AA543 - Homework #2

Problem 1:

Given the 1D heat conduction problem:

$$u_t = \alpha u_{xx} \tag{1}$$

where $u = u(x,t)$ and $\alpha = 1$, with initial conditions (ICs):

$$u(x, t=0) = \begin{cases} 2x & \text{if} \quad 0 \leq x \leq 0.5 \\ 2 - 2x & \text{if} \quad 0.5 < x \leq 1.0 \end{cases} \tag{2}$$

and boundary conditions (BCs):

$$u(x=0,t) = u(x=1,t) = 0 \tag{3}$$

in the interval $0 \leq x \leq 1$, do the following:

1.1 Derive the analytical solution of the problem stated above.

1.2 Write a computer program (and on paper the numerical algorithm) to find the numerical solution of the 1D heat conduction problem using the following schemes:

1. one-step forward in time and 2nd-order central difference in space (explicit);
2. one-step backward in time and 2nd-order central difference in space (implicit);

1.3 Compute the numerical solution for $\Delta x = 0.05$, and for $\Delta t = 0.0012$ and $\Delta t = 0.0013$, and plot the numerical solution along with the analytical solution at $t = 0,\ \Delta t,\ 10\Delta t,\ 50\Delta t$.

1.4 Discuss your solutions, and if your code does not work explain why.

1.5 Advanced (Bonus Points): Extend the 1D heat conduction problem to 2D, formulate the problem in 2D and solve 1.1 to 1.4 above in the 2D case.

Problem 2:

Given the 1D linear convection problem:

$$u_t + a u_x = 0 \tag{4}$$

where $u = u(x,t)$ and $a = 0.5$, with initial conditions (ICs):

$$u(x, t=0) = \begin{cases} 1 & \text{if} \quad -7 \leq x \leq -5 \\ 0 & \text{if} \quad x < -7, x > -5 \end{cases} \tag{5}$$

and boundary conditions (BCs):

$$u(x = -10, t) = 0 \tag{6}$$

in the interval $-10 \leq x \leq 10$, do the following:

2.1 Write a computer program (and on paper the numerical algorithm) to find the numerical solution of the 1D linear convection problem using the following schemes:

1. one-step forward in time (explicit) and first-order upwind in space;
2. Lax-Wendroff scheme.
3. an implicit scheme of your choice.

2.2 Compute the numerical solution for different CFL numbers, CFL = 0.6 and 1.2, and plot the numerical solutions along with the analytical solution [$u(x,t) = u(x - at)$] at $t = t_{max}/2$ where t_{max} is the time at which the square wave reaches the right boundary $x = 10$.

2.3 Discuss your solutions, and if your code does not work explain why.

2.4 Derive and plot the dispersion and diffusion errors of the three numerical schemes above at two different CFL numbers of your choice.

2.5 Advanced (Bonus Points): Extend the 1D linear convection problem to 2D, formulate the problem in 2D, and solve 2.1 to 2.3 above in the 2D case.

Notes:

- Do not skip any steps in the derivations.
- Present readable plots including labels, and figure numbers and captions.
- Do not place more than 2 plots in the same page, and use page space wisely.
- Include your computer codes as Appendices to your HW report.