# Reading program.

Intro:

Second order elliptic equation: $u(x_1, \cdots x_n)$:

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} + au = f$$

Define ellipticity $\Rightarrow$ $N$ equations for $N$ func $u_1, \cdots u_n$ $\to$ $x_1, x_2, \cdots x_n$.

ie: $F_i(x_1, \cdots x_n, u_1, \cdots u_N, \cdots, \frac{\partial^k u_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}) = 0$ " $i,j = 1, \cdots N$; $1 \le k_1 + \cdots + k_n \le n_j$.

for each $u_j$, there is highest $n_j$ derivitive $\to$ syst

Suppose each $F_i$ is linear in the highest order:

$$\sum_{j=1}^{N} \sum_{k_0+k_1+\cdots k_n = n_j} a_{ij}^{k_1 \cdots k_n} \frac{\partial^{n_j} u_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} + F_i(x_1, \cdots x_n, u_1, \cdots u_N, \frac{\partial^k u_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}) = 0$$

when $a_{ij} :=$ func of $x_1, \cdots x_n$ alone and $F_i$ involve mult of derivite less than $n_j$.

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Semi-linear (1.2)

Def: i) S ($n-1$ dim-surface through a pt $P$ in $(x_1, x_2 \cdots x_n)$ plane, satisfied Semi (1.2), then S is free/non-characteric"

ii) Given $u_j$ and $n_j - 1$ on S in nbhd of P. $S \equiv$ free if $u_j$ satisfied (1.2)

implies that we can calculate $n_j$ th order derivative of $u_j$ at P.

Note: ① To obtain $n_j$ th derivitive, differentiating those of order $n_{j-1}$ in directions $\to$ tangent to the surface.

② WTC: the derivitive of normal vectors

Let $\xi_1(x_1, \cdots x_n) \cdots \xi_n(x_1, \cdots x_n)$ be coordinates in the nbhd of $P$ s.t

$\xi_1 = 0$ is simple in $S$ and the Jacobian $\left|\frac{\partial \xi_i}{\partial x_j}\right| \neq 0$ in nbhd of $P \Rightarrow f$ " invertible + differential "

$$J = \left[\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_n}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad \text{" } n \times n \text{ "}$$

known that : 1) $u_j$ up to $u_j$'s derivitive of order $n_j - 1$ with $\xi_k$

2) and by differentiation can calculate $n_j$th order derivitive except " $\frac{\partial^{n_j} u_j}{\partial \xi_1^{n_j}}$ "

To calculate "Green", we have at $P$, $\frac{\partial^{n_j} u_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} = \underline{\frac{\partial^{n_j} u_j}{\partial \xi_1^{n_j}} \left(\frac{\partial \xi_1}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial \xi_1}{\partial x_n}\right)^{k_n}} + \cdots$

Inserting into (1.2), we have $\sum_{j=1}^{N} \sum_{\substack{k_1+k_2+\cdots k_n \\ = n_j}} a_{ij}^{k_1 \cdots k_n} \underline{\frac{\partial^{n_j} u_j}{\partial \xi_1^{n_j}} \left(\frac{\partial \xi_1}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial \xi_1}{\partial x_n}\right)^{k_n}} + \cdots = 0 \quad (i = 1, \cdots N).$

We need show that $\left| \sum_{k_1+k_2+\cdots k_n = n_j} a_{ij}^{k_1 \cdots k_n} \cdot \frac{\partial^{n_j} u_j}{\partial \xi_1^{n_j}} \left(\frac{\partial \xi_1}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial \xi_1}{\partial x_n}\right)^{k_n} \right| \neq 0.$

Since $\xi_1 = 0$ is the surface $S$, $\frac{\partial \xi_1}{\partial x_1}, \cdots \frac{\partial \xi_1}{\partial x_1}$ at $P$ are proportional to the direction $a_1, \cdots a_n$ of the normal to $S$ at $P$; then, Thus $S$ is free $\equiv$ $\left| \sum_{k_1+k_2+k_n = n_j} a_{ij}^{k_1 \cdots k_n} a_1^{k_1} \cdot a_2^{k_2} \cdots a_n^{k_n} \right| \neq 0.$

Def: equation, in parameter $a_1, \cdots a_n$, resulting from setting it $= 0$, " characteric determinant " is called chara-funct (1.2). If the chara-equation is satisfied by the normal $(a_1, \cdots a_n)$ to $S$ at $P$. $S$ is said to be charactertic at $P$.

$S$ is chara-surface if it's chara at every pt on it.

Def: the system of equation (1.2) is said to be elliptic at $P$, if the chara-equation has no real sol'n $(a_1, \cdots a_n)$

(Ellipticity) other than $(0, \cdots 0)$. ie: if $\forall$ surface through $P$ is free

$C_1$: for semilinear system, it's simply $\Rightarrow$ coefficient "$a_{ij}^{k_1 \cdots k_n}$"

$C_2$: for general nonlinear (1.1), the ellipticity dependent on sol'n $\rightarrow u_j$ and their deravitive $\rightarrow F_i$. The chara-equation is defined as

$$\det \left| \sum_{k_1 + \cdots + k_n = n_j} \frac{\partial F_i}{\partial \left( \frac{\partial^{n_j} u_j}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right)} a_1^{k_1} \cdots a_n^{k_n} \right| = 0$$

A surface $S$ is called chara at $P$, if for $\forall P \in S$, then $C_2$ satisfied. and for Given system of func $u_1, \cdots u_N$ if the normal $(a_1, \cdots a_n)$ to $S$ at $P$ satisfied the chara-equation with $u_1, \cdots u_N$ and their deravitives in $F_i$. And system is called elliptic if $\forall P \in S$, $(a_1, \cdots a_n) \equiv (0, \cdots, 0)$

Notion: $D$ is bounded domain in dim-$N$: $(x_1, \cdots x_n)$ and $\dot{D}$ is called the bdy and $\bar{D}$ is the closure ($\dot{D} \,/\!/\, \partial D$)

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = u_{ij} \text{ or } u_{x_i x_j} \Longrightarrow a_{ij} u_{ij} + a_i u_i + a u = f \quad \text{(ordinary)} \qquad \text{"1.3"}$$

" functions $\rightarrow (x_1, \cdots x_n)$

## Sec 1.2: Maximal Principle

consider LD: $L(u) := a_{ij} u_{ij} + a_i u_i + a u$ "elliptic"

we can write as $L(u) := M[u] + au \Rightarrow$ "BV sol'n unique" $\equiv$ "the only sol'n of $L[u] = 0$ which vanishes on $\partial D$ is identically zero.

Restrict (i) the coefficient $a < 0$, since "ex1: $u = \cos x \cos y \xrightarrow{\text{sol'n}} u_{xx} + u_{yy} + 2u = 0$ in $|x|, |y| \le \frac{\pi}{2}$, on bdy $\not\equiv 0$."

(ii) $a$ of $L[u]$ is conti over $D$ and $u$ is twice differentiable + conti in $D$ and $\bar{D}$.

Thm: If $u$ satisfies $M[u] \geq 0$ and has an interior max-pt in $D$, then $u \equiv$ constant "Maximal principle"

Weak form: the Maximal of $\forall$ func satisfying $M[u] \geq 0$ is assumed on $\dot{D}$.

Uniquness of sol'n "u" with $a \leq 0$ is derived from:

Corollary 1.1: If $u$ satisfies $L[u] \geq 0$ in $D$ ($a \leq 0$) and $u \leq 0$ in $\dot{D}$, then $u \leq 0$ in $D$

proof: Suppose $u > 0$ somewhere s.t $u$ has an interior positive max pt in $P$ (ie: $\exists \sup_{u \in D} u > 0$)

By continuity, $u > 0$ for some nbhd of $P$. $(P-\delta, P+\delta)$

But $M[u] \geq -au \geq 0$, since $a \leq 0$ and $u > 0$

By Maximal principle, it follows that $u \equiv$ const. in $P$'s nbhd.

Thus the set of Max-pt in $D$ is open.

On the other hand, by conti of $u$, the set is closed (ie. a limit pt in $D$ of Max pts $u$ is a Max pt), and is therefore all of $D$. so by continuity, $u \equiv$ positive const in $\bar{D}$ — contradiction to our original assup- $u \leq 0$ on $\partial D$

Note: $M[v] \leq 0$ at an interior max pt $P$ of $\forall$ differentiable func $v$. For at $P$ $u'(P) = 0$, and the matrix of second Deriviatives $(v_{ij})$, is that of a non-positive quadratic form $\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$

consider $M[v]$ at $P \equiv a_{ij} v_{ij}$, the trace of $(a_{ij}), (v_{ij})$ "symmetric"

Since orthogonal-transform $\Rightarrow$ invariant on the trace, then we can reduce $(a_{ij}) \rightarrow$ JC form? (Diagonal form $(p_i)$)

with $p_i > 0$, and similarly $\rightarrow (v_{ij}) \Rightarrow (\bar{v}_{ii})$, the trace $\equiv \sum_i p_i \bar{v}_{ii}$, But $(\bar{v}_{ii})$ is also matrix of a non-positive quadratic form

then $\bar{v}_{ii} \leq 0$, and $M[u]$ at $P \equiv a_{ij} v_{ij} = \sum_i p_i \bar{v}_{ii} \leq 0$.

Lemma 1.2: Let $S$ be open sphere and $P_0 \in \dot{D}$. Assume the coefficients of $M[u]$ "$a_{ij}, a_i$" are bounded in $S$ and $\exists$ a positive $m$ s.t $a_{ij}\xi_i\xi_j \geq m\sum\xi_i^2$ holds in $S$ for $\forall\ (\xi_1, \cdots \xi_n)$. Assume that $u$ is conti in $S+P_0$ and $u \in R^2(S)$, that $M[u] \geq 0$ in $S$ and $u < u(P_0)$ in $S$. Then the external normal derivitive $\frac{\partial u}{\partial N}$ at $P_0 := \lim(\inf \frac{\Delta u}{\Delta N}) > 0$

(1.4)

proof: Let $S_1$ be smaller sphere intensionally tangent to $S$ at $P_0$. Clearly the only Max-pt of $u$ in $\bar{S_1}$ is at $P_0$, choose the center of $S_1$ as origin and set $r = (\sum x_i^2)^{1/2}$. Denoted by $S'$ the intersection of $\bar{S}_1$ with a fixed closed sphere $\bar{S}_2$ having $P_0$ as center and $r_2 < r_0$, the bdy of $S'$ consist of caps of. $\bar{S}_1$ and $\bar{S}_2$ which $:= \bar{S_1}'$ and $\bar{S_2}'$

$r_0$ rad $\Rightarrow S_1$.

Introduce: $h = e^{-ar^2} - e^{-ar_0^2}$ which is postive in $S_1$ and vanish of $S_1$

consider for $a \gg 1$, s.t $M[h] = e^{-ar^2}[4a^2 a_{ij}x_ix_j - 2a\sum(a_{ii} + a_ix_i)] > 0$ in $S'$

since $r$ is bounded away from $0$ and from (1.4), $a_{ij}x_ix_j \geq mr^2$.

On $\bar{S_2}'$ (closed), the func $u < u(P_0)$ and hence bounded away from $u(P_0)$.

Thus $\exists\ \varepsilon > 0$, sufficient small the func $v = u + \varepsilon h$ also satisfied $v < v(P_0)$

then consider $v$ in $S'$. In the interior of $S'$ we have $M[v] = M[u] + \varepsilon M[h] > 0$

Therefore $v$ cannot have an interior Max pt. ie ($\max_{S'} v$ can occurs on bdy of $S'$)

But it occurs at $P_0$, for on $\bar{S_2}'$ $v < u(P_0)$, while $\bar{S_1}'$: $v = u \leq u(P_0)$, and finally $v(P_0) = u(P_0)$.

Thus $\max_{S'} v$ occurs at $P_0$, it follows that:

$$\frac{\partial v}{\partial N} = \frac{\partial u}{\partial N} + \varepsilon\frac{\partial h}{\partial N} \geq 0 \quad \text{as} \quad \frac{\partial h}{\partial N} < 0 \Rightarrow \frac{\partial u}{\partial N} > 0.$$

Application $\rightarrow$ proof of max-principle

proof: let $M[u] \geq 0$ and $u \not\equiv$ constant in $D$, and $u$ has Max-pt interior, then it is easy to find closed sphere ($S_1$ lying in $D$) having a Max pt of $u$ on bdy $\dot{D}$ (ie: $P_0$)

but none in its interior. By the lemma we have $\frac{\partial u}{\partial N} > 0$ at this pt, contradicted to first-dirative vanishes at $P_0$.

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Consequence of the Maximal principle ( $L(u)$ with $a \le 0$)

1) uniqueness of (1.3) ✓

2) Max-principle $\Rightarrow$ a bound on sol'n

Let $u \frac{\text{sol'n}}{\text{conti on } \bar{D}} \equiv \phi$ ( Given bdy value on $\dot{D}$) , Assume that $|a_{ij}|, |a_i|, |a| \le K$. (1.5)

and the equation is unif - elliptic in $D$, ie : $a_{ij} \xi_i \xi_j \ge m \cdot \sum \xi_i^2$ in $D$ (1.6)

We assert if "$g$" satisfing (1) $-L[g] \ge \max [f]$ and $g \ge \max |\phi|$ on $\dot{D}$

then $|u| \le g$

proof: WTS $v := u-g$ non-negative.

$L[v] = L[u] - L[g] = f - L[g] \ge 0$ ?, $v = \phi - g \le 0$ on $D$.

to Construct such a func "$g$" assume that the domain $D$ lies in the half plane $x_1 \ge 0$

Set $g := \max|f| (e^{a\bar{x}} - e^{ax_1}) + \max|\phi|$ $(a > 0)$ and $(\bar{x} \ge x_1$ in $D)$, clearly $g \ge \varphi$ on $\dot{D}$

and $-L[g] = \max|f| \cdot [-ae^{a\bar{x}} + ae^{ax_1} (a_{11} a^2 + a_1 a + a)] - a \max|\phi|$

$\ge \max|f| \cdot (a_{11} a^2 + a_1 a + a)$, if positive

$\ge \max|f| \cdot [ma^2 - K(a+1)]$ when $a >> \max|f|$. ( $a$ dependent on $K, m$).

Bound : A sol'n $u$ of (1.3) $(a \le 0)$ with Given $\varphi$ is bounded by

$|u| \le \max|\varphi| + \max|f| \cdot (e^{a\bar{x}} - 1)$ (1.7) and $\bar{x}$ is s.t $0 \le x_1 \le \bar{x}$ in $D$.

$\Rightarrow |u| \le k(\max|\varphi| + \max|f|)$ (1.8) $\Rightarrow \max a(e^{c\bar{x}} - 1) < 1$.

$M[u] + a^- u = (a^- - a)u + f = \tilde{f}$ where $a^- = \min(a, 0)$

$$\max|u| \le \max|\varphi| + \max|f|(e^{a\bar{x}}-1) \le \max|\varphi| + (e^{a\bar{x}}-1)(\max|f| + \max|u|\cdot \max a)$$

or $\quad \max|u| \le \dfrac{\max|\varphi| + \max|f|(e^{a\bar{x}}-1)}{1-\max a\cdot(e^{a\bar{x}}-1)}$

(B) Neumann condition : "conduct $2^{nd}$ deravitive"

Given a conti func $\varphi$ defined on bdy $\dot{D}$ (has normal directions)

WTf: a sol'n $u$ of $M[u]\equiv 0$ s.t $u$ and $u'$ are conti in $\bar{D}$ and, on $\dot{D}$ the exterior normal derivitive $\frac{du}{dN} = \varphi + C$

i) Prove uniqueness :

① restrict: $|a_{ij}|, |a_i|, |a| \le k$ and $a_{ij}\xi_i\xi_j \ge m\sum \xi_i^2$

② for $\forall P_0 \in \partial D$, we may find an open sphere $S_1 \subset D$ and $P_0$ on its bdy

By weak form of Max-principle, Max/min of $u \to M[u]=0$ happens on $\partial D$ "$\dot{D}$", by lemma we have that.

$\frac{du}{dN} > 0$ or $\frac{du}{dN} < 0$ at these $P_0$ if $u \not\equiv C$

?

if $u \xrightarrow{\text{sol'n}} M[u]=0$ which vanish $p\in\partial D$ and $\frac{\partial u}{\partial N}$ = constant on $\dot{D}$, then $C=0$ and $u\equiv 0$.

Note : (1.6) ≡ equation elliptic ? and proof → uniqueness is universal!

example : for $\forall$ sol'n $(y>0)$ of $yu_{xx}+u_{yy}=0$

elliptic (but not for $y=0$)

which has max $u$ for $y\ge 0$ on $x$-axies, has $u_y<0$ at max-pt.

If $\dot{D}$ is characteric at $P_0$, lemma $\not\equiv$ hold.

$u = -y^2\sin x,\ 0<x<\pi,\ y>0 \Rightarrow 2u_{xx}+y^2u_{yy}$ has max on $y=0$, but has also $u_y=0$ on $y=0$

(c) The Maximal principle yields also $\Rightarrow$ uniquenss sol'ns $\rightarrow$ BV (nonlinear – elliptic cases).

$$F(x_1, \cdots x_n, z, z_{x_1}, \cdots z_{x_n x_n}) = 0$$

Assume that the equation is elliptic for all $(x_1, \cdots x_n)$ in $D$ and all values of the orther

(1) ie: $\frac{\partial F}{\partial z_{x_i x_j}} \xi_i \xi_j$ is positive definite for all in $F$.

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uniquness (2) Assume $\frac{\partial F}{\partial z} \leq 0$

proof

for $w := z - \bar{z}$

As if $z$ and $\bar{z}$ are sol'ns $\Rightarrow$ equation, we have $\tilde{F}_{z_{x_i x_j}} w_{x_i x_j} + \tilde{F}_{z_{x_i}} w_{x_i} + \tilde{F}_z w = 0$

in general, $\varphi = \int_0^1 \phi(x_1, \cdots x_n, tz + (1-t)\bar{z}, t z_{x_1} + (1-t)\bar{z}_{x_1}, \cdots t z_{x_n x_n} + (1-t)\bar{z}_{x_n x_n})\, dt$.

Def: For $u \in C(\Omega)$, if $u(x) = \frac{1}{w_n r^{n-1}} \int_{\partial B_r(x)} u(y)\, dS_y$ for $\forall B_r(x) \subset \Omega$

then first mean-value-property satisfied

if $u(x) = \frac{n}{w_n r^n} \int_{B_r(x)} u(y)\, dy$ for $\forall B_r(x) \subset \Omega$ $\qquad w_n :=$ surface area of unit sphere.

then 2nd Mean-value property

Rem1: (i) $\equiv$ (ii) "by differentiale and integral."

Rem: 2: i) $u(x) = \frac{1}{w_n} \int_{|w|=1} u(x+rw)\, dS_w$ for $\forall B_r(x) \subset \Omega$

ii) $u(x) = \frac{n}{w_n} \int_{|z| \le 1} u(x+rz)\, dz$ for $\forall B_r(x) \subset \Omega$

proposition 1.1: If $u \in C(\bar{\Omega})$ satisfied MVP, then $u$ assumes its Max and Min on $\partial D$ unless $u \equiv C$.

proof: "Max" Set $\Sigma = \{x \in \Omega;\ u(x) = M = \max_{\Omega} u\} \subset \Omega$, it is obvious that it is closed

WTS: $\Sigma$ is open. For $\forall x_0 \in \Sigma$, take $\bar{B}_r(x_0) \subset \Omega$ for some $r > 0$

By MVP, we have $M = u(x_0) = \frac{n}{w_n r^n} \int_{B_r(x_0)} u(y)\, dy \le M \cdot \frac{n}{w_n r^n} \int_{B_r(x_0)} dy = M.$

Implies that $u = M$ in $B_r(x_0)$, Hence $\Sigma$ is both closed and open $\Rightarrow \Sigma = \Omega$ or $\emptyset$.

constant $u$ $\qquad$ no max $u$ inside.

Def: A function $u \in C^2(\Omega)$ is harmonic if $\Delta u = 0$ in $\Omega$.

Them 1.2: Let $u \in C^2(\Omega)$ be harmonic in $\Omega$, then $u$ satisfies MVP, take $\forall B_r(x) \subset \Omega$, for $\rho \in (0, r)$

$$0 = \int_{B_\rho} \Delta u(y)\, dy \overset{\text{div}}{=} \int_{\partial B_\rho} \frac{\partial u}{\partial \nu}\, dS = \rho^{n-1} \cdot \int_{|w|=1} \frac{\partial u}{\partial \rho}(x+\rho w)\, dS_w = \rho^{n-1}\, d/d\rho \underbrace{\int_{|w|=1} u(x+\rho w)\, dS_w}_{\equiv 0.}$$

Take $\forall B_r(x) \subset \Omega$, for $\rho \in (0, r)$

then integral over 0 to $r$, we have $\int_{|w|=1} u(x+rw)\, dS_w = \int_{|w|=1} u(x)\, dS_w = u(x) \cdot w_n$

or $u(x) = \frac{1}{w_n} \int_{|w|=1} u(x+rw)\, dS_w = \frac{1}{w_n r^{n-1}} \cdot \int_{\partial B_r} u(y)\, dS_y.$

Them 1.3: If $u \in C(\Omega)$ has MVP, then $u$ is smooth and harmonic in $\Omega$ ($\Delta u = 0$)

choose $\varphi \in C_0^\infty(B_1(0))$ with $\int_{B_1(0)} \varphi = 1$ and $\varphi(x) = \psi(|x|)$, ie

$w_n \int_0^1 r^{n-1} \psi(r)\, dr = 1$ — why equal?

we def $\varphi_\varepsilon(z) = \frac{1}{\varepsilon^n} \varphi\left(\frac{z}{\varepsilon}\right)$ for $\varepsilon > 0$. Now for $\forall x \in \Omega$, consider $\varepsilon < \operatorname{dist}(x, \partial\Omega)$. Then we have,

"Change of variable ?"

$$\int_\Omega u(y) \varphi_\varepsilon(y-x)\, dy = \int u(x+y) \varphi_\varepsilon(y)\, dy = \frac{1}{\varepsilon^n} \int_{|y|<\varepsilon} u(x+y) \varphi\left(\frac{y}{\varepsilon}\right) dy$$

$$= \int_{|y|<1} u(x+\varepsilon y) \varphi(y)\, dy$$

$$= \int_0^1 r^{n-1} dr \int_{\partial B_1(0)} u(x+\varepsilon r w) \varphi(rw)\, dS_w$$

$$= \int_0^1 \psi(r) r^{n-1} dr \int_{|w|=1} u(x+\varepsilon r w)\, dS_w$$

$$= u(x)\, w_n \cdot \int_0^1 \psi(r) r^{n-1} dr = u(x)$$

Hence we get $u(x) = (\varphi_\varepsilon * u)(x)$ for $\forall x \in \Omega_\varepsilon = \{y \in \Omega ; d(y, \partial\Omega) > \varepsilon\}$. Therefore $u$ is smooth ?

By them 1.2:

(why "0")

$$\int_{B_r(y)} \Delta u = r^{n-1} \frac{d}{dr} \int_{|w|=1} u(x+rw)\, dS_w = r^{n-1} \frac{d}{dr} (w_n u(x)) = 0$$ for $\forall B_r(x) \subset \Omega$ implies that $\Delta u = 0$ in $\Omega$

Direchlet prob:

$$\begin{cases} \Delta u = f \text{ in } \Omega & (f \in C(\Omega)) \\ u = \varphi \text{ on } \partial\Omega & (\varphi \in C(\partial\Omega)) \end{cases}$$

⚠: uniquness $\times$ hold $\to$ unbounded domain

consider $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$

"$\log|1| = 0$ + $\Delta u = 0$" $\left[ \begin{array}{l} u_{yy} = \frac{d^2}{d^2 y} \log\sqrt{x^2+y^2} = \frac{d}{dy} \log\sqrt{x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \\ \frac{d}{dx} \frac{x}{x^2+y^2} = \frac{x^2+y^2 - x \cdot 2x}{(x^2+y^2)^2} \\ u_{xx}\ \frac{d^2}{d^2 x} \log\sqrt{x^2+y^2} = \frac{d}{dx}\left(\frac{2x}{\sqrt{x^2+y^2}} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x^2+y^2}}\right) \quad \frac{y^2-x^2}{(x^2+y^2)^2} \end{array} \right.$

case I: $\Omega = \{x \in \mathbb{R}^n ; |x| > 1\}$. For $n=2$, $x \in \mathbb{R}^2$, $u(x) = \log|x| \to$ sol'n $\qquad r \to \infty$, $u \to \infty$

$n \geq 3$, $x \in \mathbb{R}^3$, $u(x) = |x|^{2-n} - 1 \longrightarrow$ sol'n $\qquad r \to \infty$ $u \to -1$.

$u$ is bounded

Case II $\Omega = \{x \in \mathbb{R}^n ; x_n > 0\}$. Then $u(x) = x_n$ is a non-trivial sol'n, which is unbounded.

Lemma 1.4: Suppose $u \in C(\bar{B}_R)$ is harmonic in $B_R = B_R(x_0)$

then $|Du(x_0)| \le \frac{n}{R} \max_{\bar{B}_R} |u|$

Q: "What is the differ betw $Du(x_0)$ and $D_{x_i} u$ ?

Proof: For simplicity we assume $u \in C^1(\bar{B}_R)$. Since $u$ is smooth, then $\Delta(D_{x_i} u) = 0$ ?

ie: $D_{x_i} u$ is also harmonic in $B_R \Rightarrow$ MVP

$$D_{x_i} u(x_0) = \frac{n}{W_n R^n} \int_{B_r(x_0)} D_{x_i} u(y)\, dy \overset{\text{How ?}}{=} \frac{n}{W_n R^n} \int_{\partial B_r(x_0)} u(y) V_i \, dS_y$$

"why equal ?"

$$\Rightarrow |D_{x_i} u(x_0)| \le \frac{n}{W_n R^n} \max_{\partial B_r} |u| \int_{\partial B_r(x_0)} V_i \, dS_y \le \frac{n}{W_n R^n} \max_{\partial B_r} |u| \cdot W_n R^{n-1} = \frac{n}{R} \max_{\partial B_r} |u|$$

Lemma 1.5: suppose $u \in C(\bar{B}_r)$ is a non-negative harmonic func in $B_r = B_r(x_0)$

then $|Du(x_0)| \le \frac{n}{R} u(x_0)$

proof: similar as 1.4, we have $|D_{x_i} u(x_0)| \le \frac{n}{W_n R^n} \int_{\partial B_r(x_0)} u(y)\, dS_y \overset{\text{"MVP"}}{=} \frac{n}{R} u(x_0)$

Cor 1.6: A harmonic f bounded from above and below is constant in $\mathbb{R}^n$.

proof: let $u \in f^n$ and $\Delta u = 0$, WTS: $u \equiv C$ if $u \ge 0$

consider i) for $u \ge 0$ in $B_R(x)$, then $|Du(x_0)| \le \frac{n}{R} u(x_0)$ by lemma 1.5, as $R \to \infty$, $|Du(x_0)| \le 0 \Rightarrow Du(x_0) \equiv 0$ for $\forall x \in \mathbb{R}^n$, then

ii) suppose $u < 0$ in $B_r(x)$, then $|-Du(x_0)| = |Du(x_0)| \le \frac{n}{R} \cdot -u(x_0)$, same ie $\Rightarrow Du(x_0) \equiv 0 \Rightarrow u \equiv C$

How to use bounded ? Is that the ie $\int_{B_r(x_0)} Du(x) \le \int_{B_r(x_0)} \frac{n}{R} u(x_0)\, d$ ?

Prop 1.7: Suppose $u \in C(\bar{B}_R)$ is harmonic in $B_R = B_R(x_0)$

The for $\forall$ multi-index $a$ with $|a| = m$

$$|D^a u(x_0)| \le \frac{n^m e^{m-1} m!}{R^m} \max_{B_R} |u|$$

Functional-Analytic $\Longleftrightarrow$ non-linear functional.

Harmonic analytic

proof by induction: i) for $m=1$, $|Du(x_0)| \le \frac{n}{R} \max_{\overline{B_r}} |u|$ by lemma 1.4 (Checked)

ii) suppose the statement holds for $m$, consider $P(m+1)$: For $0<\theta<1$, def $r=(1-\theta)R \in (0,R)$

$$|D^{m+1}u(x_0)| \le \frac{n}{r} \max_{\overline{B_r}} |D^m u| \le \frac{n}{r} \cdot \frac{n^m \cdot e^{m-1} m!}{(R-r)^m} \max_{B_R} |u| = \frac{n^{m+1} e^{m-1} m!}{R^{m+1} \theta^m (1-\theta)} \max_{\overline{B_R}} |u|$$

# Chapter I: Genesis of Fourier analysis

## Vibrating string

i) simple harmonic motion: $y(t)$ denoted as distance $\to t$

$F = -k \cdot y(t) = my''(t)$ with $c = \sqrt{k/m}$ $\Rightarrow y''(t) + c^2 y(t) = 0$ (1)

general sol'n: $y(t) = a\cos ct + b\sin ct$ $a, b$ unknown

we need two IVP $\to$ solve $a, b$ (ie $y(0)$ and $y'(0)$)

"$A = \sqrt{a^2+b^2}$" and $c$ = natural frequency" "$\varphi$: phase"

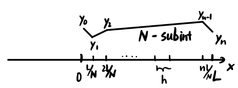
So, $y(0) = a$ ; $y'(0) = cb \Rightarrow b = \frac{y'(0)}{c}$ $\Rightarrow$ $y(t) = y(0)\cos ct + \frac{y'(0)}{c}\sin ct. = A\cos(ct - \varphi)$ where $A > 0$ and $\varphi \in R$

2) standing wave: $y = u(x,t) = \overset{u(x,0)}{\varphi(x)} \cdot \psi(t)$ "Separation of variable"

travelling wave: There is initial profile $F(x) = u(x,0)$ when $t=0 \Rightarrow u(x,t) = F(x-ct)$ "$c>0$" moving $\to$ right

$u(x,t) = F(x+ct)$ "$c>0$" moving $\to$ left.

## Sec 1.1: Derivation of wave equation




Set $y_n(t) = u(x_n, t)$ and note $x_{n+1} - x_n = h = L/N$

Suppose the tension $= (\frac{\tau}{h})(y_{n+1} - y_n)$ from right and $(\frac{\tau}{h})(y_{n-1} - y_n)$ from right

By $N_2$: $F = ma = (\frac{\tau}{h}) \cdot (y_{n+1} + y_{n-1} - 2y_n) = \rho h \cdot y_n''(t)$

$= (\frac{\tau}{h}) \cdot (u(x_n + h, t) + u(x_n - h, t) - 2u(x_n, t))$

Also for $\forall$ func $\in R^2(x)$, $\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} \xrightarrow[h \to 0]{} F''(x)$

ie: $\dfrac{\frac{F(x+h) - F(x)}{h} - \frac{F(x) - F(x-h)}{h}}{h}$

then we have $\rho \frac{\partial^2 u}{\partial t^2} = \tau \frac{\partial^2 u}{\partial x}$ $\quad \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ with $c = \sqrt{\tau/\rho}$ velocity

1-D wave equation

## Scaling: "change of units"

Think $x = aX$, for $0 \le x \le L \Rightarrow 0 \le X \le \frac{L}{a}$; $t = bT$ for $a, b > 0$.

if we set $U(X,T) = u(x,t)$, then $\frac{\partial U}{\partial X} = a\frac{\partial u}{\partial x}$ and $\frac{\partial U}{\partial T} = b\frac{\partial u}{\partial t}$

$$\frac{\partial^2 U}{\partial X^2} = a^2\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial^2 U}{\partial T^2} = b^2\frac{\partial u}{\partial t}$$

$$\frac{1}{c^2}\cdot\frac{1}{b^2}\frac{\partial^2 U}{\partial^2 T^2} = a^2\frac{\partial^2 U}{\partial X^2} \Rightarrow \text{if } a^2b^2c^2 = 1, \text{ then } \frac{\partial u}{\partial T} = \frac{\partial u}{\partial X}$$

if taking $a = L/\pi$ and $b = L/c\pi \Rightarrow \frac{L^2}{\pi^2} = 1$ " Transfrome

$0 \le x \le L \to 0 \le x \le \pi$

## Sec1.2: Sol'n to wave equation

2 ways: using traveling waves / superposition of standing wave

i) assume $c = 1$, $L = \pi$ so that $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ on $0 \le x \le \pi$, with $t \ge 0$

By traveling wave, $u(x,t) = F(x+t)$ and $u(x,t) = F(x-t)$ are sol'ns

← left → right

ie: when $t = 0$, $F(x+t) = F(x)$ and $F(x-t) = F(x)$

Since wave equations are linear, then for $u, v \to$ sol'ns, we have $u(x,t) = F(x+t) + G(x-t)$

suppose $u \in R^2(x) / R^2(t)$, suppose $\xi = x+t$ and $\eta = x-t$ and $V(\xi,\eta) = u(x,t)$

The change of variable $\Rightarrow \frac{\partial^2 v}{\partial \xi \partial \eta} = 0$ "Why" $\Rightarrow$ Integration $V(\xi,\eta) = F(\xi) + G(\eta)$

$u(x,t) = F(x+t) + G(x-t)$

Condition: $0 \le x \le \pi$, $u(x,0) = f(x)$ and $u(0,t) = u(\pi,t) = 0$ for $t \ge 0$

extend $f$ to all by $R$ making it odd $[-\pi,\pi]$ and then periodic $2\pi$.

set $u(x,t) = u(x,-t)$ whenever $t < 0$

Therefore $u(x,t) = F(x+t) + G(x-t)$ by setting $t = 0$,

$\Rightarrow F(x) + G(x) = f(x)$

Given IVP: $\frac{\partial u}{\partial t}(x,0) = g(x)$, clearly $g(0) = g(\pi) = 0$. samilarly procedure → extend $g$

$$\begin{cases} F(x)+G(x)=f(x) \\ F'(x)-G'(x)=g(x) \end{cases} \Rightarrow f'(x)=F'(x)+G'(x)$$

$$2F'(x)=f'(x)+g(x) \Rightarrow F'(x)=\frac{f'(x)+g(x)}{2} \Rightarrow F(x)=\frac{1}{2}\left[f(x)+\int_0^x g(y)\,dy\right]+C_1 \qquad 1)\ F(x)+G(x)=f(x) \Rightarrow C_1+C_2=0$$

$$2G'(x)=f'(x)-g(x) \Rightarrow G'(x)=\frac{f'(x)-g(x)}{2} \Rightarrow G(x)=\frac{1}{2}\left[f(x)-\int_0^x g(y)\,dy\right]+C_2$$

$$u(x,t)=\frac{1}{2}\left[f(x+t)+f(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy \qquad \text{" d'Alembear formula"}$$

" the extansion we choose guaratee that the string has fixed end $u(0,t)=u(\pi,t)=0$ for all $t$

setting $\bar{u}(x,t)=u(x,-t)$, we will get same

---

Method 2: Superposition of standing wave

$$\frac{d^2u}{dx^2}=\frac{d^2u}{dt^2} \Rightarrow \varphi(x)\,\psi''(t)=\varphi''(x)\cdot\psi(t) \Rightarrow \frac{\psi''(t)}{\psi(t)}=\frac{\varphi''(x)}{\varphi(x)}=\lambda_n$$

then we can set $\lambda_n$ be e-value $\rightarrow$ prob $\begin{cases} \psi''(t)-\lambda\psi(t)=0 \\ \varphi''(x)-\lambda\varphi(x)=0 \end{cases}$

Case I: $\lambda_n=-\beta^2$ for $\beta>0$ we made $\psi(x)$ oscillating,

then we have $\psi(t)=A\cos\beta t+B\sin\beta t$

$\varphi(x)=\tilde{A}\cos\beta x+\tilde{B}\sin\beta x$

consider bdy: $\varphi(0)=0$ and $\varphi(\pi)=0 \Rightarrow \tilde{A}=0$ and if $\tilde{B}\neq 0$. $\beta_n=n$

If $\beta=0$, the sol'n vanishes identically, then $\beta=n\ (1,2,\cdots)$

$u_n(x,t)=(A_n\cos nt+B_n\sin nt)\cdot\sin nx$

the graph $\rightarrow$ n=1 and n=2 :

Since $u(x,t)$ is linear: we have $u(x,t)=\sum_{n=1}^{\infty}(A_n\cos nt+B_n\sin nt)\sin nx \quad (4)$

to acquire specific sol'n, we need $f(0)=f(\pi)=0$ and $u(x,0)=f(x)$, hence

$$u(x,0)=\sum_{n=1}^{\infty}A_n\cdot\sin nx=f(x) \Rightarrow \int_0^{\pi} f(x)\sin mx\,dx=\sum_{n=1}^{\infty}A_n\int_0^{\pi}\sin nx\cdot\sin mx\,dx$$

$$=\sum_{n=1}^{\infty}A_n\int_0^{\pi}\frac{\cos(n-m)x-\cos(n+m)x}{2}\,dx$$

we have $A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$

$$= \sum_{n=1}^{\infty} A_n \cdot \Big[\underbrace{\tfrac{-2}{(n-m)}\sin(n-m)x}_{=0}\Big]\Big|_0^{\pi} + \Big[\underbrace{\tfrac{2}{n+m}\sin(n+m)x}_{=0}\Big]_0^{\pi} \quad \text{if } m \neq n$$

then $= \frac{\pi}{2}$ $(m=n)$

$$g(x) = \sum_{m=0}^{\infty} A_m' \cos mx \text{ and } F(x) = \sum_{m=1}^{\infty} A_m \sin mx + \sum_{m=0}^{\infty} A_m' \cos mx, \text{ apply } e^{ix} = \cos x + i\sin x$$

$$= \sum_{m=1}^{\infty} A_m \sin mx + A_m' \cos mx + A_0'$$

$$? = \sum_{m=-\infty}^{\infty} a_m \cdot e^{imx}$$

$$\text{Similarly } \int_{-\pi}^{\pi} F(x)\cdot e^{-inx}\,dx = \sum_{m=-\infty}^{\infty} a_m \cdot \underbrace{\int_{-\pi}^{\pi} e^{imx}\cdot e^{inx}\,dx}_{=2\pi \text{ if } n=m}$$

$$\Rightarrow a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)\cdot e^{-inx}\,dx.$$

Q: Given + reasonable func $F$ on $[-\pi, \pi]$ with $F(x) = \sum_{m=-\infty}^{\infty} a_m \cdot e^{imx}$ ? (7).

---

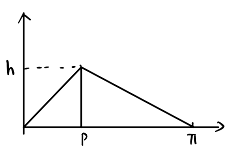
We need two IVP → find equation: $u(x,0) = f(x)$ and $\frac{du}{dt}(x,0) = g(x)$.

thus $u(x,0) = f(x) = \sum_{m=1}^{\infty} A_m \cdot \sin mt$ ; $g(x) = u_t(x,0) = \sum_{m=1}^{\infty} m\cdot B_m \cdot \sin mx.$

---

Sec 1.3: the plucked string

$$f(x) = \begin{cases} \frac{xh}{p} & \text{for } 0 \le x \le p \\ \frac{h(\pi - x)}{\pi - p} & \text{for } p \le x \le \pi \end{cases}$$




$g(x) \equiv 0 = u_t(x,0)$

$$f(x) = \sum_{m=1}^{\infty} A_m \sin mx \Rightarrow A_m = ① \frac{2}{\pi}\int_0^p \frac{xh}{p}\cdot \sin nx = \frac{2}{\pi}\cdot\Big[-\frac{1}{n}\cdot \cos nx \cdot \frac{xh}{p}\Big|_0^p + \frac{1}{n}\int_0^p \cos nx \cdot \frac{h}{p}\Big]$$

$$\frac{2}{\pi}\Big[-\frac{1}{n}\frac{h}{p}\cos np + \frac{1}{n^2}\cdot\frac{h}{p}\sin np\Big)$$

②

$$\frac{2}{\pi}\int_p^{\pi} \frac{h(\pi - x)}{\pi - p}\cdot \sin nx = \frac{2}{\pi}\cdot\Big[-\frac{1}{n}\Big(\frac{h(\pi - x)}{\pi - p}\Big)\cos nx\Big|_p^{\pi} + \frac{1}{n}\int_p^{\pi} -\frac{h}{\pi - p}\cos nx\,dx\Big]$$

$$= \frac{2}{\pi}\cdot\Big[-\frac{1}{n}\cdot\frac{h\cdot(\pi - p)}{\pi - p}\cdot\cos np + \frac{h}{n^2(\pi - p)}\sin np$$

$$= \frac{2}{\pi}\cdot\Big(\frac{h}{n^2 p} + \frac{h}{n^2(\pi - p)}\Big)\sin np$$

$$= \frac{2\,(h(\pi - p) + h(p))\sin np}{\pi\cdot n^2(p)(\pi - p)} = \frac{2h}{n^2 p(\pi - p)}\cdot \sin np$$

Thus: $u(x,t) = \sum_{n=1}^{\infty} \frac{2h}{n^2} \cdot \frac{1}{p(\pi - h)} \cdot \sin np \cdot \cos nt \cdot \sin nt.$

(8)

(9) $u(x,t) = \dfrac{f(x+t) + f(x-t)}{2}$

? $\cos v \cdot \sin u = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$ ?

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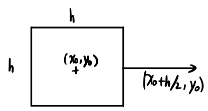
## 2. Diffusion equation:




$H(t) = 6\iint u(x,y,t)\,dx\,dy$

since Area $= h^2$

heat flow into $S$: $\frac{\partial H}{\partial t} = 6\iint_S \frac{\partial u}{\partial t}\,dx\,dy \approx 6h^2 \frac{\partial u}{\partial t}(x,y,t)$

heat flow $\to$ right: $-Kh \frac{\partial u}{\partial x}(x_0 + h/2, y_0, t)$

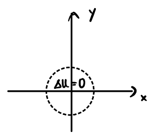
from 4 direction flow into $S$: we have $\frac{6}{K}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Sec 2.2: Steady state (ie: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$)

Laplician: $\Delta u = 0$

and sol'n is called harmonic equation

consider $D: \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$




bdy $\equiv \{x^2 + y^2 = 1\}$

$0 \le \theta < 2\pi$

polar coordinate: $D = \{(r,\theta) : 0 \le r < 1\}$ and $C = \{(r,\theta) : r = 1\}$.

Def: Dirichlet probs are $u = f$ on $C$ and $\Delta u = 0$

usually: polar coordinate easy to express for bdy: example $u(1,\theta) = f(\theta)$

$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

$x = r\cos\theta \Rightarrow dx = \cos\theta\, dr$ $\qquad \frac{\partial x}{\partial r} = \cos\theta$ and $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$ $\qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$

$= \cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y}$

$\frac{\partial u^2}{\partial r^2} = \cos\theta\left(\overset{\partial/\partial r}{\frac{\partial}{\partial x}\cdot\frac{\partial u}{\partial x}}\cdot\frac{\partial x}{\partial r} + \overset{\partial/\partial r}{\frac{\partial}{\partial y}\frac{\partial u}{\partial x}}\cdot\frac{\partial y}{\partial r}\right) + \sin\theta\left(\frac{\partial}{\partial x}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial r}\right)$

$= \cos^2\theta \frac{\partial^2 u}{\partial x} + 2\cos\theta\cdot\sin\theta\cdot\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

set $\Delta u = 0$ and $* r^2 \Rightarrow \quad r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} = -\frac{\partial^2 u}{\partial \theta^2}$

Separation of vari $\Rightarrow u(r,\theta) = F(r)\cdot G(\theta)$

$$\frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} = \lambda$$

$$\begin{cases} G''(\theta) + \lambda G(\theta) = 0 \quad \Rightarrow \text{" } G \text{ must be periodic" } \lambda \geq 0 \\ r^2F''(r) + rF'(r) - \lambda F(\lambda) = 0 \end{cases}$$

for $\lambda = m^2 > 0$

Euler identity

$G(\theta) = \tilde{A}\cos m\theta + \tilde{B}\sin m\theta \equiv G(\theta) = Ae^{im\theta} + Be^{-im\theta}$

iel: With $\lambda = m^2$ and $m \neq 0$, two special sol'n $\to$ F are $F(r) = r^m$ and $F(r) = r^{-m}$.

$m = 0$, then $r^2F''(r) + rF'(r) = 0$ $\quad$ SS: $F(r) = 1$ and $F(r) = \log(r)$

Cases: If $m > 0$, and $r \to 0$, $r^{-m} \to \infty$
If $m = 0$, and $r \to 0$, $\log(r) \to -\infty$
$\Big] \to$ unbounded for $F(r)\,G(\theta)$

therefore, by intuition, $F(r) = 1$ and $F(r) = r^m$ is suitble

$U_m(r,\theta) = r^{|m|}\cdot e^{im\theta} \quad (m \in \mathbb{Z})$

$$\downarrow$$

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|}\cdot e^{im\theta}$$

If the sol'n is general: $u(1,\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} = f(\theta)$

Q: Given $\forall$ func $f$ on $[0, 2\pi]$ with $f(0) = f(2\pi)$, can we find $a_m$? ?

Problems: #5 Verify that (i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$

(ii) $\frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n=m \end{cases}$

(iii) $\frac{1}{\pi} \cdot \int_{-\pi}^{\pi} \sin nx \cdot \sin mx \, dx = \begin{cases} 0 & \text{if } n\neq m \\ 1 & \text{if } n=m \end{cases}$

(iv) $\int_{-\pi}^{\pi} \sin nx \cdot \cos mx \, dx = 0$ for $\forall n, m$.

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\,dx = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$

if $n=0$, clearly $\frac{1}{2\pi} F(x) = 2\pi \cdot \frac{1}{2\pi} = 1$

if $n \neq 0$, then $\int_{-\pi}^{\pi} e^{inx}dx = \int_{-\pi}^{\pi} i\sin nx + \cos nx \, dx = \therefore \left[-\frac{\cos nx}{n}\Big|_{-\pi}^{\pi}\right] + \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} = 0.$

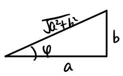
(iii) $\sin nx \cdot \sin mx = \frac{\cos(n-m)x - \cos(n+m)x}{2}$ if $n \neq m$. $\Rightarrow -\frac{\sin(n-m)x}{2(n-m)}\Big|_{-\pi}^{\pi} = 0 + 0 = 0$

if $n=m$, then $\int_{-\pi}^{\pi}(\sin nx)^2 = \int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2} = \int_{-\pi}^{\pi} \frac{1}{2} + 0 = 2\pi \cdot \frac{1}{2} = \pi$

#7: Show that if $a, b \in \mathbb{R}$, $a\cos ct + b \sin ct = A\cos(ct - \varphi)$

$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}$ $\sin\varphi = \frac{b}{\sqrt{a^2+b^2}}$

$A\cos(ct-\varphi) = A \cdot (\cos ct \cdot \cos\varphi + \sin ct \sin\varphi)$




$= \sqrt{a^2+b^2} \cdot \frac{a}{\sqrt{a^2+b^2}} \cos ct + \cdots)$ checked.

#10: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$

$x = r\cos\theta$ $y = r\sin\theta$

$$\begin{cases} \frac{\partial x}{\partial r} = \cos\theta & \frac{\partial y}{\partial r} = \sin\theta \\ \frac{\partial x}{\partial \theta} = -r\sin\theta, & \frac{\partial y}{\partial\theta} = r\cos\theta \end{cases}$$

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$

$\underline{\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta}$

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$

$= \frac{\partial u}{\partial x}\cdot(-r\sin\theta) + \frac{\partial u}{\partial y}\cdot r\cos\theta$

$\left(\frac{\partial u}{\partial r}\right)^2 = \frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial x}\cdot\cos^2\theta + 2\frac{\partial^2 u}{\partial y\partial x}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\cdot\sin^2\theta.$

$\Downarrow$

$\left(\frac{\partial u}{\partial\theta}\right)^2 = \frac{\partial^2 u}{\partial^2 x} r^2\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cdot r^2\cos^2\theta - 2r^2\frac{\partial u}{\partial x\partial y}\cos\theta\sin\theta$

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \Longleftarrow \quad \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta} = \frac{\partial^2 u}{\partial^2 x}\sin^2\theta$$

#11: $r^2 F''(r) + r F'(r) - n^2 F(r) = 0$ " $r>0$ "

Since the system is linear, let $F(r) = g(r)\cdot r^n$

then $r^2 \cdot F''(r) = r^2 \cdot \frac{d}{dr}\left(\frac{dg}{dr}\cdot r^n + ng\cdot r^{n-1}\right)$

$= r^2\cdot\left(g''(r)\cdot r^n + n\cdot r^{n-1}\cdot g'(r) + n\cdot(n-1)\cdot g\cdot r^{n-2} + n\cdot g'(r)\cdot r^{n-1}\right)$

$= r^{n+2}\cdot g''(r) + 2nr^{n+1}\cdot g'(r) + r^n\cdot n(n-1)\cdot g$

$r\cdot F'(r) = g'(r)\cdot r^{n+1} + r^n\cdot n\cdot g$ $\qquad r^{n+2}\cdot g''(r) + (2n+1)\cdot r^{n+1}\cdot g'(r) + r^n(n^2-n+n-n^2)\,g(r)$

$-n^2\cdot F(r) = -n^2\cdot r^n g$ $\qquad 0 = r^{n+2}\cdot g''(r) + (2n+1)\, r^{n+1}\cdot g'(r)$

$0 = r\cdot g'(r) + (2n+1)\, g''(r)$

Sec 2.1: Examples

$$a_n = \frac{1}{L}\int_0^L f(x)\cdot e^{-2\pi i n x/L}\,dx, \text{ for } n\in\mathbb{Z}$$

$f(\theta) \to F(e^{i\theta})$ on unit circle

Q1: If $f$ is integeral, [differentiable, conti] then $F(e^{i\theta})$ is integrable.

Main Def:

If $f$ is integrable function given on an $[a,b]$ of length $L$ (ie: $b-a=L$), then $n^{th}$ Fourier coeficient.

coeff: $\hat{f}(n) = \frac{1}{L}\int_a^b f(x)\cdot e^{-2\pi i n x/L}\,dx$, $n\in\mathbb{Z}$

$(\cos(\frac{2n\pi x}{L}) - i\sin(\frac{2n\pi x}{L}))$

$= \frac{1}{L}\int_a^b f(x)\cdot(\cos(-\frac{2n\pi x}{L}) + i\sin(-\frac{2n\pi x}{L}))$

fourier series: $\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{2\pi i n x/L}$. (不知道 convergence ?)

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x/L}$$

ex1: if $f\in R(x)$ on $[-\pi,\pi]$, then $\hat{f}(n) = a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cdot e^{-in\theta}d\theta$, $n\in\mathbb{Z}$

and $f \sim \sum_{-\infty}^{\infty} a_n\cdot e^{in\theta}$

Consider $g:[0,1]\to\mathbb{R}$, $\hat{g}(n) = a_n = \int_0^1 g(x)\cdot e^{-2\pi i n x}dx$ and $g(x) \sim \sum_{-\infty}^{\infty} a_n e^{2\pi i n x}$

$\{$Fourier series$\} \subset \{$trigonmetry series$\}$: $\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/L}$ where $c_n\in\mathbb{C}$

If $c_n = 0$ for $|n|>>1$, then $\sum_{-N}^{N} c_n e^{2\pi i n x/L}$ is finite. $\deg(F) \equiv |n|$ for which $|c_n|\neq 0$.

Def: $N^{th}$ partial sum: $N>0$, $S_N(f)(x) = \sum_{n=-N}^{N}\hat{f}(n)e^{2\pi i n x/L}$

**Prob ! : In what sense does $S_N(f)$ converge to $f$ as $N \to \infty$**

Ex 1: $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta \cdot e^{-in\theta} d\theta = \frac{1}{2\pi}\left( \left[-\theta \cdot \frac{e^{-in\theta}}{in}\right]\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{e^{-in\theta}}{in} d\theta \right)$ $= 0$

Given $f(\theta) = \theta$ for $[-\pi, \pi]$

$= \frac{1}{2\pi} \cdot \left( -\pi \cdot \frac{e^{-in\pi}}{in} + (-\pi) \cdot \frac{e^{in\pi}}{in} \right)$

$= \frac{1}{2\pi} \cdot \left( -\pi \cdot (\cos(-n\pi) + i\sin(-n\pi)) \right)$ $= 0$

If $n = 0$, then

$= \frac{1}{2\pi in}\left( -\pi \cdot (-1)^n + (-\pi) \cdot (-1)^n \right)$

$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \theta \, d\theta$

$= \frac{(-1)^{n+1}}{in}$

$= \frac{1}{2\pi} \cdot \left(\frac{1}{2}\theta^2\right)_{-\pi}^{\pi} = 0.$

Hence $f(\theta) \sim \sum_{n \neq 0} \frac{(-1)^{n+1}}{in} e^{in\theta} = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin n\theta}{n}$

$\frac{\cos n\theta + i \sin n\theta}{in}$

$= \frac{\cos n\theta}{in} + \frac{\sin n\theta}{n}$

Since $\frac{\cos(-n\theta)}{i(-n)} = -\frac{\cos(n\theta)}{in}$ (Sum $= 0$)

Ex 2: $f(\theta) = (\pi - \theta)^2/4$ for $0 \leq \theta \leq 2\pi$

$f(\theta) \sim \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2}$

Ex 3: $f(\theta) = \frac{\pi}{\sin \pi a} e^{i(\pi - \theta)a}$ in $[0, 2\pi]$ is

$f(\theta) \sim \sum_{-\infty}^{\infty} \frac{e^{in\theta}}{n+a}$

---

$D_N(x) = \sum_{-N}^{N} e^{inx} \equiv N^{th}$ Direchlet kernal

Note: $a_n = \begin{cases} 1 & \text{if } n < N \\ 0 & \text{if } n > N \end{cases}$

Formula: $D_N(x) = \frac{\sin(N + 1/2)x}{\sin(x/2)}$

proof: let $w = e^{ix}$, $\sum_{n=0}^{N} w^n = \frac{1-w^{N+1}}{1-w}$ and $\sum_{n=-N}^{-1} w^n = \frac{w^{-N}-1}{1-w}$ .

Hence $\sum_{-N}^{N} w^n = \frac{1-w^{N+1}+w^{-N}-1}{1-w} \overset{\cdot \frac{1}{w^{\frac{1}{2}}}}{=\!=} \frac{w^{-N-\frac{1}{2}} - w^{N+\frac{1}{2}}}{w^{-\frac{1}{2}} - w^{\frac{1}{2}}} = \frac{-i\sin(N+\frac{1}{2})x + i\sin(N+\frac{1}{2})x}{e^{-\frac{1}{2}ix} - e^{\frac{1}{2}ix}}$

$= \frac{}{i\sin(-\frac{1}{2}x) - i\sin(\frac{1}{2}x)}$

$= \frac{-2i(\sin(N+\frac{1}{2})x}{-2i\sin(x/2)}$

Ex: The function $P_r(\theta)$ is called Poison kernal $\theta \in [-\pi, \pi]$ and $0 \le r < 1$. by absolutely and unif-convergent

$P_r(\theta) := \sum_{-\infty}^{\infty} r^{|n|} \cdot (i\sin(n\theta) + \cos(n\theta))$

$\hat{P}_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cdot \sum_{-\infty}^{\infty} r^{|n|} \cdot e^{in\theta} \cdot e^{-in\theta}$

$= \sum_{-\infty}^{\infty} r^{|n|} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} \cdot \overset{\nearrow 1}{1}\, d\theta \quad ? = r^{|N|}$

consider $w = re^{i\theta}$, $P_r(\theta) = \sum_{n=0}^{\infty} w^n + \sum_{n=1}^{\infty} \bar{w}^n$ $\quad w = r\cdot(i\sin\theta + \cos\theta) \Rightarrow \bar{w} = r\cdot(-i\sin\theta + \cos\theta) = r\cdot(i\sin(-\theta) + \cos(-\theta)) = r\cdot e^{-i\theta}$.

$P_r(\theta) = \frac{1}{1-w} + \frac{\bar{w}}{1-\bar{w}} = \frac{1-\bar{w}+\bar{w}-\bar{w}w}{(1-w)(1-\bar{w})} = \frac{1-|w|^2}{(1-w)^2} = \frac{1-r^2}{1-2r\cos\theta + r^2}$

$|w| = \sqrt{\quad} = r$.

$w = r\cos\theta + ir\sin\theta$

---

Given $f \in R(x)$, $f: [-\pi, \pi] \to \mathbb{R}$, Is $\lim_{N\to\infty} S_N(f)(\theta) = f(\theta)$ for $\forall \theta$ ?

Fact: $f$ is conti + period $\Rightarrow \lim_{N\to\infty} S_N(f)(\theta) = f(\theta)$

and if $f$ is twice conti-differ $\Rightarrow \lim_{N\to\infty} S_N(f)(\theta) = f(\theta)$ uniformlly.

---

Sec 2.2: Uniquness of fourier series

$Q_0$: If $f, g$ have same $a_n \overset{?}{\Longrightarrow} f \equiv g$

take $h := f - g$, if $a_{hn} \equiv 0$, then $h \equiv 0$ ?

Ans: Integration may cause prob...

Them 2.1: Suppose $f \in I(x)$ on the circle with $\hat{f}_n = 0$ for all $n \in \mathbb{Z}$, then $f(\theta_0) = 0$ whenever $f$ is conti at $\theta_0$

ie: $E = \{$disconti pt on the circle$\}$ $f: [-\pi, \pi] \to \mathbb{R} \xrightarrow{\text{vanish}} f: E \to \mathbb{R}$

proof: suppose $f \in R(x)$ and WLOG, $f: [-\pi, \pi]$, that $\theta_0 = 0$, and $f(\theta_0) = f(0) > 0$

ie: construct a family of $\{P_k\}$ that peak at $0$, and $\int P_k(\theta) \cdot f(\theta)\, d\theta \to \infty$ as $k \to \infty$.

Since $f$ is conti at $0$, we can choose $\left(0 \le \delta < \pi/2\right)$, st $f(\theta) > f(0)/2$ when ever $|\theta| < \delta$ (?)

Let $p(\theta) = \varepsilon + \cos\theta$ where $\varepsilon > 0$ and $\varepsilon << 1$, s.t $|p(\theta)| < 1 - \varepsilon/2$ whenever $\left(\delta \le |\theta| \le \pi.\right)$

then choose a $\eta > 0$ with $\eta < \delta$, so that $|p(\theta)| \ge 1 + \varepsilon/2$ for $|\theta| < \eta < \delta$

Finally, let $P_k(\theta) := [p(\theta)]^k$, and $|f(\theta)| \le B$ for all $(\theta)$ "By integrable"

since $\hat{f}(n) = 0$ for all $n$, we have $\int_{-\pi}^{\pi} f(\theta) P_k(\theta)\, d\theta = 0$ for all $k$

① However, we have $\left|\int_{\delta \le |\theta|} f(\theta) \cdot P_k(\theta)\, d\theta\right| \le 2\pi \cdot B \cdot (1 - \varepsilon/2)^k$

② ?: $p(\theta)$ and $f(\theta)$ are non-negative, whenever $|\theta| > \delta$

thus $\int_{\eta \le |\theta| < \delta} f(\theta) \cdot P_k(\theta)\, d\theta \ge 0$.

③ and $\int_{|\theta| < \eta} f(\theta)\, P_k(\theta)\, d\theta \ge 2\eta \cdot \frac{f(0)}{2} (1 + \varepsilon/2)^k$

as $\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f \cdot e^{-inx} = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi}$

Hence $\int P_k(\theta) \cdot f(\theta)\, d\theta = ① + ② + ③ \longrightarrow \infty$ as $k \to \infty$ (Contridicted to the assumption that $\int P_k(\theta) \cdot f(\theta)\, d\theta = 0$ )

"Why?"

In General, we write $f(\theta) = u(\theta) + i v(\theta)$, if $\bar{f}(\theta) = \overline{f(\theta)}$, then $u(\theta) = \frac{f(\theta) + \bar{f}(\theta)}{2}$ and $v(\theta) = \frac{f(\theta) - \bar{f}(\theta)}{2i}$

Since $\hat{\bar{f}}(n) = \overline{\hat{f}(n)} = 0$, we conclude that $f = 0$ at its pts of continuities

Cor 2.2: If $f$ is conti on the circle and $\hat{f}(n) \equiv 0$ for all $n \in \mathbb{Z}$, then $f = 0$.

Cor 2.3: If $f$ is conti on circle and $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$ "Abso-conver" "有用的评意"

then Fourier series unif-conver $\to f$, that is

$$\lim_{N\to\infty} S_N(f)(\theta) = f(\theta) \text{ "uniformly in } \theta\text{."}$$

$$= \lim_{N\to\infty} \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x/L} = f(\theta)$$

proof: Since $\sum_{-\infty}^{\infty} |\hat{f}(N)| < \infty$, convergent abso $\Rightarrow$ unif-convergent

Def: $g(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e^{in\theta} = \lim_{N\to\infty} \sum_{-N}^{N} \hat{f}(n) \cdot e^{in\theta}$ is conti on the circle. "interchange the $\infty \Sigma$ with the integral."

then def: $h(\theta) = f(\theta) - g(\theta)$ is conti and $\hat{h}(n) \equiv 0$ for all $n \in \mathbb{Z}$

By cor 2.1, $h(\theta) = 0$ on circle $\Rightarrow f = g$ as desired

Intro: "$O$"

EX1: "$\hat{f}(n) = O(1/|n|^2)$ as $|n| \to \infty$" $\equiv$ "$\exists C > 0$, s.t $|\hat{f}(n)| \le C/|n|^2$ for all large $|n|$."

EX2: "$f(x) = O(g(x))$ as $x \to a$" $\equiv$ "$\exists C > 0$, s.t $|f(x)| \le C \cdot |g(x)|$ as $x \to a$".

EX3: "$f(x) = O(1)$" $\equiv$ "$|f(x)| \le C$." bounded.

Cor 2.4: Suppose $f \in R^2(x)$ on the circle, then $\hat{f}(n) = O(1/n^2)$ as $|n| \to \infty$ that is $\exists C > 0$, s.t $|\hat{f}(n)| \le C/|n|^2$ as $|n| \to \infty$.

so that Fourier series of $f$ converges (uni + abso)

Proof: If $n \neq 0$ $\quad 2\pi \hat{f}(n) = \int_0^{2\pi} f(\theta) \cdot e^{-in\theta} d\theta$

$$= \left[f(\theta) \cdot \frac{-e^{-in\theta}}{in}\right]_0^{2\pi} + \frac{1}{in} \int_0^{2\pi} f'(\theta) \cdot (e^{-in\theta}) d\theta$$

$$= \frac{1}{in} \int_0^{2\pi} f'(\theta) \cdot e^{-in\theta} d\theta$$

$$= \frac{1}{in}\Big[f'(\theta)\cdot\frac{-e^{-in\theta}}{in}\Big]_0^{2\pi} + \frac{1}{(in)^2}\int_0^{2\pi} f''(\theta)\cdot e^{-in\theta}\,d\theta$$

$$= \frac{-1}{n^2}\cdot\int_0^{2\pi} f''(\theta)\cdot e^{-in\theta}\,d\theta.$$

$f\in R^2(x)$ guaranteed.

$$2\pi\,\hat f(n)\cdot n^2 \le \Big|\int_0^{2\pi} f''(\theta)\cdot e^{-in\theta}\,d\theta\Big| \le \int_0^{2\pi}|f''(\theta)|\,d\theta \le C = 2\pi B\ (B = \text{bound for } f'')$$

"?"

Conclusion: if $f$ is differ and $f\sim\sum a_n e^{in\theta}$, then $f'\sim\sum a_n in\cdot e^{in\theta}$.

2-differ $\Rightarrow f''\sim\sum a_n (in)^2 e^{in\theta}$. ie: smoothness ↑ $\Rightarrow$ decay ↑.

Stronger version:

Ex1: If $f\in R^1(x)$, $\sum a_n\cdot e^{in\theta}$ convergent abso ✓.

!!: Hölder condition of order $a$, with $a > 1/2$.

$$\sup_\theta |f(\theta+t)-f(\theta)| \le A|t|^a \text{ for all } t.$$

Notation: $f\in C^k$ if $f$ is $k$ times conti-differ

then $f\in C^k$ / $f$ satisfied Hölder-cond $\Rightarrow$ describe smoothness.

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3: Convolution "卷积"

好玩意

⚠: Give $f, g$ on $\mathbb{R}$, $2\pi$-period. def: $f*g$ on $[-\pi,\pi]$ by $(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot g(x-y)\,dy$.

"weighted average" $\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\cdot g(y)\,dy$

⇓

ie: if we chose $g\equiv 1$. then $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot dy$ = "average value of $f$ on circle"

Note: 怎么用 $D_N(x)$ "Dirichlet kernel"

$$S_N(f)(x) = \sum_{-N}^{N}\hat f(n)\cdot e^{inx}$$

$$= \sum_{-N}^{N} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot e^{-iny} dy\right)\cdot e^{inx}.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot \left(\sum_{-N}^{N} e^{in(x-y)}\right) dy.$$

$$= (f * D_N)(x) \quad \Leftarrow \text{ Shink to this relation.}$$

prop 3.1: Suppose that $f, g$, and $h$ are $2\pi$-periodic integrable func

then (i) $f*(g+h) = (f*g) + (f*h)$

ie: $f*(g+h) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\cdot g(y) + f(x-y)\cdot h(y)\, dy \equiv$ desired. ▨

(ii) $(cf)*g = c(f*g) = f*(cg)$ for $\forall c \in \mathbb{C}$

(iii) $f*g = g*f$

(iv) $(f*g)*h = (f*h)*g$.

(v) $f*g$ is cont? "more regular"

(vi) $\widehat{f*g}(n) = \hat{f}(n)\,\hat{g}(n)$. " Fourier coeffient rule multiplication"

蓝色证明: !!! If $f$ and $g$ are continous, (Assup)

(iv) $$\widehat{f*g} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f*g)(x)\cdot e^{-inx} dx$$

$$= \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(y)\cdot g(x-y)\, dy\right) e^{-inx} dx$$

$$= \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(y)\cdot e^{-iny}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x-y)\cdot e^{-in(x-y)} dx\right) dy$$

$$= \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(y)\cdot e^{-iny}\cdot \left(\underline{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\cdot e^{-inx} dx}\right) dy$$

$$= \hat{f}(n)\cdot\hat{g}(n).$$

(iii) if $F$ is conti and $2\pi$-periodic, then

WTS: $\int_{-\pi}^{\pi} F(y)\,dy = \int_{-\pi}^{\pi} F(x-y)\,dy$ for $\forall x \in \mathbb{R}$ "?"

(V) $(f*g)(x_1) - (f*g)(x_2) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(y)\cdot[g(x_1-y)-g(x_2-y)]\,dy$ ①

Since $g$ is conti over closed $[-\pi,\pi] \Rightarrow$ unif-conti over $[-\pi,\pi]$

Since $[-\pi,\pi]$ can be extended to $\mathbb{R}$ as $g$ is $2\pi$-period.

So, given $\delta>0$, so that $|g(s)-g(t)|<\varepsilon$ whenever $|s-t|<\delta$.

then $|x_1-x_2|<\delta$ implies that $|(x_1-y)-(x_2-y)|<\delta \Rightarrow |(f*g)(x_1)-(f*g)(x_2)| \le \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(y)\cdot[g(x_1-y)-g_k(x_2-y)]\,dy\right|$

$$\le \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}|f(y)|\,dy \le \frac{\varepsilon}{2\pi}\cdot B(\text{Bound of } f(y))$$

thus $f*g$ is conti when $f,g$ are conti.

Lemma 3.2: Suppos $f$ is integrable on the circle and bounded by $B$. Then there $\exists$ a sequence $\{f_k\}_{k=1}^{\infty}$ of conti-func on circle

so that $\sup_{x\in[-\pi,\pi]}|f_k(x)| \le B$ for $k\in\mathbb{N}$

and $\int_{-\pi}^{\pi}|f(x)-f_k(x)|\,dx \to 0$ as $k\to\infty$

$\longrightarrow$ prove general (v)

proof: consider $f*g - f_k*g_k = (f-f_k)*g + f_k*(g-g_k)$

i) $|(f-f_k)*g| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y)-f_k(x-y)|\,|g(y)|\,dy$

$$\le \frac{1}{2\pi}\sup_y|g(y)|\cdot\int_{-\pi}^{\pi}|f(y)-f_k(y)|\,dy. \longrightarrow 0 \text{ as } k\to 0.$$

ii) similarly for ②

then $f*g - f_k*g_k \longrightarrow 0$ as $k\to\infty$, that is $f_k*g_k \to f*g$ uniformly.

then since each $f_k*g_k$ is conti, it follows that $f*g$ is also conti. " ? "

General (Vi) $\widehat{f_k*g_k}(n) \to \widehat{f*g}(n)$ as $k\to\infty$, and $\hat{f_k}(n)\,\hat{g_k}(n) = \widehat{f_k*g_k}(n)$ "$f_k, g_k$ conti"

then $|\hat{f}(n) - \hat{f_k}(n)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(f(x)-f_k(x))\cdot e^{-inx}\,dx\right|$

$$\le \frac{1}{2\pi}\cdot\left|\int_{-\pi}^{\pi}(f(x)-f_k(x))\,dx\right| \to 0.$$

$\Rightarrow \hat{f_k}(n) \to \hat{f}(n)$ as $k\to\infty$, similarly for $\hat{g_k}(n)\to\hat{g}(n)$, $\Rightarrow$ Get rid of continuity!

---

4. Good kernal: recover func

**Def:** A family $\{K_n(x)\}_{n=1}^{\infty}$ on the circle is said to be a family of good kernel if, "Weight distribution"

(i) For all $n \geq 1$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$$ "$K_n$ 给 $[-\pi,\pi]$ 加了 unit mass 配重"

(ii) There $\exists\, M>0$, s.t for $\forall n \geq 1$.

$$\int_{-\pi}^{\pi} |K_n(x)|\,dx \leq M$$

(iii) For $\forall\, \delta > 0$

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|\,dx \to 0 \quad \text{as } n \to \infty$$ "配重基本给圆心"

Them 4.1: Let $\{K_n\}_{n=1}^{\infty}$ be fam of good kernal, and $f$ an integrable func on the circle, then

$\lim_{n\to\infty} (f * K_n)(x) = f(x)$ whenever $x$ is conti at $x$. If $f$ is conti everywhere, then the above limit is unif-

"Approximation to the identity"

ie: $(f * K_n)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x-y)}_{\overline{\text{平均的盘}}}\, \underbrace{K_n(y)}_{\text{配重}}\,dy$ " as $n\to\infty$, $K_n$ centered mass at $y=0$"

Proof: If $f$ is conti at $x$, choose $\delta$ s.t $|y|<\delta \Rightarrow |f(x-y)-f(x)|<\varepsilon$. Then by 1st property of $\{K_n\}$

$$\begin{aligned}
(f*K_n)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\cdot f(x-y)\,dy - f(x)\cdot 1 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\cdot [f(x-y)-f(x)]\,dy
\end{aligned}$$

$$\begin{aligned}
|f*K_n(x) - f(x)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} K_n(y)\cdot [f(x-y)-f(x)]\,dy\right| \\
&\leq \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)|\cdot|f(x-y)-f(x)|\,dy \\
&\quad + \int_{\delta\leq|y|\leq\pi} |K_n(y)|\cdot|f(x-y)-f(x)|\,dy. \\
&\leq \frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} |K_n(y)|\,dy + \frac{2B}{2\pi}\int_{\delta\leq|y|\leq\pi} |K_n(y)|\,dy
\end{aligned}$$

bound for f.

$$\downarrow \text{ as } k\to\infty$$

$$< \frac{\varepsilon M}{2\pi} + \frac{2B\varepsilon}{2\pi} \qquad 0 < \varepsilon$$

$$< \left(\frac{2B+M}{2\pi}\right)\varepsilon \Rightarrow \lim_{k\to\infty} f*K_n(x) = f(x)$$

Consider $S_N(f)(x) = (f*D_N)(x)$, where $D_N(x) = \sum_{-N}^{N} e^{inx}$ is the Dirichlet kernel.

Check $\{D_N\} \to$ good kernel?

$\Uparrow$

能看是否 convergence

formula: $D_N(x) = \frac{\sin(N+1/2)x}{\sin(x/2)}$

Not interchangable

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{-N}^{N} e^{inx}\,dx = \frac{1}{2\pi}\cdot\sum_{-N}^{N}\int_{-\pi}^{\pi} e^{inx}\,dx$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(N+1/2)x}{\sin(x/2)} = ?$$

$$= \frac{1}{2\pi}\sum_{-N}^{N} \frac{1}{in}\cdot\left[e^{inx}\Big|_{-\pi}^{\pi}\right] = 1\,?$$

$$= \frac{1}{2\pi}\sum_{-N}^{N}\cdot \frac{1}{in}\cdot\left[e^{in\pi} - e^{-in\pi}\right]$$

$$= \frac{1}{2\pi}\sum_{-N}^{N} \frac{2\sin(n\pi)}{n} = 0\ ?$$

Note: for Prop 2 $\int_{-\pi}^{\pi} |D_N(x)|\,dx \geq c\log N$ as $N\to\infty$. $\Rightarrow$ X Good kernal

---

5. Cesàro and Abel sum: ie $\lim_{N\to\infty} S_N(f) = f$.

<u>Intro:</u> Suppose series of $\{C_k\}$

$C_0 + C_1 + C_2 + \cdots = \sum_{k=0}^{\infty} C_k$, def $n^{th}$ partical sum $S_n$ by $S_n = \sum_{k=0}^{n} C_k$

then if $\lim_{n\to\infty} S_n = S$ then $\{S_n\}$ converges to $s$.

EX(3): $1-1+1-1+\cdots = \sum_{k=0}^{\infty} (-1)^k$

<u>Def</u>: i) $\sigma_N = \frac{S_0+S_1+\cdots+S_{N-1}}{N}$ "Cesàro mean" of $\{S_k\}$

"$N^{th}$ Cesàro sum of the series $\sum_{n=0}^{\infty} C_n$.

ii) If $\sigma_n$ conveges to limit $\sigma$ as $N \to \infty$. then $\sum c_n$ is "Cesàro summable" to $\sigma$.

$$\lim_{N\to\infty} \frac{S_0 + S_1 + \cdots + S_{N-1}}{N} = \sigma$$

Ex3: $1 + (-1) + 1 + \cdots = \sum_{k=0}^{\infty} (-1)^k$

$$\sigma_N = \frac{S_0 + S_1 + S_2 + \cdots S_{N-1}}{N}$$

$$= \frac{\overbrace{1 + 0 + (1-1+1) + \cdots 1}^{N}}{N}$$

$\begin{cases} \text{if } N \text{ is even, then } \sigma_N = \frac{\frac{1}{2}N}{N} = \frac{1}{2} \xrightarrow[N\to\infty]{} \frac{1}{2} \\ \text{if } N \text{ is odd, then } N+1 \text{ is even } \sigma_N = \frac{\frac{1}{2}(N+1)}{N} = \frac{1}{2} + \frac{1}{N} \xrightarrow[N\to\infty]{} \frac{1}{2} \end{cases} \Rightarrow \sum c_n$ is summable to $\frac{1}{2}$.

In fact: if series is convergent to $S$, then it's also Cesàro summable to $S$

强约束 更少约束

5.2 Fejér's thm:

ie: $\{\overline{\text{Direchlet kernel}}\} \subset$ good kernal

<u>$N^{th}$ Cesàro mean of Fourier series:</u>

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + \cdots S_{N-1}(f)(x)}{N}$$

since $S_n(f) = f * D_n$, we find $\sigma_N(f)(x) = \dfrac{f * D_0(x) + \cdots + f * D_{N-1}(f)(x)}{N}$

$= f * F_N(x)$ where $F_N(x) = \dfrac{D_0(x) + \cdots + D_{N-1}(x)}{N}$

$$\frac{\frac{1-\cos nx}{2}}{\sin^2(x/2)} = \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$$

$$\| $$

$$\frac{1}{2\sin^2(x/2)} \cdot \sum_{k=0}^{N-1} \cos(kx) - \cos(k+1)x$$

$$\| $$

$$\frac{1}{\sin(x/2)} \sum_{n=0}^{N-1} \sin(nx + \tfrac{1}{2}x) = \frac{1}{2\sin^2(x/2)} \cdot \sum_{k=0}^{n-1} 2(x(k+\tfrac{1}{2})) \cdot \sin(x/2)$$

<u>Lemma 5.1</u>: $F_N(x) = \frac{1}{N} \cdot \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$

Exercise 15: ie: show that $\dfrac{\sin^2(Nx/2)}{\sin^2(x/2)} = D_0(x) + \cdots + D_{N-1}(x)$

we have $NF_N(x) = \sum_{n=0}^{N-1} \frac{w^{-n} - w^{n+1}}{1-w} = \frac{1}{1-w}\left[\sum_{n=0}^{N-1} w^{-n} - \sum_{n=0}^{N-1} w^{n+1}\right]$

where $w = e^{ix}$

$$= \frac{1}{1-w} \cdot \Big( \underset{\underset{w \cdot \frac{w^{-N}-1}{1-w}}{\shortparallel}}{\frac{1-w^{-N}}{1-w^{-1}}} - \sum_{n=0}^{N-1} w^{n+1} = w(1+w+\cdots+w^{N-1}) = w\cdot\frac{1-w^N}{1-w}\Big)$$

$$= \frac{w}{1-w} \cdot \left( \frac{w^{-N} - 1 - 1 + w^N}{1-w} \right)$$

$$= \frac{w \cdot (w^{N/2} - w^{-N/2})^2}{(1-w)^2} = \frac{(w^{N/2} - w^{-N/2})^2}{(w^{-\frac{1}{2}} - w^{\frac{1}{2}})^2} = \frac{\sin^2 Nx/2}{\sin^2(x/2)}$$

$=$

---

Part II: Check good kernel

ie $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x) = 1.$

(1) $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{D_0(x) + \cdots + D_{N-1}(x)}{N} = 1.$

(2) ? How to show $\exists M$, $\left|\int_{-\pi}^{\pi} F_N(x)\,dx\right| \le M$ for $\forall n > 1$.

$\because F_N(x) > 0$

clearly, since $\int_{-\pi}^{\pi} F_N(x)\,dx = 2\pi$, for $\forall n > 0$, $\exists M \ge 2\pi$, s.t $\int_{-\pi}^{\pi} |F_N(x)|\,dx \le M$

(3) check $\int_{\delta \le |x| \le \pi} |F_N(x)|\,dx \to 0$ as $N \to \infty$.

if $\delta \le |x| \le \pi$, then $\sin^2(x/2) \ge C_\delta$ ? "What is $C_\delta$"

Hence $F_N(x) \le 1/(N C_\delta)$, implies that $\int_{\delta \le} |F_N(x)|\,dx \to 0$

+ 4.1 thm

$\shortparallel$

Thm 5.2: If $f$ is integrable on the circle, then Fourier series of $f$ is Cesàro summable to $f$ at every pt of conti of $f$.

Moreover, if $f$ is conti, then unif- Cesàro summable

Cor 5.3: If $f$ is integrable on the circle and $\hat{f}(n) = 0$ for all $n$, then $f = 0$ at all continuity of $f$

proof: since all $\hat{f}(n) = 0 \Rightarrow \sigma_n(f)(x) = \frac{\overset{0}{\shortparallel}{S_0} + \cdots + \overset{0}{\shortparallel}{S_{N-1}}}{N} = 0$ by thm 5.2, it's Cesàro-summable to $f$.

Cor 5.4 conti-func on circle can be uniformly approximated by trigonometric polynomial

ie: If $f$ is conti on $[-\pi, \pi]$, with $f(-\pi) = f(\pi)$ and $\varepsilon > 0$, then $\exists$ trigon-poly $P$ s.t

$|f(x) - P(x)| < \varepsilon$ for all $-\pi \le x \le \pi$

5.3 Abel means and summation

Def: $\sum_{k=0}^{\infty} C_k$ is said to be "Abel summable" to $S$ if for every $0 \le r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} C_k \cdot r^k \text{ converges, and } \lim_{r \to 1} A(r) = S$$

"Abel mean"

Similarly: if series converge $\Rightarrow s$, then it is Abel summable (證)

if Cesàro summable ↗

Ex1: $1-2+3-4+5-\cdots = \sum_{k=0}^{\infty} (-1)^k \cdot (k+1)$

(v) Abel: $A(r) = \sum_{k=0}^{\infty} (-1)^k \cdot (k+1) \cdot r^k = \frac{1}{(1+r)^2} \xrightarrow[r \to 1]{} \frac{1}{4}$.

#13: (x) Cesàro:

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5.4 The poisson kernel and Dirichlet's prob in unit dist.

Def: Abel mean of $f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) \cdot e^{in\theta}$

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \cdot a_n \cdot e^{in\theta}$$

Comp: $C_0 = a_0$ and $C_n = a_n e^{in\theta} + a_{-n} e^{-in\theta}$ for $n > 0$

?: Since $f$ is integrable, $|a_n|$ is unif-bounded in $n$, so that $|a_n|$ is bounded in $n$, so $A_r(f)$ converges abso + uni

def: !!!: $A_r(f)(\theta) = (f * P_r)(\theta)$, where $P_r(\theta)$ is "Poisson kernel" given by (4): $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \cdot e^{in\theta}$

In fact,

$$A_r(f)(\theta) = \sum_{-\infty}^{\infty} r^{|n|} \cdot a_n \cdot e^{in\theta}$$

$$= \sum_{-\infty}^{\infty} r^{|n|} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \cdot e^{-in\varphi} d\varphi \right) e^{in\theta}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \cdot \left( \sum_{-\infty}^{\infty} r^{|n|} e^{-in(\varphi-\theta)} \right) d\varphi.$$

"unif-convergence of the series" $\Rightarrow$ interchange of integeral and infinite sum

Lemma. 5.5 If $0 \le r < 1$, then

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

The poisson kernel is a good kernel, as r tends to 1 from below.

Note : $1-2r\cos\theta + r^2 = (1-r)^2 + 2r(1-\cos\theta)$

if $1/2 \le r \le 1$ and $\delta \le |\theta| \le \pi$, then $1-2r\cos\theta + r^2 \ge C_\delta > 0$.

Thus $P_r(\theta) \le (1-r^2)/C_\delta$ when $\delta \le |\theta| \le \pi$ as $r \to 1$, $|P_r(\theta)| \to 0$

$$\frac{1}{2\pi} \sum_{-\infty}^{\infty} \int_{-\pi}^{\pi} r^{|n|} \cdot e^{in\theta} d\theta$$

$$= \frac{1}{2\pi} \sum_{-\infty}^{\infty} r^{|n|} \cdot \int_{-\pi}^{\pi} e^{in\theta} d\theta$$

$\frac{2\sin n\pi}{n} = 0$?

$= 0$ ? $\quad \frac{e^{in\theta}}{in}\Big|_{-\pi}^{\pi} = \frac{i\sin n\pi + \cancel{\cos n\pi} - i\sin(-n\pi) - \cancel{\cos(-n\pi)}}{in}$

+ Them 4.1

Them 5.6 : $\lim_{n\to\infty} (f * P_{r_n})(\theta) = f(\theta)$

"The fourier series integral func is Abel summable $\to f$ at its continuity.

if $f$ is conti over circle, then unif-Abel summ.

Express $\Delta u = 0$ in polar :

$$u(r,\theta) = \sum_{-\infty}^{\infty} a_m r^{|m|} \cdot e^{im\theta} \qquad a_m = \hat{f}_m \to f$$

$$\downarrow$$

$$u(r,\theta) = A_r(f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \cdot P_r(\theta - \varphi)\, d\varphi.$$

Them 5.7: In unit disc,

$$u(r,\theta) = (f * P_r)(\theta)$$

(i) $u$ has twice conti- derivative $\to$ $\Delta u = 0$

(ii) If $\theta$ ~~$\forall$~~ be pt of conti of $f$, then $\lim_{r \to 1} u(r,\theta) = f(\theta)$

(iii) If $f$ is conti, then $u(r,\theta)$ is unique-solution $\to$ steady heat with (i) + (ii)

proof: (i) $u(r,\theta) = \sum_{-\infty}^{\infty} a_n e^{in\theta} \cdot |r|^{n}$

why fix $\rho$ ? 无限卷圆 ? $\Rightarrow$ inside ?

Fix $\rho < 1$; inside each disc of $r < \rho < 1$ centered origin, the series can be differential term by term

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2}$$

(ii) clearly them 5.6 $\Rightarrow$ indicated.

(iii) def: $V \to$ steady state equation in disc(1) and converge to $f$ unif as $r \to 1$

For each fixed $0 < r < 1$, $v(r,\theta) \sim \sum_{-\infty}^{\infty} a_n(r) \cdot e^{in\theta}$ where $a_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(r,\theta) \cdot e^{-in\theta} d\theta$.

$$v \xrightarrow{\text{sol'n}} \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 v}{\partial \theta^2} = 0 = \Delta V$$

$$\frac{\partial^2}{\partial r^2} \cdot (\cdot\, a_n(r)) + \frac{1}{r} \cdot \frac{\partial}{\partial r} \cdot (\cdot\, a_n(r)) + \frac{1}{r^2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial \theta^2} \cdot e^{-in\theta} d\theta$$

How to get this $n$ ?

$$\frac{1}{r^2} \cdot \frac{\partial^2}{\partial \theta^2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} v \cdot e^{-in\theta} d\theta = \frac{-n^2}{r^2} a_n(r) \ ?$$

thus $a_n(r) = A_n r^n + B_n r^{-n}$ as $a_n''(r) + \frac{1}{r} \cdot a_n'(r) - \frac{n^2}{r^2} a_n(r) = 0$

clearly $a_n(r)$ is bounded, as integral of $v$, therefore $B_n = 0$.

find $A_n$, we let $r \to 1$, $v \to$ unif-con to $f$ as $r \to 1$

$$A_n = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} f(\theta) \cdot e^{-in\theta}.$$

consider $n=0$, then $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(r,\theta)\, d\theta$ as $r \to 1$, $A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$

ie: since $u \equiv v$ with same formula.

Note: if $u$ solves $\Delta u = 0$ in disc and converges to $0$ unif as $r \to 1$, then $u$ must $\equiv 0$.

Chapter 3: Global + local prop convergence

Sec 3.1: Them 1.1

"Suppose $f$ is integrable on the circle, then $\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 d\theta \to 0$ as $N \to \infty$

$$S_N(f)(x) = \sum_{-N}^{N} \hat{f}(\theta) \cdot e^{in\theta}$$

Def: Inner product on $V$ over $R/\mathbb{C}$

Case I: $V$ over $\mathbb{R}$ with $X, Y \in V$ $(X, Y)$

symmetric

i) $(X, Y) = (Y, X)$ and $(aX + \beta Y, Z) = a(X, Z) + \beta(Y, Z)$

ii) Def of Norm: $\|X\| = (X, X)^{1/2}$.

Def: "If $\|X\|$, then $X = 0$" $\equiv$ strickly positive definite

$\boxed{\begin{array}{l} \text{Ex: } X, Y \in R^d, \text{ then } (X, Y) = x_1y_1 + \cdots + x_dy_d \\ \qquad \|X\| = (X, X)^{1/2} = \sqrt{x_1^2 + \cdots + x_d^2} \end{array}}$

Case II: $V$ over $\mathbb{C}$ with $X, Y \in V$

i) Hermititian: $(X, Y) = (\overline{Y, X})$

ii) $(X, aY + \beta Z) = \bar{a}(X, Y) + \bar{\beta}(X, Y)$

and $(aX + \beta Y, Z) = a(X, Z) + \beta(Y, Z)$

iii) $(X, X) \geq 0$ and $\|X\| = (X, X)^{1/2}$ still strickly possitive definite

$\boxed{\begin{array}{l} \text{EX2: } Z, W \in \mathbb{C}, \\ \qquad (Z, W) := z_1 \cdot \overline{w_1} + \cdots + z_d \overline{w_d} = \sum_{i=1}^{d} \bar{z}_i w_i = (\overline{Y, X}) \\ \qquad \|Z\| = (Z, Z)^{1/2} = \sqrt{z_1 \cdot \bar{z}_1 + \cdots + z_d \bar{z}_d} = \sqrt{|z_1|^2 + \cdots + |z_d|^2} \end{array}}$

Def: orthogonal.

$X, Y$ are orthogonal iff $(X, Y) = 0$ or $X \perp Y$

3 Important result:

1) The pythagorean them: If $X \perp Y$, then $\|X+Y\|^2 = \|X\|^2 + \|Y\|^2$

2) Cauchy-Schwarz inequality: for $\forall X, Y \in V$, $|(X, Y)| \le \|X\|\|Y\|$

3) Triangle inequ: $\|X+Y\| \le \|X\| + \|Y\|$

proof: $\|X+Y\|^2 = (X+Y, X+Y) = (X, X) + (X, Y) + (Y, X) + (Y, Y)$

if $X, Y$ orthogonal, then $(X, Y) = 0 = (Y, X)$ implies that $\|X+Y\|^2 = \|X\| + \|Y\|$

(ii) If $\|Y\| = 0$, then $Y = 0$, and the equalty holds

if $y \neq 0$, then for $\forall c \in F$, we have $0 \le (x-cy, x-cy) = (x, x-cy) - c(y, x-cy) = (x, x) - \bar{c}(x, y) - c(y, x) + c^2(y, y)$

If we set $c = \frac{\langle x, y \rangle}{\langle y, y \rangle}$, then $0 \le \langle x - cy, x - cy \rangle = (x, x) - \frac{(\langle x, y \rangle)^2}{\|y\|} = \|x\|^2 - \frac{\|\langle x, y \rangle\|^2}{\|y\|^2}$ ▨

(iii) $\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle$

$= \|x\|^2 + 2\mathrm{Re}\langle x, y \rangle + \|y\|^2$

$\le \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2$

$\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$ ▨

Important example: the vector space $\ell^2(\mathbb{Z})$ over $\mathbb{C}$ is the set of all infinite sequence of complex number

$(\cdots, a_{-n}, \cdots a_{-1}, a_0, a_1, \cdots a_n, \cdots)$ s.t $\sum_{n \in \mathbb{Z}} |a_n|^2 < \infty$ (ie series convergent)

$\langle A, B \rangle = \sum_{n \in \mathbb{Z}} a_n \overline{b_n}$ and $\|A\| = (A, A)^{1/2} = \left( \sum_{n \in \mathbb{Z}} |a_n|^2 \right)^{1/2}$

Check $\ell^2(\mathbb{Z})$ is vector space

i) let $A, B \in \ell^2(\mathbb{Z})$. for each $N>0$, we define $A_N := (\cdots, 0, \cdots a_{-N}, \cdots a_0, \cdots, a_N, 0, 0 \cdots)$

with $a_n = 0$ for all $n>N$. construct $B_N$ in the same way

then by triangle inequality: $\|A_N + B_N\| \le \|A_N\| + \|B_N\| \le \|A\| + \|B\|$

$$\text{Thus} \quad \sum_{|n|\le N} |a_n + b_n|^2 \le (\|A\| + \|B\|)^2$$

$$\text{let } N \to \infty, \text{ then } \sum_{n\in\mathbb{Z}} |a_n + b_n|^2 < \infty$$

3 properties satisfied !

Def: Hilbert Space

(i) If positive strictly definite

(ii) The vector space is complete $\equiv$ $\forall$ Cauchy sequence in the norm converges to a limit in $V$.

(iii) If either failed (i), (ii), the $V$ C pre-Hilbert Space

Example 2: Let $R$ denote the set of $f_s \in R_{\mathbb{C}}(x)$ complex riemann on $[0, 2\pi]$

Def: $(f+g)(\theta) = f(\theta) + g(\theta)$

$(\lambda f)(\theta) = \lambda \cdot f(\theta)$ $\qquad \|f\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta\right)^{1/2}$

$(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) \cdot \overline{g(\theta)}\, d\theta$

Prob 1: Check Cauchy-Schwarz and triangle property $|(f,g)| \le \|f\| \cdot \|g\|$ and $\|f+g\| \le \|f\| + \|g\|$

Hint !!! : Clearly for $A, B \in R$, $2AB \le A^2 + B^2$

let $A = \lambda^{1/2} |f(\theta)|$ and $B = \lambda^{-1/2} |g(\theta)|$

then $\quad |f(\theta) \cdot g(\theta)| \le \frac{1}{2} \cdot (\lambda \cdot |f(\theta)|^2 + \lambda^{-1} \cdot |g(\theta)|^2)$

$$|(f, g)| \le \frac{1}{2}\int_0^{2\pi} |f(\theta)| \cdot |\overline{g}(\theta)|\, d\theta \le \frac{1}{2}(\lambda \|f\|^2 + \lambda^{-1} \|g\|^2)$$

then plug into $\lambda = \|g\| / \|f\|$, we have $|(f,g)| \le \frac{1}{2} \cdot (\|g\| \cdot \|f\| + \|g\| \|f\|)$ Cauchy ☒

Note: $\|f\| \neq 0$ and $\|g\| \neq 0$ in choice of $\lambda$

Prob 2: Check for Hilbert Space

(i) Strickly positive definite (X)

ie: if $\|f\| = 0$, then $f(\theta) \not\equiv 0$ (not necessary)

Ans: since $\|f\| = 0 \Rightarrow \int_0^{2\pi} |f(\theta)|^2 d\theta = 0 \Rightarrow f$ only vanish at its continuities, "measure 0"

(ii) Show that $R$ is not complete

a) $f(\theta) = \begin{cases} 0 & \theta = 0 \\ \log(1/\theta) & \theta \in (0, 2\pi] \end{cases}$

clearly it's not bounded, then $f \notin R$

b) $f_n(\theta) = \begin{cases} 0 & \text{for } 0 \le \theta \le 1/n \\ f(\theta) & \text{for } 1/n < \theta \le 2\pi \end{cases}$

" forms a cauchy sequence " ? If $\lim_{n \to N} \|f_n(\theta)\| = f(\theta) \notin R \Rightarrow$ X complete.

Lebesgue class: $L^2([0, 2\pi])$

Sec 1.2:

Prove of mean-squre convergence

consider space $R$ of integrable func on circle with inner product

$$(f,g) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cdot \overline{g(\theta)}\, d\theta \quad \text{and} \quad \|f\|^2 = (f,f) = \frac{1}{2\pi} \cdot \int_0^{2\pi} |f(\theta)|^2 d\theta$$

we must prove that $\|f - S_N(f)\| \to 0$ as $N \to \infty$

proof: $\{e_n\}_{n \in \mathbb{Z}}$ where $e_n(\theta) = e^{in\theta}$ (family of orthogonal)

that is $(e_n, e_m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$

$$\hat{f}(n) = (f, e_n) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cdot e^{-in\theta} d\theta = a_n$$

In perticular, $S_N(f) = \sum_{|n|\leq N} a_n \cdot e_n$

WTS: $f - \sum_{|n|\leq N} a_n e_n$ is ortho $\Rightarrow e_n$

$$\frac{1}{2\pi}\int_0^{2\pi} \left(f(\theta) - \underbrace{\sum_{|n|\leq N} (f, e_n)\, e^{in\theta}}_{0}\right) \cdot \cancel{e^{-in\theta}} = 0$$

Therefore, we have $(f - \sum_{|n|\leq N} a_n e_n) \perp \sum_{|n|\leq N} b_n e_n$

take $a_n = b_n$

(1) by Pythagorean thm: $f = f - \sum a_n e_n + \sum a_n e_n$

$$\|f\|^2 = \|f - \sum a_n e_n\|^2 + \|\sum a_n e_n\|^2 \Rightarrow \|f\|^2 = \|f - S_N(f)\|^2 + \sum |a_n|^2$$

since orthogonal property $\Rightarrow \|\sum a_n e_n\|^2 = \sum |a_n|^2$

Lemma 1.2: If $f$ is integrable on circle with $a_n = \hat{f}(n)$, then $\|f - S_N(f)\| \leq \|f - \sum c_n e_n\|$ for $\forall c_n$

Moreover $\|f - S_N(f)\| = \|f - \sum c_n e_n\|$ when $c_n = a_n$ for all $|n|\leq N$

proof: choose $b_n = a_n - c_n$, then $\|f - \sum c_n e_n\|^2 = \|f - S_N(f)\|^2 + \|\sum b_n e_n\|^2$

$$= \|f - S_N(f)\|^2 + \sum |b_n| \geq 0 \Rightarrow \checkmark$$

Prove of thm 1.1:

Suppose that $f$ is conti on the circle. Then, given $\varepsilon > 0$, there exist a trigonometric polynomial $P$, say of degree $M$,

s.t $|f(\theta) - P(\theta)| < \varepsilon$ for all $\theta$

$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - P(\theta)|^2 d\theta < \frac{1}{2\pi}\int_0^{2\pi} \varepsilon^2 d\theta \Rightarrow \|f - P\| < \varepsilon$ and by appro-lemma, we conclude that $\|f - S_N(f)\| < \varepsilon$

whenever $N \geq M$. ∎

(ii) if $f$ is merely integrable, then apply lemma 3.2 choose conti-func $g$ on circle, with

$$\sup_{\theta\in[0,2\pi]} |g(\theta)| \leq \sup|f(\theta)| = B, \text{ and } \int_0^{2\pi} |f(\theta)-g(\theta)|\,d\theta < \varepsilon^2$$

then we have $\|f-g\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f-g|^2\,d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} |f-g|\cdot|f-g|$$

$$\leq \frac{2B}{2\pi}\int_0^{2\pi} |f-g|\,d\theta < \underbrace{\frac{2B}{2\pi}}_{= C}\cdot\varepsilon^2$$

then approximate $g$ by $P_2$ s.t $\|g-P\|<\varepsilon \Rightarrow \|f-P\|<C'\varepsilon$

Note:

$$\|f\|^2 = \|f-S_N(f)\|^2 + \sum |a_n|^2$$

as $\|f-S_N(f)\|^2 \to 0$, we have Parseval's identity $\sum_{-\infty}^{\infty}|a_n|^2 = \|f\|^2$

## Summary

Thm 1.3: let $f\in C(x)$ with $f\sim \sum a_n e^{in\theta}$, then we have

(i) Thm 1.1

(ii) Parseval's identity.

Remark 1: If $\{e_n\}$ ortho-fam and $a_n = (f, e_n)$, then $\sum|a_n|^2 \leq \|f\|^2$ "Bessel's inequality."

and $\sum|a_n|^2 \leq \|f\|^2$ when $\|\sum_{|n|\leq N} a_n e_n - f\| \to 0$ as $N\to\infty$.

bounded

Remark 2: $\{a_n\}\in \ell^2(\mathbb{Z})$ as $\|f\|^2 = \sum |a_n|^2 < \infty$

$\ell^2(\mathbb{Z}) \subset$ Hilbert Space $\quad R$ is not complete $\equiv$ there $\exists$ $\{a_n\}_{n\in\mathbb{Z}}$ s.t $\sum|a_n|^2<\infty$, Yet no Rieman-integrable

function $F$ has $n^{th}$ fourier coefficient $= a_n$ for all $n$.

Them 1.4: (Riemann – Lebesgue lemma) if $f$ is integrable on circle, then $\hat{f}(n) \to 0$ as $|n| \to \infty$.

ie: If $f$ is integrable on $[0, 2\pi]$, then $\int_0^{2\pi} f(\theta) \sin(N\theta) \to 0$ as $N \to \infty$.

and $\int_0^{2\pi} f(\theta) \cos(N\theta) \to 0$ as $N \to \infty$.

General version: Suppose $F$ and $G$ are integrable on circle with

$$F \sim \sum a_n e^{in\theta} \text{ and } G \sim \sum b_n e^{in\theta}$$

then $\frac{1}{2\pi} \int_0^{2\pi} F(\theta) \cdot \overline{G(\theta)} \, d\theta = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}$

$$(F, G) = \frac{1}{4} [\|F+G\|^2 - \|F-G\|^2 + i(\|F+iG\|^2 - \|F-iG\|^2)$$

$$\sum (a_n+b_n)^2 - \sum (a_n-b_n)^2 + i(\sum (a_n+\overline{b_n})^2 - \sum (a_n-\overline{b_n})^2)$$

$$\frac{1}{4} \sum 4a_n b_n + i(2a_n\overline{b_n} + 2a_n\overline{b_n})$$

$$= a_n b_n + a_n \overline{b_n}$$

---

Sec 3.2: Pointwise convergence

fact: if func is differentiable at $\theta_0$, then $\sum a_n e^{in\theta}$ converges at $\theta_0$

Them 2.1: Let $f$ be integrable func on the circle which is differentiable at $\theta_0$, then $S_N(f)(\theta_0) \to f(\theta_0)$ as $N \to \infty$

proof: def
$$F(t) = \begin{cases} \frac{f(\theta_0 - t) - f(\theta_0)}{t} & \text{if } t \neq 0 \text{ and } |t| < \pi \\ -f'(\theta_0) & \text{if } t = 0 \end{cases}$$

i) WTS $F$ is integrable: ① Since $f$ is differentiable near 0, then $F$ is bounded near 0.

② for $\delta < |t| < \pi$, $F$ is integrable over $[-\pi, \delta] \cup [\delta, \pi]$.

By Prop 1.4 at Appendix, $F$ is integrable over $[-\pi, \pi]$

2) Solve $S_N(f) - f$: $S_N(f)(\theta_0) = (f * D_N)(\theta_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta_0 - t) \cdot D_N(t) dt - f(\theta_0)$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(\theta_0 - t) - f(\theta_0)] \cdot D_N(t) \, dt.$$

$$= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} F(t) \cdot t \cdot D_N(t)\, dt.$$

cont

Note : $t \cdot D_N(t) = \frac{t}{\sin(t/2)} \cdot \sin(N+\frac{1}{2})t$

$= \cdots \cdot (\sin(Nt) \cdot \cos\frac{t}{2} + \cos(Nt)\ \sin\frac{t}{2})$

$= t \cdot \cos(Nt) + t \cdot \sin Nt \frac{\cos t/2}{\sin(t/2)}$.

WTU: Riemann-Lebegue $\Rightarrow$ $\underline{F(t) \cdot t}$ and $\underline{F \cdot t \cdot \frac{\cos t/2}{\sin t/2}}$ is integrable on the circle.

Since $F(t)$, $t$, $\cos t/2$, $\sin t/2$ is integrable on the circle.

then by Riemann-Lebegue, as $N \to \infty$ $S_N(f) - f \to 0$.

Cor 2.1: If $f$ satisfied Lipschiz-condition at $\theta_0$, that is $|f(\theta) - f(\theta_0)| \leq M \cdot |\theta - \theta_0|$.

for some $M \geq 0$ and all $\theta$. if $f$ satisfied the Hölder condition for $a=1$.

Them 2.2: Suppose $f$ and $g$ are two integrable func defined on the circle, and for some $\theta_0$, $\exists$ an open interval $I$ contain $\theta_0$

s.t $f(\theta) = g(\theta)$ for all $\theta \in I$, then $S_N(f)(\theta_0) - S_N(g)(\theta_0) \to 0$ as $N \to \infty$.

proof: def $h(\theta) = f(\theta) - g(\theta)$ for all $\theta \in I$. then $h(\theta) \equiv 0$ for $\theta \in I$.

then Apply them 2.1 $\to h(\theta)$ as it's identical zero $\equiv$ continous, $S_N(h)(\theta) - h(\theta) \to 0$ as $N \to \infty$, equivalently $S_N(f)(\theta) - S_N g(\theta) = 0$.

Topic: A continous func with diverging Fourier series

"Symmetric breaking" $\Rightarrow \sum_{-\infty}^{\infty} a_n e^{in\theta} \to \sum_{n \geq 0} a_n e^{in\theta}$ and $\sum_{n<0} a_n e^{in\theta}$.

Ex: Sawtooth function $f$ which is odd in $\theta$ and $f = i(\pi - \theta)$ when $0 < \theta < \pi$

then (4) $f(\theta) \sim \sum_{n \neq 0} \frac{e^{in\theta}}{n}$

(5) $\sum_{n=-\infty}^{-1} \frac{e^{in\theta}}{n}$ is no-longer a fourier series $\to$ Riemann func

suppose (5) is "$a_n$" for some Riemann func $\tilde{f}$ and $|\tilde{f}| \le M$.

using Abel mean: $|A_r(\tilde{f})(0)| = \sum_{n=1}^{\infty} \frac{r^n}{n} \to \infty$ ( as $r \to 1$, $\sum \frac{1}{n}$ divergence)

Contridiction: $|A_r(f)(0)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{f}(\theta)| \cdot P_r(\theta) d\theta \le \sup_{\theta} |\tilde{f}(\theta)|$

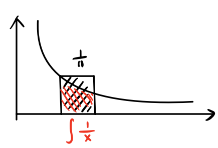
Ex: conti : For each $N \geq 1$, we define $[-\pi, \pi]$

$$f_N(\theta) = \sum_{1 < |n| < N} \frac{e^{in\theta}}{n} \quad \text{and} \quad \tilde{f}_N(\theta) = \sum_{-N \le n \le -1} \frac{e^{in\theta}}{n}$$

fact: (i) $|\tilde{f}_N(0)| \geq c \log N$

(ii) $f_N(\theta)$ is unif- bounded in $N$ and $\theta$.

(i) $|\tilde{f}_N(0)| = \sum_{n=1}^{N} \frac{1}{n} \geq \sum_{n=1}^{N-1} \int_{n}^{n+1} \frac{dx}{x} = \int_{1}^{N} \frac{dx}{x} = \log N.$




(ii). Proof of Tauber's them, if $\sum C_n$ is Abel summable to $s$ and $C_n = o(1/n)$, then $\sum C_n \to s$.

Lemma 2.3: Suppose the Abel means $A_r = \sum_{n=1}^{\infty} r^n \cdot C_n$ of the series $\sum C_n$ are bounded as $r \to 1$.

If $C_n = O(1/n)$, then $S_N = \sum_{n=1}^{N} C_n$ are bounded

proof: let $r = 1 - 1/N$ and choose $M$ s.t $n \cdot |C_n| \le M$.

consider $S_N - A_r = \sum_{n=1}^{N} C_n - \sum_{n=1}^{\infty} r^n \cdot C_n = \sum_{n=1}^{N} C_n \cdot (1 - r^n) - \sum_{n=N+1}^{\infty} C_n \cdot r^n$

$$|S_N - A_r| \le \sum_{n=1}^{N} |C_n| \cdot (1 - r^n) + \sum_{n=N+1}^{\infty} r^n \cdot |C_n|$$

$$\le M \cdot \sum_{n=1}^{N} \frac{1 - r^n}{n} + \frac{M}{N} \sum_{n=N+1}^{\infty} r^n$$

$$\searrow \sum_{n=1}^{N} 1 - r.$$

$1 - r^n = (1-r) \cdot (1 + r + \cdots + r^{n-1}) \le n(1-r)$

$$\le M \cdot N (1 - r) + \frac{M}{N} \cdot \frac{1}{1-r}$$

$= 2M.$

Hence, $|S_N - A_r| \le 2M \Rightarrow |S_N| \le 2M\text{-}K$ as $|A_r| \le K$. as $r \to 1$.

ie: Apply lemma $\Rightarrow \sum_{n\neq 0} \frac{e^{in\theta}}{n}$

i) $C_n = e^{in\theta}/n + e^{-in\theta}/(-n)$ for $n \neq 0$.

$|C_n| = |e^{in\theta}|/n + |e^{-in\theta}|/n = \frac{2}{n} \Rightarrow C_n = O(1/n)$

ii) $A_r(f)(\theta) = (f * P_r)(\theta)$, $f$ is bounded and $P_r$ is good kernel.

$\Rightarrow S_N$ is bounded $\left|\sum_{n\neq 0}^{|N|} \frac{e^{in\theta}}{n}\right| \le M$.

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要搞个东西: Topic: A continous func with diverging Fourier series

架构 : For each $N \geq 1$, we define $[-\pi, \pi]$

$$f_N(\theta) = \sum_{1<|n|<N} \frac{e^{in\theta}}{n} \quad \text{and} \quad \tilde{f}_N(\theta) = \sum_{-N\le n\le -1} \frac{e^{in\theta}}{n}$$

(i) $|\tilde{f}_N(0)| \geq c\log N$

(ii) $f_N(\theta)$ is unif- bounded in $N$ and $\theta$.

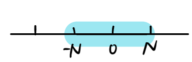
(iii) trigonometry polynomial up to $N$

Lemma 2.4: Example:

$$S_M(P_N) = \begin{cases} P_N & \text{if } M \geq 3N \\ \tilde{P}_N & \text{if } M = 2N \\ 0 & \text{if } M < N. \end{cases}$$

WTS: divergence

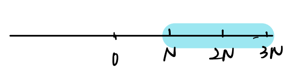
graph:

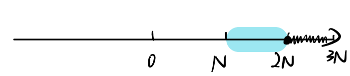



$f_N(\theta)$

$S_{2N}(e^{i2N\theta} f_N)(\theta) = e^{i(2N\theta)} \tilde{f}_N(\theta)$.

$P_N(\theta) = e^{i(2N)\theta} f_N(\theta)$



$P_N$ of degree $3N$ and $\tilde{P}_N$ up to $2N-1$ $\Rightarrow$ the $S_M(P_N)$ is non vanishing for $N$ up $3N$ ($\neq 2N$.)

ie: when $M=2N$, it breaks the symmetric of $P_N$, but for $M \neq 2N$, it's satisfied lemma. $S_M = 0 / P_N$. for $M \neq 2M$.

ie: Want to find a converage series of positive $a_k$, $\sum a_k$ and a sequence of integer $\{N_k\}$.

fast grow, s.t. (i) $N_{k+1} > 3N_k$

(ii) $a_k \log N_k \to \infty$ as $k \to \infty$.

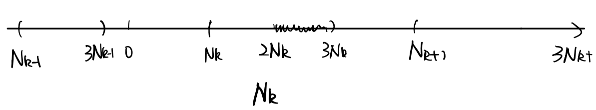
ex: choose $a_k = 1/k^2$ and $N_k = 3^{2^k}$

Def: $f(\theta) = \sum_{k=1}^{\infty} a_k \cdot P_{N_k} \cdot (\theta)$

检验: $|P_N(\theta)| = |e^{i2N\theta} \cdot f_N(\theta)| = |e^{i2N\theta}| \cdot |f_N(\theta)| = |f_N(\theta)|$

Hence the series $f(\theta)$ converages to a periodic - conti - function.

However by lemma 2.4: $|S_{2N_m} f(0)| \geq c a_m \cdot \log N_m + O(1) \to \infty$ as $m \to \infty$




Indeed, the terms $\to N_k$ with $k<m$ and $k>0$ $|\tilde{P}_N(\theta)| = |\tilde{f}_N(\theta)| \geq c \log N$. so the series of $f$ at $0$ are not bounned. To produce a function at any $\theta = \theta_0 \Rightarrow f(\theta - \theta_0) = f(0)$ same ie.

Application of Fourier series.

I: $\ell$ long curve. largest area?

II: Given $r$ (irrational), distribution of the fraction part $nr$, for $n = 1, 2, 3 \cdots$?

III: conti function + no where differentiable.

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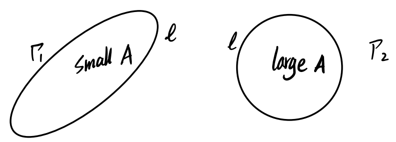
I: The isoperimetric inequality

Let $\Gamma$ denoted a closed curve, $\ell$ be the length, Area $A$.




Denotion: $r : [a, b] \to \mathbb{R}^2$, ① $Im(r) \equiv$ "curve $\Gamma$"

well-defined
- ② $\Gamma$ is simple if it's not intersect "$r(x)$ is one to one"
- ③ $\Gamma$ is closed if $r(a) = r(b)$
- ④ $\Gamma$ is smooth if $r \in C^1$ and $r'(s) \neq 0$.

⑤ orientation: $\forall$ $C^1$ bijective map $s : [c, d] \to [a, b]$ gives: $\eta(t) = r(s(t))$

"$r \equiv \eta$" if $s'(t) > 0$ for all $t$, same orientation.

$s'(t) < 0 \Rightarrow$ reversed orientazon.

ie $\eta'(t) = r'(s(t)) \cdot \underline{s'(t)}$ "orientation indicator"

⑥ length $\ell$ of $\Gamma$ is parametrized by $r(s) = (x(s), y(s))$

$$\ell = \int_a^b |r'(s)|\, ds = \int_a^b (x'(s)^2 + y'(s)^2)^{1/2}\, ds.$$

Special: $\gamma$ para by arclength if $|\gamma'(s)|=1$ " constant speed + lengh $\ell = b-a$.

$[0,\ell]$, $\ell = \int_a^b |\gamma'(s)|\, ds = b-a$.

---

Area of $\mathcal{A}$ is given by $A = \frac{1}{2}\left|\int_\Gamma (x\,dy - y\,dx)\right|$

(1) $\quad = \frac{1}{2}\left|\int_a^b x(s)y'(s) - y(s)x'(s)\, ds\right|$

Thm 1.1: Suppose that $\Gamma$ is simple + closed curve in $\mathbb{R}^2$ of length $\ell$, and $\mathcal{A} \equiv$ Area

then $\mathcal{A} \le \frac{\ell^2}{4\pi}$ with equality holds iff $\Gamma$ is circle.

ie taking $\delta = \frac{2\pi}{\ell}$, then new $\ell_1 \to 2\pi$.

then prob $\Rightarrow$ if $\ell = 2\pi$, then $\mathcal{A} \le \pi$

Note: rescale by $\delta > 0$,

map: $(x,y) \to (\delta x, \delta y)$

$\ell \Rightarrow \delta\ell$ " 放缩 $\delta \ge 1$ or $\delta \le 1$ "

proof: let $\gamma: [0,2\pi] \to \mathbb{R}^2$ with $\gamma(s) = (x(s), y(s))$ with $x'(s)^2 + y'(s)^2 = 1$ for $s \in [0,2\pi)$

ie $\frac{1}{2\pi}\int_0^{2\pi} (x'(s))^2 + y'(s)^2)\, ds = 1. = \frac{\ell}{2\pi}$

Since $x(s)$, $y(s)$ $2\pi$ - periodic, then

$x_n \sim \sum a_n e^{ins}$ and $y_n \sim \sum b_n e^{ins}$

$x_n' \sim \sum a_n \cdot (in) e^{ins}$ and $y_n' \sim \sum b_n (in) e^{ins}$

Parseval's identity:

$\|f\|^2 = \sum |a_n|^2$

① $|\gamma'(s)|^2 = 1 = \sum_{n=-\infty}^{\infty} |n|^2 (|a_n|^2 + |b_n|^2)$

② Since $x(s)$ and $y(s)$ are real valued, $a_n = \overline{a_{-n}}$, and $b_n = \overline{b_{-n}}$.

$\mathcal{A} = \frac{1}{2}\left|\int_0^{2\pi} x(s)\cdot y'(s) - y(s)x'(s)\, ds\right| = \pi \cdot \sum_{n=-\infty}^{\infty} \left| n(a_n\overline{b_n} - b_n\overline{a_n})\right|$

$$|a_n\bar{b}_n - b_n\bar{a}_n| \le 2|a_n||b_n| \le |a_n|^2 + |b_n|^2.$$

$$A \le \pi \cdot \sum_{n=-\infty}^{\infty} |n|^2 (|a_n|^2 + |b_n|^2)$$

$$\le \pi$$

when $A = \pi$, since $|n|^2 > |n|$ as soon as $n > 1$.

then $x(s) = a_{-1}e^{-is} + a_0 + a_1 e^{is} = \bar{a}_1 e^{-is} + a_0 + a_1 e^{is}$

$y(s) = b_{-1}e^{-is} + b_0 + b_1 e^{is} = \bar{b}_1 e^{-is} + b_0 + b_1 e^{is}$

$\Rightarrow$ ① $|\gamma'(s)|^2 = 1 = \sum_{n=-\infty}^{\infty} |n|^2 (|a_n|^2 + |b_n|^2)$ when $n = 1 \Rightarrow$ $(a_1)^2 + (b_1)^2 + (\bar{b}_1)^2 + (\bar{a}_1)^2 = 1$

$2(|a_1|^2 + |b_1|^2) = 1.$

$\Rightarrow |a_1| = |b_1| = \frac{1}{2}.$

thus $a_1 = \frac{1}{2}e^{i\alpha}$ and $b_1 = \frac{1}{2}e^{i\beta}$

$1 = 2|a_1\bar{b}_1 - \bar{a}_1 b_1| \Rightarrow$

$2\left|\frac{1}{4}(i\sin\alpha + \cos\alpha)\cdot(-i\sin\beta + \cos\beta) - \frac{1}{4}(-i\sin\alpha + \cos\alpha)(i\sin\beta + \cos\beta)\right|.$

$= \frac{1}{2}|\ \cancel{\sin\alpha\sin\beta} + i\sin\alpha\cos\beta - i\sin\beta\cos\alpha + \cos\alpha\cos\beta - [\sin\alpha\sin\beta - i\sin\alpha\cos\beta + i\cos\alpha\sin\beta + \cos\alpha\cos\beta)$

$1 = \frac{1}{2}|2i\sin\alpha\cos\beta - 2i\cos\alpha\sin\beta|$

$1 = |\sin\alpha\cos\beta - \cos\alpha\sin\beta| \Rightarrow |\sin(\alpha - \beta)| = 1.$

$\Rightarrow \alpha - \beta = \frac{k\pi}{2}$

$\Rightarrow x(s) = a_0 + \cos(a+s)$ and $y(s) = b_0 \pm \sin(a+s)$

$\Downarrow$

circle ✓.