AA530 HW6

Matt Russell
University of Washington
Department of Aeronautics & Astronautics
https://github.com/russellmatt66/aa530-hw/tree/main/hw6

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1 Viscoelastic material: Maxwell model

The expression for the stress in a viscoelastic material, derived with a Maxwell model, is taken from the previous homework,

$$\sigma(t) = k\epsilon_0 exp(-\frac{k}{\eta}t) \tag{1}$$

this is plotted in Figure 1 for $\epsilon_0 = 0.1, k = 0.1$ [GPa], and $\eta = 20$ [GPa·s].

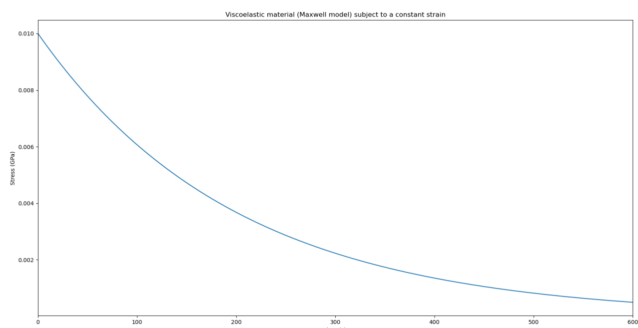

Figure 1: Time-dependent viscoelastic stress derived with a Maxwell model.

2 Viscoelastic material: Kelvin-Voigt model I

The Kelvin-Voigt model is similar to the Maxwell model in that it also captures the elastic behavior and the plastic behavior with a spring and a dashpot, respectively. The key difference is the manner in which the elements are coupled together. Kelvin-Voigt models employ a parallel coupling, illustrated in Figure 2, in contrast to the series coupling of the Maxwell model. To analyze the

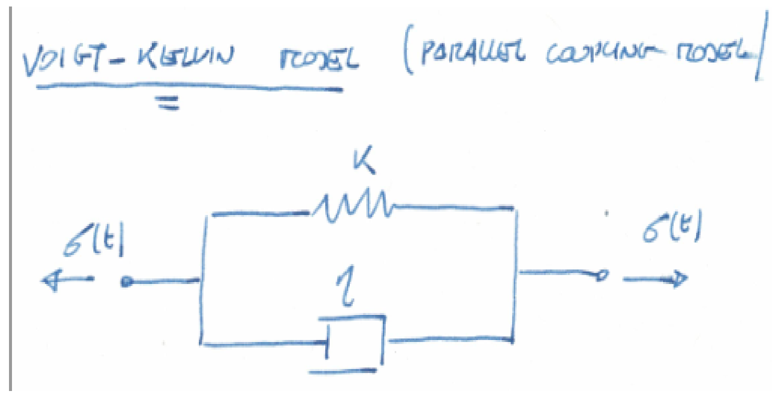

Figure 2: Circuit representation of a Kelvin-Voigt model. Illustration taken from Prof. Marco Salviato's lecture notes for AA530 - Lecture 12, pg.8.

material response we decompose the total stress into an elastic and a plastic response,

$$\sigma_{TOT} = \sigma_e + \sigma_p \tag{2}$$

the constitutive relationships for these components are,

$$\sigma_e = k\epsilon \tag{3}$$

$$\sigma_p = \eta \frac{\mathrm{d}\epsilon}{\mathrm{d}t} \tag{4}$$

the solution to the differential equation that results from plugging the above into Equation 2 is,

$$\epsilon(t) = C_1 exp(-\frac{k}{\eta}t) + \frac{\sigma_0}{k} \tag{5}$$

where our initial condition is $\sigma_{TOT}(t = 0) = \sigma_0$. To obtain the value of the constant, C_1, we perform a limiting procedure that reverses time over some initial transient period,

$$\lim_{\Delta T \to 0^+} \epsilon(\Delta T) = \frac{\sigma_0}{k} + C_1 = 0 \tag{6}$$

$$\to C_1 = -\frac{\sigma_0}{k} \tag{7}$$

$$\therefore \epsilon(t) = \frac{\sigma_0}{k}\left(1 - exp(-\frac{k}{\eta}t)\right) \tag{8}$$

3 Viscoelastic material: Kelvin-Voigt model II

The relation in Equation 8 is plotted in Figure 3 for $\sigma_0 = 1$ [MPa], $k = 0.1$ [GPa], and $\eta = 20$ [GPa·s].

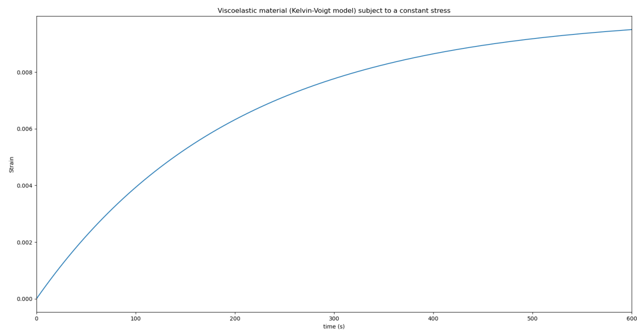

Figure 3: Equation 8 plotted for the parameters described in the problem writeup. The response illustrated by the plot is an example of retarded elastic behavior. This can be seen by noting that the plastic response, $\eta\frac{\mathrm{d}\epsilon}{\mathrm{d}t}$, will approach zero for long time and leave only the elastic response standing.

4 Constitutive relationship for viscoelastic materials

4.1 Step Displacement

In general, the stress history for a viscoelastic material is given by the expression,

$$\sigma_{ij} = \int_0^t 2G(t-\tau)\left(\dot{\epsilon}_{ij} - \dot{\epsilon}_{kk}\delta_{ij}\right) d\tau + K\epsilon_{kk}\delta_{ij} \tag{9}$$

where the strain is considered to be due to small deformations, i.e, it takes the form of the infinitesimal strain tensor,

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \tag{10}$$

with the latin indices corresponding to spatial components. For the displacement field,

$$u(t) = \begin{cases} 0 & t < 0 \\ u_0 & t > 0 \end{cases} \tag{11}$$

with uniform u_0, **all** the strain components will vanish. From Equation 9 it can be seen then that the stress does as well.

4.2 Sinusoidal Displacement

A similar situation occurs for a sinusoidal displacement field. Without a spatial dependence to $\vec{u}$ the infinitesimal strain tensor vanishes as the i,j-indices do not account for time. Note that even though the stress history depends implicitly on the strain rates the lack of spatial dependence to the displacement field causes the strain to vanish before the time derivative can act on it.

4.3 Stress Histories

Without a spatial dependence to the underlying displacement field the resulting stress components will all be zero.

4.4 Energy Dissipation

For a harmonic, spatially-independent displacement field there will be no stress or strain energy dissipated during a load cycle due to the lack of spatial dependence: $\rightarrow \epsilon_{ij} = 0 \rightarrow \sigma_{ij} = 0$.

5 Tresca Yield Surface

The Tresca yield criterion is[1],

$$max\left[|\sigma_1 - \sigma_2|, |\sigma_1 - \sigma_3|, |\sigma_2 - \sigma_3|\right] - Y(\bar{\epsilon}^p) = 0 \tag{12}$$

The yield stress is taken here to be, $Y(\bar{\epsilon}^p) = \sigma_0 = 200$ [MPa].

5.1 Numerical approach

To determine the set of points in principal stress space that satisfy Equation 12 an $\Theta(N^3)$ algorithm can be developed that simply raster scans through the input to check the above condition for all possible 3-tuples, $(\sigma_1, \sigma_2, \sigma_3)$, and then stores the corresponding 3-tuple when the condition is satisfied. This algorithm

Algorithm 1: Algorithm to compute the Tresca yield surface

```
Initialize principal stress arrays: σ1,σ2,σ3 (inputs).
Initialize empty arrays to store yield surface points: σ̃1,σ̃2,σ̃3 (outputs).
Declare tolerance, δ, for condition check, and value of yield stress, σY.
for i = 0,   i<length(σ1),   i++ do
  for j = 0,   j<length(σ2),   j++ do
    for k = 0,   k<length(σ3),   k++ do
      Compute max(abs(σ1[i]-σ2[j]),abs(σ1[i]-σ3[k]),abs(σ2[j]-σ3[k]))
      if abs(max - σY) < δ then
        Store σ1[i],σ2[j],σ3[k] in the corresponding yield surface arrays
      end if
    end for
  end for
end for
```

is implemented in the code "aa530-hw6_P5.py" using the NumPy library. For inputs of size 25 it returns 144 3-tuples that correspond to the points lying along the yield surface.

Unfortunately, this is where the approach fails. The Python library functions for 3D plotting, that the author is aware of at the time of writing, work by turning 1D coordinate vectors, representing independent variables, into 2D coordinate arrays and then using a relationship between the independent, i.e, 'x' and 'y' variables, and dependent (z) variable to create a map of the surface that can be plotted, e.g, using the output of np.meshgrid() and ax.plot_surface(). Without such an analytical relationship the 'yield surface arrays' that were obtained cannot be plotted as np.meshgrid() turns the collection into 3D coordinate arrays instead of the 2D that ax.plot_surface() requires.

5.2 Analytical approach

It turns out that there is such an analytical expression, at least in principle, but it requires the solution of the 4th-order, in σ_3, polynomial,

$$\left((\sigma_1-\sigma_2)^2-4\sigma_0^2\right)\left((\sigma_2-\sigma_3)^2-4\sigma_0^2\right)\left((\sigma_3-\sigma_1)^2-4\sigma_0^2\right)=0 \tag{13}$$

if we normalize the principal stresses by the yield stress, $\tilde{\sigma}_j=\frac{\sigma_j}{\sigma_0}$, then the above simplifies to,

$$\left((\tilde{\sigma}_1-\tilde{\sigma}_2)^2-4\right)\left((\tilde{\sigma}_2-\tilde{\sigma}_3)^2-4\right)\left((\tilde{\sigma}_3-\tilde{\sigma}_1)^2-4\right)=0 \tag{14}$$

this expression is given to the Solve[] command in the Mathematica environment. The output is shown in Figure 4, bizarrely, Mathematica gives different

```
(* Now, sigj in non-dimensional form, normalized by sig0 *)

In[•]:= Solve[((sig1 - sig2)^2 - 4) * ((sig2 - sig3)^2 - 4) * ((sig3 - sig1)^2 - 4) == 0, sig3]

Out[•]= {{sig3 → -2 + sig1 if sig1 > 4 + sig2 || -4 + sig2 < sig1 < sig2 || sig2 < sig1 < 4 + sig2 || sig1 < -4 + sig2},
  {sig3 → 2 + sig1 if sig1 > 4 + sig2 || -4 + sig2 < sig1 < sig2 || sig2 < sig1 < 4 + sig2 || sig1 < -4 + sig2},
  {sig3 → -2 + sig2 if sig1 > 4 + sig2 || -4 + sig2 < sig1 < sig2 || sig2 < sig1 < 4 + sig2 || sig1 < -4 + sig2},
  {sig3 → 2 + sig2 if sig1 > 4 + sig2 || -4 + sig2 < sig1 < sig2 || sig2 < sig1 < 4 + sig2 || sig1 < -4 + sig2}}
```

Figure 4: The output Mathematica gives as the solution to the fourth-order polynomial describing the Tresca yield surface

expressions for the exact same conditions. This leaves the question of which should be used and where open and pushes the author to move on from the problem after having tried multiple approaches without much success.

6 Von Mises Yield Surface

More success was found for this problem. The Von Mises yield criterion is written,

$$\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right]}=\sigma_0 \tag{15}$$

The above can be normalized in a suitable fashion as in Problem 5 to give,

$$(\tilde{\sigma}_1-\tilde{\sigma}_2)^2+(\tilde{\sigma}_1-\tilde{\sigma}_3)^2+(\tilde{\sigma}_2-\tilde{\sigma}_3)^2-2=0 \tag{16}$$

which can be expanded to a second-order polynomial in $\tilde{\sigma}_3$ and solved with the quadratic formula. The plotted yield surface (both positive and negative solutions) is visualized in Figure 5 the elliptical character of the surface can be

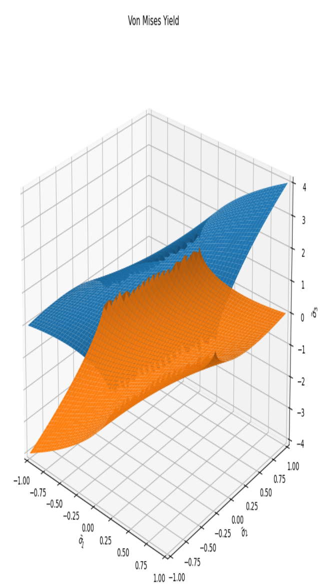

Figure 5: Von Mises yield surface

observed, but numerical artifacts from the square root in the quadratic formula and divergent behavior due to computation with inaccessible stress states $|\sigma| > \sigma_0$ distorts the surface somewhat.

References

[1] Allan F Bower. *Applied Mechanics of Solids*. eng. Baton Rouge: CRC Press, 2010. ISBN: 9781439802472.