Europ. J. Combinatorics (1999) **20**, 375–384
Article No. eujc.1998.0281
Available online at http://www.idealibrary.com on IDEAL®

A Small Aperiodic Set of Planar Tiles

CHAIM GOODMAN-STRAUSS

We give a simple set of two tiles that can only tile aperiodically—that is no tiling with these tiles is invariant under any infinite cyclic group of isometries. Although general constructions for producing aperiodic sets of tiles are finally appearing, simple aperiodic sets are fairly rare. This set is among the smallest sets ever found.

A tiling is *non-periodic* if there is no infinite cyclic group of isometries leaving the tiling invariant. In $\mathbb{E}^2$, this is equivalent to requiring that no translation leaves the tiling invariant. A set of tiles is *aperiodic* if it is possible to tile completely the plane with congruent copies of the tiles, but *only* non-periodically. For example, a pair of unit squares, one black and one white, is *not* an aperiodic set of tiles: it is possible to tile non-periodically with black and white squares but they can tile periodically as well.

Here we give a new, simple example of a set of aperiodic tiles, the T (trilobite) and C (cross) (Figure 1); in any tiling with these tiles, we will require that the 'tips' of the tiles meet as pictured at the right of the figure. (A local condition such as this is a 'matching rule'). Two variations of the tiles are given at the end of this paper. These tiles are among the simplest ever found, and are related to a family of aperiodic sets of two tiles in each $\mathbb{E}^n$, $n \geq 3$ [10].

The reader may wish to examine a photocopy of the appendix with a pair of scissors.

It has been many years since a planar aperiodic set of, say, six or fewer tiles has been found. In all, this new set is only one of a handful of known aperiodic sets of only two tiles, and only the second in which the tiles occur in only eight translation classes. On both counts, the set is tied for smallest known in $\mathbb{E}^2$ at this time.

We should list other notably small sets of tiles: Robinson gave the first small aperiodic set, requiring only six tiles [23]. The Penrose tiles occur in at least three variations with two tiles each occurring in at least 20 translation classes [6, 12, 20]. Amman's sets A2, A3, A4, and A5 have 2, 3, 2, and 2 tiles each, occurring in 8, 12, 16 and 24 translation classes [1, 12]. Kari's aperiodic set has 14 tiles, which is larger, but each occurs in only one translation class, so the number of translation classes is small [15]. This was improved on by Culik who reduced this to 13 tiles and translation classes. Very recently, Penrose found a new aperiodic set with three tiles in 30 translation classes [21]. Socolar [28] and Danzer (Section 6.3 of [19]) each have an aperiodic set of three tiles, occurring in 144 and 168 translation classes respectively. Notably, Danzer's tiles admit only tilings with local seven-fold symmetry. To the author's knowledge, this is a complete list of all known two-dimensional aperiodic sets with, say, no more than six tiles or occurring in no more than 50 translation classes.

In higher dimensions, few aperiodic sets are explicitly known—and fewer simple examples. In $\mathbb{E}^3$, Danzer has an aperiodic set of four tiles [4]. Schmitt has stated he has a method of constructing aperiodic sets of just three tiles in $\mathbb{E}^n$, $n > 2$; the three-dimensional version is given in [25]. In $\mathbb{E}^n$, $n > 2$ the author has an aperiodic set of two tiles [10].

Schmitt has produced a single tile that produces only *non-translational* tilings of $\mathbb{E}^3$; often it is said that this is an aperiodic tile. However, this example and others like it demonstrate that non-periodicity really should be defined as not being invariant under any infinite cyclic group of isometries. We would prefer to call Schmitt's tile *weakly aperiodic* [24]. A related construction, first discovered by Conway, appears in [5].

0195–6698/99/050375 + 10 $30.00/0

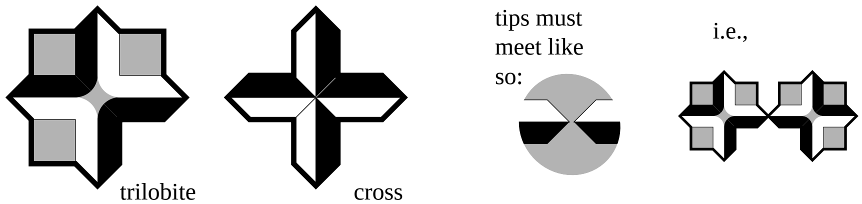


FIGURE 1. The Trilobite and Cross.

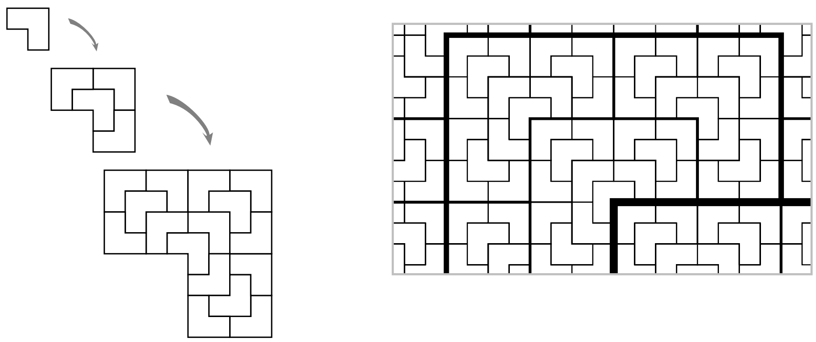


FIGURE 2. The L-substitution and a portion of an L-tiling.

We must note that Penrose has given a *single* tile that can only tile non-periodically, with simple but quite non-standard matching rules [21], and that it appears that his technique can be adapted to produce infinitely many other distinct examples. Moreover, Gummelt [13], Jeong and Steinhardt [14], and Senechal [27] have shown that a single marked *shingle* can enforce the structure of the Penrose tilings. It seems valuable to expand and synthesize our basic definitions to account for the seeming variety of possible definitions and settings (cf. [11]).

We now turn to:

THEOREM 1. *The trilobite and cross are an aperiodic set of tiles.*

We must show that they do tile the plane and that no tiling of the plane with the tiles is periodic. The proof is quite typical for a 'hierarchical' set of tiles; in broadest outline, all known proofs that a given set of tiles forms only hierarchical tilings are the same. We will present the proof in an informal style. Many of the ideas are presented in a more technical fashion in [10].

The trilobite and cross exploit the structure given by the L-*substitution* (or L-*inflation*) shown on the left of Figure 2. We begin with an L-shaped tile, and repeatedly *expand and subdivide*, as shown. Larger and larger patches of L-tiles, arranged hierarchically, emerge through this process.

An L-*tiling* is a tiling with L-tiles such that every bounded collection of tiles in the tiling is the image of a collection of tiles in some inflated L-tile—in short, every part of an L-tiling 'looks' like the interior of an inflated L-tile. That there exist well-defined tilings satisfying this condition is proven in [7, 12] and elsewhere.

In particular, note that each L-tile in each L-tiling lies in a unique inflated L-tile of any given size, as illustrated on the right of Figure 2. (The thick lines have been added to emphasize the hierarchy.)

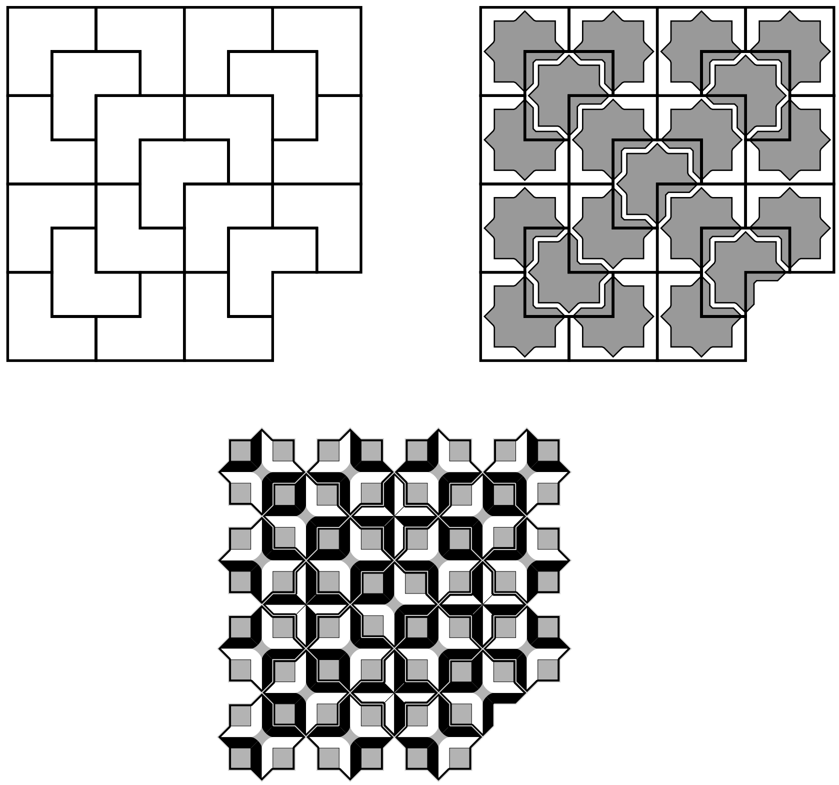


FIGURE 3. L-tilings can be recomposed into tilings with Trilobites and Crosses.

Now, suppose there is an L-tiling that is invariant under some infinite cyclic group of isometries. In the plane at least, such a group has a subgroup generated by some translation, and the L-tiling will be invariant under this translation. But then some giant inflated L-tile will intersect its translated image; any tile in this intersection will then lie in a non-unique inflated L-tile of a given size. This is a contradiction and we have proven:

LEMMA 1. *No L-tiling arising from the L-substitution system is invariant under some infinite cyclic group of isometries.*

The following Lemma serves to show that the trilobite and cross do in fact tile the plane:

LEMMA 2. *Every L-tiling can be recomposed into a tiling with trilobites and crosses.*

PROOF. Given an L-tiling, note every L-tile 'contains' a trilobite (upper right of Figure 3). We can fill in these trilobites into an L-tiling; that there are no overlaps rests on the observation that the 'elbow' of any L-tile always meets one of the outer corners of some other L-tile (upper left of Figure 3). We can be sure the tips of adjacent trilobites satisfy our matching rule by a simple inductive argument on the inflated L-tiles.

That the remaining gaps will be cross-shaped rests on the observation that if an outer corner of an L-tile does not meet the 'elbow' of some other L-tile, it meets the outer corners of three other L-tiles (see Figure 3). We only need to note that the crosses can be placed in a manner consistent with our matching rule.

Consider any string of edges lying on a straight line in any L-tiling. Such a string is to be recomposed into a string of crosses. Any such string of edges can either be propagated forever or

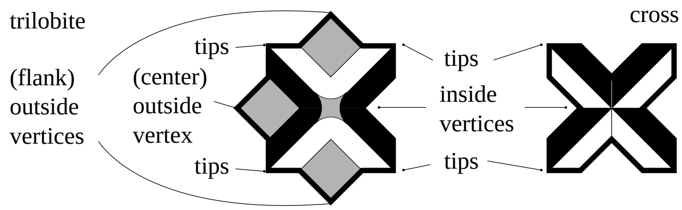


FIGURE 4. Vocabulary.

terminates at L-tiles on either end. These two tiles must be reflections of each other across a line perpendicular to the string of edges (this can be verified through induction on the inflations of the L-tiles). But then the markings propagated along this edge are fixed (by the orientations of the L-tiles at the end) and are consistent. If the string is infinite in one direction, the L-tile at the finite end fixes the marking; if the string is infinite in both directions, we have a choice of markings.

In any case, the L-tiling can be recomposed into trilobites and crosses. □

We categorize tilings with the trilobite T and cross C. To facilitate discussion, we give some terms in Figure 4. First, as tips may only meet other tips, the *inside* vertex of the trilobite tile can only meet an *outside* vertex of some other trilobite. Similarly, the outside vertices of a trilobite tile can only meet the inside vertices of either the cross or trilobite, and thus, reading off the sequence of trilobites and crosses in order across its outside vertices, a trilobite is one of six types, up to reflection: TTT, CTC, CCC, CTT, CCT, or TCT.

Note that when we recompose an L-tiling as in Lemma 2, into trilobites and crosses, the trilobites are all of the form TTT, CTC, CCC.

We can immediately show that the configurations CCT and TCT cannot occur. For if a cross is at the centre outside vertex and a trilobite on one of the flanking outside vertices, no tile can be placed between these without violating the matching rule (Figure 5).

Suppose there is a tile t of type CTT. Then the trilobite at the centre outside vertex must also be of this type, with the sequence of tiles reversed; i.e., of the form TTC. Furthermore, the inside vertex of t can only meet the outside central vertex of another trilobite, or the matching rules will be violated. This trilobite, it follows, must also be of the type TTC. So any occurrence of a trilobite of type CTT can only be in an infinite chain γ of alternating CTT and TTC tiles. Note that if there are two such chains, they cannot cross. In a tiling with a γ chain, consider the result of sliding one of the components of the complement of the chain, as illustrated in Figure 6. Our chain γ will be transformed into a chain α of alternating CTC and TTT trilobites; by a series of slides we can eliminate all γ chains and obtain a new well-formed tiling with only CCC, CTC and TTT trilobites.

So, consider tilings in which there are only CCC CTC and TTT trilobites. The reader should check that the interior vertex of any trilobite of type CCC or CTC must meet the outside vertex of a trilobite of type TTT; conversely, the outside vertices of a trilobite of type TTT can only meet the inside vertex of a trilobite of CTC or CCC. We can thus say that the trilobites must clot into clusters of four, with a tile of type TTT at the center and types CCC and CTC arranged about the outside (Figure 7). But now we are nearly done.

We now observe that our clusters of four trilobites—'2-trilobites'—are essentially large trilobites themselves when we consider how they may fit together (Figure 7). This observation is truly typical of all known proofs that establish that a set of tiles forces the emergence of a hierarchical structure.

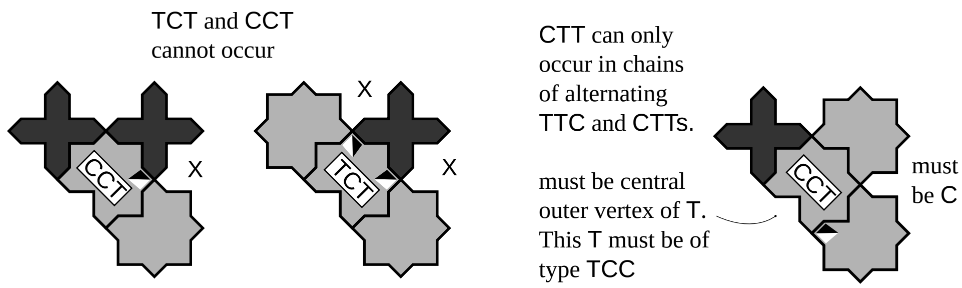


FIGURE 5. Analysis of certain configurations.

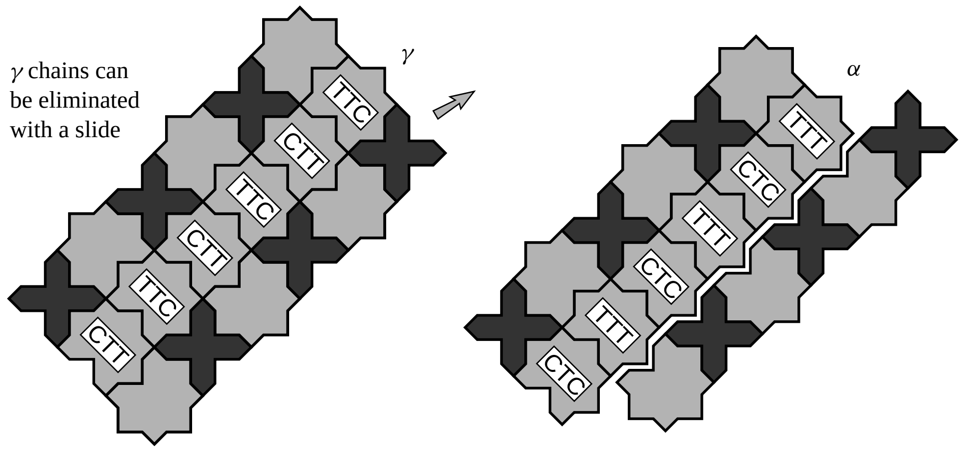


FIGURE 6. Eliminating chains of the form of γ.

In particular, the analysis of Figure 5 applies to the 2-trilobites as well, and the 2-trilobites themselves can only occur in the configurations CCC CTC TTT and CTT. (Where C stands for a cross on the central outer vertex of one of the trilobites in a 2-trilobite. Note the placement and some markings of other crosses are forced.)

Again, we find that any CTT 2-trilobite must occur in an infinite chain γ of alternating CTT and TTC 2-trilobites, that two such chains must be parallel if they occur in the same tiling, and that after eliminating all such chains with a slide, we have a tiling with only CCC CTC and TTT 2-trilobites. These must clot into clusters—3-trilobites—of four 2-trilobites, or 16 of our original trilobites. And the exact same analysis applies to 3-trilobites, and indeed continues *ad infinitum*.

In particular, consider any α chain a of n-trilobites. Such a chain contains exactly one α chain of k-trilobites, $k < n$, running down the centre of a, and itself must lie in the centre of either an α or a γ chain of $n+1$ trilobites. Recalling that γ chains can be eliminated with a slide, and that as n increases, the width of an α chain grows without bound, we observe that:

Suppose a tiling with trilobites and crosses had two distinct γ chains of trilobites. Then these chains are parallel and some distance apart. After some finite number of slides, each of these is transformed into an α chain in the centre of an α chain of width greater than the distance between our initial chains. But each α chain of n-trilobites contains only one chain of k-trilobites, $k < n$. So we have a contradiction, and so in any given tiling with trilobites and

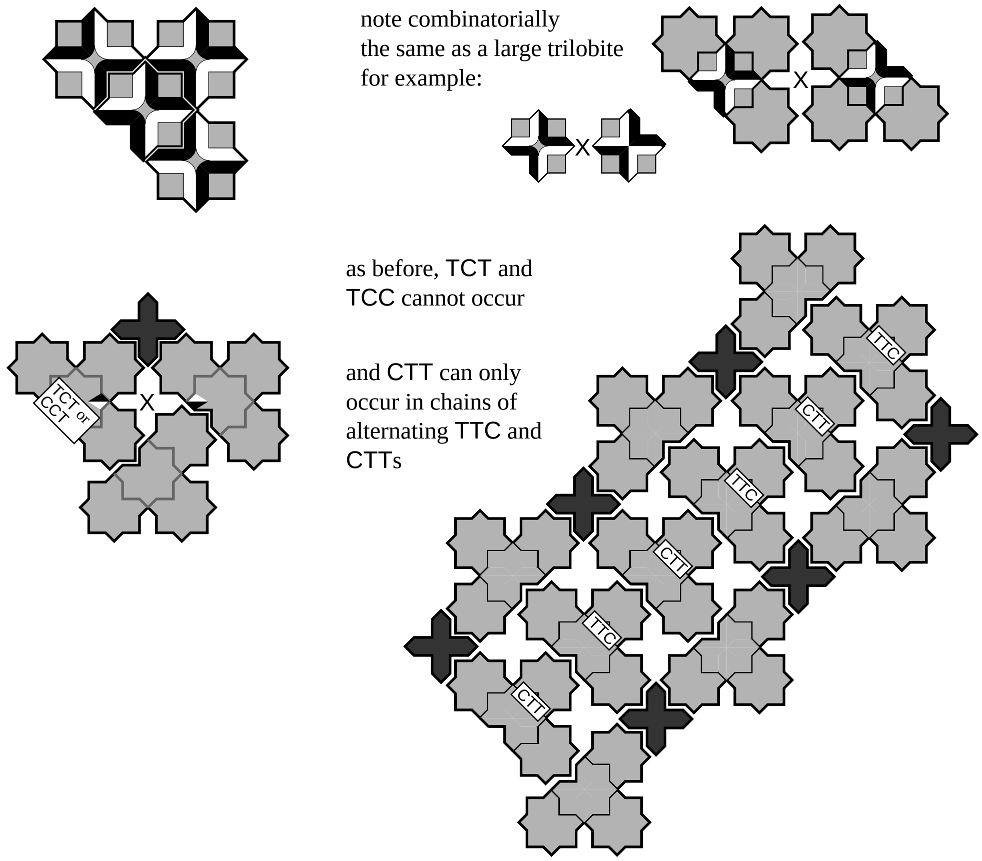


FIGURE 7. Arrangements of clusters of trilobites.

crosses, there is at most one γ chain. Similarly, after transforming all γ chains of k-trilobites, $k < n$ into α chains, one observes there can be only one γ chain of n-trilobites.

Now finally, consider a tiling with trilobites and crosses in which no n-trilobite is of type CTT or TTC. Then each n-trilobite is part of an $n+1$ trilobite. Moreover, each n-trilobite, and adjacent cross tiles, can be recomposed into an inflated L-tile (see Figure 3). And so any tiling in which no n-trilobite is of type CTT or TTC can be recomposed into an L-tiling.

We have proven:

LEMMA 3. *All tilings of the plane with the trilobite and cross tiles* T *and* C *satisfying the matching rules can be recomposed into an* L*-tiling, after a (possibly infinite) series of shifts along concentric parallel γ chains.*

We note:

PROOF OF THEOREM 1. First note that the trilobite and cross *do* tile the plane, by applying Lemma 2. Second, consider any tiling of the plane with the trilobite and cross. If there are no γ chains, then we are done by Lemma 1. So, suppose there is a series of nested γ-chains. Clearly no translation that does not leave these chains invariant leaves the tiling itself invariant, as there can only be one family of these nested chains. Now, consider any translation following

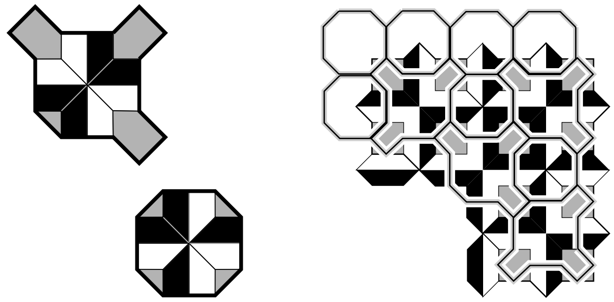


FIGURE 8. A variation of the trilobite and cross.

the chains themselves. With a finite series of shifts, we can recompose our tiling into inflated L-tiles out to any desired distance from the centre of the chains; in particular, we can recompose so that we have a string of inflated L-tiles larger than the magnitude of the translation along the centre of these chains. Now this string of L-tiles is not invariant under translation by the same reasoning as in Lemma 1. But then neither was our original tiling, as all our shifts were parallel to the original translation. □

VARIATIONS

The trilobite and crab are closely related to the Robinson tiles [23] in that 'nearly all' tilings with our set can be recomposed into a tiling by the Robinson tiles and vice-versa. Moreover, Socolar gave an aperiodic set of eight tiles that more explicitly force the structure of the L-tiling and the techniques of [8] give rise to a very large set achieving the same end. But, again, the trilobite and cross form a *small* aperiodic set.

On the other hand, how can we be sure that other small aperiodic sets are not equivalent (in particular the other very small aperiodic set, Ammans' A2). There are several invariants we can check: in particular, ratios of the occurrences of the tiles, the diffraction pattern of the tilings, and the point groups of the tilings.

We can easily show, for example, that in any tiling with the trilobites and crosses, as n goes to infinity, the ratio of trilobites to crosses in any disk of radius n goes to 2:1. On the other hand, in any tiling with the tiles in Amman's A2, the ratio of the two types of tiles goes in any disk of radius n tends to the golden ratio, $\tau = \frac{\sqrt{5}+1}{2}$, as n goes to infinity. As 2 and τ are incommensurable, it follows that there is no set of local transformations taking tilings with trilobites and crosses to tilings in of tiles in A2.

For tilings described by a 'substitution', such as the L-tilings, we have another useful invariant. The L-tilings are defined through an inflation by a factor of 2; Amman's A4 and A5 are defined through an inflation by a factor of $\sqrt{2}+1$. As 2 and $\sqrt{2}+1$ are incommensurate (or more properly, as all powers of 2 and $\sqrt{2}+1$ are incommensurate) we can be sure the L-tilings—and thus tilings with the trilobite and cross—are distinct from tilings by the sets A4 and A5.

In a similar fashion, we see that none of the other known small aperiodic sets are equivalent to ours.

We close with two variations of the trilobite and cross that have simpler matching rules, but are harder to show aperiodic. In the first variation (Figure 8) we only require that black is matched to black, white to white and grey to grey. It is clear that every tiling with the trilobite and cross can be composed into a tiling with these simpler tiles (the centres of the cross and the places where four tips meet become our new crosses and the old trilobites become our new trilobites). The

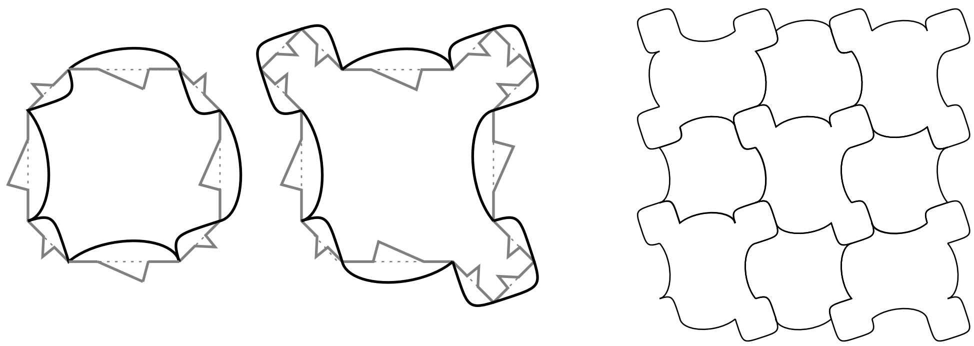


FIGURE 9. A final variation.

converse is not as clear as it may seem. The proof known to the author is a huge combinatorial argument not worth the reader's time. The active reader can easily check how much trouble there might be by trying to imitate the arguments of Figure 5 with these simpler tiles.

Finally, Figure 9 indicates an uncountable family of variations on the new trilobite and new cross. The edges of the tiles fall into two congruence classes. Each edge in each class can be changed simultaneously; one may attempt to produce Escher-like tiles in the shape of chickens, geese, shoes, or whatever else one wishes. On the right is a tiling with one set in this family.

Note that any set in this family has the advantage of purely geometrical matching rules: the only requirement is that our tiles have disjoint interiors and cover the plane.

ACKNOWLEDGEMENTS

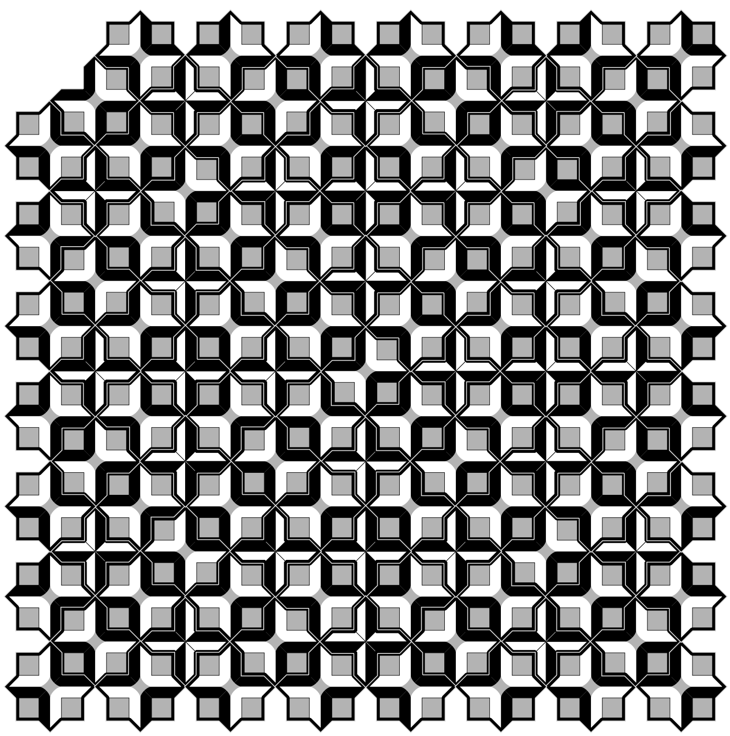
The author gratefully acknowledges the referees' helpful comments and the generous assistance of Doris Schattschneider.

REFERENCES

1. R. Amman, B. Grunbaum and G. C. Shepherd, Aperiodic tiles, *Discrete Comput. Geom.*, **8** (1992), 1–25.
2. R. Berger, The undecidability of the domino problem, *Mem. Am. Math. Soc.*, **66** (1966).
3. N. G. de Bruijn, Algebraic theory of Penrose's non-periodic tilings, *Ned. Akad. Wentensch Proc. Ser. A*, **84** (1981), 39–66.
4. L. Danzer, Three dimensional analogues of the planar Penrose tilings and quasicrystals, *Discrete Math.*, **76** (1989), 1–7.
5. L. Danzer, A family of 3D-spacefillers not permitting any periodic or quasiperiodic tiling, in: *Aperiodic '94*, World Scientific, Singapore, 1995, pp. 11–17.
6. M. Gardner, Extraordinary nonperiodic tiling that enriches the theory of tilings, *Sci. Am.*, **236** (1977), 110–121.
7. C. Goodman-Strauss, Addressing and substitution tilings, in preparation.
8. C. Goodman-Strauss, Matching rules and substitution tilings, *Ann. Math.*, **147** (1998), 181-223.
9. C. Goodman-Strauss, Aperiodic hierarchical tilings, Proceedings of the Nato Adv. Studies Inst. 'Foams, Emulsions and Cellular Materials', Kluwer (to appear).
10. C. Goodman-Strauss, An aperiodic pair of tiles in $\mathbb{E}^n$ for all $n \geq 3$, *Europ. J. Combinatorics*, **20** (1999), 385–395.
11. C. Goodman-Strauss, Matching rules and markings, in preparation.
12. B. Grünbaum and G. C. Shepherd, in: *Tilings and Patterns*, W. H. Freeman and Co, 1987.
13. P. Gummelt, Penrose tilings as coverings of congruent decagons, *Geom. Ded.*, **62** (1996), 1–17.

14. H.-C. Jeong and P. J. Steinhardt, Constructing Penrose-like tilings from a single prototile and the implications for quasicrystals, *Phys. Rev. B*, **55** (1997), 3520–3532.
15. J. Kari, A small aperiodic set of Wang tiles, *Discrete Math.*, **160** (1996), 259–264.
16. T. T. Q. Le, Local rules for quasiperiodic tilings, Proceedings of NATO Adv. Studies Inst., 'The Mathematics of Long Range Aperiodic order', (ed. R. Moody), pp. 331–366.
17. D. Levine and P. J. Steinhardt, Quasicrystals: a new class of ordered structures, *Phys. Rev. Lett.*, **53** (1984), 2477–2480.
18. S. Mozes, Tilings, substitution systems and dynamical systems generated by them, *J. D'Analyse Math.*, **53** (1989), 139–186.
19. K.-P. Nischke and L. Danzer, A construction of inflation rules based on n-fold symmetry, *Discrete Comput. Geom.*, **15** (1996), 221-236.
20. R. Penrose, The role of aesthetics in pure and applied mathematical research, *Bull. Inst. Math. Appl.*, **10** (1974) 266–271.
21. R. Penrose, Remarks on tiling: details of a $(1 + x + x^2)$-aperiodic set, in: *The Mathematics of Long-Range Aperiodic Order*, World Scientific, Singapore, 1997.
22. C. Radin, The pinwheel tilings of the plane, *Ann. Math.*, **139** (1994), 661–702.
23. R. Robinson, Undecidability and nonperiodicity of tilings in the plane, *Inverse Math.*, **12** (1971), 177–209.
24. P. Schmitt, An aperiodic prototile in space, informal notes, Vienna (1988).
25. P. Schmitt, Triples of prototiles (with prescribed properties) in space (A quasiperiodic triple in space), *Per. Math. Hung.*, **34** (1997), 143–152.
26. M. Senechal, in: *Quasicrystals and Geometry*, Cambridge University Press, Cambridge, 1995.
27. M. Senechal, private communication.
28. J. E. S. Socolar Simple octagonal and dodecagonal quasicrystals, *Phys. Rev. A*, **39** (1989), 10519–10551.

APPENDIX



Received 22 February 1998 and accepted 5 December 1998

C. GOODMAN-STRAUSS
Department of Mathematics,
University of Arkansas,
Fayetteville, AR 72701,
U.S.A.
E-mail: `cgstraus@comp.uark.edu`