Classnote 1 (1/17)

Well-function: $f: \mathbb{C} \to \mathbb{C}$

basic-properties: "1-1" "onto" "Defination of continuity"

" $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ "

!!!: being differentiable "stronger" regulation

i) If f is differentiable, then it is differentiable infinitely many times

ii) If f is differentiable, then $\forall$ line integeral $\int_\gamma f(z)\,dz$ along closed curve is 0.

iii) $\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{z-w}\,dw = f(z)$!!!

iv) If f is differentiable, then $|f(z)|$ is unbounded

Prob1: Find explicit formula $\to (a_n)_{n\in N}$, where $a_0 = 2$, $a_1 = 3$, $a_n = 4a_{n-1} + 6a_{n-2}$ for $n \geq 2$.

Answer: Intro generating function $F(z)$ "F is analytic function"

Prob2: $\forall$ non-constant single-variable poly with complex coeffient has at least one complex root. (3 ways)

$\forall\ P(x) = a_m x^m + \cdots a_1 x + a_0 \quad a_j \in \mathbb{R}\ (\mathbb{C})$, $\exists\ x_0 \in \mathbb{C}$, $P(x_0) = 0$

Prob3: $\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,dx$

tips: intro $\frac{\cos(z)}{1+z^2}$ for $z \in \mathbb{C}$, $\cos(z) > 1$.

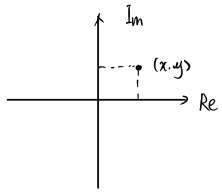
Prob 4: $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi} f(x)\,dx$. (Fourier transform)

tips: $\Rightarrow \hat{f}(z) = \int_{-\infty}^{\infty} e^{-2\pi i z} f(x)\,dx$, instead where $z \in \mathbb{C}$.

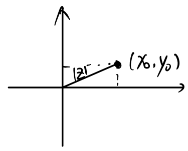
Conponent: $\forall x+iy \in \mathbb{C}$, then x is called "real part" $x = \operatorname{Re} z$

$\Updownarrow$

dot in the plane $\quad$ y is called "imaginary part" $y = \operatorname{Im} z$.



Def: The modulus or absolute value of z is defined by $|z| = \sqrt{x^2+y^2}$, $z = x+iy$



$z_0 = x_0 + iy_0$

Define: complex conjugate $z = x+iy$ is given by $\bar{z} = x - iy$

pf: $z\bar{z} = |z|^2 \Rightarrow (x+iy)(x-iy) = x^2+y^2+ixy-ixy = x^2+y^2$

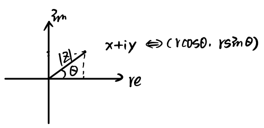
Operation: $*$ $\dfrac{z}{w} = \dfrac{z\bar{w}}{w \cdot \bar{w}} = \dfrac{(xs+yt)+i(ys-xt)}{s^2+t^2}$

" $z = x+iy$, $w = s+it$ "

$z^2+2z+3=0 \Rightarrow \dfrac{-2 \pm 2\sqrt{2}i}{2} = -1 \pm \sqrt{2}i$

Polar form: $x = r\cos(\theta)$ $\quad y = r\sin(\theta)$ where $r = \sqrt{x^2+y^2}$ and θ is the measured from positive x-axe $\to$ origin

$= |z|$



lecture note (1.19)

polar representation $\to z$: $\quad z = |z| \cdot (i\sin(\theta) + \cos(\theta))$

ex: $z = 1+i = \sqrt{2} \cdot (i \cdot \sin\frac{\pi}{4} + \cos\frac{\pi}{4})$

$z = 3i = 3 \cdot (i \cdot \sin\frac{\pi}{2} + \cos\frac{\pi}{2})$

$z = \sqrt{3}+i = 2(i\sin\frac{\pi}{6} + \cos\frac{\pi}{6})$

Why we need polar-form?

" It is suitable for computing powers! " $\quad$ ex. $(1+i)^{100} = ?$

De Moivre's Theorem

"For all $n\in N$ and $\theta\in[0,2\pi)$ we have $(i\sin(\theta)+\cos(\theta))^n = i\sin(n\theta)+\cos(n\theta)$" $\Rightarrow$ $(r(i\sin\theta+\cos\theta))^n = r^n(i\sin(n\theta)+\cos(n\theta))$

Ex sol'n $\rightarrow$ $(1+i)^{100} = (\sqrt{2}(i\sin\frac{\pi}{4}+\cos\frac{\pi}{4}))^{100} = (\sqrt{2})^{100}\cdot(i\sin 25\pi+\cos 25\pi) = -2^{50}$

Ex. Cal $(i\sin\pi/3+\cos\pi/3)^6 = (i\sin 2\pi+\cos 2\pi) = 1$ / $(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})^4 = (\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})^4 = -1$

pf: For $n=1$ (obviously) base case "By induction"

$\downarrow$

suppose formula holds for n, $n\in N$. Then

$$(i\sin\theta+\cos\theta)^{n+1} = (i\sin\theta+\cos\theta)^n\cdot(i\sin\theta+\cos\theta)$$
$$= (i\sin(n\theta)+\cos(n\theta))(i\sin\theta+\cos\theta)$$
$$= i(\sin n\theta\cos\theta+\cos n\theta\sin\theta)+(\cos n\theta\cdot\cos\theta-\sin n\theta\cdot\sin\theta)$$
$$= i(\sin(n+1)\theta)+\cos[(n+1)\theta] \quad ▨$$

Corollary De-Mo:

$$z = r_1(i\sin(\varphi_1)+\cos(\varphi_1))$$
$$w = r_2(i\sin(\varphi_2)+\cos(\varphi_2))$$
$$\Rightarrow zw = (r_1\cdot r_2)\cdot(i\sin(\varphi_1+\varphi_2)+\cos(\varphi_1+\varphi_2))$$

注意: for $|z|$ or moduli $\longrightarrow$ multiple ; augment/angle $\longrightarrow$ add.

"Use De Moives $\rightarrow$ (find formula for $\sin 2\theta$, $\cos 2\theta$.

"推公式"

sol'n: $(i\sin\theta+\cos\theta)^2 = i(2\sin\theta\cos\theta)+\cos^2(\theta)-\sin^2\theta$

$= i\sin 2\theta+\cos 2\theta$.

Def: A field F is a set with two operation $(+/\cdot)$, which satisfied field property (A), (M) and (D).

Define $\rightarrow$ complex number: $F=\{(x,y): x, y\in R\}$

"+" $(x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2)$

"·" $(x_1,y_1)+(x_2,y_2) = (x_1x_2-y_1y_2, x_1y_2+x_2y_1)$

e_0 is the field $=(0,0)$ / element 1 为 $(1,0)$.

Ex: $(0,1)^2 = (0-1, 0) = (-1,0)$

Notation: for $z = (x, y) = (x,0)(1,0) + (y,0)(0,1) = x+iy$.

"Complex # of the form $(a,0)$. $\Rightarrow$ $(a,0)+(b,0) = (a+b,0)$ $\quad (a,0)(b,0) = (ab,0)$."

Triangle inequality: For all $z, w \in \mathbb{C}$, we have $|z+w| \le |z| + |w|$

⚠: It is not possible to compare complex number! ie: $z+w \le z+w$

pf: $|z+w|^2 = |z|^2+|w|^2+2\mathrm{Re}(z\overline{w}) \le |z|^2+|w|^2+2|z\overline{w}| \le |z|^2+|w|^2+2|z||w| = (|z|+|w|)^2$

"STRAIGHT. LINE"

$y = mx+b$, m-real in term of complex number.

we know $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z) = \mathrm{Re}(mz)+b$

$\mathrm{Re}\, iz = \mathrm{Re}(-y+ix) = -y$

Circle: center $z_0 = x_0+iy_0$ with r.

then $(x-x_0)^2+(y-y_0)^2 = r^2$

$\Rightarrow |z-z_0|^2 = r^2$

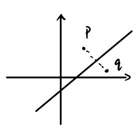
Determine what shape given by $|z-i| = \frac{1}{2}|z-1|$?

let $z = x+iy \Rightarrow x^2+(y-1)^2 = \frac{1}{4}((x-1)^2+y^2)$

$3x^2+3y+2x-8y+3 = 0 \Leftrightarrow (x+1/3)^2+(y-4/3)^2 = 8/9$

Generalization: Shape: $|z-p| = \rho|z-q|$ for some $p, q \in \mathbb{C}$ and $\rho \in (0,1)$

$\Rightarrow r = \rho|p-q|/(1-\rho^2)$ centered at

If $|z-p| = |z-q| \Rightarrow$ 

lecture note (1/24)

ex. $z^2+2z+4=0$ where $z\in\mathbb{C}$

$z=\frac{-2\pm\sqrt{-12}}{2} \Rightarrow z=-1\pm\sqrt{3}i$

Question: $P(z)=0$ for $P\in \mathrm{Poly}(n)$? $\Rightarrow$ complex number !

Def: suppose $n\in\mathbb{Z}^+$, $w\in\mathbb{C}$. A complex # $z^n=w$ is called n-th root of w.

ex. let $w=-1$, then $z^2=-1$ has two solution: $z=\pm i$ (second roots $\to$ "-1")

$\Rightarrow$ If given n th root, then there are n distinct nth root $\to w$.

Main: polar representation

let $z=|z|(i\sin\theta+\cos\theta) \xrightarrow{\text{by DeMoi}} z^n=|z|^n\cdot(i\sin(n\theta)+\cos(n\theta))$

$w=|w|\cdot(i\sin\varphi+\cos\varphi)$

$\uparrow$

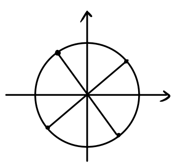
"Given"

$$\begin{cases}|z|^n=|w| \xrightarrow{\sim} |z|=|w|^{\frac{1}{n}} \\ \sin n\theta=\sin\varphi \\ \cos n\theta=\cos\varphi\end{cases} \xrightarrow{\sim} \theta=\frac{\varphi}{n} \text{ ! (Prob)}$$

$\Downarrow$

$\theta_k=\frac{\varphi}{n}+\frac{2\pi k}{n}$ for $k=0,1,2,\dots,n-1$

Geometric interpretation:

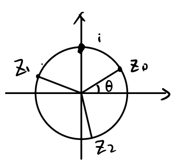


"将圆 n 等分" $\rho=|w|^{\frac{1}{n}}$

ex. for angle, Find 3rd root of i

$i=z^3$, $i=|i|(i\sin\frac{\pi}{2}+\cos\frac{\pi}{2})$

$\Rightarrow z=1\cdot(i\sin(\frac{\pi}{6}+\frac{2\pi k}{3})+\cos(\frac{\pi}{6}+\frac{2\pi k}{3}))$



ex. 2. Find 12th root of -1

ex. 3 Find 4th root of -4.

看图找!

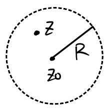
Why do we need topology ?

i) "Nice" real facture were defined on R, $[a,b]$, $(-\infty, a)$, $[b, +\infty)$, ...

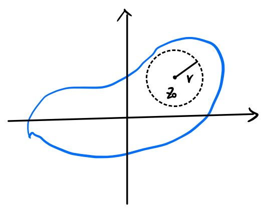
ii) for compler function: more possibility.

Def: open disc ("$D(z_0, R)$")

The set consisting of all point z that $|z - z_0| < R$ is called the open disc of radius R centered at z_0.

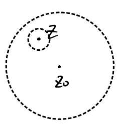


Interior point : A point z_0 in a set D in the complex plane is interior point of D if there is some open disc centered at z_0 that lies entiely in D



" open set : if all its points are interior point. "

Ex 1: $\forall D(z_0, R) = \{z : |z - z_0| < R\}$ is an open set!



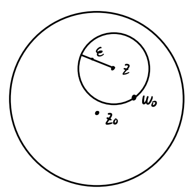
draw pic / prove formal : For $\forall w_0$ st $r = |w_0 - z_0| < R$

choose $\epsilon = \frac{1}{3}(R - |w_0 - z_0|$

Then for all $z \in D(w_0, \epsilon)$, we have $|z - z_0| \le |z - w_0| + |w_0 - z_0|$

$\le r + \frac{R - |w_0 - z_0|}{3} < R$

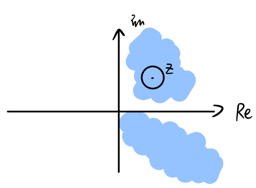
(by triangle inequality)



" $r + \frac{R}{3} - \frac{r}{3} = \frac{2r}{3} + \frac{R}{3}$ as $r < R$. "

then $r + \frac{R-r}{3} < R$

Example 2 : The set $R = \{z : \text{Re} z > 0\}$ is an open set !

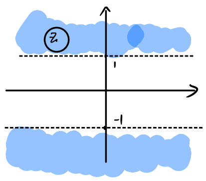


prove formal : let $w_0 \in R$ then $q = \text{Re} w_0 > 0$, let $\epsilon = \frac{q}{2}$

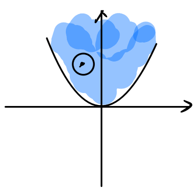
then for all $z \in D(w_0, r)$, we have

$\text{Re}\, z = \text{Re}(z - w_0) + \text{Re}(w_0) \ge -\epsilon + q > \frac{q}{2} = \epsilon$

ex 3. The set $R = \{z : |\mathrm{Im}\, z| > 1\}$ is an open set



ex4. $\{z : | z = x+iy ;\ x^2 < y\}$ is an open set.

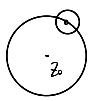


!!! : if $x^2 \le y$, then it is not a open set!

Boundary point: p is "b-p" of a set if $\forall$ open disc center at p contains both point of S and outside of S.

∩
Boundary (sets of BP)

ex. $D(z_0, R)$'s boundary $= S : \{z,\ |z - z_0| = R\}$.



Them: A set D is open iff it contains no point of its boundary.

"$\Rightarrow$" Assume D is open. Let p be a boundary point of D.

if $p \in D$, then $\exists\, D(p, r) \subsetneq D$. Thus p is not the bp.

"$\Leftarrow$" suppose $\exists$ no bp $\in D$. If $z_0 \in D$, then z_0 can not be bp $\Rightarrow D$.

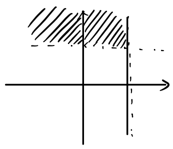
then $\exists\, D(z_0, r)$ either $\subset D$ / a $\underline{\text{subset of } D^c}$ impossible as $z_0 \in D$.

Closed set : D is closed set if it contains its boundary

Theorem : D is closed iff D^c is open. !!!

ex. Describ the interior and boundary of the set

$$\{x+iy \in \mathbb{C} : x \le 4, \; y > 3\}.$$



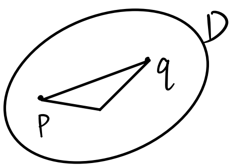
it is an closed set, boudary $\{z = 4+yi \mid y>3\}$.

$\{z = x+3i \mid x \le 4$

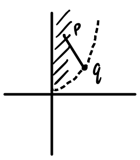
<u>lecture note 4</u>

1) Polynomial curve : Union of a finite # of directed line segment $P_1, P_2, \cdots P_n$.

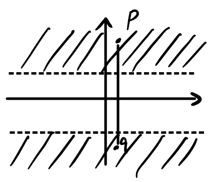
2) An open set D is connected if $\forall (p,q) \in D$ may be joined by polynomial curve lying entirely with D



ex. 1. open disc, open right half point, the set $\{x+iy \in \mathbb{C} : x^2 < y\}$.



connected ✓

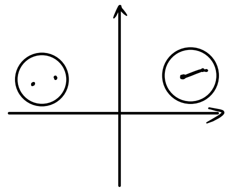


$|\mathrm{Im} z| > 1$ is not connected.

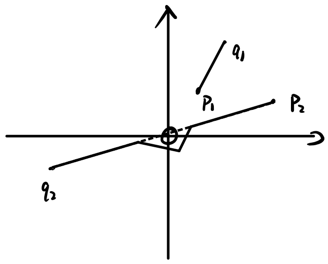
3) Domain : open + connected sets. (Natrual setting)

Example:

$D(3+3i, 1) \cup D(-3+3i, 1)$ is not connected



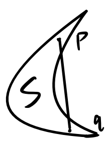
Exercise 1: Detemine if the set of all $z \in \mathbb{C}$ st $z = 0$ is domain



Exercise 2: for all $z \in \mathbb{C}$, $Re\, z \neq 0$ is not a domain

4) Covex set:

A set is convex if the line segment pq joining each pair of (p, q) is S also lies in S.



not convex: every open disc is convex. (false)

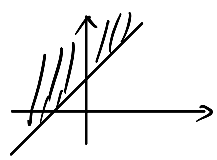
Them: $\forall$ convex open set is connected.

Q: Is the converse is true?

5) Open-half plane: ohp is defined to be points strictly to one side of a straight line

those point z for whcick $Re(az+b) > 0$

ohp is a convex domain

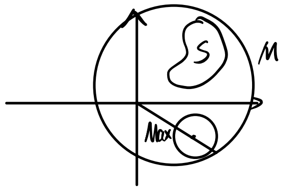


Closed half-plane : ohp + line , z with $Re(az+b) \geq 0$

convex ✓ but is not a domain ! (as it is closed)

6) Boundary set :

A set S in plane is bounded if there $\exists$ a $M \in R^*$ st $|z| \leq M$ for $\forall z \in S$. otherwise S is unbounded.



i) Open dis $D(z_0, r)$ is bounded

take $M = |z_0| + r$.

ii) open half plane is not bounded.

Ex : determine D and R $\rightarrow$ $f(z) = z^2 + 2z + 3$

soln : $D(f) = \mathbb{C} \Rightarrow R(f) = \mathbb{C}$

i) $z^2 + 2z + 3 = i \Rightarrow z = \frac{-2 \pm \sqrt{4 - 4(3-i)}}{2} = \frac{-2 \pm \sqrt{-8+4i}}{2}$

Ex2: $f(z) = \frac{1}{z-i}$ $Dom(f) = C \setminus i$

$\frac{1}{z-i} = w \Leftrightarrow \frac{1}{w} = z - i \Leftrightarrow z = \frac{1}{w} + i \Rightarrow Range = C \setminus \{0\}$.

Ex3 : Determine Dom / Ra $f(z) = |z|^4$

$Dom(f) = \mathbb{C}$ $R(f) = [0, \infty) \subset \mathbb{C}$

Ex: $f(z) = 1 + \frac{1}{Im z}$ Domain = 除了 x-axie

Range = $R \setminus \{1\} \subset \mathbb{C}$

Important example : $f(z) = \frac{1+z}{1-z}$ $R(f) \subset \{w ; |w| < 1, Re w > 0\}$

let $w = \frac{1+z}{1-z} \in Range(f)$ Dom: $\mathbb{C} - \{1\}$

then $\operatorname{Re} f(z) = \frac{1+(x+yi)}{1-(x+yi)} \cdot \frac{1+(x+yi)}{1+(x+yi)}$

$= \frac{1-|z|^2}{|1-z|^2} > 0$

impossible

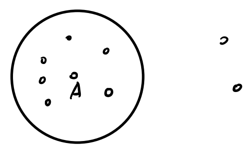
Assume $|f(z)| \geq 1$ then $|1+z|^2 \leq |1-z|^2 \Rightarrow 1+|z|^2+2\operatorname{Re}(z)+1 \leq 1+|z|^2-\operatorname{Re}(z)$

7) Graph $\rightarrow f(z)$ (impossible to draw)

Limit of a sequence:

convergence $\rightarrow$ Geometric way

$\{z_n\}_{n\in N}$ of complex # coverges to A iff whenever D is any open disc centered at A. all but a finite # of points $\{z_n\}_{n\in N}$ in D



Them $\rightarrow$ convergence:

let $z_n = x_n + iy_n$, $x_n, y_n \in R$ $A = a+bi$ Then $z_n \rightarrow A$ iff $x_n \rightarrow a$ and $y_n \rightarrow b$

If $z_n \rightarrow A$. then $|x_n - a| \leq |z_n - A|$ and $|y_n - b| \leq |z_n - A|$

$z_n = \frac{(1+i)^n}{3^n}$

lecture 6.

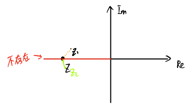
Def: $z = x+iy \in \mathbb{C} \Rightarrow e^{z} = e^{x}(i\sin(y) + \cos(y))$

The logarithm Function: for $z \neq 0 \in \mathbb{C}$, we define $\log z$ to be $\forall$ complex number w with $e^{z} = w$. (It is not unique)

we define principal branch of $\log z$: " $\log(z) = \log(|z|) + i\,\mathrm{Arg}(z)$, $\mathrm{Arg}(z) \in [-\pi, \pi]$. "

Note: $\log(z_2 z_1) = \log(z_2) + \log(z_1)$ " Not necessary equal "

! : log is continous X on the negative real axies



$\log(z) = \log(|z|) + i\,\mathrm{Arg}(z) = \log(|z|) + i\pi$

$\log(z_1) = \log(|z_1|) + i\pi(1-\varepsilon)$

$\log(z_2) = \log(|z_2|) - i\pi(1-\varepsilon)$

Jump !!! $\Rightarrow$ X conti

Ex1 $z = \log(2+2i)$ in the form $a+bi$?

$\log(z) = \log(|z|) + i\,\mathrm{Arg}(z) = \log(2\sqrt{2}) + i(\frac{\pi}{4}) = \log(2\sqrt{2}) + i\frac{\pi}{4}$

Ex2. $z = \log(3i + 3\sqrt{3}) = \log 6 + i\frac{\pi}{6}$ (✓) ?

⚠

?: $(\sqrt{2})^{\sqrt{2}} \equiv x^{x} \Rightarrow e^{\log x^{x}} = e^{x \log x}$

$\|$

$e^{\sqrt{2}\log\sqrt{2}}$

$\log(i) = \log 1 +$

ex3: find $(-1)^{i}$ values ?

we write: $(-1)^{i} = e^{\log(-1)^{i}} = e^{i\log(-1)}$

find $\log(-1) = i2\pi m + i\pi \Rightarrow e^{i(i2\pi m + i\pi)} = e^{-(2m+1)\pi}$

Calculate $(1+i)^{i}$

$(1+i)^{i} = e^{i\log(1+i)}$

then $\log(1+i) = \log\sqrt{2} + i\log\frac{\pi}{4}$. $\Rightarrow (1+i)^{i} = e^{i(\log(\sqrt{2}) + \frac{\pi}{4}i + 2\pi m i)} = e^{i\log(\sqrt{2}) - \frac{\pi}{4} - 2\pi m} = e^{\frac{-\pi}{4} - 2\pi m} \cdot (i\sin\sqrt{2} + \cos\sqrt{2})$.

Calculate $i^{i+1} \Rightarrow e^{(i+1)\log i} = e^{(i+1)\cdot(i)(\frac{\pi}{2} + 2m\pi)} = e^{(-1+i)(\frac{\pi}{2} + 2m\pi)} = e^{-(\frac{\pi}{2} + 2m\pi)} \cdot (i\sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2})) = i e^{-(\frac{\pi}{2} + 2m\pi)}$

Find at least one sol'n: $z^{1-i} = 6$

$\Rightarrow e^{(1-i)\cdot\log(z)} = e^{\log(6)}$

$\Rightarrow (1-i)\log(z) = \log(6)$

$\Rightarrow \log(z) = \frac{\log(6)}{1-i} = \frac{\log(6)(1+i)}{2}$

$\Rightarrow \frac{\log 6}{2} + i\frac{\log 6}{2} = \log(z)$

$\Rightarrow z = e^{\frac{\log 6}{2}}(i\sin(\frac{\log 6}{2}) + \cos(\log 6/2))$

Trigonometric function of z are defined in terms of the exponential function.

$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

property 1: if $z \in \mathbb{R}$, $z = x$, then $\cos z$ and $\sin z$ agrees to normal

we will check $\sin(z) = \frac{e^{ix} - e^{-ix}}{2i} = \frac{(i\sin x + \cos x) - (i\sin(-x) + \cos x)}{2i} = \sin x$.

property 2: $\sin(z) = \sin(z + 2\pi)$, $\cos(z) = \cos(z + 2\pi)$

property 3: $\sin(z)^2+\cos(z)^2=1$

Curve: A curve γ is a continuous complex-valued function $\gamma(t)$ defined for t in some $[a,b]$.

prop 4: $\sin(2z)=2\sin(z)\cos(z)$

Classification: 1) Simple curve: if $\gamma(t_1)\neq\gamma(t_2)$ whenever $a\le t_1<t_2\le b$. (without crossing)

2) not simple: 有结

Them: $\pi k,\ k\in\mathbb{Z}$ are only zeros of $\sin(z)$. $\pi k+\frac{\pi}{2}\ k\in\mathbb{Z}$ are only zeros of $\cos(z)$

$$0=\sin(z)=\frac{e^{-iz}-e^{iz}}{2i}\iff e^{-iz}=e^{iz}\Rightarrow e^{2iz}=1\iff z=\pi k.$$

(3) Curve close if $\gamma(a)=\gamma(b)$

Curve: A curve γ is a not closed continuous complex-valued closed function $\gamma(t)$ defined for t in some $[a,b]$.

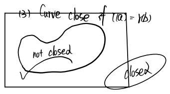
Classification: 1) Simple curve: if $\gamma(t_1)\neq\gamma(t_2)$ whenever $a\le t_1<t_2\le b$. (without crossing)

Jordan curve theorem:

2) not simple: 有结

The complement of the range of curve (simple + closed) consist of two disjoint open-connected sets. one bounded and the other unbounded. The bounded piece is inside of curve and unbounded outside.

(3) Curve close if $\gamma(a)=\gamma(b)$



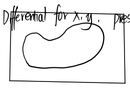
Jordan curve theorem:

The complement of the range of curve (simple + closed) consist of two disjoint open-connected sets. one bounded and the other unbounded. The bounded piece is inside of curve and unbounded outside.

Differential + Smooth Curve

If we have $\gamma:[a,b]\to\mathbb{C}$, $\gamma(t)=x(t)+iy(t)$ where $x,y:[a,b]\to\mathbb{R}$

Differential for x,y. preserved γ is differential



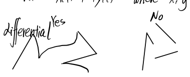
we write $\gamma'(t)=x'(t)+iy'(t)$

Differential + Smooth Curve

Piecewise smooth if it's composed of finite # of smooth curve, the end of one coinciding with

If we have $\gamma:[a,b]\to\mathbb{C}$, $\gamma(t)=x(t)+iy(t)$ where $x,y:[a,b]\to\mathbb{R}$

Differential for x,y. preserved γ is differential

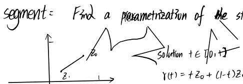


we write $\gamma'(t)=x'(t)+iy'(t)$

Note: refer range of $\gamma(t)$ as the curve $\gamma\to\gamma$. as the parametrization $\Rightarrow$ curve.

Piecewise smooth if it's composed of finite # of smooth curve, the end of one coinciding with

Ex: segment: Find a parametrization of the straight line $z_0, z_1\in\mathbb{C}$



solution $t\in[0,1]$

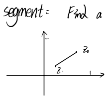
$\gamma(t)=tz_0+(1-t)z_1$

Note: refer range of $\gamma(t)$ as the curve $\gamma\to\gamma$. as the parametrization $\Rightarrow$ curve.

ex: find a parametrization of the circle of R and center $p\in\mathbb{C}$

$t\in[0,2\pi]$ $\gamma(t)=p+Re^{it}$

Ex: segment: Find a parametrization of the straight line $z_0, z_1\in\mathbb{C}$



solution $t\in[0,1]$

$\gamma(t)=tz_0+(1-t)z_1$

ex: find a parametrization of the circle of R and center $p\in\mathbb{C}$

$t\in[0,2\pi]$ $\gamma(t)=p+Re^{it}$

lecture 5.

Limit: let z_0 be a point either in S or in the boundary of S

f has limit L at the point z_0, and we write $\lim_{z \to z_0} f(z) = L$

if, given $\varepsilon > 0$, there is a $\delta > 0$, st $|f(z) - L| < \varepsilon$ if $|z - z_0| < \delta$

Ex1: The function $f(z) = |z|^2$ has a limit 1 for $z_0 = i$.

Limit at ∞: we say that the function f has a limit L at ∞, and we write $\lim_{z \to \infty} f(z) = L$, if given $\varepsilon > 0$. there is a large # M
$|f(z) - L| < \varepsilon$ if $|z| \geq M$.

Countiuity: f is defined on $S \subset \mathbb{C}$, If $z_0 \in f$. then f is conti at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

Partial Sums: if $z_i (1 \leq i \leq n)$ are complex #, define n-th partial sum $S_n = z_1 + z_2 + \cdots + z_n$.

if S_n has a limits. then $\sum_{j=1}^{n} z_j$ converges to s

Them: $\sum_{j=1}^{\infty} z_n$ converges iff $\sum_{j=1}^{\infty} x_j$ and $\sum_{j=1}^{\infty} y_n$ converge.

Them: If $\sum_{j=1}^{n} |z_j|$ converges, then $\sum_{j=1}^{\infty} z_j$ converges.

$$\sum_{j=1}^{n} |z_j| = \sum_{j=1}^{n} \sqrt{x_j^2 + y_j^2} > \sum_{j=1}^{n} |x_j| \Big/ \sum_{j=1}^{n} |y_j| \Rightarrow \sum_{j=1}^{\infty} z_j \text{ converges}$$

Property: If $\sum_{j=1}^{\infty} z_j$ converges, then $|z_n| \to 0$.

Ex3: If $\sum_{j=1}^{\infty} \frac{i^j}{j^2}$ converges?

$$\left|\frac{i^j}{j^2}\right| = \frac{1}{j^2} |i^j| = \frac{1}{j^2} \qquad \sum_{j=1}^{\infty} \frac{1}{j^2} = 0 \Rightarrow \text{converges}$$

Exponential function: for $x, y \in R$, $z = x + iy$, we define $e^z = e^x(i\sin y + \cos y)$.

Property 2: e^z is a continuous function of z.

Prop 3: $|e^z| = e^{\operatorname{Re} z}$ / $|e^{it}| = 1$.

Prop 4: $e^z \neq 0$, Check $|e^z| = e^{\operatorname{Re} z}$ and $e^x \neq 0$, for $\forall x \in R$.

Beautiful equation: $e^{i\pi} + 1 = 0$

Prop 5: let $w \in \mathbb{C}$, $e^z = w$ has infinite many sol'ns

pf: $w = r(\cos\theta + i\sin\theta) = e^{x+iy} = e^x(i\sin y + \cos y) \Rightarrow x = \log(r)$ $y = \theta + 2\pi m$ for $\forall m \in \mathbb{Z}$.

Speacial case: $S = \{z : 0 < \operatorname{Im} z \leq 2\pi\} \Rightarrow f(z) = e^z$ is 1-1.

Def: Principle branch of the log of z: $\log(z) = \log(|z|) + i\operatorname{Arg}(z)$

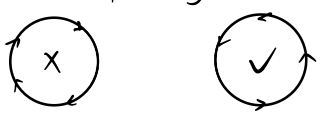
lecture 7. Green theorem:

i) orientation: Each curve r is increased $t \to$ oriented.

ii) The curve begin $r(t)$. traveled $\to$ from $a \to b$ and end at $r(b)$.

Positive oriented curve: if walk along r, direction, inside of r is on your left.

ex.



Integral: Assume $f: [a,b] \to \mathbb{C}$ if

$f = \operatorname{Re} f + \operatorname{Im} f$. we define

$$\int_a^b f(x)\,dx = \int_a^b \operatorname{Re} f(x)\,dx + i\int_a^b \operatorname{im} f(x)\,dx.$$

property:

1) $\operatorname{Re}\int_a^b f(x)\,dx = \int_a^b \operatorname{Re} f(x)\,dx$; $\operatorname{Im}\int_a^b f(x)\,dx = \int_a^b \operatorname{Im} f(x)\,dx$

Line integral of u along r:

1) Assume r is smooth and u: conti f with range r, we define u along r as:

$$\int_r u(z)\,dz = \int_a^b r'(t) \cdot u(r(t))\,dt.$$

Note: $r(t) = r_1(t) + i r_2(t)$

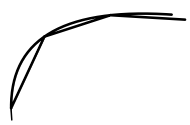
$r'(t) = r_1'(t) + i r_2'(t)$.

$$\int_r u(z)\,dz = \int_a^b r'(t) \cdot u(r(t))\,dt.$$

where $r(t) = r_1(t) + i r_2(t)$

Come from approximation ~ (useless!)

graph:



2) For a piecewise smooth curve r, we define the line integral of u along r.

$$\int_r u(z)\,dz = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} r'(t)\, u(r(t))\,dt.$$

The point t_j "piecewise smooth" by r' is continous on each segment $[t_j, t_{j+1}]$.

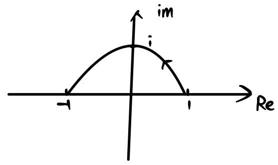
Example: i) Caculate line-integral → t^2 along 0 to $i+1$.

then $\int_\gamma u(z)dz = \int_0^1 \gamma(t)\cdot\gamma'(t) = \int_0^1 (i+1)\cdot(1+i)^2\cdot t^2\,dt = \frac{(1+i)^3}{3}(t^3|_0^1) = \frac{(1+i)^3}{3}$

para→ curve: $\gamma(t) = (1+i)\cdot t$, $t\in[0,1]$.

$\gamma'(t) = 1+i$

ii) $u = t^4$, along the circle arc from $1\to -1$ of center 0, pass i



$\int_\gamma z^4 dz = \int_0^\pi ie^{it}\cdot(e^{it})^4 = i\int_0^\pi (e^{it})^5 = \frac{i}{5i}\cdot(e^{5it}|_0^\pi) = \frac{-1-1}{5} = \frac{-2}{5}$

para→curve γ: $\gamma(t) = e^{it}$, $t\in[0,\pi]$

$\gamma'(t) = ie^{it}$

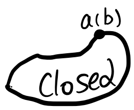
Closed

+

⚠) Assume γ is smooth curve, $n\in\mathbb{Z}$, $n\neq 1$, Then $\int_\gamma z^n dz = 0$.

Sol'n: $\int_\gamma z^m dz = \int_a^b \gamma(t)^m\cdot\gamma'(t)\,dt$, but $(\gamma^{m+1}(t))' = (m+1)\gamma(t)^m\cdot\gamma'(t)$

$\Rightarrow \quad = \frac{1}{m+1}\int_a^b (\gamma^{m+1}(t))'\,dt = \frac{\gamma^{m+1}(b)-\gamma^{m+1}(a)}{m+1}$



$\Rightarrow \int_\gamma z^m dz = \frac{0}{m+1} = 0$ ▨

Application: prove: $\frac{1}{2\pi}\int_0^{2\pi} e^{itk}\,dt = \begin{cases} 0 & \text{if } k\neq 0 \\ 1 & \text{if } k=0. \end{cases}$ "Fourier equation" $(k\in\mathbb{Z})$

Check: if $k=0$, then $\frac{1}{2\pi}\cdot\int_0^{2\pi} 1\,dt = 1$ (√)

if $k\neq 0$, as $\int_\gamma z^{k-1}dz = 0$, when γ is closed curve. ($\gamma(t) = e^{it}$, $t\in[0,2\pi]$)

In perticular we have $0 = \int_\gamma z^{k-1} = i\int_0^{2\pi} e^{ikt}\,dt$

Thm: Let g be a complex-valued conti-function on $[a,b]$, we have

$\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt$

pf: clearly, if $\int_a^b g(t)\,dt = 0$, so we may assume that $\int_a^b g(t)\,dt \neq 0$,

Let $\theta = \operatorname{Arg} \int_a^b g(t)\,dt$.

then $0 < \left|\int_a^b g(t)\,dt\right| = e^{-i\theta} \int_a^b g(t)\,dt = \int_a^b e^{-i\theta} g(t)\,dt = \int_a^b h(t)\,dt$. (as $>0 \Rightarrow h(t)$ is real number)

$e^{-i\theta} \cdot re^{i\theta} = r(\sin\theta i + \cos\theta) \cdot e^{i\theta} = r$.

where $h(t) = e^{-i\theta} g(t) \Rightarrow \left|\int_a^b g(t)\,dt\right| = Re \int_a^b h(t)\,dt = \int_a^b Re\, h(t)\,dt \le \int_a^b |h(t)|\,dt = \int_a^b |g(t)|\,dt$.

Corollary: $\left|\int_\gamma f(z)\,dz\right| \le \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\,dt$.

$= \left|\int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt\right| \le \int_a^b |f(\gamma(t))| |\gamma'(t)|\,dt$

length of the curve,

if $\gamma(t) = x(t) + iy(t)$, then the length of the curve that is the range of γ.

given by $\int_a^b \sqrt{x'(t)^2 + y'(t)^2} = \int_a^b |\gamma'(t)|\,dt$.

Cor 2: $\left|\int_\gamma f(z)\,dz\right| \le \max_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma)$.

pf: $\Rightarrow \left|\int_\gamma f(z)\,dz\right| \le \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\,dt \le \max_{t\in[a,b]} |f(\gamma(t))| \int_a^b |\gamma'(t)|\,dt = \max |f(\gamma(t))| \cdot \text{length of}(\gamma)$.

Ex. $\left|\int_\gamma z^3\,dz\right| \le 32\pi$ prove, where γ is a circle of center 0 and $r=2$.

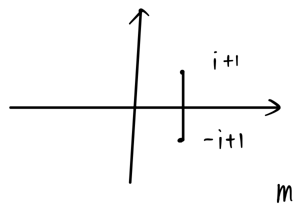
1) length of curve $= 2\pi r = 4\pi$.

2) $|z^3| = |z|^3 = 8$

$\Rightarrow \left|\int_\gamma z^3\,dz\right| \le 8 \cdot 4\pi = 32\pi$.

Ex2:

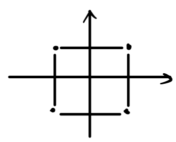
$$\left|\int_\gamma e^{-z}dz\right| \le \frac{2}{e}, \text{ where } \gamma \text{ is segment } i+1 \to -1+i$$



length = 2 as $\operatorname{Re} z = 1$

$$\max\left|e^{-z}\right| = \max e^{\operatorname{Re} z.} = e^{-1}$$

3). $\left|\int_\gamma z^2 \cdot e^{-z}dz\right| \le 16e$ where



length = $4 \cdot 2 = 8$ for all $z \in$ segment we have $|\operatorname{Re} z| \le 1$. more over $|z| \le \sqrt{2}$ for all $z \in \gamma$.

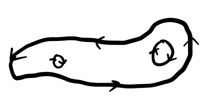
so $\max|z^2 \cdot e^{-z}| = |z|^2 \cdot e^{-\operatorname{Re} z} \le 2 \cdot e$

$\Rightarrow \left|\int_\gamma e^{-z}dz\right| \le 2e \cdot 8 = 16e.$

Different parametrization! ?: If para have influence $\to$ line integral (No. the value is alway the same).

Green theorem: fomula for domains Ω (boundary Γ consists of a finite # of disjoint piecewise curve $\gamma_1 \cdots \gamma_n$)

we oriented the boundary positive (out: counterclock; inner: clock wise)



Partial deravatives $\to$ complex plane: $(x, y) \rightsquigarrow x+iy$

Assume f is defined on $\mathbb{C}$ which $f(x,y)$

$f = p+iq$ where $p, q: \mathbb{R}^2 \to \mathbb{R}$. we define $\partial_x f(x,y) = \partial_x p(x,y) + i\partial_x q(x,y)$

$\partial_y f(x,y) = \partial_y p(x,y) + i\partial_y q(x,y)$

Green theorem: $\int_\gamma f(z)\,dz = i\iint_\Omega \partial_x f(x,y) + i\partial_y f(x,y)\,dx\,dy.$

Equivalently: $dx = \operatorname{Re}\gamma'(t)\,dt$, $dy = \operatorname{Im}\gamma'(t)\,dt$, u, v real valued.

$$\int_\gamma u\,dx + v\,dy = \iint_\Omega \partial_x v - \partial_y u\,dx\,dy$$

pf: let $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$, $f = v + iu$, so $\gamma(t) = (\gamma_1(t), \gamma_2(t))$

$$\int_\gamma f(z)\,dz = \int_a^b v(\gamma(t))\gamma_1'(t) - u(\gamma(t))\gamma_2'(t)\,dt$$

$$+\ i\int_a^b v(\gamma(t))\gamma_2'(t) + u(\gamma(t))\cdot\gamma_1'(t)\,dt.$$

$$i\iint_\Omega \partial_x f + i\,\partial_y f\,dx\,dy = i\iint_\Omega \partial_x v - \partial_y u\,dx\,dy + (-1)\iint_\Omega \partial_y v + \partial_x u\,dx\,dy.$$

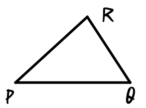
Ex. $f(z) = z^2$. then $f(z) = (x+iy)^2 = \underbrace{(x^2-y^2)}_{p(x,y)} + i\underbrace{2xy}_{q(x,y)}.$

$\Rightarrow \partial_x f(x,y) = 2x + i2y$; $\partial_y f(x,y) = -2y + 2ix$

$\Rightarrow$ Green theorem $\int_P z^2\,dz = i\iint_\Omega (2x-2y) + i(2x+2y)\,dx\,dy.$

Goal: Green's theorem $\to$ triangle   Verify!

i) $\int_P v\,dv = \iint_\Omega \partial_x v\,dx\,dy$ (for Ω being triangle with one-side horizontal.)



$$\begin{cases} P = a+ib \\ Q = c+ib \\ R = d+ie \end{cases}$$

parametrize the sides:
1) PQ: $x = t$, $y = b$ $a \le t \le c$

QR: $y = t$, $x = c + (d-c)\frac{t-b}{e-b}$ $b \le t \le e$

RP: $y = t$ $x = d + (a-d)\frac{t-e}{b-e}$ $b \le t \le e$

$A = \frac{d-c}{e-b}$

$B = c - b\frac{d-c}{e-b}$

$C = \frac{a-d}{b-e}$

$D = d - e\frac{a-d}{b-e}$

$$\int_\Gamma v\,dy = \int_{PQ} + \int_{QR} + \int_{RP}$$

$$= 0 + \int_b^e v(At+B, t)\,dt + \int_e^b v(Ct+D, t)\,dt.$$

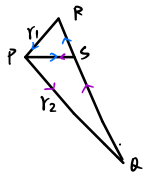
$$= \int_b^e v(At+B, t) - v((Ct+D), t)\,dt$$

$\Omega = \{(x,y): b \le y \le e,\ Cy+D \le x \le Ay+B\}$

$$\Rightarrow \iint_\Omega \partial_x v\,dx\,dy = \int_b^e \int_{Cy+D}^{Ay+B} \partial_x v\,dx\,dy.$$

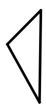
$$= \int_b^e (v(Ay+B, t) - v(Cy+D, y))\,dy.$$

Step $\to$ general triangle



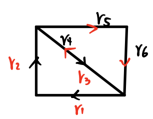
Sum up $\int_{\gamma_1} v\,dy + \int_{\gamma_2} v\,dy = \text{Area}$

Step 3: complementary integrals



$\int_\Gamma u\,dx = -\iint_\Omega \partial_y u\,dx\,dy$

Exercise: Green's theorem $\to$ rectangle



$\Rightarrow \iint_\Omega (\partial_x u - \partial_y v)\,dx\,dy = \sum_{i=1}^{6} \int_{\gamma_i} u\,dy + v\,dx.$

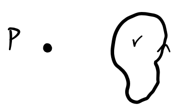
But $\int_{\gamma_3} = -\int_{4} \Rightarrow$ left 4 terms.

Bounded convex domain whose boundary consists of straight line

Important Example:

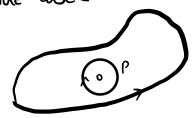
let $p \in \mathbb{C}$, γ-closed $\quad \int_\gamma \frac{dz}{z-p} = \begin{cases} 0 & \text{if } z \text{ is inside } \gamma \\ 2\pi i & \text{otherwise} \end{cases}$

$\Downarrow\ p = x + iy$



$\frac{1}{z-p} = \frac{1}{(x-x_1) + (y-y_1)i} \Rightarrow \int_\gamma \frac{1}{z-p}\,dz = i\iint_\Omega \partial_x f + i\partial_y f\,dx\,dy = 0.$

Inside case =



Harmonic function = 1) fist + Sec derivative (continous).

2) X and y is homonic on open set D if

$$\Delta u(x,y) = \partial_{xx}u + \partial_{yy}u = 0$$

Example 1. $u(x,y) = x + 2y$ is harmonic on R^2.

$u(x,y) = x^2 - y^2$ is harmonic on R^2

property = Harmonic function is determined by its values on the boundary

Them = Suppose that

pf = $u = f\partial_x f$, $v = -f\partial_y f$

in Green's theorem, then $\partial_x u = (\partial_x f)^2 + f\partial_{xx}f$, $\partial_y v = -(\partial_y f)^2 - f\partial_{yy}f$,

so $0 = \int_\gamma u\,dx + v\,dy = \iint_\Omega f \cdot \Delta f + (\partial_x f)^2 + (\partial_y f)^2 dx\,dy = \iint_\Omega (\partial_x f)^2 + (\partial_y f)^2 dx\,dy$

so $\partial_x f = \partial_y = 0$ on Ω. $\Rightarrow$ u is constant on Ω. But f is 0 on the boundary $\Rightarrow$ f is 0

harmonic-function (real-part-analysis)

Differentiable function = A function defined for z in Dom is differentiable

at point z_0 in D if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$ exist.

Analystic funiction = if f is differential $\Rightarrow$ analytic if on $\mathbb{C}$ $\Rightarrow$ entirely.

Exercise : if $f(z) = \text{Re}(z)$ is entire function ?

No ! $\lim_{\substack{h \to 0 \\ h \in R}} \frac{\text{Re}\,h - \text{Re}\,0}{h} = 1$ along $z = h \in R$

$\lim_{\substack{h \to 0 \\ h \to iR}} \frac{\text{Re}\,ih - \text{Re}\,0}{ih} = 0 \neq 1$ along $z = ih \in iR$.

Ex 2: $f(z) = z^n$ is entire and $f'(z) = nz^{(n-1)}$.

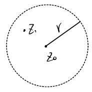
Binomial theorem: for $n \in \mathbb{N}$ and $x, y \in \mathbb{C}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k}, \quad \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

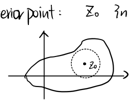
$$(z+h)^n - z_n = \sum_{k=1}^{n} \binom{n}{k} h$$

Topology:

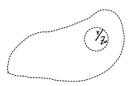
i) Open disc: The set consisting $\forall z$. st $|z-z_0|<R$ is called R centered at z_0 := "$D(z_0, r)$"



2) Interior point: z_0 in set D. st $\exists D(z_0, r) \subset D$:= interior point



3) Open set: Set D is called open if all its points are interior point.



Ex. open disc $D(z_0, R) = \{z : |z-z_0|<R\}$ is called open set

Ex2: The set $R: \{z : Re\, z > 0\}$ is an open set

Ex3: The set of all point $z = x+iy$ with $x^2 = y$ is not an open set

Boundary point:

A point p a boundary point of a set S if $\forall$ open disc centered at p contains both points of S and points not in S.

Boundary:

A set with "Bp".

Them: A set D is open iff it contains no point of its boundary.

Closed set:

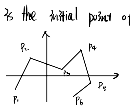
A set C is called closed if it contains its boundary

Them:

D is closed iff D^c is open.

Polynomial curve:

A "pc" is the union of a finite # of line segment $P_1, P_2, \cdots P_n$. where the terminal point of one is the initial point of the next.



Connected set: (1)

A open set D is connected if any $(p, q) \in D$ joined by a "pc" lying entirely with D. "An open connected set is called "domain"

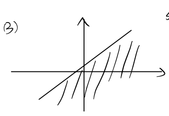
Covex set: A set S is convex if $\overline{pq}$ lies in S for $\forall (p,q) \in D$ (2)

ex1: open disc is conver

Them: $\forall$ convex open set is connected.

Open-half plane := those point strict to one side of a straight line

$Re(az+b) > 0$ (3)



Open-half-plane is a convex domain

A set is bounded if $\exists M>0$, st $|z|<M$ for all $z \in S$

otherwise is unbounded (5)

Close half-plane := $Re(az+b) \geq 0$ (4)

"Chp" is curvex, but it is not a domain.

lecture 4:

1) Domain + Range

i) ex: 1) $f(z) = z^2+2z+3$: $Dom(f) = \mathbb{C}$

Range $= \mathbb{C}$ as: let $w \in \mathbb{C}$, then $f(z) = w \Leftrightarrow z^2+2z+3 = w$ has a sol'n " $\mathbb{C} \subset Im(f)$ "

2) $f(z) = \frac{1}{z-i}$: $Dom(f) = \mathbb{C} / \{i\}$. because it can not be zero

$Im(f)$: $\frac{1}{z-i} = w \Rightarrow \frac{1}{w} = z-i \Rightarrow \frac{1}{w}+i = z \Rightarrow Im(f) = \mathbb{C}/\{0\}$.

3) $f(z) = |z|^4$ $Dom(f) = \mathbb{C}$, let $w = |z|^4 \Rightarrow Im(f) = [0, \infty) \subset C$

NM 4) $f(z) = \frac{1+z}{1-z}$ for $|z|<1$ $Im(f) \subset \{w : |w|<1, Re\,w>0\}$

sol'n: ① $Re\, f(z) = \frac{1-|z|^2}{|1-z|^2}$ as 上下 $>0 \Rightarrow Re\, f(z) > 0$

② Assume $w' \in \{w' | Re\,w' > 0\}$. let $z = \frac{w'-1}{w'+1} = \frac{\frac{1+z}{1-z}-1}{\frac{1+z}{1-z}+1} = \frac{1+z-(1-z)}{1+z+(1-z)} = \frac{2z}{2} = z$ ✓

then If $|z'| \geq 1$. that is $|w'-1|^2 \geq |w'+1|^2 \Rightarrow |w'|^2 - 2Re|w'|+1 \geq |w|^2+2Re|w'|+1$ which is impossible

thus $|z'|<1$

limit of a sequence:

let $\{z_n\}_{n\in N}$ be a sequence of complex numbers. We say $\{z_n\}_{n\in Z}$ has A as limit of that $\{z_n\}_{n\in N}$ converges to A: $\lim_{n\to\infty} z_n = A$ or $z_n \to A$

!: If given $\forall$ positive number $\varepsilon > 0$, There $\exists N \in Z$, st $|z_n - A| < \varepsilon$ for all $n \geq N$

Them: let $z_n = x_n + iy_n$, $x_n, y_n \in R$, $A = a+bi$. Then $z_n \to A$ iff $x_n \to a$ and $y_n \to b$.

Them: Assume that $\{z_n\}_{n\in N}$, $\{w_n\}_{n\in N}$ converge to A, B. then: $z_n + w_n \to A+B$

$\forall \lambda \in \mathbb{C}$, we have $\lambda z_n \to A\lambda$; $z_n w_n \to AB$; If $B \neq 0$, $\frac{z_n}{w_n} \to \frac{A}{B}$

pf: 2) $z_n w_n \to AB$. let $z_n = x_n + iy_n$ $w_n = p_n + q_n i$ $A = a+ib$ $B = c+id$

then $z_n w_n = (x_n p_n - y_n q_n) + i(y_n p_n + q_n x_n)$, As $x_n p_n - y_n q_n \to ac - bd$ $Im(z_n w_n) \Rightarrow bc+ad \Rightarrow z_n w_n \to zw$.

ex: $(1+i)^n / 3^n$ (ie: polar form) $(1+i) = \sqrt{2}(i\sin\frac{\pi}{4} + \cos\frac{\pi}{4}) \Rightarrow (1+i)^n = (\sqrt{2})^n \cdot (i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4}) / 3^n$. 错误的思路.

正解: $|z_n| = |\frac{\sqrt{2}}{3}|^n \to 0$.

ex2: $(\frac{1+\sqrt{3}i}{2})^n$: (ie: polar form) $z = (\frac{1}{2} + \frac{\sqrt{3}}{2}i)^n = (i\sin(\frac{n}{3}\pi) + \cos(\frac{n}{3}\pi))$ for $n = 6k$, then $Re(z) = 0$
$n = 6k+1$, then $Re(z) = \frac{1}{2}$

$\Rightarrow \{y_n\}_{n\in N}$ does not converge $\Rightarrow z_n$ does not converge.

Limit: let z_0 be a point either in S or on the boundary. We say that f has limit L at the point z_0

$\lim_{z\to z_0} f(z) = L$ or $f(z) \to_{z\to z_0} L$ if given $\varepsilon > 0$, there is $\delta > 0$ st $|f(z) - L| < \varepsilon$ if $|z-z_0| < \delta$

ex1: $f(z) = |z|^2$ has a limit 1 for $z_0 = i$

Limit at ∞, we say that the function f has a limit L at ∞, and we write $\lim_{z\to\infty} f(z) = L$ if. given $\varepsilon > 0$, there is a large number M st $|f(z) - L| < \varepsilon$ if $|z| \geq M$

Them: suppose that f and g are function with limits L and M. at z_0 and $\lambda \in \mathbb{C}$. then $\lim_{z\to z_0} f(z) + g(z) = L+M$.

$\lim_{z\to z_0} \lambda f(z) = \lambda L$ $\lim_{z\to z_0} f(z) g(z) = LM$. $\lim_{z\to z_0} f(z)/g(z) = L/M$ if $M \neq 0$.

Continuity: let f defined on $S \subset$ Complex plane. If $z_0 \in S$. then f is contious at z_0 if

$\lim_{z\to z_0} f(z) = f(z_0)$

! ex. check $f(z) = \frac{x-yi}{x+yi}$ is continous at $z_0 = 0$. (X conti as lim→0 however f(z) not defined!).

Sec 2.1 Analytical + Homonic + Cauchy-Riemann Equation.

Def: function $f: D \to \mathbb{Z}$ is differentiable at a point z_0 in D if

$$f'(z_0) := \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \text{ exist.}$$

If for $\forall z \in D$, f is differentiable, then f is called analytic in D.

Ex: if f and g are analytics on D, then

i) $(f+g)' = f'+g'$

ii) $(fg)' = f'g + g'f$.

iii) $(f/g)'(z_0) = \frac{g(z_0)\cdot f'(z_0) - f(z_0)\cdot g'(z_0)}{g(z_0))^2}$ for $g(z_0) \neq 0$.

iv) $[g(f(x))]' = g'(f(z))\cdot f'(z)$.

Ex.2: Show that $f(z) = e^z$ is an entire function.

Sol: $e^{z+h} - e^z = e^z \cdot (e^h - 1)$

let $h = \sigma + i\tau$

$e^h - 1 - h = \{e^\sigma \cos\tau - 1 - \sigma\} + i\{e^\sigma \sin\tau - \tau\}$

$= \{e^\sigma(\cos\tau - 1) + e^\sigma - 1 - \sigma\} + i\{e^\sigma(\sin\tau - \tau) + \tau(e^\sigma - 1)\}$.

hence $\left|\frac{e^h-1}{h} - 1\right| = \left|\frac{e^h - 1 - h}{h}\right| \le e^\sigma\left|\frac{1-\cos\tau}{\tau}\right| + \left|\frac{e^\sigma - 1 - \sigma}{\sigma}\right| + e^\sigma\left|\frac{\sin\tau - \tau}{\tau}\right| + \left|\frac{\tau(e^\sigma - 1)}{\tau}\right|$

as $\frac{1}{|h|} \le \frac{1}{|\tau|}$ and $\frac{1}{|h|} \le \frac{1}{|\sigma|}$.

Thus $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$ as $|\text{abso}| = 0$ for upper

then $\lim_{h\to 0} \frac{e^{z+h} - e^z}{h} = e^z \lim_{h\to 0} \frac{e^h - 1}{h} = e^z$.

Note: If f is analytical on $\forall$ domain D, then f is conti- on D.

$\boxed{\star}$ The Cauchy Riemann Equations.

Them 1: Cauchy-Riemann Equation:

Suppose that $f = u + iv$ is analytic on dom(D), then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

proof: take $h \in \mathbb{R}$

then $f'(z_0) = \lim_{h\to 0}\left[\frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} + i\,\frac{v(x_0+h, y_0)}{h}\right]$

$= \frac{\partial u}{\partial x}(x_0, y_0) + i\,\frac{\partial v}{\partial x}(x_0, y_0)$

then let $h = ik$, where $k \in \mathbb{R}$

$f'(z_0) = \lim_{k\to 0} \frac{u(x_0, y_0+k) - u(x_0, y_0)}{ik} + i\,\frac{v(x_0, y_0+k) - v(x_0, y_0)}{ik}$

$= \frac{1}{i}\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$

then 1) $v(x,y) = 3x^2y + p(y)$ Integral over x

2) $v(x,y) = 3x^2y - y^3 + q(x)$ Integral over y.

Application: ex: if we know $u(x,y) = x^3 - 3xy^2$, then $\frac{\partial v}{\partial x} = 6xy$, $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$.

If $f = u + iv$ be analytic on D + u, v are contin – first + second order.

then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

Hence if $u = \mathrm{Re} f$, then u is necessary $\Rightarrow$ Laplace's equation.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus for $\forall$ conti-function on D satisfying La'place equation := harmonic

Note! : for Cauchy-Riemann equation: v is called harmonic conjugate of u. Also, for v and v_1 both har-conj $\Rightarrow u$, their difference = "c" (constant)

$$\frac{\partial}{\partial y}(v - v_1) = \frac{\partial}{\partial y} v - \frac{\partial}{\partial y} v_1 = \frac{\partial}{\partial x} u - \frac{\partial}{\partial x} u = 0$$

Them2 : Suppose that $f = u + iv$ is analytic on dom(D). If either u is constant on D or $u^2 + v^2$ is constant on D, then f is constant on D.

proof : 1) Let u be constant on D, then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} = 0 \Rightarrow$ v is constant on each horizontal line as $\frac{\partial v}{\partial x} = 0$, vertical line as $\frac{\partial v}{\partial y} = 0$ } each pair of points can be lined/drawed by a curve $\Rightarrow$ v is constant.

2) let $u^2 + v^2$ be constant, suppose $|f| = \sqrt{u^2 + v^2} = c$ for some $c \in F$.

i) If $c = 0$, then f is identically zero.

ii) If $c \neq 0$. WLOG, let $c = 1$, then $|f| = 1$ Hence $f(\bar{z}) = \frac{1}{f(z)}$ on D

then as $f(z)$ is analytic on D, then $f(\bar{z}) + f(z)$ is analytic on D

then $i(f(z) + f(\bar{z}))$ is analytical, and $\mathrm{Re}(f(z) + f(\bar{z})) = 0 \Rightarrow f(z) + f(\bar{z}) =$ constant.

therefore $f(z)$ is constant.

Them3: Suppose $f = u + iv$ and $\{u, v, u_x, u_y, v_x, v_y\}$ are conti in $D(z_0, r)$. If u and v satisfied Cauchy Riemann equation at z_0, then f is differentiable at z_0

proof: Taylor series for u near (x_0, y_0)

$$u(x_0 + \delta, y_0 + v) = u(x_0, y_0) + \delta\left(\frac{\partial u}{\partial x}\right)(x_0, y_0) + v\left(\frac{\partial u}{\partial y}\right)(x_0, y_0) + \delta E_1 + v E_2$$

$\lim_{\delta \to 0} E_1(\delta) = 0$ + $\lim_{v \to 0} E_2(v) = 0$. $\quad v\left(\frac{\partial u}{\partial x}\right)(x_0, y_0) \quad -v\left(\frac{\partial v}{\partial x}\right)(x_0, y_0)$

$$v(x_0 + \delta, y_0 + v) = v(x_0, y_0) + \delta\left(\frac{\partial v}{\partial x}\right)(x_0, y_0) + v\left(\frac{\partial v}{\partial y}\right)(x_0, y_0) + \delta E_3 + v E_4.$$

$$f(z_0 + h) = f(z_0) + \left[\frac{\partial u}{\partial x}(x_0, y_0) + i\left(\frac{\partial v}{\partial x}\right)(x_0, y_0)\right] h + \delta(E_1 + iE_3) + v(E_2 + iE_4)$$

$$\| \quad\quad -h_2 + ih_1 = i(h_1 + ih_2) = i(\delta + vi).$$

$$\frac{\partial v}{\partial x}(x_0, y_0)(\delta i - v) + \frac{\partial u}{\partial x}(x_0, y_0)(\delta + vi)$$

from this + $\left|\frac{\delta}{h}\right| \le 1$ + $\left|\frac{v}{h}\right| \le 1$ $\Rightarrow \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \left[\frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)\right] + \frac{\delta}{h}(E_1 + iE_3) + \frac{v}{h}(E_2 + iE_4) + h$

Sec 2.2 Power series.

Def: A power series in z is an infinite series of the special form.

$\sum_{n=0}^{\infty} a_n (z - z_0)^n$: a_n be coefficient: $a_n \in \mathbb{C}$, z_0 is fixed := center.

ex. $f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}$, $|z| < 1$.

Them 2.1 : Suppose that there is some $z_1 \neq z_0$, st $\sum a_n (z_1 - z_0)^n$ converges. Then, for each z with $|z - z_0| < |z_1 - z_0|$, the series $\sum a_n (z - z_0)^n$ is convergent.

proof: Suppose that $|z-z_0| \leq r < |z_1 - z_0|$.

since $\sum_0^\infty a_n (z_1 - z_0)^n$ converges $\Rightarrow \lim_{n\to\infty} a_n \cdot (z_1 - z_0)^n = 0$

In pertzcular $|a_n| \cdot |(z_1 - z_0)^n| \leq M$ for all n .

then $|a_n| \cdot |z - z_0|^n = |a_n| \cdot |z_1 - z_0|^n \cdot \left(\frac{(z-z_0)}{(z_1 - z_0)} \right)^n$.

$\leq M \cdot \rho^n$ where $\rho = \frac{r}{z_1 - z_0}$ clearly $\rho < 1$.

then $\sum_0^\infty a_n \cdot (z-z_0)^n$ converges.

The radius of convergence: for $\forall$ series $\sum_0^\infty a_n (z-z_0)^n$, there are 3. possibility.

(1) The series $\sum a_n (z-z_0)^n$ converges only for $z = z_0$

(2) $\sum a_n (z-z_0)^n$ converges for all z

(3) $\sum a_n (z-z_0)^n$ converges for some $z = z_0$, but not all z.

(4) suppose that z' and z'' are two points with

$\sum_0^\infty a_n (z'-z_0)^n$ converge and $\sum_0^\infty a_n (z''-z_0)^n$ divergent.

By them 1: if $[\sum_0^\infty a_n (z'-z_0)^n > \sum_0^\infty a_n (z-z_0)^n] \Rightarrow [\sum_0^\infty a_n (z-z_0)^n$ convergent$]$.

if $[\sum_0^\infty a_n (z-z_0)^n > \sum_0^\infty a_n (z''-z_0)^n] \Rightarrow [\sum_0^\infty a_n (z-z_0)^n$ divergent$]$

By this define R (a unique #) s.t

if $|z-z_0| < R \Rightarrow \sum_0^\infty a_n (z-z_0)^n$ converges

$|z-z_0| > R \Rightarrow \sum_0^\infty a_n (z-z_0)^n$ diverges

A power series $\sum_0^n a_n (z-z_0)^n$, which converges for some $z \neq z_0$, always converges within a disc $\{z: |z-z_0| < R\}$ where R either infinite or positive.

if (2) $\sum a_n (z-z_0)^n$ converges for all z, we defin $R = 0$.

(3) $\sum a_n (z-z_0)^n$ converges for some $z = z_0$, but not all z. define $R = \infty$.

Them 2: Suppose that $\sum_0^\infty a_n (z-z_0)^n$ is a power series with positive / infinite radis convergence R

(a) If $\lim_{n\to\infty} |a_{n+1}/a_n|$ exist, then $\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

(b) if $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exist, then $\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

pf: (a) $\lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n} \right| = |z-z_0| \cdot \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |z-z_0| \cdot L$

By def of R, if $|z-z_0|L < 1 \Rightarrow$ converge, if $|z-z_0| \cdot L > 1 \Rightarrow$ diverge.

then $R \cdot L = 1 \Rightarrow L = \frac{1}{R}$.

Ex4. Find $R \rightarrow \sum_{n=0}^{\infty} 4^n z^{3n}$

Note: ratio / root test failed

$$a_k = \begin{cases} 4^{k/3} & \text{for } k = 0, 3, 6, \dots \\ 0 & \text{for otherwise} \end{cases}$$

let $w = z^3$ then

$\sum_{n=0}^{\infty} 4^n z^{3n} \Rightarrow \sum_{n=0}^{\infty} 4^n w^n$ has $\frac{1}{R} = 4 \Rightarrow R = \frac{1}{4}$.

then converge if $\frac{1}{4} > |w| = z^3$
$\frac{1}{4} < |w| = z^3$ $\Rightarrow R_* = 4^{-\frac{1}{3}}$

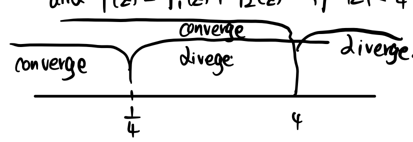
$f(z) = \sum_{n=0}^{\infty} 4^{n(-1)^n} z^n$

$$a_n = \begin{cases} 4^n, & \text{for } n = \text{even} \\ 4^{-n}, & \text{for } n = \text{odd} \end{cases}$$

then $f_1(z) = \sum_{n=0}^{\infty} 4^{2n} \cdot z^{2n} \Rightarrow R_1 = \frac{1}{4}$

$f_2(z) = \sum_{n=0}^{\infty} \frac{1}{4^{2n+1}} \cdot z^{2n+1} \Rightarrow R_2 = 4$.

and $f(z) = f_1(z) + f_2(z)$ if $|z| < \frac{1}{4}$.



$\Rightarrow R = \frac{1}{4}$.

The deravitive of a power series

let $f(z) = \sum_{0}^{\infty} a_n (z-z_0)^n$ with positive / infinite R $\Leftrightarrow$ analytic with in disc $|z-z_0| < R$ and its deravitive is given by $f'(z) = \sum_{1}^{\infty} n a_n (z-z_0)^{n-1}$. (9)

1) Show that (9)'s $R_9 \geq R$

Let $|z-z_0| = r < R$, and let $r < s < R$, then for all integer $n > N$, for some N

$n r^{n-1} \leq s^n$, $n \geq N$

Since $\lim_{n\to\infty} n r^{n-1} = 0$, then $n|a_n||z-z_0|^{n-1} \leq |a_n| s^n$ $n \geq N$.

and since $s < R$, then $|a_n| s^n$ converges. Hence $\sum |a_n||z-z_0|^{n-1}$ converges

for $|z-z_0| < R$

let $g(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$ WLOG: set $z_0 = 0$ (Suppose $z \in D(0,R)$)

let $\delta = \frac{1}{2}(R - |z|)$ and suppose $|h| < \delta$. then

$$f(z+h) - f(z) = \sum_{1}^{\infty} a_n [(z+h)^n - z^n]$$
$$= a_1 h + \sum_{2}^{\infty} a_n [(z+h)^n - z^n]$$

then
$$\frac{f(z+h) - f(z)}{h} - g(z) = a_1 + \frac{\sum_{2}^{\infty} a_n [(z+h)^n - z^n]}{h} - \sum_{1}^{\infty} n a_n (z)^{n-1}$$
$$= \sum_{2}^{\infty} a_n \left[\frac{(z+h)^n - z^n}{h} - n z^{n-1} \right]$$

Now by binomial them gives than $= \sum_{j=2}^{n} \binom{n}{j} z^{n-j} h^{j-1}$.

Hence
$$\left| \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right| \le \sum_{j=2}^{n} \binom{n}{j} |z|^{n-j} |h|^{j-1}$$
$$\le |h| \sum_{j=2}^{n} \binom{n}{j} (R-2\delta)^{n-j} \delta^{j-2}$$
$$= |h| \cdot \delta^{-2} \sum_{j=2}^{n} \binom{n}{j} (R-2\delta)^{n-j} \delta^{j}$$
$$< |h| \cdot \delta^{-2} \sum_{j=0}^{n} \binom{n}{j} (R-2\delta)^{n-j} \delta^{j}$$
$$= |h| \cdot \delta^{-2} [R - 2\delta + \delta]^n$$
$$= |h| \delta^{-2} [R - \delta]^n$$

$$\Rightarrow \quad \left| \frac{f(z+h) - f(z)}{h} - g(z) \right| = 0$$

$$|a_n| \left| \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right| \le \underbrace{|h| \cdot \delta^{-2} \sum_{n=2}^{\infty} |a_n| (R-\delta)^n}_{\to 0 \text{ when } h \to 0}$$

Thus f is differentiable with $f' = g$

$g' = q$

Them 3: If $f(z) = \sum_{0}^{\infty} a_n (z-z_0)^n$ has positive/infinite R, then with in $|z - z_0| < R$ is infinite differentiable, where

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1)\, a_n (z-z_0)^{n-k} \qquad k = 1, 2 \cdots$$

In perticular, by setting $z = z_0$, $\frac{f^{(n)}(z_0)}{n!} = a_n$, Ex. consider $\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)' = \sum_{n=1}^{\infty}\frac{n}{n!}z^{n-1} = \sum_{n=1}^{\infty}\frac{z^{n-1}}{n-1!} = \sum_{j=0}^{\infty}\frac{z^j}{j!}$

let $F(z) = \sum_{j=0}^{\infty}\frac{z^j}{j!}$ $\quad (e^{-z}F(z))' = -e^{-z}\cdot F(z) + e^{-z}F'(z) = 0$

Thus $F(z)\cdot e^{-z}$ = some constant λ But $1 = F(0) = \lambda$

so $F(z) = e^{z} \Rightarrow e^{z} = \sum_{n=0}^{\infty}\frac{z^n}{n!}$

Ex 7.

Show that $\sin z$ and $\cos z$ has power series expansions valid for all z, find the series explicitly.

Sol'n: given that $e^{z} = \sum_0^{\infty}\frac{z^n}{n!}$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} + \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$$

$$\sin z = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

Multiplication and divison of power series.

suppose $f(z) = \sum_0^{\infty}a_n z^n$ and $g(z) = \sum_0^{\infty}b_n z^n$ have radius at least R for positive/infinite.

the $f(z)\cdot g(z)$ is defined in the disc $\{z : |z| < R\}$

$$f(z)\cdot g(z) = a_0b_0 + (a_0b_1 + a_1b_0)z + (a_0b_2 + a_1b_1 + a_2b_0)z^2 + \cdots$$

$$= \sum_0^{\infty}c_n z^n \quad \text{where } c_n = \sum_{k=0}^{n}a_k\cdot b_{n-k}$$

Sec2.3 Cauchy theorem + Cauchy equation

Them1: Cauchy's theorem: Suppose that f is analytic on a dom(D), let γ be a piecewise smooth simple closed curve in D whose inside Ω also lies in D. Then $\int_\gamma f(z)\,dz = 0$

proof: by hypothesis: f' is also conti, Green theorem:

$$\int_\gamma f(z)\,dz = i\iint_\Omega \left\{\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right\} dx\,dy$$

However, as $f = u + iv$ is analytic on D, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

then $$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$= 0 + i(0) = 0 \Rightarrow \int_\gamma f(z)\,dz = 0.$$

Def: A domain D is simple-connected if, whenever γ is a simple closed curve in D the inside of γ is also a subset of D

equivalently say: " D has no holes in it "

Ex1: The disc $\{z : |z - z_0| < R\}$ is simple-connected

Ex2: Any convex domain Ω is simple connecte

Them3: If f is analytic in a simple-connected dom(D), then there is an analytic function F on D, with $F' = f$ throughout D

proof: Fix some point z_0 in D and define F at a point z of D by:

Step I: F is welldef $F(z) = \int_\gamma f(\zeta)\,d\zeta$

where γ is a polygnomial curve joining $z_0 \Rightarrow z$, which composed of a finite number of horizontal / verticle line segments

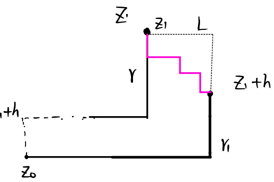
If γ_1 is another such curve, then $\Gamma = \gamma - \gamma_1$ is also a closed curve in D made of a finite number of horizontal / verticle line segments

and concequently: $0 = \int_\Gamma f(w)\,dw = \int_\gamma f(w)\,dw - \int_{\gamma_1} f(w)\,dw. \Rightarrow$ F is not dependent on the curve path choice, from $z_0 \to z \Leftrightarrow$ F is well-defined

Step II: We need $\to$ show that $F' = f$

let z_1 be a point in D, and r be a small number, st. $\{|z - z_1| < r\}$ lies in D

Now, let $|h| < r$, and let L be the curve similar $=$ T, a line (vertical or hori) joining $z_1 \to z_1 + h$



Note that $r_{max} \le 2|h|$ and that $\int_L dz = h$.

Let γ_1 be a poly-curve joining $z_0 \to z_1 + h$

the curve $T = \gamma_1 - L - \gamma$ is then a close poly-curve

Hence the line integral of $P = 0$

$$0 = \int_{\gamma_1} f(\xi)\,d\xi - \int_L f(\xi)\,d\xi - \int_\gamma f(\xi)\,d\xi = F(z_1+h) - \int_L f(\xi)\,d\xi - F(z_1) = 0$$

$$\Rightarrow \frac{F(z_1+h) - F(z_1)}{h} - f(z_1) = \int_L \frac{f(\xi)}{h}\,d\xi - f(z_1) = \int_L \frac{[f(\xi) - f(z_1)]}{h}\,d\xi$$

Given $\varepsilon > 0$, choose $\delta > 0$ small enough that $|f(\xi) - f(z_1)| < \varepsilon$ whenever $|\xi - z_1| < \delta$

$$\text{Thus } \left| \frac{F(z_1+h) - F(z_1)}{h} - f(z_1) \right| \le \frac{1}{|h|} \cdot \max\{f(\xi) - f(z_1)\} \cdot 2|h| < 2\varepsilon$$

concequently, the derivitive of F exist at z_1 and $F'(z_1) = f(z_1)$.

Cor: let f be analytic on a simple-connected domain D, and let γ be a piecewise smooth curve in D. Then

$$\int_\gamma f(z)\,dz = 0$$

proof: let F be a function on D with $F' = f$. Then $\int_\gamma f(z)\,dz = \int_\gamma F'(z) = F(\text{end point}) - F(\text{initial point}) = 0$. since γ is closed.

Thm 4: Cauchy formula: Suppose that f is analytic on D and γ is a piecewise smooth, positively oriented simple closed curve D whose inside Ω also lies in D.

then $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - \gamma}\,d\xi$ for all $z \in \Omega$.

pf: Since $z \in \Omega$. then $\exists\, \delta_0$, st $\{w : |w - z| < \delta_0\} \subset \Omega$. Define Ω_δ be the domain oriented by deleting disc $\{w : |w-z| \le \delta\}$ from Ω where $\delta < \delta_0$.

the boundary of Ω_δ consist of two curves γ, oriented positively and the circle $\{w : |w - z| = \delta\}$ oriented negatively.

Apply green theorem: $g(\xi) = \frac{f(\xi)}{\xi - z}$ on the region Ω_0, since g is analytical.

$\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y} = 0$ throughout Ω_δ. Hence $\iint_{\Omega_\delta} [\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}]\,dx\,dy = 0 = \int_{\partial\Omega_\delta} g(\xi)\,d\xi = \int_\gamma g(\xi)\,d\xi - \int_{|\xi - z| = \delta} g(\xi)\,d\xi$

replace $g(\xi) = \frac{f(\xi)}{\xi - z}$, then $\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = \int_{|\xi - z| = \delta} \frac{f(\xi)}{\xi - z}\,d\xi \xrightarrow{\text{converges}} 2\pi i f(z)$ as $\delta \to 0$.

then $\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = 2\pi i f(z)$

Application. ex 6. find the value $\to \int_0^{2\pi} \frac{1}{2+\sin\theta}\, d\theta$

let $z = e^{i\theta}$, then $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$, $z = e^{i\theta}$ $\quad \ln z = i\theta \Rightarrow \frac{\ln z}{i} = \theta \Rightarrow d\theta = \frac{1}{i}\cdot\frac{1}{z}dz = \frac{1}{iz}dz.$

$\sin\theta = \frac{1}{2i}(z-\frac{1}{z})$, $z = e^{i\theta}$ $\quad$ then $2+\sin\theta = 2+\frac{1}{2i}(z-\frac{1}{z}) = (\frac{1}{2})(4+\frac{z}{i}-\frac{1}{zi})$

$= (\frac{1}{2})(4 - iz + \frac{i}{z})$

$$\frac{d\theta}{2+\sin\theta} = (-\frac{i}{z})\cdot\frac{2}{4-iz+\frac{i}{z}}\,dz$$

$$= \frac{2}{4iz+z^2-1}\,dz = \frac{2}{[z-i(\sqrt{3}-2)][z+i(\sqrt{3}+2)]}\,dz$$

$z^2+4iz-1 = [z-i(\sqrt{3}-2)]\cdot[z+i(\sqrt{3}+2)]$

Set $p = i(\sqrt{3}-2)$ and $q = -i(\sqrt{3}+2)$, then p within $|z|=1$ as $|q| = \sqrt{3}+2$.

The function $(z-q)^{-1}$ is analytic in the disc $|z|<\sqrt{3}+2$. so by Cauchy formula.

$$\frac{1}{2\pi i}\int_{|z|}\frac{1}{(z-p)(z-q)} = \frac{1}{p-q} = \frac{1}{2\sqrt{3}i}.$$

then $\frac{1}{2\pi i}\int_{|z|}\frac{dz}{(z-q)(z-p)} = \frac{1}{2\pi i}\int_0^{2\pi}\frac{ie^{i\theta}d\theta}{2ie^{i\theta}(2+\sin\theta)} = \frac{1}{4\pi i}\int_0^{2\pi}\frac{d\theta}{2+\sin\theta}$

then $\frac{1}{2\pi i}\int_{|z|=1}\frac{dz}{(z-q)(z-p)}\cdot 4\pi i = \int_0^{2\pi}\frac{d\theta}{2+\sin\theta} = \frac{4\pi i}{2\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}.$

Ex 7. $\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{1-2a\cos\theta+a^2}$, $0<a<1$.

take $z = e^{i\theta}$

then $\cos\theta = \frac{1}{2}(\frac{1}{z}+z)$ $\quad d\theta = \frac{1}{iz}dz$

then $1-2a\cos\theta+a^2 = 1+a^2 - a\cdot(\frac{1}{z}+z)$

$$\frac{d\theta}{1-2a\cos\theta+a^2} = \frac{dz}{iz(1+a^2-a\frac{1}{z}-az)} = \frac{dz}{i(z+a^2z-a-az^2)}$$

Now $-az^2+(a^2+1)z-a = a(z^2-(a+\frac{1}{a})z+1) = -a(z-a)(z-\frac{1}{a})$.

Since $0<a<1$, then the function is analytic in the disc $|z|=1$

Hence by cauchy formula: $\frac{1}{2\pi i}\int_{|z|=1} -\frac{1}{a}\cdot\frac{1}{z-a}\cdot\frac{1}{z-\frac{1}{a}}\,dz = -\frac{1}{a}\cdot\frac{1}{(a-\frac{1}{a})}$

$= -\frac{1}{a^2-1}$

$= \frac{1}{1-a^2}$

Thus $\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{1-2a\cos\theta+a^2} = \frac{1}{1-a^2}$ $\quad 0<a<1$

$$P_a(\theta) = \frac{1-a^2}{1-2a\cos\theta+a^2}$$ is called poison kernel.

Evaluate integral:

$$\int_\gamma \frac{z}{z+1}\,dz,$$ where γ is any curve in $\{z : \operatorname{Im} z > 0\}$ which joins $-1+2i$ to $1+2i$

consider $f(z) = \frac{z}{z+1} = 1 - \frac{1}{z+1}$ then it is the deravitive of $F(z) = z - \log(z+1)$

$$\int_\gamma \frac{z}{z+1}\,dz = \int_\gamma f(z)\,dz = \int_\gamma F'(z)\,dz = F(1+2i) - F(-1+2i)$$

$$= 2 - \log(2+2i) + \log(2i)$$

$$= 2 + \log\frac{2i}{2+2i} = 2 - \log\frac{1}{1+i} = 2 + \log(1+i)$$

$$= 2 - \frac{1}{2}\log 2 + \frac{i\pi}{4}$$

Additional: Nested interval property:

For each $n \in N$, assume we are given a closed interval

$$I_n = [a_n, b_n] = \{x \in R : a_n \le x \le b_n\}$$

Assume that $I_n \supseteq I_{n+1}$ for all $n \in N$. Then the resulting nested sequence of closed interval

$I_1 \supseteq I_2 \supseteq I_3 \cdots$ has a nonempty intersection, that is $\bigcap_{n \in N} I_n \neq \emptyset$

Nested Traingle property:

for each $n \in N$, assume we are given a closed tiangle T_n. Assume also that $T_n \supseteq T_{n+1}$ for all $n \in N$

Then the resulting nested sequence of closed trangle $T_1 \supseteq T_2 \supseteq \cdots$ has a non-empty intersection $\bigcap_{n \in N} T_n \neq \emptyset$

Sec 2.4 Consequence of Cauchy's formula

Them 1: Suppose that f is analytic on D and z_0 is a point of D. If the disc $\{z : |z - z_0| < R\}$ lies in D

then f has a power series:

$$f(z) = \sum_{n=0}^{\infty} a_k (z - z_0)^k$$ vaild with $a_k = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z_0)^{k+1}}\,d\xi$ where γ is the positive oriented circle

Corollary: If f is analytic on D, then so if f'. Hence f has derivitives of all orders and each derivitive is analytic on D

Ex1. The function e^z is entire and its deravitive.

Hence $a_n = \frac{1}{n!}$ so, $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$,

$$\sin z = \sum_{n=0}^{\infty} (-1)^j \frac{z^{2n+1}}{2n+1!}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n\,!}$$

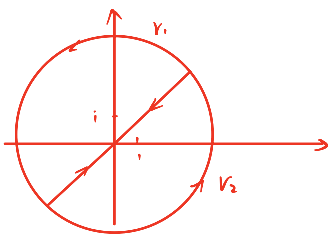
constider: $\log(1-z)$, clearly is analytic on disc $|z|<1$.

$$\frac{d^n}{dz^n}(\log(1-z)) = -\frac{(n-1)!}{(1-z)^n}.$$

then $$\log(1-z) = -\sum_{n=0}^{\infty} \frac{z^n \cdot ((1-z)-1)}{(1-z)^n}$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z)^{n+1}}\, dw \quad \text{for all } z \in \Omega.$$

ex. $$\oint_{|z|=10} \frac{\sin z}{(z-i)^2(z-1)}\, dz.$$



$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2}$$

(1) $$\frac{\sin(z)}{(z-i)^2(z-1)} = \underbrace{\frac{\sin z}{(z-i)^2}}_{f(z) \text{ inside } \gamma_2} \cdot \frac{1}{(z-1)}$$

by cauchy formula: $$\int_{\gamma_2} \frac{\sin z}{z-1}\, dz = 2\pi i \frac{\sin 1}{(1-i)^2}$$

$$\int_{\gamma_1} \frac{\sin z}{(z-i)^2} = 2\pi i\, f'(i) = 2\pi i \frac{(\cos i)(i-1) - \sin i}{(i-1)^2}$$

$$f'(z) = \frac{\cos z (z-1) - \sin z}{(z-1)^2}$$

March 2nd couses： 兔兔一定可以拿到offer哒！ 403 教点东西吧拜托了！

(1) complex number and their properties.

- de Moirve's theorem $|z| = N,\ z \,\big|\, e^{i\theta} = \sin(\theta) i + \cos(\theta) \rightarrow (Ne^{i\theta})^m = N^m \cdot e^{im\theta}$.

  $(n\ \sin(\theta) i + \cos(\theta))^m = N^m \cdot (\sin(m\theta) i + \cos(m\theta))$

  prob1: $(1-i)^{100}$.

- Various equation, e.g

  $z^2 - 2z + 4 = 0$

  $z^3 = 1$. find root $\rightarrow$ equation

  sol'n: $z^3 = 1 = (\sin 0\, i + \cos 0) = e^{2\pi i} \Rightarrow z = e^{\frac{2k\pi i}{3}}$

- exponential form $\underline{e^{i\theta} = i\sin\theta + \cos\theta}$: $e^{\frac{\pi}{4} i} = i \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2}$.

- log function, why $\text{Log}(1-i) = \log(\sqrt{2} \cdot (\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i) = \log\sqrt{2} - i\frac{\pi}{4}$. inside $[-\pi, \pi]$.

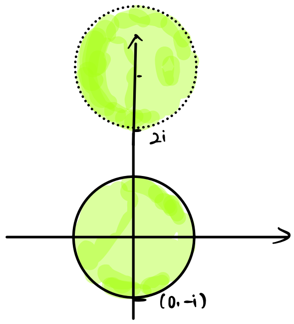
- $(-1)^{i+1} = ?$

  $(-1)^{i+1} = e^{\log(-1)(i+1)} = e^{(i+1)(-\frac{\pi}{2} i)} = e^{\frac{\pi}{2}} \cdot e^{-\frac{\pi}{2} i} = -e^{\frac{\pi}{2}}$.

(2) Topology + geometry of complex plane.

- Determine if the set $\{z \in \mathbb{C} : |z| \le 1\} \cup \{z \in \mathbb{C} : |z - 2i| < 1\}$ is open / closed / connectet.

  Find its boundary and interiors.



Openness : not open (since for instance $-i \in A$ and $-i$ belongs to the boundary)

Closeness : not close (since $2i \notin A$, but $2i$ belongs to the boundary).

Connected : Not connected, since there $\exists$ no path for $3i \to 0$, that entirely lies in A.

Domain : Not Domain Neither connected / closed.

boundary $\Rightarrow$ set $:= \{z \in \mathbb{C} : |z| = 1\} \cup \{z \in \mathbb{C} : |z - 3i| = 1\}$.

interior $\to$ set $:= \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z - 3i| < 1\}$.

(3) Series and power series

Determine if converges or diverges. e.g

(i) $\sum_{m=1}^{\infty} \frac{(i)^n}{n^3+1}$ Since $\sum_{n=1}^{\infty} \left| \frac{i^n}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3+1} = 0 < \infty$ convergent ✓

(ii) $\sum_{n=1}^{\infty} \left(\frac{i+1}{3}\right)^n \frac{1}{\sqrt{n}}$. since $\left|\left(\frac{i+1}{3}\right)\right|^n = \left(\frac{\sqrt{2}}{3}\right)^n$, $\sum_{n=1}^{\infty} \left|\frac{i+1}{3}\right|^n \cdot \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{3}\right)^n \cdot \frac{1}{\sqrt{n}} < \infty$

convergent ✓.

- find radius of convergence.

(easy) $\sum_{n=1}^{\infty} \frac{(z-i)^n}{2^n} \cdot \frac{1}{n}$

(hard) $\sum_{n=1}^{\infty} \frac{1}{3^{3n}} z^{2n}$

(4) line integrals.

- parametrization $\int_{\gamma} f(z)\, dz = \int_a^b \gamma'(t)\, f(\gamma(t))\, dt.$

eg. Calculate $\int_{\gamma_1} z^2 dz$ where γ_1 is segment from $i \to -i$.

Cauchy theorem and Cauchy fomula. (γ closed + f - anlytic)

$$\oint_{\gamma} f(z)\, dz = 0$$

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w-z} \quad (\gamma \text{ closed} + f \text{ - anlytic}) \quad z \text{ inside } \gamma.$$

Calculate $\oint_{|z|=2} \frac{z^4}{(z-3i)(z-1)} dz.$

(5) Cauchy Riemam equation + harmonic functions

(i) if $u(x.y) = x^3+y^4$ is real part of some entire function?

$\frac{\partial u}{\partial x} = 3x^2 \quad \frac{\partial u}{\partial y} = 4y^3, \qquad f: \mathbb{C} \to \mathbb{C} : \quad f(x+iy) = u(x,y) + i\, v(x,y)$

by Cauchy - riemann: $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow v = -4y^3x + f(y)$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow v = 3x^2y + g(x).$

$\Rightarrow v = -4y^3x + 3x^2y.$

(ii) find complex conjugate for $u(x,y) = 5xy.$

Def: order of zero

f has a zero of order m at z_0 if $f^{k}(z_0)=0$ for $k=0,1,2,\cdots m-1$. but $f^{(m)}(z_0)\neq 0$

Them: if f has a zero of order m at z_0, then $f(z)=(z-z_0)^m g(z)$ where $g(z_0)\neq 0$ and analytic on D

$$\text{pf: } f(z)=\sum_{k=0}^{\infty}(z-z_0)^k\frac{f^{(k)}(z_0)}{k!} = \sum_{k=m}^{\infty}(z-z_0)^k\frac{f^{(k)}(z_0)}{k!}$$

$$\text{let } g(z)=\sum_{k=m}^{\infty}(z-z_0)^{k-m}\frac{f^{(k)}(z_0)}{k!}$$

Them: If F is entire and if there is a constant M st $|f(z)|\le M$ for all z, then F is identically constant.

set $g(z)=\dfrac{F(z)-F(0)}{z}$ then $g(z)$ is entire and $z=Re^{i\theta}$, $|g(z)|\le \dfrac{|F(Re^{i\theta})|+F(0)|}{R}\le\dfrac{2M}{R}$.

$$F(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^m,\quad \frac{F(z)-F(0)}{z}=\sum_{n=0}^{\infty}\frac{f^{(m)}(0)}{m!}z^{m-1}.$$

2) let $z\in\mathbb{C}$, then $g(z)=\dfrac{1}{2\pi i}\oint_{|w|=R}\dfrac{g(z)}{w-z}\,dw$.

we estimated the integral as max of function times length of the circle $|w|=R$.

$$\left|\frac{1}{2\pi i}\oint_{|w|=R}\frac{g(z)}{w-z}\,dw\right|\le\frac{1}{2\pi}2\pi R\,\frac{1}{R-|z|}\cdot\frac{2M}{R}\xrightarrow[R\to\infty]{}0.$$

$$\text{ie: }\left|\frac{1}{w-z}\right|\le\frac{1}{|w|-|z|}=\frac{1}{R-|z|}$$

$$|g(z)|=0\Rightarrow F(z)-F(0)=0\Rightarrow F(z)=F(0)\Rightarrow \text{constant}.$$

Application:

$\forall$ non-constant polynomial p of complex coefficient has at least one root.

$p(z)=z^n(a_n+a_{n-1}/z+\cdots+a_0/z^n)$,

$$\lim_{R\to\infty,\ \theta\in[0,2\pi)}|p(Re^{i\theta})|=0$$

$\Rightarrow f(z)=\dfrac{1}{p(z)}$ is bounded Moreover, $p(z)$ has no root. so $f(z)$ is also entire $\Rightarrow f$ is entire

Fundamental theorem of Algebra:

$\forall$ complex polynomial $p: \mathbb{C} \to \mathbb{C}$ can be uniquly factored as

$$p(z) = a_n (z-z_1)^{a_1} \cdots (z-z_m)^{a_m}$$

Lecture 15^{th}

Consequence of Cauchy's formula: part II

(1) Liouville's theorem:

If F is entire and if there is a M st $|F(z)| \le M$ for all z, then F is identically constant.

Ex1: Let $f = u(z) + iv(z)$ be entire function, if $|u(z)| < M$ for $\forall z$, where $M > 0$, then WTS f is constant functio

ie: by liouville's them, then $u(z)$ is constant function, then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \Rightarrow$ by C-R formula, then

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0 \Rightarrow v$ is also constant function.

Application: let $g(z) = e^{f(z)}$, g is entire, then $|g(z)| = |e^{f(z)}| = e^{u(z)} \le e^M \Rightarrow |g(z)| \le e^M \xRightarrow{\text{Liousville}}$ then $g(z)$ is constant function.

Ex2: let f be entire function st $|f(z)| \ge 1$ for every z, WTS f is entire function?

ie. consider $g(z) = \frac{1}{f(z)} \Rightarrow g(z)$ is bounded + differentiable.

then $|g(z)| = \frac{1}{|f(z)|} \le 1 \xRightarrow{\text{Liouville}} g$ is bounded $\Rightarrow g$ is constant function $\Rightarrow f(z)$ is const.

Ex3. let $f \cdot g$ be two entire func st $|f(z)| \ge |g(z)|$ for every z. WTS $f = ag$ for $a \in F$

consider $k(z) = \frac{f(z)}{g(z)}$, then $|k(z)| \le \frac{|g(z)|}{|f(z)|} \le 1 \Rightarrow k(z)$ is constant function $\Rightarrow f = ag$.

!!!: check whether h is entire, "assume that z_0 is zero of f." $\Leftrightarrow f(z_0) = 0$

$|f(z_0)| = 0 \ge |g(z_0)| \Leftrightarrow |g(z_0)| = 0$.

Let m_1 be order $\to f$, m_2 be order $\to g$. (order of z_0), then $|(z_0 - z)^{m_1} \tilde{f}_1(z)| \ge |(z - z_0)^{m_2} \tilde{g}(z)|$

where $\tilde{f}(z_0) \ne 0$ and $\tilde{g}(z_0) \ne 0$, so $m_1 \le m_2$.

then $\frac{g(z)}{f(z)} = \frac{\tilde{g}(z)\ (z-z_0)^{m_2}}{\tilde{f}(z)\ (z-z_0)^{m_1}} = \frac{(z-z_0)^{m_2 - m_1} \cdot \tilde{g}(z)}{\tilde{f}(z)}$ where $\tilde{f}(z) \ne 0$. (解决问题).

(2) Multiplication of power series

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g(z) = \sum_{k=0}^{\infty} b_k \cdot z^k$, where $R \Rightarrow$ convergence is at least $R>0$. then $f(z)\cdot g(z)$ can be written as complex series with,

$$f(z)\,g(z) = \sum_{n=0}^{\infty} c_n z^n. \quad \left[c_n = \sum_{k=0}^{n} a_k b_{n-k}. \right]$$

informal way: $(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$

$= a_0 b_0 +$ "c_0"

$(a_0 b_1 + a_1 b_0) z +$ "c_1"

$(a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 +$ "c_2"

$\cdots\cdots$ $\Rightarrow$ "c_n" $= \sum_{k=0}^{n} a_k b_{n-k}$

Formal pf: If $f \cdot g$ have power series expansion, then $f \cdot g$ are analytic

$\Rightarrow$ $(f \cdot g)$ is analytic $\Leftrightarrow$ have power expansion "$h(z) = (f\cdot g)(z)$"

Moreover $c_n = \frac{h^{(n)}(0)}{n!}$ (Taylor expansion).

take $h^{(n)}(0) = (f\cdot g)^{(n)}(0)$ "product rule $\rightarrow$ derivative" taylor $\rightarrow f$ taylor to g.

ie: prove by induction: $= \frac{1}{n!} \sum_{k=0}^{m} \binom{n}{k} f^{(k)}(0) \cdot g^{(n-k)}(0) = \sum_{k=0}^{n}$ "$\frac{f^{(k)}(0)}{k!}$" $\cdot$ "$\frac{g^{(n-k)}(0)}{(n-k)!}$"

ex1. find power expansion $\Rightarrow z^2 \sin(z)$?

sol'n: $f(z) = z^2$ $g(z) = \sin(z)$.

$$\Rightarrow z^2 \sin(z) = z^2 \cdot \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot z^{2k+3}}{(2k+1)!}.$$

Ex2: find power series for $(z^3+z)e^z$

$$\text{then } (z^3+z)e^z = (z^3+z) \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^{k+3}}{k!} + \frac{z^{k+1}}{k!}$$

Ex 3: find the first 4 terms (power 3) ps for $\frac{1}{1-z}\sin(z)$

sol'n: we have $\frac{1}{1-z} = 1+z+z^2+z^3 \cdots +$

$$\sin(z) = z - \frac{z^3}{6} + \cdots$$

then $\frac{1}{1-z}\sin z = z + z^2 + \left(-\frac{z^3}{6}\right) + z^3 = z+z^2+\frac{5}{6}z^3$. (凑几次方).

Ex 4

find series $\to$ $\frac{1}{1-z^2} = \frac{1}{(1+z)} \cdot \frac{1}{1-z}$

ie $\begin{cases} \frac{1}{1+z} = 1 + z + z^2 + \cdots \\ \frac{1}{1-z} = 1+z+z^2+\cdots \end{cases} \Rightarrow$

Ex 5: order 4 for ps $\to$ $\frac{1}{1-z^2}e^z$.

$$\begin{cases} \frac{1}{1-z^2} = 1+z^2+z^4+\cdots \\ e^z = 1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\cdots \end{cases} \Rightarrow \frac{1}{1-z^2}e^z = 1+z+\frac{3}{2}z^2+\frac{7}{6}z^3+\frac{37}{24}z^4+\cdots$$

Find series expansion $\to$ $\cos(z^6)$

$$\cos(z^6) = \cos w = \sum_{k=0}^{\infty}\frac{(-1)^k w^{2k}}{2k!} = \sum_{k=0}^{\infty}\frac{(-1)^k z^{12k}}{(2k)!}$$

Ex 6. Find power series expansion for e^z about 1.

$$e^z = e^{z-1} \cdot e$$

$$\Rightarrow e^z = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} = \sum_{n=0}^{\infty}\frac{e(z-1)^n}{n!}$$

New types: series expansion $\to$ order of zero.

Recall: order m :: $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \cdots m-1$. but $f^{(m)}(z_0) \neq 0$.

$$f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

$$= \underbrace{f(z_0)}_{=\text{"}0\text{"}} + \underbrace{f'(z_0)(z-z_0)}_{=\text{"}0\text{"}} + \underbrace{\frac{f''(z_0)(z-z_0)^2}{2}}_{=\text{"}0\text{"}} + \cdots + \underbrace{\frac{f^{(m-1)}(z_0)(z-z_0)^{m-1}}{(m-1)!}}_{=\text{"}0\text{"}} \cdots \underbrace{\frac{f^{(m)}(z_0)(z-z_0)^m}{m!}}_{=\text{"}\neq 0\text{"}}$$

Q1: determine $z_0 = 0$ of $f(z) = \sin(z)$

1) we have $f'(z) = \cos(z)$ and $f'(0) = 1 \neq 0$, so the order $m=1$

$$f(0) = \sin(0) = 0$$

2) $\sin(z) = z - \frac{z^3}{6} + \cdots +$ so, the order of zero $= 1$.

Q2: Determine the order $\Rightarrow z=0$ $f(z) = \sin(z^3) z$

1) expansion $z \cdot \sin(z^3) = z^4 - \frac{z^{10}}{6} + \cdots +$

so the order of zero $= 4$.

Q3: Determine the order $z_0 = 0$ of $f(z) = z(e^z - 1)^2 = z(e^z-1)(e^z-1)$

we have $e^z = 1 + z + \frac{z^2}{2} + \cdots$

$$e^z - 1 = z + \frac{z^2}{2} + \cdots$$

then $f(z) = z \cdot (z + \frac{z^2}{2} + \cdots)(z + \frac{z^2}{2} + \cdots)$

$$= z^3 + \cdots \quad \Rightarrow m = 3.$$

Example: Determine order of all zeros of "$\frac{\sin z}{z}$"

1) There are zeros of the form $k\pi$, $k \in \mathbb{Z}$, consider cases $k \neq 0$ and $k = 0$ separately

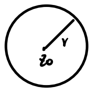
① $k \neq 0$. Then $\left(\frac{\sin(z)}{z}\right)' = \frac{\cos z \cdot z - \sin z}{z^2}$ which is $\neq 0$ when $z = k\pi$, $\Rightarrow$ the order of $k\pi = 1$.

$$= \frac{\pm k\pi}{(k\pi)^2} = \pm \frac{1}{k\pi} \neq 0.$$

② $k=0$. Then $\frac{\sin z}{z} = \frac{1}{z}(z - \frac{z^3}{6} + \cdots) = 1 - \frac{z^2}{6} + \cdots$

$\Rightarrow$ so the order of $0 \Rightarrow m = 0$.

Def: isolated singularity: An analytic func f has an isolated singularity at z_0 iff it is analytic in the punctured disc $0<|z-z_0|<r$ for some $r>0$.



ex. $\frac{1}{z-z_0}$ (everywhere except z_0).

3 examples: 1) $\frac{z_0^2-z^2}{z_0-z}$

2) $\frac{7}{(z-z_0)^2}$

3) $e^{1/z-z_0}$

Check 1) $\frac{z^2-z_0}{z-z_0} = z+z_0$ (analytical in the whole disc " x strange "

2) $\frac{7}{(z-z_0)^2}$: tend $\to \infty$ when $z \to z_0$, we can define $\frac{1}{f(z)} = (z-z_0)^2/7$.

3) $e^{\frac{1}{z-z_0}}$: the $\lim_{z\to z_0} f(z)$ does not exists, strange

Types of singularities: (1) Removable $|f(z)|$ remain bounded if $z \to z_0$

(2) Pole $|f(z)| \to \infty$ when $z \to z_0$

(3) Essential singularity Neither 1) / 2)

Removable: set: $g(z) = \begin{cases} (z-z_0)^2 f(z) & \text{for } z \neq z_0 \\ 0 & \text{otherwise.} \end{cases}$ $\Rightarrow$ Then g is conti and differential outside z_0.

since $\lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} = \lim_{z\to z_0} (z-z_0) f(z)$

$= \lim_{z\to z_0} \frac{(z-z_0)^2 f(z) - 0}{z-z_0}$

Hence: power series expansion of g at z_0.

$g(z) = b_0 + b_1(z-z_0) + \cdots$

as $b_1 = b_0 = 0$ so.

$(z-z_0)^2 f(z) = g(z) = b_2(z-z_0)^2 + \cdots$

$\Rightarrow f(z) = b_2 + b_3(z-z_0) + b_4(z-z_0)^2 + \cdots$

(⇐) Pole: Assume $|f(z)| \to \infty$ when $z \to z_0$, with $|f(z)| > 1$. Then

set: $g(z) = \frac{1}{f(z)}$ is analytic in $D(z_0, r)$ and removable singularity

$|g(z)| = \left|\frac{1}{f(z)}\right| < 1$.

Hence $g(z) = (z-z_0)^m h(z)$ where h is analytic in $D(z_0, r)$ and $h(z_0) \neq 0$ (order on 0).

$$= \frac{g^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{g^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \cdots$$

$$= (z-z_0)^m \underbrace{\left(\frac{g^{(m)}(z_0)}{m!} + \frac{g^{(m+1)}(z_0)}{(m+1)!}(z-z_0)\right)}_{h(z)}$$

$$\Rightarrow h(z_0) = (z-z_0)^m \cdot \frac{g^{(m)}(z_0)}{m!}$$

Therefore $f(z) = \frac{H(z)}{(z-z_0)^m} = \frac{1}{g(z)} = \frac{1}{(z-z_0)^m} \cdot \underbrace{\frac{1}{h(z)}}_{H(z)}$ where $H(z)$ is analytical.

Order of pole $= m$

Example: Locate each of the isolated singularity ⇒ tell types of singularity: If removable ⇒ give value
pole ⇒ give order pole.

①: $f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{6} + \cdots$

so it is analytic + removable + value at $z=0$ is 1.

② $f(z) = \frac{\sin z - z}{z^4} = \frac{\left(z - \frac{z^3}{6} + \frac{z^5}{60} \cdots\right) - z}{z^4} = -\frac{1}{6z} + \frac{z}{60} -$

$$= \frac{1}{z}\underbrace{\left(-\frac{1}{6} + \frac{z^2}{60} + \cdots\right)}_{H(z),\ H\text{-analytic about } 0 \quad H(0) \neq 0}$$

It is pole + order = 1

③ $f(z) = \frac{e^{2z}}{(z-2)^2(z-i)^5(z+2i)^6}$

3 singularity $z_1 = 2$ $z_2 = i$ $z_3 = -2i$

then 1) when $z_1 = 2$. ⇒ $f(z) = \frac{1}{(z-2)^2} \cdot \frac{e^{2z}}{(z-i)^5(z+2i)^6}$ Moreover $\frac{e^{2\cdot 2}}{(2-i)^5(2+2i)^6} \neq 0$.

so f has a pole at $z_0 = 2$ of order 2.

2) when $z_0 = i$.

$$\Rightarrow f(z) = \frac{1}{(z-i)^5} \cdot \frac{e^{2i}}{(i-2)^2 \cdot (3i)^6} \neq 0 \quad \Rightarrow \text{pole order of } 5.$$

ex3: $f(z) = \dfrac{e^z - 1}{e^{2z} - 1} = \dfrac{e^z - 1}{(e^z - 1)(e^z + 1)} \overset{①}{=} \dfrac{1}{e^z + 1}$

②: $f(z) = \dfrac{z + \frac{z^2}{2} + \cdots}{2z + \frac{(2z)^2}{2}} = \dfrac{1 + \frac{z}{2} + \cdots}{2 + 4\frac{z}{2} + \cdots} \xrightarrow{z \to 0} \dfrac{1}{2}$

removable: $z \to 0$, then $f(z_0) = \frac{1}{2}$. there fore $|f(z)|$ is bounded as $z \to 0$.

ex 4: $f(z) = \sin(1/z)$ "essential" $\Leftrightarrow$ 2 sequence $\neq$ limit

singularity $z_1 = \dfrac{1}{2\pi n}$ for $n \in \mathbb{Z}$. $z_2 = \dfrac{1}{2\pi n + \pi/2}$ Both tends to 0.

$\lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} \sin(2\pi n) = 0$.

$\lim_{n \to \infty} f(w_n) = \lim_{n \to \infty} \sin(2\pi n + \pi/2) = 1$.

$\neq$ so the $\lim_{z \to 0} f(z)$ does not exist.

$f(z) = \dfrac{z^2 + 3z - 1}{z+2}$ residue $Res(z_0$.

$= f(z) = \dfrac{e^{2z}}{z^3} = \dfrac{1 + 2z + \frac{(2z)^2}{2} + \cdots}{z^3} = \dfrac{1}{z^3} + \dfrac{2}{z^3} + \dfrac{4}{z}$

sol: $f(z) = \dfrac{e^{2z}}{z^3}$

$= \dfrac{1}{z^3} \cdot e^{2z}$ (analytic about pole $z_0 = 1$ $H(0) = e^0 = 1 \neq 0$).

so the pole is order3 $\Rightarrow$ $Res(0, f) = \dfrac{H^{(2)}(0)}{2!} = \dfrac{(2e^{2z}(0))'}{2} = \dfrac{4e^{4(0)}}{2}$

$= 4/2$

$= 2$.

(1) $f(z) = \dfrac{e^z}{(z-1)^2}$

$e^z = H(z) \Rightarrow$ pole order = 2

0 order : $H(1) = e \neq 0$

1 order : $H'(1) = e \neq 0$

$\text{Res}(1, f) = \dfrac{H^{(1)}(1)}{1} = e$

trick : $e^z = e\underline{e^{z-1}} = e \cdot \big(1 + \underset{\overset{\shortparallel}{x}}{\underline{(z-1)}} + \frac{(z-1)^2}{2} + \cdots \big).$

$f(z) = \dfrac{\sin z}{z^2+1}$

$\shortparallel$

$\dfrac{\sin z}{(z-i)(z+i)}$ pole : $i, -i$.

$\shortparallel$

$\underbrace{\dfrac{\sin z}{(z+i)}}_{H(z)} \cdot \dfrac{1}{z-i}$

$H(z)$ analytic on $z_0 = i$

$\Rightarrow$ pole order = 1.

then $\text{Res}(i, f) = \dfrac{H^{(0)}(i)}{0!} = \dfrac{\sin i}{2i}$

Find the res $f(z) = \dfrac{1}{z^2(z-1)(z-2)}$

March 30th lecture note

i) series expansion

ii) Residue theorem

Ex1: type Expansion of $\frac{1}{z-a}$

$a\in\mathbb{C}$ ① $|z|<|a|$: $\frac{1}{z-a} = \frac{-1}{a}\cdot\frac{1}{1-\frac{z}{a}} = -\frac{1}{a}\cdot\sum_{k=0}^{\infty}\left(\frac{z}{a}\right)^k = \sum_{k=0}^{\infty}\frac{-z^k}{a^{k+1}}$

② $|z|>|a|$: similarly $\frac{1}{z-a} = \frac{1}{z}\cdot\frac{1}{1-\frac{a}{z}} = \sum_{k=0}^{\infty}\frac{a^k}{z^{k+1}}$

specific example: $\frac{1}{2i-z}$ for $|z|<2$. ie $|z|<|a|=|2i|=2$.

$\frac{1}{z+3}$ for $|z|<3$, $\frac{1}{z+3} = \frac{1}{3}\cdot\frac{1}{1+\frac{z}{3}} = \frac{1}{3}\cdot\sum_{k=0}^{\infty}\left(-\frac{z}{3}\right)^k$

Ex. find series $\to z_0=0$ for $\frac{1}{z-5}+\frac{2}{z+5i}$, $|z|>5$.

$$= \frac{1}{z}\cdot\left[\frac{1}{1-\frac{5}{z}}+\frac{2}{1+\frac{5i}{z}}\right]$$

$$= \frac{1}{z}\cdot\left(\sum_{k=0}^{\infty}\left(\frac{5}{z}\right)^k+2\sum_{k=0}^{\infty}\left(-\frac{5i}{z}\right)^k\right)$$

Natrual Q: type $\frac{A}{a-z}+\frac{B}{b-z}$ for $a<|x|<b$.

ie: $-\frac{1}{z}\frac{A}{1-\frac{a}{z}}+\frac{1}{b}\frac{B}{1-\frac{z}{b}}$

$= -\frac{A}{z}\cdot\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k+\frac{B}{b}\sum_{k=0}^{\infty}\left(\frac{z}{b}\right)^k.$

ex. $\frac{z+2}{z^2-5z+4} = \frac{z+2}{(z-4)(z-1)}$ for $1<|z|<4$.

$= \frac{A}{z-4}+\frac{B}{z-1} \Rightarrow A(z-1)+B(z-4)=z+2$

$\Rightarrow \begin{cases} A+B=1 \\ -A-4B=2 \end{cases} \quad -3B=3 \Rightarrow \begin{cases} A=2. \\ B=-1 \end{cases}$

$= \frac{2}{z-4}-\frac{1}{z-1}$

$= -\frac{1}{2}\frac{1}{1-z/4}-\frac{1}{z}\cdot\frac{1}{1-1/z} = -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{4}\right)^k-\frac{1}{z}\cdot\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k$

ex.2. $\frac{z+1}{(z-2)(z-3)(z-5)}$ decompose ! $2<|z|<3$

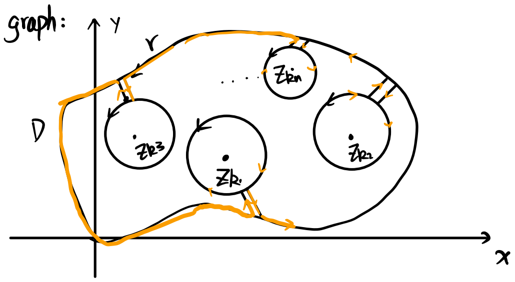
Res-theorem

suppose that f is analytic on simple-connected domain D except for a finite # of singular at $z_1, \cdots z_N$ of D. let r be positive smooth closed in D. that not pass through the above the $z_1, z_2, \cdots z_N$

ie: $\frac{g(z)}{(z-z_1)\cdots(z-z_m)}$

then $\oint_r f(z)\,dz = 2\pi i \sum_{z_k \text{ inside } r} \text{Res}(z_k, f) \Rightarrow \frac{1}{2\pi i}\oint_{|w-z_k|=s} f(w)\,dw = \text{Res}(z_k, f)$

graph:



let each disc (z^k, r_k) small enough $\Rightarrow$ disjoint.

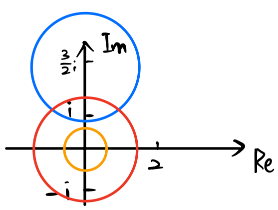
Applying Cauchy Them to a domain with

$r_1, r_2, \cdots r_N. \Rightarrow 0 = \oint_r f(z)dz - \sum_{k=1}^{N} \oint_{r_k} f(z)dz$

$\Rightarrow \oint_r f(z)dz = \sum_{k=1}^{\infty} \oint_{r_k} f(z)dz$

$= 2\pi i \sum_{z_k \in r}^{N} \text{Res}(z_k, f)$

Application $\Rightarrow$ Calculate: $\oint \frac{e^z}{(z^2+1)(z-2)^2}\,dz$ over various r.



(1) $\oint_{|z|=\frac{1}{2}} f(z) = 0$ (no singularity inside)

(2) $\oint_{|z-\frac{3}{2}i|=1} f(z)$ "ie calculate $\text{Res}(i, f)$."

$\frac{e^z}{(z^2+1)(z-2)^2} = \frac{1}{z-i} \cdot \overbrace{\frac{e^z}{(z+i)(z-2)^2}}^{H}$ pole of order 1.

then H is analytic about i, $H(i) \neq 0$

$\Rightarrow \text{Res}(i, f) = \frac{H^{(0)}(i)}{0!} = \frac{e^i}{2i\cdot(i-2)^2}$

$$\Rightarrow \oint_{|z-\frac{3}{2}i|=1} f(z) = 2\pi i\,\mathrm{Res}(i,f) = 2\pi i\cdot \frac{e^{i}}{2i(i-2)^2}$$

(3) $\oint_{|z|=3/2} f(z) = ?$ ($\pm i$ inside)

"ie: $\mathrm{Res}(i,f)$ and $\mathrm{Res}(-i,f)$"

(already done) $\Downarrow$

"similar approach"

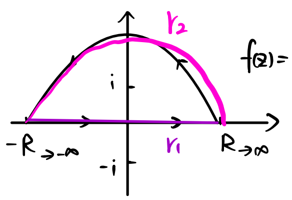
(4) $\oint_{|z|=5/2} \frac{e^z}{(z^2+1)(z-2)^2}\,dz$: singularity : $\pm i, 2$.

only need calculate $\mathrm{Res}(2,f)$

$$f(z) = \frac{1}{(z-2)^2}\cdot \underbrace{\frac{e^z}{z^2+1}}_{\text{"}H(z)\text{"}} \quad \text{pole of order 2}$$

then
$$\mathrm{Res}(2,f) = \frac{H^{(1)}(2)}{1!} = \frac{e^z(z^2+1) - e^z(2z)}{(z^2+1)^2}$$
$$= \frac{5e^2 - 4e^2}{25} = \frac{e^2}{25}.$$

Ex : $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi$ in real | Real by Complex integral |



$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

only singularity $= i \Rightarrow f(z) = \frac{1}{z-i}\cdot \underbrace{\frac{1}{z+i}}_{H(z)}$ order 1.

then $2\pi i\,\mathrm{Res}(i,f) = \frac{\frac{1}{2i}}{1} = \frac{1}{2i}\cdot 2\pi i = \pi$!!!

$$\lim_{R\to\infty} \oint_{\gamma_R} \frac{1}{z^2+1}\,dz = \underbrace{\lim_{R\to\infty}\oint_{\gamma_{R_1}} \frac{1}{z^2+1}\,d(z)}_{\text{Real integral}} + \underbrace{\lim_{R\to\infty}\oint_{\gamma_2} \frac{1}{z^2+1}\,dz}_{\text{"WTS} = 0\text{"}} = \oint_{\gamma} f(z)$$

for $z\in\gamma_{R_2}$, we have $|z| = R$, so $\left|\frac{1}{z^2+1}\right| \le \frac{1}{|z|^2-1} = \frac{1}{R^2-1}$

$$\Rightarrow \left|\oint_{\gamma_2} \frac{1}{z^2+1}\,dz\right| \le \overset{\text{"length"}}{\pi R}\,\frac{1}{R^2-1}$$

$$\oint_{\gamma_2} \frac{1}{z^2+1} dz = 0 \Leftarrow \quad \lim_{R\to\infty} \pi R \frac{1}{R^2-1} = 0$$

$$\lim_{R\to\infty} \oint_{\gamma_R} \frac{1}{z^2+1} dz = \underbrace{\lim_{R\to\infty} \oint_{\gamma_{R_1}} \frac{1}{z^2+1} d(z)}_{\text{Real integral}} + \underbrace{\lim_{R\to\infty} \oint_{\gamma_2} \frac{1}{z^2+1} dz}_{\text{"WTS} = 0\text{"}}$$

$$\int_{\gamma_{R_1}} \frac{1}{z^2+1} dz = \begin{cases} \gamma_{R_1}(t) = t \\ \text{for } t \in (-1,1) \\ R \to \infty \end{cases} = \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$$

Them: suppose P, Q are real func polynomial and $\deg(Q) \geq 2 + \deg(p)$
If $Q(x) \neq 0$ for $\forall x \in \mathbb{R}$.

then $\int_{-\infty}^{\infty} \frac{p(x)}{Q(x)} dx = 2\pi i \sum_{z_j \in U} \operatorname{Res}(z_j, \frac{P}{Q})$

"where the sum is taken over all pole of P/Q that lies in the upper-half plane."

"Lemma"

(一) let p be polyno of $\deg(p) = n$ and $p(x) = a_n z^n + \cdots + a_0$

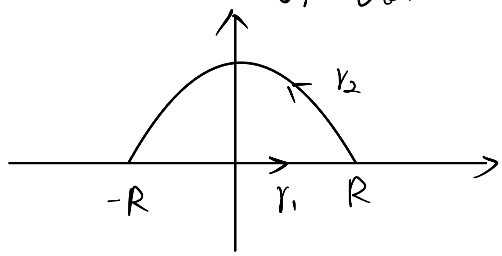
There is $R_1 > 0$, st for all $|z| = R > R_1$, we have

$$\frac{|a_n|}{2} R^n \leq |p(z)| \leq 2|a_n| R^n.$$

proof $\lim_{z\to\infty} \frac{|p(z)|}{|z^n|} = \left| \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_n}{z^n} \right|$

$$= \left| a_n + \frac{a_n}{z} + \cdots + \frac{a_n}{z^n} \right|_{z\to\infty} \to |a_n|$$

proof → them : compute $\oint_{\gamma} \frac{P(z)}{Q(z)}$ over the curve in two way



Note : "R is large enough to contains all zero of Q"

suppose $deg(Q) = m$

Fundamental : $Q(z) := a_n (z-z_1)^{m_1} \cdot (z-z_2)^{m_2} \cdots (z-z_N)^{m_N}$

them → Algebra where $m_1 + m_2 + \cdots + m_N = m$

By Res theorem : $\int_{\gamma_1} \frac{P(z)}{Q(z)} dz = 2\pi i \sum_{z_j \in U} Res(z_j, P/Q)$

Real part (Check)

$$\lim_{R\to\infty} \oint_{\gamma_R} \frac{P(z)}{Q(z)} dz = \oint_{\gamma_{R_1}} + \oint_{\gamma_{R_2}}$$

by def of line integral $\lim_{R\to\infty} \oint_{\gamma_{R,1}} \frac{P(z)}{Q(z)} dz = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$

WTS $\oint_{\gamma_{R_2}} = 0 \Rightarrow \left| \frac{P(z)}{Q(z)} \right| \leq \frac{2R^n|a_n|}{\frac{1}{2}|b_m|R^m} \leq \frac{4|a_n|}{|b_m|} \cdot \frac{1}{R^2}$ (by lemma)

ie : $P(z) = a_n z^n + \cdots + a_0$; $Q(z) = b_m z^m + \cdots + b_0$ $n \geq m+2$.

$$\frac{1}{2}|a_n|R^n \leq |P(z)| \leq 2|a_n|R^n ; \quad \frac{1}{2}|b_m|R^m \leq |Q(z)| \leq 2|b_m|R^m .$$

lenth

then $\left| \oint_{\gamma_{R_2}} \frac{P(z)}{Q(z)} dz \right| \leq \pi R \cdot \frac{4|a_n|}{|b_m|} \cdot \frac{1}{R^2}$ as $R \to \infty$ $= 0$

then $\oint_{\gamma_{R_2}} \frac{P(z)}{Q(z)} dz = 0 \implies \int_{-\infty}^{\infty} \frac{P(z)}{Q(z)} = 2\pi i \sum_{z_j \in U} Res(z_j, P/Q)$

Exercise

evaluate: $\int_{-\infty}^{\infty} \frac{\overbrace{1}^{P(z) \to deg(P)=0}}{\underbrace{(x^2+1)(x^2+9)}_{Q(z) \to deg(Q)=4}} dx$

then $\frac{1}{(z^2+1)(z^2+9)} = \frac{1}{\underline{(z-i)}(z+i)\underline{(z-3i)}(z+3i)}$

upper half

ie find $Res(z_j, P/Q)$

then $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx = 2\pi i \left(\frac{1}{2i \cdot (-2i) \cdot 4i}\right) + 2\pi i \left(\frac{1}{2i \cdot 4i \cdot 6i}\right)$

$$= \frac{\pi}{8} - \frac{\pi}{24} = \frac{\pi}{12}$$

$\boxed{\text{Check by Real-valued solution}}$

Ex2: $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx = 2\pi i \sum_{z_j \in U} Res\left(z_j, \frac{1}{(z^2+1)^3}\right) = -\frac{\pi}{4}$

$deg(1) = 0$

$deg(f(x)) = 6$

$$\frac{1}{(z^2+1)^3} = \frac{1}{(z-i)^3(z+i)^3}$$

$$= \frac{1}{(z-i)^3} \cdot \frac{1}{(z+i)^3} \quad H(z)$$

pole of order 3.

错的 ~~2~~

$$\not\Rightarrow \boxed{Res = \frac{2\pi i}{(2i)^3} = \frac{2\pi \cancel{i}}{-8\cancel{i}} = -\frac{\pi}{4}.}$$

$2\pi i \, Res(i, f) = 2\pi i \frac{H^{(2)}(i)}{2!} = \frac{3}{8}\pi$

Ex 3:

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx \neq \frac{\cos(z)}{z^2+1} \quad \text{as "} \cos(z) > 1 \text{"}$$

trap !!!

ie: we have $e^{ix} = i\sin(x) + \cos(x) \Rightarrow \cos(x) = Re(e^{ix})$ then

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{Re(e^{ix})}{x^2+1}\,dx = Re\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx.$$

then $Re\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+1} = \int_{-\infty}^{\infty} \frac{e^{iz}}{(z-i)(z+i)} = \frac{1}{z-i} \cdot \frac{e^{iz}}{(z+i)}$ order=0 analytic about i

singu=i is upper half $2\pi i\, Res(i, f) = 2\pi i \frac{e^{i\cdot i}}{2i} = \frac{\pi}{e}$

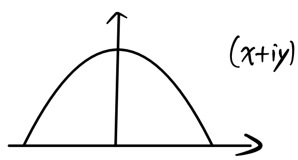
$$\lim_{R\to\infty} \oint_{\gamma_R} \frac{e^{iz}}{z^2+1}\,dz = \pi/e$$

2 part: $\lim_{R\to\infty} \oint_{\gamma_{R_1}} \frac{e^{iz}}{z^2+1}\,dz = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx$

WTS: $\lim_{R\to\infty} \oint_{\gamma_{R_2}} \frac{e^{iz}}{z^2+1}\,dz = 0$

for $z \in \gamma_{R_2}$, we have $|z| = R$, so $\left|\frac{1}{z^2+1}\right| \leq \frac{1}{|z|^2-1} = \frac{1}{R^2-1}$

Moreover for $z = x+iy$ from upper half.



$$\left|e^{i(x+iy)}\right| = e^{Re(iz)} = e^{-y}$$

for $y>0 \in$ upper half. then $|e^{-y}| \leq 1$.

then $\left|\frac{e^{iz}}{z^2+1}\right| \leq \underbrace{\frac{1}{R^2-1}}_{\text{bound for } \frac{1}{z^2+1}} \cdot 1$ ← bound by e^{-y}

$$\left|\oint_{\gamma_{R2}} \frac{e^{iz}}{z^2+1}\, dz\right| \le \pi R \cdot \frac{1}{R^2-1} \quad \text{when } R\to\infty = 0.$$

Note:

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

v $\quad$ x (x bound $Re = e^{y=R}$)

for $z = x+iy \Rightarrow \cos(x+iy) = \dfrac{e^{-y+ix}+e^{y-ix}}{2}$

18-Apri-Midterm 2

Lecture 21:

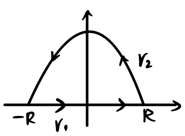
Integrals over real Axie involving Trigonometric functions.

consider: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi^2}{12}$

Recall: $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\, dx$ (Note: $\cos(z)$ is unbounded)

ie: $\cos(x) = Re\, e^{ix}$

$\Rightarrow Re \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\, dx$ (approach) $= \frac{e^{ix}}{(z+i)} \cdot \frac{1}{(z-i)}$ pole of order 1



then by Res-them: $2\pi i \cdot Res(i, f) = 2\pi i \cdot \frac{H^{(0)}(i)}{0!} = 2\pi i \frac{1}{2i \cdot e} = \frac{\pi}{e}$

as $R\to\infty$

$$\lim_{R\to\infty} \oint_{\gamma_R} \frac{e^{iz}}{z^2+1}\, dz = \underbrace{\lim_{R\to\infty} \oint_{\gamma_{R1}}}_{\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}} + \underbrace{\lim_{R\to\infty} \oint_{\gamma_{R2}}}_{\text{"}0\text{"}}$$

$\Longrightarrow |z| = R$, so,

$\left|\frac{1}{z^2-1}\right| \le \frac{1}{|z|^2-1} = \frac{1}{R^2-1}$ (bounded)

Moreover, $|e^{i(x+iy)}| = e^{Re\, iz} = e^{-y}$ $(y>0)$

then for all $|e^{i(x+iy)}| < 1 \Rightarrow \left|\frac{e^{iz}}{z^2-1}\right| \le \pi R \cdot \frac{1}{R^2-1}$

as $R\to\infty$, $\lim_{R\to\infty} \pi R \cdot \frac{1}{R^2-1} = 0 \Rightarrow \left|\frac{e^{iz}}{z^2-1}\right| \le 0$

$$\Rightarrow Re \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\, dx = \frac{\pi}{e}.$$

Jordan lemma: $\boxed{\text{上半部分}}$

$$\left|\int_{|z|=R,\ Im z>0} e^{iz}\, dz\right| < \pi \quad = \left|\int_0^{\pi} iRe^{it} \cdot e^{iRe^{it}}\right| = \text{"}\left|\int_0^{\pi} \gamma'(t) \cdot e^{i\gamma(t)}\, dt\right|\text{"}$$

ie

proof: parametrization $\gamma(t) = Re^{it}$, $t \in [0, \pi]$.

fact: $\sin\theta \ge \frac{2}{\pi}\theta$

$\le R\int_0^{\pi} e^{-R\sin(t)}\, dt = 2R \cdot \int_0^{\pi/2} e^{-R\sin(t)}\, dt$

symmetric of sin

$\le 2R \int_0^{\pi/2} e^{-2tR/\pi}\, dt = \pi(1-e^{-R}) < \pi$

$$|iRe^{it}\cdot e^{iRe^{it}}| = R\cdot e^{Re(iRe^{it})} = R\cdot e^{Re(iR(i\sin t+\cos t))}$$

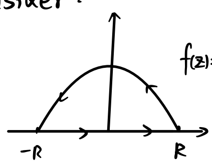
evaluate: $\int_{-\infty}^{\infty}\frac{\sin(x)\cdot x}{x^4+4}\,dx$ V.S $\int_{-\infty}^{\infty}\frac{\sin(x)}{x^4+4}\,dx$ [symmetric over [0,R]] = 0

ie: $\sin(x) = \operatorname{Im} e^{ix}$

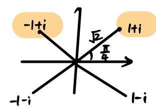
Hence Calculate:

$$\operatorname{Im}\int_{-\infty}^{\infty}\frac{e^{ix}\cdot x}{x^4+4}$$

consider:



$f(z)=\frac{e^{iz}\cdot z}{z^4+4}$ ① find singularity: roots of deno



$$z^4+4=0 = (z^2+2i)(z^2-2i)$$

then singularity in upper: $i+1$, $-1+i$

$$f(z) = \frac{1}{z-(i+1)}\underbrace{\frac{e^{-iz}\cdot z}{(z-(i-1))(z-(-1-i))(z-(-i+1))}}_{H:\text{ analytic about } i+1}$$

"pole of order 1"

$\oint_\gamma f(z) = * 2\pi i$ infront $\left[\right.$ then $\operatorname{Res}(i+1, f) = \frac{e^{-i(i+1)}\cdot(i+1)}{2(2+2i)(2i)}$

similarly, for $z_0 = -1+i$

$\operatorname{Res}(i-1, f) = \frac{e^{-i(i-1)}\quad(i-1)}{\cdots\cdots}$ $\left.\right]$ +

"Realway"

WTS: $\oint_{\gamma_{R_2}}\frac{e^{iz}\cdot z}{z^4+4}\,dz = 0$

1st, look at $\left|\frac{z}{z^4+4}\right| \le \frac{|z|}{|z|^4-4} = \frac{R}{R^4-4}$

similarly arguement in last lecture $|e^{iz}|<1$

$$\Rightarrow \left|\oint_{\gamma_{R_2}}\frac{|e^{iz}z|}{|z^4+4|}\right| \le \pi R\cdot\frac{R}{R^4-4}\xrightarrow[R\to\infty]{} 0$$

then conclude that $\operatorname{Im}\int_{-\infty}^{\infty}\frac{e^{ix}x}{x^4+4} = \frac{\pi\sin(1)}{2e} = \operatorname{Im}(2\pi i(\operatorname{Res}(i+1,f)+\operatorname{Res}(i-1,f)))$

Interesting substitution: $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ (if $z=e^{i\theta}$, $\Rightarrow \begin{cases}\cos\theta = \frac{z+\frac{1}{z}}{2}\\ \sin\theta = \frac{z-\frac{1}{z}}{2i}\end{cases}$)

substitution from parame

Look at : $\displaystyle\int_0^{2\pi} \frac{1}{2+\cos(t)}\, dt$

"since it's over 2π, a parametrization $\gamma(t)=e^{it}$, $t\in[0,2\pi)$ from complex integral." (idea)

"$dz = e^{it}dt = i\cdot e^{it} = iz$"

$$\Rightarrow \int_{|z|=1} \frac{1}{zi}\cdot\frac{1}{2+\frac{z+\frac{1}{z}}{2}}\, dz \overset{\text{Check}}{=} \int_0^{2\pi} \frac{ie^{it}}{ie^{it}}\,\frac{1}{2+\cos(t)}\, dt.$$

$$-i\oint_{|z|=1}\frac{2}{z^2+4z+1}\,dz$$

1) find singularity : $z^2+4z+1=0$

$\Delta = 16-4=12 \Rightarrow z=\frac{-4\pm 2\sqrt{3}}{2} = -2\pm\sqrt{3}.$

$$= -i\oint_{|z|=1}\frac{2}{(z-(-2-\sqrt{3}))}\cdot\frac{1}{z-(-2+\sqrt{3})} \Rightarrow -2-\sqrt{3} \text{ is outside } |z|=1$$

pole order 1

$\Rightarrow -i\cdot 2\pi i\cdot \mathrm{Res}(-2+\sqrt{3}, f)$

$$= 2\pi\cdot \mathrm{Res}(-2+\sqrt{3}, f) = \frac{2\pi\cdot 2}{\sqrt{3}-2+2+\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

Ex3: $\displaystyle\int_0^{2\pi}\frac{1}{(3+\cos\theta)(2+\cos\theta)}\,d\theta$

$$= \int_0^{2\pi}\frac{1}{2+\cos\theta}-\frac{1}{3+\cos\theta}\,d\theta$$

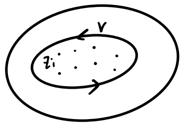
$\Rightarrow$ similar calculate $\rightarrow \displaystyle\int_0^{2\pi}\frac{1}{3+\cos\theta}\,d\theta = \oint_{|z|=1}\frac{1}{iz}\cdot\frac{1}{3+\frac{z+\frac{1}{z}}{2}}\,dz = -i\oint_{|z|}\frac{2}{z^2+6z+1}$

$$\Rightarrow 2\pi i\cdot(-i)\cdot \mathrm{Res}(-2+\sqrt{3}, f) = 2\pi i\cdot(-i)\frac{2}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \pi\left(\frac{2\sqrt{3}}{3}-\frac{\sqrt{2}}{2}\right)$$

Apr 25^{th}:

Rouche's thm: suppose f and g are analytic on open set



If $|f(z)+g(z)|<|f(z)|$ for all $z\in r$, then f and g have equal # of zeros inside r, counting multiplicity.

Ex1: $p(z)=z^5+2iz^4+3iz^2-100$ zeros # lies inside $D(0,2)$?

let $f(z)=100$, $g(z)=p(z)$. f has 0 zero inside $D(0,2)$

clearly, For $|z|=2$, we have $|g(z)+f(z)|=|z^5+2iz^4+3iz^2|\le 2^5+2\cdot 2^4+3\times 4<100=|f(z)|$

by Rouche's thm, P has 0 zeros inside $D(0,2)$.

Ex2: $P(z)=z^3+z+40$ in $D(0,3)$.

let $g(z)=p(z)$, $f(z)=-40$.

For $|z|=3$, then $|g(z)+f(z)|=|z^3+z|\le|z|^3+|z|=27+3<40=|f(z)|$

by Rouche's thm, $p(z)$ has #zero = #zero $\Rightarrow f(z)=0$

Ex3: $P(z)=z^5+2z^3+2z+1$

let $f(z)=-z^5$ $g(z)=p(z)$.

clearly for $f(z)=-z^5$ has 5 zeros. inside $D(0,4)$

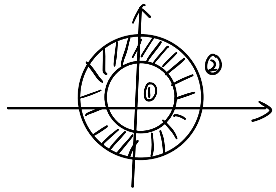
$|g(z)+f(z)|=|2z^3+2z+1|$, for $|z|=4$, $\le 2|z|^3+2|z|+1=2\cdot 4^3+9<4^5=|f(z)|$

by Rouche's thm, $p(z)$ has same # of zeros as #$f(z)$:= 5.

same ie for Q: $p(z)=z^6+iz+1$ inside $D(0,2)$

Last type:

prove all of zeros of $p(z) = 3z^3 + 2iz^2 - iz - 8$ lies inside $1<|z|<2$.



Goal: inside ①, there's no zero

②, there's 3 zero

Step1: for all zeros are outside $D(0,1)$,

let $f(z) = 8$, $g(z) = P(z)$, $f(z)$ has no zero

$|f(z) + g(z)| \le 3|z|^3 + 2|z|^2 + |z|$, for $|z|=1$ $= 3+2+1 = 6 < 8 = |f(z)|$

by Rouche's thm, there's no zero inside $D(0,1)$

Step2: let $f(z) = -3z^3$, $g(z) = P(z)$, where z has 3 zeros

$|f(z)+g(z)| \le 2|z|^2 + |z| + 8$ for $|z|=2$, $= 8+2+8 = 18 < 24 = |f(z)|$

by Rouche's thm, $p(z)$ has 3 zeros inside $D(0,2)$

Hence all zeros of $p(z)$ lies in $1<|z|<2$. by fundamental thm of algebra

(A polyno of deg(n) has exactly n zero, counting multi)

proof → Foundamental thm of algebra:

Assume $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$

take $R = 10^{100}\, n \max_j |a_j|$. Then let $f(z) = -z^n$ and $g(z) = P(z)$.

the f has n zeros inside $D(0,R)$

using Roche' thm g has same # 0 inside $D(0,R) = n$, For $|z|=R$ whave.

$|f(z) + g(z)| \le |a_{n-1}||z|^{n-1} + \cdots + |a_0| \le R^n = |f(z)|$

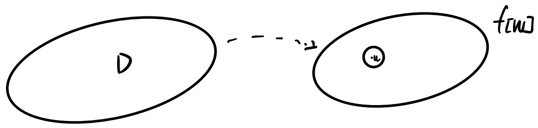
So by the Roche's thm, P has n zeros inside $D(0,R)$.

Question: suppose we have an differential func $f: (-1,1) \to R$, it's that the range is open?

Ans: no!, take $f(x) = x^2$, the range is $[0,1)$ which is not open.

Yes, for complex-system.

Them: suppose that f is a non-constant analytic func on a domain D, Then the range of $f(z)$, as z varies over D, is an open set



let $w_0 = f(z_0)$ be abitary pt $\in f[D]$

$f(z) - w_0$ is not identically zero, since $f(z)$ is not $\equiv c$

proof: let δ be $\min |f(z) - w_0|$ for all z with $|z - z_0| = r$.

and w be $\forall$ point with $|w - w_0| < \delta$. then on the circle $|z - z_0| = r$.

$$|\underbrace{f(z) - w}_{①} - \underbrace{(f(z) - w_0)}_{②}| = |w - w_0| < \delta \leq |f(z) - w_0|$$

Therefore, by Rouche them, $f - w_0$ and $f - w$ have # zero in $|z - z_0| = r$ (circle)

However, $f - w_0$ has exactly $\underset{\geq 1}{m}$ zeros, and so does $f - w$.

That implies that there is z_1, s.t $f(z_1) - w = 0 \Rightarrow f(z_1) = w$

that is to see, each point $w_0 \in f[D]$ lies in the center of small Disc lies inside Rng(f).

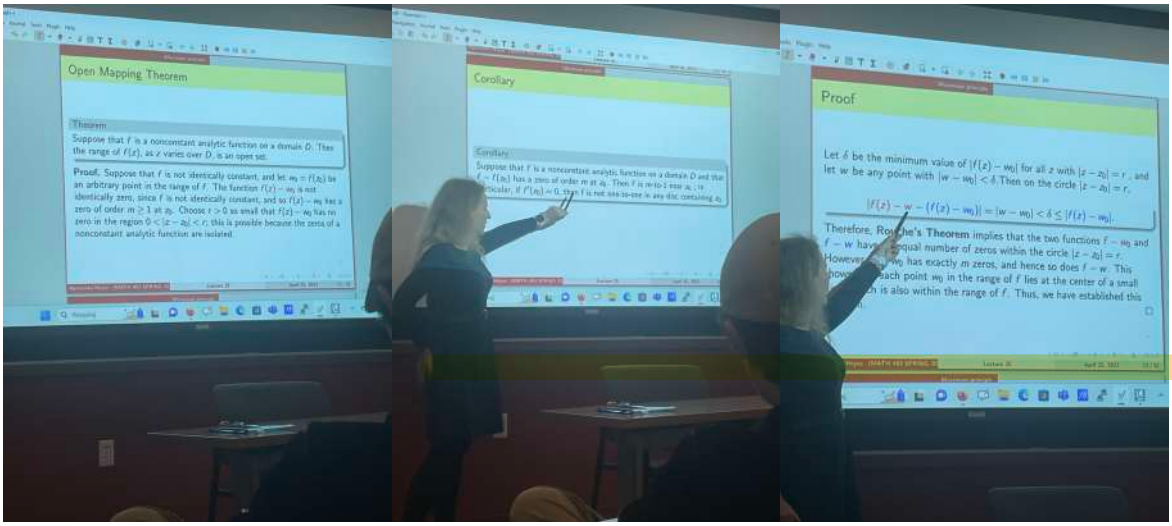
Cor: suppose that f is non-const an

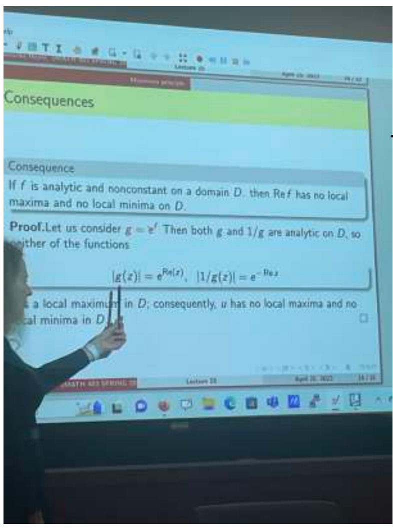
Maximal-modulo Principle:

If f is a non-const func on a domain D, then $|f|$ can have no local max on D.

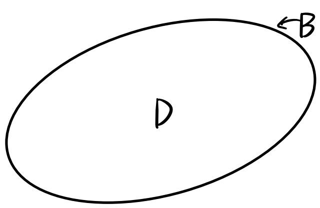
proof: suppose $|f(z_0)| \geq |f(z)|$ for all z within $|z-z_0|<r$, then $f(z_0)$ lies on the bdy of the open set $\overline{W}$, where

$W = \{f(z) : |z-z_0|<r\}$. $\Rightarrow$ contridicted to the fact W is open set with $f(z_0)$ inside.





$\Longrightarrow$ If f is analytic on a bdy dom D and contious on $D \cup B$, where B is bdy of D, then each of $|f|$, $Re f$, $-Re f$ attains its max value on B.



proof: by general thm of conti-func.

Question: is bounded a superfluous hypo ?

Ans : no. consider $f(z) = z$ on $\mathbb{C}$, $|f(z)|$ has no max over D

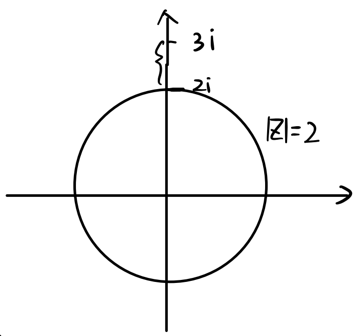
Example: let $f(z) = z^2+3z+2$. Find the max of $|f(z)|$ as z varies over disc $|z|\le 1$

by Max-principle $\to |z|=1$

$|z^2+3z+2| \le |z|^2+3|z|+2 = 6$, moreover equality hold for $z=1$, so the max is 6.

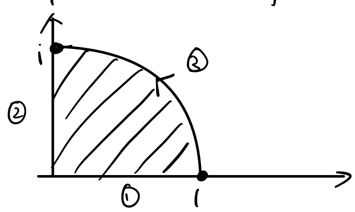
Ex2: let $f(z) = \frac{z^3}{z-3i}$, find max of $|f(z)|$ over $|z|\le 2$ and location.

max principle: "$|z|=2$" $\quad |f(z)| = \frac{|z|^3}{|z-3i|} = \frac{8}{|z-3i|}$, find point on circle $|z|=2$ which distance from $|z| \to 3i$ (smallest).



$\Rightarrow \max_{|z|\le 2}|f(z)| = \frac{8}{|2i-3i|} = 8$. and attained at $z_0=2i$

Ex3: $f(z) = z^2 \cdot e^z$, find the max of $|f(z)|$ as z varies over the region $\{z\in\mathbb{C},\ Re\, z \ge 0,\ Im\, z \ge 0,\ |z|\le 1\}$



$|f(z)| = |z^2 e^z|$ on ① part $0\le x\le 1 \Rightarrow f(x) = f(x+0i) = x^2 e^x \Rightarrow \max f(z) = e$

② part $0\le y\le 1 \Rightarrow |f(z)| = |f(0+yi)| = |(yi)^2 e^{yi}| = |y|^2$. and max=1 at $z=i$.

③ ¼ circe: $t\in[0, \pi/2]$

$|f(z)| = |f(e^{it})| = |e^{2it}| \cdot e^{Re\, e^{it}} = e^{\cos(t)}$

So, the max value is $t=0$ and $|f(z)|_{max} = 1$.

Hence $\max|f(z)| = e$.

Ex4: Use Max-principle $\Rightarrow$ Foundamental thm $\Rightarrow$ Algebra.

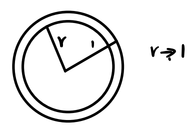
let P be a polyno deg ≥ 1, If $P(z) \neq 0$, then $1/P(z)$ is analytic and its max $|f(z)|$ in the circle $|z| \leq R$ would have occured on its bdy. We have seen, howevere $|P(z)| \to \infty$ as $z \to \infty$ could choose R so that $\left|\frac{1}{P(z)}\right| \leq \frac{1}{|P(0)|}$ for all $|z| = R$, and this is contridick

Schwarz's lemma:

suppose f is analytic in the disc $|z|<1$, that $f(0)=0$ and $|f(z)| \leq 1$ for all z in disc

Then $|f(z)| \leq |z|$, $|z|<1$.

equality $|f(z)| = |z|$ hold for some $z \neq 0$ only if $f(z) = \lambda z$ where λ is a constant of $|\lambda| = 1$.



proof: Since $f(0) = 0$, we know $g(z) = \frac{f(z)}{z}$ is analytic $|z| < 1$. For $|z| = r$.

$$|g(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r}$$

by Max-principle. the inequality $|g(z)| \leq \frac{1}{r}$ is true for $|z| < r$ as well

since $r < 1$ and near 1, $|g(z)| \leq 1$ if $|z| < 1$, thus $|f(z)| \leq |z|$, $|z| < 1$.

further, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$. then $|g(z_0)| = 1$

consequently $|g(z)|$ has interor max, and $g(z)$ is constant $\Rightarrow g(z) = \lambda f(z)$

Ex1: f is analytic on unit disc, $f(0)=0$, and $|f(z)|\le|e^z|$ whenever $|z|=1$.

Determine the $\max f(\frac{1+i}{2})$

$$\max_{|z|=1}|e^z| = \max_{|z|=1} e^{Re z} = e$$

" $|f(z)|\le e = e|z|$ "

then def: $F(z) = \frac{f(z)}{e}$ and $|F(z)| \le \frac{|f(z)|}{e} \le 1$. (satisfy shwarz lemma)

then $|f(z)|\le e|z| \Rightarrow |f(\frac{1+i}{2})|\le e\,|(i+1)/2| = \frac{e}{\sqrt{2}}$

Ex2: Explain by Schwarz' lemma why impossible to have analytic f: $f(z)$ st $f(0)=0$, $|f(z)|\le 60$ in dis $|z|<10$ and $|f(3+4i)|=31$.

Def $F(z) = \frac{f(10z)}{60}$, then $F(0)=0$ as $f(0)=0$, $F(z)$ is analytic in the dis $|z|<1$, and $|F(z)|\le 1$

by Schwarz's lemma, $|F(z)|\le|z|$

that is, as $|F(3+4i/10)| \le |3+4i|/10 = \frac{1}{2}$

then $\left|\frac{f(10\cdot\frac{3+4i}{10})}{60}\right| \le \frac{1}{2} \Rightarrow |f(3+4i)|\le 30 < 31$, thus it's impossible to have 31.

The Mean-value them for Analytic Functions

Them: if $f(z)$ is analytic in a domain contain circle $|z-z_0|=r$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt$$

proof: by Cauchy formula: $f(z_0) = \frac{1}{2\pi i}\oint_{|z-z_0|=r} \frac{f(z)}{z_0-z}\,dz$.

$$\boxed{\begin{array}{l}\text{para}: \gamma(t) = z_0 + re^{it} \\ t\in[0,2\pi]\end{array}}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{i\cancel{re^{it}}\cdot f(z_0+re^{it})}{-\cancel{re^{it}}}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt.$$

Them: if $u(x,y)$ is real part of analytic func f in a domain containing the circle $|z-z_0|=r$

$z_0 = x_0+iy_0$, then $u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0+r\cos t,\ y_0+r\sin t)\,dt$

ie : $f(z) = u(z) + iv(z)$, $u, v : \mathbb{R}^2 \to \mathbb{R}$.

proof: Exactly same as before

para: $e^{it} = \cos(t) + i\sin(t)$

lecture 26: linear fractional transformation.

Def: linear fractional transformation. T is a rational func of the sphereical form

$$T(z) = \frac{az+b}{cz+d} \quad (\text{where } a, b, c, d \in \mathbb{C} \text{ and } ad-bc \neq 0)$$

properties: (i) T is one to one func

Assume $T(z_1) = T(z_2)$

$$\frac{az_1+b}{cz_1+d} = T(z_1) = T(z_2) = \frac{az_2+b}{cz_2+d}. \Rightarrow (az_1+b)(cz_2+d) = (az_2+b)(cz_1+d)$$

$$(ad-bc)z_1 = (ad-bc)z_2.$$

(ii) T has a pole of order 1 at $z = -\frac{d}{c}$

ie: $\frac{1}{cz+d} \cdot (az+b)$

(iii) $\lim_{z \to \infty} T(z) = \frac{a}{c}$, denote as $T(\infty) = \frac{a}{c}$

Them:

LFT form a group. In perticular, if T, S are LFT, then $S \circ T$ is LFT.

ie : $S(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$ $\quad T(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$

$$S \circ T(z) = \frac{\frac{a_2 z + b}{c_2 z + d} z + b_1}{\frac{a_2 z + b}{c_2 z + d} + d_1}$$

Them: the inverse of T is given by:

$$T^{-1}(w) = \frac{-dw+b}{cw-a}$$

Analogy if: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

prop4: If $T \neq id$, then T has at most 2 fixed point, ie, points s.t $T(z) = z$.

proof: $T(z) = z$, $(az+b) = z(cz+d)$ quadratic equation has at most 2 fixed point.

Consequence: If $S.T$ are two LFT and $S(z_i) = T(z_i)$ for three disc point z_0, z_1, z_2, then $S = T$.

proof: consider $H = S^{-1} \cdot T$, then z_0, z_1, z_2 are distinct fixed pt.

$$z_i = \underset{\displaystyle H(z_i)}{\underset{\parallel}{S^{-1} \circ T(z_i)}} \quad \text{by Group - them}$$

Them: Assume z_0, z_1, z_2 and w_0, w_1, w_2 are given. There is T st $T(z_i) = w_i$ for all $i = 0, 1, 2$.

proof: $T_1(z) = \dfrac{z - z_1}{z - z_3} \quad \dfrac{z_2 - z_3}{z_2 - z_1}$

Note that $T_1(z_1) = 0$, $T_1(z_3) = \infty$, $T_1(z_2) = 1$.

$T_2(w) = \dfrac{w - w_1}{w - w_3} \quad \dfrac{w_2 - w_3}{w_2 - w_1}$ Note $T_2(w_1) = 0$ $T_2(w_3) = \infty$ $T_2(w_2) = 1$

$$z_1 \xrightarrow{T_1} 0 \xrightarrow{T_2^{-1}} w_1$$
$$z_2 \longrightarrow \infty \xrightarrow{T_2^{-1}} w_2$$
$$z_3 \longrightarrow 1 \xrightarrow{T_2^{-1}} w_3$$

Take $T = T_2^{-1} \circ T_1$

Exercise: Find the LFT that send $1, 2, 3$ to $4, 5, 6$

we write $T_1(z) = \dfrac{z-1}{z-3} \quad \dfrac{2-3}{2-1} = \dfrac{-z+1}{z-3}$

$T_2(z) = \dfrac{4-z}{z-6} = \dfrac{z-4}{z-6} \quad \dfrac{5-4}{5-6}$

$T_2^{-1}(z) = \dfrac{6z+4}{z+1}$ then $T_2^{-1} \circ T_1 = \dfrac{6\frac{-z+1}{z-3} + 4}{\frac{1-z}{z-3} + 1} = 3 + z.$

Ex2: $1, 2, 3 \to 1, -1, i$

$$T_1(z) = \frac{-z+1}{z-3}$$

$$T_2(z) = \frac{z-1}{z-i} \quad \frac{-1-1}{-1-i}$$

$T_2^{-1} \circ T_1 \Rightarrow$ plug in the formula.

!: T is actually compos of serveal simplier LFT

$$T(z) = \frac{1}{c}\left(\frac{bc-ad}{cz+d} + a\right)$$

where $U(z) = cz + d$, $W(z) = \frac{1}{z}$, $V(z) = \frac{1}{c}((bc-ad)z + a)$

$T = V \circ W \circ V$.

" U, V map line → line, circle → circle. "

" What about W ? "

$\Downarrow$

W maps circles → lines / lines → circles.

Equation: $a(x^2+y^2) + bx + cy = d$:= circle $(a \neq 0)$ / line $(a=0)$

ie: $\underset{\overset{\uparrow}{\frac{1}{x+iy}}}{1/z} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = u+iv := W(x+iy)$

replace $x+iy$ by $1/z \Rightarrow d(u^2+v^2) - bv - cu = a$. This is again either line/circle.