Complexity of Spin Glasses Model Through Mean Number of Critical Points

Russell Hua

University of Chicago

December 1, 2024

Table of Contents

Motivation

We consider a simple example from the Dean's Problem [5]:

Examples

Suppose there are N students in a group. Any two students may have either positive or negative opinions towards each other. Our task is to divide them into two groups and maximize their positive feelings.

The strategy is simple, we should put as much friends as possible in one group and their enemies in the other.

Motivation Strategy

We can model this problem by:

- Label each student as number $1, ..., N$.
- For arbitrary i-th and j-th students, we use a number g_{ij} to measure their feelings towards each other.
- We label each student $\sigma_i \in \{1, -1\}$ for separations of the groups. If i, j are friends ($g_{ij} > 0$), then $sgn(\sigma_i) = sgn(\sigma_j)$, and vice versa.

The following quantity will measure how well we finish our task,

$$\sum_{1 \le i,j \le N} g_{ij}\sigma_i\sigma_j$$

When the numbers $\{g_{ij}\}_{i<j}$ are independent standard Gaussian random variables, we have the Hamiltonian for SK(Sherrington-Kirkpatrick) model:

$$H_N^{SK}(\sigma) = \frac{1}{N} \sum_{1 \le i,j \le N} g_{ij}\sigma_i\sigma_j$$

Spherical p-spin glasses model

For our model of study spherical p-spin glasses model ($p \geq 2$):

- Generalization of the Sherrington-Kirkpatrick (SK) model.
- Considers interactions among p-tuples of spins instead of pairs.
- Useful for studying mean-field spin glasses and optimization problems.

The Hamiltonian restricted on $S^{N-1}(\sqrt{N})$ is given by:

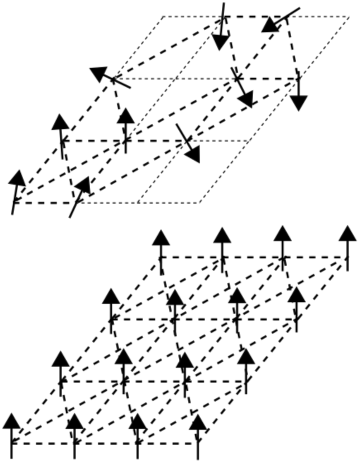
$$H_{N,p}(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\dots,i_p=1}^{N} f_{i_1,\dots,i_p} \sigma_{i_1,\dots,ip}, \quad \sigma = (\sigma_1, \dots, \sigma_N) \in R^N$$

- A configuration σ is a vector in R^N satisfied

$$\frac{1}{N} \sum_{i=1}^{N} \sigma_i = 1$$

- $f_{i_1 i_2 \dots i_p}$ are independent random variables (usually Gaussian) with zero mean and variance $\frac{p!}{2N^{p-1}}$.

Graph for p-spin glasses model



From P. Young, "Spin glasses and random fields", pp.100

Equivalent formulation

Equivalently, $H_{N,p}$ is the centered Gaussian process on the sphere $S^{N-1}(\sqrt{N})$ whose covariance is given by a (symmetric) bivariate function,

$$E[H_{N,p}(\sigma)H_{N,p}(\sigma')] = N^{1-p}(\sum_{i=1}^{N} \sigma_i \sigma_i')^p = NR(\sigma, \sigma')^p, \tag{1}$$

where the normalized inner product $R(\sigma, \sigma')^p = \frac{1}{N}\langle \sigma, \sigma' \rangle = \frac{1}{N}\sum_{i=1}^{N} \sigma_i \sigma_i'$ is usually called the overlap of the configurations σ and σ'.

Definitions

We now want to introduce the critical points of spherical spin glasses.

Definition

For any Borel set $B \subset R$ and integer $0 \leq k < N$, the (random) number $Crt_{N,k}(B)$ of critical values of the Hamiltonian $H_{N,p}$ in the set $NB := \{Nx, x \in B\}$ with index equal to k, is given by

$$Crt_{N,k}(B) = \sum_{\sigma : \nabla H_{N,p}(\sigma)=0} 1_{\{H_{N,p}(\sigma) \in NB\}} 1_{\{i(\nabla^2 H_{N,p}(\sigma))=k\}} \tag{2}$$

Here, ∇, ∇^2 are the gradient and Hessian restricted to $S^{N-1}(\sqrt{N})$ and $i(\nabla^2 H_{N,p}(\sigma))$ is the index of $\nabla^2 H_{N,p}$ at σ, that is the number of negative eigenvalues of the Hessian $\nabla^2 H_{N,p}$.

Definitions

Similarly, summing over all k, we have

Definition

We will also consider the (random) total number of critical values $Crt_N(B)$ of the Hamiltonian $H_{N,p}$ in the set NB,

$$Crt_N(B) = \sum_{\sigma : \nabla H_{N,p}(\sigma)=0} 1_{\{H_{N,p}(\sigma) \in NB\}} \tag{3}$$

Definitions

Before giving the central identity relating the GOE to the complexity of spherical spin-glass models, we fix our notations for the GOE.

Definition

(Same as in class) The Gaussian orthogonal ensemble (GOE) is a probability measure on the space of real symmetric matrices. I.e. it is the probability distribution of the $N \times N$ real symmetric random matrix M^N such that M_{ij} $(i \le j)$ are independent centered Gaussian random variables with variance

$$EM_{ij}^2 = \frac{1 + \delta_{ij}}{2N} \tag{4}$$

we will denote by $E_{GOE} = E_{GOE}^N$ the expectation under the GOE ensemble of size $N \times N$.

Special Notice

- Let $\lambda_0^N \leq \lambda_1^N \leq \lambda_2^N \leq \ldots \leq \lambda_{N-1}^N$ be increasing ordered eigenvalues of M^N, we will denote by $L_N = \frac{1}{N}\sum_{i=0}^{N-1}\delta_{\lambda_i^N}$ the (random) spectral measure of M^N, and $\rho_N(x)$ the density of the probability measure $E_{GOE}(L_N)$.
- The function $\rho_N(x)$ is called the (normalized) one-point correlation function and satisfies

$$\int_R f(x)\rho_N(x)dx = \frac{1}{N}E_{GOE}^N[\sum_{i=0}^{N-1} f(\lambda_i^N)]. \tag{5}$$

Theorem 2.1

For the following, I will introduce the main theorem in this project:

Theorem 2.1

[1] The following identity of the mean number of critical points holds for all $N, p \geq 2, k \in \{0, 1, ..., N-1\}$, and for all Borel set $B \subset R$,

$$E[Crt_{N,k}(B)] = 2\sqrt{\frac{2}{p}}(p-1)E^N_{GOE}[e^{-N\frac{p-2}{2p}(\lambda^N_k)^2} 1\{\lambda^N_k \in \sqrt{\frac{p}{2(p-1)}}B\}] \tag{6}$$

Theorem 2.2

If we take sum over all the $k \in \{0, 1, ..., N-1\}$, we will find the mean total number of critical points given a level of energy and relate it to the one-point function.

Theorem 2.2

[1] The following identity of the mean total number of critical points holds for all $N, p \geq 2, k \in \{0, 1, ..., N-1\}$, and for all Borel set $B \subset R$,

$$E[Crt_N(B)] = 2N\sqrt{\frac{2}{p}}(p-1)^{\frac{N}{2}} \int_{\sqrt{\frac{p}{2(p-1)}}B} e^{-\frac{N(p-2)x^2}{2p}} \rho_N(x)dx \qquad (7)$$

Case of p=2

For the case of $p = 2$, there is a simple connection between the 2-spin spherical model and the random matrix theory.

Remark

Let M be $N \times N$ GOE matrix, the Hamiltonian of the 2-spin spherical model is a quadratic form defined by (symmetric) M restricted to $S^{N-1}(\sqrt{N})$,

$$H_{N,2}(\sigma) = (M\sigma, \sigma)$$

Therefore, for $p = 2$, the $2N$ critical points of the Hamiltonian $H_{N,2}(\sigma)$ are the eigenvectors of M, while the critical values are eigenvalues of M. Plugging in Theorem 2.1 and Theorem 2.2 simplify the result:

$$E[Crt_{N,k}(B)] = 2E^N_{GOE}[1\{\lambda^N_k \in \sqrt{\frac{p}{2(p-1)}}B\}] = 2P^N_{GOE}[\lambda^N_k \in B] \tag{8}$$

$$E[Crt_N(B)] = 2N \int_B \rho_N(x)dx = 2N\rho_N(B) \tag{9}$$

Preparation for the proof

In order prove Theorem 2.1 (Theorem 2.2 would be a simple implication), we will need additional 3 lemmas which involves heavy calculations.

- Lemma 3.1 (Kac-Rice Formula)
- Lemma 3.2 (Covariance Estimation)
- Lemma 3.3 (Identity involve terms of λ_k^N)

Variance process $f_{N,k}$

In the paper [1], we found that it is more convenient to work with the process of variance $f_{N,p}$

Definition

Consider for any $\sigma \in S^{N-1}$, we define the process of variance $f_{N,p}$ with respect to Hamiltonian $H_{N,p}$ as

$$f_{N,p}(\sigma) = \frac{1}{\sqrt{N}} H_{N,p}(\sqrt{N}\sigma). \tag{10}$$

Again, the new function f (abuse notations for simplicity), is again the centered Gaussian process on the sphere S^{N-1} whose covariance is given by

$$E(f(\sigma)f(\sigma')) = (\sum_{i=1}^{N} \sigma_i \sigma_i')^p = R(\sigma, \sigma')^p, \quad \sigma, \sigma' \in S^{N-1}$$

where R still represents the overlap of the configurations σ and σ'.

Preparation for Kac-Rice formula

Before discussing the Kac-Rice Lemma, we want to first introduce some of the notations:

1. $\langle x, y \rangle$: the usual Euclidean inner product; Same for scaler product on the tangent space $T_\sigma(S^{N-1})$.
2. $\nabla^2 f$: The covariant Hessian of f on S^{N-1}, defined as

$$\nabla^2 f(X, Y) := XYf - \nabla_X Yf$$

   where $\nabla_X Y$ is the covariant derivative of Y in the direction of X and ∇ is the usual Riemann connection computing derivatives along the surface of the sphere.
3. Fixed orthonormal frame field $(E_i)_{1 \leq i < N}$: set of $N-1$ vector fields E_i on S^{N-1} such that $\{E_i(\sigma)\}$ is orthonormal basis to tangent space $T_\sigma(S^{N-1})$.
4. ϕ_σ: the density of the gradient vector $E_i(f(\sigma))$
5. $det\nabla^2 f(\sigma)$: determinant of the matrix $(\nabla^2 f(E_i, E_j)(\sigma))_{i,j}$

Lemma 3.1 (Kac-Rice Formula)

Lemma

Let f be a centered Gaussian field on S^{N-1} and let A be a finite atlas on S^{N-1}. Set $f^\alpha = f \circ \psi_\alpha^{-1}$: $\psi_\alpha(U_\alpha) \subset R^{N-1} \to R$ and define $f_{ij}^\alpha = \frac{\partial f^\alpha}{\partial x_i \partial x_j}$, $f_i^\alpha = \frac{\partial f^\alpha}{\partial x_i}$. Assume that for all $\alpha \in I$ and all $x, y \in \psi_\alpha(U_\alpha)$ the joint distribution of $(f_i^\alpha, f_{ij}^\alpha)$ is non-degenerate, and

$$max_{i,j}|Var(f_{ij}^\alpha(x)) + Var(f_{ij}^\alpha(y)) - 2Cov(f_{ij}^\alpha(x), f_{ij}^\alpha(y))| \leq K_\alpha |ln|x-y||^{-1-\beta} \quad (11)$$

for some $\beta > 0, K_\alpha > 0$. For a Borel set $B \subset R$, let

$$Crt_{N,k}^f(B) = \sum_{\sigma : \nabla f(\sigma) = 0} 1_{\{i(\nabla^2 f(\sigma)) = k, f(\sigma) \in B\}} \quad (12)$$

Then, using the $d\sigma$ denoted as the usual surface measure on S^{N-1}, we have

$$E(Crt_{N,k}^f(B)) = \int_{S^{N-1}} E[|det \nabla^2 f(\sigma)| 1_{\{i(\nabla^2 f(\sigma)) = k, f(\sigma) \in B\}} | \nabla f(\sigma) = 0] \phi_\sigma(0) d\sigma \quad (13)$$

Proof of Lemma

- [1] The assumptions of the lemma assure that f is a.s. a Morse function and its gradient and Hessian exist in L^2 sense.
- Direct calculation:

$$E(Crt^f_{N,k}(B)) = E\left[\sum_{\sigma:\nabla f(\sigma)=0} 1_{\{i(\nabla^2 f(\sigma))=k, f(\sigma)\in B\}}\right]$$

- Using the properties of expected values and integrals, we can rewrite this as an integral over the manifold S^{N-1},

$$E[Crt^f_{N,k}(B)] = \int_{S^{N-1}} E[\delta_{\nabla f(\sigma)}(0) 1_{\{i(\nabla^2 f(\sigma))=k, f(\sigma)\in B\}}]d\sigma$$

- For a smooth map $\phi : M \to R^N$ and an integrable function $h : M \to R$, we have

$$\int_M h(x)\delta_{\phi(x)}(y)dx = \int_{\phi^{-1}(y)} \frac{h(x)}{|detD\phi(x)|} dH_{M-n}(x)$$

where H_{M-n} is the $M-n$ dimensional Hausdorff measure.

Lemma 3.2

Lemma

Let $(f_i(\sigma))_{1\le i<N}$ and $(f_{ij}(\sigma))_{1\le i,j<N}$ be the gradient and Hessian matrix at σ. The joint distribution $f(\sigma), f_i(\sigma), f_{jk}(\sigma)$ is determined by

$$E(f(\sigma)^2) = 1, \quad E(f(\sigma)f_i(\sigma)) = E(f_i(\sigma)f_{jk}(\sigma)) = 0, \tag{14}$$

$$E(f(\sigma)f_{ij}(\sigma)) = -p\delta_{ij}, \quad E(f_i(\sigma)f_j(\sigma)) = p\delta_{ij} \tag{15}$$

$$E(f_{ij}(\sigma)f_{kl}(\sigma)) = p(p-1)[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}] + p^2\delta_{ij}\delta_{kl} \tag{16}$$

Under the conditional distribution $P[\cdot|f(\sigma) = x \in R]$, the random variables $f_{ij}(\sigma)$ satisfied

$$E[f_{ij}(\sigma)] = -xp\delta_{ij}, \quad E[f_{ij}(\sigma)^2] = (1+\delta_{ij})p(p-1) \tag{17}$$

Also, the random matrix $(f_{ij}(\sigma))$ has the same distribution as

$$M^{N-1}\sqrt{2(N-1)p(p-1)} - xpI \tag{18}$$

Proof of Lemma 3.2

We will give an example calculation here:

- Without loss of generality, we consider σ at the north pole of the sphere $n = (0, 0, ..., 1)$
- Define coordinate chart (U, ψ) on neighberhood U of n. Specifically, for $x = (x_1, ..., x_N) \in U$

$$\psi(x) := (x_1, ..., x_{N-1}) \tag{19}$$

  and the inverse map ψ^{-1} is

$$\psi^{-1}(x) = (x_1, ..., x_{N-1}, \sqrt{1 - \|x\|^2}), \quad \|x\|^2 := \sum_{i=1}^{N-1} x_i^2 \tag{20}$$

- Centered Gaussian field $f\colon S^{N-1} \to R$ with covariance function

$$Cov(f(x), f(y)) = \langle x, y \rangle^p, \tag{21}$$

- Define Gaussian process f' such that

$$f'(x) = f \circ \psi^{-1}(x)$$

Continue

- For $x \in \psi(U)$ with covariance $C(x,y)$,

$$C(x,y) = Cov(f'(x), f'(y)) = \Big(\sum_{i=1}^{N-1} x_i y_i + \sqrt{(1-\|x\|^2)(1-\|y\|^2)}\Big)^p \tag{22}$$

- Notice that we have $\psi(\sigma) = \psi(n) = 0$, so we set $x = y = 0$.
- Hence, we have

$$f_i(\sigma) = E_i f(\sigma) = \frac{\partial f'(x)}{\partial x_i}\Big|_{x=0} \tag{23}$$

and

$$E(f(\sigma)f_i(\sigma)) = \frac{\partial C(x,y)}{\partial x_i}\Big|_{x=y=0} = pS^{p-1}\frac{\partial S(x,y)}{\partial x_i}\Big|_{x=y=0} \tag{24}$$

where $S(x,y) = \sum_{i=1}^{N-1} x_i y_i + \sqrt{(1-\|x\|^2)(1-\|y\|^2)}$ and

$$\frac{\partial S(x,y)}{\partial x_i} = y_i - \frac{x_i(1-\|y\|^2)}{\sqrt{(1-\|x\|^2)(1-\|y\|^2)}} = 0$$

so that

$$E(f(\sigma)f_i(\sigma)) = p \cdot 1^{p-1} \cdot 0 = 0.$$

Lemma 3.3

Independent fact about the distribution of the eigenvalues of the GOE:

Lemma

Let M^{N-1} be a $N-1 \times N-1$ GOE matrix and X be an independent Gaussian random variable with mean μ and variance ξ^2. Then for any Borel set $G \subset R$,

$$E[|det(M^{N-1} - XI)|1_{\{i(M^{N-1}-XI)=k, X\in G\}}] \tag{25}$$

$$= \frac{\Gamma(\frac{N}{2})(N-1)^{-\frac{N}{2}}}{\sqrt{\pi\xi^2}} E^N_{GOE}[exp(\frac{N(\lambda^N_k)^2}{2} - \frac{((\frac{N}{N-1})^{\frac{1}{2}}\lambda^N_k - \mu)^2}{2\xi^2})1_{\{\lambda^N_k \in (\frac{N}{N-1})^{\frac{1}{2}} G\}}] \tag{26}$$

Condition Check for Lemma 3.1

Our goal is to have the expression:

$$E[Crt_{N,k}(B)] = 2\sqrt{\frac{2}{p}}(p-1)E_{GOE}^{N}[e^{-N^{\frac{p-2}{2p}}(\lambda_k^N)^2}1\{\lambda_k^N \in \sqrt{\frac{p}{2(p-1)}}B\}] \tag{27}$$

Notice that Lemma 1 gives indicators and integrand, Lemma 2 verify Lemma 1, and Lemma 3 gives exponential, eigenvalue, scaled set terms.

Condition Check for Lemma 3.1

The first task is to check assumptions for Lemma 1:

$(f_i(\sigma), f_{ij}(\sigma))$ is non-degenerate in the Borel set B:

- Same charts (ψ_α, U_α) and orthogonal frame $\{E_i\}$ in Lemma 2.
- $(f_i(\sigma), f_{ij}(\sigma))$ is certainly non-degenerate at least near the north pole $\sigma = n$ based on

$$Var(f_i(\sigma)) = E(f_i(\sigma)^2) = p \geq 0, \tag{28}$$

$$Var(f_{ij}(\sigma)) = E(f_{ij}(\sigma)^2) = p(2p-1)\delta_{ij}^2 + p(p-1) \geq 0 \tag{29}$$

  provided $E(f_i(\sigma) f_{ij}(\sigma))) = 0$, so that the covariance matrix is block-diagonal matrix.
- Continuity of covariance operator, then non-degenerate for $\sigma \in U_n$
- S^{N-1} can be covered by a finite number of copies of U_n (rotation around center)

Similarly for decreasing rate check.

Transfer back to Hamiltonian Version

Note that due to the rotational symmetry, the integrand over sphere does not depend on σ, we set $\sigma = n$. The definition of f in compared with the Theorem 2.1 after applying Lemma 1, we have

$$E(Crt^f_{N,k}(B)) = \omega_N E[|det\nabla^2 f_N(n)| 1_{\{i(\nabla^2 f_N(\sigma))=k\}} 1_{\{f(n)\in\sqrt{N}B\}} | \nabla f(n) = 0] \phi_n(0) \tag{30}$$

where ω_N is the volume of sphere S^{N-1},

$$\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

and the density function $\nabla f(n)$ is the same as the density of f' we used in lemma 2, we have

$$\phi_n(0) = (2\pi p)^{-\frac{N-1}{2}}$$

Condition Shift

To compute the expectation in (30), we changed the condition on $f(n)$ instead of $\nabla f(n)$ as both f and its Hessian are independent of the gradient,

$$E[|det\nabla^2 f(n)|1_{\{i(\nabla^2 f_N(\sigma))=k\}}1_{\{f(n)\in\sqrt{N}B\}}|\nabla f(n)=0] \tag{31}$$

$$= E[E[|det\nabla^2 f_N(n)|1_{\{i(\nabla^2 f_N(\sigma))=k\}}1_{\{f(n)\in\sqrt{N}B\}}|f(n)]] \tag{32}$$

Expectation in (32)

For the inside expectation we may treat as the second part of Lemma 3.2, the conditional distribution of $\nabla^2 f(n)$ given $f(n)$ is:

$$\nabla^2 f(n) \stackrel{d}{=} \sqrt{2(N-1)p(p-1)}\, M^{N-1} - p^{1/2}\left(2(N-1)(p-1)\right)^{-1/2} f(n) I,$$

where M^{N-1} is a GOE matrix of size $N-1$. Thus, the inner expectation becomes:

$$\left(2(N-1)p(p-1)\right)^{\frac{N-1}{2}} \cdot \mathbb{E}^{N-1}_{GOE}\left[\left|\det(M^{N-1} - XI)\right| 1_{\{i(M^{N-1}-XI)=k\}}\right],$$

where $X = p^{1/2}\left(2(N-1)(p-1)\right)^{-1/2} f(n)$.

Insert Lemma 3.3

Using lemma 3.3 with $\mu = 0, \xi^2 = \frac{p}{2(N-1)(p-1)},$. This is promising since $f(n)$ is a Gaussian random variable with mean 0 and variance 1, X is also Gaussian with:

- Mean: $\mu = 0$.
- Variance:

$$\xi^2 = \left(p^{1/2}\,(2(N-1)(p-1))^{-1/2}\right)^2 = \frac{p}{2(N-1)(p-1)}.$$

Continue

By Lemma 3.3, we have:

$$\frac{\Gamma\left(\frac{N}{2}\right)(N-1)^{-\frac{N}{2}}}{\sqrt{\pi\xi^2}} \cdot \mathbb{E}^N_{GOE}[\exp(\frac{N(\lambda^N_k)^2}{2} - \frac{((\frac{N}{N-1})^{\frac{1}{2}}\lambda^N_k - \mu)^2}{2\xi^2})$$
$$\cdot 1_{\{\lambda^N_k \in (\frac{N-1}{N})^{\frac{1}{2}} G\}}.$$

If we set $G = \{X : f(n) \in \sqrt{N}B\}$, then:

$$X \in G \iff p^{1/2}\left(2(N-1)(p-1)\right)^{-1/2} f(n) \in G \tag{33}$$
$$\iff f(n) \in \sqrt{\frac{2(N-1)(p-1)}{p}}G. \tag{34}$$

But since $f(n) \in \sqrt{N}B$, we have:

$$G = \sqrt{\frac{p}{2(N-1)(p-1)}}\sqrt{N}B = \sqrt{\frac{Np}{2(N-1)(p-1)}}B.$$

Substitution

Substitute the result from Lemma 3.3 into the expression for $\mathbb{E}[Crt_{N,k}(B)]$:

$$\mathbb{E}[Crt_{N,k}(B)] = \omega_N \phi_n(0) \left(2(N-1)p(p-1)\right)^{\frac{N-1}{2}} \frac{\Gamma\left(\frac{N}{2}\right)(N-1)^{-\frac{N}{2}}}{\sqrt{\pi\xi^2}} \tag{35}$$

$$\times \mathbb{E}^N_{GOE}[\exp\left(\frac{N(\lambda_k^N)^2}{2} - \frac{\left(\left(\frac{N}{N-1}\right)^{\frac{1}{2}}\lambda_k^N - \mu\right)^2}{2\xi^2}\right) \tag{36}$$

$$\times 1_{\{\lambda_k^N \in \sqrt{\frac{N-1}{N}}\sqrt{\frac{Np}{2(N-1)(p-1)}}B\}}]. \tag{37}$$

Handle terms in (36)

Check the exponential term and notice that:

$$\left(\frac{N}{N-1}\right)^{\frac{1}{2}} \lambda_k^N - \mu = \left(\frac{N}{N-1}\right)^{\frac{1}{2}} \lambda_k^N$$

since $\mu = 0$. Next, substitute $\xi^2 = \frac{p}{2(N-1)(p-1)}$:

$$\left(\left(\frac{N}{N-1}\right)^{\frac{1}{2}} \lambda_k^N\right)^2 \frac{1}{2\xi^2} = \frac{N}{N-1} \left(\lambda_k^N\right)^2 \frac{1}{2\left(\frac{p}{2(N-1)(p-1)}\right)} \tag{38}$$

$$= \frac{N(\lambda_k^N)^2(N-1)(p-1)}{pN(N-1)} = \frac{N(\lambda_k^N)^2(p-1)}{p}. \tag{39}$$

Therefore, the exponential term simplifies:

$$\exp\left(\frac{N(\lambda_k^N)^2}{2} - \frac{N(\lambda_k^N)^2(p-1)}{p}\right) = \exp\left(-N\frac{p-2}{2p}(\lambda_k^N)^2\right).$$

Conclusion

Since we have check the scaled B, indicator function, and expoential terms, recall

$$\omega_N = \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)}, \quad \phi_n(0) = (2\pi p)^{-\frac{N-1}{2}}$$

so that everything is known and match our observation with simply algebra:

$$E[Crt_{N,k}(B)] = 2\sqrt{\frac{2}{p}}(p-1)E^N_{GOE}[e^{-N^{\frac{p-2}{2p}}(\lambda^N_k)^2}1\{\lambda^N_k \in \sqrt{\frac{p}{2(p-1)}}B\}]$$

which complete our proof.

Proof of Theorem 2.2

We have shown:

$$E[Crt_{N,k}(B)] = 2\sqrt{\frac{2}{p}}(p-1)E^N_{GOE}[e^{-N\frac{p-2}{2p}(\lambda_k^N)^2}1\{\lambda_k^N \in \sqrt{\frac{p}{2(p-1)}}B\}] \tag{40}$$

Theorem 2.2 wants us to have:

$$E[Crt_N(B)] = 2N\sqrt{\frac{2}{p}}(p-1)^{\frac{N}{2}}\int_{\sqrt{\frac{p}{2(p-1)}}B} e^{-\frac{N(p-2)x^2}{2p}}\rho_N(x)dx \tag{41}$$

The proof follows from Theorem 2.1 by summing over $k \in \{0, 1, ..., N-1$. The additional N in the prefactor comes from the fact that ρ_N is normalized one point correction.

For the spherical p-spin model, understanding the asymptotics can reveal universal aspects of spin glasses and disordered systems, providing insights that are applicable across a range of similar systems.

Notations

The following are some useful expressions:

$$E_\infty = E_\infty(p) = 2\sqrt{\frac{p-1}{p}}.$$

Let $I_1 : (-\infty, -E_\infty] \to \mathbb{R}$ be given by

$$I_1(u) = \frac{2}{E_\infty^2}\int_u^{-E_\infty} (z^2 - E_\infty^2)^{\frac{1}{2}}\, dz = -\frac{u}{E_\infty^2}\sqrt{u^2 - E_\infty^2} - \log\left(-u + \sqrt{u^2 - E_\infty^2}\right) + \log E_\infty.$$

The following important functions which will describe the asymptotic complexity of the p-spin spherical spin-glass models:

$$\Theta_p(u) = \begin{cases} \frac{1}{2}\log(p-1) - \frac{p-2}{4(p-1)}u^2 - I_1(u), & \text{if } u \le -E_\infty, \\ \frac{1}{2}\log(p-1) - \frac{p-2}{4(p-1)}u^2, & \text{if } -E_\infty \le u \le 0, \\ \frac{1}{2}\log(p-1), & \text{if } 0 \le u, \end{cases}$$

Continue

For any integer $k \geq 0$, similar structure corresponding to Theorem 2.2:

$$\Theta_{k,p}(u) = \begin{cases} \frac{1}{2}\log(p-1) - \frac{p-2}{4(p-1)}u^2 - (k+1)I_1(u), & \text{if } u \leq -E_\infty, \\ \frac{1}{2}\log(p-1) - \frac{p-2}{p}u^2, & \text{if } u \geq -E_\infty. \end{cases}$$

With all these notations, the estimations will be in a good look.

Asympototic Results

Theorem

[1] For all $p \geq 2$ and $k \geq 0$ fixed,

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}[Crt_{N,k}(u)] = \Theta_{k,p}(u).$$

and the version corresponding to Theorem 2.2,

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}[Crt_{N}(u)] = \Theta_{k}(u).$$

for we fix $B = (-\infty, u)$, and abuse $Crt_{N,k}(u) := Crt_{N,k}(B)$ as same idea used in defining cumulative distribution function.

Graph for $\Theta_{k,p}(u)$

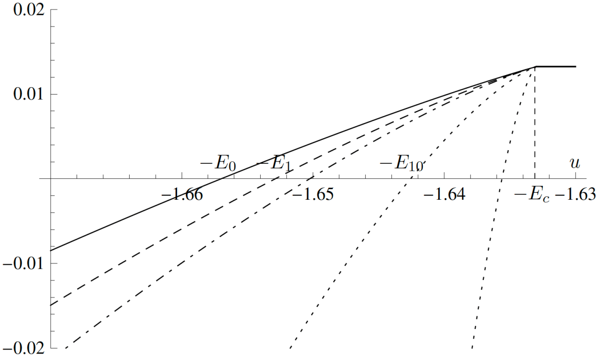


FIGURE 1. The functions $\Theta_{k,p}$ for $p = 3$ and $k = 0$ (solid), $k = 1$ (dashed), $k = 2$ (dash-dotted), $k = 10$, $k = 100$ (both dotted). All these functions agree for $u \geq -E_\infty$.

From "Random Matrices and complexity of Spin Glasses", A. Auffinger, pp.5

Remarks

[1] As an implication of Theorems, one can compute the logarithmic asymptotics of the mean total number of critical points and the mean total number of critical points of index k, $\mathbb{E}[Crt_N(\mathbb{R})]$ and $\mathbb{E}[Crt_{N,k}(\mathbb{R})]$.

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Crt_N(\mathbb{R})] = \frac{1}{2} \log(p-1),$$

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Crt_{N,k}(\mathbb{R})] = \frac{1}{2} \log(p-1) - \frac{p-2}{p}.$$

which are independent of k as in [2].

Citations

- A. Auffinger, G. Ben Arous, J. Cerny, *Random Matrices and complexity of Spin Glasses*, DOI: 10.48550/arXiv.1003.1129 (2010)
- Robert J. Adler and Jonathan E. Taylor, *Random fields and geometry*, Springer Monographs in Mathematics, Springer, New York, 2007. MR MR2319516
- M. L. Mehta, *Random matrices*, second ed., Academic Press Inc., Boston, MA, 1991. MR MR1083764 (92f:82002)
- P. Yves Gaudreu Lamarre,*Topics in Random Matrix Theory*, University of Chicago, Chicago. (2021)
- Y. Zhou, *The Spherical Spin Glass Models at Zero Temperature*, Northwestern University, Evanston. (2023)