Lecture 9
# Root Locus

Tesheng Hsiao, Associate Professor

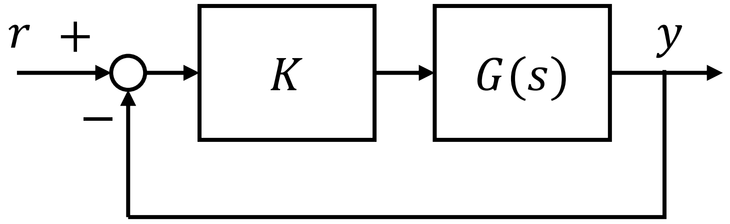
Department of Electrical and Control Engineering

National Chiao Tung University, Taiwan

# Outline

- Introduction
- Properties of Root Locus

# Closed-Loop Poles as a Function of the Controller's Gain




$K > 0 \qquad G(s) = \dfrac{Q(s)}{P(s)}$

where $P(s)$ and $Q(s)$ are $n^{\text{th}}$-order and $m^{\text{th}}$-order polynomials, respectively.

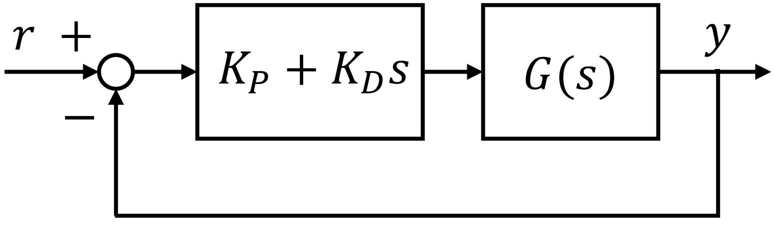
- Closed-loop Transfer Function: $M(s) = \dfrac{Y(s)}{R(s)} = \dfrac{KG(s)}{1 + KG(s)}$

$$\boxed{p \text{ is a pole of } M(s) \Leftrightarrow 1 + KG(p) = 0 \Leftrightarrow P(p) + KQ(p) = 0}$$

- For each value of $K$, we can find $n$ closed-loop poles by solving the characteristic equation $P(s) + KQ(s) = 0$.
- The loci of these $n$ closed-loop poles on the complex plane as $K$ varies from 0 to $\infty$ are called **root locus** of the closed-loop system.

# Applying Root Locus Analysis to Various Systems

- PD Controller




Let $C(s) = K_P + K_D s = K_P(1 + T_z s)$

where $T_z = \frac{K_D}{K_P}$ is chosen in advance.

Then the characteristic equation is

$$1 + K_P(1 + T_z s)G(s) = 1 + K_P G_1(s)$$

where $G_1(s) = (1 + T_z s)G(s)$

- Consider the following characteristic equation

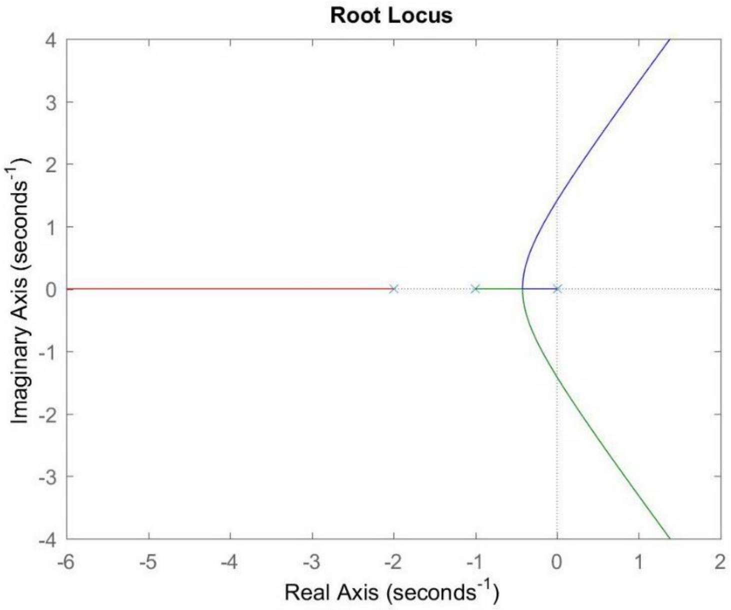
$$s(s+1)(s+2) + s^2 + (3 + 2K)s + 5 = 0$$

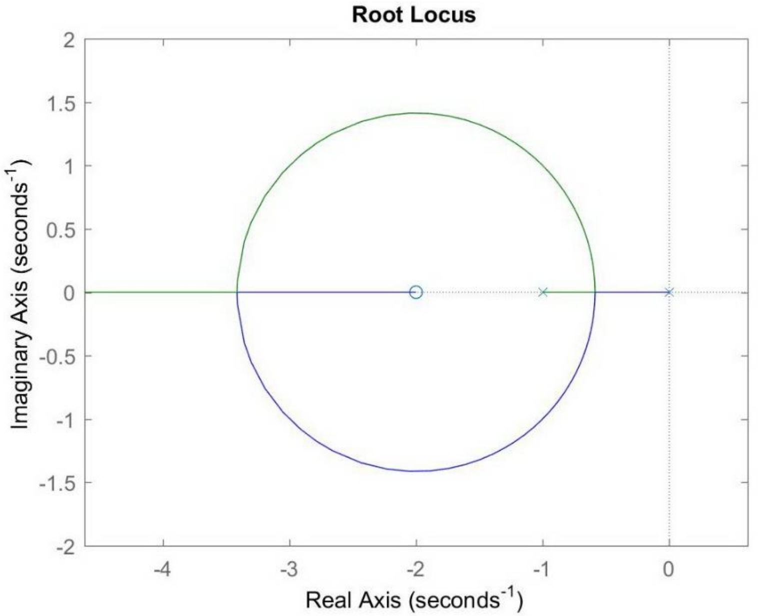
$$s(s+1)(s+2) + s^2 + 3s + 5 + 2Ks = 0$$

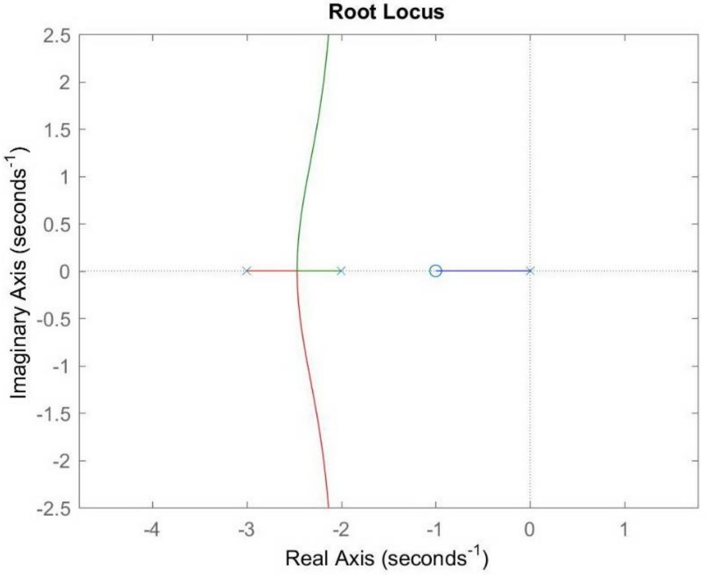
$$1 + \frac{2Ks}{s(s+1)(s+2) + s^2 + 3s + 5} = 0$$

$$1 + KG(s) = 0, \qquad \text{where } G(s) = \frac{2s}{s^3 + 4s^2 + 5s + 5}$$

# Example of Root Locus




Upper Left: $G(s) = \dfrac{1}{s(s+1)(s+2)}$

Upper Right: $G(s) = \dfrac{s+2}{s(s+1)}$

Lower Left: $G(s) = \dfrac{s+1}{s(s+2)(s+3)}$

Matlab Instruction: rlocus

# Graphical Interpretation (I)

$$1 + KG(s) = 0 \quad \Leftrightarrow \quad G(s) = -\frac{1}{K}$$
$$K > 0$$

$$\Leftrightarrow \quad \left.\begin{aligned} |G(s)| &= \frac{1}{K} \\ \angle G(s) &= (2i+1)\pi, \qquad i \in \mathbb{Z} \end{aligned}\right\} \quad (1)$$

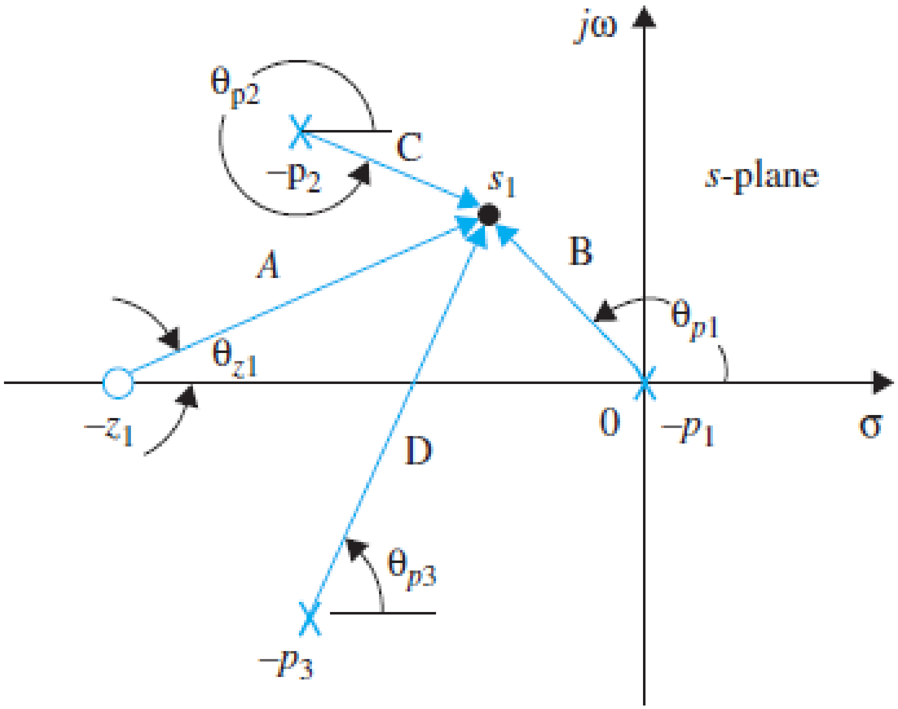
Let $G(s) = \dfrac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$ Then (1) becomes

$$\begin{cases} |G(s)| = \dfrac{\prod_{j=1}^{m} |s+z_j|}{\prod_{k=1}^{n} |s+p_k|} = \dfrac{1}{K} & (2) \\ \angle G(s) = \sum_{j=1}^{m} \angle(s+z_j) - \sum_{k=1}^{n} \angle(s+p_k) = (2i+1)\pi & (3) \end{cases}$$

- $s$ is a closed-loop pole if and only if the difference between the sums of the angles of the vectors drawn from the zeros and those from the poles of $G(s)$ to $s$ is an odd multiple of 180 degree (from (3)).
- If $s$ is a closed-loop pole, then the corresponding $K$ is determined from (2).

# Graphical Interpretation (II)

Example: $G(s) = \dfrac{K(s+z_1)}{s(s+p_2)(s+p_3)}$ $\quad K > 0$




- If any point $s_1$ on the complex plane satisfies

$$\begin{aligned} &\angle(s_1+z_1) - \angle s_1 - \angle(s+p_2) - \angle(s+p_3) \\ &= \theta_{z1} - \theta_{p1} - \theta_{p2} - \theta_{p3} \\ &= (2i+1) \times 180° \end{aligned}$$

then $s_1$ is a closed-loop pole.

- If $s_1$ is a closed-loop pole, then

$$K = \frac{|s_1||s_1+p_2||s_1+p_3|}{|s_1+z_1|} = \frac{BCD}{A}$$

# Properties of the Root Locus

- Number of branches on the Root Loci
- Symmetry of the Root Loci
- Closed-loop poles for $K = 0$ and $K = \infty$
- Root Locus on the Real Axis
- Asymptotes of the Root Locus as $|s| \to \infty$
- Angles of Departure and Arrival
- Breakaway Points
- Intersection of the Root Locus with the Imaginary Axis

# Number of Branches and Symmetry

- Number of Branches
  - A branch of the root loci corresponds to the trajectory of a closed-loop pole w.r.t. $K$ varying from 0 to $\infty$.
  - The number of branches of the root loci is equal to the order of the polynomial (i.e. the number of closed-loop poles).
- Symmetry
  - The root loci is symmetric about the real axis of the $s$-plane.

# Closed-Loop Poles for $K = 0$ and $K = \infty$

$$G(s) = \frac{Q(s)}{P(s)} \quad \Rightarrow \quad 1 + KG(s) = 0 \Leftrightarrow P(s) + KQ(s) = 0 \Leftrightarrow \frac{1}{K}P(s) + Q(s) = 0$$

- $K = 0$

$$P(s) + KQ(s) = P(s) = 0$$

$\Rightarrow$ The closed-loop poles are identical to the **open-loop poles**.

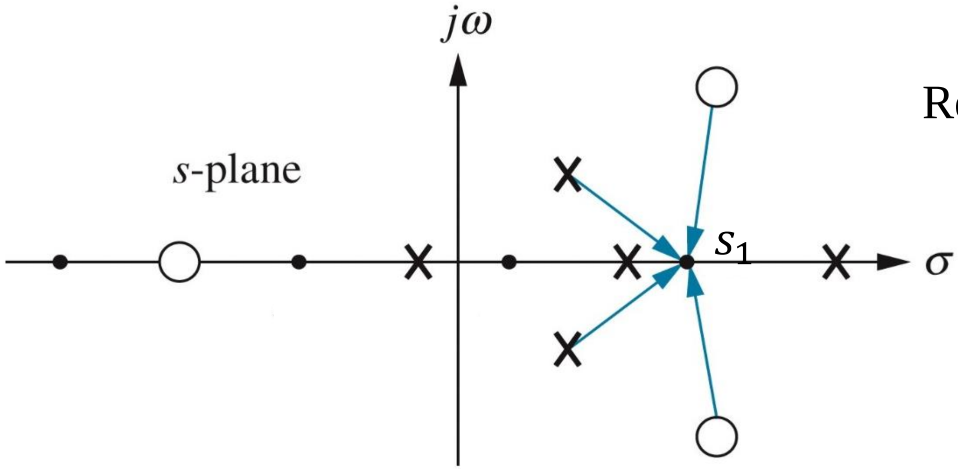
- $K = \infty$
  - Case I: $$\frac{1}{K}P(s) + Q(s) = Q(s) = 0$$

    $\Rightarrow$ The closed-loop poles are identical to the **open-loop zeros**.
  - Case II: If $G(s)$ is strictly proper, then $G(\infty) = 0 = -\dfrac{1}{K}$

    $\Rightarrow$ The closed-loop poles approach $s = \infty$ (zeros of $G(s)$ at $s = \infty$).

- The root locus starts from the open-loop poles and ends at the open-loop zeros (including the zero at $s = \infty$) as $K$ varies from 0 to $\infty$.

# Root Loci on the Real Axis



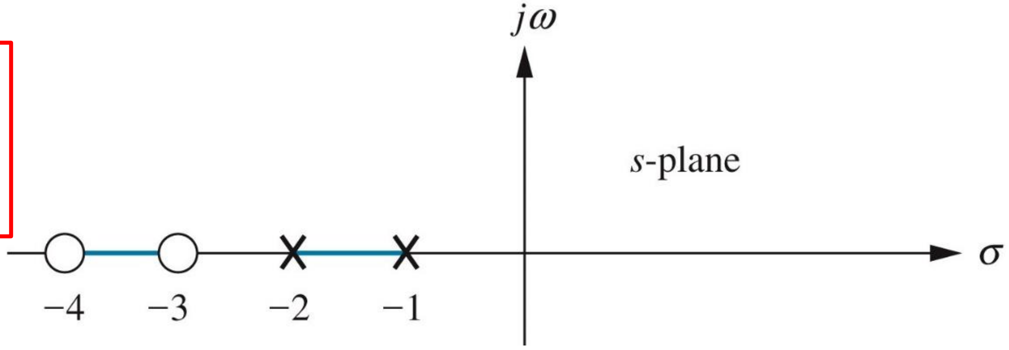

Recall that if $s_1$ is a point on the root locus, then

$$\angle G(s_1) = \sum_{j=1}^{m} \angle(s_1 + z_j) - \sum_{k=1}^{n} \angle(s_1 + p_k) = (2i+1)\pi$$

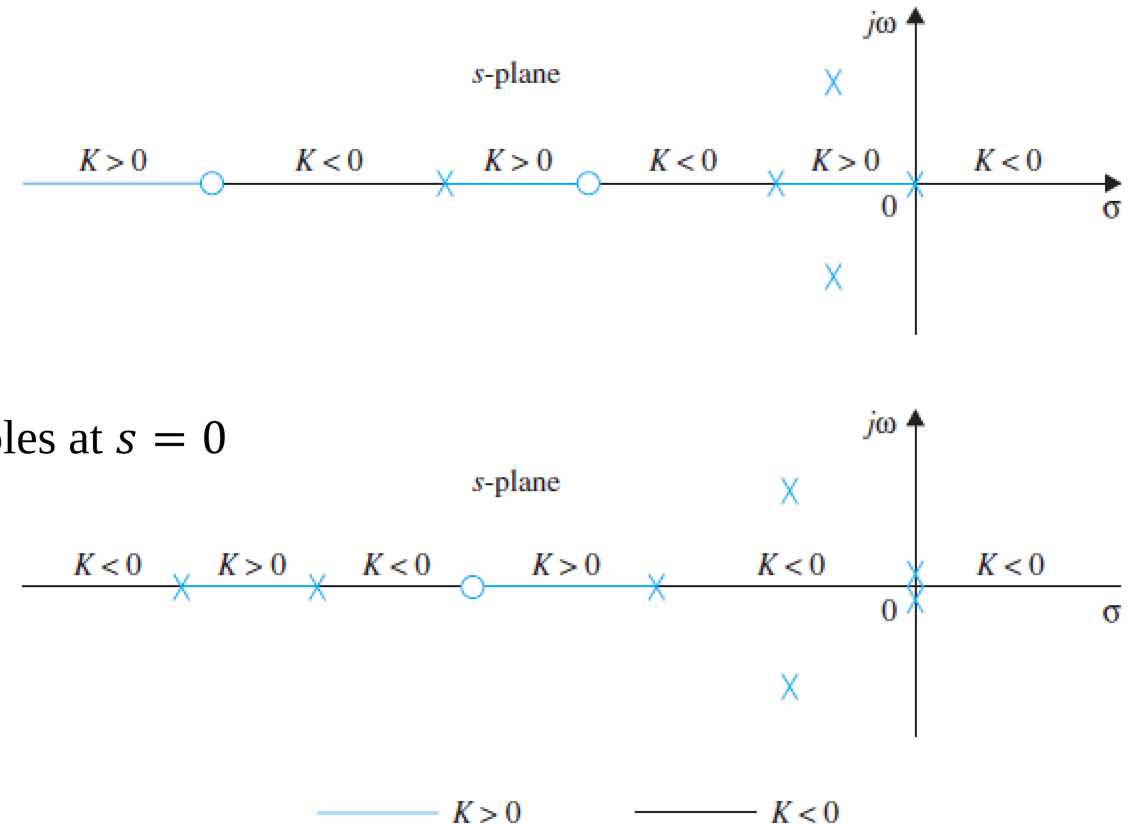
For any $s_1$ on the **real-axis**

- The angles from any **complex conjugate poles and zeros** of $G(s)$ to $s_1$ add up to zero.
- The angle contributed by any **real poles and zeros** of $G(s)$ to the **left** of $s_1$ is zero.
- Each real pole of $G(s)$ to the **right** of $s_1$ contributes $-180$ degrees.
- Each real zero of $G(s)$ to the **right** of $s_1$ contributes 180 degrees.

For $s_1$ to be a point on the root locus, there must be an **odd** number of poles and zeros of $G(s)$ to the **right** of $s_1$.



# Example 1




- Double poles at $s = 0$

# Asymptotes of the Root Loci (I)

$$G(s) = \frac{Q(s)}{P(s)} \qquad \begin{array}{l} Q(s) = b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0 \\ P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \end{array} \qquad b_m \neq 0$$

- Suppose $r = n - m > 0$. Then $G(s)$ has $r$ zeros at $s = \infty$, and there are $r$ branches of the root loci that approach $s = \infty$.

$$KG(s) = Kb_m \frac{Q(s)/b_m}{P(s)} = K' \frac{s^m + b'_{m-1}s^{m-1} + \cdots + b'_1 s + b'_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

where $K' = Kb_m$, $b'_i = \frac{b_i}{b_m}$, $i = 0,1,\cdots,m-1$.

Let $-p_i$ be the poles of $G(s)$, $i = 1,2,\cdots,n$

$$P(s) = (s+p_1)(s+p_2)\cdots(s+p_n) = s^n + \left(\sum_{i=1}^{n} p_i\right)s + \cdots$$

$$\Rightarrow \quad a_{n-1} = \sum_{i=1}^{n} p_i = -(\text{sum of poles of } G(s))$$

Similarly, $b'_{m-1} = -(\text{sum of finite zeros of } G(s))$

# Asymptotes of the Root Loci (II)

$$KG(s) = K'\frac{s^m + b'_{m-1}s^{m-1} + \cdots + b'_1 s + b'_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{K'}{\dfrac{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0}{s^m + b'_{m-1}s^{m-1} + \cdots + b'_1 s + b'_0}}$$

$$= \frac{K'}{s^{n-m} + \left(a_{n-1} - b'_{m-1}\right)s^{n-m-1} + \cdots} \approx \frac{K'}{s^r + \left(a_{n-1} - b'_{m-1}\right)s^{r-1}}, \qquad \text{as } s \to \infty$$

If $s$ is a closed-loop pole, then $KG(s) = -1$ $\Rightarrow$ $s^r + (a_{n-1} - b'_{m-1})s^{r-1} = -K'$

$$\Rightarrow \quad s^r\left(1 + \frac{a_{n-1} - b'_{m-1}}{s}\right) = -K'$$

$$\Rightarrow \quad s\left(1 + \frac{a_{n-1} - b'_{m-1}}{s}\right)^{\frac{1}{r}} = K'^{\frac{1}{r}}e^{j\theta_i}, \qquad \theta_i = \frac{(2i+1)\pi}{r}, i = 0,1,2,\cdots,r-1$$

Note: $(1+x)^t \approx 1 + tx$ Taylor expansion around $x = 0$ when $|x| \ll 1$

Then $s\left(1 + \dfrac{a_{n-1} - b'_{m-1}}{s}\right)^{\frac{1}{r}} \approx s\left(1 + \dfrac{a_{n-1} - b'_{m-1}}{rs}\right) = s + \dfrac{a_{n-1} - b'_{m-1}}{r} = K'^{\frac{1}{r}}e^{j\theta_i}$

# Asymptotes of the Root Loci (III)

$$s-\sigma_1 = K'^{\frac{1}{r}} e^{j\theta_i} \quad \text{where} \quad \sigma_1 = -\frac{a_{n-1}-b'_{m-1}}{r} \quad \theta_i = \frac{(2i+1)\pi}{r}, i = 0,1,2,\cdots,r-1$$

Let $s = \sigma + j\omega$ $\Rightarrow$ $\sigma + j\,\omega - \sigma_1 = K'^{\frac{1}{r}}(\cos\theta_i + j\sin\theta_i)$

$$\Rightarrow \begin{cases} \sigma - \sigma_1 = K'^{\frac{1}{r}}\cos\theta_i \\ \omega = K'^{\frac{1}{r}}\sin\theta_i \end{cases} \Rightarrow \frac{\omega}{\sigma-\sigma_1} = \tan\theta_i$$

$$\Rightarrow \omega = \tan\theta_i \cdot (\sigma - \sigma_1)$$

A straight line on the complex plane with slope $\tan\theta_i$ and real-axis intercept $\sigma_1$

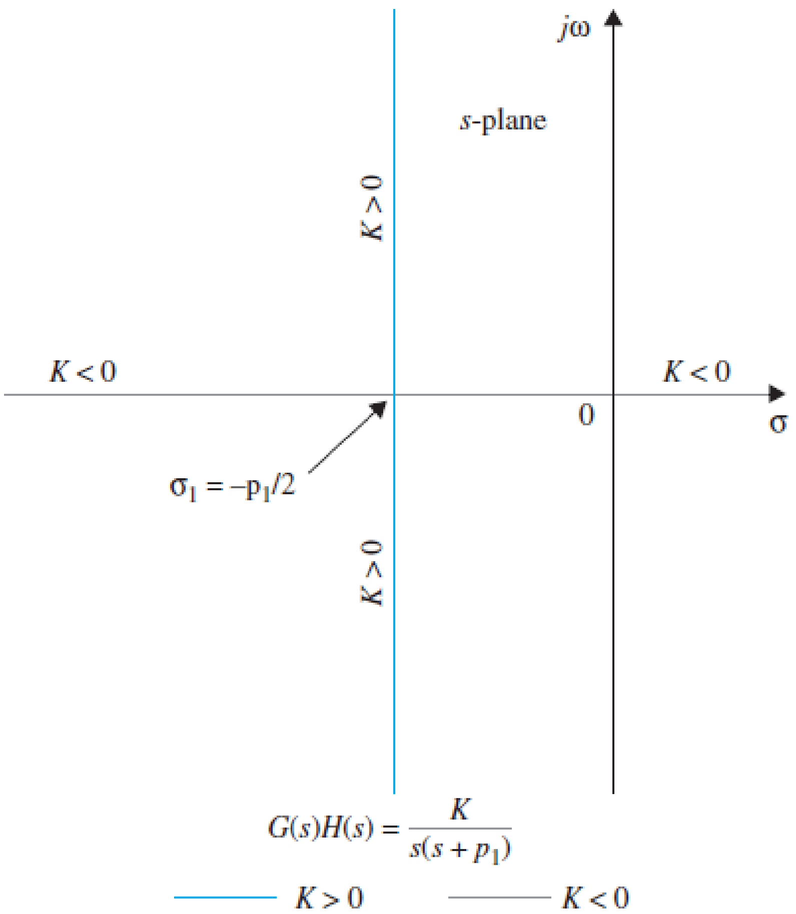
- When $r > 0$, there are $r$ asymptotes of the root loci. These $r$ asymptotes intersect the real-axis at

$$\sigma_1 = -\frac{a_{n-1}-b'_{m-1}}{r} = \frac{\sum\text{open loop poles} - \sum\text{finite open loop zeros}}{n-m}$$

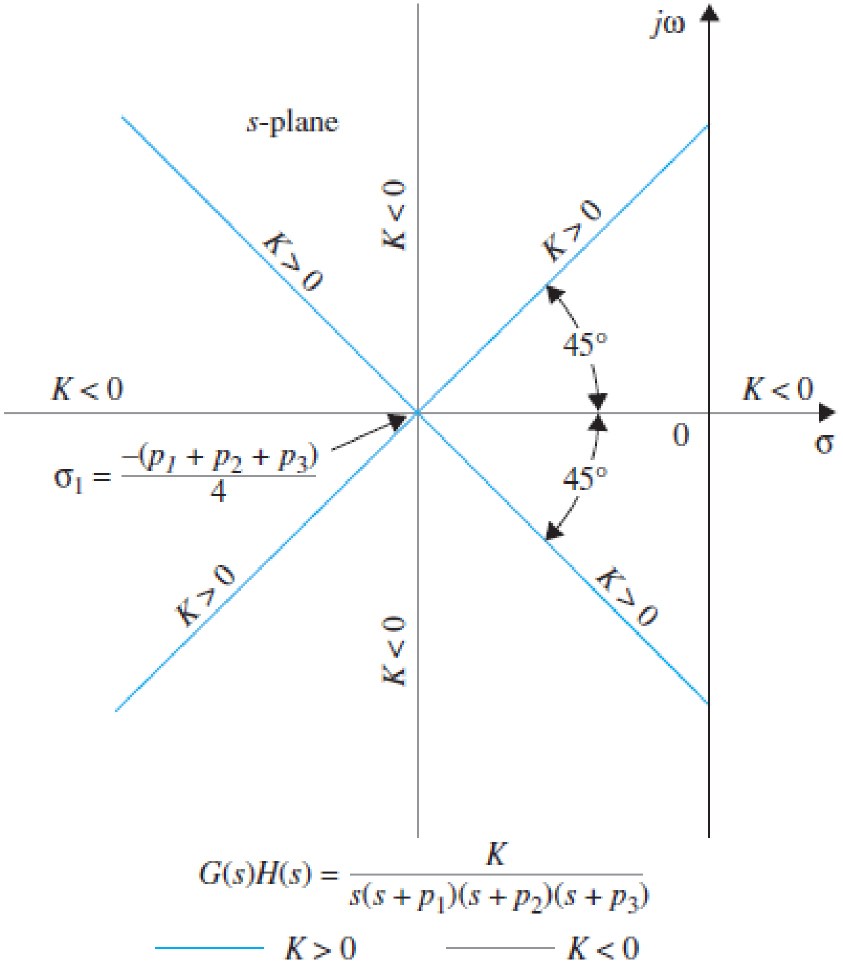
- The angles of asymptotes (w.r.t the real-axis) are

$$\theta_i = \frac{(2i+1)\pi}{r}, i = 0,1,2,\cdots,r-1$$

# Example 2 (a)

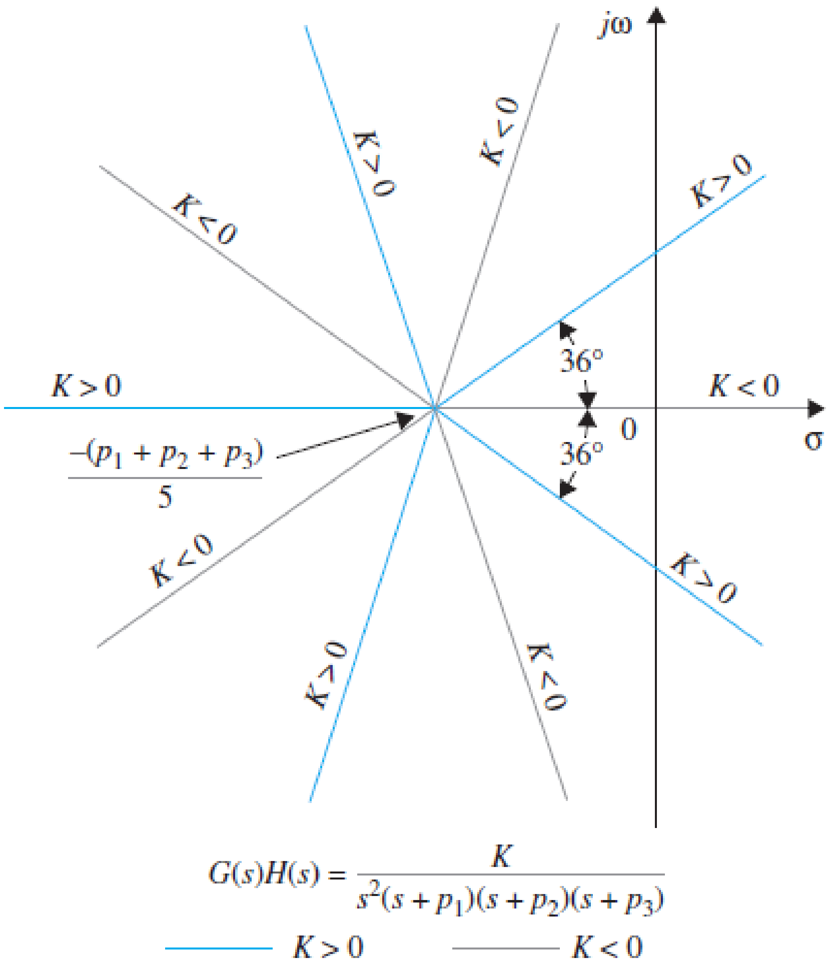


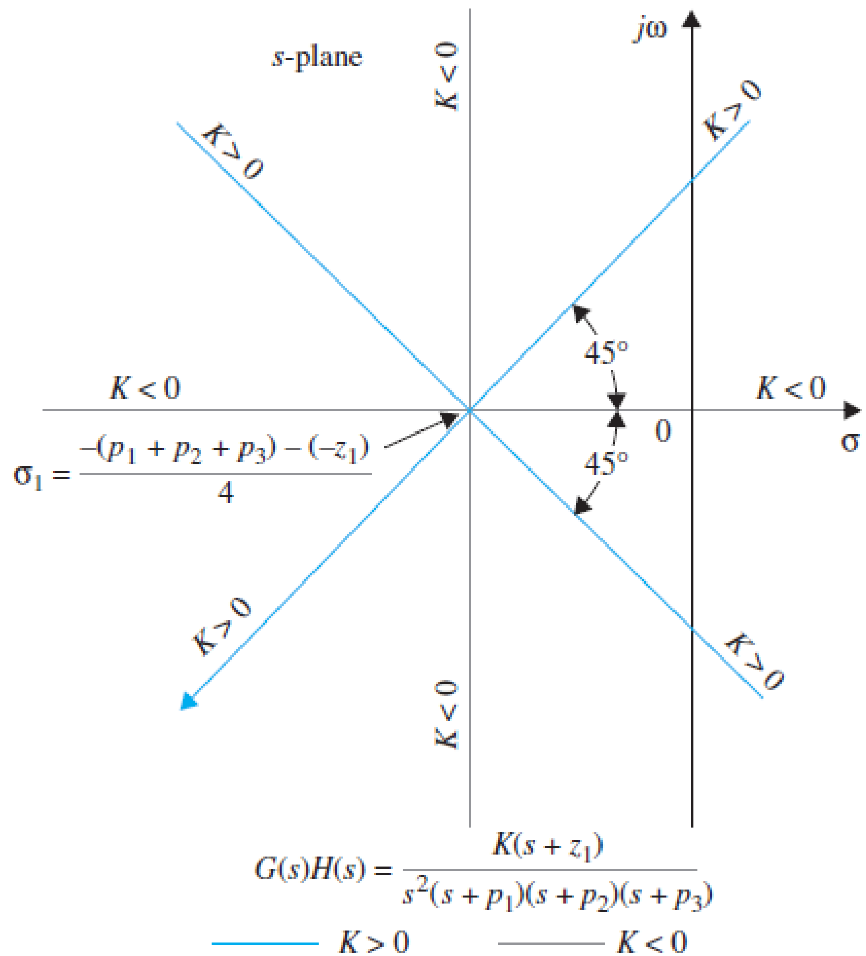
jω
s-plane
K > 0
K < 0
K < 0
0
σ
σ1 = −p1/2
K > 0
G(s)H(s) = K / s(s + p1)
K > 0
K < 0

jω
s-plane
K < 0
K > 0
K > 0
45°
K < 0
K < 0
0
σ
σ1 = −(p1 + p2 + p3) / 4
45°
K > 0
K > 0
K < 0
G(s)H(s) = K / s(s + p1)(s + p2)(s + p3)
K > 0
K < 0

# Example 2 (b)

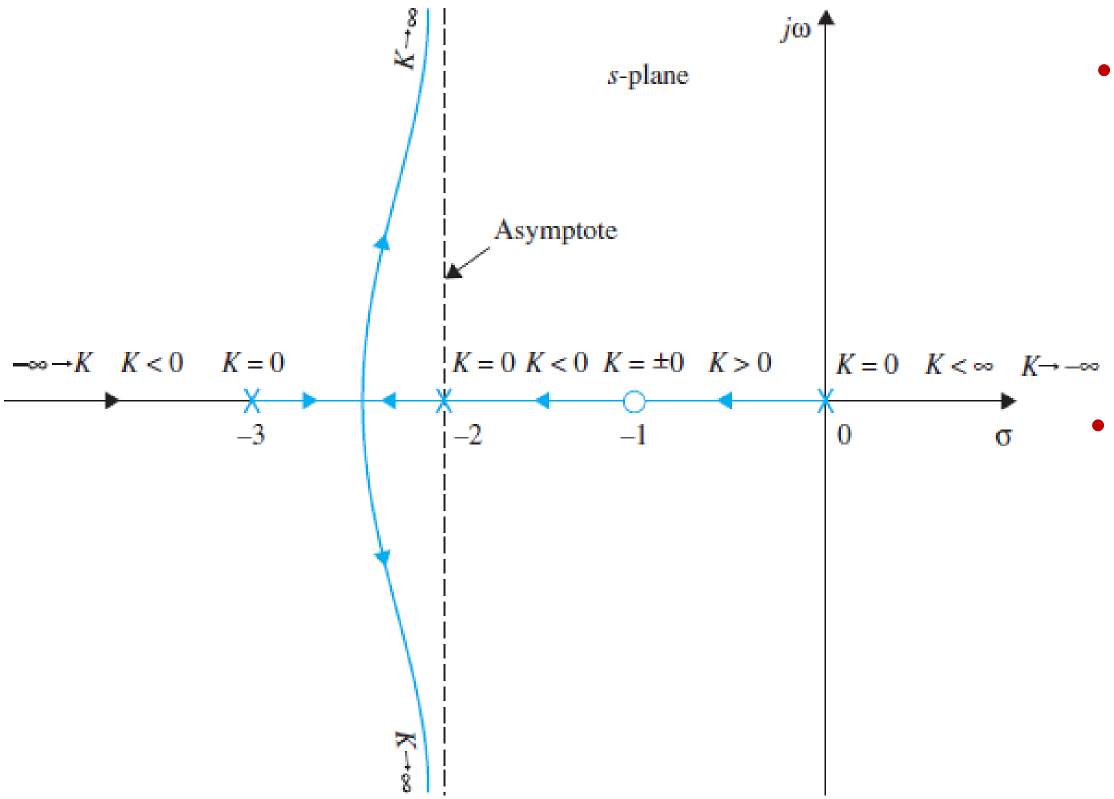

jω
K>0
K<0
K>0
K<0
K>0
K<0
K<0
K>0
K>0
K<0
36°
36°
0
σ
$\frac{-(p_1+p_2+p_3)}{5}$
$G(s)H(s) = \frac{K}{s^2(s+p_1)(s+p_2)(s+p_3)}$
K > 0
K < 0


jω
s-plane
K<0
K>0
K>0
45°
45°
K<0
K<0
0
σ
$\sigma_1 = \frac{-(p_1+p_2+p_3)-(-z_1)}{4}$
K>0
K<0
K>0
$G(s)H(s) = \frac{K(s+z_1)}{s^2(s+p_1)(s+p_2)(s+p_3)}$
K > 0
K < 0

# Example 3

$$G(s) = \frac{s+1}{s(s+2)(s+3)}$$

relative degree $r = 2 \Rightarrow$ two asymptotes




**Asymptotes**

- rea-axis intercept

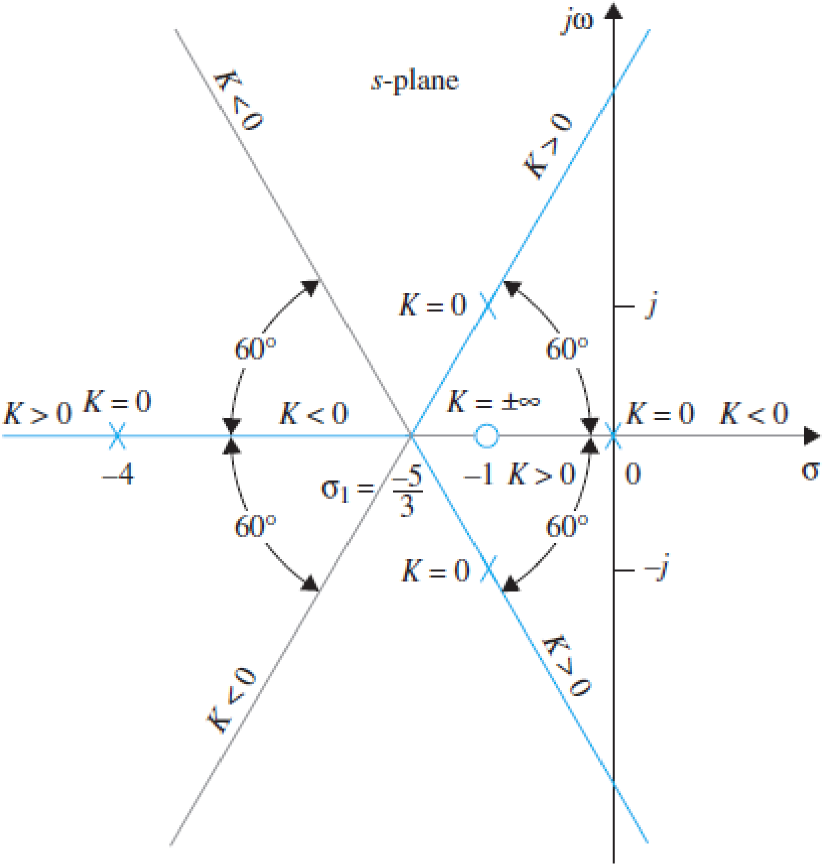
$$\sigma_1 = \frac{(0-2-3)-(-1)}{2} = -2$$

- angles

$$\theta_0 = \frac{\pi}{2} \qquad \theta_1 = \frac{3\pi}{2}$$

# Example 4

$$G(s) = \frac{s+1}{s(s+4)(s^2+2s+2)}$$ relative degree $r = 3 \Rightarrow$ three asymptotes




**Asymptotes**

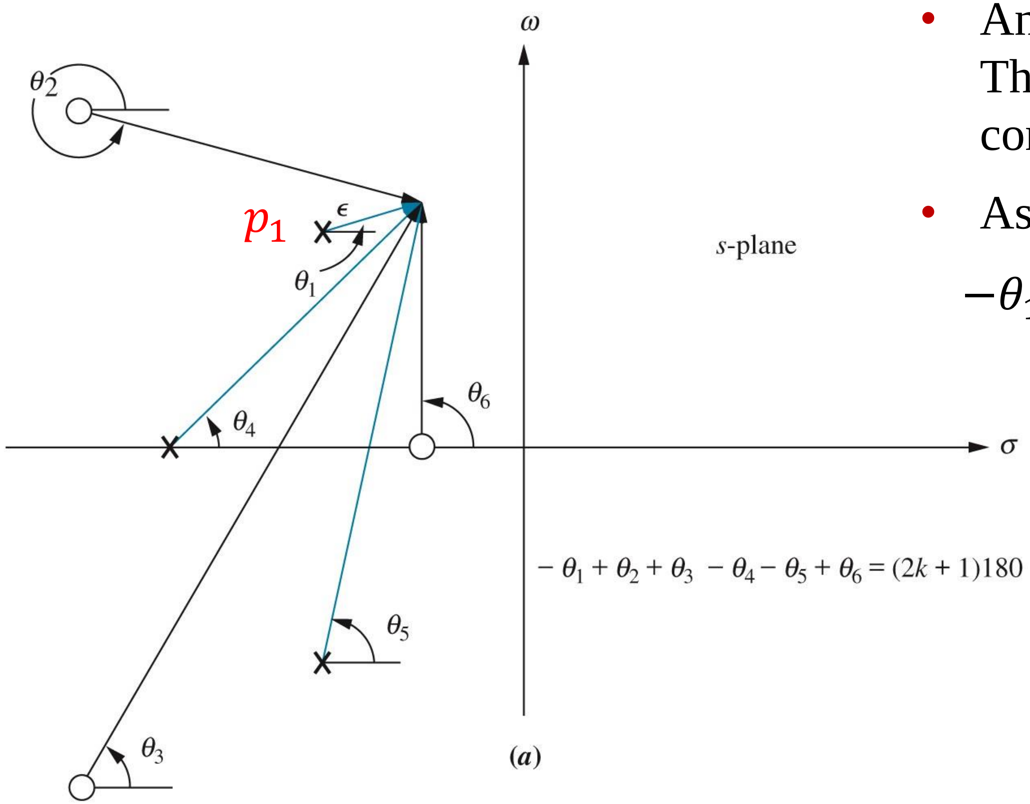
- rea-axis intercept

$$\sigma_1 = \frac{(0-4-1-1)-(-1)}{3} = -\frac{5}{3}$$

- angles

$$\theta_0 = \frac{\pi}{3} \qquad \theta_1 = \pi \qquad \theta_2 = \frac{5}{3}\pi$$

# Angles of Departure

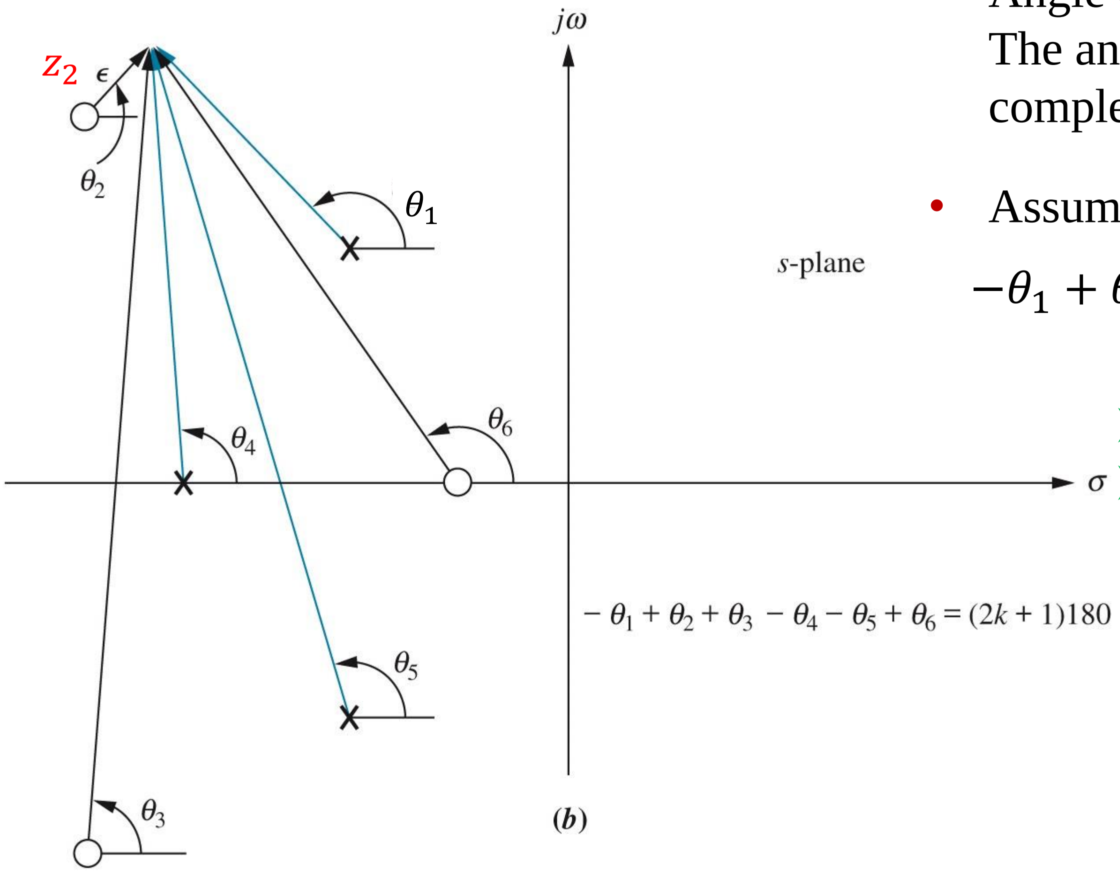



- Angle of Departure:
  The angle of a leaving root locus from a complex pole.
- Assume $0 < \epsilon \ll 1$

$$-\theta_1+\theta_2+\theta_3-\theta_4-\theta_5+\theta_6=(2i+1)\pi$$
$$i \in \mathbb{Z}$$

> - $\theta_1$ is the departure angle.
> - $\theta_i$, $i = 2, \cdots 6$, is the angle from other pole or zero to $p_1$ since we assume that $\epsilon$ is very small.

# Angle of Arrival

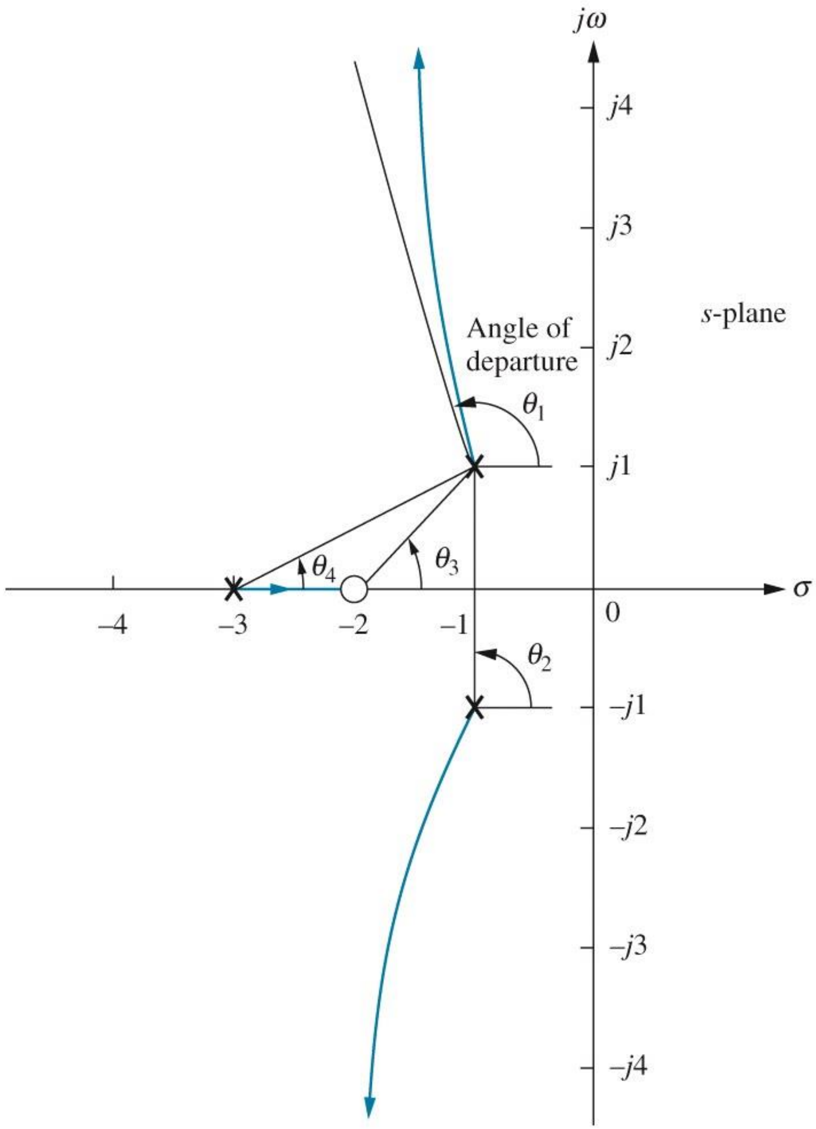



- Angle of Arrival:
  The angle of a incoming root locus to a complex pole.
- Assume $0 < \epsilon \ll 1$

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2i + 1)\pi$$

- $\theta_1$ is the arrival angle of $z_2$.
- $\theta_i$, $i = 2, \cdots 6$, is the angle from other pole or zero to $z_2$ since we assume that $\epsilon$ is very small.

# Example 5




$$G(s) = \frac{s+2}{(s+3)(s^2+2s+2)}$$

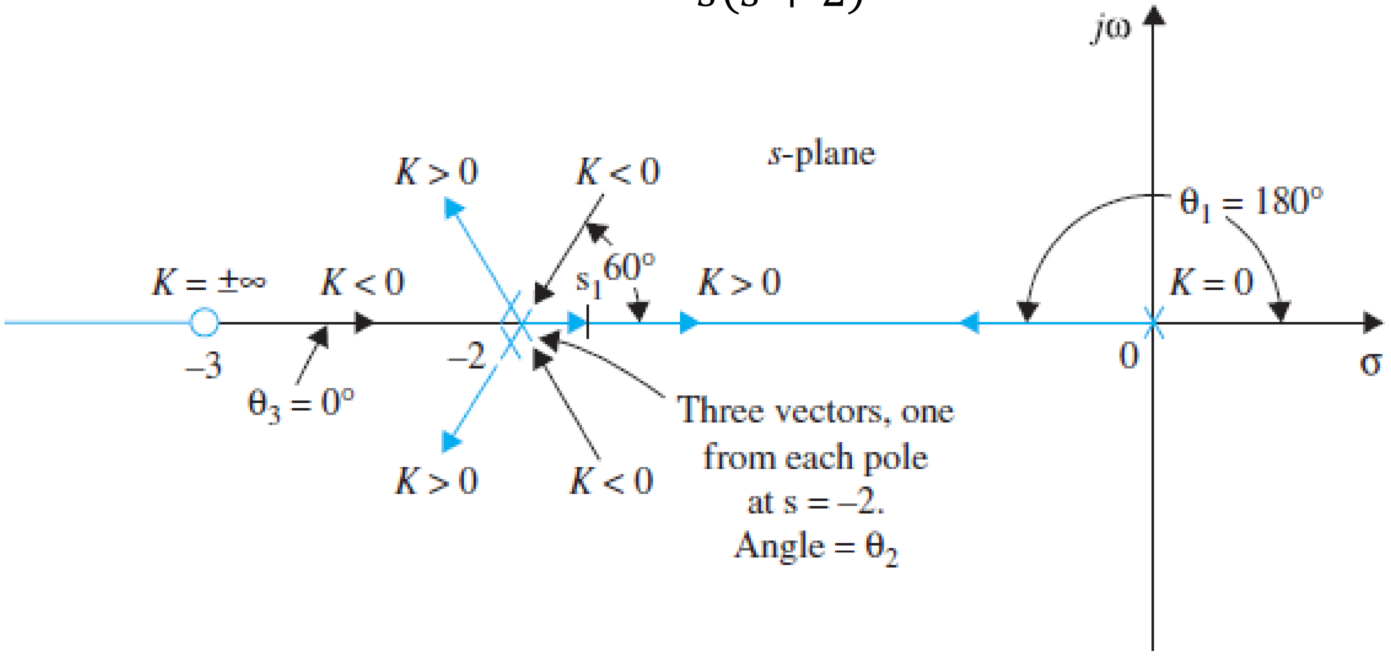
- Open-loop poles: $-3, -1 \pm j$
- Open-loop zeros: $-2$
- Angle of departure $(\theta_1)$

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4$$
$$= -\theta_1 - 90 + \tan^{-1}\frac{1}{1} - \tan^{-1}\frac{1}{2} = 180$$

$$\Rightarrow \quad \theta_1 = -251.6^\circ$$
$$\text{or} \quad \theta_1 = 108.4^\circ$$

# Example 6

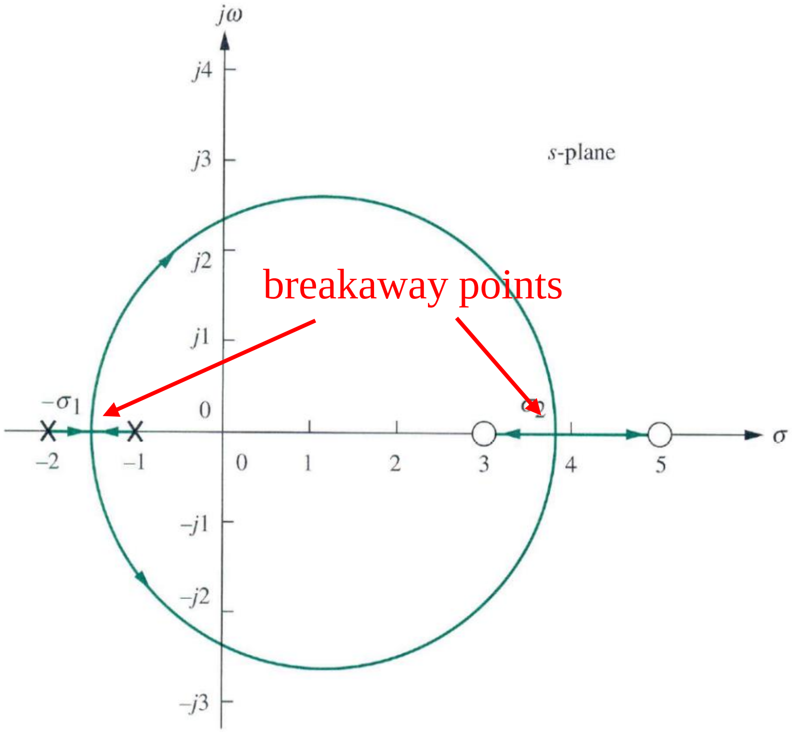
$$G(s) = \frac{s+3}{s(s+2)^3}$$

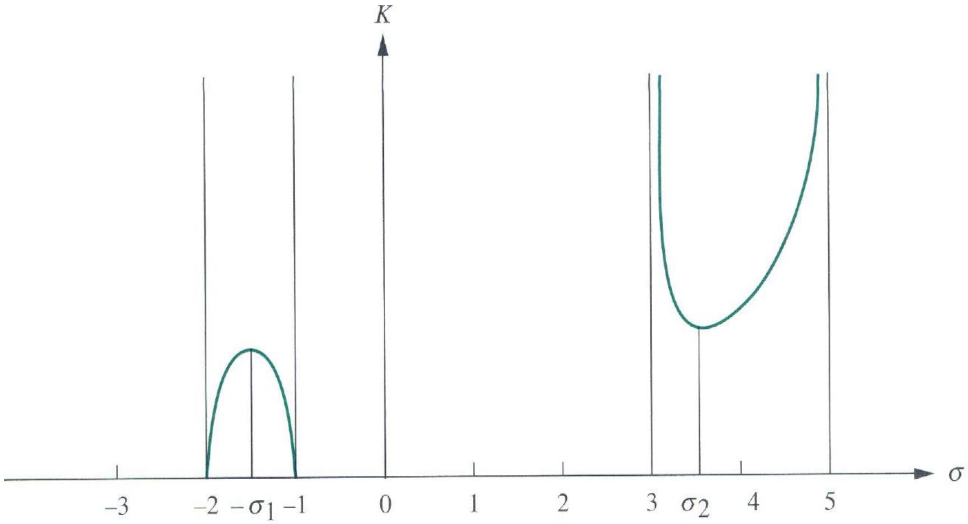



$$-\theta_1 - 3\theta_2 + 3\theta_3 = (2i-1)180°$$

$$i = 0,1,2 \quad \Rightarrow \quad \theta_2 = 0°, -120°, 120°$$

# Breakaway Points (I)




- Multiple branches meet at the breakaway point and depart in different directions.
- The breakaway point corresponds to multiple closed-loop poles.
- As $K$ increases from zero, the root loci start from the open-loop poles $(-1$ and $-2)$ . *Before* the two branches meet at $-\sigma_1$, $K$ keeps increasing. Therefore $K$ reaches maximum at $-\sigma_1$ along the root loci in the real-axis.
- *After* the two branches meet at $\sigma_2$, $K$ keeps increasing until the two branches reach the open-loop zeros. Therefore $K$ reaches minimum at $\sigma_2$ along the root loci in the real-axis.

# Breakaway Points (II)

Along the root locus, we have $G(s) = -\dfrac{1}{K}$

- A **necessary**, but **not sufficient**, **condition** for $\sigma$ to be a breakaway point is

$$\left.\frac{d}{ds}\frac{1}{G(s)}\right|_{s=\sigma} = 0 \quad \text{or} \quad \left.\frac{dG(s)}{ds}\right|_{s=\sigma} = 0$$

If $\sigma$ is a breakaway point, there must exist $K > 0$ such that $G(\sigma) = -\frac{1}{K}$

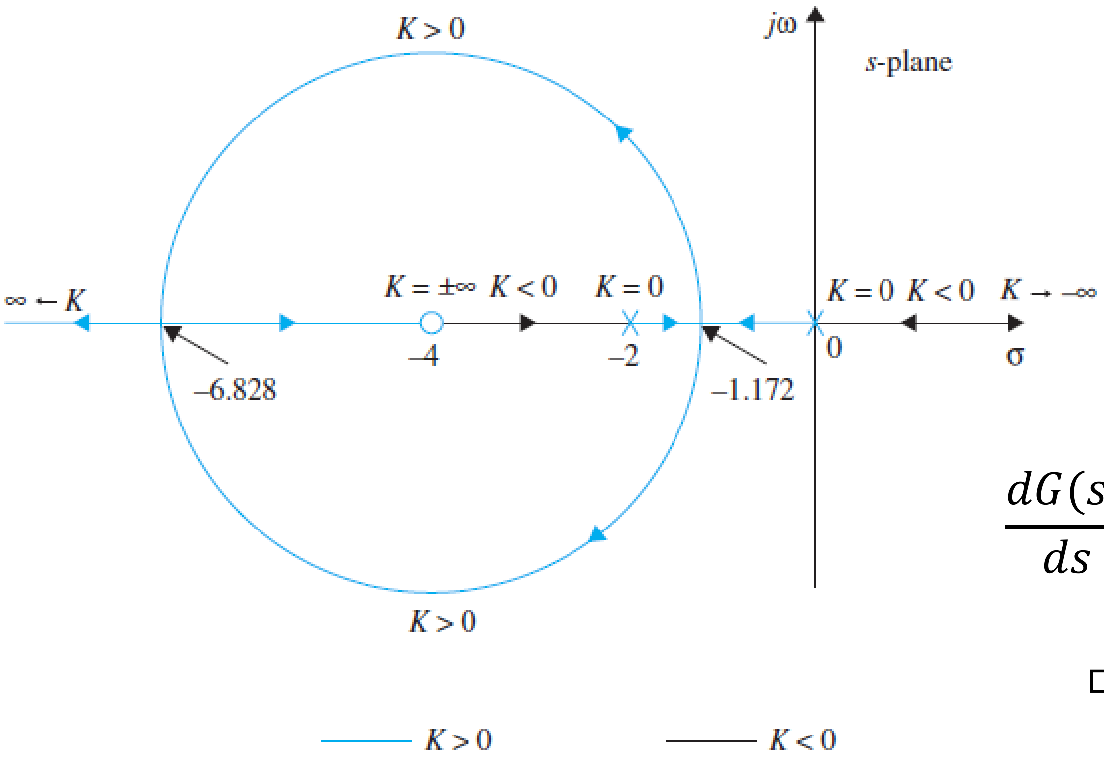
- Angle of departure and arrival at the breakaway point is $\dfrac{(2i+1)180°}{q}$, $\;i = 0,1,\cdots,q-1$

where $q$ is the number of branches meeting at the breakaway point.

# Intersection of the Root Locus with the Imaginary Axis

- The points where the root locus intersect the imaginary axis of the *s*-plane, and the corresponding values of $K$ may be determined by means of the Routh-Hurwitz criterion.

# Example 7



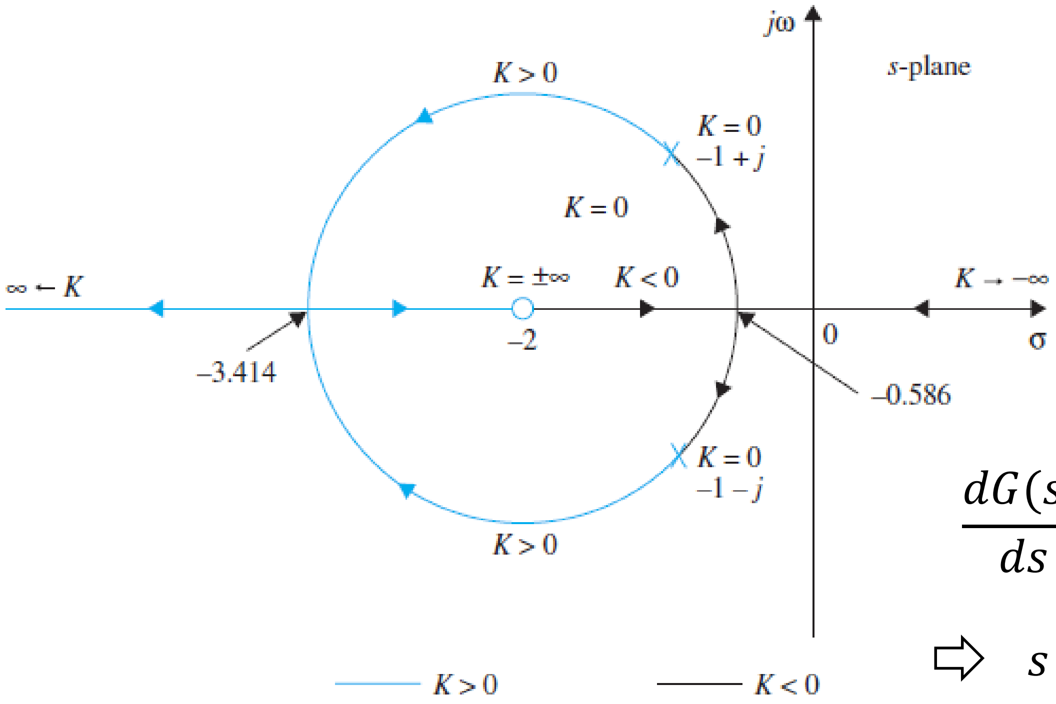

$$G(s) = \frac{s+4}{s(s+2)}$$

- Breakaway points

$$\frac{dG(s)}{ds} = \frac{s(s+2) - (s+4)(2s+2)}{s^2(s+2)^2} = 0$$

$$\Rightarrow \quad s = -1.172, -6.828$$

# Example 8




$$G(s) = \frac{s+2}{s^2+2s+2}$$

- Breakaway points

$$\frac{dG(s)}{ds} = \frac{s^2+2s+2-(s+2)(2s+2)}{(s^2+2s+2)} = 0$$

$$\Rightarrow \quad s = -3.414, 0.586$$

Since the breakaway point must be in the left of the zero $s = -2$, therefore $s = -3.414$ is the breakaway point.

# Example 9

$$G(s) = \frac{1}{s(s^2 + 2s + 2)(s + 3)}$$

- Open-loop poles: $0, -1 \pm j, -3$
- Number of asymptotes: 4
- Intersection of asymptotes:

$$\sigma_1 = \frac{-1 - 1 - 3}{4} = \frac{-5}{4}$$

- Angles of asymptotes:

$$\theta_i = \frac{(2i + 1)180°}{4} = 45°, 135°, 225°, 315°$$

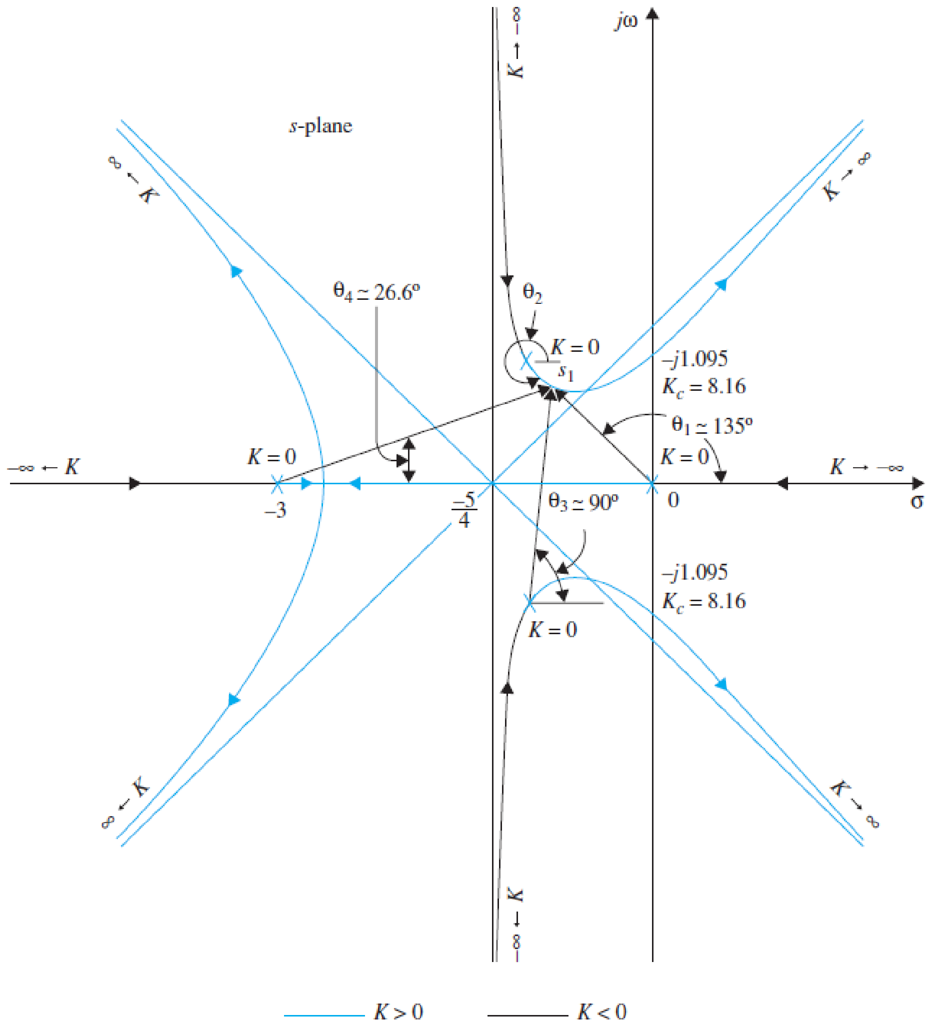
- Breakaway point

$$\frac{d}{ds}\frac{1}{G(s)} = 4s^3 + 15s^2 + 16s + 6 = 0$$

⇨ $s = -2.2886, -0.7307 \pm 0.3486j$

⇨ $K = 4.33 = \left.\frac{-1}{G(s)}\right|_{s=-2.2886}$ .no $K > 0$ for $s = -0.7307 \pm 0.3486j$

⇨ Breakaway point: $s = -2.2886$




- Angle of departure: $-(\theta_1 + \theta_2 + \theta_3 + \theta_4) = -(135° + \theta + 90° + 26.6°) = (2i + 1)180°$

⇨ $\theta_2 = -71.6° \quad (i = -1)$

# Example 9 (Continued)

- Intersection with the imaginary axis

The characteristic equation is

$$s\left(s^2+2s+2\right)(s+3)+K=s^4+5s^3+8s^2+6s+K=0$$

Routh Table

| | | | |
|---|---|---|---|
| $s^4$ | $1$ | $8$ | $K$ |
| $s^3$ | $5$ | $6$ | $0$ |
| $s^2$ | $\frac{34}{5}$ | $K$ | $0$ |
| $s^1$ | $\frac{204}{5}-5K$ | $0$ | $0$ |
| $s^0$ | $K$ | $0$ | $0$ |

- For the closed-loop poles to be on the $j\omega$-axis, the Routh table must have a row of all zeros.

$$K=\frac{204}{25}=8.16$$

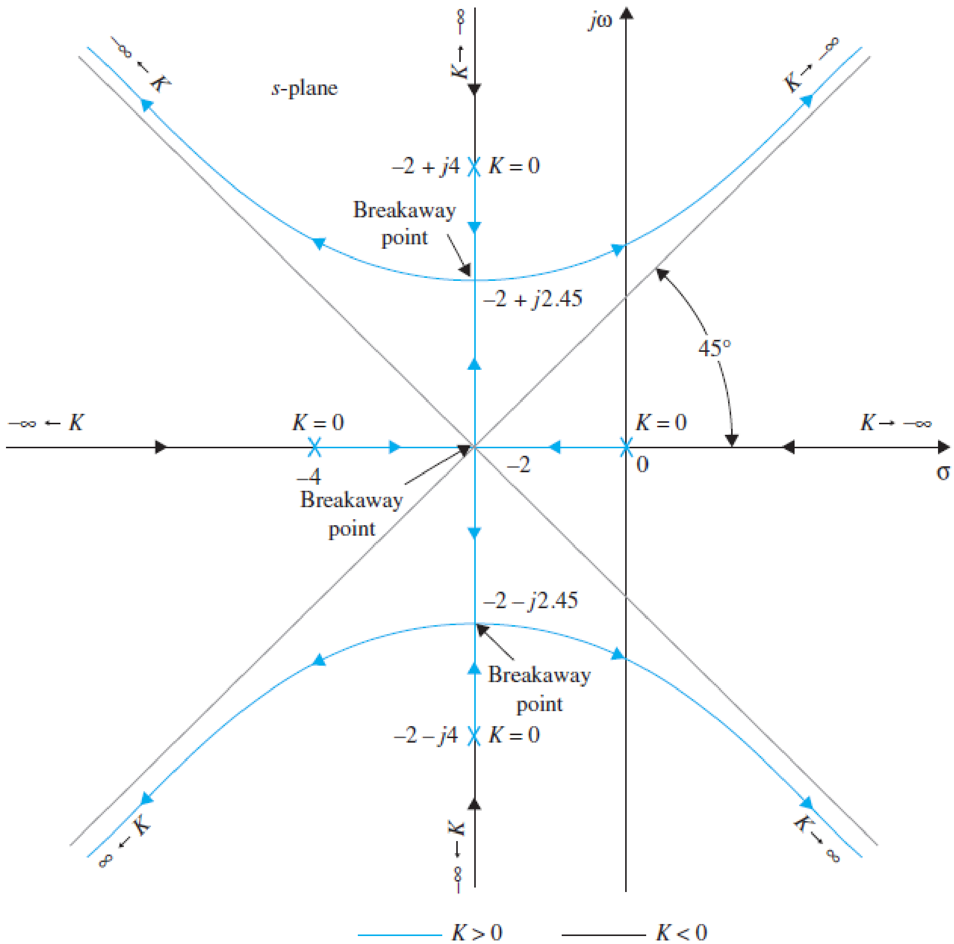
- For $K=8.16$, the characteristic polynomial contains a factor of an even polynomial

$$\frac{34}{5}s^2+K=\frac{34}{5}s^2+\frac{204}{25}=0$$

- The closed-loop poles are

$$s=\pm j\sqrt{\frac{6}{5}}=\pm 1.095j$$

# Example 10




$$G(s) = \frac{1}{s(s+4)(s^2+4s+20)}$$

- Open-loop poles: $0, -2 \pm 4j, -4$
- Number of asymptotes: 4
- Intersection of asymptotes:

$$\sigma_1 = \frac{-2-2-4}{4} = -2$$

- Angles of asymptotes:

$$\theta_i = \frac{(2i+1)180°}{4} = 45°, 135°, 225°, 315°$$

- Breakaway point

$$\frac{d}{ds}\frac{1}{G(s)} = 4\left(s^3 + 6s^2 + 18s + 20\right) = 0$$

$\Rightarrow$ $s = -2, -2 \pm 2.45j$

$\Rightarrow$ $K = 64 = \left.\frac{-1}{G(s)}\right|_{s=-2}$,

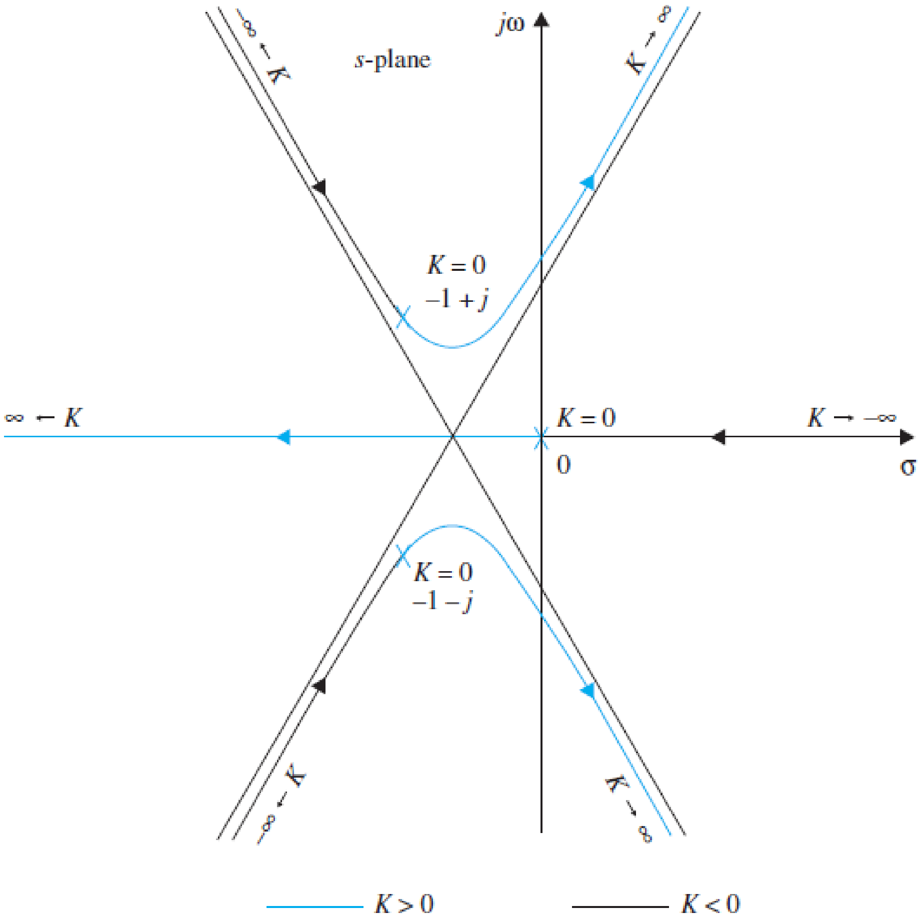
$K = 100 = \left.\frac{-1}{G(s)}\right|_{s=-2\pm 2.45j}$

(Complex breakaway point!)

- Angle of departure:

$$-\left(\theta + 90° + (180° - \tan^{-1} 2) + \tan^{-2} 2\right) = (2i+1)180° \Rightarrow \theta = -90° \quad (i = -1)$$

# Example 11




$$G(s) = \frac{1}{s(s^2 + 2s + 2)}$$

- Open-loop poles: $0, -1 \pm j$
- Number of asymptotes: 3
- Intersection of asymptotes:

$$\sigma_1 = \frac{-1-1}{3} = -\frac{2}{3}$$

- Angles of asymptotes:

$$\theta_i = \frac{(2i+1)180°}{3} = 60°, 180°, 300°$$

- Breakaway point

$$\frac{d}{ds}\frac{1}{G(s)} = 3s^2 + 4s + 2 = 0$$

⇨ $s = -0.667 \pm 0.471$

No positive $K$ for these points

⇨ No breakaway point

- Angle of departure: $-(\theta + 90° + 135°) = (2i+1)180°$

⇨ $\theta = -45° \qquad (i = -1)$

# Example 12

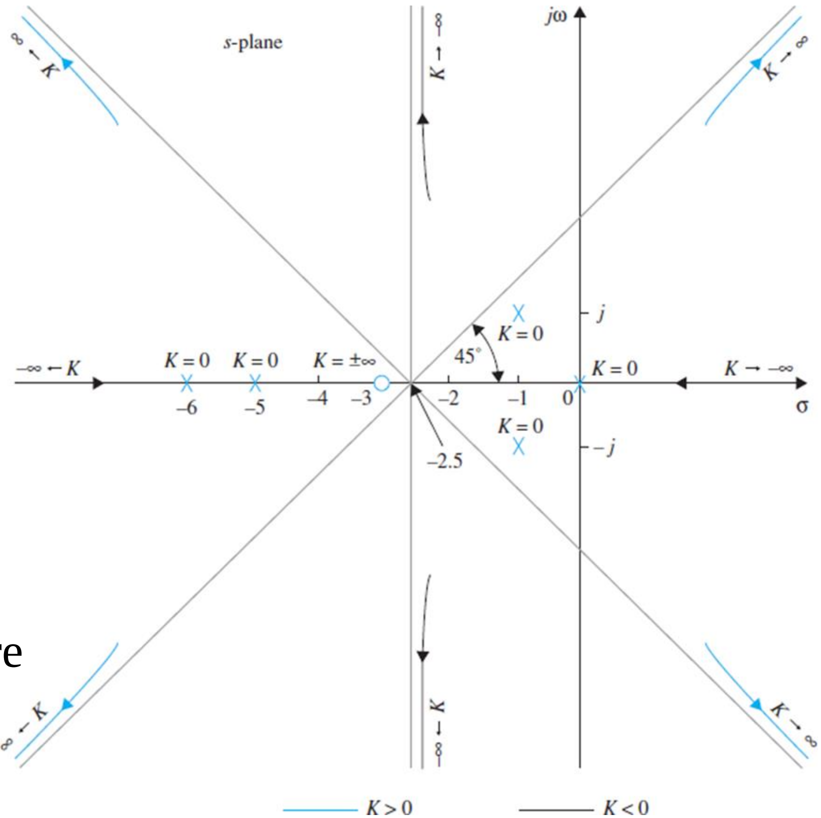
$$G(s) = \frac{s+3}{s(s+5)(s+6)(s^2+2s+2)}$$

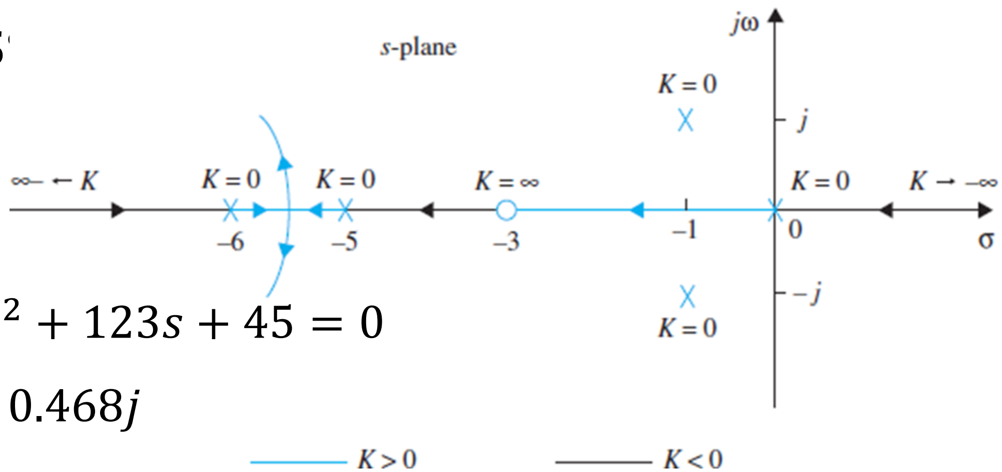
- Observation:
  - 5 branches start from 5 open-loop poles at $s = 0, -5, -6, -1 \pm j$ and end at 1 one-loop zero at $s = -3$.
  - 4 asymptotes.
- The intersection and angles of the 4 asymptotes are

$$\sigma_1 = \frac{-(5+6+1+1)+3}{4} = -2.5$$

$$\theta_i = \frac{(2i+1)180°}{4} = 45°, 135°, 225°, 315°$$

- The real-axis segments $[-3, 0]$, and $[-6, -5]$ are parts of the root loci.




- $\frac{dG(s)}{ds} = 0 \Rightarrow s^5 + 13.5s^4 + 66s^3 + 142s^2 + 123s + 45 = 0$

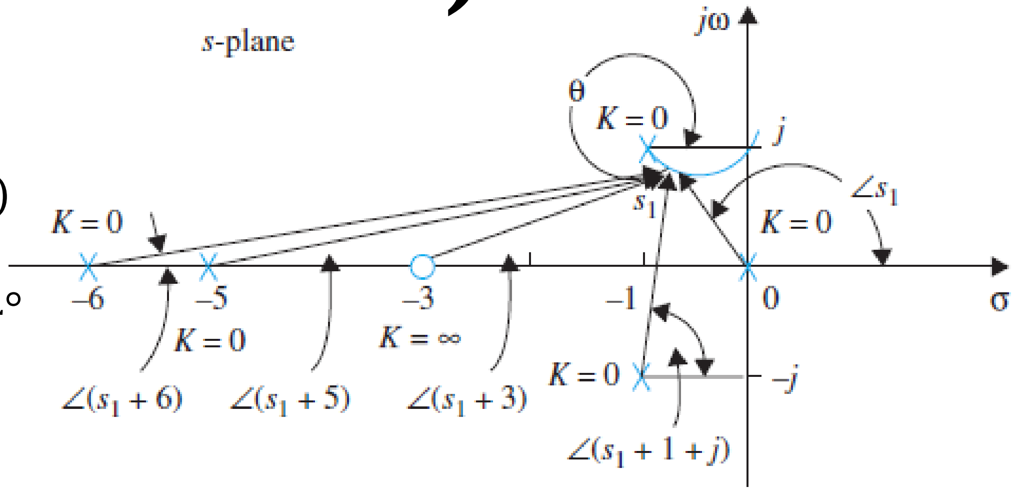
$\Rightarrow s = -5.53, -3.33 \pm 1.204j, -0.656 \pm 0.468j$

Breakaway point: $s = -5.53$

# Example 12 (Continued)

- Angle of Departure

$$-\theta - \angle s_1 - \angle(s_1 + 1 + j) + \angle(s_1 + 3)$$
$$- \angle(s_1 + 5) - \angle(s_1 + 6)$$
$$= -\theta - 135° - 90° + 26.6° - 14° - 11.4°$$
$$= (2i + 1)180°$$
$$\Rightarrow \theta = -43.8°, \qquad (i = -1)$$




- Intersection with the $j\omega$-axis

  Characteristic Equation: $s^5 + 13s^4 + 54s^3 + 82s^2 + (60 + K)s + 3K = 0$

Routh Table:

| | | | |
|---|---|---|---|
| $s^5$ | $1$ | $54$ | $60 + K$ |
| $s^4$ | $13$ | $82$ | $3K$ |
| $s^3$ | $47.7$ | $60 + 0.769K$ | $0$ |
| $s^2$ | $65.6 - 0.21K$ | $3K$ | $0$ |
| $s^1$ | $\dfrac{3940 - 105K - 0.16K^2}{65.6 - 0.21K}$ | $0$ | $0$ |
| $s^0$ | $3K$ | $0$ | $0$ |

For the closed-loop poles to be on the $j\omega$-axis: $3940 - 105K - 0.16K^2 = 0 \;\Rightarrow\; K = 35$

Even polynomial: $(65.6 - 0.21K)s^2 + 3K = 0 \;\Rightarrow\; s = \pm 1.34j$

# Example 12 (Continued)