AA530 HW4

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`https://github.com/russellmatt66/aa530-hw/tree/main/hw4`

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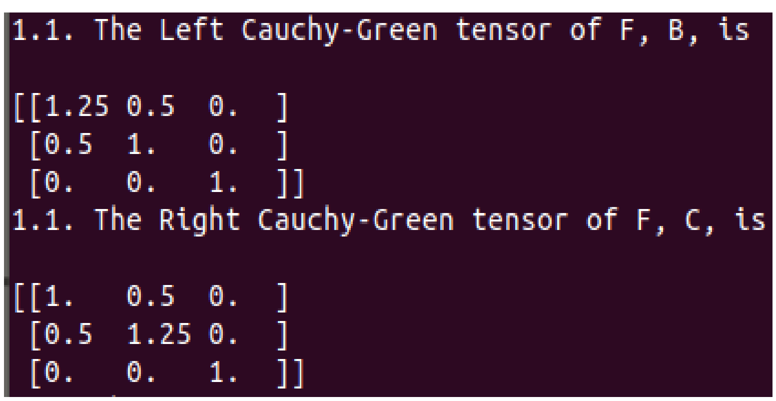
1 Stretch

To solve the parts of this problem the python code found at the repository in the title was used. There appears to be a bug somewhere that introduces a negative sign into the off-diagonal terms of B when it is constructed from the principal stretches and directions. The deformation tensor is given in the problem statement, which can be found in the assignment. A pdf copy of the assignment is located conveniently at the above repo.

1.1 Left, $\underline{\underline{B}}$, and Right, $\underline{\underline{C}}$, Cauchy-Green Deformation Tensors

$\underline{\underline{B}}$ and $\underline{\underline{C}}$ are **not** identical. This is ascribed to the asymmetry of the underlying deformation gradient tensor, $\underline{\underline{F}}$, used to build them.



```
1.1. The Left Cauchy-Green tensor of F, B, is

[[1.25 0.5  0.  ]
 [0.5  1.   0.  ]
 [0.   0.   1.  ]]
1.1. The Right Cauchy-Green tensor of F, C, is

[[1.   0.5  0.  ]
 [0.5  1.25 0.  ]
 [0.   0.   1.  ]]
```

Figure 1: Left and Right CG Deformation Tensors in matrix form.

1.2 Eigenvalues of $\underline{\underline{B}}$ and $\underline{\underline{C}}$

Despite their differences, $\underline{\underline{B}}$ and $\underline{\underline{C}}$, share the same set of eigenvalues. The order shown here is a quirk of the *eig* method that python uses to do the computation. Alternatively, the eigenvalues can be obtained from the characteristic equation for the respective tensor (matrix).

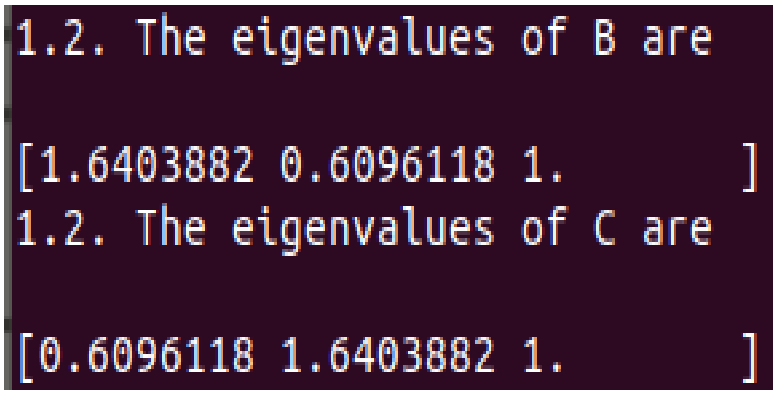


Figure 2: Eigenvalues, e_i, of the Cauchy-Green tensors.

1.3 Principal Stretches and Principal Stretch Directions of $\underline{\underline{B}}$

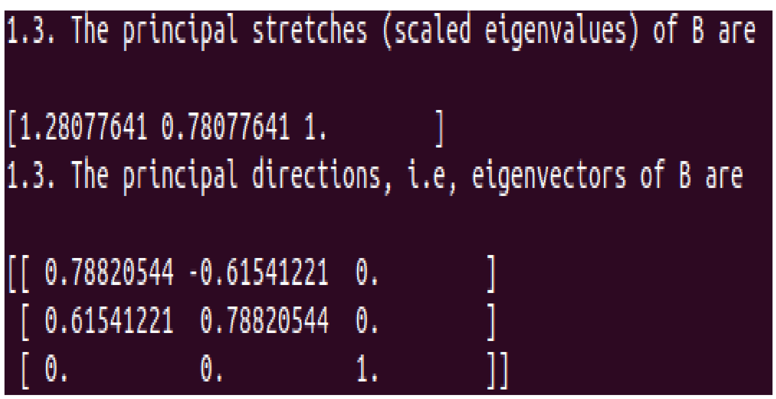


Figure 3: The principal stretches, $\lambda_i = \sqrt{e_i}$, and principal stretch directions (eigenvectors) of $\underline{\underline{B}}$.

1.4 Reconstruction

Curiously, calculating B according to,

$$\underline{\underline{B}} = \lambda_i \vec{b}_i \bigotimes \vec{b}_i \tag{1}$$

using the principal stretches and directions from above introduces negative signs into the off-diagonal terms. At the time of writing it is not known why this is the case.

```
1.4. The Left Cauchy-Green tensor of F, B, is


[[1.25 0.5  0.  ]
 [0.5  1.   0.  ]
 [0.   0.   1.  ]]
1.4. The Left Cauchy-Green tensor of F, B, that was constructed from the principal stretches and directions is


[[ 1.25 -0.5   0.  ]
 [-0.5   1.    0.  ]
 [ 0.    0.    1.  ]]
```

Figure 4: Left and Right CG Deformation Tensors in matrix form

1.5 Invariants

```
Invariant 0 is 3.250000
Invariant 1 is 3.250000
Invariant 2 is 1.000000
Normalized invariant 0 is 3.250000
Normalized invariant 1 is 3.250000
Normalized invariant 2 is 1.000000
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Figure 5: The invariants of the Left Cauchy-Green Deformation Tensor. Note that the Jacobian for this problem ("Invariant 2", i.e, the third invariant) is equal to one which means that the normalized invariants are identical to the invariants.

2 Hyperelastic Material

The equation relating the (Cauchy) stress in a hyperelastic material to the strain can be written as[1],

$$\sigma_{ij} = \frac{2}{J^{\frac{5}{3}}} \left(\frac{\partial \bar{U}}{\partial \bar{I}_1} + \bar{I}_1 \frac{\partial \bar{U}}{\partial \bar{I}_2} \right) B_{ij} - \frac{2}{3J} \left(\bar{I}_1 \frac{\partial \bar{U}}{\partial \bar{I}_1} + 2\bar{I}_2 \frac{\partial \bar{U}}{\partial \bar{I}_2} \right) \delta_{ij} - \frac{2}{J^{\frac{7}{3}}} \frac{\partial \bar{U}}{\partial \bar{I}_2} B_{ik} B_{kj} + \frac{\partial \bar{U}}{\partial J} \delta_{ij} \tag{2}$$

this is a complicated expression relating the Cauchy stress to the material strain. The latter arises as a consequence of the deformation the configuration is subject to and information about which is embedded in $\underline{\underline{B}} = \underline{\underline{F}} \cdot \underline{\underline{F}}^T$, the familiar Left Cauchy-Green tensor, and in the strain energy density (SED).

We can see from Equation (2) that σ is a tensor of rank 2. This multidimensional character means we have a number of different directions we can choose from in deciding how we approach analyzing the system of stresses and strains. The particular form of the SED that Equation (2) uses is the "normalized" form $\bar{U}$, which is a function of the normalized invariants $\bar{U} = \bar{U}(\bar{I}_1, \bar{I}_2, J)$.

The task of the physicist then is to derive various forms for $\bar{U}$ (the potential energy of the configuration) so that Equation (2) can be simplified to give the engineer insight.

2.1 Generalized neo-Hookean Solid

The SED for a generalized neo-Hookean solid is,

$$\bar{U} = \frac{\mu_1}{2}(\bar{I}_1 - 3) + \frac{K_1}{2}(J-1)^2 \tag{3}$$

the partial derivatives from Equation (2) w.r.t the invariants are read off,

$$\frac{\partial \bar{U}}{\partial \bar{I}_1} = \frac{\mu_1}{2} \tag{4}$$

$$\frac{\partial \bar{U}}{\partial \bar{I}_2} = 0 \tag{5}$$

$$\frac{\partial \bar{U}}{\partial J} = K_1(J-1) \tag{6}$$

these are inserted into Equation (2),

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}} \left[B_{ij} - \frac{1}{3} B_{kk} \delta_{ij} \right] + K_1 (J-1) \delta_{ij} \tag{7}$$

where $\bar{I}_1 = \frac{I_1}{J^{2/3}}$ is inserted into the above and $I_1 = trace(\underline{\underline{B}}) = B_{kk}$ is used. In the incompressible limit the second term is set as, $K_1(J-1) = \frac{p}{3}$. This obtains our simplified expression,

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}} \left[B_{ij} - \frac{1}{3} B_{kk} \delta_{ij} \right] + \frac{p}{3} \delta_{ij} \tag{8}$$

We can define a Nominal (first Piola-Kirchoff) stress for the deformation using the Cauchy stress,

$$S_{ij} = JF_{ik}^{-1}\sigma_{kj} \tag{9}$$

A more functional way of writing this for our purposes focuses on the strain energy density and uses the chain rule,

$$S_{ij} = \frac{\partial W}{\partial F_{ji}} = \frac{\partial \bar{U}}{\partial \bar{I}_1}\frac{\partial \bar{I}_1}{\partial F_{ji}} + \frac{\partial \bar{U}}{\partial \bar{I}_2}\frac{\partial \bar{I}_2}{\partial F_{ji}} + \frac{\partial \bar{U}}{\partial J}\frac{\partial J}{\partial F_{ji}} \tag{10}$$

the above partial derivatives can be written as[1],

$$\frac{\partial J}{\partial F_{ji}} = JF_{ji}^{-1} \tag{11}$$

$$\frac{\partial \bar{I}_1}{\partial F_{ji}} = \frac{2}{J^{2/3}}F_{ji} - \frac{2}{3}\bar{I}_1 F_{ij}^{-1} \tag{12}$$

$$\frac{\partial \bar{I}_2}{\partial F_{ji}} = \frac{2}{J^{2/3}}\bar{I}_1 F_{ji} - \frac{2}{J_{4/3}}B_{jk}F_{ki} - \frac{4}{3}\bar{I}_2 F_{ij}^{-1} \tag{13}$$

pairing these together with the partial derivatives of the SED w.r.t the adiabatic invariants from above and inserting them into Equation (10) yields,

$$S_{ij} = \frac{\mu_1}{2}\left(\frac{2}{J^{2/3}}F_{ji} - \frac{2}{3}\bar{I}_1 F_{ij}^{-1}\right) + K_1 J(J-1)F_{ij}^{-1} \tag{14}$$

These expressions, Equations (8) and (14), can be further simplified by specifying the strain state as either uniaxial tension

$$I_1 = \lambda^2 + \frac{2}{\lambda} \tag{15}$$

or equibiaxial (same load on two axes),

$$I_1 = 2\lambda^2 + \lambda^{-4} \tag{16}$$

2.2 Mooney-Rivlin

The Mooney-Rivlin SED is,

$$\bar{U} = \frac{\mu_1}{2}(\bar{I}_1 - 3) + \frac{\mu_2}{2}(\bar{I}_2 - 3) + \frac{K_1}{2}(J-1)^2 \tag{17}$$

the partial derivatives w.r.t the invariants are

$$\frac{\partial \bar{U}}{\partial \bar{I}_1} = \frac{\mu_1}{2} \tag{18}$$

$$\frac{\partial \bar{U}}{\partial \bar{I}_2} = \frac{\mu_2}{2} \tag{19}$$

$$\frac{\partial \bar{U}}{\partial J} = K_1(J-1) \tag{20}$$

S_{ij} can then be constructed from the above and the partial derivatives w.r.t to the deformation gradient tensor,

$$S_{ij} = \frac{\mu_1}{2}\left(\frac{2}{J^{2/3}}F_{ji} - \frac{2}{3}\bar{I}_1 F_{ij}^{-1}\right) + \frac{\mu_2}{2}\left[\frac{2}{J^{2/3}}\bar{I}_1 F_{ji} - \frac{2}{J^{4/3}}B_{jk}F_{ki} - \frac{4}{3}\bar{I}_2 F_{ij}^{-1}\right] + K_1 J(J-1)F_{ji}^{-1} \tag{21}$$

the stress is simplified starting from Equation (2),

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}}\left(1 + \frac{1}{J^{2/3}}B_{kk}\right)B_{ij} - \frac{\mu_2}{3J^{5/3}}\left(\bar{I}_1 + 2\bar{I}_2\right)\delta_{ij} - \frac{\mu_2}{J^{7/3}}B_{ik}B_{kj} + K_1(J-1)\delta_{ij} \tag{22}$$

$\bar{I}_1 = B_{kk}$ and $\bar{I}_2 = \frac{1}{2}\left[(B_{kk})^2 - B_{mk}B_{km}\right]$ can be substituted into the above to further simplify the expression into index notation.

2.3 Arruda-Boyce

Going to second-order in the Arruda-Boyce model gives a SED of,

$$\bar{U} = \mu\left(\frac{1}{2}(\bar{I}_1 - 3) + \frac{1}{20\beta^2}(\bar{I}_1^2 - 9)\right) \tag{23}$$

the partial derivatives w.r.t the invariants are,

$$\frac{\partial \bar{U}}{\partial \bar{I}_1} = \frac{1}{2}\mu + \frac{1}{10\beta^2}\bar{I}_1 \tag{24}$$

$$\frac{\partial \bar{U}}{\partial \bar{I}_2} = 0 \tag{25}$$

$$\frac{\partial \bar{U}}{\partial J} = K(J-1) \tag{26}$$

the stress associated with this second-order truncation is worked out to be,

$$\sigma_{ij} = \frac{2}{J^{5/3}}\left(\frac{1}{2}\mu + \frac{1}{10\beta^2}\bar{I}_1\right)B_{ij} - \frac{2}{3J}\left(\bar{I}_1\left(\frac{1}{2}\mu + \frac{1}{10\beta^2}\bar{I}_1\right)\right)\delta_{ij} + K(J-1)\delta_{ij} \tag{27}$$

and the nominal stress is,

$$S_{ij} = \left(\frac{1}{2}\mu + \frac{1}{10\beta^2}\bar{I}_1\right)\left(\frac{2}{J^{2/3}}F_{ji} - \frac{2}{3}\bar{I}_1 F_{ij}^{-1}\right) + KJ(J-1)F_{ji}^{-1} \tag{28}$$

2.4 Ogden

The Ogden SED for the isotropic solid is given in terms of a different invariant,

$$\tilde{U} = \sum_{i=1}^{N}\frac{2\mu_i}{\alpha_i^2}\left(\bar{\lambda}_1^{\alpha_i} + \bar{\lambda}_1^{\alpha_i} + \bar{\lambda}_1^{\alpha_i} - 3\right) + K_1(J-1)^2 \tag{29}$$

and the stress,

$$\sigma_{ij} = \frac{\lambda_1}{\lambda_1\lambda_2\lambda_3}\frac{\partial\tilde{U}}{\partial\lambda_1}b_i^{(1)}b_j^{(1)} + \frac{\lambda_2}{\lambda_1\lambda_2\lambda_3}\frac{\partial\tilde{U}}{\partial\lambda_2}b_i^{(2)}b_j^{(2)} + \frac{\lambda_3}{\lambda_1\lambda_2\lambda_3}\frac{\partial\tilde{U}}{\partial\lambda_3}b_i^{(3)}b_j^{(3)} \tag{30}$$

$$\frac{\partial\tilde{U}}{\partial\lambda_k} = \sum_{i=1}^{N}\frac{2\mu_i}{\alpha_i^2}\frac{1}{J^{1/3}} \tag{31}$$

therefore,

$$\sigma_{ij} = \frac{1}{J^{1/3}}\frac{1}{\lambda_1\lambda_2\lambda_3}\left(\lambda_k\frac{\partial\tilde{U}}{\partial\lambda_k}b_i^{(1)}b_j^{(1)}\right)\sum_{l=1}^{N}\frac{2\mu_l}{\alpha_l^2} \tag{32}$$

References

[1] Allan F Bower. *Applied Mechanics of Solids*. eng. Baton Rouge: CRC Press, 2010. ISBN: 9781439802472.