AA530 HW2

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https://github.com/russellmatt66/aa530-hw/tree/main/hw2

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The code performing the computations done for this assignment, the results, and the assignment containing the problem statements can all be found at the above Github repo.

1 Equilibrium & Cauchy Stress

1.1 Cauchy's Formula

The side lengths of the infinitesimal trianglular solid element are dx and dy. This gives the following trigonometric relationships w.r.t α,

$$cos(\alpha) = \frac{dy}{\sqrt{dx^2 + dy^2}} \tag{1}$$

$$sin(\alpha) = \frac{dx}{\sqrt{dx^2 + dy^2}} \tag{2}$$

Armed with this basic information the sum of force equations are now focused on and formulated for the situation where the triangular body is in static equilibrium between the traction and the normal and shear stresses applied over the corresponding length,

$$\Sigma F_x = -\sigma_1 dy - \tau_{21} dx + T_1 \sqrt{dx^2 + dy^2} = 0 \tag{3}$$

$$\Sigma F_y = -\sigma_2 dx - \tau_{12} dy + T_2 \sqrt{dx^2 + dy^2} = 0 \tag{4}$$

collecting the variables and infinitesimals together on the right yields a system of two equations,

$$T_1 = \sigma_1 \frac{dy}{\sqrt{dx^2 + dy^2}} + \tau_{21} \frac{dx}{\sqrt{dx^2 + dy^2}} \tag{5}$$

$$T_2 = \sigma_2 \frac{dx}{\sqrt{dx^2 + dy^2}} + \tau_{12} \frac{dy}{\sqrt{dx^2 + dy^2}} \tag{6}$$

invoking the trigonometric relationships in Equations (1) and (2) to simplify the tractions reveals an underlying angular dependence,

$$T_1 = \sigma_1 cos(\alpha) + \tau_{21} sin(\alpha) \tag{7}$$

$$T_2 = \sigma_2 sin(\alpha) + \tau_{12} cos(\alpha) \tag{8}$$

One way to determine the unit vector normal to the length of cut is based on examining the behavior as α is varied. When $\alpha = 0$ then $\hat{n} = \hat{x} = \hat{e}_1$ and when $\alpha = \frac{\pi}{2}$ then $\hat{n} = \hat{y} = \hat{e}_2$. These boundary conditions suggest a sinusoidal character,

$$\hat{n} = cos(\alpha)\hat{x} + sin(\alpha)\hat{y} \tag{9}$$

based on this, and the expansion of the (in-plane) stress tensor as,

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_1 & \tau_{12} \\ \tau_{21} & \sigma_2 \end{pmatrix} \tag{10}$$

the system of Equations (7) and (8) can be written as,

$$\vec{T} = \hat{n}^T \underline{\underline{\sigma}} \tag{11}$$

which is Cauchy's formula.

1.2 Calculate Normal and Shear Tractions

In general, to determine the component of a vector, $\vec{A}$, that lies in a given direction it obtains to compute the projection of $\vec{A}$ onto a unit vector pointing in the given direction. Therefore, the normal and shear tractions are computed according to,

$$T_n = \vec{T} \cdot \hat{n} \tag{12}$$

$$T_\tau = \vec{T} \cdot \hat{e}_\tau \tag{13}$$

Beginning with Equation (12), it is computed using Equation (9) and,

$$\vec{T} = [T_1, T_2]^T \tag{14}$$

to give,

$$T_n = T_1 cos(\alpha) + T_2 sin(\alpha) \tag{15}$$

To compute Equation (13) it is required to first determine $\hat{e}_\tau$. This unit vector represents the direction of shear and is known up to a negative sign, i.e, a phase of π. To compute it, the direction pointing from the upper-left vertex to the bottom-right vertex is chosen. Performing the vector algebra and using Equations (1) and (2) gives,

$$\hat{e}_\tau = \frac{dx\hat{x} - dy\hat{y}}{\sqrt{dx^2 + dy^2}} \tag{16}$$

$$\rightarrow \hat{e}_\tau = sin(\alpha)\hat{x} - cos(\alpha)\hat{y} \tag{17}$$

which can be inserted into Equation (13) so that the computation may be carried out. Doing so obtains,

$$T_\tau = T_1 sin(\alpha) - T_2 cos(\alpha) \tag{18}$$

The last step in the process is to insert the expression for the tractions, Equations (7) and (8), into Equations (15) and (18). Doing so yields,

$$T_n = \sigma_1 cos^2(\alpha) + \sigma_2 sin^2(\alpha) + (\tau_{21} + \tau_{12}) cos(\alpha) sin(\alpha) \tag{19}$$

$$T_\tau = (\sigma_1 - \sigma_2) sin(\alpha) cos(\alpha) + \tau_{21} sin^2(\alpha) - \tau_{12} cos^2(\alpha) \tag{20}$$

Now, by using the following trigonometric identities,

$$cos^2(\alpha) = \frac{1 + cos(2\alpha)}{2}$$

$$sin^2(\alpha) = \frac{1 - cos(2\alpha)}{2}$$

$$sin(\alpha)cos(\alpha) = \frac{1}{2} sin(2\alpha)$$

Equations (19) and (20) can also be written as,

$$T_n = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)cos(2\alpha) + \tau_{xy}sin(2\alpha) \tag{21}$$

$$T_\tau = \frac{1}{2}(\sigma_1 - \sigma_2)sin(2\alpha) - \tau_{xy}cos(2\alpha) \tag{22}$$

where $\tau_{12} = \tau_{21} = \tau_{xy}$.

1.3 Maximum Normal Stress

Differentiating Equation (19) w.r.t to α, using some well-known trig identities, and setting the resulting expression equal to 0 obtains,

$$(-\sigma_1 + \sigma_2)sin(2\alpha) + (\tau_{21} + \tau_{12})cos(2\alpha) = 0 \tag{23}$$

which can be rearranged to yield the general condition on α,

$$\frac{\sigma_2 - \sigma_1}{\tau_{21} + \tau_{12}} = \frac{cos(2\alpha)}{sin(2\alpha)} = cot(2\alpha) \tag{24}$$

which corresponds to the angle where the normal stress is a maximum.

Plugging in the values given for the stresses in the assignment results in,

$$\alpha_{max,n} = \frac{1}{2}cot^{-1}(\frac{30-10}{-10-10}) = \frac{1}{2}cot^{-1}(-1) = 67.5^\circ \tag{25}$$

1.4 Maximum Shear Stress

A similar analysis can be done for the shear traction (stress) to obtain the condition on α where the shear stress is a maximum,

$$\frac{\sigma_2 - \sigma_1}{\tau_{21} + \tau_{12}} = \frac{sin(2\alpha)}{cos(2\alpha)} = tan(2\alpha) \tag{26}$$

this expression and the expression for where the maximum normal stress occurs are plotted together in Figure 1.

Plugging in the values given for the stresses by the problem statement results in,

$$\alpha_{max,\tau} = \frac{1}{2}tan^{-1}(\frac{30-10}{20}) = \frac{1}{2}tan^{-1}(-1) = 67.5^\circ \tag{27}$$

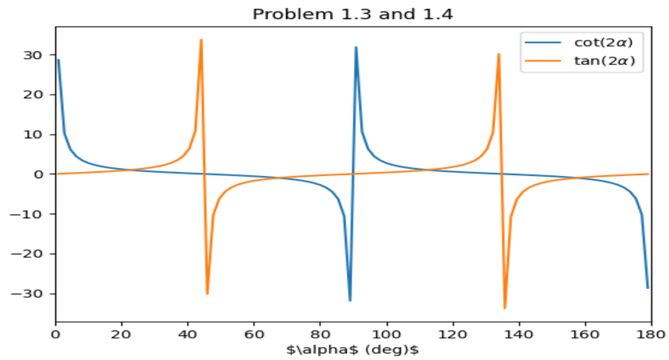

Figure 1: The characteristic line based on the relative difference and sum of the given stresses, $\sigma_1, \sigma_2, \tau_{12}, \tau_{21}$ is missing from the above graph. The intersection of this line with the appropriate curve gives the location of the extremum angle.

1.5 Relationship between the two angles

The α obtained in 1.3 is the same as the α obtained in 1.4.

1.6 Trajectory of Normal and Shear Stresses

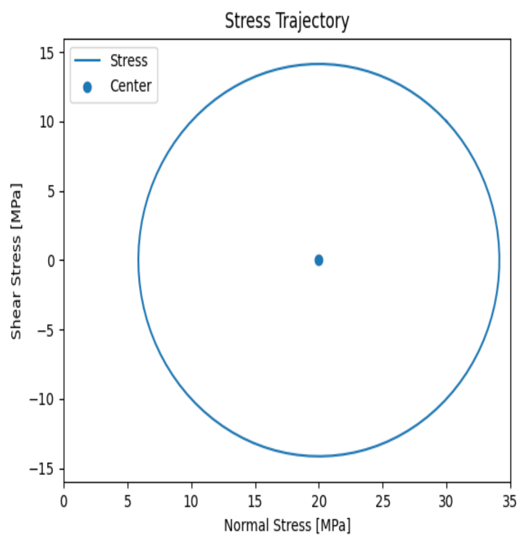

Figure 2: The circular trajectory of the normal and shear stresses for the case $\sigma_1 = 30$ [MPa], $\sigma_2 = 10$ [MPa], $\tau_{12} = \tau_{21} = -10$ [MPa].

1.7 Mohr's Circle

Starting from the parametric equations for Mohr's circle obtained from the equilibrium force balance done above we have,

$$T_n = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)cos(2\alpha) + \tau_{xy}sin(2\alpha) \tag{28}$$

$$T_\tau = \frac{1}{2}(\sigma_1 - \sigma_2)sin(2\alpha) - \tau_{xy}cos(2\alpha) \tag{29}$$

to obtain a non-parametric expression for the above in a sensible form we first label,

$$\sigma_{avg} = \frac{1}{2}(\sigma_1 + \sigma_2) \tag{30}$$

$$R^2 = \left(\frac{1}{2}(\sigma_1 - \sigma_2)\right)^2 + \tau_{xy}^2 \tag{31}$$

$$\therefore T_n - \sigma_{avg} = \frac{1}{2}(\sigma_1 - \sigma_2)cos(2\alpha) + \tau_{xy}sin(2\alpha) \tag{32}$$

$$T_\tau = \frac{1}{2}(\sigma_1 - \sigma_2)sin(2\alpha) - \tau_{xy}cos(2\alpha) \tag{33}$$

next we square both sides of each equation,

$$(T_n - \sigma_{avg})^2 = \frac{1}{4}(\sigma_1 - \sigma_2)^2 cos^2(2\alpha) + \tau_{xy}^2 sin^2(2\alpha) + (\sigma_1 - \sigma_2)\tau_{xy} cos(2\alpha) sin(2\alpha) \tag{34}$$

$$T_\tau^2 = \frac{1}{4}(\sigma_1 - \sigma_2)^2 sin^2(2\alpha) + \tau_{xy}^2 cos^2(2\alpha) - (\sigma_1 - \sigma_2)\tau_{xy} sin(2\alpha) cos(2\alpha) \tag{35}$$

and add the two expressions together which results in cancellation of the cross-terms and simplification via $\cos^2(2\alpha) + sin^2(2\alpha) = 1$ to obtain,

$$(T_n - \sigma_{avg})^2 + T_\tau^2 = \frac{1}{4}(\sigma_1 - \sigma_2)^2 + \tau_{xy}^2 \tag{36}$$

from the labeling we chose above we can recognize the RHS as R^2 which finally gives,

$$(T_n - \sigma_{avg})^2 + T_\tau^2 = R^2 \tag{37}$$

This is the equation for a circle of radius $R = \sqrt{\frac{1}{4}(\sigma_1 - \sigma_2)^2 + \tau_{xy}^2}$ centered at the coordinates $(\sigma_{avg}, 0)$ (taking T_n as the abscissa) and represents the well-known formula for Mohr's Circle. QED

The principal stresses are defined as the normal stress when the shear stress is zero. Plugging this condition into the equation for Mohr's Circle gives,

$$T_{n,principal} = \sigma_{avg} \pm \sqrt{\frac{1}{4}(\sigma_1 - \sigma_2)^2 + \tau_{xy}^2} \tag{38}$$

these are calculated to be (in units of MPa),

$$T_{n,1} = 34.142 \tag{39}$$

$$T_{n,2} = 5.858 \tag{40}$$

from Figure (2) we can observe that the maximum shear stress occurs when the normal stress is equal to the average normal stress,

$$T_{\tau,max} = \pm\sqrt{\frac{1}{4}(\sigma_1 - \sigma_2)^2 + \tau_{xy}^2} = 14.142[MPa] \tag{41}$$

where these calculations were carried out with Python.

2 Cauchy Stress

2.1 Perpendicular and Parallel Stress Components

In general, the perpendicular and parallel (to a cut plane) stress components are given via,

$$T_n = \vec{T} \cdot \hat{n} \tag{42}$$

$$T_{||} = \vec{T} \cdot \hat{e}_{||} \tag{43}$$

where $\hat{e}_{||}$ denotes a vector that is located in the plane defined by the unit normal vector, $\hat{n}$. The problem statement gives us $\hat{n} = \frac{1}{\sqrt{3}}(1, 1, 1)$ so we only need to find $\hat{e}_{||}$. The traction is found using the unit normal and the stress tensor at the material point.

To find $\hat{e}_{||}$ we recognize that there are an infinite number of candidates, all of which are described by the condition,

$$\hat{n} \cdot \hat{e}_{||} = 0 \tag{44}$$

evaluating this results in the expression for the components of $\hat{e}_{||}$,

$$e_x + e_y + e_z = 0 \tag{45}$$

together with the requirement that $\hat{e}_{||}$ be a unit vector,

$$e_x^2 + e_y^2 + e_z^2 = 1 \tag{46}$$

gives us three unknowns, but only two equations. To proceed we treat e_z as a free parameter and take $e_z = 0$. This results in,

$$e_x = -e_y = -\frac{1}{\sqrt{2}} \tag{47}$$

$$e_y = \frac{1}{\sqrt{2}} \tag{48}$$

As hinted above the tractions are computed according to,

$$\vec{T} = \hat{n} \cdot \underline{\underline{\sigma}} \tag{49}$$

which gives ($\underline{\underline{\sigma}}$ is found in the problem statement in the assignment file),

$$\vec{T} = \frac{1}{\sqrt{3}}[4\ 5\ 7] \tag{50}$$

Taking the inner product between the above and the specified unit vectors obtains the perpendicular and parallel (to the cut plane) components of the traction (stress),

$$\vec{T} \cdot \hat{n} = \frac{1}{3}(4 + 5 + 7) = \frac{16}{3} \tag{51}$$

$$\vec{T} \cdot \hat{e}_{||} = \frac{1}{\sqrt{6}}(4 - 5 + 0) = -\frac{1}{\sqrt{6}} \tag{52}$$

2.2 Principal Stresses and Directions

The principal stresses and directions are computed using the 'eig' method of the NumPy package. They are shown below in Figure 3.

```
(aa529) matt@ubuntumachine:~/aa530-hw/hw2$ python aa530-hw2.py
The eigenvalues (principal stresses) are [-1.28398772  5.20686134  8.07712638]
The eigenvectors (principal directions) are
 [[ 0.19676823 -0.77291405 -0.60322975]
 [ 0.71662868 -0.30651401  0.62649221]
 [-0.669123   -0.5555655   0.49358016]]
```

Figure 3: Eigenvalues and eigenvectors computed with the NumPy scientific computing library for the Python programming language

2.3 Maximum Shear Stress

The maximum (in-plane) shear stress is obtained when $T_n = \sigma_{avg}$ which leads to the formula,

$$T_{\tau,max} = \sqrt{\frac{1}{4}(\sigma_1 - \sigma_2)^2 + \tau_{xy}^2} \tag{53}$$

Using the principal normal stresses obtained from 2.2 and the off-diagonal term obtained from the stress tensor in the problem statement this yields a maximum shear stress of $T_{\tau,max} = 3.812$ [Pa].

2.4 Hydrostatic and Von Mises Stress

Given the stress tensor, the calculation of these two kinds of stress is fairly straightforward. We start with the hydrostatic stress,

$$\sigma_n = \frac{1}{3}TRACE(\underline{\underline{\sigma}}) = \frac{1}{3}(6+3+3) = 4 \tag{54}$$

in units of [Pa].

The Von Mises stress is defined according to,

$$\sigma_{VM} = \sqrt{\frac{3}{2}\underline{\underline{s}} : \underline{\underline{s}}} \tag{55}$$

where $\underline{\underline{s}}$ is the deviatoric stress tensor, $s_{ij} = \sigma_{ij} - \sigma_n \delta ij$. Using this it can be shown that the former expression for the Von Mises stress reduces to,

$$\sigma_{VM} = \sqrt{\frac{1}{2}\left((\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right)} \tag{56}$$

where the σ_i refer to the principal stresses previously calculated in 2.2. Inserting them into this expression and crunching the numbers obtains,

$$\sigma_{VM} = 8.306 \tag{57}$$

in units of [Pa]. This calculation was done in Python.