$$\mathcal{S} = \begin{bmatrix} 736 & 3060 & 1016 \\ 256 & 864 & 308 \\ 424 & 1068 & 428 \end{bmatrix}$$

$$\begin{bmatrix} 70 & -1151 & -380 \\ 13 & -209 & -69 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 360 \end{bmatrix} \begin{bmatrix} 106 & 267 & 107 \\ 12 & 25 & 12 \\ -1 & -2 & -1 \end{bmatrix}$$

(a)

$$\begin{bmatrix} 70 & -1 & 1 \\ 13 & 11 & -10 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 360 \end{bmatrix} \begin{bmatrix} 106 & 267 & 107 \\ -24558 & -60835 & -24558 \\ -899 & -2227 & -899 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -2 & 5 & 0 \\ -5 & -1 & -1 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 360 \end{bmatrix} \begin{bmatrix} 898 & 2865 & 1043 \\ 132 & 433 & 156 \\ -55 & -176 & -64 \end{bmatrix}$$

(c)

Fig. 3. Examples of applying the postprocessing technique of Section IV-C to the Smith canonical form $\mathcal{S} = U\Lambda V$. (a) No preprocessing. (b) Postprocessing. (c) Preprocessing and postprocessing. Comparing Fig. 3(b) with Fig. 1(b), the postprocessing technique does not perform as well as the preprocessing technique of Section III. Applying both pre- and postprocessing further reduces the size of U and makes its columns vectors closer to being orthogonal. The angles θ_{kl} between columns k and l of U are $\theta_{12} = 90.4°, \theta_{23} = 88.1°$, and $\theta_{13} = 80.4°$ in Fig. 3(c) versus $\theta_{12} = 89.2°, \theta_{23} = 84.8°$, and $\theta_{13} = 80.4°$ in Fig. 1(c).

the *Matehmatica* computer algebra environment [13] as the file "`Mathematica/LatticeTheory.m`" on the anonymous FTP site `gauss.eedsp.gatech.edu` (IP #130.207.226.24).

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A Subspace Method for Estimating Sensor Gains and Phases

V. C. Soon, L. Tong, Y. F. Huang, and R. Liu

***Abstract*—The problem of array signal processing under model errors is studied here. A signl subspace constraint is used to obtain a simple way for computing a set of sensor gains and phases consistent with a given set of DOA angles. Then the issue of uniqueness of the set of sensor gains and phases and DOA angles is addressed**

I. Introduction

Eigenstructure methods such as MUSIC [1] and ESPRIT [2] have gained considerable interest in recent years due to their high-resolution capabilities in resolving closely spaced sources. These methods are hampered, however, by the requirement of a known array manifold (i.e., the collection of array-steering vectors for all possible direction-of-arrival (DOA) angles) as in MUSIC or that the sensor subarrays as in ESPRIT be matched.

It has been observed that in practice, even after hardware calibration, sensor gain and phase errors still exist. The effects of random array model errors on the DOA estimates, in terms of their mean-square-error (MSE) and sensitivity of algorithms such as ESPRIT and MUSIC, have been a subject of many recent studies (see e.g., [3]–[7]). These studies have shown that the MSE of their DOA estimates will increase due to these errors. As a result, several methods have been proposed for DOA estimation under various array model errors. For the case of DOA estimation with unknown sensor gains and phases, a method applicable only to uniform linear arrays is proposed in [8]. In [9], [10], methods based on the orthogonality of the noise subspace to the steering vector are proposed. In [9], an iterative algorithm is proposed that involves the computation of the matrix inverse that is close to being singular when the signal-to-noise ratio (SNR) is high

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and when the estimates of the DOA angles are close to the true ones, as will be shown in Section III of this correspondence.

In this correspondence, a constraint on the set of sensor gains and phases and DOA angles consistent with the signal subspace is studied. This constraint is shown to yield a simple way of computing the sensor gains and phases for a given set of DOA angles. The issue of uniqueness of the set of sensor gains and phases and DOA angles is examined. When the DOA angles are known, the estimates of the sensor gains and phases computed using the constraint can be shown to be asymptotically unbiased. When the DOA angles are unknown, an interative procedure incorporating the constraint is proposed.

II. Model Formulation

The following notational convention is used in this correspondence. All matrix quantities will be denoted by bold-faced capital letters while vector quantities will be denoted by bold-faced small letters. Matrix transpose and matrix conjugate transpose are denoted, respectively, by $(.)^T$ and $(.)^t$. The vector norm used here is the Euclidean vector norm denoted as $||.||$ and the matrix 2-norm will be denoted as $||.||_2$, [11].

Let m be the number of sensors in the array, n be the known number of narrowband far-field sources and $\{x_i, y_i\}$ be the location of the ith sensor in the array. The nominal sensor gains and phases are assumed to be equal to unity and zero, respectively. For unknown sensor gains and phases the array model would be written as

$$\mathbf{z} = \mathbf{\Gamma}\,\mathbf{A}\mathbf{s} + \eta \tag{1}$$

where $\mathbf{z}$ denotes the observation vector from the sensor array, $\mathbf{s}$ is the source vector, η denotes the additive noise at the sensor array, and $\mathbf{\Gamma}$ is the diagonal matrix of unknown (DOA angle-independent) sensor gains and phases where $\mathbf{\Gamma} = \mathrm{diag}\,\{1, \alpha_1 e^{j\phi_1}, \cdots, \alpha_{m-1}e^{j\phi_{m-1}}\}$. Here, α_i and ϕ_i denote the ith sensor gain and phase, respectively. We assume that all sensor gains are nonzero. Without loss of generality, the first diagonal element in Γ is assumed to be unity. Furthermore, $\mathbf{A}$ is the steering matrix defined by the set of distinct DOA angles $\{\theta_k\}$ (the DOA angles are measured with reference to the y-axis of the sensor array)

$$\mathbf{A} = [\mathbf{a}(\theta_1)\cdots\mathbf{a}(\theta_n)]. \tag{2}$$

The steering vector at DOA angle θ is written

$$\mathbf{a}(\theta) = \left[1 \exp\left\{-j\frac{2\pi}{\lambda}(x_1 \sin\theta + y_1\cos\theta)\right\}\cdots \right.$$
$$\left.\cdots \exp\left\{-j\frac{2\pi}{\lambda}(x_{m-1}\sin\theta + y_{m-1}\cos\theta)\right\}\right]^T \tag{3}$$

where λ is the wavelength of the narrowband signals.

The following assumptions are made throughout this correspondence:

(A.1) The array of m sensors is *unambiguous,* [12], i.e., it satisfies the property that for any collection of $k \le m$ distinct DOA angles, $\{\theta_i\}_{i=1,\cdots,k}$, the matrix $[\mathbf{a}(\theta_1),\cdots,\mathbf{a}(\theta_k)]$ has full column rank, where $m > n$.

(A.2) The source covariance matrix $\mathbf{Q} = E\{\mathbf{s}\mathbf{s}^\dagger\}$ is positve definite.

(A.3) $\mathbf{s}$ and η are mutually uncorrelated and are stationary processes.

(A.4) η is additive white Gaussian with covariance $E\{\eta\eta^\dagger\} = \sigma^2\mathbf{I}$.

III. Computation of Sensor Gains and Phases

Computing the covariance matrix of $\mathbf{z}$ yields

$$\mathbf{R} = E\{\mathbf{z}\mathbf{z}^\dagger\} = (\mathbf{\Gamma}\,\mathbf{A})\,\mathbf{Q}\,(\mathbf{\Gamma}\,\mathbf{A})^\dagger + \sigma^2\mathbf{I}. \tag{4}$$

Taking the eigenvector-eigenvalue decomposition of $\mathbf{R}$ gives

$$\mathbf{R} = \mathbf{E}_s\mathbf{\Lambda}_s\mathbf{E}_s^\dagger + \sigma^2\mathbf{E}_n\mathbf{E}_n^\dagger \tag{5}$$

where $\mathbf{E}_s$ denotes the eigenvector matrix of $\mathbf{R}$ associated with the signal subspace spanned by $\mathbf{\Gamma}\,\mathbf{A}, \mathbf{\Lambda}_s$ is the eigenvalue matrix associated with it, and $\mathbf{E}_n$ denotes the eigenvector matrix associated with the noise or null subspace with eigenvalue σ^2. Thus, the solution set of sensor gains and phases and DOA angles are constrained such that

$$\mathbf{E_s}\mathbf{E_s}^\dagger\mathbf{\Gamma}\mathbf{A} = \mathbf{\Gamma}\mathbf{A}. \tag{6}$$

The constraint of (6) is equivalent to

$$\mathbf{E_s}\mathbf{E_s}^\dagger\mathbf{\Gamma}\mathbf{a}(\theta_i) = \mathbf{\Gamma}\mathbf{a}(\theta_i), i = 1, 2, \cdots, n. \tag{7}$$

By defining $\mathbf{D}_i = \mathrm{diag}\{\mathbf{a}(\theta_i)\}$, a diagonal matrix composed of the components of the steering vector $\mathbf{a}(\theta_i)$, and $\mathbf{v}$ as the $m \times 1$ column vector formed from the diagonal components of $\mathbf{\Gamma}$, i.e., $\mathbf{v} = [1\alpha_1 e^{j\phi_1}\cdots\alpha_{m-1}e^{j\phi_{m-1}}]^T$, we can rewrite (7) as

$$\mathbf{E_s}\mathbf{E_s}^\dagger\mathbf{D}_i\mathbf{v} = \mathbf{D}_i\mathbf{v}, i = 1, 2, \cdots, n. \tag{8}$$

Since $\mathbf{D}_i$ is a diagonal matrix composed entirely of the steering vector $\mathbf{a}(\theta_i)$ (see (3)), it is also unitary, i.e., $\mathbf{D}_i^\dagger\mathbf{D}_i = \mathbf{D}_i\mathbf{D}_i^\dagger = \mathbf{I}$ and therefore

$$\mathbf{D}_i^\dagger\mathbf{E_s}\mathbf{E_s}^\dagger\mathbf{D}_i\mathbf{v} = \mathbf{v}, i = 1, 2, \cdots, n. \tag{9}$$

In other words, the vector of the sensor gains and phases must be an eigenvector with unit eigenvalue of the matrices $\mathbf{W}_i = \mathbf{D}_i^\dagger\mathbf{E_s}\mathbf{E_s}^\dagger\mathbf{D}_i$.

We note that $\mathbf{W}_i$ has n unity eigenvalues and $m - n$ zero eigenvalues. The constraint of (6) can now be restated as the set of n eigenvector constraints of (9). This set of constraints will be used in computing the sensor gain and phase-vector $\mathbf{v}$. The following theorem shows how these n eigenvector constraints can be combined into a single eigenvector constraint.

Theorem 1: Let $\mathbf{D}_i$ and $\mathbf{W}_i$ be as defined previously. Then $(\mathbf{v}, 1)$ is an eigen-pair (eigenvector-eigenvalue pair) for $\mathbf{W}_i = \mathbf{D}_i^\dagger\mathbf{E_s}\mathbf{E_s}^\dagger\mathbf{D}_i, i = 1, 2, \cdots, n$ if and only if $(\mathbf{v}, n)$ is an eigen-pair for

$$\mathbf{W} = \sum_{i=1}^{n}\mathbf{W}_i = \sum_{i=1}^{n}\mathbf{D}_i^\dagger\mathbf{E_s}\mathbf{E_s}^\dagger\mathbf{D}_i. \tag{10}$$

where $\mathbf{v}$ is as defined in (8). Furthermore, the eigenvalue n is the largest eigenvalue of $\mathbf{W}$.

Proof: See [13].

The following corollary is an immediate consequence of the previous result.

Corollary 1: The eigenvector $\mathbf{v}$ found in Theorem 1 is also an eigenvector associated with the zero eigenvalue of $\mathbf{M}$, where

$$\mathbf{M} = \sum_{i=1}^{n}\mathbf{D}_i^\dagger\mathbf{E}_n\mathbf{E}_n^\dagger\mathbf{D}_i. \tag{11}$$

IV. Discussions

In [9], an approach was taken based on the orthogonality of $\mathbf{\Gamma}\,\mathbf{A}$ to the noise subspace, and a solution for $\mathbf{v}$ was proposed as

$$\mathbf{v} = \frac{\mathbf{M}^{-1}\mathbf{e}_0}{\mathbf{e}_o^T\mathbf{M}^{-1}\mathbf{e}_o} \tag{12}$$

where, $\mathbf{e}_0 = [1\,0\cdots 0]^T$. It is seen from the previous corollary that, at the true DOA angles, the matrix $\mathbf{M}$ is singular. In simulations, it has been observed that the estimate of this matrix tends to have a large condition number, although it is usually invertible (due to noise, errors in DOA estimates, etc.). In [10] the orthogonality of $\mathbf{\Gamma}\,\mathbf{A}$ to the noise subspace is also exploited. From the requirement that $\mathbf{E}_n^\dagger\mathbf{\Gamma}\,\mathbf{a}(\theta_k) = \mathbf{0}, k = 1, 2, \cdots, n$, the minimum singular vector of the matrix $\mathbf{G}$ is proposed as the gain/phase vector estimate, where

$$\mathbf{G} = \begin{bmatrix} \mathbf{E}_n^\dagger\mathbf{D}_1 \\ \mathbf{E}_n^\dagger\mathbf{D}_2 \\ \vdots \\ \mathbf{E}_n^\dagger\mathbf{D}_n \end{bmatrix}. \tag{13}$$

Note that the evaluation of the large matrix $\mathbf{G}$ is not necessary since the right singular vectors of $\mathbf{G}$ are the eigenvectors of $\mathbf{G}^\dagger\mathbf{G}$ and therefore

$$\mathbf{G}^\dagger\mathbf{G} = \sum_{i=1}^{n}\mathbf{D}_i^\dagger\mathbf{E}_n\mathbf{E}_n^\dagger\mathbf{D}_i = \mathbf{M}. \tag{14}$$

Thus, from Corollary 1, the proposed estimate of the gain/phase vector of [10] and the estimate obtained by the method suggested by Theorem 1 are equivalent.

If the DOA angles are known *a priori,* Theorem 1 can be applied directly. Therefore, the eigenvector associated with the largest eigenvalue of the matrix $\mathbf{W}$ is taken to be the estimated sensor gain and phase vector. Suppose that N snapshots of the observation vector $\mathbf{z}$ are taken and the covariance matrix is estimated as

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{z}(i)\mathbf{z}(i)^\dagger. \tag{15}$$

Through first-order perturbation analysis analogous to that found in [15], and assuming known DOA angles, the estimate of the sensor gain and phase vector in Theorem 1 can be shown to be asymptotically unbiased. In the case where both the set of sensor gains and phases and the DOA angles are not known, an iterative procedure that utilizes the constraint (6) is proposed.

Proposed Method:

1. The algorithm starts with the intial DOA estimates found by assuming nominal sensor gains and phases and applying an eigenstructure algorithm (e.g., MUSIC, ESPRIT, etc.)
2. Then the maximum eigenvector of the estimate of the matrix $\mathbf{W}$ is used as the sensor gain/phase vector $\mathbf{v}$.
3. The estimated sensor gains and phases are then used in the estimation of the DOA angles through appropriate modifications of ESPRIT, MUSIC, etc.
4. Iteration between computation of the gain/phase vector (Step 2) and the estimation of the DOA angles (Step 3) is performed until some convergence criterion is achieved.

Remarks: 1) For MUSIC, the incorporation of the estimated sensor gains and phases (in Step 3) is used in its cost function as $P(\theta) = (1/||\mathbf{E}_n^\dagger\,\mathbf{\Gamma}\,\mathbf{a}(\theta)||^2)$. For ESPRIT the correction for the unmatched sensor pairs may be accomplished by solving for the matrix $\mathbf{\Psi}$ from $\mathbf{E}_x\mathbf{\Psi} = \mathbf{\Gamma}_x\mathbf{\Gamma}_y^{-1}\mathbf{E}_y$, where $\mathbf{E}_x, \mathbf{E}_y$ are the eigenvector matrices associated with the two ESPRIT subarrays while $\mathbf{\Gamma}_x, \mathbf{\Gamma}_y$ are their associated sensor gain/phase matrices. This modification of ESPRIT would be referred to as M-ESPRIT (Modified ESPRIT) in the sequel.

2) Each iteration of the proposed method requires the evaluation of the largest eigenvector of the matrix $\mathbf{W}$ (Step 2) and the estimation of the DOA angles (Step 3). For M-ESPRIT, the computational complexity per iteration is on the order of $5m^3$ (for the eigenvector decomposition of $\mathbf{W}$) + $40n^3$ (for eigenvector decomposition in TLS-ESPRIT, [2]), [11]. In the simulation studies conducted with M-ESPRIT, it was found that 15 iterations of M-ESPRIT yields quite satisfactory performance. Thus, with 15 iterations, the total complexity is on the order of $15(5m^3 + 40n^3)$. The computational complexity of $5m^3$ incurred by the eigenvector decomposition of $\mathbf{W}$ refers to the computation of all the eigenvectors and eigenvalues of $\mathbf{W}$. This is really not necessary since the largest eigenvalue is known and only the eigenvector associated with it is needed. See [11] for methods with reduced complexity for the computation of the largest eigenvector.

At this point however, it is not immediately apparent that the computed eigenvector $\mathbf{v}$ is unique. This issue is investigated in the sequel.

Given the signal-subspace constraint of (6), there are three questions that should be addressed.

First, given $\mathbf{A}$, is the corresponding sensor gain and phase-matrix $\mathbf{\Gamma}$ as constrained by (6) unique? Second, given the sensor gain and phase matrix $\mathbf{\Gamma}$, is the corresponding steering matrix $\mathbf{A}$ unique? Third, is there only one pair of sensor gain and phase matrix $\mathbf{\Gamma}$ and steering matrix $\mathbf{A}$ consistent with the constraint of (6)?

We examine the first question with the following theorem that gives a sufficient condition.

Theorem 2: If $m \geq 2n - 1$ and if all size n subarrays of the array are unambiguous, then for a given $\mathbf{A}$ (as defined by the given set of DOA angles), the set of sensor gains and phases as constrained by (6) is unique.

Proof: see [13].

Remark: The condition in Theorem 2 that all size n subarrays be unambiguous is also equivalent to requiring that all $n \times n$ submatrices of $\mathbf{A}$ be nonsingular. In practice, this condition does not impose stringent constraints on the geometry of an array and common arrays such as uniform linear arrays can easily fulfill this condition. The second question is: given the sensor gain and phase matrix $\mathbf{\Gamma}$, is the corresponding set of DOA angles (i.e., $\mathbf{A}$) as constrained by the signal-subspace constraint unique? The answer is also affirmative with the condition of (A.1) proving to be sufficient in this case. This follows directly from the proof for Theorem 1 in [14].

The two questions addressed so far only guarantee that there is a one-to-one correspondence between the set of sensor gains/phases and the set of DOA angles such that the constraint is fulfilled. It leaves open the question of how many distinct pairs of sensor gains/phases and DOA angles there are. This leads to the final question, i.e., is the set of sensor gains/phases and DOA angles consistent with the constraint (6) unique? This question is still an open one. We can however, state a necessary condition which must be fulfilled.

Necessary Condition for Uniqueness of the Pair $(\mathbf{\Gamma}, \mathbf{A})$*:* As constrained by (6), it is necessary that for all diagonal matrices $\mathbf{\Lambda}$ and all steering matrices $\mathbf{A}_1 \neq \mathbf{A}_2$, we have $\mathbf{A}_1 \neq \mathbf{\Lambda}\mathbf{A}_2$.

Remarks: It can be easily shown that a linear array would violate the above necessary condition. In [8] and [9] it has been shown that a linear array would not yield unique solutions for the set of sensor gains and phases and DOA angles.

V. Simulations

A Monte-Carlo study is performed to investigate the M-ESPRIT method regarding its sensitivity to array parameters. Two parameters are varied in the study. First the DOA angle separation between two sources is varied to examine the sensitivity of the method to angle separation. The other parameter to be varied is the SNR of the sources.

Simulation 1: Varied DOA Angle Separation: There are two uncorrelated sources, both with unity power. The first is fixed at

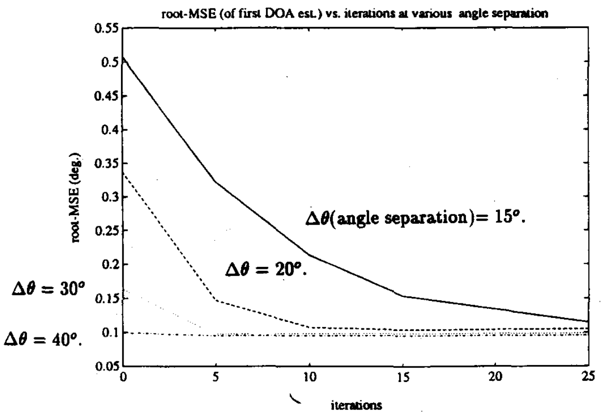

Fig. 1. MSE of estimates of DOA at −10 degrees versus iterations of M-ESPRIT for various DOA separation angles.

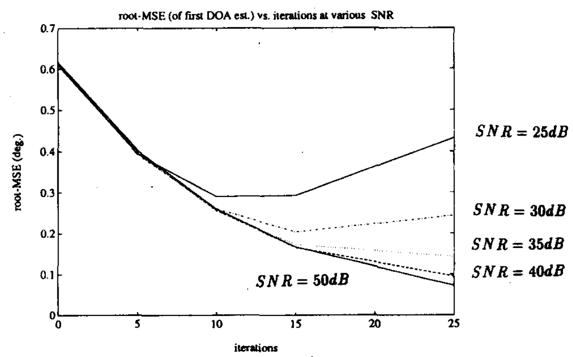

Fig. 2. MSE of estimates of DOA at −10 degrees versus iterations of M-ESPRIT for various SNR (dB).

$-10°$ while the other source location is varied according to the desired DOA angle separation. There are four sensors per subarray with the locations $\{(-.25\lambda, .5\lambda), (-.75\lambda, 0), (-.25\lambda, -.5\lambda), (+.25\lambda, 0)\}, \{(+.25\lambda, +.5\lambda), (-.25\lambda, 0), (+.25\lambda, -.5\lambda)$ $(+.75\lambda, 0)\}$, respectively. The observations are assumed to be with no additive noise and 50 Monte-Carlo trials are performed for each DOA angle separation where the sensor gains and phases are varied randomly (with uniform distribution) such that the sensor gains may be perturbed up to 4.7% of their nominal (unity) value while the phases may be perturbed by up to 0.65° from their nominal (zero) value. The array covariance matrices are estimated using 1000 array observation samples. Fig. 1 shows the results of the simulation, where the MSE of the DOA estimate of the $-10°$ source for angle separations of 15, 20, 30, and 40° are plotted. Note that for 0 iteration, the M-ESPRIT algorithm corresponds to the conventional ESPRIT algorithm since nominal sensor gains and phases are assumed. From the figure, it is clear that M-ESPRIT provides better DOA estimates than conventional ESPRIT and that after about 15 iterations, the MSE does not further appreciably improve. At any number of iterations however, M-ESPRIT performs consistently better than conventional ESPRIT (0 iteration of M-ESPRIT). Finally, it may be observed that the closer the two sources are (the smaller the angle separation), the more iterations are needed to achieve the same level of performance.

Simulation 2: Varied SNR: The same source and array scenario as in Simulation 1 are used here with the source locations fixed at $-10°$ and $+5°$, respectively. A single realization of sensor gain and phase errors is randomly generated, which yielded maximum gain errors of 4.7% (from unity) and maximum phase errors of .6° (from zero). One thousand samples from the array are used to estimate the array covariance matrix at various SNR values. Fifty Monte-Carlo trials are run to estimate the MSE of the DOA estimates. Fig. 2 shows the MSE of the DOA estimate of the $-10°$ source for the various SNR values. Fifteen iterations of M-ESPRIT are found to yield satisfactory performance. In fact, the M-ESPRIT estimates for SNR values of less than 30 dB are found to deteriorate after more than 20 iterations, most likely a result of the noise effects on the covariance. Furthermore, simulation trials using varying number of sensors within the array indicate that the trend of the MSE is to decrease with increasing number of sensors.

VI. Conclusions

A signal-subspace based method for array signal processing under unknown sensor gains and phases is proposed. The question of uniqueness of the set of sensor gains/phases and DOA angles is discussed and simulations are presented that show that the incorporation of the method may improve performance of conventional methods.

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