AA530 HW5

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https://github.com/russellmatt66/aa530-hw/tree/main/hw5

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1 Hyperelastic Material: Cauchy and Nominal Stresses - Uniaxial Tension

In the following we consider a uniform deformation, $\underline{u} = (u_1, u_2, u_3)^T$, where u_1, u_2, u_3 are all (different) constants. The deformation gradient tensor,

$$F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} \tag{1}$$

is then just the identity matrix,

$$\underline{\underline{F}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

and the left Cauchy-Green deformation tensor, $B_{ij} = F_{ik}F_{jk}$ is then the same,

$$\underline{\underline{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

giving a Jacobian of $J = det(\underline{\underline{F}}) = 1$. These relations are utilized in Problem 2 as well. For uniaxial tension the first invariant, I_1, is given as a function of the stretch rate by,

$$I_1 = \lambda^2 + 2\lambda^{-1} \tag{4}$$

The graphs for the different materials and stress states can be found in the appropriate appendices.

1.1 Neo-Hookean - Uniaxial Tension

The elements of the Cauchy stress tensor for a Neo-Hookean material are given by the expression,

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}} \left[B_{ij} - \frac{1}{3} B_{kk} \delta_{ij} \right] \tag{5}$$

where the hydrostatic pressure has been neglected for the sake of simplicity. The Cauchy (principal) stress in the $\hat{e}_1$-direction is just σ_{11}. This is computed over the span $\lambda \in (0.5, 2.0)$. The principal nominal stress for the above kind of material is[1],

$$S_1 = \mu_1 \left(\lambda - \lambda^{-2}\right) \tag{6}$$

1.2 Mooney-Rivlin - Uniaxial Tension

The elements of the Cauchy stress tensor for a Mooney-Rivlin material are given by the following expression,

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}} \left(1 + \frac{1}{J^{2/3}} B_{kk}\right) B_{ij} - \frac{\mu_2}{3J^{5/3}} \left(\bar{I}_1 + 2\bar{I}_2\right) \delta_{ij} - \frac{\mu_2}{J^{7/3}} B_{ik} B_{kj} + K_1 (J-1) \delta_{ij} \tag{7}$$

The principal nominal stress for the above kind of material is,

$$S_1 = \mu_1 \left(\lambda - \lambda^{-2}\right) + \mu_2 \left(1 - \lambda_{-3}\right) \tag{8}$$

1.3 Arruda-Boyce - Uniaxial Tension

The elements of the Cauchy stress tensor for a second-order Arruda-Boyce material are given by the following expression,

$$\sigma_{ij} = \frac{2}{J^{5/3}} \left[\frac{1}{2}\mu + \frac{1}{10\beta^2} \bar{I}_1\right] B_{ij} - \frac{2}{3J} \left[\bar{I}_1 (\frac{1}{2}\mu + \frac{1}{10\beta^2} \bar{I}_1)\right] \delta_{ij} + K(J-1)\delta_{ij} \tag{9}$$

The principal nominal stress for the above kind of material is,

$$S_1 = C \left(\lambda - \lambda^{-2}\right) \tag{10}$$

where C is,

$$C = \mu \left(1 + \frac{I_1}{5\beta^2} + \frac{33 I_1}{525\beta^4}\right) \tag{11}$$

1.4 Ogden - Uniaxial Tension

The elements of the Cauchy stress tensor for an Ogden material are given by the following expression (corrected from hw4),

$$\sigma_{ij} = \frac{1}{J^{1/3}} \frac{1}{\lambda_1 \lambda_2 \lambda_3} \left(\lambda_k \frac{\partial \tilde{U}}{\partial x_k} b_i^{(k)} b_j^{(k)}\right) \sum_{l=1}^{N} \frac{2\mu_l}{\alpha_l^2} \tag{12}$$

The principal nominal stress for the above kind of material is,

$$S_1 = \sum_n \mu_n \left(\lambda^{\alpha_n} - \lambda^{-\alpha_n/2.0}\right) / \lambda \tag{13}$$

2 Hyperelastic Material: Cauchy and Nominal Stresses - Biaxial tension

Biaxial tension corresponds to a first invariant of,

$$I_1 = 2\lambda^2 + \lambda^{-4} \tag{14}$$

with all other objects, $\underline{\underline{F}}, \underline{\underline{B}}, \sigma_{ij}, J, etc.$ the same as in Problem 1

2.1 Neo-Hookean - Biaxial Tension

For biaxial tension, the principal nominal stress in the $\hat{e}_1$-direction for a Neo-Hookean material is,

$$S_1 = \mu_1 \left(\lambda - \lambda^{-5}\right) \tag{15}$$

2.2 Mooney-Rivlin - Biaxial Tension

For biaxial tension, the principal nominal stress in the $\hat{e}_1$-direction for a Mooney-Rivlin material is,

$$S_1 = \mu_1 \left(\lambda - \lambda^{-5}\right) + \mu_2 \left(\lambda^3 - \lambda^{-3}\right) \tag{16}$$

2.3 Arruda-Boyce - Biaxial Tension

For biaxial tension, the principal nominal stress in the $\hat{e}_1$-direction for an Arruda-Boyce material is,

$$S_1 = C \left(\lambda - \lambda^{-5}\right) \tag{17}$$

with the C from Equation 11.

2.4 Ogden - Biaxial Tension

For biaxial tension, the principal nominal stress in the $\hat{e}_1$-direction for an Ogden material is,

$$S_1 = \sum_n \mu_n \left(\lambda^{\alpha_n} - \lambda^{-2.0\alpha_n}\right) / \lambda \tag{18}$$

3 Viscoelastic Material - Maxwell Model: Constant Stress

The proceeding derivation follows the development in the notes. As the name suggests, a Maxwell model for a viscoelastic material investigates the response of the system using a kind of equivalent circuit form shown below alongside two other similar types of model,

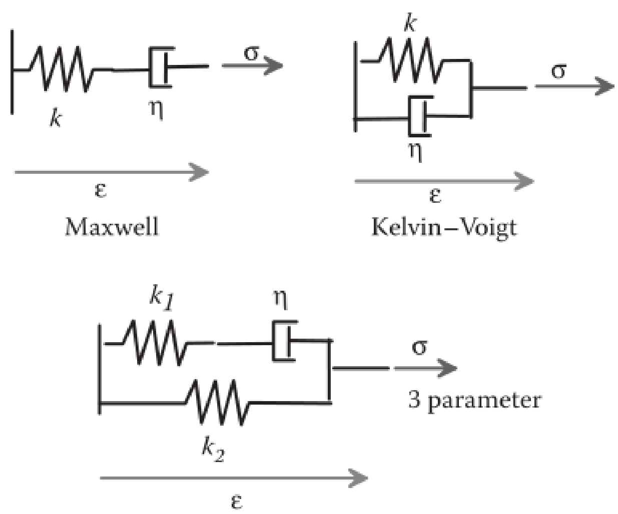

Figure 1: "Equivalent circuits" modelling the response of a viscoelastic material subject to an applied stress, $\sigma(t)$

the differential equation describing the behavior of the elastic (spring element) and plastic (dashpot) components is written,

$$\epsilon = \epsilon_e + \epsilon_p \tag{19}$$

$$= \frac{\sigma}{k} + \epsilon_p \tag{20}$$

to fully incorporate the plastic response we take the time derivative of both sides to yield,

$$\dot{\epsilon} = \frac{\dot{\sigma}}{k} + \dot{\epsilon}_p \tag{21}$$

$$= \frac{\dot{\sigma}}{k} + \frac{\sigma}{\eta} \tag{22}$$

For a constant stress the above can be integrated because the time derivative of the stress drops out. This results in an equation for the strain,

$$\epsilon = \epsilon_0 + \frac{1}{\eta}\int \sigma_0 dt \tag{23}$$

to find the constant of integration, ϵ_0, we return to the original ODE describing the dynamics of the system, and integrate both sides in time,

$$\int_{0^-}^{\Delta T} \frac{\mathrm{d}\epsilon}{\mathrm{d}t} dt = \frac{1}{k} \int_{0^-}^{\Delta T} \frac{\mathrm{d}\sigma}{\mathrm{d}t} dt + \frac{1}{\eta} \int_{0^-}^{\Delta T} \sigma dt \tag{24}$$

we assume that the material began the experiment in an unstressed state so that the above can be simplified to,

$$\epsilon(\Delta T) = \frac{1}{k}\sigma(\Delta T) + \frac{1}{\eta}\frac{\sigma_0 \Delta T}{2} \tag{25}$$

the factor of two in the above may be somewhat surprising initially, but it comes from the relation $S = \sigma t$ that holds during the initial transient period of the loading. Taking $\Delta T \to 0$ gives an expression for $\epsilon(0^+)$ which is just the constant of integration we have been looking for,

$$\epsilon_{0^+} = \epsilon_0 = \frac{\sigma_0}{k} \tag{26}$$

$$\therefore \epsilon(t) = \sigma_0 \left(\frac{1}{k} + \frac{1}{\eta} t \right) \tag{27}$$

4 Viscoelastic Material - Maxwell Model: Time-dependent Strain

For a constant loading the compliance, $J(t)$, can be calculated according to,

$$J(t) = \frac{\epsilon(t)}{\sigma_0} \tag{28}$$

taking the expression for the strain that was derived in Problem 3 it is apparent that this system exhibits a steady-state creep as the resulting compliance will be linear for long time.

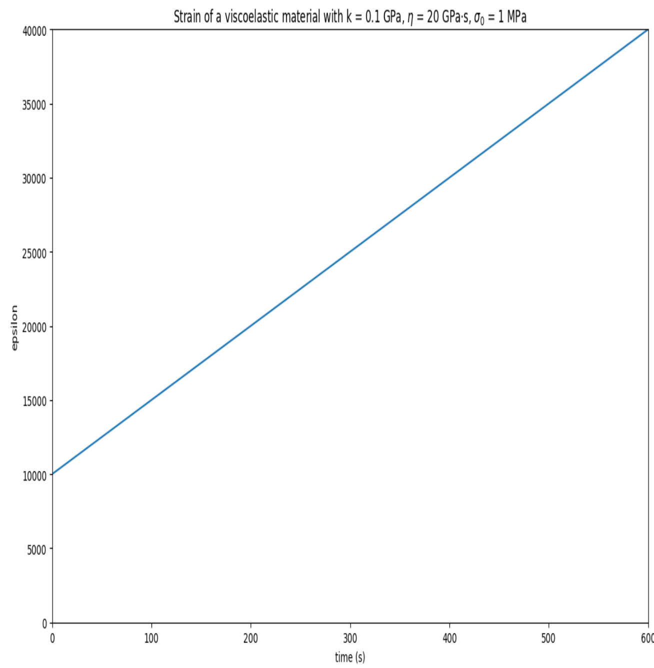

Figure 2: Material strain predicted by the Maxwell Model which captures both elastic and plastic deformations

5 Viscoelastic Material - Maxwell Model: Constant Strain

For a constant strain the 1st order ODE describing the dynamics of the strain becomes,

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 0 \tag{29}$$

$$\therefore -\frac{\sigma}{\eta} = \frac{1}{k}\frac{\mathrm{d}\sigma}{\mathrm{d}t} \tag{30}$$

$$\rightarrow \frac{1}{\sigma}d\sigma = -\frac{k}{\eta}dt \tag{31}$$

integrating this gives the well-known solution of a decaying exponential,

$$\sigma(t) = C\ exp(-\frac{k}{\eta}t) \tag{32}$$

to find the constant of integration we perform the exact same process as in Problem 4: integrate both sides of the original ODE for a short period of time and then take the limit as this timescale approaches 0 from the right (backwards w.r.t the experiment). In this situation, the integral of σ drops out during the limit and we are left with,

$$\epsilon(0^+) = \frac{1}{k}\sigma(0^+) \tag{33}$$

but $\sigma(0^+) = C = k\epsilon_0$! Inserting this back into our time-dependent stress we get,

$$\sigma(t) = k\epsilon_0\ exp(-\frac{k}{\eta}t) \tag{34}$$

Appendix - Graphs

It is not known whether these graphs are all supposed to be the same or whether this is a byproduct of the way the computation was done, e.g., because of Python's object-reference system of data declaration.

5.1 Neo-Hookean

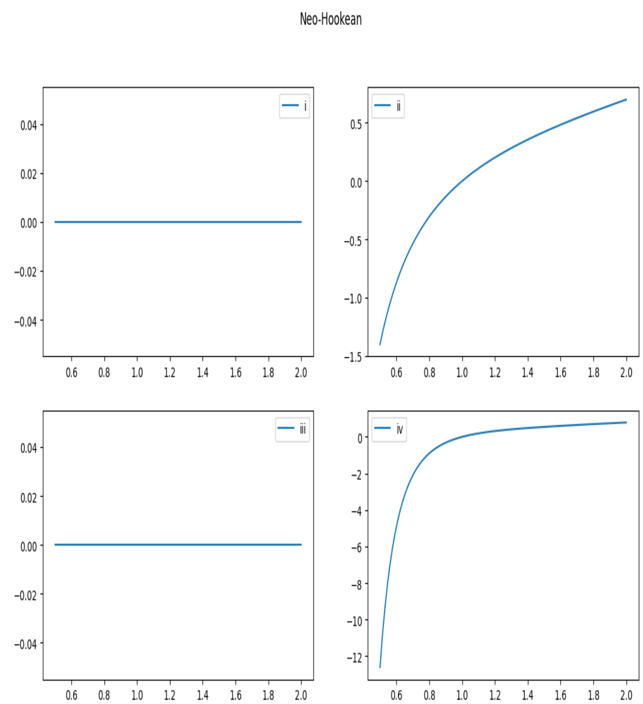

Figure 3: Computed stresses for a Neo-Hookean material as a function of stretch rate, $\lambda \in (0.5, 2.0)$. (i) and (ii) represent the Cauchy and nominal stress, respectively, under uniaxial tension and (iii) and (iv) represent the cauchy and nominal stress, respectively, under biaxial tension. The Cauchy stresses in (i) and (iii) are represented here with a value of zero, but in general they would take a uniform value.

5.2 Mooney-Rivlin

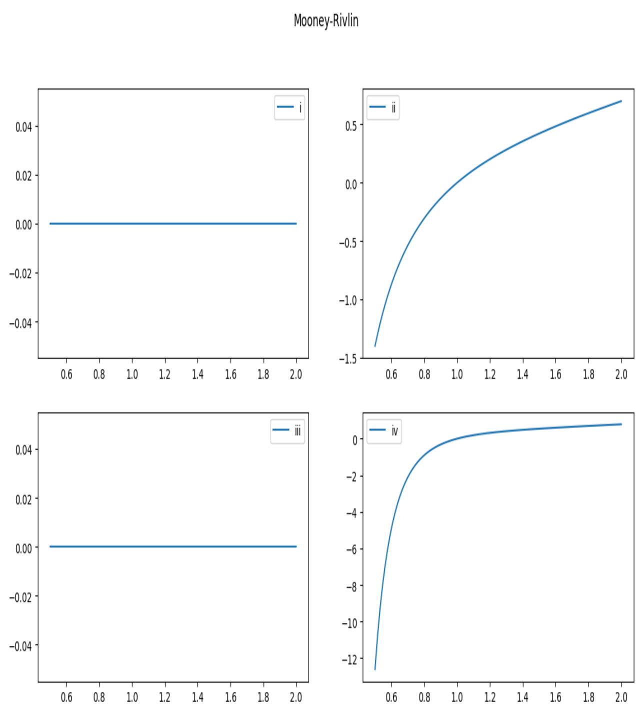

Figure 4: Computed stresses for a Mooney-Rivlin material as a function of stretch rate, $\lambda \in (0.5, 2.0)$. (i) and (ii) represent the Cauchy and nominal stress, respectively, under uniaxial tension and (iii) and (iv) represent the cauchy and nominal stress, respectively, under biaxial tension. The Cauchy stresses in (i) and (iii) are represented here with a value of zero, but in general they would take a uniform value.

5.3 Arruda-Boyce

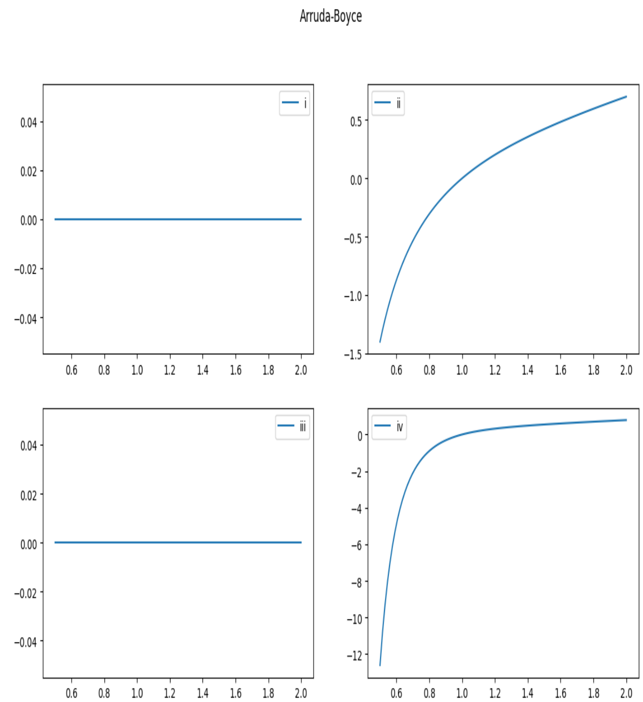

Figure 5: Computed stresses for an Arruda-Boyce material as a function of stretch rate, $\lambda \in (0.5, 2.0)$. (i) and (ii) represent the Cauchy and Nominal stress, respectively, under uniaxial tension and (iii) and (iv) represent the cauchy and nominal stress, respectively, under biaxial tension. The Cauchy stresses in (i) and (iii) are represented here with a value of zero, but in general they would take a uniform value.

5.4 Ogden

References

[1] Allan F Bower. *Applied Mechanics of Solids*. eng. Baton Rouge: CRC Press, 2010. ISBN: 9781439802472.