Differentiability

I know that all of you know what this page contains.
Nevertheless I am obliged to spell it out.
Please read it; if nothing else, it will fix notation to be used in class

Unless I am much mistaken, I believe all the results stated are found in Rudin's "Principles of Mathematical Analysis"

Definitions and Theorems V and W are finite dimensional real vector spaces. $U \subset V$ is an open subset. And $f : U \to W$ is a function.

(1) The definition of the directional derivative $D_v f(x) \in W$ as a limit (if it exists) where V and W are finite dimensional vector spaces, $U \subset V$ is open, and $f : U \to W$ is a function, $x \in U$ and $v \in V$

(2) With U, V, W, f, x as in (1) above, the definition of "f is differentiable at x" and therefore the definition of the linear transformation $f'(x) : V \to W$. The linear transformation $f'(x)$ is of course referrred to as the derivative of f.

(3) *The chain rule ...* if f is differentiable at x and if g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x))f'(x)$.

(4) When $V = \mathbb{R}^n$ and v is the i-th basis vector of $\mathbb{R}^n$ in (1) above, then the i-th partial derivative $\partial_i f(x) := D_v f(x)$ is the When $V = \mathbb{R}^n$ and v is the i-th basis vector of $\mathbb{R}^n$, the above directional derivative $D_v f$ is denoted by $\partial_i f$

(5) A function f defined on an open subset of $\mathbb{R}^n$ is C^1 if all its partial derivatives are defined and continuous. C^1 functions are often called continuously differentiable. A theorem that is easy to prove is :*f is C^1 implies f is differentiable.*

(6) The definition of C^k and C^∞ functions defined on an open subset $U \subset \mathbb{R}^n$.
The collection of C^k functions $f : U \to \mathbb{R}$ is closed under addition and multiplication.
If $U_i \subset V_i$ is open and V_i is a finite dimensional real vector space for $i = 1, 2, 3$, and
if both $f : U_1 \to U_2$ and $g : U_2 \to U_3$ are C^k then $g \circ f : U_1 \to U_3$ is C^k.

(7) The Inverse function theorem and the implicit function theorem

We will be concerned only with C^∞ functions and manifolds in this course. The above properties will be taken for granted.

Much later in this course, we will employ repeatedly C^∞ functions $f : \mathbb{R} \to [0, \infty)$ such that $f^{-1}0 = (-\infty, 0]$ when discussing manifolds with boundary.

1. First week

:

Definition 1.1. Let n be a non-negative integer. A topological space X is a n**-manifold** if every point $x \in X$ has a nbhd that is homeomorphic to an open subset of $\mathbb{R}^n$.

Example 1.2. Every open subset of $\mathbb{R}^n$ is a n-manifold.
A 0-manifold is a discrete topological space.

Definition 1.3. A subset X of $\mathbb{R}^n$ is a C^∞ submanifold of dimension k If for every $p \in X$ there is an open subset $\Omega \subset \mathbb{R}^n$ with $p \in \Omega$ and a C^∞ function $f : \Omega \to \mathbb{R}^{n-k}$ such that conditions (A) and (B) below hold:
(A) $X \cap \Omega = f^{-1}f(p)$ and
(B) the derivative $f'(p) : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a surjective linear transformation.
(The implicit function theorem then implies that X is a k-manifold in the usual sense, and it also implies the problem below).

Problem 1: With notation as in the above definition, prove $v \in \ker f'(p)$ if and only there is a C^∞ path γ in X (i.e. $\gamma : (-a, a) \to X$ is C^∞) such that $\gamma(0) = p$ and $\gamma'(0) = v$.
This vector space $\ker f'(p)$ *is the tangent-space of* X *at* p*, and is denoted by* $T_p(X)$*.*
Problem 2: $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ is a C^∞ submanifold of $\mathbb{R}^n$ and $T_p(S^{n-1} = \{v \in \mathbb{R}^n :< p, v >= 0\}$
Problem 3: A subset X of a topological space R is *locally closed* if $X = W \cap F$ where W is open and F is closed. Prove that every C^∞ submanifold of $\mathbb{R}^n$ is a locally closed subset of $\mathbb{R}^n$.
Problem 4: (a) Give "the simplest example" of a C^∞ 1-submanifold of $\mathbb{R}^2$ that is not a closed subset.
(b) Give an example of a *connected* non-closed C^∞ 1-submanifold X of $\mathbb{R}^2$ which is not contained properly in any other *connected* C^∞ 1-submanifold of $\mathbb{R}^2$.

1.1. **Homotopy versus approximation.**

Definition 1.4. continuous maps $f, g : X \to Y$ are homotopic to each other (written $f \sim g$ if there is a continuous $F : X \times [0, 1] \to Y$ such that $f(x) = F(x, 0)$ and $g(x) = F(x, 1)$ for all $x \in X$.
It is well-known (and easy to prove) that homotopy is an equivalence relation.

Definition 1.5. Let Y be a metric space and let X be an arbitrary topological space and let $\epsilon > 0$. For continuos $f, g : X \to Y$ we define $f \overset{\epsilon}{\sim} g$ if there are continuous maps $f_i : X \to Y$ for $i = 0, 1, 2, ..., n$ (for some n) such that
(i) $f_0 = f$ and $f_n = g$
(ii) $d(f_i(x), f_{i+1}(x)) < \epsilon$ for all $x \in X$ and for all $i = 0, 1, 2, ..., n$.
In the above d is the given metric on Y.
By very defintion $\overset{\epsilon}{\sim}$ *is an equivalence relation on* $C(X;Y)$*=the set of continuous maps from* X *to* Y*.*

Problem 5: If $\epsilon > 0$ and X is compact, then $f \sim g$ implies $f \overset{\epsilon}{\sim} g$.

Proof. The proof given in class assumed that X is also a metric space,
X cpt implies $X \times [0,1]$ compact implies the given $F : X \times [0,1] \to Y$ is uniformly continuous etc etc.
You are welcome to write a proof dropping this unnecessary metric assumption on X. □

Problem 6: If $f, g : X \to S^{n-1}$ are continuous and $\|f(x) - g(x)\| < 2$ for all $x \in X$, then $f \sim g$.

Proof. Given in class. □

Problem 7:: If $f \overset{\epsilon}{\sim} g$ for some continuous $f, g : X \to S^{n-1}$ and for some $0 < \epsilon \leq 2$, then $f \sim g$.

Proof. Previous problem combined with $\sim$ is an equivalence relation. □

Remark 1.6. ϵ has to be sufficiently small if one wishes $f \overset{\epsilon}{\sim} g \implies f \sim g$.
Because, for D equal to the diameter of a compact metric space Y, we see that $f \overset{D+1}{\sim} g$ for any two maps $f, g : X \to Y$.

The proposition that follows is a type of converse to problem 5.

Proposition 1.7. *Let Y be a C^∞ compact k-submanifold of $\mathbb{R}^n$. Then there is some $\epsilon = \epsilon(Y) > 0$ such that for $f \overset{\epsilon}{\sim} g \implies f \sim g$ for all continuous $f, g : X \to Y$ and for all topological spaces X.*

The proof relies on the tubular nbhd thm. for $Y \subset \mathbb{R}^n$, to state which we require the following notation:
$T_y(Y) \subset \mathbb{R}^n$=tangent-space of Y at $y \in Y$ has been defined in problem 1.
$N(Y) = \{(y, v) \in Y \times \mathbb{R}^n : \langle v, w \rangle = 0 \, \forall w \in T_y Y\}$ is the referred to as the normal bundle of Y in $\mathbb{R}^n$.
$N_c(Y) = \{(y, v) \in N(Y) : \|v\| < c\}$ (note that 'c' is quicker to type than epsilon)
$U_c(Y) = \{z \in \mathbb{R}^n : \text{there exists some } y \in Y \text{ such that } \|z - y\} < c\}$.
$a : N_c(Y) \to U_c(Y)$ is given by $a(y, v) = y + v$.

Theorem 1.8. *Tubular nbhd thm: Let Y be a C^∞ compact k-submanifold of $\mathbb{R}^n$. Then there is a positive constant c such that*
$a : N_c(Y) \to U_c(Y)$ is a homeomorphism.

This theorem will be proved later in the course. We will assume it to prove the proposition

Proof. of the proposition.
First note that $a(y, 0) = y$ for all $y \in Y$. It follows that the inverse function $a^{-1} : U_c(Y) \to N_c(Y)$ satisfies $a^{-1}(y) = (y, 0)$ for all $y \in Y$.
Next denote by $p_1 : N_c(Y) \to Y$ the projection $p_1(y, v) = y$.
So we observe that the continuous map $q = p_1 \circ a^{-1} : U_c(Y) \to Y$ is a *retraction*, i.e. $q(y) = y$ for all $y \in Y$.

We take $\epsilon(Y)$ to be the 'c' in the tubular nbhd thm. We wish to show that $f \overset{c}{\sim} g \implies f \sim g$ for $f, g : X \to Y$. Because homotopy is an equivalence relation, we may assume in fact that $\|f(x) - g(x)\| < c$ for all $x \in X$.

We then have $G(x,t) = f(x)+t(g(x)-f(x))$ and $\|G(x,t)-f(x)\| < c$ for all $t \in [0,1]$.
But $f(x) \in Y$ and therefore $G(x,t) \in U_c(Y)$.
We define $F = q \circ G : X \times [0,1] \to Y$. Because $f(x), g(x) \in Y$ and $q(y) = y$ for all $y \in Y$ we deduce that $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

□

Problem 8: Let Y be a C^∞ closed k-submanifold of $\mathbb{R}^n$. Let $z \in \mathbb{R}^n$. Because Y is closed, we have seen in an earlier course that there is some $y \in Y$ such that $\|y - z\| \leq \|y' - z\|$ for all $y' \in Y$.
Prove that $\langle y - z, w\rangle = 0$ for all $w \in T_y(Y)$. Hint: See problem 1.

Problem 9: *A Stiefel manifold*
Define $V(n,2) = \{(a,b) \in \mathbb{R}^n \times \mathbb{R}^n : \langle a,a\rangle = \langle b,b\rangle = 1 \text{ and } \langle a,b\rangle = 0\}$. Prove that $V(n,2)$ is a smooth (i.e. C^∞) (2n-3)-submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ by considering $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^3$ given by $f(a,b) = (\langle a,a\rangle, \langle b,b\rangle, \langle a,b\rangle)$ and then showing that $f'(a,b) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^3$ is a surjective linear transformation for all $(a,b) \in V(n,2)$.
In fact $f'(a,b)$ is a surjective linear transformation if and only if the vectors a and b of $\mathbb{R}^n$ are linearly independent.

1.2. **Spheres.**

Remark 1.9. Problem 6 raises the question: Take $X = S^{n-1}$ and $f(x) = x$ and $g(x) = -x$ for all $x \in S^{n-1}$. Are f and g homotopic to each other?
The above $f, g : S^{n-1} \to S^{n-1}$ are not homotopic to each other when n is odd, i.e. when we are looking at even dimensional spheres. This statement will be proved in the course by
(a) integration
(b) and also by directly defining the degree of a map– but this relies on Sard's theorem whose proof is likely to be omitted.

Problem 10: Prove that the above f, g are homotopic to each other for S^1.

Problem 11: A tangent vector field on S^{n-1} is a continuous function $f : S^{n-1} \to \mathbb{R}^n$ such that $f(x) \in T_x(S^{n-1})$ for all $x \in S^{n-1}$. Show that if n is even, there is a nowhere vanishing tangent vector field f, i.e. $\{x \in S^{n-1} : f(x) = 0\}$ is empty.

Remark 1.10. We often regard the circle as $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. For every $n \in \mathbb{Z}$, we have $f_n : S^1 \to S^1$ given by $f_n(z) = z^n$ for all $z \in S^1$. It is known that f_n is homotopic to f_m if and only if $n = m$. This is tradtionally proved by defining the winding number $n(f) \in \mathbb{Z}$ of a map $f : S^1 \to S^1$. Rudin's "Real and Complex Analysis" defines the winding number.

1.3. **Differentiable manifolds.** See page 5 of Warner.

To begin with, we are not paying any attention to separation properties, we are focusing only on differentiability.
The definition given in class is the same as that of the book (defn. 1.4, page 5), but with the following changes:

Warner considers k-times differentiable, but we restrict ourselves to infinitely differentiable.
We retain (a) and (b) of defn.1.4, but we deliberately drop part (c) of that definition.

Definition 1.11. Let X be a C^∞ manifold, let $U \subset X$ be open and let $f : U \to \mathbb{R}$ be a function. Let $p \in U$. Then the phrase " f is C^∞ at p" has been defined. PLEASE take a look at 2020Lec3. You will find it in the Files folder. Specifically, COMPARISON OF CHARTS on page 4 to Lemma 8 on page 9 of 2020Lec3 contains the initial portion of the Friday lecture.

Remark 1.12. Here is an alternative (but equivalent definition of a C^∞ n-manifold X.
NOTATION: for a set Y, the collection of all functions $f : Y \to \mathbb{R}$ is denoted by $\mathbb{R}^Y$. Given a map $h : C \to D$ we obtain $h^* : \mathbb{R}^D \to \mathbb{R}^C$ given by $h^*(f) = f \circ h$ for all $f \in \mathbb{R}^D$.
The operation h^* is referred to as "pull-back". This terminology will prevail throughout the course.
NOTE: In the above, if C is a subset of D, and $h : C \to D$ denotes the inclusion, then the standard notation for $h^*(f)$ is $f|_C$.

Definition 1.13. Let X be a topological space. A "subsheaf A of the sheaf of real-valued functions on X" consists of the following data:
For every open $U \subset X$, we have a subset $A(U) \subset \mathbb{R}^U$ that satisfies the following conditions:
(1) If $V \subset U \subset X$ are both open in X and if $f \in A(U)$, then $f|_V \in A(V)$.
(2) Assume $W = \bigcup_{i \in I} W_i$, where all the $W_i \subset X$ are open. Let $f \in \mathbb{R}^W$. If $f|_{W_i} \in A(W_i)$ for all $i \in I$, then $f \in A(W)$.

Example 1.14. A familiar example: $X = \mathbb{R}^n$ and for every open $U \subset X$, define $A(U)$ to be the collection of C^∞ functions $\{f : U \to \mathbb{R}\}$. This sheaf is denoted by $\mathrm{C}^\infty(X; \mathbb{R})$.
C^k functions also define a subsheaf. When $k = 0$ this is the sheaf of continuous rael-valued functions .

Problem 12: Let $\Omega_1 \subset \mathbb{R}^m$ and $\Omega_2 \subset \mathbb{R}^n$ be open subsets. Let $h : \Omega_2 \to \Omega_1$ be continuous. Prove the assertions (I) and (II) below are equivalent to each other:
(I) h is C^∞
(II) For every open $W \subset \Omega_1$ and for every C^∞ function $f : W \to \mathbb{R}$, its pullback $h^*(f)$ is a C^∞ real-valued function on $h^{-1}(W)$.

Definition 1.15. The alternative definition of a C^∞ n-manifold X is a pair (X, A) where X is a topological space and A is a subsheaf of the sheaf of real-valued functions on X with the property that there is an open cover $U_\lambda, \lambda \in \Lambda$ of X and homoeomorphisms $\phi_\lambda : U_\lambda \to \Omega_\lambda$ where $\Omega_\lambda \subset \mathbb{R}^n$ is open, with the following property:
for all open $W \subset \Omega_\lambda$ and all $f \in \mathbb{R}^W$, we have: f is $\mathrm{C}^\infty \iff \phi_\lambda^* f \in A(W)$.

Problem 13: Deduce Warner's condition (b) from this alternative definition of a C^∞ n-manifold.

Proof. let $\lambda, \mu \in \Lambda$. Let $\Omega_1 = \phi_\lambda(U_\lambda \cap U_\mu)$ and let $\lambda, \mu \in \Lambda$. Let $\Omega_2 = \phi_\mu(U_\lambda \cap U_\mu)$. We have the homoeomorphism $h : \phi_\lambda \circ \phi_\mu^{-1} : \Omega_2 \to \Omega_1$. Warner's condition (b) requires "h is C^∞" According to the alternative definition, if $W \subset \Omega_2$ is open and $f \in \mathbb{R}^W$, then

$$f \in C^\infty(W, \mathbb{R}) \implies \phi_\mu^* f \in A(\phi_\mu^{-1}(W)) \implies (\phi_\lambda^{-1})^* \phi_\mu^* f = h^*(f) \text{ is } C^\infty$$

thus condition II of problem 12 is satisfied. It follows that h is C^∞. □

Problem 14:

(1) Let $B \subset \mathbb{R}^n$ be an open ball in $\mathbb{R}^n$. Find a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$ such that
   (a) $f^{-1}(0, \infty)$ and $f^{-1}[0, \infty)$ are B and its closure $\overline{B}$ respectively.
   (b) The linear functional $f'(x) : \mathbb{R}^n \to \mathbb{R}$ is nonzero for every $x \in f^{-1}(0)$.

(2) Let $B_i, i = 0, 1, 2, ..., g$ be open balls in $\mathbb{R}^n$.
   Assume that $\overline{B_i} \subset B_0$ for all $i = 1, 2, ..., g$.
   Assume also that $1 \leq i < j \leq g$ implies $\overline{B_i} \cap \overline{B_j} = \emptyset$.
   Find a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$ such that
   (i) $f^{-1}(0, \infty)$ is the complement of $\overline{B_1} \cup ... \cup \overline{B_g}$ in B_0
   (ii) $f^{-1}[0, \infty)$ is the closure of $f^{-1}(0, \infty)$
   (iii) $f(x) = 0$ implies $f'(x)$ is a nonzero linear functional.
   Hint: Compare the sets $f^{-1}(0)$ in parts (1) and (2)

(3) With $f : \mathbb{R}^n \to \mathbb{R}$ as in part (2), show that
   $Z = \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^m : \|w\|^2 = f(x)\}$ is C^∞ $(m + n - 1)$-submanifold of $\mathbb{R}^{n+m}$ in the sense of defn.1.3.

Remark 1.16. Consider the Z In part (3) of the above problem.
When $n = 2$ and $m = 1$, the resulting Z is a "surface with g handles" or a "surface of genus g".
When $g = 1$ then Z is diffeomorphic to $S^{n-1} \times S^m$. Enjoy yourself by proving this fact.

CONVENTION From now on, For a C^∞ manifold X, and an open $U \subset X$ and $f \in \mathbb{R}^U$, we will write $f \in C^\infty(U; \mathbb{R})$ in place of $f \in A(U)$.

Definition 1.17. let X and Y be C^∞ manifolds. Let $f : X \to Y$ be continuous. The f is C^∞ if

$$V \subset Y \text{ is open and } g \in C^\infty(V; \mathbb{R}) \implies f^* g \in C^\infty(f^{-1}(V, \mathbb{R})$$

Remark 1.18. To check that a given $f : X \to Y$ is C^∞, one often has to appeal to problem 12("I implies II").

2. Second Week

2.1. Tangent-space of a manifold at a point.

GERMS of FUNCTIONS. The set of germs of C^∞ functions at $x \in X$ of a C^∞ manifold X will be denoted by $C^\infty_x(X;\mathbb{R})$. The definition given in class is exactly the same as in defn.1.13, page 12, Warner (but the notation there is different).
PROPERTIES:
$C^\infty_x(X;\mathbb{R})$ is a $\mathbb{R}$-algebra. For $f \in C^\infty_x(X;\mathbb{R})$ we have the value of f at x is denoted by $\mathrm{ev}_x(f)$.
$\mathrm{ev}_x : C^\infty_x(X;\mathbb{R}) \to \mathbb{R}$ is a $\mathbb{R}$-algebra homomorphism.

DEFINITION:A tangent-vector at $x \in X$ is a linear functional $L : C^\infty_x(X;\mathbb{R}) \to \mathbb{R}$ that satisfies $L(fg) = \mathrm{ev}_x(f)L(g) + \mathrm{ev}_x(g)L(f)$.
The set of tangent vectors is a real vector space. This set is denoted by $T_x(X)$ and is referred to as the tangent-space of X at x.

Definition 2.1. Let V be a finite dimensional vector space. Let f be a C^∞ real-valued function defined on an open $\Omega \subset V$. Let $v \in V$. We then have the directional derivative $D_v f : \Omega \to \mathbb{R}$ given by

$$(D_v f)(p) = (f \circ \gamma)'(0) \text{ where } \gamma(t) = p + tv \text{ assuming } p \in \Omega$$

$f \mapsto \mathrm{ev}_p(D_v f)$ gives a tangent vector of the manifold V at the point $p \in V$.
$v \in V \mapsto \mathrm{ev}_p D_v \in T_p V$ gives an isomorphism $V \to T_p V$ of vector spaces.
(The second lemma on page 13 of Warner, is crucial in proving the surjectivity of $V \to T_p V$).

Remark 2.2. All the remarks of Warner, pages 21 and 22 that involve tangent vectors *but not differentials, a topic that is postponed,* were stated and proved on Monday. In particular,
(0) The tangent-space of a C^∞ n-manifold at each point is n-dimensional.
(I) A C^∞ map $f : X \to Y$ induces the linear transformation $f'(x) : T_x X \to T_{f(x)} Y$ (we used pullbacks)
(II) The chain rule: $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$
(III) If f is a C^∞ map from an open subset $\Omega \subset \mathbb{R}^m$ to $\mathbb{R}^n$, then $f'(x)v = D_v f$

2.2. **smooth submanifolds of a manifold.**
This subsection is the material of Warner, pages 22-30, that does not involve the term "differentials". There are three equivalent definitions. We give the simplest one (defn. 2.5 below) and prove the equivalence with the other two.

Proofs are sketched or omitted when they were given in class. Proofs that I gave in class, which I thought were not optimal–or clear– are written in full

Remark 2.3. Let X be a C^∞ n-manifold and let $Y \subset X$ be an *arbitrary* subset.
We could define a subsheaf of the sheaf of $\mathbb{R}$-valued functions on Y (see defn. 1.13 of these notes) as follows:
Let $W \subset Y$ be open in the relative topology.
We first define $A_p(W) \subset \mathbb{R}^W$ for every $p \in W$, and then define $A(W) = \bigcap_{p \in W} A_p(W)$
Let $f \in \mathbb{R}^W$. Thus $f : W \to \mathbb{R}$ is a function.
Then $f \in A_p(W)$ if there is a C^∞ function g defined on a nbhd U of p in X such that $f|_{W \cap U} = g|W \cap U$.
The above defines $A_p(W) \subset \mathbb{R}^W$.
Next, $f \in A(W)$ if $f \in A_p(W)$ for every $p \in W$.

It should be clear that the topological space Y equipped with the above definition of $A(W) \subset \mathbb{R}^W$ for every open $W \subset Y$ satisfies defn.1.13, page 6, of these notes. *However what we want is a necessary and sufficient condition that the collection* $\{A(W) : W$ *open in* $Y\}$ *also satisfies defn 1.15 on the same page*

Lemma 2.4. *$X = \Omega_1 \times \Omega_2$ where the Ω_i are open in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively for $i = 1, 2$. Let $Y = \Omega_1 \times \{z\}$ where $z \in \Omega_2$. Now X is a manifold and Y is a subset. Let $f \in \mathbb{R}^W$ where W is open in Y. Then, with $A(W)$ as defined in this remark, we have:*

$$f \in A(W) \iff f : W \to \mathbb{R} \text{ is } \mathrm{C}^\infty$$

Proof. Assume f is C^∞. We have the projection $p : W \times \Omega_2 \to W$. Then $g = f \circ p$ is C^∞ and $f = g|_W$. Therefore $f \in A(W)$.
For the converse, one appeals to the fact that the inclusion of Y in X is C^∞. If $f \in A(W)$ is locally expressed as $f = g|_Y$ where g is a C^∞ function defined on an open subset of X. It follows that f is C^∞. (You are requested to spell out the "locally" in full). $\square$

Definition 2.5. A subset Y of a C^∞ n -submanifold X is a C^∞ r-submanifold if for every point $p \in Y$ there is an open $p \in U \subset X$ and a diffeomorphism $\phi : U \to B_1 \times B_2$ where B_1 and B_2 are open balls centered at 0 in $\mathbb{R}^r$ and $\mathbb{R}^{n-r}$ respectively, such that $Y \cap U = \phi^{-1}(B_1 \times \{0\})$.
Note that the above lemma shows that Y is a C^∞ manifold (in other words defn. 1.15, page 6, holds) and also that the inclusion of Y in X is a C^∞ map.

Lemma 2.6. *Let $f : X \to Z$ be a C^∞ map of C^∞ manifolds. Assume that $f'(p) : T_pX \to T_{f(p)}Z$ is surjective for some point $p \in X$. Then there is a neighborhood U of p in X such that $U \cap f^{-1}f(p)$ is a C^∞ submanifold of X, whose tangent-space at q is* $\ker f'(q)$ *for all $q \in U \cap f^{-1}f(p)$.*

Proof. The statement is local, thus we may assume that X and Z are open subsets of finite dimensional vector spaces V and W respectively, and also that $p = 0$ and $f(p) = 0$.
Let K denote the kernel of the linear transformation $f'(0) : V \to W$. Choose a linear transformation $g : V \to K$ such that $g(a) = a$ for all $a \in K$. Now consider the C^∞ map $h : V \to W \times K$ given by $h(v) = (f(v), g(v))$. Because the derivative of a linear transformation is itself, we see that $h'(0)v = (f'(0)v, g(v))$ for all $v \in V$. By very construction, we see that ker $h'(0)$ is zero. Because $f'(0)$ is surjective, we see that V and $W \times K$ have the same dimension. So $h'(0)$ is bijective. We may now apply the inverse function theorem to obtain V', W', K' which are nbhds of 0 in V, W, K respectively such that h restricts to a diffeomorphism $V' \to W' \times K'$.
Furthermore h restricts to a bijection $V' \cap f^{-1}(0) \to 0 \times K'$, satisfying the requirement of defn.2.5.
The statement on its tangent-space is left to you as an exercise (which you can deduce with or without Problem 1).

□

Lemma 2.7. *Let $f : Z \to X$ be a C^∞ map of C^∞ manifolds such that $f'(p)$ is injective for some point $p \in Z$. Then there is a nbhd U of $p \in Z$ such that $f(U)$ is a C^∞ submanifold of X. The image of $f'(q) : T_q(Z) \to T_{f(q)}(X)$ is the tangent-space of $f(U)$ at $f(q)$ for all $q \in U$.*

Proof. Once again, we may assume that Z and X are neighborhoods of 0 in vector spaces W and V respectively, and also that $p = 0$ and $f(p) = 0$. Let $C \subset V$ be a linear subspace such that $C \oplus f'(0)W = V$. Now consider $h(w, c) = f(w) + c$; this is a C^∞ map defined on a nbhd of 0 in $W \times C$ to V. Note that $h'(0)(w, c) = f'(0)w + c$ is an isomorphism. By the inverse function thm. we obtain nbhds W', C', V' of zero in W, C, V respectively such that h restricts to a diffeomorphism $W' \times C' \to V'$. Its inverse $\phi : V' \to W' \times C'$ restricts to a bijection $f(W') \to W' \times 0$ as desired by defn. 2.5.

□

From now on, we assume that all manifolds are Hausdorff.

2.3. **Important definitions and Ehresmann's theorem.**

Definition 2.8. A C^∞ map $f : X \to Z$ is a **submersion** if the linear transformation $f'(p)$ is surjective for every $p \in X$. Lemma 2.6 implies that $f^{-1}(z)$ is a closed C^∞ submanifold for every $z \in Z$.

Definition 2.9. A C^∞ map $f : Z \to X$ is an textbfimmersion if $f'(z)$ is injective for every $z \in Z$.
A C^∞ map $f : Z \to X$ is an **embedding** if $f(Z)$ is a C^∞ submanifold of X in the sense of defn.2.5, and $f : Z \to f(Z)$ is a diffeomorphism
Embedding implies immersion, but not vice versa–see Warner pages 22-23. But be careful; his defn of submanifold is not 2.5.

Definition 2.10. X and Y are locally compact Hausdorff topological spaces. A continuous $f : X \to Y$ is **proper** if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$.
Problem 15: Prove that if $f : X \to Y$ is proper, then $f(F)$ is a closed subset of Y whenever F is a closed subset of X.

Theorem 2.11. *The Ehresmann Fibration Theorem. Let $f : X \to Y$ be a proper* C^∞ *submersion. Then every point $y \in Y$ has a neighborhood U and*
a C^∞ *$r : f^{-1}(U) \to f^{-1}(y)$ such that such that $\phi(x) = (f(x), r(x))$ gives a diffeomorphism $\phi : f^{-1}(U) \to U \times f^{-1}(y)$*

The proof of this theorem is contained in problems 21-24.

Remark 2.12. Because f is a submersion, $f^{-1}y'$ is a close C^∞ submanifold of X for every $y' \in Y$. Ehresmann's theorem implies that for all $y' \in U$, the $x \mapsto r(x)$ gives a diffeomorphism $f^{-1}(y')$ is diffeomorphic to $f^{-1}(y)$.

Definition 2.13. Let $f : X \to Y$ be a C^∞ map. $x \in X$ is a *critical point of* f if $f'(x)$ is NOT surjective.
$Crit(f) \subset X$ is the set of critical points of f.
$y \in Y$ is a *regular value of* f if $f'(x)$ is surjective for every $x \in f^{-1}(y)$.
In other words, the set of regular values of f is the complement of $f(Crit(f))$.

Problem 16: (a) Prove that the set of $m \times n$ matrices T such that rank$(T) \geq r$ is an open subset of all the space of all $m \times n$ matrices.
(b) Deduce that if $f : X \to Y$ is a C^∞ map, then the collection of $x \in X$ for which rank$(f'(x)) \geq r$ is open in X.
Problem 17: With notation as in the definition, prove that
(a)$Crit(f)$ is a closed subset of X.
(b) The set of regular values of f is open in Y *under the extra assumption that f is proper.*
Problem 18: Deduce the following statement from Ehresmann's theorem:
Let y be a regular value of a C^∞ map $f : X \to Y$ .Assume f is proper. Prove that there exists an nbhd U of y such that for all $y' \in U$, their inverse images $f^{-1}(y')$ are compact submanifolds of X that are diffeomorphic to each other *under the extra assumption that f is proper.*

Problem 19: Let X, Y, S be C^∞ manifolds. Let $f : X \times S \to Y$ be C^∞. For every $s \in S$ define $f_s : X \to Y$ by $f_s(x) = f(x, s)$. (We are thinking informally of C^∞ maps $f_s : X \to Y$ parametrized by $s \in S$).
Assume that X is compact.
(a) Let $U = \{s \in S : f_s \text{ is an immersion }\}$. Prove that U is open in S.
(b) Let $U' = \{s \in S : f_s \text{ is a submersion }\}$. Prove that U' is open in S.

Problem 20: With notation and assumptions as in the previous problem, show that $W = \{s \in S : f_s \text{ is an embedding }\}$ is open in S.
NOTE: Because X is compact, f_s is an embedding if it is (a) an immersion and (b) one-to-one.

Problem 21: Let $K \subset X \subset \mathbb{R}^n$. Both K and X are C^∞ submanifolds of $\mathbb{R}^n$, and in addition K is assumed to be compact. Prove that there is a pair (W, r) where
(i) $K \subset W \subset X$ and W is open in X, and
(ii) $r : W \to K$ is C^∞ and $r(x) = x$ for all $x \in K$.
Hint: First prove it for $X = \mathbb{R}^n$ using the tubular nbhd thm. and the deduce the general case from this special case.

Problem 22: Y is a C^∞ manifold and X is a C^∞ submanifold of $\mathbb{R}^n$.
$f : X \to Y$ is C^∞. Assume that y is a regular value of f (see defn. 2.13). Let $K = f^{-1}(y)$. Assume that K is compact.
(a) Is K a C^∞ submanifold? What is the relation between T_xK and $f'(x)$?
(b) We have the pair (W, r) from the preceding problem. Prove that $\phi : W \to K \times Y$ given by $\phi(x) = (r(x), f(x))$ induces an isomorphism $\phi'(x)$ of tangent-spaces for every $x \in K$.

Problem 23: With notation and assumptions as in the previous problem, show there is an open W' with $K \subset W' \subset W$ and a nbhd U of y in Y such that $\phi : W \to K \times Y$ restricts to a diffeomorphism $W' \to K \times U$.
Ehresmann's theorem has the properness of f in its hypothesis. The hypothesis of Problem 22 is implied by properness, but is weaker than it. Problem 23 shows how far you can go with this hypothesis. However, that W' is strictly contained in $f^{-1}(U)$ remains a possibility! The properness hypothesis is essential to complete the proof of Ehresmann's thm

Problem 24: Prove Ehresmann's theorem assuming that X is a C^∞ submanifold of $\mathbb{R}^n$.

2.4. **Proof of the tubular nbhd thm.**
The statement is given in Theorem 1.8, page 4, of these notes. We use the same notation: $Y \subset \mathbb{R}^n$ is a compact C^∞ submanifold, and
$N(Y) = \{(y, v) \in Y \times \mathbb{R}^n : \langle v, w\rangle = 0 \, \forall w \in T_yY\}$.

STEP 1: *Prove that the normal bundle $N(Y)$ is a C^∞ submanifold of $Y \times \mathbb{R}^n$. Show that the tangent-space of $N(Y)$ at $(y, 0)$ is $T_yY \times N_y(Y)$*

Proof. $N(Y) = \{(y, v) \in Y \times \mathbb{R}^n : \langle v, w\rangle = 0 \, \forall w \in T_yY\}$.
It suffices to show that $N(Y) \cap U \times \mathbb{R}^n$ is a C^∞ submanifold, for a collection of open U's that cover Y.
Therefore we choose an open $U \subset Y$ that is equipped with a diffeomorphism $f : \Omega \to U$ where $\Omega \subset \mathbb{R}^k$ is open.
Let $e_1, e_2, ..., e_k$ be the standard basis of $\mathbb{R}^k$. If $p \in \Omega$, then $f'(p)e_1, f'(p)e_2, ..., f'(p)e_k$ form a basis for $T_{f(p)}Y$.
(Other standard notation for $f'(p)e_i$ is $\frac{\partial f}{\partial x_i}(p) = (\frac{\partial f_1}{\partial x_i}(p), \frac{\partial f_2}{\partial x_i}(p), ..., \frac{\partial f_k}{\partial x_i}(p)) \in \mathbb{R}^n$)
Denoting by $g : U \to \Omega$ the inverse of the given f, we define $h : U \times \mathbb{R}^n \to \mathbb{R}^k$ by $h(y, v) = (\langle v, f'(g(y))e_1\rangle, \langle v, f'(g(y)e_2\rangle, ..., \langle v, f'(g(y))e_k\rangle)$ and observe that $N(Y) \cap U \times \mathbb{R}^n = h^{-1}(0)$. Therefore it suffices to show that h induces a surjection on tangent-spaces.
Note that the tangent-space of $U \times \mathbb{R}^n$ at (y, v) equals $T_yY \times T_v\mathbb{R}^n = T_yY \times \mathbb{R}^n$.
Fix $y \in U$. Note that $a \mapsto h(y, a)$ is a surjective linear transformation $\mathbb{R}^n \to \mathbb{R}^k$ with kernel$=N_yY$. It follows that $h'(y, a)(0, v) = h(y, v)$ for all $v \in \mathbb{R}^n$. This proves the desired surjectivity of $h'(y, a)$ for all $(y, a) \in U \times \mathbb{R}^n$. This completes the proof of the first assertion.

Now the tangent-space of at (y, v) of $N(Y)$ is $\ker h'(y, v)$. We have seen that $\ker h'(y, v) \cap 0 \times \mathbb{R}^n$ is precisely $0 \times N_y(Y)$. But we also have to consider the restriction of $h'(y, v)$ to $T_yY \times 0$. We specialise to the case $v = 0$. For this purpose, we consider the composite $y \mapsto h(y, 0)$ is a constant map and therefore its derivative is zero. It follows that $T_Y \times 0$ is contained in $\ker h'(y, 0)$. Therefore the tangent-space of $N(Y)$ at $(y, 0)$ is precisely $T_yY \oplus N_yY$. This completes the proof of Step 1.

□

WARNING: *Irrelevant for us, but the tangent-space of $N(Y)$ at $(y, v) \in N(Y)$ with $v \neq 0$ is NOT $T_yY \oplus N_yY$. It leads to the definition of the second fundamental form*

STEP 2: *We have $a : N(Y) \to \mathbb{R}^n$ given by $a(y, v) = y + v$.
Let $D = \{(y, v) \in N(Y) : a'(y, v) : T_{(y,0)}N(Y) \to \mathbb{R}^n$ is an isomorphism$\}$. Then D is an open subset of $N(Y)$ that contains $Y \times 0$. The inverse function theorem shows that $a|_D$ is an open map.*

Proof. The open-ness of D is left as an exercise.
By considering the restriction of a to $Y \times 0$ and to $0 \times N_yY$ separately, we see that $a'(y, 0)(w, 0) = w$ for all $w \in T_yY$ and $a'(y, 0)(0, v)$ for all $v \in N_yY$. Because $\mathbb{R}^n = T_yY \oplus N_yY$, we see that $a'(y, 0)$ is an isomorphism.

□

STEP 3: *Appeal to the compactness of Y (and the inverse function theorem once again) to complete the proof of the tubular nbhd theorem*

Proof. We wish to show that there is a positive constant c such that
a restricts to a diffeomorphism from $N_c(Y) = \{(y, v) \in N(Y) : \|v\| < c\}$
to $U_c(Y) = \{z \in \mathbb{R}^n : \inf\{\|y - z\| : y \in Y\} < c\}$.
That $a(N_c(Y))$ equals $U_c(Y)$ has been shown (see Problem 4).
The compactness of Y and the open-ness of the D of step 2 shows that D contains N_C for some $C > 0$. In particular the restriction of a to N_C is open.

We wish to find a positive c such that the restriction of a to N_c is one-to-one. If such a c does not exist, we have two sequences (y_n, v_n) and (y'_n, v'_n) in N such that
(i) $(y_n, v_n) \neq (y'_n, v'_n)$ for every $n \in \mathbb{N}$
(ii) $a(y_n, v_n) = y_n + v_n = y'_n + v'_n = a(y'_n, v'_n)$ for every $n \in \mathbb{N}$
(iii) both sequences v_n and v'_n approach zero as $n \to \infty$.

Th compactness of Y enables us to assume that $y_n \to y$ as $n \to \infty$, by replacing by a subsequence. Now (ii) shows that $y'_n \to y$ as well as $n \to \infty$. This shows that the restriction of a to every nbhd of $(y, 0)$ in $N(Y)$ is *not* injective, and that contradicts the inverse function theorem.

We have now found a positive c such that a restricts to an injective function on $N_c(Y)$. Taking $c \leq C$, we see that a is also open. The inverse function $a^{-1} : U_c \to N_c$ is C^∞ by the inverse function theorem. $\square$

3. Third week

3.1. Monday. Review of problems 21-23

3.2. Wednesday.

The tangent bundle and cotangent bundle as in Warner, page 19,
Both TM and T^*M are C^∞ $2n$-manifolds when M is a C^∞ n-manifold.

3.2.1. *Vector fields.*

A C^∞ section of TM is referred to as a "vector field on M"; it is a C^∞ function $v : M \to TM$ such that $p(v(x)) = x$, in other words $v(x) \in T_xM$ for all $x \in M$.

Given vector fields v, w on M and C^∞ functions $f, g : M \to \mathbb{R}$, we obtain the vector fields $fv + gw$ on M by pointwise scalar multiplication and addition:
$(fv + gw)(x) = f(x)v(x) + g(x)w(x) \in T_xM$ for every $x \in M$

For an open subset $\Omega \subset V$ where V is a finite dimensional vector space, we have seen that $T_v\Omega$ is canonically identified with V.
Thus a vector field v on Ω is a C^∞ function $\Omega \to V$.
When $V = \mathbb{R}^n$, we have the n vector fields $\partial_i = \frac{\partial}{\partial x_i}$, which when evaluated at a point $a \in \Omega$, give a basis of $T_a\Omega$.
Every vector field v on $\Omega \subset \mathbb{R}^n$ is uniquely expressed as $f_1\partial_1 + f_2\partial_2 + ... + f_n\partial_n$ where the $f_i : \Omega \to \mathbb{R}$ are C^∞.

3.2.2. *Differential forms or one-forms.* ω on M are C^∞ sections of T^*M:

i.e. $\omega : M \to T^*M$ is a C^∞ map with the property that $\omega(x) \in T^*_xM$ for every $x \in M$.
A one-form ω on $\Omega \subset \mathbb{R}^n$ is C^∞ if $\omega(\partial_i) : M \to \mathbb{R}$ is C^∞ for every $i = 1, 2, ..., n$.
Given one-forms ω, η on M and C^∞ functions $f, g : M \to \mathbb{R}$ we obtain the one-form $f\omega + g\eta$ on M.

Definition 3.1. A C^∞ function $f : M \to \mathbb{R}$ gives rise to the differentlal form df on M, defined as follows:
First fix a point $x \in M$.
Recall that T_xM is the collection of linear functionals L on the space of germs $C^\infty_x(M; \mathbb{R})$ that satisfies

$$L(uv) = u(x)L(g) + v(x)L(u) \tag{1}$$

for all germs u, v.
Abusing notation, denoting the germ of f at x by f itself, $L \mapsto L(f)$ gives a linear functional on T_xM, i.e. an element of T^*_xM which we denote by $df(x)$.
We have defined $df(x) \in T^*xM$ for every $x \in M$ and this defines the *section* df of T^*M. That it is C^∞ will be seen in part (iv) of the example below. From equation (1) and the linearlty of the L's in T_xM we deuce

$$d(fg) = fdg + gdf \text{ and } d(f + g) = df + dg \tag{2}$$

for all C^∞ functions $f, g : M \to \mathbb{R}$.

Example 3.2. Let $\Omega \subset \mathbb{R}^n$ be open. Let $f : \Omega \to \mathbb{R}$ be C^∞. Then $df(\partial_i) = \partial_i(f)$ (by the definition of df).
(i) Because the $\partial_i f$ are also C^∞ we see that df is C^∞.
(ii) The x_i are then C^∞ functions $\Omega \to \mathbb{R}$. By the above definition, we get one-forms dx_i on M. Plugging $f = x_i$ in (i) we see that $dx_j(\partial_i) = \delta_{i,j}$.
This shows that $dx_1, dx_2, ..., dx_n$ is the dual basis of $T_a^*\Omega$ produced by the basis $\partial_1, ..., \partial_n$ of $T_x\Omega$
(iii) We deduce $df = (\partial_1 f)dx_1 + ... + (\partial_n f)dx_n$ from (i) and (ii).
(iv) Verification of "df is C^∞ on a smooth manifold M" is local–therefore it follows from (i).

Remark 3.3. Every one-form ω on Ω open in $\mathbb{R}^n$ can be uniquely expressed as $g_1 dx_1 + ... + g_n dx_n$ where all the g_i are C^∞ functions on Ω.

Problem 25: Is it true that every one-form on $\mathbb{R}$ equals df for some C^∞ function $f : \mathbb{R} \to \mathbb{R}$?
Problem 26: Is it true that every one-form on $\mathbb{R}^2$ equals df for some C^∞ function $f : \mathbb{R}^2 \to \mathbb{R}$?
Hint: $\partial_1\partial_2 = \partial_2\partial_1$
Problem 27: Let $a \in M$. Assume that C^∞ functions $f_1, f_2, ..., f_k : M \to \mathbb{R}$ have the property that $df_1(a), , ..., df_k(a) \in T_a^*M$ are linearly independent. Define $f = (f_1, f_2, ..., f_k) : M \to \mathbb{R}^k$. Prove that there is a nbhd U of a in M such that $f|_U : U \to \mathbb{R}^k$ is a submersion.
Problem 28: $f_1, ..., f_n : M \to \mathbb{R}$ are C^∞, and M is a C^∞ n-manifold. We have $f = (f_1, ..., f_n) : M \to \mathbb{R}^n$. Let $a \in M$. Prove that there is some neighborhood U of a in M such that $f|_U$ gives a chart $\iff$ the cotangent-space T_a^*M has $df_1(a), ..., df_n(a)$ as a basis.

3.2.3. *Friday.*

Definition 3.4. Let $f : X \to Y$ be a C^∞ map of C^∞ manifolds. Given ω is a one-form on Y, its pull-back $f^*\omega$ is the one-form on X defined by

$$f^*(\omega)(x) = f'(x)^*\omega(f(x)) \text{ i.e.} f^*(\omega)(x)(L) = \omega(f(x)) \circ f'(x)L \text{ for all } L \in T_xX$$

Lemma 3.5. *With f, X, Y as in the above defn, $f^*(d\varphi) = d(f^*\varphi)$ for every C^∞ function $\varphi : Y \to \mathbb{R}$*

Example 3.6. Given a 1-form ω on Y and a path γ in Y, i.e. a smooth $\gamma; [a, b] \to Y$, then we have the 1-form $\gamma^*(\omega)$ on $[a, b]$. Thus $\gamma^*(\omega) = \psi(t)dt$ where $\psi : [a, b] \to \mathbb{R}$ is C^∞. $\int_\gamma \omega$ is then *defined as* $\int_a^b \gamma^*\omega$ which in turn is the Riemann integral $\int_a^b \psi(t)dt$. Now take $\omega = d\phi$. the above lemma (together with the fundamental theorem of calculus) shows that $\int_\gamma d\phi = \phi(\gamma(b)) - \phi(\gamma(a))$

Problem 29: $\omega = (x^2 + y^2)^{-1}(xdy - ydx)$ is a one-form defined on the complement of the origin in the plane. Is there a real-valued function ψ defined on this region such that $\omega = d\psi$?
Problem 30: If $d\varphi = 0$ where $\varphi : B \to \mathbb{R}$ is C^∞ and B is an open ball in $\mathbb{R}^n$, prove that φ is a constant function.
Problem 31:
(a) What hypothesis on a C^∞ manifold M is required for the truth of the statement "If $d\varphi = 0$ where $\varphi : M \to \mathbb{R}$ is C^∞, then φ is a constant function ?"
(b) Give an example to show that the condition you stated in (a) is necessary.
(c) Show that the condition you stated in (a) is sufficient.

3.2.4. *The exterior algebra of the dual vector space.*
WARNER, Chapter 2, page 56, Defn. 2.5 defines $A_r(V)$ for any real vector space V.
For a *finite dimensional* vector space V over $\mathbb{R}$, we defined $\Lambda^r(V^*) = A_r(V)$.
NOTE: For $r = 1$, we get $\Lambda^1(V^*) = V^*$.
Given $\omega \in \Lambda^r(V^*)$ and $\eta \in \Lambda^s(V^*)$, their wedge product $\omega \wedge \eta \in \Lambda^{r+s}$ was defined
The following properties were clear from the definition:
$\omega \wedge \eta = (-1)^{rs}\eta \wedge \omega$ with notation as above
(A) The wedge product defined above is left and right distributive: equivalently the resulting map

$$\Lambda^r(V^*) \times \Lambda^s(V^*) \to \Lambda^{r+s}(V^*)$$

is bilinear.
(B) Associativity of the wedge product was stated but not proved
(C) If $v_1, v_2, ..., v_n$ form a basis for V and if $e_1, e_2, ..., e_n$ is the dual basis for V^*, it was shown that the set $\{e_{i_1} \wedge ... \wedge e_{i_r} | 1 \le i_1 < i_2 < ... < i_r \le n\}$ forms a basis for $\Lambda^r(V^*)$.

THREE MORE PROBLEMS: Warner, chapter two exercises:9,10 on page 78 and exrecise 15, page 80. The validity of these problems do not require the finite dimensionality of V, but you may assume that V IS finite dimensional.

4. Fourth week

Chapter 2 of Warner in progress.
Anything that involves vector fields (e.g. Lie derivatives) will be postponed.

Definition 4.1. A k-form ω on a n-manifold assigns to every point $a \in M$, a member $\omega(a) \in \Lambda^k(T_a^*M)$ "in a C$^\infty$ manner" made precise in the following manner:
If $v_1, v_2, ..., v_k$ are C$^\infty$ vector fields defined on an open subset $U \subset M$, then $a \mapsto \omega(a)(v_1(a), v_2(a), ..., v_k(a))$ gives a C$^\infty$ function $U \to \mathbb{R}$.
This was not the definition given in class on Monday, but can you check that is is equivalent to it?
NOTATION $E^k(M)$ is the collection of all C$^\infty$ k-forms on M. A "differential form of degree k on M " is the same as a k-form on M for some $k \geq 0$

4.1. The exterior derivative.

We consider maps $d^k(M) : E^k(M) \to E^{k+1}(M)$ for all n-manifolds M and for all $k \geq 0$. We list some desirable properties of these maps.
(P1) $d^k(M) : E^k(M) \to E^{k+1}(M)$ is a linear transformation for all $k \geq 0$.
(P2) $d^0(\phi) = d\phi$ and $d^1 d^0 \phi = 0$ for all $\phi \in E^0(M)$
(P3) $d^{p+q}(\omega \wedge \eta) = d^p\omega \wedge \eta + (-1)^p \omega \wedge d^q\eta$ for all $\omega \in E^p(M)$ and $\eta \in E^q(M)$
(P4) $d^{k+1}d^k\omega = 0$ for all $\omega \in E^k(M)$ for all $k \geq 0$.
(P5) $f^* d^k(N)\omega = d^k(M)(f^*\omega)$ for all $\omega \in E^k(N)$ and for all $k \geq 0$ and for all C$^\infty$ maps $f : M \to N$
(P5') Same as (P5) but only in cases (1) f is a diffeomorphism or (2) M is an open subset of N and $f(x) = x$ for all $x \in M$.

NOTATION: Every subset $I \subset \{1, 2, ..., n\}$ is uniquely written as$\{1 \leq i_1 < i_2 < ... < i_k\}$. dx_I is the k-form on $\mathbb{R}^n$ given by

$$dx_I = dx_{i_1} \wedge ... \wedge dx_{i_k}$$

LEMMA 1 There is a unique system of $d^k(M)$ satisfying P1,P2,P3 when M is an open subset of $\mathbb{R}^n$.

Proof. P1 implies $d^0 f = df$ and $d^1(df) = 0$ for all $f \in E^0(M)$. In particular, $d^1(dx_i) = 0$ for all i.
Use P3 to show $d^k(dx_I) = 0$ for subset $I \subset \{1, 2, ..., n\}$ of cardinality$k$ (where $k > 0$).
Every $\omega \in E^k(M)$ equals $\Sigma_I \phi_I dx_I$ where the ϕ_I are C$^\infty$ functions on M.
Apply P3 again to deduce

$$d^k \Sigma_I \phi_I dx_I = \Sigma_I d\phi_I \wedge dx_I \tag{3}$$

This proves the uniqueness. To complete the proof, we have no option but to verify that the system of d^k given by the formula in equation (3) satisfies P1,P2,P3. P1 is clear,
P2 is equivalent to $\partial_i \partial_j = \partial_j \partial_i$ and
P3 holds for $p = q = 0$ and one may deduce the general case of P3 from this fact. ☐

The uniqueness in lemma 1 implies
COROLLARY: If $f : M_1 \to M_2$ is a diffeomorphism , then$f^* d^k(M_2)\omega = d^k(M_1) f^*\omega$ for all $\omega \in E^k(M_2)$ and all $k \geq 0$.

LEMMA 2: There is a unique system of $d^k(M), k \geq 0$, defined for all C^∞ manifolds that satisfies P1,P2,P3,P5'.
Quick sketch:
(A) Lemma 1 gives existence and uniqueness for all open subsets of $\mathbb{R}^n$.
(B): Case (1) of P5' plus the Corollary of lemma 1 now give existence of uniqueness for all manifolds diffeomorphic to an open subset of $\mathbb{R}^n$.
(C) Next, let's check uniqueness and existence in the general case: Let $\omega \in E^k(M)$. Let $\mathcal{U}$ be the collection of all open subsets that are diffeomorphic to open subsets of $\mathbb{R}^n$.
By (B), $d^k(U)(\omega_| U)$ is defined for all $U \in \mathcal{U}$.
Let $U, V \in \mathcal{U}$. Then $W = U \cap V \in \mathcal{U}$. By case (2) of P5', we see that the restrictions of both $d^k(U)(\omega|_U)$ and $d^k(V)(\omega|_V)$ to W are the same as $d^k(W)(\omega|_W)$ (and in particular they are equal to each other). By the sheaf property, we obtain a unique element $\alpha \in E^{k+1}(M)$ such that $\alpha|_U = d^k(U)(\omega|_U)$ for every $u \in \mathcal{U}$. Case (2) of P5' shows that if a system of $d^k(M)$ for all k, M then $d^k(M)\omega = \alpha$. Thus we *define* $d^k(M)\omega = \alpha$.
(D) It only remains to show that this definition of $d^k(M)$ satisfies P1,P2,P3. But these properties hold locally (e.g. for all $U \in \mathcal{U}$, from which we deduce their validity on M.

LEMMA 3: The system of $d^k(M), k \geq 0$ defined by lemma 2 satisfies P4.
This can be deduced directly from $\partial_i \partial_j = \partial_j \partial_i$ but it is good to note the identity given in the formal proof below.

Proof. Let $D^k = d^{k+1}d^k : E^k(M) \to E^{k+2}(M)$. Apply P3 twice to show

$$D^{p+q}\omega \wedge \eta = D^p\omega \wedge \eta + \omega \wedge D^q\eta \text{ for all } \omega \in E^p(M) \text{ and } \eta \in E^q(N)$$

Taking $\omega = \phi$ with $p = 0$ and $\eta = dx_I$ in the above where M is open in $\mathbb{R}^n$ we deduce that $D^q(\phi dx_I) = 0$ Taking linear combinations, we see that $d^{q+1}d^q(\eta) = 0$ for all q-forms on open subsets of $\mathbb{R}^n$. P5' does the rest.

(The proof could be much shorter, but it is good to note that the square D of an antiderivation d is a derivation).

□

From now on, $d^k(M)$ will be abbreviated to d
LEMMA 4: The system of $d^k(M), k \geq 0$ defined by lemma 2 satisfies P5

Proof. Let $\phi \in E^0(N)$. P5 has been proved when $\omega = \phi$ (page 15 of these notes, Wednesday 3rd week). Check that P5 is valid when $\omega = d\phi$ as well (both sides of the desired equality are zero invoking P2).
Now use P3 to deduce that P5 holds for all ω are sums of wedge products of various ϕ' s and $d\phi$'s.
But every ω is expressible in this manner on open subsets of $\mathbb{R}^n$, and therefore on an open cover $U_i, i \in I$, of N.
The desired equality, namely $f^*d\omega = d)(f^*\omega)$, holds when restricted to $f^{-1}(U_i)$ for all i. These open subsets cover M, so we deduce LHS=RHS.

□

Definition 4.2.
$Z^k(M) = \{\omega \in E^k(M) : d\omega = 0\}$ is the vector space of **closed** k-forms on M.
$B^k(M) = \{d\eta : \eta \in E^{k-1}(M) = 0$ is the vector space of **exact** k-forms on M.
P4 implies: $0 \subset B^k(M) \subset Z^k(M) \subset E^k(M)$.
$\mathrm{H}^k_{DR}(M) = Z^k(M)/B^k(M)$ is the **De Rham cohomology** of M. This is the quotient of a real vector space by a linear subspace. Thus $\mathrm{H}^k_{DR}(M)$ is a vector space over the real numbers.

Remark 4.3. Given $f : M \to N$ we get $f^* : E^k(N) \to E^k(M)$. Thanks to P5, we see that $f^*(Z^k(N) \subset Z^k(M)$ and $f^*(B^k(N)) \subset B^k(M)$. Thus f^* induces a linear transformation on quotient spaces as well, i.e. we have $f^* : \mathrm{H}^k_{DR}(N) \to \mathrm{H}^k_{DR}(M)$.

Problem 32: If M is a nonempty connected manifold, then $\mathrm{H}^0_{DR}(M) = Z^0(M) = \mathbb{R}$ (constant functions).
Problem 33: Provide diffeomorphisms (i) $(0, \infty) \to \mathbb{R}$, (ii) $(a, b) \to (0, \infty)$, (iii) $(a, b) \to \mathbb{R}$
Problem 34: Provide diffeomorphisms (i) $\Pi_{i=1}^{i=n}(a_i, b_i) \to \mathbb{R}^n$, (ii) open balls to $\mathbb{R}^n$, and (iii) from $\Pi_{i=1}^{i=n}(a_i, b_i)$ to open balls.
POINCARE LEMMA: Warner page 55.
In class, we employed integration in the first variable to deduce the Poincare lemma – in fact we proved the proposition below which proves the Poincare lemma by induction on dimension.

Proposition 4.4. *Let $\pi : \mathbb{R} \times M \to M$ denote the projection. Then $\pi^* : \mathrm{H}^k_{DR}(M) \to \mathrm{H}^k_{DR}(\mathbb{R} \times M)$ is an isomorphism for all $k \geq 0$.*

Proof. Let $i_0 : M \to \mathbb{R} \times M$ be the inclusion $i_0(p) = (0, p)$. Because $\pi(i_0(p)) = p$ for all $p \in M$ we deduce that the composite

$$\mathrm{H}^k_{DR}(M) \xrightarrow{\pi^*} \mathrm{H}^k_{DR}(\mathbb{R} \times M) \xrightarrow{i_0^*} \mathrm{H}^k_{DR}(M)$$

is also the identity. It follows that $\pi^* : \mathrm{H}^k_{DR}(M) \to \mathrm{H}^k_{DR}(\mathbb{R} \times M)$ is an **injection**.
So it remains to prove that π^* induces a **surjection** on De Rham cohomology. In other words, it has to be proved that
if ω is a closed k-form on $\mathbb{R} \times M$, then there is a $(k-1)$ form η on $(\mathbb{R} \times M)$ and a closed k-form θ on M such that $\omega = d\eta + \pi^*(\theta)$. Denoting by $cl(\omega)$ and $cl(\theta)$ the corresponding elements of $\mathrm{H}^k_{DR}(\mathbb{R} \times M)$ and $\mathrm{H}^k_{DR}(M)$ respectively, the equation of k-forms stated above implies that $\pi^* cl(\theta) = cl(\omega)$.

In what follows, we will use the notation $x \in \mathbb{R}$ and $p \in M$. Recall that $\nu \in E^k(\mathbb{R} \times M)$ is a section of Λ^k of the cotangent-bundle, i.e. $\nu(x, p) \in \Lambda^k(T^*(\mathbb{R} \times M)_{(x,p)}$.
The projection $\pi : \mathbb{R} \times M \to M$ yields the inclusion of $T^*_p M$ in $T_{(x,p)}(\mathbb{R} \times M)$ and therefore induces the inclusion of $\Lambda^k T^*_p M$ in $\Lambda^k T^*_{(x,p)}(\mathbb{R} \times M)$.
We define $F^k(\mathbb{R} \times M) = \{\nu \in E^k(\mathbb{R} \times M) : \nu(x, p) \in \Lambda^k T^*_p M$ for all $(x, p) \in \mathbb{R} \times M\}$
The claims 1,2,3 below are' local on M' and thus it suffices to check them in the case when M is open in $\mathbb{R}^n$
CLAIM 1: Every $\omega \in E^k(\mathbb{R} \times M)$ equals $dx \wedge \alpha + \beta$ for a unique choice of $\alpha \in F^{k-1}(M \times R)$ and $\beta \in F^k(M \times R)$.
Let us concentrate on the α. First fix $p \in M$. Then we obtain the function

$\mathbb{R} \to \Lambda^{k-1}T_p^*M$ given by $x \mapsto \alpha(x,p)$. Thus we may integrate this function from 0 to x to define $\eta(x,p) = \int_0^x \alpha(t,p)dt \in \Lambda^{k-1}T_p^*M$. This yields $\eta \in F^{k-1}(\mathbb{R} \times M)$.
CLAIM 2: With ω, α, β as in Claim 1, and η as above, put $\omega_1 = \omega - d\eta$. The $\omega_1 \in F^k(\mathbb{R} \times M)$.
By Claim 1, we have $d\omega_1 = dx \wedge \alpha_1 + \beta_1$. Note that $\omega_1 \in F^k$, thus for a fixed $p \in M$, we have the function $\mathbb{R} \to \Lambda^k T_p^*M$ given by $x \mapsto \omega_1(x,p)$ and its derivative
CLAIM 3: $\frac{d}{dx}\omega_1(x,p) = \alpha_1(x,p)$
(The proof of claim 3 is similar to that of claim 2. It would be more elegant to formulate both those claims into a single statement)
Now assume that ω is a closed form. Then $\omega_1 = \omega - d\eta$ is also closed. In particular, $\alpha_1 = 0$. By claim 3, this says:
for every $p \in M$, the function $x \mapsto \omega_1(x,p)$ is a constant function $\mathbb{R} \to \Lambda^k T_p^*M$. This proves Claim 4 below.
CLAIM 4: There is a (unique) k-form θ on M such that $\omega_1 = \pi^*\theta$.
$\pi^*(d\theta) = d\pi^*\theta = d\omega_1 = 0$, and this says that $d\theta = 0$ (WHY?)
This finishes the proof of the surjectivity of $\pi^* : \mathrm{H}^k_{DR}(M) \to \mathrm{H}^k_{DR}(\mathbb{R} \times M)$

□

Problem 35:
(i) Show that ω and η are closed forms of degrees p and q on M respectively, then $\omega \wedge \eta$ is also a closed $(p+q)$-form on M.
(ii) Assume in addition that ω or η is an exact form. Show that $\omega \wedge \eta$ is exact.
(iii) Combine (i) and (ii) to define a product $\mathrm{H}^p_{DR}(M) \times \mathrm{H}^q_{DR}(M) \to H^{p+q}_{DR}(M)$.
In other words, if ω, ω' are closed p-forms that differ by an exact form,
and if η, η' are closed q-forms that differ by an exact form,
then show that $\omega \wedge \eta$ and $\omega' \wedge \eta'$ differ by an exact form.

Problem 36: Compute $\mathrm{H}^q_{DR}(S^1)$ for $q = 0, 1$ in the following manner. Regarding S^1 as complex numbers of absolute value one, we have $\mathbb{R}/Z \to S^1$ given by $t \mapsto \exp(2\pi i t)$. This identifies $E^0(S^1)$ with C^∞ functions $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(t+1) = \phi(t)$ for all $t \in \mathbb{R}$.
Now the function $t \mapsto t$ is not periodic, but its differential dt is periodic. Thus $dt \in E^1(S^1)$ is a nowhere vanishing one-form.
Every $\omega \in E^1(S^1)$ is uniquely expressed as $\phi(t)dt$ where $\phi(t) \in E^0(S^1)$.
Problem 37: Compute the De Rham cohomologies of $M = S^1 \times S^1$ using the following identifications.
$E^0(M)$ is identified withthe collection of C^∞ functions $\phi : \mathbb{R}^2 \to R$ such that $\phi(t_1 + m_1, t_2 + m_2) = \phi(t_1, t_2)$ for all $(m_1.m_2) \in \mathbb{Z}]times\mathbb{Z}$.
Every $\omega \in E^1(M)$ is expressed uniquely as $\phi_1 dt_1 + \phi_2 dt_2$ where $\phi_1, \phi_2 \in E^0(M)$
Every $\omega \in E^2(M)$ is expressed uniquely as $\phi dt_1 \wedge dt_2$ where $\phi \in E^0(M)$.

4.2. **Compactly supported k-forms.**

Definition 4.5. The **support** of a k-form ω on a n-manifold M, denoted by suppω, is the closure of the open set $\{x \in M : \omega(x) \neq 0\}$. If $U \subset M$ is an open subset , then

$$\omega|_U = 0 \iff U \cap \operatorname{supp}(\omega) = 0$$

Extension by zero Let ω be a k-form on an open subset U of M. Assume that $F = \operatorname{supp}(\omega)$ is a *closed subset of* M. Then there is a unique k-form ω_e on M such that $\omega_e|_U = \omega$ and $\operatorname{supp}(\omega_e) = \operatorname{supp}(\omega)$.

Proof. If M is covered by its open sets U_1 and U_2 and we have k-forms ω_i on U_i for $i = 1, 2$ such that the restrictions of both ω_1 and ω_2 to $U_1 \cap U_2$ agree with each other, then we get a unique k-form η on M such that $\eta|_{U_i} = \omega_i$ for $i = 1, 2$.
We take $U_1 = U$, $\omega_1 = \omega$, $U_2 = M \setminus F$, $\omega_2 = 0$. Both ω_i restrict to zero on $U_1 \cap U_2$. We denote the resulting η by ω_e. $\square$

$E^k_c(M) = \{\omega \in E^k(M) : \operatorname{supp}(\omega) \text{ is compact}\}$ is a linear subspace of $E^k(M)$, and is referred to as the space of *compactly supported forms on* M.
One of the many reasons why the Hausdorffness assumption on M is important because *compact subsets of M are closed*:
In particular, if $\omega \in E_c(U)$ and U is open in M, then $\operatorname{supp}(\omega)$ is *closed in* M and we obtain $\omega_e \in E^k_c(M)$.
NOTE: $E^k(M) \to E^k(U)$ given by $\omega \mapsto \omega|_U$ is a special case of "pullback", whereas $\omega \mapsto \omega_e$ gives an inclusion $E^k_c(U) \hookrightarrow E^k_c(M)$ in the opposite direction.

4.3. **Orientation of Manifolds.** See Warner, Chapter 4.

Definition 4.6. (i) Let M be a C^∞ n-manifold. A $\omega \in E^n(M)$ is *nowhere vanishing* if for every $x \in M$, the element $\omega(x) \in \Lambda^n T^*_x M$ is nonzero.
(ii) If ω, ω' are both nowhere anishing n-forms on M then we have a $\phi : M \to \mathbb{R} \setminus \{0\}$ such that $\omega' = \phi\omega$. If $\phi(x) > 0$ for all $x \in M$, then ω and ω' are *compatible* with each other. Compatibility of nowhere vanishing n-forms on M is an equivalence relation.
(iii) An *orientation* of M is (an equivalence class of) a nowhere vanishing n-form on M. We will refer to one such as "an orientation form on M" and often denote it by θ_M or siimply by θ.
(iv) An open subset U of an oriented manifold (M, θ_M) acquires the structure of an oriented manifold by setting $\theta_U = \theta_M|_U$.
(v) Given oriented n-manifolds M and N, C^∞ map $f : M \to N$ is *orientation preserving* if $f^*\theta_N$ is compatible with θ_M.
(vi) Every open $\Omega \subset \mathbb{R}^n$ acquires its standard orientation: $dx_1 \wedge dx_2 \wedge ... \wedge dx_n$.

See WARNER, page 139, Prop.,4.2. This was proved under the (unnecessary) assumption that M is compact
Stokes theorem I, page 144 of WARNER, you have seen before in an earlier course.
Stokes theorem, second form, has been postponed.

5. Fifth week

5.1. **Partitions of Unity.** Everybody should read pages 8-11 of Warner's book for Partitions of Unity, even though we employ it in a special case, and prove it there (under a compactness assumption). The C^∞ function $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t) = \exp(-1/t) \text{ for } t > 0 \text{ and } \phi(t) = 0 \text{ for } t \leq 0$$

has the following application:

Corollary 5.1. *Let $x \in U \subset M$ where U is open and M is a C^∞ manifold. Then there is a C^∞ function $\phi : M \to \mathbb{R}$ such that*
(i) $\phi(x) > 0$, (ii) $\operatorname{supp}(\phi) \subset U$, and (iii) $\phi(y) \geq 0$ for all $y \in M$.

Remark 5.2. The existence of a C^∞ partition of unity subordinate to a given finite cover of a compact manifold is a simple consequence of Cor.5.1 ans was proved in class.

5.2. **Integration of compactly supported n-forms on oriented n-manifolds.**

Definition 5.3. (i) Let $\omega \in E^n_c(\mathbb{R}^n)$. Then $\omega = \phi dx_1 \wedge ... \wedge dx_n$ where $\phi \in E^0_c(\mathbb{R}^n)$. Then $\int_{\mathbb{R}^n} \omega$ is defined to be the Riemann integral $\int_{\mathbb{R}^n} \phi(x_1, ..., x_n) dx_1 dx_2 ... dx_n$.
(ii) Let $\omega \in E^n_c(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is open. We then have $\omega_e \in E^n_c(\mathbb{R}^n)$,
and $\int_\Omega \omega$ is defined to be $\int_{\mathbb{R}^n} \omega_e$.

Proposition 5.4. *(Change of Variables formula) Let $f : \Omega \to \Omega'$ be an orientation preserving diffeomorphism of open subsets of $\mathbb{R}^n$ (i.e. a diffeomorphism that satisfies $\det f'(x) > 0$ for every $x \in \Omega$). Let $\omega \in E^n_c(\Omega')$. Then $\int_{\Omega'} \omega = \int_\Omega f^*\omega$*

Proof. Proof hopefully seen earlier. We assume it and proceed. □

Proposition 5.5. *Let M be an oriented C^∞ (second countable Hausdorff) n-manifold. There is a unique linear functional $\int_M : E^n_c(M) \to \mathbb{R}$ that has the following property: For every orientation preserving diffeomorphism $f : \Omega \to U$, where $\Omega \subset \mathbb{R}^n$ and $U \subset M$ are both open, and for every $\omega \in E^n_c(M)$ with $\operatorname{supp}(\omega) \subset U$, we have: $\int_M \omega = \int_\Omega f^*\omega$*

Proof. This was proved in class only when M itself is **compact**. That proof is reproduced below.
M is covered by open subsets U_i for where there are orientation preserving diffeomorphisms $f_i : \Omega_i \to U_i$, where the $\Omega_i \subset \mathbb{R}^n$ are open.
The compactness assumption on M permits us to assume that $i = 1, 2, 3, ..., m$.
Choose a C^∞ partition of unity $\phi_1, ..., \phi_m$ subordinate to the given cover $U_1, ..., U_m$.
Let $\omega \in E^n(M)$. Then $\operatorname{supp}(\phi_i\omega) \subset U_i$. Define

$$I(\omega) = \sum_{i=1}^{i=m} \int_{\Omega_i} f_i^*(\phi_i\omega)$$

Assume there is a linear functional $\int_M$ satisfying the property stated in the proposition. We will show first that $\int_M \omega = I(\omega)$.
Because $\phi_i\omega$ has its support contained in U_i , we see that

$$\int_M \phi_i\omega = \int_{\Omega_i} f_i^*(\phi_i\omega) \tag{4}$$

Summing equation(4) over $i = 1, 2, ..., m$ we see that

$$\int_M \omega = \sum_{i=1}^{i=m} \int_M \phi_i \omega = \sum_{i=1}^{i=m} \int_{\Omega_i} f_i^*(\phi_i \omega) = I(\omega)$$

It remains to show that if $f : \Omega \to U$ is an orientation preserving diffeomorphism and if $\omega \in E^n(M)$ has supp$(\omega) \subset U$, then

$$I(\omega) = \int_\Omega f^* \omega \tag{5}$$

It suffices to prove, for every $i = 1, 2, 3, ..., m$ that

$$\int_{\Omega_i} f_i^*(\phi_i \omega) = \int_\Omega f^*(\phi_i \omega) \tag{6}$$

because the LHS (resp. RHS) of equation (5) is the sum of the LHS's (resp. RHS's) of equation (6) taken over $i = 1, 2, ..., m$.
(6) however is a special case of (change of variables) Prop.5.4 in the following manner.
(i) Let $V_i = f_i^{-1}(U_i \cap U)$. Then f_i restricts to an orientation preserving diffeomorphism $a_i : V_i \to U_i \cap U$.
(ii) Let $W_i = f^{-1}(U_i \cap U)$. Then f restricts to an orientation preserving diffeomorphism $b_i : W_i \to U_i \cap U$.
(iii) The form $\phi_i \omega$ has support contained in $U_i \cap U$. Therefore we obtain
(iii') a (compactly supported) n-form $a_i^*(\phi_i \omega)$ on V_i and
(iii") a (compactly supported) n-form $b_i^*(\phi_i \omega)$ on W_i, and an orientation preserving diffeomorphism $h_i := b_i^{-1} a_i : V_i \to W_i$ such that $h_i^*(b_i^*(\phi_i \omega)) = a_i^*(\phi_i \omega)$.
The change of variables formula proves the equality of the two middle terms below

$$\int_{\Omega_i} f_i^*(\phi_i \omega) = \int_{V_i} a_i^*(\phi_i \omega) = \int_{W_i} b_i^*(\phi_i \omega) = \int_\Omega f^*(\phi_i \omega)$$

□

Problem 38: Modify the proof to cover the proof of the proposition in all generality.
Problem 39: Prove that if M is a nonempty oriented n-manifold, there is a compactly supported $\omega \in E_c^n(M)$ such that $\int_M \omega = 1$. *(In particular, this is true when M is replaced by a nonempty open subset of U).*
Problem 40: If $\eta \in E_c^{n-1}(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} d\eta = 0$. This simple observation has the corollary stated below.

Proof. Proved in class. For $n = 1$ this just says that $\int_{\mathbb{R}} \phi'(x)dx = 0$ when $\phi : \mathbb{R} \to \mathbb{R}$ is compactly supported. □

Corollary 5.6. *$\int_M d\eta = 0$ where $\eta \in E^{n-1}(M)$ and M is a compact oriented n-manifold. Thus $\int_M : E^n(M) \to \mathbb{R}$ gives rise to a linear functional on $H_{DR}^n(M) = E^n(M)/dE^{n-1}(M) \to \mathbb{R}$; by abuse of notation, this linear functional is also denoted $\int_M : \mathrm{H}_{DR}^n(M) \to \mathbb{R}$.*

Theorem 5.7. *The linear functional $\int_M : \mathrm{H}_{DR}^n(M) \to \mathbb{R}$ of the corollary is an isomorphism when M is connected nonempty oriented.*

Proof. The proof of the theorem does rely on proposition 5.9 below: If $\omega \in E^n_c(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \omega = 0$, then there is some $\eta \in E^{n-1}_c(\mathbb{R}^n)$ satisfying $\omega = d\eta$. We postpone the proof of this statement and proceed.
Quick sketch: it was covered in elaborate detail in class.
We consider open subsets $W \subset M$ which have the property:
(P)$\omega \in E^n_c(W)$ *and* $\int_M \omega_e = 0$ *implies* $\omega = d\eta$ *for some* $\eta \in E^{n-1}_c(W)$'
CLAIM: If W_1, W_2 have (P), and if $W_1 \cap W_2$ is nonempty, then $W_1 \cup W_2$ also has (P).
Employing partitions of unity, express ω as the sum $\omega_1 + \omega_2$ where $\omega_i \in E^n_c(W_i)$ for $i = 1, 2$.
Choose $\alpha \in E^n_c(W_1 \cap W_2)$ such that $\int_M \alpha_e = 1$.
Let $c_i = \int_M \omega_i$. Thus the integral of $\omega_i - c_i\alpha_i$ is zero, and because this is a compactly supported form on the open set W_i which has the property in question, we have $\eta_i \in E^{n-1}_c(W_i)$ such that $d\eta_i = \omega_i - c_i\alpha$. Noting that $c_1 + c_2 = \int_M \omega = 0$ we see that $\eta = \eta_1 + \eta_2$ is a compactly supported form on $W_1 \cup W_2$ for which $d\eta = \omega$.
Next, cover M by open subsets U_i, for $i = 1, 2, ..., m$ such that each U_i is diffeomorphic to $\mathbb{R}^n$. Employing the connectedness of M we see that we may renumber these open subsets so that $V_h = U_1 \cup U_2 \cup ... \cup U_h$ is connected (WHY?).
We prove by induction that V_h has (P). All the U_i have (P) by the proposition below.
In particylar, the case $h = 1$ has been covered.
For $h > 1$, put $V_{h-1} = W_1$ and W_2. The inductive hypothesis shows both of the W_i have (P). The connectedness of V_h implies that $W_1 \cap W_2$ is nonempty. The claim now implies that $V_h = W_1 \cup W_2$ also has (P). Taking $h = m$, we see that M itself has (P).
□

Definition 5.8. $Z^k_c(M) = \{\omega \in E^k_c(M) : d\omega = 0$ and $B^k_c(M) = \{d\eta : \eta \in E^{k-1}_c(M)\}$.
We have the inclusions $0 \subset B^k_c(M) \subset Z^k_c(M) \subset E^k_c(M)$.
The **compactly supported** k**-th De Rham cohomology of** M, denoted by $\mathrm{H}^k_{DR,c}(M)$, is the vector space $Z^k_c(M)/B^k_c(M)$.

Problem 41: $f : M \to N$ be a C^∞ map of C^∞ manifolds.
(a) Give an example of M, N, f and $\omega \in E^k_c(N)$ such that $f^*\omega$ is NOT compactly supported.
(b) What additional hypothesis on f ensures the validity of" $\omega \in E^k_c(N) \implies f^*(\omega) \in E^k_c(M)$ " ?
(c) Show that the additional hypothesis produces $f^* : \mathrm{H}^k_{DR,c}(N) \to \mathrm{H}^k_{DR,c}(M)$

Proposition 5.9. $\int_{\mathbb{R}^n} : \mathrm{H}^n_{DR,c}(\mathbb{R}^n) \to \mathbb{R}$ *is an isomorphism. In other wirds, if* ω *is a compactly supported* n*-form on* $\mathbb{R}^n$ *whose integral is zero, then* $\omega = d\eta$ *for some compactly supported* $(n-1)$*-form* η *on* $\mathbb{R}^n$.

Proof. Let $\omega \in E^n_c(\mathbb{R}^n)$. So $\omega = \phi.dx_1 \wedge ... \wedge dx_n$ where $\phi : R^n \to \mathbb{R}$ is a C^∞ compactly supported function.
Assume that $\int_{\mathbb{R}^n} \omega = 0$.
(1) Let $\psi(x_1, x_2, ..., x_n) = \int_{-\infty}^{x_1} \phi(t, x_2, ..., x_n)dt$ and let $\eta = \psi.dx_2 \wedge ... \wedge dx_n$.
(2) We know that $\omega = d\eta$. When $n = 1$, the assumption $\int_{\mathbb{R}} \omega = 0$ shows that $\psi = \eta$ is compactly supported as well.
So the proposition has been proved when $n = 1$.
However η is not necessarily compactly supported when $n > 1$.

Next, we will modify η to get a compactly supported form.
(3) Let $\widetilde{\psi}(x_2, x_3, ..., x_n) = \int_{-\infty}^{\infty} \phi(t, x_2, ..., x_n)dt$.
The $(n-1)$-form $\omega' = \widetilde{\psi}.dx_2 \wedge ... \wedge dx_n$ on $\mathbb{R}^{n-1}$ is compactly supported.
We note that $\int_{\mathbb{R}^{n-1}} \omega' = \int_{\mathbb{R}^n} \omega = 0$.
The proof given is by induction on n.
Thus we have $\eta' \in E_c^{n-2}(\mathbb{R}^{n-1})$ such that $\omega' = d\eta'$.
$\pi(x_1, x_2, ..., x_n) = (x_2, x_3, ..., x_n)$ denotes the projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$
(4) Let $v : \mathbb{R} \to \mathbb{R}$ be a compactly supported C^∞ function such that $\int_{\mathbb{R}} v(t)dt = 1$.
Let $u(x) = \int_{-\infty}^{x} v(t)dt$.
Let $q(x_1, ..., x_n) = x_1$.
(5) We see that $\psi_1(x_1, x_2, ..., x_n) = \psi(x_1, x_2, ..., x_n) - u(x_1)\widetilde{\psi}(x_2, ..., x_n)$ is compactly supported. So $\eta_1 = \psi_1.dx_2 \wedge ... \wedge dx_n$ is also compactly supported.
$\omega - d\eta_1 = \omega - d\eta + d(q^*u.\pi^*\omega') = d(q^*u.\pi^*\omega')$ by recalling (2)
$= d(q^*u) \wedge \pi^*\omega'$ because $\pi^*\omega'$ is closed
$=d(q^*u) \wedge d\pi^*\eta'$ (see (3))
$=\pm d\pi^*\eta' \wedge d(q^*u)$
$=\pm d(\pi^*\eta' \wedge q^*du)$
Because η' and du are compactly supported forms on $\mathbb{R}^{n-1}$ and $\mathbb{R}$ respectively, we see that $\pi^*\eta' \wedge q^*du$ is compactly supported on $\mathbb{R}^n$. Thgus ω is the exterior derivative of the sum of the compactly supported forms η_1 and $\pi^*\eta' \wedge q^*du$ on $\mathbb{R}^n$.

□

The proposition is actually is a speciial case of

Theorem 5.10. *$\pi_* : \mathrm{H}^m_{DR,c}(\mathbb{R} \times M) \to \mathrm{H}^{m-1}_{DR,c}(M)$ is an isomorphism for all $m \in \mathbb{Z}$. In particular, $\mathrm{H}^k_{DR,c}(\mathbb{R}^n) = 0$ for all $k \neq n$*

Definition 5.11. Let $f : M \to N$ be a C^∞ map, where both M and N are compact connected nonempty oriented n-manifolds. Then "the degree of f", denoted by $\deg(f)$ is the (unique) real number with the property that $\int_M f^*\omega = \deg(f) \int_N \omega$ for all $\omega \in E^n(N)$.

5.3. **Friday.** Stokes theorem for compact oriented manifolds with boundary was proved:
$\int_{\partial M} \eta = \int_M d\eta$ for all $\eta \in E^{n-1}(M)$ where $N = \dim(M)$.

274 takehome midterm due Friday April 26.
Problem 1: Show that there is no submersion from a nonempty compact manifold M to a non-compact connected manifold N.
Problem 2: Let M be a compact C^∞ n-manifold.
(A) Let $x \in M$. Show that there is a C^∞ function $f : M \to \mathbb{R}^n$ for which $f'(x) : T_xM \to \mathbb{R}^n$ is an isomorphism.
(B)Show that there is a C^∞ *immersion* $g : M \to \mathbb{R}^m$ for some m (i.e. $\ker(g'(x)) = 0$ for all $x \in M$.
(C) Show that there is a C^∞ *embedding* $h : M \to \mathbb{R}^p$ for some p (i.e. in addition to being an immersion, it also has to be one-to-one).
Problem 3: Let $C \subset \mathbb{R}^2$ be a C^∞ submanifold, and let $h : S^1 \to C$ be a diffeomorphism. Prove that C has a unit normal vector field v, i.e.
(i) $v : C \to \mathbb{R}^2$ is C^∞, (ii) $\langle v(x), T_xC\rangle = 0$ for all $x \in C$, (iii) $\|v\| = 1$
(ii) Show there is a positive constant ϵ such that $U_\epsilon(C) \setminus C$ has exactly two connected components (Notation for $U_\epsilon(C)$ is taken from the tubular nbhd thm)
(iii) Show that $\mathbb{R}^2 \setminus C$ has at most two connected components
A hint which you may ignore: given $y \in \mathbb{R}^2, y \notin C$, select a point $x \in C$ and consider the straight line segment that joins x and y.
(iv) Show that there is a C^∞ function $\phi : U_\epsilon(C) \to \mathbb{R}$ such that
(a) $\phi^{-1}(0) = C$
(b) $\phi^{-1}(0, \infty)$ and $\phi^{-1}(-\infty, 0)$ are the two connected components of $U_\epsilon(C) \setminus C$.
(c) $d\phi$ is compactly supported, i.e. $d\phi \in E^1_c(U_\epsilon)$
(v) In particular, we get $(d\phi)_e \in E^1_c(\mathbb{R}^2)$ (see section 4.2, "extension by zero"). Prove that there is a function $\psi : \mathbb{R}^2 \to \mathbb{R}$ for which $d\psi = (d\phi)_e$.
(vi) Prove that ψ can be chosen so that $\psi(x) = \phi(x)$ for all $x \in U_\epsilon(C)$.
(vii) Prove that $C = \psi^{-1}(0)$.
(viii) Deduce (the Jordan Curve theorem for C^∞ Jordan curves C): the complement of C in $\mathbb{R}^2$ has exactly two connected components.
Problem 4: Use Thm.5.7 to deduce that there is a real number denoted by $\deg(f)$ that satisfies the requirements of defn. 5.11.
Problem 5: With $f : M \to N$ as in defn. 5.11, let θ_M and θ_N be their orientation forms. Assume that there is some $p \in N$ such that $f^{-1}(p) = \{q_1, q_2, ..., q_r\}$ and $f'(q_i) : T_{q_i}M \xrightarrow{\cong} T_pN$ for all $i = 1, 2, ..., r$. It follows that $f^*\theta_N(p) = \epsilon_i a_i \theta_M(q_i)$ where $a_i > 0$ and $\epsilon_i \in \{1, -1\}$ for all i (Equivalently we are assuming that p is a regular value of f (see defn.2.13 of these notes for critical points and regular values) in the view of the properness of f).
Prove that $\deg(f) = \epsilon_1 + \epsilon_2 + ... + \epsilon_r$.
NOTE: Such p always exist in plenty thanks to Sard's theorem, so it follows that $\deg(f)$ is *always* an integer.

Problem 5 carries 15 points. Each part of the first four problems carries 5 points. Total Score:80.
If your deduce part Z from parts X and Y correctly, you will get full score for part Z, whether your solutions to X and Y are correct.
SEE NEXT PAGE FOR COMMENTS ON PROBLEM 5

MIDTERM CONTINUED

If you wish, you may do the three parts below of problem 5 separately, worth five points each

5(i) Show that there is a nbhd U of p in N, and pairwise disjoint neighbourhoods V_i of q_i in M such that
(a) f restricts to a diffeomorphism $V_i \to U$ for $i = 1, 2, ..., r$.
(b) $f^{-1}(U) = V_1 \cup ... \cup V_r$

(ii) If $\omega \in E^n(M)$ is such that supp(ω) is contained in U,
then $(\epsilon_1 + ... + \epsilon_r) \int_N \omega = \int_M f^*(\omega)$

(iii) Now deduce problem 5 from problem 4.

Remark 5.12. This is a remark on the proof of the Jordan Curve Theorem suggested on the previous page. It is not relevant for your midterm.
Let $C \subset \mathbb{R}^n$ be a C^∞ CLOSED submanifold of dimension $(n-1)$.
Assume Part (i) of problem 3, the existence of a unit normal vector field v on C.
Assume that C has r connected components.
Then the complement of C in $\mathbb{R}^n$ has exactly $(r+1)$ connected components.
This is how far one can get by pushing each step of Problem 3.
Compactness of C was appealed to in the tubular nbhd thm but it can be got rid off by replacing the position constant ϵ by a continuous function $\epsilon : C \to (0, \infty)$.

However, in Algebraic Topology, one learns that a normal vector field always exists, and therefore the analogue of the Jordan curve theorem stated (i.e. $r+1$ connected components) is *true*.

6. Sixth Week

6.1. **Mon and Wed.** WARNER pages 173-176 on **Cochain complexes**

The ∂ you see in Proposition 5.17, page 174, is referred to as *the connecting homomorphism*

Lemma 6.1. *Let $\mathcal{U} = \{U_1, U_2\}$ be an open cover of a C^∞ manifold M.*
Define $i : E^k(M) \to E^k(U_1) \oplus E^k(U_2)$ by $i(\omega) = (\omega|_{U_1}, \omega|_{U_2})$.
Define $p : E^k(U_1) \oplus E^k(U_2) \to E^k(U_1 \cap U_2)$ by $p(\omega_1, \omega_2) = \omega_2|V - \omega_1|V$ where $V = U_1 \cap U_2$. Then

$$0 \to E^k(M) \xrightarrow{i} E^k(U_1) \oplus E^k(U_2) \xrightarrow{p} E^k(U_1 \cap U_2) \to 0$$

is a short exact sequence, i.e.
$\ker(i) = 0, \ker(p) = \mathrm{image}(i)$*, and p is surjective.*

Proof. The first two assertions follow from "the sheaf property".
To see why p is surjective, let ϕ_1, ϕ_2 be partition of unity subordinate to the open cover $\mathcal{U} = \{U_1, U_2\}$.
Thus $\mathrm{supp}(\phi_i) = F_i$ is contained in U_i.
It follows that $\{U_1 \cap U_2, U_2 \setminus F_1\}$ is an open cover of U_2.
Given $\omega \in E^k(U_1 \cap U_2)$, we note that $\phi_1\omega \in E^k(U_1 \cap U_2)$ extends by zero to a k-form $(\phi_1\omega)_e$ on U_2:
indeed $\phi_1\omega \in E^k(U_1 \cap U_2)$ and $0 \in E^k(U_2 \setminus F_1)$ agree with each other on their intersection, which is $U_2 \cap (U_1 \setminus F_1)$.
Similarly, we get $(\phi_2\omega_1)_e \in E^k(U_1)$.
Denote these k-forms on U_2 and U_1 by ω_2 and $-\omega_1$ respectively. Because $\phi_1 + \phi_2 = 1$, we see that $p(\omega_1, \omega_2) = \omega$.

□

Remark 6.2. There is an analogous property for *an arbitrary open cover* :
"The q-th cohomology of the Cech complex $C(\mathcal{U}, \mathcal{F})$ of a fine sheaf $\mathcal{F}$ vanishes for all $q > 0$." proved in WARNER, section 5.33, pages 200-204. "Fine Sheaves" on p.170. The special case when there are just two open sets is lemma 6.1.

Problem 42: $E^k(U_1) \to E^k(U_1 \cap U_2)$ is not necessarily surjective. Find an example.

Proposition 6.3. *The Mayer-Vietoris sequence. M, U_1, U_2 are as in lemma 6.1. Then there are linear transformations $\delta^k : \mathrm{H}^k_{DR}(U_1 \cap U_2) \to \mathrm{H}^{k+1}_{DR}(M)$ that give rise to the long exact sequence of cohomology*

$$\to \mathrm{H}^{k-1}_{DR}(U_1 \cap U_2) \xrightarrow{\delta^{k-1}} \mathrm{H}^k_{DR}(M) \xrightarrow{i} \mathrm{H}^k_{DR}(U_1) \oplus \mathrm{H}^k_{DR}(U_2) \xrightarrow{p} \mathrm{H}^k_{DR}(U_1 \cap U_2) \xrightarrow{\delta^k} \mathrm{H}^{k+1}_{DR}(M) \to$$

where i and p as given as follows.
Let $i_1 : U_1 \to M$ and $i_2 : U_2 \to M$ and $j_1 : U_1 \cap U_2$ and $j_2 : U_1 \cap U_2 \to U_2$ denoted the given inclusions. The corresponding homomorphisms on De Rham cohomology are denoted by $\mathrm{H}^k(i_1) : \mathrm{H}^k_{DR}(M) \to \mathrm{H}^k_{DR}(U_1)$ etc. Then

$$i(a) = (\mathrm{H}^k(i_1)a, \mathrm{H}^k(i_2)a) \text{ and } p(b_1, b_2) = \mathrm{H}^k(j_2)b_2 - \mathrm{H}^k(j_1)b_1$$

Proof. Lemma 6.1 provides the short exact sequence of cochain complexes:

$$0 \to E(M) \xrightarrow{i} E(U_1) \oplus E(U_2) \xrightarrow{p} E(U_1 \cap U_2) \to 0$$

The lemma then follows from part (1) of Proposition 5.17, page 174, of Warner. □

Problem 43: Let $0 \to U \xrightarrow{i} V \xrightarrow{j} W \to 0$ is a short exact sequence of vector spaces.
(a) If $U = 0$, what properties does j have?
(b) If $W = 0$, what properties does i have?
(c) If $V = 0$, what can be said about U and W?

Problem 44: In the set-up of Warner's Proposition 5.17, page 174,
(a) assume that the homomorphisms $\mathrm{H}^k(C) \to \mathrm{H}^k(D)$ and $\mathrm{H}^{k+1}(C) \to \mathrm{H}^{k+1}(D)$ are injective, for some integer k.
Show then that $0 \to H^k(C) \to \mathrm{H}^k(D) \to \mathrm{H}^k(E) \to 0$ is a short exact sequence.
(b) If $\mathrm{H}^k(D) = \mathrm{H}^{k+1}(D) = 0$, what conclusion can you draw about $\delta^k : \mathrm{H}^k(E) \to \mathrm{H}^{k+1}(C)$?
(c) What does the hypothesis $\mathrm{H}^{k-1}(E) = \mathrm{H}^k(E) = 0$ yield about $\mathrm{H}^k(C) \to \mathrm{H}^k(D)$?

NOTE: $\mathrm{H}^k_{DR}(M)$ is defined for all $k \in \mathbb{Z}$.
$E^k(M) = 0$ for all $k < 0$ by definition.

Problem 45: $\mathrm{H}^k(S^n) = 0$ for $k \neq 0, n$; proved in class on Monday.

Problem 46: Prove that there is an isomorphism $\mathrm{H}^k(M \times S^n) \cong \mathrm{H}^k(M) \oplus \mathrm{H}^{k-n}(M)$ for all $k \in \mathbb{Z}$.
Hint: Apply Mayer-Vietoris to the covering $M \times U_1, M \times U_2$ of $M \times S^n$, for a suitable covering U_1, U_2 of S^n. Let $U'_n = U_1 \cap U_2$.
The connecting homomorphism is denoted by $\delta^k(n) : \mathrm{H}^k(M \times U'_n) \to \mathrm{H}^{k+1}(M \times S^n)$
Projections are denoted by p_1, p_2.

Problem 47: Let $n > 0$ and let $0 \neq e_n \in \mathrm{H}^n_{DR}(S^n)$.
$T : \mathrm{H}^k_{DR}(M) \oplus \mathrm{H}^{k-n}_{DR}(M) \to \mathrm{H}^k_{DR}(M \times S^n)$ is given by
$T(a, b) = \mathrm{H}^k(p_1)a + \mathrm{H}^{k-n}(p_1)b \wedge \mathrm{H}^n(p_2)e$.
Show that T is an isomorphism.

NOTATION: Let $i : K \to M$ denote the inclusion of a compact oriented k-submanifold of M.
$\int_K : \mathrm{H}^k_{DR}(M) \to \mathbb{R}$ is the composite $\mathrm{H}^k_{DR}(M) \xrightarrow{H^k(i)} \mathrm{H}^k_{DR}(K) \xrightarrow{\int_K} \mathbb{R}$.

Problem 48: Let M be a n product of spheres, possibly of different dimensions.
Show that for every k there are compact oriented submanifolds $K_1, K_2, ..., K_r$ of M such that the linear functionals $\int_{K_1}, \int_{K_2}, ..., \int_{K_r}$ form a basis for the dual vector space $(\mathrm{H}^k_{DR}(M))^*$.
Hint: Use the previous problem to proceed by induction on the number of spheres.

Problem 49: Let M be a compact oriented connected (nonempty) n-manifold. Let M' be the complement of a point a of M.
(A)Show that $\mathrm{H}^k_{DR}(M) \to H^k_{DR}(M')$, given by restriction, is an isomorphism for all $k < n$.
(B) Show that $\mathrm{H}^n_{DR}(M') = 0$.

Hint: Apply the M.V. long exact sequence to the covering M', U of M where U is a nice nbhd of $a \in M$. For part (B) and also for $k = n-1$ in part (A), you will have to prove that $\delta^{n-1} : \mathrm{H}^{n-1}_{DR}(U \cap M') \to \mathrm{H}^{n}_{DR}(M)$. Whereas deducing the next problem from this one requires less work!

Problem 50: M is as in the previous problem, but now M_r is the complement of a finite subset of M of cardinality r, with $r > 1$.
(A) Show that $\mathrm{H}^k_{DR}(M) \to H^k_{DR}(M_r)$, given by restriction, is an isomorphism for all $k < n-1$.
(B) $\mathrm{H}^{n-1}_{DR}(M_r) \cong \mathrm{H}^{n-1}_{DR}(M) \oplus \mathbb{R}^{r-1}$
(C) $\mathrm{H}^{n}_{DR}(M_r) = 0$.
Hint: Proceed by induction on r.

Problem 51: Find compact oriented $(n-1)$-submanifolds $K_1, ..., K_{r-1}$ of the manifold M_r of the previous problem such that

$$0 \to \mathrm{H}^{n-1}_{DR}(M) \xrightarrow{\mathrm{H}^{n-1}(i)} \mathrm{H}^{n-1}_{DR}(M_r) \xrightarrow{T} \mathbb{R}^{r-1} \to 0$$

is a short exact sequence, where $i : M_r \to M$ is the inclusion, and
$T(a) = (\int_{K_1} a, ..., \int_{K_{r-1}} a)$

6.2. **Friday.** We talked a bit about projective space after working out problem 49.

Proof. of problem 49.
We cover M by two open sets M' and U where U is a neighborhood of $a \in M$ that is diffeomorphic to $\mathbb{R}^n$. Thus $U' = M' \cap U$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$ which in turn is diffeomorphic to $S^{n-1} \times \mathbb{R}$. Thus $\mathrm{H}^q(U') \cong \mathrm{H}^q(S^{n-1})$ which we know is zero if $q \neq 0, n-1$. This is enough to prove part (A) for $k < n-1$ (it will not be repeated here–but you can ask me or Frank about it). Part (B) plus the remaining assertion of (A) follows that the connecting homomorphism $\delta^{n-1} : \mathrm{H}^{n-1}_{DR}(U') \to \mathrm{H}^n_{DR}(M)$ is an isomorphism. Note that both source and target of δ^{n-1} are one-dimensional vector spaces.
We recall the definition of $\delta^{n-1}(\eta)$ where $\eta \in E^{n-1}(U')$ is a closed form. We find $\eta_1 \in E^{n-1}(M')$ and $\eta_2 \in E^{n-1}(U)$ such that $\eta = \eta_2|_{U'} - \eta_1|_{U'}$ (see the notation for p in the statement of the M.V. Theorem). Thus the exact forms $d\eta_i$ for $i = 1, 2$ on the regions M' and U respectively agree with each other on U' because $d\eta = 0$. Thus we get a n-form ω on M that restricts to $d\eta_1$ and $d\eta_2$ on M' and U respectively. See lemma 5.17, page 174 of Warner, to see that $\delta^{n-1}\overline{\eta} = \overline{\omega}$ where represents the image in De Rham cohomology of a closed form.

Recall that $\int_M : \mathrm{H}^n_{DR}(M) \to \mathbb{R}$ is an isomorphism. Thus it is good to compute $\int_M \omega$. Fix an orientation-preserving diffeomorphism $\phi : \mathbb{R}^n \to U$. Then ϕ takes the closed unit ball $D^n \subset \mathbb{R}^n$ to M_2. which is a *closed submanifold with boundary of* M. Let M_1 be the closure of the complement $M \setminus M_2$. Thus M is expressed as the union of submanifolds M_1 and M_2 which intersect only on their boundaries.
It follows that:
$\int_M \omega$
$= \int_{M_2} \omega + \int_{M_1} \omega$
$= \int_{M_2} d\eta_2 + \int_{M_1} d\eta_1$ note that $M_2 \subset U$ and $M_1 \subset M'$
$=\int_{\partial M_2} \eta_2 + \int_{\partial M_1} \eta_1$ by Stokes Theorem for manifolds with boundary
$=\int_{\partial M_2} \eta_2 - \int \partial M_2 \eta_1 = \int_{\partial M_2} \eta$ by noting that the submanifolds ∂M_i of M are equal to each other, but the orientations they derive by being the boundaries of M_i are negative of each other

We use the standard notation set up between problems 47 and 48 to define $\int_{\partial M_2} : \mathrm{H}^{n-1}_{DR}(U') \to \mathbb{R}$ and appeal to homotopy invariance (Proposition 4.4) and Theorem 5.7 to show that $\int_{\partial M_2}$ is an isomorphism. We have now shown that

$$\text{the composite } \mathrm{H}^{n-1}_{DR}(U') \xrightarrow{\delta^{n-1}} \mathrm{H}^n_{DR}(M) \xrightarrow{\int_M} \mathbb{R} \text{ equals } H^{n-1}_{DR}(U') \xrightarrow{\int_{\partial M_2}} \mathbb{R}$$

Because $\int_M$ and $\int_{\partial M_2}$ are isomorphisms, it follows that δ^{n-1} is also an isomorphism. $\square$

Manifolds with boundary again?
Problem 52:
Projective spaces?

7. Seventh Week: vector fields

7.1. **integral curves of a chosen vector field and local one-parameter groups.** All objects (manifolds, vector fields, maps) are assumed to be C^∞.
WARNER, page 36, defn.1.46 has the definition of an integral curve γ of a vector field X on a manifold M.

Example 7.1. (A)"Constant vector field on $\mathbb{R}^n$, i.e. $v \in \mathbb{R}^n$, $M = \mathbb{R}^n$ and $X(p) = v$ for all $p \in \mathbb{R}^n$.
For every $q \in \mathbb{R}^n$, we see that $\gamma(t) = q + tv$ is an integral curve.
(B) "Radial vector field": Here M is the complement of the origin of $\mathbb{R}^n$
and $X(p) = \frac{p}{\|p\|}$ Here, every $p \in S^{n-1}$ gives rise to an integral curve $\gamma(t) = tp$ for all $t \in (0, \infty)$.

Problem 53: Let $a \in \mathbb{R}$. Take $M = \mathbb{R}^n$ and $X(v) = av$ for all $v \in \mathbb{R}^n$.
Find all the integral curves of the vector field X.
Problem 54: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $M = \mathbb{R}^n$ and $X(p) = Tp$ for all $p \in M$.
Find all the integral curves of X.
NOTE that the previous problem suggests a series expansion for the answer. You should check that it converges, and prove that it provides an integral curve.

Theorem 7.2. *EXISTENCE: Given a vector field X on M and $p \in M$ there exists a nbhd U of $p \in M$, an open interval $(-a, a)$, and a C^∞ function $\varphi : (-a, a) \times U \to M$ such that*
(i) $\varphi(0, q) = q$ for all $q \in U$ and
(ii) $\gamma_g(t) = \varphi(t, q)$ defines an integral curve $\gamma_g : (-a, a) \to M$.

Proof. Warner refers to a book by Hurewicz for the proof □

Lemma 7.3. *UNIQUENESS Let $\gamma_i : I_i \to M$ for $i = 1, 2$ are integral curves of a vector field X on a manifold M (where the $I_i \subset \mathbb{R}$ are open intervals) assume that $\gamma_1(t_0) = \gamma_2(t_0)$ for some point $t_0 \in I_1 \cap I_2$.*
Then there is an open interval $I_3 \subset I_1 \cap I_2$ such that $\gamma_1(t) = \gamma_2(t)$ for all $t \in I_3$.

Corollary 7.4. *With assumptions as in the uniqueness lemma above, deduce that the restrictions of the γ_i to $I_1 \cap I_2$ are equal to each other.*

Definition 7.5. and **notation**
(a) Given a vector field X on M and $p \in M$, there is a greatest integral curve $\gamma_p : I_p \to M$ satisfying $\gamma_p(0) = p$. Precisely, if $\gamma : I \to M$ is an integral curve for the same vector field X with initial point $\gamma(0) = p$, then $I \subset I_p$ and $\gamma = \gamma_p|_I$.
(b) Furthermore if $t \in I_p$ and $s \in I_q$, where $q = \gamma_p(t)$,
then $s + t \in T_p$ and $I_p(s + t) = I_q(s)$.
(c) $\mathcal{D} = \{(t, p) \in \mathbb{R} \times M : t \in I_p\}$. $\mathcal{D}$ is open in $\mathbb{R} \times M$.
The $\mathcal{D}_t$ of Warner, page 37, is $\{p \in M : (t, p) \in \mathcal{D}\}$
(d) We have $\varphi : \mathcal{D} \to M$ given by $\varphi(t, p) = \gamma_p(t)$.
(e) φ_t is not a function from M to itself, rather, it is defined on the open subset $\mathcal{D}_t$; thus we have $\varphi_t : \mathcal{D}_t \to M$ given by $\varphi_t(p) = \varphi(t, p) = \gamma_p(t)$
(f) We have $\varphi_s(\varphi_t(p)) = \varphi(s + t)(p)$ wherever the LHS is defined.
(g) If M is compact then $\mathcal{D} = \mathbb{R} \times M$.

We observe that when M is compact, we obtain a group homomorphism from $\mathbb{R}$ to the group of diffeomorphisms of M: given by $t \mapsto \varphi_t$

Proof. (a) and (b) of 7.5 follow directly from cor.7.4.
The open-ness of 7.5(c) is an immediate consequence of the "U of the existence theorem 7.2.
For (g), one notes that the compactness of M and the open-ness of $\mathcal{D}$ implies that $(-a, a) \times M \subset \mathcal{D}$ for some $a > 0$. The rest follows from part (f) and induction. □

WARNER, PAGE 40, Prop.1.53 was crushed into the last five minutes of the Monday lecture. Please go through it carefully. It will be employed on Wednesday, immediately after
(i) the Lie bracket defined by Warner page 36, 1.44, is checked to be a vector field
(ii) the formula of the Lie bracket is found for vector fields open subsets $\Omega \subset \mathbb{R}^n$: see problem 58.

Problem 55: Re: Newton's second law.
Ω is open in $\mathbb{R}^n$ and $a : \Omega \to \mathbb{R}^n$ is a C^∞ vector field ('a' is acceleration).
Consider the vector field X on $M = \Omega \times \mathbb{R}^n$ given by
$X(p, v) = (v, a(p))$ for all $(p, v) \in M$ to deduce:
For all $(p, v) \in M$ there is some constant $c(p, v) > 0$ and $\gamma : [0, c(p, v)) \to \Omega$ satisfying $\gamma(0) = p$, $\gamma'(0) = v$, $\gamma''(t) = a(\gamma(t))$.

Problem 56: Re: Inverse Square Law. E=Kinetic + Potential Energy.
Application: Escape Velocity.
Take $n = 1, \Omega = (0, \infty)$ and $a(x) = \frac{-1}{x^2}$ in the previous problem.
Define $E(p, v) = \frac{1}{2}v^2 - \frac{1}{p}$ for all $(p, v) \in M = \Omega \times \mathbb{R}$.
The γ below is exactly the same as in the previous problem.
(i) Show that $E(p, v) = E(\gamma(t), \gamma'(t))$ for all $t \in [0, c(p, v))$ with $p, v, \gamma, c(p, v)$ as in the previous problem.
(ii) Assume that the initial point (p, v) satisfies (a) $v > 0$ and (b) $E(p, v) \geq 0$.
(A) Prove that γ is defined on $[0, \infty)$ and
(B) $\gamma(t)$ approaches ∞ as $t \to \infty$
(iii) Assume that $v \leq 0$ or $E(p, v) < 0$. Show that γ is defined on $[0, c)$ for some $0 < c < \infty$ with $\gamma(t) \to \infty$ as $t \to c$.

7.2. **Very preliminary Lie bracket discussion and The Frobenius Theorem.**

Definition 7.6. A is an associative $\mathbb{R}$-algebra. A linear transformation $D : A \to A$ is a **derivation** if $D(ab) = (Da)b + a(Db)$ for all $a, b \in A$.
If D_1, D_2 are derivations, and $t_1, t_2 \in \mathbb{R}$, then $t_1 D_1 + t_2 D_2$ is also a derivation.
If D_1, D_2 are derivations, then their **Lie bracket** $[D_1, D_2] := D_1 D_2 - D_2 D_1$ is a derivation.
In particular, if M is a manifold, we take $A = \mathrm{C}^\infty(M; \mathbb{R})$.
Then a vector field X on M is identified with the operator $X : \mathrm{C}^\infty(X; \mathbb{R}) \to \mathrm{C}^\infty(X; \mathbb{R})$, which is a derivation.
If X, Y are derivations, then $[X, Y]$ is also a derivation of $\mathrm{C}^\infty(M; \mathbb{R})$. Going to the level of germs at $p \in M$, one sees that $[X, Y]$ is also a vector field on M –see WARNER, page 36, 1.44.

Problem 57: If X, Y are vector fields on M and $f : M \to \mathbb{R}$ is C^∞, then $[X, fY] = X(f)Y + f[X, Y]$.
Problem 58: Let $\Omega \subset \mathbb{R}^n$ be open. Let $v : \Omega \to \mathbb{R}^n$ and let $w : \Omega \to \mathbb{R}^k$ be C^∞. The directional derivative $D_v w : \Omega \to \mathbb{R}^k$ is defined in the usual manner: $(D_v w)p$ is the limit of $\frac{w(p+tv)-w(p)}{t}$ as $t \to 0$.
Let X, Y be vector fields on Ω. Prove that $[X, Y] = D_X Y - D_Y X$.

Theorem 7.7. *If ϕ_t and ξ_s are the local one-parameter groups corresponding to vector fields X and Y on M, then $\phi_t \xi_s = \xi_s \phi_t$.*

This theorem was proved under the extra assumption: $X(p) \neq 0$ for all $p \in M$. This sufficed for the proof of the Frobenius theorem.
The definition of a Rank r bundles of the tangent bundle was given.
The definition of an involutive bundle was given.
The Frobenius theorem was proved in class following Volterra's method to be found in "Analysis of Real and Complex manifolds" by R.Narasimhan.
Problem 59: Let X be a vector field on M and let $f : M \to \mathbb{R}$, where M, X, f are all C^∞. Let γ be an integral curve of X and let δ be an integral curve of fX with $\gamma(0) = \delta(0 = p \in M$. Deduce from the existence and uniqueness theorem of integral curves that there is a C^∞ function real valued function defined on some open interval I such that (i) $0 \in I$, (ii) $g(0)$, (iii) $\delta(t) = \gamma(g(t))$ for all $t \in I$.

Problem: Let Ω be open in $\mathbb{R}^n$. Let $v : \Omega \to \mathbb{R}^n$ and $w : \Omega \to \mathbb{R}^n$ be C^∞. Then we have $D_v w : \Omega \to \mathbb{R}^n$

Problem: A $\mathbb{R}$-derivation of a R-algebra A is a $\mathbb{R}$-linear transformation $D : A \to A$ that satisfies $D(fg) = (Df).g + f.(Dg)$ for all $f, g \in A$.

Prove that if D_1, D_2 are $\mathbb{R}$-derivations of A, prove that $[D_1, D_2] : D_1 D_2 - D_2 D_1$ is also a $\mathbb{R}$-derivation of A.

Note that the commutativity of the multiplication in A is not required– associativity suffices.