Plane Curve and Contact Geometry

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Abstract

The goal of this expository paper is to differentiate immersions with contact geometry. This document covers some basic definitions and practice problems about immersions, Legendrian, and calculus techniques about forms. It also includes an example of a contactomorphism between two different 3-manifolds and an example of an algorithmic way of obtaining the conormal knot of a given plane curve. Additionally, it explains Legendrian isotopy and the Legendrian invariants $J_1(S^1)$. Finally, it shows some properties of the invariants for the conormal knot of plane curves and how dangerous self-tangency during the regular homotopy process affects the invariants.

Contents

1. Calculus Preliminaries

Before starting to show the algorithm of finding the conormal knot and use it to show more properties of conormal knot of plane curves, we first want to go over some preliminary background of calculus on manifold and contact geometry with some definitions and practice problems.

Definition 1.1 (vector space)**.** A *vector space* is a set of elements that can interact with scalars and whose elements satisfy the 8 axioms of vector space.
The 8 axioms of vector space are:

(1) Associativity of vector addition
(2) Commutativity of vector addition
(3) Identity element of vector addition
(4) Inverse elements of vector addition
(5) Compatibility of scalar multiplication with field multiplication
(6) Identity element of scalar multiplication
(7) Distributivity of scalar multiplication with respect to vector addition.
(8) Distributivity of scalar multiplication with respect to field addition

Definition 1.2 (Linear)**.** Consider function F that map element from vector space V to vector space W is *Linear*.($F : V \to W$ is *Linear*)
Then the function have to satisfy the following:
$\forall v_1, v_2 \in V, F(v_1 + v_2) = F(v_1) + F(v_2)$ and $\forall v \in V, \forall c \in \mathbb{R}, F(cv) = cF(v)$

Definition 1.3 (Multilinear)**.** A function from $T : V^k \to \mathbb{R}$ is multilinear if for any $i \in [k]$(set of all integers from 1 to k) we have [Spi68]

$$T(v_1, \ldots, v_i + v_i', \ldots, v_k) = T(v_1, \ldots, v_i, \ldots, v_k) + T(v_1, \ldots, v_i', \ldots, v_k),$$
$$T(v_1, \ldots, av_i, \ldots, v_k) = aT(v_1, \ldots, v_i, \ldots, v_k).$$

Notice when k=1 the definition is identical as being linear.

Definition 1.4 (Dual vector space)**.** A *Dual vector space* is the set of linear functions that map the element in original vector space to $\mathbb{R}$.
The *dual vector space* of V can be written as V^* and you can describe it in set notation as $V^* = \{f : V \to \mathbb{R} | \text{f is } Linear\}$

Example 1.0.1. Show $V^* = \{f : V \to \mathbb{R} | f\, linear\}$ is a vector space.
In order to check whether V^* is a vector space we have to check the 8 axioms.

(1) associative addition $(u + (v + w) = (u + v) + w)$
let function u, v ,w ,which map vectors in $V \to \mathbb{R}, x \to u(x), v(x), w(x)$
then $[u + (v + w)](x) = u(x) + (v(x) + w(x)) = u(x) + v(x) + w(x) = [(u + v) + w](x)$
(2) commutative addition $(u + v = v + u)$
similarly $[u + v](x) = u(x) + v(x) = v(x) + u(x) = [v + u](x)$
(3) identity element$(0 + v = v)$
the zero element is $\mathbf{0}(x) = 0$ for any input x
thus any $[v + \mathbf{0}](x) = v(x) + 0 = v(x)$
(4) additive inverse$(v + -v = 0)$
the additive inverse of $v : x \to v(x)$ is $-v : x \to -v(x)$
it is within V^* because multiply -1 to a linear function is still linear,
and $v + -v$ sums up to the zero element because $(v + -v)(x) = v(x) + -v(x) = 0$
(5) identity element for scalar multiplication$(1v = v)$
the identity element for scalar multiplication is just 1 in $\mathbb{R}$ such that $1v(x) = v(x)$

(6) compatibility of scalar and field multiplication ($(ab)v = a(bv)$)
let a,b be real numbers. then we can show that $bv : x \to bv(x)$ because it is linear. thus $a(bv)(x) = a(bv(x)) = abv(x) = (ab)v(x) = ((ab)v)(x)$
(7) distributive law with vector addition ($a(u + v) = au + av$)
$\{u + v : x \to [u + v](x) = u(x) + v(x)\}$ thus
$a[u + v](x) = a(u(x) + v(x) = au(x) + av(x) = [au](x) + [av](x)$
(8) distributive law with field addition ($(a + b)v = av + bv$)
$[(a + b)v](x) = (a + b)v(x) = av(x) + bv(x) = [av](x) + [bv](x)$

Thus we verified the 8 axioms for V^* and it is indeed a vector space

Definition 1.5 (1-tensor)**.** A k-tensor on a vector space V is a multilinear function that maps from $V^k \to \mathbb{R}$.
In the special case of *1-tensor*, it is reduces to a linear function on the vector space V that map to $\mathbb{R}$.

Example 1.0.2. Find a basis for $(\mathbb{R}^3)^*$

$(\mathbb{R}^3)^*$ is the vector space for all linear functions that maps a vector in $\mathbb{R}^3$ to $\mathbb{R}$
These linear functions can be written as $v : \mathbf{x} = [a, b, c] \to v(x) = v([a, b, c]) = ma + nb + pc$ where m,n,p are real numbers.

Claim. A basis for $(\mathbb{R}^3)^*$ is $\phi_1, \phi_2, \phi_3 \in (\mathbb{R}^3)^*$ which represent $\phi_1 : [a, b, c] \to a$
ϕ_2, ϕ_3 are defined similarly where $\phi_2 : [a, b, c] \to b$, $\phi_3 : [a, b, c] \to c$.

To prove that these three element form a basis for $(\mathbb{R}^3)^*$ we have to check whether they are all linearly independent and yet they span the whole vector space $(\mathbb{R}^3)^*$
If ϕ_1, ϕ_2, ϕ_3 are linearly independent then $m\phi_1 + n\phi_2 + p\phi_3 = 0$ if and only if $m = n = p = 0$
notice that multiply the expression $m\phi_1 + n\phi_2 + p\phi_3 = 0$ by e_1 we can get $m*1 + n*0 + p*0 = 0$
Thus we can show $m = 0$, similarly we can also show $n = 0, p = 0$ by multiplying e_2, e_3
Therefore we can prove that it have to be linearly independent.

ϕ_1, ϕ_2, ϕ_3 also span the entire $(\mathbb{R}^3)^*$ because any function $f([a, b, c])$ can be decomposed into $f(a[1, 0, 0] + b[0, 1, 0] + c[0, 0, 1])$ and because of its linear property it is equal to
$f([1, 0, 0]) * \phi_1([a, b, c]) + f([0, 1, 0]) * \phi_2([a, b, c]) + f([0, 0, 1]) * \phi_3([a, b, c])$
Thus we can show any element in $(\mathbb{R}^3)^*$ can be written as a linear combination of ϕ_1, ϕ_2, ϕ_3 thus make them spanning the vector space.

Example 1.0.3. Give examples of 1-tensors in $\mathbb{R}^2$ and in $\mathbb{R}^3$. Find their kernel, what can you say about the dimension of their kernels in general .

a 1-tensor in $\mathbb{R}^2$ is a linear function send a vector in $\mathbb{R}^2$ to $\mathbb{R}$, it has the form of $f([a, b]) = ma + nb$, similarly a 1-tensor in $\mathbb{R}^3$ has the form $f([a, b, c]) = ma + nb + pc$

The kernel space for the linear functional is all the input that will lead to a output of 0.

for $\mathbb{R}^2$ we can solve for the kernel space which is the line where $a = \frac{-nb}{m}$ and $b = 0$ if $m = 0$ which have a dimension of 1.

For $\mathbb{R}^3$ we can use the rank nullity theorem $n = rank(f) + nul(f)$, apply $n = 3$ and $rank(f) = 1$(because f is a linear function and we can see it as multiplying a 1*3 matrix) we can get the $nul(f)$ which is the dimension of the kernel is equal to 2.

Definition 1.6 (1-form)**.** A *1-form* is an object that at every point p in vector space V, it represent a linear function that map the element in the tangent space V_p to $\mathbb{R}$.
In another sense, it is a 1-tensor at every point in V_p that changes based on the point in V.

Example 1.0.4. What is the dimension of the vector space of 1 tensors in $\mathbb{R}^2$ and $\mathbb{R}^3$? How about 1 forms?

A 1 tensor in $\mathbb{R}^2$ is a linear mapping that map an element in the tangent space $\mathbb{R}^2_p$ at point p in the $\mathbb{R}^2$ to $\mathbb{R}$.

The general expression of such 1-tensor can be expressed as $\alpha([dx, dy]) = mdx + ndy$. And we can find the vector space of 1-tensor in $\mathbb{R}^2$ can be spanned by two linearly independent one tensor $\phi_1([dx, dy]) = dx$ and $\phi_2([dx, dy]) = dy$.

The dimension of the vector space of 1 form in $\mathbb{R}^2$ is 2 because its basis have 2 element.

We can use similar method to find the basis for 1 tensor in $\mathbb{R}^3$ and thus show the basis contain 3 linearly independent elements thus the dimension of the vector space is 3.

But When it comes to 1-forms, We can not find a finite number of element to form a basis for 1-form. Since the one form can include any function possible thus the dimension is actually infinite.

Example 1.0.5. The set of 1-forms whose kernel is a fixed subspace. Does it form a vector subspace? Why or why not? If not, could you describe this set in another way?

It doesn't form a vectorspace because this set doesn't include an zero element since a form that always map to zero doesn't have a fixed kernel.

This set could be described as a vector space formed by the basis of the fixed subspace that does not include zero.

Definition 1.7 (Pushforward)**.** Consider a differentiable function f: $\mathbb{R}^n \to \mathbb{R}^m$, The pushforward of the function came from the derivative matrix Df. It is a map with the same linear transformation as Df but happens in the tangent space at p and $f(p)$.

The pushforward $f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ is defined as

$$f_*(v_p) = (D(f(p)(v))_{f(p)}$$

Definition 1.8 (Pullback)**.** For the same function f:$\mathbb{R}^n \to \mathbb{R}^m$, and any form $\alpha : \mathbb{R}^m \to \mathbb{R}$, we define the pullback as $f^*(\alpha) = \alpha \circ f$ or more specifically $f^*(\alpha)](p) = \alpha(f(p))$. It pull forms from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Example 1.0.6. Spivak's calculus on manifold question 4.13(a).
If $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$, show that $(g \circ f)_* = g_* \circ f_*$ and $(g \circ f)^* = f^* \circ g^*$.
(a)

$$\begin{aligned}(g \circ f)_*(v_p) &= D(g \circ f(p)(v))(\text{by definition})\\ &= Dg(f(p)) \circ Df(p)(v)(\text{chain rule})\\ &= g_* \circ (Df(p)(v)) = g_* \circ f_*(v_p)\end{aligned}$$

(b)

$$\begin{aligned}(g \circ f)^*(\omega) &= w((g \circ f)_*)\\ &= \omega(g_* \circ f_*)\\ &= g^*\omega(f_*) = f^* \circ g^*(\omega)\end{aligned}$$

Example 1.0.7. Spivak's calculus on manifold question 4.14
Let c be a differentiable curve in $\mathbb{R}^n$, that is, a differentiable function $c : [0,1] \to \mathbb{R}^n$. Define the tangent vector v of c at t as $c_*((e_1)_t) = ((c^1)'(t), \ldots, (c^n)'(t))_{c(t)}$. If $f : \mathbb{R}^n \to \mathbb{R}^m$, show that the tangent vector of $f \circ c$ at t is $f_*(v)$.

Proof. v is the tangent vector of c(t) where v is in $\mathbb{R}^n$ and every entry v^i is the derivative of the parameterization c^i
and the output of f is in $\mathbb{R}^m$ thus each entry of the output can be donoted as $f^i(x), x \in \mathbb{R}^n$
Thus tangent vec of $f \circ c$ is $((f^1 \circ c)'(t), ..., (f^m \circ c)'(t))$
the derivative can be calculated as the sum of the partial and using chain rule:
so each entry is $\sum_1^n D_{xi} f'(c(t)) c^{i\prime}(t)$, then replace $c^{i\prime}(t)$ with v^i
we will have $(f^i \circ c)'(t) = \sum_1^n D_{xi} f^{i\prime}(c(t)) * v^i$ it is exactly $f_*(v)$, so every term of them are equal. ☐

Example 1.0.8. Describe all $\{T : \mathbb{R}^n \to \mathbb{R}^n | linear\}$ satisfying $T^*\phi_1 = \phi_1$
when n=1 and 2.

When n=1, T is a function from $\mathbb{R}^1$ to $\mathbb{R}^1, x = T(x)$, and the 1-form is $\phi_1(dx) = dx$
The pullback of T applied to the 1-form is $dx = \phi_1 \circ T(x) * \phi_1'(x) * dx = \phi_1 \circ T(x) * dx$
Therefore $\phi_1 \circ T(x) = 1$ thus T(x)=1 is the only transformation that satisfy $T^*\phi_1 = \phi_1$

When n=2, T is a function from $\mathbb{R}^2$ to $\mathbb{R}^2, x = T_1(x,y), y = T_2(x,y)$, and the 1-form is $\phi_1(dx, dy) = dx$ Then the $dx = 1 * (\frac{\partial T_1(x,y)dx}{\partial x} + \frac{\partial T_1(x,y)dy}{\partial y}) + 0 * DT_2(x,y)$, Therefore $T_1 = dx + 0dy$ and T_2 can be any function based on x,y.

Example 1.0.9. Let V be a vector space and $\alpha, \beta \in V^*$ such that $\ker\alpha = \ker\beta$, prove $\alpha = a\beta$ for some real value $a \neq 0$

Proof. V is a vector space and V^* is its dual space, and α, β are two elements from the dual space. Since they have the same kernel space, consider any element $x \in V$ such that $\alpha(x) = 0 \iff \beta(x) = 0$.

Now lets consider vectors $v \in V, \notin \ker\alpha$ (if it doesn't exist then both function have full kernel thus is the zero function we simply take a=1), let $\alpha(v) = 1, \beta(v) = a \neq 0$(we can always assume this because α is linear and we are able to divide a constant to get 1, $\beta(v) \neq 0$ is because v is not in the kernel space).

Also since the dual space is linear, we know that the column space for v where $\alpha(v) = 1$ have dimension of 1. Thus the solution set of v is spanned by 1 vector thus using the linear property we can show when $\alpha(v) = 1, \beta(v) = a$ is always true.

Since the kernel and the span of v is the entire vector space space V, we are able to write any vector in V as a linear combination of the element in the two spaces.
Consider element $v_1 \in \ker(\alpha), v_2 \in \mathbf{span}(v), C_1, C_2 \in \mathbb{R}$
$\alpha(C_1v_1 + C_2v_2) = C_1\alpha(v_1) + C_2\alpha(v_2) = C_2, \beta(C_1v_1 + C_2v_2) = C_1\beta(v_1) + C_2\beta(v_2) = C_2a$ it satisfy $\beta(v) = a\alpha(v)$, more generally since every element in V can be written as a linear combination of element in the two spaces this property hold for every element in V, thus we can conclude $\alpha = a * \beta$ where $a \neq 0$ ☐

Example 1.0.10. T is a transformation that rotate element in $\mathbb{R}^2$ by θ_0 radians.
a) Describe $T_\theta(a,b)$ in terms of a, b, θ.

Lets first write it in polar coordinate: $T(r,\theta) = (r, \theta + \theta_0)$, Therefore since $(a,b) = (rcos(\theta), rsin(\theta))$

$T(a,b) = (rcos(\theta + \theta_0), rsin(\theta + \theta_0))$, plug in $r = \sqrt{a^2+b^2}, \theta = arctan(b/a)$

$$\begin{aligned} T(a,b) &= (\sqrt{a^2+b^2}(cos(\theta)cos(\theta_0) - sin(\theta)sin(\theta_0)), \sqrt{a^2+b^2}(sin(\theta)cos(\theta_0) + cos(\theta)sin(\theta_0))) \\ &= (acos(\theta_0) - bsin(\theta_0), bcos(\theta_0) + asin(\theta_0) \end{aligned}$$

Thus $T_\theta(a,b) = (acos(\theta) - bsin(\theta), bcos(\theta) + asin(\theta))$

b) Compute $T_\theta^*\phi_1$ and $T_\theta^*\phi_2$

we can calculate DT matrix which is $\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$,The pullback $T_\theta^*\phi_1$ is just $T_\theta^*dx = cos(\theta)da - sin(\theta)db$ and $T_\theta^*\phi_2 = T_\theta^*dy = sin(\theta)da + cos(\theta)db$

c) Find $\ker T_\theta^*\phi_1$

the kernel $\ker(T_\theta^*\phi_1)$ is located in the space like $\mathbb{R}^2$ (because $T_\theta^*\phi_1$ is a 1-form of $\mathbb{R}^2$)
Since we already have the expression written out we can simply solve for the kernel.
$cos(\theta_0)da - sin(\theta_0)db = 0$ thus $da = sin(\theta_0)/cos(\theta_0)db = tan(\theta_0)db$
The kernel is the space spanned by $(1, tan(\theta_0))$

2. Contact geometry Preliminaries

Definition 2.1 (Contact form). A *contact form* is a 1-form that never vanish under wedge product operation with its derivatives.
Consider α is a contact form we have $\alpha \wedge \mathrm{d}\alpha^k \neq 0$, where $\mathrm{d}\alpha^k = \mathrm{d}\alpha \wedge ... \wedge \mathrm{d}\alpha$
In our study we are mainly focused on 3-manifolds where 2k+1=3, so $\alpha \wedge \mathrm{d}\alpha \neq 0$ is enough.

Definition 2.2 (Standard contact form). The *standard contact form* in $\mathbb{R}^{2n+1}$ with coordinate $(x_1, y_1, x_2, y_2, ..., x_n, y_n, z)$ can be written as the 1-form $\alpha_1 = \mathrm{d}z + \sum_{j=1}^n x_j \mathrm{d}y_j$.
In particular, in $\mathbb{R}^3$ the standard contact form is $\alpha_1 = \mathrm{d}z + x\mathrm{d}y$

Definition 2.3 (Standard contact structure). The *standard contact structure* is the hyperplane field that represent the kernel space of the standard contact form at each point in the vector space. In $\mathbb{R}^3$ it can be described as $\ker(\mathrm{d}z + x\mathrm{d}y)$

Definition 2.4 (Diffeomorphism). A diffeomorphism is a mapping between two different manifold that is a differentiable bijection, s.t $f : M \to N$ is a differentiable bijection and its inverse $f^{-1} : N \to M$ is also differentiable.

Definition 2.5 (Contactomorphism). A contactomorphism is a diffeomorphism between two contact manifolds such that the diffeomorphism's differential(pushforward) which is a linear transformation that map the kernel of contact form of the first manifold as a bijection to the kernel of the second contact form. Or in another way of describing, the contactomorphism pullback on the second form is the first form.

If their are two contact manifold defined as $[M, \ker(\alpha_1)], [N, \ker(\alpha_2)]$, and exist a diffeomorphism $f : M \to N$, and $f^*\alpha_2 = \alpha_1$, it is a contactomorphism. [Han08]

Definition 2.6 (Immersed curve). A immersed curve is a mapping from S^1 to another manifold that satisfy its derivative is never zero. It can be considered as a mapping from a segment in a circle to a continuous curve on the manifold.

$$f : S^1 \to M, s.t \forall \theta \in S^1, f'(\theta) \neq 0$$

Definition 2.7 (Embedded curve). A Embedded curve is an immersion that is also one to one, Where there are no intersection in the curve. Consider the same immersion above it would be a embedding if $\forall\theta_1, \theta_2 \in S^1$, $\theta_1 \neq \theta_2 \Rightarrow f(\theta_1) \neq f(\theta_2)$.

Definition 2.8 (Plane curve). A plane curve is a immersion from $C : S^1 \to \mathbb{R}^2$, we denote the orientation of the plane curve on x with two vectors v_x, w_x where v_x is the direction of C on x and w_x can be obtained by rotating v_x counterclockwise 90°

Definition 2.9 (regular homotopy). A regular homotopy between two plane curve γ_0, γ_1 is a continuously differentiable homotopy via plane curves $\gamma_t : S \to \mathbb{R}^2, t \in [0, 1]$.

The homotopy process can be described as $H : S \times [0, 1] \to \mathbb{R}^2$ where $H(\theta, t) = \gamma_t(\theta), t \in [0, 1]$, it describe curve γ_0 deform into curve γ_1 in a continuous process.

Theorem 2.1 (Whitney–Graustein Theorem). *Regular homotopy classes of regular closed curves $\gamma : S \to \mathbb{R}^2$ are in one-to-one correspondence with the integers, the correspondence being given by $rot(\gamma)$.*
Where $rot(\gamma)$ represent the number of complete turns by the velocity vector γ'.

Definition 2.10 (Dangerous self-tangency). A self-tangency of a plane curve is dangerous if the orientations on the tangent directions to the curve agree at the tangency.

Definition 2.11 (Safely homotopic). Two plane curves without dangerous self-tangencies are safely homotopic if they are homotopic through plane curves without dangerous self-tangencies.

Definition 2.12 (Conormal knot). The conormal knot of a plane curve C is a unique subset of the unit cotangent bundle $ST\mathbb{R}^2$ where the element are given by

$$\{\xi \in ST\mathbb{R}^2 | \, \xi \text{ lies over some } x \in C \text{ and } \langle \xi, v_x \rangle = 0, \langle \xi, w_x \rangle = 1\}$$

This conormal knot preserved information about the orientation of the original plane curve because at each point of curve C in $\mathbb{R}^2$ the element is determined.

If the original plane curve does not have dangerous self-tangency. Then for all safely isotopic plane curve the conormal knot is going to be an embedding because no two points will intersect on the same point and have the same tangent direction during all the homotopy process(no dangerous self-tangency) thus the conormal knot never intersect itself.

Definition 2.13 (Legendrian knot). A *legendrian knot* is an embedding that map $\gamma : S^1 \to M$ (in this study M is mainly going to be $J^1(S^1)$. That have the property $\gamma'(\theta) \in \ker(\alpha_1(\gamma(\theta))), \forall\theta \in S^1$

In other word, the legendrian knot is a curve in the vector space M such that the derivative of the curve at a point always sits within the Kernel space of the contact form at that point.

Remark 2.1 (Legendrian equation for $\mathbb{R}^3$). The legendrian equation for $\mathbb{R}^3$ can be derived with the derivative requirement from the definition.
Lets say the curve is a mapping between $\gamma : S^1 \to \mathbb{R}^3$ and $\gamma(t) = (x(t), y(t), z(t))$

Since the Derivative of the curve is within the kernel of the contact form $\alpha_1 = \mathrm{d}z + x\mathrm{d}y$, we know that $(\mathrm{d}z + x\mathrm{d}y) * (x'(t), y'(t), z'(t)) = 0$ Therefore $z'(t) + x(t)y'(t) = 0$ and $x(t) = -\frac{z'(t)}{y'(t)} = -\dfrac{\mathrm{d}z}{\mathrm{d}y}$

Now since we know this relation is always true for Legendrian curves, we can look at the front projection on the YZ plane and recover the full information about the curve because the value of x can be interpreted with y and z.

3. Example of Contactomorphism between $ST\mathbb{R}^2$ and $J^1(S^1)$

The contact form on $ST\mathbb{R}^2$ given in the paper was $\alpha_1 = p_1 \mathrm{d}q_1 + p_2 \mathrm{d}q_2$ (the kernel describe the contact structure that we are interested in).

This is actually an abuse of notation where this form is actually in $T\mathbb{R}^2$ where the coordinate are (q_1, q_2, p_1, p_2). The q_1, q_2 represent coordinate in $\mathbb{R}^2$ and p_1, p_2 are two coordinate in the tangent space at the point in $\mathbb{R}^2$. But in $ST\mathbb{R}^2$ the coordinate living in the tangent space is dictated with only 1 variable which live in the space isomorphic to S, thus the coordinates are (q_1, q_2, θ).

We will introduce a simple transformation from $ST\mathbb{R}^2$ to $T\mathbb{R}^2$ and find the pullback of the given form in $ST\mathbb{R}^2$, Where $q_1 = q_1, q_2 = q_2, p_1 = \cos\theta, p_2 = \sin\theta$. Thus the contact form living in $ST\mathbb{R}^2$ is $\alpha_1 = \cos\theta \mathrm{d}q_1 + \sin\theta \mathrm{d}q_2$

The contact form on $J^1(S^1)$ given in the paper was $\alpha_2 = \mathrm{d}z - y\mathrm{d}\theta$. Thus, we can use the given parameterization in Lenhard's paper as a mapping to describe contactomorphism between the two contact form in $ST\mathbb{R}^2$ and $J^1(S^1)$.
With $\theta = arg(p_1, p_2), y = -q_1 \sin\theta + q_2 \cos\theta, z = q_1 \cos\theta + q_2 \sin\theta$ as the mapping $f : ST\mathbb{R}^2 \to J^1(S^1)$, we can plug it in the form α_2 to check whether $f^*(\alpha_2) = \alpha_1$

$$
\begin{aligned}
\alpha_2 &= \mathrm{d}z - y\mathrm{d}\theta \\
f^*(\alpha_2) &= \mathrm{d}(q_1 \cos\theta + q_2 \sin\theta) - (-q_1 \sin\theta + q_2 \cos\theta)\mathrm{d}\theta \\
&= (q_1 \mathrm{d}\cos\theta + q_2 \mathrm{d}\sin\theta + \cos\theta \mathrm{d}q_1 + \sin\theta \mathrm{d}q_2) + q_1 \sin\theta \mathrm{d}\theta - q_2 \cos\theta \mathrm{d}\theta \\
&= q_1(-\sin\theta \mathrm{d}\theta) + q_2(\cos\theta \mathrm{d}\theta) + q_1 \sin\theta \mathrm{d}\theta - q_2 \cos\theta \mathrm{d}\theta + \cos\theta \mathrm{d}q_1 + \sin\theta \mathrm{d}q_2 \\
&= 0 + \cos\theta \mathrm{d}q_1 + \sin\theta \mathrm{d}q_2 = \alpha_1
\end{aligned}
$$

Now that we've verified $f^*(\alpha_2) = \alpha_1$, by the definition of contactomorphism, the mapping f is a contactomorphism between The contact structure on $ST^*\mathbb{R}^2$ and $J^1(S^1)$.

The θ, z projection in $J^1(S^1)$
Since the contact 1-form on $J^1(S^1)$ is $\mathrm{d}z - y\mathrm{d}\theta$, it is clear that the Y coordinate information of a legendrian can be recovered with $y = \dfrac{\mathrm{d}z}{\mathrm{d}\theta}$(similar to Rmk 1.1)

Now Consider the θ, z projection of the conormal knot of a plane curve C. Suppose at a point on C its coordinate is (q_1, q_2) and its tangent vector is $(\cos(\phi), \sin(\phi))$.

We know that its θ-coordinate can be expressed as $arg(p1, p2)$, But since $(p1, p2)$ is conormal to the curve its argument is $\theta = \phi + \pi/2$

Similarly, the z-coordinate can be expressed as
$z = q_1 \cos\theta + q_2 \sin\theta = q_1 \cos(\phi + \pi/2) + q_2 \sin(\phi + \pi/2) = -q_1 \sin\phi + q_2 \cos\phi$

4. Legendrian Isotopy and Legendrian invariants

Two Legendrian knots are legendrian isotopic if and only if you are able to smoothly transform one to another where in each stage between it is always a legendrian knot, i.e the derivative at each point of the knot still lives in the kernel space of the contact form. It is different than Homotopy because Legendrian are embedding which does not allow self intersections.

If two curves are safely-homotopic then their conormal knot is legendrian isotopic because at any process of the homotopy the conormal is always a legendrian embedding which match exactly to the definition of legendrian isotopic.

Legendrian invariants are tools that we use to identify different legendrian curves. If two legendrian curves are legendrian isotopic the invariants will not be changed.
There are 3 classic invariants for Legendrian curves. Which are the knot type, tb(Thurston–Bennequin number) and r(rotation number).

The knot type is the family of knots that are legendrian isotopic to the knot. Which is of course invariant under legendrian isotopic.

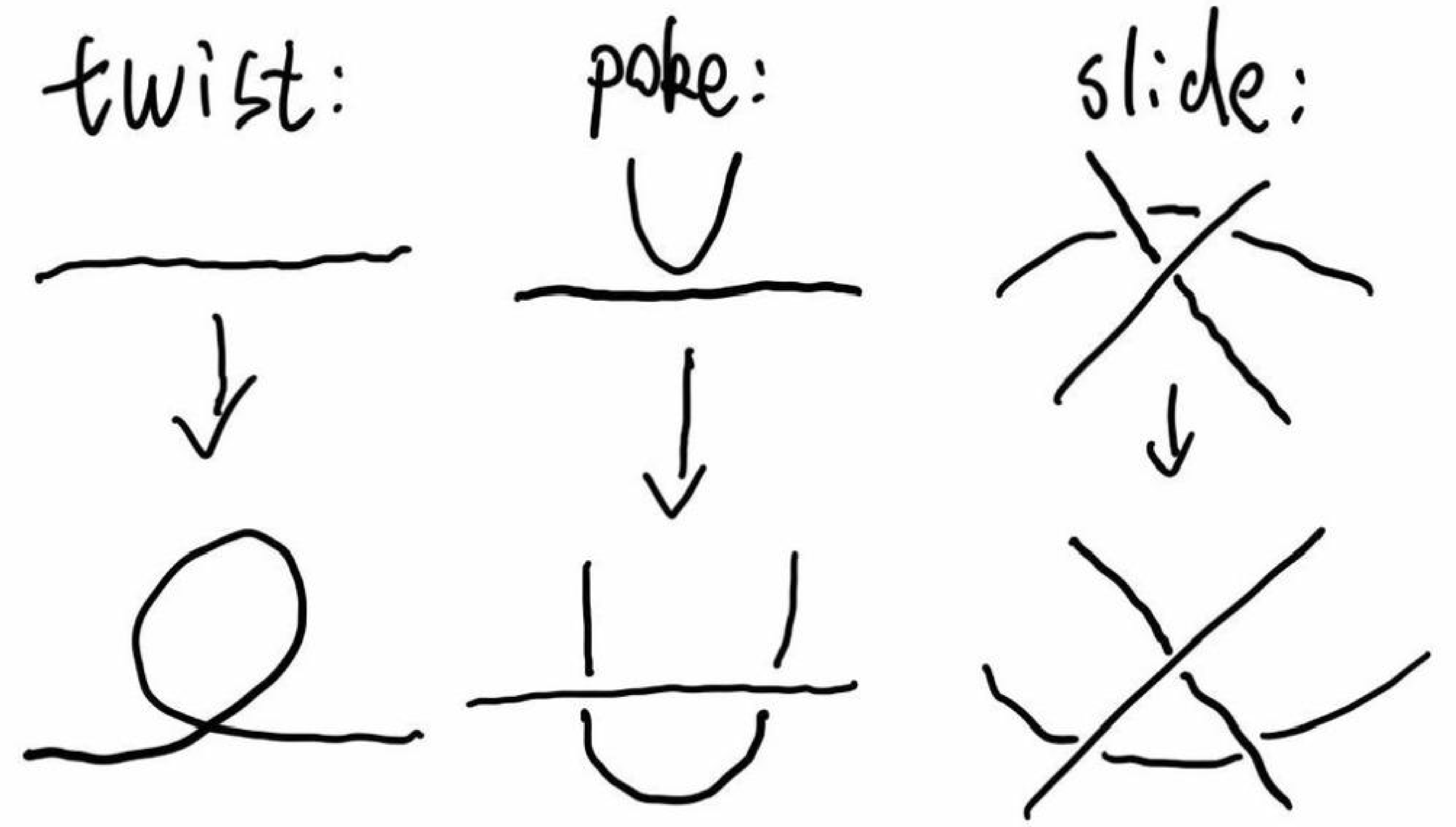
The Thurston-Bennequin number can be calculated by counting the positive crossing minus the negative crossing minus the number of right cusps. Since in $J^1(S^1)$ we are able know the coordinate of the y axis from the front projection by taking the slope $dz/d\theta$, we know which curve is above and below for the crossing in the front projection. The positive crossing is where you stand from the direction of the strand that is above and the lower strand is going from right to left,and the negative crossing is wise versa.

The rotation number can be calculated by adding the number of cusps going from down to up minus the cusps going from up to down divided by two. It captures the total number of rotation of the derivative that lives in the kernel of the contact form. Here is a picture from Lenhard's paper to explain how to calculate tb and r with the front projection of a legendrian. And there is a section that explain why this works on this paper from Geiges. [Gei07]

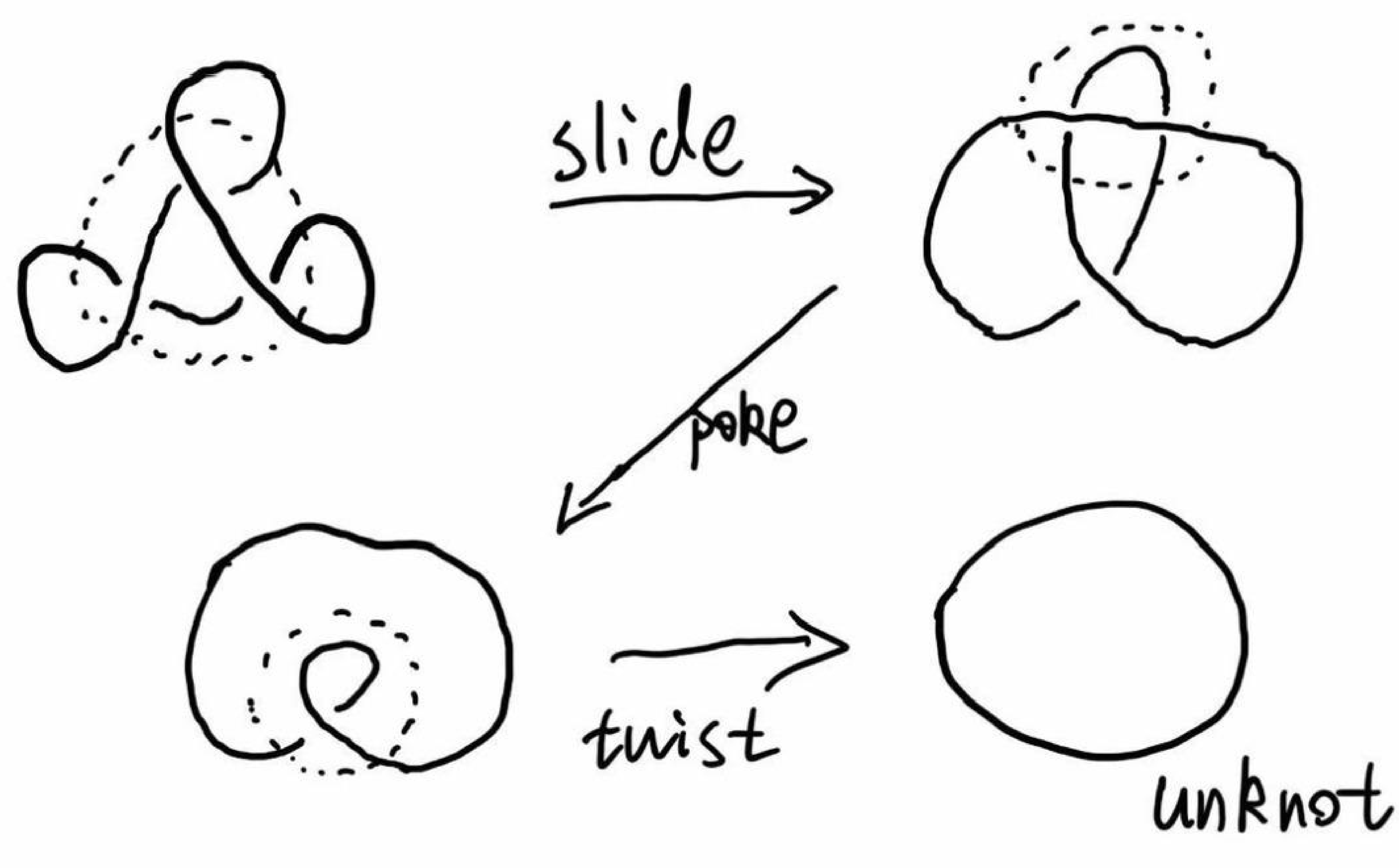
However tb and r as invariants cannot determine legendrian isotopy completely. Two legendrian curves can have the same tb and r but not be legendrian isotopic. But under some circumstances tb and r are sufficient to show legendrian isotopy.

Theorem 4.1 (Eliashberg-Fraser). *Let K and K' be two Topologically trivial Legendrian knots in $(\mathbb{R}^3, \xi_{st})$, They are Legendrian isotopic if and only if $tb(K) = tb(K')$, $r(K) = r(K')$* [Han08]

A knot is called topologically trivial knot if it can be reduced to a unknot under Reidemeister moves.
There are three reidemeister moves, twist, poke and slide, the reidemeister moves can be used in any rotation or symmetric scenarios. Here is a image help understand the concept.

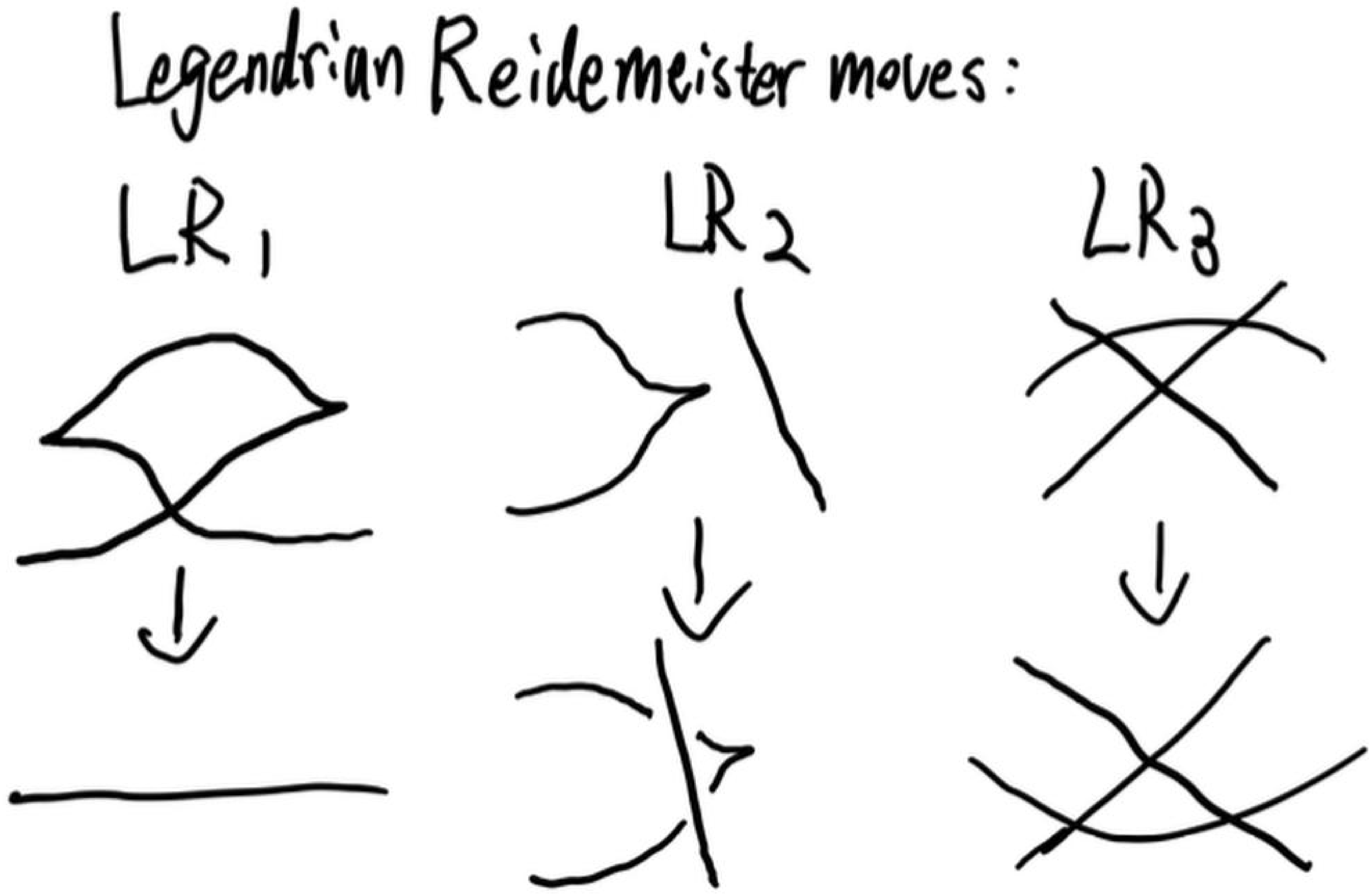


Here is an example of using Reidemeister moves to show a knot is topologically trivial. There are of course other ways to reduce it to unknot such as using three twist moves.



Definition 4.1 (Legendrian Reidemeister moves)**.** Legendrian Reidemeister moves are similar to Reidemeister moves where any order or amount of Legendrian Reidemeister moves on a legendrian will still be Legendrian isotopic to the original Legendrian.

There are also three moves as well as the upside down version(rotated by π).



Proposition 4.1. *Two Legendrian K_1 and K_2 are Legendrian isotopic if and only if their front projections are related by a sequence of Legendrian Reidemeister moves.*

Lemma 4.1. *Tb and r are invariant under legendrian isotopy.*

Proof. As we know from the proposition, if two legendrian K_1 and K_2 are legendrian isotopic we can find a sequence of legendrian Reidemeister moves and apply it to the front projection of K_1 that change it to K_2.

For LR1, the the twist increase one up and one down cusp so r don't change, the crossing is always positively oriented thus tb don't change either.

For LR2, the number of cusps don't change therefore r don't change, the two crossing have opposite sign to each other thus tb don't change.

For LR3, there are no cusps involved thus r don't change, the middle crossing stays the same, the upper left crossing will always have the same sign as the lower right crossing and the same thing goes for the upper right and lower left. Therefore the total tb don't change.

Thus for any of the 3 Legendrian Reidemeister moves, the tb and r are not changed in the process. Thus the final tb and r for legendrian isotopy after sequence of Legendrian Reidemeister moves will also not be changed. Therefore K_1 and K_2 being Legendrian isotopic means they have same tb and r. ☐

5. Example of algorithm to find conormal knot

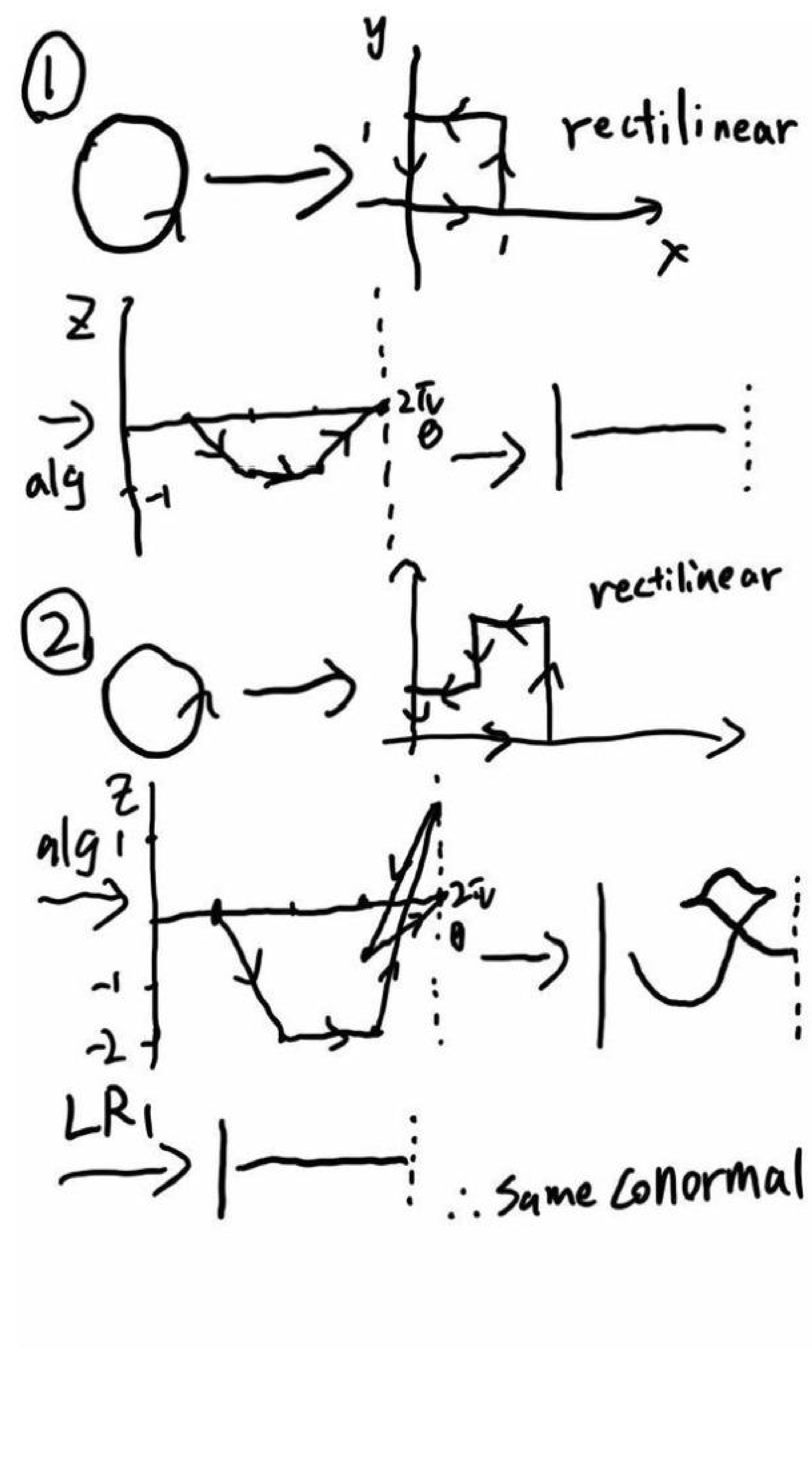
Definition 5.1 (Rectilinear)**.** We call a plane curve rectilinear if it is composed only with line segments that is parallel to the coordinate axis and does not have two line segment lie on the same line.

Since any plane curve without dangerous self tangency is isotopic to a curve that is Rectilinear, we can consider the θ, z projection of the Rectilinear curve which is going to be isotopic to the actual conormal knot and carries the same information about the invariants.

Consider the Following algorithm.[Ng05]
For any plane curve, first find its rectilinear isotopy. For each line-segment of the rectilinear curve it is associated with a point in the front of $J^1(S^1)$ which is equivalent to $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$. One can think of it as $\mathbb{R}^2$ but 0 and 2π are glued together such that they are the same point.
For each segment L of the rectilinear curve:

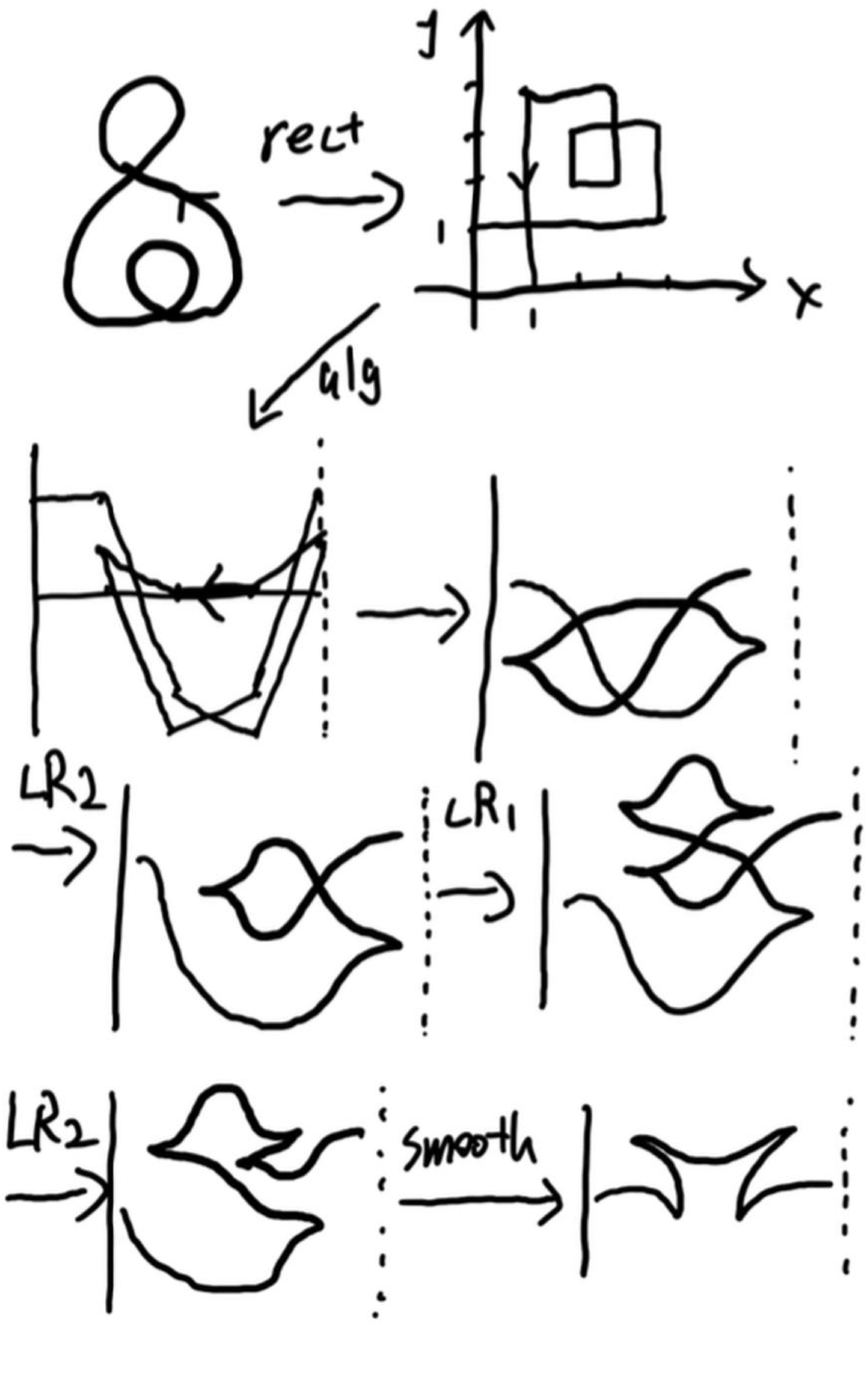
(1) $(\pi/2, y)$ if L is in the +x direction and y is the y coordinate of L;
(2) $(\pi, -x)$ if L is in the +y direction and x is the x coordinate of L;
(3) $(3\pi/2, -y)$ if L is in the +x direction and y is the y coordinate of L;
(4) $(0, x)$ if L is in the +y direction and x is the x coordinate of L;

And the conormal front projection is the connecting curve of all the points produced by the algorithm. Place cusps correspond to inflection points of the rectilinear curve where the second derivative change sign.
A simple plane curve with two different rectilinear isotopy will achieve the same conormal:



As we can see in the picture above, in the front projection of second example, both point $(2\pi, 1), (3/2\pi, -1)$ are cusps because they represent inflection points in the smoothed rectilinear curve near $(1, 1), (0, 1)$ where the second derivative changed sign. Thus we can apply Legendrian Reidemeister move 1 and it will achieve the same result as example 1.

Another example of drawing conormal knot front projection with more rigorous use of Legendrian Reidemeister moves:



6. Why this algorithm works

Before showing why this algorithm works we will prove some of the non-trivial result that was omitted during the explaining in previous sections.

1.Why the Conormal of any plane curve is a legendrian in $ST\mathbb{R}^2$?

Lets say the plane curve is a function $C: S \to \mathbb{R}^2$ and $C(\phi) = (q_1(\phi), q_2(\phi))$.

Thus by the definition of conormal knot we can think of it as a immersion from $L: S \to ST\mathbb{R}^2$, Where $f(\phi) = (q_1(\phi), q_2(\phi), (\cos(\theta), \sin(\theta)), \theta = arg(\mathrm{d}q_1(\theta), \mathrm{d}q_2(\theta)) + \pi/2$.

To show it is a legendrian knot we have to show that at each point of the immersion the derivative sits in the kernel space of the contact structure $\alpha_1 = \cos\theta \mathrm{d}q_1 + \sin\theta \mathrm{d}q_2$.

we know that $L(\phi) = (q_1(\phi), q_2(\phi), (\cos(\theta), \sin(\theta))$
Lets plug in into the form α_1

$$\begin{aligned}
\alpha_1(L'(\phi)) &= \cos\theta(\phi)\mathrm{d}q_1 + \sin\theta \mathrm{d}q_2 \\
&= \cos(arg(\mathrm{d}q_1(\phi), \mathrm{d}q_2(\phi)) + \pi/2))\mathrm{d}q_1 + \sin(arg(\mathrm{d}q_1(\phi), \mathrm{d}q_2(\phi)) + \pi/2))\mathrm{d}q_2 \\
&= -\mathrm{d}q_2\mathrm{d}q_1 + \mathrm{d}q_1\mathrm{d}q_2 = 0
\end{aligned}$$

From here we can also see an intuitive explanation for why the conormal is a legendrian. Since from the definition of conormal knot that the derivative v_x is perpendicular to $(\cos\theta, \sin\theta)$. We know the form $\alpha_1(L'(\phi)) = \langle(\mathrm{d}q_1, \mathrm{d}q_2), (\cos\theta, \sin\theta)\rangle = 0$
Thus the conormal knot is a legendrian.

2. let $f: (M, \alpha_1) \to (N, \alpha_2)$ be a contactomorphism and $L \subset M$ be a Legendrian then show that $f(L)$ is Legendrian.

Since we know L is a legendrian on M, then at every point of $L(t), t \in [0, 2\pi]$ we have $\alpha_1(L'(t)) = 0$, $\forall t$. Also by definition of contactomorphism α_1 is equivalent to the pullback of α_2 over f. Thus $\alpha_2 \circ Df = \alpha_1$.
Now lets consider $0 = \alpha_{1L(t)}(L'(t))$ and work backward.

$$\begin{aligned}
0 &= \alpha_{2f(L(t))}(f_*(L'(t))) \\
&= \alpha_{2f(L(t))}(Df(L(t))_{f(L)})
\end{aligned}$$

From here we are able to see that $f(L)$ is a immersion on N such that at every t its derivative sits in the kernel space of $\alpha_{2f}(L(t))$. Thus $f(L)$ is a legendrian on N.

Now we can look into why the algorithm showed in previous section work.

Proof of algorithm

The rectilinear curve can be separated into different line segments that goes into four different directions. $+x, +y, -x, -y$.

And to find the front projection of the conormal knot passed to $J^1(S^1)$ we need to first do the conormal transformation to $ST\mathbb{R}^2$ then use the contactomorphism to pass it to $J^1(S^1)$.

Lets say the plane curve is $c(t) = (q_1(t), q_2(t)$, the conormal knot is $f(c(t)) = (q_1(t), q_2(t), (\cos\theta, \sin\theta))$ where $\theta = arg(\mathrm{d}q_1, \mathrm{d}q_2) + \pi/2$, and the immersion passed to J^1S^1 is $g(t) = (\theta, y, z)$, where θ is not changed and $y = -q_1 \sin\theta + q_2 \cos\theta$, $z = q_1 \cos\theta + q_2 \sin\theta$.
At any point (x, y) in curve C going in direction v we know $\theta = v + \pi/2$. Now since rectilinear curves only have four direction we are able to check that the front projection (θ, z) of any such line segments.

In the +x direction, $v = 0, \theta = \pi/2, z = q_1 \cos\theta + q_2 \sin\theta = q_2(t) = y$, thus the image is $(\pi/2, y)$.
In the +y direction, $v = pi/2, \theta = \pi/2, z = q_1 \cos\theta + q_2 \sin\theta = -q_1(t) = -x$, thus the image is $(\pi, -x)$.
In the -x direction, $v = \pi, \theta = \pi/2, z = q_1 \cos\theta + q_2 \sin\theta = -q_2(t) = -y$, thus the image is $(3\pi/2, -y)$.
In the -y direction, $v = 3\pi/2, \theta = 2\pi$ or 0, $z = q_1 \cos\theta + q_2 \sin\theta = q_1(t) = x$, thus the image is $(0, x)$.

Now we showed that the front projection of the rectilinear line segments can be mapped to the points given in the algorithm. If we change the corner of the rectilinear curve into a smoothed differentiable corner, the resulting curve in $J^1(S^1)$ will also be smooth and continuous and will pass through the two points found by our algorithm.

Now lets check smoothed corner of the rectilinear curve approximated by the quarter circle with a relatively small radius r.
Since the conormal angle points to the center of the quarter circle if the corner is positively oriented and point in inverse direction if the corner is oriented negatively.
Let the center of the quarter circle be (a, b) then the parameterization of the quarter circle is $(a - r\cos\theta, b - r\sin\theta)$ if it is positive oriented corner, $(a + r\cos\theta, b + r\sin\theta)$ if negatively oriented.

Thus the z coordinate of front projection of corner can be expressed as a function of a, b, θ, $\theta \in [\theta, \theta + \pi/2]$, $z = a\cos\theta \pm r\cos^2\theta + b\sin\theta \pm r\sin^2(\theta) = a\cos\theta + b\sin\theta \pm r$

Since r can be decreased to infinitely small we can ignore it in our calculation thus the front projection of smoothing corner at (a,b) is $(\theta, a\cos\theta + b\sin\theta)$. It is clear that it pass through the two points representing the two line segments of the corner given in previous steps.
Lets say the front projection of two corner located at (a,b) and (c,d) intersect at point (θ, z). From the parameterization we can show $(a - c)\cos\theta = (d - b)\sin\theta$.

We can separate it into two different situations:

- $a = c$ or $b = d$ (not both because it will be same corner).
- $a \neq c$ and $b \neq d$.

The first situation imply either $\cos\theta = 0$ or $\sin\theta = 0$, It is obvious there will not be another solution in when θ change within $\pi/2$ because it have to change exactly by π

For the second situation we can actually divide it then $\tan\theta = (d - b)/(a - c)$. Then because tangent function never takes same value between $[0, 2\pi]$ we know there are no other intersection point.

Therefore we proved any front projection of conormal of smoothed corner can only intersect at most once. Also it is easy to show if two line segment intersect each other on the conormal front projection, there exist a intersection angle in the smoothing corner too. Because if they intersect as line it must be in a $\pi/2$ window and the sign of $(d-b)/(a-c)$ is the same as tangent in that interval, thus it exist exactly 1 theta between the interval that the corner intersect.

Thus we are able to approximate the smoothing corner with a line segment because the intersection pattern does not change. So the connected curve that the algorithm provide is indeed legendrian isotopic to the conormal front of the original plane curve.

Proposition 6.1. *The cusps of the front projection produced in this algorithm should be placed correspond to the inflection point of the rectilinear curve.*

Proof. Lets say the plane curve is parameterized as $C(t) = (x(t), y(t))$ where $x(t), y(t)$ describe the behavior of the plane curve in $\mathbb{R}^2$. Since the inflection points happen when the curvature changes sign, and the curvature $\mathrm{d}\phi/\mathrm{d}s$ represent the rate of change of the tangent vector angle $(\cos\phi, \sin\phi)$ with respect to arc length s. Since the curvature is

$$\mathrm{d}\phi/\mathrm{d}s = (\mathrm{d}\phi/\mathrm{d}t)/(\mathrm{d}s/\mathrm{d}t) = (\mathrm{d}\phi/\mathrm{d}t)/(\sqrt{(x')^2 + (y')^2})$$

The denominator is always positive, thus we can show that it always have the same sign as $\mathrm{d}\phi/\mathrm{d}t$ which is the change of tangent direction with respect to time. Notice the conormal direction θ is just the tangent direction plus $\pi/2$, thus $\mathrm{d}\theta/\mathrm{d}t = 0$ when the curvature is 0 and change sign when inflection point happen.

Now we can use the legendrian equation obtained from the form $\mathrm{d}z - y\mathrm{d}\theta$ which is $z'(t) - y(t)\theta'(t) = 0$, we are able to see that $\mathrm{d}z = 0$ when $\mathrm{d}\theta = 0$. Since we know curve C is an immersion thus the first derivative $(y'(t), z'(t), \theta'(t)) \neq 0$, Thus $y'(t) \neq 0, z'(t) = \theta'(t) = 0$ Thus it must be a cusp on the front projection. □

Proposition 6.2. *The right cusps of the conormal knot of a plane curve traverse upward and left cusps traverse downward.*

Proof. Lets consider $\mathrm{d}y$ near the neighborhood of the cusp where $\mathrm{d}\theta = 0$, we know $y = -q_1\sin\theta + q_2\cos\theta$, thus we have:

$$\begin{aligned}
\mathrm{d}y &= -\sin\theta\mathrm{d}q_1 + \cos\theta\mathrm{d}q_2 - q_1\cos\theta\mathrm{d}\theta - q_2\sin\theta\mathrm{d}\theta \\
&= -\cos(\phi)\mathrm{d}q_1 - \sin(\phi)\mathrm{d}q_2 - q_1\cos\theta\mathrm{d}\theta - q_2\sin\theta\mathrm{d}\theta \\
&= -\cos(\phi)\mathrm{d}q_1 - \sin(\phi)\mathrm{d}q_2 - (q_1\cos\theta + q_2\sin\theta)\mathrm{d}\theta
\end{aligned}$$

Since $\cos\phi, \mathrm{d}q_1$ and $\sin\phi, \mathrm{d}q_2$ are pointing in the same direction thus the product must be positive. And because it is around the neighborhood of $\mathrm{d}\theta = 0$, we know that $\mathrm{d}y$ have to be negative near the cusp. Therefore no matter θ is increasing and then decreasing(right cusp) or vise versa the y value decrease after passing through the cusp. Recall that $y = \frac{\mathrm{d}z}{\mathrm{d}\theta}$, the slope of the front projection decrease after cusp means at the cusp it always have a positive orientation thus right cusp always traverse up and left cusp always traverse down. □

Lemma 6.1. *The rotation number of conormal knot of a plane curve is 0.*

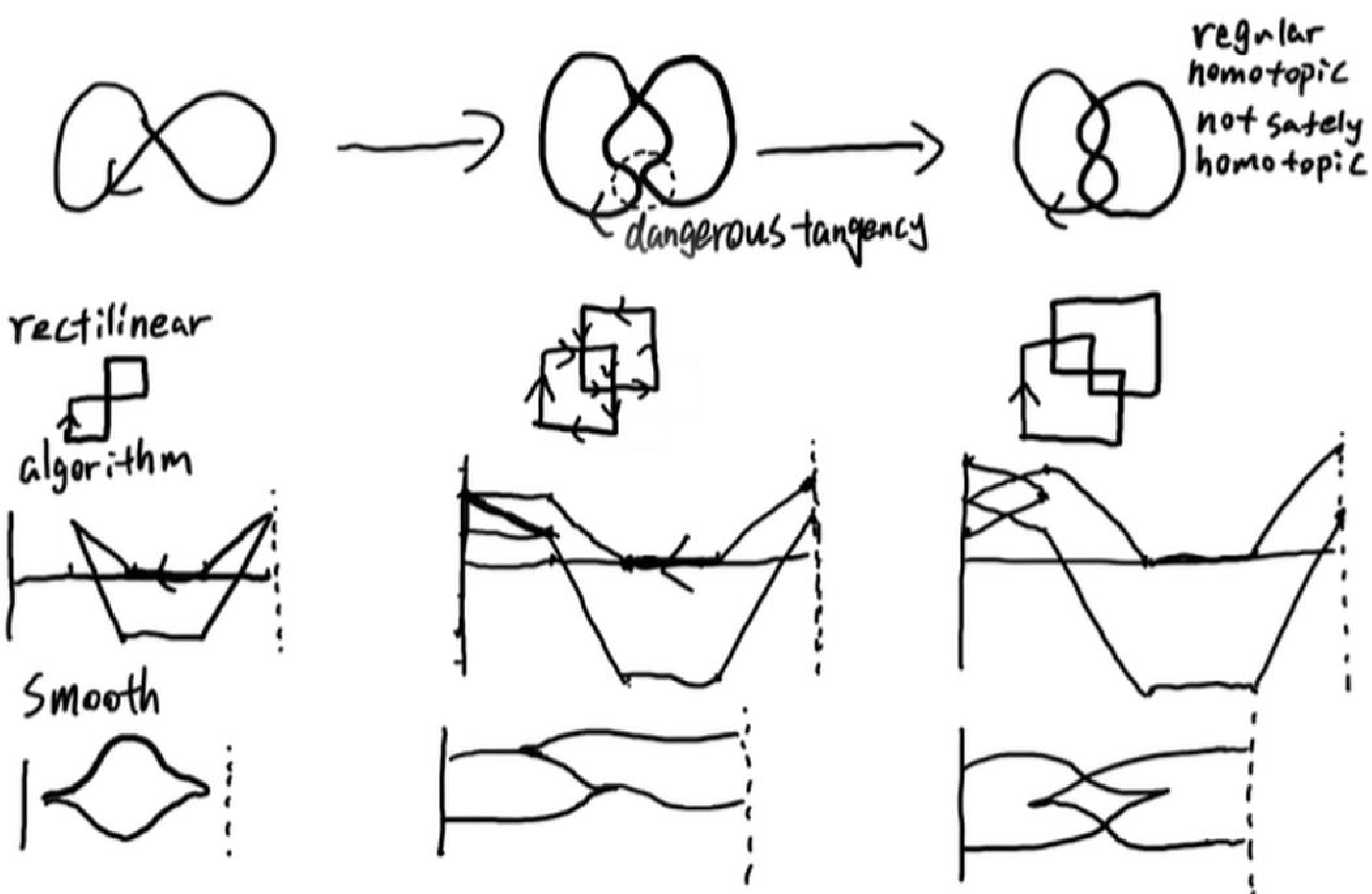
Proof. This lemma follows closely with the definition of how to calculate r with front projection and the previous proposition.
The rotation number is calculated by adding the total number of downward cusps and minus the number of upward cusp divided by 2. With the result of proposition 6.2 we only have to show that the front projection have equal number of left and right cusps.

Since the legendrian is an immersion it is connected from $t = 0$ and $t = 2\pi$, its start and end point must have the same derivative. And there are no vertical tangent line in the front projection. Therefore all instance of change in sign of $d\theta$ must be a cusps. So the total instance of changing of sign of $d\theta$ from positive to negative must be equal to negative to positive (which represent right and left cusps), because the sign of $d\theta$ is the same at $t = 0$ and $t = 2\pi$.

Therefore we know their exist same number of left and right cusps thus with proposition 6.2 we are able to show the number of upward and downward cusps are equal thus rot=0. □

7. Dangerous self tangencies relation with TB

Here is an example of two plane curves that are regularly homotopic but not safely homotopic. And how the conormal knot change during the homotopy process.



Because of the dangerous self tangency, the rectilinear curve will always have two line segment that lies on the same line or have repeated line segment. Thus we are not able to find a rectilinear curve that strictly follows the definition. However we can still apply the algorithm to see what happens to the conormal knot. As we can see in the picture, the conormal knot when the self-tangency occurred have two cusps that are connected to each other and it is not a legendrian anymore because of the restriction of being an embedding.
But we can compare the conormal before and right after the dangerous self tangencies we can show that the tb is actually increased by 2 from -1 to 1.

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References

[1] Hansjörg Geiges. *A contact geometric proof of the Whitney-Graustein theorem*, Dec 2007.

[2] Hansjörg Geiges. *An introduction to contact topology*, Cambridge University Press, 2008.

[3] Lenhard Ng. *Plane curves and contact geometry*, Nov 2005.

[4] Michael Spivak. *Calculus on Manifolds: A Modern Approach to Classical Theorems of Advanced Calculus*, Benjamin/Cummings, 1968.