Applied Functional Analysis

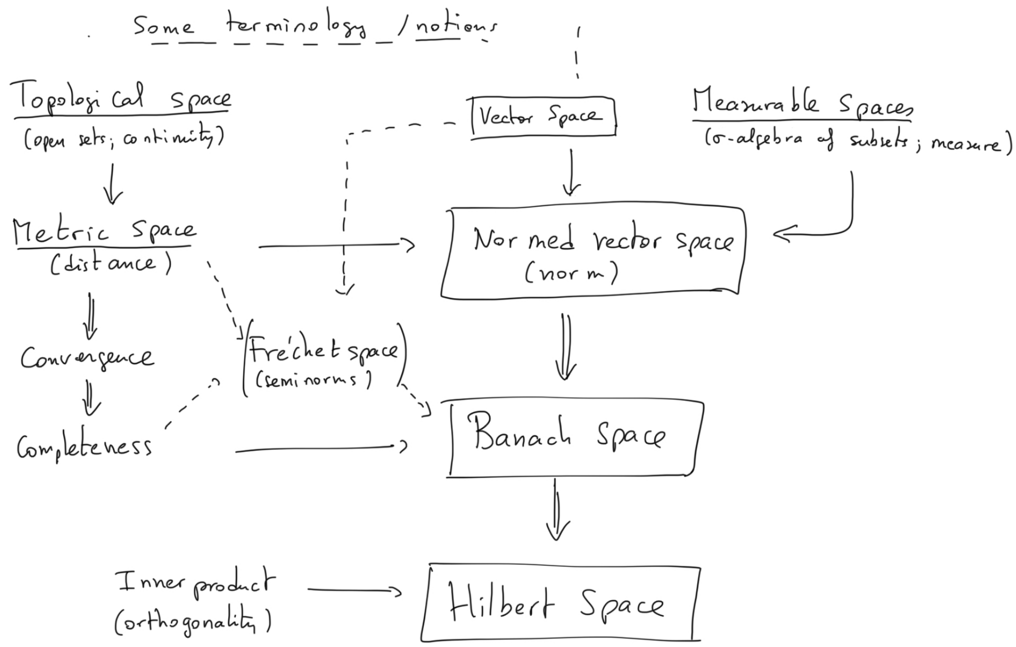
Russell Hua

Jan 2024

1 Classnote 1/3/2024

Topological space (Chapter 4)

Big pic:



Def 1.1: A topology is a collection τ of open subsets of X such that:

- $\emptyset$ and X are open;
- **Family of open sets** that closed under arbitray union and finite intersections.

We call the pair (X, τ) a topology and **def A is closed** $\iff$ $A^c = X - A$ is open.

Note: Topology here plays an role just like algebra!

Def 1.2: V is a **neighborhood** of x if for some open set G we have $x \in G \subset V \subset X$.

Def 1.3: τ is **Hausdorff(or seperated)** if $\forall x$, y distinct, there $\exists V_x$ and V_y, nbhds of x and y, such that $V_x \cap V_y = \emptyset$

Example 1.4: some topology spaces,

- Discrete topology: every point is open, i.e. $\tau = P(X) = 2^X$, which is too rich!
- Trivial topology: $\emptyset$ and X are open open set, too small! (it is not Hausdorff if X has ≥ 2 elements.
- Generated topology by τ_0: $\tau_0 = \bigcap_{\alpha \in I} \tau_\alpha$, the smallest topology, you can think of it as the Borel sigma algebra for open sets.

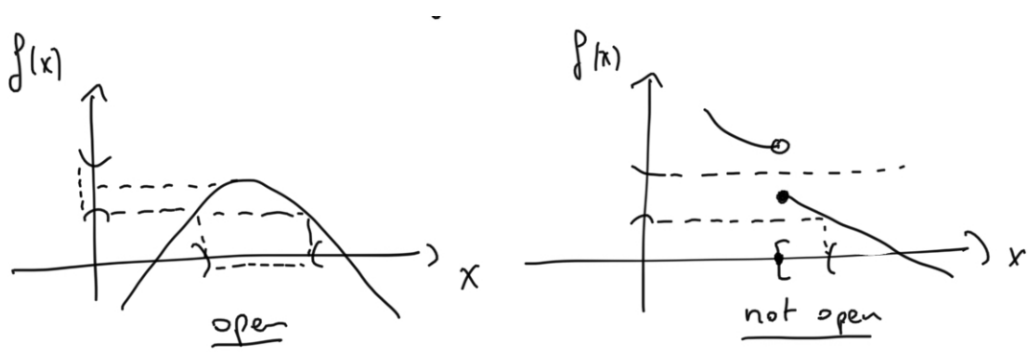
Def 1.5 (Convergence): $x_n \to x \in X$ if for all nbhd V_x of x, $x_n \in V_x$ for n large enough.

Def 1.6 (Continuity): $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous at x if for all nbhd $W_{f(x)}$ of $f(x)$, there is an nbhd V_x of x s.t $f(V(x)) \subset W_{f(x)}$.

Theorem 1.7:

$$f : (X, \tau_X) \to (Y, \tau_Y) \text{ continuous iff } f^{-1}(G) \in \tau_x \text{ for all G} \in \tau_y.$$

i.e. f^{-1}(open) is open.



Def 1.8 Compactness: $K \subset X$ is compact, if every open cover of K admits a finite sub-cover.

Example 1.9:

- On $\mathbf{R}$, let τ be the topology generated by open interval (a, b), then $(0, 1)$ is not compact, since

  $$(0,1) = \bigcup_{n \geq 3}^{\infty} (\frac{1}{n}, 1 - \frac{1}{n})$$

  which exists no finite subconver.
- Alternatively, for sequences with limit 1, the $(0, 1)$ does not contain its limit point.

- $[0,1]$ is compact by Heine-Borel theorem.
- $[0,1]^5$ is compact, however, $[0,1]^\infty$ is not compact by the incompleteness of infinity Euclidean space.

Def 1.10: A metric on X is $d : X \times X \to \mathbf{R}$ such that

1. $d(x,y) = d(y,x)$ symmetricity;
2. $d(x,z) \leq d(x,y) + d(y,z)$ trangular inequality;
3. $d(x,y) \geq 0$;
4. $d(x,y) = 0 \iff x = y$.

Example 1.11:

- On R, $|x - y| = d(x,y)$ is the usual Euclidean distance.
- For Cartesian product, we may define the L^1 **norma** as

$$d_{X\times Y}((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2)$$

Def 1.12: The **natural topology** on a metric space is the topology τ generated by open balls $B_\epsilon(x) = \{y \in X | d(x,y) < \epsilon\}$; for closed ball denoted as $\overline{B_\epsilon(x)}$.

Def 1.13: A vector space V over a field F is the space under 8 rules, in short:

1. $(V,+)$ is abelian group;
2. $\times$ is a multiplication $\lambda f \in V$

Example 1.14:

- R^n;
- For $f : (0,1) \to R$, the $L^2(0,1)$ with

$$\int_0^1 |f|^2 dx < \infty;$$

- $\mathbf{C^0}[0,1]$ continuous function with compact supp$[0,1]$;
- The space of bounded operators;
- limit sphere: $\{x \in R^N, |x| = (\sum_{i=1}^N {x_i}^2)^{\frac{1}{2}} = 1\}$ is not a vector space instead a metric space.

Note: Vector space usually a topological or metrical, so it is rich in structure!

Def 1.15: Norm $\|\cdot\| : V \to R$ is a function which satisfies:

1. $\|x\| \geq 0$;
2. $\|\lambda x\| = |\lambda| \|x\|$ for $\lambda \in F$;
3. $\|x + y\| \leq \|x\| + \|y\|$;
4. $\|x\| = 0 \Rightarrow x = 0$

Notes:

- $(V, \|\cdot\|)$ is normed vector space of V vector space;
- On $(V, \|\cdot\|), d(x, y) = \|x - y\|$ makes (V, d) a metric space;
- The precondition for the above two is V is finite dimensional; For infinite dimension V, no conclusion can draw.
- $\overline{B} = \{x \in X, \|x\| \leq 1\}$ and B $= \{x \in X, \|x\| < 1\}$
- **C is convex subset** when $x, y \in C \Rightarrow tx + (1-t)y \in C$
- $\|\cdot\|$ and $|||\cdot|||$ are **equivalent** if $\exists C > 0$ s.t $C^{-1}\|\cdot\| \leq |||\cdot||| \leq C\|\cdot\|$;
- We can certainly proof that $\|\cdot\|_p$ norms are equivalent on R^n

Def 1.16 (Convergence in metric): $\{x_n\}_{n \in N} \in X$ converges to x if $\forall \epsilon > 0, \exists N \in \mathbf{N}$ s.t $\forall n > N$, $d(x_n, x) \leq \epsilon$.

Cauchy in metric sense $\Rightarrow$ C.V in topological sense.

Def 1.17: sequence $\{x_n\}_{n \in N} \in X$ is **Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbf{N}$ s.t $\forall n, m > N$, $d(x_n, x_m) \leq \epsilon$.

Note:

- Clearly, convergence implies Cauchy;
- However, Cauchy not always implies convergence. Given in Q,

$$x_n = \frac{p_n}{q_n} \to \sqrt{2} \notin Q$$

  x_n is Cauchy in Q, but not converges in Q.

Def 1.18: A metric space is complete $\iff$ all its Cauchy sequence converges.

Notes:

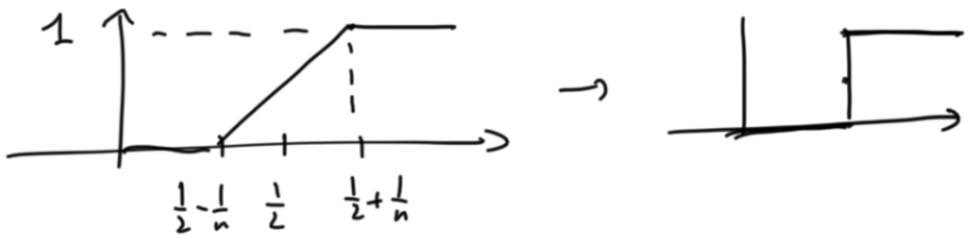
1. **A normed vector space that is complete is called a Banach Space;**
2. $X \in R^N$, then $C^0(X); C^{k,\alpha}(X); L^p(X); W^{m,p}(X)$ are Banach Spaces;
3. $C^\infty(X) = \bigcap_{k \geq 1} C^k(X); \phi(R^N); \phi'(R^N)$ are Frechet spaces (Not Banach).

1.19 Theorem: Every metric space (X,d) has a completion $(\bar{X}, \bar{d})$ such that $d(x,y) = \bar{d}(x,y)$ for $x, y \in X$ and X is dense in $\bar{X}$,

$$\textbf{Dense means } \forall \bar{x} \in \bar{X}, \forall \epsilon > 0, \exists x, \text{s.t, } \bar{d}(x, \bar{x}) \le \epsilon$$

Example 1.20: $(C[0,1], \|\cdot\|_2)$ is normed vector space but not complete:

1. Cauchy sequence of $\|\cdot\|_2$ but limit not in C.



2. Completion of $(C[0,1], \|\cdot\|_2)$ is $(L^2(0,1), \|\cdot\|_2)$;
3. In general, Completion of $(C[0,1], \|\cdot\|_p)$ is $(L^p(0,1), \|\cdot\|_p)$ for p finite.
4. For p inifite, $(C[0,1], \|\cdot\|_\infty)$ is Banach with uniform norm.

Def 1.21: A function is continuous...

1. continuous at x_0 if $\exists \delta(\epsilon)$ such that $d_X(x, x_0) \le \delta$ implies $d_Y(f(x_0), f(x)) < \epsilon$;
2. uniformly continuous if δ does not depend on x_0;
3. sequentially continuous at x if $x_n \to x \Rightarrow f(x_n) \to f(x)$.

Note:

1. f is continuous $\iff$ f is sequentially continuous.
2. For $F \subset X$ is closed $\iff$ for all $x_n \to x$, we have $x \in F$. i.e. Contain all its limit points!

Def 1.22:

- The **closure of A** is $\bar{A} := \{x \in X | \forall x_n \in A, x_n \to x\}$.
- $A \subset X$ is **dense** in X if $\bar{A} = X$.
- A subset is **separable** if it has a countable dense subset.

Def 1.23: A space is **sequentially compact** if every sequence in K admits a converging sub-sequence in K.

Theorem 1.24: $K \subset X$ in a metric space, K is compact $\iff$ K is sequentially compact. (We call a set pre-compact if its closure is compact!)

Theorem 1.25: K is compact $\Rightarrow$ K is boundedd and closed.

Theorem 1.26(Heine Borel): Subset of R^n are compact iff they are closed and bounded.

Theorem 1.27(Bolzano-Weierstrass): Bounded sequence of R^n admits a convergent subsequence.

Def 1.28: For abstract definitions for arbitary compact subset of metric spaces:

1. $\{G_\alpha, \alpha \in I\}$ is **cover** of A if $A \subset \bigcup_{\alpha \in I} G_\alpha$.
2. $\{\chi_\alpha, \alpha \in I\}$ is ϵ**-net** of A if $A \subset \bigcup_{\alpha \in I} B_\epsilon(\chi_\alpha)$.
3. $A \subset X$ is **totally bounded** if it has finite ϵ-net for all $\epsilon > 0$

Theorem 1.29: $A \subset X$ is sequentially compact iff it is compact and totally bounded.

2 Classnote 1/8/2024

2.1 Review

Def 0.1.(Normed vector space): V over $\mathcal{F} = \mathbf{R}$ or $\mathbf{C}$ with $\|\cdot\|$ is called Normed VSP if it agrees with the normed def (4 properties).

Note 0.2.Given norm $\|\cdot\| \to d(x,y) = \|x-y\|$ with topology generated by $B_\epsilon(x) = \{y \in V, d(x,y) < \epsilon\}$. i.e the intersection of all such topology, we get an metric.

Def 0.3.(Cauchy sequence) $\|x_n - x_m\| \xrightarrow{n,m\to\infty} 0$

Def 0.4.(Banach Space): Given normed vector space $(V, \|\cdot\|)$ with all Cauchy sequence converges $\Rightarrow$ Banach Space (The second condition implies complete)

Example 0.5.$(C_{[0,1]}, \|\cdot\|_1)$ a normed vsp where $\|f\|_1 = \int_0^1 |f| dx$, is same for Example 1.20 with completion equals to $L^1_{(0,1)}$.

2.2 Continuity

Def 1.1. $f : X \to Y$, and two metric space (X, d_x) and (Y, d_y), is continous at $x_0 \in X$ if $\epsilon > 0, \exists \delta = \delta(\epsilon, x_0)$ s.t $d_x(x,y) < \delta \to d_y(f(x), f(y)) < \epsilon$.

Def 1.2. (Sequential continuous): $x_n \to x \Rightarrow f(x_n) \to f(x)$.

Prop 1.3. $\rho : X \to Y$ is continuous iif ρ is sequentially continuous.

Prop 1.4. $F \subset X$ is closed iif $(x_n \to x) \Rightarrow (x \in F)$

Def 1.5.(Closure): $\overline{A}$ is the smallest closed set containing A

$$= \{x \in X, \exists x_n \in A, x_n \to x\}$$

Def 1.6.(Density): $A \subset X$ is dense when $\overline{A} = X$ e.g.($\overline{Q} = R$)

Def 1.7.(Separable): (X, d) is separable if it has a countable many dense set.

2.3 Compactness

Def 2.1. $K \in X$ is sequentially compact if every sequence in K $(x_n \in K)$ admits a convergent subsequence. e.g: $x_n = (-1)^n$

Def 2.2.(subsequence): $\phi : N^* \to N^*, \phi(N+1) \geq \phi(N) + 1$.

Them 2.3. If $K \subset X$, a metric sp, is compact iif it is sequentially compact.

Note 2.4.

1. A precompact when $\overline{A}$ is compact
2. e.g: $K \subset X = \mathbf{R}$
   - K is unbounded $\Rightarrow$ not compact;
   - $(0, 1]$ is not compact, but $[0, 1]$ is compact.

Prop 2.5. K is compact $\Rightarrow$ K is closed and bounded.

Prop 2.6.(Heine-Borel): Subsets in R^n are compact iif it is bounded and closed.

Prop 2.7.(Bolzano-Weistrass): Every bounded sequence admits a converges subsequence.

Note 2.8. (H.B) does not apply to R^∞ consider example of Banach space:

$$l^2 = \{x = (x_n)_{n \in N^*}, (\sum_n^\infty (x_n)^2)^{\frac{1}{2}} < \infty\}$$

(e^k) is the basis for l^2 w/o $e^k_j = \delta_{j,k}$ and we certainly have $\|e^k\|_{l^2} = 1$. If we set $K := \overline{B}_{(0,1)}$ and $x_k = e^k$, then

$$\|x_k - x_{k+m}\| = \sqrt{2}$$

which implies it is (x_n) is not Cauchy, thus can not be compact.

2.4 Abstract results

Def 3.1. $\{G_a, a \in A\}$ is called cover of A if $A \subset \bigcup_{a\in A} G_a$

Def 3.2. $\{x_a, a \in A\}$ is ϵ-net of A if $A \subset \bigcup_{a\in A} B_\epsilon(x_a)$

Def 3.3. A is totally bounded if A has finite ϵ-net for any $\epsilon > 0$

Them 3.4. $A \subset X$ is sequentially compact iif it is complete + totally bounded.

Them 3.5. $f : K \to Y$ is continuous w/o K compact $\Rightarrow$ $f(K)$ is compact. (Proof by seq-compact)

Them 3.6. Let K be compact metric space, $f : K \to Y$ is continuous, then f attains its max and min by

$$\sup_{x\in K} |f(x)| = \max_{x\in K} |f(x)|$$

and same for $\inf_{x\in K} |f(x)| = \min_{x\in K} |f(x)|$.

3 Classnote 1/10/2024

3.1 Continuous function on metric space

Given that (X, d) metric space $f : X \to \mathcal{R}$ is continuous.

1. $(X, \|\cdot\|_2$ or $\|\cdot\|_p)$ is not strong enough to preserve completeness of its image.
2. $(X, \|\cdot\|)$ uniform norm can address the above problem:
   - e.g. $\|f\| = \|f\|_\infty = \sup_{x\in X} |f(x)|$
   - f_n C.V uniformly when $\|f - f_n\| \to 0$

Theorem 1.1. Let f_n be sequence bounded and continuous functions, and $\|f - f_n\| \to 0$. Then f is continuous.

Proof: Since $\|f - f_n\| \to 0$, there exists $N >> 1$ such that $n > N \Rightarrow \|f - f_n\| \leq \frac{\epsilon}{3}$

Consider $|f(x) - f(y)|$ for $x, y \in X$,

$$|f(x)-f(y)| = |f(x)-f_n(x)|+|f_n(x)-f_n(y)|+|f_n(y)-f(y)| \leq \frac{2\epsilon}{3}+|f_n(x)-f_n(y)|$$

And by continuity of the f_n, we have for $n >> 1$, there exists $\delta = \delta(x, \epsilon)$ such that $d(x, y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$. Thus we have proved our result. □

Example 1.2. Suppose the following holds:

- $f : K \to \mathcal{R}$ is cont. and bounded;
- $\|f\|$ is defined as the unif-norm;
- $K \subset X$ is compact;
- $\mathcal{C}(K, \|\cdot\|)$ is normed vector space.

Then $(\mathcal{C}(K), \|\cdot\|)$ is Banach space.

Proof: (WTS: every cauchy sequence converges in $\mathcal{C}(K)$)
Let $(f_n)_{n\geq 1}$ be cauchy in $(\mathcal{C}(K), \|\cdot\|)$

$$\|f_n - f_m\| < \epsilon \text{ for } n, m > N \tag{1}$$

$$\sup_{x\in K} |f_n(x) - f_m(x)| < \epsilon \text{ for } n, m >> 1 \tag{2}$$

For fixed x, f_n continuous w/ K compact, by Them 3.5(pp.10). we have that $f(K)$ is compact. Thus, $f_n \to f \in f(K) \subset \mathcal{R}$.
We have two things remain to check:

1. $f \in \mathcal{C}(K)$?

2. $\|f - f_n\| \to 0$?

Check:

1. Partial right direction:

$$\sup_{x\in K} \|f(x) - f_n(x)\| = \sup_{x\in K} \lim_{m\to\infty} |f_m(x) - f_n(x)| \tag{3}$$
$$\leq \lim_{m\to\infty} \sup_{x\in K} |f_m(x) - f_n(x)| < \epsilon \text{ for } n >> 1 \tag{4}$$

2. Right direction:

$$\sup_{x\in K} \|f(x) - f_n(x)\| = \sup_{x\in K} \liminf_{m\to\infty} |f_m(x) - f_n(x)| \tag{5}$$
$$\leq \liminf_{m\to\infty} \sup_{x\in K} |f_m(x) - f_n(x)| < \epsilon \text{ for } n >> 1 \tag{6}$$

3. Since the unif-norm preserves continuity by Them 1.1(pp.10), we have that $\|f(x) - f_n(x)\| \to 0$ and $f_n : K \to \mathcal{R}$ is cont. and bounded, then $\lim_n f_n(x) := f(x) \in C(K)$

4. Notes: Something remarkable here is that $\sup_{x\in K} \liminf_n \leq \liminf_n \sup_{x\in K}$ holds in general, but $\sup_{x\in K} \lim_n \leq \lim_n \sup_{x\in K}$ does not hold in some cases that lim does not exists! One can think of lim inf are the greatest lower bounds (steady-state), taking sup we are finding the long-term lower bounded over K. And taking liminf after using sup, we're determining the lowest point that the peaks of the sequences eventually settle down to.

3.2 Weierstrass first approximation

Def 2.1. support of f $supp(f) = \overline{\{x \in X, f(x) \neq 0\}}$

Def 2.2. $\mathcal{C}_c(X) = \{$Continuous functions with compact-supp$\}$

Def 2.3. $\mathcal{C}_0(X) = \overline{\mathcal{C}_c(X)} = \{$Continuous functions on X such that $f \to 0$ at $\infty\}$

Def 2.4. $\mathcal{C}_b(X) = \{$Bounded continuous functions on X$\}$

In general,
$$\mathcal{C}_c(X) \subset \mathcal{C}_0(X) \subset \mathcal{C}_b(X) \subset \mathcal{C}(X)$$

1. $\mathcal{C}_c(X)$ is not complete, and thus not Banach (Too small to hold)

2. $\mathcal{C}_0(X)$ and $\mathcal{C}_b(X)$ are Banach

3. $\mathcal{C}(X)$ is not a normed-vp for the infinite norm and thus not Banach. (Too rich)

Theorem 2.5.(Weistrass): Polynomials are dense in $\mathcal{C}([a,b], \|\cdot\|)$.

Sketch of the proof:

- Firstly we need to construct a mapping from $[a,b] \to [0,1]$ and we only focus on $\mathcal{C}[0,1]$ (by change of variable)
- Consider Bernstein polynomials:

$$B_n(x;f) = \sum_{k=0}^{n} f(\frac{k}{n})x^k(1-x)^{n-k}\binom{n}{k} \in P[x]$$

where we can check,

$$B_n(\frac{k}{n};f) = f(\frac{k}{n})$$

and

$$\|B_n(\cdot;f) - f\| \leq \epsilon + \frac{\|f\|}{2n\delta^2} < 2\epsilon$$

for n large enough.

3.3 Ascolli-Arzela Theorem

In many metric spaces H.B(Heine-Borel) failed due to the incompleteness of the metric space, even when space is complete, infinite-dim Banach space also failed.

Def 3.1. $\mathcal{F} := \{$the family of continuous function$\}$, $f : (X, dx) \to (Y, dy)$ is called equicontinuous if,

$\forall x \in X, \forall \epsilon > 0, \exists \delta := \delta(\epsilon, x)$ such that $d_x(x,y) < \delta \Rightarrow d_y(f(x), f(y)) < \epsilon$ for $\forall f \in \mathcal{F}$

Note that cont. on compact set $\Rightarrow$ unif-cont. where,

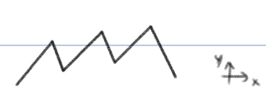
$$\text{Cont.: } \delta(\epsilon, x, \delta) \tag{7}$$

$$\text{Unif cont.: } \delta(\epsilon, \delta) \tag{8}$$

Theorem 3.2. (Ascolli-Arzela): Given K compact, a subset $(\mathcal{C}(K), \|\cdot\|)$ is compact iif it's closed, bounded, and equicontinous.

Def 3.3. $f : X \to \mathbf{R}$ is Lipschitz cont. if $\exists c$ s.t $|f(x) - f(y)| \leq cd_x(x-y)$ (i.e. $\delta = \frac{\epsilon}{c}$, linear relation between $\epsilon - \delta$)

"bounded slope c"

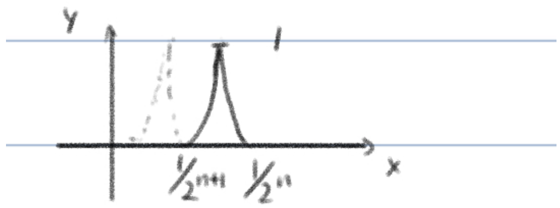


Def 3.4. $Lip(f) = \sup_{x \neq y} \frac{|f(x)-f(y)|}{d_x(x,y)} < \infty$

Prop 3.5. If $\mathcal{F}_n$:={f cont. and lipschitz $lip(f) \leq n$}, then $\mathcal{F}_n$ is equi-continuous

Example 3.6. From (Ascolli-Arzela), we require a 3 properties:

1. Violation of equiconti.: $\mathcal{F}_n = \{f_n, n \in N\}$



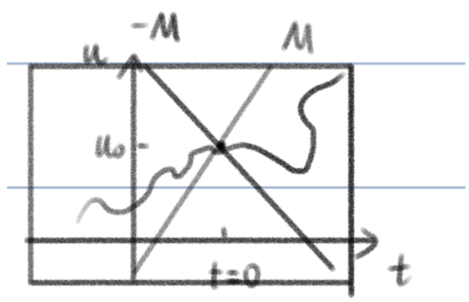
   $f_n \to 0 \forall x \in [0,1]$, but f_n is not cauchy(not in unif-c.v) for $\|f_n - f\| = 1$ thus not equicontiunous $\Rightarrow$ not compact.

Applications: (K, d_x) metric $\to (Y, d_y)$ complete metric w/ $\mathcal{C}(K,Y) : d(f,g) = \sup_{x \in K} d_y(f(x), g(x)) \Rightarrow (\mathcal{C}, d)$ is compact metric space $\iff$ closed + bounded + equi-conti.

3.4 Applications to ODE(Peano-construction)

$$\dot{u}(t) = f(t, u(t)) \; t \in I.0 \in I \tag{9}$$
$$u(0) = u_0 \tag{10}$$

f is cont. on $R \times R \to R$ / $R \times R^n \to R^n$ w/ $|f(t,u)| \leq M$



Assume that $|f(t,u)| \leq M$ in box around $(0, u_0)$, we want to get uniqueness and existence of solution by the strategies:

- Construction approximation;

- Pass to limit;
- Check ODE

(i):

1. Since f is cont., we can use MVT to derivive that

$$\dot{u}(t) = f(t, u(t)) \approx \frac{u(t+h) - u(t)}{h} \Rightarrow u(t+h) \approx u(t) + hf(t, u)$$

2. Let $t = kh$, then we have $u_{k,h} = u_{h,k-1} + hf((k-1)h, u_{k-1})$ series of discrete solutions.

3. interpolate pinearly, we get $u_h(t) = \lim_k u_{k,h}$

4. Clearly, $u_h(t)$ is continuous and for any k, we have slope is bounded by M implies u_h is bounded. Combined with its closeness, by Ascoli-Arzela, we have $\{u_h\}$ is compact, and thus converges to $u_{\phi(h)}(t) \to u(t) \in C(I_1)$ for $0 \in I_1$

5. We want to check whether $u(t)$ we defined solve ODE, but we only have the information from $u_h(t)$, we want to show:

$$u(t) = u_0 + \int_0^t f(s, u(s))ds$$

6. transform our $u_h(t)$ in the same manner,

$$u_h(t) - \frac{u_0}{u_h(0)} = \int_0^t \dot{u}_h(s)ds = \int_0^t \dot{f}(s, u_h(s))ds + \int_0^t \dot{u}_h(s) - f(s, u_h(s))ds$$

where LHS converges to $u(t) - u_0$ and the last term converges to 0 as h is small, we have

$$u(t) - u_0 = \int_0^t \dot{f}(s, u_h(s))ds + r_n(t) \to \int_0^t \dot{f}(s, u(s))ds$$

Note that we only get the existence of the solution, but not uniqueness (hard to show)!!
Consider the example:

$$u(t) = t^2 \text{ for } t > 0; \dot{u} = 2t = 2\sqrt{u(t)} = f(t, u)$$

$f(t, u)$ is continuous certainly, along with $u(0) = 0$, we also have $u = 0$ is another solution (uniqueness breakout!)

Thus we may consider more constraint on f (lipschtiz continuous):
Let u, v be solutions to the system:

$$\dot{u}(t) = f(t, u(t)) \ t \in I \tag{11}$$
$$u(0) = u_0 \tag{12}$$

we basiclly want to show whether $w := u - v = 0$, with

$$\begin{cases} \dot{w}(t) & = f(t, u(t)) - f(t, v(t)) \\ w(0) & = 0 \end{cases}$$

we have that

$$w(t) = \int_0^t \dot{w}(s)ds = \int_0^t f(s, u(s)) - f(s, v(s))ds$$

$$\Rightarrow |w(t)| \leq \int_0^t |f(s, u(s)) - f(s, v(s))|ds \tag{13}$$
$$\leq \int_0^t M|u(s) - v(s)|ds = \int_0^t M|w(s)|ds \tag{14}$$

Given that f is M-lips, and by Gronwall's Lemma $\Rightarrow w(s) \equiv 0$

Lemma 1.1.(Gronwall's inequality):
Define $h_\epsilon(t) = \epsilon + \int_0^t |w(s)|ds > 0$;

$$\dot{h}_\epsilon(t) = |w(s)| \leq M \int_0^t |w(s)|ds \leq M h_\epsilon(t) \tag{15}$$

By Gronwall's inequality we have

$$h_\epsilon(t) \leq \epsilon e^{\int_0^t M ds} = \epsilon e^{Mt} \Rightarrow |w(s)| \leq \epsilon M e^{Mt} \Rightarrow |w(s)| \to 0$$

4 Classnote 1/17/2024

Def 1.1. (X,d) complete metric space, $T : X \to X$ is a contraction mapping if $\exists 0 < c < 1$ s.t
$$d(T(x), T(y)) \leq cd(x,y)$$
for any $x, y \in X$.

Note 1.2.

- T may be nonlinar, the definition states: $T(B_r(x)) \subset B_{cr}(T(x))$.
- X is not necessary a vector space; often refer to a ball in a vector space
- Them 1.3.(Contraction mappings): $T : X \to X$ contraction mapping on X complete. Then $T(x) = x$ admits a unique solution!
- Proof: Let $x_0 \in X$, construct $x_{n+1} = T(x_n)$ for $n \geq 0$ for each $x_n \in X$.
  1. we want to show that x_n is Cauchy:
  $$d(x_{n+1}, x_n) \leq cd(x_n, x_{n-1}) \leq ... \leq c^n d(x_1, x_0)$$
  and
  $$d(x_{n+m}, x_n) \leq d(x_{n+m}, x_{n+m-1}) + ... + d(x_{n+1}, x_n) \tag{16}$$
  $$\leq (c^{n+m} + ... + c^n) d(x_1, x_0) \leq \frac{c^n}{1-c} d(x_1, x_0) \tag{17}$$
  thus we have $d(x_{n+m}, x_n) \to 0$ as $n \to \infty \Rightarrow x_n \to x \in X$(X complete)
  2. T is c-lipschtiz $\Rightarrow$ continuous, so $x_{n+1} = T(x_n) \to T(x) = x$
  3. (uniqueness):
  $$\begin{cases} T(x) & = x \\ T(y) & = y \end{cases} 0 \leq d(T(x), T(y)) \leq cd(x,y) = cd(T(x), T(y)) \Rightarrow d(T(x), T(y)) = 0$$
  since $0 < c < 1$.
- Comtraction mapping is useful to handle "small" perturbation:
  $$f(x) = g(x) + \int_0^b k(x,y) f(y) dy$$
  where $(Kf)(x) := \int_0^b k(x,y) f(y) dy$ for $g \in C[a,b], k \in C([a,b]^2)$
  Find $f \in C([a,b])$ reset above, $f = g + Kf$.

Show $K : C([a,b]) \to C([a,b])$. $(I-K)f = g$; $f = (I-K)^{-1}g$?
We want to write f as the solution of $f = T(f)$, given which we have

$$d(T(f),T(h)) = \sup_x \|T(f) - T(h)\| = \sup_x |\int_a^b k(x,y)(f(y)-h(y))dy| \tag{18}$$

$$\leq (\sup_x \int_a^b |k(x,y)|dy)\|f-h\|_\infty \tag{19}$$

for which if we assume that $(\sup_x \int_a^b |k(x,y)|dy) = c < 1$ then we know that there's unique solution, since each time we get a cont. function.
Our question can be turned into: suppose we have

$$(A+B)f = h$$

then it implies

$$(I + A^{-1}B)f = A^{-1}h = g \Rightarrow f + A^{-1}Bf = g$$

which is our original function. Illustrating this in that way, let's write $f = g + Kf$, which mean $f = (I-K)^{-1}g$ and that the matrix has an inverse because K is small (Let B be sufficicent small), now by geometric expansion

$$f = (I-K)^{-1}g = \sum_{n=1}^{\infty} K^n g$$

which is converges series since

$$\|K^n g\|_\infty \leq c^n \|g\|_\infty \to \text{ convergent sum in sup-norm}$$

Related to the proof of theorem:

$$f_0 = g, f_{n+1} = Tf_n = g + Kf_n = g + Kg + K\phi_{n-1} = \sum_{k=0}^{n+1} K^k g$$

Thus $f_n = \sum_{k=0}^{n} K^k g \to \sum_{k=0}^{\infty} K^k g = f \equiv (I-K)^{-1}g$ (Neumann series expansion)

Application to ODE system:

$$\begin{cases} \dot{u}(t) & = f(t,u(t)) \\ u(0) & = u_0 \end{cases}$$

for $t > 0$, we can come up with another way for showing unique solution from Banach contraction theorem.

Them 1.4. (Picard-Lindelof) If f is lipschitz w.r.t u, then there exist a unique solution to the above system of ODE.

Proof: The idea is nothing but the non-linear cases:
Let,

$$u(t) = u(t_0) + \int_{t_0}^{t} f(s, u(s))ds := T(u)(t)$$

We try to show that T is a contraction on $C(I_\delta)$ where $I_\delta = [t_0 - \delta, t_0 + \delta]$, from the expression, T is integral of an continuous function, thus it is C^1, so self-mapping.
Now we have

$$\|T(u) - T(v)\|_\infty = \sup_{t\in I_\delta} |\int_{t_0}^{t} f(s, u(s)) - f(s, v(s))ds| \leq \delta L \|u - v\|_\infty$$

and since $\delta > 0$ can be arbitary, we have $c = \delta L \in (0, 1)$

5 Classnote 1/22/2024

Def 1.1. (Banach Space): Complete normed vector space $(V, \|\cdot\|)$

Example 1.2.

- $C(K, \|\cdot\|_\infty)$, continuous functions from $K \to Y$ complete.
- $C^k(K, \|\cdot\|_{k,\infty})$, space of continuous functions from $K \subset R^n \to R$ w/ derivative up to order k, bounded and continuous.

$$\|f\|_{k,\infty} = \sum_{j=0}^{\infty} \|f^{(j)}\|_\infty$$

- **(Note!)** $C^\infty = \bigcap_{k\geq 0} C^k$ is not a normed vector space. This is indeed a frechet-space w/ complete for norm based on semi-norms $\|f^{(j)}\|_\infty$ and given explicitly by

$$d(f,g) = \sum_{k\geq 0} 2^{-k} \frac{\|f^{(k)} - g^{(k)}\|_\infty}{1 + \|f^{(k)} - g^{(k)}\|_\infty}$$

Example 1.3.

- L^p, $W^{m,p}$(up to m-th derivative of functions in L^p)
- $l^p(N)$ space of infinite sequences $x = (x_n)_{n\geq 1}$ w/

$$\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} \text{ and } \|x\|_\infty = \sup_{n\geq 1} |x_n|$$

<u>**Prop 1.4.**: $(l^p(N), \|\cdot\|_p)$ is Banach Space for $1 \leq p \leq \infty$</u>

Proof:

1. Clearly, for $p = \infty$, it is same as continuous functions w/ sup-norm, so we only consider $1 \leq p < \infty$
2. We want to first show that $\|\cdot\|_p$ is a norm:
   - $\|x\|_p = 0 = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} \Rightarrow x_n = 0$ for any n $\Rightarrow x = 0$
   - $\|ax\|_p = (\sum_{n=1}^{\infty} |ax_n|^p)^{\frac{1}{p}} = |a|\|x\|_p$
   - $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ (Minkowski-ineq)

<u>**Important Inequalities:**</u>

1. (Young's ineq:) For $a, b \geq 0$ and $1/p + 1/q = 1$

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

From my view, it is much easy to use graph to show that ineq. But following the proof from textbook we get:

$$\log(ta^p + (1-p)b^q) \geq t\log(a^p) + (1-t)\log(b^q) = \log(ab)$$

by setting $t = 1/p$ and monotonicity of log $\Rightarrow (1/p)a^p + (1/q)b^q \geq ab$

2. (Holder ineq:) $\|ab\|_1 \leq \|a\|_p\|b\|_q$
3. (Minkovski ineq.) $\|x+y\|_p \leq \|x\|_p + \|y\|_q$

5.1 Material from textbook:

Def 5.1. A Banach Space is a normed linear space that is complete w.r.t norm.

Example 5.2.

- $(R^n, \|\cdot\|_p)$ is Banach w.r.t to p-norm for $1 \leq p \leq \infty$

$$\|(x_1, ..., x_n)\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$$

and

$$\|(x_1, ..., x_n)\|_\infty = \max\{|x_i|\}$$

- $(C([a, b]), \|\cdot\|_{\sup})$ is Banach $\equiv (C(K), \|\cdot\|_{\sup})$ is Banach w/ K compact.

$$\|f\|_{\sup} = \sup_{x \in K} |f(x)|$$

- $C^k([a, b])$ w/ k-th continuously differentiable is not Banach w.r.t $\|\cdot\|_\infty$, since the limit of continuously differentiable need not to be differentiable. $(\lim_n f_n = f \notin C^k([a, b]))$. However, for C^k norm defined as

$$\|f\|_{C^k} = \|f\|_\infty + \|f'\|_\infty + ... + \|f^{(k)}\|_\infty$$

is a Banach space, guarantee the limit exists.

- $l^p(N)$ w/ $1 \leq p \leq \infty$ consists of all infinite sequence $x = (x_n)_{n \geq 1}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty,$$

with the p norm,

$$\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$$

is Banach Space.

- $L^p([a,b])$ for $1 \leq p \leq \infty$ is Banach space w/ function p-norm. We only need to notice that

$$\|f\|_\infty = \inf\{M | |f(x) \leq M \text{ a.e in } [a,b]\}$$

defined as essentially supreme, and

$$\|f\|_p = (\int_a^b |f(x)|^p)^{1/p}$$

is the function p-norm.

Note: A closed linear subspace of a Banach space is complete and thus Banach, since closed subset of a complete space is complete; Infinite dimsional subspace need not to be closed, however, it has proper dence subspaces:

Example 5.3. The space of polynomial is a linear subspace of $C([0,1])$, since linear combinations of polynomials are still polynomial. However, it is not closed, and theorem 2.9 on the textbook implies that it is dense in the $C([0,1])$. But, consider $\{f \in C([0,1]) | f(0) = 0\}$ is closed linear subspace of $C([0,1])$, thus it is Banach w.r.t usual sup-norm!

5.2 Bounded Linear maps

Def 5.2.1. A linear map/operator between X, Y linear spaces is function $T : X \to Y$ such that

$$T(ax + by) = aT(x) + bT(y)$$

for any $x, y \in X$ and $a, b \in R/C$.

We say that T is invertible / non-singular if T is one to one and onto, and define the inverse map $T^{-1} : Y \to X$ by $T^{-1}y = x$ iff $Tx = y$. The linearity of T implies the linearity of T^{-1}.

Note: If X, Y are normed spaces then we can defined the notion of bounded linear map, and it essentially implies the continuity of T!

Def 5.2.2. Let X, Y be normed linear spaces. A linear map $T : X \to Y$ is said to be bounded if there exists $M > 0$ such that

$$\|Tx\| \leq M\|x\| \text{ for } \forall x \in X$$

If no such M, we say that T is unbounded. Moreover, we can indeed define operator norm / uniform norm $\|T\|$ of T by

$$\|T\| = \inf\{M | \|Tx\| \leq M\|x\| \text{ for } \forall x \in X\}$$

equivalently,

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|$$

We have special classification of all operators:

$$\mathcal{L}(X,Y) = \{T|T : X \to Y\} \text{ and } \mathcal{B}(X,Y) = \{T|T : X \to Y \text{ is bounded}\}$$

Example 5.2.3. (Easy to Hard)

1. Linear map $A : R \to R$ defined by $Ax = ax$ for $a \in R$ fixed is BLF, w/ operator norm $\|A\| = |a|$.
2. The identity map $I : X \to X$ is BLF on any normed space space X, w/ operator norm $\|I\| = 1$. Similarly for zero-map.
3. Consider $X := C^\infty([0,1])$ smooth functions on $[0,1]$ equipped with sup-norm is normed linear space. However, it is not complete w.r.t sup-norm. We can defined the differential operator D as $Du = u'$ for $u, u' \in C^\infty([0,1])$, is certainly unbounded operator, since for example, $u = e^{ax} \Rightarrow Du = au$, and $\|D\| = \frac{\|Du\|}{\|x\|} = |a|$ can be arbitarily large. (In contrast to the first one!)

Note: The most common example of linear operator is matrix! (we can thus redefine linear algebra!):

- $\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|$
- $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$
- $\|A\|_2 = \sqrt{\lambda_{max}(AA^*)} = \sigma_{\max}(A)$ the largest singular value.

Them 5.2.4. A linear map is bounded iif it is continuous.
Proof: Let $T : X \to Y$ be linear map.
$(\Rightarrow)$ Suppose that it is bounded, we have

$$\|T(x) - T(y)\| = \|T(x-y)\| \le M\|x-y\|$$

by linearity and moreover, we can pick $\delta = \epsilon/M \Rightarrow \|x-y\| < \delta \Rightarrow \|T(x) - T(y)\| < \epsilon$, thus continuous.
$(\Leftarrow)$ Suppose that T is continuous, then for any $\epsilon > 0$ there exist δ such that

$$\|x-y\| < \delta \Rightarrow \|T(x) - T(y)\| < \epsilon$$

want to show that $\exists M > 0$, such that $\|Tx\| \le M\|x\|$ for any $x \in X$
First, suppose that T is continuous at 0. Since T is linear, we have $T(0) = 0$. Choose $\epsilon = 1$, we can conclude that there exists $\delta > 0$ such that $\|Tx - 0\| \le 1$, whenever $\|x\| < \delta$. For any $x \in X$ not equal to 0, we define

$$\bar{x} = \delta \frac{x}{\|x\|}$$

such that, $\|\bar{x}\| \le \delta \Rightarrow \|Tx\| \le 1$. So it follow from linearity of T that

$$\|Tx\| = \frac{\|x\|}{\delta}\|T\bar{x}\| \le M\|x\|$$

Thus T is bounded.

Them 5.2.5. Let X be NLS and Y be Banach. If M is dense linear subspace of X and

$$T : M \subset X \to Y$$

is a bounded linear map, then there is unique bounded linear map $\bar{T} : X \to Y$ such that $\bar{T}x = Tx$ for all $x \in M$ and $\|\bar{T}\| = \|T\|$.

Them 5.2.6.(Open Mapping theorem): $T : X \to Y$ is **1-1, onto, bounded linear map, X,Y Banach spaces**, then $T^{-1} : Y \to X$ is bounded. Basically saying is T(open) is open.

Application: Suppose that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces. Then there exists $\|\cdot\|_1 \cong \|\cdot\|_2$.
Consider $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$, 1-1, onto, bounded linear map. We know that $I^{-1} : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ is bounded. That is $\|\cdot\|_1 \leq M\|\cdot\|_2$; similarly for the other direction.

Def 5.2.7. $T : X \to Y$ linear is said to be closed, if $Graph(T)$ is closed, meaning that

$$\{(x_n, Tx_n) \to (x, y)\} \Rightarrow \{y = Tx\}$$

where Graph(T) is defined as

$$Graph(T) = \cup_{x \in X}[x, Tx] \subset X \times Y$$

for X, Y Banach spaces.

Them 5.2.8. $T : X \to Y$ linear closed map, then T is bounded.

Def 5.2.9. $T : X \to Y$ linear

$$Ker(T) = \{x \in X, Tx = 0\}$$

and

$$Ran(T) = \{y \in Y, \exists x \in X, Tx = y\}$$

Them 5.2.10. $T : X \to Y$ linear and $Ker(T) \subset X, Ran(T) \subset Y$:

- If T is bounded, then $Ker(T)$ is closed.
- T is 1-1 iif $Ker(T) = \{0\}$
- T is onto iif $Ran(T) = Y$

Them 5.2.11. $T : X \to Y$ linear bounded, X, Y Banach. Then

$$\{\exists c > 0, c\|x\| \leq \|Tx\|, \forall x \in X\} \iff \{Ran(T) \text{ is closed and } Ker(T) = \{0\}\}$$

Notes 5.2.12.

- $(X, \|\cdot\|)$ has dimension n. Then $\|\cdot\| \cong \|\cdot\|_1$
- Every finite dimensional n.v.sp is Banach.
- Every finite dimensional subspace of n.v.sp is Banach.
- Every linear operator T on finite dim-space is bounded.
- $\mathcal{B}(X,Y) = \{T | T : X \to Y \text{ is bounded}\}$, $(\mathcal{B}(X,Y)), \|\cdot\|)$ is Banach w.r.t $\|\cdot\| = \sup_{\|x\|=1} \|Tx\|$
- **Them 5.1.13**: $T \in \mathcal{B}(X,Y), S \in \mathcal{B}(Y,Z) \Rightarrow ST \in \mathcal{B}(X,Z)$ with $\|ST\| \leq \|S\|\|T\|$

Def 5.2.14. $T_n \to T$ uniformly(in operator norm) if $\|T_n - T\| \to 0$ as $n \to \infty$

Them 5.2.15. X is n.v.sp and Y Banach. Then $(\mathcal{B}(X,Y), \|\cdot\|)$ is Banach.
Proof: Show that every Cauchy converges in $\mathcal{B}(X,Y)$

(i) Let T_n Cauchy, $\|T_n - T_m\| \to 0 \Rightarrow \|T_n(x) - T_m(x)\|_y \to 0$ for any $x \in X$ fixed, $n, m > N$. Thus $T_n(x)$ is Cauchy in Y and $T_n(x) \to y \in Y$ since Y is Banach. Define T as $x \to T(x) = y$ and checked that T is linear;

(ii) By above convergence, we have that $\forall \epsilon > 0$, there is M, such that $n > M \Rightarrow \|Tx - T_m x\| \leq \frac{\epsilon}{2}\|x\|$. Thus we have

$$\|T_n x - Tx\| \leq \|T_m x - Tx\| + \|T_n x - T_m x\| \leq \frac{\epsilon}{2}\|x\| + \frac{\epsilon}{2}\|x\| = \frac{\epsilon}{\|x\|}$$

Moreover we have

$$T(x) \leq \|T_n x - Tx\| + \|T_n\| \leq C\|x\|$$

Then T is bounded map and $\|T_n x - Tx\| \leq \frac{\epsilon}{\|x\|} \Rightarrow T_n \to T$ uniformly.

6 Classnote 1/29/2024

6.1 Compact operator

Def 6.1.1. $T : X \to Y$ is compact if $T(B)$ is precompact in Y, where B is unit ball in X centered at 0. (N.B: precompact means compact closure.)

Them 6.1.2. T is compact iif for each sequence $(x_n)_{n \in N} \in X$ with $\|x_n\|_X \leq c$, there is a subsequence x_{ϕ_n} such that Tx_{ϕ_n} converges in Y. (useful)

Notes 6.1.3.

1. T compact maps bounded families to compact families.
2. Let X, Y Banach spaces, $(\mathcal{B}(x, y), \|\cdot\|)$ is Banach space and is an algebra.
3. Let $K(X, Y)$ be the subspace of compact operators in $\mathcal{B}(X, Y)$
4. $\dim Ran(T) < \infty \Rightarrow$ T is compact.
5. if $S \in K(X, Y), T \in \mathcal{B}(X, Y) \Rightarrow$ ST and TS are compact (when defined)

Them 6.1.4.(useful) $K(X, Y)$ is a closed subspace of $\mathcal{B}(X, Y)$.

This means that $aT + bS$ compact when T, S compact but mostly that $T_n \to T$ uniformly and T_n compact $\Rightarrow$ T is compact.

Def 6.1.5. $T_n \in \mathcal{B}(X, Y)$ converges to T strongly if $\lim_n T_n = Tx$ for all $x \in X$.

This means that: $\|T_n x - Tx\|_Y \to 0$ for all $x \in X$.

Them 6.1.6.(useful): If $T_n \to T$ uniformly, then $T_n \to T$ strongly.
Proof: Since $T_n \to T$ uniformly, we have

$$\|T_n x - Tx\|_Y \; leq \|T_n - T\|\|x\| \to 0$$

since $\|T_n - T\| \to 0$.

Them 6.1.6.(Uniform Boundness Theorem) Let X, Y Banach Spaces, $(T_i)_{i \geq 1} \in \mathcal{B}(X, Y)$. Assume that $\sup_i \|T_i x\|_Y < \infty$ for all x. Then there exists $c > 0$ such that

$$\|T_i x\|_Y \leq c\|x\|_X$$

for all $x \in X$ and $i \in I$

Cor 6.1.7. Let X, Y Banach Spaces, $T_n \in \mathcal{B}(X, Y)$ and $T_n \to T$ strongly. Then $\sup_n \|T_n\| < \infty$ and $T \in \mathcal{B}(X, Y)$ w/

$$\|T\| \leq \liminf_n \|T_n\|$$

6.2 Dual Spaces

Def 6.2.1. Coordinates:

$$x_i : \begin{cases} R^n & \to R \\ x & \to x_i(x) = \langle e_i, x \rangle \end{cases}$$

Def 6.2.2. Let X be vector space. The space of all conti. linear functional from X to R is called the <u>dual</u> to the X.

Notes 6.2.3.

- Notations: $X^* = \mathcal{B}(X, R)$
- Let X be n.v.sp on X^*, then we have $\|\phi\| := \sup_{x \neq 0} \frac{|\phi(x)|}{\|x\|}$ unif-norm
- ϕ is bounded implies that $|\phi| \leq \|\phi\| \|x\| < \infty$
- Since R is Banach, then X^* is automaticlly Banach

Them 6.2.4.(Hahn-Banach Theorem) Let $Y \subset X$, X is n.v.sp and $\phi : Y \to R$ bounded linear functional with $\|\phi\|_{Y^*} = M < \infty$. Thene there exists $\psi : X \to R$ bounded linear, such that $\psi|_Y = \phi$ and $\|\phi\|_{X^*} = \|\psi\|_{Y^*}$

Cor 6.2.5. $\forall x \in X$, there is $f_0 \in X^*$ such that $\|f_0\|_{X^*} = \|x_0\|_X$ and $\langle f_0, x_0 \rangle = \|x_0\|_X^2$

6.3 Weak and Weak* convergence

Def 6.3.1. $x_n \in X$ <u>converges weakly to X</u> if $\phi(x_n) \to \phi(x)$ for any $\phi \in X^*$.

Notation: $x_n \xrightarrow{n \to \infty} x$

Note that $x_n \to x$ strongly implies weak c.v.Since $\|\phi(x) - \phi(x_n)\| \leq \|\phi\| \|x - x_0\| \to 0$

Def 6.3.2. $\phi_n \in X^*$ converges weak* to ϕ if $\phi_n(x) \to \phi(x)$ for any x.

Notation: $\phi_n \xrightarrow{*} \phi$

Them 6.3.3.(Banach-Alaoglu Theorem) The closed unit ball in X^* is weak* compact

Them 6.3.4.(Kakutami Theorem) X reflective $(X = X^{**})$ iif

$$B_X = \{x \in X, \|x\| \leq 1\}$$

is compact for (X, C_w).