Master Thesis Reading for Muyi Chen's Paper

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Contents

1 Introduction

In this paper, he mainly wants to convey the existence of solutions to the nonlinear Helmonltz equation and nonlinear Direc equation by finding the scattering matrix and using fixed point arguments to analyze the operators' spectrum.

2 Definitions and the free Schrödinger equation

2.1 Definitions

In quantum mechanics setting, a particle in R^3 is characterized by

$$\psi(x,t), \quad (x,t) \in R^3 \times R$$

a wave equation. Moreover, consider

$$\rho_t(x) = |\psi(x,t)|^2$$

defined as **probability density**, described the likehood of finding particle at position x at time t. We also denote the position x as the **observables** and its **expectation** as

$$\mathbb{E}_\psi(x) = \int_{R^3} x|\psi|^2 d^3x$$

It is useful to check whether particle is inside of certain area Ω, the probability of finding the particle in Ω

$$\mathbb{E}_\psi(\mathbb{1}_\Omega) = \int_{R^3} \mathbb{1}_\Omega |\psi|^2 d^3x = \int_\Omega |\psi|^2 d^3x$$

We also define the **square-mean derivation(variance)** as

$$\Delta_\psi(x)^2 = \mathbb{E}_\psi(x^2) + \mathbb{E}_\psi(x)^2$$

Note that quantum mechanics system is described as the normalized vectors in a Hilbert space (complete w.r.t to inner product). Measurable quantities are called observables and correspond to self-adjoint operators in the Hilbert space. Moreover, the expectation is

$$\mathbb{E}_\psi(A) = \langle A\psi, \psi \rangle = \langle \psi, A\psi \rangle$$

given A self-adjoint operator and in state ψ, a real number. Similarly for variance, we have

$$\Delta_\psi(A)^2 = \mathbb{E}_\psi(A^2) + \mathbb{E}_\psi(A)^2 = \|(A - \mathbb{E}_\psi)\psi\|^2$$

where we require A defined on the dense subset $\mathcal{D}(A) \in \mathcal{H}$, called domain of A.

The time evolution of a quantum mechanical system: Given initial state $\psi_(0)$, there should be a unique $\psi(t) = U(t)\psi(0)$. Moreover, the **superposition of states** hold,

$$U(t)[a\psi_1(0) + b\psi_2(0)] = a\psi_1(t) + b\psi_2(t)$$

This implies that U is a linear operator, and consider that

$$\|U(t)\psi(0)\| = \|\psi(t)\| = 1$$

since ψ is a state, we conclude that $U(t)$ is unitary. Since we assume the uniqueness of the solution to the following

$$U(0) = I, \quad U(t+s) = U(t)U(s)$$

the family of the $U(t)$ is called a **one-parameter unitary group**. We also assume the strong continuity,

$$\lim_{t \to t_0} U(t)\psi(0) = \psi(t_0)$$

to help us pass the limit. Given the fact that each of such unitary group admits an **infinitesimal generator** defined as

$$H\psi = \lim_{t \to 0} \frac{i}{t}(U(t)\psi - \psi), \quad \mathcal{D}(H) := \{\psi \in H | \lim_{t \to 0} \frac{i}{t}(U(t)\psi - \psi) \text{ exists}\}$$

The operator is called **Hamiltonian**, which gives the total energy of the system. If $\psi(0) \in \mathcal{D}(H)$, then we have the solution $\psi(t)$ of the **Schrödinger equation**

$$i\frac{\partial}{\partial t}\psi(t) = H\psi(t)$$

2.2 Schrödinger equation

Following the idea, we define the particle in R^d as

$$\psi(x,t), \quad (x,t) \in R^d \times R$$

and if we set $-\Delta_x = H$, we have the new version of Schrödinger equation,

$$i\frac{\partial}{\partial t}\psi(x,t) = -\Delta_x\psi(x,t)$$

Using separation of variable, let $\psi(x,t) = T(t)X(x)$, we have

$$i\frac{\partial T(t)}{\partial t}X(x) = -\frac{\partial^2 X}{\partial x^2}T(t) \implies \frac{iT'(t)}{T(t)} = -\frac{X''(x)}{X(x)} := \lambda$$

the equation

$$-\frac{X''(x)}{X(x)} := \lambda$$

is called **time-independent Schrödinger equation.**

Now, we know that $\psi(t,x) = e^{-i\lambda t}X(x)$ is a solution to our system, the remaining is to solve $X(x)$ explicitly, note that

$$X_k(x) = e^{ikx} = e^{i(k_1x_1+k_2x_2+\ldots+k_dx_d)} \tag{1}$$

are solutions to $X(x)$ in general. Moreover, we have

$$\Delta_x X_k(x) = |k|^2 X_k(x)$$

we deduce that $\lambda = |k|^2$, and by plugging into the general solution, we have

$$\psi_k(t,x) = e^{-i\lambda t}X(x) = e^{-ik^2t+ikx} \tag{2}$$

However for any fixed t, we have

$$\psi_k(t,x) \notin L^2(x) \tag{3}$$

So we aim to solve this issue by considering the equivalent class of our solution, i.e linear combination with certain properties to jump this issue.

$$\begin{cases} \psi(t,x) = \int_{R^d} \rho(k)\psi_k(x,t) = \int_{R^d} \rho(k)e^{-ik^2t+ikx} \\ \psi(0,x) = \psi_0(x) = \int_{R^d} \rho(k)e^{ikx} \end{cases} \tag{4}$$

subject to the initial condition. The problem left is to check what class of $\rho(k)$ would make sense for $\psi(x,t) \in L^2(R^d;R)$;

Using Fourier Transform to solve the equation. We get

$$i\hat{\psi}_t(\xi,t) = |\xi|^2\hat{\psi}(\xi,t); \quad \hat{\psi}(0,\xi) = \hat{\psi}_0(\xi) \tag{5}$$

The solution to this ODE subject to initial condition is

$$\hat{\psi}(\xi,t) = e^{-ik^2t}\hat{\psi}_0(\xi) \tag{6}$$

Taking Fourier inverse, we have

$$\psi(x,t) = (\mathcal{F}^{-1}e^{-i|k|^2t}\mathcal{F}\psi_0)(x) \tag{7}$$

By calculation, we have the following result subject to some regularity requirement.

Theorem 1. *Let $\psi_0 \in \mathcal{S}(R^d)$. Then, there exists a global solution $\psi \in C(R_t, \mathcal{S}(R^d))$ of the free Schroedinger equation for $t \neq 0$, given by*

$$\psi(t,x) = \frac{1}{(2\pi it)^{\frac{d}{2}}}\int_{R^d} e^{\frac{i|x-y|^2}{t}}\psi_0(y)dy \tag{8}$$

Moreover, $\|\psi(t,\cdot)\|_{L^2(R^d)} = \|\psi_0\|_L^2(R^d)$

Remark 1. *The Theorem 1 is nothing but a convolution form of what we did in PDE course and Equation (8) implies that*

$$\sup_{x \in R^d} |\psi(x,t)| \leq \frac{\|\psi_0\|_L^1}{(2\pi i t)^{\frac{d}{2}}} \to 0 \tag{9}$$

as $t \to \infty$. However, theorem implies that L^2 norm stays constants. This implies the solution is flattened in space

Remark 2. *Let N be the set of non-negative integers, and for any $n \in N$, let $N^n := N \times \cdots \times N$ be the n-fold Cartesian product. The Schwartz space or space of rapidly decreasing functions on $\mathbb{R}^n$ is the function space*

$$\mathcal{S}(\mathbb{R}^n, \mathbb{C}) := \{f \in C^\infty(\mathbb{R}^n, \mathbb{C}) \mid \forall \alpha, \beta \in N^n, \quad \|f\|_{\alpha,\beta} < \infty\},$$

where $C^\infty(\mathbb{R}^n, \mathbb{C})$ is the function space of smooth functions from $\mathbb{R}^n$ into $\mathbb{C}$, and

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha (D^\beta f)(x)|.$$

Moreover, we have some results:

- *If $1 \leq p \leq \infty$, then $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$.*
- *If $1 \leq p \leq \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.*
- *The space of all bump functions, $C_c^\infty(\mathbb{R}^n)$, is included in $\mathcal{S}(\mathbb{R}^n)$.*

3 General solution to normal Schrödinger equation

In finite-dimensional case, the solution of the Schrödinger equation

$$i\frac{d}{dt}\psi(t) = H\psi(t) \tag{10}$$

is given by

$$\psi(t) = e^{itH}\psi(0) \tag{11}$$

we aim to to extend this to self-adjoint operators using the specturm theory.

3.1 Existence and uniqueness of the solution

Definition 1. *For self-adjoint operator H, we define*

$$U(t) = e^{-iHt} := \int e^{-i\lambda t} dP(\lambda) \tag{12}$$

where P is the projection-value measure associate to $(H, \mathcal{D}(H))$

Theorem 2. *Let H be a self-adjoint operator and $U(t) = e^{-iHt}$. Then*

1. *U(t) is a strong continuous one-parameter unitary group*
2. *the $lim_{t\to 0}\frac{1}{t}(U(t)\psi - \psi)$ exists if and only if $\psi \in \mathcal{D}(H)$. In the case, $lim_{t\to 0}\frac{1}{t}(U(t)\psi - \psi) = -iH\psi$*
3. *$U(t)\mathcal{D}(H) = \mathcal{D}(H)$ and $HU(t) = U(t)H$*

This theorem implies that the $U(t)\psi(0) = \psi(t)$ is indeed the solution to (10). In fact it is also the unique solution.

Lemma 1. *Let $\psi_0 \equiv \psi(0) \in \mathcal{D}(H)$, and $\psi(t)$ the solution to (10), then $U(t)\psi(0) = \psi(t)$*

Now we know $U(t)$ corresponding to self-adjoint operator H is one-parameter unitary group, the Stone's theorem gives us the converse.

Theorem 3. *(Stone) Let $U(t)$ be a weakly continuous one-parameter unitary group. Let $H : \mathcal{D}(H)$ be the generator of $U(t)$. Then H is self-adjoint and $U(t) = e^{-iHt}$*

3.2 The RAGE Theorem (Chapter 5)

In this part, he wants to show the asymptotic behavior of a quantum system. Using the knowledge in Applied Functional Analysis, we want to convey the properies of the self-adjoint operator H.

Remark 3. *Recall that the* ***spectrum*** *of H represents the complement of its resolvent set that operator $(H - \lambda I)$ is 1-1 and onto, denoted by $\rho(H)$, and $\sigma(H)$ denoted as the spectrum of $H \in \mathcal{B}(H)$, i.e. $(H - \lambda I)$ failed to be invertible.*

Problem 1. *The notation for $\sigma(H)$ as $\mathcal{H}$.(???)*

Let $\mathcal{H}_{ac}, \mathcal{H}_{sc}, \mathcal{H}_{pp}$ represent absolutely continuous, singularly continuous, pure point spectral subspace of $\mathcal{H}$ respectively, and we know $\mathcal{H} = \mathcal{H}_{ac} \bigoplus \mathcal{H}_{sc} \bigoplus \mathcal{H}_{pp}$.

Firstly, observe that let $\psi \in \mathcal{H}_{ac}$, then the measure μ_ψ is absolutely continuous. For all $\phi \in \mathcal{H}$, we have

$$|\mu_{\phi,\psi}(\Omega)| = |\langle \phi, P(\Omega)\psi\rangle| \overset{C.S}{\leq} \|\langle \phi, P(\Omega)\rangle\|^{\frac{1}{2}} \|\langle \psi, P(\Omega)\rangle\|^{\frac{1}{2}} = \mu_\psi^{\frac{1}{2}}(\Omega)\mu_\phi^{\frac{1}{2}}(\Omega) \quad (13)$$

Thus $\mu_{\psi,\phi}$ is also absolutely continuous. Define $U(t) = e^{-iHt}$ as above, by Riemann-Lebesgue Lemma(???), we have

$$\langle \phi, P(\Omega)\psi\rangle = \int e^{-i\lambda t} d\mu_{\psi,\phi} \quad (14)$$

This implies, if we start from absolutely continuous spectral subspace, then the probability of finding the system at any ϕ is 0. However, if ψ is an eigenvector of H, then

$$|\langle \phi, U(t)\psi\rangle| = |\langle \phi, \psi\rangle| \quad (15)$$

show that the state is unchanged as time evolves.

Problem 2. *Riemann-Lebesgue Lemma(???):*

- *Version 1: Fourier transform of* L^1 *function goes to 0*
- *Version 2:* $\int_{R^+} f(t)e^{itz}dz \to 0$ *as* $|z| \to \infty$ *with* $Re(z) \geq 0$

For the long-term behavior of the system, we have the following theorem.

Theorem 4. *(Wiener) Let* μ *be a finite complex Borel measure on* $\mathbb{R}$ *and*

$$\hat{\mu}(t) := \int_{\mathbb{R}} e^{-it\lambda} d\mu(\lambda). \tag{16}$$

Then,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\hat{\mu}(t)|^2 dt = \sum_{\lambda\in\mathbb{R}} |\mu(\{\lambda\})|^2, \tag{17}$$

where the sum on the right-hand side is finite.

Remark 4. *Since every Borel measure can be decomposed as* $\mu = \mu_{ac}+\mu_{sc}+\mu_{pp}$*, and since* μ_{ac} *and* μ_{sc} *are continuous,* $\mu(\{\lambda\}) = \mu_{pp}(\lambda)$*. Then,* $\sum_{\lambda\in\mathbb{R}} |\mu(\{\lambda\})|^2 = \sum_{\lambda\in\mathbb{R}} |\mu_{pp}(\{\lambda\})|^2$*. Note the support of* μ_{pp} *is a countable set.*

Let's apply this theorem to study the long-term behavior of the system. Let $\psi \in H_{ac} \oplus H_{sc}$, and $\phi \in H$. Then, the measure $\mu_{\phi,\psi}(\{\lambda\}) = 0$. By Theorem 4, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\langle\phi, e^{-iHt}\psi\rangle|^2 dt = 0. \tag{18}$$

This implies the average of probability finding the system at state ϕ tends to zero.
We can also consider $\langle\psi, e^{-iHt}\psi\rangle$ as $P_U(t)\psi|^2$, where P denotes the orthogonal projection onto U. Inspired by this, we extend to a more general class of operators, the compact operators, which generalize the finite rank operators. An operator $K \in \mathcal{L}(H)$ is called a **finite rank operator** if its range is of finite dimension. Every finite rank operator can be written as a linear combination of projection operators.

Definition 2. *An operator* $K \in \mathcal{L}(H)$ *is called compact if* K *maps every unit ball in* H *to a pre-compact set.*

Definition 3. *An operator* $K : D(K) \to H$ *is called relatively compact with respect to an self-adjoint operator* H *if there exists* $z \in H$*, such that* $KR_z(H) = K(z - H)^{-1}$ *is compact.*

The notion of a relatively compact operator gives us the following theorem, which is handy to our goal, the RAGE theorem.

Theorem 5. *Let* H *be a self-adjoint operator,* K *be relatively compact with respect to* H*. Then, for all* $\psi \in D(H)$*,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |Ke^{-iHt}P_c(H)\psi|^2 \, dt = 0, \tag{19}$$

and

$$\lim_{T\to\infty} |Ke^{-iHt}P_c(H)\psi|^2 = 0, \tag{20}$$

where $P_c(H) = P_{ac}(H) + P_{sc}(H)$ is the orthogonal projection onto $H_{ac} \oplus H_{sc}$. Furthermore, if K is bounded, then the result holds true for all $\psi \in H$.

Finally, we have the RAGE theorem, which tells us $|Ke^{-iHt}\psi|$ can be used to identify the spectral subspaces H_c and H_{pp} of H.

Theorem 6 (RAGE)**.** *Let H be a self-adjoint operator and K_n be a sequence of relatively compact operators with respect to H, converging strongly to the identity. Then,*

$$H_c = \{\psi \in H \mid \lim_{T\to\infty} \frac{1}{T}\int_0^T |K_n e^{-iHt}\psi| = 0\} \tag{21}$$

and

$$H_{pp} = \{\psi \in H \mid \limsup_{t\to\infty} |(I-K_n)e^{-iHt}\psi| = 0\}. \tag{22}$$

4 Nonlinear stationary Schrödinger equation in 1-D

In this section, we consider the Schrödinger equation defined on $\mathbb{R}$ in the following form:

$$(-\Delta - k^2)u = f(u) \tag{23}$$

where f has the nonlinear form:

$$f(u) = V(x)u + w(x)|u|^2 u \tag{24}$$

where $V(x)$ and $w(x)$ are compactly supported and bounded. We aim to show the existence of solution with proper regularities and approximate the scattering matrix.

4.1 Existence of solution

Let's first consider the case $V(x) = 0$. The solution outside the support of the perturbation is just the superposition of plane waves. We decompose u as $u = u_{\text{out}} + u_{\text{in}}$, where $u_{\text{in}} = Ae^{ikx}$ is the incoming solution that solves the unperturbed equation, and u_{out} is a correction. Plugging into (23):

$$(-\Delta - k^2)(u_{\text{out}} + u_{\text{in}}) = f(u_{\text{out}} + u_{\text{in}}). \tag{25}$$

Rewrite our problem,

$$(-\Delta - k^2)(\tilde{u}) = f(u_{\text{out}} + Ae^{ikx}). \tag{26}$$

for $\tilde{u} \in X$, for some complete metric space X. $\tilde{u}$ is then given by

$$\tilde{u} = (-\Delta - k^2)^{-1}_{\text{out}} f(u_{\text{out}} + Ae^{ikx}) \tag{27}$$

where we used the outgoing condition $(-\Delta - k^2)^{-1}_{\text{out}} = \lim_{\epsilon \to 0^+}(-\Delta - (k^2 + i\epsilon))^{-1}$. If we show $T(u_{\text{out}}) := (-\Delta - k^2)^{-1}_{\text{out}} f(u_{\text{out}} + Ae^{ikx})$ is a contraction X, then the above equation has a unique fixed point in X satisfying $u_{\text{out}} = (-\Delta - k^2)^{-1} f(u_{\text{out}} + Ae^{ikx})$. (???)

Problem 3. *If we try to show the solution exists, is that to show $T(\tilde{u})$ exists?*

The function u must be twice differentiable, and u plus u^2 must be bounded on any compact domain. This implies that our nonlinear term $V(x)u + u(x)|u|^2$ is bounded on any compact domain. Outside the compact domain of perturbation, the solution is a plane wave. Since every term is in $L^\infty(\mathbb{R})$, we consider $u \in L^\infty(\mathbb{R})$.

Problem 4. *Can function setting can be $W^{1,2}$? or $L^2(\Omega)$?*

Now we define the outgoing Green's function. If $k^2 \notin \sigma(-\Delta)$ is in $(0, \infty)$, then the solution is:

$$\tilde{u} = [(-\Delta - k^2)^{-1}(f(u_{\text{out}} + Ae^{ikx}))](x) \tag{28}$$

with the corresponding Green's function:

$$G(x; k^2) = \frac{1}{2\sqrt{-k^2}} e^{-|x|\sqrt{-k^2}}. \tag{29}$$

The outgoing Green's function when $k \in (0, \infty)$ is

$$G_{\text{out}}(x; k^2) = \lim_{\epsilon \to 0^+} G(x; k^2 + i\epsilon) = \frac{1}{2ik} e^{i|kx|}. \tag{30}$$

We will use a variance of the fixed point argument.

Theorem 7. *Let (X, d) be a complete metric space, and $A : X \to X$. Furthermore, assume there exists a $x \in X$ and $r > 0$ such that*

1. *the ball $B(a, r)$ is an invariant set for A.*
2. *the map A is a contraction on $B(a, r)$.*

Then there exist a unique fixed point of A inside ball $B(a, r)$

Remark 5. *This theorem is one of the most important theorem to deduce the unique solution, if we can show the contraction and the invariant set properties, we can get a fixed point. If moreover, we can show that the fixed pt is non-trivial, then it is the unique solution to our problem.*

Now, the first existence theorem for the equation.

Theorem 8. *For an incoming solution $u_{in} = Ae^{ikx}$ with the amplitude A small enough, there exists an outgoing solution u_{out} for equation 5.1.*

Proof. Consider $B_{L^\infty}(-Ae^{ikr}, r)$, we want to show T is invariant on this ball for suitable r. Let $v \in B_{L^\infty}(-Ae^{ikr}, r)$, we have

$$\begin{aligned}
\|T(v)\|_\infty &\leq \text{ess sup}_x \left| \frac{1}{2ik} e^{-ikx} \int_{\mathbb{R}} w(y)v(y) + Ae^{iky} dy \right| \\
&\leq \frac{1}{2k} \int_{\mathbb{R}} |w(y)v + Ae^{iky}| dy \\
&\leq \frac{1}{2k} \left(\|w\|_1 \|v\|_\infty + \|Ae^{iky}\|_1 \right) \\
&\leq \frac{\|w\|_1}{2k} \left(\|v\|_\infty + |A| \right).
\end{aligned}$$

Thus r needs to satisfy

$$\frac{\|w\|_1}{2k} r^3 + |A| \leq r. \tag{31}$$

For contraction, let $v, v' \in B_{L^\infty}(-Ae^{ikr}, r)$ and set $a(x) = v(x) + Ae^{ikx}$, $b(x) = v'(x) + Ae^{ikx}$. Note $\|a\|_\infty \leq r$ and $\|b\|_\infty \leq r$. Then,

$$\begin{aligned}
\|T(v) - T(v')\|_\infty &\leq \frac{\|w\|_1}{2k} \|v - v' + Ae^{ikx}(v - v')\|_\infty \\
&= \frac{\|w\|_1}{2k} \left(\|a^2 - b^2\|_\infty \right) \\
&= \frac{\|w\|_1}{2k} \left(\|a + b\|_\infty (a - b) + ab(a - b) \right)_\infty \\
&\leq \frac{\|w\|_1^3}{2k} \|v - v'\|_\infty.
\end{aligned}$$

We then require

$$r < \sqrt[3]{\frac{2k}{\|w\|_1}}. \tag{32}$$

For suitable $|A| \neq 0, M$ and $\|u\|_\infty$ such that there exists a r satisfying the inequalities above, the contraction mapping gives us a non-trivial outgoing solution, since $u_{\text{out}} = 0$ is not a fixed point. The global solution $u_{\text{in}} + u_{\text{out}}$ is also not trivial since $u_{\text{out}} = Ae^{ikr}$ is not a fixed point. □

Problem 5. *I don't understand the last line, is that mean $\tilde{u} = u_{in} + u_{out} = Ae^{ikr}$ is not a fixed point?*

Remark 6. *Now, we move our focus to a general requirement for when (31) and (32) satisfied. I.e., when there exists such r makes the fixed point arguement satisfied.*
Define

$$p := -\frac{2k}{\|w\|_1} \quad \text{and} \quad q := -\frac{2k}{\|w\|_1}\sqrt{\frac{3}{27}} \tag{33}$$

We want to find the intersection of

$$r^3 + p + q = -pr \tag{34}$$

In order to have a real $r > 0$ to $\frac{2k}{\|w\|_1}$, we must have two intersections in the first quadrant. The last intersection must lay in the third quadrant. This implies that all three roots must be real. Thus p and q must satisfy $4p^3 + 27q^2 \leq 0$.
In our case,

$$-\left(\frac{3\sqrt{3}|A|}{\|w\|_1}\right)^3 - \left(\frac{|A|}{\|w\|_1}\right)^2 = -D \leq 0 \tag{35}$$

for some $D \geq 0$. Equivalently, $|A| \leq \sqrt{\frac{8k}{27\|w\|_1}}$. In addition, using the Cardano formula, the root of 5.11 is given by

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q^2}{4} + \frac{p^3}{27}\right)}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q^2}{4} + \frac{p^3}{27}\right)}}$$

This reduces to

$$x_n = 2R^3 \cos\left(\frac{\theta + 2\pi n}{3}\right), \quad n = 0, 1, 2$$

where $R = \sqrt{\frac{|A|^2k^2}{\|w\|_1^2} + D} = \sqrt{\left(\frac{2k}{3\|w\|_1}\right)^3}$ and $\theta = \tan^{-1}\left(-\frac{\sqrt{D}}{k|A|}\right)$. Then $\theta \in \left[\frac{1}{2}\pi, \frac{3}{2}\pi\right]$ and we have two roots greater than zero.

If we apply the constraint (32), the second largest root $x_1 = 2R^3\cos\left(\frac{\theta - \frac{2\pi}{3}}{3}\right)$ must be less than $\sqrt{\frac{2k}{3\|w\|_1}}$, so there exists a r to have a contraction. Noticing $\cos\left(\frac{\theta - \frac{2\pi}{3}}{3}\right) < \frac{1}{2}$, so

$$x_1 < R^3\sqrt{\frac{2k}{3\|w\|_1}}.$$

To summarize, our fixed point argument works as long as $|A| \leq \sqrt{\frac{8k}{27\|w\|_1}}$ or a bound on a form of energy

$$|\lambda|^2\|w\|_1 = |A|^2\|w\|_1 \leq \frac{8k}{27}.$$

where λ corresponding to eigenvalue of A.

Proof. (For case of $V \not\equiv 0$)
The proof strategy is the same as proof for Theorem 8, we divided into 2 steps:
Step 1: Show that T is invariant on $B_{L^\infty}(-Ae^{ikx}, r)$.
Considering $v \in B_{L^\infty}(-Ae^{ikx}, r)$, then

$$T(v) \leq \frac{\|V\|_1}{2k} r + \frac{\|w\|_1}{2k} r^3.$$

To have T invariant on $B_{L^\infty}(-Ae^{ikx}, r)$, we require

$$\frac{\|V\|_1}{2k} r + \frac{\|w\|_1}{2k} r^3 + |A| \leq r. \tag{36}$$

Step 2: Show that T is contraction on $B_{L^\infty}(-Ae^{ikx}, r)$.
Let $v, \tilde{v} \in B_{L^\infty}(-Ae^{ikx}, r)$, then

$$\|T(v) - T(\tilde{v})\|_\infty \leq \frac{\|V\|_1}{2k} \|v - \tilde{v}\|_\infty + \frac{3r^2 \|w\|_1}{2k} \|v - \tilde{v}\|_\infty.$$

This simplifies to

$$\|T(v) - T(\tilde{v})\|_\infty = \left(\frac{\|V\|_1}{2k} + \frac{3r^2 \|w\|_1}{2k} \right) \|v - \tilde{v}\|_\infty.$$

To have T contraction, we require

$$\left(\frac{\|V\|_1}{2k} + 3r^2 \frac{\|w\|_1}{2k} \right) \leq 1. \tag{37}$$

For A small, there exists an r that satisfies the inequalities.

□

4.2 Analysis on Scattering Matrix (Chapter 12)

Consider the case $V = 0$. We approximate the scattering matrix by approximating the fixed point, assuming the incoming wave Ae^{ikx} we sent in is small (i.e., $|A|$ small). Let's start from $u_0 = 0$, the first iteration gives

$$u_1(x) = T(u_0)(x) = -\frac{1}{2ik} |A|^2 \int_{\mathbb{R}} e^{ik|x-y|} w(y) A e^{iky} \, dy. \tag{5.16}$$

For the next iteration, defining

$$h(y) = e^{-iky} \int_{\mathbb{R}} e^{ik|y-x|} w(x) e^{ikx} \, dx, \tag{36}$$

and keeping order up to $|A|^5$, we use the approximation

$$u_2 = u_1 + \frac{1}{4k^2} |A|^4 \int_{\mathbb{R}} e^{ik|x-y|} w(y) A e^{iky} (h(y) - 2h(y)) \, dy + O(|A|^7).$$

We will use

$$u_{\text{out}} = u_1 + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} e^{ik|x-y|} w(y) A e^{iky} (h(y) - 2h(y)) dy \tag{37}$$

as an approximation to the outgoing solution. When x is at the right of the perturbation which corresponds to transmission, we have

$$u_{\text{out}} = Ae^{ikx} \left(-\frac{1}{2ik}|A|^2 \int_{\mathbb{R}} w(y)\, dy + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} w(y)(h(y) - 2h(y))\, dy \right) \tag{38}$$

$$=: Ae^{ikx} T_-. \tag{39}$$

When x is at the left of the perturbation which corresponds to reflection, we have

$$u_{\text{out}} = Ae^{-ikx} \left(-\frac{1}{2ik}|A|^2 \int_{\mathbb{R}} w(y) e^{2iky}\, dy + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} w(y) e^{2iky} (h(y) - 2h(y))\, dy \right) \tag{40}$$

$$=: Ae^{-ikx} R_-. \tag{41}$$

Now assume incoming wave from right $u_{\text{in}} = e^{-ikx}$. Note that the Green's function is unchanged. Then,

$$u_{\text{out}} = -\frac{1}{2ik}|A|^2 \int_{\mathbb{R}} e^{ik|x-y|} w(y)(Ae^{-iky}\, dy + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} e^{ik|x-y|} w(y)(Ae^{-iky}(g(y) - 2g(y))\, dy$$

where

$$g(y) = e^{iky} \int_{\mathbb{R}} e^{ik|y-x|} w(x) e^{-ikx} dx. \tag{42}$$

When x is at the left of the perturbation which corresponds to transmission, we have

$$u_{\text{out}} = Ae^{-ikx} \left(-\frac{1}{2ik}|A|^2 \int_{\mathbb{R}} w(y)\, dy + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} w(y)(g(y) - 2g(y))\, dy \right) \tag{43}$$

$$=: Ae^{-ikx} T_+. \tag{44}$$

When x is at the right of the perturbation which corresponds to reflection, we have

$$u_{\text{out}} = Ae^{ikx} \left(-\frac{1}{2ik}|A|^2 \int_{\mathbb{R}} w(y) e^{-2iky}\, dy + \frac{1}{4k^2}|A|^4 \int_{\mathbb{R}} w(y) e^{-2iky} (g(y) - 2g(y))\, dy \right) \tag{45}$$

$$=: Ae^{ikx} R_+. \tag{46}$$

The scattering matrix is given by the coefficients given above.

In the case $V \not\equiv 0$, we compute the linearized solution:

$$u_{\text{out}} = -\frac{1}{2ik} \int_{\mathbb{R}} e^{ik|x-y|} [V(y) + w(y)|A|^2] A e^{iky}\, dy.$$

Following the same idea as above, we conclude that the scattering coefficients are:

$$\begin{cases} T_- &= -\frac{1}{2ik}\int_{\mathbb{R}}[V(y)+w(y)|A|^2]\,dy, \\ R_- &= -\frac{1}{2ik}\int_{\mathbb{R}}e^{2iky}[V(y)+w(y)|A|^2]\,dy, \\ T_+ &= -\frac{1}{2ik}\int_{\mathbb{R}}[V(y)+w(y)|A|^2]\,dy, \\ R_+ &= -\frac{1}{2ik}\int_{\mathbb{R}}e^{-2iky}[V(y)+w(y)|A|^2]\,dy. \end{cases}$$

In the linear approximation, We see that the reflection coefficient is given by the Fourier transform of $V(y)+w(y)|A|^2$. Using two different incoming solutions, we have a system of equations. We can recover potential by taking the inverse Fourier transform of the reflection coefficients.

5 Nonlinear stationary Dirac Equation with a linear domain wall

In this section we try to do the same thing for the non-linear stationary Dirac equation with a linear domain wall, including the existence of solution and scattering matrix analysis as before, the function we consider is:

$$(H-E)\psi(x,y) = f(\psi)(x,y), \quad H = D_x\sigma_3 - D_y\sigma_2 + y\sigma_1, \quad f(\psi) = w(x,y)(\psi^*C\psi)\psi. \tag{47}$$

where $w(x,y)$ is compactly supported and bounded, and C is a constant matrix. We decompose $\psi = \psi_{\text{in}} + \psi_{\text{out}}$ as before.

5.1 1-D Dirac-equation

We consider the simplier case for our function,

$$(D_x\sigma_3 - k)\psi(x) = w(x)(\psi^*C\psi)\psi \tag{48}$$

Outside the support of $w(x)$, the solution is

$$\psi_{\text{in}} = \begin{pmatrix} A_1 e^{-ikx} \\ A_2 e^{ikx} \end{pmatrix} \tag{49}$$

Observe that

$$(D_x\sigma_3 - k)(D_x\sigma_3 + k) = (-\Delta - k^2)I_2, \tag{50}$$

so

$$(D_x\sigma_3 - k)^{-1}_{\text{out}} = (D_x\sigma_3 + k)(-\Delta - k^2)^{-1}_{\text{out}}, \tag{51}$$

where applying $(D_x\sigma_3 + k)$ does not change the outgoing condition. Thus the outgoing Green's function of $D_x\sigma_3 - k$ is given by:

$$G_{\text{out}}(x;k) = \frac{1}{2ik}\begin{pmatrix} D_x + k & 0 \\ 0 & -D_x + k \end{pmatrix}\begin{pmatrix} -\frac{1}{2ik}e^{ik|x|} & 0 \\ 0 & -\frac{1}{2ik}e^{ik|x|} \end{pmatrix} \tag{52}$$

$$= \begin{pmatrix} \frac{i}{2}e^{ik|x|}(1-\text{sgn}(k)\text{sgn}(x)) & 0 \\ 0 & \frac{i}{2}e^{ik|x|}(1+\text{sgn}(k)\text{sgn}(x)) \end{pmatrix}. \tag{53}$$

Define T:

$$T(\psi) = (D_x\sigma_3 - k)^{-1}_{\text{out}} f(\psi + \psi_{\text{in}}).$$

for $\psi \in L^\infty(\mathbb{R}, \mathbb{C}^2)$.

Remark 7. *The function setting is basiclly the same reason as in chap 4, the following is the illustration. Note that $L^\infty(\mathbb{R}, \mathbb{C}^2)$ is a Banach space with respect to $\|\psi\| = ess\ sup_x \|\psi(x)\|$, where the latter norm can be any ℓ_p norm. Indeed, let ψ^n be a Cauchy sequence in $L^\infty(\mathbb{R}, \mathbb{C}^2)$, and let $\begin{pmatrix}\psi_1^n \\ \psi_2^n\end{pmatrix}$ be the elementwise limit of $\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$ in $L^\infty(\mathbb{R}, \mathbb{C}^2)$, then obviously $\begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$ is also the limit of ψ^n in $L^\infty(\mathbb{R}, \mathbb{C}^2)$.*

Theorem 9. *For an incoming solution $\psi_{in} = \begin{pmatrix}A_1 e^{-ikx} \\ A_2 e^{ikx}\end{pmatrix}$ with amplitude small enough, there exists an outgoing solution ψ_{out} for equation (48)*

Proof. The idea is based on theorem 7, we just need to check the invariant set and contraction properties for certain constraint on r.
We first show T is invariant on $B_{\|\cdot\|}(-\psi_{\text{in}}, r)$ for suitable r. Let $\psi \in B_{\|\cdot\|}(-\psi_{\text{in}}, r)$ and σ_c be the largest singular value of C, we have:

$$\begin{aligned}
\|T(\psi)\| &= \|G_{\text{out}}(x;k) * w(x)[(\psi + \psi_{\text{in}})^* C(\psi + \psi_{\text{in}})(\psi + \psi_{\text{in}})\| && (54)\\
&\le [(\int_{\mathbb{R}} |w(y)||(\psi + \psi_{\text{in}}^* C(\psi + \psi_{\text{in}})(\psi + \psi_{\text{in}})||\psi_1 + A_1 e^{-iky}|\ dy)^2 && (55)\\
&+ (\int_{\mathbb{R}} |w(y)||(\psi + \psi_{\text{in}}^* C(\psi + \psi_{\text{in}})(\psi + \psi_{\text{in}})||\psi_1 + A_2 e^{iky}|\ dy)^2]^{\frac{1}{2}} && (56)\\
&\le \sqrt{2}\sigma_c \|w\|_1 r^2. && (57)
\end{aligned}$$

We then require $\sqrt{2}\sigma_c\|w\|_1 r^2 + \sqrt{|A_1|^2 + |A_2|^2} \le r$
Let $\psi, \phi \in B_{\|\cdot\|}(-\psi_{\text{in}}, r)$, we show T is a contraction. For simplicity, let's write $a := \psi + \psi_{\text{in}}$ and $b := \phi + \psi_{\text{in}}$, then by the Hölder inequality, $\|T(\psi) - T(\phi)\|$ is bounded by

$$\left\| \int_{\mathbb{R}} |w(y)|(|Ca| - |Cb|)\, dy \right\|^2 + \left(\text{ess sup}_y(|a^*Ca| - |b^*Cb|)\right)^{1/2}$$

The bound for $T(\psi) - T(\phi)$ is expressed as:

$$\sqrt{2\|w\|_1}\,\text{ess sup}_y\,|(a^*Ca) - (b^*Cb)| \le \sqrt{2\|w\|_1}\,\text{ess sup}_y\,(|(a^*Ca) - (b^*Cb)|)$$

which simplifies to

$$\sqrt{2\|w\|_1}\,\text{ess sup}_y\,(|(a^*Ca) - (b^*Cb)|)\,. \tag{58}$$

Since we want $\|T(\psi) - T(\phi)\|$ to be bounded by some factor of $\|\psi - \phi\|$, we rewrite it as

$$\sqrt{2\|w\|_1}\,\text{ess sup}_y \left(|(a^*Ca + b^*Cb)(a-b) + (a^*Ca)b - (b^*Cb)a|\right) \tag{59}$$

where the first part,

$$|(a^*Ca + b^*Cb)(a-b)| \leq \sigma_c(|a|^2 + |b|^2)|a-b|, \tag{60}$$

and expanding the second part,

$$|(a^*Ca)b - (b^*Cb)a| = |a||b|\,||C^*a| - |C^*b|| \leq \sigma_c|a||b||a-b|. \tag{61}$$

Thus,

$$|(a^*Ca)b - (b^*Cb)a| \leq \sigma_c\|a\|\|b\||a-b|. \tag{62}$$

Plugging equations (60) and (62) into (59), we obtain

$$\begin{aligned} \|T(\psi) - T(\phi)\| &\leq \sqrt{2}\|w\|_1\,\text{ess sup}_y \left(|a|^2 + |b|^2\right)|a-b| + \sigma_c|a||b||a-b| \\ &\leq 3\sqrt{2}\sigma_c\|w\|_1\, r^2\|a-b\| = 3\sqrt{2}\sigma_c\|w\|_1\, r^2\|\psi - \phi\|. \end{aligned}$$

We thus require the parameters to satisfy:

$$3\sqrt{2}\sigma_c\|w\|_1\, r^2 \leq 1, \quad \text{and} \quad |A|^2\|w\|_1 \leq \frac{4\sqrt{2}}{27\sigma_a}.$$

Moreover, we compute its scattering matrix and get,

$$\begin{cases} T_+ &= \begin{pmatrix} i\int_{\mathbb{R}} w(y) \begin{pmatrix} A_1e^{-iky} \\ A_2e^{iky} \end{pmatrix}^* C \begin{pmatrix} A_1e^{-iky} \\ A_2e^{iky} \end{pmatrix} dy \\ i\int_{\mathbb{R}} w(y)(\begin{pmatrix} A_1e^{-iky} \\ A_2e^{iky} \end{pmatrix}^* C \begin{pmatrix} A_1e^{-iky} \\ A_2e^{iky} \end{pmatrix} dy \end{pmatrix} \\ T_+ &= \begin{pmatrix} i\int_{\mathbb{R}} w(y) \begin{pmatrix} A_1e^{iky} \\ A_2e^{-iky} \end{pmatrix}^* C \begin{pmatrix} A_1e^{iky} \\ A_2e^{-iky} \end{pmatrix} dy \\ i\int_{\mathbb{R}} w(y)(\begin{pmatrix} A_1e^{iky} \\ A_2e^{-iky} \end{pmatrix}^* C \begin{pmatrix} A_1e^{iky} \\ A_2e^{-iky} \end{pmatrix} dy \end{pmatrix} \\ R_+ &= 0 \\ R_- &= 0 \end{cases}$$

□

6 2D nonlinear Dirac-equation

In this section, we follow the same idea to show the existence of solution to (47), by finding unperturbed solution, constructing outgoing Green's function, and using theorem 7 (Check two properties for suitable r). *"Asymmetric transport computations in dirac models of topological insulators"*, the first two part are in here.

6.1 unperturbed solution

The operator H in (47) is translation invariant in the x direction, thus we take the Fourier transform in x. Denoting ξ as the Fourier variable,

$$\hat{H}(\xi) - E = \xi\sigma_3 - D_y\sigma_2 + y\sigma_1 - E = \begin{pmatrix} \xi & \partial_y + y \\ -\partial_y + y & -\xi \end{pmatrix} - E.$$

Note $a = \partial_y + y$ is the creation operator, and $a^* = -\partial_y + y$ is the annihilation operator. It is useful to look at the block diagonal matrix:

$$\hat{H}(\xi)^2 = \begin{pmatrix} \xi^2 + aa^* & 0 \\ 0 & \xi^2 + a^*a \end{pmatrix}.$$

We define $\phi_n(y) = a_n(a^*)^n\phi_0(y)$ (where a_ns are the normalizing constants), then $\phi_n(y)$ are Hermite functions that form an orthonormal basis of $L^2(\mathbb{R}_y)$, and satisfy the following properties:

$$a^*a\phi_n = 2n\phi_n, \quad a\phi_n = \sqrt{2n}\phi_{n-1}, \quad a^*\phi_n = \sqrt{2(n+1)}\phi_{n+1}, \quad \phi_0 = \pi^{-1/4}e^{-y^2/2}.$$

We define a set M consisting of indices $m = (n, \epsilon_m)$, where $N \geq 1$ and $\epsilon_m = \pm 1$. In the case $n = 1$, we define $m = (0, -1)$. The eigenvalue of $\hat{H}(\xi)$ are $E_m = \epsilon_m(2n + \xi^2)$. Now for any $\xi \in \mathbb{R}$ and m, we define

$$\phi_m = c_m \begin{pmatrix} a\phi_n \\ (E_m - \xi)\phi_n \end{pmatrix}, \quad n \geq 1,$$

where $c_m = \frac{1}{\sqrt{2n+(E_m-\xi)^2}}$ is the normalizing constant. In the case $n = 0$, $m = (0, -1)$ and $E_0(\xi) = -\xi$, we define

$$\phi_0 = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}.$$

The above family of eigenvectors ϕ_m form a basis of $L^2(\mathbb{R}, \mathbb{C}^2)$. The above discussion characterized the spectrum decomposition of $\hat{H}(\xi)$.
To solve equation (47) for a fixed $E \in \mathbb{R}$, we need to reverse the map $E_m = \epsilon_m(2n + \xi^2)\frac{1}{2}$:

$$\xi_m = \epsilon_m\sqrt{E^2 - 2n} = \begin{cases} \epsilon_m\sqrt{E^2 - 2n}, & E^2 \geq 2n, \\ i\epsilon_m\sqrt{2n - E^2}, & E^2 < 2n. \end{cases}$$

Then, $\phi_m(x, y; E)$ satisfy:

$$(\hat{H}(\xi_m) - E)\phi_m = \begin{pmatrix} \xi_m - E & a \\ a^* & -(\xi_m + E) \end{pmatrix}\phi_m = 0.$$

In the physical domain, the generalized eigenvectors

$$\psi_m(x, y; E) = e^{i\xi_m x}\phi_m(y; E)$$

satisfy $(H - E)\psi_m = 0$. Linear combinations of these eigenvectors then give the solutions to the unperturbed equation.

6.2 Outgoing Green's function

To construct the Outgoing Green's function for $(H - E)$, we need to solve $(H - E)G = \delta(x - x_0)\delta(y - y_0)$. Since H is translation invariant in the x direction, we assume $x_0 = 0$. Note that

$$(H + E)(H - E)G = (H + E)\delta(x)\delta(y - y_0),$$

then

$$G = (H + E)(H^2 - E^2)^{-1}\delta(x)\delta(y - y_0).$$

As in the one-dimensional case, we need to first find $(H^2 - E^2)^{-1}\delta(x)\delta(y - y_0)$, then apply $(H + E)$ on it. Noting $H^2 - E^2$ is diagonalized to:

$$\begin{pmatrix} D_x^2 - E^2 + aa^* & 0 \\ 0 & D_x^2 - E^2 + a^*a \end{pmatrix},$$

we thus need to solve

$$(-\partial_x^2 - \partial_y^2 + y^2 + 1 - E^2)G_+ = \delta(x)\delta(y - y_0).$$

Recall that $\phi_n(y)$ are the eigenfunctions to $-\partial_y^2 + y^2 = 1$, we expand G_- in the basis of Hermite functions $\phi_n(y)$:

$$G_- = \sum_n G_{-,n}(x)\phi_n(y).$$

Then,

$$(-\partial_x^2 - E^2 + 2n)G_{-,n}(x) = \delta(x)\phi_n(y_0)$$

is the Helmholtz equation, and the outgoing Green's function when $2n < E^2$ is given in 5.8. Assuming $E^2 \neq 2n$ for $n \in \mathbb{N}$, we have:

$$G_{-,n}(x) = \frac{\phi_n(y_0)}{2\sqrt{E^2 - 2n}} e^{-\sqrt{2n-E^2}|x|}, \quad 2n > E^2.$$

$$G_{-,n}(x) = \frac{\phi_n(y_0)}{2i\sqrt{2n - E^2}} e^{i\sqrt{E^2-2n}|x|}, \quad 2n < E^2.$$

The computation of G_+ is similar by replacing $2n$ to $2n + 2$. Define $\theta_n = i\sqrt{E - 2n}$, then

$$G_-(x, y; y_0) = \sum_{n\geq 0} \frac{e^{-|\theta_n||x|}}{2|\theta_n|}\phi_n(y)\phi_n(y_0),$$

$$G_+(x, y; y_0) = \sum_{n\geq 0} \frac{e^{-|\theta_{n+1}||x|}}{2|\theta_{n+1}|}\phi_n(y)\phi_n(y_0).$$

Since applying $H + E$ does not change the outgoing condition, the outgoing Green's function of $H - E$ is:

$$G(x, y; y_0) = ((D_x + E)G_+ \quad aG_-a^*G_+ \quad (-D_x + E)G_-).$$

Remark 8. *As stated in "Asymmetric transport computations in dirac models of topological insulators", the Green's function G has $\frac{1}{r}$ singularity.*

6.3 Contraction mapping and invariant set check

Let's first notice that the Green's function blows up like $\frac{1}{r}$ near the singularity, which implies G is integrable over any compact interval for fixed x and y. Now let's introduce the main theorem.

Theorem 10. *For an incoming solution with amplitude small enough, there exists an outgoing solution of (47)*

Proof. As in the previous proof, we define the solution operator

$$T(\psi) = G * f(\psi + \psi_{\text{in}})(x, y).$$

Also, we use the uniform bound on the integral of G

$$\max\left(\|(D_x + E)G\|_{L^1}, \|a^*G\|_{L^1}, \|(-D_x + E)G\|_{L^1}, \|aG\|_{L^1}\right) < C,$$

where the integral is taken over the compact support of $w(x, y)$. The incoming solution is

$$\psi_{\text{in}} = \sum_{m \in M} a_m \psi_m.$$

We first show T is invariant in $B(-\psi_{\text{in}}, r)$ for some $r > 0$. Using the norm we introduced in the one-dimensional case, we have

$$\|T(\psi)\| \leq \sigma_c \text{ ess sup}_{(x,y)} \left(\|(D_x + E)G\| + \|aG\| + \|(-D_x + E)G\| + \|a^*G\|\right) \text{ ess sup}_{(x,y)}(\|\psi + \psi_{\text{in}}\|^2),$$

using the Hölder inequality and rearranging terms, denoting σ_c as the largest singular value of C, we bound

$$\|T(\psi)\| \leq \sigma_c \text{ ess sup}_{(x,y)} \left(\|(D_x + E)G\| + \|a^*G\| + \|(-D_x + E)G\| + \|aG\|\right) \text{ ess sup}_{(x,y)}(\|\psi + \psi_{\text{in}}\|^2).$$

We apply the Hölder inequality again, and use the bound c to obtain:

$$\|T(\psi)\| \leq \sigma_c \|w\|_{L^\infty} \left(\text{ess sup}_{(x,y)}[\|\psi + \psi_{\text{in}}\|^2]^{1/2} + \text{ess sup}_{(x,y)}[\|\psi + \psi_{\text{in}}\|^2]^{1/2}\right)^2 \leq 2\sigma_c \|w\|_{L^\infty} r^3.$$

Thus, we need:

$$20\sigma_c \|w\|_{L^\infty} r^3 + \sum_{m \in M} |\lambda_m| \leq r.$$

Let $\psi, \phi \in B(-\psi_{\text{in}}, r)$, we show T is a contraction. Similarly, let's write

$a(x_0, y_0) := (\psi + \psi_{\text{in}})(x_0, y_0)$, then:

$$\|T(\psi) - T(\phi)\| = \text{ess sup}_{(x,y)} \left| \int [(D_x + E)G_+(x, y; x_0, y_0)] \, (a(x_0, y_0)(a^*Ca)_1 - (b^*Cb)_1) \, dx \, dy \right| \tag{63}$$

$$+ \left| \int aG_-(x, y; x_0, y_0)(a(x_0, y_0)(a^*Ca)_2 - (b^*Cb)_2) \, dx \, dy \right| \tag{64}$$

$$+ \left| \int (-D_x + E)G_-(x, y; x_0, y_0)(a(x_0, y_0)(a^*Ca)_2 - (b^*Cb)_2) \, dx \, dy \right| \tag{65}$$

$$+ \left| \int a^*G_+(x, y; x_0, y_0)(a(x_0, y_0)(a^*Ca)_1 - (b^*Cb)_1) \, dx \, dy \right|. \tag{66}$$

Similar to calculating the operator norm of T, we obtain:

$\|T(\psi) - T(\phi)\| \leq 2\|c\|\|w\|_{L^\infty} \text{ ess sup}_{(x_0,y_0)} [(|a^*Ca_1 - b^*Cb_1| + |a^*Ca_2 - b^*Cb_2|)]$,

which we have already shown the bound in 1-D case. Thus,

$$\|T(\psi) - T(\phi)\| \leq 6\sigma_c\|w\|_{L^\infty} r^2 \|\psi - \phi\|.$$

We require:

$$60\sigma_c\|w\|_{L^\infty} r^2 < 1.$$

Combining two inequalities, we have the bound:

$$\left(\sum_m |\lambda_m| \right)^2 \|w\|_{L^\infty} \leq \frac{2}{27\sigma_c}.$$

□

7 Summary

From this paper, I have learned a general steps for solving a Quantum system equation by first solving the unperturbed/homogeneous (serving for u_{in}), and second finding the outgoing Green's function, and finally using the fixed point theorem and checking the nontrivial property of the solution (Also scattering matrix ?).

7.1 Technical details

The following includes the Technical details show up in this paper:

- Fixed Point Theorems: The core mathematical tool used in the thesis is fixed point theory, which is crucial for proving the existence of solutions to nonlinear PDEs. Specific fixed point theorems mentioned include:
  1. Banach Fixed Point Theorem: Used to show the existence of unique solutions under contraction mappings.
  2. Contraction Mapping Principle: Applied to ensure the uniqueness and existence of solutions within defined conditions.
- Fourier Transforms: using in solving the unperturbed case for nonlinear PDES
- Spectral Theory: decomposing the operator subspace.
- Scattering Theory: Used to examine how wave functions evolve and interact under the influence of a scattering potential, particularly in the analysis of the scattering matrix.
- Green's Functions: Using to get the nonlinear PDEs general solution (same idea as in Chap 2 in Evans)
- Unitary Transformations and Operators: These are utilized to describe time evolution in quantum mechanics and to ensure that solutions respect the conservation of probability and other physical properties.
- Hermite Functions: Used in the expansion of solutions, for harmonic oscillator where orthogonality and specific eigenvalue problems are discussed. (HW 7 Functional Analysis)
- Checking the fixed point theorem require two thing:
  1. Define the $T(\psi)$, the solution contraction map and we want a bound like the form
  $$\|T(\psi)\| \leq C\|w\|r^{\cdot} \tag{67}$$
  2. Moreover, we need the contraction mapping by showing that given $\psi, \hat{psi}$, we have the form like
  $$\|T(\psi) - T(\hat{\psi})\| \leq c\|\psi - \hat{\psi}\| \tag{68}$$
  for constant $c < 1$. Recall in Functional Analysis Course HW set, we have used mean value theorem, Lipschitz continuous, time invariant translation, finite iteration, etc to decude this property.

7.2 Auxillary Reading

There's some auxiliary reading needed for some techinical issue:

1. *Guillaume Bal, Jeremy G. Hoskins, and Zhongjian Wang. Asymmetric transport computations in dirac models of topological insulators. Journal of Computational Physics, 487:112151, 2023.* (The strategy to deducing the unperturbed solution and Green's function and singularity $\frac{1}{r}$)

2. Chapter 5 and Chapter 12 of Mathematical Methods in Quantum Mechanics: With Applications to Schrodinger Operators.

3. The reason why we need to check scattering matrix (transmission and reflection coefficient with respect to perturbation)

4. The function space, we are using. I.e., Consider the $W^{1,2}$, the $|u|^2$ term can also be bounded by Poincare lemma. (Check more on *Leoni, A First Course in Sobolev Space)*

5. The RAGE Theorem, I actually somehow lose sense of what this section want us to do.