Automatic Control Systems
Lecture 5
Block Diagram and Signal Flow Graph

Tesheng Hsiao, Associate Professor

Department of Electrical and Control Engineering

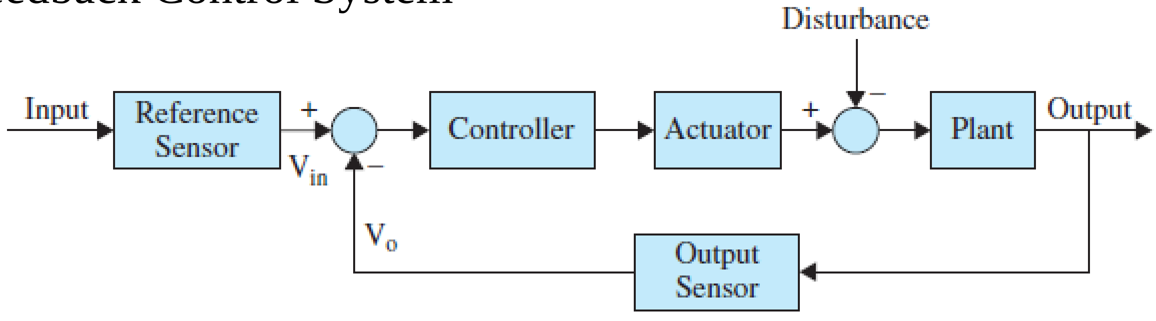
National Chiao Tung University, Taiwan

Outline

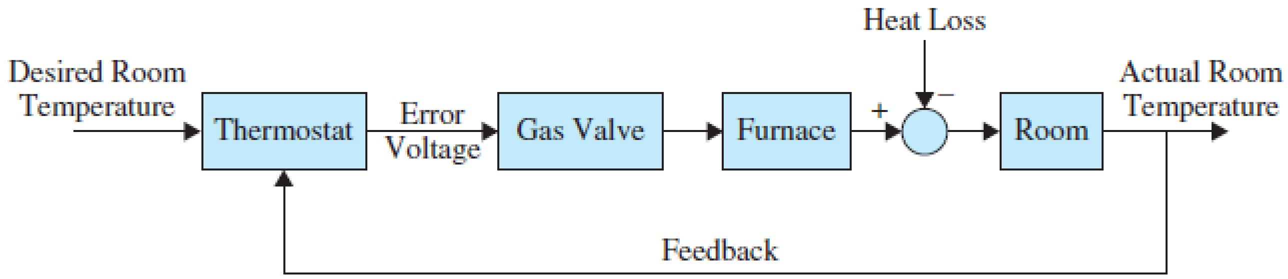
- Basic Structures of Block Diagrams
- Differential Equations and Block Diagrams
- Block Diagram Reduction
- Block Diagrams for Multiple Variable Systems
- Signal Flow Graph

Control Systems Represented by Block Diagrams

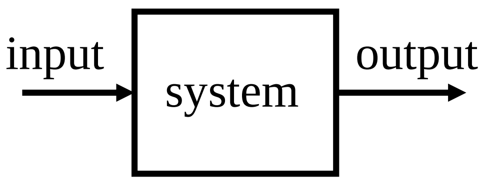
General Feedback Control System



Indoor Temperature Control System

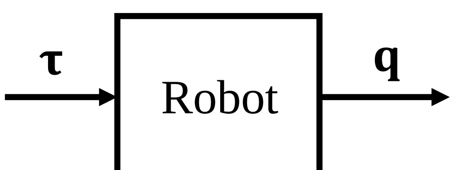


Block: Input-Output Relation



Economic System

Nonlinear System



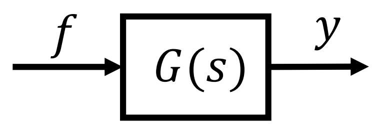
$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) + \mathbf{F}(\dot{\mathbf{q}}) = \boldsymbol{\tau}$$

LTI System

$$M\ddot{y} + B\dot{y} + Ky = f$$

Transfer Function

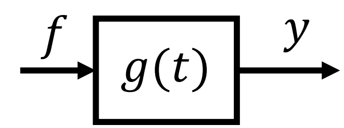
$$Y(s) = G(s)F(s)$$



$$G(s) = \frac{\frac{1}{M}}{s^2 + \frac{B}{M}s + \frac{K}{M}}$$

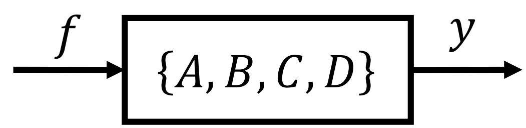
Impulse Response

$$y(t) = g(t) * f(t)$$



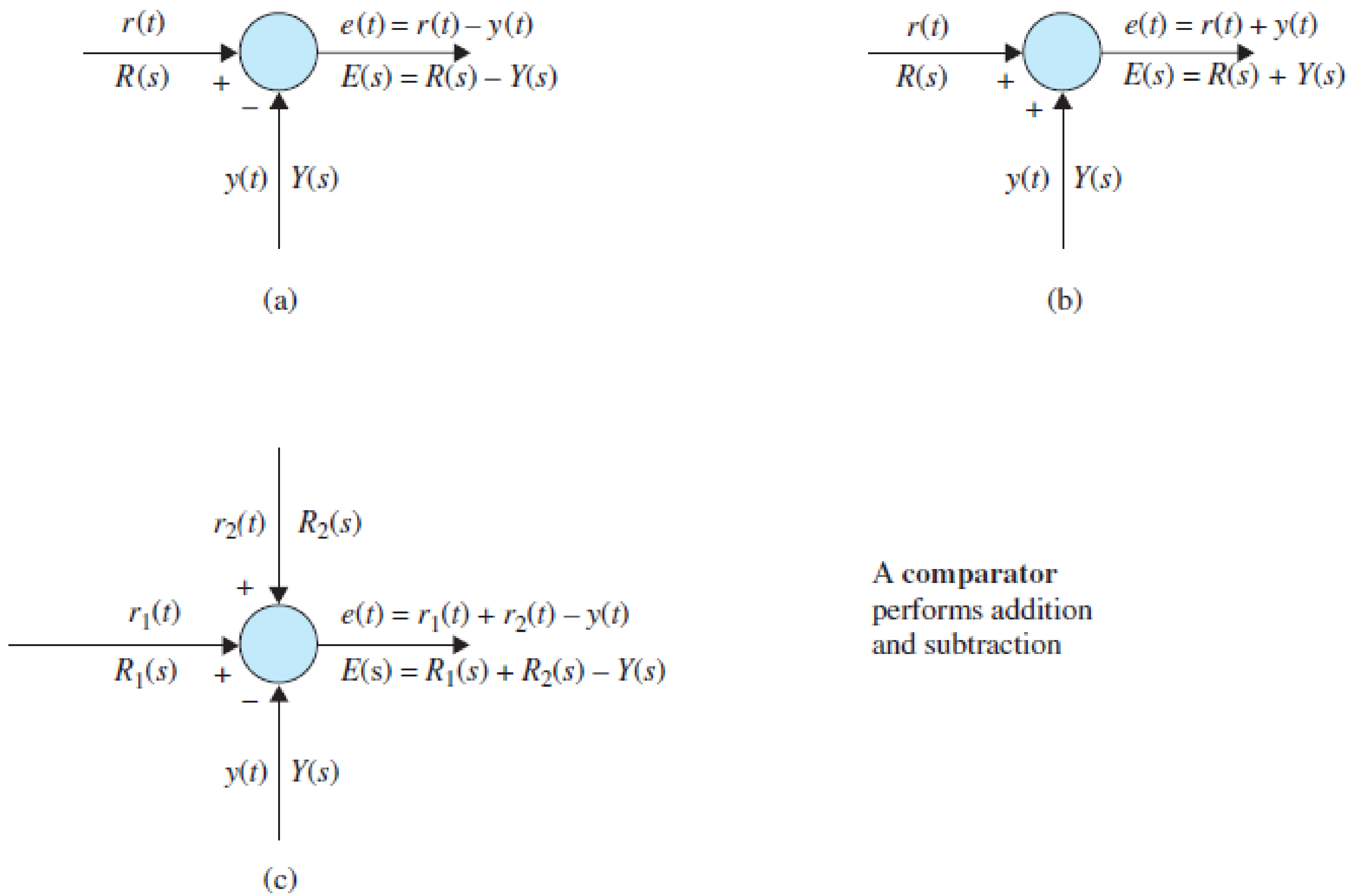
$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$

State Space



$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

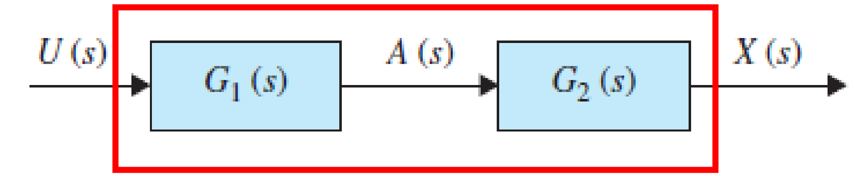
Comparator: Addition/Subtraction



A comparator performs addition and subtraction

Connection of Blocks

- Cascade System

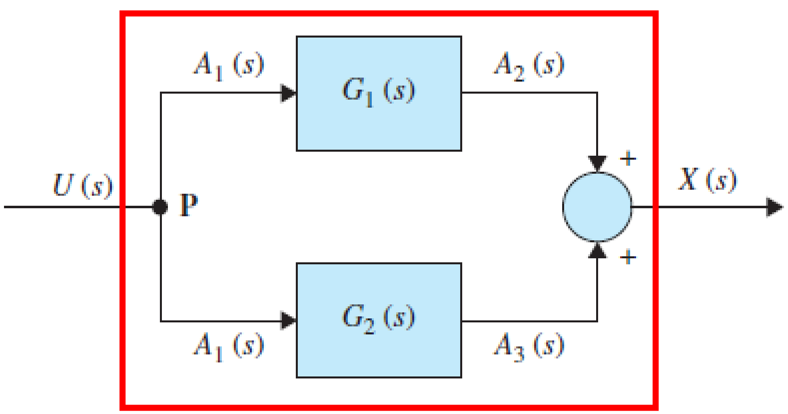


$$G(s) = G_1(s)G_2(s)$$

$$A(s) = G_1(s)U(s)$$
$$X(s) = G_2(s)A(s) = G_2(s)G_1(s)U(s)$$

$$\Rightarrow \frac{X(s)}{U(s)} = G_1(s)G_2(s) = G_2(s)G_1(s)$$

- Parallel System

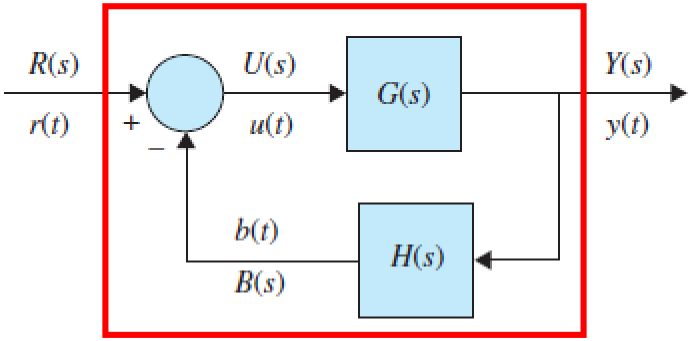


$$G(s) = G_1(s) + G_2(s)$$

$$A_2(s) = G_1(s)A_1(s) = G_1(s)U(s)$$
$$A_3(s) = G_2(s)A_{1(s)} = G_2(s)U(s)$$
$$X(s) = A_2(s) + A_3(s) = \big(G_1(s) + G_2(s)\big)U(s)$$
$$\Rightarrow \frac{X(s)}{U(s)} = G_1(s) + G_2(s) = G_2(s) + G_1(s)$$

Connection of Blocks (Continued)

- Feedback System



$$Y(s) = G(s)U(s) \quad \text{(feedforwd path)}$$
$$B(s) = H(s)Y(s) \quad \text{(feedback path)}$$
$$U(s) = R(s) - B(s) = R(s) - H(s)Y(s)$$
$$\Rightarrow Y(s) = G(s)[R(s) - H(s)Y(s)]$$
$$\Rightarrow [1 + G(s)H(s)]Y(s) = G(s)R(s)$$
$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

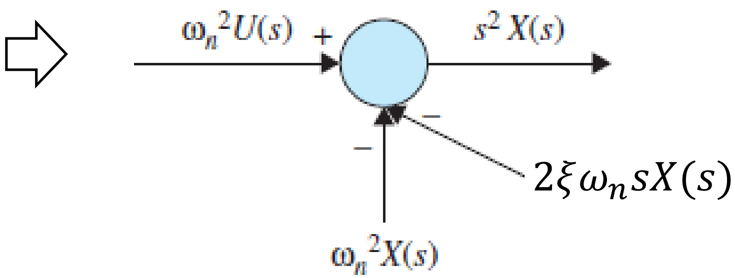
Closed-loop Transfer Function $$M(s) = \frac{G(s)}{1 + G(s)H(s)}$$

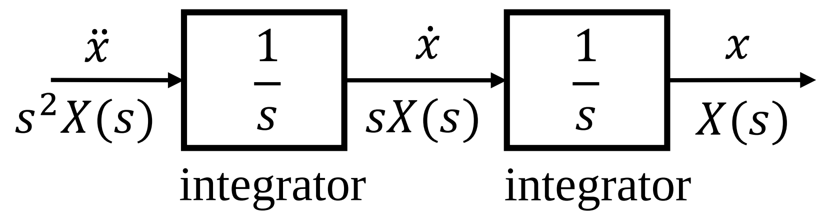
- **Remarks:**
  - Negative Feedback: $u = r - b$
  - Positive Feedback: $u = r + b$
  - Unity Feedback Loop: $H(s) = 1$
  - Open Loop: $H(s) = 0$

Differential Equations and Block Diagrams

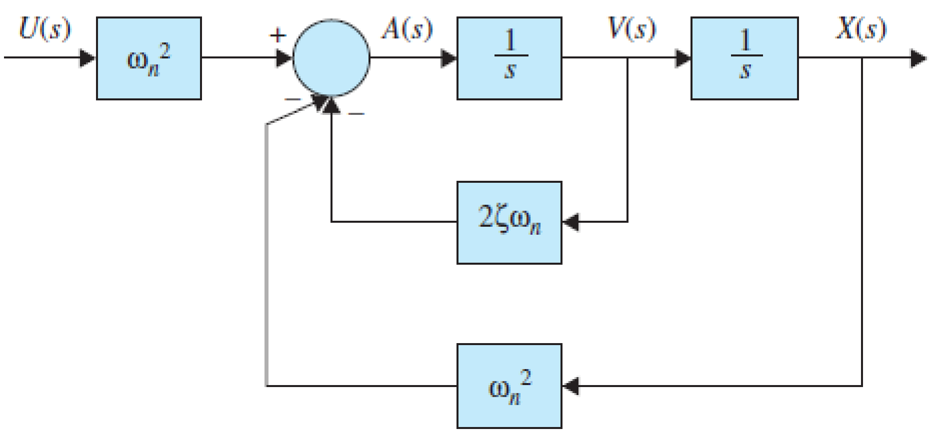
Prototype 2nd –order System $\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 u$

Assume zero initial conditions



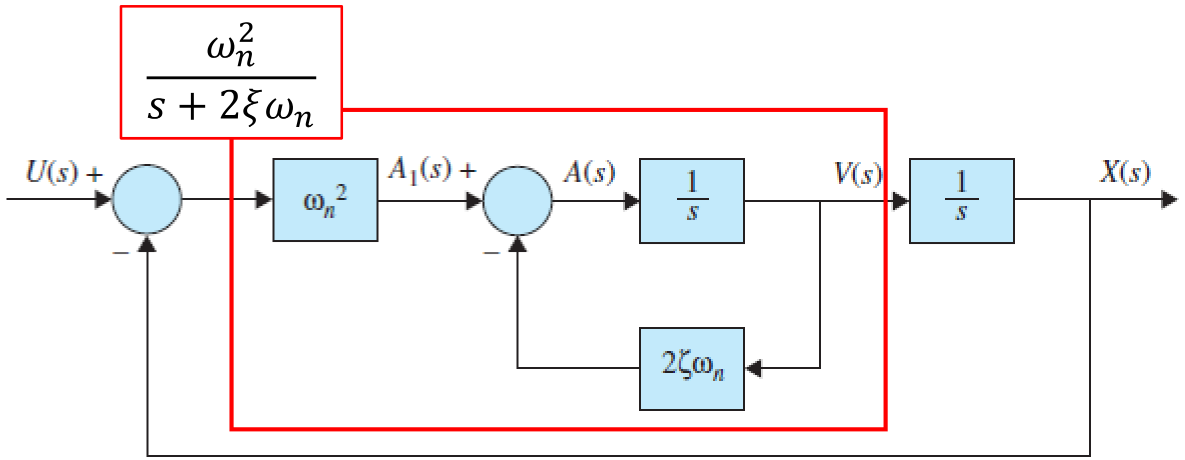


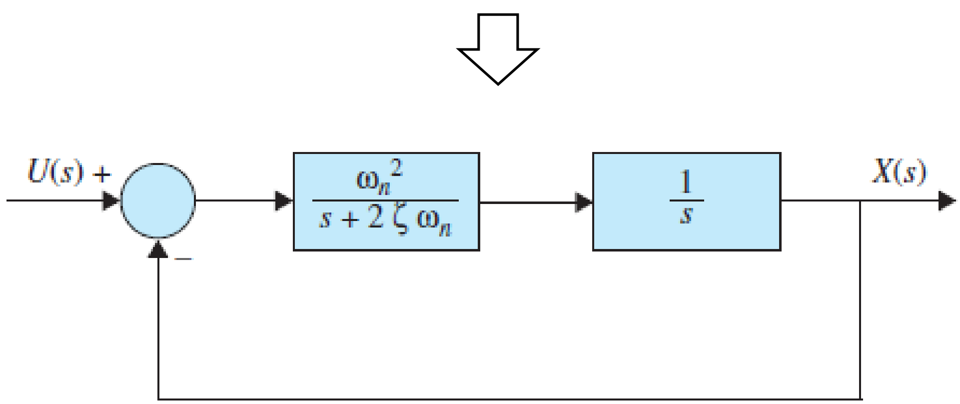
$$s^2X(s) = \omega_n^2 U(s) - 2\xi\omega_n sX(s) - \omega_n^2 X(s)$$



$$G(s) = \frac{X(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Alternative Representation

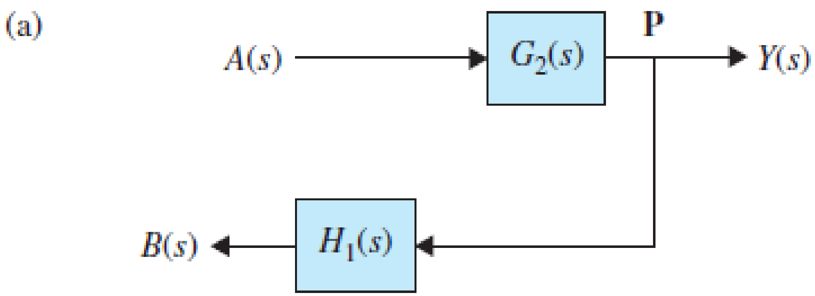




$$G(s) = \frac{X(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

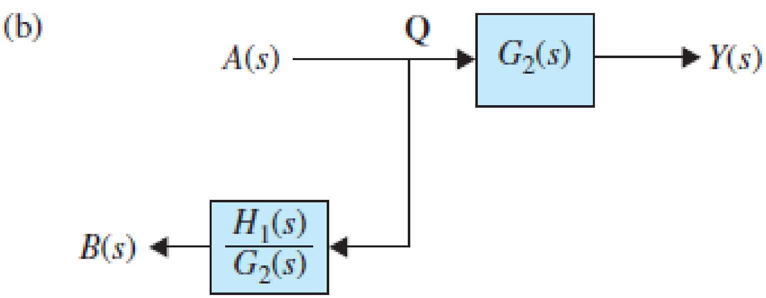
Block Diagram Reduction

- Moving a *branch point* from P to Q



(a) $Y(s) = G_2(s)A(s)$

$$B(s) = H_1(s)Y(s) = \frac{H_1(s)}{G_2(s)}A(s)$$

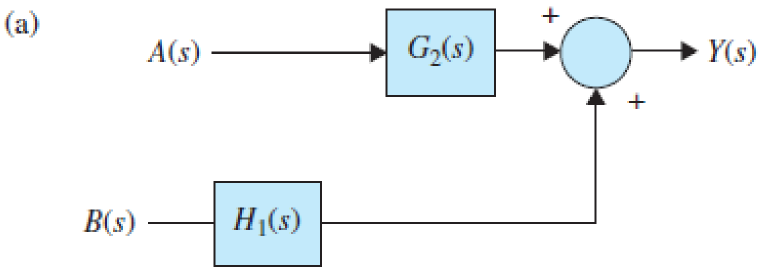


(b) $Y(s) = G_2(s)A(s)$

$$B(s) = \frac{H_1(s)}{G_2(s)}A(s)$$

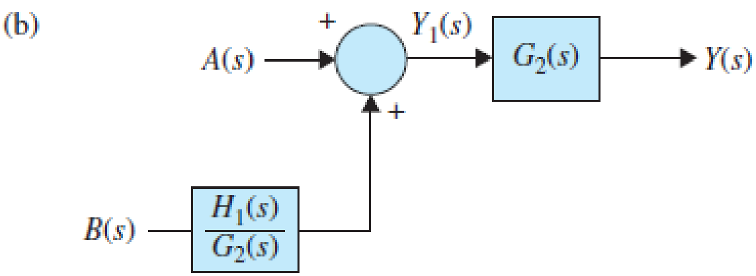
Block Diagram Reduction (Continued)

- Moving a comparator



(a)

$$Y(s) = G_2(s)A(s) + H_1(s)B(s)$$

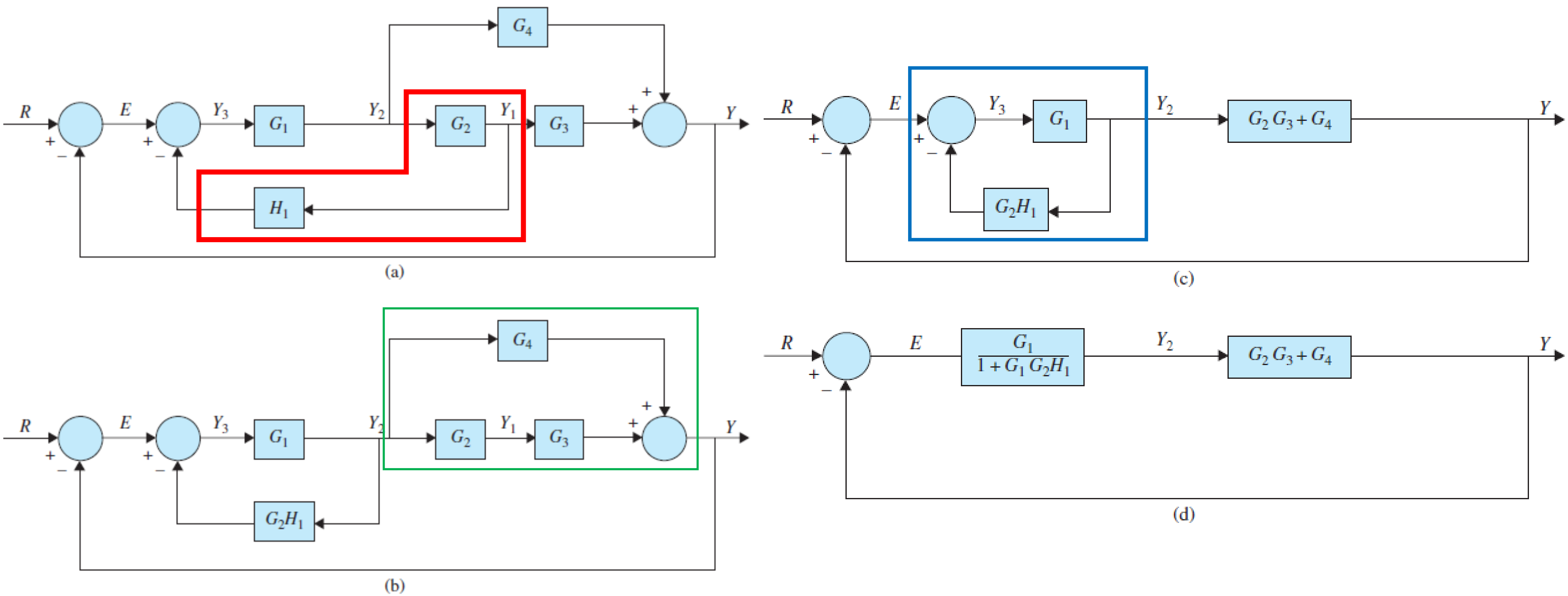


(b)

$$Y_1(s) = A(s) + \frac{H_1(s)}{G_2(s)}B(s)$$

$$Y(s) = G_2(s)Y_1(s) = G_2(s)A(s) + H_1(s)B(s)$$

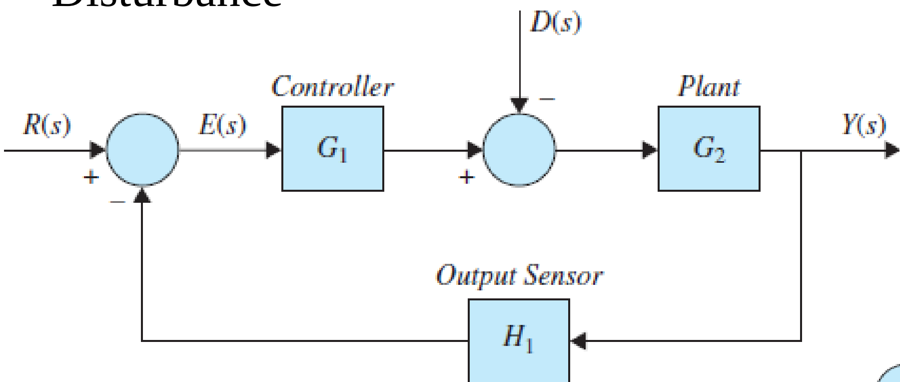
Example 1



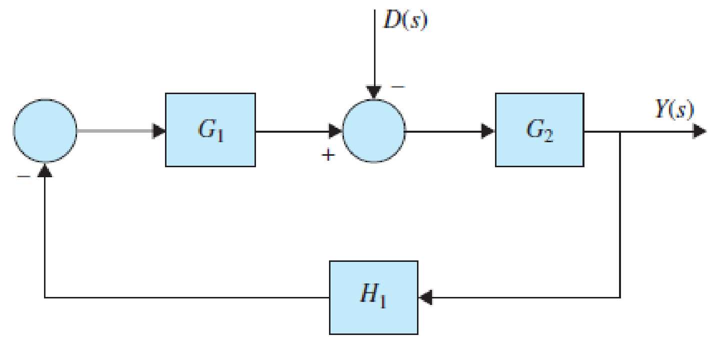
$$\frac{Y(s)}{R(s)} = \frac{\dfrac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1}}{1 + \dfrac{G_1(G_2G_3 + G_4)}{1 + G_1G_2H_1}} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_1G_2G_3 + G_1G_4}$$

Block Diagrams with Multiple Inputs

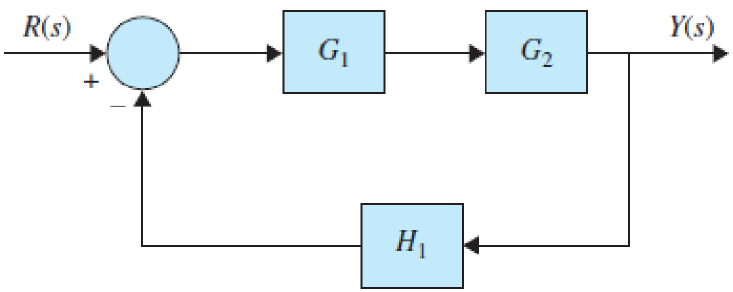
Feedback Control System with Disturbance



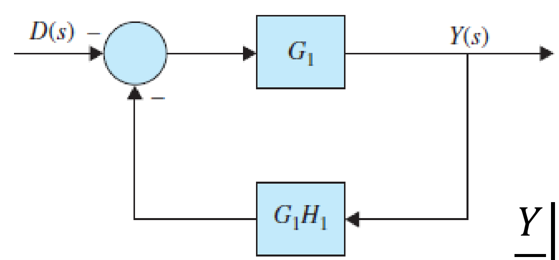
$$Y = G_2(-D + G_1E) = -G_2D + G_1G_2(R - H_1Y)$$
$$\Rightarrow (1 + G_1G_2H_1)Y = -G_2D + G_1G_2R$$
$$\Rightarrow Y = \frac{G_1G_2}{1 + G_1G_2H_1}R - \frac{G_2}{1 + G_1G_2H_1}D$$



(a)



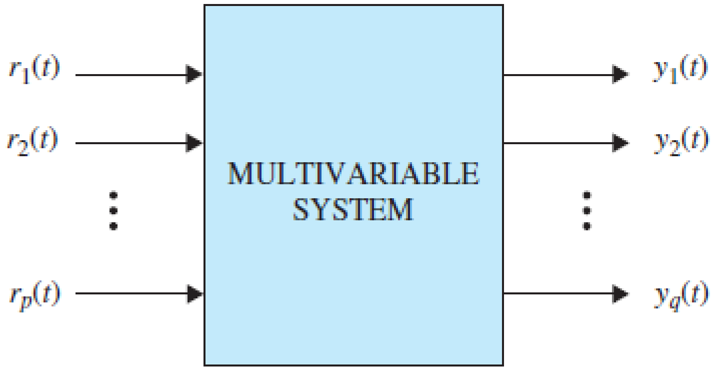
$$\left.\frac{Y}{R}\right|_{D=0} = \frac{G_1G_2}{1 + G_1G_2H_1}$$



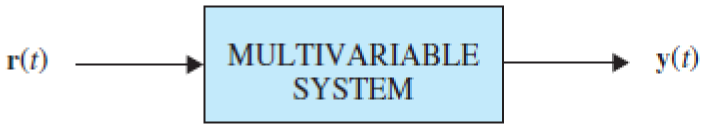
(b)

$$\left.\frac{Y}{D}\right|_{R=0} = -\frac{G_2}{1 + G_1G_2H_1}$$

Multivariable Systems



(a)



(b)

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{R}(s)$$

$$\mathbf{Y}(s) \in \mathbb{C}^q, \mathbf{R}(s) \in \mathbb{C}^p$$

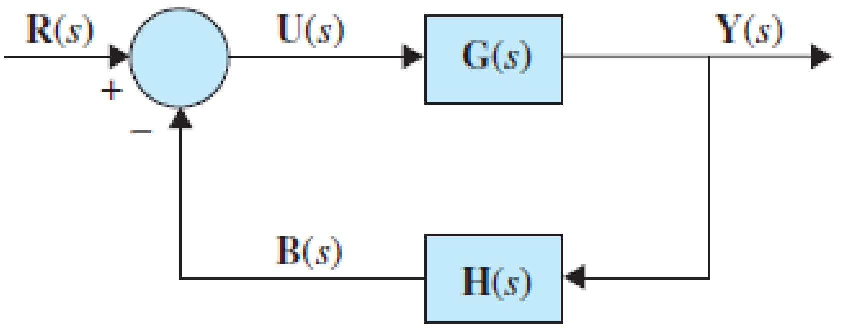
$\mathbf{G}(s) \in \mathbb{C}^{q \times p}$: **Transfer Function Matrix**

In the previous example $p = 1$ and $q = 1$

$$Y(s) = \frac{G_1 G_2}{1 + G_1 G_2 H_1} R - \frac{G_2}{1 + G_1 G_2 H_1} D$$

$$= \frac{G_2(s)}{1 + G_1(s) G_2(s) H_1(s)} \begin{bmatrix} G_1(s) & -1 \end{bmatrix} \begin{bmatrix} R(s) \\ D(s) \end{bmatrix}$$

Closed-Loop Systems



$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$
$$\mathbf{U}(s) = \mathbf{R}(s) - \mathbf{B}(s) = \mathbf{R}(s) - \mathbf{H}(s)\mathbf{Y}(s)$$

$\mathbf{Y}(s) \in \mathbb{C}^q, \mathbf{U}(s), \mathbf{R}(s), \mathbf{B}(s) \in \mathbb{C}^p$

$\mathbf{G}(s) \in \mathbb{C}^{q\times p}, \mathbf{H}(s) \in \mathbb{C}^{p\times q}$

(1)

$$\mathbf{Y}(s) = \mathbf{G}(s)[\mathbf{R}(s) - \mathbf{H}(s)\mathbf{Y}(s)]$$
$$\Rightarrow \left[\mathbf{I}_q + \mathbf{G}(s)\mathbf{H}(s)\right]\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{R}(s)$$
$$\Rightarrow \mathbf{Y}(s) = \left[\mathbf{I}_q + \mathbf{G}(s)\mathbf{H}(s)\right]^{-1}\mathbf{G}(s)\mathbf{R}(s)$$

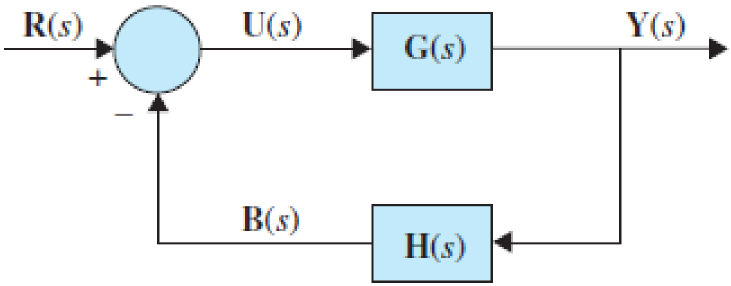
(2)

$$\mathbf{U}(s) = \mathbf{R}(s) - \mathbf{H}(s)\mathbf{G}(s)\mathbf{U}(s)$$
$$\Rightarrow \left[\mathbf{I}_p + \mathbf{H}(s)\mathbf{G}(s)\right]\mathbf{U}(s) = \mathbf{R}(s)$$
$$\Rightarrow \mathbf{U}(s) = \left[\mathbf{I}_p + \mathbf{H}(s)\mathbf{G}(s)\right]^{-1}\mathbf{R}(s)$$
$$\Rightarrow \mathbf{Y}(s) = \mathbf{G}(s)\left[\mathbf{I}_p + \mathbf{H}(s)\mathbf{G}(s)\right]^{-1}\mathbf{R}(s)$$

The closed-loop transfer function matrix is

$$\mathbf{M}(s) = \left[\mathbf{I}_q + \mathbf{G}(s)\mathbf{H}(s)\right]^{-1}\mathbf{G}(s) = \mathbf{G}(s)\left[\mathbf{I}_p + \mathbf{H}(s)\mathbf{G}(s)\right]^{-1}$$

Example 2



$$\mathbf{G}(s) = \begin{bmatrix} \dfrac{1}{s+1} & -\dfrac{1}{s} \\ 2 & \dfrac{1}{s+2} \end{bmatrix}, H(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

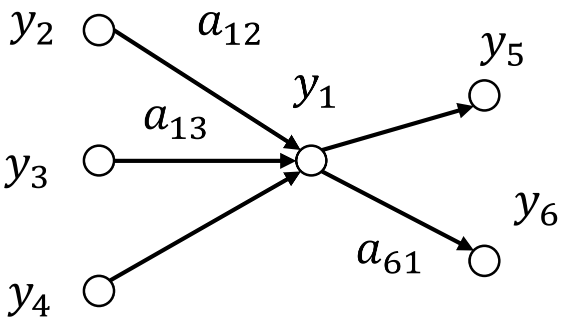
$$\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s) = \begin{bmatrix} 1 + \dfrac{1}{s+1} & -\dfrac{1}{s} \\ 2 & 1 + \dfrac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \dfrac{s+2}{s+1} & -\dfrac{1}{s} \\ 2 & \dfrac{s+3}{s+2} \end{bmatrix}$$

$$\mathbf{M}(s) = [\mathbf{I} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s) = \frac{1}{\Delta}\begin{bmatrix} \dfrac{s+3}{s+2} & \dfrac{1}{s} \\ -2 & \dfrac{s+2}{s+1} \end{bmatrix}\begin{bmatrix} \dfrac{1}{s+1} & -\dfrac{1}{s} \\ 2 & \dfrac{1}{s+2} \end{bmatrix}$$

$$= \frac{s(s+1)}{s^2+5s+2}\begin{bmatrix} \dfrac{3s^2+9s+4}{s(s+1)(s+2)} & -\dfrac{1}{s} \\ 2 & \dfrac{3s+2}{s(s+1)} \end{bmatrix} \qquad \left(\Delta = \frac{s+2}{s+1}\cdot\frac{s+3}{s+2} + \frac{2}{s} = \frac{s^2+5s+2}{s(s+1)}\right)$$

Signal Flow Graph

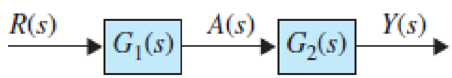
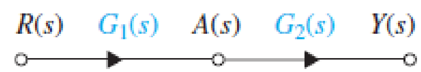
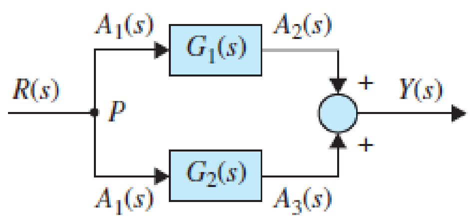
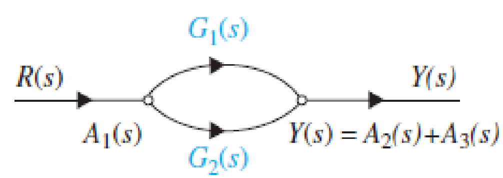
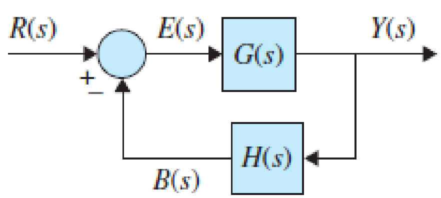
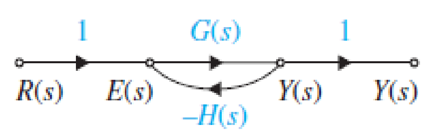
- The signal flow graph (SFG) is an alternative representation of the block diagram.
- The SFG consists of *nodes* and *directional branches*.
  - Node: variable or signal
  - Branch: (linear) operation (e.g. transfer functions, gains, etc.). The operation is denoted by the branch. If there is no notation associated with a branch, it denotes a unity gain.
- The branch connects two nodes, with the arrow denotes the direction of the signal flow.
- A node can have several entering and leaving branches.
- **The value of a node is the sum of all entering branches.**



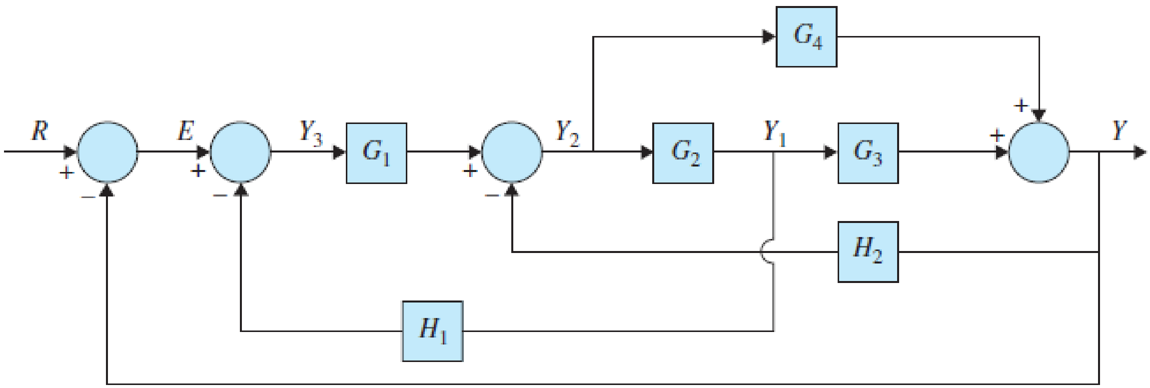
$$y_1 = a_{12}y_2 + a_{13}y_3 + y_4$$
$$y_5 = y_1$$
$$y_6 = a_{61}y_1$$

Input node (source): node without entering branches
Output node (sink): node without leaving branches

Basic Structure

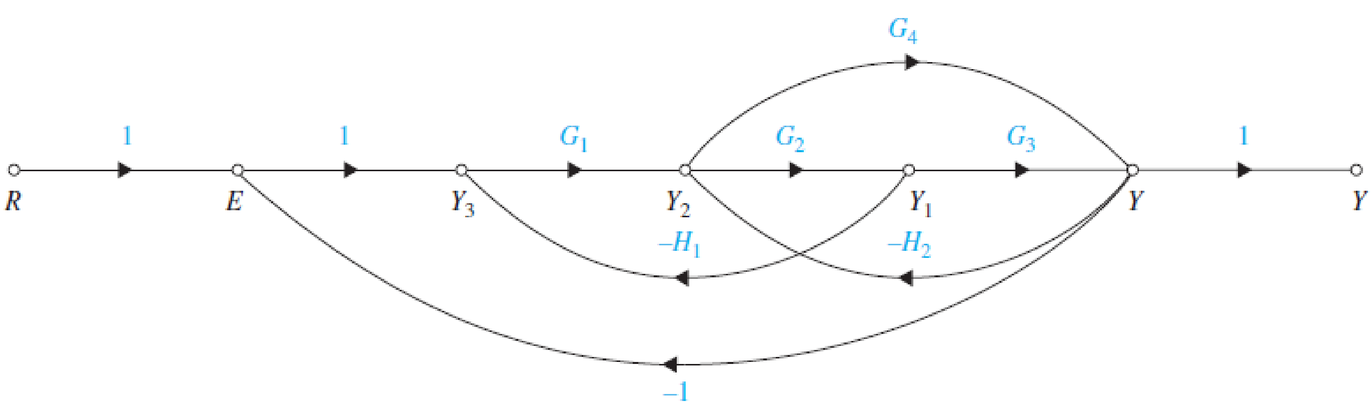
Transfer Function	Block Diagram	Signal Flow Diagram
One block System $\frac{Y(s)}{R(s)} = G(s)$	(a) 	(b) 
Cascade $\frac{Y(s)}{R(s)} = G_1(s)\,G_2(s)$	(c) 	(d) 
Parallel $\frac{Y(s)}{R(s)} = G_1(s) + G_2(s)$	(e) 	(f) 
Feedback $\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$	(g) 	(h) 

Example 3



(a)

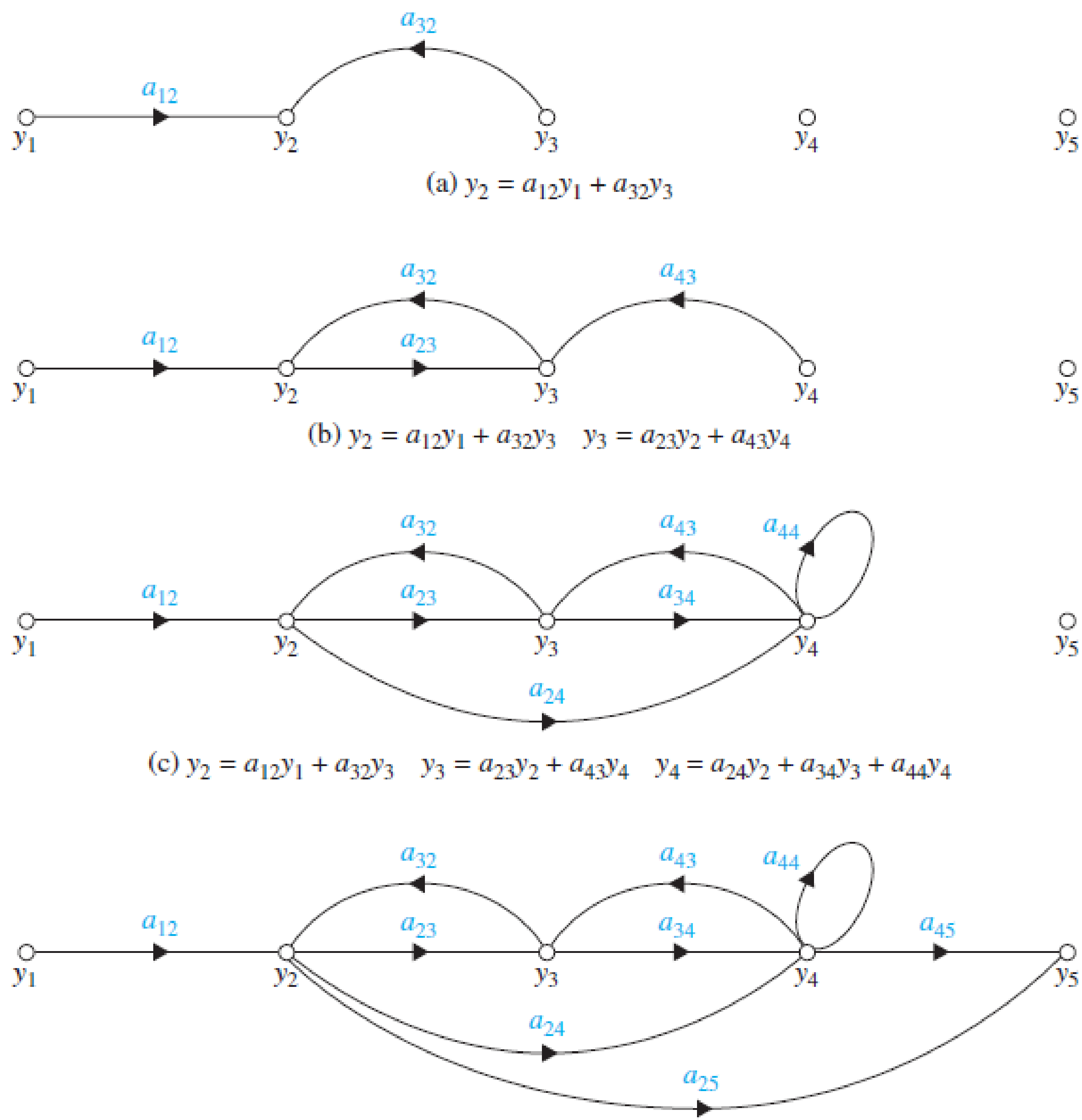
(b)



(c)

Example

$$y_2 = a_{12}y_1 + a_{32}y_3$$
$$y_3 = a_{23}y_2 + a_{43}y_4$$
$$y_4 = a_{24}y_2 + a_{34}y_3 + a_{44}y_4$$
$$y_5 = a_{25}y_2 + a_{45}y_4$$



(d) Complete signal-flow graph