Geometry and Topology Math Qual Exam Review

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Introduction

This project is written for my Ph.D. qualifying exam in the Geometry and Topology track. The material covers most topics from "Introduction to Differential Manifolds" by John Lee. The course "Differential Manifolds" in Spring covered only a small portion of the exam, and preparation is time-consuming. I dropped that course but resulted in a pass grade in this related exam. The notes are handwritten but readable in a sense that it helps many of my colleagues, so I decided to post it online. This also resulted in a small project for applications in De Rham cohomology parts and involves many interesting theorems' proofs.

Contents

Lecture 2: (Mar 20^{th}) Russell-Hua

Def. $U \underset{open}{\subset} \mathbb{R}^n$ and $f: U \to \mathbb{R}$ conti-func. we say that f is C^k-func if all its partial deravitive of order at least k

$$\partial^a f = \frac{\partial^{|a|} f}{(\partial x^1)^{a_1} \cdots (\partial x^n)^{a_n}} \quad \text{exist and conti on } \bar{U}.$$

same def $\Rightarrow$ C^∞-func

Def 1.1 : A smooth map $f: U \to V$ is diffeomorphism if f is 1-1 and onto and $f^{-1}: U \to V$ is alsosmooth.

OBS : i) If $f: U \to V$ diffeomorphism $\Rightarrow$ f^{-1} is also diffeo.

ii) If $f: U \to V$ and $g: V \to W$ are diffeo, so it is $g \circ f: U \to W$.

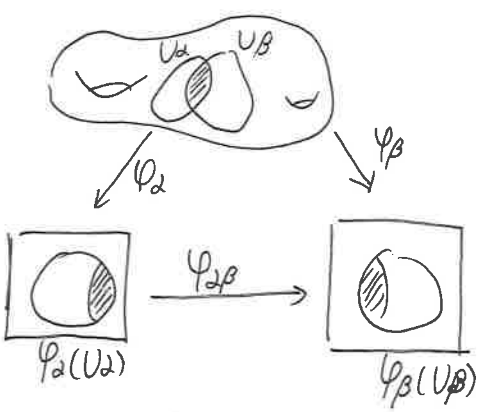
Def 1.2 : Chart $\{\varphi, U, V\}$: $U \underset{open}{\subset} M$, $V \underset{open}{\subset} \mathbb{R}^n$

It is natrual to identify $\forall f$ on $\bar{U}$ w/ $f \circ \varphi^{-1}$ on V $\longrightarrow$ define f is smooth/not.

Def 1.3 : Let M be topological manifold of dim n.

$\{\varphi_\alpha, U_\alpha, V_\alpha\}$ and $\{\varphi_\beta, U_\beta, V_\beta\}$ of M are compatible if the transition map

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \text{ is a diffeomorphism}$$



Def 1.3 : (1) An atlas $\mathcal{A}$ on M is a collection of charts $\{\varphi_\alpha, U_\alpha, V_\alpha\}$ w/ $\bigcup U_\alpha = M$ s.t all charts are compatible w/ each other.

(2) Two atlas are said to be equivalent if their union is again an altas.

eg. we define 3 altas on $\mathbb{R}$ by $\mathcal{A}_i = \{\varphi_i, \mathbb{R}, \mathbb{R}\}$ $(1 \le i \le 3)$, where $\varphi_1(x) = x$, $\varphi_2(x) = 2x$, $\varphi_3(x) = x^3$

then $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent, but $\mathcal{A}_1$, $\mathcal{A}_3$ are non-equivalent since

$$\varphi_{31}(x) = \varphi_1 \circ \varphi_3^{-1}(x) = x^{1/3} \text{ is not smooth on } \mathbb{R}.$$

Def 1.4 : An n-dim smooth manifold is an n-dim topo manifold w/ $\underbrace{\text{equivalent class of atlas}}_{\text{"smooth structure"}}$.

$\Rightarrow$ Pair $(M, \mathcal{A})$

Some results : (i) There $\exists$ topo-manifold that do not admit smooth structure. "Kervaire 10-dim-manifold"

(ii) If M admits a C^1 structure $\Rightarrow$ admits a C^∞ structure.

(iii) $\forall$ Manifold M admits a finite atlas consisting of dim $M+1$ charts.

Prop 1.5 : If a topological manifold M can be covered by a single chart, then $\{\varphi_0, U_0, V_0\}$ determines a smooth structure on M.

Concequenty : $\mathbb{R}^n$ and $\forall \{U \underset{\text{open}}{\subset} \mathbb{R}^n\}$ is a smooth manifold

Example (Graph): For $\forall\, U \underset{\text{open}}{\subset} \mathbb{R}^m$ and $\forall$ conti-func $f: U \to \mathbb{R}^n$, the graph of f is the subset in $\mathbb{R}^{n+m} = \mathbb{R}^m \times \mathbb{R}^n$.

defined by

$$\Gamma(f) = \{(x, y) \mid x \in U,\ y \in f(x)\} \subset \mathbb{R}^{m+n}$$

w/ subspace topo (i.e topo basis $\cap\ U$) inheritated from $\mathbb{R}^{m+n}$.

We have: (1) $\Gamma(f)$ is Hausdorff and 2^{nd} countable

(2) $\Gamma(f)$ is locally-Euclidean since $\{\varphi, \Gamma(f), U\}$

where $\varphi: \Gamma(f) \to U$, $\varphi(x, y) = x$ "projection onto"

i.e φ is homeo: φ is conti. invertible and

$\varphi^{-1}: U \to \Gamma(f) \Rightarrow \varphi^{-1}(x) = (x, f(x))$ is conti.

$\Rightarrow$ $\Gamma(f)$ is topo manifold of dim-m. Since it can be covered by 1-chart.

$\Rightarrow$ $\Gamma(f)$ of $\forall$ conti func $f: U \subset \mathbb{R}^n \to \mathbb{R}$ admits an intrinstic structure $\to$ smooth manifold

"It is possible that $\Gamma(f)$ is not smooth submanifold of $\mathbb{R}^{n+1}$!"

(The sphere as smooth manifold): (Spheres)

For each $n \geq 0$, the unit n-sphere

$$S^n = \{(x^1, \dots x^n, x^{n+1}) \mid (x^1)^2 + \dots + (x^{n+1})^2 = 1\} \subset \mathbb{R}^{n+1}$$

w/ sub.sp. topo is 2^{nd} countable + Hausdorff + "locally Euclidean"(?)

ie We can cover S^n by $U_+ = S^n \setminus \{(0, \dots 0, -1)\}$, $U_- = S^n \setminus \{(0, \dots 0, 1)\}$.

and define $\{\varphi_+, U_+, \mathbb{R}^n\}$ and $\{\varphi_-, U_-, \mathbb{R}^n\}$ by sterographic proj $\nearrow$

$$\varphi_\pm(x^1, \dots x^{n+1}) = \frac{1}{1 \pm x^{n+1}}(x^1, \dots x^n).$$

RMK: We can also cover S^n by 2^{n+2} charts using hemisphere.

$U_i^\pm = \{(x^1, \dots x^{n+1}) \in S^n, x_i >< 0$

$\varphi_i^+: U_i^+ \to B^n(1)$

$\varphi_i^+(x^1, \dots x^n) = (x^1, \dots x^i, x^{i+1} \dots x^n)$

Check more (compatible).

$\varphi_{-\pm}(y^1, \dots y^n) = \varphi_+ \circ \varphi_-^{-1}(y_1, \dots y_n)$

$= \varphi_+\left(\frac{1}{1+|y|^2}(2y_1, \dots 2y_n, -1+|y|^2)\right)$

$= \frac{1}{|y|^2}(y^1, \dots y^n)$

which is differmorphism from $\mathbb{R}^n \setminus \{0\}$.

$\Uparrow$

Check: $\varphi_\pm$ are conti and invertible, and

$$\varphi_\pm^{-1}(y^1, \dots y^n) = \frac{1 \cdot (2y^1, \dots 2y^n, \pm(1-(y^1)^2 - \dots - (y^n)^2)}{1 + (y^1)^2 + \dots + (y_n)^2}$$

is also continous.

Example: (The set of all straight line in $\mathbb{R}^2$)

i.e $ax+by+c=0$ (form of straight line)

(a,b,c) and (a',b',c') defines same line iff $[a:b:c]=[a':b':c']$

Note: $[0:0:1]$ will not given $\forall$ line, while for the others in $\mathbb{RP}^2$, we have.

bijective map [Möbius band!]

$$\varphi: \{\text{the set of all lines in } \mathbb{R}^2\} \to \mathbb{RP}^2 \setminus \{[0:0:1]\}$$

$$ax+by+c=0 \longmapsto [a:b:c]$$

$\Rightarrow$ a smooth manifold structure

Lec 3 (Mar 22th) Russell-Hua

Def 1.1: Let $(M, \mathcal{A})$ be a smooth manifold, and $f: M \to \mathbb{R}$ a function

(1) we say that f is smooth at $p \in M$ if there $\exists (\varphi_\alpha, U_\alpha, V_\alpha) \in \mathcal{A}$ w/ $p \in U_\alpha$

s.t the function $f \circ \varphi_\alpha^{-1}: V_\alpha \to \mathbb{R}$ is smooth at $\varphi_\alpha(p)$

(2) we say that f is a smooth func on M if hold for all $x \in M$.

RMK: we let $(\varphi_\beta, U_\beta, V_\beta)$ be another chart in $\mathcal{A}$ w/ $p \in U_\beta$, by compatibility, the function.

$$f \circ \varphi_\beta^{-1} = (f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1}) \text{ must be smooth at } \varphi_\beta(p)$$

So, we solve in "Lecture 2" the indep problem.

RMK2: According to chain rule, it's easy to see that if $f: M \to \mathbb{R}$ is smooth at $p \in M$ and $h: \mathbb{R} \to \mathbb{R}$ is smooth at $f(p)$, then $h \circ f$ is smooth at p.

Example: Each coordinate function $f_i(x^1, \dots, x^{n+1}) = x^i$ is smooth on S^n since.

$$f_i \circ \varphi_\pm^{-1}(y^1, \cdots y^n) = \begin{cases} \dfrac{2y^i}{1+|y|^2} & 1 \le i \le n \\ \pm \dfrac{1-|y|^2}{1+|y|^2} & i = n+1 \end{cases} \quad \text{are smooth func on } \mathbb{R}^n$$

Notation: We will denote all functions (smooth) on M by $C^\infty(M)$ "commutative algebra"

If f, g smooth $\Rightarrow$ $\alpha f + \beta g$ and fg are smooth

Def 1.2: Suppose $f \in C^\infty(M)$. the support of f is by definition the set

$$\operatorname{supp}(f) = \overline{\{p \in M \mid f(p) \neq 0\}}$$

and f is comppactly supported: $f \in C_0^\infty(M)$, if the support of f is comp in M.

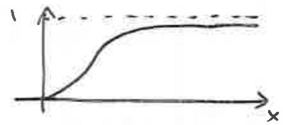
Obviously: if $f, g \in C_0^\infty(M) \Rightarrow af+bg \in C_0^\infty(M)$, $f\cdot g \in C_0^\infty(M)$ $\Rightarrow$ $C_0^\infty(M)$ is an ideal of $C^\infty(M)$

N.B: If M is cpt, then $\forall$ func (smooth) is cpt-supported.

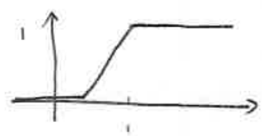
Bump-func: (test function) "cpt support + smooth + non-negative + "≤ 1" + "$=1$ on cpt set" ".

Example:

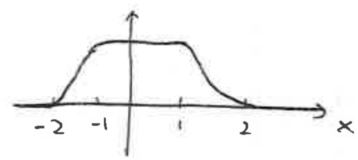
$$f_1(x) = \begin{cases} e^{-1/x} & (x>0) \\ 0, & (x \leq 0) \end{cases} \Rightarrow f_1(x) = \begin{cases} \in (0,1) & x>0 \\ 0, & x \leq 0 \end{cases}$$



$$f_2(x) = \frac{f_1(x)}{f_1(x)+f_1(1-x)} \Rightarrow f_2(x) = \begin{cases} 0 & x \leq 0 \\ \in (0,1) & 0<x<1 \\ 1 & x \geq 1 \end{cases}$$



$$f_3(x) = f_2(2-|x|) \Rightarrow f_3(x) = \begin{cases} 0 & |x| \geq 2 \\ \in (0,1) & 1<|x|<2 \\ 1 & |x| \leq 1 \end{cases}$$



Thm 1.2: Let M be smooth manifold, $A \subset M$ is cpt and $U \subset M$ is open w/ $A \subset U \subset M$

then $\exists$ a bump function $\varphi \in C_0^\infty(M)$ s.t $|\varphi| \leq 1$ and $\varphi \equiv 1$ on A and $\mathrm{supp}(\varphi) \subset U$.

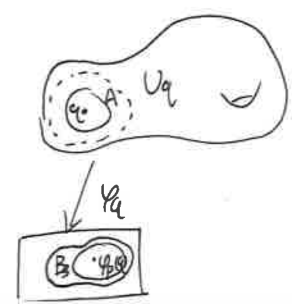
proof: [idea: cover A by finite small pieces $\in (\varphi, U, V)$, so that one can copy the above example].

For each $q \in A$, there $\exists$ (φ_q, U_q, V_q) near q s.t $U_q \subset U$ and V_q contains the open ball $B_3(0)$ "$r=3$"

Let $\tilde{U}_q = \varphi_q^{-1}(B_1(0))$ and let

$$f_q(p) = \begin{cases} f_3(\varphi_q(p)) & p \in U_q \\ 0 & p \notin U_q \end{cases}$$

then $f_q \in C_0^\infty(M)$, $\mathrm{supp}(f_q) \subset U_q$ and $f_q \equiv 1$ on $\tilde{U}_q$



Now the family of open sets $\{\tilde{U}_q\}_{q \in A}$ is open cover of A.

$\exists \{\tilde{U}_q\}_{1 \leq N}$

Let $\psi = \sum_{i=1}^{N} f_{q_i}$

Then ψ is cpt-supp on M.

$\psi \geq 1$ on A and $\mathrm{supp}(\psi) \subset U$.

$\Rightarrow \varphi(p) = f_2(\psi(p)) \equiv 1$. $\square$

Partition of unity:

So far, we can always for $K \underset{cpt}{\subset} M$ covered by nbhds on which we can construct nice "local func"

By adding these funcs, we can find behavior nicely on K. $\Rightarrow$ same thing for M.

Def 2.1: Let M be a smooth manifold, $\{U_\alpha\}$ be open cover $\to M$. A partition of unity (P.O.U) subcoordinate to $\{U_\alpha\}$ is collection of smooth functions $\{\beta_\alpha\}$ "On whole M !!!"

(1) $0 \leq \beta_\alpha \leq 1$ for all α

(2) $\mathrm{supp}(\beta_\alpha) \subset U_\alpha$

(3) each $p \in M$ has nbhd which intersects only finitely many $\mathrm{supp}(\beta_\alpha)$'s.

(4) $\sum_\alpha \beta_\alpha(p) = 1$ for all $p \in M$.

RMKs: (i) denote W_p a nbhd of p w/ finite intersect w/ $\mathrm{supp}(\beta_\alpha)$'s

(1) we have $\{W_p\}$ open cover $\to \underset{\text{"2nd countable"}}{M} \Rightarrow \{W_p\}_{i \leq N}$ covers M w/ W_{p_i} intersect only finite many $\mathrm{supp}(\beta_\alpha)$

$\Rightarrow$ Only finite many $\mathrm{supp}(\beta_\alpha)$ is non-empty.

(2) For each p on open W_p, a sum like (4) is actually finite sum by (1)

Them 2.2 (Existance) Let M be a smooth manifold, and $\{U_\alpha\}$ an open cover of M. Then $\exists$ a P.O.U subordinate to $\{U_\alpha\}$.

i.e Locally, each manifold $\trianglelefteq \mathbb{R}^n$ so that one have lot's of things to operate

P.O.U $\xrightarrow{\text{"glue"}}$ local $\hookrightarrow$ global smooth.

Application: (1) Appromx conti func via smooth

(2) define integrals of differential forms

(3) construct Riemann metric / linear connection. etc

Cor 2.3: Let M be a smooth manifold. $A \underset{\text{closed}}{\subset} M$, $A \subset U \underset{\text{open}}{\subset} M$

then $\exists$ bump function $\varphi \in C_0^\infty(M)$ s.t $0 \le \varphi \le 1$, $\varphi \equiv 1$ on A and $\mathrm{supp}(\varphi) \subset U$.

proof: $\{U, M \setminus A\}$ is an open cover $\to M$. Let $\{\rho_1, \rho_2\}$ be P.O.U.

Then $\varphi := \rho_1$ is what we need, as ρ_1 is smooth, $0 \le \rho_1 \le 1$, $\mathrm{supp}(\rho_1) \subset U$ and $\rho_1 = 1$ on A since $\rho_2 = 0$ on A

"Urysohn's Lemma" smooth-version

proof of them 2.2:

Lemma 2.4: For $\forall$ open cover $\mathcal{U} = \{U_\alpha\} \to M$, one can find two countable family of open covers $\{V_j\} = \mathcal{V}$ and $\mathcal{W} := \{W_j\}$ of M.

(1) each j, $\bar{V}_j$ is cpt and $\bar{V}_j \subset W_j$

(2) $\mathcal{W}$ is a refinement of $\mathcal{U}$, for each j, $\exists\, \alpha = \alpha(j)$ s.t $W_j \subset U_\alpha$.

(3) $\mathcal{W}$ is locally finite. $\forall$ nbhd of $p \in M$, $W_p \cap W_j \neq \emptyset$ for only finitely many W_j's.

Since $\bar{V}_j \subset W_j$ is cpt and $W_j \subset M$ open

Them 1.2 $\Rightarrow \exists\, \varphi_j \in C_0^\infty(M)$ s.t

$$0 \le \varphi_j \le 1, \ \varphi_j \equiv 1 \text{ on } \bar{V}_j, \quad \mathrm{supp}(\varphi_j) \subset W_j.$$

Since $\mathcal{W}$ is locally finite covering,

$$\varphi = \sum_j \varphi_j \text{ is well defined smooth func on } M.$$

Since φ_j is non-negative, $\mathcal{V}$ covering $\to M$, $\varphi > 0$ on M.

Let $\psi_j = \dfrac{\varphi_j}{\varphi}$ are smooth and $0 \le \psi_j \le 1$ and $\sum_j \psi_j = 1$.

For each j, we fixed index $\alpha(j)$ so that $W_j \subset U_{\alpha(j)}$ and define

$$\rho_\alpha = \sum_{\alpha(j)=\alpha} \psi_j \quad (\text{finite sum near each pt})$$

By local finiteness $\to \mathcal{W}$.

$$\mathrm{supp}\, \rho_\alpha = \overline{\bigcup_{\alpha(j)=\alpha} \mathrm{supp}\,\psi_j} = \bigcup_{\alpha(j)=\alpha} \overline{\mathrm{supp}(\psi_j)}$$

$$= \bigcup_{\alpha(j)} \mathrm{supp}\,\psi_j \subset \bar{U}_\alpha.$$

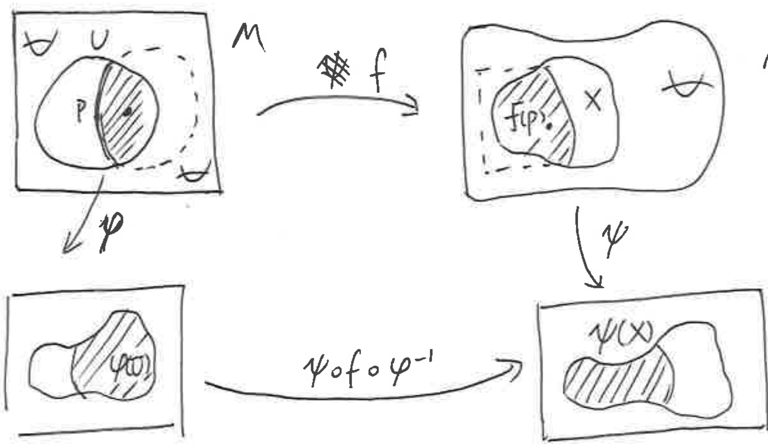
$\Rightarrow \{\rho_\alpha\}$ is P.O.U subordinate to $\{U_\alpha\}$.

Lecture 4. (Mar 25^{th}) Russell-Hua.

Smooth map between manifold

Def 1.1: Let M, N be manifold (smooth). We say a continuous map $f: M \to N$ is smooth if for $\forall$ chart $(\varphi_\alpha, U_\alpha, V_\alpha)$ of M and ~~$(\psi_\beta, U_\beta, V_\beta)$~~ $(\psi_\beta, X_\beta, Y_\beta)$ of N the map:

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1}: \varphi_\alpha(U_\alpha \cap f^{-1}(X_\beta)) \to \psi_\beta(X_\beta) \text{ is smooth.}$$



RMK: We require f to be conti. in definition.

$\Rightarrow$ Guarantee $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is defined on $\varphi_\alpha(p)$'s nbhd.

In general: smooth $(\psi \circ f \circ \varphi^{-1}) \not\Rightarrow f$ is conti. (Problem set)

prop 1.2: If $f: (M, \mathcal{A}) \mapsto (N, \mathcal{B})$ is smooth, $\mathcal{A}_1$, $\mathcal{B}_1$ are atlas on M and N that is compatible w.r.t $\mathcal{A}$ and $\mathcal{B}$. then $f: (M, \mathcal{A}_1) \to (N, \mathcal{B}_1)$ is also smooth "Indep of choice of charts" Check for same as PP. 5.

The set of all smooth map from $M \mapsto N$: $C^\infty(M, N)$.

If $f \in C^\infty(M, N)$, and $g \in C^\infty(N, P) \Rightarrow g \circ f \in C^\infty(M, P)$.

As a concequence, $\forall$ smooth map $f: M \mapsto N$ induces a "pull-back" map

$$f^*: C^\infty(N) \to C^\infty(M) \quad g \to g \circ f$$

Examples of smooth map.

1) Consider $\mathbb{R}$ equipped with $\{(\varphi_1(x)=x, \mathbb{R}, \mathbb{R})\}$, a map $f: M \to \mathbb{R}$ is a smooth map iff it's smooth function.

Moregenerally, a map

$$f = (f_1, \cdots f_k): M \to \mathbb{R}^k \text{ is a smooth map iff } f_i \in C^\infty(M) \text{ for } \forall M.$$

2) The inclusion map i : ~~\#~~ $S^n \to \mathbb{R}^{n+1}$ is smooth, since

$$i \circ \varphi_\pm^{-1}(y^1, \cdots y^n) = \frac{1}{1+|y|^2}(2y^1, \cdots 2y^n, \pm(1-|y|^2))$$

are smooth maps from $\mathbb{R}^n \to \mathbb{R}^{n+1}$

N.B: If g is $\forall$ smooth func on $\mathbb{R}^{n+1}$. the pullback i^*g is just the restriction of g to S^n:

$$i^*g = g|_{S^n}$$

So the restriction of $\forall$ smooth func on $\mathbb{R}^{n+1} \to S^n$, is a smooth func on S^n.

Ex 2: The proj map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is smooth, since

$$\varphi_i \circ \pi(x^1, \cdots x^n) = \left(\frac{x^1}{x^i}, \cdots \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \cdots \frac{x^{n+1}}{x^i}\right)$$

is smooth on $\pi^{-1}(U_i) = \{(x^1, \cdots x^{n+1}) : x^i \neq 0\}$ for each i.

Def 1.3: Let M, N be smooth manifold. A map $f: M \to N$ is a diffeomorphism if it is smooth, bijective, and f^{-1} is smooth.

"$M \cong N$"

prop: (i) the identity map $Id: M \to M$ is a differeomorphism

(ii) If $f: M \mapsto N$; $g: N \to N$ is diffeo $\Rightarrow$ $g \circ f$ is diffeomorphism.

(iii) If $f: M \to N$ is diffeomorphism $\Rightarrow$ f^{-1} is also moreover $\dim M = \dim N$.

$\{f: M \to N \mid f \text{ is a diffeomorphism}\} := Diff(M)$ is a group of M.

Examples: (1) For $M = \mathbb{R}$, the two atlas $\mathcal{A} := \{(\varphi_1(x) = x, \mathbb{R}, \mathbb{R})\}$ and $\mathcal{B} = \{(\varphi_2(x) = x^3, \mathbb{R}, \mathbb{R})\}$ defines non-equivalent smooth structions

However: $f: (\mathbb{R}, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$, $f(x) = x^{1/3}$ is a diffeomorphism

<u>Differential of Eucld-smooth map:</u>

Let U, V be Euclidean open, and $f: U \to V$ a smooth map. The differential of f assign to each pt $a \in U$, defines a linear map $df_a: \mathbb{R}^n \to \mathbb{R}^m$ whose matrix is <u>Jacobian matrix</u> of f at a.

$$df_a = \begin{pmatrix} \frac{\partial f_1}{\partial x^1}(a), & \cdots & \frac{\partial f_1}{\partial x^n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x^1}(a), & \cdots & \frac{\partial f_n}{\partial x^n}(a) \end{pmatrix}$$

Since the "linearization" of the map f near the point x:

$$\lim_{x \to a} \frac{\|f(x) - f(a) - df_a(x-a)\|}{\|x - a\|} = 0.$$

(Chain rule): if $f: U \to V$ and $g: V \to W$ are smooth maps. so is the map $g \circ f: U \to W$ and

$$d(g\circ f)_x = dg_{f(x)} \circ df(x)$$

Them 2.1 (Invariance of Dimension) If $f: U \to V$ is a diffeo, then for each $x \in U$, the differential df_x is a linear isomorphism $\dim U = \dim V$.

proof: Apply Chain rule to $f^{-1} \circ f = Id_U = Id_{\mathbb{R}^n}: \mathbb{R}^n \to \mathbb{R}^n$

$$\Rightarrow (df^{-1})_{f(x)} \circ df_x = Id_{\mathbb{R}^n}$$

similarly

$$\Rightarrow \cancel{(df)_{f(x)}} \; df_x \circ (df^{-1})_{f(x)} = Id_{\mathbb{R}^m}$$

$\to$ $m = n$ and df_x is isomorphism.

RMK on Inverse / implicit func theorem:

ex. consider the map

$$f: \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\} \quad z \to f(z) = z^2.$$

Then $f(z) = f(-z)$. So f is x diffeomorphism since it is not invertible

However at each point $z = (x, y) \in \mathbb{R}^2 \setminus \{0\}$.

$$df_z = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

which is an isomorphism for each $z = (x,y) \neq (0,0)$.

i.e for $\forall x \in \mathbb{C}\setminus\{0\}$ one can find U_x of x, s.t $f|_{U_x}: U_x \to f(U_x)$ is diffeomorphism

Def 2.2: Let $f: U \to V$ smooth map. We say f is a local diffeomorphism near $x \in U$ if there $\exists$ a nbhd U_x and $V_{f(x)}$

s.t $f|_{U_x}: U_x \to V_{f(x)}$ is diffeomorphism.

"Inverse func them"

Them 2.3: If $f: U \to V$ is a smooth map and df_x is an isomorphism

$\Rightarrow$ f is a local diffeomorphism near x.

Them (2.4) "Implicit": let W be an open set in $\mathbb{R}^n_x \times \mathbb{R}^m_y$ and $F = (F_1, \cdots F_m): W \to \mathbb{R}^m$ a smooth map. Let $(x_0, y_0) \in W$ so that

$$\begin{pmatrix} \frac{\partial F_1}{\partial y^1}(x_0,y_0) & \cdots & \frac{\partial F_1}{\partial y^m}(x_0,y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y^1}(x_0,y_0) & \cdots & \frac{\partial F_m}{\partial y^m}(x_0,y_0) \end{pmatrix} \text{ non singular}$$

then, there $\exists$ nbhd $U_0 \times V_0$ of (x_0, y_0) in W and smooth $f: U_0 \to V_0$ so that

- $f(x_0) = y_0$
- $C := F(x_0, y_0)$, then $F^{-1}(c) \cap (U_0, Y_0) = \text{Grah}(f)$

i.e $F(x, f(x)) = C$ for all $x \in U_0$

Lecture 5: Differential of a smooth map (Mar 27^{th}) Russell-Hua

Motivation: $f: M \to N$ smooth. In Euclidean S.P. df_p is a linear map between "Tangent Space"

Q: What's the tangent S.P $\to$ Manifold?

If M is concrete manifold in $\mathbb{R}^N$, then we choose (φ, U, V) near p, so that $\varphi^{-1}: V \to U$ is a diffeomorphism.
If we denote the embedding of M into $\mathbb{R}^N$ to be $i: M \hookrightarrow \mathbb{R}^N$, then we have $i \circ \varphi^{-1}: V \to \mathbb{R}^N$ between Euclid open set, and def!: T_pM to be the image of $d(i\circ\varphi^{-1})_{\varphi(p)}(\mathbb{R}^n)$ of the differential $d(i\circ\varphi^{-1})_{\varphi(p)}: \mathbb{R}^n \to \mathbb{R}^N$

Sec: The Euclid-differential / directional derivative: "Algebraic characteristic."

Recall: for $\forall \vec{v} \in \mathbb{R}^n$, the directional derivative is

$$D^a_{\vec{v}} f := df_x(\vec{v}) = \lim_{h\to 0} \frac{f(x+h\vec{v}) - f(x)}{h} = \frac{\partial}{\partial t}\Big|_{t=0} f(a+t\vec{v})$$

So given $\vec{v} \Rightarrow D^a_{\vec{v}}: C^\infty(\mathbb{R}^n) \to \mathbb{R}$. In coordinate, if $\vec{v} = \langle v^1, \cdots, v^n\rangle^T$, then

$$D^a_{\vec{v}} f = \sum v^i \frac{\partial f}{\partial x^i} \quad / \quad D^a_{\vec{v}} = \sum_i v^i \frac{\partial \cancel{f}}{\partial x^i}$$

Some properties: (1) $D^a_{\vec{v}}$ is linear operator:

$$D^a_{\vec{v}}(af+bg) = a D^a_{\vec{v}} f + b D^a_{\vec{v}} g$$

(2) satisfied Leibniz law at a:

$$D^a_{\vec{v}}(fg) = f(a) \cdot D^a_{\vec{v}} g + g(a) \cdot D^a_{\vec{v}} f$$

conversely also.

$$\boxed{\begin{array}{l}\text{☆ prop. 1.1. If } D: C^\infty(\mathbb{R}^n) \to \mathbb{R} \text{ is linear and satisfied Lebniz law } a, \\ D(fg) = f(a)D(g) + g(a)D(f) \\ \text{then } D = D^a_{\vec{v}} \text{ for some vector } \vec{v} \text{ at } a.\end{array}}$$

proof: $\forall f \in C^\infty(\mathbb{R}^n)$, we have

$$f(x) = f(a) + \int_0^1 \frac{\partial}{\partial t} f(a+t(x-a))\,dt$$
$$= f(a) + \sum_{i=1}^n (x^i - a^i) h_i(x)$$

where $h^i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(a+t(x-a))\,dt$

By Lebnitz-Law $\Rightarrow D(1) = D(1\cdot 1) = 2D(1) \Rightarrow D(1) = 0$
$\Rightarrow D(C) = 0$

So, $D(f) = 0 + \sum_{i=0}^n D(x^i) h_i(a) + \sum_{i=0}^n (a^i - a^i) D(h_i) = \sum_{i=0}^n D(x^i) \frac{\partial f}{\partial x^i}(a)$

#9.

Cont. on proof $\Rightarrow$ Prop 1.1

It follows that as an operator on $C^\infty(\mathbb{R}^n)$

$$D = \sum_{i=1}^{n} D(x^i) \frac{\partial}{\partial x_i}\Big|_{x=a}$$

i.e: if we let $\vec{v} = (D(x^1), \cdots D(x^n)) \Rightarrow D = D^a_{\vec{v}}$

Def 1.2: Any linear operator $D^a : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$ satisfied Lebnitz law at a is called a derivative at a

Now, consider that $\mathcal{D}$ of all derivative at a is a V.P,

$$\vec{v} \rightsquigarrow D^a_{\vec{v}}$$

we have

(i) It is a linear map: $\mathbb{R}^n \rightarrow \mathcal{D}^a_{\vec{v}}$

$$\mathcal{D}^a_{\alpha\vec{v}+\beta\vec{w}} = \alpha \mathcal{D}^a_{\vec{v}} + \beta D^a_{\vec{w}}.$$

(ii) injective: if $\vec{v}_1 \neq \vec{v}_2$, then $D^a_{\vec{v}_1} \neq D^a_{\vec{v}_2}$

(iii) surjective: Follow from Prop 1.1 (i.e $\forall$ linear lebnitz operator $\exists\, \vec{v} \in \mathbb{R}^n$, s.t $D = D^a_{\vec{v}}$)

$$\Rightarrow \mathcal{D}^a_{\vec{v}} \cong T^a_{\vec{v}}$$

Tangent vector on manifolds

Def 1.3: Let M be an n-dim smooth mf

A tangent vector at $p \in M$ is a $\mathbb{R}$-linear map $X_p : C^\infty(M) \rightarrow \mathbb{R}$ satisfied Labnitz law

$$X_p(fg) = f(p) \cdot X_p(g) + g(p)\, X_p(f)$$

for $\forall\, f, g \in C^\infty(M)$

$\{X_p\} := T_pM$, similarly $\Rightarrow X_p(C) = 0$

Lemma 1.4: If $f = c$ in a nbhd of p. then $X_p(f) = 0$.

proof: Let φ be smooth func (bump func) equals to 1 near p and 0 at points $f \neq c$

Then $(f-c)(\varphi) \equiv 0$

$$\Rightarrow 0 = X_p((f-c)\varphi) = (f(p) - c)\, X_p(\varphi) + X_p(f)\,\varphi(p) = X_p(f)$$

i.e If $f = g$ in nbhd of p', then $X_p(f) = X_p(g)$

$\Rightarrow X_p(f)$ is determined by f near p.

So one can replace $C^\infty(M)$ is Def 1.3 by $C^\infty(U)$ and $U \underset{\text{open}}{\subseteq} M$ contains p.

Prop 1.5 : If M is a smooth manifold, $p \in U \subset M$, where U is open, then

$$T_pM \cong T_pU$$

" Summary of "germ" properties "

Sec: The differential of smooth map between smooth manifold:

Recall: The differential of a smooth map $f: U \to V$ between open sets in Euclid S.P at $a \in U$

is a linear map $df_a : T_aU \to T_{f(a)}V$ whose matrix is Jacobian $\left(\frac{\partial f^i}{x^j}\right)(a)$

$$\begin{matrix} T_aU & \to & T_{f(a)}V \\ \| & & \| \\ \mathbb{R}^n_x & \longrightarrow & \mathbb{R}^m_y \end{matrix}$$

certainly, we can define $\vec{v} \in \mathbb{R}^n$ at a w/ derivative $D^a_{\vec{v}} = \sum v^i \frac{\partial}{\partial x^i}$

$$\Rightarrow df_a(\vec{v}) = \left(\frac{\partial f^i}{\partial x^j}\right)\vec{v} = \left\langle \sum_j \frac{\partial f_1}{\partial x^j} v^j, \dots, \sum_j \frac{\partial f_n}{\partial x^j} v^j \right\rangle^T \in \mathbb{R}^m_y$$

when interpreted as a derivative on V at $f(a)$, it's a map : $g \in C^\infty(\mathbb{R}^m_y)$ to

$$\sum_i \sum_j v^j \frac{\partial f_i}{\partial x^j} \cdot \frac{\partial g}{\partial y^i} = \sum_j v^j \frac{\partial}{\partial x^j}(g \circ f) = D^a_{\vec{v}}(g \circ f)$$

$\Rightarrow$ Def 2.1 : Let $f: M \to N$ be a smooth map. Then for each $p \in M$, the differential of f is the linear map $df_p : T_pM \to T_{f(p)}N$ defined by

$$df_p(X_p)(g) = X_p(g \circ f)$$

for all $X_p \in T_pM$ and $g \in C^\infty(N)$

prop of differentials:

Them 2.2 (Chain rule) Suppose $f: M \to N$ and $g: N \to P$ are smooth maps, then

$$d(g\circ f)_p = dg_{f(p)} \circ df_p$$

proof: For $\forall X_p \in T_pM$ and $h \in C^\infty(P)$

$$d(g\circ f)_p(X_p)(h) = X_p(h\circ g\circ f) = df_p(X_p)(h\circ g) = dg_{f(p)}(df_p(X_p))(h)$$

Cor 2.3: If $f: M \to N$ is diffeomorphism, then $df_p: T_pM \to T_{f(p)}N$ is a linear isomorphism

(Lec 4, Them 2.1)

In particular,

Cor 2.4: If $\dim M = n$, then T_pM is an n-dim linear space.

proof: Let (φ, U, V) be chart near p. Then $\varphi: U \to V$ is diffeomorphism

$$\Rightarrow \dim T_pM \overset{*}{=} \dim T_{f(p)}N = n$$

$$\dim T_pU = \dim T_{f(p)}V$$

N.B: We really show that: For $\forall$ local chart (φ, U, V) around p

$$T_pM = \operatorname{span}\{\partial_1, \cdots \partial_n\}.$$

where $\partial_i := d\varphi^{-1}\left(\frac{\partial}{\partial x^i}\right)$ where $x^i \circ \varphi^{-1} := x^i$ actually

$$\partial_i: C^\infty(U) \to \mathbb{R}, \quad \partial_i f = \frac{\partial(f\circ\varphi^{-1})}{\partial x^i}(\varphi(p))$$

Lecture 6: (Mar 29th) Russell Hua

The inverse function theorem:

Recall: If $f: M \to N$ a diffeomorphism $\Rightarrow$ $df_p: T_pM \to T_pN$ is linear isomorphism.

Conversely:

Them 1.1 (IFT): Let $f: M \to N$ be a smooth map s.t $df_p: T_pM \to T_{f(p)}N$ is linear isomorphism, then there exists a nbhd U_1 of p and a nbhd X_1 of $q = f(p)$ s.t $f|_{U_1}: U_1 \to X_1$ is diffeomorphism (local diffeo)

proof: Take a chart (φ, U, V) near p and a chart (ψ, X, Y) near $f(p)$ s.t $f(U) \subset X$ (possible by shrinking U and V)

Since $\varphi: U \to V$ and $\psi: X \to Y$ are diffeomorphism.

$$d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} = d\psi_q \circ df_p \circ d\varphi^{-1}_{\varphi(p)} : T_{\varphi(p)}V = \mathbb{R}^n \to T_{\psi(q)}Y = \mathbb{R}^n$$

is an linear isomorphism. It follows from the IVP on $\mathbb{R}^n$ that (Lec 4) There $\exists$ V_1 of $\varphi(p)$ and Y_1 of $\psi(q)$ so that $\psi \circ f \circ \varphi^{-1}$ is linear diffeomorphism from $V_1 \to Y_1$. Take $U_1 = \varphi^{-1}(V_1)$ and $X_1 = \psi^{-1}(Y_1)$

Then $f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$ is a diffeomorphism from $U_1 \to X_1$.

Sec: Global diffeo V.S local diffeo

Def: We say a smooth map $f: M \to N$ is a local-diffeo near p, if it maps $U_p \xrightarrow{\simeq} V_{f(p)}$

Example (local diffeo, X Global diffeo)

Let $f: S^1 \to S^1$ be given by $f(e^{i\theta}) = e^{2i\theta}$, then it's a local diffeo everywhere, but not global diffeo since it's not invertible (Lec 4 pp.5)

Prop 1.3 : Suppose $f: M \to N$ is a local diffeomorphism near every $p \in M$. If f is invertible, then f is a global diffeomorphism.

proof : It's enough to show f^{-1} is smooth (Globally). Fixed $q = f(p)$. Smoothness of f^{-1} is depend on f^{-1} near q.

Since f is diffeo from $U_p \xrightarrow{\text{onto}} V_{f(p)=q}$, f^{-1} is smooth at q. //

Constant rank theorem.

Submersion / immersion : Motivation If df_p is not linear isomorphism ?

Def 2.1 : Let $f: M \to N$ be a smooth map

(1) f is a <u>submersion</u> at p : if $df_p : T_pM \xrightarrow{\text{onto}} T_pN$ is surjective

(2) f is a <u>immersion</u> at p : if $df_p : T_pM \xrightarrow{1\text{-}1} T_pN$ is injective

$\Rightarrow$ at $\forall p \in M \Rightarrow f$ is sub/im-mersion

OBS : (1) If f is a submersion : $\dim M \geqslant \dim N$

(2) If f is a immersion : $\dim M \leq \dim N$.

examples : (1) $\pi : TM \to M$ is a submersion

(2) "zero section" $i : M \to TM : p \to (p, 0)$ is immersion.

(3) Canonical submer : If $m \geqslant n$, then

$$\pi : \mathbb{R}^m \to \mathbb{R}^n, \quad (x^1, \cdots x^m) \to (x^1, \cdots x^n)$$

(4) Canonical immer : If $m \leq n$, then inclusion map

$$i : \mathbb{R}^m \hookrightarrow \mathbb{R}^n, \quad (x^1, \cdots x^m) \to (x^1, \cdots x^m, 0, 0, \cdots 0)$$

N.B : $\forall$ submersion/immersion Looks like (3), (4) locally.

Thm 2.2 (Canonical Submersion/Immersion)

Let $f: M \to N$ be submersion/immersion at $p \in M$, then obviously $m = \dim M \geq \dim N = n$ / $m = \dim M \leq \dim N = n$ then there $\exists$ charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) around $q = f(p)$ s.t

$$\psi_1 \circ f \circ \varphi_1^{-1} = \pi|_{V_1}$$

or

$$\psi_1 \circ f \circ \varphi_1^{-1} = i|_{V_1}$$

Def 2.4: We say that a smooth map $f: M \to N$ is a constant rank map near $p \in M$, if there $\exists$ U_p so that df_p has constant rank r. (i.e $\exists r \in \mathbb{N}$, so that $\operatorname{rank}(df)_q \equiv r$) for all $q \in U$.

Example: (1) If f is submersion/immersion at p, then it's sub/im near p, and thus constant rank map near p.

(2) "Canonical" constant rank map. Generally, by composing suitable canonical $\underbrace{\text{and}}_{\text{submmersion}}$ immersion $\Rightarrow$ constant rank map

$$\mathbb{R}^m = \mathbb{R}^{r+m-r} \xrightarrow{\pi} \mathbb{R}^r \xrightarrow{i} \mathbb{R}^{r+n-r} = \mathbb{R}^n$$

$$(x^1, \dots x^r, x^{r+1}, \dots x^m) \longrightarrow (x^1, \dots x^r, 0, 0, \dots 0)$$

Thm 2.5: (Constant Rank theorem) Let $f: M \to N$ be smooth map so that $\underbrace{\ }_{\text{rank}}(df) = r$ near p. Then there $\exists$ charts (φ_1, U_1, V_1) around p and (ψ_1, X_1, Y_1) near $f(p)$ s.t

$$\psi_1 \circ f \circ \varphi_1^{-1}(x^1, \dots x^m) = (x^1, \dots x^r, 0, 0, \dots 0).$$

proof: Step 1 (Euclidean S.P)

Conti on proof of CRT :

Assume $\underset{\text{open}}{U \subset} \mathbb{R}^n$ and $f: U \to \mathbb{R}^n$ smooth so that df_x has constant rank r near $x \in U$.

By translation, we may assume $0 \in U$ and $f(0) = 0$. Since rank $(df)_0 = r$. By switching coordinate, we may assume that.

the upper left of Jacobian $df = \underset{①}{\left(\frac{\partial f^i}{\partial x^j}\right)_{1 \le i \le n,\, 1 \le j \le m}} \xrightarrow{\text{restrict}} \underset{②}{\left(\frac{\partial f^i}{x^j}\right)_{1 \le i,j \le r}}$ is non-singular at $x = 0$. (non-singular near $x=0$)

i.e we want to take $f_1, \cdots f_r$ as coordinate, and keep r^{th} term unchanged)

Rank ① = rank ② (clearly)

Now define $\varphi: U \to \mathbb{R}$ by $\varphi(x) = (f_1(x), \cdots f_r(x), x^{r+1}, \cdots x^m)$. Then $\varphi(0) = 0$ and the differential

$$\partial\varphi = \begin{pmatrix} \left(\frac{\partial f^i}{x^j}\right)_{1 \le i,j \le r} & * \\ 0 & Id_{n-r} \end{pmatrix}$$ is non-singular at $x=0$

By IFT, φ is a local diffeo near 0. There $\exists\, 0 \in U_1 \underset{\text{open}}{\subset} \mathbb{R}^m$ and $\underset{f(0)}{\overset{}{V_1}} \underset{\text{open}}{\subset} \mathbb{R}^m$ s.t. $\varphi: U_1 \to V_1$ is diffeomorphism.

By def. we have $f \circ \varphi^{-1}(f_1, \cdots f_r, x^{r+1}, \cdots x^m) = f \circ \varphi^{-1}(\varphi(x)) = f(x) = (f_1, \cdots f_n(x))$

i.e locally, we have $f \circ \varphi^{-1}(x) = (x^1, \cdots x^r, g_{r+1}(x), \cdots g_n(x))$ for some smooth $g_{r+1}, \cdots g_n$ ($g_i(0) = 0$)

Chain rule : $$df_{\varphi^{-1}(x)} \circ (d\varphi^{-1})_x = \begin{pmatrix} Id_r & 0 \\ * & \frac{\partial g^i}{\partial x^j}\ {}_{r+1 \le i,j \le n} \end{pmatrix}$$

OBS: Since $(\partial\varphi^{-1})_x$ is linear isomorphism "rank (df_x) $= r$ near 0" $\Rightarrow$ "rank $(df_{\varphi^{-1}(x)} \circ (d\varphi^{-1})_x) = r$ near 0."

and thus $$\frac{\partial g^i}{\partial x^j} = 0 \quad \forall\, r+1 \le i, j \le m. \text{ near } 0.$$

It follows that in a small nbhd of 0, we have

(Cont. on proof of Constant rank them)

$$g_i(x) = g_i(x^1, \cdots x^r) \quad \forall\, r+1 \le i \le n.$$

and near 0

$$f \circ \varphi^{-1}(x) = (x^1, \cdots x^r, g_{r+1}(x^1, \cdots x^r), \cdots g_n(x^1, \cdots x^r))$$

we still need to kill the g_i's terms. So we define

$$\psi(y) = (y^1, \cdots y^r, y^{r+1} - g_{r+1}(x^1, \cdots x^r), \cdots, y^n - g_n(y^1, \cdots y^r))$$

"Check it is indeed local diffeomorphism"

$$\Rightarrow \psi \circ f \circ \varphi^{-1}(x^1, \cdots x^r, x^{r+1}, \cdots x^n) = (x^1, \cdots x^r, 0, 0, \cdots 0)$$

"Again Follow from IFT and $\partial \psi_0 = \begin{pmatrix} Id & 0 \\ * & Id_{n-r} \end{pmatrix}$

Step 2: (General case)

It follows from what we usually do from maps between manifold. Taking (φ, U, V) (near p) and (ψ, X, Y) near $f(p)$ so that $f(U) \subset X$ and df_q has constant rank r on U. The $\psi \circ f \circ \varphi^{-1}$ has rank r, sice.

$$\partial(\psi \circ f \circ \varphi^{-1})_x = \partial\psi_{f(\varphi^{-1}(x))} \circ df_{\varphi^{-1}(x)} \circ (\partial\varphi^{-1})_x$$

linear isomorphism

and by Euclidean case we are done $\square$

Concequence; map is a constant rank map iff it can be locally written as $j \circ s$

immersion

submersion.

Generally $\Rightarrow$ If a constant rank map is surjective $\Rightarrow$ submersion

injective $\Rightarrow$ immersion.

Lec 7 (Apr 1st) Sard's theorem Russell Hua

Recall from calculus:

(1) $a \in \mathbb{R}$ is called a critical point of smooth func $f: \mathbb{R} \to \mathbb{R}$ if $f'(a) = 0$

(2) $a \in \mathbb{R}^n$ is called a critical point of smooth func $f: \mathbb{R}^m \to \mathbb{R}$ if $\frac{\partial f}{\partial x^i} = 0$ for all i.

(i.e $df_a: \mathbb{R}^m \to \mathbb{R}$ is not surjective)

(3) $a \in \mathbb{R}^m$ is a critical point of a smooth map $f: \mathbb{R}^m \to \mathbb{R}^n$ between Euclidean open sets if $df_a: \mathbb{R}^m \to \mathbb{R}^n$ is not surjective.

Def 1.1: Let M, N be smooth manifold and $f: M \to N$ a smooth map.

(a) We say $p \in M$ is a critical point of f if $df_p: T_pM \to T_{f(p)}N$ is not surjective $\Rightarrow \{p\}_f := \mathrm{Crit}(f)$.

(b) $p \in M$ is a regular point if it's not critical point.

(c) $q \in N$ is a regular value of f if $\forall p \in f^{-1}(q)$ is a regular point,

We also say $q \in N$ is a critical value if it's not regular value.

RMK: Let $f: M \to N$ smooth map

(i) $q \in N \setminus \mathrm{Im}(f)$ is a regular value

(ii) $\mathrm{Im}(\mathrm{Crit}(f))$ = "set of critical values" but $f^{-1}(\text{crit value}) \not\equiv \mathrm{Crit}(f)$.

Ex. $f \in C^\infty(M)$, $p \in M$ is max / min of f. Then f has p as critical pt.

proof. For $\forall X_p = \sum a_i \partial_i|_p \in T_pM$, we have

$$X_p(f) = \sum a_i \partial_i|_p(f) = \sum a_i \frac{\partial(f\circ\varphi^{-1})}{\partial x_i}(\varphi(p)) = 0.$$

so $g(t) = t \in C^\infty(\mathbb{R})$

$$df_p(X_p)(g) = X_p(g\circ f) = X_p(f) = 0 \Rightarrow df_p(X_p) = 0 \Rightarrow p \text{ is critical pt.}$$

Lemma 1.3: If we identify $T_t\mathbb{R}$ w/ $\mathbb{R}$, then

$$df_p(X_p) = X_p(f), \quad \forall f \in C^\infty(\mathbb{R}), \quad \forall X_p \in T_p(\mathbb{R})$$

example. $f: S^n \to \mathbb{R}$, $f(x_1, \cdots x^{n+1}) = x^{n+1}$

has "pole" $(0, \cdots 1)$ and $(0, \cdots -1)$ as critical value! and all other pts are regular value.

Extreme/trivial case: $f: M \to N$ const. map $f(p) \equiv q_0 \in N$,

$f: M \to N$ smooth, but $\dim(M) < \dim(N)$

Sard's thm

I.e the set of critical pts → values is negligible in N. (Sometimes failed "const. map")

<u>Thm</u>: For $\forall$ smooth map $f: M \to N$. the set of critical pts has measure zero. in N.

N.B: Since we may cover M by $\# m < \infty$, (U_i, φ_i) charts, and union of meas-zero set is still $\mu(E) = 0$.
It's sufficient to prove in $\mathbb{R}^m \to \mathbb{R}^n$. case.

proof. first, we have for $m < n$, the result trivially hold.

we will proof by induction, for $m = 0$. then clearly meas-0 set → meas set. Let C be the set of critical pts

WTS: $f(C)$ meas-0 in N.

Let $C_j := \{x \in U \mid \partial^a f(x) = 0$ for all $|a| \leq j\}$.

OBS:

$$f(C) = f(C \setminus C_1) \cup f(C_1 \setminus C_2) \cup \cdots \cup f(C_{k-1} \setminus C_k) \cup f(C_k)$$

Step 1: $f(C \setminus C_1)$ meas-0 in N

for $\forall x \in C \setminus C_1$, $\exists U_x \ni x$, s.t $f(C \cap U_x)$ has meas-0. Since $C \setminus C_1$ can be covered by at most countable many such sets $\Rightarrow f(C \setminus C_1)$ has meas 0.

Step 2: $f(C_i \setminus C_{i+1})$ meas-0 in N.

Let $x \in C_i \setminus C_{i+1}$, there is α w/ $|\alpha| = i$, s.t $w := \partial^\alpha f$ vanish on C_i, at least $\frac{\partial w}{\partial x^1}$ does not vanish at x.

By IFT, we have

$$h: U_x \xrightarrow[\text{onto}]{} \mathbb{R}^m, \quad h(x) = (w(x), x^2, \dots x^m)$$

diffeomorphic on $V \subseteq \mathbb{R}^m$

By construction, $h: C_i \cap U_x \to \{0\} \times \mathbb{R}^{m-1}$

Let $g = f \circ h^{-1}$ and $\bar{g}: (\{0\} \times \mathbb{R}^{m-1}) \cap V \to \mathbb{R}^n$

$\Rightarrow$ the set of critical pt of $\bar{g}$ is meas in $\mathbb{R}^n$.

Setp 3: $f(C_k)$ has meas-0 for $k >> 1$, say $k \geq m/n$. (Milner)

Let $Q \subset U$ be cube w/ side δ-length. WTS: $k > m/n - 1$, $f(C_k \cap Q)$ has meas-0.

By Taylor's thm, Q cpt and C_k critical pt set, we have.

$$f(x+h) = f(x) + R(x, h).$$

where $|R(x,h)| < a|h|^{k+1}$ for $x \in C_k \cap Q$, $x+h \in Q$ and a depends only on f and Q.

Now, we divide Q into r^m cubes w/ side δ/r. $x \in Q_1$ can be written as $x+h$ w/ $|h| < \sqrt{m}\,\delta/r$.

By this we can show that $f(C_k \cap Q) \subset \bigcup_{m=1}^{r^m} r_m$ cubes and.

$$Vol \leq r^m \left(\frac{b}{r^{k+1}}\right)^n = b^n r^{m-(k+1)n} \quad \text{where } b = 2a(\sqrt{m}\,\delta)^{k+1}.$$

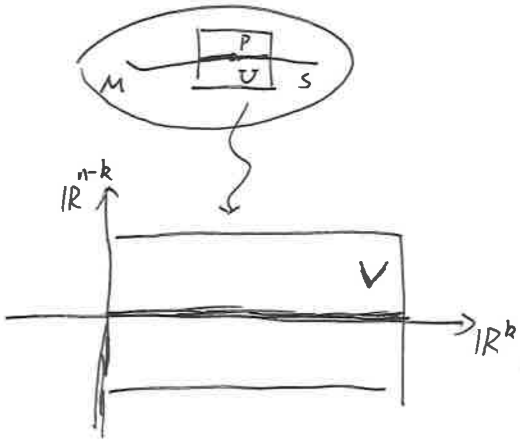
Since $k > m/n - 1$, we see $Vol \to 0$ as $r \to \infty$. $\Rightarrow f(C_k \cap Q)$ is meas-0 in N. $\square$

Lec 8: (Apr 3rd) Russell - Hua.

Def. $S \subset M$ is a k-dim submanifold (smooth) of M if $\forall p \in S$, there $\exists$ (φ, U, V) of p s.t.

$$\varphi(U \cap S) = V \cap (\mathbb{R}^k \times \{0\}) = \{x \in \varphi(U) \mid x^{k+1} = \cdots = x^n = 0\}.$$

we call $\operatorname{codim}(S) = n-k$ "codimension of S."



ex. M, N smooth mfs, and $f: M \to N$ smooth. Then

$$\Gamma(f) = \{(p,q) \mid q = f(p)\} \subset M \times N$$

is smooth submanifold of $M \times N$.

Let (φ, U, V) be arbitary chart near p. and (ψ, X, Y) near $q = f(p)$.

$\Rightarrow$ $(\varphi \times \psi,\ U \times X,\ V \times Y)$ is chart $M \times N$ near (p,q)

Not enough; $q = f(p)$ as $\psi^{-1}(y) = f(\varphi^{-1}(x))$

$\Rightarrow$ $y = \psi \circ f(\varphi^{-1}(x))$

Let $\Psi: V \times Y \to \mathbb{R}^m \times \mathbb{R}^n$ as $(a,b) \to (a,\ b - \psi \circ f(\varphi^{-1}(a)))$

$\Rightarrow$ Ψ is 1-1 and local diffeomorphism e.s $\Rightarrow$ Global diffeo and $V \times Y \xrightarrow{\text{onto}} \Psi(X \times Y)$

$(p,q) \in \Gamma_f \cap (U \times X)$

$\Downarrow$

$\Psi(q) = \psi(f(\varphi^{-1}(\varphi(p)))) = \Psi(\varphi(p), \psi(q))$
$= (\varphi(p), 0).$

$(\Psi \circ (\varphi \times \psi),\ U \times X,\ V \times Y)$ charts.

near (p,q).

$\Uparrow$

RMK: 1) $\forall f \in C(U)$, $U \subset \mathbb{R}^n$. there is smooth structure on $\Gamma_f \rightarrow$ smooth manifold dim n.

2) f is not smooth func w/ Γ_f smoot submf:

$$f(x) = x^{1/3}, \quad y = f(x) \text{ "="} \ x = y^3 \text{ which is of a smooth submanifold of } \mathbb{R}^2$$

ex 2. S^n is smooth submf of $\mathbb{R}^{n+1}$.

The induced smooth structure:

Let

$\pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$, $(x^1, \dots x^n) \mapsto (x^1, \dots x^k)$

$i : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$, $(x^1, \dots x^k) \mapsto (x^1, \dots x^n)$

then we have

Prop 1.2: (U, φ, V) on M. that satisfied $\varphi(U \cap S) = V \cap (\mathbb{R}^k \times \{0\})$

Let $X = U \cap S$, $Y = \pi \circ \varphi(X)$ and $\psi = \pi \circ \varphi|_X$. Then (ψ, X, Y) smooth and $\forall$ charts of this form are compatible $\Rightarrow$ S smooth manifold, and $i : S \hookrightarrow X$ is immersion.

proof. By def: ψ is invertible, $\psi^{-1} = \varphi^{-1} \circ i$. So (ψ, X, Y) charts on S. Let ψ_β, ψ_α be maps.

$$\psi_\beta \circ \psi_\alpha^{-1} = \pi \circ \varphi_\beta \circ \varphi_\alpha^{-1} \circ i = \pi \circ \varphi_{\alpha,\beta} \circ i, \text{ smooth again.}$$

the map $i : S \hookrightarrow M$ is smooth immersion,

$$\varphi \circ i \circ \psi^{-1} = i$$

Now, let $S \subset M$ be submanifold. $p \in S$. Since $i : S \hookrightarrow M$ is an embedding

$di_p : T_pS \to T_pM$ is injective.

We can identify $\forall\ X_p \in T_pS$ w/ $\tilde{X}_p = di_p(X_p)$ in $T_p(M)$ so that $\forall f \in C^\infty(M)$.

$$\tilde{X}_p(f) = (di_p(X_p))f = X_p(f \circ i) = X_p(f|_S)$$

<u>THM</u> : If $S \subset M$ submanifold, $p \in S$. Then

$$T_pS = \{X_p \in T_pM \mid X_p(f) = 0 \text{ for all } f \in C^\infty(M) \text{ w/ } f|_S = 0\}.$$

proof. ($\Rightarrow$) Let $X_p \in T_pS$. then for $f \in C^\infty(M)$ w/ $f|_S = 0$, $\tilde{X}_p(f) = X_p(f|_S) = 0$.

($\Leftarrow$) Let $X_p \in T_pM$ satisfied $X_p(f) = 0$ for $f|_S = 0$.

Take (φ, U, V) on M near p. S is given by $x^{k+1} = \cdots = x^n = 0$

Then T_pM is span$\{\partial_1, \dots \partial_n\}$ while T_pS is span$\{\partial_1, \cdots \partial_k\}$.

I.e $X_p = \sum X^i \partial_i \in T_pS$ iff $X^i = 0$ for all $i > k$.

Now let h be bump func on U supported ($=1$)

For $j > k$, consider $f_j(x) = h(x) \cdot x^j(\varphi(x))$. $f_j(x) = 0$ on $M \setminus U$.

Then $f_j|_S = 0$. So

$$0 = X_p(f_j) = \sum \frac{X^i \cdot \partial(h(\varphi^{-1}(x))x^j)}{\partial X^i}(\varphi(p)) = X^j \text{ for } \forall j > k \Rightarrow X_p \in T_p(S) \quad \square$$

We have shown the result that : preimage of a regular value of $\forall$ smooth map is smooth submanifold.

By Sard thm : if $f(M)$ is not negligible in N, then for most $q \in f(M)$, $f^{-1}(q)$ is a smooth submanifold of M.

Thm 2.1: Let $f: M \to N$, $q \in N$ regular value.

Then $f^{-1}(q)$ is smooth submf of M w/ $\dim(f^{-1}(q)) = \dim(M) - \dim(N)$.

Moreover, $p \in S$, $T_pS = \ker(df_p : T_pM \to T_pN)$

Thm 2.2: (Recall) Let M, N smooth mf, $f: M \to N$ has const. rank r. Then each level set of f is closed mf of codim $= r$.

Moreover, $p \in S$. $T_pS = \ker(df_p : T_pM \to T_pN)$

proof (2.1): Let $p \in S := f^{-1}(q)$. Then by const. rank thm. $\exists (\varphi_1, U_1, V_1)$ centered at p and (ψ_1, X_1, Y_1) centered at q s.t $f(U_1) \subset X_1$ and

$$\psi_1 \circ f \circ \varphi_1^{-1}(x^1, \cdots x^m) = (x^1, \cdots x^r, 0, \cdots 0)$$

It follows that

φ_1 maps $U_1 \cap f^{-1}(q)$ onto $V_1 \cap \{(0, 0, \cdots 0, x^{r+1} \cdots x^m)\}$. So $f^{-1}(q)$ is submf of codim r.

Let $i: S \hookrightarrow M$. Then for $\forall p \in S$, $f \circ i(p) = q$. In.e $f \circ i$ is const map on S.

So $df_p \circ di_p = 0$ i.e $df_p = 0$ on $\mathrm{Im}(di_p)$ / $T_pS \subset \ker(df_p)$

But $\dim T_pS = \dim S = m - r$ and

$$\dim \ker(df_p) = \dim \ker((d\psi_1)_{f(p)} \circ df_p \circ (d\varphi_1^{-1})_{\varphi(p)}) = m - r.$$

$$\Rightarrow \dim T_pS = \dim \ker(df_p) \Rightarrow T_pS = \ker(df_p). \quad \square$$

Some examples:

(i) S^n is smooth submf of $\mathbb{R}^{n+1}$

(ii) $SL(n,\mathbb{R})$ is smooth submf of $GL(n,\mathbb{R})$

(iii) $O(n,\mathbb{R}) \subset GL(n,\mathbb{R})$.

N.B. the level set of critical value may failed to be smooth mf.

$f: \mathbb{R}^2 \to \mathbb{R}$. $f(x,y) = x^2 - y^2$

$df_{(x,y)} = (2x, 2y)$ and critical pt is $(0,0)$.

But $f^{-1}(0) = \{(x,y),\ x^2-y^2=0\}$ is not a mf!!!

Difference between smooth mf and immersed mf ?

prop 2.3. Let S be a smooth submf of M. Then $\iota: S \hookrightarrow M$ is a homeomorphism from $S \to \iota(S)$

Def 2.4: $f: M \to N$ immersion. f is called embedding if it's a homeomorphism onto its image $f(N)$

topology on $f(N)$ is subsp topo.

By def: $i: S \hookrightarrow M$ is an embedding. so each smooth submanifold is the image of an embedding!!!

Them 2.5. Let $f: N \to M$ embedding. Then $f(N)$ is smooth submf of M.

proof. Let $p \in N$, $q = f(p) \in M$. f is immersion, by Canonical immersion them

$\exists\ (\varphi_1, U_1, V_1),\ (\psi_1, X_1, Y_1)$ near p, q. s.t. on V_1, $\psi_1 \circ f \circ \varphi_1^{-1}$ is the canonical embedding $J: \mathbb{R}^m \to \mathbb{R}^n$

i.e

$\psi_1 \circ f = J \circ \varphi_1$ on U_1, Since f is homoemphism on $f(U_1) \Rightarrow f(U_1) \underset{\text{open}}{\subset} f(N) \cap X$.

Replace X_1 by $X_1 \cap X \Rightarrow$ $\psi_1(X_1 \cap f(N)) = Y_1 \cap \psi_1(f(U_1)) = Y_1 \cap J(\varphi_1(U_1)) = Y_1 \cap (\mathbb{R}^m \times \{0\})$.

Summary: (immersion and embedding)

- If $f: M \to N$ an immersion, then by Canonical immersion thm, $\forall p \in N$ has nbhd in N whose image is "nice" in M.
- If $f: M \to N$ an embedding, $\forall q \in f(N)$ have nbhd in $f(N)$ that is "nice" in M (Thm 2.5).

Lec 9 (Apr 5^{th}) Russell Hua

Thm (Whitney - embedding thm)

Any m-dim smooth mf M can be embedded into $\mathbb{R}^{2m+1}$ (and immersed into $\mathbb{R}^{2m}$).

We will only proof for cpt case, for the non-cpt case, the steps are same.

(1) Injective immerse M into $\mathbb{R}^k$ for $k>>1$.

(2) For $k > 2m+1$, project $\mathbb{R}^k$ to some $\mathbb{R}^{k-1}$

(3) cpt conditions, injective immersion $\Rightarrow$ embeddings

Thm 1.1: $\forall$ cpt mf M admits an injective immersion into $\mathbb{R}^k$ for $k>>1$.

proof. Let $\{(\varphi_i, U_i, V_i)\}_k$ be finite charts set on M. (cover)

Let $\{\rho_i \mid i \le k\}$ be P.O.U subord to $\{U_i \mid 1 \le i \le k\}$.

Define $\Phi: M \to \mathbb{R}^{k(m+1)}$, $p \to (\rho_1(p)\varphi_1(p), \dots \rho_k(p)\varphi_k(p), \rho_1(p), \dots \rho_k(p))$

1) Φ is injective. if $\Phi(P_1) = \Phi(P_2)$. Let i be s.t $\rho_i(P_1) = \rho_i(P_2) \neq 0$. Then $P_1, P_2 \in supp(\rho_i) \subset \bar{U}_i$. $\Rightarrow \varphi_i(P_1) = \varphi_i(P_2)$.

since φ is bijective $\Rightarrow P_1 = P_2$

2) Φ is immersion, For $\forall X_p \in T_pM$, by Leibnitz law,

$$d\Phi_p(X_p) = (X_p(\rho_1)\varphi_1(p) + \rho_1(p)(d\varphi_1)_p(X_p), \dots X_p(\rho_k)\varphi_k(p) + \rho_k(p)(d\varphi_k)_p(X_p), X_p(\rho_1), \dots X_p(\rho_k))$$

if $d\Phi_p(X_p) = 0 \Rightarrow X_p(\rho_i) = 0$ for $\forall i$. $\Rightarrow \rho_i(p)(d\varphi_i)_p(X_p) = 0$.

Similarly, pick i s.t $\rho_i(p) \neq 0$. $(d\varphi_i)_p(X_p) = 0$ since φ_i diffeomorphism $\Rightarrow X_p = 0$ so $d\Phi_p$ is injective.

Them 1.3. If M, $\dim(M)=n$, admits an injective immersion from $M \to \mathbb{R}^k$ for some $k > 2m+1$, then it admits an injective immer-sion into $\mathbb{R}^{k+1}$. (N.B. the Step 2 does not require cpt - property)

(I skipped this proof, but basically we ganna use Sard's them to show that $\Phi_{[v]}$ is not immersion is of measure-zero)

Them 1.5: If $f: M \to N$ is an injective immersion, and N is cpt, then f is an embeddeding

proof. f injective. $f: N \to f(N)$ is invertible. Since $f: N \to M$ conti. $f: N \to f(N)$ conti.

let $A \underset{\text{closed}}{\subset} N$, Since N cpt $\Rightarrow$ A cpt. So $f(A)$ is cpt. Since $f(N)$ is Hausdorff, $f(A)$ is closed in $f(N)$.

So we have f conti + closed and $f^{-1}: f(N) \to N$ is conti. $\square$

Immediate Concequence: $\forall$ smooth cpt M of dim m can be immersed into $\mathbb{R}^{2m}$ and embedded into $\mathbb{R}^{2m+1}$.

Lec (10) Apr 5^{th} Russell Hua

Recall lec 6 if df_p is a linear isomorphism $\Rightarrow f$ is local diffeo near p.

Thm 1.1 (Generalized IVFT)

Let $f: M \to N$, $X \subset M$ submanifold. Suppose $df_p: T_pM \to T_{f(p)}N$ is linear isomo for every $p \in X$.

And (i) If X is cpt, then assume f is 1-1 on X

(ii) If X is non-cpt, we assume $f(X)$ is submanifold of N and f maps X diffeo onto $f(X)$

Then f maps $U \subset X$ in M diffeo onto $V \subset f(X) \subset N$.

RMK: f: 1-1 on X $\Leftrightarrow$ X diffeomorphically onto $f(X)$ when X cpt!

$f|_X: X \to N$ is immersion. So if $f|_X$ is 1-1 and X cpt, then $f|_X: X \to N$ is an embedding, thus $f|_X: X \to f(X)$ is diffeomorphism.

Example for (ii) need additional assup:

e.g. $f: \mathbb{R}^2 \to T^2$, $(t,s) \to (e^{it}, e^{is})$

f is locally diffeo near any $(t,s) \in \mathbb{R}^2$. Let X be "irrational slope line"

$X = \{(t, \sqrt{2}t), t \in \mathbb{R}\} \subseteq \mathbb{R}^2$ and $f|_X$ is injective but there's $\nexists\, U \subset X$ in $\mathbb{R}^2$ and $V \subseteq T^2$ so that.

f maps U diffeo onto V !!

proof. (We only do cpt case here)

By IFT, f is a local diffeomophism near $\forall x \in X$. According to Prop 1.3 Lec 6, it's sufficient to show f is injective on nbhd of X.

We embed M into $\mathbb{R}^k$, and let ε-nbhd of $X \subset M$ as

$$\underset{\substack{\cap \text{ open} \\ \mathbb{R}^k}}{X^{\varepsilon}} = \{x \in M \mid d(x, X) < \varepsilon\}, \quad d(x, X) = \inf\{(x, y),\ y \in X\}.$$

Since X is cpt, then X^{ε} is bdd. Moreover, we have $X = \bigcap_{k>0} X^{1/k}$ since X is closed.

Suppose f is not 1-1 on each $X^{1/k}$, then we can find $a_k \neq b_k \in X^{1/k}$ s.t $f(a_k) = f(b_k)$.

Since $a_k \in \underset{\substack{\text{bdd+cpt} \\ (\text{can be } X^{1/k})}}{A_k} \subset \mathbb{R}^k$, we can find $a_{k_i} \to a_{\infty} \in X$. Similarly, $b_{k_{i_j}} \to b_{\infty} \in X$, since $f(a_{\infty}) = f(b_{\infty})$

one must have f is 1-1 on X (f is smooth). So $\forall$ nbhd of a_{∞}, f is not 1-1. But $df_{a_{\infty}}$ is linear isomorphism implies f is a local diffeomorphism near a_{∞}. (contradiction).

Non-cpt case: [Any non-cpt manifold can be written as union of countablely many "cpt stripes"].

Tubular NBHD them.

$X \subset M$. smooth-submanifold. X always admits a "tubular" nbhd inside M!

Them 2.1: (ε-nbhd them) Let $\iota: X \hookrightarrow \mathbb{R}^k$ be smooth submf. Then $\exists$ a conti positive value func $\varepsilon: X \to \mathbb{R}^+$, s.t if we let X^ε be ε-nbhd of X.

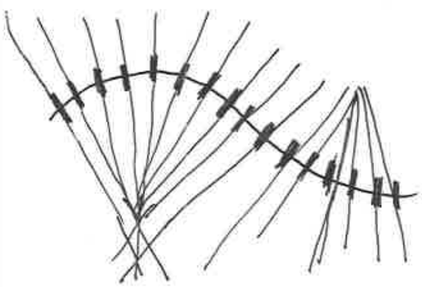
then,

(1) $\forall y \in X^\varepsilon$ possesed a unique closest pt $\pi_\varepsilon(y) \in X$ (impressive! Just like projection them in Functional analysis)

(2) the map $\pi_\varepsilon : X^\varepsilon \to X$ is a submersion

(N.B. if X is cpt. then ε can be const. func).

Before moving on. we note that we need to find an isomorphism between nbhd X inside its "normal bundle" (some lines crowed into submanifold) skinny lines can form sth !!



Let $\iota: X \hookrightarrow \mathbb{R}^k$ of dim r. For $\forall x \in X$. we can identify $T_x X$ w/ r-dim VP in $\mathbb{R}^k$

I.e: $T_x X \cong d\iota_x(T_x X) \subset T_x \mathbb{R}^k \cong \mathbb{R}^k$.

Let $N_x(X, \mathbb{R}^k) := (T_x X)^\perp$ in $\mathbb{R}^k$.

$:= \{ v \in T_x \mathbb{R}^k \cong \mathbb{R}^k \mid v \perp T_x X \}$. $\dim(N_x) = k - r$

and define $N(X, \mathbb{R}^k) := \{ (x, v) \in \mathbb{R}^k \times \mathbb{R}^k \mid x \in X, v \in N_x(X, \mathbb{R}^k) \} \subset T\mathbb{R}^k$

Actually $N(X, \mathbb{R}^k)$ is k-dim submf of $T\mathbb{R}^k$.

And the canonical projection map

$\pi: N(X, \mathbb{R}^k) \to X, \ (x, v) \to x$ is a submersion.

Def 2.2. Let $\iota: X \hookrightarrow \mathbb{R}^k$ be smooth submf. We call $N(X, \mathbb{R}^k)$ abovea as <u>normal bundle</u> of $X \subseteq \mathbb{R}^k$.

proof of extension them:

Let $h: N(X, \mathbb{R}^k) \to \mathbb{R}^k, \ (x, v) \to x+v$.

$\forall (x,0) \in N(X, \mathbb{R}^k)$, dh is non-singular, since $dh_{(x,0)}: T_{(x,0)}(X \times \{0\}) \subset T_{(x,0)} N(X, \mathbb{R}^k)$ bijectively onto $T_x X \subset T_x \mathbb{R}^k$.

$T_{(x,0)}(\{x\} \times N_x(X, \mathbb{R}^k)) \xrightarrow[\text{bijective onto}]{dh_{(x,0)}} N_x(X, \mathbb{R}^k) \subset T_x \mathbb{R}^k$.

Also, $\begin{array}{c} X \times \{0\} \xrightarrow[h]{\text{diffeo}} X \subset \mathbb{R}^k \\ \cap \\ N(X, \mathbb{R}^k) \end{array}$

By Generalize IFT,

$$\begin{array}{ccc} U & \xrightarrow[\text{diffeo}]{h} & V \\ \cap & & \cap \\ X \times \{0\} & & X \\ \cap & & \cap \\ N(X, \mathbb{R}^k) & \xrightarrow{h} & \mathbb{R}^k \end{array}$$

Now for each $x \in X$, we define $\varepsilon(x) = \sup\{r \leq 1, B_r(x) \subset V\}$.

Check: ε is positive conti on X!

N. B: $X^\varepsilon \underset{\text{open}}{\subset} V$ submf. and consider the map

$$\pi_\varepsilon: X^\varepsilon \to X, \quad y \to \pi_\varepsilon(y) = \pi \circ h^{-1}(y).$$

It's a submersion since π is a submersion and h^{-1} is diffeo on V. WTS: $\pi_\varepsilon(y)$ is unique

Let $z \in X$, represent $d(z,y) = \inf_x d(x,y) \Rightarrow$ $y-z$ is perpendicular to X at z, $y - z \in N_z(X, \mathbb{R}^k)$.

$y = z + (y-z) = h(z, y-z)$.

i.e $\pi_\varepsilon(y) = z \Rightarrow z$ is unique $\Rightarrow \pi_\varepsilon(y)$ is unique in X. $\square$

In general, not necessary $\mathbb{R}^k$, we can still define the normal bundle:

$$N_x(X, M) := T_x M / T_x X. \qquad X \subset M$$

and $N(X,M) := \{(x,v),\ x \in X,\ v \in N_x(X,M)\}$. (x)

In fact, $\dim(N(X,M)) = \dim(M)$
$\Uparrow$
smooth manifold.

Geometric speaking: WTS: $M \xrightarrow{\text{embed}} \mathbb{R}^k$. i ?: $T_x X \subset T_x M \subset T_x \mathbb{R}^k$ and $T_x M / T_x X$ can be indentifyed as

$$N(X,M) \cong \{(x,v) \mid x \in X,\ v \in T_x M \text{ and } v \perp T_x X\}. \ (v)$$

and we have seen

$$T_{(x,0)} N(X,M) \cong T_x X \oplus T_x^{\perp} X$$

Thm 2.3 (Tubular nbhd thm) Let $X \subset M$ be smooth submf. Then $\exists$ a diffeomorphisms from $W \subset X \subset N(X,M)$ $\downarrow$ onto $V_\varepsilon \subset X \subset M$.

proof. Embedded M into $\mathbb{R}^K$. Let $\pi_\varepsilon : M^\varepsilon \to M$ be ε-nbhd thm
($\iota : M \hookrightarrow \mathbb{R}^k$ embedding)

Again, consider $h: N(X,M) \to \mathbb{R}^K$, $h(x,v) \to x+v$

Then $W := h^{-1}(M^\varepsilon)$ is open nbhd of X in $N(X,M)$. Now consider the composition.

$$h_\varepsilon = \pi_\varepsilon \circ h : W \to M.$$

Then h_ε smooth and the identity map on $X \subset N(X,M)$. We also have $T_{(x,0)} N(X,M) \xRightarrow[\text{onto bijectives}]{(dh_\varepsilon)_{(x,0)}} T_x M$
So the theorem followed from Genealized IFT. $\square$

#22

Lec 13 (Apr 8^{th}) Russell-Hua

Smooth vector field.

Review: a tangent vector X_p at $p \in M$ is a linear map: $X_p : C^\infty(M) \to \mathbb{R}$ s.t.

$$X_p(fg) = X_p(f)g(p) + X_p(g)f(p)$$

We can also replace $\underset{\text{open}}{U \subset M}$ by $C^\infty(M) \to C^\infty(U)$. And let (φ, U, V) be chart, then $\forall p \in U$, the tangent vectors $\{(\partial_i)_p \mid 1 \le i \le m\}$ form a basis $\to T_pM$.

$$(\partial_i)_p = (\partial\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right) : C^\infty(U) \mapsto \mathbb{R}$$

$$f \mapsto (\partial_i f)_p = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))$$

where $\left\{\frac{\partial}{\partial x^i}\Big|_{\varphi(p)} \mid 1 \le i \le n\right\}$ is std basis $\to T_{\varphi(p)}\mathbb{R}^n$. I.e $\forall X_p \in T_pM$, we have

$$X_p = \sum_{i=1}^n a_i (\partial_i)_p$$

Geometric view: If we embed $M \to \mathbb{R}^k$, then we can visualize $(\partial_i)_p$ as TV

$$\gamma_i(t) = \varphi^{-1}((0, \cdots, 0, t, 0, \cdots 0)), \quad t \in (-\varepsilon, \varepsilon)$$

at p, where t at i^{th} position, ε chosen that $(0, \cdots 0, t, 0, \cdots 0) \in V$ for all $-\varepsilon < t < \varepsilon$.

and $\varphi(p) = 0 \in V \subset \mathbb{R}^n$.

Def. 1.1 Let $\gamma : \mathbb{R} \to M$ be curve on M w/ $\gamma(0) = p$ and d/dt be unit tv on $\mathbb{R}$. Then

$$\dot{\gamma}(0) = \partial\gamma_0\left(\frac{\partial}{\partial t}\right) \text{ the tv of curve } \gamma \text{ at } p = \gamma(0).$$

N.B. By definition, $\forall f \in C^\infty(M)$, we have

$$\dot{\gamma}(0)(f) = d\gamma_0\left(\frac{\partial}{\partial t}\right)(f) = \frac{\partial}{\partial t}\Big|_{t=0}(f\circ\gamma)$$

This coincide w/ our former definition.

Def 1.2. A smooth vector field X on a M manifold is smooth-assignment that $\forall p \in M$ a tangent vector $X_p \in T_p(M)$

We say X is smooth on U (φ, U, V), if all coefficient X^i's are smooth on U.

$$Xf: M \to \mathbb{R}, \quad p \longmapsto Xf(p) := X_p(f)$$

prop 1.3. X is smooth iff $f \in C^\infty(M) \Rightarrow Xf \in C^\infty(M)$

And we denote the space of smooth vector field as

$\Gamma^\infty(TM)$ Algebraic structure:

$$X_1, X_2 \in \Gamma^\infty(TM),\ f_1, f_2 \in C^\infty(M) \Rightarrow f_1X_1 + f_2X_2 \in \Gamma^\infty(TM)$$

"modulo" space

There are also alternative way of definition.

"sign $p \in M$ to a $X_p \in T_pM$" $\equiv$ "Give $X: M \to TM$ s.t $\pi \circ \cancel{M} = Id$" "Def. "section" "

where $\pi: TM \to M$ is canonical projection $\pi(X_p) = p$. (submersion)

Prop 2.1: (Smooth Vector field = Smooth section on tangent bundle)

$$X : M \to TM \text{ s.t } \pi \circ X = Id.$$

proof. $(\varphi, U, V) \Rightarrow M$ and $(T\varphi, \pi^{-1}(U), V \times \mathbb{R}^n)$ for TM, where $T\varphi$ is given by

$$T\varphi(p, X_p) = (\varphi(p), d\varphi_p(X_p))$$

By def. $d\varphi(\partial_i) = \frac{\partial}{\partial x^i} \Rightarrow X_p = \sum X^i(p)(\partial_i)_p \Rightarrow d\varphi_p(X_p) = \sum X^i(\varphi^{-1}(x)) \frac{\partial}{\partial x^i}\Big|_{\varphi(p)}$

So in the charts, we have

$$T\varphi \circ X \circ \varphi^{-1}(x^1, \dots x^n) = (x^1, \dots x^n, X^1(\varphi^{-1}(x)), \dots X^n(\varphi^{-1}(x)))$$

By def $X : M \to TM$ is smooth map iff X^i's are smooth $\square$.

Let X be smooth VF on M. By Lebnitz-rule: Given $X_p : C^\infty(M) \to \mathbb{R}$, we have

$$X(fg) = fX(g) + X(f)g, \quad \forall f, g \in C^\infty(M)$$

Def 2.2: A derivation of the algebra $C^\infty(M)$ is a map $D: C^\infty(M) \to C^\infty(M)$ that

$$D(fg) = fD(g) + D(f)g, \quad \forall f, g \in C^\infty(M)$$

So $\forall X$ smooth vector field on M is a deraviation, Conversely,

prop 2.3. $\forall$ Deraviation D on $C^\infty(M)$, there $\exists$ a VF X, s.t $Df = Xf$.

Recall that differential operator of n on $U \subset \mathbb{R}^m$ is an operator of the form

$$P = \sum_{|j|<n} a_j(x^1, \dots x^m) \left(\frac{\partial}{\partial x^1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial x^m}\right)^{j_m}, \quad \text{Assume } a_j \text{ are smooth.}$$

Def 2.4. A differential operator of order n on M is map $P: C^\infty(M) \to C^\infty(M)$ "local form"

$\forall$ local chart (φ, U, V), P can be written as

$$P = \sum_{|j| \le n} a_j(\varphi(p))(\partial_1)^{j_1} \cdots (\partial_m)^{j_m}$$

RMK: N.B $\forall$ differential operator is local:

$\operatorname{supp}(Pf) \subset \operatorname{supp}(f)$, $\forall f \in C^\infty(M)$.

Apr 10th Lec 14th Russell

Integral curves : Integral curves are sort of vector field from ODE view (parametric curve)

Recall : smooth curve γ on M is $\gamma: I \to M$, $I \subset \mathbb{R}$ (interval). For $\forall a \in I$, the tv of at $\gamma(a)$ is

$$\dot{\gamma}(a) = \frac{\partial \gamma}{\partial t}(a) = d\gamma_a\left(\frac{\partial}{\partial t}\right)$$

Def 1.1 : Let $X \in \Gamma^\infty(TM)$ be VF on M. smooth curve $\gamma: I \to M$ is called integral curve of X if $\forall t \in I$

$$\dot{\gamma}(t) = X_{\gamma(t)}.$$

example :

Lemma 1.2. If $\gamma: I \to M$ an integral curve on X, then

(1) $I_a = \{t \mid t+a \in I\}$, then

$$\gamma_a : I_a \to M, \quad \gamma_a(t) := \gamma(t+a) \text{ is also intergral curve.}$$

(2) Let $I^a = \{t \mid ta \in I\}$, then

$$\gamma^a : I^a \to M, \quad \gamma^a(t) := \gamma(at), \text{ its an integral curve for } X^a = aX.$$

example : $X = \frac{\partial}{\partial x^1}$ on $\mathbb{R}^n$. we have integral curves of X are straight lines parallel to the x^1-axies, as

$$\gamma(t) = (C_1 + t, C_2, \cdots C_n)$$

Check.

$$d\gamma\left(\frac{\partial}{\partial t}\right)f = \frac{\partial}{\partial t}(f\circ\gamma) = \nabla f \cdot \frac{\partial \gamma}{\partial t} = \frac{\partial f}{\partial x^1}.$$

N.B $\tilde{\gamma}(t) := (C_1 + 2t, C_2, \cdots C_n)$ is integral curve of $2X$, since $\dot{\tilde{\gamma}}(t) = 2\frac{\partial}{\partial x^1}$

example. VF: $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Then if $\gamma(t) = (x, y)$ is an integral curve, we must have $\forall f \in C^\infty(\mathbb{R}^2)$

$$x'(t)\frac{\partial f}{\partial x} + y'(t)\frac{\partial f}{\partial y} = \nabla f \cdot \frac{\partial \gamma}{\partial t} = X_{\gamma(t)} f = x(t)\frac{\partial f}{\partial y} - y(t)\frac{\partial f}{\partial x}$$

$$\Leftrightarrow \begin{cases} x'(t) = -y(t) \\ \dot{y}(t) = x(t). \end{cases} \Rightarrow \begin{cases} x(t) = a\cos t - b\sin t \\ y(t) = a\sin t + b\cos t. \end{cases}$$

As defined, $\forall$ integral curve should have an maximal interval ($\mathbb{R}$)

Def 2.1. VF X on M is complete if for $\forall p \in M$, there $\exists \gamma : \mathbb{R} \to M$, s.t $\gamma(0) = p$.

N.B. $\forall$ smooth curve VF is complete!

example. Let $X := t^2 \frac{\partial}{\partial t}$ on $\mathbb{R}$. Let $\gamma(t) = (x(t))$ be inte-curve

$$\dot{x}(t)\frac{\partial}{\partial t} = X_{\gamma(t)} = x^2\frac{\partial}{\partial t} \Rightarrow x'(t) = x(t)^2$$

If given $x(0) = c$, then

$$x_c(t) = \frac{1}{-t + 1/c} \quad \text{for } c \neq 0$$

and

$$x_0(t) = 0 \quad \text{for } c = 0$$

N.B. $\begin{cases} I_c = (-\infty, 1/c), \; c > 0 \\ I_c = (1/c, \infty), \; c < 0 \end{cases}$ are maximal interval for ODE.

But since $\forall$ integral curve starting at $c \neq 0$ is not defined for all $t \in \mathbb{R}$.

$\Rightarrow X$ is not complete.

Compactly supp VF are complete!

$\mathrm{supp}(X) = \overline{\{p \in M \mid X(p) \neq 0\}}$ be support of VF X

Them 2.2. If X is a cpt supported VF, then complete.

proof. (skipped)

Lemma 2.3. Any smooth VF on cpt manifold is complete.

By definition, we know that $\forall p \in M$, $\exists \gamma_p : \mathbb{R} \to M$ s.t $\gamma(0) \in M$

From this, we can define a map, $\forall t \in \mathbb{R}$,

$$\phi_t : M \to M, \quad p \to \gamma_p(t).$$

And we know that φ_t is smooth in t, $\forall$ fixed pt p

$\{\varphi_t \mid t \in \mathbb{R}\}$ would satisfied some interesting group law.

prop 2.4. $\forall t, s \in \mathbb{R}$, we have $\phi_t \circ \varphi_s = \phi_{t+s}$

proof. $\forall p \in M$, fixed $s \in \mathbb{R}$.

$\gamma_1(t) = \varphi_t \circ \varphi_s(p)$ and $\gamma_2(t) = \varphi_{t+s}(p)$ are both integral curve for X.

$\gamma_1(0) = \varphi_s(p) = \gamma_2(0)$

By uniquness of integral curves, we have.

$$\varphi_t \circ \varphi_s = \varphi_{t+s}. \qquad \square$$

Let $\varphi_0 = Id$, we conclude that.

Cor. 2.5 $\varphi_t : M \to M$ is bijective, and $\varphi_t^{-1} = \varphi_{-t}$.

As a concequence,

$$t \to \varphi_t \quad \text{is a group homomorphism from } \mathbb{R} \to \mathrm{Diff}(M)$$

We call the family $\{\varphi_t, t \in \mathbb{R}\}$ one-parameter subgroup of differomorphisms.

Stronger version: $\Phi := \{\varphi_t\}$.

$$\Phi : \mathbb{R} \times M \to M, \quad (t, p) \mapsto \varphi_t(p) \quad \text{is a smooth map on joint } (t, p).$$

Application: Let $\varphi(x,t)$ defined as particle in $\mathbb{R}^3$. (Quantum)

we can see that $U(t)$ (unitary opeator for particle)

is actually a $\{U(t)\}$ one parameter-subgroup in Quantum physics !!!

$$\begin{cases} U(0)\,\varphi(x,t) = \varphi(0) \\ U(t)\,\varphi(x,t) = \varphi(t) \end{cases}$$

Lec 17th Apr 10th Russell Lie group and lie algebra

A lie group is simply a smooth manifold w/ group structure + manifold structure

"Symmetricity" "geometry" "physics"

Def 1.1. A lie group G is a smooth manifold equipped w/ group structure, so that

$$\mu: G \times G \to G, \quad (g_1, g_2) \mapsto g_1 \cdot g_2 \text{ is a smooth map.}$$

example. $\cdot (\mathbb{R}^n, +)$ $\quad$ $\cdot$ $GL(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) \mid \det X \neq 0\}$

$\cdot (\mathbb{R}^*, *)$ $\quad$ $SL(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}), \det X = 1\}$

$\cdot (S^1, \cdot)$ $\quad$ $O(n) = \{X \in M(n,\mathbb{R}), XX^T = I_n\}$.

$\cdot$ If G_1, G_2 are lie Groups then so is $G_1 \times G_2$

$(T^n = S_1 \times S^1 \times S^1 \times \dots S^1)$

N.B. Not all smooth manifold admits a lie group structure, consider that S^4 (S^0, S^1, S^3) okey!

Here are some topo restriction to smooth manifold:

(1) underlying s.p of lie group must be oriented

(2) $\forall G$, lie group, the fundamental group $\pi_1(G)$ must be abelian

(3) TG, tangent bundle is trivial $TG \cong G \times \mathbb{R}^n$

Fact: All dim-2 closed manifold, the only one has lie group is $T^2 = S^1 \times S^1$

Left/right multiplication:

Let G be lie group. $a, b \in G$. there are two natural maps.

$$L_a: G \to G, \quad g \to a \cdot g$$

$$R_b: G \to G, \quad g \mapsto g \cdot b$$

OBV: $L_a^{-1} = L_{a^{-1}}$

$R_b^{-1} = R_{b^{-1}}$

so both L_a and R_b are diffeomorphisms.

And we have.

$$L_a R_b = R_b L_a$$

L_a is smooth since $L_a: G \xhookrightarrow{j_a} G \times G \xrightarrow{\mu} G$ both smooth

$$g \longmapsto (a, g) \longrightarrow a \cdot g$$

$$R_b = \mu \circ i_b \text{ for } i_b: G \hookrightarrow G \times G$$

$$g \mapsto (g, b)$$

Some application of left multiplication

prop 1.2. $\forall$ Lie Group G, we have $TG \cong G \times \mathbb{R}^n$

proof. We identify $T_eG = \mathbb{R}^n$, and

$$\varphi : G \times T_eG \to TG, \quad \varphi(a, \xi) = (a, dL_a(\xi))$$

Clearly, bijective w/ inverse $\varphi^{-1}(a, \xi) = (x, dL_{-a}(\xi))$

N.B. when we fixed x, φ, φ^{-1} are linear isomorphism

$$\{a\} \times T_eG \cong T_aG$$

$\Rightarrow$ φ and φ^{-1} are smooth $\Rightarrow$ diffeomorphism.

Lemma 1.3. The differential of $u : G \times G \to G$ is given by

$$du_{a,b}(X_a, Y_b) = (dR_b)_a(X_a) + (dL_a)_b(Y_b)$$

For $\forall$ $(X_a, Y_b) \in T_aG \times T_bG \cong T_{(a,b)}(G \times G)$

proof. Let $f \in C^\infty(G)$, we have.

$$\begin{aligned}(du_{a,b}(X_a, Y_b))(f) = (X_a, X_b)(f \circ u) &= X_a(f \circ u \circ i_b) + Y_b(f \circ u \circ j_a) \\ &= X_a(f \circ R_b) + Y_b(f \circ L_a) \\ &= (dR_b)_a(X_a)(f) + (dL_a)_b(Y_b)(f)\end{aligned}$$

Prop 1.4: For $\forall$ Lie Group G, the group inverse map.

$$i : G \to G, \quad g \mapsto g^{-1}$$

is smooth and $\quad di_a(X_a) = -(dL_{a^{-1}})_e (dR_{a^{-1}})_a(X_a)$, $\forall$ $a \in T_aG$

proof. Let $f : G \times G \to G \times G$, $(a, b) \to (a, ab)$ bijective map. By Lemma 1.3, we have.
$\Rightarrow$ Global diffeo

$$df_{(a,b)} = T_aG \times T_bG \to T_aG \times T_{ab}G,$$

$$(X_a, Y_b) \to (X_a, (dR_b)_a(X_a) + (dL_a)_b(Y_b))$$

$df_{(a,a^{-1})} = (X_a, 0)$

$\Downarrow$

$Y_{a^{-1}} = -(dL_a)_{a^{-1}}(dR_{a^{-1}})_a(X_a$

$= -(dL_{a^{-1}})_e(dR_{a^{-1}})_a(X_a$

$dL_e = dR_e = Id \Rightarrow (di)_e(X_e) =$

bijective since dR_b, dL_a are. And By IFT, f is locally a diffeomorphism near (a,b)

$$f^{-1} : G \times G \to G \times G, \quad (a, c) \to (a, a^{-1}c) \text{ is diffeo}$$

So the inverse map i

$$\begin{aligned} G &\hookrightarrow G \times G \xrightarrow{f^{-1}} G \times G \xrightarrow{\pi_2} G \quad \text{smooth} \\ a &\longrightarrow (a, e) \longrightarrow (a, a^{-1}) \longrightarrow a^{-1} \end{aligned}$$

$d\pi_2 \circ (df^{-1})_{(a,e)}(X_a, 0)$

$\Rightarrow \quad di_a(X_a) = -(dL_{a^{-1}})_e (dR_{a^{-1}})_a(X_a).$

Def 2.1: A Lie algebra is a real VP V toghther w/ binary bracket operator

$$[\cdot,\cdot] : V \times V \longrightarrow V,$$

s.t $\forall X, Y, Z \in V$, we have

(1) $[aX_1 + bX_2, Y] = a[X_1, Y] + b[X_2, Y]$

(2) $[X, Y] = -[Y, X]$

(3) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$

ex. · $\forall$ Vector space admits a trivial : $[X, Y] \equiv 0$

· the set of all smooth VF $\Gamma^\infty(TM)$ admits a Lie algebra

· $M(n, \mathbb{R})$ is a Lie algebra is we define the Lie braket as.

$$[A, B] := AB - BA.$$

Left invariant VF on Lie group.

G Lie group. L_a left transition. For $\forall X_e \in T_eG$. we can define VF X on G by

$$X_a = (dL_a)_e(X_e)$$

Consider that $(dL_a)_b(X_b) = (dL_a)_b \circ dL_b(X_e) = dL_{ab}(X_e) = X_{ab}$

Def. A left-invariant VF on Lie Group G is X on G s.t.

$$(dL_a)_b(X_b) = X_{ab} \quad \forall a, b \in G.$$

$\mathfrak{g} := \{X \in \Gamma^\infty(TM) \mid X \text{ is left invariant}\}$.

Clearly $\mathfrak{g}$ is vector S.P of $\Gamma^\infty(TG)$

$$\mathfrak{g} \cong T_eG. \text{ and } \dim \mathfrak{g} = \dim G.$$

prop 2.3

If $X, Y \in \mathfrak{g}$, so is there Lie bracket

proof. let $f \in C^\infty(G)$

$Y(f \circ L_a)(b) = Y_b(f \circ L_a) = (dL_a)_b(Y_b) f = Y_{ab} f = (Yf)(L_a b) = (Yf) \circ L_a(b)$.

then

$X_{ab}(Yf) = (dL_a)_b(X_b)(Yf) = X_b Y(f \circ L_a)$

Similarly, $Y_{ab} Xf = Y_b X(f \circ L_a)$. Thus $dL_a([X, Y]_b) f = X_b Y(f \circ L_a) - Y_b X(f \circ L_a) = [X, Y]_{ab}(f)$

Lie algebra w/ Lie group

It follows that $\mathfrak{g}$ on G w/ $[\cdot,\cdot]$ is an n-dim Lie subalgebra of All $T^\infty(TG)$!

Def. The Lie algebra $\mathfrak{g}$ of G is called Lie algebra of G.

example.

1. $(\mathbb{R}^n, +)$; for $\forall a \in \mathbb{R}^n$, we have L_a is defined as usual. "dL_a is the Id"

   we have
$$X_v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n} \quad \text{for } (v_1, \cdots v_n) \in T_0\mathbb{R}^n.$$
   Since $\frac{\partial}{\partial x_i}$ commute with $\frac{\partial}{\partial x_j}$ $\Rightarrow$ $G = \mathbb{R}^n$ is $\mathfrak{g} = \mathbb{R}^n$ w/ vanishing Lie braket

2. $GL(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}),\ \det X \neq 0\}$. $\subseteq \mathbb{R}^{n^2}$

   n^2-dim Lie Group, so if we define $GL_+(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}),\ \det X > 0\}$

   $GL_-(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}),\ \det X < 0\}$.

   $\mathfrak{gl}(n,\mathbb{R}) = \{A \mid A \in M_{n\times n}\}$,

   So we if we consider $\begin{cases} A = (A_{ij})_{n\times n} \in \mathfrak{gl}(n,\mathbb{R}) \\ X^{ij} \in GL(n,\mathbb{R}) \end{cases}$ we have $\sum A_{ij} \frac{\partial}{\partial X^{ij}} \in T_{I_n} GL(n,\mathbb{R})$

   and Lie bracket $[A,B]$, $A, B \in \mathfrak{g}$ is the matric

$$\left[\sum X^{ik} A_{kj} \frac{\partial}{\partial X^{ij}},\ \sum X^{pq} B_{qr} \frac{\partial}{\partial X^{pr}} \right] = \sum X^{ik} A_{kj} B_{jr} \frac{\partial}{\partial X^{ir}} - \sum X^{pq} B_{qr} A_{rj} \frac{\partial}{\partial X^{pj}}$$

$$= \sum X^{ik} (A_{kr} B_{rj} - B_{kr} A_{rj}) \frac{\partial}{\partial X^{ij}}$$

$$\Rightarrow [A,B] = AB - BA.$$

Lec 15^{th} Lie Homomorphis + exponential map.

Def 1.1. let G, H be Lie groups.

(1) φ is called group homomorphism if smooth and

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2), \quad \forall g_1, g_2 \in G.$$

(2) $\varphi : G \to H$ is isomorphism if it's invertible and $\varphi^{-1}: H \to G$ is also homo.

N.B. $G \underset{\varphi}{\overset{\cong}{\to}} H \Rightarrow$ they r diffeo as manifold and isomorphic as group.

example: $\forall G$. lie group $a \in G$, the conjugation map.

$$c(a) := L_a \circ R_{a^{-1}} : G \to G, \quad g \to aga^{-1} \text{ is Group isomorphism.}$$

Def 1.2. Let g, h be Lie algebras

(1) $L: g \to h$ is Lie homo if

$$L([X_1, X_2]) = [L(X_1), L(X_2)], \quad \forall X_1, X_2 \in g$$

(2) $L: g \to h$ is Lie isomo if it's invertible.

ex. $\forall X \in GL(n, \mathbb{R})$, the adjoint map

$$Ad_X : gl(n, \mathbb{R}) \to gl(n, \mathbb{R}), \quad A \to XAX^{-1}$$

is a Lie algebra isomorphism

w/ $(Ad_X)^{-1} = X^{-1}AX$ also a lie algebra isomorphism.

Group homo $\Rightarrow$ Algebra homo.

$$\begin{tikzcd} g \arrow[r, "d\varphi"] \arrow[d, phantom, "\cong" rotate=90] & h \arrow[d, phantom, "\cong" rotate=90] \\ T_eG \arrow[r, "d\varphi_e"'] & T_eH \end{tikzcd}$$

Let $\varphi: G \to H$ group homo, then differential at e. $d\varphi_e : T_eG \to T_eH$.

In other word, we have: $(d\varphi(X))_h = dL_h(d\varphi_e(X_e))$

the image of $d\varphi(X)$ is Left invariant VF of H whose value at $e \in H$ is $d\varphi_e(X_e)$.

example. Consider $c(X): GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$ on I_n, we get.

$$(\partial c(X))_{I_n}(A) = \frac{\partial}{\partial t}\Big|_{t=0} c(X)(I+tA) == \frac{\partial}{\partial t}\Big|_{t=0} X(I+tA)X^{-1} = XAX^{-1}$$

I.e. the induced map is lie algebra homo.

$$d\,c(X) = Ad_X : gl(n,\mathbb{R}) \to gl(n,\mathbb{R})$$

Lemma 1.3. $\forall X \in g$ is φ-related to $d\varphi(X) \in h$.

Let $X \in g$. write $h = \varphi(g)$. Since φ group homo, we have

$$\varphi \circ L_g = L_h \circ \varphi \quad \text{and} \quad d\varphi_g(X_g) = d\varphi_g \circ (dL_g)_e(X_e)$$

$$= dL_h \circ d\varphi_e(X_e)$$

$$= (d\varphi(X))_h \quad \square$$

Them 1.4. If $\varphi: G \to H$ is group homo, then $d\varphi: g \to h$ is lie algebra homo.

proof. Idea (1) X φ-related to $d\varphi(X)$, $Y \xrightarrow{\varphi} d\varphi(Y)$

$$\Rightarrow [X,Y] \xrightarrow{\varphi} [d\varphi(X), d\varphi(Y)]$$

(2) $[X,Y] \xrightarrow{\varphi} d\varphi([X,Y])$ ①

It follows that $[d\varphi(X), d\varphi(Y)]_e = d\varphi_e([X,Y]_e) = (d\varphi[X,Y])_e$

Since $d\varphi([X,Y])$ and $[d\varphi(X), d\varphi(Y)]$ are left invariant VF on H.

we conclude that ① holds.

example. $GL(n,\mathbb{R})$ Lie group — $gl(n,\mathbb{R})$ w/ $[A,B] = AB - BA$

$\Rightarrow$ $GL(n,\mathbb{R}) \to \mathbb{R}^*$ is a Lie group homo

$$\det(XY) = \det(X)\cdot\det(Y), \quad \forall X,Y \in GL(n,\mathbb{R})$$

and as mentioned in lec 2, we have

$$d\det_X(A) = \det(X)\,tr(X^{-1}A), \quad \forall X \in GL(n,\mathbb{R}),\ A \in gl(n,\mathbb{R})$$

taking $X = I_n$, we get Lie algebra homo for det is

$$d\det = tr : gl(n,\mathbb{R}) \to \mathbb{R}, \quad A \to tr(A)$$

$tr(AB) = tr(BA)$, $\forall A,B \in gl(n,\mathbb{R})$ since $\mathbb{R}$ is trivial lie algebra.

Exponential map.

Our goal is to explain the following map

Given $\varphi: G \to H$ group homo, the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\partial\varphi} & \mathfrak{h} \\ \downarrow \exp G & & \downarrow \exp H \\ G & \xrightarrow{\varphi} & H \end{array} \qquad \star$$

commutes. i.e $\varphi \circ \exp_G = \exp_H \circ (\partial\varphi)$

And sense that: If G connected, $\forall$ lie group $\varphi: G \to H$ is determined by lie algebra $d\varphi: \mathfrak{g} \to \mathfrak{h}$.

Def. 2.2. The exponential map of G is

$$\exp: \mathfrak{g} \to G. \quad X \to \varphi_1^X(e)$$

N.B. $\varphi_{ts}^X = \varphi_s^{tX}$, so we have

$$\exp(tX) = \varphi_1^{tX}(e) = \varphi_t^X(e).$$

Also, we can check that $\{\exp(tX), t \in \mathbb{R}\}$ is one-parameter subgroup of G.

$$\exp(tX) \cdot \exp(sX) = \exp(t+s)X$$

However,

$$\exp(tX)\exp(tY) \neq \exp(t(X+Y)) \text{ in general.}$$

example. for $G = \mathbb{R}^*$, we can see $T_1 G = \mathbb{R}$

Let $x \in \mathbb{R} = T_1 G$, Left invariant VF $\to$ $X = x\frac{\partial}{\partial t} \in T_1 G$. at $a \in G$.

$$X_a = ax\frac{\partial}{\partial t}$$

integral curve $\Rightarrow$ ODE $(e=1)$ $\gamma_e^X(t) = e^{tx}$. $\left(\dot{\gamma}_e^X(t) = xe^{tx}\frac{\partial}{\partial t} = X_{\gamma_e^X(t)}\right)$

$$\Rightarrow \exp(x) = \varphi_1^X(e) = \gamma_e^X(1) = e^x.$$

Similarly, we have sth like

$G = (S^1, \cdot)$: $\exp : i\mathbb{R} = T_e S^1 \to S^1$, $\exp(ix) = e^{ix}$

$G = (\mathbb{R}^n, +)$: $\exp : \mathbb{R}^n = T_0 \mathbb{R}^n \to \mathbb{R}^n$, $\exp(x) = x$.

$G = GL(n, \mathbb{R})$, $\exp : gl(n, \mathbb{R}) \to GL(n, \mathbb{R})$, $\exp(A) = e^A = I + A + \frac{A^2}{2!} + \cdots$

The differential of the exponential map.

Lemma 2.3. $\exp : g \to G$ is smooth and if we identify $T_0 g$ w/ g, we get.

$$d\exp_0 = Id.$$

proof. Consider

$$\tilde{\Phi} : \mathbb{R} \times G \times g \to G \times g \quad (t, g, x) \to (g \cdot \exp(tX), X)$$

Check that this is the flow on $G \times g$ ~ left invariant VF $(X, 0)$ smooth!
and exp is decomposited as

$$g \hookrightarrow \mathbb{R} \times G \times g \xrightarrow{\tilde{\Phi}} G \times g \xrightarrow{\pi_1} G,$$

$$X \to (1, e, X) \to (\exp(tX), X) \to \exp(tX) \qquad \text{all smooth maps.}$$

Also $\exp(tX) = \phi_t^X(e) = \gamma_e^X(t)$, we get

$$\frac{d}{dt}\Big|_{t=0} \exp(tX) = X \quad \text{and} \quad \frac{d}{dt}\Big|_{t=0} \exp \circ tX = (d\exp)_0 \frac{d(Xt)}{dt}$$

$$= (d\exp)_0 X$$

$$\Rightarrow (d\exp)_0 = Id \qquad \square$$

Cor 2.4. exp is local diffeo near 0. i.e. it's diffeo $U_0 \in T_e G \xrightarrow{\exp} e \in G$ $V_0 \subset$

Since $(d\exp)_0$ is bijective!

properties: For $\forall X, Y \in g$. there $\exists$ smooth $Z : (-\varepsilon, \varepsilon) \to g$ s.t for all $t \in (-\varepsilon, \varepsilon)$.

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + t^2 Z(t))$$

Application:

$$Z(t) = \frac{1}{2}[X, Y] + \frac{t}{12}([X, [X, Y]] - [Y, [X, Y]]) + \frac{t^2}{24}[X, [Y, [X, Y]]] + \cdots$$

As seen in Dynamic system !!!

Lec 16th Russell (Tensor and differential form)

Multi linear map.

Let $V_1, \dots V_k$ be finite dim VSP.

Def 1.1. A func $T: V_1 \times \dots \times V_n \to \mathbb{R}$ is call multi-linear if linear in each entry.

$\forall i \in I$ (index) and $v_1 \in V_1, \dots v_{i-1} \in V_{i-1}, v_{i+1} \in V_{i+1}, \dots v_k \in V_k$

$T_i : V_i \to \mathbb{R} \cdot v_i \to T(v_1, \dots v_i \dots v_k)$ is linear.

N.B. If T_1, T_2 are multi-linear map, then span $\{T_1, T_2\}$ same.

ex. $f^1 \in V_1^* \times \dots f_k \in V_k^* \to \mathbb{R}$, we define $(v_1, \dots v_k) \to f^1(v_1) \dots f^k(v_k)$

$f^1 \otimes \dots \otimes f^k$ is a multi-linear map. By define $i \le k$, and $\lambda \in \mathbb{R}$, we have

$f^1 \otimes \dots f^{i-1} \otimes \lambda f^i \otimes \dots \otimes f^k = \lambda f^1 \otimes \dots \otimes f^i \otimes \dots \otimes f^k$.

Them 1.2: Let $\{f_i^1, \dots f_i^{n(i)}\}$ be basis of V_i^*. Then the set

$\{f_1^{i_1} \otimes \dots \otimes f_k^{i_k} \mid 1 \le i_j \le n(j)\}$ form a basis of VP on multi-map $V_1 \times \dots V_k$

In particular, $\dim \otimes^k V^* = n^k$.

Notation: We denote the VP of multi-linear maps on $V_1 \times V_2 \times \dots V_k$ by $V_1^* \otimes V_2^* \dots \otimes V_k^* := E$

$e \in E$ would be called k-tensor.

N.B. If $T \in V_1^* \otimes \dots \otimes V_k^*$
$S \in V_{k+1}^* \otimes \dots \otimes V_{k+e}^*$ $\Rightarrow$ tensor product $T \otimes S(v_1, \dots v_{k+e}) := T(v_1 \dots v_k) S(v_{k+1}, \dots v_{k+e})$

And by def. we know that

$(T \otimes S) \otimes \mathbb{R} = T \otimes (S \otimes \mathbb{R})$ associativity.

Now, let V be n-dim VP, and V^* its dual. We will call

$$\bigotimes^{k,l} V := (\otimes^l V) \otimes (\otimes^k V^*)$$

the space of (l,k)-tensors on V. I.e $T \in \otimes^{l,k} V$ iff $T = T(\beta^1, \cdots \beta^l, v_1, \cdots v_k)$ for $\beta^i \in V^*$, $v_j \in V$

N.B. $\otimes^{1,0} V = V$ and $\otimes^{0,1} V = V^*$

$\otimes^{k,0} V = \otimes^k V$ and $\otimes^0 V = \mathbb{R}$.

Def 1.3. For $\forall\ 1 \le r \le l$, and $1 \le s \le k$, we define <u>the (r,s)-contraction</u> $C^r_s : \otimes^{l,k} V \to \otimes^{l-1,k-1} V$ as

$$C^r_s(T)(\beta^1, \cdots \beta^{k-1}, v_1, \cdots v_{l-1}) = \sum_i T(\beta_1, \cdots \beta^{r-1}, f^i, \beta^r, \cdots \beta^{k-1}, v_1, \cdots v_{s-1}, e_i, v_s, \cdots v_{l-1})$$

where $\{e_1, \cdots e_n\}$ basis $\Rightarrow V$

$\{f_1, \cdots f^n\}$ dual basis.

Lemma 1.4. Let T be (l,k) tensor. For $1 \le r \le l$, $1 \le s \le k$, we have.

(1) $\{e_i\}$ independent

(2) For $\forall\ v_1, \cdots v_l \in V$ and $\beta^1, \cdots \beta^k \in V^*$

"$\hat{}$ means remove the entries"

$$C^r_s(v_1 \otimes \cdots \otimes v_l \otimes \beta^1 \otimes \cdots \otimes \beta^k) = \beta^s(v_r) v_1 \otimes \cdots \otimes \hat{v_r} \otimes \cdots \otimes v_l \otimes \beta^1 \otimes \cdots \otimes \hat{\beta^s} \otimes \cdots \beta^k$$

example. if $v, w \in V$ and $\alpha, \beta, \gamma \in V^*$

$$C^1_2(v \otimes w \otimes \alpha \otimes \beta \otimes \gamma) = \beta(v) w \otimes \alpha \otimes \gamma$$

Check:

$$C^1_2(v \otimes w \otimes \alpha \otimes \beta \otimes \gamma)(\beta^1, v_1, v_2) = \sum_i v \otimes w \otimes \alpha \otimes \beta \otimes \gamma(f^i, \beta^1, v_1, e_i, v_2)$$

$$= \sum_i f^i(v) \beta^1(w) \alpha(v_1) \beta(e_i) \gamma(v_2)$$

$$= \left[\sum_i f^i(v) \cdot \beta(e_i)\right] \beta^1(w) \alpha(v_1) \gamma(v_2)$$

$$= \beta(v) \beta^1(w) \alpha(v_1) \gamma(v_2)$$

$$= \beta(v)\, w \otimes \alpha \otimes \gamma(\beta^1, v_1, v_2)$$

linear p-form:

Now, we fixed a VP V, and consider k-tensor T on V. $T \in \otimes^k V^*$

Def. 2.1 Let $T \in \otimes^k V^*$ be k-tensor on V

(1) we say T is symmetric if $\forall$ permutation σ of $(1, 2, \cdots k)$

$$T(v_1, \cdots v_k) = T(v_{\sigma(1)}, \cdots v_{\sigma(k)})$$

(2) we say T is alternating (linear k-form) if

$$T(v_1, \cdots v_i, \cdots v_j, \cdots v_k) = -T(v_1, \cdots v_j, \cdots v_i, \cdots v_k)$$

for all $v_1, \cdots v_k \in V$ and $\forall\ 1 \leq i \neq j \leq k$

ex. $\langle \cdot, \cdot \rangle$ on V is positive symmetric 2-tensor

det is linear n-form on $\mathbb{R}^n$

Notation. we let VP of k-forms by $\Lambda^k V^* \underset{LSP}{\subset} \otimes^k V^*$

Recall: permutation $\sigma \in S_k$ is even/odd is determined by # simple transpositions.

Given k-tensor T and $\sigma \in S^k$, we define k-tensor T^σ by

$$T^\sigma(v_1, \cdots v_k) = T(v_{\sigma(1)}, \cdots v_{\sigma(k)})$$

Anti-Symmetricity: (map)

$$Alt(T) := \frac{1}{k!} \sum_{\pi \in S_k} (-1)^\pi T^\pi$$

Lemma: Alt is projection from $\otimes^k V^* \to \Lambda^k V^*$

Check (1) $\forall T \in \otimes^k V^*$, $Alt(T) \in \Lambda^k V^*$

(2) $\forall T \in \Lambda^k V^*$, $Alt(T) = T$

(1): let $\sigma \in S^k$, we have $[Alt(T)]^\sigma = \frac{1}{k!} \sum_{\pi \in S^k} (-1)^\pi (T^\pi)^\sigma = \frac{1}{k!} (-1)^\sigma \sum_{\pi} (-1)^{\pi \circ \sigma} T^{\sigma \circ \pi} = (-1)^\sigma Alt(T)$

(2): Each $(-1)^\pi (T)^\pi$ equals to T $\Rightarrow$ $Alt(T) = T$ since $|S_k| = k!$

we also need sth like

Lemma 2.3. Let T, S, R be k-, ℓ-, and m-form.

(1) $\mathrm{Alt}(T\otimes S) = (-1)^{k\ell}\,\mathrm{Alt}(S\otimes T)$

(2) $\mathrm{Alt}(\mathrm{Alt}(T\otimes S)\otimes R) = \mathrm{Alt}(T\otimes S\otimes R) = \mathrm{Alt}(T\otimes \mathrm{Alt}(S\otimes T))$

Wedge product.

Def 2.4. The wedge product of $T\in\Lambda^k V^*$ and $S\in\Lambda^\ell V^*$ is $(k+\ell)$-form

$$T\wedge S = \frac{(k+\ell)!}{k!\,\ell!}\mathrm{Alt}(T\otimes S)$$

This is a kind of operation !!

prop. The wedge prod operation $\wedge : (\Lambda^k V^*)\times(\Lambda^\ell V^*) \longrightarrow \Lambda^{k+\ell} V^*$ is

(1) Bilinear: $(T,S)\to T\wedge S$ is linear in T and S

(2) Anti-commu: $T\wedge S = (-1)^{k\ell} S\wedge T$.

(3) Asso: $(T\wedge S)\wedge S' = (T\wedge S\wedge S') = T\wedge(S\wedge S')$

In general, we have.

$$(f^1\wedge\cdots\wedge f^k)(v_1,\dots v_k) = k!\,\mathrm{Alt}(f^1\otimes\cdots\otimes f^k)(v_1,\cdots v_k)$$

Prop 2.6. $\forall f^1,\cdots f^k\in V^*$ and $v_1,\cdots v_k\in V$,

$$(f^1\wedge\cdots\wedge f^k)(v_1,\cdots v_k) = \det(f^i(v_j))$$

proof. $(f^1\wedge\cdots f^k)(v_1,\cdots v_k) = k!\,\mathrm{Alt}(f^1\otimes\cdots\otimes f^k)(v_1,\cdots v_k)$

$$= \sum_{\sigma\in S_k}(-1)^\sigma f^1(v_{\sigma(1)})\cdots f^k(v_{\sigma(k)})$$

$$= \det((f^i(v_j))).$$

VP of linear k-forms

<u>Thm 2.7</u> Let $\{f^1, \cdots f^n\}$ be basis of V^*. Then the set of k-forms

$$\{f^{i_1} \wedge \cdots \wedge f^{i_k} \mid 1 \le i_1 \le i_2 \le \cdots i_k \le n\} \text{ form a basis } \rightarrow \wedge^k V^*$$

and $\dim \wedge^k V^* = \binom{n}{k}$

RMK: $\dim (\wedge^n V^*) = \binom{n}{n} = 1$

- $\forall$ n-form of n dim V is a multiple of non-trivial n-form "det"

for $k > n$, $\wedge^k (V^*) = 0$

The interior product and pull-back

Def 2.8. Interior product of $v \in V$ w/ k-form $a \in \wedge^k(V^*)$ is the $(k-1)$-covector.

$$i_v a(X_1, \cdots X_{k-1}) := a(v, X_1, \cdots X_{k-1})$$

Def 2.9. Let $L: W \rightarrow V$ linear. The pull back $L^*: \wedge^k(V^*) \rightarrow \wedge^k(W^*)$ is defined to be.

$$(L^* a)(X_1, \cdots X_k) := a(L(X_1), \cdots, L(X_k))$$

prop 2.10. Let a be linear k-form on V, β a l-form on V

then

(1) $\forall v \in V$, $i_v i_v a = 0$

(2) $\forall v \in V$, $i_v(\alpha \wedge \beta) = (i_v \alpha) \wedge \beta + (-1)^k a \wedge i_v \beta$.

(3) $\forall L: W \rightarrow V$, $L^*(\alpha \wedge \beta) = L^* \alpha \wedge L^* \beta$.

Tensor field and Differential forms on smooth manifold.

Cotangent S.P

Let M be a smooth manifold. we asso to each $p \in M$ a T_pM, vector space.
If we take $\forall (\varphi, U, V)$ ~~about~~ on p, the we can write down a basis for T_pM.

$$\partial_i|_p : C^\infty(U) \rightarrow \mathbb{R}, \quad \partial_i|_p(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p)) \quad (1 \le i \le n)$$

N.B. for each $1 \le i \le n$, $x^i \circ \varphi : U \to \mathbb{R}$ is smooth on U

The differential of this, dx^i for simplicity. (restricted to $\forall q \in U$)

$$dx^i|_q : T_qM = T_qU \to T_{x^i\circ\varphi(q)}\mathbb{R} = \mathbb{R}.$$

I.e $dx^i|_q \in T_q^*M$ and by def, we have

$$dx^i|_q(\partial_j|_q) = \partial_j|_q(x^i\circ\varphi) = \delta^i_j$$

So we conclude,

prop 3.1. In $\forall$ local chart (φ, U, V), $\{dx^i|_q : 1 \le i \le n\}$ is basis $\to T_q^*M$. Moreover, this is the dual basis of $\{\partial_i|_q : 1 \le i \le n\}$ of T_qM.

In fact, $\forall f \in C^\infty(U)$, by same way, we can identify $df_q : T_qM \to \mathbb{R}$.
By def, $df_p(\partial_i|_p) = \partial_i|_p(f)$. It follows that

$$df_p = (\partial_1|_p f)\,dx^1|_p + \cdots (\partial_n|_p f)\,dx^n|_p$$

and $\forall X \in \Gamma^\infty(TU)$,

$$df(X) = Xf,$$ we call <u>df a 1-form on U.</u>

Def. 3.2. A k-form w on M is an assignment to p that $w_p \in \wedge^k T_p^*M := \Omega^k(M)$
A k-form w is smooth if locally,

$$w = \sum_I w_I\, dx^I = \sum_I w_{i_1\ldots i_k}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k},$$

$I = \{1 \le i_1 < \cdots < i_k \le n\}$ and each $w_I \in C^\infty(U)$

N.B: $\Omega^0(M) = C^\infty(M)$, and since there's no k-form on T_pM if $k > n$, we have

$$\Omega^k(M) = 0, \quad \forall k > n = \dim(M)$$

operations on k-forms.

- wedge prod: $\Omega^k(U) \times \Omega^\ell(U) \to \Omega^{k+\ell}(U)$

ex.

$$(dx^1 + 2dx^2)\wedge(dx^1\wedge dx^2 - dx^2\wedge dx^3 + 3dx^1\wedge dx^3) = -7\,dx^1\wedge dx^2\wedge dx^3$$

- Interior product:

$$\iota_X(dx^{i_1}\wedge\cdots\wedge dx^{i_k}) = \sum_r (-1)^{r-1} dx^{i_r}(X)\, dx^{i_1}\wedge\cdots\wedge \widehat{dx^{i_r}}\wedge\cdots\wedge dx^{i_k}$$

- pull back: $U' \to U$ (φ)

$$(\varphi^*w)_p(X_1, \ldots X_k) = w_{\varphi(p)}(d\varphi_p(X_1), \ldots d\varphi_p(X_k))$$

prop 3.5. Let $w \in \Omega^k(U)$, $\eta \in \Omega^\ell(U)$, $X \in \Gamma^\infty(TU)$, $\varphi \in C^\infty(U', U)$, $\psi \in C^\infty(\tilde{U}, \bar{U})$
then
(1) $w\wedge\eta = (-1)^{k\ell}\eta\wedge w$
(2) $\varphi^*(w\wedge\eta) = \varphi^*w\wedge\varphi^*\eta$
(3) $\iota_X(w\wedge\eta) = (\iota_X w)\wedge\eta + (-1)^k w\wedge\iota_X\eta$
(4) $\iota_X\circ\iota_X = 0$
(5) $(\psi\circ\varphi)^* = \varphi^*\circ\psi^*$

Lec 17th (Russell) Exterior derivative (Lie-derivative)

Unlike wedge, interior, and pull back, exterior derivative is locally operator !!

Consider $f \in \Omega^0(U) = C^\infty(U)$, we seen that $df \in \Omega^1(U)$

$$d : \Omega^0(U) \to \Omega^1(U), \quad f \mapsto df$$

Locally on each coordinate, we have

$$df = \sum_i (\partial_i f) dx^i$$

we also have "invariant def" of $df \in \Omega^1(U)$ via

$$df(X) = Xf, \quad \forall X \in \mathcal{T}^\infty(TU).$$

Now, suppose w is k-form on M, so that locally

$$w = \sum_I w_{i_1, \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

we want to define dw as $(k+1)$-form as

Def 1.1. The exterior derivative of w is the $(k+1)$-form dw given by

(1) $$dw = \sum_I dw_{i_1, \cdots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{I, i} \partial_i (w_{i_1, \cdots i_k}) dx^i \wedge dx^{i_1} \wedge \cdots dx^{i_k}$$

We can also define this operation in coordinate free system !

For small k, we have

- $k=0$, $w = f \in C^\infty(U)$, $df : \mathcal{T}^\infty(TU) \to C^\infty(U)$, $df(X) = Xf$.
- $k=1$, $w \in \Omega^1(U)$. dw as $C^\infty(U)$-bilinear map. $dw : \mathcal{T}^\infty(TU) \times \mathcal{T}^\infty(TU) \to C^\infty(U)$

$$w = \sum_i w_i dx^i, \quad X = \sum_k X^k \partial_k \text{ and } Y = \sum_\ell Y^\ell \partial_\ell. \text{ we have the}$$

calculations :

$$dw(X, Y) = \sum_{i,j,k,\ell} (\partial_j w_j) dx^j \wedge dx^i (X^k \partial_k, Y^\ell \partial_\ell)$$

$$= \sum_{i,j} ((\partial_j w_i) X^j Y^i - (\partial_j w_i) X^i Y^j)$$

$$= X(w(Y)) - Y((w(X)) - w([X, Y])$$

$$\Rightarrow dw(X, Y) = X(w(Y)) - Y(w(X)) - w([X, Y])$$

Induction on this idea, we have.

Thm 1.2 : For $\forall\, w \in \Omega^k(U)$, the $(k+1)$-form dw, viewed as a $C^\infty(U)$-multilinear map

$$dw : \mathcal{T}^\infty(TU) \times \cdots \times \mathcal{T}^\infty(TU) \to C^\infty(U)$$

is given by

$$dw(X_1, \cdots X_{k+1}) = \sum (-1)^{i-1} X_i (w(X_1, \cdots \hat{X_i}, \cdots X_{k+1})) + \sum_{i<j} (-1)^{i+j} w([X_i, X_j], X_1, \cdots \hat{X_i}, \cdots \hat{X_j}, \cdots X_{k+1})$$

prop 1.3. Suppose $w \in \Omega^k(U)$, $\eta \in \Omega^\ell(U)$, $X \in \Gamma^\infty(TU)$, $\varphi \in C^\infty(U, U')$

then (1) $d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$

(2) $d \circ d = 0$.

(3) $\varphi^* \circ d = d \circ \varphi^*$

proof.

(1). Since d is linear, it's sufficient to show $w = f dx^{i_1} \wedge \cdots dx^{i_k}$ and $\eta = g\, dx^{j_1} \wedge \cdots dx^{j_t}$ w/ $I \cap J = \emptyset$.

then, $d(w \wedge \eta) = d(fg\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_t})$

$$= \sum_i d_i(fg)\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots dx^{j_t}$$

$$= \sum_i (\partial_i f) dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \eta + (-1)^k w \wedge \sum_i (\partial_i g)\, dx^i \wedge dx^{j_1} \wedge \cdots dx^{j_t}$$

$$= dw \wedge \eta + (-1)^k w \wedge d\eta.$$

(2). we frist check $k=0$:

$$d(df)(X, Y) = X(df(Y)) - Y(df(X)) - df([X,Y]) = X(Y(f)) - Y(X(f)) - [X,Y]f = 0.$$

for $k>0$, by linearity, we assume $w = f dx^1 \wedge \cdots dx^k$, since $ddf = 0$ and $dd x^i = 0$. then

$$d(dw) = d(df \wedge dx^1 \cdots \wedge dx^k)$$

$$= \underline{d(df)} \wedge dx^1 \wedge \cdots \wedge dx^k + \sum_i (-1)^i df \wedge dx^1 \wedge \cdots \wedge \underline{d(dx^i)} \wedge \cdots \wedge dx^k$$

$$= 0$$

(3) Again check $k=0$, $(\varphi^* df)_p(X_p) = df_{\varphi(p)}(d\varphi_p(X_p)) = d(\varphi^* f)_p(X_p)$

By (1), (2), Prop 3.5(2), $\Rightarrow$ $\varphi^* dw = d(\varphi^* w)$ $\square$

The lie deravitives !

Def 2.1. The lie deravitive of a $f \in C^\infty(M)$ w.r.t $X \in T^\infty(TU)$ is

$$\mathcal{L}_X(f) := \frac{\partial}{\partial t}\Big|_{t=0} \varphi_t^* f = \left(\lim_{t\to 0} \frac{\varphi_t^* f - f}{t}\right)$$

φ is local flow geneated by VF X !!

Def 2.2. The lie deravite of a k-form $\omega \in \Omega^k(M)$ w.r.t $X \in T^\infty(TM)$ is

$$\mathcal{L}_X(\omega) := \frac{\partial}{\partial t}\Big|_{t=0} \varphi_t^* \omega \left(= \lim_{t\to 0} \frac{\varphi_t^*\omega - \omega}{t}\right)$$

N.B. $\mathcal{L}_X f$ that defined in Lec 5 is just special case since $C^\infty(M) = \Omega^0(M)$ and we have seen that

$$\mathcal{L}_X f = Xf \quad \text{and} \quad \mathcal{L}_{[X,Y]} f = \mathcal{L}_X \mathcal{L}_Y(f) - \mathcal{L}_Y \mathcal{L}_X(f)$$

prop 2.2. Let $\omega \in \Omega^k(U)$, $\eta \in \Omega^l(U)$, $X_1, X_2, X \in T^\infty(TM)$. then

(1) $d\mathcal{L}_X\omega = \mathcal{L}_X(d\omega)$

(2) $\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X\omega \wedge \eta + \omega \wedge \mathcal{L}_X(\eta)$

(3) [Cartan's Magic formula]: $\mathcal{L}_X\omega = d i_X \omega + i_X d\omega$

(4) $\mathcal{L}_{[X_1, X_2]}\omega = \mathcal{L}_{X_1}\mathcal{L}_{X_2}\omega - \mathcal{L}_{X_2}\mathcal{L}_{X_1}\omega$

(5) $(\mathcal{L}_X\omega)(X_1, \cdots X_k) = \mathcal{L}_X(\omega(X_1, \cdots X_k)) - \sum_i \omega(X_1, \cdots \mathcal{L}_X X_i, \cdots X_k)$

Basically follow the same proof in above. starting from $k=0$. and (3) is based on induction, and we map locally assume that $\omega = f dx^1 \wedge \cdots dx^k = dx^1 \wedge \omega_1$, where $\omega_1 = f dx^2 \wedge \cdots \wedge dx^k$.

Lec 18th (Russell) Integration on Manifold.

Top form: Let M be smooth mf of dim m, $\Omega^k(M)=0$ if $k>m$.

then we call $\forall$ smooth m-form a top form on M. Now let $p\in M$ and $(\varphi, U, V)_p$, then

$dx^1\wedge\cdots\wedge dx^m$ a top form on U. N.B for $\forall q\in U$, $(dx^1\wedge\cdots\wedge dx^m)_q\neq 0$ since $\forall q\in U$

$$(dx^1\wedge\cdots\wedge dx^m)_q(\partial_1,\cdots\partial_m)=\det(dx^i(\partial_j))_{1\le i,j\le m}=1$$

Moreover, since $\dim \Lambda^m T_pM=1$, we see that $\forall$ top form ω on U and $\forall q\in U$, $\exists\lambda_q$ s.t

$$\omega_q=\lambda_q(dx^1\wedge\cdots\wedge dx^m)_q.$$

We aim to find coordinate-change factor!

Lemma 1.1. If $\varphi:\mathbb{R}^m\to\mathbb{R}^m$ is diffeomorphism and $y=\varphi(x)$, then

$$\varphi^*(dy^1\wedge\cdots\wedge dy^m)=\det(d\varphi_x)\,dx^1\wedge\cdots\wedge dx^m$$

proof. Let $\varphi=(\varphi^1,\cdots\varphi^m)$, then $\varphi^*y^i=y^i\circ\varphi=\varphi^i$. So

$$\varphi^*(dy^1\wedge\cdots\wedge dy^m)=d\varphi^1\wedge\cdots d\varphi^m$$

But since

$$d\varphi^1\wedge\cdots d\varphi^m(\partial_1^x\cdots\partial_m^x)=\det(d\varphi_x)$$

we have desired result. Ω

Now let $(\varphi_\alpha, U_\alpha, V_\alpha)$, $(\varphi_\beta, V, V_\beta)$ be two coordinate system. The the coordinate change map is $\varphi_{\alpha\beta}=\varphi_\beta\circ\varphi_\alpha^{-1}$

so we have $\qquad \varphi_\alpha(x)\to y=\varphi_\beta(x)$

$$(\varphi_{\alpha\beta})^*(\varphi_\beta^{-1})^*dx_\beta^1\wedge\cdots\wedge dx_\beta^m=\det(d\varphi_{\alpha\beta})(\varphi_\alpha^{-1})\,dx_\alpha^1\wedge\cdots dx_\alpha^m$$

since $(\varphi_{\alpha\beta})^*(\varphi_\beta^{-1})^*=(\varphi_\beta^{-1}\circ\varphi_{\alpha\beta})^*\Rightarrow dx_\beta^1\wedge\cdots\wedge dx_\beta^m=\det(d\varphi_{\alpha\beta})\,dx_\alpha^1\wedge\cdots\wedge dx_\alpha^m$!!!

Orientability(!)

Let M be mf of $\dim(M) = n$, $w \in \Omega^m(M)$ a m-form. We want to define $\int_M w$. For simplicity, we let w is supp on (φ, U, V) w/ coordinate $\{x^1, \cdots x^m\}$. Then we can write,

$$w = f(\varphi(x))\, dx^1 \wedge \cdots \wedge dx^m, \quad f \in C^\infty(V)$$

It's natural to define in Euclidean diff-form,

$$\int_U w := \int_V f(x)\, dx^1 \cdots dx^m$$

But we need to check RHS is indep choice of coordinate: let $(\varphi_\alpha, U, V_\alpha)$, $(\varphi_\beta, U, V_\beta)$ on U w/ $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : V_\alpha \to V_\beta$.

Then $w = f_\beta(x_\beta)\, dx_\beta^1 \wedge \cdots dx_\beta^m = f_\beta(\varphi_{\alpha\beta}(x_\alpha)) \det(d\varphi_{\alpha\beta})\, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^m$, (we need to make it well-defined), so

$$\int_{V_\beta} f_\beta(x_\beta)\, dx_\beta^1 \cdots dx_\beta^m = \int_{V_\alpha} f_\beta(\varphi_{\alpha\beta}(x_\alpha)) \det(\varphi_{\alpha\beta})(x_\alpha)\, dx_\alpha^1 \cdots dx_\alpha^m \quad \text{(Not necessary true !!)}$$

Recall in Calculus 3, we only have

$$\int_{V_2} f(x)\, dy^1 \cdots dy^m = \int_{V_1} f(\varphi(x)) |\det(d\varphi)(x)|\, dx^1 \cdots dx^m \quad \text{if } \varphi : V_1 \xrightarrow{\text{diffe}} V_2$$

Compare to our condition, we need additional assume that "$\det(\varphi_{\alpha\beta}) > 0$"

Def 1.2. Let M be smooth mf of dim m.

(1) $(\varphi_\alpha, U_\alpha, V_\alpha)$, $(\varphi_\beta, U_\beta, V_\beta)$ two charts are orientational compatible if $\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}$ satisfied $\det(d\varphi_{\alpha\beta})_p > 0$ for all $p \in \varphi_\alpha(U_\alpha \cap U_\beta)$

(2) An orientation of M is atlas $\mathcal{A} = \{(\varphi_\alpha, U_\alpha, V_\alpha) \mid \alpha \in \Lambda\}$ whose charts are pairwise orientation compactable.

(3) If M has an orientation $\Rightarrow$ M is orientable

RMK: Let U be a chart w/ $\{x^1, \cdots x^m\}$. we use notation $-U := \{-x^1, x^2, \cdots x^m\}$ and $U/-U$ are orientation compatible.

Let $\tilde{U}$ be another chart s.t $\tilde{U} \cap U \neq \emptyset$ connected. Then either

- $\tilde{U}$ and U orien-compatible or
- $\tilde{U}$ and $-U$ are orient-compactible.

As a concequence, If M is connected and orientable, then M admits exactly two different orientations

ex. $\mathbb{RP}^n$: we have an altas has # $n+1$ charts, we have seen that $\mathbb{RP}^n$ is orientable for n odd and turn out that $\mathbb{RP}^n$ is n even is not orientable.

<u>Integration on Smooth manifold</u>:

Now assume M smooth, orientable mf of dim m, and $\mathcal{A}$ orientation on M. Let w be m-form on M.

To define $\int_M w$, we assume w supp on (φ, U, V) v.s $\mathcal{A}$. I.e. $\exists f \in U(\text{supp})$ s.t

$$w = f(\varphi(x))\, dx^1 \wedge \cdots \wedge dx^m \implies \int_U w := \int_V f(x)\, dx^1 \cdots dx^m \quad (3)$$

To define a general m-form $w \in \Omega^m(M)$, we take locally finite cover $\{U_\alpha\}$ of M that are compatible w/ $\mathcal{A}$.

Let $\{U_\alpha\}$ w/ $\{\rho_\alpha\}$ P.O.U. Now suppose ρ_α is supp in U_α, $\rho_\alpha w$ is supp in U_α. We define

$$\int_M w := \sum_\alpha \int_{U_\alpha} \rho_\alpha w. \quad (4)$$

If RHS. ABS-CV then we say it's <u>integrable</u>.

<u>Thm2.1</u>: Let w is cpt supp / w is integrable. then above expression is indep of $\{U_\alpha\}$ and $\{\rho_\alpha\}$.

proof. We will first show that (3) is well-defined.

Let $\{X_\alpha^i\}$ and $\{X_\beta^i\}$ be two oritation compatible system on U. then

$$\omega := f_\alpha dX_\alpha^1 \wedge \cdots \wedge dX_\alpha^m = f_\beta dX_\beta^1 \wedge \cdots \wedge dX_\beta^m$$

WTS: $\int_{V_\alpha} f_\alpha dX_\alpha^1 \cdots dX_\alpha^m = \int_{V_\beta} f_\beta dX_\beta^1 \cdots dX_\beta^n$ which is true since $dX_\beta^1 \wedge \cdots dX_\beta^n = \det(d\varphi_{\alpha\beta}) dX_\alpha^1 \wedge \cdots \wedge dX_\alpha^n$

implies $f_\alpha = \det(d\varphi_{\alpha\beta}) f_\beta$, and $\det(d\varphi_{\alpha\beta}) > 0 \Rightarrow$ conclusion (in Euclidean S.P) change of coordinate

To show (4) is well defined, we need $\{U_\alpha\}, \{U_\beta\}$ be local finite cover
$\{\rho_\alpha\}, \{\rho_\beta\}$ be P.O.U $\Rightarrow$ $\{U_\alpha \cap U_\beta\}$ new local finite cover $\to M$
$\{\rho_\beta \rho_\alpha\}$ new P.O.U $\to \{U_\alpha \cap U_\beta\}$

It's sufficient to show: $\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_{\alpha,\beta} \int_{U_\alpha \cap U_\beta} \rho_\alpha \rho_\beta \omega$ (*)

for each fixed α, we have $\int_{U_\alpha} \rho_\alpha \omega = \int_{U_\alpha} (\sum_\beta \rho_\beta) \rho_\alpha \omega = \sum_\beta \int_{U_\alpha \cap U_\beta} \rho_\beta \rho_\alpha \omega$.

Change of coordinate ($\mathbb{R}^m \to$ manifolds)

Def. 2.2. Let M, N be orientation smooth mfs, w/ $\mathcal{A}, \mathcal{B}$ be orientations resp. A diffeomorphism $\varphi: M \to N$ is called <u>orientation preserving</u> if each $(\psi_\beta, U_\beta, V_\beta) \in \mathcal{B}$, the charts $(\psi_\beta \circ \varphi, \varphi^{-1}(U_\beta), V_\beta)$ on M is orienti compatible w/ $\mathcal{A}$.
And <u>orientation revertible if incompactible w/ $\mathcal{A}$</u>

<u>Thm 2.3</u> Let M, N be n-dim orientation mfs, and $\varphi: M \to N$ diffeomorphism

(1) If φ - orienti-preserving, then
$$\int_M f^* \omega = \int_N \omega.$$

proof. Nth different than on $\mathbb{R}^m$ case... (change of coordinate)

(2) If φ - orienti - reverting, then
$$\int_M f^* \omega = -\int_N \omega$$

Check: "PP427-432 J. Lee" to see integrate via density

Lec19th: Stoke them (Russell)

Manifold w/ boundary: Recall that manifolds w/ boundary, denote $\mathbb{R}^m_+ := \{(x^1, \cdots x^n) \mid x^m \geq 0\}$

Def 1.1. A topo manifold is called <u>m-dim mf w/ bdy</u> if it's Hausdorff, 2^{nd} countable, and $\forall p \in M$, $\exists U_p$ which is homeomorphic to either $\mathbb{R}^m / \mathbb{R}^m_+$.

We can also define the boundary of M: $\partial M = \{p \in M,\ p$ has no nbhd in M s.t is homeomphic to $\mathbb{R}^m\}$

interior of M: $\operatorname{int}(M) = M \setminus \partial M$.

ex. Closed ball: $B^m(1) = \{x \in \mathbb{R}^m \mid |x| \leq 1\}$ w/ $\partial B^m(1) = S^{m-1}$

ex2. Let M any smooth mf, $f \in C^\infty(M)$. If a is singular-value of f, then the sub-level set

$$M_a = f^{-1}((-\infty, a)) \text{ is Smooth Mf w/ } \partial M_a = f^{-1}(a)$$

N.B. If M, N are smooth mfs w/ bdy $\not\Rightarrow$ $M \times N$ is smooth mf w/bdy

ex. $[0,1] \times [0,1]$ "smooth mf w/ corners"

However If M w/bdy, N w/o bdy $\Rightarrow$ $M \times N$ is smooth mf w/o bdy. (with bdy)

Lemma 1.2. Let M be m-dim smooth mf w/bdy, then ∂M is smooth mf of dim $(m-1)$ "unless $\partial M = \emptyset$" that is properly embedded into M.

proof. (Sketch) Let $(U, x^1, \cdots x^m)_p \in \partial U$ that is homeomorphic to $\mathbb{R}^m_+$. Then $U \cap \partial M = \{(x^1, \cdots x^m) \mid x^m = 0\}$.

Then $(U \cap \partial M, x^1, x^2, \cdots x^m)$ is a charts on ∂M (Check) $\square$.

N.B. we can also define orientation on M w/bdy, $\mathcal{A}$, $\det(\varphi_{\alpha\beta}) > 0$ for two charts and M w/ bdy is orientable iff it admits a nowhere vanishing top form !!!

Thm 1.3: If M an orientable manifold w/bdy of dim m, then ∂M is an orientable $m-1$ mfu submf of M.

proof. Let $(U_\alpha, x_\alpha^1, \cdots x_\alpha^m)$ and $(U_\beta, x_\beta^1, \cdots x_\beta^m)$ be orien-charts of M near $p \in M$.

$M \cap U_\alpha$ is chara by $x_\alpha^m \geq 0$ and $M \cap U_\beta$ w/ $x_\beta^m \geq 0$.

WTS: $(U_\alpha \cap \partial M, x_\alpha^1, \cdots x_\alpha^{m-1})$ and $(U_\beta \cap \partial M, x_\beta^1, \cdots x_\beta^{m-1})$ are orien-compatible.

Let $\varphi_{\alpha\beta}: U_\alpha \to U_\beta$ by $(\varphi_1, \cdots \varphi^m)$, then on $\partial M \cap U_\alpha \cap U_\beta$, we have $x_\alpha^m = x_\beta^m = 0$.

$\varphi^m(x^1, \cdots x^{m-1}, 0) = 0$ on $U_\alpha \cap U_\beta \cap \partial M$ and $\varphi^m(x^1, \cdots x^m) > 0$ on $U_\alpha \cap U_\beta \cap \mathrm{Int}(M)$.

It follows that $\frac{\partial \varphi^m}{\partial x^i}(x^1, \cdots x^{m-1}, 0) = 0$, $i = 1, \cdots m-1$ and $\frac{\partial \varphi^m}{\partial x^m}(x^1, \cdots x^{m-1}, 0) \geq 0$

Since $(U_\alpha, x_\alpha^1, \cdots x_\alpha^m)$ and $(U_\beta, x_\beta^1, \cdots x_\beta^m)$ are orien-compatible, we have $\det(\frac{\partial \varphi^i}{\partial x^j}) > 0$ in $U_\alpha \cap U_\beta$.

$$\det\left(\frac{\partial \varphi^i}{\partial x^j}(x^1, \cdots x^{m-1}, 0)\right) = \det\begin{pmatrix} \underbrace{\left[\frac{\partial \varphi^i}{\partial x^j}(x^1, \cdots x^{m-1}, 0)\right]_{1 \leq i \leq j \leq m-1}}_{①} & * \\ 0 & \frac{\partial \varphi^m}{\partial x^m}(x^1, \cdots x^{m-1}, 0) \end{pmatrix} > 0$$

$\Rightarrow \det ① > 0$ so that $(U_\alpha \cap \partial M, x_\alpha^1, \cdots x_\alpha^{m-1})$ and the other are compatible.

RMK: The bdy of non-orientable manifold could be either (oriented: Mobiüs bound)

(non-orientable: $[0,1] \times M$, M non-orientable)

Them 2.1 (Stoke's them) Let M be m-dim manifold w/ ∂M. For $\forall \omega \in \Omega^{m-1}(M)$ w/ cpt supp, we have

$$\int_{\partial M} \iota_{\partial M}^* \omega = \int_M d\omega, \quad \text{where } \iota_{\partial M} : \partial M \hookrightarrow M \text{ "inclusion map"}$$

RMK: (1) If M w/o bdy, $\partial M = \emptyset \Rightarrow$ RHS = LHS = 0

(2) If M w/ coners (ex. $[0,1]\times[0,1]$) still Holds! (J. Lee PP 415-421).

proof. Case I (ω is supported in $U \underset{\text{diffe}}{\cong} \mathbb{R}^m$)

Since $\omega = 0$ on ∂M, we have $\int_{\partial M} \iota_{\partial M}^* \omega = 0$. let $\omega := \sum_i (-1)^{i-1} f_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^m$; where f_i's cpt supp funcs.
then $d\omega = \sum_i \frac{\partial f_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^m$, by definition we have

$$\int_M d\omega = \int_{\mathbb{R}^m} \sum_i \frac{\partial f_i}{\partial x^i} dx^1 \cdots dx^m = \sum_i \int_{\mathbb{R}^{m-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x^i} dx^i\right) dx^1 \cdots \widehat{dx^i} \cdots dx^m = 0$$

Case II (ω is supp in $U \underset{\text{diffeo}}{\cong} \mathbb{R}^m_+$), same formula except last term.

$$\int_{\mathbb{R}^{m-1}} \left(\int_0^{\infty} \frac{\partial f_m}{\partial x^m} dx^m\right) dx^1 \cdots dx^{m-1} = -\int_{\mathbb{R}^{m-1}} f_n(x^1, \cdots x^{m-1}, 0)\, dx^1, \cdots dx^{n-1}$$

On the other hand since $x^m = 0$ on ∂M, we see that $\iota_{\partial M}^* \omega = (-1)^{m-1} f_m (x^1, \cdots x^{m-1}, 0)\, dx^1 \wedge \cdots \wedge dx^{m-1}$

implies that $\int_{\partial M} \iota_{\partial M}^* \omega = \int_{\mathbb{R}^{m-1}} (-f_m(x^1, \cdots x^{m-1}, 0)) dx^1 \cdots dx^{m-1}$ $\quad = -f_m(x^1, \cdots x^{m-1}, 0) \cdot (-1)^m dx^1 \wedge \cdots \wedge dx^{m-1}$

$$= -\int_{\mathbb{R}^{m-1}} f_m(x^1, \cdots x^{m-1}, 0)\, dx^1 \cdots dx^{m-1}$$

result follows

Case III. (In general) we cover the set $\mathrm{supp}(\omega)$ by finitely many Charts, and take P.O.U. Then

$$\int_{\partial M} \iota_{\partial M}^* \omega = \sum_i \int_{\partial M} \iota_{\partial M}^* (\rho_i \omega) = \sum_i \int_M d(\rho_i \omega) = \sum_i \int_M d\rho_i \wedge \omega + \int_M d\omega = \int_M \underbrace{d(\textstyle\sum \rho_i)}_{=0} \wedge \omega + \int_M d\omega$$

$\square$

Lec 24th The de RHAM - cohomology. (Russell)

Def 1.1. Let M be smooth mf, and $\omega \in \Omega^k(M)$

(1) ω is closed if $d\omega = 0$

(2) ω is exact if $\exists$ a $(k-1)$-form $\eta \in \Omega^{k-1}(M)$ s.t $\omega = d\eta$

Notation:

$Z^k(M) := \ker(d : \Omega^k(M) \to \Omega^{k+1}(M))$ "Closed forms"

$B^k(M) := \operatorname{Im}(d : \Omega^{k-1}(M) \to \Omega^k(M))$ "exact forms"

As we have seen: $d : \Omega^k(M) \to \Omega^{k+1}(M)$ linear that $\forall k$ and $\forall \omega \in \Omega^k(M)$

we have $d^2\omega = d(d\omega) = 0$, then $B^k(M) \subset Z^k(M) \subset \Omega^k(M)$

RMK: Let $\dim(M) = m$, the by def

(1) $k > m$: $B^k(M) = Z^k(M) = \{0\}$.

(2) $k = 0$: $B^0(M) = \{0\}$; $Z^0(M) = \{f \in C^\infty(M) \mid df = 0\} \simeq \mathbb{R}^k$

(3) $k = m$: $Z^m = \Omega^m(M)$ "top form"

ex. Consider $M = \mathbb{R}$. We have

$B^0(\mathbb{R}) = \{0\}$, $Z^0(\mathbb{R}) \simeq \mathbb{R}$ and $\Omega^0(\mathbb{R}) = C^\infty(\mathbb{R})$

And $\forall$ 1-form $g(t)\,dt$ on $\mathbb{R}$, we have

$$\omega = g(t)\,dt \Leftrightarrow \omega = dG, \text{ where } G(t) = \int_0^t g(\tau)\,d\tau$$

$$\Rightarrow \Omega^1(\mathbb{R}) = B^1(\mathbb{R}) = Z^1(\mathbb{R})$$

The De Rham cohomology group.

Since $d^2 = 0$, we have the following

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \cdots \xrightarrow{d} \Omega^m(M) \xrightarrow{d} 0$$

N.B each composition of consecutive maps is zero map. is called "cochain complex"

depending on the "direction" of the maps.

Def 1.2. The quotient group (VSP)

$$H^k_{dR}(M) := Z^k(M)/B^k(M)$$

is called k^{th} de Rham cohomology group of M.

ex. For $M = \mathbb{R}$, $H^0_{dR}(\mathbb{R}) \cong \mathbb{R}$ and $H^k_{dR}(\mathbb{R}) = \{0\}$ for $k \geq 1$.

Given $\forall\, w \in Z^k(M)$, we will call $[w]$, the corresponding cohomology class.

RMK: Suppose $\dim M = m$, we have $H^k_{DR}(M) = \{0\}$, $\forall k > m$ and $H^0_{dR}(M) \cong \mathbb{R}^K$

There's lots of smooth manifold that $\dim H^k_{dR}(M) < \infty$ for all k. and

$$\dim H^0_{DR}(\mathbb{Z}) = \infty$$

ex. $H^1_{dR}(\mathbb{R}^2/\mathbb{Z}^2)$

$\dim(H^1_{dR}(\mathbb{R}^2/\mathbb{Z}^2)) = +\infty$.

Def 1.3. In the case $\dim H^k_{dR}(M) < \infty$ for each k, we will call the number

$$b_k(M) := \dim H^k_{dR}(M)$$

the k^{th} Betti number of M, and $\chi_M = \sum_{k=0}^{m} (-1)^k b_k(M)$ the Euler characterstics of M.

The de Rham cohomology of S^1.

Consider $M = S^1$. As we have seen,

$$H^0_{dR}(S^1) \cong \mathbb{R} \text{ and } H^k_{dR}(S^1) = 0 \text{ for } k \geq 2$$

N.B. on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, θ is not smooth on S^1, but d on $\mathbb{R} \Rightarrow d\theta$ is a Globally defined 1-form on S^1.

So we can write

$$Z^1(S^1) = \Omega^1(S^1) = \{f\,d\theta,\ f \in C^\infty(S^1)\} \cong \{f \in C^\infty(\mathbb{R}).\ f \text{ is } 2\pi\text{-periodic}\}$$

Also by Fundamental thm of Calculus,

w is an exact 1-form $\iff w = df,\ f$ is 2π-periodic $\iff w = g(\theta)\,d\theta,\ g$ is 2π-periodic and $\int_0^{2\pi} g(\theta)\,d\theta = 0.$

$$\Rightarrow H^1_{dR}(S^1) \cong \frac{\{f \in C^\infty(\mathbb{R}).\ f\ 2\pi\text{-periodic}\}}{\{g \in C^\infty(\mathbb{R}).\ g\ 2\pi\text{-periodic and } \int_0^{2\pi} g(\theta)\,d\theta = 0\}} \Rightarrow H^1_{dR}(S^1) \cong \mathbb{R}$$

Since the linear map

$\varphi : H^1_{dR}(S^1) \to \mathbb{R},\ [f] \to \int_0^{2\pi} f(\theta)\,d\theta$ is linear isomorphism $\Rightarrow \varphi$ is well defined

$[f_1] = [f] \Rightarrow \int_0^{2\pi} f_1(\theta)\,d\theta = \int_0^{2\pi} f(\theta)\,d\theta$

φ is injective

$[f_1] \neq [f] \Rightarrow \quad \dots \neq \dots$

φ is surjective

$\forall\, c \in \mathbb{R},\ f(\theta) = c \in Z^1(S^1) \Rightarrow \varphi([f]) = 2\pi c$
$= \int_0^{2\pi} f(\theta)\,d\theta$

Operations on deRham class:

let $w \in Z^k(M)$ and $\eta \in Z^\ell(M)$, then

$$d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta = 0$$

i.e. $w \wedge \eta \in Z^{k+\ell}(M)$

And let $\xi_1 \in \Omega^{k-1}(M),\ \xi_2 \in \Omega^{\ell-1}(M)$

$$(w + d\xi_1) \wedge (\eta + d\xi_2) = w \wedge \eta + d\,[(-1)^k w \wedge \xi_2 + (-1)^{k-1} \xi_1 \wedge \eta + (-1)^{k-1} (\xi_1 \wedge \overset{d}{\xi_2})]$$

I.e. $[w \wedge \eta]$ is indep of choice of w and η in $[w]$ and $[\eta]$.

Def 1.4. The cup product $[w] \in H^k_{dR}(M)$ and $[\eta] \in H^\ell_{dR}(M)$ is

$$[w] \cup [\eta] := [w \wedge \eta] \in H^{k+\ell}_{dR}(M).$$

Similarly, $\varphi : M \to N$ smooth map. Then $d\varphi^* = \varphi^* d \Rightarrow \varphi^*(Z^k(N)) \subset Z^k(M)$ and $\varphi^*(B^k(N)) \subset B^k(M)$.

OBV: φ^* is a group homomorphism

Check:
① $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$
② $Id^* = Id$

It follows that $\varphi^* : \Omega^k(N) \to \Omega^k(M)$ descends to pull back $\varphi^* : H^k_{dR}(N) \to H^k_{dR}(M)$,

$\varphi^*([w]) := [\varphi^* w]$.

Cor 1.5 (Diffeomorphism invariance). If $\varphi: M \to N$ is a diffeomorphism, then

$$\varphi^*: H^k_{dR}(N) \to H^k_{dR}(M).$$

is linear isomorphism for all k. In fact $b_k(N) = b_k(M)$ for all k and $\chi(N) = \chi(M)$.

RMK: $\forall$ smooth map $\varphi: M \to N$, the cup prod makes: $H^*_{dR}(M) = \bigoplus_{k=0}^{M} H^k_{dR}(M)$ a graded ring and inclusion map

φ^* is in fact a ring diffeomorphism $\varphi^*: H^*_{dR}(M) \to H^*_{dR}(N)$ since $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$.

Moreover, if φ is a diffeo, then $\varphi^*: H^*_{dR}(N) \to H^*_{dR}(M)$ is a ring isomorphism.

Homotopic invariance: (I.e if two manifolds are equivalent, then they have the same de Rham cohomolgy groups).

Def 2.1. M, N two topological spaces are homotopic equivalent if there exist contious map $\varphi: M \to N$ and $\psi: N \to N$ so that $\varphi \circ \psi$ is homotopic to Id_N and $\psi \circ \varphi$ is homotopic to Id_M

In fact, homotopic equivalence is much weaker relation than homeomorphism/diffeomorphism.

ex. (1) S^{n-1} is homo-equiv to $\mathbb{R}^n \setminus \{0\}$

(2) $\forall$ star-shaped region is homo-equiv to a single point set $\{x_0\}$

Thm 2.2. (Homotopic invariance) Let M, N smooth manifolds. If M, N are homotopic equivalence, then

$$H^k_{dR}(M) \simeq H^k_{dR}(N), \ \forall k$$

OBS: If indeed M, N are diffeomorphism $\Rightarrow H^k_{dR}(M) \simeq H^k_{dR}(N), \forall k$

Def. (Singular cohomology group $H^k_{sing}(X, \mathbb{R})$ dependes only on its topology)

Them 2.3 (The de Rham theorm). $H^k_{dR}(M) = H^k_{sing}(M, \mathbb{R})$ for all k.

Another concequence of them 2.2 is

Cor 2.4 (Poincare's Lemma) If U is star-shaped region in $\mathbb{R}^n$, then $\forall k \geq 0/1$, we have $H^k_{dR}(U) = 0$.

In pertiaular, $H^k_{dR}(\mathbb{R}^m) = 0$, $\forall k \geq 1$.

Since $\forall p \in M$, $\exists U_p \cong U$ (star shaped) $\subseteq \mathbb{R}^n \Rightarrow \forall$ closed form is locally exact !

Cor 2.5. Suppose $k \geq 1$. Then $\forall$ closed k-form $w \in Z^k(M)$ and $\forall p \in M$, there $\exists U_p$ and $(k-1)$ form $\eta \in \Omega^{k-1}(U)$ so that

$w = d\eta$ on U.

proof. (Thm 2.2) :

Step 1. (Thm 2.6) Let $f, g \in C^\infty(M, N)$ be homotopic, then

$$f^* = g^* : H^k_{dR}(N) \to H^k_{dR}(M)$$

proof(2.6) : define cochain homotopy betwen f^* and g^* is a sequence of maps $h_k : \Omega^k(N) \to \Omega^{k-1}(M)$ s.t on $\Omega^k(N)$

$g^* - f^* = d_M h_k + h_{k+1} d_N$. (Cochain homotopy)

Suppose there $\exists$ such a cochain homotopy: $[w] \in H^k_{dR}(N)$. Then $dw = 0$ since w is closed. It follows that $g^*w - f^*w = (dh + hd)\,w = dhw \in B^k(M) \Rightarrow f^*([w]) = [f^*w] = [g^*w] = g^*([w])$

Graph I.e.

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \Omega^{k-1}(N) \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, shift right, "g^*"'] \arrow[d, shift left, "f^*"] & \Omega^k(N) \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, shift right, "g^*"'] \arrow[d, shift left, "f^*"] & \Omega^{k+1}(N) \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, shift right, "g^*"'] \arrow[d, shift left, "f^*"] & \cdots \arrow[dl, "h"'] \\
{} \arrow[r, "d"] & \Omega^{k-1}(M) \arrow[r, "d"] & \Omega^k(M) \arrow[r, "d"] & \Omega^{k+1}(M) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

Step2. Let $\varphi: M \to N$ and $\psi: N \to M$ be conti. maps so that $\varphi\circ\psi \sim Id_N$ and $\psi\circ\varphi \sim Id_M$

Then one can find $\varphi_1 \in C^\infty(M,N)$ and $\psi' \in C^\infty(N,M)$ so that $\varphi_1 \sim \varphi$, $\psi_1 \sim \psi$. It follows that

both $\varphi_1\circ\psi_1$ and $\psi_1\circ\varphi_1$ are smooth and $\varphi_1\circ\psi_1 \sim Id_N$ w/ $\psi_1\circ\varphi_1 \sim Id_M$.

Applying Step 1. we get

$$\begin{matrix} \varphi_1^*\circ\psi_1^* = Id : H^k_{dR}(M) \to H^k_{dR}(N) \\ \psi_1^*\circ\varphi_1^* = Id : H^k_{dR}(M) \to H^k_{dR}(M) \end{matrix} \Rightarrow \text{both } \varphi^* \text{ and } \psi^* \text{ are linear isomophism } \square$$

Step3. It's remain to show that existence of the cochain homotopy in them 2.6 !

Recall if X is complete VF on M, then X generates a flow $\varphi_t : M \to M$.

Lemma 2.8: Let X be complete VF on M, and φ_t the flow geneated by X. Then $\exists$ linear operator $Q: \Omega^k(M) \to \Omega^{k-1}(M)$ so that for $\forall\, \omega \in \Omega^k(M)$

$$\varphi_1^*\omega - \omega = dQ(\omega) + Q(d\omega).$$

proof. If we let $Q_t(\omega) = \iota_X(\varphi_t^*\omega)$. then $Q_t : \Omega^k(M) \to \Omega^{k-1}(M)$ and

$$\frac{d}{dt}\varphi_t^*\omega = \frac{d}{ds}\Big|_{s=0} \varphi_{t+s}^*\omega = \frac{d}{ds}\Big|_{s=0} \varphi_s^*\varphi_t^*\omega.$$

$$= \iota_X(\varphi_t^*\omega) = d\iota_X(\varphi_t^*\omega) + \iota_X d(\varphi_t^*\omega) = d(Q_t\omega) + \iota_X\varphi_t^*(d\omega) = d(Q_t\omega) + Q_t(d\omega)$$

Let $Q(\omega) = \int_0^1 Q_t(\omega)\,dt$

then $Q: \Omega^k(M) \to \Omega^{k-1}(M)$

and $\varphi_1^*\omega - \omega = \int_0^1 \left(\frac{d}{dt}\right)\varphi_t^*\omega\,dt = dQ(\omega) + Q(d\omega).$ $\square$

It follows that $\forall \omega \in \Omega^k(N)$

$g^*\omega - f^*\omega = (d\iota^*QF^* + \iota^*QF^*d)$

so if we set

$h: \Omega^k(N) \to \Omega^{k-1}(M)$ w/

$g^*\omega - f^*\omega = (dh + hd)\omega$

then we are done

Construction: (h_k) Let $W = M\times\mathbb{R}$, then $X = d/dt$ is complete VF on X w/ $\varphi_t(p,a) = (p, a+t)$

By Lemma 2.8, $\exists\, Q : \Omega^k(W) \to \Omega^{k-1}(W)$ s.t

$$\varphi_1^*\omega - \omega = dQ(\omega) + Q(d\omega)$$

let $F: W \to N$ homotopy and $\iota: M \hookrightarrow W$ w/ $\iota(p) = (p,0)$ $\Rightarrow$ $f = F\circ\iota$ and $g = F\circ\varphi_1\circ\iota$

Lec 25^{th} The Mayer-Vietoris sequence

The Mayer-Vietoris seq is algebraic tool that help us to compute homology / cohomology group in proper subspace

It's sort of analogy of the van Kampan theorem of the fundamental group and probably the abelian property

the assumption is much weaker.

1. Exact Sequence.

Suppose we have a cochain complex $(\mathcal{A}, d)$ i.e

$$\cdots A^{k-1} \xrightarrow{d_{k-1}} A^k \xrightarrow{d_k} A^{k+1} \xrightarrow{d_{k+1}} A^{k+2} \cdots$$

where A^ks are vector S.P and d_k's are linear maps s.t $d_k \circ d_{k-1} = 0 \ \forall k$.

OBV: $Im(d_k) \subset ker(d_{k+1})$, we can define cohomology groups in the same way for de Rham cochain :

$$H^k(\mathcal{A}) := ker(d_k) / Im(d_{k-1})$$

Such a sequence is called an <u>exact-seqence</u> if $H^k(\mathcal{A}, d) = 0$ for all k. I.e.

$$Im(d_{k-1}) = ker(d_k), \ \forall k$$

N.B. if the sequence start w/ 0,

$$(\cdots \to)\ 0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} V^2 \xrightarrow{d_2} V^3 \xrightarrow{d_3} \cdots$$

then $d_1 : V^1 \to V^2$ is injective, and if sequence end w/ 0,

$$V^{k-1} \xrightarrow{d_{k-1}} V^k \xrightarrow{d_k} V^{k+1} \xrightarrow{d_{k+1}} 0\ (\to \cdots)$$

then $d_k : V^k \to V^{k+1}$ is surjective

$\Rightarrow$

A more strick condition :

$$0 \longrightarrow V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} V_3 \longrightarrow 0$$

then as mensioned on Left side,

$$V^2 \cong ker(d_2) \oplus Im(d_2) \cong Im(d_1) \oplus Im(d_2) \cong V^1 \oplus V$$

Another condition for finite sequence,

$0 \to A^1 \to A^2 \to A^3 \to \cdots \to A^k \to 0$ is exact

$$\Rightarrow \sum (-1)^i \dim A^i = 0.$$

A general principle in homological algebra : Give 3 cochain complexes $\mathcal{A}, \mathcal{B}, \mathcal{C}$, w/ $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$. (exact)

in the sense that $\forall k$: $0 \to A^k \to B^k \to C^k \to 0$ is short exact sequence. Then we can construct a long exact sequence

$$\cdots \to H^{k-1}(\mathcal{C}) \to H^k(\mathcal{A}) \to H^k(\mathcal{B}) \to H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A}) \to \cdots$$

Now, suppose M is smooth manifold, $U, V \underset{\text{open}}{\subseteq} M$ so that $M = U \cup V$. Since U, V, M, $U \cap V$ both smooth manifolds.

we have 4 de Rham cohomologies complexes :

(1) $\Omega^*(M)$: $0 \to \Omega^0(M) \to \Omega^1(M) \to \cdots$

(2) $\Omega^*(U)$: $0 \to \Omega^0(U) \to \Omega^1(U) \to \cdots$

(3) $\Omega^*(V)$: $0 \to \Omega^0(V) \to \Omega^2(V) \to \cdots$

(4) $\Omega^*(U \cap V)$: $0 \to \Omega^0(U \cap U) \to \Omega^2(U \cap V) \to \cdots$

$\Rightarrow 0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V) \to 0$ (1)

(1) : consider $\iota_1 : U \hookrightarrow M$, $\iota_2 : V \hookrightarrow M$.

and $J_1 : U \cap V \hookrightarrow U$, $J_2 : V \cap U \hookrightarrow V$.

induce linear maps between Ω^k and let

$$\alpha_k = \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V), \quad \omega \to (\iota_1^*\omega, \iota_2^*\omega$$

$$\beta_k : \Omega^k(U) \oplus \Omega^k(U) \to \Omega^k(U \cap U), \quad (\omega_1, \omega_2) \mapsto J_1^*\omega_1 - J_2^*\omega_2$$

prop 1.1. For $\forall k$, the sequence

$$0 \to \Omega^k(M) \xrightarrow{\alpha_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{\beta_k} \Omega^k(U \cap U) \to 0$$

is short exact sequence.

By these, we should be able to construct a long exact sequence consisting of the deRham cohomology group.

In (1), $\alpha_k : H^k_{dR}(M) \to H^k_{dR}(U) \oplus H^k_{dR}(V)$

$\beta_k : H^k_{dR}(U) \oplus H^k_{dR}(V) \to H^k_{dR}(U \cap V)$

We need sth called <u>connecting homomorphism</u> to get long sequence :

$$\delta_k : H^k_{dR}(U \cap V) \to H^{k+1}_{dR}(M).$$

The map δ_k's can be constructed by "diagram chasing method". So fixed $\{\rho_U, \rho_V\}$ P.O.U subordinate to $\{U, V\}$.

For $\forall\, w \in Z^k(U \cap V)$, we define

$$\delta_k([w]) = [\eta], \quad \eta := \begin{cases} d(\rho_V w) & \text{on } U \\ -d(\rho_U w) & \text{on } V. \end{cases} \quad \text{"}(k+1)\text{ form on } M\text{"}$$

In fact, we just try to "force $w = 0$ near the ∂M" as zero-extension of w is not smooth map k-form on M.

Lemma 1.2. The map δ_k is well-defined.

proof. Several things to check (ignore details)

(i) $\rho_V w \in \Omega^k(U)$ i.e. w is not defined on $U \setminus U \cap V$.

(ii) $\eta \in \Omega^{k+1}(M)$

(iii) $\eta \in Z^{k+1}(M)$

(iv) $[\eta]$ is indep of choice of ρ_U and ρ_V

(v) $[\eta]$ is indep choice of w:

(iv): Let ρ_U and $\tilde{\rho}_U$, ρ_V and $\tilde{\rho}_V$ be P.O.U's to $\{U, V\}$ and $\tilde{\eta}$ be the resulting $(k+1)$-form. Then $\tilde{\rho}_V - \rho_V = \rho_U - \tilde{\rho}_U$ is supp in $U \cap V$.

If we set $\xi := (\tilde{\rho}_V - \rho_V)\, w$, then smooth k-form on M. By construction $\tilde{\eta} - \eta = d\xi$ on both $U / V \to$ on M.

(v). Let $\tilde{w} = w + d\xi$ and $\tilde{\eta}$ be resulting $(k+1)$ form by $\tilde{\eta}$.

we have
$$\tilde{\eta} - \eta = \begin{cases} d(\rho_V d\xi) & \text{on } U \\ -d(\rho_U d\xi) & \text{on } V \end{cases} \quad \text{define } \tau := \begin{cases} -d\rho_V \wedge \xi & \text{on } U \\ d\rho_U \wedge \xi & \text{on } V \end{cases} \overset{(4)}{\Longrightarrow} d\tau = \begin{cases} d\rho_V \wedge d\xi = d(\rho_V d\xi), U, \\ -d\rho_U \wedge d\xi = -d(\rho_U d\xi)\ V \end{cases}$$

$$= \tilde{\eta} - \eta \Rightarrow [\tilde{\eta}] = [\eta]. \quad \square$$

Thm 1.3 (Main theorem). Let U, V be open sets in M so that $M = U \cup V$, then we have long exact sequence:

$$\cdots \xrightarrow{\delta_{k-1}} H^k_{dR}(M) \xrightarrow{\alpha_k} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{\beta_k} H^k_{dR}(U \cap V) \xrightarrow{\delta_k} H^{k+1}_{dR}(M) \xrightarrow{\alpha_{k+1}} \cdots$$

proof. Check (1) $Im(\alpha_k) = ker(\beta_k)$

(2) $Im(\beta_k) = ker(\delta_k)$

(3) $Im(\delta_k) = ker(\alpha_{k+1})$

There's 6 relation need to verify: sample ($Im(\beta_k) \subset ker(\delta_k)$)

Let $\omega_1 \in Z^k(U)$, $\omega_2 \in Z^k(V)$. let $\omega := \beta_k(\omega_1, \omega_2) = J_1^*\omega_1 - J_2^*\omega_2 \in \Omega^k(U \cap V)$

Then $\delta_k([\omega]) = [\eta]$, where

$$\eta = \begin{cases} d(\rho_V \omega) = d(\rho_V \omega - \omega_1) \text{ on } U, \\ -d(\rho_U \omega) = -d(\rho_U \omega + \omega_2) \text{ on } V. \end{cases}$$

Note that on $U \cap V$, $\rho_V \omega - J_1^*\omega_1 = -\rho_U \omega - J_2^*\omega_2$

So there $\exists$ a smooth k-form ξ on M s.t

$$\xi = \begin{cases} \rho_V \omega - \omega_1 & \text{on } U \\ -\rho_U \omega - \omega_2 & \text{on } V \end{cases}$$

Consequently, $\eta = d\xi$ and thus $[\eta] = 0$.

RMK: As mentioned van kampen's thm. It's known that $H^k_{dR}(U)$ and $H^k_{dR}(V)$ and $H^k_{dR}(U \cap V) \overset{\text{determine}}{\not\to} H^k_{dR}(M)$

we need " homomorphism connecting them " to move further !!

Application: de-Rham cohomology $\Rightarrow$ sphere

Thm 2.1. For $n \geq 1$, $H^k_{dR}(S^n) \cong \begin{cases} \mathbb{R} & k=0 \\ 0, & 1 \leq k \leq n-1 \end{cases}$

proof. we have shown that

$$H^0_{dR}(S^n) \cong \mathbb{R} \text{ and } H^1_{dR}(S^1) \cong \mathbb{R}.$$

Just 2 things to reveal:

(1) For $n \geq 2$, $H^1_{dR}(S^n) = 0$

(2) For $n \geq 2$, $k \geq 2$,

$\Rightarrow$ For $n \geq 2$, we let

$U = S^n - \{(0, \cdots 0, -1)\}$ and $V = S^n - \{(0, 0, \cdots 0, 1)\}$

Then $M = U \cup V$. U and V diffeo to $\mathbb{R}^n$; $U \cap V$ homotopy $\cong S^{n-1}$

(1). $0 \longrightarrow H^0_{dR}(S^n) \longrightarrow H^0_{dR}(U) \oplus H^0_{dR}(V) \longrightarrow H^0_{dR}(U \cap V) \longrightarrow H^1(S^n) \rightarrow \cdots$

$\Rightarrow 0 \rightarrow \mathbb{R} \xrightarrow{\alpha_0} \mathbb{R}^2 \xrightarrow{\beta_0} \mathbb{R} \xrightarrow{\delta_0} H^1_{dR}(S^n) \rightarrow 0$

Since α_0 is injective, $\dim(\ker(\beta_0)) = \dim(Im(\alpha_0)) = 1 \Rightarrow \dim Im(\beta_0) = \dim \mathbb{R}^2 - \dim \ker(\beta_0) = 1$

i.e β_0 is surjective. So $\ker(\delta_0) = \mathbb{R}$, $\Rightarrow \delta_0 \equiv 0$. But by exactness, δ_0 is surjective $\Rightarrow H^1_{dR}(S^n) = 0$

(2). $H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) \xrightarrow{\beta_{k-1}} H^{k-1}_{dR}(U \cap V) \xrightarrow{\delta_{k-1}} H^k_{dR}(S^n) \xrightarrow{\alpha_k} H^k_{dR}(U) \oplus H^k_{dR}(V)$,

$\Rightarrow 0 \xrightarrow{\beta_{k-1}} H^{k-1}_{dR}(S^{n-1}) \xrightarrow{\delta_{k-1}} H^k_{dR}(S^n) \xrightarrow{\alpha_k} 0.$

By exactness. δ_{k-1} is 1-1 and onto, and thus must be linear isomorphism!

As a consequence, we can show an very early conclusion for this series of lectures. (Topological invariant of dimension)

(In UCB lecture videoes, I guess?)

If $m \neq n$, then $\mathbb{R}^n$ is not homomorphic to $\mathbb{R}^m$.

proof. If $f: \mathbb{R}^n \to \mathbb{R}^m$ is homeomorphism, then $f: \mathbb{R}^n / \{0\} \to \mathbb{R}^m / \{f(0)\}$ is a homeomorphism.

So $H^k_{dR}(\mathbb{R}^n \setminus \{0\}) = H^k_{dR}(\mathbb{R}^m \setminus \{f(0)\})$, $\forall k$

But $\mathbb{R}^n \setminus \{0\}$ is homotopy equivalent to S^{n-1}, while $\mathbb{R}^m / \{f(0)\}$ is homotopy equivalent to S^{m-1}

So $H^k_{dR}(S^{m-1}) = H^k_{dR}(S^{n-1})$, $\forall k$. This contradicts w/ the fact that $m \neq n$. $\square$