Existence of Solution to the Nonlinear Dirac Equation

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Abstract

For nonlinear partial differential equations with compactly supported nonlinearity, we can study the scattering solutions. In this paper, for the compactly supported cubic nonlinearity, I will use fixed point arguments to show the existence of solutions to the nonlinear Helmholtz equation and the nonlinear Dirac equation.

1 Introduction

In this paper, I will discuss the existence of solutions to several partial differential equations (PDEs). The first four sections are devoted to notes about some basics taken from [5]. In section five, I will show the existence of an outgoing solution of the nonlinear stationary Schrödinger equation in 1-D. The leading order scattering matrix will also be shown. In section six, I first generalize the argument to the nonlinear Dirac equation in 1-D, then to the nonlinear Dirac equation in 2-D.

2 Postulate of quantum mechanics

In quantum mechanics, a particle in $\mathbb{R}^3$ is described by a complex-valued function, the **wavefunction**:

$$\psi(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.1}$$

The square of the wave equation $\rho_t(x) = |\psi(x,t)|^2$ is interpreted as the **probability density** of the particle in time t. So ψ must be normalized. The position x of the particle is called observables, and its **expectation** is

$$\mathbb{E}_\psi(x) = \int_{\mathbb{R}^3} x|\psi(x,t)|^2 d^3x. \tag{2.2}$$

In real life, x can not be measured directly, and one will be able to measure certain functions of x. For example, check whether the particle is inside a certain area Ω. The probability of finding the particle in this area is

$$\mathbb{E}_\psi\left(\chi_\Omega\right) = \int_{\mathbb{R}^3} \chi_\Omega(x)|\psi(x,t)|^2 d^3x = \int_\Omega |\psi(x,t)|^2 d^3x. \tag{2.3}$$

The **mean-square deviation(variance)** is given by $\Delta_\psi(x)^2 = \mathbb{E}_\psi\left(x^2\right) - \mathbb{E}_\psi(x)^2$, and is always non-zero.

In general, quantum mechanical systems are described by normalized vectors in Hilbert spaces. Measurable quantities are called observables and correspond to self-adjoint operators in the Hilbert space. The expectation of a self-adjoint operator A if the system is in state ψ is given by a real number

$$\mathbb{E}_\psi(A) = \langle \psi, A\psi \rangle = \langle A\psi, \psi \rangle \tag{2.4}$$

Similarly, the mean-square deviation is

$$\Delta_\psi(A)^2 = \mathbb{E}_\psi\left(A^2\right) - \mathbb{E}_\psi(A)^2 = \left\|\left(A - \mathbb{E}_\psi(A)\right)\psi\right\|^2 \tag{2.5}$$

We require that A is defined on the dense subset $\mathcal{D}(A) \in \mathcal{H}$, called the **domain** of A.

Now let's investigate the time evolution of a quantum mechanical system. Given initial state $\psi(0)$, there should be a unique $\psi(t) = U(t)\psi(0)$. Moreover, it follows from the experimental results that **superposition of states** holds: $U(t)\left(\alpha_1\psi_1(0) + \alpha_2\psi_2(0)\right) = \alpha_1\psi_1(t) + \alpha_2\psi_2(t)$. This implies $U(t)$ is a linear operator. In addition, $||U(t)\psi|| = ||\psi|| = 1$ since $\psi(t)$ is a state for all t. So $U(t)$ is unitary. Since we assumed the uniqueness of the solution,

$$U(0) = \mathbb{I}, \quad U(t+s) = U(t)U(s) \tag{2.6}$$

A family of $U(t)$ is called a **one-parameter unitary group**. We also assume strong continuous:

$$\lim_{t\to t_0} U(t)\psi = U\left(t_0\right)\psi, \quad \psi \in \mathcal{H}. \tag{2.7}$$

Each such group has an **infinitesimal generator** defined by

$$H\psi = \lim_{t\to 0} \frac{\mathrm{i}}{t}(U(t)\psi - \psi), \quad \mathcal{D}(H) = \left\{\psi \in \mathcal{H} \mid \lim_{t\to 0} \frac{1}{t}(U(t)\psi - \psi) \text{ exists }\right\}. \tag{2.8}$$

The operator H is called the **Hamiltonian**, which gives the energy of the system. If $\psi(0) \in \mathcal{D}(H)$, then $\psi(t)$ is a solution of the **Schrödinger equation**

$$i\frac{d}{dt}\psi(t) = H\psi(t). \tag{2.9}$$

3 The free Schrödinger equation

The free Schrödinger equation, corresponding to a free particle in $\mathbb{R}^d$, is given by

$$i\partial_t\psi(t,x) = -\Delta_x\psi(t,x) \tag{3.1}$$

where we set $\hbar = 1$ and $m = 1/2$. This is a special case of the Schrödinger equation when $H \equiv -\Delta$. We can find a special solution by separating $\psi(t,x) = \phi(x)T(t)$. Plugging into the equation, we have $-\frac{\partial^2\phi(x)}{\partial x^2}T(t) = i\phi(x)\frac{\partial T(t)}{\partial t}$. Putting all x dependent on the left side and all t dependent on the right side, we have $\frac{-\frac{\partial^2\phi(x)}{\partial x^2}}{\phi(x)} = i\frac{\frac{\partial T(t)}{\partial t}}{T(t)} \equiv \lambda$. The equation

$$-\frac{\partial^2\phi(x)}{\partial x^2} = \lambda\phi(x) \tag{3.2}$$

is called the time-independent Schrödinger equation.

Now $\psi(t,x) = e^{-i\lambda t}\phi(x)$ is solution to equation 3.1. To obtained solutions to $\phi(x)$, note plane waves on $\mathbb{R}^d$ are solutions to equation 3.2:

$$\phi_k(x) = e^{ik\cdot x} = e^{i(k_1x_1+\cdots+k_dx_d)} \tag{3.3}$$

In addition,

$$-\Delta_x\phi_k(x) = (k_1^2+\cdots k_d^2)e^{ik\cdot x} = |k|^2\phi_k(x) \tag{3.4}$$

Thus $\lambda = |k|^2$.

Thus a solution to The free Schrödinger equation is

$$\psi_k(t,x) = e^{-ik^2t+ikx} \tag{3.5}$$

However, notice that $\psi(t,\cdot) \notin L^2(\mathbb{R}^d)$ for all t,

$$\int dx|\psi_k(t,x)|^2 = +\infty \tag{3.6}$$

This does not make sense in quantum mechanics.

Since the Schrödinger equation is a linear equation, a linear combination of Eq. 3.5 is also a solution. We consider solutions of the form:

$$\psi(t,x) = \int_{\mathbb{R}^d}\rho(k)\psi_k(x,t)dk \equiv \int_{\mathbb{R}^d}\rho(k)e^{-i(k^2t-k\cdot x)}dk \tag{3.7}$$

subject to the initial condition

$$\psi(0,x) = \psi_0(x) = \int_{\mathbb{R}^d}\rho(k)e^{ikx}dk \tag{3.8}$$

The question is for what class of $\rho(k)$ the function $\psi(t,x)$ makes sense in quantum mechanics, that is $\psi(t,\cdot) \in L^2(\mathbb{R}^d)$?

Now we use the Fourier transform to solve the free Schrödinger equation. We get

$$i\partial_t\hat{\psi}(t,k) = |k|^2\hat{\psi}(t,k) \tag{3.9}$$

The solution of this ordinary differential equation is

$$\hat{\psi}(t,k) = e^{-i|k|^2t}\hat{\psi}(0,k) \tag{3.10}$$

Then we take the inverse Fourier transform to get:

$$\psi(t,x) = (\mathcal{F}^{-1}e^{-i|k|^2t}\mathcal{F}\psi_0)(x) \tag{3.11}$$

Theorem 3.1. *(Existence of a unique global solution for the free Schrödinger equation) Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$($\mathcal{S}$ denotes the Schwartz space). Then, there exists a global solution $\psi \in \mathcal{C}(\mathbb{R}_t, \mathcal{S}(R)^d)$ of the free Schrödinger equation for $t \neq 0$, given by the expression*

$$\psi(t,x) = \frac{1}{(2\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{t}} \psi_0(y) dy \tag{3.12}$$

with initial condition $\psi(0,x) = \psi_0(x)$. Moreover, $||\psi(t,\cdot)||_{L^2(\mathbb{R}^d)} = ||\psi_0||_{L^2(\mathbb{R}^d)}$

Remark 3.1. *Eq. 3.12 implies that*

$$\sup_{x\in\mathbb{R}^d} |\psi(t,x)| \leq \frac{||\psi_0||_{L^1}}{(2\pi t)^{d/2}} \to 0 \tag{3.13}$$

as $t \to \infty$. However, the above theorem tells us that the L^2 norm stays constant. This means the solution is flattened in space.

4 Quantum dynamics

In the finite-dimensional case, the solution of the Schrödinger equation

$$i\frac{d}{dt}\psi(t) = H\psi(t) \tag{4.1}$$

is given by

$$\psi(t) = e^{-itH}\psi(0) \tag{4.2}$$

We want to generalize this to self-adjoint operators using the spectral theorem.

4.1 Existence and uniqueness of the solution

For self-adjoint operator H, we define

$$U(t) = e^{-iHt} := \int e^{-i\lambda t} dP(\lambda) \tag{4.3}$$

where P is the projection-valued measure associated to $(H, \mathcal{D}(H))$.

Theorem 4.1. *Let H be a self-adjoint operator and let $U(t) = e^{-iHt}$. Then*

(i) $U(t)$ is a strongly continuous one-parameter unitary group.

(ii) the limit $\lim_{t\to 0} \frac{1}{t}(U(t)\psi - \psi)$ exists if and only if $\psi \in \mathcal{D}(H)$. In this case $\lim_{t\to 0} \frac{1}{t}(U(t)\psi - \psi) = -iH\psi$.

(iii) $U(t)\mathcal{D}(H) = \mathcal{D}(H)$ and $HU(t) = U(t)H$

This theorem implies $U(t)\psi(0)$ is indeed the solution to 4.1. In fact it is also the only solution.

Lemma 4.2. *Let $\psi_0 \equiv \psi(0) \in \mathcal{D}(H)$, and $\psi(t)$ be the solution to 4.1. Then $\psi(t) = U(t)\psi_0$*

Now we know self-adjoint corresponds to a one-parameter unitary group. The Stone's theorem tells us the converse.

Theorem 4.3. *(Stone) Let $U(t)$ be a weakly continuous one-parameter unitary group. Let $H : \mathcal{D}(H)$ be the generator of $U(t)$. Then H is self-adjoint and $U(t) = e^{-iHt}$.*

4.2 The RAGE theorem

In this section, we want to understand the asymptotic behavior of a quantum system, based on the spectral properties of a self-adjoint operator H. Let $\mathcal{H}_{ac}$,$\mathcal{H}_{sc}$, and $\mathcal{H}_{pp}$ be the absolutely continuous, singularly continuous, pure point spectral subspaces of H, respectively. We know $\mathcal{H} = \mathcal{H}_{ac} \bigoplus \mathcal{H}_{sc} \bigoplus \mathcal{H}_{pp}$.

Firstly, let's make some simple observation. Let $\psi \in \mathcal{H}_{ac}$, then the measure μ_ψ is absolutely continuous. For all $\varphi \in \mathcal{H}$, we have

$$|\mu_{\varphi,\psi}(\Omega)| = |\langle \varphi, P(\Omega)\psi\rangle| \leq ||\langle \varphi, P(\Omega)\varphi\rangle||^{1/2}||\langle \psi, P(\Omega)\psi\rangle||^{1/2} = \mu_\varphi(\Omega)^{1/2}\mu_\psi(\Omega)^{1/2} \tag{4.4}$$

Hence, $\mu_{\varphi,\psi}$ is also absolutely continuous. Define $U(t) = e^{-iHt}$ as before, by the Riemann-Lebesgue lemma, we have

$$\langle \varphi, U(t)\psi\rangle = \int e^{-i\lambda t} d\mu_{\varphi,\psi} \to 0 \tag{4.5}$$

as $t \to 0$. This implies, if we start from the absolutely continuous spectral subspace, then the probability of finding the system in any state φ is zero. However, if ψ is an eigenvector of H, then

$$|\langle \varphi, U(t)\psi\rangle| = |\langle \varphi, \psi\rangle| \tag{4.6}$$

shows the state is unchanged as time evolves.

We have a theorem that gives us more comprehensive understanding of the long-term behavior of the system.

Theorem 4.4 (Wiener)**.** *Let μ be a finite complex Borel measure on $\mathbb{R}$ and*

$$\hat{\mu}(t) := \int_{\mathbb{R}} e^{-it\lambda} d\mu(\lambda). \tag{4.7}$$

Then,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\hat{\mu}(t)|^2 dt = \sum_{\lambda\in\mathbb{R}} |\mu(\{\lambda\})|^2, \tag{4.8}$$

where the sum on the right-hand side is finite.

Remark 4.1. *Since every Borel measure can be decomposed as $\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$, and since μ_{ac} and $\mu_s c$ are continuous, $\mu(\{\lambda\}) = \mu_{pp}(\lambda)$. Then, $\sum_{\lambda\in\mathbb{R}} |\mu(\{\lambda\})|^2 = \sum_{\lambda\in\mathbb{R}} |\mu_{pp}(\{\lambda\})|^2$. Note the support of μ_{pp} is a countable set.*

Now let's apply this theorem to study the long-term behavior of the system. Let $\psi \in \mathcal{H}_{ac} \bigoplus \mathcal{H}_{sc}$, and $\varphi \in \mathcal{H}$. Then, the measure $\mu_{\varphi,\psi}(\{\lambda\}) = 0$. By the Theorem 4.4, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\langle \varphi, e^{-iHt}\psi\rangle|^2 dt = 0 \tag{4.9}$$

The average of probability finding the system in φ tends to zero.

We can also consider $|\langle\varphi, e^{-iHt}\psi\rangle|^2$ as $|P_\varphi U(t)\psi|^2$, where P_φ denotes the orthogonal projection onto φ. Inspired by this, we extend to a more general class of operators, the compact operators, which generalize the finite rank operators. An operator $K \in \mathcal{L}(\mathcal{H})$ is called a **finite rank operator** if its range is of finite dimension. Every finite rank operator can be written as a linear combination of projection operators.

Definition 4.1. *An operator $K \in \mathcal{L}(\mathcal{H})$ is called compact if K maps every unit ball in $\mathcal{H}$ to a pre-compact set.*

Definition 4.2. *An operator $K : \mathcal{D}(K) \to \mathcal{H}$ is called relatively compact with respect to an self-adjoint operator H if there exists $z \in \rho(H)$, such that $KR_z(H) = K(z-H)^{-1}$ is compact.*

The notion of a relatively compact operator gives us the following theorem, which is handy to our goal, the RAGE theorem.

Theorem 4.5. *Let H be a self-adjoint operator, K be relatively compact with respect to H. Then, for all $\psi \in \mathcal{D}(H)$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |Ke^{-iHt}P_c(H)\psi|^2 dt = 0, \tag{4.10}$$

and

$$\lim_{t\to\infty} ||Ke^{-iHt}P_{ac}(H)\psi||^2 = 0, \tag{4.11}$$

where $P_c(H) = P_{ac}(H) + P_{sc}(H)$ is the orthogonal projection onto $\mathcal{H}_{ac} \bigoplus \mathcal{H}_{sc}$. Furthermore, if K is bounded, then the result holds true for all $\psi \in \mathcal{H}$.

Finally, we have the RAGE theorem, which tells us $||Ke^{-iHt}\psi||$ can be used to identify the spectral subspaces $\mathcal{H}_c$ and $\mathcal{H}_{pp}$ of H.

Theorem 4.6. *(RAGE) Let H be a self-adjoint operator and K_n be a sequence of relatively compact operators with respect to H, converging strongly to the identity. Then,*

$$\mathcal{H}_c = \{\psi \in \mathcal{H} | \lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T ||K_n e^{-iHt}\psi|| = 0\} \tag{4.12}$$

$$\mathcal{H}_{pp} = \{\psi \in \mathcal{H} | \lim_{n\to\infty}\sup_{t\geq 0} ||(\mathbb{I} - K_n)e^{-iHt}\psi|| = 0\} \tag{4.13}$$

5 Nonlinear stationary Schrödinger equation in 1-D

In this section, we consider the nonlinear stationary Schrödinger equation defined on $\mathbb{R}$:

$$(-\Delta - k^2)u = f(u) \tag{5.1}$$

where f represents the nonlinearity of the form

$$f(u) = V(x)u + w(x)|u|^2u \tag{5.2}$$

with $w(x)$ and $V(x)$ compactly supported and bounded. Our first task is to show the existence of a solution by contraction mapping. The second task is to approximate the scattering matrix.

5.1 Existence of the outgoing solution

Let's first consider the case $V(x) \equiv 0$. The solution outside the support of the perturbation is just the superposition of plane waves. We decompose $u = u_{out} + u_{in}$, where $u_{in} = Ae^{ikx}$ is the incoming solution that solves the unperturbed equation, and u_{out} is a correction. Plugging into 5.1:

$$(-\Delta - k^2)(u_{out} + u_{in}) = f(u_{out} + u_{in}). \tag{5.3}$$

Let's now rewrite our problem:

$$(-\Delta - k^2)\tilde{u} = f(u_{out} + Ae^{ikx}) \tag{5.4}$$

for $\tilde{u} \in X$, for some complete metric space X. $\tilde{u}$ is then given by

$$\tilde{u} = (-\Delta - k^2)^{-1}_{out} f(u_{out} + Ae^{ikx}) \tag{5.5}$$

where we used the outgoing condition $(-\Delta - k^2)^{-1}_{out} = \lim_{\varepsilon \to 0^+}(-\Delta - (k^2 + i\varepsilon))^{-1}$. If we show $T(u_{out}) := (-\Delta - k^2)^{-1}_{out} f(u_{out} + Ae^{ikx})$ is a contraction X, then the above equation has a unique fixed point in X satisfying $u_{out} = (-\Delta - k^2)^{-1}_{out} f(u_{out} + Ae^{ikx})$.

We need to first discuss the function space we should work with. The function u must be twice differentiable, and thus must be bounded on any compact domain. This implies that our nonlinear term $V(x)u + w(x)|u|^2 u$ is bounded on any compact domain. Outside the compact domain of perturbation, the solution is a plane wave. Since every term is in $L^\infty(\mathbb{R})$, we consider $u \in L^\infty(\mathbb{R})$.

Now we define the outgoing Green's function. If $k^2 \notin \sigma(-\Delta) \equiv [0, \infty)$, then the solution is:

$$\tilde{u} = \left[(-\Delta - k^2)^{-1}(f(u_{out} + Ae^{ikx}))\right](x) \tag{5.6}$$

with the corresponding Green's function:

$$G(x; k^2) = \frac{1}{2\sqrt{-k^2}} e^{-|x|\sqrt{-k^2}}. \tag{5.7}$$

The outgoing Green's function when $k \in (0, \infty)$ is

$$G_{out}(x; k^2) = \lim_{\varepsilon \to 0^+} G(x; k^2 + i\varepsilon) = -\frac{1}{2ik} e^{i|k||x|}. \tag{5.8}$$

We will use a variance of the fixed point argument.

Theorem 5.1. *Let (X, d) be a complete metric space, and $A : X \to X$. Furthermore, assume there exists $a \in X$ and $r > 0$ such that*

(i) the ball $B(a, r)$ is an invariant set for A.

(ii) the map A is a contraction on $B(a, r)$.

Then, there exists a unique fixed point of A inside $B(a,r)$.

Now let me introduce the first existence theorem.

Theorem 5.2. *For an incoming solution $u_{in} = Ae^{ikx}$ with the amplitude A small enough, there exists an outgoing solution u_{out} for eqn. 5.1.*

Proof. Consider $B_{L^\infty}(-Ae^{ikx}, r)$, we want to show T is invariant on this ball for suitable r. Let $v \in B_{L^\infty}(-Ae^{ikx}, r)$, we have

$$\begin{aligned}
\|T(v)\|_{L^\infty} &= \operatorname*{ess\,sup}_x \left| \int_{\mathbb{R}} -\frac{1}{2ik} e^{ik|x-y|} w(y) \left|v + Ae^{iky}\right|^2 (v + Ae^{iky}) dy \right| \\
&\leq \frac{1}{2k} \int_{\mathbb{R}} |w(y)| \left|v + Ae^{iky}\right|^3 dy \\
&\leq \frac{\|w\|_{L^1}}{2k} \left\|v + Ae^{iky}\right\|_{L^\infty}^3 \\
&\leq \frac{\|w\|_{L^1}}{2k} r^3.
\end{aligned}$$

Thus r needs to satisfy

$$\frac{\|w\|_{L^1}}{2k} r^3 + |A| \leq r. \tag{5.9}$$

For contraction, let $v, \tilde{v} \in B_{L^\infty}(-Ae^{ikx}, r)$ and set $a(x) = v(x) + Ae^{ikx}$, $b(x) = \tilde{v}(x) + Ae^{ikx}$. Note $\|a\|_{L^\infty} \leq r$ and $\|b\|_{L^\infty} \leq r$. Then,

$$\begin{aligned}
\|T(v) - T(\tilde{v})\|_{L^\infty} &\leq \frac{\|w\|_{L^1}}{2k} \left\| |v + Ae^{ikx}|^2 (v + Ae^{ikx}) - |\tilde{v} + Ae^{ikx}|^2 (\tilde{v} + Ae^{ikx}) \right\|_{L^\infty} \\
&= \frac{\|w\|_{L^1}}{2k} \left\| |a|^2 a - |b|^2 b \right\|_{L^\infty} \\
&= \frac{\|w\|_{L^1}}{2k} \left\| \left(|a|^2 + |b|^2\right)(a - b) + ab\left(\overline{a} - \overline{b}\right) \right\|_{L^\infty} \\
&\leq \frac{\|w\|_{L^1}}{2k} 3r^2 \|v - \tilde{v}\|_{L^\infty}.
\end{aligned}$$

We then require

$$r < \sqrt{\frac{2k}{3\|w\|_{L^1}}}. \tag{5.10}$$

For suitable $|A| \neq 0$, M and$\|w\|_\infty$ such that there exists a r satisfies 5.9 and 5.10, the contraction mapping gives us a non-trivial outgoing solution, since $u_{out} = 0$ is not a fixed point. The global solution $u_{in} + u_{out}$ is also not trivial since $u_{out} = Ae^{ikx}$ is not a fixed point. □

Remark 5.1. *When there exists a r such that 5.9 and 5.10 are satisfied? Consider*

$$r^3 - \frac{2k}{\|w\|_{L^1}} r + \frac{2k}{\|w\|_{L^1}} |A| = 0. \tag{5.11}$$

Define $p := -\frac{2k}{\|w\|_{L^1}}$ *and* $q := \frac{2k}{\|w\|_{L^1}}|A|$. *We want to find the intersection of* $r^3 + q = -pr$. *In order to have a real* $r > 0$ *to exist, we must have two intersections in the first quadrant. The last intersection must lay in the third quadrant. This implies that all three roots must be real. Thus* p *and* q *must satisfy* $4p^3 + 27q^2 \leq 0$. *In our case,* $-\left(\frac{2k}{3\|w\|_{L^1}}\right)^3 + \left(\frac{k|A|}{\|w\|_{L^1}}\right)^2 = -D \leq 0$, *for some* $D \geq 0$. *Equivalently,* $|A| \leq \sqrt{\frac{8k}{27\|w\|_{L^1}}}$.

In addition, using the Cardano formula, the root of 5.11 is given by

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \tag{5.12}$$

This reduces to

$$x_n = 2R^{\frac{1}{3}} \cos(\frac{\theta}{3} + \frac{2\pi}{3}n), \qquad n = 0, 1, 2 \tag{5.13}$$

where $R = \sqrt{\frac{|A|^2k^2}{(\|w\|_{L^1})^2} + D} = \sqrt{\left(\frac{2k}{3\|w\|_{L^1}}\right)^3}$ *and* $\theta = \tan^{-1}\left(\frac{\sqrt{D}}{-\frac{k|A|}{\|w\|_{L^1}}}\right)$. *Then* $\theta \in (\frac{1}{2}\pi, \frac{3}{2}\pi]$ *and we have two roots greater then zero.*

If we apply the constraint 5.10, the second largest root $x_1 = 2R^{\frac{1}{3}} \cos(\frac{\theta}{3} - \frac{2\pi}{3})$ *must be less than* $\sqrt{\frac{2k}{3\|w\|_{L^1}}}$, *so there exists a* r *to have a contraction. Noticing* $\cos(\frac{\theta}{3} - \frac{2\pi}{3}) < \frac{1}{2}$, *so* $x_1 < R^{\frac{1}{3}} \leq \sqrt{\left(\frac{2k}{3\|w\|_{L^1}}\right)}$.

To summarize, our fixed point argument works as long as $|A| \leq \sqrt{\frac{8k}{27\|w\|_{L^1}}}$, *or a bound on a form of energy*

$$|\lambda|^2\|w\|_{L^1} \leq \frac{8k}{27}.$$

Now consider $V(x) \not\equiv 0$. The fixed argument gives us a similar result. Considering $v \in B_{L^\infty}(-Ae^{ikx}, r)$, then

$$T(v) \leq \frac{\|V\|_{L^1}}{2k}r + \frac{\|w\|_{L^1}}{2k}r^3.$$

To have T invariant on $B_{L^\infty}(-Ae^{ikx}, r)$, we require

$$\frac{\|V\|_{L^1}}{2k}r + \frac{\|w\|_{L^1}}{2k}r^3 + |A| \leq r. \tag{5.14}$$

Let $v, \tilde{v} \in B_{L^\infty}(-Ae^{ikx}, r)$,

$$\begin{aligned}\|T(v) - T(\tilde{v})\|_{L^\infty} &\leq \frac{\|V\|_{L^1}}{2k}\|v - \tilde{v}\|_{L^\infty} + \frac{\|w\|_{L^1}}{2k}3r^2\|v - \tilde{v}\|_{L^\infty} \\ &= \left(\frac{\|V\|_{L^1}}{2k} + 3r^2\frac{\|w\|_{L^1}}{2k}\right)\|v - \tilde{v}\|_{L^\infty}.\end{aligned}$$

To have T contraction, we require

$$\left(\frac{\|V\|_{L^1}}{2k} + 3r^2 \frac{\|w\|_{L^1}}{2k} \right) \leq 1. \tag{5.15}$$

For A small, there exists a r satisfies the inequalities.

5.2 Scattering matrix

Consider the case $V \equiv 0$. We approximate the scattering matrix by approximating the fixed point, assuming the incoming wave Ae^{ikx} we sent in is small (i.e. $|A|$ small).

Let's start from $u_0 = 0$, the first iteration gives

$$u_1(x) = T(u_0)(x) = -\frac{1}{2ik}|A|^2 \int_{\mathbb{R}} e^{ik|x-y|} w(y) A e^{iky} dy. \tag{5.16}$$

For the next iteration, defining $h(y) := e^{-iky} \int_{\mathbb{R}} e^{ik|x-y|} w(x) e^{ikx} dx$, and keeping order up to $|A|^5$, we use the approximation

$$u_2 = u_1 + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} e^{ik|x-y|} w(y) A e^{iky} \left(\overline{h(y)} - 2h(y) \right) dy + \mathcal{O}(|A|^7). \tag{5.17}$$

We will use $u_{out} = u_1 + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} e^{ik|x-y|} w(y) A e^{iky} \left(\overline{h(y)} - 2h(y) \right) dy$ as an approximation to the outgoing solution. When x is at the right of the perturbation which corresponds to transmission, we have

$$\begin{aligned} u_{out} &= Ae^{ikx} \left\{ -\frac{1}{2ik}|A|^2 \int_{\mathbb{R}} w(y) dy + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} w(y) \left(\overline{h(y)} - 2h(y) \right) dy \right\} \\ &:= Ae^{ikx} T_-. \end{aligned} \tag{5.18}$$

When x is at the left of the perturbation which corresponds to reflection, we have

$$\begin{aligned} u_{out} &= Ae^{-ikx} \left\{ -\frac{1}{2ik}|A|^2 \int_{\mathbb{R}} w(y) e^{2iky} dy + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} w(y) e^{2iky} \left(\overline{h(y)} - 2h(y) \right) dy \right\} \\ &:= Ae^{-ikx} R_-. \end{aligned} \tag{5.19}$$

Now assume incoming wave from right $u_{in} = e^{-ikx}$. Note that the Green's function is unchanged. Then,

$$u_{out} = -\frac{1}{2ik}|A|^2 \int_{\mathbb{R}} e^{ik|x-y|} w(y) A e^{-iky} dy + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} e^{ik|x-y|} w(y) A e^{-iky} \left(\overline{g(y)} - 2g(y) \right) dy \tag{5.20}$$

where $g(y) = e^{iky} \int_{\mathbb{R}} e^{ik|x-y|} w(x) e^{-ikx} dx$. When x is at the left of the perturbation which corresponds to transmission, we have

$$u_{out} = Ae^{-ikx} \left\{ -\frac{1}{2ik}|A|^2 \int_{\mathbb{R}} w(y) dy + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} w(y) \left(\overline{g(y)} - 2g(y) \right) dy \right\}$$

$$:= Ae^{-ikx}T_+. \tag{5.21}$$

When x is at the right of the perturbation which corresponds to reflection, we have

$$\begin{aligned} u_{out} &= Ae^{ikx}\left\{-\frac{1}{2ik}|A|^2\int_{\mathbb{R}} w(y)e^{-2iky}dy + \frac{1}{4k^2}|A|^4\int_{\mathbb{R}} w(y)e^{-2iky}\left(\overline{g(y)} - 2g(y)\right)dy\right\} \\ &:= Ae^{ikx}R_+. \end{aligned} \tag{5.22}$$

The scattering matrix is given by the coefficients given above.

In the case $V \not\equiv 0$, we compute the linearized solution:

$$u_{out} = -\frac{1}{2ik}\int_{\mathbb{R}} e^{ik|x-y|}\left[V(y) + w(y)|A|^2\right]Ae^{iky}. \tag{5.23}$$

The scattering coefficients are:

$$\begin{aligned} T_- &= -\frac{1}{2ik}\int_{\mathbb{R}}\left[V(y) + w(y)|A|^2\right]dy \\ R_- &= -\frac{1}{2ik}\int_{\mathbb{R}} e^{2iky}\left[V(y) + w(y)|A|^2\right]dy \\ T_+ &= -\frac{1}{2ik}\int_{\mathbb{R}}\left[V(y) + w(y)|A|^2\right]dy \\ R_+ &= -\frac{1}{2ik}\int_{\mathbb{R}} e^{-2iky}\left[V(y) + w(y)|A|^2\right]dy \end{aligned}$$

We see, in the linear approximation, that the reflection coefficient is given by the Fourier transform of $V(y) + w(y)|A|^2$. Using two different incoming solutions, we have a system of equations. We can recover potential by taking the inverse Fourier transform of the reflection coefficients.

6 Non-linear Stationary Dirac equation

We now turn our focus to the non-linear stationary Dirac equation with a linear domain wall:

$$(H - E)\,\psi(x,y) = f(\psi)(x,y), \qquad H = D_x\sigma_3 - D_y\sigma_2 + y\sigma_1, \qquad f(\psi) = w(x,y)\,(\psi^* C\psi)\,\psi \tag{6.1}$$

where $w(x,y)$ is compactly supported and bounded, and C is a constant matrix. We decompose $\psi = \psi_{in} + \psi_{out}$ as before.

6.1 One-dimensional non-linear Dirac equation

We consider Eqn. 6.1 in 1-D:

$$(D_x\sigma_3 - k)\psi = w(x)\,(\psi^* C\psi)\,\psi. \tag{6.2}$$

Outside the support of $w(x)$, solution is $\psi_{in} = \begin{pmatrix} A_1 e^{-ikx} \\ A_2 e^{ikx} \end{pmatrix}$. Observe that $(D_x\sigma_3 - k)(D_x\sigma_3 + k) = (-\Delta - k^2)I_2$, so $(D_x\sigma_3 - k)^{-1}_{out} = (D_x\sigma_3 + k)\left[(-\Delta - k^2)^{-1}_{out} I_2\right]$ (applying $(D_x\sigma_3 + k)$ does not change the outgoing condition). Thus the outgoing Green's function of $D_x\sigma_3 - k$ is given by:

$$\begin{aligned} G_{out}(x;k) &= \begin{bmatrix} D_x + k & 0 \\ 0 & -D_x + k \end{bmatrix} \begin{bmatrix} -\frac{1}{2ik} e^{i|k||x|} & 0 \\ 0 & -\frac{1}{2ik} e^{i|k||x|} \end{bmatrix} \\ &= \begin{bmatrix} \frac{i}{2} e^{i|k||x|}(1 - sign(k)sign(x)) & 0 \\ 0 & \frac{i}{2} e^{i|k||x|}(1 + sign(k)sign(x)) \end{bmatrix}. \end{aligned} \tag{6.3}$$

Define T:

$$T(\psi) = (D_x\sigma_3 - k)^{-1}_{out} f(\psi + \psi_{in}) \tag{6.4}$$

for $\psi \in L^\infty(\mathbb{R}, \mathbb{C}^2)$. $L^\infty(\mathbb{R}, \mathbb{C}^2)$ is a Banach space with respect to $\|\psi\| = \operatorname{ess\,sup}_x \|\psi\|$, where the later norm can be any l_p norm. Indeed, let ψ^n be a Cauchy sequence in $L^\infty(\mathbb{R}, \mathbb{C}^2)$, and let $\begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}$ be the elementwise limit of $\begin{pmatrix} \psi_1^n \\ \psi_2^n \end{pmatrix}$ in L^∞, then obviously $\begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}$ is also the limit of ψ^n in $L^\infty(\mathbb{R}, \mathbb{C}^2)$.

Theorem 6.1. *For an incoming solution $\psi_{in} = \begin{pmatrix} A_1 e^{-ikx} \\ A_2 e^{ikx} \end{pmatrix}$ with amplitude small enough, there exists an outgoing solution ψ_{out} of eqn. 6.2.*

Proof. We first show T is invariant on $B_{\|\cdot\|}(-\psi_{in}, r)$ for suitable r. Let $\psi \in B_{\|\cdot\|}(-\psi_{in}, r)$ and σ_c be the largest singular value of C, we have:

$$\begin{aligned} \|T(\psi)\| &= \|G_{out}(x;k) * w(x)\left[(\psi + \psi_{in})^* C(\psi + \psi_{in})\right](\psi + \psi_{in})\| \\ &\le \Bigg[\left(\int_{\mathbb{R}} |w(y)|\,|(\psi + \psi_{in})^* C(\psi + \psi_{in})|\,|\psi_1 + A_1 e^{-iky}| dy\right)^2 \\ &\qquad + \left(\int_{\mathbb{R}} |w(y)|\,|(\psi + \psi_{in})^* C(\psi + \psi_{in})|\,|\psi_2 + A_2 e^{iky}| dy\right)^2\Bigg]^{\frac{1}{2}} \\ &\le \sigma_c \|w\|_{L^1} \left[\operatorname*{ess\,sup}_y \left(|(\psi + \psi_{in})|^2\,|\psi_1 + A_1 e^{-iky}|\right)^2 + \operatorname*{ess\,sup}_y \left(|(\psi + \psi_{in})|^2\,|\psi_2 + A_2 e^{iky}|\right)^2\right]^{\frac{1}{2}} \\ &\le \sigma_c \|w\|_{L^1} \left[2 \operatorname*{ess\,sup}_y \left[\left(|(\psi + \psi_{in})|^2\,|\psi_1 + A_1 e^{-iky}|\right)^2 + \left(|(\psi + \psi_{in})|^2\,|\psi_2 + A_2 e^{iky}|\right)^2\right]\right]^{\frac{1}{2}} \\ &\le \sqrt{2}\sigma_c \|w\|_{L^1} r^3. \end{aligned}$$

We then require $\sqrt{2}\sigma_c \|w\|_{L^1} r^3 + \sqrt{|A_1|^2 + |A_2|^2} \le r$.

Let $\psi, \phi \in B_{\|\cdot\|}(-\psi_{in}, r)$, we show T is a contraction. For simplicity, let's write $a := \psi + \psi_{in}$ and $b := \phi + \psi_{in}$, then by the Hölder inequality, $\|T(\psi) - T(\phi)\|$ is bounded by

$$\|w\|_{L^1} \left[\left(\operatorname*{ess\,sup}_y |((a^* C a)\, a_1 - (b^* C b)\, b_1|\right)^2 + \left(\operatorname*{ess\,sup}_y |(a^* C a)\, a_2 - (b^* C b)\, b_2|\right)^2\right]^{\frac{1}{2}}$$

$$
\begin{aligned}
\leq&\sqrt{2}\|w\|_{L^1} \operatorname*{ess\,sup}_y \left[\left|(a^*Ca)\, a_1 - (b^*Cb)\, b_1\right|^2 + \left|(a^*Ca)\, a_2 - (b^*Cb)\, b_2\right|^2\right]^{\frac{1}{2}} \\
=&\sqrt{2}\|w\|_{L^1} \operatorname*{ess\,sup}_y |(a^*Ca)a - (b^*Cb)b|. \qquad (6.5)
\end{aligned}
$$

Since we want $\|T(\psi) - T(\phi)\|$ to be bounded by some factor of $\|\psi - \phi\| = \|a - b\|$, we rewrite 6.5 as $\sqrt{2}\|w\|_{L^1} \operatorname{ess\,sup}_y |(a^*Ca + b^*Cb)(a-b) + (a^*Ca)b - (b^*Cb)a|$ which is bounded by

$$
\sqrt{2}\|w\|_{L^1} \operatorname*{ess\,sup}_y \left[|(a^*Ca + b^*Cb)(a-b)| + |(a^*Ca)b - (b^*Cb)a|\right]. \qquad (6.6)
$$

Note the first part,

$$
|(a^*Ca + b^*Cb)(a-b)| \leq \sigma_c(|a|^2 + |b|^2)|a-b|. \qquad (6.7)
$$

Expanding the second part, $|(a^*Ca)b - (b^*Cb)a|$, we obtain

$$
\begin{aligned}
&\left[|a^*Ca|^2|b|^2 + |b^*Cb|^2|a|^2 - a^*C^*ab^*Cbb^*a - b^*C^*ba^*Caa^*b\right]^{\frac{1}{2}} \\
=&\left[|a^2||b|^2(a^*CC^*a) + |a|^2|b|^2(b^*CC^*b) - |a^2||b|^2(b^*CC^*a) - |a^2||b|^2(a^*CC^*b)\right]^{\frac{1}{2}} \\
=&|a||b||C^*a - C^*b|. \qquad (6.8)
\end{aligned}
$$

Thus,

$$
|(a^*Ca)b - (b^*Cb)a| \leq \sigma_c|a||b||a-b|. \qquad (6.9)
$$

Plug 6.7 and 6.9 into 6.6, we obtain

$$
\begin{aligned}
\|T(\psi) - T(\phi)\| \leq&\sqrt{2}\sigma_c\|w\|_{L^1} \operatorname*{ess\,sup}_y \left[(|a|^2 + |b|^2)|a-b| + |a||b||a-b|\right] \\
\leq&3\sqrt{2}\sigma_c\|w\|_{L^1}r^2\|a-b\| \\
\equiv&3\sqrt{2}\sigma_c\|w\|_{L^1}r^2\|\psi - \phi\|.
\end{aligned}
$$

Thus we require $3\sqrt{2}\sigma_c\|w\|_{L^1}r^2 \leq 1$.

We restrict the parameters to be

$$
|\lambda|^2\|w\|_{L^1} \leq \frac{4\sqrt{2}}{27\sigma_A}.
$$

$\square$

We compute the leading order approximation of the scattering matrix. The reflection coefficients are both zero. The transmission coefficients are:

$$
\begin{aligned}
T_+ &= \begin{pmatrix} i\int_{\mathbb{R}} w(y)\left(\begin{pmatrix} A_1e^{-iky} \\ A_2e^{iky}\end{pmatrix}^* C \begin{pmatrix} A_1e^{-iky} \\ A_2e^{iky}\end{pmatrix}\right) dy \\ i\int_{\mathbb{R}} w(y)\left(\begin{pmatrix} A_1e^{-iky} \\ A_2e^{iky}\end{pmatrix}^* C \begin{pmatrix} A_1e^{-iky} \\ A_2e^{iky}\end{pmatrix}\right) dy \end{pmatrix} \\
T_- &= \begin{pmatrix} i\int_{\mathbb{R}} w(y)\left(\begin{pmatrix} A_1e^{iky} \\ A_2e^{-iky}\end{pmatrix}^* C \begin{pmatrix} A_1e^{iky} \\ A_2e^{-iky}\end{pmatrix}\right) dy \\ i\int_{\mathbb{R}} w(y)\left(\begin{pmatrix} A_1e^{iky} \\ A_2e^{-iky}\end{pmatrix}^* C \begin{pmatrix} A_1e^{iky} \\ A_2e^{-iky}\end{pmatrix}\right) dy \end{pmatrix}.
\end{aligned}
$$

6.2 Two-dimensional non-linear Dirac equation

We now consider eqn. 6.1 in two dimensions. The solutions satisfying the unperturbed equation and the Green's function to $(H-E)$ are given in [2], and we will introduce them here.

6.2.1 Solution to the unperturbed equation

The operator H in 6.1 is translational invariable in x direction, thus we take the Fourier transform in x. Denoting ξ the Fourier variable,

$$\hat{H}(\xi) - E = \xi\sigma_3 - D_y\sigma_2 + y\sigma_3 - E = \begin{pmatrix} \xi & \partial_y + y \\ -\partial_y + y & -\xi \end{pmatrix} - E.$$

Note $\mathbf{a} = \partial_y + y$ is the creation operator, and $\mathbf{a}^* = -\partial_y + y$ is the annilation operator. It is useful to look at the block diagonal matrix:

$$\hat{H}(\xi)^2 = \begin{pmatrix} \xi^2 + \mathbf{a}\mathbf{a}^* & 0 \\ 0 & \xi^2 + \mathbf{a}^*\mathbf{a} \end{pmatrix}.$$

We define $\varphi_n(y) = a_n(\mathbf{a}^*)^n\varphi_0(y)$ (a_ns are the normalizing constants), then $\varphi_n(y)$ are Hermite functions that form an orthonormal basis of $L^2(\mathbb{R}_y)$, and satisfy the following properties:

$$\mathbf{a}^*\mathbf{a}\varphi_n = 2n\varphi_n, \quad \mathbf{a}\varphi_n = \sqrt{2n}\varphi_{n-1}, \quad \mathbf{a}^*\varphi_n = \sqrt{2n+2}\varphi_{n+1}, \qquad \varphi_0 = \pi^{-\frac{1}{4}}e^{-\frac{1}{2}y^2}.$$

We define a set M consisting of indices $m = (n, \epsilon_m)$, where $\mathbb{N} \ni n \geq 1$ and $\epsilon_m = \pm 1$. In the case $n = 1$, we define $m = (0, -1)$. The eigenvalue of $\hat{H}(\xi)$ are $E_m = \epsilon_m(2n + \xi^2)^{\frac{1}{2}}$. Now for any $\xi \in \mathbb{R}$ and m, we define

$$\phi_m = c_m \begin{pmatrix} \mathbf{a}\varphi_n \\ (E_m - \xi)\varphi_n \end{pmatrix}, \qquad n \geq 1,$$

where $c_m = \frac{1}{\sqrt{(2n+(E_m-\xi)^2)}}$ is the normalizing constant. In the case $n = 0$, $m = (0, -1)$ and $E_0(\xi) = -\xi$, we define

$$\phi_0 = \begin{pmatrix} 0 \\ \varphi_0 \end{pmatrix}.$$

The above family of eigenvectors ϕ_m form a basis of $L^2(\mathbb{R}, \mathbb{C}^2)$.

The above discussion characterized the spectrum decomposition of $\hat{H}(\xi)$. While to solve eqn. 6.1 for fixed a $E \in \mathbb{R}$, we need to reverse the map $E_m = \epsilon_m(2n + \xi^2)^{\frac{1}{2}}$:

$$\xi_m = \epsilon_m\sqrt{(E^2 - 2n)} = \begin{cases} \epsilon_m\sqrt{(E^2 - 2n)}, & E^2 \geq 2n \\ i\epsilon_m\sqrt{(2n - E^2)}, & E^2 \leq 2n. \end{cases}$$

Then, $\phi_m(x, y; E)$ satisfy:

$$(\hat{H}(\xi_m) - E)\phi_m = \begin{pmatrix} \xi_m - E & \mathbf{a} \\ \mathbf{a}^* & -(\xi_m + E) \end{pmatrix} \phi_m = 0.$$

In the physical domain, the generalized eigenvectors

$$\psi_m(x, y; E) = e^{i\xi_m x}\phi_m(y; E) \tag{6.10}$$

satisfy $(H - E)\psi_m = 0$. Linear combinations of these eigenvectors then give the solutions to the unperturbed equation.

6.2.2 Outgoing Green's function

To construct the Outgoing Green's function to $(H - E)$, we need to solve $(H - E)G = \delta(x - x_0)\delta(y - y_0)I$. Since H is translational invariable in the x direction, we assume $x_0 = 0$. Note that $(H + E)(H - E)G = (H + E)\delta(x)\delta(y - y_0)I$, then

$$G = (H + E)(H^2 - E^2)^{-1}\delta(x)\delta(y - y_0)I.$$

As in the one-dimensional case, we need to first find $(H^2 - E^2)^{-1}\delta(x)\delta(y - y_0)I$, then apply $(H + E)$ on it. Noting $H^2 - E^2$ is diagonal

$$H^2 - E^2 = \begin{pmatrix} D_x^2 - E^2 + \mathbf{a}\mathbf{a}^* & 0 \\ 0 & D_x^2 - E^2 + \mathbf{a}^*\mathbf{a} \end{pmatrix},$$

we thus need to solve

$$(-\partial_x^2 - \partial_y^2 + y^2 \pm 1 - E^2)G_\pm = \delta(x)\delta(y - y_0).$$

Recall that $\varphi_n(y)$ are the eigenfunctions to $-\partial_y^2 + y^2 \pm 1$, we expand G_- in the basis of Hermite functions $\varphi_n(y)$:

$$G_- = \sum_n G_{-,n}(x)\varphi_n(y).$$

Then,

$$(-\partial_x^2 - E^2 + 2n)G_{-,n}(x) = \delta(x)\varphi_n(y_0)$$

is the Helmholtz equation, and the outgoing Green's function when $2n < E^2$ is given in 5.8. Assuming $E \neq 2n$ for $n \in \mathbb{N}$, we have

$$G_{-,n}(x) = \frac{\varphi_n(y_0)}{2\sqrt{|E^2 - 2n|}} \begin{cases} e^{-\sqrt{2n-E^2}|x|}, & 2n > E^2 \\ ie^{i\sqrt{E^2-2n}|x|}, & 2n < E^2 \end{cases}.$$

The computation of G_+ is similar by replacing $2n$ to $2n + 2$. Define $\theta_n = i\sqrt{E - 2n}$, then

$$G_-(x, y; y_0) = \sum_{n\geq 0} \frac{-e^{\theta_n|x|}}{2\theta_n}\varphi_n(y)\varphi_n(y_0)$$
$$G_+(x, y; y_0) = \sum_{n\geq 0} \frac{-e^{\theta_{n+1}|x|}}{2\theta_{n+1}}\varphi_n(y)\varphi_n(y_0).$$

Since applying $H + E$ does not change the outgoing condition, the outgoing Green's function of $H - E$ is:

$$G(x, y; y_0) = \begin{pmatrix} (D_x + E)G_+ & \mathbf{a}G_- \\ \mathbf{a}^*G_+ & (-D_x + E)G_- \end{pmatrix}(x, y; y_0). \tag{6.11}$$

Remark 6.1. *As stated in [2], the Green's function G has $\frac{1}{r}$ singularity.*

6.2.3 Contraction mapping

Let's first notice that the Green's function blows up like $\frac{1}{r}$ near the singularity, which implies G is integrable any compact interval for fixed x and y. Now let's introduce the main theorem.

Theorem 6.2. *For an incoming solution with amplitude small enough, there exists an outgoing solution of eqn. 6.1.*

Proof. As in the previous proof, we define the solution operator

$$T(\psi) := G * f(\psi + \psi_{in})(x, y).$$

Also, we use the uniform bound on the integral of G

$$\max\left\{\left(\|(D_x + E)G_+\|_{L^1(x_0,y_0)} + \|\mathbf{a}^* G_+\|_{L^1(x_0,y_0)}\right), \left(\|(-D_x + E)G_-\|_{L^1(x_0,y_0)} + \|\mathbf{a}G_-\|_{L^1(x_0,y_0)}\right)\right\} < c$$

where the integral is taken over the compact support of $w(x, y)$. The incoming solution is the superposition of 6.10

$$\psi_{in} = \sum_{m \in M} \alpha_m \psi_m.$$

We first show T is invariant in $B(-\psi_{in}, r)$ for some $r > 0$. Using the norm we introduced in the one-dimensional case, we have

$$\begin{aligned}
\|T(\psi)\| = \operatorname*{ess\,sup}_{(x,y)} \Bigg[\Bigg| & \int (D_x + E)G_+(x, y; x_0, y_0)[f(\psi)(x_0, y_0)]_1 dx_0 dy_0 \\
& + \int \mathbf{a}G_-(x, y; x_0, y_0)[f(\psi)(x_0, y_0)]_2 dx_0 dy_0 \Bigg| \\
& + \Bigg| \int (-D_x + E)G_-(x, y; x_0, y_0)[f(\psi)(x_0, y_0)]_2 dx_0 dy_0 \\
& + \int \mathbf{a}^* G_+(x, y; x_0, y_0)[f(\psi)(x_0, y_0)]_1 dx_0 dy_0 \Bigg| \Bigg].
\end{aligned}$$

Using the Hölder inequality and rearranging terms, denoting σ_c as the largest singular value of C, we bound

$$\begin{aligned}
\|T(\psi)\| \le \sigma_c \operatorname*{ess\,sup}_{(x,y)} \Bigg[& \|(D_x + E)G_+(x, y; x_0, y_0)w(x_0, y_0)\|_{L^1(x_0,y_0)} \operatorname*{ess\,sup}_{(x_0,y_0)} \left[|\psi + \psi_{in}|^2 |\psi_1 + \psi_{in,1}|\right] \\
& + \|\mathbf{a}^* G_+(x, y; x_0, y_0)w(x_0, y_0)\|_{L^1(x_0,y_0)} \operatorname*{ess\,sup}_{(x_0,y_0)} \left[|\psi + \psi_{in}|^2 |\psi_1 + \psi_{in,1}|\right] \\
& + \|(-D_x + E)G_-(x, y; x_0, y_0)w(x_0, y_0)\|_{L^1(x_0,y_0)} \operatorname*{ess\,sup}_{(x_0,y_0)} \left[|\psi + \psi_{in}|^2 |\psi_2 + \psi_{in,2}|\right] \\
& + \|\mathbf{a}G_-(x, y; x_0, y_0)w(x_0, y_0)\|_{L^1(x_0,y_0)} \operatorname*{ess\,sup}_{(x_0,y_0)} \left[|\psi + \psi_{in}|^2 |\psi_2 + \psi_{in,2}|\right] \Bigg].
\end{aligned}$$

We apply the Hölder inequality again, and use the bound c to obtain

$$\begin{aligned}\|T(\psi)\| &\leq \sigma_c c\|w\|_{L^\infty}\left[\operatorname*{ess\,sup}_{(x_0,y_0)}\left[|\psi+\psi_{in}|^2|\psi_1+\psi_{in,1}|\right]+\operatorname*{ess\,sup}_{(x_0,y_0)}\left[|\psi+\psi_{in}|^2|\psi_2+\psi_{in,2}|\right]\right]\\ &\leq 2\sigma_c c\|w\|_{L^\infty}r^3.\end{aligned}$$

Thus we need

$$2\sigma_c c\|w\|_{L^\infty}r^3+\sum_{m\in M}|\alpha_m|\leq r.$$

Let $\psi,\phi\in B(-\psi_{in},r)$, we show T is a contraction. Similarly, let's write $a(x_0,y_0):=(\psi+\psi_{in})(x_0,y_0)$ and $b(x_0,y_0):=(\phi+\psi_{in})(x_0,y_0)$, then

$$\begin{aligned}\|T(\psi)-T(\phi)\| = \operatorname*{ess\,sup}_{(x,y)}\Bigg[\Bigg|&\int(D_x+E)G_+(x,y;x_0,y_0)w(x_0,y_0)[(a^*Ca)a_1-(b^*Cb)b_1]dx_0dy_0\\ &\int \mathbf{a}G_-(x,y;x_0,y_0)w(x_0,y_0)[(a^*Ca)a_2-(b^*Cb)b_2]dx_0dy_0\Bigg|\\ \Bigg|&\int(-D_x+E)G_-(x,y;x_0,y_0)w(x_0,y_0)[(a^*Ca)a_2-(b^*Cb)b_2]dx_0dy_0\\ &\int \mathbf{a}^*G_+(x,y;x_0,y_0)w(x_0,y_0)[(a^*Ca)a_1-(b^*Cb)b_1]dx_0dy_0\Bigg|\Bigg].\end{aligned}$$

Similar to calculating the operator norm of T, we obtain

$$\begin{aligned}\|T(\psi)-T(\phi)\| &\leq 2c\|w\|_{L^\infty}\operatorname*{ess\,sup}_{(x_0,y_0)}\left[|(a^*Ca)a_1-(b^*Cb)b_1|+|(a^*Ca)a_2-(b^*Cb)b_2|\right]\\ &= 2c\|w\|_{L^\infty}\operatorname*{ess\,sup}_{(x_0,y_0)}\|(a^*Ca)a-(b^*Cb)b\|.\end{aligned}$$

which we have already shown the bound in 1-D case. Thus,

$$\|T(\psi)-T(\phi)\|\leq 6\sigma_c c\|w\|_{L^\infty}r^2\|\psi-\phi\|.$$

We require

$$6\sigma_c c\|w\|_{L^\infty}r^2<1.$$

Combining two inequalities, we have the bound

$$\left(\sum_m|\alpha_m|\right)^2\|w\|_{L^\infty}\leq\frac{2}{27\sigma_c c}.$$

□

6.3 Conclusion

We have successfully shown the existence of a solution to the nonlinear Helmholtz equation and nonlinear Dirac equation using fixed point arguments. The solutions are not trivial, as $u_{out} = -u_{in}$ is not a fixed point. This argument generalizes to other PDEs if the Green's functions are integrable on a compact domain.

In this paper, we assumed that nonlinearity is compactly supported. In future work, I want to apply a similar argument when the nonlinearity is fast-decaying to zero.

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