

DIRECTION OF ARRIVAL ESTIMATION BY EIGENSTRUCTURE METHODS WITH UNKNOWN SENSOR GAIN AND PHASE

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Abstract

Direction of arrival estimation by eigenstructure methods requires knowledge of the array covariance matrix and an exact characterization of the array in terms of geometry, sensor gain and phase, etc. It often happens that the actual sensor gain and phase are perturbed from their assumed nominal values. If eigenstructure methods are applied with incorrect sensor parameters, the method essentially breaks down or at best gives poor results. We propose a new approach which uses information in the observed covariance matrix to correct for these effects. This method yields substantially improved performance, a fact illustrated by the results of computer simulations.

I. Introduction

Eigenstructure methods were first introduced in time series analysis for extracting harmonics embedded in white noise. They were generalized in a significant way by Schmidt (1980) and applied to the direction of arrival (DOA) estimation problem. Their chief advantages over conventional methods are that they produce estimates possessing apparently higher resolution and yield asymptotically exact estimates for the DOA and other signal and noise field parameters. These methods are however known to be sensitive to sampling and modeling errors. We are concerned here with modeling errors in the sensor gain and phase. Sources of such errors include the sensor itself and its related electronics. Sonar arrays are a typical example, where, errors of 1 db RMS in gain and 6 ° RMS in phase may be routinely expected.

The degradation due to sensor errors has previously received some attention. Cox (1973) studied them in the context of optimal beamforming and several others such as Quazi (1982) have considered their effect on conventional array processing. As yet, no results have been reported on the effect of such errors on the performance of eigenstructure methods for the DOA estimation problem nor have means for mitigating these been proposed.

In this paper, we propose an algorithm for estimating the unknown sensor gain and phase by exploiting the structure of the array covariance matrix. The resulting estimates are then incorporated into the array model to reduce the DOA estimation problem to the standard form. Simulation results are presented which substantiate the performance of our method.

II. Problem Formulation

Like all problems in array processing, we can formulate the present problem in a wideband or a narrowband version. Here, we consider only the narrowband problem. In the following, we use lower and upper case to represent scalars, lower case bold font to represent vectors and upper case bold font to represent matrices.

Consider an uniformly spaced sensor line array with m sensors. The uniformly spaced line geometry assumption is essential to our method for the phase estimation problem but not for the gain estimation. The sensors have an unknown gain and phase response. Assume that there are d narrowband stationary zero mean and mutually uncorrelated sources centered at frequency ω_0 and sufficiently far from the array to allow a planar wavefront approximation. Additive noise is present at each sensor and is assumed to be stationary zero mean random processes and independent from sensor to sensor.

The received signal of the i -th sensor $r_i(t)$ is given by

$$r_i(t) = \psi_i e^{j\phi_i} \left\{ \sum_{k=1}^d s_k(t) e^{-j\omega_0(i-1) \frac{\Delta}{c} \sin \theta_k} + n_i(t) \right\} \quad (1)$$

where

- $s_k(\cdot)$ = the signal emitted by the k -th source
- θ_k = the direction of arrival of the k -th source.
- Δ = the spacing between the sensors
- c = the speed of propagation
- $n_i(\cdot)$ = the additive noise at the i -th sensor
- ψ_i = the gain of the i -th sensor

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ϕ_i = the phase of the i -th sensor

We assume $d < m$. Rewriting (1) in matrix notation, we have

$$\mathbf{r}(t) = \Psi\Phi\{\mathbf{A}\mathbf{s}(t) + \mathbf{n}(t)\} \quad (2a)$$

where $\mathbf{r}(t)$ and $\mathbf{n}(t)$ are the $m \times 1$ vectors

$$\begin{aligned} \mathbf{r}^T(t) &= [r_1(t) \cdots r_m(t)], \\ \mathbf{n}^T(t) &= [n_1(t) \cdots n_m(t)] \end{aligned} \quad (2b)$$

and $\mathbf{s}(t)$ is the $d \times 1$ vector of impinging wavefronts with

$$\mathbf{s}^T(t) = [s_1(t) \cdots s_d(t)] \quad (2c)$$

\mathbf{A} is a $m \times d$ matrix with a columns $\{\mathbf{a}_k, k = 1, \dots, d\}$ defined by

$$\mathbf{a}_k^T = [1, \dots, e^{-j\omega_d(i-1)\tau_k}, \dots, e^{-j\omega_d(m-1)\tau_k}] \quad (2d)$$

and

$$\tau_k = \frac{\Delta}{c} \sin \theta_k \quad (2e)$$

Therefore, \mathbf{A} is a Vandermonde matrix whose columns are the steering vectors of the impinging planar wavefronts. Finally Ψ and Φ are $m \times m$ diagonal matrices with the diagonal entries given by

$$[\Psi]_{ii} = \psi_i, \quad [\Phi]_{ii} = e^{j\phi_i} \quad (2f)$$

Multiplying (2a) by its conjugate transpose and taking expectations, assuming the additive sensor noises have variance σ^2 , and are uncorrelated with the source signals, we can write

$$\begin{aligned} \mathbf{P} &= E[\mathbf{r}(t)\mathbf{r}^+(t)] \\ &= \Psi\Phi\{\mathbf{A}\mathbf{S}\mathbf{A}^+ + \sigma^2\mathbf{I}\}\Phi^+\Psi^+ \end{aligned} \quad (3)$$

where \mathbf{P} is the $m \times m$ covariance matrix of the array measurements. We note that since Ψ and Φ are diagonal matrices with real and complex entries respectively, we have the relations

$$\begin{aligned} \Psi^+ &= \Psi, \quad \Phi^+ = \text{conj}\{\Phi\}, \\ \Phi\Phi^+ &= \mathbf{I}, \quad \Phi^{-1} = \Phi^+ \end{aligned} \quad (4)$$

Defining $\mathbf{G} = \Psi\Phi$, we have

$$\mathbf{P} = \mathbf{G}\mathbf{R}\mathbf{G}^+ = \Psi\mathbf{Q}\Psi^+ \quad (5)$$

where

$$\mathbf{R} = \mathbf{A}\mathbf{S}\mathbf{A}^+ + \sigma^2\mathbf{I} \quad (6)$$

is the array covariance when all sensors have uniform gain and phase and

$$\mathbf{Q} = \Phi\mathbf{A}\mathbf{S}\mathbf{A}^+\Phi^+ + \sigma^2\mathbf{I} = \Phi\mathbf{R}\Phi^+ \quad (7)$$

is the array covariance when all sensors have uniform gain but with possibly varying phase ϕ_i .

This completes the definition of the model.

Our approach to this problem is predicated on first learning the array covariance matrix \mathbf{P} and then extract-

ing from this the sensor gain and phase $\{\psi_i\}$ and $\{\phi_i\}$. We then use these estimates to find the covariance (as it turns out to within some unknown constants) for a hypothetical uniform sensor gain and phase array. Thereafter, the normal eigenstructure methods are directly applicable to obtain the DOAs.

III Estimation of Sensor Gain and Phase

From observations $\mathbf{r}(t_j)$ at time t_j , the array covariance matrix is estimated by

$$\hat{\mathbf{P}} = \frac{1}{N} \sum_{j=1}^N \mathbf{r}(t_j)\mathbf{r}^+(t_j) \quad (8)$$

In the sequel, in the interest of notational convenience, we use \mathbf{P} instead of $\hat{\mathbf{P}}$ with the understanding that we have only an estimate for \mathbf{P} . This situation will motivate us to formulate least squares solution to gain and phase determination problems rather than seek exact solutions. We also note that we can make $\hat{\mathbf{P}}$ arbitrarily exact by making N large.

We begin with the sensor gain estimation problem and note that if the sensors had uniform gain and phase, the true array covariance \mathbf{R} will be Toeplitz. This follows from the assumptions of uniform linear array and uncorrelated sources.

It follows that

$$|[\mathbf{P}]_{ij}| = |[\mathbf{R}]_{ij}| \psi_i \psi_j \quad (9)$$

with $|\cdot|$ denoting modulus. Also, since we have only an estimate of \mathbf{P} , (9) will be only approximately satisfied. Our aim here is to exploit the Toeplitz property of \mathbf{R} to eliminate $|[\mathbf{R}]_{ij}|$ from (9) to determine ψ_i .

With this intent, we introduce a quantity determined from \mathbf{P}

$$\mu_{ijk} = \ln\{|[\mathbf{P}]_{ij}| / |[\mathbf{P}]_{kl}|\} \quad (10)$$

If we choose i, j, k and l such that $i-j = k-l$, i.e., $|[\mathbf{P}]_{ij}|$ and $|[\mathbf{P}]_{kl}|$ lie on the same diagonal, then from the Toeplitz property of \mathbf{R} and (9), (10) we get

$$\mu_{ijk} = \ln\psi_i - \ln\psi_j - \ln\psi_k + \ln\psi_l \quad (11)$$

Note that (11) will not be exactly satisfied due to errors in estimating \mathbf{P} .

Taking all such (nonredundant) relations when i, j and k, l pairs that lie on the main/super diagonals, we get a total of $k_\psi = \sum_{i=2}^m \{i(i-1)/2\}$ equations of the type (11). We can write these compactly as

$$\begin{aligned} \mathbf{B} [\ln\psi_1, \ln\psi_2, \dots, \ln\psi_m]^T \\ = [\dots, \mu_{ijk}, \dots]^T, \quad i-j = k-l \end{aligned} \quad (12)$$

Where, \mathbf{B} is a $k_\psi \times m$ matrix whose rows have one of the following forms:

- (a) $[0, \dots, 0, 2, 0, \dots, 0, -2, 0, \dots]$ when $i=j$ and $k=l$. All entries in this row are zero except for a 2 and -2 at the i th and the k th positions respectively.

- (b) $[0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots]$ when $i \neq j$ and $j = k$. All entries in this row are zero except for a 1 and -1 at the i th and the l th positions respectively.
- (c) $[\dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0, -1, 0, \dots, 0, 1, 0, \dots]$ when i, j, k and l are all distinct. All entries in this row are zero except for 1, -1, -1 and 1 at the i th, j th, k th and the l th positions respectively.

Examining \mathbf{B} shows that the $1 \times m$ row vector $[1, 1, 1, \dots, 1]^T$ is the lone vector spanning its row null space.

We can solve (12) with a least squares approach using the singular value decomposition of \mathbf{B} . i.e.,

$$\begin{aligned} & [\ln \psi_1, \ln \psi_2, \dots, \ln \psi_m]^T \\ &= \mathbf{B}^\# [\dots, \nu_{ijkl}, \dots]^T \end{aligned} \quad (13)$$

Where $\mathbf{B}^\#$ is the pseudo inverse defined in terms of the $m-1$ singular values and the corresponding left and right singular vectors of \mathbf{B} . i.e.,

$$\mathbf{B}^\# = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m] \Sigma^{-1} [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k]^T \quad (14)$$

With \mathbf{u}_i and \mathbf{v}_i being the left and right singular vectors of \mathbf{B} respectively. Σ is a $m-1 \times m-1$ diagonal matrix of the $m-1$ singular values of \mathbf{B} . We note that (12) yields the minimum norm least squares solution. A general solution can be obtained by adding an arbitrary scalar times the null space vector. This amounts to saying that we can only determine the sensor gain to within an arbitrary multiplicative constant.

Now, having found ψ_i , we can use (3) to determine \mathbf{Q} . Hence, starting with the measured array covariance matrix \mathbf{P} in which both gain and phase were unknown, we have found the matrix \mathbf{Q} (to within a scalar constant) - a covariance matrix with uniform sensor gain but with the phase unknown.

We estimate sensor phase by noting from (7)

$$\text{angle}([\mathbf{Q}]_{ij}) = \text{angle}([\mathbf{R}]_{ij}) + \phi_i - \phi_j \quad (15)$$

Where

$$\text{angle}(y) = \arctan(\text{imag}(y)/\text{real}(y))$$

Now, defining

$$\nu_{ijkl} = \text{angle}([\mathbf{Q}]_{ij}) - \text{angle}([\mathbf{Q}]_{kl}) \quad (16)$$

Since $\text{angle}([\mathbf{Q}]_{ii})$ is zero, (16) is meaningful only when i, j, k and l lie on the super diagonals.

Again invoking the Toeplitz property of \mathbf{R} , $\text{angle}([\mathbf{R}]_{ij})$ depends on $i-j$ alone and it follows that

$$\nu_{ijkl} = \phi_i - \phi_j - \phi_k + \phi_l, \quad i-j = k-l \quad (17)$$

Taking all nonredundant relations from the super diagonals excepting the singleton $[\mathbf{P}]_{1m}$, we get $k_\phi = \sum_{i=2}^{m-2} \{i(i-1)/2\}$ equations. And we can write these compactly as

$$\begin{aligned} & \mathbf{C} [\phi_1, \phi_2, \phi_3, \dots, \phi_m]^T \\ &= [\dots, \nu_{ijkl}, \dots]^T, \quad i-j = k-l \end{aligned} \quad (18)$$

Where $[\phi_1, \phi_2, \phi_3, \dots, \phi_m]$ is a $m \times 1$ vector of unknowns $\{\phi_i\}$, $[\dots, \nu_{ijkl}, \dots]$ is a $k_\phi \times 1$ vector of arctan differences computed from \mathbf{Q} and \mathbf{C} is a $k_\phi \times m$ matrix whose rows have one of the following forms:

- (a) $[\dots, 0, 1, 0, \dots, 0, -2, 0, \dots, 0, 1, 0, \dots]$ when $j=k$. All entries are zero except for a 1, -2 and 1 at the i th, j ($=k$)th and the l th positions respectively.
- (b) $[\dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0, -1, 0, \dots, 0, 1, 0, \dots]$ when i, j, k and l are all distinct. All entries are zero except for a 1, -1, -1 and 1 at the i th, j th, k th and the l th positions respectively.

Again, by inspecting the rows of \mathbf{C} , it is clear that there are two row vectors viz $[1, 1, 1, \dots, 1]^T$ and $[1, 2, 3, \dots, m]^T$ in the null space of \mathbf{C} . The first null space vector implies that phase can only be determined to within an arbitrary reference, an matter of no consequence in the DOA problem. The second null space vector is significant and implies that the phase can only be determined to within an arbitrary progressive phase factor. This in turn means that the best we can hope to do is to find the DOAs to within an arbitrary rotation factor.

Once again, we need to find the least squares solution and this is

$$\begin{aligned} & [\phi_1, \phi_2, \phi_3, \dots, \phi_m]^T \\ &= \mathbf{C}^\# [\dots, \nu_{ijkl}, \dots]^T \end{aligned} \quad (19)$$

With $\mathbf{C}^\#$ being the pseudo inverse of \mathbf{C} and defined as earlier in terms of the singular values and vectors.

IV Simulation Results

Computer simulations were carried out to verify the performance of the proposed approach. The simulation model consisted of an 8 element array with half wavelength inter element spacing. Three equal power signal sources were generated to arrive at angles -35° , -63° and -25° . Further, additive and uncorrelated sensor noise was injected with a sensor level SNR of 0 db referenced to the signal sources. The sample covariance \mathbf{P} was accumulated from 300 snapshots of array data. Fig. 1, plot A shows a 'spectral' plot obtained by eigenstructure methods for the known (identical) sensor gain and phase model. As expected, we get peaks in the true directions and all the sources are clearly resolved. Plot B is for the case when sensor gain and phase are perturbed from their assumed uniform values. 0.8db and 6° (both RMS) gain and phase errors respectively were used. An examination shows peaks skewed from their true directions and a resolution performance that is clearly poor. This plot is only representative of the degradation to be expected, since it is sensitive to the actual gain and phase error values

used. It is clear that eigenstructure methods are badly degraded in several practical situations when sensor data is not known exactly.

Fig. 2, Plot A is the 'spectral' plot after the sensor phase and gain have been estimated as per procedure detailed in section III and then plugged into the model before the eigenstructure algorithm was applied. The new procedure has obviously succeeded in resolving the three sources. However, a rotation has evidently occurred due to the null space vector described earlier. Plot B represents the corresponding result after the rotation uncertainty has been resolved by means of an additional input such as the phase of any one sensor with respect to another. The plot now has three peaks which occur in the true directions. No bias is evident. It is clear that sensor phase and gain estimation procedures have succeeded in restoring the resolution and accuracy of the eigenstructure methods.

V Concluding Remarks

In this paper we have shown that sensor gain and phase errors significantly degrade the performance of eigenstructure methods in the DOA estimation problem. This can result in poorer performance than the low resolution but more robust conventional angular spectrum estimators. We then showed for a uniformly spaced line array and uncorrelated sources, there is enough information in the covariance matrix for estimating the sensor gain and phase. These estimates can then be used to fully restore the performance of the eigenstructure method. However, since the gain and phase can only be determined to within certain unknown factors, source powers can only be estimated to within an arbitrary scale factor. More significantly, the estimation of the DOAs can be only be made to within an arbitrary rotation factor.

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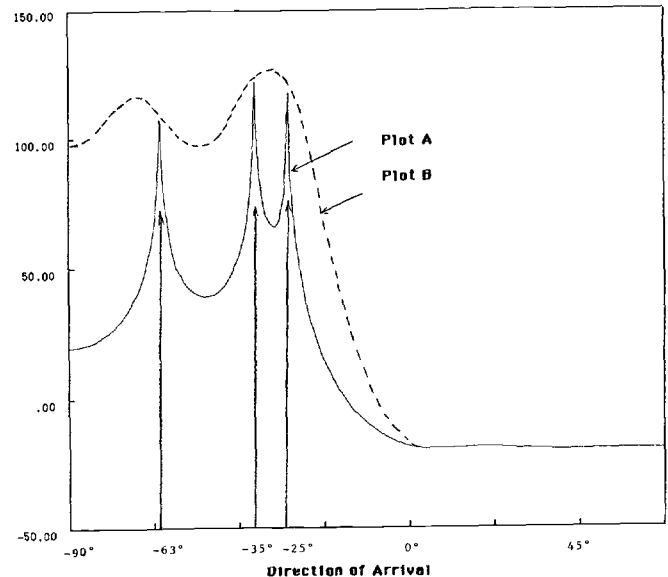


Fig. 1 DDA Estimation 'Spectral' Plot - Standard Algr.
A : With No Sensor Errors
B : With Sensor Errors

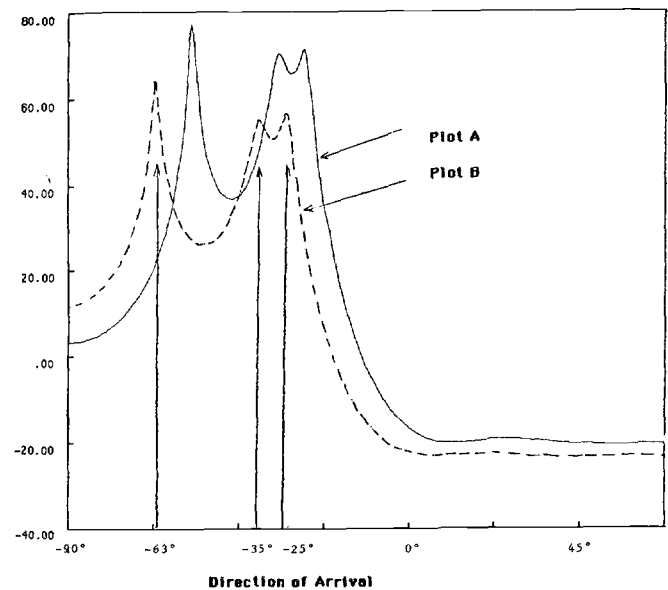


Fig. 2 DDA Estimation 'Spectral' Plot - Modified Algr.
A : With Unresolved Rotational Ambiguity
B : With Resolved Ambiguity