FOURIER TRANSFORM WITH ITS APPLICATION

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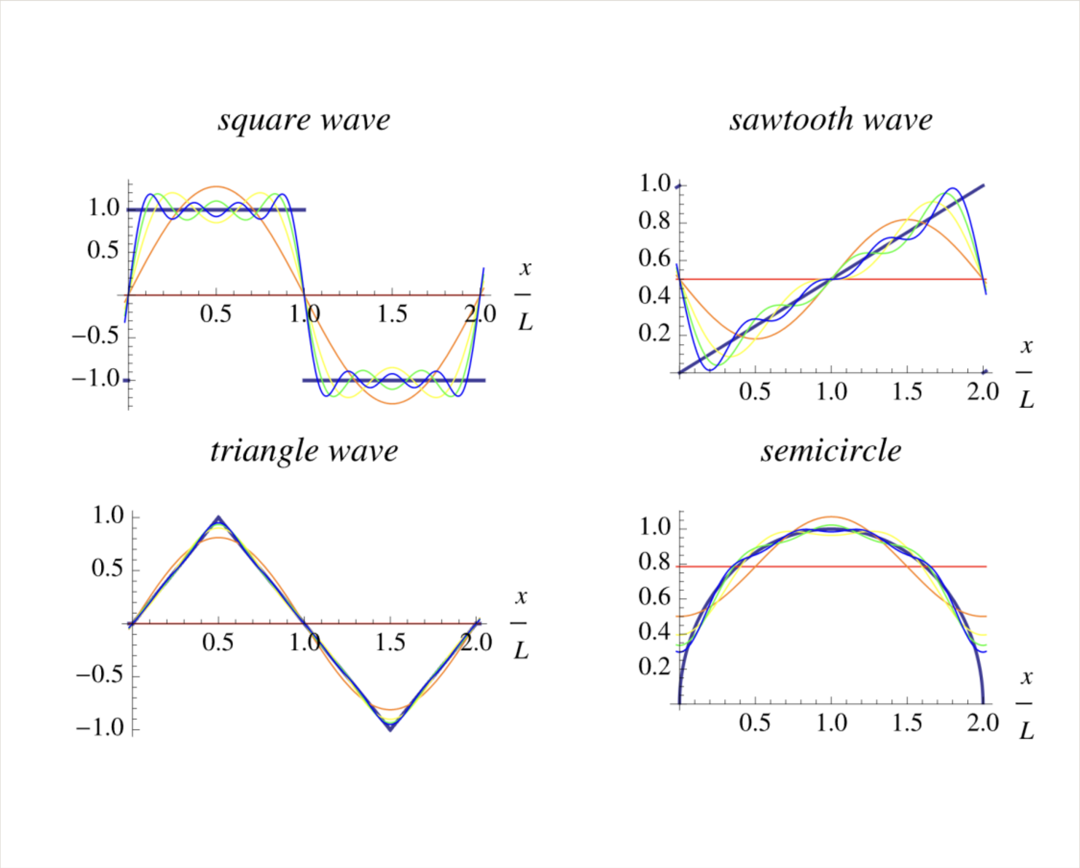
TEXTBOOK: FOURIER ANALYSIS, STEIN

- Reading process:
  - Chapters: 1-6
- Why do we need Fourier analysis?
  - In Taylor's theorem, we can expand arbitrarily "good" functions as the sum of polynomials, $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. In Fourier analysis, by the same analogy, we want to expand functions as the sum of sin and cos!

“FOURIER TRANSFORM” VS “TAYLOR EXPANSION”

- 1. ”Good” functions have a wide range of general functions that varied based on the functions. However, Fourier analysis only uses sin/cos functions, well-known and lots of good properties (ex. Periodic, continuous,…)
- 2. Sin and Cos functions have property of ocillation, making the phenomenoms more intensified than Taylor expansion.
- 3. By restraint of Taylor expansion, f should be belongs to class of c^⊗. But by Fourier transform, the functions are less depend on the “smoothness” and thus less restricted.

WHY WE NEED FOURIER TRANSFORM



- A tool for estimating equations and feedback images
- A tool that can be used to restore images
- A simplifying tool for complicated functions

HOW THIS TOOL WORKS

Given a function f on $[0, \pi]$ (with $f(0) = f(\pi) = 0$), can we find coefficients A_m so that

$$f(x) = \sum_{m=1}^{\infty} A_m \sin mx \ ?$$

can ask if an even function $g(x)$ on $[-\pi, \pi]$ can be expressed as a cosine series, namely

$$g(x) = \sum_{m=0}^{\infty} A'_m \cos mx.$$

More generally, since an arbitrary function F on $[-\pi, \pi]$ can be expressed as $f + g$, where f is odd and g is even,[3] we may ask if F can be written as

$$F(x) = \sum_{m=1}^{\infty} A_m \sin mx + \sum_{m=0}^{\infty} A'_m \cos mx,$$

$$\begin{aligned} \int_0^\pi f(x) \sin nx \, dx &= \int_0^\pi \left(\sum_{m=1}^{\infty} A_m \sin mx \right) \sin nx \, dx \\ &= \sum_{m=1}^{\infty} A_m \int_0^\pi \sin mx \, \sin nx \, dx = A_n \cdot \frac{\pi}{2}, \end{aligned}$$

where we have used the fact that

$$\int_0^\pi \sin mx \, \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi/2 & \text{if } m = n. \end{cases}$$

Therefore, the guess for A_n, called the n^{th} Fourier sine coefficient of f, is

$$A_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx. \qquad (6)$$

DEFINITION AND BACKGROUND

Fourier series are part of a larger family called the **trigonometric series** which, by definition, are expressions of the form $\sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L}$ where $c_n \in \mathbb{C}$. If a trigonometric series involves only finitely many nonzero terms, that is, $c_n = 0$ for all large $|n|$, it is called a **trigonometric polynomial**; its **degree** is the largest value of $|n|$ for which $c_n \neq 0$.

For instance, if f is an integrable function on the interval $[-\pi, \pi]$, then the n^{th} Fourier coefficient of f is

$$\hat{f}(n) = a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta, \qquad n \in \mathbb{Z},$$

and the Fourier series of f is

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

NATURAL Q: PARTIAL SUM = FUNCTION?

Problem: In what sense does $S_N(f)$ converge to f as $N \to \infty$?

$$(1) \qquad \lim_{N\to\infty} S_N(f)(\theta) = f(\theta) \quad \text{for every } \theta?$$

Note:

1. It is natural to assume f is continuous, as the Riemann integrable is mainly based on partitions. (i.e., we can change the integral function without changing Sn)
2. But, Is continuous property enough? Ans: Not always!

CONTINUITY OF FUNCTIONS

Corollary 2.3 *Suppose that f is a continuous function on the circle and that the Fourier series of f is absolutely convergent, $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$. Then, the Fourier series converges uniformly to f, that is,*

$$\lim_{N\to\infty} S_N(f)(\theta) = f(\theta) \qquad \textit{uniformly in } \theta.$$

Note: For certain functions, continuity is enough, but what if its Fourier series diverges?

A CONTINUOUS FUNCTION WITH DIVERGENT FOURIER SERIES

Finally, we need to find a convergent series of positive terms $\sum \alpha_k$ and a sequence of integers $\{N_k\}$ which increases rapidly enough so that:

(i) $N_{k+1} > 3N_k$,

(ii) $\alpha_k \log N_k \to \infty$ as $k \to \infty$.

We choose (for example) $\alpha_k = 1/k^2$ and $N_k = 3^{2^k}$ which are easily seen to satisfy the above criteria.

Finally, we can write down our desired function. It is

$$f(\theta) = \sum_{k=1}^{\infty} \alpha_k P_{N_k}(\theta).$$

Theorem 4.1 *Let* $\{K_n\}_{n=1}^{\infty}$ *be a family of good kernels, and* f *an integrable function on the circle. Then*

$$\lim_{n\to\infty}(f * K_n)(x) = f(x)$$

whenever f *is continuous at* x. *If* f *is continuous everywhere, then the above limit is uniform.*

SOME THEOREM

EASY APPROACH: SPECIAL CONSIDERATION AND DEF

Dirichlet Kernel

EXAMPLE 4. The trigonometric polynomial defined for $x \in [-\pi, \pi]$ by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

is called the N^{th} **Dirichlet kernel** and is of fundamental importance in the theory (as we shall see later). Notice that its Fourier coefficients a_n have the property that $a_n = 1$ if $|n| \leq N$ and $a_n = 0$ otherwise. A closed form formula for the Dirichlet kernel is

Def for Good Kernel

(a) For all $n \geq 1$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)\, dx = 1.$$

(b) There exists $M > 0$ such that for all $n \geq 1$,

$$\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M.$$

(c) For every $\delta > 0$,

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|\, dx \to 0, \qquad \text{as } n \to \infty.$$

Note: No restriction on continuity!
Integribility is enough!

Fourier series

$$\begin{aligned}
S_N(f)(x) &= \sum_{n=-N}^{N} \hat{f}(n) e^{inx} \\
&= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy \right) e^{inx} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^{N} e^{in(x-y)} \right) dy \\
&= (f * D_N)(x),
\end{aligned}$$

Convolution

Given two 2π-periodic integrable functions f and g on $\mathbb{R}$, we define their **convolution** $f * g$ on $[-\pi, \pi]$ by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y)\, dy. \tag{2}$$

"Major way to do approximation"

LOGIC: IF DN(X) IS GOOD KERNEL AND F IS CONTINUE, THEN WE ARE DONE.

- However, Dirichlet Kernel is not a good kernel!

continuous at x. Unfortunately, this is not the case. Indeed, an estimate shows that D_N violates the second property; more precisely, one has (see Problem 2)

$$\int_{-\pi}^{\pi} |D_N(x)|\, dx \geq c \log N, \qquad \text{as } N \to \infty.$$

- We need to pose more constraints on f(x)!

DIFFERENT CONSTRAIN ON F(X)

- Case 1: twice differential continues

Corollary 2.4 *Suppose that f is a twice continuously differentiable function on the circle. Then*

$$\hat{f}(n) = O(1/|n|^2) \quad \text{as } |n| \to \infty,$$

so that the Fourier series of f converges absolutely and uniformly to f.

DIFFERENT CONSTRAIN ON F(X)

- Case II: Holder condition

a **Hölder condition** of order α, with $\alpha > 1/2$, that is,

$$\sup_{\theta} |f(\theta + t) - f(\theta)| \leq A|t|^{\alpha} \quad \text{for all } t.$$

- The main idea of these two contraint, is to make the function “smooth”. (i.e., to make the function decay rapidly so that the Fourier coefficient can catch up with it)

EXTANSION TO THE WHOLE REAL LINE

- Q: What if function is not periodic?
- Ans: We can still use Fourier series to describe!

- Note:

"Just as we discussed before, we still need our function to be smooth enough"

The **Fourier transform** of a function $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx.$$

Some simple properties of the Fourier transform are gathered in the following proposition. We use the notation

$$f(x) \longrightarrow \hat{f}(\xi)$$

to mean that $\hat{f}$ denotes the Fourier transform of f.

are **rapidly decreasing**, in the sense that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(\ell)}(x)| < \infty \quad \text{for every } k, \ell \geq 0.$$

We denote this space by $\mathcal{S} = \mathcal{S}(\mathbb{R})$, and again, the reader should verify that $\mathcal{S}(\mathbb{R})$ is a vector space over $\mathbb{C}$. Moreover, if $f \in \mathcal{S}(\mathbb{R})$, we have

$$f'(x) = \frac{df}{dx} \in \mathcal{S}(\mathbb{R}) \quad \text{and} \quad x f(x) \in \mathcal{S}(\mathbb{R}).$$

EXAMPLE FOR HOW TO ANALYSIS ”FOURIERSLY”

Theorem 2.1 *Suppose that f is an integrable function on the circle with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(\theta_0) = 0$ whenever f is continuous at the point θ_0.*

Thus, in terms of what we know about the set of discontinuities of integrable functions,[5] we can conclude that f vanishes for “most” values of θ.

KEY IDEA AND INDICATOR

- Main idea: Proof by Contradiction.
  - ie. If f not equal to 0 its continuity, then it can not have a Fourier coefficients equal to 0
- Notable construction:
  - A trigonomtric family of peak index. (Magnifiers, to zoom in the behavior of function as 0)
  - Division for certain interval.

FUNCTION SETTING AND PEAK CONSTRUCTION

1. Suppose f is real-valued.

2. WLOG, f is defined on $[-\pi, \pi]$, that $\theta_0 = 0$, and $f(0) > 0$

3. $p(\theta) = \varepsilon + \cos\theta$, $p_k(\theta) = [p(\theta)]^k$.

image : θ on $[0, \pi]$.

Since f is continous at 0, we can choose $0 < \delta \le \pi/2$, so that $f(\theta) > f(0)/2$ whenever $|\theta| < \delta$.

ie. take $\varepsilon = f(0)/2$, we have $|f(\theta) - f(0)| < f(0)/2 \Rightarrow f(\theta) > f(0)/2 > 0$.

Let $p(\theta) = \varepsilon + \cos\theta$ (trigonometric function), "continous"

where $\varepsilon > 0$ is small enough that $|p(\theta)| < 1 - \varepsilon/2$, whenever $\delta \le |\theta| \le \pi$.

Then, choose a positive η with $\eta < \delta$, so that $p(\theta) \ge 1 + \varepsilon/2$ for $|\theta| < \eta$.

Let $p_k(\theta) = [p(\theta)]^k$, and select B, so that $|f(\theta)| \le B$ for all θ. "$f \in R(x)$ Riemann-integrable"

$\Uparrow$

trigonometric polynomial.

EXAMPLE

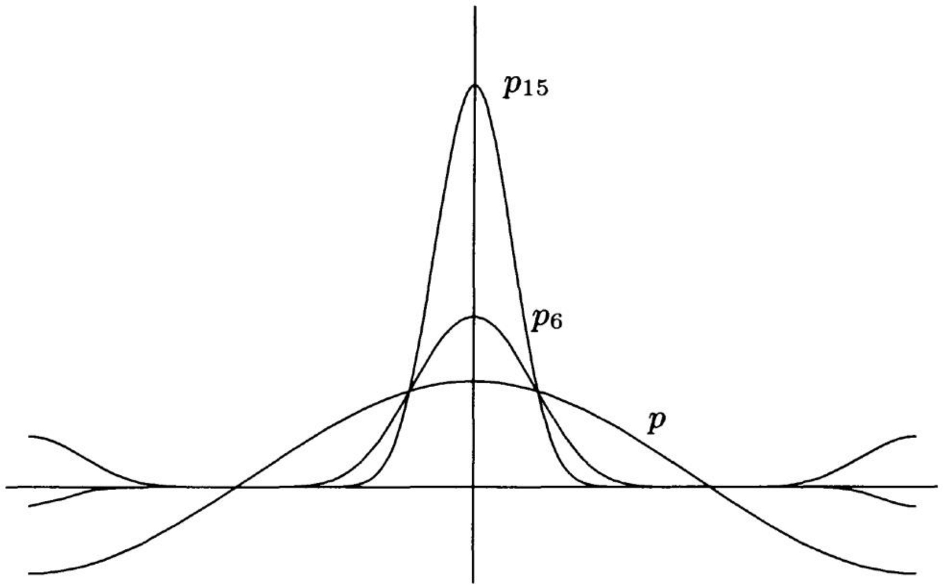


Figure 3. The functions p, p_6, and p_{15} when $\epsilon = 0.1$

ANALYSIS PROCESS/ESTIMATION

Key point, since $\hat{f}(n) = 0$ by Assumption for all n, we must have $\int_{-\pi}^{\pi} f(\theta)\, P_k(\theta)\, d\theta = 0$ for all k.

Estimate: ① $\left| \int_{\delta \le |\theta|} f(\theta) \cdot P_k(\theta)\, d\theta \right| \le \overset{\text{"length"}}{2\pi}\ \underline{B \cdot (1 - \epsilon/2)^k}$. (upper bound).

② $\int_{\eta \le |\theta| < \delta} f(\theta) \cdot P_k(\theta)\, d\theta \ \ge 0.$

ie, $f(\theta) > f(0)/2 > 0$ $\quad P_k(\theta) = [p(\theta)]^k = [\epsilon + \cos\theta]^k > 0$ "Since $|\delta| \le \pi/2$, $\cos\theta \ge 0$"

③ $\int_{|\theta| < \eta} P_k(\theta) \cdot f(\theta) d\theta \ge 2\eta \cdot \frac{f(0)}{2} \cdot (1 + \epsilon/2)^k \xrightarrow[k \to \infty]{} \infty$

Therefore, $\int_{|\theta| \le \pi} P_k(\theta) \cdot f(\theta) = ① + ② + ③ \longrightarrow \infty$, where ① bounded ② non-negative, ③ close to infinity.

hence, contridicted to the fact that $\hat{f}(n) = 0$ for all n.

EXTENDED TO COMPLEX FIELD

when f is real-valued. In general, write $f(\theta) = u(\theta) + iv(\theta)$, where u and v are real-valued. If we define $\overline{f}(\theta) = \overline{f(\theta)}$, then

$$u(\theta) = \frac{f(\theta) + \overline{f}(\theta)}{2} \quad \text{and} \quad v(\theta) = \frac{f(\theta) - \overline{f}(\theta)}{2i},$$

and since $\hat{\overline{f}}(n) = \overline{\hat{f}(-n)}$, we conclude that the Fourier coefficients of u and v all vanish, hence $f = 0$ at its points of continuity. The idea

THANKS FOR YOUR LISTENING!!!

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