EXISTENCE OF SOLUTION TO THE WEAKLY NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We consider the cubic non-linear stationary Schrödinger equation with an external potential V that compactly supported or satisfied the fast decay condition. For the fast decay potential with V in the class of SR satisfied decay rate $V(x) = \mathcal{O}(1/|x|^{1+\epsilon})$, we will investigate the existence of a solution of the nonlinear Schrödinger equation defined on $\mathbb{R}^1$ using the Banach Fix Point Theorem and approximate a modified scattering behavior corresponding to the solution. For the compactly supported potential, we will adapt the same method to verify the existence of a solution to the NLS equation defined on $\mathbb{R}^3$.

1. Introduction

In this paper, I will discuss the existence of solutions corresponding to different potentials in nonlinear stationary Schrödinger equation. The first four sections serve as a summary of techniques and theorems taken from [1], [2], [3], [4], [5] corresponding to our problem settings. In Section 5, I will show the existence of an outgoing solution of the nonlinear stationary Schröodinger equation with a fast decay potential defined in 1-D. The leading-order scattering matrices will also be shown in Section 6. In Section 7, we will first calculate the general Green's function for stationary Schrödinger equation defined on general n-dimensions, and then we will extend our results by showing the existence of the outgoing solution corresponding to the bounded compactly supported potential in 3-D.

2. Outgoing and Incoming solutions

We consider the following class of operators:

$$H := D_x^2 + V(x), \quad D_x := \frac{1}{i}\partial x, \quad V \in \mathcal{L}^\infty_{comp}(\mathbb{R}) \tag{2.1}$$

the stationary Schrödinger equation is then the

$$(H - z)u = f, \quad z \in \mathbb{C}, f \in \mathcal{L}^2(\mathbb{R}) \tag{2.2}$$

While the dynamical equation is given by

$$(i\partial_t - H)v = F, \quad v|_{t=0} = v_0, \quad v_0 \in \mathcal{L}^2(\mathbb{R}), \quad F \in \mathcal{L}^1_{loc}(\mathbb{R}_t, \mathcal{L}^2(\mathbb{R}_x)) \tag{2.3}$$

Outside the support of V and f, say $|x| \geq R$ (assume that f is compactly supported), we have a homogeneous equation and the solution to (2.2) is given by

$$u(x) := a_\pm e^{i\sqrt{z}|x|} + b_\pm e^{-i\sqrt{z}|x|}, \quad \pm x \geq R \tag{2.4}$$

Consider the well-known fact that $\sigma(H_0) := \sigma(-\Delta) = [0, \infty)$ [1], we may consider that $\sqrt{z}$ defined on $\mathbb{C}/[0, \infty)$ with $Im(z) > 0$ lies in the upper half plane, so that

$$\lim_{\epsilon \to 0^+} \sqrt{z + i\epsilon} =: \pm\sqrt{z \pm i0} > 0, \quad z \in (0, \infty). \tag{2.5}$$

When consider $z \in (0, \infty)$, we write $\sqrt{z} = \sqrt{z + i0}$

Definition 2.1. *A solution to (2.2) with $z > 0$ is called outgoing if*

$$u(x) = a_- e^{-i\sqrt{z}x}, \quad x < -R, u(x) = a_+ e^{i\sqrt{z}x}, \quad x > R, \tag{2.6}$$

Similarly, the solution to (2.2) is called incoming if

$$u(x) = b_- e^{i\sqrt{z}x}, \quad x < -R, u(x) = b_+ e^{-i\sqrt{z}x}, \quad x > R, \tag{2.7}$$

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We can also consider the initial valued wave equation:

$$(-\partial_t^2 - H)v = F, \quad v|_{t=0} = v_0, \quad \partial_t v|_{t=0} = v_1 \tag{2.8}$$

Note that the stationary Schrödinger equation, formally obtained by taking the Fourier transform in t, is given by

$$(H - k^2)u = f, \quad k \in \mathbb{C} \tag{2.9}$$

In the case of convention, we choose

$$k^2 = z, \quad k = \sqrt{z} \tag{2.10}$$

so that it is consistent with Definition 2.1 that is

$$u_{in}(x) = a_{sgn(x)}e^{-ik|x|}, \quad u_{out}(x) = b_{sgn(x)}e^{ik|x|}, \quad |x| > R. \tag{2.11}$$

representing the outgoing solution and incoming solution.

In scattering theory, we compare the incoming and outgoing waves that can be captured by the scattering matrix, which is defined as

$$S : \begin{bmatrix} b_- \\ b_+ \end{bmatrix} \to \begin{bmatrix} a_- \\ a_+ \end{bmatrix} \tag{2.12}$$

To describe $S := S(k)$ at frequency k, we aim to find solutions to (2.9) in form of

$$u^\pm(x) = e^{\pm ikx} + v^\pm(x,k) \tag{2.13}$$

where $v^\pm(x,k)$ are outgoing and can be founded by outgoing Resolvent $R_V(k)$:

$$v^\pm(x,k) = -R_V(k)(Ve^{\pm ikx}) \tag{2.14}$$

where $R_V(k)$ is the resolvent operator of (2.9) and is well defined away from the pole of $R_V(k)$. Specifically, in the self-adjoint case that means $u^\pm(x)$ exists for $k \in \mathbb{R}/\{0\}$.

If we write

$$v^\pm_{sgn(x)}(k) := e^{-ik|x|}v^\pm(x,k), \quad |x| > R, \tag{2.15}$$

then (2.12) shows that

$$S(k) : \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 + v^+_+(k) \\ v^+_-(k) \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 + v^-_+(k) \\ v^-_-(k) \end{bmatrix}, \tag{2.16}$$

which means that

$$S(k) = I + A(k), \quad A(k) = \begin{pmatrix} v^+_+(k) & v^-_+(k) \\ \\ v^+_-(k) & v^-_-(k) \end{pmatrix} \tag{2.17}$$

Theorem 2.1. [2] *The coefficient of $A(k)$ are meromorphic functions of k given by*

$$v^\omega_\theta(k) = \frac{1}{2ik}\int_{\mathbb{R}} e^{ik(\omega-\theta)x}V(x)(1 - e^{-ik\omega x}R_V(k)(e^{ik\omega\cdot}V(x))dx, \tag{2.18}$$

where $\theta, \omega \in \{+,-\}$.

The coefficients $v^\omega_\theta(k)$ have important physical interpretations:

$$T_\pm(k) = 1 + v^\pm_\pm(k) \quad \textit{are the transition coefficients} \tag{2.19}$$

$$R_\pm(k) = v^\mp_\pm(k) \qquad \textit{are the right and left reflection coefficients} \tag{2.20}$$

where those coefficients relate particle states before and after undergoing a scattering process and are proven effective in recovering the potentials.

Moreover, we can also derive the scattering matrix in the operator perspective.

Definition 2.2. *The wave operators of pair (H, H_0) are defined as*

$$W_\pm = \lim_{t\to\infty} e^{itH}e^{-itH_0} \tag{2.21}$$

where we denote H_0 as the self-adjoint realization of $-\Delta$ in $\mathcal{L}^2(\mathbb{R}^n)$ with domain $D(H_0) = \mathcal{H}^2(\mathbb{R}^n)$. For V is H_0 compact, we denote H as the self-adjoint realization of $-\Delta + V(x)$ in $\mathcal{L}^2(\mathbb{R}^n)$ with the same domain $D(H_0) = \mathcal{H}^2(\mathbb{R}^n)$ and essential spectrum $[0,\infty)$.

Definition 2.3. *The wave operator W is said to be complete if*

$$Range(W_+) = Range(W_-) = \mathcal{L}^2_{ac}(\mathbb{R}^n) \tag{2.22}$$

provided the operator exists and $\mathcal{L}^2_{ac}(\mathbb{R}^n)$ denoted as the absolute continuous subspace of $\mathcal{L}^2(\mathbb{R}^n)$ with respect to H.

The interpretation for this is simple: $Range(W_\pm)$ is the set of all solutions that have an incoming/outgoing asymptotic behavior.

When the wave operator exists and is complete, we may define the scattering operator S by

$$S = W_+^* W_-, \quad S \textit{ is unitary operator on } \mathcal{L}^2(\mathbb{R}^n). \tag{2.23}$$

Taking Fourier Transform and introducing polar coordinate $\xi = (k, \omega)$ we get

$$\mathcal{F}(Sf)(k,\omega) = \mathcal{S}(k)(\mathcal{F}f)(k,\cdot)(\omega), \tag{2.24}$$

where $f \in \mathcal{L}^2(\mathbb{R}^n)$ and $\mathcal{S}(k)$ is an operator valued function defined for $k > 0$ referring to the scattering matrix.

On the other hand, we also need to determine when the wave operator exists. Note that for $u \in D(W_\pm)$, the domain of the wave operators is the set of incoming/outgoing asymptotics states.

Lemma 2.1. *(Cook) Suppose $D(H) \subseteq D(H_0)$. If*

$$\int_0^\infty \|(H - H_0)e^{\mp itH_0}u\|dt < \infty, \quad u \in D(H_0) \tag{2.25}$$

then $u \in D(W_\pm)$, respectively. We even have

$$\|(W_\pm - I)u\| \le \int_0^\infty \|(H - H_0)e^{\mp itH_0}u\|dt \tag{2.26}$$

in this case

As a simple consequence, we will have an application in the Schrödinger operator defined on $\mathbb{R}^3$

Theorem 2.2. *Suppose H_0 is the free Schrödinger operator and $H = H_0 + V$ with $V \in \mathcal{L}^2(\mathbb{R}^3)$, then the wave operators exists and $D(W_\pm) = \mathcal{L}^2(\mathbb{R}^3)$.*

3. LIMIT ABSORPTION PRINCIPLE

From previous discussion, we can see that the potential V always admits some strong conditions to guarantee the existence and completeness of the wave operator. However, from Agmon's paper [2], a minimal decay assumption would be enough, which is closely related to the limit absorption principle.

Definition 3.1. *A real function $V(x) \in \mathcal{L}^2_{loc}(\mathbb{R}^n)$ is said to belong to Short Range (SR) potential if, for some $\epsilon > 0$, the multiplication map*

$$u(x) \to (1 + |x|)^{1+\epsilon}V(x)u(x) \tag{3.1}$$

defines a compact operator from $\mathcal{H}_2(\mathbb{R}^n)$ into $\mathcal{L}_2(\mathbb{R}^n)$

Remark 3.1. *Here, the short range definition serves for a minimal decay requirement:*

$$\sup_{x\in\mathbb{R}^n}[(1 + |x|)^{2+2\epsilon}\int_{|y-x|\le 1} |V(y)|^2|y - x|^{-n+\mu}dy] < \infty \tag{3.2}$$

for some $\epsilon > 0$ and $0 < \mu < 4$. Specifically, this holds for V which at infinity verifies

$$V(x) = \mathcal{O}(|x|^{-1-\epsilon}) \quad \textit{as } |x| \to \infty \tag{3.3}$$

and which locally satiesfied $V \in \mathcal{L}^p_{loc}(\mathbb{R}^n)$ with $p = 2$ for $n \le 3$ which satisfied the first equation we will conduct in Section 5.

The assumption that V is of class SR will be used in an essential way in the following results:

Theorem 3.1. *Let $\mathrm{e}^+(H)$ be the set of all positive eigenvalues of H, we have $\mathrm{e}^+(H)$ is a discrete set on the real line. Moreover, the only possible limit points of $\mathrm{e}^+(H)$ on the extended real line are $\lambda = 0$ and $\lambda = +\infty$ with every point in $\mathrm{e}^+(H)$ of finite multiplicity.*

We shall formulate now the limiting absorption principle for general Schrödinger operators.

Theorem 3.2. *Let $H = -\Delta + V(x)$ be the Schrödinger operator with potential V of class SR. Consider $R(z) := (H - z)^{-1}$, the resolvent of H, as analytic operator value function defined on $\mathbb{C}/\sigma(H)$ with values in $B(\mathcal{L}^{2,s}, \mathcal{H}^{2,-s})$ for any $s > \frac{1}{2}$. Let $\lambda \in R^+/\sigma(H)$, the following limit exists in uniform topology of $B(\mathcal{L}^{2,s}, \mathcal{H}^{2,-s})$*

$$\lim_{\substack{z \to \lambda \\ \pm Im(z) > 0}} R(z) = R^{\pm}(\lambda). \tag{3.4}$$

Moreover, for any $f \in \mathcal{L}^{2,s}(\mathbb{R}^n)$, we have

$$R^{\pm}(\lambda)f = R_0^{\pm}(\lambda)f - R_0^{\pm}(\lambda)VR_0^{\pm}(\lambda)f. \tag{3.5}$$

Remark 3.2. *In the theorem, we use $\mathcal{L}^{2,s}(\mathbb{R}^n)$ and $\mathcal{H}^{2,-s}(\mathbb{R}^n)$ defined by*

$$\mathcal{L}^{2,s}(\mathbb{R}^n) = \{u(x) : (1 + |x|^2)^{s/2}u(x) \in \mathcal{L}^2(\mathbb{R}^n)\}$$
$$\mathcal{H}^{2,-s}(\mathbb{R}^n) = \{u(x) : D^{\alpha}u(x) \in \mathcal{L}^{2,-s}(\mathbb{R}^n), \quad 0 \leq |\alpha| \leq 2\}$$

for $s > \frac{1}{2}$. The expression of $R_0^{\pm}(z)$ followed the same construction as in the theorem by

$$\lim_{\substack{z \to \lambda \\ \pm Im(z) > 0}} R_0(z) = R_0^{\pm}(\lambda) \tag{3.6}$$

for $\lambda \in (0, \infty)$. If we set $u^+ = R^+(\lambda)f$ as outgoing solution and $u^- = R^-(\lambda)f$ as incoming solution, we will find that this is followed the convention of (2.12) defined on 1-D and solve the differential equation

$$(-\Delta - \lambda + V(x))u = f, \quad \text{in } \mathbb{R}^n \tag{3.7}$$

Moreover, as we shall see later in approximation of scattering matrix, Theorem 3.3 can be improved to $\{V_j\}$ satisfied [2]*:*

(1) $\lim_{j\to\infty} V_j(x) = V(x)$ *for almost all x;*
(2) $|V_j(x)| \leq W(x)$ *for almost all x and W is function of class SR.*

We have

$$\lim_{j\to\infty} R_j^{\pm}(\lambda) = R^{\pm}(\lambda). \tag{3.8}$$

for any $\lambda \in \mathcal{K}$, for any compact set $\mathcal{K} \subseteq \mathbb{R}^+/e^+(H)$.

Combined the constructions of families of generalized eigenfunctions (see [2], p.187) with condition (3.2), we are able to show that

Theorem 3.3. *Let $H = -\Delta + V$ be a Schrödinger equation with potential V satisfied condition (3.2). Then the wave operator:*

$$W_{\pm} = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}, \tag{3.9}$$

exists and complete.

The key idea is to approximate V by sequence compactly support potentials V_j defined by

$$V_j(x) = \begin{cases} V(x), & \text{if } |x| \leq j \text{ and } |V_j(x)| \leq j \\ 0, & \text{otherwise} \end{cases} \tag{3.10}$$

combined with extensive estimations.

Hence, for potentials satisfying condition (3.2), we have checked that it satisfied our problem setting for the bounded potential $V = \mathcal{O}(|x|^{-1-\epsilon})$ defined in 1D, the wave operator exists and is complete so that we can approximate the scattering matrix based on this theorem.

4. 2-D SCATTERING PROBLEM

In this section, I want to show some of the results for the scattering problem in R^2[3]. Let $D \subseteq \mathbb{R}^2$ be a bounded domain containing the origin with connected complement such that ∂D is in class C^2. We consider the impedance boundary value problem:

$$\Delta u + k^2 u = 0 \quad in\ \mathbb{R}^2/\overline{D} \tag{4.1}$$

$$u(x) = e^{ikx\cdot d} + u^s(x) \tag{4.2}$$

$$\lim_{r\to\infty} \sqrt{r}(\frac{\partial u^s}{\partial r} - iku^s) = 0 \tag{4.3}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0 \tag{4.4}$$

where $\lambda \in C(\partial D)$, $\lambda(x) > 0$ for any $x \in \partial D$, ν is the unit outward vector normal to ∂D. We shall see later in section 7 that the fundamental solution to this situation is

$$\Phi(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|) \tag{4.5}$$

and note that limit absorption principle holds uniformly in x and y and as $|x-y| \to 0$, we have

$$\Phi(x,y) := \frac{1}{2\pi}\log(\frac{1}{|x-y|}) + \mathcal{O}(1) \tag{4.6}$$

Theorem 4.1. *Let $u^s \in C^2(\mathbb{R}^2/\overline{D}) \cap C^1(\mathbb{R}^2/D)$ be a solution of the stationary Schrödinger equation in the exterior of D satisfied limit absorption principle. Then for $x \in \mathbb{R}^2/D$ we have that*

$$u^s(x) = \int_{\partial D} (u^s(y)\frac{\partial}{\partial \nu(y)}\Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y)\Phi(x,y))ds(y) \tag{4.7}$$

This theorem can be extended to (4.2) by letting $u \in C^2(D) \cap C^1(\overline{D})$ be the solution in D and for $x \in D$, we have

$$u(x) = \int_{\partial D} (\frac{\partial u}{\partial \nu}(y)\Phi(x,y) - u(y)\frac{\partial}{\partial \nu(y)}\Phi(x,y))ds(y) \tag{4.8}$$

that is real-analytical functions of its independent variable. In addition, if we further assume that u^s satisfies line (4.4), we will have the result for $u^s = 0$ in $\mathbb{R}^2/\overline{D}$ with extension for $u = 0$ in D.[3]

The next theorem is a classic result in scattering theory.

Theorem 4.2. *(Rellich's Lemma). Let $u \in \mathbb{R}^2/\overline{D}$ be a solution to the (4.1) satisfied*

$$\lim_{R\to\infty} \int_{y=|R|} |u|^2 ds = 0. \tag{4.9}$$

Then $u = 0$ in $\mathbb{R}^2/\overline{D}$

Moreover, we shall see that the solution of system (4.1)-(4.4) is controlled by the incident field u^i.

Theorem 4.3. *There exists a unique solution to the scattering problem (4.1)-(4.4) which depends continuously on $u^i(x) = e^{ikx\cdot d}$ in $C^1(\partial D)$.*

5. NONLINEAR STATIONARY SCHRÖDINGER EQUATION WITH FAST DECAY POTENTIAL

We first consider the standard perturbated Stationary Schrödinger Equation in the given form:

$$(\Delta + k^2 + V(x))u = 0 \tag{5.1}$$

we have for u_{in} as solutions to the homogeneous equation satisfied $(\Delta + k^2)u_{in} = 0$, and decompose $u := u_{out} + u_{in}$ we have

$$\begin{aligned}(\Delta + k^2 + V)(u_{in} + u_{out}) &= 0\\ (\Delta + k^2 + V)u_{out} &= -Vu_{in}\\ (\Delta + k^2)u_{out} &= -V(u_{in} + u_{out})\\ u_{out} &= -(\Delta + k^2)^{-1}_{out}V(u_{in} + u_{out})\end{aligned}$$

If everything is well-defined and if we can show the right hand side is indeed a contraction for $u_{out} \in X$ (some complete metric space), then we can show there exist unique global solution $u = u_{out} + u_{in}$.

Inspired by above, we consider the stationary Schrödinger equation defined on $\mathbb{R}$ with the potential V and w:

$$f(u) = V(x)u + w(x)|u|^2u \tag{5.2}$$

where for any $\epsilon > 0$, we let $V(x) := \mathcal{O}(\frac{1}{|x|^{1+\epsilon}})$ and $w(x) := \mathcal{O}(\frac{1}{|x|^{1+\epsilon}})$ which are essentially bounded and short-range potentials. Consider that

$$(-\Delta - k^2)u = f(u) \tag{5.3}$$

We decompose u as $u = u_{\text{out}} + u_{\text{in}}$, where $u_{\text{in}} = Ae^{ikx}$ is the incoming solution that solves the homogeneous equation, and u_{out} is a correction. Plugging into (5.3):

$$(-\Delta - k^2))(u_{out} + Ae^{ikx}) = f(u_{out} + Ae^{ikx}) \tag{5.4}$$

which is equivalent to

$$u_{out} = (-\Delta - k^2))^{-1}_{out} f(u_{out} + Ae^{ikx}) \tag{5.5}$$

where as the limit absorbtion principle, we used the outgoing condition:

$$(-\Delta - k^2))^{-1}_{out} = \lim_{\epsilon \to 0^+} (-\Delta - (k^2 + i\epsilon))^{-1} \tag{5.6}$$

If we can show that for $Tu_{out} := (-\Delta - k^2))^{-1}_{out} f(u_{out} + Ae^{ikx})$ is a contraction map on X, where X is some complete metric space, then the above equation admits an unique fixed point satisfied $u_{out} = (-\Delta - k^2))^{-1}_{out} f(u_{out} + Ae^{ikx})$ by contraction mapping theorem.

Remark 5.1. *The function u must be twice differentiable, u and u^2 must be bounded in any bounded domain. This implies that our non-linear term $V(x)u + u(x)|u|^2$ is bounded in any bounded domain except at finitely many points. As $x \to \pm\infty$, the solution is a plane wave due to the fast decay condition on perturbations. Since every term is in $L^\infty(\mathbb{R})$, we consider $u \in L^\infty(\mathbb{R})$.*

Now we define the outgoing Green's function. If $k^2 \notin \sigma(-\Delta) = [0, \infty)$, then the solution is:

$$u_{out}(x) = [(-\Delta - k^2)^{-1}(f(u_{\text{out}} + Ae^{ikx}))](x) \tag{5.7}$$

with the corresponding Green's function

$$G(x; k^2) = \frac{1}{2\sqrt{-k^2}} e^{-|x|\sqrt{-k^2}}. \tag{5.8}$$

The outgoing Green's function when $k \in (0, \infty)$ is

$$G_{\text{out}}(x; k^2) = \lim_{\epsilon \to 0^+} G(x; k^2 + i\epsilon) = -\frac{1}{2ik} e^{ik|x|}. \tag{5.9}$$

We will use a variance of the fixed point argument.

Theorem 5.1. *Let (X, d) be a complete metric space, and $A : X \to X$. Furthermore, assume there exists a $x \in X$ and $r > 0$ such that*

(1) *the ball $B(a, r)$ is an invariant set for A.*
(2) *the map A is a contraction on $B(a, r)$.*

Then there exist a unique fixed point of A inside ball $B(a, r)$

We now introducing the existence theorem.

Theorem 5.2. *For any incoming solution $u_{in} = Ae^{ikx}$ with amplitude A small enough, there exist an outgoing solution satisfied (5.7)*

We first consider $V \equiv 0$ to simplify the calculation. I.e. $f(u) = w(x)|u|^2u$; for $V \neq 0$ the proof is the same just with some additional calculations and we will show later.

Proof. Consider $B_{\mathcal{L}^\infty(\mathbb{R})}(-Ae^{ikx}, r)$, we want to show that T is invariant on this ball with suitable r. Let $u \in B_{\mathcal{L}^\infty(\mathbb{R})}(-Ae^{ikx}, r)$, we have

$$\begin{aligned}
\|T(u)\|_{\mathcal{L}^\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}} |\int_{\mathbb{R}} \frac{1}{2ik} e^{ik|x-y|} w(y)|u + Ae^{ikx}|^2(u + Ae^{ikx})dy| \\
&\leq \frac{1}{2k}\int_{\mathbb{R}} |w(y)||u + Ae^{ikx}|^3 dy \\
&\leq \frac{1}{2k}\|u + Ae^{ikx}\|^3_{\mathcal{L}^\infty} \int_{\mathbb{R}} |w(y)|dy \\
&= \frac{1}{2k}\|u + Ae^{ikx}\|^3_{\mathcal{L}^\infty}\|w\|_{\mathcal{L}^1}
\end{aligned}$$

where the last line is due to the condition that $w(x) \in \mathcal{L}^1(\mathbb{R})$. Note that for $w(x) = \mathcal{O}(1/|x|^{1+\epsilon})$, for any $\epsilon > 0$, there exists M, C such that for all $|x| \geq M$, we have $|w(x)| \leq C/|x|^{1+\epsilon}$, and we can decompose the integral into

$$\begin{aligned}
\int_{\mathbb{R}} |w(y)|dy &= \int_{|y|\geq M} |w(y)|dy + \int_{|y|<M} |w(y)|dy \\
&\leq C\int_{|y|\geq M} \frac{1}{y^{1+\epsilon}}dy + \int_{|y|<M} |w(y)|dy \\
&= C[\int_M^\infty \frac{1}{|y|^{1+\epsilon}}dy + \int_{-\infty}^{-M} \frac{1}{|y|^{1+\epsilon}}dy] + \int_{|y|<M} |w(y)|dy \\
&= 2CM^{-\epsilon} + \int_{|y|<M} |w(y)|dy \\
&\leq 2CM^{-\epsilon} + 2M\|w\|_{\mathcal{L}^\infty}
\end{aligned}$$

where the last line we used w is bounded, and we have checked that $w(x) \in \mathcal{L}^1(\mathbb{R})$. Moreover, we have that

$$\|T(u)\|_{\mathcal{L}^\infty} \leq \frac{1}{2k}\|u + Ae^{ikx}\|^3_{\mathcal{L}^\infty}\|w\|_{\mathcal{L}^1} \leq \frac{r^3}{2k}\|w\|_{\mathcal{L}^1} \tag{5.10}$$

To make T is invariant, by above consideration, we require

$$\frac{r^3}{2k}\|w\|_{\mathcal{L}^1} + |A| \leq r \tag{5.11}$$

For contraction map condition, we consider $u, v \in B_{\mathcal{L}^\infty(\mathbb{R})}(-Ae^{ikx}, r)$, and we set

$$\alpha(x) := u(x) + Ae^{ikx}, \quad \beta(x) := v(x) + Ae^{ikx} \tag{5.12}$$

for simplication. Note that as $u, v \in B_{\mathcal{L}^\infty(\mathbb{R})}(-Ae^{ikx}, r)$, we have $\|\alpha\|_{\mathcal{L}^\infty} \leq r$ and $\|\beta\|_{\mathcal{L}^\infty} \leq r$, we then have

$$\begin{aligned}
\|T(u) - T(v)\|_{\mathcal{L}^\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}} |\int_{\mathbb{R}} \frac{1}{2ik} e^{ik|x-y|} w(y)(|u + Ae^{ikx}|^2(u + Ae^{ikx}) \\
&\quad + |v + Ae^{ikx}|^2(v + Ae^{ikx}))dy| \\
&\leq \frac{1}{2k}\||\alpha|^2\alpha - |\beta|^2\beta\|_{\mathcal{L}^\infty} \int_{\mathbb{R}} |w(y)|dy \\
&\leq \frac{1}{2k}\|w\|_{\mathcal{L}^1}\||\alpha|^2\alpha - |\beta|^2\beta\|_{\mathcal{L}^\infty}
\end{aligned}$$

as shown in the first condition. We then consider that

$$\begin{aligned}
\||\alpha|^2\alpha - |\beta|^2\beta\|_{\mathcal{L}^\infty} &= \|(|\alpha|^2 + |\beta|^2)(\alpha - \beta) + \alpha\beta(\bar{\alpha} - \bar{\beta})\|_{\mathcal{L}^\infty} \\
&\leq 3r^2\|u - v\|_{\mathcal{L}^\infty}
\end{aligned}$$

provided that $\|\alpha\|_{\mathcal{L}^\infty} \leq r$ and $\|\beta\|_{\mathcal{L}^\infty} \leq r$. Thus our calculation becomes

$$\|T(u) - T(v)\|_{\mathcal{L}^\infty} \leq \frac{3r^2}{2k}\|w\|_{\mathcal{L}^1}\|u - v\|_{\mathcal{L}^\infty} \tag{5.13}$$

To make T an contraction, we need

$$\frac{3r^2}{2k}\|w\|_{\mathcal{L}^1} < 1 \tag{5.14}$$

which is equivalent to require r satisfied

$$r < \sqrt{\frac{2k}{3\|w\|_{\mathcal{L}^1}}} \tag{5.15}$$

For suitable $|A| \neq 0$ such that there exists a r satisfies (5.15) and (5.11), the contraction mapping gives us a non-trivial outgoing solution, since $u_{out} = 0$ is not a fixed point. The global solution $u = u_{in} + u_{out}$ is also not trivial since $u_{out} = Ae^{ikx}$ is not a fixed point. □

Remark 5.2. *Note that there is no guarantee that condition (5.11) and (5.15) will coexists for a positive real r, we need to to determine that whether it indeed can be satisfied for $|A|$ sufficient small.*

Theorem 5.3. *There exists $r \in \mathbb{R}^+$ for $|A|$ sufficient small such that satisfied both (5.15) and (5.11)*

Proof. Since (18) gives relation of r in terms of $|A|$, we first consider that equation

$$r^3 - \frac{2k}{\|w\|_{\mathcal{L}^1}}r + \frac{2k}{\|w\|_{\mathcal{L}^1}}|A| = 0 \tag{5.16}$$

which is equivalent to

$$r^3 + pr + q = 0 \tag{5.17}$$

for $p := -\frac{2k}{\|w\|_{\mathcal{L}^1}}$ and $q := \frac{2k}{\|w\|_{\mathcal{L}^1}}|A|$.

In order to obtain a positive real $r > 0$, we need to have two of the intersections of the cubic function and the line lay in the first quadrant, and the last one lay in the third quadrant to guarantee that we have three real value r. Moreover, it is necessary to show the discriminant $\Delta(Poly(r)) = -4p^3 - 27q^2 \geq 0$, and we can simplified as

$$4p^3 + 27q^2 \leq 0 \tag{5.18}$$

Plugging into our expression we get

$$\begin{aligned} 4(-\frac{2k}{\|w\|_{\mathcal{L}^1}})^3 + 27(\frac{2k}{\|w\|_{\mathcal{L}^1}}|A|)^2 &\leq 0 \\ 27\frac{4k^2|A|^2}{\|w\|^2_{\mathcal{L}^1}} &\leq \frac{32k^3}{\|w\|^3_{\mathcal{L}^1}} \\ |A|^2 &\leq \frac{8k}{27\|w\|_{\mathcal{L}^1}} \\ |A| &\leq \sqrt{\frac{8k}{27\|w\|_{\mathcal{L}^1}}} \end{aligned}$$

So now, we get a contraint for $|A|$ that guarantee we have 3 real roots, we still to further proceed the solutions for positive r and indeed whether this contraint works for (5.15).

We use simplified expression for calculations to roots r_i, consider that

$$(-\frac{2k}{3(\|w\|_{\mathcal{L}^1})})^3 + (\frac{2k|A|}{\|w\|_{\mathcal{L}^1}})^2 := -z \leq 0 \quad (\textit{for some } z \geq 0) \tag{5.19}$$

by rewriting restriction of $\Delta(Poly(r))$ in this form, we can use Cardano formulation, and the roots of (29) is given by

$$r = \sqrt[3]{-q/2 + \sqrt{q^2/4 + p^3/27}} + \sqrt[3]{-q/2 - \sqrt{q^2/4 + p^3/27}} \tag{5.20}$$

which can indeed have some explicit expression as

$$r_i = 2R^{1/3}cos(\theta/3 + 2\pi i/3), \quad i = 0, 1, 2 \tag{5.21}$$

where

$$R = \sqrt{(\frac{2k|A|}{\|w\|_{\mathcal{L}^1}})^2 + z} = \sqrt{(\frac{2k}{3(\|w\|_{\mathcal{L}^1})})^3} \tag{5.22}$$

and

$$\theta = \tan^{-1}\left(\frac{\sqrt{z}}{\frac{-2k|A|}{\|w\|_{\mathcal{L}^1}}}\right) \tag{5.23}$$

Then we have $\theta \in (\frac{1}{2}\pi, \frac{3}{2}\pi]$ and we must have two roots greater than 0.

To further check whether the $|A|$ contraint works for (5.15), we first consider that

$$r_1 = 2R^{1/3}cos(\theta/3 - 2\pi/3) < R^{1/3} \leq \sqrt{\frac{2k}{3\|w\|_{\mathcal{L}^1}}} \tag{5.24}$$

the second largest root r_1 which satisfied the condtion (5.15) as desired.

To summarize, as long as $|A| \leq \sqrt{\frac{8k}{27(\|w\|_{\mathcal{L}^1})}}$, the bound we used, or in terms of energy form

$$|\lambda|^2\|w\|_{\mathcal{L}^1} \leq \frac{8k}{27} \tag{5.25}$$

We will have the suitable $r > 0$. □

Now, we start to prove the case for $V \not\equiv 0$.

Corollary 5.1. *For any incoming solution $u_{in} = Ae^{ikx}$ with the amplitude $|A|$ small enough, there exists an outgoing solution satisfied (5.7) with short range potential $V \not\equiv 0$.*

Proof. Considering $B_{\mathcal{L}^\infty(\mathbb{R})}(-Ae^{ikx}, r)$, we want to show that T is invariant on this ball with a suitable r. where $V(x) = \mathcal{O}(\frac{1}{|x|^{1+\epsilon}})$ and $w(x) = \mathcal{O}(\frac{1}{|x|^{1+\epsilon}})$ which are short-range potentials and essentially bounded, we denote

$$\exists M_w, C_w \text{ such that for all } |x| \geq M_w, \text{ we have } |w(x)| \leq C_w/|x|^{1+\epsilon} \tag{5.26}$$

and

$$\exists M_v, C_v \text{ such that for all } |x| \geq M_v, \text{ we have } |V(x)| \leq C_v/|x|^{1+\epsilon} \tag{5.27}$$

As mentioned above, these verifies $V, w \in \mathcal{L}^1(\mathbb{R})$. Let $u \in B_{\mathcal{L}^\infty(\mathbb{R})}(-Ae^{ikx}, r)$, we have

$$\begin{aligned}\|T(u)\|_{\mathcal{L}^\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left|\int_{\mathbb{R}} \frac{1}{2ik}e^{ik|x-y|}w(y)|u+Ae^{ikx}|^2(u+Ae^{ikx}) + V(y)(u+Ae^{ikx})dy\right| \\ &\leq \frac{1}{2k}\int_{\mathbb{R}} |w(y)||u+Ae^{ikx}|^3 dy + \int_{\mathbb{R}} |V(y)||u+Ae^{ikx}|dy \\ &\leq \frac{1}{2k}r^3(\|w\|_{\mathcal{L}^1}) + \frac{r}{2k}(\|V\|_{\mathcal{L}^1})\end{aligned}$$

We thus require,

$$\frac{1}{2k}r^3(\|w\|_{\mathcal{L}^1}) + \frac{r}{2k}(\|V\|_{\mathcal{L}^1}) + |A| \leq r \tag{5.28}$$

To check T is contraction, we again let $u, v \in B_{\mathcal{L}^\infty(\mathbb{R})}(-Ae^{ikx}, r)$, and consider that

$$\begin{aligned}\|T(u) - T(v)\|_{\mathcal{L}^\infty} &\leq \frac{1}{2k}\|V\|_{\mathcal{L}^1}\|u-v\|_{\mathcal{L}^\infty} + \frac{3r^2}{2k}\|w\|_{\mathcal{L}^1}\|u-v\|_{\mathcal{L}^\infty} \\ &\leq \frac{\|V\|_{\mathcal{L}^1} + 3r^2\|w\|_{\mathcal{L}^1}}{2k}\|u-v\|_{\mathcal{L}^\infty}\end{aligned}$$

We thus require

$$\frac{\|V\|_{\mathcal{L}^1} + 3r^2\|w\|_{\mathcal{L}^1}}{2k} < 1 \tag{5.29}$$

Hence for suitable $|A| \neq 0$ small enough such that for some positive r satisfied (5.29) and (5.28), we will have a non-trivial outgoing solution and a non-trivial global solution. □

6. Scattering Matrix

We will approximate the scattering matrix by the contraction map we just constructed for $|A|$ sufficient small. First, consider $V \equiv 0$, and set $u_0 = 0$ we begin our iteration

$$u_1(x) = T(u_0)(x) = -\frac{1}{2ik}\int_{\mathbb{R}} e^{ik|x-y|}w(y)|A|^2 A e^{iky}dy \tag{6.1}$$

$$= -\frac{1}{2ik}|A|^2\int_{\mathbb{R}} e^{ik|x-y|}w(y) A e^{iky}dy \tag{6.2}$$

for simplicity, we use $h(x) := h(y) = e^{-ikx}\int_{\mathbb{R}} e^{ik|x-y|}w(x)e^{iky}dy$ and keeping up to order of $|A|^5$, we have

$$u_2 = T(u_1) = -\frac{1}{2ik}\int_{\mathbb{R}} e^{ik|x-y|}w(y)|u_1 + Ae^{iky}|^2(u_1 + Ae^{iky})dy \tag{6.3}$$

$$= u_1 - \frac{1}{2ik}\int_{\mathbb{R}} e^{ik|x-y|}w(y)[u_1^3 + 3Au_1^2 e^{iky} + 2|A|^2 u_1 e^{-2ky} + |A|^2 u_1]dy \tag{6.4}$$

since we keep up to order 5 and each u_1 counts for $|A|^3$, then we have

$$u_2 = T(u_1) = u_1 - \frac{1}{2ik}\int_{\mathbb{R}} e^{ik|x-y|}w(y)(2|A|^2 u_1 e^{-2ky} + |A|^2 u_1)dy + \mathcal{O}(|A|^7) \tag{6.5}$$

$$= u_1 + \frac{|A|^4}{4k^2}\int_{\mathbb{R}} e^{ik|x-y|}w(y)Ae^{iky}(\tilde{h}(y) - 2h(y))dy + \mathcal{O}(|A|^7) \tag{6.6}$$

Since $|A|$ is small, we can use the above result to approximate our outgoing solution,

$$u_{out} \approx u_1 + \frac{|A|^4}{4k^2}\int_{\mathbb{R}} e^{ik|x-y|}w(y)Ae^{iky}(\tilde{h}(y) - 2h(y))dy \tag{6.7}$$

Remark 6.1. *Since we are now dealing with the fast decay potential cases, the scattering coefficient is not clear on the position of x with different sides of the perturbations. However, we can multiply a characteristic function $\mathbb{1}_n := \mathbb{1}_{(\{|y|\le n\})}$, which makes transmission and reflection conditions valid as usual scattering.*

When x is at the right of the discretized perturbation which corresponding to transmission, we have

$$u_{out} = Ae^{ikx}\left(-\frac{1}{2ik}|A|^2\int_{\mathbb{R}} \mathbb{1}_n w(y)\,dy + \frac{1}{4k^2}|A|^4\int_{\mathbb{R}} \mathbb{1}_n w(y)(\tilde{h}(y) - 2h(y))\,dy\right) \tag{6.8}$$

$$:= Ae^{ikx}T_{\{n,-\}}; \tag{6.9}$$

and when x is at the left of the purturbation which corresponding to reflection, we have

$$u_{out} = Ae^{-ikx}\left(-\frac{1}{2ik}|A|^2\int_{\mathbb{R}} \mathbb{1}_n w(y)e^{2iky}dy + \frac{1}{4k^2}|A|^4\int_{\mathbb{R}} \mathbb{1}_n w(y)e^{2iky}(\tilde{h}(y) - 2h(y))\,dy\right) \tag{6.10}$$

$$:= Ae^{ikx}R_{\{n,-\}}; \tag{6.11}$$

Now, change the direction of our incoming wave from the right $u_{in} = e^{-ikx}$, note that the Green's function does not change. Since we are changing all expression with $u_{in} = e^{-ikx}$, except $h(y)$ replaced by $g(y) = e^{ikx}\int_{\mathbb{R}} e^{ik|x-y|}w(x)e^{-iky}dy$, we have

$$u_{out} \approx u_1 + \frac{|A|^4}{4k^2}\int_{\mathbb{R}} e^{ik|x-y|}w(y)Ae^{-iky}(\tilde{g}(y) - 2g(y))dy \tag{6.12}$$

If we consider x at the left of the purturbations which corresponding to transmission, we have

$$u_{out} = Ae^{-ikx}\left(-\frac{1}{2ik}|A|^2\int_{\mathbb{R}} \mathbb{1}_n w(y)\,dy + \frac{1}{4k^2}|A|^4\int_{\mathbb{R}} \mathbb{1}_n w(y)(\tilde{g}(y) - 2g(y))\,dy\right) \tag{6.13}$$

$$:= Ae^{-ikx}T\{n,+\} \tag{6.14}$$

and x at the right of the purturbations which corresponding to reflection, we have

$$u_{out} = Ae^{ikx}\left(-\frac{1}{2ik}|A|^2\int_{\mathbb{R}} \mathbb{1}_n w(y)e^{-2iky}dy + \frac{1}{4k^2}|A|^4\int_{\mathbb{R}} \mathbb{1}_n w(y)e^{-2iky}(\tilde{h}(y) - 2h(y))\,dy\right) \tag{6.15}$$

$$:= Ae^{ikx}R_{\{n,+\}}; \tag{6.16}$$

and we have the following relation:

$$T_{\{n,\pm\}} \xrightarrow{n\to\infty} T_{\pm} \quad \text{and} \quad R_{\{n,\pm\}} \xrightarrow{n\to\infty} R_{\pm} \tag{6.17}$$

The scattering matrix is given by the coefficients given above.
The case for $V \not\equiv 0$ also turns into

$$\begin{aligned}
T_- &:= \lim_{n\to\infty} T_{\{n,-\}} = \lim_{n\to\infty} -\frac{1}{2ik}\int_{\mathbb{R}} \mathbb{1}_n[V(y)+w(y)|A|^2]\,dy,\\
R_- &:= \lim_{n\to\infty} R_{\{n,-\}} = \lim_{n\to\infty} -\frac{1}{2ik}\int_{\mathbb{R}} e^{2iky}\mathbb{1}_n[V(y)+w(y)|A|^2]\,dy,\\
T_+ &:= \lim_{n\to\infty} T_{\{n,-\}} = \lim_{n\to\infty} -\frac{1}{2ik}\int_{\mathbb{R}} \mathbb{1}_n[V(y)+w(y)|A|^2]\,dy,\\
R_+ &:= \lim_{n\to\infty} R_{\{n,+\}} = \lim_{n\to\infty} -\frac{1}{2ik}\int_{\mathbb{R}} e^{-2iky}\mathbb{1}_n[V(y)+w(y)|A|^2]\,dy.
\end{aligned}$$

We see that in the linear approximation, using two different incoming solutions, we have a system of equations. The reflection coefficient is given by the Fourier transform of $V(y)+w(y)|A|^2$; On the other hand, we can recover the potential by taking the inverse Fourier transform of the reflection coefficients.

7. NONLINEAR 3-D STATIONARY SCHRÖDINGER EQUATION WITH COMPACTLY SUPPORTED POTENTIAL

In this section, we still consider the stationary Schrödinger equation

$$(\Delta+|k|^2)u = f(u) \tag{7.1}$$

where u is defined on $\mathbb{R}^3$ and k is a wave vector, and

$$f(u) = V(x)u(x) + w(x)|u|^2u \tag{7.2}$$

for $w(x), V(x)$ are functions defined on $\mathbb{R}^3$ that is bounded and compactly supported. Before moving on to the calculations, we first look at the general representations of the outgoing Green's functions of the n-dimensional stationary Schrödinger equation. Consider that

$$(\Delta+|k|^2)u = 0$$

where u defined on $\mathbb{R}^n$. The solutions to this homogeneous equation is our desired incoming solutions,

$$u_{in} := Ae^{ik\cdot r}$$

where $r\in\mathbb{R}^n$ and k defined as a wave vector. Spherical symmetry implies that we can rewrite our equation

$$(\Delta+|k|^2)G = -\delta(r,r') \tag{7.3}$$

as the following form

$$\frac{\partial^2 G}{\partial r^2} + \frac{n-1}{r}\frac{\partial G}{\partial r} + k^2 G = 0 \tag{7.4}$$

Without loss of generality, we assume that the point mass is at $r' = \vec{0} \in R^n$ and define:

$$G(r, r'=0) := r^{1-\frac{n}{2}}g(r)$$

plugging in (7.4), we have

$$r^2g''(r) + rg'(r) + ((kr)^2 - (\frac{n}{2}-1)^2)g(r) = 0 \tag{7.5}$$

This is nothing but a Bessel's differential equation with order $(\frac{n}{2}-1)$ and argument kr, and we have the general solution of the form

$$G(r,r'=0) = Ar^{1-\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(kr) + Br^{1-\frac{n}{2}}H^{(2)}_{\frac{n}{2}-1}(kr) \tag{7.6}$$

where $H^{(i)}_{\frac{n}{2}-1}(kr)$ represent first and second kind Hankel functions. To calculate A, we need

$$\lim_{\epsilon\to 0}\oint_{r=\epsilon}\frac{\partial G}{dr}dS_{n-1} = -1 \tag{7.7}$$

where the integration is over the surface of the S^{n-1} the sphere of radius ϵ. The surface area of $S_{n-1} = \frac{2\pi^{\frac{n}{2}}\epsilon^{n-1}}{\Gamma(\frac{n}{2})}$. Plugging (7.6) to (7.7), we get

$$\lim_{\epsilon\to 0}(S_{n-1}A(k\epsilon^{1-\frac{n}{2}}H^{(1)'}_{\frac{n}{2}-1}(k\epsilon)+(1-\frac{n}{2})\epsilon^{-\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(k\epsilon)\big)=-1 \tag{7.8}$$

We can use small argument asymptotic approximation for the Hankel function and its derivative. These are

$$H^{(1)}_{n/2-1}(k\epsilon)\sim -i\frac{\Gamma(n/2-1)}{\pi}\left(\frac{2}{k\epsilon}\right)^{n/2-1}$$

$$H^{(1)'}_{n/2-1}(k\epsilon)\sim i\frac{\Gamma(n/2-1)}{2\pi}\left(\frac{2}{k\epsilon}\right)^{n/2}$$

and plug into (7.8), we have

$$A=\frac{i}{4}\left(\frac{k}{2\pi}\right)^{n/2-1}$$

Thus, since in our case $B=0$ from outgoing condition, we have

$$G(r,r'=0)=Ar^{1-\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(kr)=\frac{i}{4}(\frac{k}{2\pi})^{n/2-1}r^{1-\frac{n}{2}}H^{(1)}_{\frac{n}{2}-1}(kr) \tag{7.9}$$

By shifting the point source to r', we have

$$G(r,r')=\frac{i}{4}(\frac{k}{2\pi|\vec{r}-\vec{r'}|})^{n/2-1}H^{(1)}_{\frac{n}{2}-1}(k|\vec{r}-\vec{r'}|) \tag{7.10}$$

For u defined on $\mathbb{R}^3$, the above consideration remains valid, except for the expression of the incoming solution is $u_{in}=Ae^{ik\cdot x}$ for $x\in\mathbb{R}^3$. We define the outgoing Green's function: if $|k|^2\notin\sigma(-\Delta)=[0,\infty)$, then the solution to (7.1) is:

$$u_{out}(x)=[(-\Delta-|k|^2)^{-1}(f(u_{\text{out}}+Ae^{ik\cdot x}))](x) \tag{7.11}$$

Plugging into the general expression for $n=3$, we have outgoing Green's function when $|k|^2\in(0,\infty)$ is

$$G_{\text{out}}(x;y;|k|^2)=\lim_{\epsilon\to 0^+}G(x;y;|k|^2+i\epsilon)=\frac{1}{4\pi|x-y|}e^{i|k||x-y|}. \tag{7.12}$$

We still first consider $V\equiv 0$ to simplify the calculation.

Theorem 7.1. *For any incoming solution $u_{in}=Ae^{ik\cdot x}$ with amplitude A small enough, there exist an outgoing solution satisfied (7.11) for $V\equiv 0$*

Proof. Consider $B_{\mathcal{L}^\infty(\mathbb{R}^3)}(-Ae^{ik\cdot x},r)$, we want to show that T is invariant on this ball with suitable r. Let $u\in B_{\mathcal{L}^\infty(\mathbb{R}^3)}(-Ae^{ik\cdot x},r)$, we have

$$\begin{aligned}\|T(u)\|_{\mathcal{L}^\infty}&=\operatorname*{ess\,sup}_{x\in\mathbb{R}}\left|\int_{\mathbb{R}^3}\frac{1}{4\pi|x-y|}e^{ik|x-y|}w(y)|u+Ae^{ik\cdot y}|^2(u+Ae^{ik\cdot y})dy^3\right|\\&\le\frac{1}{4\pi}\operatorname*{ess\,sup}_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}|w(y)||u+Ae^{ik\cdot y}|^3dy\\&\le\frac{1}{4\pi}r^3\operatorname*{ess\,sup}_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}|w(y)|dy\end{aligned}$$

Note that from the last line we have

$$\begin{aligned}\int_{\mathbb{R}^3}\frac{1}{|x-y|}|w(y)|dy&=\int_{\mathbb{R}^3}\frac{1}{|y|}|w(x-y)|dy\quad(\textit{change of variable})\\&=\int_{\mathbb{R}^3}\frac{1}{|y|}|w(x-|y|\hat{y})||y|^2d|y|d\hat{y}\quad(\textit{spherical coordinate})\\&=\int_{\mathbb{R}^3}|y||w(x-|y|\hat{y})|d|y|d\hat{y}\end{aligned}$$

where $\hat{y}=\frac{y}{|y|}$ is the unit vector. For calculation, we set $|y|:=r$ and we have explicit expression as

$$\int_{\mathbb{R}^3}\frac{1}{|x-y|}|w(y)|dy=\int_0^\infty\int_{S^2}r|w(x-r\hat{y})|d\hat{y}dr$$

Since we know that w is bounded and compactly supported, we consider $supp(W) \subseteq B_R(0)$ closed ball of radius R, and we have

$$\begin{aligned}\int_0^\infty \int_{S^2} r|w(x-r\hat{y})|d\hat{y}dr &\leq \|w\|_\infty \int_0^R \int_{S^2} rd\hat{y}dr \\ &= 4\pi\|w\|_\infty \int_0^R rdr = 2\pi\|w\|_\infty R^2\end{aligned}$$

plugging back to our original expression, we have

$$\|T(u)\|_{\mathcal{L}^\infty} \leq \frac{R^2}{2}\|w\|_\infty r^3 \tag{7.13}$$

To make T is invariant, by above consideration, we require

$$\frac{R^2}{2}\|w\|_\infty r^3 + |A| \leq r \tag{7.14}$$

To make T an contraction map, we consider $u, v \in B_{\mathcal{L}^\infty(\mathbb{R}^3)}(-Ae^{ik\cdot x}, r)$, and again we set

$$\alpha(x) := u(x) + Ae^{ik\cdot x}, \quad \beta(x) := v(x) + Ae^{ik\cdot x} \tag{7.15}$$

for simplication. And we have

$$\begin{aligned}\|T(u) - T(v)\|_{\mathcal{L}^\infty} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} |\int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} e^{i|k||x-y|} w(y)(|u + Ae^{ik\cdot y}|^2(u + Ae^{ik\cdot y}) \\ &+ |v + Ae^{iky}|^2(v + Ae^{iky}))dy| \\ &\leq \frac{1}{4\pi}\||\alpha|^2\alpha - |\beta|^2\beta\|_{\mathcal{L}^\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}^3} \int_{\mathbb{R}} \frac{|w(y)|}{|x-y|}dy \\ &\leq \frac{3}{2}(Rr)^2\|w\|_\infty\|u-v\|_{\mathcal{L}^\infty}\end{aligned}$$

Thus to make T an contraction mapping, we need

$$\frac{3}{2}(Rr)^2\|w\|_\infty < 1 \tag{7.16}$$

which is equivalent to require

$$r < \sqrt{\frac{2}{3R^2\|w\|_\infty}} \tag{7.17}$$

For suitable $|A| \neq 0$ such that there exists a r satisfies (7.30) and (7.23), the contraction mapping gives us a non-trivial outgoing solution, since if $u_{out} = 0$, we have $T(u_{out}) \neq u_{out} = 0$ which is impossible. Also, the global solution $u = u_{in} + u_{out}$ is also nontrivial, since if $u = 0$, we will have $u_{out} = Ae^{ik\cdot x}$ which is not a fixed point for the same reason.

□

Remark 7.1. *We still need to verify that condition (7.30) and (7.23) will coexist for a positive real r, we need to determine whether it can be satisfied for $|A|$ sufficiently small.*

Corollary 7.1. *For $|A| \leq \sqrt{\frac{8}{27(R^2\|w\|_\infty)}}$, there exists $r \in \mathbb{R}^+$ such that condition (7.30) and (7.23) hold.*

Proof. This is a direct application of the Theorem 5.3. where we have the equation

$$r^3 - \frac{2}{R^2\|w\|_\infty}r + \frac{2|A|}{R^2\|w\|_\infty} \tag{7.18}$$

by setting

$$p := -\frac{2}{R^2\|w\|_\infty} \quad \textit{and} \quad q = \frac{2|A|}{R^2\|w\|_\infty} \tag{7.19}$$

Again, if we apply the constraint (7.30), and notice that $\cos\left(\frac{\theta}{3} - \frac{2\pi}{3}\right) < \frac{1}{2}$, then the second largest root $x_1 = 2R^{\frac{1}{3}}\cos\left(\frac{\theta}{3} - \frac{2\pi}{3}\right)$ must satisfy

$$x_1 < R^{\frac{1}{3}} \leq \sqrt{\frac{2}{3R^2\|w\|_\infty}}.$$

so there exists an r to have a contraction. $\square$

We now proceed the case for $V \not\equiv 0$ and V, w are bounded and compactly supported as before.

Corollary 7.2. *For any incoming solution $u_{in} = Ae^{ik\cdot x}$ with amplitude A small enough, there exist an outgoing solution satisfied (7.11) for $V \not\equiv 0$*

Proof. We can directly obtain from Theorem 7.1 that for $u \in B_{\mathcal{L}^\infty(\mathbb{R}^3)}$, we have

$$\|T(u)\|_{\mathcal{L}^\infty} \leq \frac{R^2}{2}\|w\|_\infty r^3 + \frac{R^2}{2}\|V\|_\infty r \tag{7.20}$$

so that we need

$$\frac{R^2}{2}\|w\|_\infty r^3 + \frac{R^2}{2}\|V\|_\infty r + |A| \leq r \tag{7.21}$$

Moreover, for $u, v \in B_{\mathcal{L}^\infty(\mathbb{R}^3)}(-Ae^{ik\cdot x}, r)$, we have

$$\|T(u) - T(v)\|\mathcal{L}^\infty \leq (\frac{R^2\|V\|_\infty}{2} + \frac{3(rR)^2\|w\|_\infty}{2})\|u - v\|_{\mathcal{L}^\infty} \tag{7.22}$$

To have T contraction, we require

$$\frac{R^2\|V\|_\infty}{2} + \frac{3(rR)^2\|w\|_\infty}{2} < 1 \tag{7.23}$$

Thus, as shown above, for $|A|$ sufficiently small such that there exists positive r satisfied (7.36) and (7.34), we will have a unique non-trivial outgoing solution u_{out}. $\square$

8. SUMMARY

We have successfully shown the existence of a solution to the nonlinear Schrödinger equations in 1D with a fast decay potential and in 3D with compactly supported potential using fixed point arguments. The solutions are not trivial, as $u_{out} = -u_{in}$ is not a fixed point. We also notice that the approximation for the scattering matrix is well posed in 1D. However, for higher-dimensional scatterings, it is hard to get an explicit expression or approximation. (Future Work)

Since the limit absorption principle in 3D with fast decay potential is checked in section 3, the fix point argument should also work with only difficulties in finding a suitable bound for the singular integral and checking for the positive real r. My current work also focus on finding this bound with techniques in harmonic analysis by changing the integral into series form, or finding a good estimation by Hardy-Littlewood Maximal Function and relating the singularity to averages or maximal functions.

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