AA530 HW1

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1 Locally Resonant Sonic Materials

(i) The Problem

Constructing a crystal with an acoustic band-gap is a chaotic and difficult process.

(ii) The Objective

Come up with a new way of constructing acoustic band-gap crystals and then compare experimental data from the system with theoretical calculations.

(iii) The Approach

Embed a periodic microstructure of dense, hard spheres in a material substrate to create localized sonic resonances that lead to a bandgap.

(iv) Findings

The team found relatively good agreement between their calculations and the experimental data. Some of the striking differences between data from the composite and the epoxy reference material were explained as resulting from the microstructure.

(v) Conclusion

An interesting and unique material with scientific value was constructed in a simple and clever fashion. This material exhibits a sonic band-gap as a consequence of its microstructure and extensions to other frequency ranges could lead to possible applications.

2 Rotation of a Homogeneous, Isotropic Circular Shaft

2.1 Deformation Gradient Tensor

The Deformation Gradient Tensor (DeGT) is defined by,

$$\underline{\underline{F}} \triangleq \underline{\underline{I}}^{(2)} + \underline{u} \otimes \underline{\nabla} \tag{1}$$

where $\underline{\underline{I}}^{(2)}$ is the identity tensor of second-rank. Written in summation notation Equation 1 corresponds to,

$$F_{ik} = \delta_{ik} + \frac{\partial u_i}{\partial x_k} \tag{2}$$

The Displacement Gradient Tensor (DGT), $\underline{u} \otimes \underline{\nabla}$, is expanded according to the displacement field given in the problem statement found in the assignment and the assumption that $[\underline{u}, \underline{\nabla}] = 0$, i.e, the displacement field and gradient operator commute. This yields,

$$\underline{u} \otimes \underline{\nabla} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \tag{3}$$

$$\underline{u} \otimes \underline{\nabla} = \begin{pmatrix} cos(\frac{\alpha x_3}{L}) - 1 & -sin(\frac{\alpha x_3}{L}) & -\frac{\alpha}{L}\left[x_1 sin(\frac{\alpha x_3}{L}) + x_2 cos(\frac{\alpha x_3}{L})\right] \\ sin(\frac{\alpha x_3}{L} & cos(\frac{\alpha x_3}{L}) - 1 & -\frac{\alpha}{L}\left[-x_1 cos(\frac{\alpha x_3}{L}) + x_2 sin(\frac{\alpha x_3}{L})\right] \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

Which gives,

$$\underline{\underline{F}} = \begin{pmatrix} cos(\frac{\alpha x_3}{L}) & -sin(\frac{\alpha x_3}{L}) & -\frac{\alpha}{L}\left[x_1 sin(\frac{\alpha x_3}{L}) + x_2 cos(\frac{\alpha x_3}{L})\right] \\ sin(\frac{\alpha x_3}{L} & cos(\frac{\alpha x_3}{L}) & -\frac{\alpha}{L}\left[-x_1 cos(\frac{\alpha x_3}{L}) + x_2 sin(\frac{\alpha x_3}{L})\right] \\ 0 & 0 & 1 \end{pmatrix} \tag{5}$$

2.2 Lagrange Strain Tensor

The Lagrange Strain Tensor (LST) is expressed as,

$$\underline{\underline{E}} \triangleq \frac{1}{2}\left(\underline{\underline{F}}^T \underline{\underline{F}} - \underline{\underline{I}}^{(2)}\right) \tag{6}$$

and the components calculated according to,

$$E_{ij} = \frac{1}{2}\left(F_{ki}F_{kj} - \delta_{ij}\right) \tag{7}$$

Using the general form of $\underline{\underline{F}}$ we can write the components of the LST in terms of the displacement field and its partial derivatives. These are tabulated in the Appendix. The components of the DGT, Equation 4, give us the information we need to simplify these expressions. Fortuitously, when we perform this process we find that significant cancellation occurs amongst the sin and cos terms and, after some algebra, yields,

$$\underline{\underline{E}} = \begin{pmatrix} 0 & 0 & -\frac{\alpha}{2L}x_2 \\ 0 & 0 & \frac{\alpha}{2L}x_1 \\ -\frac{\alpha}{2L}x_2 & \frac{\alpha}{2L}x_1 & \frac{\alpha^2}{2L^2}(x_1^2+x_2^2) \end{pmatrix} \tag{8}$$

From the above, it is evident that $\underline{\underline{E}}$ is not a function of x_3. This is due to the fact that all x_3-dependence is contained in the phase angle, $\frac{\alpha x_3}{L}$, which drops out of the calculation when the sin and cos terms do.

2.3 Increase in Length of a Material Fiber on the Surface of the Cylinder

A fiber located a distance $R^2 = x_1^2 + x_2^2$ from the axis of the cylinder and of initial length, $l_0 = dl$, will expand or contract in a given direction, $\underline{n}$, according to the formula,

$$\epsilon_E(\underline{n}) = \frac{l^2 - l_0^2}{2l_0^2} = \underline{n} \cdot \underline{\underline{E}}^* \cdot \underline{n} \tag{9}$$

Substituting dl into the above and re-arranging we can obtain,

$$l - dl = \Delta l = \frac{2dl^2}{l+dl}\underline{n} \cdot \underline{\underline{E}}^* \cdot \underline{n} \tag{10}$$

To proceed, we take $\underline{n} = \hat{e}_3$ and expand $\underline{\underline{E}}^*$ according to its definition,

$$\underline{\underline{E}}^* \triangleq \frac{1}{2}\left(\underline{\underline{I}}^{(2)} - \underline{\underline{F}}^{-T}\underline{\underline{F}}^{-1}\right) \tag{11}$$

Using our expression for the DeGT, Equation 5, we can compute $\underline{\underline{F}}^{-1}$ and $\underline{\underline{F}}^{-T}$. These are done using Mathematica to streamline the algebra,

$$\underline{\underline{F}}^{-1} = \begin{pmatrix} cos(\frac{\alpha x_3}{L}) & sin(\frac{\alpha x_3}{L}) & \frac{\alpha x_2}{L} \\ -sin(\frac{\alpha x_3}{L} & cos(\frac{\alpha x_3}{L}) & -\frac{\alpha x_1}{L} \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

$$\underline{\underline{F}}^{-T} = \begin{pmatrix} cos(\frac{\alpha x_3}{L}) & -sin(\frac{\alpha x_3}{L}) & 0 \\ sin(\frac{\alpha x_3}{L} & cos(\frac{\alpha x_3}{L}) & 0 \\ \frac{\alpha x_2}{L} & -\frac{\alpha x_1}{L} & 1 \end{pmatrix} \tag{13}$$

With $\underline{n} = \hat{e}_3$ then $\underline{n} \cdot \underline{\underline{E}}^* \cdot \underline{n}$ serves to pick out the E^*_{33} component,

$$E^*_{33} = \frac{1}{2}\left[1 - F^{-T}_{3k} F^{-1}_{k3}\right] \tag{14}$$

where repeated indices indicate summation. The RHS of the above can be found from inspecting the previous expressions for the two tensors,

$$E^*_{33} = \frac{1}{2}\left[1 - (\frac{\alpha x_2}{L})^2 - (\frac{\alpha x_1}{L})^2 + 1\right] \tag{15}$$

Leveraging the constraint that $R^2 = x_1^2 + x_2^2$ and plugging into Equation 9 we obtain,

$$\Delta l = -\frac{dl^2}{l + dl}(\frac{\alpha R}{L})^2 \tag{16}$$

2.4 Material Fibers in the $\hat{e}_1$ and $\hat{e}_2$ Directions

We repeat the same calculation as in (2.3) with $\underline{n} = \hat{e}_1$ and $\underline{n} = \hat{e}_2$ instead of $\hat{e}_3$. The result is to pick out the E^*_{11} and E^*_{22} components, respectively. From inspection of Equations 12 and 13, these are both equal to 0 (in both cases $cos^2() + sin^2() = 1$ cancels the contribution from the identity tensor). QED

2.5 Principal Values and Directions of the LST

For the given point,

$$(x_1, x_2, x_3, \alpha) = (\frac{L}{10}, 0, 0, 5^\circ) \tag{17}$$

The LST becomes,

$$\underline{\underline{E}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.25 \\ 0 & 0.25 & 0.125 \end{pmatrix} \tag{18}$$

The eigenvalues and eigenvectors of this matrix are found using the eig method of the NumPy library. The output is shown in Figure 1. It is clear from

```
The eigenvalues of the LST in 2.5 are %f [ 0.3201941 -0.1951941  0.        ]
The eigenvectors of the LST in 2.5 are %f [[ 0.          0.          1.        ]
 [ 0.61541221  0.78820544  0.        ]
 [ 0.78820544 -0.61541221  0.        ]]
```

Figure 1: Eigenvalues and eigenvectors of the Lagrangian Strain Tensor, $\underline{\underline{E}}$, at the point $(x_1, x_2, x_3, \alpha) = (\frac{L}{10}, 0, 0, 5^\circ)$

the figure that the $\hat{e}_3$ direction corresponds to the largest expansion in length and the orientation $(0.615, 0.788, 0.0)$ corresponds to the greatest contraction in length.

2.6 Infinitesimal Strain Tensor

The infinitesimal strain tensor (IST) is defined according to,

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left[\underline{u} \otimes \underline{\nabla} + (\underline{u} \otimes \underline{\nabla})^T\right] \tag{19}$$

expanding this into matrix form,

$$\underline{\underline{\epsilon}} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) & \frac{1}{2}(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}) \\ \frac{1}{2}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) \\ \frac{1}{2}(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}) & \frac{1}{2}(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}) & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \tag{20}$$

We use Equation 4 to write these partial derivatives out and simplify them. Doing so yields,

$$\underline{\underline{\epsilon}} = \begin{pmatrix} cos(\frac{\alpha x_3}{L}) - 1 & 0 & -\frac{\alpha}{2L}(x_1 sin(\frac{\alpha x_3}{L}) + x_2 cos(\frac{\alpha x_3}{L})) \\ 0 & cos(\frac{\alpha x_3}{L}) - 1 & -\frac{\alpha}{2L}(-x_1 cos(\frac{\alpha x_3}{L}) + x_2 sin(\frac{\alpha x_3}{L})) \\ -\frac{\alpha}{2L}(x_1 sin(\frac{\alpha x_3}{L}) + x_2 cos(\frac{\alpha x_3}{L})) & -\frac{\alpha}{2L}(-x_1 cos(\frac{\alpha x_3}{L}) + x_2 sin(\frac{\alpha x_3}{L})) & 0 \end{pmatrix} \tag{21}$$

Clearly the above is different from Equation 8 therefore the two tensors differ for finite rotations. To address the other part of the question we consider the general form of the LST which is enumerated in the Appendix. The condition of small rotations amounts to neglecting any 2nd-order or higher terms in these expressions. For an example consider E_{11},

$$\begin{aligned} E_{11} &= \frac{1}{2}\left[(1 + \frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_1})^2 + (\frac{\partial u_3}{\partial x_1})^2 - 1\right] \\ &= \frac{1}{2}\left[1 + 2\frac{\partial u_1}{\partial x_1} + (\frac{\partial u_1}{\partial x_1})^2 - 1\right] \\ &= \frac{\partial u_1}{\partial x_1} \end{aligned}$$

the other diagonal terms follow a similar pattern to yield,

$$\begin{aligned} E_{22} &= \frac{\partial u_2}{\partial x_2} \\ E_{33} &= \frac{\partial u_3}{\partial x_3} \end{aligned}$$

and the off-diagonal terms are even simpler as they do not involve the expansion of a second-order polynomial,

$$\begin{aligned} E_{12} &= \frac{1}{2}\left[(1 + \frac{\partial u_1}{\partial x_1})\frac{\partial u_1}{\partial x_2} + (1 + \frac{\partial u_2}{\partial x_2})\frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right] \\ &= \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right] \end{aligned}$$

and so on and so forth to yield,

$$E_{13} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right]$$
$$E_{21} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right]$$
$$E_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right]$$
$$E_{31} = \frac{1}{2}\left[\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right]$$
$$E_{32} = \frac{1}{2}\left[\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right]$$

which can be compared visually with the elements of $\underline{\underline{\epsilon}}$ in Equation 20 to be in exact agreement.

2.7 IST - Length Estimates

2.8 Bonus - Visualization

3 Strain Compatibility Problem

I have as the output of the six compatibility equations,

$$
\begin{aligned}
2k - 2k'z &= 0 \\
0 + 0 - 0 &= 0 \\
2k + 0 + 0 &= 0 \\
0 + 0 - 0 - k'y &= 0 \\
0 + 0 - k'x - 0 &= 0 \\
0 + 0 - 0 - 0 &= 0
\end{aligned}
$$

For the above, three are satisfied with no argument. Two more are satisfied outright by the specification that k, and k' are small constants. The final one, which also happens to be the first in the system of equations, upon further thought is also satisfied provided that $k, k' << z$ be true. Therefore, this represents a possible state of strain for the continuum.

Appendix

LST Components - General Form

These calculations were done by hand and checked using Mathematica. For space considerations only the results are quoted,

$$
\begin{aligned}
E_{11} &= \frac{1}{2}\left((1+\frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_1})^2 + (\frac{\partial u_3}{\partial x_1})^2 - 1\right) \\
E_{12} &= \frac{1}{2}\left((1+\frac{\partial u_1}{\partial x_1})\frac{\partial u_1}{\partial x_2} + (1+\frac{\partial u_2}{\partial x_2})\frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right) \\
E_{13} &= \frac{1}{2}\left((1+\frac{\partial u_1}{\partial x_1})\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\left(\frac{\partial u_3}{\partial x_2} + 1\right)\right) \\
E_{21} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}\left(1+\frac{\partial u_1}{\partial x_1}\right) + (1+\frac{\partial u_2}{\partial x_2})\frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_1}\right) \\
E_{22} &= \frac{1}{2}\left((\frac{\partial u_1}{\partial x_2})^2 + (1+\frac{\partial u_2}{\partial x_2})^2 + (\frac{\partial u_3}{\partial x_2})^2 - 1\right) \\
E_{23} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} + (1+\frac{\partial u_2}{\partial x_2})\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\left(1+\frac{\partial u_3}{\partial x_3}\right)\right) \\
E_{31} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}\left(1+\frac{\partial u_1}{\partial x_1}\right) + \frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_1} + \left(1+\frac{\partial u_3}{\partial x_3}\right)\frac{\partial u_3}{\partial x_1}\right) \\
E_{32} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\left(1+\frac{\partial u_2}{\partial x_2}\right) + (1+\frac{\partial u_3}{\partial x_3})\frac{\partial u_3}{\partial x_2}\right) \\
E_{33} &= \frac{1}{2}\left((\frac{\partial u_1}{\partial x_3})^2 + (\frac{\partial u_2}{\partial x_3})^2 + (1+\frac{\partial u_3}{\partial x_3})^2 - 1\right)
\end{aligned}
$$