Fourier Analysis

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January 2024

Contents

General summary:
This course will mostly discuss the materials from *Fourier analysis and its application* by Folland, ranging from Chapter 2 (Fourier series) to Chapter 9 (Generalized Function), with special topics in signal processing, frquency analysis, and wavelets.

1 Classnotes: 1/4/2024

1.1 Motivations:

1. **Vibrating string (Bernulli)**
2. $u_{tt} - u_{xx} = 0$

Qes 1: What do specific solution should look like for vibrating string?

Ans 1:

1. "simple harmonic motion" ($L = \pi$)
$$u(t, x) = Asin(nx)cos(nt - \alpha)$$
2. "multiple harmonic motion"
$$u(t, x) = \sum_{n=1}^{N} A_n sin(nx)cos(nt - \alpha_n)$$

Qes 2: If given the solution format above, initial condition should be like?
$$u|_{t=0} = f \ , \ u_t|_{t=0} = g.$$

Ans 2:

For Mult-Motion, it is okey to expressed f and g as finite sums:
$$\sum_{n=1}^{N} a_n sin(nx)$$

Here is an possible example:
$$f : x \to sin^3(x) \Rightarrow sin^3(x) = \frac{1}{4}[3sin(x) - sin(3x)]$$

But sometimes it is not:

Prop 1.1.1. Suppose $(a_k)_{k\geq 1}$ is such that

$$\sum_{n=1}^{\infty} a_n sin(nx)$$

converges on $[0, \pi]$, with

$$\sum_{n=1}^{\infty} a_n sin(nx) = sin^2(x).$$

Then a_k is non-zero for inf. many values of k.

Pf: we can set:

$$sin^2(x) = \sum_{n=1}^{N} a_n sin(nx)$$

Then for fixed N, we know that

$$\int_0^{\pi} sin(mx)sin(nx) = 0 \text{ for m} \neq \text{n}$$

Thus for $k > N$, we have

$$\int_0^{\pi} sin^2(x)sin(kx) = \sum_{n=1}^{N} a_n \int_0^{\pi} sin(nx)sin(kx) = 0$$

But this contridicted that

$$\int_0^{\pi} sin^2(x)sin(kx) = \frac{2(cos(k\pi) - 1)}{k(k^2 - 4)}$$

non-zero whenever k is odd. $\square$

Similarly for heat equation, we have the same format of solutions:

1. "simple solution":

$$u(x,t) = (A_n cos(nx) * B_n sin(nx))e^{n^2 t}$$

2. "general solution":

$$u(x,t) = \sum_{n=1}^{N} (A_n cos(nx) * B_n sin(nx))e^{n^2 t}$$

Which corresponding to an inital temperature possible for

$$u(0, x) = f(x)$$

if and only if

$$f(x) = \sum_{n=1}^{N} (A_n cos(nx) + B_n sin(nx))$$

Our question then evolves as: when can we represent a periodic function (w/period 2π) as a series of the form:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (A_n cos(nx) + B_n sin(nx)) \tag{F}$$

From Fourier idea: extend this pattern to infinite series!

Note:

1. Recall, $cos(nx) = \frac{e^{inx}+e^{-inx}}{2}$, $sin(nx) = \frac{e^{inx}-e^{-inx}}{2i}$, and thus (F) can be written as

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{F'}$$

with $c_0 = \frac{1}{2}a_0$, $c_n = \frac{1}{2}(a_n - ib_n)$, and $c_{-n} = \frac{1}{2}(a_n - ib_n)$

2. IF a periodic function f has an representation as (F'), then it is easy to compute (c_n). Indeed the general formula for 3 Fourier coefficient can be computed explicitly by fixing $k \in \mathbf{Z}$ and taking integral over $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} f(x)e^{ikx}dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{inx} * e^{ikx}dx$$

If $n \neq k$, then we have RHS is zero; Otherwise, for $n = k$, RHS gives us the formula for c_n:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{inx}dx$$

We therefore also get the formula for a_n, b_n:

$$a_0 = 2c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \tag{1}$$

$$a_n = c_n + c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)[e^{inx} + e^{-inx}]dx \tag{2}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx \tag{3}$$

$$b_n = c_n + c_{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx \tag{4}$$

1.2 Course contend

Def 1.2.1. Suppose f is periodic w/period 2π, and integrable over $[-\pi, \pi]$. The numbers $(c_n), (a_n), (b_n)$ are called **Fourier coefficient** with the formula:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{5}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx \tag{6}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx \tag{7}$$

And the **Fourier series** of f are

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(A_n cos(nx) + B_n sin(nx)) \text{ or } \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Notation 1.2.2. One can certainly integrate over $[0, 2\pi]$ and get the same result, since

$$\alpha \to \int_{\alpha}^{\alpha+P} F dx$$

is independent from α given the period is P. Indeed we denoted this relations as

$$\Pi : R/2\pi\mathbf{Z} \iff [0, 2\pi]$$

Notes: Some property of f can give us some easy way to calculate its Fourier coefficient,

1. if f is odd, then

$$a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx \tag{8}$$

$$b_n = 0 \tag{9}$$

2. if f is even, then

$$b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx \tag{10}$$

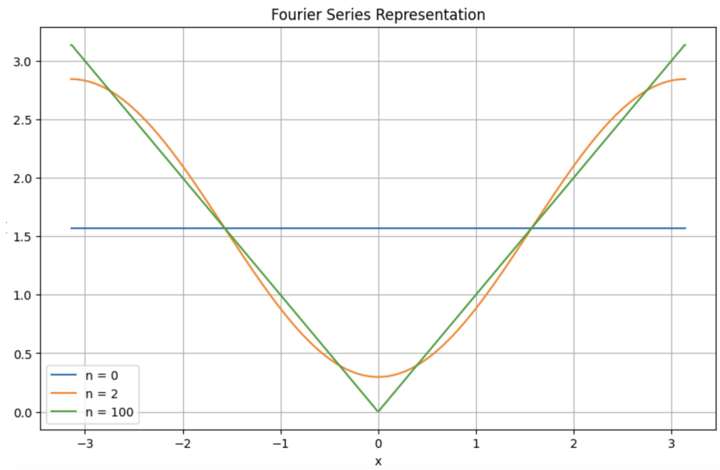
$$a_n = 0 \tag{11}$$

3. $c_0 = \frac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$ represent the converge value of f over any interval of length 2π.

Example 1.2.3.

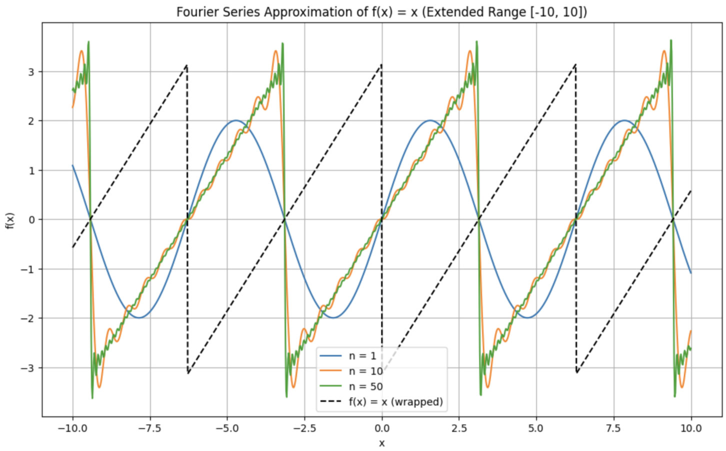
1. $f : x \to |x|$ in $[-\pi, \pi]$ has the Fourier series (even function):

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\text{n odd}} \frac{cos(nx)}{n^2}$$



2. $f : x \to x$ in $[-\pi, \pi)$ has the Fourier series (odd function):

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}sin(nx)}{n}$$



and more importantly, the violation of piecewise continous is crucial in this case.

Proposition 1.2.4. **(Bessel's ineq)** If f is 2π periodic and integrable with c_n F-coefficient, then $\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$.

Corollary 1.2.5. Let f is 2π periodic and integrable with c_n F-coefficient, and $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$, then $c_n \to 0$ as $n \to \infty$

<u>Notes:</u>

1. Bessel's ineq $\Rightarrow (a_n), (b_n)$

$$c_n = \frac{a_n + b_n}{2} \tag{12}$$

$$c_{-n} = \frac{a_n - b_n}{2} \tag{13}$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{4}(a_0)^2 + \sum_{n=1}^{\infty} \frac{(a_n)^2 + (b_n)^2}{2} \leq \int_{-\pi}^{\pi} |f(x)|^2 dx \tag{14}$$

2. The corollary still hold for $a_n, b_n \to 0$ as $n \to \infty$ if $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$.

3. Stronger corollary only requires $f \in L^1$ by **Riemann-Lebesgue Lemma!** (for $f \in L^1$)

Proof to Riemann-Lebesgue Lemma:

- Firstly, if $f(x) = \mathbb{1}_{[a,b)}(x)$ for $-\pi \leq a < b \leq \pi$, then we have

$$c_n = \int_{-\pi}^{pi} f(x)e^{-inx}dx = \frac{e^{-ibx} - e^{-iax}}{-in} \xrightarrow{n\to\infty} 0$$

By linearility, this obs holds for simple functions.

- since $f \in L^1$, by approximation theorem, lebesgue integrable function f on R, there exists g, continuous simple function with compact support, such that $\|f - g\|_1 < \epsilon$

- Now, consider

$$c_n = \int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{15}$$

$$= \int_{-\pi}^{\pi} [f(x) - g(x)]e^{-inx}dx + \int_{-\pi}^{\pi} g(x)e^{-inx}dx \tag{16}$$

By theorem, the first term on RHS is bounded by ϵ and the second term converges to 0 as $n \to \infty$. Therefore,

$$\limsup_{n\to\infty} |\int_{-\pi}^{\pi} f(x)e^{-inx}dx| < \epsilon$$

Since ϵ arbitary, we finish the proof $\square$

Def 1.2.6. $a, b \in R$, $a < b$, $f : [a, b] \to R$ is **piecewise continuous** if f is continuous on $[a, b]$ except for finite many discontinuity. Moreover, at each discontinuity, LH-limit and RH-limit must exist.

$$f(x; -) = \lim_{y\to x_i^-} f(y) \tag{17}$$

$$f(x; +) = \lim_{y\to x_i^+} f(y) \tag{18}$$

In such a case, we let **PC(a,b)** denoted as the class of all such functions.

2 Classnotes: 1/9/2024

2.1 Piecewise smooth and piecewise continuous

Def 2.1. $f : [a, b] \to \mathbf{F}$ is piecewise smooth if f and f' are piecewise cont. on $[a, b]$, and we let $PS[a, b]$ denoted as all such functions.

Note: More generally, $f \in PS[a, b] \iff f \in PC(a, b)$ and f is differentiable on (a, b) except for at most finite many points $x_1, ...x_n$ with $f'(x_i^-)$ and $f'(x_i^+)$ exsits for all i, along with the existence of the limit of $f'(b^-)$ and $f'(a^+)$

Def 2.2. $f : R \to R$ is pw-cont.if $f \in PC(a,b)$ for all $a, b \in R$ w/ $a < b$ ($f \in PC(R)$); Similarly, $f : R \to R$ is ps-cont.if $f \in PS(a,b)$ for all $a, b \in R$ w/ $a < b$ ($f \in PS(R)$);

2.2 Partial sum to approximate f

Given that

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

we will work with the partial sum $S_N^f(x)$ for $N \geq 1$:

$$S_N^f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N} a_n cos(nx) + b_n sin(nx) = \sum_{|n|\leq N} c_n e^{inx}$$

Our goal is try to show $S_N^f(x) \to f$ as $N \to \infty$!!!

Let's first look at the case of piecewise converges for $f \in PS(R - \bigcap C(R))$(In what sense is important!) ?: That maybe an typo from the lecture notes I guess

Def 2.3. (Dirichlet Kernel):

$$\begin{aligned} S_N^f(x) = \sum_{|n|\leq N} c_n e^{inx} &= \sum_{|n|\leq N} [\frac{1}{2\pi}\int_{\pi} f(\psi)e^{-in\psi}d\psi]e^{inx} \text{ (Formula of } c_n) && (19)\\ &= \frac{1}{2\pi}\sum_{|n|\leq N}\int_{\pi} f(\psi)e^{in(x-\psi)}d\psi && (20)\\ &= \frac{1}{2\pi}\sum_{|n|\leq N}\int_{\pi} f(\psi)e^{in(\psi-x)}d\psi \text{ (by changing } n \to -n) && (21)\\ &= \frac{1}{2\pi}\sum_{|n|\leq N}\int_{\pi} f(x+\psi)e^{in\psi}d\psi \text{ (by change of variable } u = x+\psi) && (22)\\ &= \int_{\pi} f(x+\psi)(\frac{1}{2\pi}\sum_{|n|\leq N} e^{in\psi})d\psi && (23) \end{aligned}$$

Where $\frac{1}{2\pi}\sum_{|n|\leq N} e^{in\psi} = D_N(\psi)$ is defined as Nth Dirichlet Kernel.

Note:

1. $D_N(\psi) = \frac{1}{2\pi}\frac{sin(N+\frac{1}{2})\psi}{sin(\frac{1}{2}\psi)}$ (useful formula to calculate)

$$D_N(\psi) = \frac{1}{2\pi}\sum_{|n|\leq N} e^{in\psi} = \frac{1}{2\pi}[\sum_{n=0}^{N} e^{in\psi} + \sum_{n=-N}^{-1} e^{in\psi}] \tag{24}$$

$$\sum_{n=0}^{N} e^{in\psi} + \sum_{n=-N}^{-1} e^{in\psi} = \sum_{n=0}^{N}\omega^n + \sum_{n=-N}^{-1}\omega^n \text{ (letting } \omega^n = e^{in\psi}) \tag{25}$$

$$= \frac{1-\omega^{N+1}}{1-\omega} + \frac{\omega^{-N}-1}{1-\omega} \tag{26}$$

$$= \frac{\omega^{-N}-\omega^{N+1}}{1-\omega} \tag{27}$$

$$= \frac{\omega^{-N-\frac{1}{2}}-\omega^{N+\frac{1}{2}}}{\omega^{-\frac{1}{2}}-\omega^{\frac{1}{2}}} \tag{28}$$

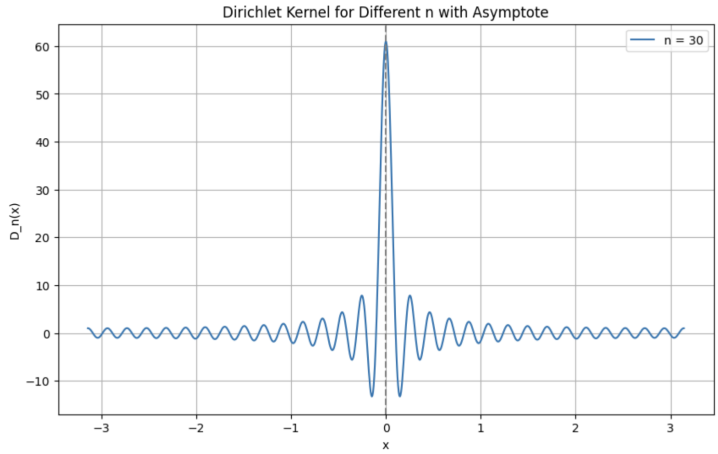
$$= \frac{sin((\frac{1}{2}+N)\psi)}{sin(\frac{1}{2}\psi)} \tag{29}$$

2. The geometric series has some conclusions need to remember:

$$\sum_{k=m}^{n} ar^k = \begin{cases} a(m-n+1) & \text{if } r = 1 \\ \frac{r^m - r^{n+1}}{1-r} & \text{if } r \neq 1 \end{cases}$$

$$\sum_{k=m}^{n} ar^{k-1} = \begin{cases} a(m-n+1) & \text{if } r = 1 \\ \frac{r^{m-1} - r^{n}}{1-r} & \text{if } r \neq 1 \end{cases}$$

3. One can plot the graph for D_n:



The interpretion is that: from the definition of D_n, the sharp spike of D_n at $\psi = 0$ picks out the value of $f(x)$, while the oscillations leads to carry out the rest of the terms in the integral. (eliminite the error)

4. For any $N \geq 1$,

$$\int_0^{\pi} D_N(\psi)d\psi = \int_{-\pi}^{0} D_N(\psi)d\psi = \frac{1}{2}$$

Recalling that $cos(x) = \frac{e^{ix}+e^{-ix}}{2}$, we have

$$D_N(\psi) = \frac{1}{2\pi}\sum_{|n|\leq N} e^{in\psi} = \frac{1}{2\pi}[1 + \sum_{n\leq N} 2cos(n\psi)]$$

which leads to

$$\int_0^{\pi} D_N(\psi)d\psi = \frac{1}{2\pi}[\psi + \sum_{n\leq N}\frac{2sin(n\psi)}{n}]_{\psi=0}^{\psi=\pi} = \frac{1}{2}$$

and $\int_{-\pi}^{0} D_N(\psi)d\psi = \frac{1}{2}$ the same.

Them 2.4. Let $f \in PS(R)$ and $2\pi-$periodic be given, let S_N^f be defined as above. Then for every $x \in R$, we have

$$\lim_{N\to\infty} S_N^f(x) = \frac{1}{2}[f(x^-) + f(x^+)]$$

In particular, $S_N^f \to f$ for f cont. (stronger condition)

Proof($*$): Clearly.

$$\frac{1}{2}[f(x^-) + f(x^+)] = f(x^-)\int_{-\pi}^{0} D_N(\psi)d\psi + f(x^+)\int_0^{\pi} D_N(\psi)d\psi$$

Meanwhile,

$$S_N^f(x) = \int_{-\pi}^{\pi} f(x+\psi) D_N(\psi) d\psi = \int_{-\pi}^{0} f(x+\psi) D_N(\psi) d\psi + \int_{0}^{\pi} f(x+\psi) D_N(\psi) d\psi$$

Thus

$$\begin{aligned} S_N^f(x) - \frac{1}{2}[f(x^-) + f(x^+)] &= \int_{-\pi}^{0} [f(x+\psi) - f(x^-)] D_N(\psi) d\psi && (30) \\ &+ \int_{0}^{\pi} [f(x+\psi) - f(x^+)] D_N(\psi) d\psi && (31) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\psi)[e^{i(N+1)\psi} - e^{-iN\psi}] \text{ given by (27)} && (32) \end{aligned}$$

where $g(\psi)$ is defined as:

$$g(\psi) = \begin{cases} \frac{f(x+\psi) - f(x^-)}{e^{i\psi} - 1} & \text{if } \psi \in (-\pi, 0) \\ \frac{f(x+\psi) - f(x^+)}{e^{i\psi} - 1} & \text{if } \psi \in (0, \pi) \end{cases}$$

Note that g inherits the regularity property of f on $[-\pi, \pi]$ except possibly at $\psi = 0$ where $e^{i\psi} - 1 = 0$. By L'Hospital rule,

$$\lim_{\psi \to 0^+} g(\psi) = \lim_{\psi \to 0^+} \frac{f'(x+\psi)}{ie^{i\psi}} = \frac{f'(x+)}{i} \in \mathbf{C}$$

and similar for $g(o^-)$. Thus $g \in PC(-\pi, \pi) \Rightarrow \|g\|_{L^2} < \infty$, and we have

$$c_n(g) = \frac{1}{2\pi} \int_{\pi} g(\psi) e^{-in\psi} d\psi$$

Fourier coefficient of g converges to 0 as $n \to \pm\infty$. To conclude, (32) is exactly $c_{N+1} - c_{-N} \to 0$ as $N \to \infty$. □

Example 2.5.

1. $f : x \to |x|$ w/ $f \in PS(R) \cap C(R) \Rightarrow S_N^f(x) \to f(x) \forall x$ and thus

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \geq 1} \frac{cos((2n-1)x)}{(2n-1)^2}$$

Taking x = 0, we have something interesting:

$$\frac{\pi}{2} = \frac{4}{\pi} \sum_{n \geq 1} (2n-1)^2 \Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + ...(= 1 + \frac{1}{3^2} + \frac{1}{5^2} + ...)$$

2. $g : x \to x$ on $(\pi, \pi]$, $g \in PS(R)$, g is cont. except at $x = k\pi$ for k odd. $(\frac{1}{2}[g(k\pi^-) + g(k\pi^+)] = 0)$

$$\hookrightarrow \begin{cases} S_N^g(x) & \to g(x) \text{ when } x \neq k\pi \text{ for } \forall k \text{ odd} \\ S_N^g(k\pi) & \to 0 \end{cases}$$

In particular,

$$2\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} sin(nx) = x \text{ for } x \in (\pi, \pi)$$

Looking at $x = \frac{\pi}{2}$,

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - ...$$

3. (Exercise): using F series for $f : x \to x^2$, $x \in (-\pi, \pi)$ to show that

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(Hint: look at $x = \pi$)

Remark:

1. If we always use the convention that $f \in PS(R)$ is redefined at the discontinuities by $f(x) = \frac{1}{2}[f(x^-) + f(x^+)]$, then theorem 2.4. implies that $f \in PS(R)$, 2π periodic $\Rightarrow S_N^f \to f$ pointwise everywhere.

2. In my view, maybe we can say that $S_N^f \to f$ a.e(or in measure ?), since those discontinuities have $\lambda(x_i) = 0$
(Update: There is the Kolmogorov example of an L^1 function with a.e. divergent Fourier series, which implies that $f \in L^1 \not\Rightarrow S_N^f \to f$ in measure.)

Cor 2.6: f,g 2π-periodic and $f, g \in PS(R)$ as redefined above. If f, g has the same F-coefficient then f=g.

Proof: Since $S_N^f(x) = S_M^g(x)$ and $f, g \in PS(R)$, we have that $S_N^f \to f$ and $S_M^g \to g$ pointwisely, thus f=g.

3 Classnotes: 1/12/2024

3.1 F-series: derivatives, integral, and unif-c.v

RMK: In this section, we focus on the case of f 2π-periodic and $f \in PS(R) \cap C(R)$. In here, Fundamental Theorem of Calculus remains valid:

$$f(b) - f(a) = \int_a^b f'(x)dx.$$

If $f \in C^1([a,b])$ and $f \in C^1([b,c])$, then

$$\int_a^c f'(x)dx = \int_a^b f'(x)dx + \int_b^c f'(x)dx \tag{33}$$

$$= f(b) - f(a) + f(c) - f(b) = f(c) - f(a). \tag{34}$$

Prop 1.1. f is 2π-periodic and $f \in PS(R) \cap C(R)$. If $(a_n), (b_n), (c_n)$ are F-coefficients of f and $(a'_n), (b'_n), (c'_n)$ are the F-coefficient of f', then

$$a'_n = nb_n; b'_n = -na_n; c'_n = inc_n.$$

Pf (idea): we only need to consider c_n:

$$c'_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx \tag{35}$$

$$= \frac{1}{2\pi}[f(x)e^{-inx}|_{-\pi}^{\pi} + in\int_{-\pi}^{\pi} f(x)e^{-inx}dx] \tag{36}$$

$$= \frac{1}{2\pi}[0 + inc_n] \text{ (f is periodic and } c_n \text{ definition)} \tag{37}$$

$$= inc_n \tag{38}$$

Moreover b_n and a_n can be proved by exactly same procudure $\square$

Cor 1.2. f is 2π-periodic and $f \in PS(R) \cap C(R)$. Suppose $f' \in PS(R)$, If $(a'_n), (b'_n), (c'_n)$ are F-coefficients of f', then for all $x \in R$ where f' exists we have

$$f'(x) = \sum_{|n| \in \mathbf{Z}} inc_n e^{inx} = \sum_{n=1}^{\infty} nb_n cos(nx) - na_n sin(nx)$$

The disappearance of a_0 is due to $a_0 = nb_n = 0$. Moreover, for fits for the same jump of discontinuities $\frac{1}{2}[f'(x^+) + f'(x^-)]$

Ex. $f(x) = 1$ periodic, but $\int f dx = x + c$ is not for any c!
However, if we integrate a F-series term by term, we get

$$\sum_{n\in Z} c_n e^{inx} \to \int c_0 dx + \sum_{n\in Z-\{0\}} \frac{c_n}{in} e^{inx} + c$$

which is periodic if $c_0 = 0$.

Prop 1.3. Suppose $f \in PS(R)$, If $(a_n), (b_n), (c_n)$ are F-coefficients of F, set $F(x) = \int_0^x f(\psi)d\psi$ and set

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)dx.$$

If $c_0 = \frac{1}{2}a_0 = 0$, then for $\forall x \in R$ we have,

$$F(x) = C_0 + \sum_{n\neq 0} \frac{c_n}{in} e^{inx} = C_0 + \sum_{n\geq 1} \frac{a_n}{n} sin(nx) - \frac{b_n}{n} cos(nx)$$

If $c_0 \neq 0$, this is simply $F(x) - c_0 x$

<u>RMK:</u> The RHS of the equation is only about the integral over F-series, and this does not assure that F-series for f converges! Moreover, F is integral at a $PC(R)f_n$ and so belongs to $PS(R \cap C(R)$

Proof is nothing tricky, show that F is 2π-periodic and use the theorem to show that it is indeed converges to F-series, and using the same anology as differentiation, we have our desired results:

$$c_0 = 0 \Rightarrow F(x+2\pi) - F(x) = \int_0^{x+2\pi} f(\psi) - \int_0^x f(\psi) = \int_{-\pi}^{\pi} f(\psi) = 2\pi c_0 = 0.$$

3.2 C.V-unif and C.V absolutly

<u>Recall:</u> we say that a convergent series $\sum_{n\geq 1} g_n(x)$ which converges to g(x), <u>converges absolutly</u> if

$$\sum_{n\geq 1} |g_n(x)|$$

converges. And we say that $\sum_{n\geq 1} g_n(x)$ <u>converges uniformly</u> to g on S if

$$\sup_{x\in S} |g(x) - \sum_{n=1}^{N} g_n(x)| \to 0 \text{ as } N \to \infty$$

Prop 2.1.(Weierstrass M-test): If $(M_n) \subset R_+$ is s.t $|g_n(x)| \leq M_n$ for all $x \in X$ and $\sum_{n=1}^{\infty} M_n < \infty$, then

$$x \to \sum_{n=1}^{\infty} g_n(x) \text{ is absolutly converges and uniformly converges on S}$$

<u>RMK:</u> Applying this to F-series, we write

$$|a_n cos(nx)| \leq |a_n|; |b_n sin(nx)| \leq |b_n|; |c_n e^{inx}| = |c_n|$$

Thus the W.M-test applies when:

1. $\sum_{n=1}^{\infty} |a_n| < \infty$ and $\sum_{n=1}^{\infty} |b_n| < \infty$ or
2. $\sum_{n \in Z} |c_n| < \infty$

(Since $|c_{\pm n}| \leq |a_n| + |b_n|$; $|a_n| \leq |c_n| + |c_{-n}|$; $|b_n| \leq |c_n| + |c_{-n}|$; and thus 1,2 are equivalent)

Theorem 2.3. f is 2π-periodic and $f \in PS(R) \cap C(R)$. Then F series of f converges to f absolutely and uniformly on R.

Proof: it is suffices to show that $\sum_{n \in Z} |c_n| < \infty$ by W.M-test. Let (c'_n) be the F-coefficient of f' so that $c_n = \frac{c'_n}{in}$ by for $n \neq 0$, we then have that

$$\sum_{n \in Z} |c_n| = |c_0| + \sum_{n \in Z - \{0\}} |\frac{c'_n}{in}| \leq |c_0| + (\sum_{n \in Z - \{0\}} \frac{1}{n^2})^{\frac{1}{2}} (\sum_{n \in Z - \{0\}} |c'_n|^2)^{\frac{1}{2}}$$

By Cauchy-schwarz inequality. Moreover,

$$\leq |c_0| + (\sum_{n \in Z - \{0\}} \frac{1}{n^2})^{\frac{1}{2}} (\frac{1}{2\pi} \int_{\pi} |f'(x)|^2)^{\frac{1}{2}}$$

by Bessel's inequality. Clearly, $(\sum_{n \in Z - \{0\}} \frac{1}{n^2})^{\frac{1}{2}} = (2 \sum_{n \geq 1} \frac{1}{n^2})^{\frac{1}{2}} = (\frac{\pi^2}{3})^{\frac{1}{2}}$; $f \in PS(R) \cap C(R) \Rightarrow (\frac{1}{2\pi} \int_{\pi} |f'(x)|^2)^{\frac{1}{2}} < \infty$; so do as $|c_0|$. Then we are done!

<u>RMK:</u> There is a close connection between the smoothness of f and the rate of convergence of its F-series, that is sth like

$$f \to f' \to f'' \to ... \tag{39}$$

$$c_n \to inc_n \to -n^2 c_n \to ... \tag{40}$$

By considering the Ratio test(Radius of convergence), in order to make f twice(or even more smooth) differentiable, we need c_n must converges "pretty fast", for example, faster that n^2.

Theorem 2.4. Suppose f is 2π-periodic.

1. if $f \in C^{k-1}$ and $f^{k-1} \in PS(R)$ for some $k \geq 1$, then

$$\sum_{n\geq 0} n^{2k}|a_n|^2 < \infty; \sum_{n\geq 1} n^{2k}|b_n|^2 < \infty; \sum_{n\in Z} |n|^{2k}|c_n|^2 < \infty$$

   and thus $n^k a_n; n^k|b_n|; |n|^k|c_n| \to 0$ as $n \to \infty$

2. if c_n is s.t $|c_n| \leq C|n|^{-(k+a)}$ for $n \neq 0$ and some $C > 0, a > 1$ independent of n, then $f : x \mapsto \sum c_n e^{inx} \in C^k(\mathbb{R})$

Proof:

1. The first prove it relative easy, we can see that $(c^{(k)_m})$ of $f^{(k)}$ are given by $c_n^k = (in)c^k$. The desired result follow from Bessel's ineq. applied to $f^{(k)}$.(convergence as followed from Cor.1.2.5 pp.6)

2. Suppose that c_n is defined as above, and then for $j \leq k$ we have

$$\sum_{n\neq 0} |n^j c_n| \leq C \sum_{n\neq 0} |n|^{j-k-a} \tag{41}$$

$$\leq 2C \sum_{n\geq 1} |n|^{-a} < \infty \text{ (since } a > 1) \tag{42}$$

   and by W.M-test, by setting $M_n = |n^j c_n|$

$$|(in)^j c_n e^{inx}| \leq |n^j c_n|; \sum |n^j c_n| < \infty$$

   and $\sum_{n=-\infty}^{\infty} (in)^j c_n e^{inx}$ is unif-C.V and therefore is continuous.
   By setting $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, we have that $f \in C(K)$.

<u>RMK:</u>

1. The above (2) implies that:
   f has cont. derivatives of all orders ($f \in C^\infty$) $\iff$ (c_n) decays w.r.t n faster than any power of n

2. all RMKs above can be extend to F.series on general intervals, some useful notes:

- $f(-\pi,\pi] \to \mathbb{R}$ can be extend periodically to $f_{\text{ext}} : \mathbb{R} \to \mathbb{R}$
- $f : [0,\pi] \to \mathbb{R}$
  two natural ways of extending f to a $2\pi-$periodic f_n :
  (a) $f_{\text{even}} : [-\pi,\pi] \to \mathbb{R}$ with
  $$f_{\text{even}}(-x) = f(x) \text{ for } x \in [0,\pi]$$
  (b) $f_{\text{odd}} : [-\pi,\pi] \to \mathbb{R}$ with
  $$f_{\text{odd}}(-x) = -f(x) \text{ for } x \in [0,\pi]$$
  (c) F series for $\begin{cases} f_{\text{even}} & \text{has only cos terms} \\ f_{\text{odd}} & \text{has only sin terms} \end{cases}$

Def 2.5. Given $f : [0,\pi] \to \mathbb{R}$, $a_n = \frac{2}{\pi}\int_0^\pi f(x)cos(nx)dx, b_n = \frac{2}{\pi}\int_0^\pi f(x)sin(nx)dx$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(nx) \text{ is the } \underline{\text{Fourier cosine series of f;}} \tag{43}$$

$$\sum_{n=1}^{\infty} b_n sin(nx) \text{ is the } \underline{\text{Fourier sine series of f}} \tag{44}$$

Theorem 2.6. $f \in PS(0,\pi)$, then the F.cosine series and F.sine series converges to $\frac{1}{2}(f(x^+) + f(x^-))$ at every $x \in (0,\pi)$ (and thus they converges to f(x) at every $x \in (0,\pi)$ where f is cts.) Moreover, the F.cosine series converges to $f(o^+)$ at 0 and to $f(\pi^-)$ at π; the F.sine series converges to $f(o)$ at 0 and to $f(\pi)$ at π.

RMK(cont.):

1. $$f : [0,l] \to \mathbb{R} \Rightarrow g : [0,\pi] \to \mathbb{R} \tag{45}$$
   $$\text{determined by } g(x) = f(\frac{lx}{\pi}) \tag{46}$$
   see pp.46 47 of Folland.

2. (pp.60 at Folland):
   $f : R \to R$ $2\pi-$periodic. If f has a discty. at some pt $x_0 \in R$, then F.series can not converges uniformly on any closed and bdd interval containing x_0.(unif.C.V of cont.funs is cont.)

3.3 Preview to the following

1. Fourier series as a transform we have

$$f : R \to C(2\pi\text{-periodic}) \Rightarrow \hat{f} : \mathbb{Z} \to C \text{ w/ } \hat{f}(n) = c_n$$

2. What about more general f (beyond the $PS(R)$ class?)

   - $f : [-\pi, \pi] \Rightarrow \mathbb{R}$ of bdd variation

     $$(\iff f = f_1 - f_2, \text{ see e.g. Wheedem-Zygmund})$$

     $\Rightarrow$ point wise convergence.

   - $\exists f \in C(R)$ periodic s.t F.series diverges at some pts(e.g 1877 du Beis-Raymend, 1966 Kahane-Katzrelson)
   - $f \in L^2 \Rightarrow$ F.series of f converges to f a.e (actually it is true for $f \in Lp, 1 < p < \infty$)
   - Resolution: look at other types of convergence, e.g. in L^p norm!

     $$\lim_{N\to\infty} \int_a^b |\sum_{n=1}^{N} f_n(x) - f(x)|^p dx \to 0 (\|S_N^f - f\|_{L^p} \to 0, N \to \infty)$$

3. Important definitions:

   (a) $a \in C^k$ w/ $a : \{a_1, ..., a_k\}$

   $$< a, b >= \sum a_i \overline{b_i} = \sum \sum a_{ij} \overline{b_{ij}} \; ; \; \|a\| = (\sum |a_i|^2)^{\frac{1}{2}} = \sum |a_{ij}|^2)^{\frac{1}{2}}$$

   Now, consider the vector space $PC(a, b)$ and define

   $$< f, g >= \int_a^b f(x)\overline{g(x)}dx \; , \; \|f\| = (\int_a^b |f(x)|^2 dx)^{\frac{1}{2}}$$

   (b) RMK that Cauchy-Schwartz ineq. $< f, g > \leq \|f\|\|g\|$; the triangular ineq. $\|f + g\| \leq \|f\| + \|g\|$; and the pythagorean theorem $\| \sum_{i=1}^n f_i\|^2 = \sum_{i=1}^n \|f_i\|^2$ when $< f_i, f_j >= 0$ for $i \neq j$ (orthogonal space)

   (c) To make $< f, g >$ an inner product, we require that $f \neq 0 \Rightarrow \|f\| = \sqrt{< f, f >} > 0$. But $\|f\| = (\int_a^b |f(x)|^2 dx)^{\frac{1}{2}}$ does not see the value of f at certain points. Two resolutions;

      i. use the convention that $f \in PC(a, b) \Rightarrow f(x) = \frac{1}{2(f(x^+) + f(x^-))}$
      ii. regard two functions f,g as equal if they agree except at finite many pts (equal a.e)

4 Classnotes: 1/16/2024

4.1 Course contend

Def 1.1. $f, g \in PC(a,b)$ orthogonal if $< f, g >= 0$; $\{f_i\}$ orthonormal if

$$< f_i, f_j >= \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if else} \end{cases}$$

Question: If $\{\phi_n\}$ is orthonormal set in $PC(a,b)$ can we write

$$f = \sum_n < f, \phi_n > \phi_n$$

Two things to notice:

- Does $\{\phi_n\}$ span the whole space?
- Does $\sum_n < f, \phi_n > \phi_n$ converges?

Example 1.2. Let ϕ_n be given by

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$$

in $PC(a,b)$, the consider

$$< \phi_n, \phi_m >= \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} e^{inx} * e^{-imx} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} e^{i(n-m)x} = \delta_{nm}$$

thus we have shown that $\{\phi_n\}$ is an orthonormal set. Letting c_n follow the usual definition from Fourier coefficient, we have

$$c_n = \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{\sqrt{2\pi}} < f, \phi_n >$$

Moreover, Fourier series w.r.t c_n turn into:

$$\sum_{n \in \mathbf{Z}} c_n e^{inx} = \sum_{n \in \mathbf{Z}} < f, \phi_n > \phi_n$$

However, our condition is slightly different from what $f = \sum c_n e^{inx}$, which required $PS(R)$, this lead to further discussion...

Def 1.3. $\{f_n\}_{n\geq 1} \subset PC(a,b)$ <u>converges to f in norm</u> if $\|f_n - f\| \to 0$

$$\int_a^b |f_n - f| dx \to 0$$

as $n \to \infty$.

RMK 1.4.

1. converges in norm $\neq$ ptwise cv.
   - $f_n(x) = \begin{cases} 1 & 0 \leq x \leq \frac{1}{n} \\ 0 & \text{elsewise} \end{cases}$
     $\Rightarrow \|f\|^2 = \int_0^{\frac{1}{n}} 1 dx = \frac{1}{n} \to 0$, converges in norm, however, $f_n(0) = 1$ for any n, we have that $f_n \not\to 0$ pts.
   - $g_n(x) = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & \text{elsewise} \end{cases}$
     we certainly have that $g_n \to 0$ pts on $[0,1]$, however,
     $$\|g_n - 0\|^2 = \int_0^{\frac{1}{n}} n^2 dx = n \to \infty$$
     we have that $g_n \not\to 0$ in norm!
2. Prop 1.5. $f_n \to f$ uniformly on $[a,b]$, then $f_n \to f$ in norm.
3. Proof: From unif.CV, we have that
   $$|f_n - f| \leq M_n; M_n \to 0$$
   for convergent sequence of non-negative $\{M_n\}$ for any x. Therefore,
   $$\|f_n - f\|^2 = \int_a^b |f_n - f| dx \leq \int_a^b M_n dx = M_n(b-a) \to 0$$
4. Norm and inner product agree with CV.in norm: for $f_n \to f$:
   $$\|f_n\| \to \|f\| \tag{47}$$
   $$< f_n, g > \to < f, g > \tag{48}$$
   $$< f, g_n > \to < f, g > \tag{49}$$

5. $PC(a,b)$ is incomplete! Consider that:

$$[0,1], f_n(x) = \begin{cases} 0 & x \leq \frac{1}{n} \\ \frac{1}{x^{\frac{1}{4}}} & \frac{1}{n} < x < 1 \end{cases}$$

for $m > n$,

$$\|f_n - f_m\| = \int_{\frac{1}{m}}^{\frac{1}{n}} x^{\frac{1}{4}} = 2(n^{-\frac{1}{2}} - n^{-\frac{1}{2}}) \to 0$$

as $n, m \to \infty$. Thus $\{f_n\}$ is Cauchy, but it converges to f:

$$f(x) = \frac{1}{x^{\frac{1}{4}}}$$

for $x \in (0,1]$ and $f(0) = 0$, since it turns into unbounded, we have that $f \notin PC(0,1)$.

Def 1.6. $L^2(a,b)$ = space of square-integrable functions on $[a,b]$,

$$L^2(a,b) = \{f : \int_a^b |f(x)|^2 dx < \infty\}$$

which is natrual def, since we know that

$$st \leq \frac{1}{2}(s^2 + t^2)$$

thus we can mimic this idea as

$$|f(x)\overline{g(x)}| \leq \frac{1}{2}(|f|^2 + |g|^2)$$

and for inner product,

$$< f, g >= \int_a^b f(x)\overline{g(x)}dx \leq \frac{1}{2}\int_a^b |f(x)|^2 dx + \frac{1}{2}\int_a^b |g(x)|^2 dx < \infty$$

for $f, g \in L^2(a,b)$. Therefore, we still hold the properties from norms and inner product (Well-defined). As before, we still want $\|f\| = 0 \Rightarrow f = 0$, but here we are dealing with measure, so the augment should turn into 0 a.e. ($f = 0$ except at some set of measure 0)

Prop 1.7. $a, b \in R$, $L^2 := L^2(a,b)$. then

1. L^2 is complete w.r.t converges in norm.

2. $\forall f \in L^2, \exists (f_n) \subset C([a,b])$ s.t $\|f_n - f\| \to 0$

Moreover, the sequence of (f_n) in (2) can be chosen to consist of function in C^∞, that is infinitely continuously differentiable w/ period are $(b-a)$ or vanish outside a bdd set.

Prop 1.8. (Bessel's ineq): If (ϕ_n) is an orthonormal set in $L^2(a,b)$ and $f \in L^2(a,b)$, then

$$\sum_{n=1}^{\infty} | < f, \phi_n > |^2 \leq \|f\|^2$$

Proof: Fix $N \geq 1$. By orthogonality, (Recall that $\|\phi_n\| = 1$)

$$\| \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n \|^2 = \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2$$

so that

$$0 \leq \|f - \sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n \|^2 \tag{50}$$

$$= \|f\|^2 - 2Re\langle f, \phi_n \rangle \phi_n \rangle + \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 \tag{51}$$

$$= \|f\|^2 - 2Re(\underbrace{\overline{\langle f, \phi_n \rangle} \langle f, \phi_n \rangle}_{=|\langle f, \phi_n \rangle|^2}) + \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 \tag{52}$$

$$= \|f\|^2 - \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 \tag{53}$$

Letting $N \to \infty$ we have our desired result.

Now switching back to our original focus:

$$f \xrightarrow{?} \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n$$

Lemma 1.9. If (ϕ_n) is an orthonormal set in $L^2(a,b)$ and $f \in L^2(a,b)$. Then

1. $\sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n$ converges in norm

2. $\| \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \| \leq \|f\|$

Proof 1.9.1: By Bessel's ineq, $\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \leq \|f\|^2 < \infty$, by $f \in L^2(a,b)$. so that by orthogonality we have

$$\| \sum_{n=m}^{N} \langle f, \phi_n \rangle \phi_n \|^2 = \sum_{n=m}^{N} |\langle f, \phi_n \rangle|^2 \to 0$$

for all $N \geq m$ as $m \to \infty$. Thus $(\sum_{n=1}^{N} \langle f, \phi_n \rangle \phi_n)_{n \geq 1}$ is Cauchy, and since $L^2(a,b)$ is complete, thus converges in norm w/ the same bound as $\|f\|$

Them 1.10. If (ϕ_n) is an orthonormal set in $L^2(a,b)$, then following augments are TFAE:

1. $\langle f, \phi_n \rangle = 0 \forall n \Rightarrow f = 0 \forall f \in L^2(a,b)$
2. $\forall f \in L^2(a,b)$, $f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n$ w/ convergence of the series taken in norm.
3. $\forall f \in L^2(a,b)$,

$$\|f\|^2 = \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 \text{ (Paseval's identity)}$$

<u>RMK:</u> An orthonormal set satisfying (1) (3) is a <u>Complete orthonormal set</u> or <u>orthonormal basis</u>. In this case, $\sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n$ can be seen as F. series and $\langle f, \phi_n \rangle$ can be seen as the F. coeffts.

<u>Notation:</u> If (ψ_n) is an orthogonal set, s.t. $(\frac{\psi_n}{\|\psi_n\|})$ is an o'normal basis. Then (ψ_n) is said to be <u>Complete orthogonal set</u> or <u>orthogonal basis</u> <u>Proof to Them 1.10.</u>
$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$
Given $f \in L^2(a,b)$ and let $\{\phi_n\}$ be orthonormal set. By Lemma 1.9 (1), we have that $\sum_{n \geq 1} \langle f, \phi_n \rangle \phi_n$ converges in norm. We can see that its norm is f by showing that the difference $g = f - \sum_{n \geq 1} \langle f, \phi_n \rangle \phi_n$ is zero. But

$$\langle g, \phi_m \rangle = \langle f, \phi_m \rangle - \sum_{n \geq 1} \langle f, \phi_n \rangle \langle \phi_n, \phi_m \rangle = \langle f, \phi_m \rangle - \langle f, \phi_m \rangle = 0$$

Thus, if (i) hold, then (ii) hold.
Suppose that $f = \sum_{n \geq 1} \langle f, \phi_n \rangle \phi_n$, then by pythogerean theorem, we have

$$\|f\|^2 = \lim_{n} \sum_{n=1}^{N} |\langle f, \phi_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2$$

(iii)⇒(i) is obvious, since

$$\|f\| = (\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2)^{\frac{1}{2}}$$

if $\langle f, \phi_n \rangle = 0 \Rightarrow \|f\| = 0 \Rightarrow f = 0$

4.2 Textbook auxillary:

Def 0.1. antilinear/ conjugate linear:

$$\langle za + wb, c \rangle = z\langle a, c \rangle + w\langle b, c \rangle; \langle a, zb + wc \rangle = \bar{z}\langle a, b \rangle + \bar{w}\langle a, c \rangle$$

Hermitian:

$$\langle b, a \rangle = \overline{\langle a, b \rangle}$$

Lemma 0.2. For $\forall a, b \in C^k$,

$$\|a + b\|^2 = \|a\|^2 + 2Re|\langle a, b \rangle| + \|b\|^2$$

Them 0.3. (Pythagorean): If $a_1, a_2, ..., a_n$ are mutually orthogonal, then

$$\|a_1 + a_2 + ... + a_n\|^2 = \|a_1\|^2 + \|a_2\|^2 + ... + \|a_n\|^2$$

For Cauchy-Schtwirz, triangular, and pythagorean theorem, we only required 0.1 and 0.2 to be satisfied for any inner product!!!

Let $\{u_1, ...u_k\}$ be orthonormal set in C^k, if a vector $a \in C^k$, we can expressed

$$a = c_1 u_1 + ... + c_k u_k.$$

linear combination of $\{u_k\}$. To express c_k, we can have

$$c_j = \langle a, u_j \rangle$$

which is well defined, for otherwise if we define $b = a - \tilde{a}$, then we would force b = 0, for otherwise, $\{u_1, ...u_k, b\}$ would be orthonormal basis.

Theorem 0.4. Let $\{u_k\}$ be orthonormal basis in C_k. for any $a \in C^k$, we have

$$a = \sum_{j=1}^{k} \langle a, u_j \rangle u_j$$

Moreover,

$$\|a\|^2 = \sum_{j=1}^{k} |\langle a, u_j\rangle|^2$$

Def 0.5. In addition to what we have for F.series for c_n, the orthornormal set defined as

$$\psi_0(x) = (\frac{1}{\pi})^{\frac{1}{2}}, \psi_n(x) = (\frac{2}{\pi})^{\frac{1}{2}} \cos(nx)$$

is an orthonormal set in $PC(0, \pi)$. Moreover, for a_n F.coeffits for $f \in PC(0, \pi)$,

$$a_n = \begin{cases} 2(\frac{1}{\pi})^{\frac{1}{2}}\langle f, \psi_0\rangle & \text{for } n = 0 \\ (\frac{2}{\pi})^{\frac{1}{2}}\langle f, \psi_n\rangle & \text{for } n > 0 \end{cases}$$

we have

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = \sum_{n=0}^{\infty}\langle f, \psi_n\rangle\psi_n(x)$$

5 Classnotes 1/18/2023

5.1 Cont. on previous lecture

Them 1.1. $\{e^{inx} : n \in Z\}$ and $\{\cos(nx) : n \geq 0\} \cup \{\sin(nx) : n \geq 1\}$ are orthogonal bases for $L^2(-\pi, \pi)$.

<u>RMK:</u> The set $\{\cos(nx) : n \geq 0\}$ and $\{\sin(nx) : n \geq 1\}$ are orthogonal bases for $L^2(0, \pi)$

Pf to Them 1.1: We focus on $\{e^{inx} : n \in Z\}$. By prop 4.1.7, there exists $\bar{f} \in C^\infty(R)$ and 2π-periodic s.t $\|f - \bar{f}\| \leq \frac{\epsilon}{3}$. Let $c_n, \bar{c}_n$ be F.coeffficts for $f, \bar{f}$ resp. Since $\bar{f} \in PS \cap C(R)$, then by theorem, we have that $S_N^f \to f$ unif. and by theorem, we have unif.cv $\Rightarrow$ converges in norm.

$$\|\sum_{|n|\leq N} \bar{c}_n e^{inx} - \bar{f}\| \to 0$$

as $N \to \infty$. We can choose N large enough such that

$$\|\sum_{|n|\leq N} \bar{c}_n e^{inx} - \bar{f}\| < \frac{\epsilon}{3}$$

Moreover, by pythagorean theorem and Bessel's ineq, we have

$$\|\sum_{|n|\leq N} \overline{c_n} e^{inx} - \sum_{|n|\leq N} c_n e^{inx}\| = 2\pi \sum_{|n|\leq N} |c_n - \bar{c}_n|^2 \leq 2\pi \sum_{n\in Z} |c_n - \bar{c}_n|^2$$

Thus by Bessel's ineq. we have

$$\| \sum_{|n|\leq N} \overline{c_n} e^{inx} - \sum_{|n|\leq N} c_n e^{inx} \|^2 \leq \|\bar{f} - f\|^2 < (\frac{\epsilon}{3})^2$$

Therefore by triangle ineq. we have

$$\|f - \sum_{|n|\leq N} c_n e^{inx}\| \leq \|f - \bar{f}\| + \| \sum_{|n|\leq N} \bar{c}_n e^{inx} - \bar{f}\| + \| \sum_{|n|\leq N} \overline{c_n} e^{inx} - \sum_{|n|\leq N} c_n e^{inx}\| < \epsilon$$

By them 4.1.10 (2), we have that $\{c_n e^{inx}\}$ is orthonormal basis.

5.2 Summary for Chapter 2

Summery: $f : R \to R$ periodic, then

1. if $f \in PS(R) \cap C(R)$, then we have $S_N^f \to f$ uniformly, absolutely, and in norm;
2. if $f \in PS(R)$, then $S_N^f \to f$ pointwisely, and in norm.
3. if $f \in L^2(a,b)$, then only in norm
4. Define $f \in L^p(a,b)$, then (3) still hold, where

$$L^p(a,b) = (\int_a^b |f|^p dx)^{\frac{1}{p}} < \infty$$

for $p > 1$. (false for the example discussed in the pp.18/HW1 Problem 5(d))

Additional Remark:

- Weighted L^2 space ($dx \to w(x)dx$)
- General domain $D \in R^n \to L^2(D)$ similar properties (Folland pp.81 82)
- Dominant Convergence Theorem: Consider couterexample that

$$f_n(x) = \begin{cases} 1 & n \leq x \leq n+1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_n(x) = \begin{cases} n & 0 \le x \le 1/n \\ 0 & \text{otherwise} \end{cases}$$

From the graph, we can clearly see that f_n move all the way to the right, and g_n move all the way up. However,

$$\lim_n \int f_n dx = \lim_n \int g_n dx = 1$$

and they both converges to 0 point wisely. Hence, we need a dominant convergence theorem w/

- **Dominant Convergence Theorem:** $D \in R^n$, suppose that $(g_n), \phi$ are functions on D, s.t
  1. $\phi(x) \ge 0, \int_D \phi(x)dx < \infty$
  2. $|g_n| \le \phi(x)$ for any $n, x \in D$.
  3. $g_n(x) \to g(x)$ as $n \to \infty$ for any $x \in D$

  Then we have $\int_D g_n dx \to \int_D g dx$
- **Cor:** $(f_n) \subset L^2(D), f_n \to f$ pointwisely. Suppose that $\psi \in L^2(D)$ s.t $|f_n(x)| \le |\psi(x)|$ for all $n, x \in D$, then

$$\|f_n - f\|_{L^2} \to 0 \text{ as } n \to \infty \ (f_n \to f \text{ in norm!})$$

Proof to Corollary: Certainly we have,

$$|f_n - f|^2 \le (|f_n| + |f|)^2 \le (|\psi| + |f|)^2$$

Moreover, since $lim_n |f_n| = |f| \le |\psi|$, we have

$$|f_n - f|^2 \le (|\psi| + |\psi|)^2 = 4|\psi|^2 \leftarrow \int_D |\psi|^2 < \infty$$

By DCT (setting $\phi(x) = 4|\psi|^2$) we have that

$$\|f_n - f\|_{L^2} = \int_D |f_n - f|^2 \to \int_D 0 = 0$$

thus convergence in norm!

5.3 Best Approximation and isopermetric-ineq.

Best Approx in L^2: Let $\{\phi_n\}$ be o'normal basis for $L^2(D)$

$$\hookrightarrow f = \sum \langle f, \phi_n \rangle \phi_n \ \forall f \in L^2$$

However, what happened if c is not a basis? (o'normal set not complete). By our lemma, $\sum \langle f, \phi_n \rangle \phi_n$ converges to what?

Prop 5.3.1: $\{\phi_n\}$ o'normal set in $L^2(D), f \in L^2$. Then for all $(c_n)_{n\geq 1}$ w/ $\sum |c_n|^2 < \infty$, we have

$$\|f - \sum \langle f, \phi_n \rangle \phi_n\| \leq \|f - \sum c_n \phi_n\|$$

equality hold iff $c_n = \langle f, \phi_n \rangle$ for all n.

Proof: Fixed $N \leq 1$, we have $\|f - \sum c_n \phi_n\|^2 =$

$$\|f - \sum \langle f, \phi_n \rangle \phi_n\|^2 + 2Re\langle f - \sum \langle f, \phi_n \rangle \phi_n, \sum (\langle f, \phi_m \rangle - c_m)\phi_m \rangle + \| \sum (\langle f, \phi_m \rangle - c_m)\phi_m\|^2 \tag{54}$$

$$= \|f - \sum \langle f, \phi_n \rangle \phi_n\|^2 + \sum |\langle f, \phi_m \rangle - c_m|^2 \tag{55}$$

by pythagonean theorem w/ 4.1.10(2), and RHS' second term equal to 0 iff given the condition as stated in the theorem.

Corollary: $\{\phi_n\}$ o'normal basis in $L^2(D)$. For $f \in L^2$, $N \geq 1$. $\sum \langle f, \phi_n \rangle \phi_n$ is the best approx in L^2 norm to f among all linear combinations of $(\phi_n)_{n \leq N}$

2D isopermetric inequlity: Given $\Omega \subset R^2$, Area:= A, perimeter = L,

$$L^2 \geq 4\pi A$$

w/ equality hold iff Ω is circle!

Lemma (Poincare-writnger): $f \in C^2(\mathbf{R})$ periodic / period $2 - \pi$,

$$f_{av} = \frac{1}{2\pi} \int_0^{2\pi} f dx.$$

Then

$$\int_0^{2\pi} (f - f_{av})^2 dx \leq \int_0^{2\pi} f^2 dx.$$

w/ equality hold iff $f = f_{a,v} + acos(x) + bsin(x)$ for some $a, b \in R$.

Proof to isop-ineq:
WLOG, suppose that Ω has perimeter 2π, and curve bounded Ω, defined by $(x(s), y(x))$ parametrized by arclength s on the curve, so that $(\frac{dx}{ds})^2 + (\frac{dx}{ds})^2 = 1$. Since the curve is closed and has length 2π, x,y has period 2π. Now, recall Green's theorem:

$$\int_{\partial D} pdx + qdy = \iint_D qx - pydxdy$$

Apply this w/ $(p, q) = (0, x)$. Then

$$\int_{\partial D} xdy = \iint_D dxdy = A,$$

so,

$$A = \int_0^{2\pi} xy'ds = \int_0^{2\pi} xy'ds - \int_0^{2\pi} x_{av}y'ds \tag{56}$$

$$= \frac{1}{2}\int_0^{2\pi} (x - x_{av})^2 + (y')^2 - (x - x_{av} - y)^2 dx \tag{57}$$

$$\leq \frac{1}{2}\int_0^{2\pi} (x')^2 + (y')^2 = \pi \tag{58}$$

For general Ω, set $\bar{\Omega} = \frac{2\pi}{L}\Omega = \{\frac{2\pi x}{L} : x \in \Omega\}$. Then $\bar{\Omega}$ has perimeter

$$\frac{2\pi}{L}Per(\Omega) = \frac{2\pi}{L}L = 2\pi$$

and area

$$(\frac{2\pi}{L})^2 Area(\Omega) = \frac{4\pi^2 A}{L}$$

We then get

$$\frac{4\pi^2 A}{L} \leq \pi \iff 4\pi A \leq L^2$$

Proof to Poincare Lemma:
Expand f, f' as F.series,

$$f(x) = \frac{1}{2}a_0 + \sum_{n\geq 1} a_n cos(nx) + b_n sin(nx)$$

w/ $\frac{1}{2}a_0 = f_{av}$.
Now, by parseval identity, we have

$$\int_0^{2\pi} (f - f_{av})^2 = (const)\sum_{k=1}^{\infty} a_k^2 + b_k^2 = \pi \sum_{k=1}^{\infty} a_k^2 + b_k^2$$

while the f.series to f' is

$$\sum_{n=1}^{\infty} -na_n sin(nx) + nb_n cos(nx)$$

By Bessel's ineq. we have,

$$\pi \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2) \leq \|f'\|^2 = \int_0^{2\pi} (f'(x))^2$$

Moreover, combined these results,

$$\int_0^{2\pi} f'^2 - \int_0^{2\pi} (f - f_{av})^2 \geq \pi(\sum_{n=1}^{\infty} n^2(a_n^2+b_n^2) + \sum_{k=1}^{\infty} a_k^2+b_k^2) = \pi \sum_{n\geq 2}(n^2-1)(a_n^2+b_n^2) \geq 0$$

6 Classnotes 1/23/2024

6.1 General Fourier transform

Motivation: $f: R \to R$, for all $l > 0$, look at $f|_{[-l,l]}$ and expand as a Fourier series,

$$f(x) = \frac{1}{2l} \sum_{n\in Z} c_{n,l} e^{in\pi x/l}$$

and F.coeffi,

$$c_{n,l} = \int_{-l}^{l} f(y) e^{-in\pi y/l} dy$$

For convention, we write: $\Delta\xi = \frac{\pi}{l}$; $\xi_n = \frac{n\pi}{l}$, thus the above becomes,

$$f(x) = \frac{1}{2\pi} \sum_{n\in Z} c_{n,l} e^{i\xi_n x} \Delta\xi$$

and

$$c_{n,l} = \int_{-l}^{l} f(y) e^{-i\xi_n y} dy$$

Suppose that f vanishes rapidly as $x \to \pm\infty$. Then

$$c_{n,l} \approx \int_{-l}^{l} f(y) e^{-i\xi_n y} dy := \hat{f}(\xi_n) \hookrightarrow f(x) \approx \frac{1}{2\pi} \sum \hat{f}(\xi_n) e^{i\xi_n x} \Delta\xi$$

Now let $l \to \infty$, we have the formal definition that...

Def 6.1.1. $f \in L^1(R)$, then $\hat{f} : R \to \mathbf{C}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-i\xi x} dx$$

is the Fourier Transform of f. Also we may see that as a functional on f by $\mathcal{F}(f(x)) = \hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-i\xi x} dx$

RMK:

1. $L^1 = L^1(R) = \{f : R \to R | \int |f| dx < \infty\}$
2. $L^1 \not\subset L^2$ and $L^2 \not\subset L^1$ (also for general p,q): Consider

$$f(x) = \begin{cases} x^{-2/3} & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

and

$$g(x) = \begin{cases} x^{-2/3} & 1 < x < \infty \\ 0 & \text{else} \end{cases}$$

we have $f \in L^1, f \notin L^2$; $g \in L^2, g \notin L^1$

3. $f \in L^1$ and f is bdd ($f \in L^\infty$), then $f \in L^2$:

$$|f|^2 \leq M|f| \Rightarrow \int |f|^2 \leq M \int |f| < \infty$$

4. $f \in L^2$, and f vanishes outside a bdd interval $[a, b]$, then $f \in L^1$:

$$\int |f| dx = \int_a^b |f| dx \leq (b-a)^{\frac{1}{2}} (\int_a^b |f|^2)^{\frac{1}{2}} \text{ (Cauchy-Schwartz)}$$

Def 6.1.2.(Convolution): Let $f, g : R \to R$, the Convolution of f and g is defined as

$$(f * g)(x) = \int f(x-y) g(y) dy$$

provided that the integral exists!

The following condition would ensure the definition is well-defined
i.e $\int |f(x-y)g(y)| < \infty$:

- $f \in L^1$, g is bdd;
- f is bdd, $g \in L^1$;
- $f, g \in L^2$:

$$\int |f(x-y)g(y)| \leq (\int |f(x-y)|^2 dy)^{\frac{1}{2}} + (\int |g(y)|^2 dy)^{\frac{1}{2}}$$

- $f \in PC(R)$, g is bdd w/cpt support;
- $f, g \in L^1$, then $(f * g)(x)$ exists a.e (From Fubini's theorem)

Prop 6.1.3.

1. $f * (ag + bh) = a(f * g) + b(f * h)$
2. $f * g = g * f$
3. $f * (g * h) = (f * g) * h$
4. f is differentiable, $f * g$, $f' * g$ are well-defined $\Rightarrow$ $f * g$ is differentiable w/

$$(f * g)' = f' * g$$

(Note:) There is no such $g : R \to R$ such that $f * g = f$ for all f! (Chap 9 Folland)

But we certainly have some good approx to f in such cases:
For $g \in L^1$, $\epsilon > 0$: set

$$g_\epsilon(x) = \frac{1}{\epsilon} g(\frac{x}{\epsilon})$$

somehow play a role like compress in x-direction and stretch in y-direction. Then we have

$$\int g_\epsilon(x)dx = \int g(y)dy \text{ (Change of variable)}$$

Prop 6.1.4. If

- $g \in L^1$, $g \geq 0$, and $\int g dy = 1$.
- $\alpha = \int_{-\infty}^0 g dy$, and $\beta = \int_0^\infty g dy$
- $f \in PC(R)$, and either f is bdd or g has cpt-support ($f * g$ is well-defined).

Then

$$\lim_{\epsilon\to 0^+}(f * g_\epsilon)(x) = \alpha f(x^+) + \beta f(x^-)$$

for any x. Moreover, if f is cts on $[a, b]$, then the convergence is uniformly on $[a, b]$

Translated in normal case: If f is cts at x in the above, then

$$\lim_{\epsilon\to 0}(f * g_\epsilon)(x) = f(x)$$

<u>Fact:</u>

- $g \in L^1(R)$ bdd w/ $g \geq 0$, and $\int g dy = 1$. If $f \in L^2(R)$, then $(f * g)$ is well-defined for all x, and $\|f - f * g_\epsilon\|_{L^2} \to 0$ as $\epsilon \to 0$
- For (g_ϵ) family, such that $f*g_\epsilon \to 0$ as $\epsilon \to 0$ is said to be <u>approximate identity</u>
- (Gaussion): example of approx-identity,

  $$G : y \to \frac{1}{\sqrt{\pi}}e^{-y^2}$$

  and

  $$G_\epsilon : x \to \frac{1}{\epsilon}G(\frac{x}{\epsilon})$$

  If f is bdd and in $PC(R)$ implies that $f * G_\epsilon$ is smooth (C^∞) and a good approx for f when ϵ is small.
- $K : R \to R$ def. by

  $$K(y) = \begin{cases} \frac{1}{c}e^{-i/(1-y^2)} & -1 < y < 1 \\ 0 & \text{else} \end{cases}$$

  Where

  $$c = \int_{-1}^{1} e^{-i/(1-y^2)}dy$$

  similarly as Gaussian, we have that K is even functions, w/ derivitive exists and nice bounds. Moreover, K has cpt supports! Thus $f * K_\epsilon$ is well defined for every $f \in PC(R)$ not necessary bounded, gives also good approx to f.

7 Classnotes 1/25/2024

7.1 Properties for Fourier Transform

Now back to the original topics about Fourier Transform:
RMK:

- for all ξ,

$$|\hat{f}(\xi)| = |\int_R e^{-i\xi x} f(x)dx| \leq \int_R |e^{-i\xi x}||f(x)|dx \leq \int_R |f|dx$$

  so that $\hat{f}$ is a bdd function. Moreover,

$$|\hat{f}(\xi) - \hat{f}(\eta)| \leq \int_R |e^{-i\xi x} - e^{-i\eta x}||f(x)|dx \leq 2\int_R |f|dx$$

  so that by LDCT, setting $\phi(x) = 2|f|$, we get

$$\leq \int_R |e^{-i\xi x} - e^{-i\eta x}||f(x)|dx \to 0$$

  as $\xi \to \eta$, since $g(y) = e^{-iyx}$ is continuous functions, and thus we reach the conclusion that $\hat{f} \in C(R) \cap L^\infty$

Prop 7.1.1 Suppose that $f \in L^1$, for any $a \in R$,

- $\mathcal{F}_x[f(x-a)] = \int e^{-i\xi x} f(x-a)dx = \int e^{-i\xi(y+a)} f(y)d(y) = e^{-ia\xi}\hat{f}(\xi)$
- $\mathcal{F}_x[e^{iax} f(x)] = \hat{f}(\xi - a)$ for the same procedure as above.
- $\mathcal{F}_x[\frac{1}{\delta} f(\frac{x}{\delta})] = \hat{f}(\delta\xi)$
- $\mathcal{F}_x[f(\delta\xi)] = \frac{1}{\delta}\hat{f}(\frac{\xi}{\delta})$
- $f \in PS(R) \cap C(R)$, $f' \in L^1(R)$ implies

$$\mathcal{F}[f'(x)] = (i\xi)\hat{f}(\xi)$$

  (Question: Why $f \in L^1 \Rightarrow f(x) \to 0$?)
- $f \in L^1$ and $xf \in L^1$, since $xe^{-i\xi x} = i(d/d\xi)e^{-i\xi x}$, by previous one, we have

$$\mathcal{F}[xf(x)] = i\frac{d}{d\xi}\int e^{-i\xi x} f(x)dx = i(\hat{f})'(\xi)$$

- $f, g \in L^1$, we have

$$\mathcal{F}[(f * g)(x)] = \int e^{-i\xi x}(\int f(x-y)g(y)dy) \tag{59}$$
$$= \iint e^{-i\xi(x-y)} f(x-y) e^{-i\xi y)} g(y)dydx \tag{60}$$
$$= (\int e^{-i\xi u} f(u)du)(\int e^{-i\xi y)} g(y)dy) \tag{61}$$
$$= \hat{f}(\xi)\hat{g}(\xi) \tag{62}$$

Example 7.1.2

1. Fix $a > 0$, we have that for $\mathbb{1}_a(x)$ has FT as:

$$\hat{\mathbb{1}_a}(\xi) = \int_{-a}^{a} e^{-i\xi x} dx = \frac{2sin(a\xi)}{\xi}$$

2. $a > 0$, $f(x) = e^{-ax^2}$, we have that its FT as:

$$\hat{f}(\xi) = \sqrt{\frac{2\pi}{a}} e^{-\xi^2/2a}$$

(From Folland pp.215-216)

7.2 General Theorem for Fourier Transform

Prop 7.2.1 (Riemann-Lebesgue Lemma): $f \in L^1 \Rightarrow \lim_{\xi \to \pm\infty} \hat{f}(\xi) = 0$ (proof in pp.7)

Prop 7.2.2(Fourier Invertion Formula): $f \in L^1(R) \cap PC(R)$, $f(x) = \frac{1}{2}(f(x^+) + f(x^-)$. Then we have

$$f(x) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_R e^{i\xi x - \epsilon^2 \xi^2/2} \hat{f}(\xi) d\xi$$

Prop 7.2.3(Fourier Invertion Formula): $f \in L^1(R) \cap PC(R)$, $f(x) = \frac{1}{2}(f(x^+) + f(x^-)$ and $\hat{f} \in L^1$, then we have

$$f(x) = \frac{1}{2\pi} \int_R e^{i\xi x} \hat{f}(\xi) d\xi$$

with f is continuous! (useful)

Prop 7.2.4: If $\hat{f}, \hat{g} \in L^1$ with $\hat{f} = \hat{g}$, then $f = g$. (Proof is clear, we have $(f-g)\hat{}(\xi) = 0 \Rightarrow f - g = 0$)

RMK:

- When $\phi = \hat{f}$ for some $f \in L^1$, we say f is the reverse F.transform of ϕ w/ $f = \mathcal{F}^{-1}[\phi]$, $f = \check{\phi}$

8 What I think is real notes!

Fundamentally speaking, I don't see any usefulness for the previous lecture notes would make sense to anyone try to completely understand this course is about. I will rewrite what I learned in this fantastic subject in my own language.

8.1 Discrete case of Fourier

Def 1.1 (Fourier coefficient):

Consider that f is 2π periodic, integrable, let $\begin{cases} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx \\ c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \end{cases}$

are called the F-coefficient of f, and

$$\lim_N S_N^f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx) = \sum_{n=1}^{\infty} c_n e^{inx}$$

are called the F-series of f.

The reason that is important is because for $f \in PS(R)\bigcap C(R)$, 2π periodic, we would see that $\lim_N S_N^f(x) = f$, even at those finite discontinuous points.

One thing to note that by calculation of S_N^f, we have a general formula for

$$S_N^f(x) = \int_{\mathbb{T}} f(x+\psi)(\frac{1}{2\pi}\sum_{|n|\leq N} e^{in\psi})d\psi$$

where we can get Dirichlet Kernel,

$$D_N(x) = \frac{1}{2\pi}\sum_{|n|\leq N} e^{in\psi} = \frac{sin(N+1/2)\psi}{sin(1/2)\psi}$$

and

$$\int_0^{\pi} D_N(x)dx = \int_{-\pi}^{0} D_N(x)dx = \frac{1}{2}$$

have some good properties for us to derive certain bound like $\log(N)$ or something more faster (Details in Homework 1)

However, it is not a 'good kernel'/'Approximation identity' followed by the word from *Stein, Fourier Analysis*, that can not best approximate f in certain restraint.

In general we request $\sum_{n=-\infty}^{\infty} |c_n|^2$, or $\sum_{n=1}^{\infty} |b_n|^2$ and $\sum_{n=1}^{\infty} |a_n|^2$ less than infinity (i.e converges) to make the Fourier coefficient make sense