# AA530 HW3

Matt Russell
University of Washington
Department of Aeronautics & Astronautics
https://github.com/russellmatt66/aa530-hw/tree/main/hw3

November 2, 2021

## 1 Linear elastic isotropic material

In general, the relationship between the stresses and strains in a linear material is given by Hooke's Law,

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\epsilon}} \tag{1}$$

where the 4th-rank elastic stiffness tensor, $\underline{\underline{\underline{\underline{C}}}}$, has components,

$$C_{ijkl} = [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] \tag{2}$$

with $\lambda$ and $\mu$ being the Lame coefficients. Distributing the strain tensor inside and continuing in index notation obtains,

$$\sigma_{ij} = \lambda \delta_{ij}\delta_{kl}\epsilon_{kl} + \mu\delta_{ik}\delta_{jl}\epsilon_{kl} + \mu\delta_{il}\delta_{jk}\epsilon_{kl} \tag{3}$$

Proceeding term-by-term, the first is only non-zero when $i = j$ and $k = l$, the second when $i = k$ and $j = l$, and the third when $i = l$ and $j = k$. This simplifies the above,

$$\sigma_{ij} = \lambda \delta_{ij}\epsilon_{kk} + \mu\left(\epsilon_{ij} + \epsilon_{ji}\right) \tag{4}$$

The symmetric property of the stress tensor allows this to be simplified one step further and we take the opportunity to identify the contraction of the stress tensor along the diagonal as the trace operator,

$$\sigma_{ij} = \lambda \delta_{ij} Tr(\underline{\underline{\epsilon}}) + 2\mu\delta_{ij} \tag{5}$$

so that in tensor notation the resulting expression is,

$$\underline{\underline{\sigma}} = \lambda \underline{\underline{I}} Tr(\underline{\underline{\epsilon}}) + 2\mu\underline{\underline{\epsilon}} \tag{6}$$

Re-arranging to isolate $\underline{\underline{\epsilon}}$ on the left yields,

$$\underline{\underline{\epsilon}} = \frac{1}{2\mu}\left(\underline{\underline{\sigma}} - \lambda Tr(\underline{\underline{\epsilon}})\underline{\underline{I}}\right) \tag{7}$$

to proceed we take the trace of Equation 6,

$$Tr(\underline{\underline{\sigma}}) = \lambda Tr(\underline{\underline{\epsilon}})Tr(\underline{\underline{I}}) + 2\mu Tr(\underline{\underline{\epsilon}}) \tag{8}$$

the trace of the identity tensor of 2nd rank outputs a value of 3. This allows the above to be written as a direct relationship between the trace of $\underline{\underline{\sigma}}$ and $\underline{\underline{\epsilon}}$. From this relationship the trace of the strain can be isolated and inserted into the above expression for the strain. This yields,

$$\underline{\underline{\epsilon}} = \frac{1}{2\mu}\left(\underline{\underline{\sigma}} - \frac{\lambda}{3\lambda + 2\mu} Tr(\underline{\underline{\sigma}})\underline{\underline{I}}\right) \tag{9}$$

from the class notes an equivalent expression using the familiar elastic constants can be written for the case where thermal expression is neglected,

$$\underline{\underline{\epsilon}} = \frac{1+\nu}{E}\underline{\underline{\sigma}} - \frac{\nu}{E} Tr(\underline{\underline{\sigma}})\delta_{ij} \tag{10}$$

and the two expressions can be compared to show that,

$$\frac{1}{2\mu} = \frac{1+\nu}{E} \tag{11}$$

which can be trivially rearranged,

$$\mu = \frac{E}{2(1+\nu)} \tag{12}$$

where $\mu$'s identity as the shear modulus, $G$, is now revealed. QED

## 2 Constitutive relationship

For a linear, isotropic, elastic material subject to a thermal expansion process, the constitutive relationship between stress and strain can be written in the following equivalent forms,

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta ij + \alpha\Delta T\delta_{ij} \tag{13}$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right] - \frac{E\alpha\Delta T}{1-2\nu}\delta_{ij} \tag{14}$$

### 2.1 Plane Strain

The condition of plane strain states that all transverse strain components are considered to be equal to zero,

$$\epsilon_{33} = \epsilon_{32} = \epsilon_{31} = \epsilon_{13} = \epsilon_{23} = 0 \tag{15}$$

We use this as a starting point to evaluate the individual components of the stress tensor using Equation (14),

$$\sigma_{11} = \frac{E}{1+\nu}\left[\epsilon_{11} + \frac{\nu}{1-2\nu}(\epsilon_{11}+\epsilon_{22})\right] - \frac{E\alpha\Delta T}{1-2\nu} \tag{16}$$

$$\sigma_{22} = \frac{E}{1+\nu}\left[\epsilon_{22} + \frac{\nu}{1-2\nu}(\epsilon_{11}+\epsilon_{22})\right] - \frac{E\alpha\Delta T}{1-2\nu} \tag{17}$$

$$\sigma_{33} = \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}(\epsilon_{11}+\epsilon_{22})\right] - \frac{E\alpha\Delta T}{1-2\nu} \tag{18}$$

$$\sigma_{12} = \frac{E}{1+\nu}\epsilon_{12} \tag{19}$$

with the remaining transverse components all expanding to be 0. Then we evaluate the individual components of the strain tensor using Equation (13),

$$\epsilon_{11} = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}) + \alpha\Delta T \tag{20}$$

$$\epsilon_{22} = \frac{1+\nu}{E}\sigma_{22} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}) + \alpha\Delta T \tag{21}$$

$$\epsilon_{33} = \frac{-\nu}{E}(\sigma_{11}+\sigma_{22}) + \alpha\Delta T \tag{22}$$

$$\epsilon_{12} = \frac{1+\nu}{E}\sigma_{12} \tag{23}$$

From the matrix multiplication in Sec 3.2.3 of Bower's *Applied Mechanics of Solids* we can obtain the following system of equations for the stresses,

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_{11} + \nu\epsilon_{22}\right] - \frac{E\alpha\Delta T}{1-2\nu} \tag{24}$$

$$\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}\left[(\nu)\epsilon_{11} + (1-\nu)\epsilon_{22}\right] - \frac{E\alpha\Delta T}{1-2\nu} \tag{25}$$

$$\sigma_{12} = \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{1-2\nu}{2}\right)2\epsilon_{12} \tag{26}$$

$$\sigma_{33} = \frac{E\nu(\epsilon_{11}+\epsilon_{22})}{(1-2\nu)(1+\nu)} + \frac{E\alpha\Delta T}{1-2\nu}, \sigma_{13} = \sigma_{23} = 0 \tag{27}$$

and strains,

$$\epsilon_{11} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{11} - \nu\sigma_{22}\right] + (1+\nu)\alpha\Delta T \tag{28}$$

$$\epsilon_{22} = \frac{1+\nu}{E}\left[-\nu\sigma_{11} + (1-\nu)\sigma_{22}\right] + (1+\nu)\alpha\Delta T \tag{29}$$

$$2\epsilon_{12} = \frac{1+\nu}{E}(2\sigma_{12}) \tag{30}$$

$$\epsilon_{33} = \frac{1+\nu}{E}(\sigma_{33} - \nu\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}) + (1+\nu)\alpha\Delta T \tag{31}$$

The correctness of some of the components, e.g, the planar shear, is immediately obvious. Verifying the others requires multiplication through and re-arrangement. The transverse-normal strain expressions do not appear to be equivalent. The transverse-normal stress expressions; however, do, as well as the transverse-inplane stresses.

## 2.2 Plane Stress

The condition of plane stress states that all transverse stress components are considered to be equal to zero,

$$\sigma_{33} = \sigma_{32} = \sigma_{31} = \sigma_{13} = \sigma_{23} = 0 \tag{32}$$

We use this as a starting point to evaluate the individual components of the strain tensor using Equation (13),

$$\epsilon_{11} = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}) + \alpha\Delta T \tag{33}$$

$$\epsilon_{22} = \frac{1+\nu}{E}\sigma_{22} - \frac{\nu}{E}(\sigma_{11}+\sigma_{22}) + \alpha\Delta T \tag{34}$$

$$\epsilon_{33} = \frac{-\nu}{E}(\sigma_{11}+\sigma_{22}) + \alpha\Delta T \tag{35}$$

$$\epsilon_{12} = \frac{1+\nu}{E}\sigma_{12} \tag{36}$$

with the remaining transverse components all expanding to 0. Then we evaluate the individual components of the stress tensor using Equation (14),

$$\sigma_{11} = \frac{E}{1+\nu}\left[\epsilon_{11} + \frac{\nu}{1-2\nu}(\epsilon_{11}+\epsilon_{22})\right] - \frac{E\alpha\Delta T}{1-2\nu} \tag{37}$$

$$\sigma_{22} = \frac{E}{1+\nu}\left[\epsilon_{22} + \frac{\nu}{1-2\nu}(\epsilon_{11}+\epsilon_{22})\right] - \frac{E\alpha\Delta T}{1-2\nu} \tag{38}$$

$$\sigma_{33} = \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}(\epsilon_{11}+\epsilon_{22})\right] - \frac{E\alpha\Delta T}{1-2\nu} \tag{39}$$

$$\sigma_{12} = \frac{E}{1+\nu}\epsilon_{12} \tag{40}$$

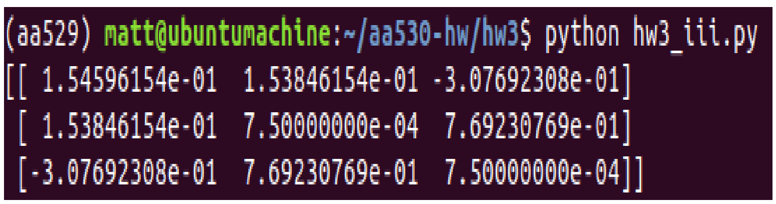
The next step is to matrix multiply Bower's matrix form expression for the plane stress situation and carry out a similar process as the one that was done in depth for the plane strain case.

# 3 Constitutive relationship

The components of the stress tensor in index notation for an isothermal measurement of strain state are,

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\epsilon_{ij} + \frac{\nu}{1-2\nu}Tr(\underline{\epsilon})\delta_{ij}\right] \tag{41}$$

Given $E = 200$ [GPa] and $\nu = 0.3$ it is a light matter to instruct a modern computer to perform this calculation for you. The corresponding stress tensor is shown in Figure 1 and the units of the stress tensor determined here are in [GPa].



```
(aa529) matt@ubuntumachine:~/aa530-hw/hw3$ python hw3_iii.py
[[ 1.54596154e-01  1.53846154e-01 -3.07692308e-01]
 [ 1.53846154e-01  7.50000000e-04  7.69230769e-01]
 [-3.07692308e-01  7.69230769e-01  7.50000000e-04]]
```

Figure 1: Output from the program $hw3_iii.py$ which carried out the computation

## 4 Coordinate transformation

The transformations between the given strain gauge readings and the strain in the xy-axes are given by the system of equations[1],

$$\epsilon_{0^\circ} = \epsilon_x \tag{42}$$

$$\epsilon_{60^\circ} = \frac{\epsilon_{0^\circ} + \epsilon_y}{2} + \frac{\epsilon_{0^\circ} - \epsilon_y}{2} cos(120^\circ) + \epsilon_{xy} sin(120^\circ) \tag{43}$$

$$\epsilon_{120^\circ} = \frac{\epsilon_{0^\circ} + \epsilon_y}{2} + \frac{\epsilon_{0^\circ} - \epsilon_y}{2} cos(240^\circ) + \epsilon_{xy} sin(240^\circ) \tag{44}$$

Many computational tools for solving systems of equations exist, to solve this one I used Wolfram Alpha. The resulting strains are,

$$\epsilon_x = 0.005 \tag{45}$$

$$\epsilon_y = -0.001 \tag{46}$$

$$\epsilon_{xy} = \frac{3}{1732} \tag{47}$$

Using the reduced stress-strain relations and assuming an isothermal measurement the stress components are computed to be,

$$\sigma_x = 1.231 \tag{48}$$

$$\sigma_y = 0.308 \tag{49}$$

$$\sigma_{xy} = 0.266 \tag{50}$$

in units of [GPa].

## References

[1] efunda. *Rosette Strain Gage*. https://www.efunda.com/formulae/solid_mechanics/mat_mechanics/strain_gage_rosette.cfm.