Introduction

Consider the minimization problem with or without small perturbation in the following form:

$$\inf\left\{\int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2 + \alpha \int_0^1 |u|^2\right\}$$

Given that

1. u : function defined on (0,1);
2. a_i : k points in (0,1) with k greater than 1;
3. f_i : k real numbers;
4. α : a parameter taking values in $\{0,1\}$

The natural functional setting is the Sobolev space $W^{1,1}(0,1)$. We know that when $\alpha = 0$ (without perturbation) the minimizer is achieved. However, for the case with small perturbation, it is not likely to have a minimizer in our original function space.

Case for $\alpha = 0$

When $\alpha = 0$, WLOG consider

$$0 < a_1 < a_2 < ... < a_k < 1,$$

we aim to find $u(a_i)$ approximate f_i best and maintain regularity of u at the same time so that we can achieve our minimal. So, we define functional F:

$$F(u) = \int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2,$$

and our goal is to find $m = \inf_{u \in W^{1,1}(0,1)} F(u)$ always admits minimizers. (Note: F is well-defined on $W^{1,1}(0,1) \subset C([0,1])$ so that $u(a_i)$ make sense)

Case for $\alpha = 0$

Consider a finite dimensional auxiliary problem: Given

$$\lambda = (\lambda_1, ..., \lambda_k) \in \mathbb{R}^k$$

set

$$\phi(\lambda) := \sum_{i=1}^{k-1} |\lambda_{i+1} - \lambda_i| + \sum_{i=1}^{k} |\lambda_i - f_i|^2.$$

by convexity, we have

$$m := min_{\lambda \in \mathbb{R}^k} \phi(\lambda) := \phi(u)$$

where $u = (u_1, ..., u_k)$ is uniquely achieved.

Case for $\alpha = 0$

Theorem 2.1(T.Sznigir[6,7]). We have

$$m = inf_{u \in W^{1,1}} F(u)$$

and $u \in W^{1,1}(0,1)$ is a minimizer iff the following three conditions hold:

- u is monotone on each interval $(a_i, a_{i+1}), i = 1, 2, ...k - 1$;
- $u(a_i) = u_i, i = 1, ...k$;
- $u(x) = u_1, \forall x \in [0, a_1]$ and $u(x) = u_k, \forall x \in [a_k, 1]$

where m is defined as our previous slide and the **Inf** is achieved. This provides us a description of what the minimizers should look like.

Proof of Theorem 2.1

($\Leftarrow$): Given that $u \in W^{1,1}(0,1)$, we have

$$\int_0^1 |u'| \geq \sum_{i=1}^{k-1} \int_{a_i}^{a_{i+1}} |u'| \geq \sum_{i=1}^{k-1} |u(a_{i+1}) - u(a_i)|$$

with equalities iff

- u is monotone on each interval (a_i, a_{i+1});
- u is constant on $(0, a_1)$ and $(a_k, 1)$

Thus, adding both side of $\sum_{i=1}^{k} |u(a_i) - f_i|^2$, we have

$$F(u) \geq \sum_{i=1}^{k-1} |u(a_{i+1}) - u(a_i)| + \sum_{i=1}^{k} |u(a_i) - f_i|^2$$

Let $\lambda_i := u(a_i)$, we see that for every $u \in W^{1,1}(0,1)$,

$$F(u) \geq min_{\lambda \in \mathbb{R}^k} \phi(\lambda) = m$$

Proof of Theorem 2.1 (Cont.)

and if u satiesfied the 3 conditions, we would have

$$F(u) = \sum_{i=1}^{k-1} |u_{i+1} - u_i| + \sum_{i=1}^{k} |u_i - f_i|^2 = m$$

and u is indeed a minimizer for our problem.
($\Rightarrow$): If $F(u) = m$ for some $u \in W^{1,1}(0,1)$, then as mentioned above, condition 1 and 3 are naturally hold by requirement. Moreover, since $u(a_i) = \lambda_i$ is a minimizer by assumption, by uniqueness of $\min_{\lambda \in \mathbb{R}^k} \phi(\lambda)$, we have $u(a_i) = u_i$.

Remarks on minimizers

Before moving on to the case of $\alpha = 1$, let's consider some minimizers that is relative "stable" w.r.t to different perturbations. (i.e some type of perturbed functionals):

- $F_{1,\epsilon}(u) = \epsilon \int_{0,1} |u'|^2 + F(u),\ u \in H^1(0,1), \epsilon > 0$
- $F_{2,p}(u) = \int_{0,1} |u'|^p + \sum_{i=1}^{k} |u(a_i) - f_i|^2,\ u \in W^{1,p}(0,1), p > 1$
- $F_{3,\epsilon}(u) = \epsilon \int_{0,1} |u''|^2 + F(u),\ u \in H^2(0,1), \epsilon > 0$

The conclusion is easy to check that each admits a unique minimizers, the difference is some minimizers u_l is linear in each interval (a_i, a_{i+1}) that can be obtained by linear interpolations, and third one piece-wise cubic function.

Case for $\alpha = 1$

However, for our case of $\alpha = 1$, we aim to minimize the following functional

$$G(u) := F(u) + \int_0^1 |u|^2 = \int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2 + \int_0^1 |u|^2,$$

$W^{1,1}$ is not a good function space to detect minimizer. Indeed, we choose a larger our space to $BV(0,1)$ that we can find minimizers and can be viewed as general solution. The only problem left is, for $u \in BV(0,1)$, our functional is not well-defined since $u(a_i)$ does not make sense, when it is not continuous.

Case for $\alpha = 1$ that has no minimizer

Corollary 5.2: Assume the problem $A = \inf_{u \in W^{1,1}(0,1)} G(u) := G(u_0)$ has minimizer, then necessarily

$$u_0 = K_0 = \frac{1}{k+1} \sum_{i=1}^{k} f_i.$$

and moreover,

$$|f_i - K_0| \leq 1 \tag{1}$$

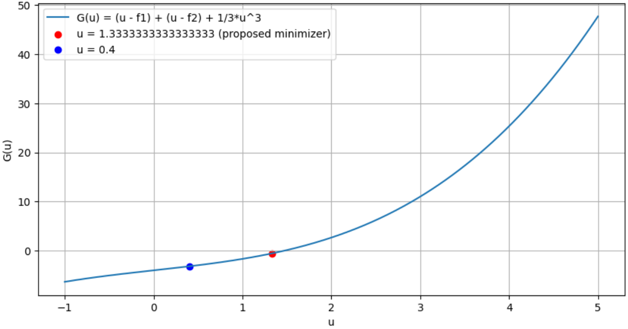
$$(a_{i+1} - a_i)|K_0| \leq 1 \tag{2}$$

$$|K_0| \leq k + 1 \tag{3}$$

Given this fact, for example like $k = 2$, $f_1 = 3$, $f_2 = 1$ such that $|2f_1 - f_2| > 3$ would violate the (1) and thus admits no minimizer.

Case for $\alpha = 1$ that has no minimizer

The graph illustrate that if u_0 is the minimizer by Cor 5.2, and from some analysis fact that we know $u_0 = K_0 = \frac{4}{3}$ for $\forall x \in (0, 1)$, $f_1 = 3$; $f_2 = 1$ we would have a graph



And it is easy to see that clear u_0 is not the minimizer!

Construct relax functional

Back to our case for seeking such minimizer on $BV(0,1)$, the key is to find relax functional so consider that

$$F_r(v) := \inf \liminf_{n\to\infty} F(v_n)$$

where the *Inf* taking over all sequences $(v_n) \subset W^{1,1}(0,1)$ and $v_n \xrightarrow{\mathbb{L}^2} v$. Moreover, some definitions for further construction: Given $v \in BV(0,1), a \in (0,1)$

- $j(v)(a) := [\min(v(a-0), v(a+0)), \max(v(a-0), v(a+0))]$ denoted as the jump interval;
- $\phi(t) = \begin{cases} t^2 & \text{if } 0 \le t \le 1 \\ 2t-1 & \text{if } t > 1 \end{cases}$

Theorem 3.1

We will introduce a new formulation of functional F_r which is lower-semi-continuous and convex so that is useful to make sense of $v \in BV(0,1)$ at certain discontinuities.
Theorem 3.1: For every $v \in BV(0,1)$, we have

$$F_r(v) = \int_0^1 |v'|^2 + \sum_{i=1}^{k} \phi(dist(f_i, j(v)(a_i)))$$

where *dist* denoted as the distance of points f_i to our set defined as before. It is a long proof involving two steps and we will introduce 3 lemmas to explain.

Lemma 3.1

Lemma 3.1: Let $v_n \in BV(a,b)$ such that $v_n \xrightarrow{\mathbb{L}^1} v$, $v_n(a) \to \alpha, v_n(b) \to \beta$ as $n \to \infty$. Then $v \in BV(a,b)$ and

$$\liminf_{n\to\infty} \int_a^b |v_n'| \geq \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|$$

Proof of Lemma 3.1

Proof: fixed any $h \in C_c^\infty(\mathbb{R})$ such that $h(a) = \alpha, h(b) = \beta$. Consider the functions

$$w_n(t) := \begin{cases} h(t), & \text{if } t < a, \\ v_n(t), & \text{if } a \leq t \leq b, \\ h(t), & \text{if } t > b, \end{cases} \quad \text{and} \quad w(t) := \begin{cases} h(t), & \text{if } t < a, \\ v(t), & \text{if } a \leq t \leq b, \\ h(t), & \text{if } t > b. \end{cases}$$

Clearly $w_n, w \in BV(\mathbb{R})$ (Recall $h \in C_c^\infty(\mathbb{R}) \Rightarrow h$ is bounded and thus $h \in BV(\mathbb{R})$; Moreover, by *Lecture 5 "Compactness of $BV(\Omega)$, pp.34"* we get that $w \in BV(\mathbb{R})$) and

$$\int_{\mathbb{R}} |w_n'| = \int_{-\infty}^{a} |h'| + \int_a^b |v_n'| + \int_b^\infty |h'| + |v_n(a) - \alpha| + |v_n(b) - \beta|,$$

$$\int_{\mathbb{R}} |w'| = \int_{-\infty}^{a} |h'| + \int_a^b |v'| + \int_b^\infty |h'| + |v(a) - \alpha| + |v(b) - \beta|.$$

Proof of Lemma 3.1 Cont.

Since $w_n \to w$ in $L^1(\mathbb{R})$ it is well-known that

$$\liminf_{n\to\infty} \int_{\mathbb{R}} |w_n'| \geq \int_{\mathbb{R}} |w'|.$$

Combining this with fact $h(a) = \alpha, h(b) = \beta$ we have reached our conclusion.

Remark: The well known fact comes from total variation is lsc w.r.t to $\mathbb{L}^1$ convergence, I just follow the proof in *Lecture 8, pp60*:

For $\phi \in C_c^\infty(\Omega, \mathbb{R}), \|\phi\| \leq 1$, we have

$$\int_\Omega v div\phi dx = \lim_{n\to\infty} \int v_n div\phi dx \tag{4}$$

$$\leq \liminf_{n\to\infty} \sup\{\int_\Omega v_n div\phi dx, \phi \in C_c^\infty(\Omega, \mathbb{R}), \|\phi\| \leq 1\} \tag{5}$$

$$= \liminf_{n\to\infty} |Dv_n|(\Omega) \tag{6}$$

Taking supreme over all ϕ, we have our desired result.

Proof of Lemma 3.2

Lemma 3.2: Given any $v \in BV(a,b)$ and constants $\alpha, \beta \in \mathbb{R}$, there exists a sequence $(v_n) \subset W^{1,1}(a,b)$ such that $v_n \to v$ in $L^2(a,b)$, $v_n(a) = \alpha$, $v_n(b) = \beta$, $\forall n$, and

$$\lim_{n\to\infty} \int_a^b |v_n'| = \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|.$$

Proof of Lemma 3.2

Let $w_n = \rho_n * w$ where (ρ_n) is a sequence of mollifiers (For regularity purpose). Clearly

$$\int_{\mathbb{R}} |w_n'| \leq \int_{\mathbb{R}} |w'| = \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|.$$

Moreover $w_n(t) = \alpha$ if $t < a - (1/n)$ and $w_n(t) = \beta$ if $t > b + (1/n)$. Rescaling the sequence (w_n) by a change of variables we obtain a sequence (v_n) of smooth functions such that $v_n \to v$ in $L^2(a, b)$, $v_n(a) = \alpha$, $v_n(b) = \beta$, $\forall n$, and

$$\limsup_{n \to \infty} \int_a^b |v_n'| \leq \int_a^b |v'| + |v(a) - \alpha| + |v(b) - \beta|.$$

Applying Lemma 3.1 we conclude that statement holds.

Lemma 3.2 Cont.

Remark: We may notice that there is some trouble when we try to use Lemma 3.1 as it required $\mathbb{L}^1$ convergence. However, consider that

$$\|v_n - v\|_{\mathbb{L}^1} = \int_a^b |v_n - v| dx \overset{\text{Hölder}}{\leq} (b-a)^{\frac{1}{2}} \|v_n - v\|_{\mathbb{L}^2} \to 0$$

Moreover, the reason we need the mollifier is to add the regularity since we require our function space in $W^{1,1}(a,b)$.

Lemma 3.3. Given any $\alpha, \beta, f \in \mathbb{R}$ we have

$$\inf_{t\in\mathbb{R}}\{|t-\alpha| + |t-\beta| + |t-f|^2\} = |\alpha-\beta| + \phi(\text{dist}(f, J)),$$

where $J = [\min(\alpha, \beta), \max(\alpha, \beta)]$ and ϕ has been defined before.
This is nothing but calculation, one may naturally think that, the inf over first two term would implies $t \in [\alpha, \beta]$ and if f is outside our interval, the distance of f with J w.r.t to ϕ is linear, and inside J is just like graph of $|t-f|^2$.

Proof of Theorem 3.1

Step 1. Given any $v \in BV(0,1)$ there exists a sequence $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ and

$$\lim_{n\to\infty} F(v_n) = F_r(v),$$

where $F_r(v)$ is defined as before.
Proof. Applying Lemma 3.3 with $\alpha = v(a_i - 0)$, $\beta = v(a_i + 0)$, and $f = f_i$, $1 \le i \le k$, we obtain some t_i (the inf value in Lemma 3.3) such that

$$|t_i - v(a_i - 0)| + |t_i - v(a_i + 0)| + |t_i - f_i|^2 = |v(a_i - 0) - v(a_i + 0)| + \phi(\mathrm{dist}(f_i, j(v(a_i)))). \tag{7}$$

We next apply Lemma 3.2 successively on $(0, a_1), (a_i, a_{i+1})$, and $(a_k, 1)$. First on $(0, a_1)$ with $\alpha = v(0+)$ and $\beta = t_1$. This yields a sequence $(v_n) \subset W^{1,1}(0, a_1)$ such that $v_n(0) = v(0+)$, $v_n(a_1) = t_1$, $\forall n$, $v_n \to v$ in $L^2(0, a_1)$ and

$$\int_0^{a_1} |v_n'| = \int_0^{a_1} |v'| + |v(a_1 - 0) - t_1| + o(1).$$

.

Step 1 Cont.

Same thing on $(a_k, 1)$, we have

$$\int_{a_k}^{1} |v_n'| = \int_{a_k}^{1} |v'| + |v(a_k + 0) - t_k| + o(1).$$

Next on (a_i, a_{i+1}), $1 \leq i \leq k-1$, this yields a sequence $(v_n) \subset W^{1,1}$ such that $v_n(a_i) = t_i$, $v_n(a_{i+1}) = t_{i+1}$, $v_n \to v$ in L^2, and

$$\int_{a_i}^{a_{i+1}} |v_n'| = \int_{a_i}^{a_{i+1}} |v'| + |v(a_i + 0) - t_i| + |v(a_{i+1} - 0) - t_{i+1}| + o(1).$$

Togather, we have

$$\int_0^1 |v_n'| = \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^{k} [|v(a_i - 0) - t_i| + |v(a_i + 0) - t_i|] + o(1).$$

since $v_n(a_i) = t_i$ for $\forall n, i$, and result from Lemma 3.3, we have,

$$\int_0^1 |v_n'| = \int_0^1 |v'| - \sum_{i=1}^{k} |t_i - f_i|^2 + \sum_{i=1}^{k} \phi(\mathrm{dist}(f_i, j(v(a_i)))) + o(1)$$

Step 2. Let (v_n) be a bounded sequence in $W^{1,1}(0,1)$ such that $v_n \to v$ in $L^1(0,1)$. Then

$$\liminf_{n\to\infty} F(v_n) \geq F_r(v).$$

Proof. Passing to a subsequence we may always assume that, for every $i = 0, 1, \ldots, k+1$, there exists some ℓ_i such that

$$v_n(a_i) \to \ell_i \quad \text{as} \quad n \to \infty.$$

From Lemma 3.1 we know that for every $i = 0, 1, \ldots, k$,

$$\int_{a_i}^{a_{i+1}} |v_n'| \geq \int_{a_i}^{a_{i+1}} |v'| + |v(a_i+0) - \ell_i| + |v(a_{i+1}-0) - \ell_{i+1}| + o(1).$$

Adding these inequalities yields

$$F(v_n) \geq \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^{k} (|v(a_i-0)-\ell_i| + |v(a_i+0)-\ell_i| + |\ell_i - f_i|^2) + o(1).$$

Proof of Lemma 2

Applying Lemma 3.3 we find that

$$F(v_n) \geq \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} |v'| + \sum_{i=1}^{k} (|v(a_i+0) - v(a_i-0)|) \tag{8}$$

$$+ \phi(\text{dist}(f_i, j(v(a_i)))) + o(1) \tag{9}$$

$$= F_r(v) + o(1) \tag{10}$$

which completes the proof of Step 2, and thereby the proof of Theorem 3.1.

Properties of Relax functional

Lemma 4.1. The functional F_r is convex on $BV(0,1)$ and it is lower semicontinuous in the sense that for every sequence $(v_n) \subset BV(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ as $n \to \infty$, we have

$$\liminf_{n\to\infty} F_r(v_n) \geq F_r(v).$$

Proof of Lemma 4.1

Proof. Given $v, w \in BV(0,1)$ there exist (by Step 1 above) sequences $(v_n), (w_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$, $w_n \to w$ in $L^2(0,1)$ and $F(v_n) \to F_r(v)$, $F(w_n) \to F_r(w)$. By convexity of F we have

$$F(tv_n + (1-t)w_n) \leq tF(v_n) + (1-t)F(w_n), \quad \forall t \in [0,1].$$

Passing to the limit and using Step 2 we see that

$$F_r(tv + (1-t)w) \leq tF_r(v) + (1-t)F_r(w).$$

By Step 1 applied to v_n with n fixed we may find some $w_n \in W^{1,1}(0,1)$

$$\|v_n - w_n\|_{L^2} < \frac{1}{n} \quad \text{and} \quad |F_r(v_n) - F(w_n)| < \frac{1}{n}.$$

Thus $w_n \to v$ in $L^2(0,1)$ and from the definition we conclude that

$$F_r(v) \leq \liminf_{n\to\infty} F(v_n) = \liminf_{n\to\infty} F_r(v_n).$$

Thus, we complete the proof.

Small perturbation case

For our case of $\alpha = 1$, we aim to minimize the following functional

$$G(u) := F(u) + \int_0^1 |u|^2 = \int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2 + \int_0^1 |u|^2,$$

We may set for simplicity

$$A := \inf_{v \in W^{1,1}} G(v)$$

It turns out that the infimum in need not be achieved and we will replace it by a relaxed problem defined on BV(0,1) as we have done. For every $v \in BV(0,1)$ set

$$G_r(v) = \inf \liminf_{n\to\infty} G(v_n), \tag{5.3}$$

where the Inf in (5.3) is taken over all sequences $(v_n) \subset W^{1,1}(0,1)$ such that $v_n \to v$ in $L^2(0,1)$. It is easy to check that

$$G_r(v) = F_r(v) + \int_0^1 |v|^2, \quad \forall v \in BV(0,1). \tag{5.4}$$

Small perturbation case

G_r is strictly convex on BV(0,1) and it is lower semicontinuous in the sense that for every sequence $(v_n) \subset BV(0,1)$ such that $v_n \to v$ in $L^2(0,1)$ as $n \to \infty$, we have

$$\liminf_{n\to\infty} G_r(v_n) \geq G_r(v). \tag{5.5}$$

Consequently,

$$B = \inf_{v \in BV} G_r(v) := G_r(\tilde{v}). \tag{5.6}$$

is uniquely achieved, and we denote by $\tilde{v} \in BV(0,1)$ its unique minimizer.

A=B

It is clear that $G_r \leq G$ on $W^{1,1}(0,1)$, and thus

$$B = \inf_{v \in BV} G_r(v) \leq \inf_{v \in W^{1,1}} G_r(v) \leq \inf_{v \in W^{1,1}} G(v) = A. \tag{5.8}$$

On the other hand, we have by (5.2)

$$A \leq G(u) = F(u) + \int_0^1 |u|^2, \quad \forall u \in W^{1,1}(0,1). \tag{5.9}$$

From (5.9) and Step 1 in Section 3 we deduce that

$$A \leq F_r(v) + \int_0^1 |v|^2 = G_r(v), \quad \forall v \in BV(0,1). \tag{5.10}$$

And thus

$$A \leq \inf_{v \in BV} G_r(v) = B. \tag{5.11}$$

Combining (5.11) with (5.8) yields $A = B$.

Conclusion

From above consideration, we have successfully conduct the case of small perturbation and without perturbation of the problem

$$\inf \left\{ \int_0^1 |u'| + \sum_{i=1}^{k} |u(a_i) - f_i|^2 + \alpha \int_0^1 |u|^2 \right\}$$

When $W^{1,1}$ does not admits a solution, we can find convergent sequence $(v_n) \subset W^{1,1}$ to $v \in BV$. Moreover, if it indeed admits a minimizer, for $v_0 \in W^{1,1}(0,1)$, then

$$B \leq G_r(v_0) \leq G(v_0) = A,$$

so that $G_r(v_0) = B$, again v_0 is a minimizer by uniqueness $v_0 = \tilde{v}$. Hence, the relax functional and associated $\tilde{v}$ can be seen as a general solution.

Reference

REFERENCES

1 T. Sznigir, Various Minimization Problems Involving the Total Variation in One Dimension, PhD Rutgers University, Sept 2017.

2 T. Sznigir, A Regularized Interpolation Problem Involving the Total Variation, (to appear).

3 Brezis, H. (2019). Regularized interpolation driven by total variation. Analysis in Theory and Applications, 35(4), 335-354. https://doi.org/10.4208/ata.OA-0017

4 Baer, E. (2024). Lecture notes, 1-60. https://canvas.uchicago.edu/courses/56670/files/folder/Lecture