Homoclinic Bifurcation Theorem 6.1 (Andronov & Leontovich [1939])

Consider a two-dimensional system

$$\dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}, \tag{1}$$

with smooth f, having at $\alpha = 0$ a saddle equilibrium point $x_0 = 0$ with eigenvalues $\lambda_1(0) < 0 < \lambda_2(0)$ and a homoclinic orbit Γ_0. Assume the following genericity conditions hold:

(H.1) $\sigma_0 = \lambda_1(0) + \lambda_2(0) \neq 0$;

(H.2) $\beta'(0) \neq 0$, where $\beta(\alpha)$ is the previously defined split function.

Then, for all sufficiently small $|\alpha|$, there exists a neighborhood U_0 of $\Gamma_0 \cup x_0$ in which a unique limit cycle L_β bifurcates from Γ_0. Moreover, the cycle is stable and exists for $\beta > 0$ if $\sigma_0 < 0$, and is unstable and exists for $\beta < 0$ if $\sigma_0 > 0$.

Homoclinic Bifurcation Analysis

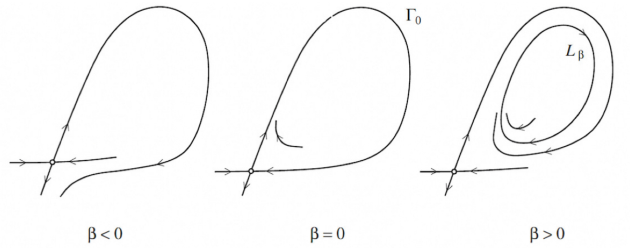


FIGURE 6.7. Homoclinic bifurcation on the plane ($\sigma_0 < 0$).

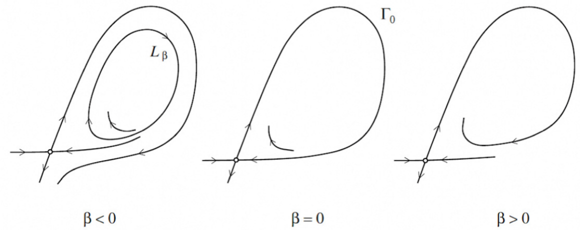


FIGURE 6.8. Homoclinic bifurcation on the plane ($\sigma_0 > 0$).

Summary: The sign of σ_0 determines the direction of bifurcation and stability of L_β

- If $\sigma_0 < 0$, the homoclinic orbit at $\beta = 0$ is stable from the inside. By theorem, we know that there exists $L_\beta \subset U_0$ for $\beta > 0$, but no periodic orbits in U_0 for $\beta < 0$ (L_β stable limit cycle).
- If $\sigma_0 > 0$, the homoclinic orbit at $\beta = 0$ is unstable from the inside. By theorem, we know that there exists $L_\beta \subset U_0$ for $\beta < 0$, but no periodic orbits in U_0 for $\beta > 0$ (L_β unstable limit cycle).

Definitions of Homoclinic and Heteroclinic Orbits

Def 1.1 Homoclinic Orbit:
An orbit Γ starting at a point $x \in \mathbb{R}^n$ is called *homoclinic* to the equilibrium point x_0 of system (1) if $\phi^t x \to x_0$ as $t \to \pm\infty$.
Def 1.2 Heteroclinic Orbit:
An orbit Γ starting at a point $x \in \mathbb{R}^n$ is called *heteroclinic* to the equilibrium points $x^{(1)}$ and $x^{(2)}$ of system (1) if $\phi^t x \to x^{(1)}$ as $t \to -\infty$ and $\phi^t x \to x^{(2)}$ as $t \to +\infty$.

- Homoclinic orbits connect an equilibrium point to itself, creating a loop in the phase space.
- Heteroclinic orbits connect two different equilibrium points, forming a path between them in the phase space.

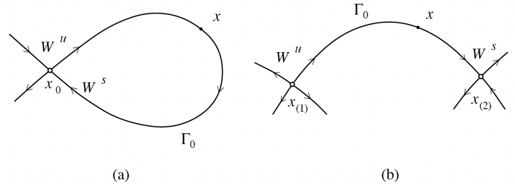


FIGURE 6.1. (a) Homoclinic and (b) heteroclinic orbits on the plane.

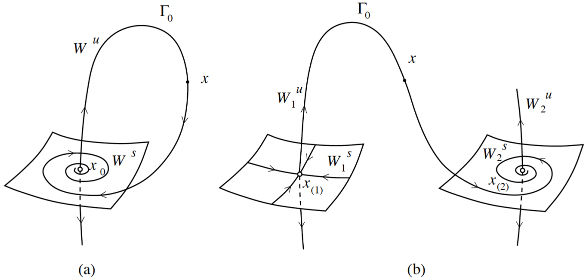


FIGURE 6.2. (a) Homoclinic and (b) heteroclinic orbits in three-dimensional space.

Examples and Connection with $W^{(u,s)}$

- It is clear that a homoclinic orbit Γ_0 to the equilibrium x_0 belongs to the intersection of its unstable and stable manifolds (i.e. $\Gamma_0 \subseteq W^u(x_0) \cap W^s(x_0)$).
- Similarly, a heteroclinic orbit Γ_0 to the equilibria $x^{(1)}$ and $x^{(2)}$ satisfies $\Gamma_0 \subseteq W^u(x^{(1)}) \cap W^s(x^{(2)})$.

Remark:
Definitions 1.1 and 1.2 do not require the equilibria to be hyperbolic.
Homoclinic orbit to a saddle-node point with an eigenvalue $\lambda_1 = 0$.

Motivation for Lemma 1.3:
Orbits homoclinic to hyperbolic equilibria can result in structural instability while the equilibria themselves are structurally stable.

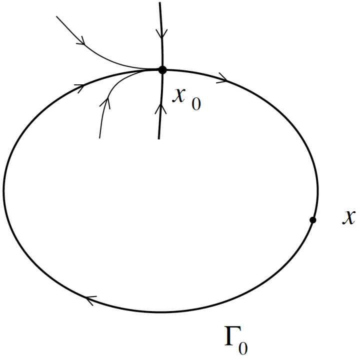


FIGURE 6.3. Homoclinic orbit Γ_0 to a saddle-node equilibrium.

Lemma 1.3: A homoclinic orbit to a hyperbolic equilbrium is structually unstable.

Proof to Lemma 1.3

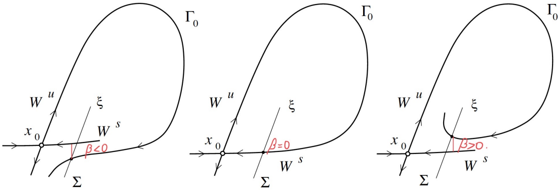
Definiation 1.4: Two smooth manifolds M,N $\subset \mathbb{R}^n$ *intersect transversally* if there exist n linearly independent vectors that are tangent to at least one of these manifolds at any intersection point

Remark: Since we are most interest in 2d system, the hyperbolic equilibrum is saddle. Stable manifold and Unstable manifold are smooth invariant manifolds

Proof to Lemma 1.3

Proof of Lemma 1.3:
Suppose that system has a hyperbolic equilibrium x_0 with a eigenvalues having positive real parts and b eigenvalues having negative real parts, a + b = n (dimension of the system). Assume that the corresponding stable and unstable manifolds $W^u(x_0)$ and $W^s(x_0)$ intersect along a homoclinic orbit. We shall show that the intersection cannot be transversal.
Since at any point x of this homoclinic orbit, the vector $f(x)$ is tangent to both manifolds $W^u(x_0)$ and $W^s(x_0)$. Therefore, we can find no more than a + b -1 = n -1 independent tangent vectors to these manifolds, and intersect transversally requires n vectors.(contridiction)
Moreover, we can find *split function* splits the manifolds in that remaining direction and they do not intersect anymore near Γ_0.

Split function



$\beta < 0$ $\beta = 0$ $\beta > 0$

FIGURE 6.4. Split function in the planar case ($n = 2$).

The manifolds split either "down" or "up" and the force preventing W^u come back to the saddle is a quantity denoted by ξ^u, the ξ value of the intersection of W^u with Σ.(Σ denoted as cross-section to W^s near saddle)
And we may define the scalar $\beta = \xi^u$ a split function.

Example of Homoclinic Bifurcation theorem

Consider the following system :

$$\begin{cases} \dot{x} = -x + 2y + x^2 \\ \dot{y} = (2-\alpha)x - y - 3x^2 + (3/2)xy^2 \end{cases}$$

The origin $(0,0)$ is clearly a saddle for $|\alpha|$ sufficient small.

To use our theorem, we need to check the required 3 conditions in this example!

Condition Check

At $a = 0$,

$$Df = \begin{bmatrix} -1 + x & 2 \\ 2 - 3x + 3y^2 & -1 + \frac{3}{2}xy \end{bmatrix} \Rightarrow Df(0,0) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

then, this saddle has eigenvalues

$$(-1-\lambda)^2 - 4 = 0 \Rightarrow \lambda^2 + 2\lambda - 3 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -3.$$

with $\sigma_0 = -2 < 0$ implies that (H.1) check.

Condition Check

Step 2: We need to show that there exist a homoclinic orbit to the origin (0,0). Equivalently, we can introduce Cartesian leaf: $H(x, y) \equiv x^2(1-x) - y^2 = 0$, consisting of one of the orbit, which is homoclinic to $(0, 0)$ and show that vector field at $a = 0$:

$$V(x, y) = \left[-x + 2y + x^2, \quad 2x - y - 3x^2 + 3/2xy\right]^\top .$$

is tangent to the curve $H(x, y) = 0$ at all nonequilibrum point

$$\begin{aligned}(\nabla H)(x, y) &= (2x - 3x^2, -2y)^T \quad \text{"Normal Vector of } H\text{"} \\ \langle V \cdot \nabla H \rangle &= (2x - 3x^2) \cdot (-x + 2y + x^2) + (-2y) \cdot (2x - y - 3x^2 + \frac{3}{2}xy) \\ &= -2x^2 + 4xy + 2x^3 + 3x^3 - 6x^2y - 3x^4 - 4xy + 2y^2 + 6x^2y - 3xy^2 \\ &= -2x^2 + 5x^3 - 3x^4 + 2y^2 - 3xy^2\end{aligned} \tag{2}$$

plug info $(0, 0)$ and $(1, 0)$, we get $\langle V, \nabla H \rangle = 0$ along $H(x, y) = 0$.

Graph of Step 2

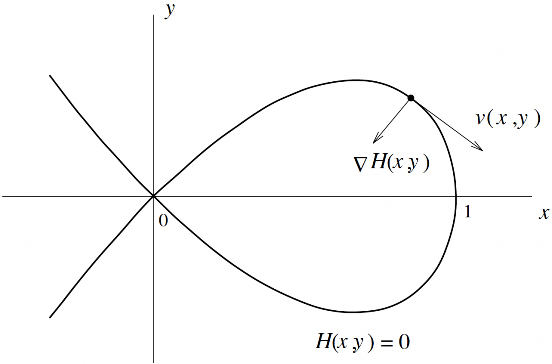


FIGURE 6.18. The homoclinic orbit of (6.8) at $\alpha = 0$.

Condition Check

Step 3: we need to check H.2: $\beta' \neq 0$ at $a = 0$ to use our theorem. This is equivalent to show, known as "Melnikov integral",

$$Ma(0) = \int_{-\infty}^{\infty} \exp\left[-\int_0^t \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\partial\tau\right]\left(f_1\frac{\partial f_2}{\partial a} - f_2\frac{\partial f_1}{\partial a}\right)\partial t \neq 0$$

Check Condition

Now, given

$$f_1 = -x + 2y + x^2 \quad \text{and} \quad f_2 = (2-\alpha)x - y - 3x^2 + 3xy.$$

The divergence div$f$ is given by

$$\text{div} f = \frac{\partial(-x + 2y + x^2)}{\partial x} + \frac{\partial((2-\alpha)x - y - 3x^2 + 3xy)}{\partial y}$$

= (-1 + 2x) + (-1 + 3x) = 2x - 2.

The partial derivatives of f with respect to α are

$$\frac{\partial f_1}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial f_2}{\partial \alpha} = -x.$$

We have the Melnikov integral $M_\alpha(0)$ as

$$M_\alpha(0) = \int_{-\infty}^{+\infty} \exp\left(-\int_0^t (2x(\tau) - 2)d\tau\right) \cdot f_2(x(t), y(t)) \cdot x(t)dt.$$

Check condition

The remaining unknown is time t, and we have the system

$$\begin{cases} \frac{dx}{dt} = -x + 2y + x^2, \\ \frac{dy}{dt} = 2x - y - 3x^2 + 3xy, \end{cases}$$

which implies

$$\frac{dy}{dx} = \frac{2x - y - 3x^2 + 3xy}{-x + 2y + x^2} \Rightarrow y = \frac{3x-2}{3x-1} \cdot x \quad \text{(numerical result)}.$$

Then, $t(x)$ is given by

$$t(x) = \int \frac{dx}{-x + 2g(x) + x^2} \quad \text{where} \quad g(x) := \frac{3x-2}{3x-1} - x.$$

We may compute the $M_\alpha(0)$ numerically with the result.

Conclusion

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Melnikov integral result: 0.1284727278352124
Estimated error: 1.0808023516821339e-08
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Hence, we can see that integral is not equal to 0. Equivalently, H.2 condition checked. Thus, by Homoclinic Bifurcation Theorem, we would have the following graph corresponding to figure 6.7:

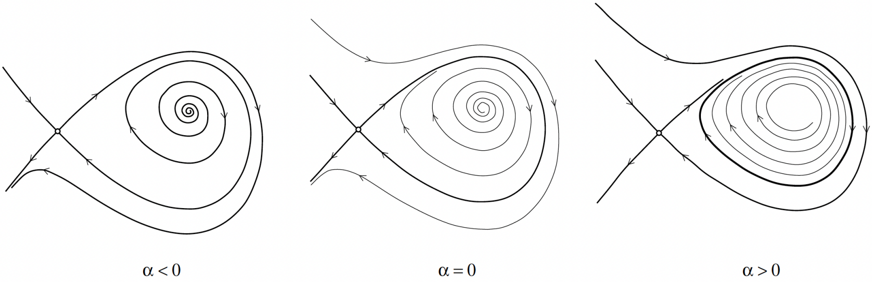

FIGURE 6.19. Homoclinic bifurcation in (6.8): A stable limit cycle exists for small $\alpha > 0$.

Introduction to the Proof of Homoclinic Bifurcation Theorem

The intuition of the proof is to introduce two local cross section near the saddle equilibrium (Σ, Π), that is transversal to the local stable and unstable manifold.

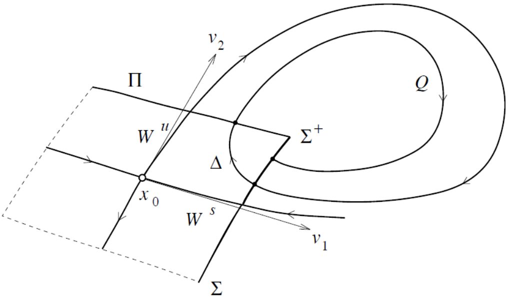

FIGURE 6.11. Poincaré map for homoclinic bifurcation on the plane.

Then construct two mapping $\Delta : \Sigma^+ \to \Pi$ and $Q : \pi \to \Sigma$.

Introduction to the Proof of Homoclinic Bifurcation Theorem

We can derive that with the transformed dynamic $(\dot{\xi}, \dot{\eta}) = (\lambda_1 \xi, \lambda_2 \eta)$ We have

$$\Delta : \xi = \eta^{\frac{-\lambda_1}{\lambda_2}} \qquad , Q : \eta = \beta + a\xi + o(\xi^2)$$

Then we can define a poincare map P on a half-section Σ^+ which is

$$P = Q \circ \Delta, \qquad P : \Sigma^+ \to \Sigma$$

And it is clear that the fix point of P is corresponding to a trajectory of limit cycle. and we can analyse the existence and uniqueness of the fix point of P easily with the derived equation for P easily around neighborhood of small $|\beta|$

Step 1: Introduction of eigenbasis coordinates

Without loss of generality, we consider the system in eigenbasis coordinates where we move the saddle equilibrium to the origin and write $f(x) = Ax + g(x)$ as usual, where g is smooth and $O(\|x^2\|)$.

Let the EVD of A be $A = PJP^{-1}$ Thus by transforming $y = P^{-1}x$, we get:

$$\dot{y} = Jy + P^{-1}g(Py)$$

Step 1: Introduction of eigenbasis coordinates

Thus, We can express the original system as:

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + g_1(x_1, x_2), \\ \dot{x}_2 = \lambda_2 x_2 + g_2(x_1, x_2), \end{cases}$$

where g_1, g_2 are smooth functions. where, $x = (x_1, x_2)^T$ and g1,g2 are $O(\|x^2\|)$. Note that this transformation is smooth and invertible.

Step 2: Local linearization of the invariant manifolds

According to the Local Stable Manifold Theorem, the stable and unstable manifolds W^s and W^u can be represented as:

$$W^s : x_2 = S(x_1), \quad S(0) = S'(0) = 0,$$
$$W^u : x_1 = U(x_2), \quad U(0) = U'(0) = 0,$$

where S and U are smooth functions.

Step 2: Local linearization of the invariant manifolds

We can then introduce new variables $y = (y_1, y_2)^T$ in the neighbor of the saddle.

$$\begin{cases} y_1 = x_1 - U(x_2), \\ y_2 = x_2 - S(x_1) \end{cases}$$

This change of coordinate is smooth and invertible in the neighborhood of the saddle since U and S are smooth and invertible locally by the local stable manifold theorem.

Additionally we can assume this neighborhood contains a unit square $\Omega = \{y : -1 < y_1, y_2 < 1\}$ by adding an additional linear transform scaller to the eigenbasis system in step 1 such as $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

Step 2: Local linearization of the invariant manifolds

The system found in part 1 can be written as below,by substituting the coordinate change.

$$\begin{cases} \dot{y}_1 = \lambda_1 y_1 + y_1 h_1(y_1, y_2), \\ \dot{y}_2 = \lambda_2 y_2 + y_2 h_2(y_1, y_2), \end{cases}$$

And that $h_1, h_2 = O(\|y\|)$, we can always factor out a y_1 and y_2 in the derivative because $y_1, y_2 = 0$ are on the local unstable/stable manifold.

Step 2: Local linearization of the invariant manifolds

Note that this system is now a nonlinear smooth system still with a saddle at the origin but its invariant manifold now is exactly the coordinate axis within the unit square Ω.

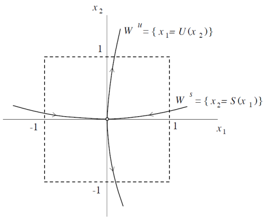

FIGURE 6.12. Local stable and unstable manifolds in x-coordinates.

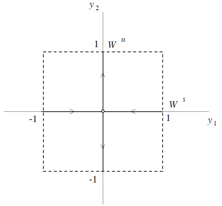

FIGURE 6.13. Locally linearized stable and unstable manifolds in y-coordinates.

And this coordinate from x to y basically pushed the local manifolds to the coordinate axis within the unit square Ω where the local manifold are smooth and invertible.