

Geometry and Topology Review

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Introduction

This project is written for my Ph.D. qualifying exam in the Geometry and Topology track. The material covers most topics from *Introduction to Differential Manifolds* by John Lee. The notes are handwritten but readable in a sense that it helps many of my colleagues, so I decided to post it online. This also resulted in a small project for applications in De Rham cohomology parts and involves many interesting theorems' proofs.

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Def. $U \subset_{\text{open}} \mathbb{R}^n$ and $f: U \rightarrow \mathbb{R}$ conti-func. we say that f is C^k -func if all its partial derivative of order at least k .

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{(\partial x^1)^{a_1} \dots (\partial x^n)^{a_n}} \text{ exist and conti on } U.$$

same def $\Rightarrow C^\infty$ -func

Def 1.1 = A smooth map $f: U \rightarrow V$ is diffeomorphism if f is 1-1 and onto and $f^{-1}: U \rightarrow V$ is also smooth.

OBS: i) If $f: U \rightarrow V$ diffeomorphism $\Rightarrow f^{-1}$ is also diffeo.

ii) If $f: U \rightarrow V$ and $g: V \rightarrow W$ are diffeo, so it is $g \circ f: U \rightarrow W$.

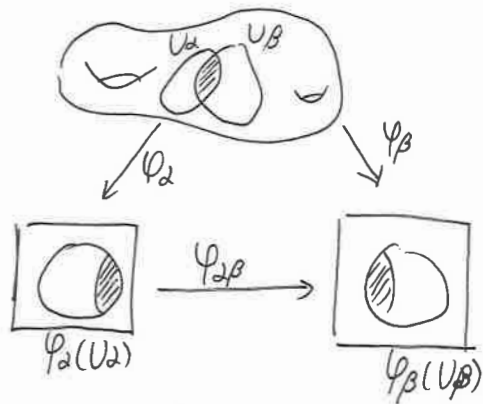
Def 1.2: Chart $\{\varphi, U, V\}: U \subset_{\text{open}} M, V \subset_{\text{open}} \mathbb{R}^n$

It is natural to identify f on U w/ $f \circ \varphi^{-1}$ on $V \rightarrow$ define f is smooth/not.

Def 1.3: Let M be topological manifold of dim n .

$\{\varphi_\alpha, U_\alpha, V_\alpha\}$ and $\{\varphi_\beta, U_\beta, V_\beta\}$ of M are compatible if the transition map.

$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is a diffeomorphism



Def 1.3: (1) An atlas \mathcal{A} on M is a collection of charts $\{\varphi_1, U_1, \varphi_2, U_2, \dots\}$ w/ $\bigcup U_i = M$ s.t. all charts are compatible w/ each other.

(2) Two atlas are said to be equivalent if their union is again an atlas.

eg. we define 3 atlas on \mathbb{R} by $\mathcal{A}_i = \{\varphi_i, \mathbb{R}, \mathbb{R}\}$ ($1 \leq i \leq 3$), where $\varphi_1(x) = x$, $\varphi_2(x) = 2x$, $\varphi_3(x) = x^3$

then \mathcal{A}_1 and \mathcal{A}_2 are equivalent, but $\mathcal{A}_1, \mathcal{A}_3$ are non-equivalent since

$$\varphi_{31}(x) = \varphi_1 \circ \varphi_3^{-1}(x) = x^{1/3} \text{ is not smooth on } \mathbb{R}.$$

Def 1.4: An n -dim smooth manifold is an n -dim topo manifold w/ equivalent class of atlas.
"Smooth structure"

\Rightarrow Pair (M, \mathcal{A})

Some results: (i) There \exists topo-manifold that do not admit smooth structure. "Kervaire 10-dim-manifold"

(ii) If M admits a C^1 structure \Rightarrow admits a C^∞ structure.

(iii) \forall Manifold M admits a finite atlas consisting of $\dim M + 1$ charts.

Prop 1.5: If a topological manifold M can be covered by a single chart, then $\{\varphi_0, U_0, \varphi_0\}$ determines a smooth structure on M .

Consequently: \mathbb{R}^n and $\forall \{ \bigcup_{\text{open}} \mathbb{R}^n \}$ is a smooth manifold

Example (Graph): For $\forall U \subset_{\text{open}} \mathbb{R}^m$ and \forall conti-func $f: U \rightarrow \mathbb{R}^n$, the graph of f is the subset in $\mathbb{R}^{n+m} = \mathbb{R}^m \times \mathbb{R}^n$.

defined by

$$T(f) = \{(x, y) \mid x \in U, y \in f(x)\} \subset \mathbb{R}^{m+n}$$

w/ subspace topo (i.e topo basis $\cap U$) inherited from \mathbb{R}^{m+n} .

We have: (1) $T(f)$ is Hausdorff and 2^{nd} countable

(2) $T(f)$ is locally- Euclidean since $\{\varphi, T(f), U\}$

where $\varphi: T(f) \rightarrow U$, $\varphi(x, y) = x$ "projection onto"

i.e φ is homeo: φ is conti, invertible and

$$\varphi^{-1}: U \rightarrow T(f) \Rightarrow \varphi^{-1}(x) = (x, f(x)) \text{ is conti.}$$

$\Rightarrow T(f)$ is topo manifold of dim - m . Since it can be covered by 1-chart.

$\Rightarrow T(f)$ of \forall conti func $f: U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ admits an intrinsic structure \rightarrow smooth manifold

"It is possible that $T(f)$ is not smooth submanifold of \mathbb{R}^{n+1} !"

(The sphere as smooth manifold): (Spheres)

For each $n \geq 0$, the unit n -sphere

$$S^n = \{(x^1, \dots, x^n, x^{n+1}) \mid (x^1)^2 + \dots + (x^{n+1})^2 = 1\} \subset \mathbb{R}^{n+1}$$

w/ sub.sp. topo is 2^{nd} countable + Hausdorff + "locally Euclidean" (?)

i.e We can cover S^n by $U_+ = S^n \setminus \{(0, \dots, 0, -1)\}$, $U_- = S^n \setminus \{(0, \dots, 0, 1)\}$.

and define $\{\varphi_+, U_+, \mathbb{R}^n\}$ and $\{\varphi_-, U_-, \mathbb{R}^n\}$ by stereographic proj

$$\varphi_{\pm}(x^1, \dots, x^{n+1}) = \frac{1}{1 \pm x^{n+1}} (x^1, \dots, x^n).$$

RMK: We can also cover S^n by

2^{n+2} charts using hemisphere.

$$U_i^{\pm} = \{(x^1, \dots, x^{n+1}) \in S^n, x_i > < 0\}$$

$$\varphi_i^{\pm}: U_i^{\pm} \rightarrow B^n(1)$$

$$\varphi_i^{\pm}(x^1, \dots, x^n) = (x^1, \dots, x^i, x^{i+1}, \dots, x^n)$$

(check more (compatible)).

$$\varphi_{-}(y^1, \dots, y^n) = \varphi_{+} \circ \varphi_{-}^{-1}(y^1, \dots, y^n)$$

$$= \varphi_{+} \left(\frac{1}{1+|y|^2} (2y^1, \dots, 2y^n, -1+|y|^2) \right)$$

$$= \frac{1}{|y|^2} (y^1, \dots, y^n)$$

which is diffeomorphism from

$$\uparrow \uparrow \mathbb{R}^n \setminus \{0\}.$$

Check: φ_{\pm} are conti and invertible, and

$$\varphi_{\pm}^{-1}(y^1, \dots, y^n) = \frac{1 \cdot (2y^1, \dots, 2y^n, \pm(1-(y^1)^2 - \dots - (y^n)^2))}{1 + (y^1)^2 + \dots + (y^n)^2}$$

is also continuous.

Example: (The set of all straight line in \mathbb{R}^2)

i.e. $ax + by + c = 0$ (form of straight line)

(a, b, c) and (a', b', c') defines same line iff $[a : b : c] = [a' : b' : c']$

Note: $[0 : 0 : 1]$ will not given \forall line, while for the others in \mathbb{RP}^2 , we have.

bijective map [Möbius band!]

$$\varphi: \{\text{the set of all lines in } \mathbb{R}^2\} \rightarrow \mathbb{RP}^2 \setminus \{[0 : 0 : 1]\}$$

$$ax + by + c = 0 \mapsto [a : b : c]$$

\Rightarrow a smooth manifold structure

Def 1.1: Let (M, \mathcal{A}) be a smooth manifold, and $f: M \rightarrow \mathbb{R}$ a function

(1) we say that f is smooth at $p \in M$ if there $\exists (\varphi_\alpha, U_\alpha, V_\alpha) \in \mathcal{A}$ w/ $p \in U_\alpha$
s.t. the function $f \circ \varphi_\alpha^{-1}: V_\alpha \rightarrow \mathbb{R}$ is smooth at $\varphi_\alpha(p)$

(2) we say that f is a smooth func on M if hold for all $x \in M$.

RMK: we let $(\varphi_\beta, U_\beta, V_\beta)$ be another chart in \mathcal{A} w/ $p \in U_\beta$, by compatibility, the function

$$f \circ \varphi_\beta^{-1} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1}) \text{ must be smooth at } \varphi_\beta(p)$$

So, we solve in "Lecture 2" the index problem.

RMK2: According to chain rule, it's easy to see that if $f: M \rightarrow \mathbb{R}$ is smooth at $p \in M$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ is smooth at $f(p)$, then $h \circ f$ is smooth at p .

Example: Each coordinate function $f_i(x^1, \dots, x^{n+1}) = x^i$ is smooth on S^n since

$$f_i \circ \varphi_\pm^{-1}(y^1, \dots, y^n) = \begin{cases} \frac{2y^i}{1+|y|^2} & 1 \leq i \leq n \\ \pm \frac{1-|y|^2}{1+|y|^2} & i = n+1 \end{cases} \quad \text{are smooth func on } \mathbb{R}^n$$

Notation: We will denote all functions (smooth) on M by $C^\infty(M)$ "commutative algebra"

If f, g smooth $\Rightarrow \alpha f + \beta g$ and $\alpha f g$ are smooth

Def 1.2: Suppose $f \in C^\infty(M)$. the support of f is by definition the set

$$\text{supp}(f) = \overline{\{p \in M \mid f(p) \neq 0\}}$$

and f is compactly supported: $f \in C_c^\infty(M)$, if the support of f is comp in M .

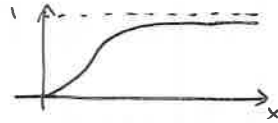
Obviously: if $f, g \in C_0^\infty(M) \Rightarrow af + bg \in C_0^\infty(M) \Rightarrow C_0^\infty(M)$ is an ideal of $C^\infty(M)$
 $f \cdot g \in C_0^\infty(M)$

N.B: If M is cpt, then \forall func (smooth) is cpt-supported.

Bump-func: (test function) "cpt support + smooth + non-negative + " ≤ 1 " + " $= 1$ on cpt set"

Example:

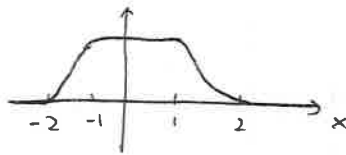
$$f_1(x) = \begin{cases} e^{-1/x} & (x > 0) \\ 0 & (x \leq 0) \end{cases} \Rightarrow f_1(x) = \begin{cases} \in (0, 1) & x > 0 \\ 0 & x \leq 0 \end{cases}$$



$$f_2(x) = \frac{f_1(x)}{f_1(x) + f_1(1-x)} \Rightarrow f_2(x) = \begin{cases} 0 & x \leq 0 \\ \in (0, 1) & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$



$$f_3(x) = f_2(2 - |x|) \Rightarrow f_3(x) = \begin{cases} 0 & |x| \geq 2 \\ \in (0, 1) & 1 < |x| < 2 \\ 1 & |x| \leq 1 \end{cases}$$



Thm 1.2: Let M be smooth manifold, $A \subset M$ is cpt and $U \subset M$ is open w/ $A \subset U \subset M$

then \exists a bump function $\varphi \in C_0^\infty(M)$ s.t $|\varphi| \leq 1$ and $\varphi \equiv 1$ on A and $\text{supp}(\varphi) \subset U$.

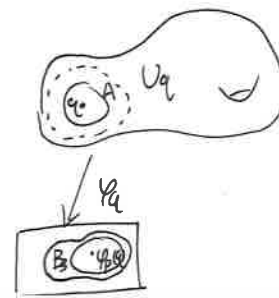
proof: [idea: cover A by finite small pieces $\in (U, U, U)$, so that one can copy the above example].

For each $q \in A$, there $\exists (V_q, U_q, V_q)$ near q s.t $U_q \subset U$ and V_q contains the open ball $B_r(0)$ "r=3"

Let $\tilde{U}_q = \varphi_q^{-1}(B_r(0))$ and let

$$f_q(p) = \begin{cases} f_3(\varphi_q(p)) & p \in U_q \\ 0 & p \notin U_q \end{cases}$$

then $f_q \in C_0^\infty(M)$, $\text{supp}(f_q) \subset U_q$ and $f_q \equiv 1$ on \tilde{U}_q



Now the family of open sets $\{\tilde{U}_q\}_{q \in A}$ is open cover of A .

$$\exists \{ \tilde{U}_q \}_{i=1}^N$$

$$\text{Let } \psi = \sum_{i=1}^N f_{q_i}$$

Then ψ is cpt-supp on M .

$\psi \geq 1$ on A and

$\text{supp}(\psi) \subset U$.

$$\Rightarrow \varphi(p) = f_2(\psi(p)) \equiv 1. \quad \square$$

Partition of unity:

So far, we can always for $K \subset M$ covered by nbhds on which we can construct nice "local func"
cpt

By adding these funcs, we can find behavior nicely on K . \Rightarrow same thing for M .

Def 2.1: Let M be a smooth manifold, $\{U_\alpha\}$ be open cover $\rightarrow M$. A partition of unity (P.O.U) subordinate to $\{U_\alpha\}$ is collection of smooth functions $\{\beta_\alpha\}$ "On whole M !!!"

- (1) $0 \leq \beta_\alpha \leq 1$ for all α
- (2) $\text{supp}(\beta_\alpha) \subset U_\alpha$
- (3) each $p \in M$ has nbhd which intersects only finitely many $\text{supp}(\beta_\alpha)$'s.
- (4) $\sum_\alpha \beta_\alpha(p) = 1$ for all $p \in M$.

RMKS = (i) Denote W_p a nbhd of p w/ finite intersect w/ $\text{supp}(\beta_\alpha)$'s

(1) we have $\{W_p\}$ open cover $\rightarrow M \Rightarrow \{W_p\}_{i \in \mathbb{N}}$ covers M w/ W_{p_i} intersect only finite many $\text{supp}(\beta_\alpha)$
"2nd countable"

\Rightarrow Only finite many $\text{supp}(\beta_\alpha)$ is non-empty.

(2) For each p on open W_p , a sum like (4) is actually finite sum by (1)

Thm 2.2 (Existence) Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M . Then \exists a P.O.U subordinate to $\{U_\alpha\}$.

i.e. locally, each manifold $\cong \mathbb{R}^n$ so that one have lot's of things to operate

P.O.U $\xrightarrow{\text{"glue"}}$ local \hookrightarrow global smooth.

Application: (1) Approxx conti func via smooth

(2) define integrals of differential forms

(3) construct Riemann metric / linear connection. etc

Cor 2.3: Let M be a smooth manifold, $A \subset_{\text{closed}} M$, $A \subset_{\text{open}} U \subset M$

then \exists bump function $\varphi \in C_0^\infty(M)$ s.t. $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on A and $\text{supp}(\varphi) \subset U$.

proof: $\{U, M \setminus A\}$ is an open cover $\rightarrow M$. Let $\{p_1, p_2\}$ be P.O.U.

Then $\varphi := p_1$ is what we need, as p_1 is smooth, $0 \leq p_1 \leq 1$, $\text{supp}(p_1) \subset U$ and $p_1 \equiv 1$ on A since $p_2 = 0$ on A .

"Urysohn's Lemma" smooth-version

proof of thm 2.2:

Lemma 2.4: For \forall open cover $U = \{U_\alpha\} \rightarrow M$, one can find two countable family of open covers $\{V_j\} = \mathcal{V}$ and $W := \{W_j\}$ of M .

(1) each j , \bar{V}_j is cpt and $\bar{V}_j \subset W_j$

(2) W is a refinement of U , for each j , $\exists \alpha = \alpha(j)$ s.t. $W_j \subset U_\alpha$.

(3) W is locally finite, \forall nbhd of $p \in M$, $W_j \cap W_k \neq \emptyset$ for only finitely many W_j 's.

Since $\bar{V}_j \subset W_j$ is cpt and $W_j \subset M$ open

Thm 1.2 $\Rightarrow \exists \varphi_j \in C_0^\infty(M)$ s.t.

$$0 \leq \varphi_j \leq 1, \varphi_j \equiv 1 \text{ on } \bar{V}_j, \text{supp}(\varphi_j) \subset W_j.$$

Since W is locally finite covering,

$$\varphi = \sum_j \varphi_j \text{ is well defined smooth func on } M.$$

Since φ_j is non-negative, \mathcal{V} covering $\rightarrow M$, $\varphi > 0$ on M .

$$\text{Let } \psi_j = \frac{\varphi_j}{\varphi} \text{ are smooth and } 0 \leq \psi_j \leq 1 \text{ and } \sum_j \psi_j = 1.$$

For each j , we fixed index $\alpha(j)$

so that $W_j \subset U_{\alpha(j)}$ and define

$$p_\alpha = \sum_{\alpha(j)=\alpha} \psi_j \text{ (finite sum near each pt)}$$

By local finiteness $\rightarrow W$.

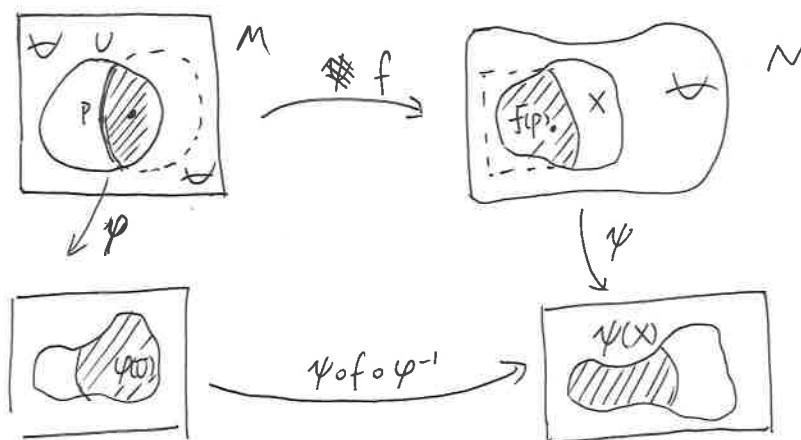
$$\begin{aligned} \text{supp } p_\alpha &= \overline{\bigcup_{\alpha(j)=\alpha} \text{supp } \psi_j} = \bigcup_{\alpha(j)=\alpha} \overline{\text{supp}(\psi_j)} \\ &= \bigcup_{\alpha(j)=\alpha} \text{supp } \psi_j \subset U_\alpha. \end{aligned}$$

$\Rightarrow \{p_\alpha\}$ is P.O.U subordinate to $\{U_\alpha\}$.

Smooth map between manifold

Def 1.1: Let M, N be manifold (smooth). We say a continuous map $f: M \rightarrow N$ is smooth if for \forall chart (φ_2, U_2, ψ_2) of M and ~~$(\psi_\beta, U_\beta, \varphi_\beta)$~~ of N the map:

$$\psi_\beta \circ f \circ \varphi_2^{-1}: \varphi_2(U_2 \cap f^{-1}(X_\beta)) \rightarrow \psi_\beta(X_\beta) \text{ is smooth.}$$



RMK: We require f to be conti. in definition.

\Rightarrow Guarantee $\psi_\beta \circ f \circ \varphi_2^{-1}$ is defined on $\varphi_2(p)$'s nbhd.

In general: $\text{smooth}(\psi \circ f \circ \varphi^{-1}) \Rightarrow f$ is conti. (Problem set)

prop 1.2: If $f: (M, \mathcal{A}) \rightarrow (N, \mathcal{B})$ is smooth, \mathcal{A}, \mathcal{B} are atlas on M and N that is compatible w.r.t \mathcal{A} and \mathcal{B} .

then $f: (M, \mathcal{A}_1) \rightarrow (N, \mathcal{B}_1)$ is also smooth "Indep of choice of charts" Check for same as PP. 5.

The set of all smooth map from $M \rightarrow N: C^\infty(M, N)$.

If $f \in C^\infty(M, N)$, and $g \in C^\infty(N, P) \Rightarrow g \circ f \in C^\infty(M, P)$.

As a consequence, \forall smooth map $f: M \rightarrow N$ induces a "pull-back" map

$$f^*: C^\infty(N) \rightarrow C^\infty(M) \quad g \rightarrow g \circ f$$

Examples of smooth map.

1) Consider \mathbb{R} equipped with $\{(\varphi_i(x)=x, \mathbb{R}, \mathbb{R})\}$, a map $f: M \rightarrow \mathbb{R}$ is a smooth map iff it's smooth function.

More generally, a map

$f = (f_1, \dots, f_k): M \rightarrow \mathbb{R}^k$ is a smooth map iff $f_i \in C^\infty(M)$ for $\forall M$.

2) The inclusion map $i: S^n \rightarrow \mathbb{R}^{n+1}$ is smooth, since

$$i \circ \varphi_\pm^{-1}(y^1, \dots, y^n) = \frac{1}{1+|y|^2} (2y^1, \dots, 2y^n, \pm(1-|y|^2))$$

are smooth maps from $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$

N.B: If g is a smooth func on \mathbb{R}^{n+1} , the pullback i^*g is just the restriction of g to S^n :

$$i^*g = g|_{S^n}$$

So the restriction of a smooth func on $\mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is a smooth func on S^n .

Ex 2: The proj map $\pi: \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$ is smooth, since

$$\varphi_i \circ \pi(x^1, \dots, x^{n+1}) = \left(\frac{x^1}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i} \right)$$

is smooth on $\pi^{-1}(U_i) = \{(x^1, \dots, x^{n+1}) : x^i \neq 0\}$ for each i .

Def 1.3: Let M, N be smooth manifold. A map $f: M \rightarrow N$ is a diffeomorphism if it is smooth, bijective, and f^{-1} is smooth.

" $M \cong N$ "

prop: (i) the identity map $\text{Id}: M \rightarrow M$ is a diffeomorphism

(ii) If $f: M \rightarrow N$; $g: N \rightarrow N$ is diffeo $\Rightarrow g \circ f$ is diffeomorphism.

(iii) If $f: M \rightarrow N$ is diffeomorphism $\Rightarrow f^{-1}$ is also moreover $\dim M = \dim N$.

$\{f: M \rightarrow N \mid f \text{ is a diffeomorphism}\} := \text{Diff}(M)$ is a group of M .

Examples: (1) For $M = \mathbb{R}$, the two atlas $\mathcal{A} := \{(U(x)=x, \mathbb{R}, \mathbb{R})\}$ and $\mathcal{B} = \{(U(x)=x^3, \mathbb{R}, \mathbb{R})\}$ defines non-equivalent smooth structure.
However: $f: (\mathbb{R}, \mathcal{A}) \rightarrow (\mathbb{R}, \mathcal{B})$, $f(x) = x^{1/3}$ is a diffeomorphism

Differential of Eucl-smooth map:

Let U, V be Euclidean open, and $f: U \rightarrow V$ a smooth map. The differential of f assign to each pt $a \in U$, defines a linear map $df_a: \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose matrix is Jacobian matrix of f at a .

$$df_a = \begin{pmatrix} \frac{\partial f_1}{\partial x^1}(a) & \cdots & \frac{\partial f_1}{\partial x^n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x^1}(a) & \cdots & \frac{\partial f_m}{\partial x^n}(a) \end{pmatrix}$$

Since the "linearization" of the map f near the point x :

$$\lim_{x \rightarrow a} \frac{\|f(x) - f(a) - df_a(x-a)\|}{\|x-a\|} = 0.$$

(Chain rule): if $f: U \rightarrow V$ and $g: V \rightarrow W$ are smooth maps, so is the map $g \circ f: U \rightarrow W$ and

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

Thm 2.1 (Invariance of Dimension) If $f: U \rightarrow V$ is a diffeo, then for each $x \in U$, the differential df_x is a linear isomorphism $\dim U = \dim V$.

proof: Apply Chain rule to $f^{-1} \circ f = Id_U = Id_{\mathbb{R}^n} = \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\Rightarrow (df^{-1})_{f(x)} \circ df_x = Id_{\mathbb{R}^n}$$

similarly

$$\Rightarrow df_x \circ (df^{-1})_{f(x)} = Id_{\mathbb{R}^m}$$

$\rightarrow m=n$ and df_x is isomorphism.

"Inverse func them"

Thm 2.3: If $f: U \rightarrow V$ is a smooth map

and df_x is an isomorphism

$\Rightarrow f$ is a local diffeomorphism near x .

RMK on Inverse/implicit func theorem:

ex. consider the map

$$f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\} \quad z \rightarrow f(z) = z^2.$$

Then $f(z) = f(-z)$. So f is not a diffeomorphism since it is not invertible

However at each point $z = (x, y) \in \mathbb{R}^2 \setminus \{0\}$.

$$df_z = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

which is an isomorphism for each $z = (x, y) \neq (0, 0)$.

i.e. for $\forall x \in \mathbb{C} \setminus \{0\}$ one can find U_x of x , s.t. $f|_{U_x}: U_x \rightarrow f(U_x)$ is diffeomorphism

Def 2.2: Let $f: U \rightarrow V$ smooth map. We say f is a local diffeomorphism near $x \in U$ if there \exists a nbhd U_x and $V_{f(x)}$

s.t. $f|_{U_x}: U_x \rightarrow V_{f(x)}$ is diffeomorphism.

Thm (2.4) "Implicit": Let W be an open set

in $\mathbb{R}_x^n \times \mathbb{R}_y^m$ and $F = (F_1, \dots, F_m): W \rightarrow \mathbb{R}^m$

a smooth map. Let $(x_0, y_0) \in W$ so that

$$\begin{pmatrix} \frac{\partial F_1}{\partial y^1}(x_0, y_0) & \dots & \frac{\partial F_1}{\partial y^m}(x_0, y_0) \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y^1}(x_0, y_0) & \dots & \frac{\partial F_m}{\partial y^m}(x_0, y_0) \end{pmatrix} \text{ non singular}$$

then, there \exists nbhd $U_0 \times V_0$ of (x_0, y_0) in W and smooth $f: U_0 \rightarrow V_0$ so that

- $f(x_0) = y_0$
- $C := F(x_0, y_0)$, then $F^{-1}(C) \cap (U_0, y_0) = \text{Graph}(f)$

i.e. $F(x, f(x)) = C$ for all $x \in U_0$

Lecture 5: Differential of a smooth map (Mar 27th) Russell-Hua

Motivation = $f: M \rightarrow N$ smooth. In Euclidean S.P. df_p is a linear map between "Tangent Space"

Q: What's the tangent S.P. \rightarrow Manifold?

If M is concrete manifold in \mathbb{R}^n , then we choose (φ, U, V) near p , so that $\varphi^{-1}: V \rightarrow U$ is a diffeomorphism.

If we denote the embedding of M into \mathbb{R}^n to be $i: M \hookrightarrow \mathbb{R}^n$, then we have $i \circ \varphi^{-1}: V \rightarrow \mathbb{R}^n$ between Euclid open set, and def!: $T_p M$ to be the image of $d(i \circ \varphi^{-1})_{\varphi(p)}(\mathbb{R}^n)$ of the differential

$$d(i \circ \varphi^{-1})_{\varphi(p)}: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

Sec: The Euclid-differential / directional derivative: "Algebraic characteristic."

Recall: for $\forall \vec{v} \in \mathbb{R}^n$, the directional derivative is

$$D_{\vec{v}}^a f := df_x(\vec{v}) = \lim_{h \rightarrow 0} \frac{f(x+h\vec{v}) - f(x)}{h} = \left. \frac{d}{dt} \right|_{t=0} f(a+t\vec{v})$$

So given $\vec{v} \Rightarrow D_{\vec{v}}^a: C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$. In coordinate, if $\vec{v} = \langle v^1, \dots, v^n \rangle^T$, then

$$D_{\vec{v}}^a f = \sum v^i \frac{\partial f}{\partial x^i} \quad / \quad D_{\vec{v}}^a = \sum v^i \frac{\partial}{\partial x^i}$$

Some properties: (1) $D_{\vec{v}}^a$ is linear operator:

$$D_{\vec{v}}^a (af + bg) = a D_{\vec{v}}^a f + b D_{\vec{v}}^a g$$

(2) satisfied Leibniz law at a :

$$D_{\vec{v}}^a (fg) = f(a) \cdot D_{\vec{v}}^a g + g(a) \cdot D_{\vec{v}}^a f$$

conversely also.

★ prop. 1.1. If $D: C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ is linear and satisfied Leibniz law at a ,

$$D(fg) = f(a) D(g) + g(a) D(f)$$

then $D = D_{\vec{v}}^a$ for some vector \vec{v} at a .

proof: $\forall f \in C^\infty(\mathbb{R}^n)$, we have

$$f(x) = f(a) + \int_0^1 \frac{d}{dt} f(a+t(x-a)) dt$$

$$= f(a) + \sum_{i=1}^n (x^i - a^i) h_i(x)$$

$$\text{where } h_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(a+t(x-a)) dt$$

$$\begin{aligned} \text{By Leibnitz-Law} \Rightarrow D(1) &= D(1 \cdot 1) = 2D(1) \Rightarrow D(1) = 0 \\ &\Rightarrow D(c) = 0 \end{aligned}$$

$$\text{So, } D(f) = 0 + \sum_{i=1}^n D(x^i) h_i(a) + \sum_{i=1}^n (a^i - a^i) D(h_i) = \sum_{i=1}^n D(x^i) \frac{\partial f}{\partial x^i}(a)$$

Cont. on proof \rightarrow Prop 1.1

It follows that as an operator on $C^\infty(\mathbb{R}^n)$

$$D = \sum_{i=1}^n D(x^i) \frac{\partial}{\partial x_i} \Big|_{x=a}$$

i.e.: if we let $\vec{v} = (D(x^1), \dots, D(x^n)) \Rightarrow D = D_{\vec{v}}^a$

Def 1.2: Any linear operator $D^a: C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ satisfied Leibnitz law at a is called a derivative at a

Now, consider that \mathcal{D} of all derivative at a is a V.P.

$$\vec{v} \mapsto D_{\vec{v}}^a$$

we have

(i) It is a linear map: $\mathbb{R}^n \rightarrow \mathcal{D}_{\vec{v}}^a$

$$D_{a\vec{v}+\beta\vec{w}}^a = aD_{\vec{v}}^a + \beta D_{\vec{w}}^a.$$

(ii) injective: if $\vec{v}_1 \neq \vec{v}_2$, then $D_{\vec{v}_1}^a \neq D_{\vec{v}_2}^a$

(iii) surjective: Follow from Prop 1.1 (i.e. \forall linear Leibnitz operator $\exists \vec{v} \in \mathbb{R}^n$, s.t. $D = D_{\vec{v}}^a$)

$$\Rightarrow \mathcal{D}_{\vec{v}}^a \cong T_{\vec{v}}^a$$

Tangent vector on manifolds

Def 1.3: Let M be an n -dim smooth mf

A tangent vector at $p \in M$ is a

\mathbb{R} -linear map $X_p: C^\infty(M) \rightarrow \mathbb{R}$
satisfied Leibnitz law

$$X_p(fg) = f(p) \cdot X_p(g) + g(p) \cdot X_p(f)$$

for $\forall f, g \in C^\infty(M)$

$$\{X_p\} := T_p M, \text{ similarly } \Rightarrow X_p(c) = 0$$

Lemma 1.4: If $f = c$ in a nbhd of p ,
then $X_p(f) = 0$.

proof: Let φ be smooth func (bump func)
equals to 1 near p and 0 at points $f \neq c$

$$\text{Then } (f-c)(\varphi) \equiv 0$$

$$\begin{aligned} \Rightarrow 0 &= X_p((f-c)\varphi) = (f(p)-c)X_p(\varphi) \\ &\quad + X_p(f)\varphi(p) \\ &= X_p(f) \end{aligned}$$

i.e. If $f=g$ in nbhd of p , then $X_p(f) = X_p(g)$
 $\Rightarrow X_p(f)$ is determined by f near p .

So one can replace $C^\infty(M)$ in Def 1.3
by $C^\infty(U)$ and $U \subset_{\text{open}} M$ contains p .

prop 1.5 = If M is a smooth manifold, $p \in U \subset M$, where U is open, then

$$T_p M \supseteq T_p U$$

"Summary of 'germ' properties"

Sec: The differential of smooth map between smooth manifold:

Recall: The differential of a smooth map $f: U \rightarrow V$ between open sets in Euclid S.P at $a \in U$

is a linear map $df_a = T_a U \rightarrow T_a V$ whose matrix is Jacobian $(\frac{\partial f_i}{\partial x_j})(a)$
" " "
 $\mathbb{R}_x^n \rightarrow \mathbb{R}_y^m$

certainly, we can define $\vec{v} \in \mathbb{R}^n$ at a w/ derivative $D_{\vec{v}}^a = \sum v_i \frac{\partial}{\partial x_i}$

$$\Rightarrow df_a(\vec{v}) = \left(\frac{\partial f_i}{\partial x_j} \right) \vec{v} = \left(\sum_j \frac{\partial f_1}{\partial x_j} v_j, \dots, \sum_j \frac{\partial f_m}{\partial x_j} v_j \right)^T \in \mathbb{R}_y^m$$

when interpreted as a derivative on V at $f(a)$, it's a map: $g \in C^\infty(\mathbb{R}_y^m)$ to

$$\sum_i \sum_j v_j \frac{\partial f_i}{\partial x_j} \cdot \frac{\partial g}{\partial y_i} = \sum_j v_j \frac{\partial}{\partial x_j} (g \circ f) = D_{\vec{v}}^a (g \circ f)$$

\Rightarrow Def 2.1: Let $f: M \rightarrow N$ be a smooth map. Then for each $p \in M$, the differential of f is the linear map $df_p = T_p M \rightarrow T_p N$ defined by

$$df_p(X_p)(g) = X_p(g \circ f)$$

for all $X_p \in T_p M$ and $g \in C^\infty(N)$

prop of differentials:

Theorem 2.2 (Chain rule) Suppose $f: M \rightarrow N$ and $g: N \rightarrow P$ are smooth maps, then

$$d(g \circ f)_p = dg_{f(p)} \circ df_p$$

proof: For $\forall X_p \in T_p M$ and $h \in C^\infty(P)$

$$\partial(g \circ f)_p(X_p)(h) = X_p(h \circ g \circ f) = \partial f_p(X_p)(h \circ g) = \partial g_{f(p)}(\partial f_p(X_p)(h))$$

Cor 2.3: If $f: M \rightarrow N$ is diffeomorphism, then $df_p: T_p M \rightarrow T_{f(p)} N$ is a linear isomorphism

↖ (Lec 4, Thm 2.1)

In particular,

Cor 2.4: If $\dim M = n$, then $T_p M$ is an n -dim linear space.

proof: Let (φ, U, V) be chart near p . Then $\varphi: U \rightarrow V$ is diffeomorphism.

$$\Rightarrow \dim T_p M^* = \dim T_{f(p)} N = n$$

$$" \dim T_p U = \dim T_{f(p)} V "$$

N.B: We really show that: For \forall local chart (p, U, ψ) around p

$$T_p M = \text{span} \{ \alpha_1, \dots, \alpha_n \}.$$

where $\mathcal{L}_i := \mathcal{L} \varphi^{-1}(\frac{\partial}{\partial x^i})$ where $x^i \circ \varphi^{-1} = x^i$ actually

$$d_i: C^\infty(U) \rightarrow \mathbb{R}, \quad d_i f = \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))$$

Lecture 6: (Mar 29th) Russell Hua

The inverse function theorem:

Recall: If $f: M \rightarrow N$ a diffeomorphism $\Rightarrow df_p: T_p M \rightarrow T_p N$ is linear isomorphism.

Conversely:

Thm 1.1 (IFT): Let $f: M \rightarrow N$ be a smooth map s.t. $df_p: T_p M \rightarrow T_p N$ is linear isomorphism, then

there exists a nbhd U_1 of p and a nbhd X_1 of $q = f(p)$ s.t. $f|_{U_1}: U_1 \rightarrow X_1$ is diffeomorphism (local diffeo)

proof: Take a chart (φ, U, V) near p and a chart (ψ, X, Y) near $f(p)$ s.t. $f(U) \subset X$ (possible by shrinking U and V)
Since $\varphi: U \rightarrow V$ and $\psi: X \rightarrow Y$ are diffeomorphism.

$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} = d\psi_q \circ df_p \circ d\varphi_p^{-1}: T_{\varphi(p)} V = \mathbb{R}^n \rightarrow T_{\psi(q)} Y = \mathbb{R}^n$$

is an linear isomorphism. It follows from the IVP on \mathbb{R}^n that (Lec 4) There $\exists V_1$ of $\varphi(p)$ and Y_1 of $\psi(q)$
so that $\psi \circ f \circ \varphi^{-1}$ is linear diffeomorphism from $V_1 \rightarrow Y_1$. Take $U_1 = \varphi^{-1}(V_1)$ and $X_1 = \psi^{-1}(Y_1)$

Then $f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$ is a diffeomorphism from $U_1 \rightarrow X_1$.

Sec: Global diffeo v.s local diffeo

Def: We say a smooth map $f: M \rightarrow N$ is a local-diffeo near p , if it maps $U_p \xrightarrow{\sim} V_{f(p)}$

Example (local diffeo, X Global diffeo)

Let $f: S^1 \rightarrow S^1$ be given by $f(e^{i\theta}) = e^{2i\theta}$, then it's a local diffeo everywhere, but not global diffeo since it's not invertible (Lec 4 pp.5)

Prop 1.3: Suppose $f: M \rightarrow N$ is a local diffeomorphism near every $p \in M$. If f is invertible, then f is a global diffeomorphism.

proof: It's enough to show f^{-1} is smooth (globally). Fixed $q = f(p)$. Smoothness of f^{-1} is depend on f^{-1} near q .

Since f is diffeo from $U_p \xrightarrow{\text{onto}} V_{f(p)=q}$, f^{-1} is smooth at q .

Constant rank theorem.

Submersion/Immersion: Motivation If df_p is not linear isomorphism?

Def 2.1: Let $f: M \rightarrow N$ be a smooth map

(1) f is a submersion at p : if $df_p: T_p M \xrightarrow{\text{onto}} T_p N$ is surjective

(2) f is a immersion at p : if $df_p: T_p M \xrightarrow{\text{inl}} T_p N$ is injective

\Rightarrow at $\forall p \in M \Rightarrow f$ is sub/immersion

OBS: (1) If f is a submersion: $\dim M \geq \dim N$

(2) If f is a immersion: $\dim M \leq \dim N$.

examples: (1) $\pi: TM \rightarrow M$ is a submersion

(2) "zero section" $\zeta: M \rightarrow TM: p \rightarrow (p, 0)$ is immersion.

(3) Canonical submer: If $m \geq n$, then

$$\pi: \mathbb{R}^m \rightarrow \mathbb{R}^n, (x^1, \dots, x^m) \rightarrow (x^1, \dots, x^n)$$

(4) Canonical immer: If $m \leq n$, then inclusion map

$$\zeta: \mathbb{R}^m \hookrightarrow \mathbb{R}^n, (x^1, \dots, x^m) \rightarrow (x^1, \dots, x^m, 0, 0, \dots, 0)$$

N.B: \forall submersion/immersion looks like (3), (4) locally.

Thm 2.2 (Canonical Submersion/Immersion)

Let $f: M \rightarrow N$ be submersion/immersion at $p \in M$, then obviously $m = \dim M \geq \dim N = n$ then there \exists charts
 $m = \dim M \leq \dim N = n$

(φ_i, U_i, V_i) around p and (ψ_i, X_i, Y_i) around $q = f(p)$ s.t

$$\psi_i \circ f \circ \varphi_i^{-1} = \pi|_{V_i}$$

or

$$\psi_i \circ f \circ \varphi_i^{-1} = \tilde{\pi}|_{V_i}$$

Def 2.4: We say that a smooth map $f: M \rightarrow N$ is a constant rank map near $p \in M$, if there $\exists U_p$ so that df_p has constant rank r .
 (i.e $\exists r \in \mathbb{N}$, so that $\text{rank}(df)_q \equiv r$ for all $q \in U$).

Example: (1) If f is submersion/immersion at p , then it's sub/im near p , and thus constant rank map near p .

(2) "Canonical" constant rank map. Generally, by composing suitable canonical and immersion \Rightarrow constant rank map

$$\mathbb{R}^m = \mathbb{R}^{r+m-r} \xrightarrow{\pi} \mathbb{R}^r \xrightarrow{i} \mathbb{R}^{r+n-r} = \mathbb{R}^n$$

$$(x^1, \dots, x^r, x^{r+1}, \dots, x^m) \longrightarrow (x^1, \dots, x^r, 0, 0, \dots, 0)$$

Thm 2.5: (Constant Rank theorem) Let $f: M \rightarrow N$ be smooth map so that $\underbrace{(df)}_{\text{rank}} = r$ near p . Then there \exists charts (φ_i, U_i, V_i) around p and (ψ_i, X_i, Y_i) near $f(p)$ s.t

$$\psi_i \circ f \circ \varphi_i^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, 0, \dots, 0).$$

proof: Step 1 (Euclidean S.P)

Conti on proof of CRT:

Assume $U \subset \mathbb{R}^n$ and $f: U \rightarrow \mathbb{R}^m$ smooth so that df_x has constant rank r near $x \in U$.

By translation, we may assume $0 \in U$ and $f(0) = 0$. Since $\text{rank}(df)_0 = r$. By switching coordinate, we may assume that.

the upper left of Jacobian $df = \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i \leq n, 1 \leq j \leq m} \xrightarrow{\text{restrict}} \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i, j \leq r} \text{ is non-singular at } x=0. \text{ (non-singular near } x=0)$

i.e. we want to take f_1, \dots, f_r as coordinate, and keep r^{th} term unchanged)

Rank ① = rank ② (clearly)

Now define $\varphi: U \rightarrow \mathbb{R}^m$ by $\varphi(x) = (f_1(x), \dots, f_r(x), x^{r+1}, \dots, x^m)$. Then $\varphi(0) = 0$ and the differential

$$d\varphi = \begin{pmatrix} \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i, j \leq r} & * \\ 0 & I_{m-r} \end{pmatrix} \text{ is non-singular at } x=0$$

By IFT, φ is a local diffeo near 0. There $\exists 0 \in U_1 \subset_{\text{open}} \mathbb{R}^m$ and $V_1 \subset_{\text{open}} \mathbb{R}^m$ s.t. $\varphi: U_1 \rightarrow V_1$ is diffeomorphism.

By def, we have $f \circ \varphi^{-1}(f_1, \dots, f_r, x^{r+1}, \dots, x^m) = f \circ \varphi^{-1}(\varphi(x)) = f(x) = (f_1, \dots, f_n(x))$

i.e. locally, we have $f \circ \varphi^{-1}(x) = (x^1, \dots, x^r, g_{r+1}(x), \dots, g_n(x))$ for some smooth g_{r+1}, \dots, g_n ($g_i(0) = 0$)

Chain rule:

$$df_{\varphi^{-1}(x)} \circ (d\varphi^{-1})_x = \begin{pmatrix} Id_r & 0 \\ * & \frac{\partial g_i}{\partial x^j} \quad r+1 \leq i, j \leq m-r \end{pmatrix}$$

OBS: Since $(d\varphi^{-1})_x$ is linear isomorphism "rank $(df_x) = r$ near 0" \Rightarrow "rank $(df_{\varphi^{-1}(x)} \circ (d\varphi^{-1})_x) = r$ near 0."

and thus

$$\frac{\partial g_i}{\partial x^j} = 0 \quad \forall r+1 \leq i, j \leq m. \text{ near } 0.$$

It follows that in a small nbhd of 0, we have

(Cont. on proof of Constant rank thm)

$$g_i(x) = g_i(x', \dots, x^n) \quad \forall r+1 \leq i \leq n.$$

and near 0

$$f \circ \varphi^{-1}(x) = (x', \dots, x^r, g_{r+1}(x', \dots, x^r), \dots, g_n(x', \dots, x^r))$$

We still need to kill the g_i 's terms. So we define

$$\psi(y) = (y', \dots, y^r, y^{r+1} - g_{r+1}(x', \dots, x^r), \dots, y^n - g_n(y', \dots, y^r))$$

"Check it is indeed local diffeomorphism"

$$\Rightarrow \psi \circ f \circ \varphi^{-1}(x', \dots, x^r, x^{r+1}, \dots, x^n) = (x', \dots, x^r, 0, 0, \dots, 0)$$

"Again Follow from IFT and

$$d\psi_0 = \begin{pmatrix} \text{Id} & 0 \\ * & \text{Id}_{n-r} \end{pmatrix}$$

Step 2: (General case)

It follows from what we usually do from maps between manifold. Taking (φ, U, V) and (ψ, X, Y) near $f(p)$ (near p)

so that $f(U) \subset X$ and df_q has constant rank r on U . The $\psi \circ f \circ \varphi^{-1}$ has rank r , since.

$$d(\psi \circ f \circ \varphi^{-1})_x = d\psi_{f(\varphi^{-1}(x))} \circ df_{\varphi^{-1}(x)} \circ (d\varphi^{-1})_x \quad \text{and by Euclidean case we are done } \square$$

linear isomorphism

Consequence; map is a constant rank map iff it can be locally written as $j \circ s$

immersion

submersion.

Generally \Rightarrow If a constant rank map is surjective \Rightarrow submersion

injective \Rightarrow immersion.

Recall from calculus:

- (1) $a \in \mathbb{R}$ is called a critical point of smooth func $f: \mathbb{R} \rightarrow \mathbb{R}$ if $f'(a) = 0$
- (2) $a \in \mathbb{R}^n$ is called a critical point of smooth func $f: \mathbb{R}^m \rightarrow \mathbb{R}$ if $\frac{\partial f}{\partial x_i} = 0$ for all i .
(i.e. $df_a: \mathbb{R}^m \rightarrow \mathbb{R}$ is not surjective)
- (3) $a \in \mathbb{R}^m$ is a critical point of a smooth map $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ between Euclidean open sets if $df_a: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is not surjective.

Def 1.1: Let M, N be smooth manifold and $f: M \rightarrow N$ a smooth map.

- (a) we say $p \in M$ is a critical point of f if $df_p: T_p M \rightarrow T_{f(p)} N$ is not surjective $\Rightarrow \{p\}_f := \text{Crit}(f)$.
- (b) $p \in M$ is a regular point if it's not critical point.
- (c) $q \in N$ is a regular value of f if $\forall p \in f^{-1}(q)$ is a regular point.

We also say $q \in N$ is a critical value if it's not regular value.

RMK: Let $f: M \rightarrow N$ smooth map

- (i) $q \in N \setminus \text{Im}(f)$ is a regular value
- (ii) $\text{Im}(\text{Crit}(f)) = \text{"set of critical values"}$ but $f^{-1}(\text{crit value}) \neq \text{Crit}(f)$.

Ex. $f \in C^\infty(M)$, $p \in M$ is max/min of f . Then f has p as critical pt.

proof. For $\forall X_p = \sum a_i d_i|_p \in T_p M$, we have

$$X_p(f) = \sum a_i d_i|_p(f) = \sum a_i \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) = 0.$$

so $g(t) = t \in C^\infty(\mathbb{R})$

$$df_p(X_p)(g) = X_p(g \circ f) = X_p(f) = 0 \Rightarrow df_p(X_p) = 0 \Rightarrow p \text{ is critical pt.}$$

Lemma 1.3: If we identify $T_t \mathbb{R}$ w/ \mathbb{R} , then

$$df_p(X_p) = X_p(f), \quad \forall f \in C^\infty(\mathbb{R}), \quad \forall X_p \in T_p(\mathbb{R})$$

example. $f: S^n \rightarrow \mathbb{R}$, $f(x_1, \dots, x_{n+1}) = x_{n+1}$

has "pole" $(0, \dots, 1)$ and $(0, \dots, -1)$ as critical value! and all other pts are regular value.

Extreme/trivial case: $f: M \rightarrow \mathbb{N}$ const. map $f(p) \equiv z_0 \in \mathbb{N}$,

$f: M \rightarrow \mathbb{N}$ smooth, but $\dim(M) < \dim(\mathbb{N})$

Sard's theorem

I.e. the set of critical pts \rightarrow values is negligible in N . (Sometimes failed "const. map")

Thm: For \forall smooth map $f: M \rightarrow N$, the set of critical pts has measure zero in N .

N.B: Since we may cover M by $\#m < \infty$, (U_i, φ_i) charts, and union of meas-zero set is still $\mu(E) = 0$.
It's sufficient to prove in $\mathbb{R}^m \rightarrow \mathbb{R}^n$ case.

proof first, we have for $m < n$, the result trivially hold.

We will proof by induction, for $m=0$, then clearly meas-0 set \rightarrow meas set. Let C be the set of critical pts
WTS: $f(C)$ meas-0 in N .

Let $G_j := \{x \in U \mid \partial^a f(x) = 0 \text{ for all } |a| \leq j\}$.

OBS:
 $f(C) = f(C \setminus C_1) \cup f(C_1 \setminus C_2) \cup \dots \cup f(C_{k-1} \setminus C_k) \cup f(C_k)$

Step 1: $f(C \setminus C_1)$ meas-0 in N

for $\forall x \in C \setminus C_1$, $\exists U_x \ni x$, s.t. $f(C \cap U_x)$ has meas-0. Since $C \setminus C_1$ can be covered by at most countable many such sets $\Rightarrow f(C \setminus C_1)$ has meas 0.

Step 2: $f(C_1 \setminus C_{i+1})$ meas-0 in N .

Let $x \in C_i \setminus C_{i+1}$, there is a w w/ $|a| = i$, s.t. $w := \partial^a f$ vanish on C_i , at least $\frac{\partial w}{\partial x^i}$ does not vanish at x .

By IFT, we have

$h: U_x \rightarrow \mathbb{R}^m$, $h(x) = (w(x), x^1, \dots, x^m)$
onto
diffeomorphic on $V \subseteq \mathbb{R}^m$

By construction, $h: C_i \cap U_x \rightarrow \{0\} \times \mathbb{R}^{m-1}$

Let $g = f \circ h^{-1}$ and $\bar{g}: (\{0\} \times \mathbb{R}^{m-1}) \cap V \rightarrow \mathbb{R}^n$

\Rightarrow the set of critical pt of \bar{g} is meas in \mathbb{R}^n .

Step 3: $f(C_k)$ has meas-0 for $k \gg 1$, say $k \geq \frac{m}{n}$. (Milner)

Let $Q \subset \mathbb{U}$ be cube w/ side δ -length. WTS: $k > \frac{m}{n} - 1$, $f(C_k \cap Q)$ has meas-0.

By Taylor's thm, Q cpt and C_k critical pt set, we have

$$f(x+h) = f(x) + R(x, h).$$

where $|R(x, h)| < a|h|^{k+1}$ for $x \in C_k \cap Q$, $x+h \in Q$ and a depends only on f and Q .

Now, we divide Q into r^m cubes w/ side $\frac{\delta}{r}$. $x \in Q_1$ can be written as $x+h$ w/ $|h| < \sqrt{m} \frac{\delta}{r}$.

By this we can show that $f(C_k \cap Q) \subset \bigcup_{m=1}^{r^m}$ cubes and

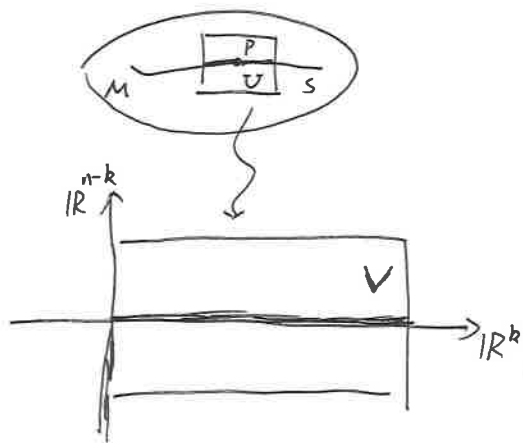
$$\text{Vol} \leq r^m \left(\frac{b}{r^{k+1}} \right)^n = b^n r^{m - (k+1)n} \quad \text{where } b = 2a(\sqrt{m}\delta)^{k+1}.$$

Since $k > \frac{m}{n} - 1$, we see $\text{Vol} \rightarrow 0$ as $r \rightarrow \infty$. $\Rightarrow f(C_k \cap Q)$ is meas-0 in N . \square

Def. $S \subset M$ is a k -dim submanifold (smooth) of M if $\forall p \in S$, there $\exists (\varphi, U, V)$ of p s.t.

$$\varphi(U \cap S) = V \cap (\mathbb{R}^k \times \{0\}) = \{x \in \varphi(U) \mid x^{k+1} = \dots = x^n = 0\}.$$

we call $\text{codim}(S) = n - k$ "codimension of S ".



ex. M, N smooth mfs, and $f: M \rightarrow N$ smooth. Then

$$T(f) = \{(p, q) \mid q = f(p)\} \subset M \times N$$

is smooth submanifold of $M \times N$.

Let (φ, U, V) be arbitrary chart near p . and (ψ, X, Y) near $q = f(p)$.

$\Rightarrow (\varphi \times \psi, U \times X, V \times Y)$ is chart $M \times N$ near (p, q)

Not enough: $q = f(p)$ as $\psi^{-1}(y) = f(\varphi^{-1}(x))$

$$\Rightarrow y = \psi \circ f(\varphi^{-1}(x))$$

Let $\gamma: V \times Y \rightarrow \mathbb{R}^m \times \mathbb{R}^n$ as $(a, b) \rightarrow (a, b - \psi \circ f(\varphi^{-1}(a)))$

$\Rightarrow \gamma$ is 1-1 and local diffeomorphism es \Rightarrow Global diffeo and $V \times Y \xrightarrow{\text{onto}} \psi(X \times Y)$

$$(p, q) \in T_f \cap (U \times X)$$

\Downarrow

$$\begin{aligned} \psi(q) &= \psi(f(\varphi^{-1}(\varphi(p)))) = \psi(\varphi(p), \psi(p)) \\ &= (\varphi(p), 0). \end{aligned}$$

$(\psi \circ (\varphi \times \psi), U \times X, V \times Y)$ charts.

near (p, q) .

\Uparrow

RMK: 1) $\forall f \in C(U)$, $U \subset \mathbb{R}^n$, there is smooth structure on $P_f \rightarrow$ smooth manifold dim n .

2) f is not smooth func w/ P_f smooth submf:

$$f(x) = x^{1/3}, \quad y = f(x) \text{ "}" \not\equiv \not\equiv x = y^3 \text{ which is of a smooth submanifold of } \mathbb{R}^2.$$

ex 2. S^n is smooth submf of \mathbb{R}^{n+1} .

The induced smooth structure:

Let

$$\pi: \mathbb{R}^n \rightarrow \mathbb{R}^k, (x^1, \dots, x^n) \mapsto (x^1, \dots, x^k)$$

$$i: \mathbb{R}^k \hookrightarrow \mathbb{R}^n, (x^1, \dots, x^k) \mapsto (x^1, \dots, x^n)$$

then we have

Prop 1.2: (U, φ, V) on M , that satisfied $\varphi(U \cap S) = V \cap (\mathbb{R}^k \times \{0\})$

Let $X = U \cap S$, $Y = \pi \circ \varphi(X)$ and $\psi = \pi \circ \varphi|_X$. Then (ψ, X, Y) smooth and \forall charts of this form are compatible $\Rightarrow S$ smooth manifold, and $i: S \hookrightarrow X$ is immersion.

proof. By def: ψ is invertible, $\psi^{-1} = \varphi^{-1} \circ i$. So (ψ, X, Y) charts on S . Let ψ_β, ψ_α be maps.

$$\psi_\beta \circ \psi_\alpha^{-1} = \pi \circ \varphi_\beta \circ \varphi_\alpha^{-1} \circ i = \pi \circ \varphi_{\alpha, \beta} \circ i, \text{ smooth again.}$$

the map $i: S \hookrightarrow M$ is smooth immersion,

$$\varphi \circ i \circ \psi^{-1} = i$$

Now, let $S \subset M$ be submanifold, $p \in S$. Since $i: S \hookrightarrow M$ is an embedding

$di_p: T_p S \rightarrow T_p M$ is injective.

We can identify $\forall X_p \in T_p S$ w/ $\tilde{X}_p = di_p(X_p)$ in $T_p(M)$ so that $\forall f \in C^\infty(M)$.

$$\tilde{X}_p(f) = (di_p(X_p))f = X_p(f \circ i) = X_p(f|_S)$$

THM: If $S \subset M$ submanifold, $p \in S$. Then

$$T_p S = \{X_p \in T_p M \mid X_p(f) = 0 \text{ for all } f \in C^\infty(M) \text{ w/ } f|_S = 0\}.$$

proof. (\Rightarrow) Let $X_p \in T_p S$. then for $f \in C^\infty(M)$ w/ $f|_S = 0$, $\tilde{X}_p(f) = X_p(f|_S) = 0$.

(\Leftarrow) Let $X_p \in T_p M$ satisfied $X_p(f) = 0$ for $f|_S = 0$.

Take (φ, U, V) on M near p . S is given by $x^{k+1} = \dots = x^n = 0$

Then $T_p M$ is $\text{span}\{\partial_1, \dots, \partial_n\}$ while $T_p S$ is $\text{span}\{\partial_1, \dots, \partial_k\}$.

I.e $X_p = \sum X^i \partial_i \in T_p S$ iff $X^i = 0$ for all $i > k$.

Now let h be bump func on U supported $\equiv 1$

For $j > k$, consider $f_j(x) = h(x) \cdot x^j(\varphi(x))$. $f_j(x) = 0$ on $M \setminus U$.

Then $f_j|_S = 0$. So

$$0 = X_p(f_j) = \sum \frac{X^i \cdot \partial(h(\varphi^{-1}(x))x^j)}{\partial x^i}(\varphi(p)) = X^j \text{ for } \forall j > k \Rightarrow X_p \in T_p(S) \quad \square$$

We have shown the result that: preimage of a regular value of \forall smooth map is smooth submanifold.

By Sard theorem: if $f(M)$ is not negligible in N , then for most $q \in f(M)$, $f^{-1}(q)$ is a smooth submanifold of M .

Thm 2.1: Let $f: M \rightarrow N$, $q \in N$ regular value.

Then $f^{-1}(q)$ is smooth submf of M w/ $\dim(f^{-1}(q)) = \dim(M) - \dim(N)$.

Moreover, $p \in S$, $T_p S = \ker(df_p: T_p M \rightarrow T_p N)$

Thm 2.2: Let M, N smooth mf, $f: M \rightarrow N$ has const. rank r . Then each level set of f is closed mf of codim $= r$.
(Recall!)

Moreover, $p \in S$, $T_p S = \ker(df_p: T_p M \rightarrow T_p N)$

proof (2.1): Let $p \in S := f^{-1}(q)$. Then by const. rank thm, $\exists (\varphi_i, U_i, V_i)$ centered at p and (ψ_i, X_i, Y_i) centered at q s.t. $f(U_i) \subset X_i$ and

$$\psi_i \circ f \circ \varphi_i^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^r, 0, \dots, 0)$$

It follows that

φ_i maps $U_i \cap f^{-1}(q)$ onto $V_i \cap \{(0, 0, \dots, 0, x^{r+1}, \dots, x^m)\}$. So $f^{-1}(q)$ is submf of codim r .

Let $i: S \hookrightarrow M$. Then for $\forall p \in S$, $f \circ i(p) = q$. In.e $f \circ i$ is const map on S .

So $df_p \circ di_p = 0$ i.e. $df_p = 0$ on $\text{Im}(di_p)$ / $T_p S \subset \ker(df_p)$

But $\dim T_p S = \dim S = m - r$ and

$$\dim \ker(df_p) = \dim \ker(df_p) = \dim \ker(df_p) = m - r.$$

$$\Rightarrow \dim T_p S = \dim \ker(df_p) \Rightarrow T_p S = \ker(df_p). \quad \square$$

Some examples:

(i) S^n is smooth submf of \mathbb{R}^{n+1}

(ii) $SL(n, \mathbb{R})$ is smooth submf of $GL(n, \mathbb{R})$

(iii) $O(n, \mathbb{R}) \subset GL(n, \mathbb{R})$.

N.B. the level set of critical value may failed to be smooth mf.

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = x^2 - y^2$$

$$df_{(x,y)} = (2x, -2y) \text{ and critical pt is } (0, 0).$$

But $f^{-1}(0) = \{(x, y), x^2 - y^2 = 0\}$ is not a mf!!!

Difference between smooth mf and immersed mf?

prop 2.3. Let S be a smooth submf of M . Then $i: S \hookrightarrow M$ is a homeomorphism from $S \rightarrow i(S)$

Def 2.4: $f: M \rightarrow N$ immersion. f is called embedding if it's a homeomorphism onto its image $f(N)$
topology on $f(N)$ is subsp topo.

By def: $i: S \hookrightarrow M$ is an embedding. so each smooth submanifold is the image of an embedding!!!

Thm 2.5. Let $f: N \rightarrow M$ embedding. Then $f(N)$ is smooth submf of M .

proof. Let $p \in N$, $q = f(p) \in M$. f is immersion, by canonical immersion then

$\exists (\varphi_i, U_i, V_i), (\psi_i, X_i, Y_i)$ near p, q . s.t. on V_i , $\psi_i \circ f \circ \varphi_i^{-1}$ is the canonical embedding $J: \mathbb{R}^m \rightarrow \mathbb{R}^n$

i.e. $\psi_i \circ f = J \circ \varphi_i$ on U_i , since f is homeomorphism on $f(U_i) \Rightarrow f(U_i) \subset_{\text{open}} f(N) \cap X_i$.

Replace X_i by $X_i \cap X \Rightarrow \psi_i(X_i \cap f(N)) = Y_i \cap \psi_i(f(U_i)) = Y_i \cap J(\varphi_i(U_i)) = Y_i \cap (\mathbb{R}^m \times \{0\})$.

Summary: (Immersion and embedding)

- If $f: M \rightarrow N$ an immersion, then by Canonical Immersion theorem, $\forall p \in N$ has nbhd in N whose image is "nice" in M .
- If $f: M \rightarrow N$ an embedding, $\forall p \in f(N)$ have nbhd in $f(N)$ that is "nice" in M (Thm 2.5).

Thm (Whitney - embedding thm)

Any m - $2m$ smooth mf M can be embedded into \mathbb{R}^{2m+1} (and immersed into \mathbb{R}^{2m}).

We will only proof for cpt case, for the non-cpt case, the steps are same.

- (1) Injective immerse M into \mathbb{R}^k for $k \gg 1$.
- (2) For $k > 2m+1$, project \mathbb{R}^k to some \mathbb{R}^{k-1}
- (3) cpt conditions, injective immersion \rightarrow embeddings

Thm 1.1: \forall cpt mf M admits an injective immersion into \mathbb{R}^k for $k \gg 1$.

proof. Let $\{\varphi_i, U_i, \nu_i\}_k$ be finite charts set on M . (cover)

Let $\{p_i | 1 \leq i \leq k\}$ be P.O.U subord to $\{U_i | 1 \leq i \leq k\}$.

Define $\Phi: M \rightarrow \mathbb{R}^{k(m+1)}$, $p \rightarrow (p_1(p)\varphi_1(p), \dots, p_k(p)\varphi_k(p), p_1(p), \dots, p_k(p))$

1) Φ is injective. If $\Phi(p_1) = \Phi(p_2)$. Let i be s.t $p_i(p_1) = p_i(p_2) \neq 0$. Then $p_1, p_2 \in \text{supp}(p_i) \subset U_i \Rightarrow \varphi_i(p_1) = \varphi_i(p_2)$.
since φ is bijective $\Rightarrow p_1 = p_2$

2) Φ is immersion, For $\forall X_p \in T_p M$, by Leibnitz law,

$$d\Phi_p(X_p) = (X_p(p)\varphi_1(p) + p_1(p)(d\varphi_1)_p(X_p), \dots, X_p(p_k)\varphi_k(p) + p_k(p)(d\varphi_k)_p(X_p), X_p(p_1), \dots, X_p(p_k))$$

$$\text{If } d\Phi_p(X_p) = 0 \Rightarrow X_p(p_i) = 0 \text{ for } \forall i. \Rightarrow p_i(p)(d\varphi_i)_p(X_p) = 0.$$

Similarly, pick i s.t $p_i(p) \neq 0$. $(d\varphi_i)_p(X_p) = 0$ since φ_i diffeomorphism $\Rightarrow X_p = 0$ so $d\Phi_p$ is injective.

Thm 1.3. If M , $\dim(M) = n$, admits an injective immersion from $M \rightarrow \mathbb{R}^k$ for some $k > 2n+1$, then it admits an injective immersion into \mathbb{R}^{2n+1} . (N.B. the step 2 does not require cpt-property)

(I skipped this proof, but basically we gonna use Sard's theorem to show that $\Phi_{[n]}$ is not immersion is of measure-zero)

Thm 1.5: If $f: M \rightarrow N$ is an injective immersion, and N is cpt, then f is an embedding.
proof. f injective. $f: N \rightarrow f(N)$ is invertible. Since $f: N \rightarrow M$ conti. $f: N \rightarrow f(N)$ conti.

Let $A \subset_{\text{closed}} N$, Since N cpt $\Rightarrow A$ cpt. So $f(A)$ is cpt. Since $f(N)$ is Hausdorff, $f(A)$ is closed in $f(N)$.

So we have f conti + closed and $f^{-1}: f(N) \rightarrow N$ is conti. \square

Immediate Consequence: \forall smooth cpt M of dim n can be immersed into \mathbb{R}^{2n} and embedded into \mathbb{R}^{2n+1} .

Recall lec 6 $\exists f$ df_p is a linear isomorphism $\Rightarrow f$ is local diffeo near p .

Thm 1.1 (Generalized IVFT)

Let $f: M \rightarrow N$, $X \subset M$ submanifold. Suppose $df_p: T_p M \rightarrow T_{f(p)} N$ is linear isomorphism for every $p \in X$.

And (i) If X is cpt, then assume f is 1-1 on X

(ii) If X is non-cpt, we assume $f(X)$ is submanifold of N and f maps X diffeo onto $f(X)$

Then f maps $U \subset X \subset M$ diffeo onto $V \subset f(X) \subset N$.

RMK: $f: 1-1$ on $X \Leftrightarrow X$ diffeomorphically onto $f(X)$ when X cpt!

$f|_X: X \rightarrow N$ is immersion. So if $f|_X$ is 1-1 and X cpt, then $f|_X: X \rightarrow N$ is an embedding, thus $f|_X: X \rightarrow f(X)$ is diffeomorphism.

Example for (ii) need additional assup:

e.g. $f: \mathbb{R}^2 \rightarrow T^2, (t, s) \rightarrow (e^{it}, e^{is})$

f is locally diffeo near any $(t, s) \in \mathbb{R}^2$. Let X be "irrational slope line"

$X = \{t, \sqrt{2}t\}, t \in \mathbb{R} \subseteq \mathbb{R}^2$ and $f|_X$ is injective but there's $\nexists U \subset X \subset \mathbb{R}^2$ and $V \subseteq T^2$ so that

f maps U diffeo onto V !!

proof. (We only do cpt case here)

By IFT, f is a local diffeomorphism near $\forall x \in X$. According to Prop 1.3 Lec 6, it's sufficient to show f is injective on nbhd of X .

We embed M into \mathbb{R}^k , and let ε -nbhd of $X \subset M$ as

$$X^\varepsilon = \{x \in M \mid d(x, X) < \varepsilon\}, \quad d(x, X) = \inf \{d(x, y), y \in X\}.$$

\cap open
 \mathbb{R}^k

Since X is cpt, then X^ε is bdd. Moreover, we have $X = \bigcap_{k>0} X^{1/k}$ since X is closed.

Suppose f is not 1-1 on each $X^{1/k}$, then we can find $a_k \neq b_k \in X^{1/k}$ s.t. $f(a_k) = f(b_k)$.

Since $a_k \in A_k \subset \mathbb{R}^k$, we can find $a_{k_i} \rightarrow a_\infty \in X$. Similarly, $b_{k_i} \rightarrow b_\infty \in X$, since $f(a_\infty) = f(b_\infty)$
bdd + cpt
(can be $X^{1/k}$)

one must have f is 1-1 on X (f is smooth). So \forall nbhd of a_∞ , f is not 1-1. But df_{a_∞} is linear isomorphism implies f is a local diffeomorphism near a_∞ . (contradiction).

Non-cpt case: [Any non-cpt manifold can be written as union of countably many "cpt stripes"].

Tubular NBHD them.

$X \subset M$, smooth-submanifold. X always admits a "tubular" nbhd inside M !

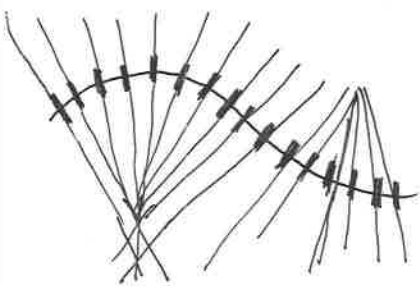
Thm 2.1: (ε -nbhd them) Let $\iota: X \hookrightarrow \mathbb{R}^k$ be smooth submf. Then \exists a conti positive value func $\varepsilon: X \rightarrow \mathbb{R}^+$, s.t if we let X^ε be ε -nbhd of X .

then,

- (1) $\forall y \in X^\varepsilon$ possessed a unique closest pt $\pi_\varepsilon(y) \in X$ (impressive! Just like projection them in Functional analysis)
- (2) the map $\pi_\varepsilon: X^\varepsilon \rightarrow X$ is a submersion

(N.B. if X is cpt, then ε can be const. func).

Before moving on. we note that we need to find an isomorphism between nbhd X inside its "normal bundle" (some lines crowded into submanifold) skinny lines can form sth !!



Let $\iota: X \hookrightarrow \mathbb{R}^k$ of dim r . For $\forall x \in X$. we can identify $T_x X$ w/ r -dim VP in \mathbb{R}^k

$$\text{I.e: } T_x X \cong d_{\iota_x}(\bar{T}_x X) \subset T_x \mathbb{R}^k \cong \mathbb{R}^k.$$

$$\text{Let } N_x(X, \mathbb{R}^k) := (T_x X)^\perp \text{ in } \mathbb{R}^k.$$

$$:= \{v \in T_x \mathbb{R}^k \cong \mathbb{R}^k \mid v \perp T_x X\}. \quad \dim(N_x) = k - r$$

$$\text{and define } N(X, \mathbb{R}^k) := \{(x, v) \in \mathbb{R}^k \times \mathbb{R}^k \mid x \in X, v \in N_x(X, \mathbb{R}^k)\} \subset T\mathbb{R}^k$$

Actually $N(X, \mathbb{R}^k)$ is k -dim submf of $T\mathbb{R}^k$.

And the canonical projection map

$\pi: N(X, \mathbb{R}^k) \rightarrow X, (x, v) \rightarrow x$ is a submersion.

Def 2.2. Let $\iota: X \hookrightarrow \mathbb{R}^k$ be smooth submf. We call $N(X, \mathbb{R}^k)$ above as normal bundle of $X \subseteq \mathbb{R}^k$.

proof of extension them:

Let $h: N(X, \mathbb{R}^k) \rightarrow \mathbb{R}^k, (x, v) \rightarrow x + v$.

$\forall (x, 0) \in N(X, \mathbb{R}^k)$, dh is non-singular, since $dh_{(x, 0)}: T_{(x, 0)}(X \times \{0\}) \subset T_{(x, 0)}N(X, \mathbb{R}^k)$ bijectively onto $T_x X \subset T_x \mathbb{R}^k$.

Also, $X \times \{0\} \xrightarrow[h]{\text{diffeo}} X \subset \mathbb{R}^k$
 \cap
 $N(X, \mathbb{R}^k)$

$T_{(x, 0)}(X \times \{0\} \cup N_x(X, \mathbb{R}^k)) \xrightarrow[\text{bijective onto}]{dh_{(x, 0)}} N_x(X, \mathbb{R}^k) \subset T_x \mathbb{R}^k$

By Generalize IFT, $\bigcup_{\cap} X \times \{0\} \xrightarrow[\text{diffeo}]{h} \bigcup_{\cap} V$. Now for each $x \in X$, we define $\epsilon(x) = \sup \{r \leq 1, B_r(x) \subset V\}$.
 \cap
 $N(X, \mathbb{R}^k) \xrightarrow{h} \mathbb{R}^k$
 \cap
 X
 \cap
 $N(X, \mathbb{R}^k) \xrightarrow{h} \mathbb{R}^k$

Check: ϵ is positive conti on X !

N.B: $X^\epsilon \subset V$ submf. and consider the map

$\pi_\epsilon: X^\epsilon \rightarrow X, y \rightarrow \pi_\epsilon(y) = \pi \circ h^{-1}(y)$.

It's a submersion since π is a submersion and h^{-1} is diffeo on V . WTS: $\pi_\epsilon(y)$ is unique

Let $z \in X$, represent $d(z, y) = \inf_x d(x, y) \Rightarrow y - z$ is perpendicular to X at z , $y - z \in N_z(X, \mathbb{R}^k)$.

$y = z + (y - z) = h(z, y - z)$.

$\therefore \pi_\epsilon(y) = z \Rightarrow z$ is unique $\Rightarrow \pi_\epsilon(y)$ is unique in X . \square

In general, not necessarily \mathbb{R}^k , we can still define the normal bundle:

$$N_x(X, M) := T_x M / T_x X. \quad X \subset M$$

$$\text{and } N(X, M) := \{(x, v), x \in X, v \in N_x(X, M)\}. \quad (x)$$

In fact, $\dim(N(X, M)) = \dim(M)$
 \uparrow
 smooth manifold.

Geometric speaking: WTS: $M \xrightarrow{\text{embed}} \mathbb{R}^k$. i.e.: $T_x X \subset T_x M \subset T_x \mathbb{R}^k$ and $T_x M / T_x X$ can be identified as

$$N(X, M) \cong \{(x, v) \mid x \in X, v \in T_x M \text{ and } v \perp T_x X\}. \quad (v)$$

and we have seen

$$T_{(x,0)} N(X, M) \cong T_x X \oplus T_x^\perp X$$

Theorem 2.3 (Tubular nbhd theorem) Let $X \subset M$ be smooth submf. Then \exists a diffeomorphism from $W \subset X \subset N(X, M)$

proof. Embedded M into \mathbb{R}^k . Let $\pi_\varepsilon: M^\varepsilon \rightarrow M$ be ε -nbhd them
 $(\iota: M \hookrightarrow \mathbb{R}^k \text{ embedding})$

\downarrow onto
 $V_\varepsilon \subset X \subset M$.

Again, consider $h: N(X, M) \rightarrow \mathbb{R}^k$, $h(x, v) \rightarrow x + v$

Then $W := h^{-1}(M^\varepsilon)$ is open nbhd of X in $N(X, M)$. Now consider the composition.

$$h_\varepsilon = \pi_\varepsilon \circ h: W \rightarrow M.$$

Then h_ε smooth and the identity map on $X \subset N(X, M)$. We also have $T_{(x,0)} N(X, M) \xrightarrow{(\partial h_\varepsilon)_{(x,0)}} T_x M$

So the theorem followed from Generalized IFT. \square

onto
 bijectives

Smooth vector field.

Review: a tangent vector X_p at $p \in M$ is a linear map: $X_p: C^\infty(M) \rightarrow \mathbb{R}$ s.t.

$$X_p(fg) = X_p(f)g(p) + X_p(g)f(p)$$

We can also replace $U \subset M$ by $C^\infty(M) \rightarrow C^\infty(U)$. And let (φ, U, V) be chart, then $\forall p \in U$, the tangent vectors $\{(\partial_i)_p \mid 1 \leq i \leq m\}$ form a basis $\rightarrow T_p M$.

$$(\partial_i)_p = (\partial \varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) : C^\infty(U) \mapsto \mathbb{R}$$

$$f \mapsto (\partial_i f)_p = \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} (\varphi(p))$$

where $\left\{ \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \mid 1 \leq i \leq n \right\}$ is std basis $\rightarrow T_{\varphi(p)} \mathbb{R}^n$. I.e. $\forall X_p \in T_p M$, we have

$$X_p = \sum_{i=1}^n a_i (\partial_i)_p$$

Geometric view: If we embed $M \rightarrow \mathbb{R}^k$, then we can visualize $(\partial_i)_p$ as TV

$$\gamma_i(t) = \varphi^{-1}((0, \dots, 0, t, 0, \dots, 0)), \quad t \in (-\varepsilon, \varepsilon)$$

at p , where t at i^{th} position, ε chosen that $(0, \dots, 0, t, 0, \dots, 0) \in V$ for all $-\varepsilon < t < \varepsilon$.

and $\varphi(p) = 0 \in V \subset \mathbb{R}^n$.

Def. 1.1 Let $\gamma: \mathbb{R} \rightarrow M$ be curve on M w/ $\gamma(0) = p$ and $\frac{d}{dt}$ be unit tv on \mathbb{R} . Then $\dot{\gamma}(0) = d_{\gamma_0} \left(\frac{d}{dt} \right)$ the tv of curve γ at $p = \gamma(0)$.

N.B. By definition, $\forall f \in C^\infty(M)$, we have

$$\dot{\gamma}(t)(f) = \partial_{\gamma(t)}(\frac{d}{dt})(f) = \frac{d}{dt}|_{t=0}(f \circ \gamma)$$

This coincide w/ our former definition.

Def 1.2. A smooth vector field X on a M manifold is smooth-assignment that $\forall p \in M$ a tangent vector $X_p \in T_p(M)$

We say X is smooth on $U \subset (M, \varphi, U, \psi)$, if all coefficient X^i 's are smooth on U .

$$Xf: M \rightarrow \mathbb{R}, p \mapsto Xf(p) := X_p(f)$$

prop 1.3. X is smooth iff $f \in C^\infty(M) \Rightarrow Xf \in C^\infty(M)$

And we denote the space of smooth vector field as

$T^\infty(TM)$ Algebraic structure:

$$X_1, X_2 \in T^\infty(TM), f_1, f_2 \in C^\infty(M) \Rightarrow f_1 X_1 + f_2 X_2 \in T^\infty(TM)$$

"modulo" space

There are also alternative way of definition.

"sign $p \in M$ to a $X_p \in T_p M$ " \equiv "Give $X: M \rightarrow TM$ s.t $\pi \circ X = Id$ "

"Def. section"

where $\pi: TM \rightarrow M$ is canonical projection $\pi(X_p) = p$. (submersion)

Prop 2.1: (Smooth Vector field = Smooth section on tangent bundle)

$$X: M \rightarrow TM \text{ s.t. } \pi \circ X = \text{Id}.$$

Proof. $(\varphi, U, V) \rightarrow M$ and $(T\varphi, \pi^{-1}(U), V \times \mathbb{R}^n)$ for TM , where $T\varphi$ is given by

$$T\varphi(p, X_p) = (\varphi(p), d\varphi_p(X_p))$$

$$\text{By def. } d\varphi(d_i) = \frac{\partial}{\partial x^i} \Rightarrow X_p = \sum X^i(p) (d_i)_p \Rightarrow d\varphi_p(X_p) = \sum X^i(\varphi^{-1}(x)) \frac{\partial}{\partial x^i} \Big|_{\varphi(p)}$$

So in the charts, we have

$$T\varphi \circ X \circ \varphi^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^n, X^1(\varphi^{-1}(x)), \dots, X^n(\varphi^{-1}(x)))$$

By def $X: M \rightarrow TM$ is smooth map iff X^i 's are smooth \square .

Let X be smooth VF on M . By Leibnitz-rule: Given $X_p: C^\infty(M) \rightarrow \mathbb{R}$, we have

$$X(fg) = fX(g) + X(f)g, \quad \forall f, g \in C^\infty(M)$$

Def 2.2: A derivation of the algebra $C^\infty(M)$ is a map $D: C^\infty(M) \rightarrow C^\infty(M)$ that

$$D(fg) = fD(g) + D(f)g, \quad \forall f, g \in C^\infty(M)$$

So $\forall X$ smooth vector field on M is a derivation, Conversely.

Prop 2.3. \forall Derivation D on $C^\infty(M)$, there \exists a VF X , s.t. $Df = Xf$.

Recall that differential operator of order n on $U \subset \mathbb{R}^m$ is an operator of the form

$$P = \sum_{|j| \leq n} a_j(x^1, \dots, x^m) \left(\frac{\partial}{\partial x^1} \right)^{j_1} \dots \left(\frac{\partial}{\partial x^m} \right)^{j_m}, \quad \text{Assume } a_j \text{ are smooth.}$$

Def 2.4. A differential operator of order n on M is map $P: C^\infty(M) \rightarrow C^\infty(M)$ "local form"

\forall local chart (φ, U, ψ) , P can be written as

$$P = \sum_{|j| \leq n} a_j(\varphi(p)) (\partial_1)^{j_1} \dots (\partial_m)^{j_m}$$

RMK: N.B \forall differential operator is local:

$$\text{supp}(Pf) \subset \text{supp}(f), \quad \forall f \in C^\infty(M).$$

Integral curves: Integral curves are sort of vector field from ODE view (parametric curve)

Recall: smooth curve γ on M is $\gamma: I \rightarrow M$, $I \subset \mathbb{R}$ (interval). For $\forall a \in I$, the tv of γ at $\gamma(a)$ is

$$\dot{\gamma}(a) = \frac{\partial \gamma}{\partial t}(a) = \partial_{\gamma(a)} \left(\frac{\partial}{\partial t} \right)$$

Def 1.1: Let $X \in T^\infty(TM)$ be VF on M . smooth curve $\gamma: I \rightarrow M$ is called integral curve of X if $\forall t \in I$

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$

example:

Lemma 1.2. If $\gamma: I \rightarrow M$ an integral curve on X , then

(1) $I_a = \{t \mid t+a \in I\}$, then

$\gamma_a: I_a \rightarrow M$, $\gamma_a(t) := \gamma(t+a)$ is also integral curve.

(2) Let $I^a = \{t \mid ta \in I\}$, then

$\gamma^a: I^a \rightarrow M$, $\gamma^a(t) := \gamma(at)$, its an integral curve for $X^a = aX$.

example: $X = \frac{\partial}{\partial x^1}$ on \mathbb{R}^n . we have integral curves of X are straight lines parallel to the x^1 -axis, as

$$\gamma(t) = (c_1 + t, c_2, \dots, c_n)$$

Check.

$$\partial \gamma \left(\frac{\partial}{\partial t} \right) f = \frac{\partial}{\partial t} (f \circ \gamma) = \nabla f \cdot \frac{\partial \gamma}{\partial t} = \frac{\partial f}{\partial x^1}.$$

N.B $\tilde{\gamma}(t) := (c_1 + 2t, c_2, \dots, c_n)$ is integral curve of $2X$, since $\dot{\tilde{\gamma}}(t) = 2 \frac{\partial}{\partial x^1}$

example. VF: $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ on \mathbb{R}^2 . Then if $\gamma(t) = (x, y)$ is an integral curve, we must have $\forall f \in C^\infty(\mathbb{R}^2)$

$$x'(t) \frac{\partial f}{\partial x} + y'(t) \frac{\partial f}{\partial y} = \nabla f \cdot \frac{\partial \gamma}{\partial t} = X_{\gamma(t)} f = x(t) \frac{\partial f}{\partial y} - y(t) \frac{\partial f}{\partial x}$$

$$\Leftrightarrow \begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases} \Rightarrow \begin{cases} x(t) = a \cos t - b \sin t \\ y(t) = a \sin t + b \cos t \end{cases}$$

As defined, \forall integral curve should have an maximal interval (\mathbb{R})

Def 2.1. VF X on M is complete if for $\forall p \in M$, there $\exists \gamma: \mathbb{R} \rightarrow M$, s.t. $\gamma(0) = p$.

N.B. \forall smooth curve VF is complete!

example. Let $X := t^2 \frac{\partial}{\partial t}$ on \mathbb{R} . Let $\gamma(t) = (x(t))$ be inte-curve

$$x'(t) \frac{\partial}{\partial t} = X_{\gamma(t)} = x^2 \frac{\partial}{\partial t} \Rightarrow x'(t) = x(t)^2$$

If given $x(0) = c$, then

$$x_c(t) = \frac{1}{-t + 1/c} \quad \text{for } c \neq 0$$

and

$$x_0(t) = 0 \quad \text{for } c = 0$$

$$\text{N.B. } \begin{cases} I_c = (-\infty, 1/c), c > 0 \\ I_c = (1/c, \infty), c < 0 \end{cases} \quad \text{are maximal interval for ODE.}$$

But since \forall integral curve starting at $c \neq 0$ is not defined for all $t \in \mathbb{R}$,

$\Rightarrow X$ is not complete.

Compactly supp VF are complete!

$$\text{supp}(X) = \overline{\{p \in M \mid X(p) \neq 0\}}$$
 be support of VF X

Thm 2.2. If X is a cpt supported VF, then complete.

proof. (skipped)

Lemma 2.3. Any smooth VF on cpt manifold is complete.

By definition, we know that $\forall p \in M, \exists \gamma_p: \mathbb{R} \rightarrow M$ s.t. $\gamma_p(0) = p$

From this, we can define a map, $\forall t \in \mathbb{R}$,

$$\phi_t: M \rightarrow M, p \rightarrow \gamma_p(t).$$

And we know that ϕ_t is smooth in t , \forall fixed pt p

$\{\phi_t \mid t \in \mathbb{R}\}$ would satisfied some interesting group law.

prop 2.4. $\forall t, s \in \mathbb{R}$, we have $\phi_t \circ \phi_s = \phi_{t+s}$

proof. $\forall p \in M$, fixed $s \in \mathbb{R}$.

$\gamma_1(t) = \phi_t \circ \phi_s(p)$ and $\gamma_2(t) = \phi_{t+s}(p)$ are both integral curve for X .

$$\gamma_1(0) = \phi_s(p) = \gamma_2(0)$$

By uniqueness of integral curves, we have.

$$\phi_t \circ \phi_s = \phi_{t+s}.$$

□

Let $\varphi_0 = \text{Id}$, we conclude that,

Cor. 2.5 $\varphi_t : M \rightarrow M$ is bijective, and $\varphi_t^{-1} = \varphi_{-t}$.

As a consequence,

$t \rightarrow \varphi_t$ is a group homomorphism from $\mathbb{R} \rightarrow \text{Diff}(M)$

We call the family $\{\varphi_t, t \in \mathbb{R}\}$ one-parameter subgroup of diffeomorphisms.

Stronger version: $\Phi := \{\varphi_t\}$.

$\Phi : \mathbb{R} \times M \rightarrow M, (t, p) \mapsto \varphi_t(p)$ is a smooth map on point (t, p) .

Application: Let $\varphi(x, t)$ defined as particle in \mathbb{R}^3 . (Quantum)

we can see that $U(t)$ (unitary operator for particle)

is actually a $\{U(t)\}$ one parameter-subgroup in Quantum physics !!!

$$\begin{cases} U(0) \varphi(x, t) = \varphi(0) \\ U(t) \varphi(x, t) = \varphi(t) \end{cases}$$

A lie group is simply a smooth manifold w/ group structure + manifold structure

"Symmetry" "geometry" "physics"

Def 1.1. A lie group G is a smooth manifold equipped w/ group structure, so that
 $u: G \times G \rightarrow G, (g_1, g_2) \mapsto g_1 \cdot g_2$ is a smooth map.

example: $(\mathbb{R}^n, +)$ $GL(n, \mathbb{R}) = \{X \in M(n, \mathbb{R}) \mid \det X \neq 0\}$ $(\mathbb{R}^*, *)$ $SL(n, \mathbb{R}) = \{X \in M(n, \mathbb{R}), \det X = 1\}$ (S^1, \cdot) $O(n) = \{X \in M(n, \mathbb{R}), XX^T = I_n\}$

If G_1, G_2 are Lie Groups then so is $G_1 \times G_2$
 $(T^n = S^1 \times S^1 \times S^1 \times \dots \times S^1)$

N.B. Not all smooth manifold admits a lie group structure, consider that S^4 (S^0, S^1, S^2)
 Here are some topo restriction to smooth manifold:
 okey!

- (1) underlying s.p of lie group must be oriented
- (2) $\forall G$, lie group, the fundamental group $\pi_1(G)$ must be abelian
- (3) TG , tangent bundle is trivial $TG \cong G \times \mathbb{R}^n$

Fact: All dim-2 closed manifold, the only one has lie group is $T^2 = S^1 \times S^1$

Left/right multiplication:

Let G be lie group. $a, b \in G$, there are two natural maps.

$$L_a: G \rightarrow G, g \rightarrow a \cdot g$$

$$R_b: G \rightarrow G, g \mapsto g \cdot b$$

$$OBLV: L_a^{-1} = L_{a^{-1}}$$

$$R_b^{-1} = R_{b^{-1}}$$

so both L_a and R_b are diffeomorphisms.

And we have.

$$L_a R_b = R_b L_a$$

$$L_a \text{ is smooth since } L_a: G \xrightarrow{j_a} G \times G \xrightarrow{u} G \quad \text{both smooth}$$

$$g \mapsto (a, g) \mapsto a \cdot g$$

$$R_b = u \circ \tau_b \text{ for } \tau_b: G \hookrightarrow G \times G$$

$$g \mapsto (g, b)$$

Some application of left multiplication

prop 1.2. \forall Lie Group G , we have $TG \cong G \times \mathbb{R}^n$

proof. We identify $T_e G = \mathbb{R}^n$, and

$$\varphi: G \times T_e G \rightarrow TG, \varphi(a, \xi) = (a, dL_a(\xi))$$

Clearly, bijective w/ inverse $\varphi^{-1}(a, \xi) = (a, dL_{a^{-1}}(\xi))$

N.B. when we fixed x , φ, φ^{-1} are linear isomorphism

$$\{a\} \times T_e G \cong T_a G$$

$\Rightarrow \varphi$ and φ^{-1} are smooth \Rightarrow diffeomorphism.

Lemma 1.3. The differential of $u: G \times G \rightarrow G$ is given by

$$d_{u,a,b}(X_a, Y_b) = (dR_b)_a(X_a) + (dL_a)_b(Y_b)$$

For $\forall (X_a, Y_b) \in T_a G \times T_b G \cong T_{(a,b)}(G \times G)$

proof. Let $f \in C^\infty(G)$, we have.

$$\begin{aligned} (d_{u,a,b}(X_a, Y_b))(f) &= (X_a, Y_b)(f \circ u) = X_a(f \circ u \circ i_b) + Y_b(f \circ u \circ j_a) \\ &= X_a(f \circ R_b) + Y_b(f \circ L_a) \\ &= (dR_b)_a(X_a)(f) + (dL_a)_b(Y_b)(f) \end{aligned}$$

Prop 1.4: For \forall Lie Group G , the group inverse map.

$$i: G \rightarrow G, g \mapsto g^{-1}$$

is smooth and $d_{i,a}(X_a) = -(dL_{a^{-1}})_e (dR_{a^{-1}})_a(X_a), \forall a \in T_a G$

proof. Let $f: G \times G \rightarrow G \times G$
 $(a,b) \mapsto (a, ab)$ bijective map. By Lemma 1.3, we have.
 \Rightarrow Global diffeo

$$df_{(a,b)}: T_a G \times T_b G \rightarrow T_a G \times T_{ab} G,$$

$$(X_a, Y_b) \mapsto (X_a, (dR_b)_a(X_a) + (dL_a)_b(Y_b))$$

bijective since dR_b, dL_a are. And By IFT, f is locally a diffeomorphism near (a,b)

$$f^{-1}: G \times G \rightarrow G \times G, (a, c) \mapsto (a, a^{-1}c) \text{ is diffeo}$$

So the inverse map is

$$\begin{aligned} G &\hookrightarrow G \times G \xrightarrow{f^{-1}} G \times G \xrightarrow{\pi_2} G \text{ smooth} \\ a &\mapsto (a, e) \mapsto (a, a^{-1}) \mapsto a^{-1} \Rightarrow d_{i,a}(X_a) = -(dL_{a^{-1}})_e (dR_{a^{-1}})_a(X_a). \end{aligned}$$

$$\begin{aligned} df_{(a,a^{-1})} &= (X_a, 0) \\ \downarrow \\ Y_{a^{-1}} &= -(dL_a)_a + (dR_{a^{-1}})_a(X_a) \\ &= -(dL_{a^{-1}})_e (dR_{a^{-1}})_a(X_a) \\ dL_e &= dR_e = \text{Id} \Rightarrow (dL_e)_e(X_e) = X_e \end{aligned}$$

Def 2.1: A Lie algebra is a real VP V together w/ binary bracket operator

$$[\cdot, \cdot] : V \times V \rightarrow V,$$

s.t $\forall X, Y, Z \in V$, we have

$$(1) [aX_1 + bX_2, Y] = a[X_1, Y] + b[X_2, Y]$$

$$(2) [X, Y] = -[Y, X]$$

$$(3) [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

ex. \forall Vector space admits a trivial: $[X, Y] \equiv 0$

the set of all smooth VF $T^\infty(TM)$ admits a Lie algebra

$M(n, \mathbb{R})$ is a Lie algebra as we define the Lie bracket as.

$$[A, B] = AB - BA.$$

Left invariant VF on Lie group.

G Lie group. L_a left translation. For $\forall X_e \in T_e G$, we can define VF X on G by

$$X_a = (dL_a)_e(X_e)$$

Consider that $(dL_a)_b(X_b) = (dL_a)_b \circ dL_b(X_e) = dL_{ab}(X_e) = X_{ab}$

Def. A left-invariant VF on Lie Group G is X on G s.t.

$$(dL_a)_b(X_b) = X_{ab} \quad \forall a, b \in G.$$

$$\mathfrak{g} := \{X \in T^\infty(TM) \mid X \text{ is left invariant}\}.$$

Clearly \mathfrak{g} is vector s.p of $T^\infty(TG)$

$$\mathfrak{g} \cong T_e G. \text{ and } \dim \mathfrak{g} = \dim G.$$

prop 2.3

If $X, Y \in \mathfrak{g}$, so there Lie bracket

proof. let $f \in C^\infty(G)$

$$Y(f \circ L_a)(b) = Y_b(f \circ L_a) = (dL_a)_b(Y_b)f = Y_{ab}f = (Yf)(L_a b) = (Yf) \circ L_a(b).$$

then

$$X_{ab}(Yf) = (dL_a)_b(X_b)(Yf) = X_b Y(f \circ L_a)$$

$$\text{similarly, } Y_{ab}Xf = Y_b X(f \circ L_a). \text{ Thus } dL_a([X, Y]_b)f = X_b Y(f \circ L_a) - Y_b X(f \circ L_a) = [X, Y]_{ab}(f)$$

Lie algebra w/ Lie group

It follows that \mathfrak{g} on G w/ $[\cdot, \cdot]$ is an n -dim Lie subalgebra of $\text{All } T^\infty(TG)$!

Def. The Lie algebra \mathfrak{g} of \sqrt{G} is called Lie algebra of G .

example.

1. $(\mathbb{R}^n, +)$; for $\forall a \in \mathbb{R}^n$, we have L_a is defined as usual. " dL_a is the Id"

we have

$$X_v = v_1 \frac{\partial}{\partial x_1} + \dots + v_n \frac{\partial}{\partial x_n} \text{ for } (v_1, \dots, v_n) \in T_0 \mathbb{R}^n.$$

Since $\frac{\partial}{\partial x_i}$ commute with $\frac{\partial}{\partial x_j} \Rightarrow G = \mathbb{R}^n$ is $\mathfrak{g} = \mathbb{R}^n$ w/ vanishing Lie bracket

2. $GL(n, \mathbb{R}) = \{X \in M(n, \mathbb{R}), \det X \neq 0\} \cong \mathbb{R}^{n^2}$

n^2 -dim Lie Group, so if we define $GL_+(n, \mathbb{R}) = \{X \in M(n, \mathbb{R}), \det X > 0\}$

$$GL_-(n, \mathbb{R}) = \{X \in M(n, \mathbb{R}), \det X < 0\}.$$

$$\mathfrak{gl}(n, \mathbb{R}) = \{A \mid A \in M_{n \times n}\},$$

So we if we consider $\begin{cases} A = (A_{ij})_{n \times n} \in \mathfrak{gl}(n, \mathbb{R}) \\ X^{ij} \in GL(n, \mathbb{R}) \end{cases}$ we have $\sum A_{ij} \frac{\partial}{\partial x^{ij}} \in T_{In} GL(n, \mathbb{R})$

and Lie bracket $[A, B]$, $A, B \in \mathfrak{g}$ is the matrix

$$\begin{aligned} \left[\sum X^{ik} A_{kj} \frac{\partial}{\partial x^{ij}}, \sum X^{pq} B_{qr} \frac{\partial}{\partial x^{pr}} \right] &= \sum X^{ik} A_{kj} B_{jr} \frac{\partial}{\partial x^{ir}} - \sum X^{pq} B_{qr} A_{rj} \frac{\partial}{\partial x^{pj}} \\ &= \sum X^{ik} (A_{kr} B_{rj} - B_{kr} A_{rj}) \frac{\partial}{\partial x^{ij}} \end{aligned}$$

$$\Rightarrow [A, B] = AB - BA.$$

Lec 15th Lie Homomorphisms + exponential map.

Def 1.1. Let G, H be Lie groups.

(1) φ is called group homomorphism if smooth and

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2), \quad \forall g_1, g_2 \in G.$$

(2) $\varphi: G \rightarrow H$ is isomorphism if it's invertible and $\varphi^{-1}: H \rightarrow G$ is also homo.

N.B. $G \xrightarrow[\varphi]{\cong} H \Rightarrow$ they are diffeomorphic as manifold and isomorphic as group.

example: $\forall G$, Lie group $a \in G$, the conjugation map.

$$c(a) := L_a \circ R_{a^{-1}} : G \rightarrow G, \quad g \mapsto aga^{-1} \text{ is Group Isomorphism.}$$

Def 1.2. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras

(1) $L: \mathfrak{g} \rightarrow \mathfrak{h}$ is Lie homo if

$$L([X_1, X_2]) = [L(X_1), L(X_2)], \quad \forall X_1, X_2 \in \mathfrak{g}$$

(2) $L: \mathfrak{g} \rightarrow \mathfrak{h}$ is Lie isomo if it's invertible.

ex. $\forall X \in \mathfrak{gl}(n, \mathbb{R})$, the adjoint map

$$\text{Ad}_X : \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathfrak{gl}(n, \mathbb{R}), \quad A \mapsto XAX^{-1}$$

is a Lie algebra isomorphism

$$\text{Ad}_X^{-1} = X^{-1}AX \text{ also a Lie algebra isomorphism.}$$

Group homo \rightarrow Algebra homo.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\varphi} & \mathfrak{h} \\ \parallel & & \parallel \\ \mathfrak{g} & & \mathfrak{h} \end{array}$$

Let $\varphi: G \rightarrow H$ group homo, then differential at e , $d\varphi_e: T_e G \rightarrow T_e H$.

In other words, we have: $(d\varphi(X))_h = dL_h(d\varphi_e(X_e))$

the image of $d\varphi(X)$ is left invariant $\forall \mathfrak{h} \in \mathfrak{h}$ whose value at $e \in H$ is $d\varphi_e(X_e)$.

example. Consider $c(X) : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ on I_n , we get.

$$(\partial c(X))_{I_n}(A) = \left. \frac{\partial}{\partial t} \right|_{t=0} c(X)(I + tA) = \left. \frac{\partial}{\partial t} \right|_{t=0} X(I + tA)X^{-1} = XAX^{-1}$$

I.e. the induced map is Lie algebra homo.

$$\partial c(X) = \text{Ad}_X : \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathfrak{gl}(n, \mathbb{R})$$

Lemma 1.3. $\forall X \in \mathfrak{g}$ is φ -related to $\partial\varphi(X) \in \mathfrak{h}$.

Let $X \in \mathfrak{g}$. write $\mathfrak{h} = \varphi(\mathfrak{g})$. since φ group homo, we have

$$\begin{aligned} \varphi \circ L_g &= L_h \circ \varphi \quad \text{and} \quad \partial\varphi_g(X_g) = \partial\varphi_g \circ (\partial L_g)_e(X_e) \\ &= \partial L_h \circ \partial\varphi_e(X_e) \\ &= (\partial\varphi(X))_h \quad \square \end{aligned}$$

Thm 1.4. If $\varphi : G \rightarrow H$ is group homo, then $\partial\varphi : \mathfrak{g} \rightarrow \mathfrak{h}$ is Lie algebra homo.

proof. Idea (1) X φ -related to $\partial\varphi(X)$, $Y \xrightarrow{\varphi} \partial\varphi(Y)$

$$\Rightarrow [X, Y] \xrightarrow{\varphi} [\partial\varphi(X), \partial\varphi(Y)]$$

$$(2) [X, Y] \xrightarrow{\varphi} \partial\varphi([X, Y]) \quad \textcircled{1} \quad \nwarrow$$

It follows that $[\partial\varphi(X), \partial\varphi(Y)]_e = \partial\varphi_e([X, Y])_e = (\partial\varphi([X, Y]))_e$

Since $\partial\varphi([X, Y])$ and $[\partial\varphi(X), \partial\varphi(Y)]$ are left invariant VF on H , we conclude that $\textcircled{1}$ holds.

example. $GL(n, \mathbb{R})$ Lie group $\longrightarrow \mathfrak{gl}(n, \mathbb{R})$ w/ $[AB] = AB - BA$

$\Rightarrow GL(n, \mathbb{R}) \rightarrow \mathbb{R}^*$ is a Lie group homo

$$\det(XY) = \det(X) \cdot \det(Y), \quad \forall X, Y \in GL(n, \mathbb{R})$$

and as mentioned in lec 2, we have

$$d\det_X(A) = \det(X) \text{tr}(X^{-1}A), \quad \forall X \in GL(n, \mathbb{R}), A \in \mathfrak{gl}(n, \mathbb{R})$$

taking $X = I_n$, we get Lie algebra homo for \det is

$$d\det = \text{tr} : \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathbb{R}, \quad A \rightarrow \text{tr}(A)$$

$\text{tr}(AB) = \text{tr}(BA)$, $\forall A, B \in \mathfrak{gl}(n, \mathbb{R})$ since \mathbb{R} is trivial Lie algebra.

Exponential map.

Our goal is to explain the following map

Given $\varphi: G \rightarrow H$ group homo, the diagram

$$\begin{array}{ccc} g & \xrightarrow{d\varphi} & h \\ \downarrow \exp_G & & \downarrow \exp_H \\ G & \xrightarrow{\varphi} & H \end{array} \quad \star$$

commutes, i.e. $\varphi \circ \exp_G = \exp_H \circ (d\varphi)$

And sense that: If G connected, \forall Lie group $\varphi: G \rightarrow H$ is determined by Lie algebra $d\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$.

Def. 2.2. The exponential map of G is

$$\exp: \mathfrak{g} \rightarrow G, X \mapsto \varphi_1^X(e)$$

N.B. $\varphi_{ts}^X = \varphi_s^{tX}$, so we have

$$\exp(tX) = \varphi_1^{tX}(e) = \varphi_t^X(e).$$

Also, we can check that $\{\exp(tX), t \in \mathbb{R}\}$ is one-parameter subgroup of G .

$$\exp(tX) \cdot \exp(sX) = \exp((t+s)X)$$

However,

$$\exp(tX) \exp(tY) \neq \exp(t(X+Y)) \text{ in general.}$$

example, for $G = \mathbb{R}^*$, we can see $T_1 G = \mathbb{R}$

Let $x \in \mathbb{R} = T_1 G$, Left invariant VF $\rightarrow x = x \frac{\partial}{\partial t} \in T_1 G$ at $a \in G$.

$$X_a = ax \frac{\partial}{\partial t}$$

$$\text{integral curve} \Rightarrow \text{ODE } (e=1) \quad \gamma_e^X(t) = e^{tX}. \quad (\dot{\gamma}_e^X(t) = Xe^{tX} \frac{\partial}{\partial t} = X_{\gamma_e^X(t)})$$

$$\Rightarrow \exp(x) = \varphi_1^X(e) = \gamma_e^X(1) = e^x.$$

Similarly, we have sth like

$$G = (S', \cdot) : \exp: i\mathbb{R} = T_e S' \rightarrow S', \exp(ix) = e^{ix}$$

$$G = (\mathbb{R}^n, +) : \exp: \mathbb{R}^n = T_0 \mathbb{R}^n \rightarrow \mathbb{R}^n, \exp(x) = x.$$

$$G = GL(n, \mathbb{R}), \exp: gl(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}), \exp(A) = e^A = I + A + \frac{A^2}{2!} + \dots$$

The differential of the exponential map.

Lemma 2.3. $\exp: g \rightarrow G$ is smooth and if we identify $T_0 g \cong g$, we get.
 $d\exp_0 = \text{Id}.$

proof. Consider

$$\tilde{\Phi}: \mathbb{R} \times G \times g \rightarrow G \times g \quad (t, g, x) \rightarrow (g \cdot \exp(tX), X)$$

Check that this is the flow on $G \times g \sim$ left invariant VP $(X, 0)$ smooth!
 and \exp is decomposed as

$$g \hookrightarrow \mathbb{R} \times G \times g \xrightarrow{\tilde{\Phi}} G \times g \xrightarrow{\pi_1} G,$$

$$X \hookrightarrow (1, e, X) \rightarrow (\exp(tX), X) \rightarrow \exp(tX) \quad \text{all smooth maps.}$$

Also $\exp(tX) = \phi_t^X(e) = \gamma_{e^X(t)}$, we get

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \exp(tX) &= X \quad \text{and} \quad \frac{d}{dt} \Big|_{t=0} \exp \circ tX = (d\exp)_0 \frac{d(Xt)}{dt} \\ &\Rightarrow (d\exp)_0 = \text{Id} \quad \square \quad \quad \quad = (d\exp)_0 X \end{aligned}$$

Cor 2.4. \exp is local diffeo near 0. i.e. it's diffeo $U_0 \in T_e G \xrightarrow[\underbrace{V_0 \subset}_{V_0 \subset}]{\exp} e \in G$

Since $(d\exp)_0$ is bijective!

properties: For $\forall X, Y \in g$, there \exists smooth $Z: (-\varepsilon, \varepsilon) \rightarrow g$ s.t for all $t \in (-\varepsilon, \varepsilon)$.
 $\exp(tX) \exp(tY) = \exp(t(X+Y) + t^2 Z(t))$

Application:

$$Z(t) = \frac{1}{2} [X, Y] + \frac{t}{12} ([X, [X, Y]] - [Y, [X, Y]]) + \frac{t^2}{24} [X, [Y, [X, Y]]] + \dots$$

As seen in Dynamic system !!!

Lec 16th Russell (Tensor and differential form)

Multi-linear map.

Let V_1, \dots, V_k be finite dim VSP.

Def 1.1. A func $T: V_1 \times \dots \times V_n \rightarrow \mathbb{R}$ is call multi-linear if linear in each entry.

$\forall i \in I$ (index) and $V_i \in V_1, \dots, V_{i-1} \in V_{i-1}, V_{i+1} \in V_{i+1}, \dots, V_k \in V_k$

$T_i: V_i \rightarrow \mathbb{R} \cdot V_i \rightarrow T(V_1, \dots, V_i, \dots, V_k)$ is linear.

N.B. If T_1, T_2 are multi-linear map, then span $\{T_1, T_2\}$ same.

ex. $f' \in V_1^* \times \dots \times f_k \in V_k^* \rightarrow \mathbb{R}$, we define

$$(V_1, \dots, V_k) \rightarrow f'(V_1) \dots f_k(V_k)$$

$f' \otimes \dots \otimes f^k$ is a multi-linear map. By define $i \leq k$, and $\lambda \in \mathbb{R}$, we have

$$f' \otimes \dots \otimes f^{i-1} \otimes \lambda f^i \otimes \dots \otimes f^k = \lambda f' \otimes \dots \otimes f^i \otimes \dots \otimes f^k.$$

Thm 1.2: Let $\{f_1^i, \dots, f_i^{n_i}\}$ be basis of V_i^* . Then the set

$\{f_1^{i_1} \otimes \dots \otimes f_k^{i_k} \mid 1 \leq i_j \leq n_{i_j}\}$ form a basis of VP on multi-map $V_1 \times \dots \times V_k$

In particular, $\dim \otimes^k V^* = n^k$.

Notation: We denote the VP of multi-linear maps on $V_1 \times V_2 \times \dots \times V_k$ by $V_1^* \otimes V_2^* \dots \otimes V_k^* := E$
 $e \in E$ would be called k -tensor.

N.B. If $T \in V_1^* \otimes \dots \otimes V_k^*$

$S \in V_{k+1}^* \otimes \dots \otimes V_{k+e}^* \Rightarrow$ tensor product $T \otimes S (V_1, \dots, V_{k+e}) := T(V_1, \dots, V_k) S(V_{k+1}, \dots, V_{k+e})$

And by def. we know that

$$(T \otimes S) \otimes IR = T \otimes (S \otimes IR) \text{ associativity.}$$

Now, let V be n -dim VP, and V^* its dual. We will call

$$\otimes^{l,k} V := (\otimes^l V) \otimes (\otimes^k V^*)$$

the space of (l, k) -tensors on V . I.e. $T \in \otimes^{l,k} V$ iff $T = T(\beta^1, \dots, \beta^l, v_1, \dots, v_k)$
for $\beta^i \in V^*, v_j \in V$

N.B. $\otimes^{1,0} V = V$ and $\otimes^{0,1} V = V^*$

$$\otimes^{k,0} V = \otimes^k V \text{ and } \otimes^{0,0} V = \mathbb{R}.$$

Def 1.3. For $\forall 1 \leq r \leq l$, and $1 \leq s \leq k$, we define the (r, s) -contraction $C_s^r : \otimes^{l,k} V \rightarrow \otimes^{l-1, k-1} V$
as
$$C_s^r(T)(\beta^1, \dots, \beta^{k-1}, v_1, \dots, v_{l-1}) = \sum_i T(\beta^1, \dots, \beta^{r-1}, f^i, \beta^r, \dots, \beta^{k-1}, v_1, \dots, v_{s-1}, e_i, v_s, \dots, v_{l-1})$$

where $\{e_1, \dots, e_n\}$ basis $\rightarrow V$
 $\{f_1, \dots, f^n\}$ dual basis.

Lemma 1.4. Let T be (l, k) tensor. For $1 \leq r \leq l$, $1 \leq s \leq k$, we have.

(1) $\{e_i\}$ independent

(2) For $\forall v_1, \dots, v_l \in V$ and $\beta^1, \dots, \beta^k \in V^*$

$$C_s^r(v_1 \otimes \dots \otimes v_l \otimes \beta^1 \otimes \dots \otimes \beta^k) = \beta^s(v_r) \otimes \dots \otimes \hat{v}_r \otimes \dots \otimes v_l \otimes \beta^1 \otimes \dots \otimes \hat{\beta^s} \otimes \dots \otimes \beta^k$$

" means remove the entries"

example. if $v, w \in V$ and $\alpha, \beta, r \in V^*$

$$C_2^1(v \otimes w \otimes \alpha \otimes \beta \otimes r) = \beta(v) w \otimes \alpha \otimes r$$

Check:

$$\begin{aligned} C_2^1(v \otimes w \otimes \alpha \otimes \beta \otimes r)(\beta^1, v_1, v_2) &= \sum_i v \otimes w \otimes \alpha \otimes \beta \otimes r(f^i, \beta^1, v_1, e_i, v_2) \\ &= \sum_i f^i(v) \beta^1(w) \alpha(v_1) \beta(e_i) r(v_2) \\ &= \left[\sum_i f^i(w) \beta(e_i) \right] \beta^1(v) \alpha(v_1) r(v_2) \\ &= \beta(v) \beta^1(w) \alpha(v_1) r(v_2) \\ &= \beta(v) w \otimes \alpha \otimes r(\beta^1, v_1, v_2) \end{aligned}$$

linear p-form:

Now, we fixed a VP V , and consider k -tensor T on V , $T \in \otimes^k V^*$

Def. 2.1 Let $T \in \otimes^k V^*$ be k -tensor on V

(1) we say T is symmetric if \forall permutation σ of $(1, 2, \dots, k)$

$$T(v_1, \dots, v_k) = T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

(2) we say T is alternating (linear k -form) if

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

for all $v_1, \dots, v_k \in V$ and $\forall 1 \leq i \neq j \leq k$

ex. $\langle \cdot, \cdot \rangle$ on V is positive symmetric 2-tensor

\det is linear n -form on \mathbb{R}^n

Notation. we let VP of k -forms by $\underbrace{\wedge^k V^*}_{\text{LSP}} \subset \otimes^k V^*$

Recall: permutation $\sigma \in S_k$ is even/odd is determined by #simple transpositions.

Given k -tensor T and $\sigma \in S_k$, we define k -tensor T^σ by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

Anti-Symmetry: (map)

$$\text{Alt}(T) := \frac{1}{k!} \sum_{\pi \in S_k} (-1)^\pi T^\pi$$

Lemma: Alt is projection from $\otimes^k V^* \rightarrow \wedge^k V^*$

Check (1) $\forall T \in \otimes^k V^*$, $\text{Alt}(T) \in \wedge^k V^*$

(2) $\forall T \in \wedge^k V^*$, $\text{Alt}(T) = T$

(1): let $\sigma \in S_k$, we have $[\text{Alt}(T)]^\sigma = \frac{1}{k!} \sum_{\pi \in S_k} (-1)^\pi (T^\pi)^\sigma = \frac{1}{k!} (-1)^\sigma \sum_{\pi} (-1)^\pi T^{\sigma \circ \pi} = (-1)^\sigma \text{Alt}(T)$

(2): Each $(-1)^\pi (T)^\pi$ equals to $T \Rightarrow \text{Alt}(T) = T$ since $|S_k| = k!$

We also need sth like

Lemma 2.3. Let T, S, R be k -, l -, and m -form.

$$(1) \text{Alt}(T \otimes S) = (-1)^{kl} \text{Alt}(S \otimes T)$$

$$(2) \text{Alt}(\text{Alt}(T \otimes S) \otimes R) = \text{Alt}(T \otimes S \otimes R) = \text{Alt}(T \otimes \text{Alt}(S \otimes T))$$

Wedge product.

Def 2.4. The wedge product of $T \in \wedge^k V^*$ and $S \in \wedge^l V^*$ is $(k+l)$ -form

$$T \wedge S = \frac{(k+l)!}{k!l!} \text{Alt}(T \otimes S)$$

This is a kind of operation !!

Prop. The wedge prod operation $\wedge : (\wedge^k V^*) \times (\wedge^l V^*) \rightarrow \wedge^{k+l} V^*$ is

(1) Bilinear: $(T, S) \rightarrow T \wedge S$ is linear in T and S

(2) Anti-commu: $T \wedge S = (-1)^{kl} S \wedge T$.

(3) Asso: $(T \wedge S) \wedge S' = (T \wedge S \wedge S') = T \wedge (S \wedge S')$

In general, we have.

$$(f^1 \wedge \dots \wedge f^k)(v_1, \dots, v_k) = k! \text{Alt}(f^1 \otimes \dots \otimes f^k)(v_1, \dots, v_k)$$

Prop 2.6. $\forall f^1, \dots, f^k \in V^*$ and $v_1, \dots, v_k \in V$,

$$(f^1 \wedge \dots \wedge f^k)(v_1, \dots, v_k) = \det(f^i(v_j))$$

proof. $(f^1 \wedge \dots \wedge f^k)(v_1, \dots, v_k) = k! \text{Alt}(f^1 \otimes \dots \otimes f^k)(v_1, \dots, v_k)$

$$= \sum_{\sigma \in S_k} (-1)^\sigma f^1(v_{\sigma(1)}) \dots f^k(v_{\sigma(k)})$$

$$= \det(f^i(v_j)).$$

VP of linear k-forms

Thm 2.7 Let $\{f^1, \dots, f^n\}$ be basis of V^* . Then the set of k-forms

$$\{f^{i_1} \wedge \dots \wedge f^{i_k} \mid 1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n\} \text{ form a basis } \rightarrow \wedge^k V^*$$

and $\dim \wedge^k V^* = \binom{n}{k}$

RMK: $\dim(\wedge^n V^*) = \binom{n}{n} = 1$

- \forall n-form of $\dim V$ is a multiple of non-trivial n-form "det"

for $k > n$, $\wedge^k(V^*) = 0$

The interior product and pull-back

Def 2.8. Interior product of $v \in V$ w/ k-form $a \in \wedge^k(V^*)$ is the $(k-1)$ -covector.

$$i_v a(X_1, \dots, X_{k-1}) := a(v, X_1, \dots, X_{k-1})$$

Def 2.9. Let $L: W \rightarrow V$ linear. The pull back $L^*: \wedge^k(V^*) \rightarrow \wedge^k(W^*)$ is defined to be.

$$(L^*a)(X_1, \dots, X_k) := a(L(X_1), \dots, L(X_k))$$

Prop 2.10. Let α be linear k-form on V , β a l-form on V

then

(1) $\forall v \in V, i_v i_v \alpha = 0$

(2) $\forall v \in V, i_v (L \wedge \beta) = (i_v L) \wedge \beta + (-1)^k L \wedge i_v \beta.$

(3) $\forall L: W \rightarrow V, L^*(L \wedge \beta) = L^*L \wedge L^*\beta.$

Tensor field and Differential forms on smooth manifold.

Cotangent S.P

Let M be a smooth manifold. we asso to each $p \in M$ a $T_p M$, vector space.

If we take $\forall (p, U, V)$ ~~about~~ ^{on} p , then we can write down a basis for $T_p M$.

$$d_i|_p : C^\infty(U) \rightarrow \mathbb{R}, \quad d_i|_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) \quad (1 \leq i \leq n)$$

N.B. for each $1 \leq i \leq n$, $X^i \circ \varphi: U \rightarrow \mathbb{R}$ is smooth on U

The differential of this, dx^i for simplicity. (restricted to $\forall q \in U$)

$$dx^i|_q: T_q M = T_q U \rightarrow T_{X^i \circ \varphi(q)} \mathbb{R} = \mathbb{R}.$$

I.e $dx^i|_q \in T_q^* M$ and by def, we have

$$dx^i|_q (d_j|_q) = d_j|_q (X^i \circ \varphi) = \delta_j^i$$

So we conclude,

prop 3.1. In \forall local chart (φ, U, V) , $\{dx^i|_q: 1 \leq i \leq n\}$ is basis $\rightarrow T_q^* M$. Moreover,

this is the dual basis of $\{d_j|_q: 1 \leq j \leq n\}$ of $T_q M$.

In fact, $\forall f \in C^\infty(U)$, by same way, we can identify $df_q: T_q M \rightarrow \mathbb{R}$.

By def, $df_p(d_j|_p) = d_j|_p(f)$. It follows that

$$df_p = (d_1|_p f) dx^1|_p + \dots + (d_n|_p f) dx^n|_p$$

and $\forall X \in T^\infty(TU)$,

$$df(X) = Xf, \text{ we call } df \text{ a 1-form on } U.$$

Def. 3.2. A k -form ω on M is an assignment to p that $\omega_p \in \wedge^k T_p^* M := \Omega^k(M)$

A k -form ω is smooth if locally,

$$\omega = \sum_I \omega_I dx^I = \sum_I \omega_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k},$$

$I = \{1 \leq i_1 < \dots < i_k \leq n\}$ and each $\omega_I \in C^\infty(U)$

N.B: $\Omega^0(M) = C^\infty(M)$, and since there's no k -form on $T_p M$ if $k > n$, we have,

$$\Omega^k(M) = 0, \forall k > n = \dim(M)$$

operations on k -forms.

• wedge prod: $\Omega^k(U) \times \Omega^l(U) \rightarrow \Omega^{k+l}(U)$

ex.

$$(dx^1 + 2dx^2) \wedge (dx^1 \wedge dx^2 - dx^2 \wedge dx^3 + 3dx^1 \wedge dx^3) = -7dx^1 \wedge dx^2 \wedge dx^3.$$

• Interior product:

$$L_X(dx^{i_1} \wedge \dots \wedge dx^{i_k}) = \sum_r (-1)^{r-1} X^{i_r}(X) dx^{i_1} \wedge \dots \wedge \widehat{dx^{i_r}} \wedge \dots \wedge dx^{i_k}$$

prop 3.5. Let $\omega \in \Omega^k(U)$
 then $\eta \in \Omega^l(U)$
 (1) $X \in T^\infty(TU)$
 $\omega \wedge \eta$
 $= (-1)^{kl} \eta \wedge \omega$
 $\varphi \in C^\infty(U', U)$
 $\psi \in C^\infty(U, \bar{U})$
 (2) $\varphi^*(\omega \wedge \eta)$
 $= \varphi^* \omega \wedge \varphi^* \eta$
 (3) $L_X(\omega \wedge \eta) = (L_X \omega) \wedge \eta + (-1)^k \omega \wedge L_X \eta$
 (4) $L_X \circ L_X = 0$
 (5) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$

Lec 17th (Russell) Exterior derivative (Lie-derivative)

Unlike wedge, interior, and pull back, exterior derivative is locally operator !!

Consider $f \in \Omega^0(U) = C^\infty(U)$, we seen that $df \in \Omega^1(U)$

$$d: \Omega^0(U) \rightarrow \Omega^1(U), f \mapsto df$$

Locally on each coordinate, we have

$$df = \sum_i (df/dx^i) dx^i$$

we also have "invariant def" of $df \in \Omega^1(U)$ via

$$df(X) = Xf, \quad \forall X \in T^\infty(U).$$

Now, suppose w is k -form on M , so that locally

$$w = \sum_I w_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

we want to define dw as $(k+1)$ -form as

Def 1.1. The exterior derivative of w is the $(k+1)$ -form dw given by

$$(1) \quad dw = \sum_I dw_{i_1, \dots, i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{I, i} d(w_{i_1, \dots, i_k}) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

We can also define this operation in coordinate free system!

For small k , we have

• $k=0$, $w = f \in C^\infty(U)$, $df: T^\infty(U) \rightarrow C^\infty(U)$, $df(X) = Xf$.

• $k=1$, $w \in \Omega^1(U)$. dw as $C^\infty(U)$ -bilinear map. $dw: T^\infty(U) \times T^\infty(U) \rightarrow C^\infty(U)$

$w = \sum_i w_i dx^i$, $X = \sum_k X^k \partial_k$ and $Y = \sum_\ell Y^\ell \partial_\ell$. we have the calculations:

$$dw(X, Y) = \sum_{i,j,k,\ell} (d_j w_i) dx^j \wedge dx^i (X^k \partial_k, Y^\ell \partial_\ell)$$

$$= \sum_{i,j} ((d_j w_i) X^j Y^i - (d_j w_i) X^i Y^j)$$

$$= X(w(Y)) - Y(w(X)) - w([X, Y])$$

$$\Rightarrow dw(X, Y) = X(w(Y)) - Y(w(X)) - w([X, Y])$$

Induction on this idea, we have.

Thm 1.2: For $\forall w \in \Omega^k(U)$, the $(k+1)$ -form dw , viewed as a $C^\infty(U)$ -multilinear map $dw: T^\infty(U) \times \dots \times T^\infty(U) \rightarrow C^\infty(U)$ is given by

$$dw(X_1, \dots, X_{k+1}) = \sum (-1)^{i-1} X_i(w(X_1, \dots, \hat{X}_i, \dots, X_{k+1})) + \sum_{i < j} (-1)^{i+j} w([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1})$$

prop 1.3. Suppose $w \in \Omega^k(U)$, $\eta \in \Omega^l(U)$, $X \in T^{\infty}(U)$, $\varphi \in C^{\infty}(U, U')$

then (1) $d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$

(2) $d \circ d = 0$.

(3) $\varphi^* \circ d = d \circ \varphi^*$

proof.

(1). Since d is linear, it's sufficient to show $w = f dx^{i_1} \wedge \dots \wedge dx^{i_k}$ and $\eta = g dx^{j_1} \wedge \dots \wedge dx^{j_l}$ w / $I \cap J = \emptyset$.

then, $d(w \wedge \eta) = d(fg dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l})$

$$= \sum_i d_i(fg) dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$= \sum_i (d_i f) dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge \eta + (-1)^k w \wedge \sum_i (d_i g) dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}$$

$$= dw \wedge \eta + (-1)^k w \wedge d\eta.$$

(2). we first check $k=0$:

$$d(df)(X, Y) = X(df(Y)) - Y(df(X)) - df([X, Y]) = X(Y(f)) - Y(X(f)) - [X, Y]f = 0.$$

for $k > 0$, by linearity, we assume $w = f dx^1 \wedge \dots \wedge dx^k$, since $d^2 f = 0$ and $d^2 x^i = 0$. then

$$d(dw) = d(df \wedge dx^1 \wedge \dots \wedge dx^k)$$

$$= \underline{d(df)} \wedge dx^1 \wedge \dots \wedge dx^k + \sum_i (-1)^i df \wedge dx^1 \wedge \dots \wedge \underline{d(dx^i)} \wedge \dots \wedge dx^k$$

$$= 0$$

(3) Again check $k=0$, $(\varphi^* df)_p(X_p) = df_{\varphi(p)}(d\varphi_p(X_p)) = d(\varphi^* f)_p(X_p)$

By (1), (2), Prop 3.5(2), $\Rightarrow \varphi^* dw = d(\varphi^* w)$ \square

The Lie derivatives!

Def 2.1. The Lie derivative of a $f \in C^\infty(M)$ w.r.t $X \in T^\infty(TU)$ is

$$L_X(f) := \left. \frac{d}{dt} \right|_{t=0} \phi_t^* f = \left(\lim_{t \rightarrow 0} \frac{\phi_t^* f - f}{t} \right)$$

Def 2.2. The Lie derivative of a k -form $w \in \Omega^k(M)$ w.r.t $X \in T^\infty(TM)$ is

$$L_X(w) := \left. \frac{d}{dt} \right|_{t=0} \phi_t^* w = \left(\lim_{t \rightarrow 0} \frac{\phi_t^* w - w}{t} \right)$$

ϕ is local flow generated by VF X !!

N.B. $L_X f$ that defined in Lec 5 is just special case since $C^\infty(M) = \Omega^0(M)$ and we have seen that

$$L_X f = Xf \text{ and } L_{[X,Y]} f = L_X L_Y f - L_Y L_X f$$

prop 2.2. Let $w \in \Omega^k(U)$, $\eta \in \Omega^l(U)$, $X_1, X_2, X \in T^\infty(TM)$, then

$$(1) \quad dL_X w = L_X(dw)$$

$$(2) \quad L_X(w \wedge \eta) = L_X w \wedge \eta + w \wedge L_X(\eta)$$

$$(3) \quad [\text{Cartan's magic formula}]: L_X w = d i_X w + i_X dw$$

$$(4) \quad L_{[X_1, X_2]} w = L_{X_1} L_{X_2} w - L_{X_2} L_{X_1} w$$

$$(5) \quad (L_X w)(X_1, \dots, X_k) = L_X(w(X_1, \dots, X_k)) - \sum_i w(X_1, \dots, L_X X_i, \dots, X_k)$$

Basically follow the same proof as above. starting from $k=0$. and (3) is based on induction, and we may locally assume that $w = f dx^1 \wedge \dots \wedge dx^k = dx^1 \wedge w_1$, where $w_1 = f dx^2 \wedge \dots \wedge dx^k$.

Lec 18th (Russell) Integration on Manifolds.

Top form: Let M be smooth mf of dim m , $\Omega^k(M) = 0$ if $k > m$.

then we call \forall smooth m -form a top form on M . Now let $p \in M$ and $(\varphi, U, \nu)_p$, then

$dx^1 \wedge \dots \wedge dx^m$ a top form on U . N.B for $\forall q \in U$, $(dx^1 \wedge \dots \wedge dx^m)_q \neq 0$ since $\forall q \in U$

$$(dx^1 \wedge \dots \wedge dx^m)_q (d_1, \dots, d_m) = \det(dx^i(d_j))_{1 \leq i, j \leq m} = 1$$

Moreover, since $dx^1 \wedge \dots \wedge dx^m|_p = 1$, we see that \forall top form ω on U and $\forall q \in U$, $\exists \lambda q$ s.t

$$\omega_q = \lambda q (dx^1 \wedge \dots \wedge dx^m)_q.$$

We aim to find coordinate-change factor!

Lemma 1.1. If $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is diffeomorphism and $y = \varphi(x)$, then

$$\varphi^*(dy^1 \wedge \dots \wedge dy^m) = \det(d\varphi_x) dx^1 \wedge \dots \wedge dx^m$$

proof. Let $\varphi = (\varphi^1, \dots, \varphi^m)$, then $\varphi^* y^i = y^i \circ \varphi = \varphi^i$. So

$$\varphi^*(dy^1 \wedge \dots \wedge dy^m) = d\varphi^1 \wedge \dots \wedge d\varphi^m$$

But since

$$d\varphi^1 \wedge \dots \wedge d\varphi^m (d_1^x \dots d_m^x) = \det(d\varphi_x)$$

we have desired result. \square

Now let $(\varphi_\alpha, U_\alpha, \nu_\alpha)$, $(\varphi_\beta, U_\beta, \nu_\beta)$ be two coordinate system. The the coordinate change map is $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$

so we have

$$\varphi_\alpha(x) \rightarrow y = \varphi_\beta(x)$$

$$(\varphi_{\alpha\beta})^*(\varphi_\beta^{-1})^* dx_\beta^1 \wedge \dots \wedge dx_\beta^m = \det(d\varphi_{\alpha\beta})(\varphi_\alpha^{-1})^* dx_\alpha^1 \wedge \dots \wedge dx_\alpha^m$$

$$\text{since } (\varphi_{\alpha\beta})^*(\varphi_\beta^{-1})^* = (\varphi_\beta^{-1} \circ \varphi_{\alpha\beta})^* \Rightarrow dx_\beta^1 \wedge \dots \wedge dx_\beta^m = \det(d\varphi_{\alpha\beta}) dx_\alpha^1 \wedge \dots \wedge dx_\alpha^m !!!$$

Orientability(!)

Let M be mf of $\dim(M) = n$, $\omega \in \Omega^n(M)$ a n -form. We want to define $\int_M \omega$. For simplicity, we let ω is supp on (φ, U, V) w/ coordinate $\{x^1, \dots, x^n\}$. Then we can write,

$$\omega = f(\varphi(x)) dx^1 \wedge \dots \wedge dx^n, \quad f \in C^\infty(V)$$

It's natural to define in Euclidean diff-form,

$$\int_U \omega := \int_V f(x) dx^1 \dots dx^n$$

But we need to check RHS is indep choice of coordinate: let $(\varphi_\alpha, U, V_\alpha), (\varphi_\beta, U, V_\beta)$ on U w/ $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}: V_\alpha \rightarrow V_\beta$.

Then $\omega = f_\beta(x_\beta) dx_\beta^1 \wedge \dots \wedge dx_\beta^n = f_\beta(\varphi_{\alpha\beta}(x_\alpha)) \det(d\varphi_{\alpha\beta}) dx_\alpha^1 \wedge \dots \wedge dx_\alpha^n$, (we need to make it well-defined), so

$$\int_{V_\beta} f_\beta(x_\beta) dx_\beta^1 \dots dx_\beta^n = \int_{V_\alpha} f_\beta(\varphi_{\alpha\beta}(x_\alpha)) \det(\varphi_{\alpha\beta})(x_\alpha) dx_\alpha^1 \dots dx_\alpha^n \quad (\text{Not necessary true!!})$$

Recall in Calculus 3, we ^{only} have

$$\int_{V_2} f(x) dy^1 \dots dy^n = \int_{V_1} f(\varphi(x)) |\det(d\varphi)_x| dx^1 \dots dx^n \quad \text{if } \varphi: V_1 \xrightarrow{\text{diffe}} V_2$$

Compare to our condition, we need additional assume that " $\det(\varphi_{\alpha\beta}) > 0$ ".

Def 1.2. Let M be smooth mf of $\dim n$.

(1) $(\varphi_\alpha, U_\alpha, V_\alpha), (\varphi_\beta, U_\beta, V_\beta)$ two charts are orientation compatible if $\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}$ satisfied $\det(d\varphi_{\alpha\beta})_p > 0$ for all $p \in \varphi_\alpha(U_\alpha \cap U_\beta)$

(2) An orientation of M is atlas $\mathcal{A} = \{(\varphi_\alpha, U_\alpha, V_\alpha) \mid \alpha \in \Lambda\}$ whose charts are pairwise orientation compatible.

(3) If M has an orientation $\Rightarrow M$ is orientable

RMK: Let U be a chart w/ $\{x^1, \dots, x^m\}$, we use notation $-U := \{-x^1, x^2, \dots, x^m\}$ and U/\bar{U} are orientation compatible.

Let \tilde{U} be another chart s.t $\tilde{U} \cap U \neq \emptyset$ "connected". Then either

- \tilde{U} and U orien-compatible or
- \tilde{U} and $-U$ are orien-compatible.

As a consequence, If M is connected and orientable, then M admits exactly two different orientations

ex. $\mathbb{R}P^n$: we have an atlas has $\#n+1$ charts, we have seen that $\mathbb{R}P^n$ is orientable for n odd and turn out that $\mathbb{R}P^n$ is n even is not orientable.

Integration on smooth manifold:

Now assume M smooth, orientable mf of $\dim m$, and A orientation on M . Let w be m -form on M .

To define $\int_M w$, we assume w supp on (φ, U, V) v.s A . I.e. $\exists f \in U(\text{supp})$ s.t

$$w = f(\varphi(x)) dx^1 \wedge \dots \wedge dx^m \Rightarrow \int_U w = \int_V f(x) dx^1 \dots dx^m \quad (3)$$

To define a general m -form $w \in \Omega^m(M)$, we take locally finite cover $\{U_\alpha\}$ of M that are compatible w/ A .

Let $\{U_\alpha\}$ w/ $\{\rho_\alpha\}$ P.O.U. Now suppose ρ_α is supp in U_α , $\rho_\alpha w$ is supp in U_α . We define

$$\int_M w := \sum_\alpha \int_{U_\alpha} \rho_\alpha w. \quad (4)$$

If RHS, ABS-CV then we say it's integrable.

Thm 2.1: Let w is cpt supp / w is integrable. then above expression is indep of $\{U_\alpha\}$ and $\{\rho_\alpha\}$.

proof. We will first show that (3) is well-defined.

Let $\{X_\alpha^i\}$ and $\{X_\beta^j\}$ be two orientation compatible system on U . then

$$\omega := f_\alpha dx_\alpha^1 \wedge \dots \wedge dx_\alpha^m = f_\beta dx_\beta^1 \wedge \dots \wedge dx_\beta^m$$

WTS: $\int_{U_\alpha} f_\alpha dx_\alpha^1 \dots dx_\alpha^m = \int_{U_\beta} f_\beta dx_\beta^1 \dots dx_\beta^m$ which is true since $dx_\beta^1 \wedge \dots \wedge dx_\beta^m = \det(d\psi_{\alpha\beta}) dx_\alpha^1 \wedge \dots \wedge dx_\alpha^m$

implies $f_\alpha = \det(d\psi_{\alpha\beta}) f_\beta$, and $\det(d\psi_{\alpha\beta}) > 0 \Rightarrow$ conclusion (in Euclidean S.P.)

change of
coordinate

To show (4) is well defined, we need $\{U_\alpha\}, \{U_\beta\}$ be local finite cover
 $\{P_\alpha\}, \{P_\beta\}$ be p.o. U

$\Rightarrow \{U_\alpha \cap U_\beta\}$ new local finite cover $\rightarrow M$

$\{P_\beta P_\alpha\}$ new p.o. $U \rightarrow \{U_\alpha \cap U_\beta\}$

It's sufficient to show: $\sum_\alpha \int_{U_\alpha} P_\alpha \omega = \sum_{\alpha, \beta} \int_{U_\alpha \cap U_\beta} P_\alpha P_\beta \omega$ (1)

for each fixed α , we have $\int_{U_\alpha} P_\alpha \omega = \int_{U_\alpha} (\sum_\beta P_\beta) P_\alpha \omega = \sum_\beta \int_{U_\alpha \cap U_\beta} P_\beta P_\alpha \omega$.

Change of coordinate ($\mathbb{R}^m \rightarrow$ manifolds)

Def. 2.2. Let M, N be orientation smooth mfs, $\omega/\mathbb{A}, \mathcal{B}$ be orientations resp. A diffeomorphism $\varphi: M \rightarrow N$ is called orientation preserving if each $(\psi_\beta, U_\beta, V_\beta) \in \mathcal{B}$, the charts $(\psi_\beta \circ \varphi, \varphi^{-1}(U_\beta), V_\beta)$ on M is orienti compatible w/ \mathbb{A} .
And orientation revertible if incompatible w/ \mathbb{A}

Thm 2.3 Let M, N be n -dim orientation mfs, and $\varphi: M \rightarrow N$ diffeomorphism

(1) If φ - orienti - preserving, then

$$\int_M \varphi^* \omega = \int_N \omega.$$

(2) If φ - orienti - revertting, then

$$\int_M \varphi^* \omega = - \int_N \omega$$

proof. Nth different than on \mathbb{R}^m case...
(change of coordinate)

Check: "PP427-432 J. Lee" to see integrate via density

Lec 19th: Stoke them (Russell)

Manifold w/ boundary: Recall that manifolds w/ boundary, denote $\mathbb{R}_+^m := \{ (x^1, \dots, x^m) \mid x^m \geq 0 \}$

Def 1.1. A topo manifold is called m-dim mf w/ bdy if it's Hausdorff, 2nd countable, and $\forall p \in M, \exists U_p$ which is homeomorphic to either $\mathbb{R}^m / \mathbb{R}_+^m$.

We can also define the boundary of M : $\partial M = \{ p \in M, p \text{ has no nbhd in } M \text{ s.t. is homeomorphic to } \mathbb{R}^m \}$
interior of M : $\text{int}(M) = M \setminus \partial M$.

ex. Closed ball: $B^n(1) = \{ x \in \mathbb{R}^n \mid |x| \leq 1 \}$ w/ $\partial B^n(1) = S^{n-1}$

ex2. Let M any smooth mf, $f \in C^\infty(M)$. If a is singular-value of f , then the sub-level set $M_a = f^{-1}((-\infty, a))$ is smooth mf w/ $\partial M_a = f^{-1}(a)$

N.B. If M, N are smooth mfs w/ bdy $\Rightarrow M \times N$ is smooth mf w/ bdy

ex. $[0, 1] \times [0, 1]$ "smooth mf w/ corners"

However If M w/ bdy, N w/o bdy $\Rightarrow M \times N$ is smooth mf w/o bdy. (w/ bdy)

Lemma 1.2. Let M be m-dim smooth mf w/ bdy, then ∂M is smooth mf of dim $(m-1)$ "unless $\partial M = \emptyset$ " that is properly embedded into M .

proof. (Sketch) Let $(U, x^1, \dots, x^m)_p \in \partial U$ that is ^{map} homeomorphic to \mathbb{R}_+^m . Then $U \cap \partial M = \{ (x^1, \dots, x^m) \mid x^m = 0 \}$.
Then $(U \cap \partial M, x^1, x^2, \dots, x^{m-1})$ is a charts on ∂M (check) \square .

N.B. we can also define orientation on M w/ bdy, Δ , $\det(\varphi_{ap}) > 0$ for two charts and M w/ bdy is orientable iff it admits a nowhere vanishing top form !!!

Thm 1.3: If M an orientable manifold w/bdy of $\dim m$, then ∂M is an orientable $m-1$ mf a submf of M .

proof. Let $(U_\alpha, x_\alpha^1, \dots, x_\alpha^m)$ and $(U_\beta, x_\beta^1, \dots, x_\beta^m)$ be orient-charts of M near $p \in M$.

$M \cap U_\alpha$ is chara by $x_\alpha^m \geq 0$ and $M \cap U_\beta$ w/ $x_\beta^m \geq 0$.

WTS: $(U_\alpha \cap \partial M, x_\alpha^1, \dots, x_\alpha^{m-1})$ and $(U_\beta \cap \partial M, x_\beta^1, \dots, x_\beta^{m-1})$ are orient-compatible.

Let $\varphi_{\alpha\beta}: U_\alpha \rightarrow U_\beta$ by $(\varphi^1, \dots, \varphi^m)$, then on $\partial M \cap U_\alpha \cap U_\beta$, we have $x_\alpha^m = x_\beta^m = 0$.

$\varphi^m(x^1, \dots, x^{m-1}, 0) = 0$ on $U_\alpha \cap U_\beta \cap \partial M$ and $\varphi^m(x^1, \dots, x^m) > 0$ on $U_\alpha \cap U_\beta \cap \text{Int}(M)$.

It follows that $\frac{\partial \varphi^m}{\partial x^i}(x^1, \dots, x^{m-1}, 0) = 0$, $i = 1, \dots, m-1$ and $\frac{\partial \varphi^m}{\partial x^m}(x^1, \dots, x^{m-1}, 0) \geq 0$.

Since $(U_\alpha, x_\alpha^1, \dots, x_\alpha^m)$ and $(U_\beta, x_\beta^1, \dots, x_\beta^m)$ are orient-compatible, we have $\det(\frac{\partial \varphi^i}{\partial x^j}) > 0$ in $U_\alpha \cap U_\beta$.

$$\det\left(\frac{\partial \varphi^i}{\partial x^j}(x^1, \dots, x^{m-1}, 0)\right) = \det\left(\underbrace{\left[\frac{\partial \varphi^i}{\partial x^j}(x^1, \dots, x^{m-1}, 0)\right]_{1 \leq i, j \leq m-1}}_{\textcircled{1}} \quad * \quad \frac{\partial \varphi^m}{\partial x^m}(x^1, \dots, x^{m-1}, 0)\right) > 0$$

$\Rightarrow \det \textcircled{1} > 0$ so that $(U_\alpha \cap \partial M, x_\alpha^1, \dots, x_\alpha^{m-1})$ and the other are compatible.

RMK: The bdy of non-orientable manifold could be either (oriented: Möbius band)

(non-orientable: $[0, 1] \times M$, M non-orientable)

Thm 2.1 (Stoke's thm) Let M be m -dim manifold w/ ∂M . For $\forall w \in \Omega^{m-1}(M)$ w/ cpt supp, we have

$$\int_{\partial M} \iota_{\partial M}^* w = \int_M dw, \quad \text{where } \iota_{\partial M} : \partial M \hookrightarrow M \text{ "inclusion map"}$$

Rmk: (1) If M w/o bdy, $\partial M = \emptyset \Rightarrow \text{RHS} = \text{LHS} = 0$

(2) If M w/ coners (ex. $[0,1] \times [0,1]$) still holds! (J. Lee PP 415-421).

proof. Case I (w/ ∂M supp in $U \xrightarrow[\text{diffeo}]{\hookrightarrow} \mathbb{R}^m$)

Since $w=0$ on ∂M , we have $\int_{\partial M} \iota_{\partial M}^* w = 0$. Let $w := \sum_i (-1)^{i-1} f_i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^m$, where f_i 's cpt supp func's.
then $dw = \sum_i \frac{\partial f_i}{\partial x^i} dx^1 \wedge \dots \wedge dx^m$, by definition we have

$$\int_M dw = \int_{\mathbb{R}^m} \sum_i \frac{\partial f_i}{\partial x^i} dx^1 \dots dx^m = \sum_i \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x^i} dx^i \right) dx^1 \dots \widehat{dx^i} \dots dx^m = 0$$

Case II (w/ ∂M supp in $U \xrightarrow[\text{diffeo}]{\hookrightarrow} \mathbb{R}_+^m$), same formula except last term.

$$\int_{\mathbb{R}^{m-1}} \left(\int_0^{\infty} \frac{\partial f_m}{\partial x^m} dx^m \right) dx^1 \dots dx^{m-1} = - \int_{\mathbb{R}^{m-1}} f_m(x^1, \dots, x^{m-1}, 0) dx^1 \dots dx^{m-1}$$

On the other hand since $x^m=0$ on ∂M , we see that $\iota_{\partial M}^* w = (-1)^{m-1} f_m(x^1, \dots, x^{m-1}, 0) dx^1 \wedge \dots \wedge dx^{m-1}$

$$\begin{aligned} \text{implies that } \int_{\partial M} \iota_{\partial M}^* w &= \int_{\mathbb{R}^{m-1}} (-f_m(x^1, \dots, x^{m-1}, 0) dx^1 \dots dx^{m-1}) \\ &= - \int_{\mathbb{R}^{m-1}} f_m(x^1, \dots, x^{m-1}, 0) dx^1 \dots dx^{m-1} \end{aligned}$$

result follows

Case III. (In general) we cover the set $\text{supp}(w)$ by finitely many charts, and take P.O.U. Then

$$\int_{\partial M} \iota_{\partial M}^* w = \sum_i \int_{\partial M} \iota_{\partial M}^* (p_i w) = \sum_i \int_M d(p_i w) = \sum_i \int_M dp_i \wedge w + \int_M dw = \int_M d(\sum_i p_i) \wedge w + \int_M dw$$

" 0 " □

Lec 14th The de RHAM - cohomology. (Russell)

Def 1.1. Let M be smooth mf, and $\omega \in \Omega^k(M)$

(1) ω is closed if $d\omega = 0$

(2) ω is exact if \exists a $(k-1)$ -form $\eta \in \Omega^{k-1}(M)$ s.t $\omega = d\eta$

Notation:

$$Z^k(M) := \ker(d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)) \quad \text{"closed forms"}$$

$$B^k(M) := \text{Im}(d: \Omega^{k-1}(M) \rightarrow \Omega^k(M)) \quad \text{"exact forms"}$$

As we have seen: $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ linear that $\forall k$ and $\forall \omega \in \Omega^k(M)$
we have $d^2\omega = d(d\omega) = 0$, then $B^k(M) \subset Z^k(M) \subset \Omega^k(M)$

RMK: Let $d\Omega^m(M) = 0$, then by def

(1) $k > m: B^k(M) = Z^k(M) = \{0\}$

(2) $k = 0: B^0(M) = \{0\}; Z^0(M) = \{f \in C^\infty(M) \mid df = 0\} \cong \mathbb{R}^k$

(3) $k = m: Z^m = \Omega^m(M)$ "top form"

ex. Consider $M = \mathbb{R}$, we have

$$B^0(\mathbb{R}) = \{0\}, \quad Z^0(\mathbb{R}) \cong \mathbb{R} \quad \text{and} \quad \Omega^2(\mathbb{R}) = C^\infty(\mathbb{R})$$

And \forall 1-form $g(t)dt$ on \mathbb{R} , we have

$$\omega = g(t)dt \Leftrightarrow \omega = dG, \text{ where } G(t) = \int_0^t g(s)ds$$

$$\Rightarrow \Omega^1(\mathbb{R}) = B^1(\mathbb{R}) = Z^1(\mathbb{R})$$

The de Rham cohomology group.

Since $d^2 = 0$, we have the following

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \rightarrow \dots \xrightarrow{d} \Omega^m(M) \xrightarrow{d} 0$$

N.B each composition of consecutive maps is zero map, is called

"cochain complex"

depending on the "direction" of the maps.

Def 1.2. The quotient group (VSP)

$$H_{\text{DR}}^k(M) := Z^k(M) / B^k(M)$$

is called k^{th} de Rham cohomology group of M .

ex. For $M = \mathbb{R}$, $H_{\text{DR}}^0(\mathbb{R}) \cong \mathbb{R}$ and $H_{\text{DR}}^k(\mathbb{R}) = \{0\}$ for $k \geq 1$.

Given $\forall w \in Z^k(M)$, we will call $[w]$, the corresponding cohomology class.

RMK: Suppose $\dim M = m$, we have $H_{\text{DR}}^k(M) = \{0\}$, $\forall k > m$ and $H_{\text{DR}}^0(M) \cong \mathbb{R}^1$

There's lots of smooth manifold that $\dim H_{\text{DR}}^k(M) < \infty$ for all k , and

$$\dim H_{\text{DR}}^0(\mathbb{Z}) = \infty$$

ex. $H_{\text{DR}}^1(\mathbb{R}^2/\mathbb{Z}^2)$

$$\dim(H_{\text{DR}}^1(\mathbb{R}^2/\mathbb{Z}^2)) = +\infty.$$

Def 1.3. In the case $\dim H_{\text{DR}}^k(M) < \infty$ for each k , we will call the number

$$b_k(M) := \dim H_{\text{DR}}^k(M)$$

the k^{th} Betti number of M , and $\chi_M = \sum_{k=0}^m (-1)^k b_k(M)$ the Euler characteristics of M .

The de Rham cohomology of S^1 .

Consider $M = S^1$. As we have seen,

$$H_{\text{DR}}^0(S^1) \cong \mathbb{R} \text{ and } H_{\text{DR}}^k(S^1) = 0 \text{ for } k \geq 2$$

N.B. on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, θ is not smooth on S^1 , but d on $\mathbb{R} \Rightarrow d\theta$ is a Globally Defined 1-form on S^1 .

So we can write

$$Z^1(S^1) = \Omega^1(S^1) = \{f \cdot d\theta, f \in C^\infty(S^1)\} \subset \{f \in C^\infty(\mathbb{R}), f \text{ is } 2\pi\text{-periodic}\}$$

Also by Fundamental thm of Calculus,

$$w \text{ is an exact 1-form} \Leftrightarrow w = df, f \text{ is } 2\pi\text{-periodic} \Leftrightarrow w = g(\theta) d\theta, g \text{ is } 2\pi\text{-periodic and } \int_0^{2\pi} g(\theta) d\theta = 0.$$

$$\Rightarrow H_{dR}^1(S^1) \cong \frac{\{f \in C^\infty(\mathbb{R}), f \text{ } 2\pi\text{-periodic}\}}{\{g \in C^\infty(\mathbb{R}), g \text{ } 2\pi\text{-periodic and } \int_0^{2\pi} g(\theta) d\theta = 0\}} \Rightarrow H_{dR}^1(S^1) \cong \mathbb{R}$$

Since the linear map

$$\varphi: H_{dR}^1(S^1) \rightarrow \mathbb{R}, [f] \mapsto \int_0^{2\pi} f(\theta) d\theta \text{ is linear isomorphism} \Rightarrow \varphi \text{ is well defined}$$

Operations on deRham class:

Let $w \in Z^k(M)$ and $\eta \in Z^l(M)$, then

$$d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta = 0$$

$$\text{i.e. } w \wedge \eta \in Z^{k+l}(M)$$

And let $\xi_1 \in \Omega^{k+1}(M)$, $\xi_2 \in \Omega^{l+1}(M)$

$$(w + d\xi_1) \wedge (\eta + d\xi_2) = w \wedge \eta + d[(-1)^k w \wedge \xi_2 + (-1)^{k+1} \xi_1 \wedge \eta + (-1)^{k+1} \xi_1 \wedge \xi_2]$$

I.e. $[w \wedge \eta]$ is indep of choice of w and η in $[w]$ and $[\eta]$.

Def 1.4. The cup product $[w] \in H_{dR}^k(M)$ and $[\eta] \in H_{dR}^l(M)$ is

$$[w] \cup [\eta] := [w \wedge \eta] \in H_{dR}^{k+l}(M).$$

Similarly, $\varphi: M \rightarrow N$ smooth map. Then $d\varphi^* = \varphi^* d \Rightarrow \varphi^*(Z^k(N)) \subset Z^k(M)$ and $\varphi^*(B^k(N)) \subset B^k(M)$.

$$[f_1] = [f] \Rightarrow \int_0^{2\pi} f_1(\theta) d\theta = \int_0^{2\pi} f(\theta) d\theta$$

φ is injective

$$[f_1] \neq [f] \Rightarrow \dots \neq \dots$$

φ is surjective

$$\forall c \in \mathbb{R}, f(\theta) = c \in Z^1(S^1) \Rightarrow \varphi([f]) = 2\pi c = \int_0^{2\pi} f(\theta) d\theta$$

OBV: φ^* is a group homomorphism

Check:

$$\textcircled{1} (\varphi \circ \psi)^* = \varphi^* \circ \psi^*$$

$$\textcircled{2} Id^* = Id$$

It follows that $\varphi^*: \Omega^k(N) \rightarrow \Omega^k(M)$

depends to pull back $\varphi^*: H_{dR}^k(M) \rightarrow H_{dR}^k(N)$

$$\varphi^*([w]) := [\varphi^* w].$$

Cor 1.5 (Diffeomorphism invariance). If $\varphi: M \rightarrow N$ is a diffeomorphism, then

$$\varphi^*: H_{dR}^k(N) \rightarrow H_{dR}^k(M).$$

is linear isomorphism for all k . In fact $b_k(N) = b_k(M)$ for all k and $\chi(N) = \chi(M)$.

RMK: \forall smooth map $\varphi: M \rightarrow N$, the cup prod makes: $H_{dR}^*(M) = \bigoplus_{k=0}^M H_{dR}^k(M)$ a graded ring and inclusion map

φ^* is in fact a ring diffeomorphism $\varphi^*: H_{dR}^*(M) \rightarrow H_{dR}^*(N)$ since $\varphi^*(a \wedge b) = \varphi^*a \wedge \varphi^*b$.

Moreover, if φ is a diffeo, then $\varphi^*: H_{dR}^*(N) \rightarrow H_{dR}^*(M)$ is a ring isomorphism.

Homotopic invariance: (I.e if two manifolds are equivalent, then they have the same de Rham cohomology groups).

Def 2.1. M, N two topological spaces are homotopic equivalent if there exist continuous map $\varphi: M \rightarrow N$ and $\psi: N \rightarrow M$ so that $\varphi \circ \psi$ is homotopic to Id_N and $\psi \circ \varphi$ is homotopic to Id_M .

In fact, homotopic equivalence is much weaker relation than homeomorphism/diffeomorphism.

ex. (1) S^{n-1} is homo-equiv to $\mathbb{R}^n \setminus \{0\}$

(2) \forall star-shaped region is homo-equiv to a single point set $\{x_0\}$

Thm 2.2. (Homotopic invariance) Let M, N smooth manifolds. If M, N are homotopic equivalence, then

$$H_{dR}^k(M) \cong H_{dR}^k(N), \forall k$$

OBS: If indeed M, N are diffeomorphism $\Rightarrow H_{dR}^k(M) \cong H_{dR}^k(N), \forall k$

Def. (Singular cohomology group $H_{sing}^k(X, \mathbb{R})$ depends only on its topology)

Thm 2.3 (The de Rham theorem). $H_{dR}^k(M) = H_{sing}^k(M, \mathbb{R})$ for all k .

Another consequence of thm 2.2 is

Cor 2.4 (Poincaré's Lemma) If U is star-shaped region in \mathbb{R}^n , then $\forall k \geq 0, 1$, we have $H_{dR}^k(U) = 0$.

In particular, $H_{dR}^k(\mathbb{R}^m) = 0, \forall k \geq 1$.

Since $\forall p \in M, \exists U_p \subseteq U$ (star shaped) $\subseteq \mathbb{R}^n \Rightarrow \forall$ closed form is locally exact!

Cor 2.5. Suppose $k \geq 1$. Then \forall closed k -form $w \in Z^k(M)$ and $\forall p \in M$, there $\exists U_p$ and $(k-1)$ -form $y \in \Omega^{k-1}(U)$ so that $w = dy$ on U .

proof. (Thm 2.2):

Step 1. (Thm 2.6) Let $f, g \in C^\infty(M, N)$ be homotopic, then

$$f^* = g^* : H_{dR}^k(N) \rightarrow H_{dR}^k(M)$$

proof(2.6): Define cochain homotopy between f^* and g^* is a sequence of maps $h_k : \Omega^k(N) \rightarrow \Omega^{k-1}(M)$ s.t on $\Omega^k(N)$

$$g^* - f^* = d_M h_k + h_{k+1} d_N. \quad (\text{Cochain homotopy})$$

Suppose there \exists such a cochain homotopy: $[w] \in H_{dR}^k(N)$. Then $dw = 0$ since w is closed. It follows

that
$$g^*w - f^*w = (d_M h + h d_N)w = d_M h w \in B^k(M) \Rightarrow f^*([w]) = [f^*w] = [g^*w] = g^*([w])$$

Graph I.e.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d} & \Omega^{k-1}(N) & \xrightarrow{d} & \Omega^k(N) & \xrightarrow{d} & \Omega^{k+1}(N) & \xrightarrow{d} & \cdots \\ & \searrow h & \downarrow g^* \parallel f^* & & \downarrow g^* \parallel f^* & & \downarrow g^* \parallel f^* & & \searrow h \\ & \xrightarrow{d} & \Omega^{k-1}(M) & \xrightarrow{d} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \xrightarrow{d} & \cdots \end{array}$$

Step 2. Let $\varphi: M \rightarrow N$ and $\psi: N \rightarrow M$ be conti. maps so that $\varphi \circ \psi \in Id_N$ and $\psi \circ \varphi \in Id_M$

Then one can find $\varphi_1 \in C^\infty(M, N)$ and $\psi_1 \in C^\infty(N, M)$ so that $\varphi_1 \in \varphi$, $\psi_1 \in \psi$. It follows that both $\varphi_1 \circ \psi_1$ and $\psi_1 \circ \varphi_1$ are smooth and $\varphi_1 \circ \psi_1 \in Id_N$ w/ $\psi_1 \circ \varphi_1 \in Id_M$.

Applying Step 1. we get

$$\begin{aligned} \varphi_{1*} \circ \psi_{1*} &= Id: H_{2R}^k(M) \rightarrow H_{2R}^k(N) \\ \psi_{1*} \circ \varphi_{1*} &= Id: H_{2R}^k(N) \rightarrow H_{2R}^k(M) \end{aligned} \Rightarrow \text{both } \varphi^* \text{ and } \psi^* \text{ are linear isomorphisms } \square$$

Step 3. It's remain to show that existence of the cochain homotopy in them 2.6 !

Recall if X is complete VF on M , then X generates a flow $\varphi_t: M \rightarrow M$.

Lemma 2.8: Let X be complete VF on M , and φ_t the flow generated by X . Then \exists linear operator $Q: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ so that for $\forall w \in \Omega^k(M)$

$$\varphi_t^* w - w = dQ(w) + Q(dw).$$

proof. If we let $Q_t(w) = L_X(\varphi_t^* w)$, then $Q_t: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ and

$$\frac{d}{dt} \varphi_t^* w = \frac{d}{ds} \Big|_{s=0} \varphi_{t+s}^* w = \frac{d}{ds} \Big|_{s=0} \varphi_s^* \varphi_t^* w.$$

$$\text{Let } Q(w) = \int_0^1 Q_t(w) dt$$

$$\text{then } Q: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$$

$$\text{and } \varphi_t^* w - w = \int_0^1 \left(\frac{d}{dt} \right) \varphi_t^* w dt = dQ(w) + Q(dw). \quad \square$$

It follows that $\forall w \in \Omega^k(M)$

$$g^* w - f^* w = (dL^* \circ F^* + L^* \circ dF^*) w$$

so if we set

$$\begin{aligned} h: \Omega^k(N) &\rightarrow \Omega^{k-1}(M) \text{ w/} \\ g^* w - f^* w &= (dh + h d) w \end{aligned}$$

then we are done

let $F: W \rightarrow N$ homotopy and $\iota: M \hookrightarrow W$ w/ $\iota(p) = (p, 0) \Rightarrow f = F \circ \iota$ and $g = F \circ \varphi_1 \circ \iota$

Construction: Let $W = M \times \mathbb{R}$, then $X = \frac{d}{dt}$ is complete VF on X w/ $\varphi_t(p, a) = (p, a+t)$

(h_k)

By Lemma 2.8, $\exists Q: \Omega^k(W) \rightarrow \Omega^{k-1}(W)$ s.t

$$\varphi_t^* w - w = dQ(w) + Q(dw)$$

Lec 25th The Mayer-Vietoris sequence

The Mayer-Vietoris seq is algebraic tool that help us to compute homology/cohomology group in proper subspace

It's sort of analogy of the van Kampen theorem of the fundamental group and probably the abelian property the assumption is much weaker.

1. Exact Sequence.

Suppose we have a cochain complex (A, d) i.e

$$\cdots A^{k-1} \xrightarrow{d_{k-1}} A^k \xrightarrow{d_k} A^{k+1} \xrightarrow{d_{k+1}} A^{k+2} \cdots$$

where A^k 's are vector s.p and d_k 's are linear maps s.t $d_k \circ d_{k-1} = 0 \ \forall k$.

OBV: $\text{Im}(d_k) \subset \ker(d_{k+1})$, we can define cohomology groups in the same way for de Rham cochain:

$$H^k(A) := \ker(d_k) / \text{Im}(d_{k-1})$$

Such a sequence is called an exact-sequence if $H^k(A, d) = 0$ for all k . I.e.

$$\text{Im}(d_{k-1}) = \ker(d_k), \ \forall k$$

N.B. if the sequence start w/ 0,

$$(\cdots \rightarrow) 0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} V^3 \xrightarrow{d_3} \cdots \Rightarrow$$

then $d_1: V^1 \rightarrow V^2$ is injective, and if sequence end w/ 0,

$$V^{k-1} \xrightarrow{d_{k-1}} V^k \xrightarrow{d_k} V^{k+1} \xrightarrow{d_{k+1}} 0 (\rightarrow \cdots)$$

then $d_k: V^k \rightarrow V^{k+1}$ is surjective

A more strict condition:

$$0 \rightarrow V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} V^3 \rightarrow 0$$

then as mentioned on left side,

$$V^2 \supseteq \ker(d_2) \oplus \text{Im}(d_2) \subseteq \text{Im}(d_1) \oplus \text{Im}(d_2) \subseteq V^1 \oplus V^2$$

Another condition for finite sequence,

$$0 \rightarrow A_1 \rightarrow A^2 \rightarrow A^3 \rightarrow \cdots \rightarrow A^k \rightarrow 0 \text{ is exact}$$

$$\Rightarrow \sum (-1)^i \dim A^i = 0.$$

A general principle in homological algebra: Give 3 cochain complexes A, B, C , w/ $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ (exact)

in the sense that $\forall k: 0 \rightarrow A^k \rightarrow B^k \rightarrow C^k \rightarrow 0$ is short exact sequence. Then we can construct a long exact sequence

$$\dots \rightarrow H^{k-1}(C) \rightarrow H^k(A) \rightarrow H^k(B) \rightarrow H^k(C) \rightarrow H^{k+1}(A) \rightarrow \dots$$

Now, suppose M is smooth manifold, $U, V \subseteq M$ so that $M = U \cup V$, Since U, V, M both smooth manifolds,

we have 4 de Rham cohomology complexes:

$$(1) \Omega^*(M): 0 \rightarrow \Omega^0(M) \rightarrow \Omega^1(M) \rightarrow \dots$$

$$(2) \Omega^*(U): 0 \rightarrow \Omega^0(U) \rightarrow \Omega^1(U) \rightarrow \dots$$

$$(3) \Omega^*(V): 0 \rightarrow \Omega^0(V) \rightarrow \Omega^1(V) \rightarrow \dots$$

$$(4) \Omega^*(U \cap V): 0 \rightarrow \Omega^0(U \cap V) \rightarrow \Omega^1(U \cap V) \rightarrow \dots$$

$$\Rightarrow 0 \rightarrow \Omega^*(M) \rightarrow \Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(U \cap V) \rightarrow 0 \quad (1)$$

prop 1.1. For $\forall k$, the sequence

$$0 \rightarrow \Omega^k(M) \xrightarrow{\alpha_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta_k} \Omega^k(U \cap V) \rightarrow 0$$

is short exact sequence.

$$U \cap V$$

(1): consider $i_1: U \hookrightarrow M, i_2: V \hookrightarrow M$.

and $j_1: U \cap V \hookrightarrow U, j_2: U \cap V \hookrightarrow V$.

induce linear maps between Ω^k and let

$$\alpha_k: \Omega^k(M) \rightarrow \Omega^k(U) \oplus \Omega^k(V), w \mapsto (i_1^* w, i_2^* w)$$

$$\beta_k: \Omega^k(U) \oplus \Omega^k(V) \rightarrow \Omega^k(U \cap V), (w_1, w_2) \mapsto j_1^* w_1 - j_2^* w_2$$

By these, we should be able to construct a long exact sequence consisting of the de Rham cohomology group.

$$\text{In (1), } \alpha_k: H_{dR}^k(M) \rightarrow H_{dR}^k(U) \oplus H_{dR}^k(V)$$

$$\beta_k: H_{dR}^k(U) \oplus H_{dR}^k(V) \rightarrow H_{dR}^k(U \cap V)$$

We need sth called connecting homomorphism to get long sequence:

$$\delta_k: H_{dR}^k(U \cap V) \rightarrow H_{dR}^{k+1}(M).$$

The map δ_k 's can be constructed by "diagram chasing method". So fixed $\{P_U, P_V\}$ P.O.U subordinate to $\{U, V\}$.

For $\forall w \in Z^k(U \cap V)$, we define

$$\delta_k([w]) = [\eta], \quad \eta := \begin{cases} d(P_U w) & \text{on } U \\ -d(P_V w) & \text{on } V. \end{cases} \quad \text{"(k+1)-form on } M \text{"}$$

In fact, we just try to "force $w = 0$ near the ∂M " as zero-extension of w is not smooth map k -form on M .

Lemma 1.2. The map δ_k is well-defined.

proof. Several things to check (ignore details)

(i) $P_U w \in \Omega^k(U)$ i.e. w is not defined on $U \setminus U \cap V$.

(ii) $\eta \in \Omega^{k+1}(M)$

(iii) $\eta \in Z^{k+1}(M)$

(iv) $[\eta]$ is indep of choice of P_U and P_V

(v) $[\eta]$ is indep choice of w .

(iv): Let P_U and \tilde{P}_U be P.O.U's to $\{U, V\}$ and $\tilde{\eta}$ be the resulting $(k+1)$ -form. Then $\tilde{P}_V - P_V = P_U - \tilde{P}_U$ is supp in $U \cap V$.

If we set $\xi := (\tilde{P}_V - P_V)w$, then smooth k -form on M . By construction $\tilde{\eta} - \eta = d\xi$ on both $U/V \rightarrow$ on M .

(v). Let $\tilde{w} = w + d\xi$ and $\tilde{\eta}$ be resulting $(k+1)$ form by \tilde{w} .

we have

$$\tilde{\eta} - \eta = \begin{cases} d(P_U d\xi) & \text{on } U \\ -d(P_V d\xi) & \text{on } V \end{cases}$$

$$\text{define } \zeta := \begin{cases} -dP_U \wedge \xi & \text{on } U \\ dP_V \wedge \xi & \text{on } V \end{cases}$$

$$\stackrel{(4)}{\Rightarrow} d\zeta = \begin{cases} dP_U \wedge d\xi = d(P_U d\xi) & \text{on } U \\ -dP_V \wedge d\xi = -d(P_V d\xi) & \text{on } V \end{cases}$$

$$= \tilde{\eta} - \eta \Rightarrow [\tilde{\eta}] = [\eta]. \quad \square$$

Thm 1.3 (Main theorem). Let U, V be open sets in M so that $M = U \cup V$, then we have long exact sequence:

$$\dots \xrightarrow{\delta_{k-1}} H_{dR}^k(M) \xrightarrow{d_k} H_{dR}^k(U) \oplus H_{dR}^k(V) \xrightarrow{\beta_k} H_{dR}^k(U \cap V) \xrightarrow{\delta_k} H_{dR}^{k+1}(M) \xrightarrow{d_{k+1}} \dots$$

proof. Check (1) $\text{Im}(d_k) = \ker(\beta_k)$
 (2) $\text{Im}(\beta_k) = \ker(\delta_k)$
 (3) $\text{Im}(\delta_k) = \ker(d_{k+1})$

There's 6 relations need to verify: sample ($\text{Im}(\beta_k) \subset \ker(\delta_k)$)

Let $w_1 \in Z^k(U)$, $w_2 \in Z^k(V)$. Let $w := \beta_k(w_1, w_2) = J_1^* w_1 - J_2^* w_2 \in \Omega^k(U \cap V)$

Then $\delta_k([w]) = [\eta]$, where

$$\eta = \begin{cases} d(p_U w) = d(p_U w - w_1) \text{ on } U, \\ -d(p_V w) = -d(p_V w + w_2) \text{ on } V. \end{cases}$$

Note that on $U \cap V$, $p_U w - J_1^* w_1 = -p_V w - J_2^* w_2$

So there \exists a smooth k -form ξ on M s.t

$$\xi = \begin{cases} p_U w - w_1 & \text{on } U \\ -p_V w - w_2 & \text{on } V \end{cases}$$

Consequently, $\eta = d\xi$ and thus $[\eta] = 0$.

RMK: As mentioned van Kampen's thm. It's known that $H_{dR}^k(U)$ and $H_{dR}^k(V)$ and $H_{dR}^k(U \cap V) \xrightarrow{\text{determine}} H_{dR}^k(M)$

we need "homomorphism connecting them" to move further!!

Application: de-Rham cohomology \rightarrow sphere

Thm 2.1. For $n \geq 1$, $H_{dR}^k(S^n) \cong \begin{cases} \mathbb{R} & k=0 \\ 0 & 1 \leq k \leq n-1 \end{cases}$

proof. we have shown that

$$H_{dR}^0(S^n) \cong \mathbb{R} \text{ and } H_{dR}^1(S^1) \cong \mathbb{R}.$$

Just 2 things to reveal:

$$(1) \text{ For } n \geq 2, H_{dR}^1(S^n) = 0$$

\Rightarrow

For $n \geq 2$, we let

$$U = S^n - \{(0, \dots, 0, -1)\} \text{ and } V = S^n - \{(0, 0, \dots, 0, 1)\}$$

$$(2) \text{ For } n \geq 2, k \geq 2,$$

Then $M = U \cup V$. U and V diffeo to \mathbb{R}^n , $U \cap V$ homotopy $\cong S^{n-1}$.

$$(1). 0 \rightarrow H_{dR}^0(S^n) \rightarrow H_{dR}^0(U) \oplus H_{dR}^0(V) \rightarrow H_{dR}^0(U \cap V) \rightarrow H^1(S^n) \rightarrow \dots$$

$$\Rightarrow 0 \rightarrow \mathbb{R} \xrightarrow{\alpha_0} \mathbb{R}^2 \xrightarrow{\beta_0} \mathbb{R} \xrightarrow{\delta_0} H_{dR}^1(S^n) \rightarrow 0$$

$$\text{Since } \alpha_0 \text{ is injective, } \dim(\ker(\beta_0)) = \dim(\text{Im}(\alpha_0)) = 1 \Rightarrow \dim \text{Im}(\beta_0) = \dim \mathbb{R}^2 - \dim \ker(\beta_0) = 1$$

$$\text{i.e. } \beta_0 \text{ is surjective. So } \ker(\delta_0) = \mathbb{R}, \Rightarrow \delta_0 \equiv 0. \text{ But by exactness, } \delta_0 \text{ is surjective } \Rightarrow H_{dR}^1(S^n) = 0$$

$$(2). H_{dR}^{k-1}(U) \oplus H_{dR}^{k-1}(V) \xrightarrow{\beta_{k-1}} H_{dR}^{k-1}(U \cap V) \xrightarrow{\delta_{k-1}} H_{dR}^k(S^n) \xrightarrow{\alpha_k} H_{dR}^k(U) \oplus H_{dR}^k(V),$$

$$\Rightarrow 0 \xrightarrow{\beta_{k-1}} H_{dR}^{k-1}(S^{n-1}) \xrightarrow{\delta_{k-1}} H_{dR}^k(S^n) \xrightarrow{\alpha_k} 0.$$

By exactness, δ_{k-1} is 1 and onto, and thus must be linear isomorphism!

As a consequence, we can show an very early conclusion for this series of lectures. (Topological invariant of dimension)
(In UCB lecture videos, I guess?)

If $m \neq n$, then \mathbb{R}^n is not homeomorphic to \mathbb{R}^m .

proof. If $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is homeomorphism, then $f: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^m \setminus \{0\}$ is a homeomorphism.

$$\text{So } H_{\mathbb{R}}^k(\mathbb{R}^n \setminus \{0\}) = H_{\mathbb{R}}^k(\mathbb{R}^m \setminus \{f(0)\}), \forall k$$

But $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1} , while $\mathbb{R}^m \setminus \{f(0)\}$ is homotopy equivalent to S^{m-1}

So $H_{\mathbb{R}}^k(S^{n-1}) = H_{\mathbb{R}}^k(S^{m-1}), \forall k$. This contradicts w/ the fact that $m \neq n$. \square