RELAX FUNCTION IN MINIMIZATION PROBLEM WITH SMALL PERTURBATION

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Abstract. My project mostly focus on the minimization problem with the following given form:

$$\inf \{ \int_0^1 |u'| + \sum_{i=1}^k |u(a_i) - f_i|^2 + \alpha \int_0^1 |u|^2 \}$$

Given that

(1) u : function defined on (0,1);
(2) a_i : k points in (0,1) with k greater than 1;
(3) f_i : k real numbers;
(4) α : a parameter taking values in $\{0, 1\}$

When $\alpha = 0$ the minimizer is achieved within $W^{1,1}(0,1)$. However, for the case of $\alpha = 1$, minimizer need not to be achieved in our original function setting and the paper introduce an relaxed functional defined on $BV(0,1)$, whose minimizers always exists and can be viewed as generalized solution for the problem in the given form.

1. Introduction

Given k points, with $k \geq 2$, $\{a_i\}_{1 \leq i \leq k}$ increasing sequence, we aim to find $u(a_i)$ approximate f_i at each check point best and maintain the regularity of u at the same time in total variation sense. For this purpose, we define functional F:

$$F(u) = \int_0^1 |u'| + \sum_{i=1}^k |u(a_i) - f_i|^2,$$

and our goal is to find $m = \inf_{u \in W^{1,1}(0,1)} F(u)$. Note that F is well defined in $W^{1,1}(0,1) \subset C([0,1])$ so that $u(a_i)$ makes sense. By convention, we know that $W^{1,1}(0,1)$ is not a good function space to detect minimizer from Functional Analysis aspect, and we aim to find a better space for our minimization problem. In variation method setting it is often to look up the minimizer in larger space $BV(0,1)$, the space of functions of bounded variation. However, in contrast to the $W^{1,1}(0,1)$, $u(a_i)$ need not to make sense for our defined functional F when u has jump discontinuity at a_i. [3]

In Section 2 we establish that the problem

$$\inf_{u \in W^{1,1}(0,1)} F(u)$$

which always admits minimizers and Theorem 1 provides an brief insight for some properties of our minimizer. We will also give an example of $W^{1,1}(0,1)$ failed to detect an minimizer corresponding to small perturbation of F ($\alpha = 1$), and the key is to relax the problem, search for minimizer in $BV(0,1)$, and make sense of

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minimization functional at discontinuities by Functional Analysis tool.

In Section 3 we introduce the relaxed functional F_r of F, which is better applied to our minimization problem involving small perturbation version of F. We start with abstract formulation, F_r is defined for every $v \in BV(0,1)$ by

$$F_r(v) := \inf \liminf_{n\to\infty} F(v_n)$$

for inf is taking over all sequence $\{v_n\} \in W^{1,1}(0,1)$ and $v_n \xrightarrow{\mathbb{L}^2} v$. The main theorem, Theorem 2 gives us an explicit formula for the F_r. The problem of discontinuities can be resolved by considering the left and right limits of $u \in BV(0,1)$ for every data point $a_i \in (0,1)$, which enters into the formula for F_r and Theorem 3 provides some properties of minimizers of F_r on $BV(0,1)$.

In Section 4 we consider the mild perturbation case of F, and we show that the corresponding minimization functional turn out to be

$$G(u) := F(u) + \int_0^1 |u|^2 = \int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2 + \int_0^1 |u|^2$$

where $u \in W^{1,1}(0,1)$. As we will showed in the Section 1, the example of failing to find minimizer in $W^{1,1}(0,1)$, our goal turn out to be find minimizer in $BV(0,1)$. It is easy to check that the relax version of G, G_r is given by

$$G_r(v) = F_r(v) + \int_0^1 |v|^2, \quad \forall v \in BV(0,1),$$

so that it inherits the convexity and lower semicontinuity as we shown in Theorem 3 in the sense that for any $\{v_n\} \in BV(0,1)$, $v_n \xrightarrow{\mathbb{L}^2} v$, we have

$$\liminf_{n\to\infty} G_r(v_n) \geq G_r(v)$$

Consequently set

$$B = G_r(\hat{v}) := \inf_{v \in BV(0,1)} G_r(v)$$

where $\hat{v}$ is uniquely achieved and can be viewed as generalized solution even when minimizer is achieved in $W^{1,1}(0,1)$. Moreover, we will also briefly introduce some minimizers that is preferable in the sense that are stable with respect to perturbations, and the generalized solution we just constructed also applied in these scenarios.[3]

2. Minimization Problem in $W^{1,1}(0,1)$

Consider a finite dimensional auxiliary problem: Given

$$\lambda := (\lambda_1, \ldots \lambda_k) \in R^k \tag{1}$$

if we set

$$\Phi(\lambda) := \sum_{i=1}^{k-1} |\lambda_{i+1} - \lambda_i| + \sum_{i=1}^{k} |\lambda_i - f_i|^2. \tag{2}$$

by convexity, we have

$$m := \min_{\lambda \in \mathbb{R}^k} \phi(\lambda) := \phi(u) \tag{3}$$

where $u = (u_1, ..., u_k)$ is uniquely achieved.

We claim that this auxiliary problem classified all minimizers for the problem,

$$\inf_{u \in W^{1,1}(0,1)} F(u) \tag{4}$$

and the problem indeed always admits minimizers which is triggered by the main theorem in this Section.

2.1. Main theorem for no perturbation.

Theorem 1. *(T.Szniger [1, 2]) We have*

$$m = \inf_{u \in W^{1,1}} F(u) \tag{5}$$

and $u \in W^{1,1}(0,1)$ is a minimizer if and only if the following three conditions hold:

- *u is monotone on each interval $(a_i, a_{i+1}), i = 1, 2, ...k-1$;*
- *$u(a_i) = u_i, i = 1, ...k$;*
- *$u(x) = u_1, \forall x \in [0, a_1]$ and $u(x) = u_k, \forall x \in [a_k, 1]$*

where u_i and m are defined as in the auxiliary problem (3) for our functional, and the inf *is achieved.*

Proof. [3] Given that $u \in W^{1,1}(0,1)$, we have

$$\int_0^1 |u'| \geq \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} |u'| \geq \sum_{i=1}^{k-1} |u(a_{i+1}) - u(a_i)|$$

with equalities if and only if

- u is monotone on each interval (a_i, a_{i+1});
- u is constant on $(0, a_1)$ and $(a_k, 1)$

Thus, adding both side of $\sum_{i=1}^{k} |u(a_i) - f_i|^2$, we have

$$F(u) \geq \sum_{i=1}^{k-1} |u(a_{i+1}) - u(a_i)| + \sum_{i=1}^{k} |u(a_i) - f_i|^2$$

Let $\lambda_i := u(a_i)$, we see that for every $u \in W^{1,1}(0,1)$,

$$F(u) \geq \min_{\lambda \in \mathbb{R}^k} \phi(\lambda) = m$$

If $u \in W^{1,1}(0,1)$ satisfied the 3 conditions we have

$$F(u) = \sum_{i=1}^{k-1} |u_{i+1} - u_i| + \sum_{i=1}^{k} |u_i - f_i|^2 = m$$

and u is a minimizer for (5). On the other hand, if $u \in W^{1,1}(0,1)$ such that $F(u) = m$, then by if and only if condition we have first and third condition naturally hold. Moreover, $u(a_i) = \lambda_i$ is a minimizer in (3), and by uniqueness we have $u(a_i) = u_i$ for $i = 1, ...k$. $\square$

2.2. **Example for no minimizer in $W^{1,1}$.**

However, for our case of $\alpha = 1$, we aim to minimize the following functional

$$G(u) := F(u) + \int_0^1 |u|^2 = \int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2 + \int_0^1 |u|^2, \tag{6}$$

where $W^{1,1}$ is not a good function space to detect minimizer. The following Theorem and graph will help us to acknowledge this fact.

Proposition 1. *(H.Brezis [3]) Suppose that problem (6) admits a minimizer $\hat{u} \in W^{1,1}(0,1)$, then necessarily*

$$\hat{u} = \hat{K} = \frac{1}{k+1}\sum_{i=1}^{k} f_i.$$

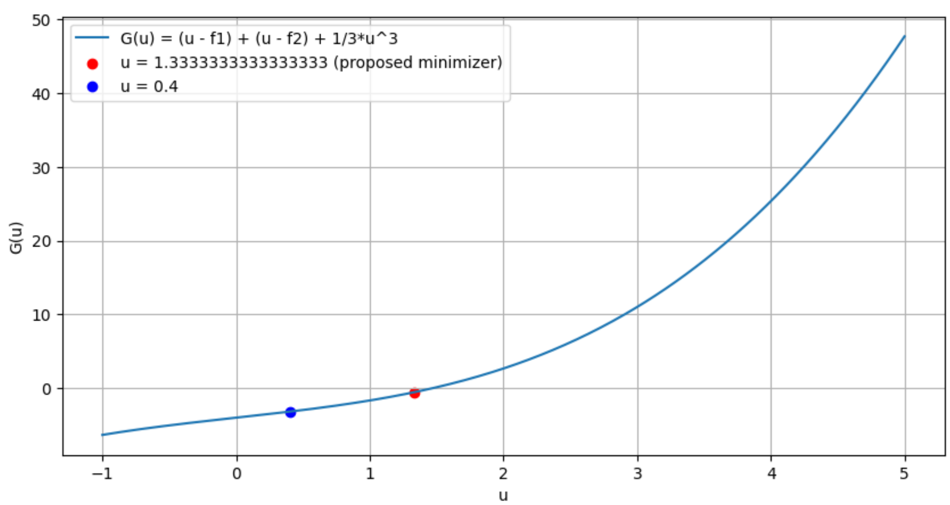
and moreover,

$$|f_i - K_0| \leq 1 \tag{7}$$

$$(a_{i+1} - a_i)|K_0| \leq 1 \tag{8}$$

$$|K_0| \leq k+1 \tag{9}$$

Given this fact, consider example for $k = 2$, $f_1 = 3$, $f_2 = 1$ such that $|2f_1 - f_2| > 3$ would violate the (7) and thus admits no minimizer. The graph below illustrates that if $\hat{u}$ is the minimizer by assumption, and from some analysis fact that we know $\hat{u} = \hat{K} = \frac{4}{3}$ for $\forall x \in (0,1)$, $f_1 = 3$, $f_2 = 1$ we would have the graph,



and it is easy to see that $\hat{u}$ is not the minimizer. Therefore, we need to find minimizer in a more suitable space, $BV(0,1)$.

3. The relaxed functional F_r on $BV(0,1)$

3.1. **Explicit formulation for relaxed functional.**

Returning to our case for seeking such minimizer in $BV(0,1)$, the key is to find a relaxation functional, so consider that

$$F_r(v) := \inf \liminf_{n\to\infty} F(v_n) \tag{10}$$

where inf takes over all sequences $(v_n) \subset W^{1,1}(0,1)$ and $v_n \xrightarrow{\mathbb{L}^2} v$. The main result in this section is to deduce the new formulation for relaxed functional F_r, but first some notations. Given $v \in BV(0,1), a \in (0,1)$, we denote by $j(v)(a)$ the jump interval of v at a,

$$j(v)(a) := [\min(v(a-0), v(a+0)), \max(v(a-0), v(a+0))] \tag{11}$$

We also set

$$\phi(t) = \begin{cases} t^2 & \text{if } 0 \le t \le 1 \\ 2t-1 & \text{if } t > 1 \end{cases} \tag{12}$$

Theorem 2. *(H.Brezis [3]) For every $v \in BV(0,1)$, we have*

$$F_r(v) = \int_0^1 |v'|^2 + \sum_{i=1}^{k} \phi(dist(f_i, j(v)(a_i))) \tag{13}$$

where dist denoted as the distance of points f_i to jump interval of v at a_i defined as before.

It is a long proof involving two steps and we will introduce 3 lemmas to explain.

Lemma 1. *(H.Brezis [3]) Let $v_n \in BV(a,b)$ such that $v_n \xrightarrow{\mathbb{L}^1} v$, $v_n(a) \to \alpha, v_n(b) \to \beta$ as $n \to \infty$. Then $v \in BV(a,b)$ and*

$$\liminf_{n\to\infty} \int_a^b |v_n'| \ge \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta| \tag{14}$$

Proof. Fixed any $h \in C_c^\infty(\mathbb{R})$ such that $h(a) = \alpha, h(b) = \beta$. Consider the functions

$$w_n(t) := \begin{cases} h(t), & \text{if } t < a, \\ v_n(t), & \text{if } a \le t \le b, \\ h(t), & \text{if } t > b, \end{cases} \quad \text{and} \quad w(t) := \begin{cases} h(t), & \text{if } t < a, \\ v(t), & \text{if } a \le t \le b, \\ h(t), & \text{if } t > b. \end{cases} \tag{15}$$

Since $v_n \in BV(0,1)$, by [4] *Lecture 5 "Compactness of $BV(\Omega)$, pp.34"*, we have $v \in BV(0,1)$. Moreover, since $h \in C_c^\infty(\mathbb{R})$ implies h is bounded and thus $h \in BV(\mathbb{R})$, we have $w_n, w \in BV(\mathbb{R})$. Consider that

$$\int_{\mathbb{R}} |w_n'| = \int_{-\infty}^a |h'| + \int_a^b |v_n'| + \int_b^\infty |h'| + |v_n(a) - \alpha| + |v_n(b) - \beta|, \tag{16}$$

and

$$\int_{\mathbb{R}} |w'| = \int_{-\infty}^a |h'| + \int_a^b |v'| + \int_b^\infty |h'| + |v(a) - \alpha| + |v(b) - \beta|. \tag{17}$$

Since $w_n \to w$ in $L^1(\mathbb{R})$, we claim that

$$\liminf_{n\to\infty} \int_{\mathbb{R}} |w_n'| \ge \int_{\mathbb{R}} |w'|. \tag{18}$$

For $\phi \in C_c^\infty(\Omega, \mathbb{R}), \|\phi\| \le 1$, we have

$$\int_\Omega v \nabla\phi dx = \lim_{n\to\infty} \int v_n \nabla\phi dx \tag{19}$$

$$\le \liminf_{n\to\infty} \sup\{\int_\Omega v_n \nabla\phi dx, \phi \in C_c^\infty(\Omega, \mathbb{R}), \|\phi\| \le 1\} \tag{20}$$

$$= \liminf_{n\to\infty} |Dv_n|(\Omega) \tag{21}$$

Taking supreme over all ϕ, we have our desired result.
Combined (16), (17), (18) with assumption that $v_n(a) \to \alpha$ and $v_n(b) \to \beta$, we have our desired result. $\square$

Lemma 2. *(H.Brezis [3]) Given any $v \in BV(a,b)$ and constants $\alpha, \beta \in \mathbb{R}$, there exists a sequence $(v_n) \subset W^{1,1}(a,b)$ such that $v_n \to v$ in $L^2(a,b)$, $v_n(a) = \alpha$, $v_n(b) = \beta$, for $\forall n$, and*

$$\lim_{n\to\infty} \int_a^b |v_n'| = \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|. \tag{22}$$

Proof. Set

$$w(t) := \begin{cases} \alpha, & \text{if } t < a, \\ v(t), & \text{if } a \le t \le b, \\ \beta, & \text{if } t > b. \end{cases} \tag{23}$$

Let $w_n = \rho_n * w$ where (ρ_n) is a sequence of mollifiers (regularity purpose). Clearly

$$\int_{\mathbb{R}} |w_n'| \le \int_{\mathbb{R}} |w'| = \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|. \tag{24}$$

Moreover $w_n(t) = \alpha$ if $t < a - (1/n)$ and $w_n(t) = \beta$ if $t > b + (1/n)$. Rescaling the sequence (w_n) by a change of variables we obtain a sequence (v_n) of smooth functions such that $v_n \to v$ in $L^2(a,b)$, $v_n(a) = \alpha$, $v_n(b) = \beta$, $\forall n$, and

$$\limsup_{n\to\infty} \int_a^b |v_n'| \le \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|. \tag{25}$$

On the other hand, consider that

$$\|v_n - v\|_{\mathbb{L}^1} = \int_a^b |v_n - v| dx \overset{\text{Hölder}}{\le} (b-a)^{\frac{1}{2}} \|v_n - v\|_{\mathbb{L}^2} \to 0 \tag{26}$$

Applying Lemma 1, we have

$$\liminf_{n\to\infty} \int_a^b |v_n'| \ge \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta| \ge \limsup_{n\to\infty} \int_a^b |v_n'| \tag{27}$$

implies that

$$\lim_{n\to\infty} \int_a^b |v_n'| = \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta| \tag{28}$$

Hence, we have our desired result. $\square$

Lemma 3. *(H.Brezis [3]) Given any $\alpha, \beta, f \in \mathbb{R}$ we have*

$$\inf_{t\in\mathbb{R}} \{|t-\alpha| + |t-\beta| + |t-f|^2\} = |\alpha - \beta| + \phi(dist(f, J)) \tag{29}$$

where $J = [\min(\alpha,\beta), \max(\alpha,\beta)]$ and ϕ is defined as in (12).

Proof. Without lost of generality, let $\alpha \le \beta$ and set

$$h(t) := |t-\alpha| + |t-\beta| + (t-f)^2$$

For simplicity consider that

- for $t \le \alpha$, we have

  $$h(t) = \beta + \alpha - 2t + (t-f)^2 \implies h'(t) = 2(t - (f+1)), \quad h''(t) = 2$$

  if $f + 1 \le a$, then it is a critial point and local minimum;

- for $t \in [\alpha, \beta]$, we have

$$h(t) = \beta - \alpha + (t-f)^2 \implies h'(t) = 2(t-f), \quad h''(t) = 2$$

  if $f \in [\alpha, \beta]$, then it is a critial point and local minimum;
- for $t \geq \alpha$, we have

$$h(t) = -\beta - \alpha + 2t + (t-f)^2 \implies h'(t) = 2(t-(f-1)), \quad h''(t) = 2$$

  if $f - 1 \geq \beta$, then it is a critial point and local minimum.

We have divided our interval corresponding to position of f as

$$f \leq \alpha - 1, \quad f \in [\alpha-1, \alpha], \quad f \in [\alpha, \beta], \quad f \in [\beta, \beta+1], \quad f \geq \beta + 1$$

compare the results of local minimum at each interval with the $h(\alpha)$ and $h(\beta)$, we have

$$\inf_{t\in R} h(t) = \begin{cases} h(f) & = |f-\alpha| + |f-\beta| = |\alpha-\beta| + \phi(0) \text{ if } f \in [\alpha,\beta] \\ h(\alpha) & = |\alpha-\beta| + |\alpha-f|^2 = |\alpha-\beta| + \phi(|f-\alpha|) \text{ if } f \in [\alpha-1,\alpha] \\ h(\beta) & = |\alpha-\beta| + |\beta-f|^2 = |\alpha-\beta| + \phi(|f-\beta|) \text{ if } f \in [\beta,\beta+1] \\ h(f+1) & = |f+1-\alpha| + |f+1-\beta| = |\alpha-\beta| + [2(\alpha-f)-1] \text{ if } f \leq \alpha-1 \\ h(f-1) & = |f-1-\alpha| + |f-1-\beta| = |\alpha-\beta| + [2(f-\beta)-1] \text{ if } f \geq \beta+1 \end{cases}$$

Thus we have shown that

$$\inf_{t\in\mathbb{R}} h(t) = |\alpha - \beta| + \phi(\text{dist}(f, J))$$

and hence we are done $\square$

Based on above three Lemmas, we can proceed the proof of Theorem 2.

Proof of Theorem 2. It consists of two steps.

Step 1. Given any $v \in BV(0,1)$ there exists a sequence $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ and

$$\lim_{n\to\infty} F(v_n) = F_r(v), \tag{30}$$

where $F_r(v)$ is defined as in (13).

Proof. [3] Applying Lemma 3 with $\alpha = v(a_i - 0)$, $\beta = v(a_i + 0)$, and $f = f_i$, $1 \leq i \leq k$, we obtain some t_i (the inf value in (29)) such that

$$|t_i - v(a_i-0)| + |t_i - v(a_i+0)| + |t_i - f_i|^2 = |v(a_i-0) - v(a_i+0)| + \phi(\text{dist}(f_i, j(v(a_i)))). \tag{31}$$

We next apply Lemma 2 successively on

$$(0, a_1), \quad (a_i, a_{i+1}), \text{ and } (a_k, 1) \tag{32}$$

for $1 \leq i \leq k-1$. First on $(0, a_1)$ with $\alpha = v(0+)$ and $\beta = t_1$. This yields a sequence $(v_n) \subset W^{1,1}(0, a_1)$ such that $v_n(0) = v(0+)$, $v_n(a_1) = t_1$, $\forall n$, $v_n \to v$ in $L^2(0, a_1)$ and

$$\int_0^{a_1} |v_n'| = \int_0^{a_1} |v'| + |v(a_1 - 0) - t_1| + o(1). \tag{33}$$

Similarly on $(a_k, 1)$ with $\alpha = t_k$ and $\beta = v(1-)$. This yields a sequence $(v_n) \subset W^{1,1}(a_k, 1)$ such that $v_n(a_k) = t_k$, $v_n(1) = v(1-)$, $\forall n$, $v_n \to v$ in $L^2(a_k, 1)$ and

$$\int_{a_k}^{1} |v_n'| = \int_{a_k}^{1} |v'| + |v(a_k + 0) - t_k| + o(1). \tag{34}$$

Finally on (a_i, a_{i+1}) for each $1 \leq i \leq k-1$, such that $(v_n) \subset W^{1,1}$, $v_n(a_i) = t_i$, $v_n(a_{i+1}) = t_{i+1}$, $v_n \to v$ in L^2, and

$$\int_{a_i}^{a_{i+1}} |v_n'| = \int_{a_i}^{a_{i+1}} |v'| + |v(a_i + 0) - t_i| + |v(a_{i+1} - 0) - t_{i+1}| + o(1). \tag{35}$$

Combined (33), (34), and (35), this yield sequence $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$, and

$$\int_0^1 |v_n'| = \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^{k} [|v(a_i - 0) - t_i| + |v(a_i + 0) - t_i|] + o(1). \tag{36}$$

with the convention that $a_0 = 0$ and $a_{k+1} = 1$. Inserting (31) to (36), we get

$$\int_0^1 |v_n'| = \int_0^1 |v'| - \sum_{i=1}^{k} |t_i - f_i|^2 + \sum_{i=1}^{k} \phi(\mathrm{dist}(f_i, j(v(a_i)))) + o(1) \tag{37}$$

where the left and right limit resolved discontinuities for every data point $a_i \in (0,1)$. Additional, we know that $v_n(a_i) = t_i$ for $\forall n, i$, we conclude that

$$\int_0^1 |v_n'| + \sum_{i=1}^{k} |v_n(a_i) - f_i|^2 = \int_0^1 |v'| + \sum_{i=1}^{k} \phi(\mathrm{dist}(f_i, j(v(a_i)))) + o(1) \tag{38}$$

and

$$F(v_n) = \int_0^1 |v_n'| - \sum_{i=1}^{k} |v_n(a_i) - f_i|^2 = F_r(v) + o(1) \tag{39}$$

This finished Step 1. □

Step 2. Let (v_n) be a bounded sequence in $W^{1,1}(0,1)$ such that $v_n \to v$ in $L^1(0,1)$. Then

$$\liminf_{n\to\infty} F(v_n) \geq F_r(v). \tag{40}$$

Proof. Passing to a subsequence, we may always assume that, for every $i = 0, 1, \ldots, k+1$, there exists some ℓ_i such that

$$v_n(a_i) \to \ell_i \quad \text{as} \quad n \to \infty. \tag{41}$$

From Lemma 1 we know that for every $i = 0, 1, \ldots, k$,

$$\int_{a_i}^{a_{i+1}} |v_n'| \geq \int_{a_i}^{a_{i+1}} |v'| + |v(a_i + 0) - \ell_i| + |v(a_{i+1} - 0) - \ell_{i+1}| + o(1). \tag{42}$$

Adding these inequalities corresponding to each i gives

(43)

$$F(v_n) \geq \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^{k} (|v(a_i - 0) - \ell_i| + |v(a_i + 0) - \ell_i| + |\ell_i - f_i|^2) + o(1).$$

Applying Lemma 3 we find that

$$F(v_n) \geq \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^{k} (|v(a_i+0) - v(a_i-0)|) \tag{44}$$

$$+ \phi(\mathrm{dist}(f_i, j(v(a_i)))) + o(1) \tag{45}$$

$$= F_r(v) + o(1) \tag{46}$$

This end of the proof of Step 2, and thereby the proof of Theorem 2. □

Before moving on to the minimization problem with small perturbation, we need to convey some important properties of the relaxed functional we just constructed.

3.2. **Important properties of the relaxed functional.**

Theorem 3. *The functional F_r defined in (13) is convex on $BV(0,1)$ and is lower semicontinuous in the sense that for every sequence $(v_n) \subset BV(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ as $n \to \infty$, we have*

$$\liminf_{n\to\infty} F_r(v_n) \geq F_r(v). \tag{47}$$

Proof. Given $v, w \in BV(0,1)$ there exist (by Step 1 (30)) sequences $(v_n), (w_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$, $w_n \to w$ in $L^2(0,1)$ and $F(v_n) \to F_r(v)$, $F(w_n) \to F_r(w)$. By convexity of F we have

$$F(tv_n + (1-t)w_n) \leq tF(v_n) + (1-t)F(w_n), \quad \forall t \in [0,1]. \tag{48}$$

Passing to the limit in (48) and using Step 2 (40) we see that

$$F_r(tv + (1-t)w) \leq tF_r(v) + (1-t)F_r(w). \tag{49}$$

By Step 1 (30) applied to v_n with n fixed, we may find some $w_n \in W^{1,1}(0,1)$ such that

$$\|v_n - w_n\|_{L^2} < \frac{1}{n} \quad \text{and} \quad |F_r(v_n) - F(w_n)| < \frac{1}{n}. \tag{50}$$

Thus $w_n \to v$ in $L^2(0,1)$ and from the definition we conclude that

$$F_r(v) \leq \liminf_{n\to\infty} F(v_n) = \liminf_{n\to\infty} F_r(v_n). \tag{51}$$

This complete the proof. □

4. Minimization Problem with Small perturbation in $BV(0,1)$

4.1. **Relaxed functional acts on small perturbation case.**

For our goal of small perturbation case, we aim to minimize the problem

$$\inf \Big\{ \int_0^1 |v'| + \sum_{i=1}^{k} |v(a_i) - f_i|^2 + \int_0^1 |v|^2 \Big\}. \tag{52}$$

[3]: Set

$$G(u) := F(u) + \int_0^1 |u|^2 = \int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2 + \int_0^1 |u|^2. \tag{53}$$

We may additionally set for simplicity

$$A := \inf_{v \in W^{1,1}} G(v). \tag{54}$$

It turns out to be that the inf need not to be achieved since we have shown an example of $W^{1,1}(0,1)$ failed to admit minimizers, and we will replace it by a relaxed problem defined on $v \in BV(0,1)$ as we have done in Section 3.

For every $v \in BV(0,1)$, set

$$G_r(v) = \inf \liminf_{n\to\infty} G(v_n), \tag{55}$$

where the Inf in (55) is taken over all sequences $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$. It is easy to check that

$$G_r(v) = F_r(v) + \int_0^1 |v|^2, \quad \forall v \in BV(0,1). \tag{56}$$

Moreover, G_r is strictly convex on BV(0,1) and it is lower semicontinuous in the sense that for every sequence $(v_n) \subset BV(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ as $n \to \infty$, we have

$$\liminf_{n\to\infty} G_r(v_n) \geq G_r(v). \tag{57}$$

Additionally, convexity implies that

$$B = \inf_{v\in BV} G_r(v) := G_r(\tilde{v}). \tag{58}$$

is uniquely achieved, and we denote by $\tilde{v} \in BV(0,1)$ its unique minimizer. It is clear that $G_r \leq G$ on $W^{1,1}(0,1)$, and thus

$$B = \inf_{v\in BV} G_r(v) \leq \inf_{v\in W^{1,1}} G_r(v) \leq \inf_{v\in W^{1,1}} G(v) = A. \tag{59}$$

On the other hand, by (54) we have

$$A \leq G(v) = F(v) + \int_0^1 |v|^2, \quad \forall v \in W^{1,1}(0,1). \tag{60}$$

as A is taking Inf over all $v \in W^{1,1}(0,1)$. Combined with Step 1 (39), we have

$$A \leq F_r(v) + \int_0^1 |v|^2 = G_r(v), \quad \forall v \in BV(0,1). \tag{61}$$

Thus,

$$A \leq \inf_{v\in BV} G_r(v) = B \tag{62}$$

Combined (59) and (62), we conclude that $A = B$.

Concequently, if (54) admits a minimizer $v_0 \in W^{1,1}(0,1)$, then

$$B \leq G_r(v_0) \leq G(v_0) \leq A, \tag{63}$$

so that by $A = B$, $G_r(v_0) =$ B. i.e., v_0 is a minimizer for (58). By uniqueness, $v_0 = \tilde{v}$.

The key point is that we have replaced our problem (54) which not necessarily have a solution by (58) which always admits unique solution $\tilde{v}$. Therefore $\tilde{v}$ can be viewed as the generalized solution of Problem (54).

4.2. **Stable minimizer on mild perturbations.**

[3] In light of the abundance of minimizers for F in $W^{1,1}(0,1)$ we may consider some of the preferable minimizer in the sense that they are stable with respect to perturbations.

$$F_{1,\varepsilon}(u) = \varepsilon \int_0^1 |u'|^2 + F(u), \qquad u \in H^1(0,1),\ \varepsilon > 0,$$

$$F_{2,p}(u) = \int_0^1 |u'|^p + \sum_{i=1}^k |u(a_i) - f_i|^2, \qquad u \in W^{1,p}(0,1),\ p > 1,$$

$$F_{3,\varepsilon}(u) = \varepsilon \int_0^1 |u''|^2 + F(u), \qquad u \in H^2(0,1),\ \varepsilon > 0.$$

The minimizer u_ℓ of F obtained by linear interpolation, for which u_ℓ is linear on each interval (a_i, a_{i+1}), is certainly good candidate. Additionally, each functional admits a unique minimizer. [3]

By T. Sznigir [1, 2], as $\varepsilon \to 0$ (resp. $p \to 1$) the minimizers of $F_{1,\varepsilon}$ (resp. $F_{2,p}$) converges u_ℓ. Moreover, the minimizer of $F_{3,\epsilon}$ converges as $\epsilon \to 0$ to the solution $\hat{u}$ of a variational inequality corresponding to

$$\min \left\{ \int_0^1 |u''|^2 \,:\, u \in H^2(0,1) \text{ and satisfies Theorem 1} \right\}. \tag{64}$$

The function $\hat{u}$ belongs to $C^1([0,1])$ (while $u_\ell \notin C^1$) and $\hat{u}$ is a piecewise cubic function on each interval (a_i, a_{i+1}), $i = 1, \dots, k-1$. [1, 2]

In addition, if we consider the small perturbation case with respect to the different functions above,

$$G_{1,\varepsilon}(u) = F_{1,\varepsilon}(u) + \int_0^1 |u|^2 \tag{65}$$

$$G_{2,p}(u) = F_{2,p}(u) + \int_0^1 |u|^2 \tag{66}$$

$$G_{3,\varepsilon}(u) = F_{3,\varepsilon}(u) + \int_0^1 |u|^2, \tag{67}$$

their unique minimizers also converge in $L^2(0,1)$ to $\tilde{v}$ (58) as what we finished in early this Section. I.e., if $(u_n) \subseteq W^{1,1}(0,1)$ is a minimizing sequence for (54) in different setting, then $u_n \to \tilde{v}$ as $n \to \infty$ in $L^2(0,1)$. Indeed, we have

$$G_r(u_n) \le G(u_n) \le A + o(1),$$

and a subsequence (u_{n_k}) converges in $L^2(0,1)$ to some $\tilde{u} \in BV(0,1)$ satisfying

$$B \le G_r(\tilde{u}) = A,$$

so that $G_r(\tilde{u}) = B$ and by uniqueness $\tilde{u} = \tilde{v}$. Combined with the fact that $C^\infty(0,1)$ is dense in $W^{1,1}(0,1)$, we can conclude that the unique minimizer $\tilde{v}$ as we constructed, also encoded with these mild perturbation minimization problems.

References

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