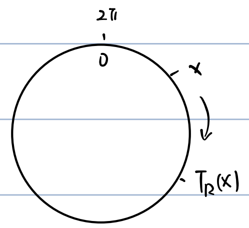
Ex: unif $([0,2\pi])$ "distribution"

for $\forall x \in [0,2\pi]$. $T_R(x) = \begin{cases} x+1, & x+1 \le 2\pi \\ x+1-2\pi, & x+1 > 2\pi. \end{cases}$



Notation: $T_R(T_R(x)) = T_R^2(x)$

$T_R^{(\ell)}(x) =$

Axiom of choice

for each $\lambda \in \Lambda$. select $x_\lambda \in X_\lambda$

$A = \{x_\lambda : \lambda \in \Lambda\}$.

$\underline{P(A)}$? "motivation:"

Define: $\underline{\text{equivalence relation}}$

$x \sim y$ if $y = T_R^{(\ell)}(x)$

or $y = T_L^{(\ell)}(x)$

for some $\ell = 1, 2, \dots$

similarly, define $T_L(x)$ "reverse order"

$[0,2\pi] = \bigcup_{\lambda \in \Lambda} X_\lambda$ disjoint union.

X_λ is an $\underline{\text{equivalence class}}$

Ans: $[0,2\pi] = \bigcup_{\lambda\in\Lambda} X_\lambda = A \cup \left(\bigcup_{i=1}^{\infty}(A+i)\right) \cup \left(\bigcup_{i=1}^{\infty}(A-i)\right) = A \cup \left(\bigcup_{i=1}^{\infty} T_R^{(i)}(x)\right) \cup \left(\bigcup_{i=1}^{\infty} T_L^{(i)}(x)\right)$ "more formal"

$\mathbb{P}([0,2\pi]) = 1 = \mathbb{P}(A) + \sum_{i=1}^{\infty}\mathbb{P}(A+i) + \sum_{i=1}^{\infty}\mathbb{P}(A-i) = 2\sum_{i=1}^{\infty}\mathbb{P}(A) + \mathbb{P}(A) = 1$ (contradiction)

Non-measurable, no Ans !!!

Set limits:

$A_n \uparrow \quad A_n \subseteq A_{n+1} \qquad \lim_n A_n = \bigcup_{n=1}^{\infty} A_n$

$A_n \downarrow \quad A_n \supseteq A_{n+1} \qquad \lim_n A_n = \bigcap_{n=1}^{\infty} A_n$

for arbitrary sequence of A_n:

$\liminf_n A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k$

$\limsup_n A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$

$\begin{cases} \liminf_n A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k \\ \limsup_n A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \end{cases}$ + $\begin{cases} \liminf_n f_n = \lim_{n\to\infty} \inf_{k\ge n} f_k. \\ \limsup_n f_n = \lim_{n\to\infty} \sup_{k\ge n} f_k. \end{cases}$

$\Downarrow$

for a set A,

$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$

HW1: 1) $\liminf_n \mathbb{1}_{A_n}(x) = \mathbb{1}_{\liminf A_n}$

2) $\limsup_n \mathbb{1}_{A_n}(x) = \mathbb{1}_{\limsup A_n}$

$x \in \liminf_n A_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right)$ for fixed n. [$x \in A_n$ except finite sets]

$x \in \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \Rightarrow x \in$ infinitely many set.

$\Rightarrow \liminf A_n \subset \limsup A_n$ (?)

Def: (6-field/Algebra) Ω sample space, $\mathcal{A} \subseteq 2^{\Omega}$ (i.e. the set of all subsets). "mathcal" - latex.

i) $\emptyset$ and Ω are in $\mathcal{A}$.

ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

iii) $(A_i)_{i\geq 1} \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. $\Longrightarrow A \cap B = (A^c \cup B^c)^c \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$.

Def: $\mathcal{E} \subseteq 2^{\Omega}$, $\underline{6(\mathcal{E})} :=$ the smallest 6-field that contains $\mathcal{E}$. (Borel - sigma - Algebra)

$$\begin{cases} 6(\mathcal{E}) \text{ is a } 6\text{-Algebra} \\ \forall\ 6\text{-field}, \mathcal{F} \supseteq \mathcal{E} \Rightarrow 6(\mathcal{E}) \subset \mathcal{F} \end{cases}$$

easy to check!

Lemma: If $\mathcal{A}_\lambda$ is 6-field so is $\bigcap_{\lambda \in \Lambda} \mathcal{A}_\lambda$.

i.e Given $\mathcal{E} \in 2^{\Omega}$ (power set), $6(\mathcal{E}) = \bigcap_{i=1}^{\infty} \mathcal{F}_i$ s.t $\mathcal{E} \in \mathcal{F}_i$.

But we have to show that $\bigcap \mathcal{F}_i$ is indeed an 6-Algebra, countable union may not! (HW1).

Given a subset if $\forall A, B \in \mathcal{P}$, we have $A \cap B \in \mathcal{P} \Rightarrow \mathcal{P}$ is $\underline{\pi\text{-system}}$!

Def: ($\underline{\pi\text{-system}}$): $\mathcal{P} \subseteq 2^{\Omega}$ is a π-syst if $A, B \in \mathcal{P} \Rightarrow A \cap B \in \mathcal{P}$. (i)

Def: (field): $\mathcal{A} \subseteq 2^{\Omega}$ is called a field if
$$\begin{cases} \emptyset \in \mathcal{A}. \\ A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}. \\ A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}. \text{ (finite)}. \end{cases}$$

only different!

"Def: (λ-system): $\mathcal{L} \subseteq 2^{\Omega}$ is $\underline{\lambda\text{-system}}$ if."
$$\begin{cases} \Omega \in \mathcal{L} \\ A, B \in \mathcal{L},\ A \supseteq B \Rightarrow A - B \in \mathcal{L} \\ A_n \uparrow \in \mathcal{L} \Rightarrow \bigcup_{i=1}^{\infty} A_n \in \mathcal{L} \end{cases}$$

$\equiv$ A subset of 2^{Ω} w/

1) $\emptyset \in \mathcal{L}$

2) if $D \in \mathcal{L}$, then $\Omega - D \in \mathcal{L}$

3) if $A_1, \cdots A_n$ disjoint $\Rightarrow \bigcup A_n \in \mathcal{L}$.

equivalent

Def: (monotone-class): $\mathcal{M} \subseteq \Omega$ is a monotone-class if $A_n \uparrow / A_n \downarrow \in \mathcal{M} \Rightarrow \cancel{A} \in \mathcal{M}$.

$\underline{\mathcal{M} \subseteq \Omega \text{ is said to be mono-class if } A_n \uparrow / A_n \downarrow \in \mathcal{M}, \Rightarrow \bigcup A_n \in \mathcal{M} / \bigcap A_n \in \mathcal{M}.}$

Def: For $\mathcal{E} \subseteq 2^{\Omega}$, $\ell(\mathcal{E})$ is a smallest λ-system that contains $\mathcal{E}$, $m(\mathcal{E})$ is the smallest mono system that contains $\mathcal{E}$.

given $\mathcal{E} \subseteq 2^{\Omega}$, $\ell(\mathcal{E}) = \bigcap \{\mathcal{L}(\mathcal{E})\}$ $m(\mathcal{E}) = \bigcap \{M(\mathcal{E})\}$.

$\underline{\text{Thm: (Mono-class): If } \mathcal{A} \text{ is a field, then } 6(\mathcal{A}) = m(\mathcal{A})}$

Thm: (π-λ) if $\mathcal{P}$ is a π-system. then $\sigma(\mathcal{P}) = \ell(\mathcal{P})$

proof: "$\Leftarrow$" $m(\mathcal{A}) \subseteq \sigma(\mathcal{A})$, (clearly)

"$\Rightarrow$" WTS "$m(\mathcal{A})$ is a field"

(i) $\emptyset \in \mathcal{A} \subset m(\mathcal{A})$

(ii) let $A \in \mathcal{A}$, $\Rightarrow A^c \in \mathcal{A} \subset m(\mathcal{A})$ "Wrong direction"

$\mathcal{U}_0 := \{A \in m(\mathcal{A});\ A^c \in m(\mathcal{A})\}$ WTS: $\mathcal{U}_0 = m(\mathcal{A})$

$\mathcal{U}_0 \subseteq m(\mathcal{A})$, WTS: $m(\mathcal{A}) \subseteq \mathcal{U}_0$. $\Rightarrow \begin{cases} \mathcal{U}_0 \supseteq \mathcal{A}. \text{ (I)} \\ \text{and } \mathcal{U}_0 \text{ is monotone. (II).} \end{cases}$

(I) let $A \in \mathcal{A}$, $\Rightarrow A^c \in \mathcal{A} \subset m(\mathcal{A}) \Rightarrow A \in \mathcal{U}_0 \Rightarrow \mathcal{A} \subset \mathcal{U}_0$

(II) $A_n \uparrow \in \mathcal{U}_0$ $A_n^c \in m(\mathcal{A}) \Rightarrow A_n^c \downarrow \in m(\mathcal{A}) \Rightarrow \lim_n (A_n)^c \in m(\mathcal{A}) \Rightarrow \left(\bigcup_{i=1}^{\infty} A_i\right)^c = \left(\lim_n A_n\right)^c \in m(\mathcal{A})$

$\Rightarrow A_n \uparrow \in \mathcal{U}_0$.

Review: $\mathcal{A} \subseteq 2^{\Omega}$ is σ-Algebra

(i) $\emptyset \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$.

(iii) $A_n \in \mathcal{A} \Rightarrow \bigcup_{n \geq 1}^{\infty} A_n \in \mathcal{A}$.

called $(\Omega, \mathcal{A})$ is called meas-s.p.

Def: $\mathcal{A} \subseteq 2^{\Omega}$ is an algebra if

(i) $\emptyset \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$.

(iii) $A_n \in \mathcal{A} \Rightarrow \bigcup_{n \geq 1}^{N} A_n \in \mathcal{A}$.

Def: Monotone-class if

$A_n (\uparrow \text{ or } \downarrow) \in m \Rightarrow \lim_n A_n \in m$

Thm: $\mathcal{A}$ is algebra $\Rightarrow \sigma(\mathcal{A}) = m(\mathcal{A})$

proof: "$\Leftarrow$" $m(\mathcal{A}) \subseteq \sigma(A)$ clearly

"$\Rightarrow$" $\sigma(A) \subseteq m(A) \Rightarrow$ "$m(\mathcal{A})$ is σ-field"

(i) $\mathcal{A}$ is field $\Rightarrow \emptyset \in \mathcal{A} \subseteq m(\mathcal{A})$

To check certain property define a set !!!

(ii) Def $m_0 = \{A \subseteq m(\mathcal{A}) : A^c \in m(\mathcal{A})\}$. "Good method"

WTS: $m(\mathcal{A}) \subseteq m_0$ + $m_0 \subseteq m(\mathcal{A}) \Rightarrow m_0 = m(\mathcal{A})$

Check $\begin{cases} \mathcal{A} \subseteq m_0 & (i) \\ m_0 \text{ is monotone} & (ii) \end{cases}$

(i) let $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A} \subset m(\mathcal{A}) \Rightarrow A \in m_0$

(ii)

Step 3: $m_1 := \{A \in m(\mathcal{A})$ w/ $A \cup B \in m(\mathcal{A})$ for any $B \in m(\mathcal{A})\}$.

WTS: $m_1 = m(\mathcal{A})$

clearly $m_1 \subseteq m(\mathcal{A})$.

WTS: $m_1 \supseteq m(\mathcal{A})$

(i) $\begin{cases} \mathcal{A} \in m_1 & (1) \\ m_1 \text{ is monotone.} & (2) \end{cases}$

(1) let $A \in \mathcal{A}$ and $B \in m(\mathcal{A})$ Ktm?+ process ~

$A \in \mathcal{A} \subset m(\mathcal{A})$ $\underline{A \cup B \in \mathcal{A} \subseteq m(\mathcal{A}) \text{ for } \forall B \in \mathcal{A}}$.

1) If $B \in \mathcal{A}$ or $B \in m(\mathcal{A}) - \mathcal{A}$. $\Rightarrow B \in m(A) \cap \mathcal{A}^c$

(V) $B \in m(A)$ and $B \in \mathcal{A}^c$.

let Step 5: $A_n \in m(\mathcal{A})$

$\bigcup_{n=1}^{\infty} A_n \in m(\mathcal{A})$.

Def: $P \subseteq 2^{\Omega}$ is π-system if $A, B \in P \Rightarrow A \cap B \in P$

Def $\mathcal{L} \subseteq 2^{\Omega}$ is λ-system if

$\begin{cases} \Omega \in \mathcal{L} \\ A, B \in \mathcal{L},\ A \supseteq B \Rightarrow A - B \in \mathcal{L} \\ A_n \uparrow \in \mathcal{L} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{L} \end{cases}$

Thm: P is π, then $\mathcal{L}(P) = \sigma(P)$

prove $\sigma(P) \subseteq \mathcal{L}(P)$

$\mathcal{L}(P)$ is σ-Algebra.

(i) $\Omega \subset \mathcal{L}(P)$ and $\Omega - \Omega \in \mathcal{L}(P) \Rightarrow \emptyset \in \mathcal{L}(P)$

(ii) let $A \in \mathcal{L}(P)$ $A^c = \Omega - A \in \mathcal{L}(P)$

(iii) }
(iiii) } $\Rightarrow \forall A, B \in \mathcal{L}(P) \Rightarrow A \cup B \in \mathcal{L}(P)$

(iv) Closed under ~~limit~~. $\bigcup_{n=1}^{\infty} A_n \uparrow \in \mathcal{L}(P)$

Measurable-func: $(X, \mathcal{A}), (Y, \mathcal{B})$ are meas-s.p

$f: (X, \mathcal{A}) \to (Y, \mathcal{B})$ is meas if $\forall B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$.

$f^{-1}(B) = \{x \in X, : f(x) \in B\}$.

$(\Leftrightarrow)$ $f^{-1}(\mathcal{B}) \subseteq \mathcal{A} = \{f^{-1}(B), B \in \mathcal{B}\}$.

rules: (i) $f^{-1}(\emptyset) = \emptyset$

(ii) $f^{-1}(A^c) = (f^{-1}(A))^c$

(iii) $f^{-1}(\bigcup_{\lambda \in \Lambda} A_\lambda) = \bigcup_{\lambda \in \Lambda} f^{-1}(A_\lambda)$

general rules: $\begin{cases} f^{-1}(\emptyset) = \emptyset \\ f^{-1}(A^c) = (f^{-1}(A))^c \\ f^{-1}(\bigcup_i A_i) = \bigcup_i f^{-1}(A_i) \end{cases}$

Consequence: $\left.\begin{array}{l} f: (X, \mathcal{A}) \to (Y, \mathcal{B}) \\ g: (Y, \mathcal{B}) \to (Z, \mathcal{C}) \end{array}\right\}$ meas

then $g \circ f: (X, \mathcal{A}) \to (Z, \mathcal{C})$ is meas

let $C \in \mathcal{C}$, $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)) \in \mathcal{B}$

Conposition $g \circ f$ is meas:

WTS: $(g \circ f)^{-1} \in \mathcal{A}$.

let $C \in \mathcal{C}$, $f^{-1}(g(C)) = f^{-1}(B)$ for $B \in \mathcal{B}$.

$= A$ for $A \in \mathcal{A}$.

Def: $\mathcal{B}_{\mathbb{R}} = \sigma(\{A : A \text{ open}\}) = \sigma(\{[a, +\infty) : a \in \mathbb{R}\})$

$= \sigma(\{(a, +\infty) : a \in \mathbb{R}\})$

$= \sigma(\{[a, +\infty) : a \in \mathbb{Q}\})$

Def: $f: (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \Rightarrow$ random variable

f: maps $\to$ Borel is R.V.

How to check f is R.V

WTS: $\forall A \in \mathcal{B}_{\mathbb{R}} \Rightarrow f^{-1}(A) \in \mathcal{A}$. (Common way "Not useful"!)

Lemma: $f: (X, \mathcal{A}) \to (Y, \sigma(\mathcal{E}))$ is meas iff $f^{-1}(\mathcal{E}) \subseteq \mathcal{A}$

proof: Check $6(f^{-1}\mathcal{E}) \subseteq \mathcal{A}$. "$\Rightarrow$" f is meas if $\mathcal{E} \in 6(\mathcal{E}) \Rightarrow f^{-1}(\mathcal{E}) \subseteq \mathcal{A}$ clearly fit for

WTS: $f^{-1}(6(\mathcal{E})) \subseteq \mathcal{A}$. suffices to show $6(f^{-1}\mathcal{E}) = f^{-1}(6(\mathcal{E}))$

"$\Leftarrow$" if $f^{-1}(\mathcal{E}) \subseteq \mathcal{A} \Rightarrow$ for $\forall E \subseteq 6(\mathcal{E}) \Rightarrow f^{-1}(E) \subseteq \mathcal{A}$?

Step 1) $f^{-1}(\mathcal{E}) \subseteq f^{-1}(6(\mathcal{E}))$ clearly

if $\mathcal{E} \in 6(\mathcal{E})$.

$f^{-1}(\mathcal{E}) \in \mathcal{A}$.

$f^{-1}(6(\mathcal{E})) \subseteq \mathcal{A}$

$6(f^{-1}(\mathcal{E})) = \cap \{A \supseteq f^{-1}(\mathcal{E})$

then claim $6(f^{-1}\mathcal{E}) \subseteq f^{-1}6(\mathcal{E})$

check $f^{-1}6(\mathcal{E})$ is 6-field.

Step2: $\mathcal{B} := \{B \in Y = f^{-1}(B) \in 6(f^{-1}(\mathcal{E}))$

(i) $\mathcal{E} \subseteq \mathcal{B}$, since $f^{-1}\mathcal{E} \subseteq 6(f^{-1}\mathcal{E})$

can check $\mathcal{B}$ is 6-field

$\Rightarrow 6(\mathcal{E}) \subseteq \mathcal{B}$

$\Rightarrow f^{-1}(6(\mathcal{E})) \subseteq 6(f^{-1}(\mathcal{E}))$

Cor: f is R.V by check $\{f < a\} \in \mathcal{A}$ for $\forall a \in \mathbb{R}$

$\{f \leq a\} \in \mathcal{A}$

$\{f > a\} \in \mathcal{A}$.

- $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is meas if $f^{-1}(\mathcal{B}) \subseteq \mathcal{A}$
- $f : (X, \mathcal{A}) \to (Y, 6(\mathcal{E}))$ is meas if $f^{-1}\mathcal{E} \subseteq \mathcal{A}$
- $f : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is meas (Random variable) if $\{f > a\} \in \mathcal{A}$ for $\forall a \in \mathbb{R}$

Q: If f, g are R.V. then $f+g$ is R.V i.e $\{f+g > a\} \in \mathcal{A}$?

Ans: $\bigcup_{c \in \mathbb{R}} (\{f > c\} \cap \{g > a-c\}) = \bigcup_{c \in \mathbb{Q}} (\{f > c\} \cap \{g > a-c\})$ is meas (since $c \in \mathbb{Q}$ is countable)

Q: f_n are RVs, $\sup_n f_n$ and $\inf_n f_n$ are both R.V

Ans: Check $\{\sup_n f_n > a\} = \bigcup_n \{f_n > a\}$ is meas

wrong direction

$\{\inf_n f_n > a\} = \cancel{\bigcup_n \{-f_n > a\} \text{ is meas (?)}} = \bigcup_{t=1}^{\infty} \bigcap_n \{f_n > a + 1/t\}$ is meas (✓).

N.B: $\inf_n f_n > a = \bigcap \{f_n > a\}$ is not countable $\{1/n\}$.

Q: $\liminf_n f_n$ and $\limsup_n f_n$ are meas ?

Ans: $\{\liminf_n f_n > a\} = \{\sup_k \inf_{n \geq k} f_n > a\} = \bigcup_{k=1}^{\infty} \bigcup_{t=1}^{\infty} \bigcap_{n=k}^{\infty} \{f_n > a + 1/t\}$. is meas.

Q: What are meas-function ?

Ans: (Extension of meas-set)

e.g. $\mathbb{1}_A(w) = \begin{cases} 1 & w \in A \\ 0 & w \notin A \end{cases}$ $\Longrightarrow$ simple func:

$\{A_n\} \in \mathcal{A}$ and $\bigcup_{i=1}^{n} A_i = \Omega$, $S_n(x) = \sum_{i=1}^{n} a_i \cdot \mathbb{1}_{A_i}(x)$ (L.C)

$\searrow$ disjoint (form partition) $(a_i \in \mathbb{R})$

Check $\{\mathbb{1}_A(w) \leq a\} \in \mathcal{A}$?

$= \begin{cases} \Omega & \text{if } a \geq 1. \\ A^c & \text{if } 0 \leq a < 1. \\ \emptyset & \text{if } a < 0. \end{cases} \in \mathcal{A}$?

$\mathbb{1}_A$ is meas $\Leftrightarrow$ A is meas.

$\Downarrow$

Non-negative R.V $f \geq 0$

Lemma: if f is non-negative R.V, there $\exists$ non-negative $\{S_n\}$ that $S_n \uparrow f$, moreover if f is bounded ($\exists M$, s.t $f(w) \leq M$, for $\forall x \in \Omega$) $\Rightarrow S_n \overset{unif}{\rightarrow} f$

proof $\Rightarrow$ Lemma: $S_n := \underline{\sum_{k=0}^{n2^n - 1} k/2^n \, \mathbb{1}\{k/2^n \leq f < \frac{k+1}{2^n}\} + n \, \mathbb{1}\{f \geq n\}}$.

(i) Check S_n is simple ? (1) certainly

(ii) whether it's increasing ? (2) since we are getting finer and finer Approximation. (L.B).

Since $\mathbb{1}\{k/2^n \leq f < \frac{k+1}{2^n}\} \leq \mathbb{1}\{k'/2^n \leq f < \frac{k'+1}{2^n}\}$. for $k' > k$

(iii) Case I: $f(w) < n$: $0 \le f(w) - S_n(w) \le 1/2^n$

$n \gg 1 \Rightarrow f_n - f$ unif.

Case II: $f(w) \ge n$: $S_n(w) = n$.

$\Downarrow$

R.V: $f = f^+ - f^-$

for $f^+ = \max(f, 0)$

$f^- = -\min(f, 0)$

Lemma: a r.v of f

if $\exists$ simple func $f_n \to f$

If $|f| \le M$, then $f_n \to f$ unif

Measure:

Def: $\emptyset \in \mathcal{E} \subseteq 2^{\Omega}$, $\mu(\cdot)$ is a measure if $\mu: \mathcal{E} \to [0, \infty]$

$$\begin{cases} \mu(\emptyset) = 0 \\ \{A_n\} \subset \mathcal{E}, \text{ disjoint, and } \bigcup_{n=1}^{\infty} A_n \in \mathcal{E}, \text{ then } \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \end{cases}$$

(i) If $\forall A \in \mathcal{E}$, $\mu(A) < \infty \Rightarrow \mu$ is finite.

(ii) If $\forall A \in \mathcal{E}$, $\exists A_n \in \mathcal{E}$ s.t $A \subseteq \bigcup_{n=1}^{\infty} A_n$ and $\mu(A_n) < \infty$ is finite for each n $\Rightarrow \mu$ is σ-finite.

Def: $(\Omega, \mathcal{A}, \mu)$ is called measure-space given $\mathcal{A}$ is σ-field.

If $\mu(\Omega) = 1$, then $(\Omega, \mathcal{A}, \mu)$ is probability space

Def: λ is lebesgue-meas on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ if $\begin{cases} \lambda(J) = \text{"length of } J \text{ if } J \text{ is an interval"} \\ \lambda(J + x) = \lambda(J) \text{ any } J \in \mathcal{B}_{\mathbb{R}} \end{cases}$

Useless Part !!!

Q: Is λ exists/unique (Yes!)

$\Downarrow$

Q': How to construct Lebesgue measure on $([0,1], \mathcal{B}_{[0,1]})$ "unif-distribution".

Strategy: Construct outer-meas ⇒ meas

$\mathcal{A} = \{A \subseteq [0,1] : A$ is a finite union of intervals$\}$.

show (i) $\mathcal{A}$ is field
(ii) easy to construct λ on $\mathcal{A}$.

(Reading assignment) "Lalley's note"

if so: ⇒ we know $\mathcal{B}_{[0,1]} = \sigma(\mathcal{A})$

⇓

$Q' \Rightarrow Q'' :=$ measure on field $\xrightarrow[\text{extend}]{?}$ $\sigma(\mathcal{A})$. If so whether it's unique?

Focus on uniqueness: (Cathodonomy-extension-thm)

Thm: Suppose $\mathcal{A}$ is a field, μ and ν are two finite-meas on $\sigma(\mathcal{A})$, s.t $\mu(A) = \nu(A)$ for $\forall A \in \sigma(\mathcal{A})$

proof: $\mathcal{H} = \{A \in \sigma(\mathcal{A}) : \mu(A) = \nu(A)\}$.

$\mathcal{A} \subseteq \mathcal{H} \subseteq \sigma(\mathcal{A})$

WTS: $\mathcal{H}$ is the smallest mono class $m(\mathcal{A})$, by (Mono-class-thm) → $\mathcal{H} = \sigma(\mathcal{A})$

let $A_n \uparrow$ in $\mathcal{H}$ (WLOG)

$\Rightarrow \mu(A_n) = \nu(A_n)$

by lemma(ii) below.

$\lim_n \mu(A_n) = \lim_n \nu(A_n) \Rightarrow \mu(\lim_n A_n) = \nu(\lim_n A_n) \Rightarrow \lim_n A_n \in \mathcal{H}$

Lemma:

μ is finite

(i) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

(ii) $A_n \uparrow \Rightarrow \lim_n \mu(A_n) = \mu(\lim_n A_n)$

$A_n \downarrow \Rightarrow \lim_n \mu(A_n) = \mu(\lim_n A_n)$

(iii) $\mu(\Omega) = \mu(A) + \mu(A^c)$

Qes?: should $A_n \downarrow$ be finite

proof: (1)

$\mu(B) = \mu(A) + \mu(B-A)$ and $\mu(B-A) \geq 0 \Rightarrow \mu(A) \leq \mu(B)$

(2) let $A_n \uparrow$ in $\mathcal{A}$, Define $B_n = A_n - A_{n-1} \Rightarrow \{B_n\}$ disjoint.

then $\mu(\underset{\lim_n A_n}{\overset{\shortparallel}{\bigcup B_n}}) = \sum \mu(B_n) = \sum \mu(A_n - A_{n-1}) = \lim_n \mu(\bigcup_{n=1}^{n} A_n - A_{n-1})$

$= \lim_n \mu(A_n)$

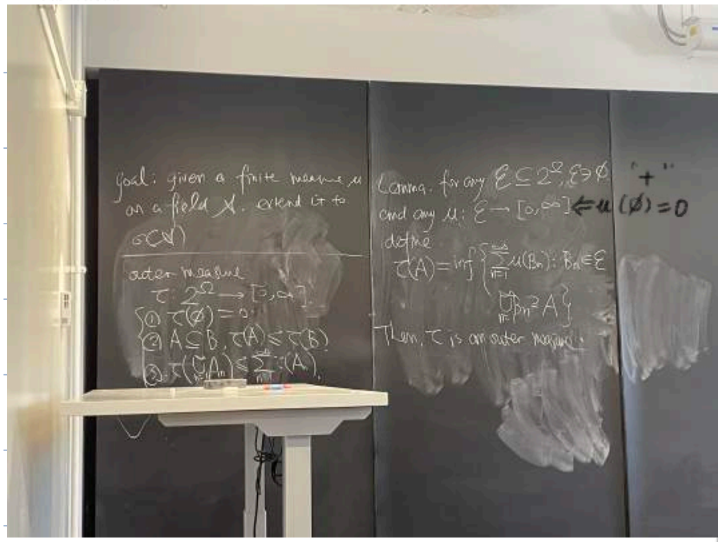
$A_n \downarrow \Rightarrow \Omega - A_n \uparrow$ $\lim_n \mu(\Omega - A_n) = \mu(\lim_n (\Omega - A_n))$

$= \mu(\Omega) - \lim_n \mu(A_n) = \mu((\lim_n A_n)^c)$

$\mu(\Omega) - \lim_n \mu(A_n) = \mu(\Omega) - \mu(\lim_n A_n)$

Def of outer meas.

Jan 16th



(2)

Note: $\inf(\emptyset) = \infty$

proof: 1) $\tau(\emptyset) \geq 0$

$$\tau(\emptyset) \leq \sum_{n=1}^{\infty} u(\emptyset) = 0.$$

2) $\forall B_n$, s.t $\bigcup_n B_n \supseteq B \supseteq A \Rightarrow \tau(A) \leq \sum_{n=1}^{\infty} u(B_n)$

taking inf over $\{B_n\}$

$$\tau(A) \leq \tau(B)$$

3) "$\tau(\bigcup A_n) \leq \sum \tau(A_n)$":

for each A_n, $\exists \{B_{n,k}\}$ cover of A_n s.t $\bigcup_k B_{n,k} \supseteq A_n$

and $\tau(A_n) \leq \sum_k u(B_{n,k})$

$\leq \tau(A_n) + \varepsilon/2^n$

$$\bigcup_n \bigcup_k B_{n,k} \supseteq \bigcup_n A_n$$

$$\Rightarrow \tau(\bigcup_n A_n) \leq \sum_n \sum_k u(B_{n,k})$$

$$\leq \sum_n \tau(A_n) + \varepsilon \cdot \sum_n \frac{1}{2^n}$$

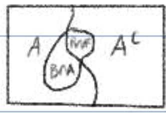
send $\varepsilon \to 0$ $\quad \tau(\bigcup_n A_n) \leq \sum_n \tau(A_n)$

2 stuff to check

Goal: for $\forall$ outer measure τ, find a σ-field $\mathcal{F}_\tau \subseteq 2^\Omega$

s.t τ is a measure on $\mathcal{F}_\tau$.

Them (Caratheodory):



(1) For $\forall$ outer measure τ,

define $\mathcal{F}_\tau = \{A \subseteq \Omega : \tau(B) = \tau(B \cap A) + \tau(B \cap A^c) \text{ for } \forall B \subseteq \Omega\}$

Then $(\Omega, \mathcal{F}_\tau, \tau)$ is a meas-space. (construct an σ-algebra).

(proof): check $\mathcal{F}_\tau$ is a σ-field

1) $\tau(B) = \tau(B \cap \emptyset) + \tau(B)$ (clearly for $\forall B$).

2) let $A \in \mathcal{F}_\tau$, then $\tau(B) = \tau(B \cap A) + \tau(B \cap A^c)$

//

wts. $\tau(B) = \tau(B \cap A^c) + \tau(B \cap A)$

$\Rightarrow A^c \in \mathcal{F}_\tau$.

Basically: for $\forall \tau$ outer meas.
we can find σ-Algebra for
making it a meas.

3) let $A_1, A_2 \in \mathcal{F}_\tau$.

$$\tau(B \cap (A_1 \cap A_2)) + \tau(B \cap (A_1 \cap A_2)^c)$$

$$= \tau((B \cap A_1) \cap (B \cap A_2)) + \tau(B \cap A_1^c \cup B \cap A_2^c)$$

$$\tau(B) = \tau(B \cap A_1) + \tau(B \cap A_1^c)$$

$$= \tau(B \cap A_1 \cap A_2) + \tau(B_1 \cap A_1 \cap A_2^c) + \tau(B \cap A_1^c \cap A_2) + \tau(B \cap A_1^c \cap A_2^c)$$

$= \tau(B \cap (A_1 \cap A_2)) + \tau(B \cap (A_1 \cap A_2)^c)$ $\Rightarrow$ closed under finite intersect. $\Rightarrow$ closed under finite union.

where $\tau(B \cap (A_1 \cap A_2)^c) = \tau(B \cap (A_1^c \cup A_2^c)) = \tau(B \cap (A_1^c \cup A_2^c) \cap A_1)$

$+ u(B \cap (A_1^c \cup A_2^c) \cap A_1^c)$

$= \tau(B \cap A_2^c \cap A_1) + \tau(B \cap A_1^c) = \tau(B \cap A_2^c \cap A_1) + \tau(B \cap A_1^c \cap A_2) + \tau(B \cap A_1^c \cap A_2^c)$

4) WTS if $A_n \in \mathcal{F}_c \Rightarrow \bigcup_n A_n \in \mathcal{F}_c$

Claim: WLOG, Assume $\{A_n\}_{n\geq 1}$ are mutually disjoint

Since $A_1 \cup A_2 \cdots$

$= A_1 \cup (A_2 \cap A_1^c) \cup (A_3 \cap (A_2 \cup A_1)^c) \cup \cdots$

mutually disjoint

clearly $\tau(B) \leq \tau(B \cap (\bigcup A_n)) + \tau(B \cap (\bigcup A_n)^c)$

only need to check other direction:

$\tau(B) = \tau(B \cap (\bigcup_{n=1}^{m} A_n)) + \tau(B \cap (\bigcup_{n=1}^{m} A_n)^c)$

$\geq \tau(B \cap (\bigcup_{n=1}^{m} A_n)) + \tau(B \cap (\bigcup_{n=1}^{\infty} A_n)^c)$

Claim: $= \sum_{n=1}^{m} \tau(B \cap A_n) + \tau(B \cap (\bigcup_{n=1}^{\infty} A_n)^c)$ let $m \to \infty \Rightarrow \tau(B) \geq \sum_{n=1}^{\infty} \tau(B \cap A_n) + \tau(B \cap (\bigcup_{n=1}^{\infty} (A_n))^c)$ (*)

$\geq \tau(B \cap (\bigcup_{n=1}^{\infty} A_n)) + \tau(B \cap (\bigcup_{n=1}^{\infty} (A_n^c)$

e.g. $\tau(B \cap (A_1 \cup A_2 \cup A_3))$

$= \tau(B \cap (A_1 \cup A_2 \cup A_3) \cap A_1) + \tau(B \cap (A_1 \cup A_2 \cup A_3) \cap A_1^c)$

$= \tau(B \cap A_1) + \tau(B \cap (A_2 \cup A_3))$ ← A_2 cut

$= \tau(B \cap A_1) + \tau(B \cap A_2) + \tau(B \cap A_3)$

Now, check τ is a meas on $\mathcal{F}_c$

1) $\tau(\emptyset) = 0$

2) $A_n \in \mathcal{F}_c$ disjoint, $\tau(\bigcup_n A_n) \leq \sum_n \tau(A_n)$ for the outer meas

take $B = \bigcup_n A_n \Rightarrow \tau(\bigcup A) \geq \sum \tau(A_n) + 0$ for (*). $\square$.

different as above

Thm: Suppose u is a finite measure on a field $\mathcal{A}$. we can uniquely extended it to $\sigma(\mathcal{A})$.

uniqueness has already shown (✓).

Have a measure on algebr

$\Rightarrow$ σ-algebra.

measure on.

$\tau(A) = \inf\{\sum_{n=1}^{\infty} u(B_n) : B_n \in \mathcal{A}, \bigcup_{n=1}^{\infty} B_n \supseteq A\}$ is an outer measure

(1) $\tau(A) = u(A)$ for $\forall A \in \mathcal{A}$ (field)

clearly $\tau(A) \leq u(A)$ "$(A \subseteq A \cup \emptyset \cup \emptyset \ldots)$"

WTS: $\tau(A) \geq u(A)$

By def. $\exists \{B_n\}, \bigcup_n B_n \subset A$. $\tau(A) + \varepsilon \geq \sum u(B_n) \geq u(\bigcup B_n) \geq u(A)$

(2) need to show $\sigma(A) \subseteq \mathcal{F}_c$. satisfie to show $\mathcal{A} \subset \mathcal{F}_c$ $\forall A \subset \mathcal{A}$.

$\tau(B) \le \tau(B\cap A) + \tau(B\cap A^c)$ need the other direction.

$\exists B_n \in \mathcal{A},\ \cup B_n \supseteq B$

$$\tau(B)+\epsilon \ge \sum_n \mu(B_n) = \sum_n \left(\mu(B_n\cap A) + \mu(B_n\cap A^c)\right)$$
$$= \sum_n \left(\tau(B_n\cap A) + \tau(B_n\cap A^c)\right)$$
$$\ge \tau(\cup B_n\cap A) + \tau(\cup B_n\cap A^c)$$
$$\ge \tau(B\cap A) + \tau(B\cap A^c)$$

Jan 18th

Def: $(\Omega, \mathcal{A}, \mu)$ meas-sp is complete if,

(*) $A\in\mathcal{A},\ \mu(A)=0 \Rightarrow B\in\mathcal{A}$ for all $B\subseteq A$.

Them: For $\forall\tau$, outer meas $(\Omega, \mathcal{F}_\tau, \tau)$ is a complete meas.sp.

Recall: $\mathcal{F}_\tau = \{A\subseteq\Omega : \tau(B) = \tau(B\cap A)+\tau(B\cap A^c) \text{ for } \forall B\subseteq\Omega\}$

WTS: $\tau(A)=0 \Rightarrow A\in\mathcal{F}_\tau.\ \Rightarrow B\subseteq A\in\mathcal{F}_\tau.$

(i) $\tau(B) \le \tau(B\cap A)+\tau(B\cap A^c)$

(ii) $\tau(B) \ge \tau(B\cap A^c) = \tau(B\cap A^c) + \underbrace{\tau(B\cap A)}_{=0}$

e.g. $([0,1], \mathcal{B}_{[0,1]}, \lambda) \longrightarrow ([0,1], \mathcal{F}_\lambda, \lambda)$

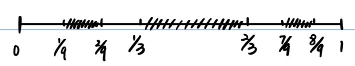
we know that $\mathcal{B}_{[0,1]} \subseteq \mathcal{F}_\lambda$ (complete), but $\mathcal{B}_{[0,1]}$ is not complete.

$\Downarrow$

remove $[1/3, 2/3]$

C_1 :

C_2 :



⋮

(Cantor set) $C := \lim_{n\to\infty} C_n$, we have ① $C\in\mathcal{B}_{[0,1]}$ ② $\lambda(C)=0$ ③ $\text{card}(C) = |C| = |[0,1]|$

? $|\mathcal{B}_{[0,1]}| = |[0,1]|$

Suppose $\mathcal{B}_{[0,1]}$ is complete, $\lambda(C)=0,\ B\subseteq C \Rightarrow B\in\mathcal{B}_{[0,1]} \Rightarrow |\mathcal{B}_{[0,1]}| \ge |2^C| \overset{C_2}{=} |2^{[0,1]}| \overset{C_1}{>} |[0,1]| = |\mathcal{B}_{[0,1]}|$ (contradiction)

Completion → measure s.p:

Them: $(\Omega, \mathcal{A}, \mu)$ is a meas.sp, we define $\bar{\mathcal{A}} = \{A\cup N : A\subseteq\mathcal{A},\ \exists B\in\mathcal{A} \text{ s.t } N\subseteq B \text{ and } \mu(B)=0\}$. is σ-Algebra. Moreover, $\forall A\cup N\in\bar{\mathcal{A}},\ \tilde{\mu}(A\cup N) := \mu(A)$

then $(\Omega, \bar{\mathcal{A}}, \tilde{u})$ is complete meas sp (completion). w/ for $\forall A \in \mathcal{A}$, $\tilde{u}(A) = u(A)$

"smallest s.p containing original"

proof: (σ-algebra)

(i) $\emptyset = \emptyset_1 \cup \emptyset_2$, $\emptyset_1 \in \mathcal{A}$ and $\emptyset_2 \subseteq \emptyset_2$ and $u(\emptyset_2) = 0$.

(ii) $A \cup N \in \bar{\mathcal{A}}$, $\Rightarrow (A \cup N)^c \in \bar{\mathcal{A}} \equiv A^c \cap N^c \in \bar{\mathcal{A}}$.

$$A^c \cap N^c = A' \cup N' = (A^c \cap N^c \cap B^c) \cup (A^c \cap N^c \cap B)$$

$$= (A^c \cap B^c) \cup \underbrace{(A^c \cap N^c \cap B)}_{\subseteq B} = A' \cup N' \in \bar{\mathcal{A}}$$

$A \in \mathcal{A}$, $B \in \mathcal{A} \Rightarrow (A^c \cap B^c) \in \mathcal{A}$.

(iii) suppose $\underset{\in \mathcal{A}}{A_n} \cup \underset{\subseteq B_n}{N_n} \in \bar{\mathcal{A}}$ $\begin{cases} B_n \in \mathcal{A} \\ u(B_n) \in \mathcal{A} \end{cases}$

$$\bigcup_n (A_n \cup N_n) = \underset{\in \mathcal{A}.}{\left(\bigcup_n A_n\right)} \cup \left(\bigcup_n N_n\right)$$

$\subseteq \bigcup_n B_n$ and $u(\bigcup_n B_n) \le \sum u(B_n) = 0$.

(二) WTS: Show $\tilde{u}$ is well defined

$$\underset{\in \mathcal{A}}{A_1} \cup \underset{\subseteq B_1}{N_1} = \underset{\in \mathcal{A}}{A_2} \cup \underset{\subseteq B_2}{N_2}$$

$u(A_1) + u(B_1) = u(A_1) \ge u(A_1 \cup B_1)$

$$\tilde{u}(A_1 \cup N_1) = u(A_1) = u(A_1 \cup B_1) \ge u(A_2) = \tilde{u}(A_2 \cup N_2)$$

similarly $\tilde{u}(A_1 \cup N_1) \le \tilde{u}(A_2 \cup N_2)$.

(三) Show $\forall A \in \mathcal{A}$, $\tilde{u}(A) = u(A)$

$$\tilde{u}(A \cup \emptyset) = u(A)$$

(iv) Show $\tilde{u}$ is a measure.

(1) $\tilde{u}(\emptyset) = \tilde{u}(\emptyset \cup \emptyset) = u(\emptyset) = 0$.

(2) suppose $\{A_n \cup N_n\} \in \bar{\mathcal{A}}$ "disjoint" $\Rightarrow \tilde{u}(\bigcup_n (A_n \cup N_n)) = u(\bigcup_n A_n) = \sum_n u(A_n) = \sum_n \tilde{u}(A_n \cup N_n)$

if $\{A_n \cup N_n\}$ disjoint.
$\Rightarrow \{A_n\}$ disjoint

(v) Show completness

$\forall A \cup N \in \bar{\mathcal{A}}$ and $\tilde{u}(A \cup N) = u(A) = 0$, WTS $\forall D \subseteq A \cup N$, $D \in \bar{\mathcal{A}}$

$\mathcal{A}$ $\subseteq B$ $\begin{cases} B \in \mathcal{A} \\ u(B) = 0 \end{cases}$

consider $D = \emptyset \cup D$
$\subseteq A \cup B$ where 1) $A \cup B \in \mathcal{A}$
2) $u(A \cup B) = 0$

Motivation: $([0,1], \mathcal{B}_{[0,1]}, \lambda)$ ← Borel-σ algebra (小的)　　true also for μ-σ-finite　　$(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$

$([0,1], \mathcal{F}_\lambda, \lambda)$ ← Lebesgue-σ algebra (大的)　　$(\mathbb{R}, \mathcal{F}_\mu, \mu)$

Goal: Show ② is completion of ①

Them: If μ-finite meas on a field $\mathcal{A}$, τ is the outer meas generated from μ, then $(\Omega, \mathcal{F}_\tau, \tau)$ is the completion of $(\Omega, \sigma(\mathcal{A}), \tau)$

proof: (1) Lemma: under same setting, $\forall A \in \mathcal{F}_\tau$, $\exists B \in \sigma(\mathcal{A})$ s.t $A \subseteq B$ and $\tau(A) = \tau(B)$

proof: $\tau(A) = \inf\{\sum_{k=1}^{\infty} \mu(B_k) : B_k \in \mathcal{A}, \bigcup_{k=1}^{\infty} B_k \supseteq A\}$.　def ⇒ out measure generated by μ.

WTS: $\exists B_{n,k} \in \mathcal{A}$ s.t $\bigcup_k B_{n,k} \supseteq A$.　$\tau(A) \le \sum_k \mu(B_{n,k}) \le \tau(A) + 1/n$.

$B = \bigcap_n \bigcup_k B_{n,k} \in \sigma(\mathcal{A})$

$\bigcup_k B_{n,k} \supseteq B \subseteq A$.

$\tau(A) + 1/n \ge \sum_k \mu(B_{n,k}) = \sum_k \tau(B_{n,k}) \ge \tau(\bigcup_k B_{n,k}) \ge \tau(B)$ $\ge \tau(A)$ taking $\lim_n$.

(2) $\mathcal{F} = \sigma(\mathcal{A})$, show $\mathcal{F}_\tau = \hat{\mathcal{F}}$

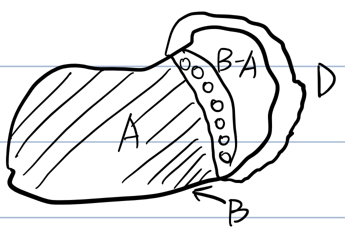
1) $\hat{\mathcal{F}} \subseteq \mathcal{F}_\tau$ (σ-field) (complete)

1) $A \cup N \in \hat{\mathcal{F}} \Rightarrow A \cup N \in \mathcal{F}_\tau$

$A \in \mathcal{F}$, $N \subseteq B$, $B \in \mathcal{F}$, $\tau(B) = 0$

(3) Show $\mathcal{F}_\tau \subseteq \hat{\mathcal{F}}$: $\forall A \in \mathcal{F}_\tau$, $\exists B \in \mathcal{F}$ s.t $A \subseteq B$ $\tau(A) = \tau(B)$.

$\exists D \in \mathcal{F}$, s.t $B - A \subseteq D$. $\tau(D) = \tau(B-A) = 0$.



$A = (D^c \cap B) \cup (A \cap D)$

$D^c \cap B \in \mathcal{F}$　　$A \cap D \subseteq D \in \mathcal{F}$, $\tau(D) = 0$

$\Rightarrow A \cap D \in \hat{\mathcal{F}}$.

Claim: $A \cup N = A \triangle N' = A \triangle (N \cap A^c)$ where $N \cap A^c \subseteq N \subseteq B \Rightarrow \mu(B) = 0$.

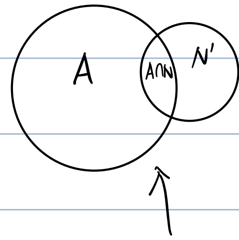
$A \cup N = A \cup (N - A)$

$= A \cup N'$

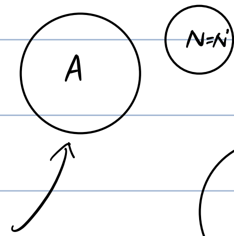
$= ((A \cup N') \cap A) \cup (A \cup N' \cap A^c)$

$= (A \cup (N' \cap A)) \cup (N' \cap A^c)$

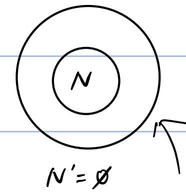
$=$



$A \triangle N' = A \cup N$



$A \triangle N' = A \cup N$



$N' = \emptyset$

$A \cup N = A = (A - N') \cup (N' - A)$

$= A$.

Integral

$f : (\Omega, \mathcal{A}, \mu) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

define $\int f \, d\mu$?

int = { indicator $\Downarrow$ L-C, Simple $\Downarrow$ Approx, non-negati f $\Downarrow$ $f = f^+ - f^-$, f.

1. $f = \mathbb{1}_A$ for $\int f \, d\mu = \mu(A)$

2. nongative simple func $\{A_1, \cdots A_n\}$ partition of Ω $a_i \geq 0$.

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i} \quad \int f \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

3. $f \geq 0$. $\int f \, d\mu = \sup \{ \int g \, d\mu : f \geq g, \ g$ non-negative simple $\}$ formal default

OR $f_n \uparrow f$ (dependent on sequence of value)

4. $f = f^+ - f^-$. $\int f^+ d\mu - \int f^- d\mu = \int f \, d\mu$

i) $\min(\int f^+ d\mu, \int f^- d\mu) < \infty \Rightarrow \int f \, d\mu$ exists

ii) $\max(\int f^+ d\mu, \int f^- d\mu) < \infty \Rightarrow \int f \, d\mu < \infty$ integrable ($f \in \mathcal{L}^1$)

$\Updownarrow$ $\int |f| \, d\mu < \infty$ (useful).

$X: (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

↘ Random-variable

$\mathbb{E}X = \int X \, d\mathbb{P}$ (expectation)

exists if $(\mathbb{E}X^+, \mathbb{E}X^-) < \infty \iff \mathbb{E}|X| < \infty$.

Goal: Check def + properties

For (2) above: is well-defined

$\Downarrow$

2. nongative simple func $\{A_1, \cdots A_n\}$ partition of Ω $a_i \geq 0$.
$f = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$ $\int f \, d\mu = \sum_{i=1}^n a_i \mu(A_i)$

proof: let $\{A_1, \cdots A_n\}$ part $\to \Omega$

$\{B_1, \cdots B_m\}$ part $\nearrow$ WLOG: $(m \leq n)$

(Questionable) $f = \sum_{i=1}^m b_j \mathbb{1}_{\{f = b_j\}}$

$A_i \cap \{f = b_j\} = \emptyset$

OR

$A_i \subseteq \{f = b_j\}$

$$\sum_{i=1}^n a_i \mu(A_i) = \sum_{i=1}^n \sum_{j=1}^m a_i \mu(A_i \cap \{f = b_j\})$$

$$= \sum_{i=1}^n \sum_{j: A_i \subseteq \{f = b_j\}} b_j \mu(A_i \cap \{f = b_j\})$$

$$= \sum_{i=1}^n \sum_{j=1}^m b_j \mu(A_i \cap \{f = b_j\})$$

$$= \sum_{j=1}^m b_j \mu(\{f = b_j\})$$

Lemma: If f, g, f_n are non-negative simple. Then

1) $\int f \, d\mu \geq 0$

2) $a \geq 0$ $\int (af) d\mu = a \int f \, d\mu$

3) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

4) $f \geq g \Rightarrow \int f \, d\mu \geq \int g \, d\mu$

5) $f_n \uparrow$, $\lim_n f_n \geq g \Rightarrow \lim_n \int f_n \, d\mu \geq \int g \, d\mu$.

proof (3): $f = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$

$g = \sum_{j=1}^m b_j \mathbb{1}_{B_j}$

$$f + g = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) \mathbb{1}_{\{A_i \cap B_j\}}$$

$$\int f + g \, d\mu = \sum_{i=1}^n \sum_{j=1}^m (a_i + b_j) \mu(A_i \cap B_j) = \sum_{i=1}^n a_i \mu(A_i) + \sum_{j=1}^m b_j \mu(B_j)$$

(4) $\int f-g+g \, d\mu = \int f-g \, d\mu + \int g \, d\mu$

$\int f \, d\mu$ (= the left side); $\int f-g \, d\mu \geq 0$

$\geq \int g \, d\mu$

(5) $A_n(\alpha) = \{f_n \geq \alpha g\}$

$\alpha \in [0,1]$

$\int f_n \, d\mu \geq \int f_n \mathbb{1}_{A_n(\alpha)} \, d\mu$

$\geq \alpha \int g \mathbb{1}_{A_n(\alpha)} \, d\mu \qquad g = \sum_{j=1}^{m} b_j \mathbb{1}_{B_j} \qquad g \mathbb{1}_{A_n(\alpha)} = \sum_{j=1}^{m} b_j \mathbb{1}_{B_j \cap A_n(\alpha)}$

$= \alpha \sum_{j=1}^{m} b_j \mathbb{1}_{A_n(\alpha) \cap B_j}$

$\lim_n \int f_n \, d\mu \geq \alpha \sum_{j=1}^{m} b_j \lim_n \mathbb{1}_{A_n(\alpha) \cap B_j} \uparrow B_j$

$= \alpha \sum_{j=1}^{m} b_j \lim \mathbb{1}_{B_j} = \alpha \int g \, d\mu$ send $\alpha \to 1$.

3. $f \geq 0 \quad \int f \, d\mu = \sup \{\int g \, d\mu : f \geq g.\ g$ non-negative simple$\}$.

Prop: If f is non-negative simple, then two define agrees

$\int f \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i) \overset{?}{=} \sup \{\int g \, d\mu : \sum_{i=1}^{n} a_i \mu(A_i) \geq g,\ g$ non-negative simple$\}$.

for A_i partition

clearly 1) $\sum_{i=1}^{n} a_i \mu(A_i) \leq \sup \{\}$

$\int f \, d\mu \geq \sup \{\int g \, d\mu\}$

Lemma: 1) $f_n \uparrow f$ (non-negative)

$\Rightarrow \int f_n \, d\mu \uparrow \int f \, d\mu$

2) $\int f \, d\mu \geq 0$

3) $\forall a \geq 0 \quad \int af \, d\mu = a \int f \, d\mu$.

4) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$.

5) $f \geq g \Rightarrow \int f \geq \int g$.

proof: i) $f_n \leq f$

$\int f_n \, d\mu \leq \int f$

$\lim_n \int f_n \, d\mu \leq \int f = \sup \{\int g \, d\mu\}$.

+ we have $\lim_n \int f_n \, d\mu \geq \int g \, d\mu$

$\Rightarrow \lim_n \int f_n \, d\mu = \int f$

$\lim_n \int f_n = \int f$

let $h_n \uparrow (f+g) \Rightarrow \lim_n \int h_n \, d\mu = \lim_n \int f_n + g_n = \lim_n \int f_n + \lim_n \int g_n$
$= \int f + \int g.$

---

for $A \in \mathcal{A}$, we use $\int_A f \, d\mu = \int f \, \mathbb{1}_A \, d\mu$

Lemma: 1) $A \in \mathcal{A}$ has $\mu(A) = 0$, then

$$\int_A f \, d\mu = 0$$

2) If $f \geq g$ a.e, then $\int f \, d\mu \geq \int g \, d\mu$

3) $f = g$ a.e then $\int f = \int g$. (If f is integrable $\Rightarrow$ g is integrable)

proof: $\int_A f_n d\mu = \int f_n \mathbb{1}_A \, d\mu = \int f_n \, \mathbb{1}_A \, d\mu = \int \sum_{i=1}^n a_i \cdot \mathbb{1}_{A_i \cap A} = \int \sum_{i=1}^n a_i \, \mathbb{1}_{A_i \cap A}$
$\leq \sum_{i=1}^n a_i \, \mu(A) = 0.$

$f \geq 0$ $f_n \uparrow f$

$\Rightarrow f_n \mathbb{1}_A \uparrow f \cdot \mathbb{1}_A$

$\int_A f \, d\mu = \lim_n \int_A f_n \, d\mu = 0$

for $f \geq 0$, $\int f \, d\mu = \int_A f \, d\mu + \int_{A^c} f \, d\mu$

$\geq \int_A g \, d\mu + \int_{A^c} f \, d\mu \; {=}0$

$= \int_A g \, d\mu + \underbrace{\int_{A^c} g \, d\mu}_{=0}$

$= \int g \, d\mu.$

In general $f = f^+ - f^-$ $f^+ \geq g^+$ and $f^- \geq g^-$ for the negative part.

$\int f^+ \geq \int g^+$
$\int f^- \geq \int g^-$ $\Rightarrow$ $f \geq g$.

Lemma: f, g integrable

1) $a \in \mathbb{R}$ $\int (af) d\mu = a \int f d\mu$.

2) $\int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu$.

$\int f^+ - \int f^- + \int g^+ - \int g^- \Rightarrow \int f^+ + g^+ = \int f^- + g^- + \int (f+g)^+$

$f+g = (f+g)^+ - (f-g)^-$

Thm (Levi) Monotone-convergent thm.

$f_n, f \geq 0$, $f_n \uparrow f$ a.e, then $\int \lim_n f_n \, d\mu = \lim_n \int f_n$

For each f_n. $\exists$ non-negative, simple, $f_{n,k} \uparrow f_n$ for each n

$\underline{f_{11}}, \underline{f_{12}}, \dots \uparrow f_1$

$f_{21}, \underline{f_{22}} \cdot \cdot \uparrow f_2$

$\vdots \qquad \vdots$

$f_{n,1}, f_{n2}, \dots \uparrow f_n \uparrow f$

take $g_1 = f_{1,1}$

$g_2 = \max\{f_{1,2}, f_{22}\}$

$g_3 = \max\{f_{1,3}, f_{2,3}\, f_{3,3}\}$.

$\vdots$

$g_k = \max\{f_{1,k}, \dots f_{k,k}\}$. $\Rightarrow$ $g_k \uparrow$ non-negative simple.

w/ $g_1 = f_1 \leq \max(f_{1,2}, f_{2,2}) = g_2$

$\leq \max(f_{1,3}\, f_{2,3})$

$\leq \max(f_{1,3}, f_{2,3}, f_{3,3}) = g_3 \dots$

Clearly, $g_k \leq f_k \Rightarrow \lim_k g_k \leq \lim_k f_k = f$

$(\Rightarrow)$ Claim $\lim_k g_k \geq f_1$. since $g_1 \geq f_{1,1}, \dots \Rightarrow g_k \geq f_{1,k}$. $\Rightarrow \lim_k g_k \geq f_1$

and $g_2 \geq f_{2,2}, \dots \Rightarrow g_k \geq f_{2,k} \Rightarrow \lim_k g_k \geq f_2$

$\vdots$

$\lim_k g_k \geq \lim_k f_k = f$. $\Rightarrow g_k \uparrow f \Rightarrow \lim_k \int g_k d\mu = \int \lim_k g_k \, d\mu$.

Also: $\lim_k \int g_k \, d\mu \leq \lim_n \int f_n \, d\mu$.

$\leq \int f \, d\mu \leq \lim_k \int g_k \, d\mu$

$\Rightarrow \lim_n \int f_n d\mu = \int f \, d\mu$. ▨

Thm (Fatou's Lemma)

$\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n$ $\quad$ ($\liminf_n f_n = \lim_{k \to \infty} \inf_{n \geq k} f_n$

define $g_k = \inf_{n \geq k} f_n \Rightarrow g_k \uparrow \liminf_n f_n$ w/ (MCT))

By MCT $\Rightarrow \int \liminf_n f_n \, d\mu = \lim_k \int g_k \, d\mu$

$\forall n \geq k \quad g_k \leq f_n \Rightarrow \int g_k \leq \int f_n$

$\Rightarrow \int g_k \leq \inf_{n \geq k} \int f_n$

$\Rightarrow \lim_k \int g_k \leq \lim_k \inf_{n \geq k} \int f_n = \liminf_n \int f_n$ ▨

DCT (Dominate converges them)

$f_n \to f, \ |f_n| \leq g$ w/ $\int g \, d\mu < \infty$ (integrable)

$S_n \circ g \uparrow f \circ g$?

Then $\int \lim_n f_n \, d\mu = \lim_n \int f_n \, d\mu$

$S_n(g(B)) \uparrow f \circ g$

proof: $-g \leq f_n \leq g$

$g(\ )$

$f_n + g \geq 0 \Rightarrow \liminf \int f_n + \int g = \liminf_n \int f_n + g \leq \int \liminf (f_n + g) = \int g + \int \liminf f_n$

$\liminf \int f_n \leq \int \liminf f_n$

$g - f_n \geq 0 \Rightarrow \liminf \int -f_n \leq \int \liminf -f_n$

$\Downarrow$

$\limsup \int f_n \leq \int \limsup f_n. \Rightarrow \limsup_n \int f_n \leq \int f \leq \liminf \int f_n.$

$\int \liminf (g - f_n \, d\mu) \leq \liminf \int g - f_n \, d\mu. \Uparrow$

$\Rightarrow \lim_n \int f_n = \int f.$

$\Leftrightarrow \int g - \int \limsup f_n \leq \int g - \limsup_n \int f_n$

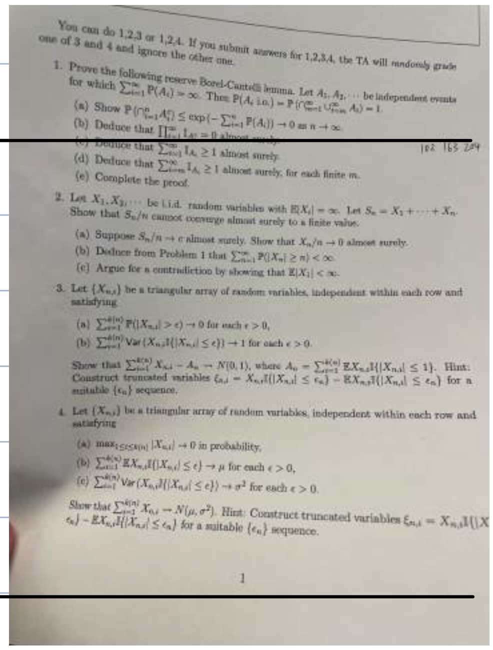
Corollary: μ is finite (Probability measure)

if $f_n \to f$ a.e

$|f_n| \leq M$

Then $\int \lim_n f_n \, d\mu = \lim_n \int f_n \, d\mu.$

proof: taking $g = M$.


You can do 1,2,3 or 1,2,4. If you submit answers for 1,2,3,4, the TA will randomly grade one of 3 and 4 and ignore the other one.

1. Prove the following reserve Borel-Cantelli lemma. Let $A_1, A_2, \cdots$ be independent events for which $\sum_{i=1}^{\infty} P(A_i) = \infty$. Then $P(A_i \text{ i.o.}) = P(\cap_{m=1}^{\infty} \cup_{i=m}^{\infty} A_i) = 1$.
   (a) Show $P(\cap_{i=1}^{n} A_i^c) \leq \exp(-\sum_{i=1}^{n} P(A_i)) \to 0$ as $n \to \infty$.
   (b) Deduce that $\prod_{i=1}^{\infty} I_{A_i^c} = 0$ almost surely.
   (c) Deduce that $\sum_{i=1}^{\infty} I_{A_i} \geq 1$ almost surely.
   (d) Deduce that $\sum_{i=m}^{\infty} I_{A_i} \geq 1$ almost surely, for each finite m.
   (e) Complete the proof.

102 163 204

2. Let $X_1, X_2, \cdots$ be i.i.d. random variables with $\mathbb{E}|X_i| = \infty$. Let $S_n = X_1 + \cdots + X_n$. Show that S_n/n cannot converge almost surely to a finite value.
   (a) Suppose $S_n/n \to c$ almost surely. Show that $X_n/n \to 0$ almost surely.
   (b) Deduce from Problem 1 that $\sum_{n=1}^{\infty} P(|X_n| \geq n) < \infty$.
   (c) Argue for a contradiction by showing that $\mathbb{E}|X_1| < \infty$.

3. Let $\{X_{n,i}\}$ be a triangular array of random variables, independent within each row and satisfying
   (a) $\sum_{i=1}^{k(n)} P(|X_{n,i}| > \epsilon) \to 0$ for each $\epsilon > 0$,
   (b) $\sum_{i=1}^{k(n)} Var(X_{n,i} I\{|X_{n,i}| \leq \epsilon\}) \to 1$ for each $\epsilon > 0$.

   Show that $\sum_{i=1}^{k(n)} X_{n,i} - A_n \to N(0,1)$, where $A_n = \sum_{i=1}^{k(n)} \mathbb{E}X_{n,i} I\{|X_{n,i}| \leq 1\}$. Hint: Construct truncated variables $\xi_{n,i} = X_{n,i} I\{|X_{n,i}| \leq \epsilon_n\} - \mathbb{E}X_{n,i} I\{|X_{n,i}| \leq \epsilon_n\}$ for a suitable $\{\epsilon_n\}$ sequence.

4. Let $\{X_{n,i}\}$ be a triangular array of random variables, independent within each row and satisfying
   (a) $\max_{1 \leq i \leq k(n)} |X_{n,i}| \to 0$ in probability,
   (b) $\sum_{i=1}^{k(n)} \mathbb{E}X_{n,i} I\{|X_{n,i}| \leq \epsilon\} \to \mu$ for each $\epsilon > 0$,
   (c) $\sum_{i=1}^{k(n)} Var(X_{n,i} I\{|X_{n,i}| \leq \epsilon\}) \to \sigma^2$ for each $\epsilon > 0$.

   Show that $\sum_{i=1}^{k(n)} X_{n,i} \to N(\mu, \sigma^2)$. Hint: Construct truncated variables $\xi_{n,i} = X_{n,i} I\{|X$ $\epsilon_n\} - \mathbb{E}X_{n,i} I\{|X_{n,i}| \leq \epsilon_n\}$ for a suitable $\{\epsilon_n\}$ sequence.

1

Independence:

$X, Y: (X, \mathcal{A}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

(*)

$P(X \in A, Y \in B) = P(X \in A) \cdot P(X \in B) \quad \forall A, B \in \mathcal{B}_{\mathbb{R}}$

i.e $\mu(\cap A_i) = \prod \mu(A_i)$

Def. $\{A_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{A}$ are independent if $\forall A_1, \dots A_n \in \{A_\lambda, \lambda \in \Lambda\}$ w/ $P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i)$

Def. $\{\mathcal{E}_\lambda, \lambda\in\Lambda\}$, $\mathcal{E}_\lambda \subseteq \mathcal{A}$ are independent if $A_\lambda \in \mathcal{E}_\lambda$ for $\{A_\lambda, \lambda\in\Lambda\}$ are independent.

Concept Note: $f:(\Omega, \mathcal{A}, \mu) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \Rightarrow f$ is meas $\Leftrightarrow f^{-1}\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{A}$.

Def: $\sigma(f)$ is the smallest σ-algebra s.t f is meas.

$\hookrightarrow \sigma(f) = f^{-1}\mathbb{B}_{\mathbb{R}}$

e.g $f = \mathbb{1}_A \Rightarrow \sigma(f) = \{\emptyset, \Omega, A, A^c\}$.

Def: $\{X_\lambda, \lambda\in\Lambda\}$ are independent if $\{\sigma(X_\lambda): \lambda\in\Lambda\}$ are independent $\equiv$ (*)

$\Rightarrow \sigma(X) \perp\!\!\!\perp \sigma(Y)$

Lemma: Suppose X, Y independent $X \perp\!\!\!\perp Y$ w/ $\mathbb{E}|X|, \mathbb{E}|Y| < \infty$

then,

$$\mathbb{E}(XY) = \mathbb{E}(X)\cdot\mathbb{E}(Y)$$

proof: 1) $X = \mathbb{1}_A$, $Y = \mathbb{1}_B$,

$\mathbb{E}[XY] = P(A\cap B) = P(A)\cdot P(B) = \mathbb{E}X \cdot \mathbb{E}Y$

2) $X = \sum_{i=1}^n a_i \mathbb{1}_{A_i}$, $Y = \sum_{j=1}^m b_j \mathbb{1}_{B_j}$

$XY = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \mathbb{1}_{A_i\cap B_j}$

$E(XY) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j P(A_i\cap B_j) \not\Longrightarrow E(XY) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j P(A_i)P(B_j)$

e.g. $X:(\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

$X = \mathbb{1}_{[0,1]} = \mathbb{1}_{[0,\frac{1}{2})} + \mathbb{1}_{[\frac{1}{2},1]}$

$\sigma(X) = \{\emptyset, \mathbb{R}, [0,1], [0,1]^c\}$.

RA $\Rightarrow \mathbb{E}(X, Y) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j P(X=a_i, Y=b_j)$

$= \sum_{i=1}^n \sum_{j=1}^m a_i b_j P(X=a_i)P(Y=b_j)$

$= \left(\sum_{i=1}^n a_i P(X=a_i)\right)\left(\sum_{j=1}^m b_j P(Y=b_j)\right) = \mathbb{E}X\,\mathbb{E}Y.$

3) $X, Y \geq 0$, $X_n \uparrow X$, $Y_n \uparrow Y$ $X_n \perp\!\!\!\perp Y_n$ ← Hard

$X_n Y_n \uparrow XY$

$\mathbb{E}XY = \lim_n \mathbb{E}X_n Y_n$

$= \lim_n \mathbb{E}X_n \,\mathbb{E}Y_n$

$= \mathbb{E}X\,\mathbb{E}Y$.

Midterm

(Stong law of large number)

Them (Komolgvov)

$X_1, \cdots X_3 \cdots$ independent and identically distributed $\quad \mathbb{P}(X_1 \le t) = \mathbb{P}(X_2 \le t) = \cdots$

$\mathbb{E}|X_1| < \infty, \ \mathbb{E}X_1 = u \in \mathbb{R}$

$$\Rightarrow \frac{X_1 + \cdots + X_n}{n} \to u \text{ a.e}$$

(**)

Them: $X_1, \cdots$ independent, $\mathbb{E}X_i < \infty$ and $E(X_i) = u \in \mathbb{R}$, $\max \mathbb{E}X_i^4 < \infty$

Then, $\frac{X_1 + \cdots X_n}{n} \to u$ a.e.

Lemma: (Markov): $X \ge 0, \ t > 0, \ \mathbb{E}|X| < \infty \Rightarrow \mathbb{P}(X > t) \le \frac{\mathbb{E}X}{t}$

proof: $P(X \ge t) = \int \mathbb{1}_{\{x \ge t\}} \, dP$

$$\le \int \mathbb{1}_{\{x \ge t\}} \frac{x}{t} \, dP$$

$$\le \int \frac{x}{t} \, dP = \frac{\mathbb{E}X}{t}.$$

Lemma (Borel-Cantelli): $\sum_n \mathbb{P}(A_n) < \infty$

$\Rightarrow P(A_n$ inf.often$) = 0$

$\Leftrightarrow \mathbb{P}\left(\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{m} A_n\right) = 0$

Proof: $\sum_n P(A_n) < \infty$ (converges)

$\Rightarrow \forall \varepsilon > 0, \ \exists m, \ \text{s.t} \ \sum_{n \ge m} \mathbb{P}(A_n) < \varepsilon$

$\bigcap_{m=1}^{\infty} \bigcup_{n \ge m} A_n \subseteq \bigcup_{n \ge m} A_n$

$\Downarrow$

$\mathbb{P}\left(\bigcap_{m=1}^{\infty} \bigcup_{n \ge m} A_n\right) \le P\left(\bigcup_{n \ge m} A_n\right) \le \sum_{n \ge m} \mathbb{P}(A_n) < \varepsilon \to 0$ ∎

proof ⇒ (**): WLOG, $(u = 0)$ $\quad S_n = \sum_{i=1}^{n} X_i$

It is suffices to show $\forall \varepsilon > 0$.

$$\sum_n P\left(\frac{|S_n|}{n} > \varepsilon\right) < \infty.$$

B-C-T $\Rightarrow$ $\mathbb{P}(\frac{|S_n|}{n} > \varepsilon \text{ i.o}) = 0$

I.e $\Rightarrow$ $P(\exists M, \text{ s.t } \frac{|S_n|}{n} \le \varepsilon \ \forall n \ge m) = 1.$

A.I.e $\Rightarrow$ $\mathbb{P}(\limsup_n \frac{|S_n|}{n} \le \varepsilon) = 1.$

send $\varepsilon \to 0$ $\Rightarrow$ $\mathbb{P}(\limsup_n \frac{|S_n|}{n} = 0) = 1.$

Step 1: $\sum_n \mathbb{P}(\frac{|S_n|}{n} > \varepsilon) \le \sum_n \frac{\mathbb{E} S_n^4}{(n\varepsilon)^4}$ (to be continue...)

Strong law of large number:

Thm: independent of integrable X_i, $\mathbb{E}X_i = 0$, $\sum_{i=1}^{\infty} \frac{\mathbb{E}X_i^2}{i!} < \infty$

Then! : $\overline{X} \to 0$ a.e

Thm 2: iid X_i integrable. $\mathbb{E}X_i = 0$. Then $\overline{X} \to 0$ a.e

proof: $Y_i := X_i \mathbb{1}\{|X_i| \le i\}$

$u_i = \mathbb{E}X_i \mathbb{1}\{|X_i| \le i\}$. $\Rightarrow$ $\frac{1}{n}\sum_{i=1}^{n} X_i = \underbrace{\frac{1}{n}\sum_{i=1}^{n}(X_i - Y_i)}_{①} + \underbrace{\frac{1}{n}\sum_{i=1}^{n}(Y_i - u_i)}_{②} + \underbrace{\frac{1}{n}\sum_{i=1}^{n} u_i}_{③}$

$= \mathbb{E}X \cdot \mathbb{1}\{|X| \le i\}$

③: $|\frac{1}{n}\sum_{i=1}^{n} u_i| \le \frac{1}{n}\sum_{i=1}^{n} |u_i|$

since $\mathbb{E}X = 0$

$= \frac{1}{n}\sum_{i=1}^{n} |\mathbb{E}X \cdot \mathbb{1}\{|X| > i\}|$

$\le \frac{1}{n}\sum_{i=1}^{n} |\mathbb{E}|X| \cdot \mathbb{1}\{|X| > i\}.$

$= \mathbb{E}|X| \ \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{|X| > i\}.$

$\le \underbrace{\mathbb{E}|X| \min(1, \frac{|X|}{n})}_{\text{DCT}: \ \le |X|} \to 0.$

$\mathbb{E}$ is somehow like integral operator

① By B-C, suffices to show that $\sum_{i=1}^{\infty} \mathbb{P}(X_i \ne Y_i) < \infty.$

$\sum_{i=1}^{\infty} \mathbb{P}(X_i \ne Y_i) \le \sum_{i=1}^{\infty} \mathbb{P}(|X_i| > i)$

$= \sum_{i=1}^{\infty} \mathbb{P}(|X| > i)$

$= \mathbb{E}(\sum_{i=1}^{\infty} \mathbb{1}\{|X| > i\})$

$\le \mathbb{E}|X| < \infty$

$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{|X| > i\} \le \min(1, \frac{|X|}{n})$

$\Downarrow$

$\sum_{i=1}^{n} \mathbb{1}\{|X| > i\} \le |X|$

② $\sum_{i=1}^{\infty} \frac{Var(Y_i)}{i^2} \le \sum_{i=1}^{\infty} \frac{\mathbb{E}(Y_i)^2}{i^2}$ (Note: $Var^2(X_i) = \mathbb{E}X_i^2 - (\mathbb{E}X_i)^2$)

$= \sum_{i=1}^{\infty} \frac{1}{i^2} \mathbb{E}X_i^2 \mathbb{1}\{|X_i| \le i\}.$

$= \frac{1}{i^2} \sum_{i=1}^{\infty} \mathbb{E}X_i^2 \ \mathbb{1}\{|X_i| \le i\}$

$\le \sum_{i=1}^{\infty}\sum_{j=0}^{i} \frac{1}{i^2} \cdot \mathbb{E}X^2 \mathbb{1}\{j-1 < |X| \le i\}$

$= \sum_{j=1}^{\infty}(\sum_{i=j}^{\infty} \frac{1}{i^2}) \cdot \mathbb{E}X^2 \cdot \mathbb{1}\{j-1 < |X| \le i\} \le 100000 \sum_{j=1}^{\infty} \frac{1}{j} \mathbb{E}X^2 \mathbb{1}\{j-1 < |X| \le j\}$

$\leq 100000 \sum_{j=1}^{\infty} \mathbb{E}X \, \mathbb{1}\{j-1 < |X| \leq j\}$ $\leftarrow$ $|x|/j \leq 1$.

$\leq 100000 \, \mathbb{E}|X| < \infty$

Central limit them

iid X_i, $\mathbb{E}X_i = 0$, $Var(X_i) = 1$

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \rightsquigarrow N(0,1)$$

Weak convergence:

1) metric space: (X, d)

metric: $d(x,y)$ w/ 3 property

2) Borel σ-field $\mathcal{B}(X)$ + Meas s.p $(X, \mathcal{B}(X))$

random element

3) $(\Omega, \mathcal{A}, \mathbb{P}) \to (X, \mathcal{B}(X))$

$\forall B \in \mathcal{B}(X)$

Def: push forward meas $:= P(B) = \mathbb{P}(X \in B) = \mathbb{P}(X^{-1}B)$
of $\mathbb{P}$ induced by X

!!! Distribution of X (i.e a meas.s.p)

$(X, \mathcal{B}(X), P)$

Def: bounded Lipschitz function

$f: X \to \mathbb{R}$ is Lipschitz if $|f(x) - f(y)| \leq k \, d(x,y)$ for $\forall x, y \in X$, $0 < k < \infty$

$BL(X) = \{ f: X \to \mathbb{R} \mid \sup_{x \in X} |f(x)| < \infty$ and $\sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty \}$.

Def: Weak convergence $\{P_n\}$ sequence of Prob-measures on $(X, \mathcal{B}(X))$ $P_n \xrightarrow{weak} P$ if

$\forall f \in BL(X)$, $P_n f \to Pf$

$P_n f = \int f \, dP_n$, $Pf = \int f \, dP$.

Def: random elements $\{X_n\}$, X. $X_n \xrightarrow{weak} X$ if $P_n \xrightarrow{weak} P$ (distribution) $\Leftrightarrow$ $\mathbb{E}f(X_n) \to \mathbb{E}f(X)$ for $\forall f \in BL(X)$.

$\mathbb{E}f(X_n) = \int f(X_n) \, d\mathbb{P} = \int f \, dP_n$

$\mathbb{E}f(X) = \int f(x) \, d\mathbb{P} = \int f \, dP$

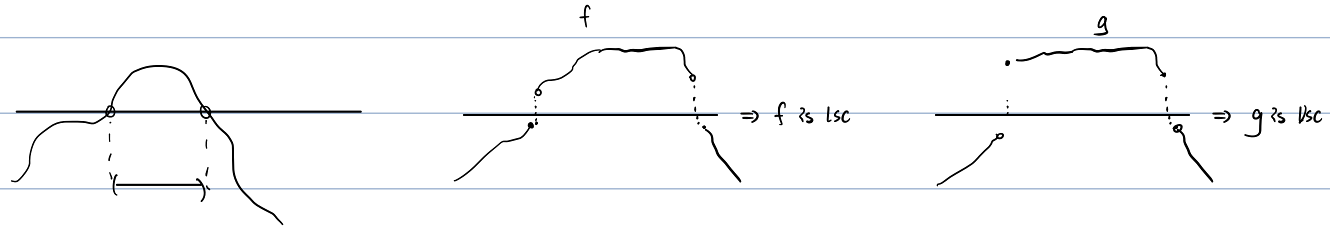
Notations: $P_n \xrightarrow{weak} P$, $X_n \xrightarrow{weak} X$, $X_n \xrightarrow{weak} P$

$X_n \xrightarrow{d} X$, $X_n \xrightarrow{L} X$, $X_n \Rightarrow X$

Def: $g: X \to \mathbb{R}$ is <u>lower semi-conti</u> (lsc) if

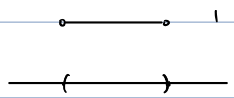
$$\{x \in X,\ g(x) > t\} = \{g > t\} \text{ is open for all } t.$$

equivalently, f <u>upper-semi-conti</u> (usc) if $-f$ is lower semi-conti $\iff$ $\{f < t\}$ is open $\forall t \in \mathbb{R}$



e.g: $\mathbb{1}_B$ is LSC if B is open

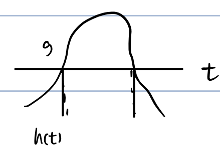
$\mathbb{1}_B$ is USC if B is closed



Lemma: g is lsc and is bounded from below, then there $\exists\ \ell_n \in BL(X)$ such that $\ell_n(x) \uparrow g(x)$ (i.e lsc function can be approx by BL)

proof: WLOG, $g \geq 0$

define $h_t := t\,\mathbb{1}\{g > t\}$

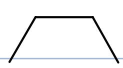


$\Rightarrow g = \sup_t h_t$ ↳ over countable

$\Uparrow$

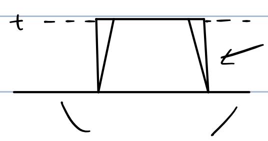
$$\boxed{\begin{aligned} &\text{Check } h_t \leq g: \quad h_t = t\,\mathbb{1}\{g>t\} + 0\,\mathbb{1}\{g \leq t\} \\ &\qquad\qquad\qquad \leq g\,\mathbb{1}\{g>t\} + g\,\mathbb{1}\{g \leq t\} \\ &\qquad\qquad\qquad = g \\ &g(x) = \sup_t \{t : t < g(t)\} = \sup_t h_t(x) \end{aligned}}$$

Step2:



$\nearrow$ $\ell_{t,k} \in BL(X)$

Define $\ell_{t,k}(x) = t \wedge (k\,d(x, \{g \leq t\}))$ $\quad d(x, \{g \leq t\}) := \inf_{y \in B} d(x,y)$.



$g \leq t$ $\quad$ $g > t$ is open

let $k \uparrow \infty$, we have

$$\lim_{k \to \infty} \ell_{t,k}(x) = t\,\mathbb{1}\{d(x,\{g \leq t\}) \neq 0\}$$

Claim $:= t\,\mathbb{1}\{g = t\}$

Step3: Combine:

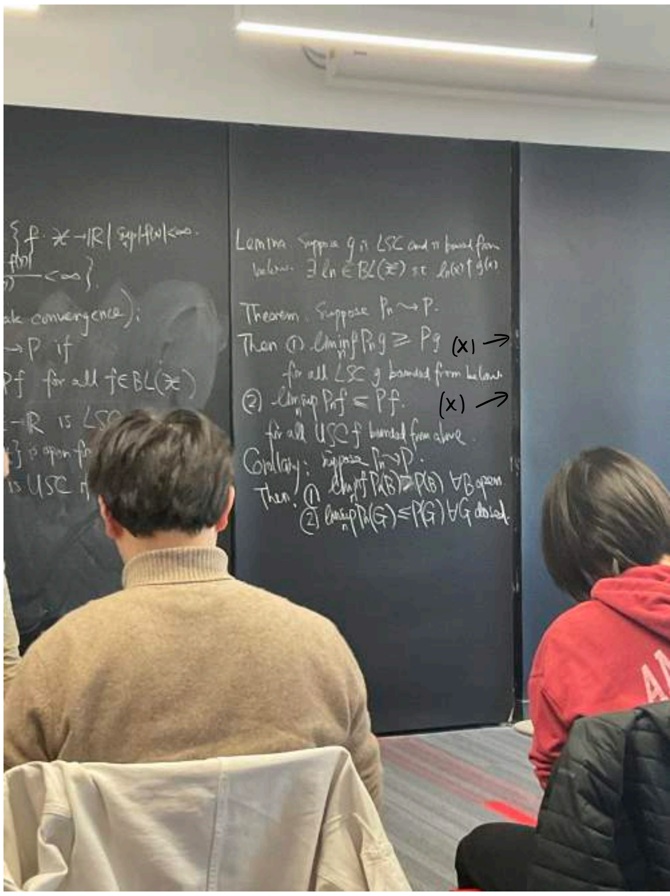
$g = \sup_{t,k} \ell_{t,k}$ ↳ (overcountable set)

list $\{\ell_{t,k}\}$ as

$f_1 \cdots f_n \cdots$

$\ell_1 = f_1 \quad \ell_2 = f_2 \vee f_1 \cdots \ell_n = f_n \vee \cdots \vee f_1$

$\Rightarrow \ell_n \uparrow g$, $\ell_n \in BL(X)$.



proof: WLOG $g \ge 0$, $\exists\, \ell_m \in BL(\mathcal{X})$ s.t $\ell_m \uparrow g$ and assume $\ell_m \ge 0$.

$$\liminf_n P_n g \ge \liminf_n P_n \ell_m$$

$$= P\ell_m$$

$\underline{P\ell_m \to Pg}$ (Monotone convergence) $\Rightarrow \liminf_n P_n g \ge Pg$.

bounded function $(0 \le f \le 1)$

$\underline{\mathring{f} = \sup\{g \le f : g \text{ is LSC}\}}$. i.e largest LSC below f.

Check $\mathring{f}$ is LSC: $\{\mathring{f} > t\} = \bigcup_g \{g > t\}$ is open.

For a set

$\mathring{\mathbb{1}}_B = \mathbb{1}_{\mathring{B}}$ (HW) "interior indicator"

\mathring{B}.

$\underline{\bar{f} = \inf\{g \ge f.\ g \text{ is USC}\}}$. "Closure" (N.B.) $\bar{f} \ge f \ge \mathring{f}$

$\bar{\mathbb{1}}_B = \mathbb{1}_{\bar{B}}$.

$\underline{\text{Claim: } \bar{f}(x) = \mathring{f}(x) \Leftrightarrow f \text{ is continous on } X.}$

$\forall \varepsilon$, $\exists$ open nbhd V_x s.t $|f(y) - f(x)| \le \varepsilon$ for $\forall y \in B$.

"$\Rightarrow$" $\mathring{f} = f \Rightarrow$ define:

$U = \{y : \mathring{f}(y) > f(x) - \varepsilon\}$. $\Rightarrow$ U is open and $x \in U$

$\bar{f} = f \Rightarrow$ define

$V = \{y : \bar{f}(y) < f(x) + \varepsilon\} \Rightarrow V$ is open and $x \in V$

Let $B = U \cap V$ and $x \in U \cap V$

open

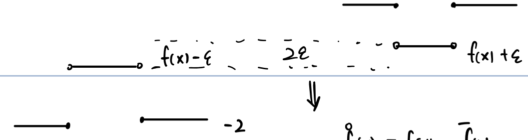
$$\Rightarrow \forall y \in B \Rightarrow \begin{cases} f(y) \ge \mathring{f}(y) > f(x) - \varepsilon \\ f(y) \le \bar{f}(y) < f(x) + \varepsilon \end{cases} \Rightarrow d_Y(f(y), f(x)) < \varepsilon.$$

"$\Leftarrow$" Since $|f(y) - f(x)| < \varepsilon$ for $y \in B$ Assume $\varepsilon \in (0,1)$

we have

USC

$$\underbrace{(f(x) - \varepsilon)\mathbb{1}_B(y) - 2\mathbb{1}_{B^c}(y)}_{\text{LSC}} \le f(y) \le \underline{(f(x) + \varepsilon)\mathbb{1}_B(y) + 2\mathbb{1}_{B^c}(y)}$$



$\Downarrow$

$\mathring{f}(x) = f(x) = \bar{f}(x)$ since they differ at x by at most 2ε.

For $\forall$ bounded f

$$P\bar{f} \geq \limsup_n P_n \bar{f} \geq \limsup_n P_n f \geq \liminf_n P_n f \geq \liminf_n P_n \mathring{f} \geq P\mathring{f}$$

suppose $P(\bar{f} - \mathring{f}) = 0 \Rightarrow P_n f \to Pf$.

$\Updownarrow$

$P(C_f) = 1$

Thm: $P_n f \to Pf$ for all bounded f there is conti: a.s. $-P$.

Cor: If $P_n \rightsquigarrow P$ Then $P_n(B) \to P(B)$ for all Borel B s.t $P(\partial B) = 0$ $\qquad \partial(2,3] = \{2,3\}$.

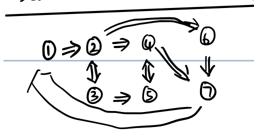
$\bar{\mathbb{1}}_B - \mathring{\mathbb{1}}_B = \mathbb{1}_{\partial B}$

$$\begin{cases} \bar{\mathbb{1}}_B = \mathbb{1}_{\bar{B}} \\ \mathring{\mathbb{1}}_B = \mathbb{1}_{\mathring{B}} \end{cases} \qquad \bar{B} \cap (\mathring{B})^c = \partial B$$

Thm: The follow TFAE

1) $P_n \ell \to P\ell$ for all $\ell \in BL(*)$
2) $\liminf_n P_n g \geq Pg$ for all LSC g.
3) $\limsup_n P_n f \leq Pf$ for all USC f.
4) $\liminf P_n(B) \geq P(B)$ (B open)
5) $\limsup_n P_n(G) \leq P(G)$ (G closed) $\qquad C_f = \{x : \bar{f} = \mathring{f}\}$
6) $\lim_n P_n f = Pf$ for all f bounded and $P(C_f) = 1$
7) $\lim_n P_n(B) = P(B)$ for all B s.t $P(\partial B) = 0$

Scratch: ⑦ $\Rightarrow$ ① is enough



$\int_{E^c} P(f > t)\,dt$ and $P_n f = \int_0^1 P_n(f > t)\,dt = \int_{E^c} P_n(f > t)\,dt$

proof: $\quad Pf = \int_0^1 P(f > t)\,dt$ (How to write in this form)

$$\lambda \mathbb{1}\{f > t\} = \int_0^f dt = f \Rightarrow Pf = P\lambda \mathbb{1}\{f > t\} \underset{\uparrow}{=} \lambda P \mathbb{1}\{f > t\}. = \int_0^1 P(f > t)\,dt.$$

Fubini thm

$\Uparrow$

them: $u \otimes v$: $\int |f| \, du \otimes v$ then $\iint f(x,y)\, du(x)\, dv(y) = \iint f(x,y)\, dv(y)\, du(x)$.

Ques: $P_n(f \to t) \xrightarrow{\quad} P(f \to t)$ a.e.-$\lambda \Rightarrow P_n f \to Pf$ by DCT.

$\Uparrow$

if $P(\partial\{f>t\}) = 0 \Leftrightarrow P(f=t) = 0$.

$E: \{t \in [0,1] : P(f=t) > 0\}$ has meas-zero $\Rightarrow u(E) = 0$.

$= \bigcup_{n=1} \{t \in [0,1] : P(f=t) > 1/n\}$

show Card(E) is countable

$\{f>t\}$ = open (t, ∞)

$\{f>t\}^\circ = \{f>t\}$

$\overline{\{f>t\}} = \{f \geq t\}$

$\Rightarrow \overline{\{f-t\}} - \{f>t\}^\circ = \{t\}$.

$\# (\{t \in [0,1] : P(f=t) > 1/n\}) \leq n$.

Weak convergence in $\mathbb{R}$

i) $P_n \ell \to P\ell \quad \forall \ell \in BL(\mathbb{R})$

$\Updownarrow$

ii) $P_n(-\infty, x] \to P(-\infty, x] \quad \forall x \in \mathbb{R}$ w/ $P\{x\} = 0$

$\Updownarrow$

iii) $P_n f \to Pf \quad \forall f \in BC^\infty(\mathbb{R})$

Bounded func w/ derivative of all order

i) $\Rightarrow$ iii) $C^\infty(\mathbb{R}) \subseteq BL(\mathbb{R})$

iii) $\Rightarrow$ i) Def. $\ell_\sigma(x) := \mathbb{E}\ell(x+\sigma Z) \quad Z \sim N(0,1)$

$= \int \ell(x+\sigma t) \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2} \, dt$

$\Rightarrow \ell_\sigma \in C^\infty(\mathbb{R})$ (Hint)

$\sup_{x \in \mathbb{R}} |\ell(x) - \ell_\sigma(x)| \leq \sup_{x \in \mathbb{R}} \mathbb{E}|\ell(x+\sigma z) - \ell(x)|$

$\leq C\sigma \mathbb{E}|Z|$

$\leq C'\sigma \longrightarrow 0$ as $\sigma \to 0$.

Now consider

$|P_n\ell - P\ell| \leq \underbrace{|P_n\ell_\sigma - P\ell|}_{\to 0} + 2C'\sigma \longrightarrow 0$ as $\sigma \to 0$.

Centre-limit theorem.

e.g. independent X_i w/ $\mathbb{E}X_i = 0$ $\mathrm{Var}(X_i) = 1$

WTS: $\frac{X_1 + \cdots + X_n}{\sqrt{n}} \rightsquigarrow N(0,1) / Z \sim N(0,1)$

Lindeberg swapping: writing RHS as $\frac{Z_1 + \cdots + Z_n}{\sqrt{n}}$

and i.e $\frac{X_1 + \cdots + Z_n}{\sqrt{n}} = \frac{\xi_1 + \cdots + \cdots X_n}{\sqrt{n}}$ (change 1 at a time)

$S := \underbrace{\xi_1 + \cdots + \xi_k}_{\text{independent}} \quad \mathbb{E}\xi_i^2 < \infty$

w/ $\begin{cases} \mathbb{E}\eta_i = \mathbb{E}\xi_i^2 < \infty \\ \mathrm{Var}(\eta_i) = \mathrm{Var}(\xi_i) \end{cases}$

$T := \underbrace{\eta_1 + \cdots + \eta_k}_{\text{independent Gaussians}}$

Goal: $\underbrace{|\mathbb{E}f(S) - \mathbb{E}f(T)|}_{\text{find bound for } \forall f \in C^\infty(\mathbb{R})}$

Let $X = \xi_1 + \xi_2 + \cdots + \xi_{i-1} + \cdots + \eta_{i+1} + \cdots + \eta_k$

$Y = \cdots\cdots \xi_{i-1} + \xi_i$

$Z = \cdots\cdots \xi_{i-1} + \eta_i$

$\mathbb{E}f(X+Y) - \mathbb{E}f(X+Z) \qquad f \in C^\infty(\mathbb{R}) \Rightarrow \sup_{x \in \mathbb{R}} |f'''(x)| < \infty$

Taylor expansion:

$f(x+y) = f(x) + f'(x)y + \frac{1}{2}f''(x)y^2 + R(x,y)$

where $R(x,y) = \frac{1}{6}f'''(x^*)y^3$ where x^* between x and $x+y$.

$|R(x,y)| \leq C|y|^3$

$\mathbb{E}f(X+Y) = \mathbb{E}f(X) + \underbrace{\mathbb{E}f'(X)Y}_{\mathbb{E}f(x)\mathbb{E}(Y)} + \frac{1}{2}\mathbb{E}f''(x)\mathbb{E}Y^2 + \mathbb{E}R(x,y)$

independent $\Rightarrow$ same analogy

$\longrightarrow$ first 3 terms equal $\Rightarrow |\mathbb{E}f(X+Y) - \mathbb{E}f(X+Z)| \le |\mathbb{E}R(X+Y) - \mathbb{E}(X+Z)|$

$\mathbb{E}f(X+Y) = \mathbb{E}f(X) + \mathbb{E}f(x)\mathbb{E}Z + \frac{1}{2}\mathbb{E}f''(x)\mathbb{E}Z^2 + \mathbb{E}R(x,Z)$

$\le C(\mathbb{E}|Y|^3 - \mathbb{E}Z^3)$

N.B we need to assume $\mathbb{E}|\xi|^3 < \infty$ exists.

Assume $Z \sim N(u, \sigma^2) \Leftrightarrow Z = u + \sigma W \quad W \sim N(0,1) \longrightarrow \le M\mathbb{E}|Y|^3$

$\mathbb{E}(|Z|^3) = \mathbb{E}|u+\sigma W|^3$

Apply n times: $|\mathbb{E}f(S) - \mathbb{E}f(T)| \le M \sum_{i=1}^{k} \mathbb{E}|\xi_i|^3$

$\le \mathbb{E}(|u| + \sigma|W|)^3 \le 10\,\mathbb{E}[|u|^3 + \sigma^3|W|^3]$

$\le 10000\,(|u|^3 + \sigma^3)$

$= 10000\,((\mathbb{E}Y)^3 + |Var(Y)|^3) \le 10000\,((\mathbb{E}Y)^3 + ((\mathbb{E}Y^2)^{3/2}))$

triangular array.

By Hölder ineq. $\begin{cases} \mathbb{E}(Y) \le (\mathbb{E}|Y|^3)^{1/3} \\ \mathbb{E}|Y|^2 \le (\mathbb{E}|Y|^3)^{2/3} \end{cases}$

$$\begin{cases} \xi_{1,1}, \xi_{1,2} \cdots \xi_{1,k(1)} & P_1 \\ \xi_{2,1}, \xi_{2,2} \cdots\cdots \xi_{2,k(2)} & P_2 \\ \xi_{3,1}, \xi_{3,2}, \cdots - - - - \xi_{3,k(3)} & \ddots \\ \vdots & P_n \end{cases}$$

At each row, sum of independent R.Vs $k(n) \to \infty$ as $n \to \infty$ (Meaning of Centre 极限 theorem).

Theorem: In the setting of triangular array

1) $\sum_{i=1}^{k(n)} \mathbb{E}\xi_{n,i} \to u \in \mathbb{R}$

2) $\sum_{i=1}^{k(n)} Var(\xi_{n,i}) \to \sigma^2 \in (0,\infty)$

3) $\sum_{i=1}^{k(n)} \mathbb{E}|\xi_{n,i}|^3 \to 0$

then, $\sum_{i=1}^{k(n)} \xi_{n,i} \rightsquigarrow N(u, \sigma^2)$

prove: Def $u_n = \sum_{i=1}^{k(n)} \mathbb{E}\xi_{n,i}$

$\sigma_n^2 = \sum_{i=1}^{k(n)} Var(\xi_{n,i})$

$\forall f \in C^\infty(\mathbb{R})$, WTS: $|\mathbb{E}f(\sum_{i=1}^{k(n)} \xi_{n,i}) - \mathbb{E}f(N(u,\sigma))| \to 0$

$\le \underbrace{|\mathbb{E}f(\sum_{i=1}^{k(n)} \xi_{n,i}) - \mathbb{E}f(N(u_n, \sigma_n^2))|}_{①}$

$+ \underbrace{|\mathbb{E}f(N(u_n, \sigma_n^2)) - \mathbb{E}f(N(u,\sigma))|}_{②}$.

①: By Lindeberg swapping $\Rightarrow$ ① $\le M \sum_{i=1}^{k(n)} \mathbb{E}|\xi_{n,i}|^3 \longrightarrow 0$.

②: $|\mathbb{E}f(N(u_n, \sigma_n^2)) - \mathbb{E}f(N(u,\sigma))|$. (since $u_n \to u$ and $\sigma_n^2 \to \sigma^2$)

$= |\mathbb{E}f(u_n + \sigma_n W) - \mathbb{E}f(u + \sigma W)|$ (where $W \rightsquigarrow N(0,1)$)

$\le |\mathbb{E}f(u_n + \sigma_n W) - \mathbb{E}f(u + \sigma_n W)| + |\mathbb{E}f(u + \sigma_n W) - \mathbb{E}f(u + \sigma W)|$

$\le C|u_n - u| + C|\sigma_n - \sigma|\,\mathbb{E}|W| \to 0$

Example: $X_1, \cdots X_n \overset{iid}{\sim} \text{Bernoilli}(p_n)$

$$\frac{S_n - p_n}{\sqrt{np_n(1-p_n)}} \rightsquigarrow N(0,1)$$

as long as $np_n(1-p_n) \to \infty$.

$P = P_n$

$$\xi_{n,i} = \frac{X_i - P}{\sqrt{np(1-p)}}$$

① $\sum_{i=1}^n \mathbb{E}\xi_{n,i} = 0$

② $\sum_{i=1}^n \text{Var}\,\xi_{n,i} = 1$

③ $\sum_{i=1}^n \mathbb{E}|\xi_{n,i}|^3 = \sum_{i=1}^n \mathbb{E}\left|\frac{x_i - p}{\sqrt{np(1-p)}}\right|^3 = n \cdot \frac{\mathbb{E}|X_i - p|^3}{(np(1-p))^{3/2}} = \frac{np(1-p)\overbrace{(p^2+(1-p)^2)}^{\text{at most } 2}}{(np(1-p))^{3/2}} \asymp \frac{1}{\sqrt{np(1-p)}} \to 0$

$\boxed{\text{Try to remove 3 condition} \to \text{CLT}}$

Cor: $X_1, \cdots X_n$ iid

$\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = 1$

then $\frac{X_1 + \cdots + X_n}{\sqrt{n}} \rightsquigarrow N(0,1)$

$\forall f \in C^\infty(\mathbb{R})$.

$$\left| \mathbb{E}f\Big(\underline{\sum_{i=1}^n \xi_{n,i}}\Big) - \mathbb{E}f\Big(\underline{\sum_{i=1}^n \frac{X_i}{\sqrt{n}}}\Big)\right|$$

$$= \mathbb{E}\left|f\Big(\sum_{i=1}^n \xi_{n,i}\Big) - f\Big(\sum_{i=1}^n \frac{X_i}{\sqrt{n}}\Big)\right| \cdot \mathbb{1}\Big\{\sum_{i=1}^n \xi_{n,i} \neq \sum_{i=1}^n \frac{X_i}{\sqrt{n}}\Big\}$$

proof: $\xi_{n,i} = \frac{X_i}{\sqrt{n}} \mathbb{1}\{|X_i \leq \sqrt{n}\}$.

$$\leq C\,P\Big(\sum_{i=1}^n \xi_{n,i} \neq \sum_{i=1}^n \frac{X_i}{\sqrt{n}}\Big) \leq C\sum_{i=1}^n \mathbb{P}\Big(\xi_{n,i} \neq \frac{X_i}{\sqrt{n}}\Big) \to 0.$$

1) $\sum_{i=1}^n \mathbb{E}(\xi_{n,i}) = \sqrt{n}\,\mathbb{E}X\mathbb{1}\{|X| \leq \sqrt{n}\} = -\sqrt{n}\mathbb{E}X\mathbb{1}\{|X| > \sqrt{n}\}. \Rightarrow \left|\sum_{i=1}^n \mathbb{E}\xi_{n,i}\right| \leq \sqrt{n}\,\mathbb{E}|X|\mathbb{1}\{|X| > \sqrt{n}\}.$

$$\leq \mathbb{E}X^2\mathbb{1}\{|X| > \sqrt{n}\} \to 0.$$

DCT: $\mathbb{E}X^2 = 1 < \infty$

2) $\sum_{i=1}^n \text{Var}(\xi_{n,i}) = n\,\text{Var}\big(\frac{X}{\sqrt{n}}\mathbb{1}\{|X| \leq \sqrt{n}\}\big)$

$$= \text{Var}(X\mathbb{1}\{|X| \leq \sqrt{n}\}) = \underbrace{\mathbb{E}X^2\mathbb{1}\{|X| \leq \sqrt{n}\}}_{\substack{\text{DCT} \\ \mathbb{E}X^2 = 1}} - \underbrace{(\mathbb{E}X\mathbb{1}\{|X| \leq \sqrt{n}\})^2}_{\to 0} \to 1$$

3) $\sum_{i=1}^n \mathbb{E}|\xi_{n,i}|^3 = n\,\mathbb{E}\frac{|X|^3}{n^{3/2}}\mathbb{1}\{|X| \leq \sqrt{n}\}.$

$$= \mathbb{E}\frac{|X|}{\sqrt{n}}\mathbb{1}\{|X| \leq \sqrt{n}\} \cdot |X|^2$$

$$\leq \mathbb{E}X^2 \min\Big(\frac{|X|}{\sqrt{n}}, 1\Big) \to 0 \text{ by DCT.} \Rightarrow \sum_{i=1}^n \xi_{n,i} \rightsquigarrow N(0,1)$$

Stein method:

Lemma: $Z \sim N(0,1)$ f is absolutely cont. and $\mathbb{E}|f'(Z)| < \infty$

$\Rightarrow \mathbb{E} f'(Z) = \mathbb{E} Z f(Z)$

N.B. $P(B) = \int_B \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, d\lambda(x)$

The reverse is also true

find f s.t

$0 = \mathbb{E}(f'(Z) - Zf(Z)) = \mathbb{E}(\mathbb{1}\{Z \le t\} - \Phi(t))$

where $\Phi(t) = \mathbb{P}(Z \le t)$ $Z \sim N(0,1)$

find Stein differential equation: $f'(w) + wf(w) = \mathbb{1}\{w \le t\} - \Phi(t)$

Gaussian density

$$\Rightarrow e^{-w^2/2} f'(w) + e^{-w^2/2} wf(w) = e^{-w^2/2} (\mathbb{1}\{w \le t\} - \Phi(t))$$

$$\Rightarrow (e^{-w^2/2} f(w))' = e^{-w^2/2} (\mathbb{1}\{w \le t\} - \Phi(t))$$

$$\Rightarrow e^{-w^2/2} f(w) = \int_{-\infty}^{w} e^{-x^2/2} \, \mathbb{1}\{w \le t\} - \Phi(t) \quad + C \quad ?$$

$$\Rightarrow f(w) = e^{w^2/2} \cdot \int_{-\infty}^{w} e^{-x^2/2} \, \mathbb{1}\{x \le t\} - \Phi(t) \, dx \quad + Ce^{w^2/2}$$

N.B: $\Phi(t)$ is the control of $\underline{0 = \mathbb{E}(f'(Z) - Zf(Z))}$ make it finite ! $w \uparrow \Rightarrow f(x) \le e^{w^2/2} \uparrow$

C is zero inorder to make well defined !

Application to prove:

$$\boxed{\begin{array}{l}\text{Centre-limit theorem.} \\ \text{e.g. independent } X_i \text{ w/ } \mathbb{E}X_i = 0 \quad Var(X_i) = 1 \\ \text{WTS: } \dfrac{X_1 + \cdots + X_n}{\sqrt{n}} \rightsquigarrow N(0,1) / Z \sim N(0,1)\end{array}}$$

If $\mathbb{E}f'(w) - \mathbb{E}wf(w) \overset{?}{=} \mathbb{E}h(w) - \mathbb{E}h(Z)$

then $\Rightarrow$ LHS $<<1$ and for enough f $\Rightarrow$ RHS is small

Stein differential equation:

$f'(w) - wf(w) = h(w) - Nh$ where $Nh = \mathbb{E}h(Z)$, $Z \sim N(0,1)$

$$e^{-w^2/2} f'(w) - e^{-w^2/2} wf(w) = e^{-w^2/2} (h(w) - Nh)$$

$$(e^{-w^2/2} f(w))' = e^{-w^2/2} (h(w) - Nh)$$

$$f(w) = e^{w^2/2} \int_{-\infty}^{w} e^{-x^2/2} (h(x) - Nh) \, dx$$

$$\Rightarrow f_h(w) = e^{w^2/2} \int_{-\infty}^{w} e^{-x^2/2} (h(x) - Nh)$$

Lemma:

$\max\{\|f_h\|_\infty, \|f_h'\|_\infty, \|f_h''\|_\infty\} \le 2\, Lip(h)$

where $Lip(h) := \inf\{\ell > 0, \ |h(x) - h(y)| \le \ell |x - y|\}$.

Wasserstein distance

$W(P, Q) = \sup_{Lip(h) \le 1} |P_h - Q_h|$ where P, Q are $\forall$ prob-meas on $\mathbb{R}$

$Z \sim N(0,1)$.

$W(w, Z) = \sup_{Lip(h) \le 1} |\mathbb{E}h(w) - \mathbb{E}h(Z)| = \sup_{Lip(h) \le 1} |\mathbb{E}h(w) - Nh|$

setting:

$$W = \frac{X_1+\cdots+X_n}{\sqrt{n}}$$ indep

$\mathbb{E}X_i = 0 \quad \mathbb{E}X_i^2 = 1$

$\mathbb{E}|X_i|^3 < \infty$

$$= \sup_{Lip(h)\le 1} |\mathbb{E} f_h'(w) - \mathbb{E}\, w f_h(w)|$$

$$\le \sup_{\substack{\|f\|_\infty \le 2 \\ \|f_h'\|_\infty \le 2 \\ \|f_h''\|_\infty \le 2}} |\mathbb{E} f'(w) - \mathbb{E}\, w f(w)|$$

consider that

$$\mathbb{E} w f(w) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{E} X_i f(W)$$

trick ☆: let $W_i = W - \frac{X_i}{\sqrt{n}}$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{E} X_i (f(w) - f(w_i)) + \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{E} X_i f(w_i)}_{=0}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{E} X_i (f(w) - f(w_i) - (w-w_i) f'(w_i)) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{E} X_i (w-w_i) f'(w_i)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{E} X_i \frac{1}{2} (w-w_i)^2 f''(\xi_i) + \frac{1}{n} \sum_{i=1}^n \mathbb{E} X_i^2 f'(w_i)$$

for some $\xi_i \in (w, w_i)$

$$= \frac{1}{2}\frac{1}{\sqrt{n}} \cdot \frac{1}{n} \sum_{i=1}^n \mathbb{E} X_i^3 f''(\xi_i) + \frac{1}{n} \sum_{i=1}^n \mathbb{E} f'(w_i)$$

$\Rightarrow |\mathbb{E} f'(w) - \mathbb{E}\, w f(w)|$

$$\le \frac{1}{\sqrt{n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i|^3 + |\mathbb{E} f'(w) - \frac{1}{n} \sum_{i=1}^n \mathbb{E} f'(w_i)|$$

$$\le \frac{1}{n} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^n \mathbb{E}|X_i|^3 + \frac{1}{n} \sum_{i=1}^n \mathbb{E}\, | f'(w) - f'(w_i)|$$

$$\le \frac{1}{\sqrt{n}} \cdot \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i|^3 + \frac{2}{\sqrt{n}} \cdot \frac{1}{n} \cdot \sum_{i=1}^n \underbrace{\mathbb{E}|X_i| \le (\mathbb{E}|X_i|^3)^{1/3} \le \mathbb{E}|X_i|^3}_{\text{since } \mathbb{E}|X_i|^2 = 1 \Rightarrow \mathbb{E}|X_i|^3 \ge (\mathbb{E}|X_i|^2)^{3/2} \ge 1.}$$

$$\le \frac{3}{\sqrt{n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i|^3$$

Thm: $X_1, \cdots X_n$ indep $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = 1$, $\mathbb{E}|X_i|^3 < \infty$.

Then,

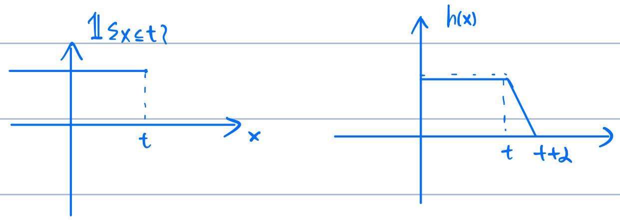
$$W\left(\frac{X_1+X_2+\cdots+X_n}{\sqrt{n}}, N(0,1)\right) \le \frac{3}{\sqrt{n}} \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i|^3$$

Def: $KS(P,Q) = \sup_{t\in\mathbb{R}} |P \mathbb{1}_{\{\cdot \le t\}} - Q \mathbb{1}_{\{\cdot \le t\}}|$

$KS(W,Z) = \sup_{t\in\mathbb{R}} |\mathbb{P}(W \le t) - \mathbb{P}(Z \le t)|$ important!

$TV(P,Q) = \sup_B |P(B) - Q(B)|$

Since $Lip(\mathbb{1}) = \infty$, we can not direct derive the same procedure as before.



$$h(x) = \begin{cases} 1 & x \le t \\ \text{linear} & x \in (t, t+d) \\ 0 & x \ge t+d \end{cases}$$

Clearly: $\mathbb{1}\{x\le t\} \le \underbrace{h(x)}_{\le \mathbb{1}\{w\le t+d\}}$; Moreover $\mathbb{1}\{x\le t\} \ge h(x+d)$. $\Rightarrow$

consider $Z \sim N(0,1)$.

$P(W\le t) - P(Z\le t)$

$\le Eh(W) - Eh(Z) + Eh(Z) - P(Z\le t)$

$\le \frac{1}{d} W(W,Z) + P(Z\le t+d) - P(Z\le t)$

$\le \frac{1}{d} W(W,Z) + P(t < Z \le t+d)$

$\int_t^{t+d} \phi(x)\,dx \le d$

$P(Z\le t) - P(W\le t)$

$\le P(Z\le t) - Eh(W+d)$

$= P(Z\le t) - Eh(Z+d) + Eh(Z+d) - Eh(W+d)$

$\le P(Z\le t) - P(Z\le t-d) + \frac{1}{d} W(W,Z)$

$= P(t-d < Z \le t) + \frac{1}{d} W(W,Z)$

$\le d + \frac{1}{d} W(W,Z) \Rightarrow KS(W,Z) \le \frac{1}{d} W(W,Z) + d$

take $a = \sqrt{W(W,Z)}+d$

Lemma: If $Z \sim N(0,1)$

$KS(W,Z) \le 2\sqrt{W(W,Z)}+d$

Cor: indep $X_1, \dots, X_n$, $EX_i = 0$, $EX_i^2 = 1$, $E|X_i|^3 < \infty$

then

$$KS\left(\frac{X_1+\dots+X_n}{\sqrt{n}}, N(0,1)\right) \le 2\sqrt{\frac{3}{\sqrt{n}} \frac{1}{n} \sum_{i=1}^n E|X_i|^3}$$

Berry-Essence: iid $X_1, \dots X_n$ $EX_i = 0$, $EX_i^2 = 1$ $E|X_i|^3 < \infty$

$$KS\left(\frac{X_1+\dots+X_n}{\sqrt{n}}, N(0,1)\right) \le C \frac{E|X_i|^3}{\sqrt{n}}$$

N.B

$f'(w) - wf(w) = h(w) - Nh$

$$f_h(w) = e^{w^2/2} \int_{-\infty}^{w} (h(x) - Nh) \cdot e^{-x^2/2}\, dx$$

Lemma: $\|f_h\|_\infty$, $\|f_h'\|_\infty$, $\|f_h''\|_\infty \le 2 Lip(h)$

$Z \sim N(0,1)$ $(t>0)$ $P(Z>t)$?? (behave)

① $P(Z>t) = \int_t^\infty \phi(x)\,dx = \int_t^\infty \frac{1}{x} x\phi(x)\,dx \le \frac{1}{t}\int_t^\infty x\phi(x)\,dx$

$= \frac{\phi(t)}{t} = \frac{e^{-t^2/2}}{\sqrt{2\pi}\, t}$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$\phi'(x) = -x\phi(x)$

② $P(Z>t) = P(e^{\lambda Z} > e^{\lambda t}) = e^{-\lambda t} E e^{\lambda Z} = e^{-\lambda t + \frac{1}{2}\lambda^2} := e^{f(\lambda)}$

$f'(\lambda) = -t + \lambda = 0 \Rightarrow \lambda = t \Rightarrow P(Z>t) \le e^{-\frac{1}{2}t^2}$

Cont. on N.B

$$f_h(w) = e^{w^2/2}\int_{-\infty}^{w} e^{-x^2/2}(h(x) - Nh)\,dx \quad \text{is bounded}$$

$$= -e^{w^2/2}\int_{w}^{\infty} e^{-x^2/2}(h(x) - Nh)\,dx$$

$$|f_h(w)| \le \begin{cases} e^{w^2/2}\int_{-\infty}^{w} e^{-x^2/2}|h(x) - Nh|\,dx & w<0 \\ -e^{w^2/2}\int_{w}^{\infty} e^{-x^2/2}|h(x) - Nh|\,dx & w\ge 0. \end{cases}$$

$w \ge 0$

$$|f_h| \le \underbrace{e^{w^2/2}\int_{w}^{\infty} e^{-x^2/2}|h(x)-h(0)|\,dx}_{(1)} + \underbrace{|h(0)-Nh|\,e^{w^2/2}\int_{w}^{\infty} e^{-x^2/2}\,dx}_{(2)}$$

$$\le (1+\sqrt{2\pi}\,\mathbb{E}|Z|)\,\mathrm{Lip}(h)$$

① $\le e^{w^2/2}\cdot \mathrm{Lip}(h)\int_{w}^{\infty} e^{-x^2/2}\,dx = \mathrm{Lip}(h)$

② $|h(0)-Nh|\sqrt{2\pi}\cdot e^{w^2/2}\cdot e^{-w^2/2} = \sqrt{2\pi}\,|h(0)-Nh| \le \sqrt{2\pi}\cdot \mathbb{E}|h(Z)-h(0)| \quad Z\sim(0,1)$

$$\le \sqrt{2\pi}\,\mathrm{Lip}(h)\,\mathbb{E}|Z|$$

③

$$\mathbb{P}(Z>t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{(x+t)^2}{2}}\,dx$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}}\cdot e^{-x^2/2}\, e^{-t^2/2}\cdot e^{-xt}\,dx$$

$$= \frac{e^{-t^2/2}}{t\sqrt{2\pi}}\int_0^\infty e^{-x^2/2}\cdot \boxed{t\cdot e^{-xt}}\,dx$$

← exponential distribution

$$= \frac{e^{-t^2/2}}{t\sqrt{2\pi}}\,\mathbb{E}e^{-X^2/2}, \quad X\sim \mathrm{Exp}(t)$$

$x \ge 0$:

$$1-x \le e^{-x} \le 1-x+\frac{x^2}{2}$$

$$1-x^2/2 \le e^{-x^2/2} \le 1-\frac{x^2}{2}+\frac{x^4}{8}$$

$\mathbb{E}X^2 = \frac{2}{t^2}$, $\quad \mathbb{E}X^4 = \frac{24}{t^4}$

$$\left(1-\frac{1}{t^2}\right)\frac{e^{-t^2/2}}{t\sqrt{2\pi}} \le \mathbb{P}(Z>t) \le \left(1-\frac{1}{t^2}+3/t^4\right)\frac{e^{-t^2/2}}{t\sqrt{2\pi}}$$

Poisson- approxmation

$\xi_{n,1}, \dots \xi_{n,n} \overset{iid}{\sim} \mathrm{Bernolli}(p_n)$

if $np_n \longrightarrow \lambda$

$\sum_{i=1}^{n} \xi_{n,i} \rightsquigarrow \mathrm{Poisson}(\lambda)$

Louis Chen - Stein method

Def. X, Y, $(\hat X, \hat Y)$ is a coupleny of (X, Y) make it in the same space.

if $\hat X \overset{d}{=} X$, $\hat Y \overset{d}{=} Y$.

$$\mathbb{E}f(X) - \mathbb{E}f(Y) = \mathbb{E}f(\hat X) - \mathbb{E}f(\hat Y)$$

$$= \mathbb{E}(f(\hat X) - f(\hat Y))$$

Def: $(\hat X, \hat Y)$ is maximal caupling if $\mathbb{P}(\hat X = \hat Y)$ is maximized.

Them: X, Y discrete

$\{ \mathbb{P}(X=i) = p_i$

$\mathbb{P}(Y=i) = q_i$

then $\max(\hat{X},\hat{Y})$ exists and $\mathbb{P}(\hat{X}=\hat{Y}) = \sum_i^{\infty} \min(p_i, q_i)$ 1- "total variation distance" / tv-affinity.

proof: $A = \{i.\ p_i < q_i\}$

for $\forall$ coupling $(\hat{X},\hat{Y})$,

$$\mathbb{P}(\hat{X}=\hat{Y}) = \mathbb{P}(\hat{X}=\hat{Y}\in A) + \mathbb{P}(\hat{X}=\hat{Y}\in A^c)$$

$$\le P(\hat{X}\in A) + \mathbb{P}(\hat{Y}\in A^c)$$

$$= \sum_{i\in A} p_i + \sum_{i\in A^c} q_i = \sum_{i=1}^{\infty} \min(p_i, q_i)$$

Construct of Max-coupling to achieve

$$\alpha = \sum_i \min(p_i, q_i)$$

$d_i = \frac{\min(p_i, q_i)}{\alpha}$ $\rightarrow$ Probability

$b_i = p_i - \min(p_i, q_i) / 1-\alpha$

$c_i = q_i - \min(p_i, q_i) / 1-\alpha$

$$\begin{cases} p_i = \alpha d_i + (1-\alpha) b_i \\ q_i = \alpha d_i + (1-\alpha) c_i \end{cases}$$

consider R.Vs D,B,C

$$\begin{cases} \mathbb{P}(D=i) = d_i \\ \mathbb{P}(B=i) = b_i \\ \mathbb{P}(C=i) = c_i \end{cases}$$

$\boxed{I \sim \text{Bernolli}}$

$$\hat{X} = \begin{cases} D & I=1 \\ B & I=0 \end{cases}$$

$$\hat{Y} = \begin{cases} D & I=1 \\ C & I=0 \end{cases}$$

$$\mathbb{P}(\hat{X}=\hat{Y}) \ge \mathbb{P}(I=1) = \alpha$$

$$\mathbb{P}(\hat{X}=i) = \alpha \mathbb{P}(D=i) + (1-\alpha)\mathbb{P}(B=i) = p_i = \mathbb{P}(X=i)$$

$$\mathbb{P}(\hat{Y}=i) = q_i = \mathbb{P}(Y=i)$$

$$\frac{(\sqrt{x}-\sqrt{\lambda})(\sqrt{x}+\sqrt{\lambda})}{(\sqrt{x}+\sqrt{\lambda})}$$

$N(\lambda, \lambda)$.