Google DeepMind

AlphaProof

When RL meets Formal Maths

Thomas HUBERT, March 2025

Google DeepMind


Mathematics
AlphaProof

We have used maths to both describe, predict and shape the natural world

Mathematics, a root node to intelligence?

Reasoning & Planning

Generalisation & Abstraction

Knowledge & Creativity

Open ended & Unbounded complexity

Even requires an eye for beauty...

A Brief History of Formalisation in Mathematics

Proofs & Axiomatisation

Importance of proofs discovered by the ancient Greeks

Since then, what constitutes a mathematical proof has evolved.

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From words to symbols

A square and ten times its root are equal to thirty-nine.

$$x^2 + 10x = 39$$

Take the coefficient of the root, divide it into two parts, and multiply one of them by itself.
Add this to thirty-nine
Take the square root of sixty-four which is eight
Subtract from this one of the halves of the coefficient of the root.
The result is the value of the root which is three.

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$$-\frac{b}{2} \pm \sqrt{\frac{b}{2}^2 - c}$$

A Brief History of Formalisation in Mathematics

1. Rigor and clarity
2. Efficiency and Communication
3. Abstraction and Generalisation
4. Unification
5. Created new fields

Computer Formalisation of Mathematics

Mathematics in code

```
/-- Euclid's theorem on the **infinitude of primes**.
Here given in the form: for every `n`, there exists a prime number `p ≥
n`. -/
theorem exists_infinite_primes (n : ℕ) : ∃ p, n ≤ p ∧ Prime p :=
  let p := minFac (n ! + 1)
  have f1 : n ! + 1 ≠ 1 := ne_of_gt <| succ_lt_succ <| factorial_pos _
  have pp : Prime p := minFac_prime f1
```

▼ Expected type

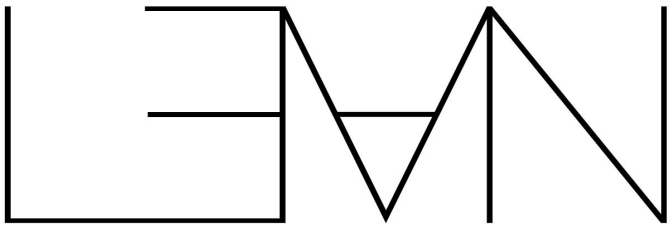
```
n : ℕ
p : ℕ := (n ! + 1).minFac
f1 : n ! + 1 ≠ 1
pp : Prime p
⊢ ∃ p, n ≤ p ∧ Prime p
```

Lean

Lean is a programming language, theorem prover, and interactive proof assistant.

It has a vibrant open source community of mathematicians.

Lean has been used to formalize fields medal mathematics.
A huge quest the community is on it to build the math library of the future.



Programming Language and Theorem Prover

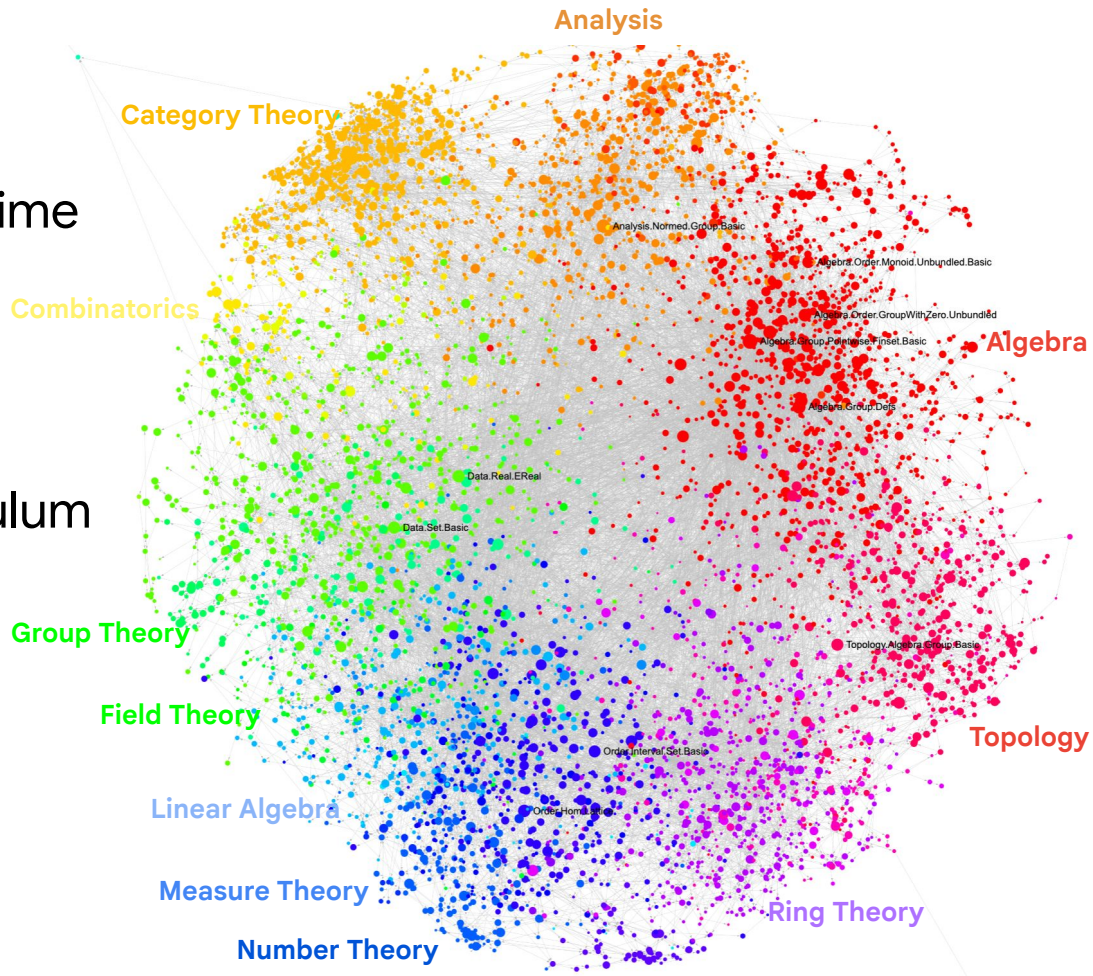
Mathlib

Built **open source** on own free time of mathematicians!

It aims to be **General** & **Unified.**

Covers ~ 80% undergrad curriculum

Most of the library is above undergraduate but has irregular coverage with big holes.



Computer Formalisation unlocks enormous synergies

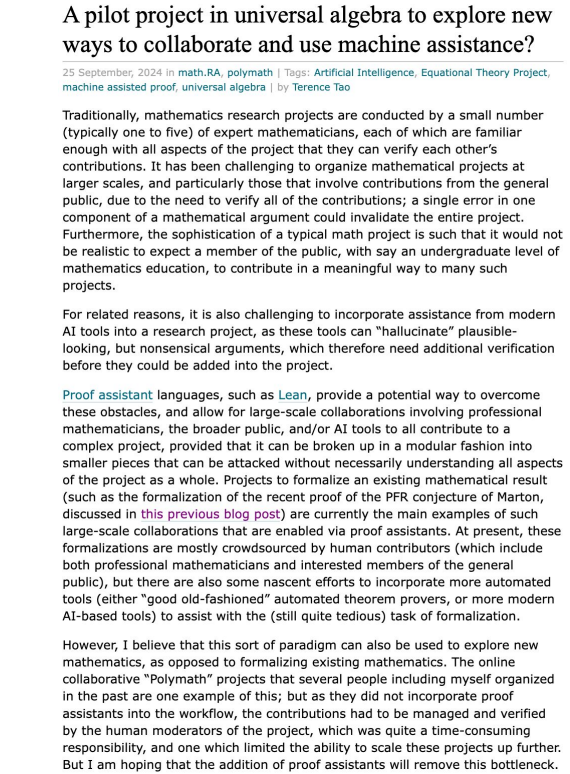
Perfect Verification + Software Stack

Instant Wins

- Correctness concerns disappear
- Giant Proofs can be trusted
- Proof checking can be delegated entirely to the computer

Game Changer

- Transforms mathematics into a video game 🎉!
- For Education
- All the tools of Software Engineering for Maths
- Unlocks Massive Collaboration



A pilot project in universal algebra to explore new ways to collaborate and use machine assistance?

25 September, 2024 in math.RA, polymath | Tags: Artificial Intelligence, Equational Theory Project, machine assisted proof, universal algebra | by Terence Tao

Traditionally, mathematics research projects are conducted by a small number (typically one to five) of expert mathematicians, each of which are familiar enough with all aspects of the project that they can verify each other's contributions. It has been challenging to organize mathematical projects at larger scales, and particularly those that involve contributions from the general public, due to the need to verify all of the contributions; a single error in one component of a mathematical argument could invalidate the entire project. Furthermore, the sophistication of a typical math project is such that it would not be realistic to expect a member of the public, with say an undergraduate level of mathematics education, to contribute in a meaningful way to many such projects.

For related reasons, it is also challenging to incorporate assistance from modern AI tools into a research project, as these tools can "hallucinate" plausible-looking, but nonsensical arguments, which therefore need additional verification before they could be added into the project.

Proof assistant languages, such as Lean, provide a potential way to overcome these obstacles, and allow for large-scale collaborations involving professional mathematicians, the broader public, and/or AI tools to all contribute to a complex project, provided that it can be broken up in a modular fashion into smaller pieces that can be attacked without necessarily understanding all aspects of the project as a whole. Projects to formalize an existing mathematical result (such as the formalization of the recent proof of the PFR conjecture of Marton, discussed in this previous blog post) are currently the main examples of such large-scale collaborations that are enabled via proof assistants. At present, these formalizations are mostly crowdsourced by human contributors (which include both professional mathematicians and interested members of the general public), but there are also some nascent efforts to incorporate more automated tools (either "good old-fashioned" automated theorem provers, or more modern AI-based tools) to assist with the (still quite tedious) task of formalization.

However, I believe that this sort of paradigm can also be used to explore new mathematics, as opposed to formalizing existing mathematics. The online collaborative "Polymath" projects that several people including myself organized in the past are one example of this; but as they did not incorporate proof assistants into the workflow, the contributions had to be managed and verified by the human moderators of the project, which was quite a time-consuming responsibility, and one which limited the ability to scale these projects up further. But I am hoping that the addition of proof assistants will remove this bottleneck.

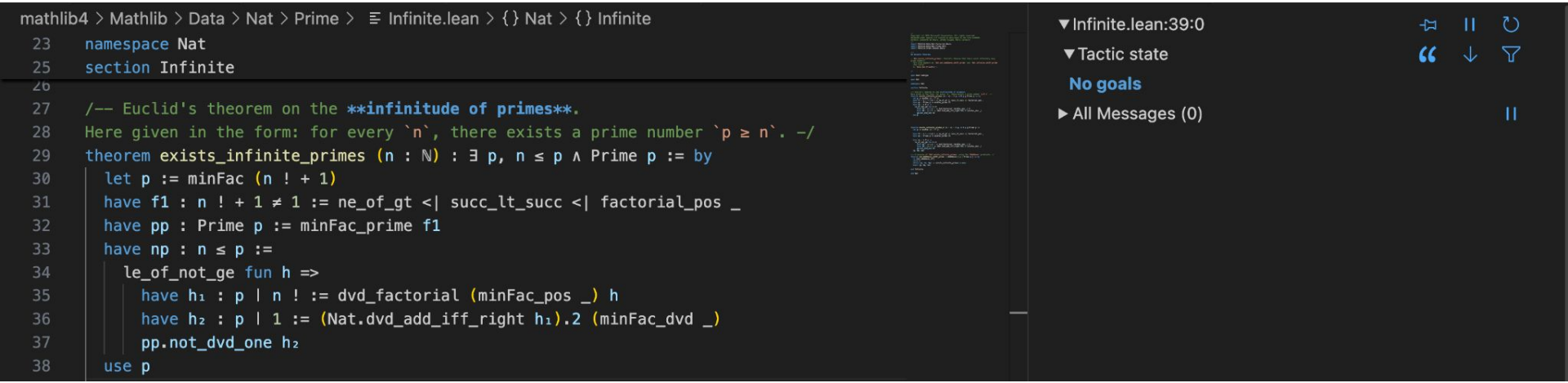
Challenges for Computer Formalisation

Only 0.1%-1% of mathematicians have adopted Lean.

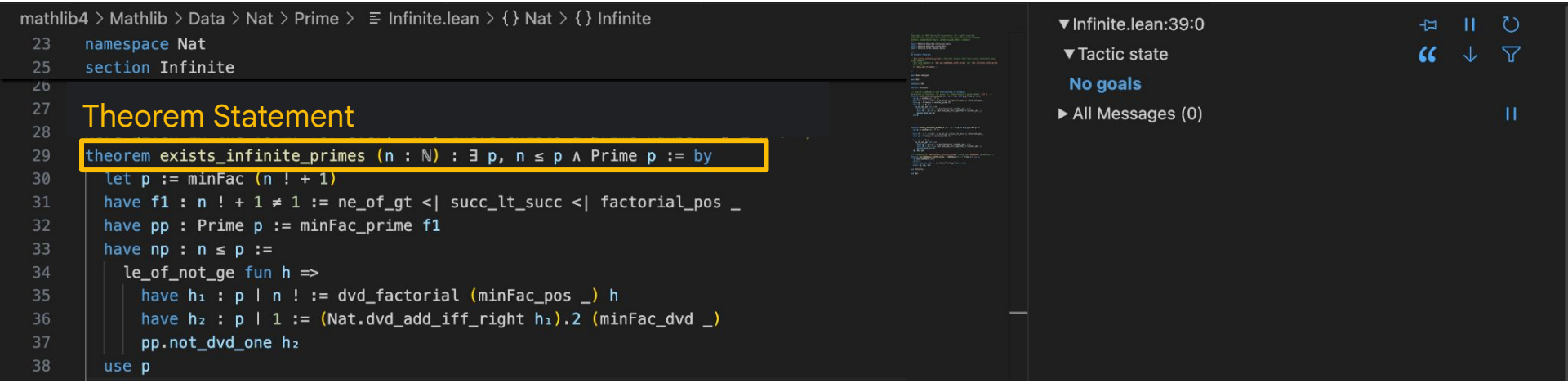
Not yet a good tool for mainstream research

- Steep Learning Curve
- Significant Time Investment
- Tooling and Library maturity (though rapidly growing)
- Perceived lack of necessity for research.

Demo - Infinitude of Primes



Demo - Infinitude of Primes



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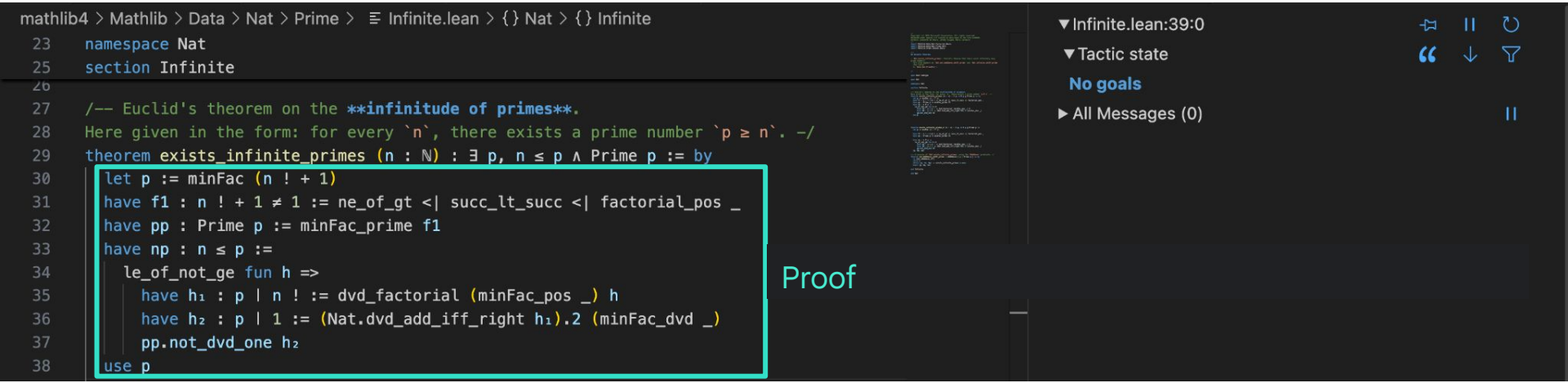

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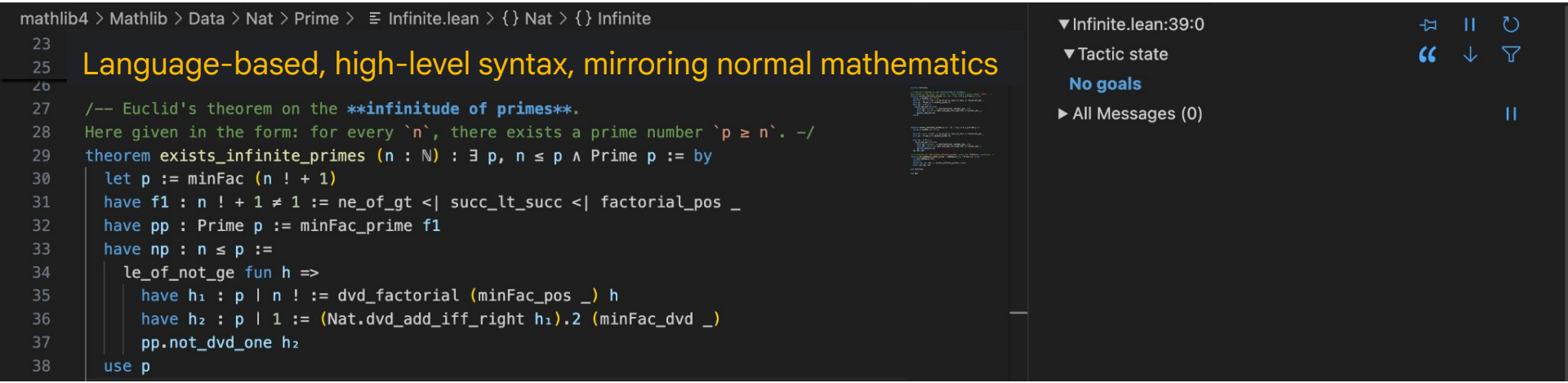

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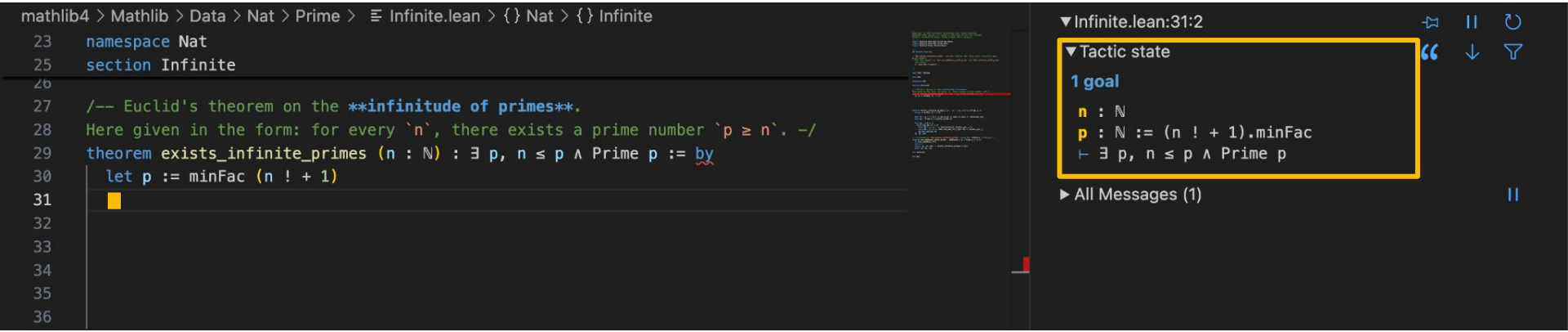

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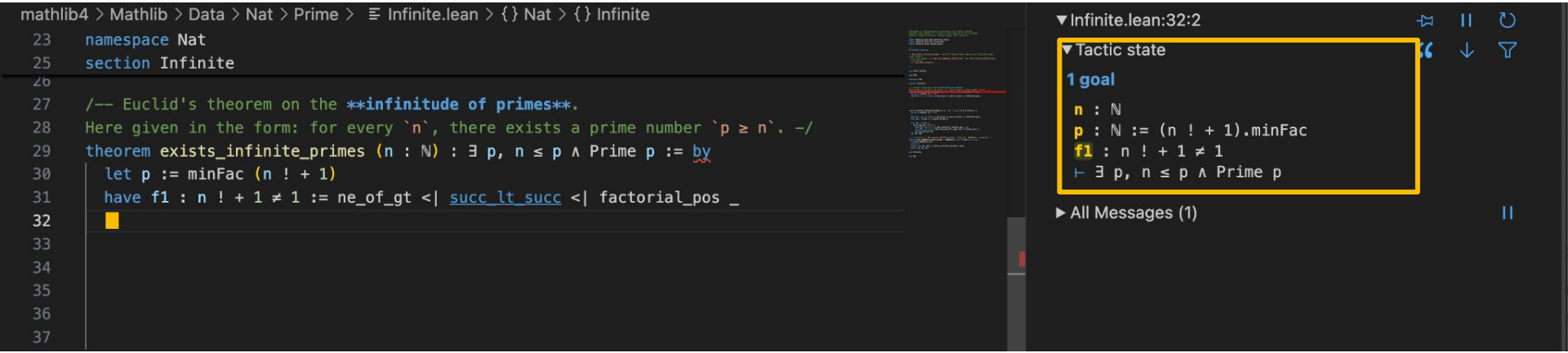

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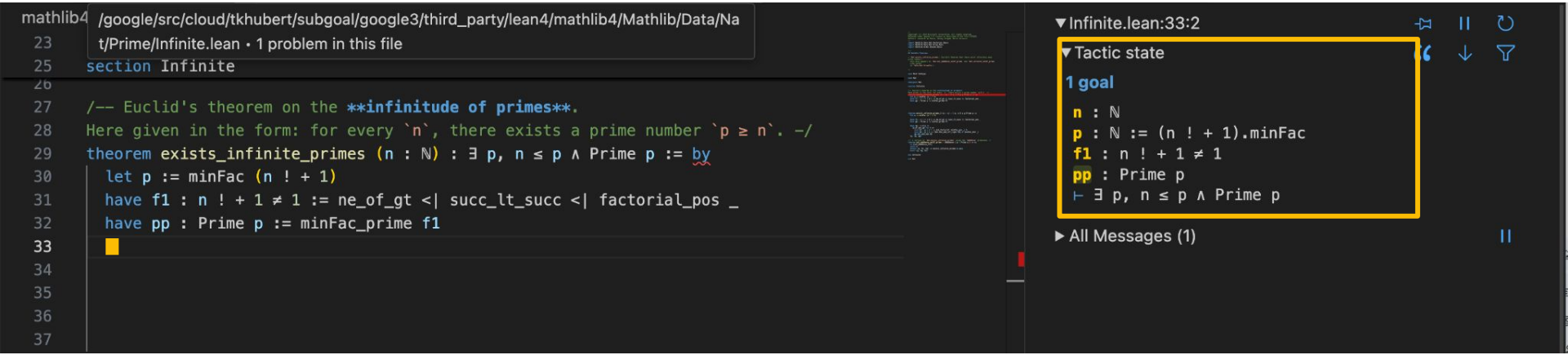

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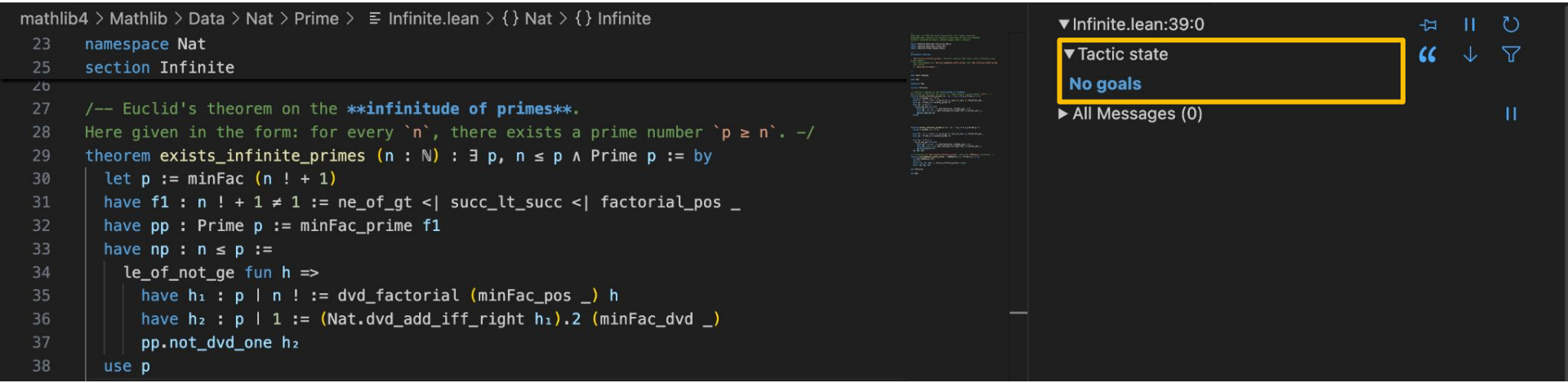


Computer Formalisation

Huge enthusiasm from a vibrant community

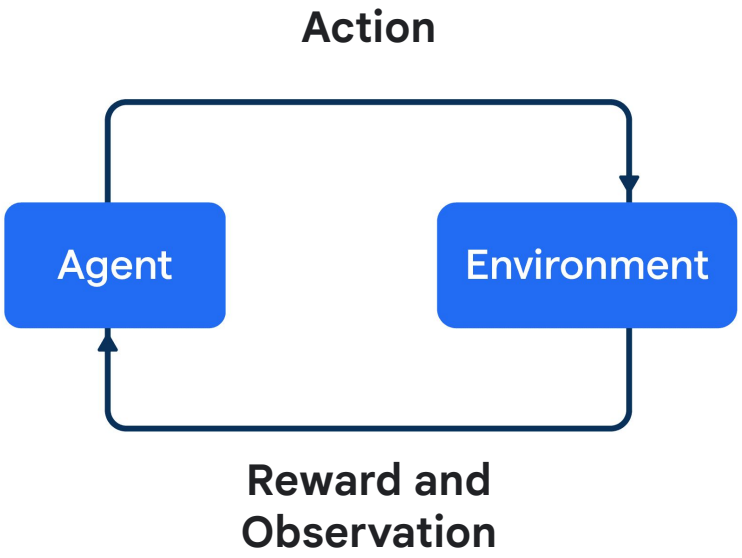
Building the library of the future

Unlocks enormous synergies

Still has some way to go

Belief this will inevitably play an integral role in the future of Mathematics

Reinforcement Learning



What's RL

RL is trial and error learning.

An **agent** learns by **interacting** with an **environment** to maximize **reward**.

SuperScale RL: A Proven Recipe to Superintelligence

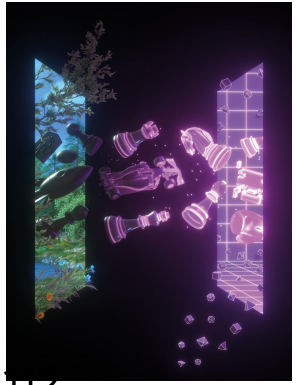
Atari DQN

AlphaGo

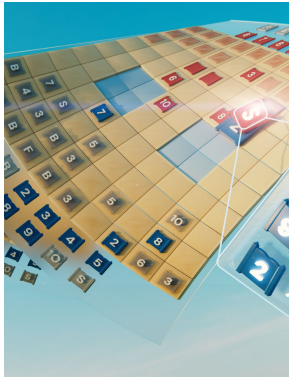
AlphaGoZero

AlphaZero

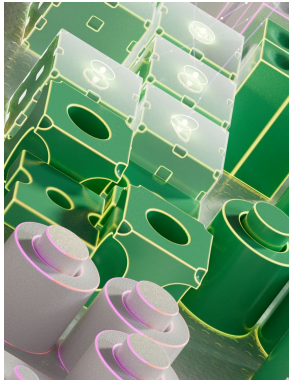
AlphaStar



MuZero



Stratego



AlphaDev

AlphaTensor

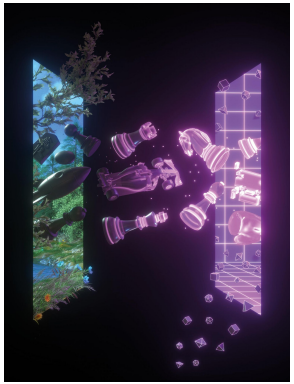
Fusion

The Zero Series

AlphaGo

AlphaGoZero

AlphaZero



MuZero

AlphaTensor

The Zero Philosophy

If an agent can learn to master an environment **tabula rasa**, then you demonstrably have a system that is **discovering and learning new knowledge by itself**.

It also means that this algorithm is **general** and should apply to other domains.

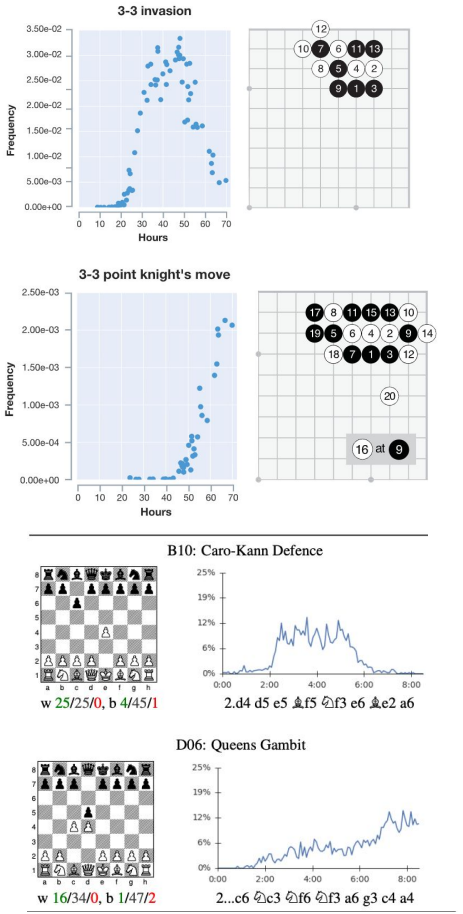
This is particularly important to **advance science**.

AlphaGoZero & AlphaZero

Learned simply by playing games against itself, starting from completely random play.

Rediscovered human patterns, accumulating thousands of years of human knowledge in a matter of days.

Ultimately discarded some in preference of **new discoveries** that humans don't even know about.



AlphaTensor

Not only board games!

AlphaTensor **discovered novel**, efficient, and provably correct **algorithms** for fundamental tasks such as **matrix multiplication**.

Bonus: Can tackle huge action spaces $O(10^{30})$

Size (n, m, p)	Best method known	Best rank known	AlphaTensor rank Modular	AlphaTensor rank Standard
$(2, 2, 2)$	(Strassen, 1969)[2]	7	7	7
$(3, 3, 3)$	(Laderman, 1976)[15]	23	23	23
$(4, 4, 4)$	(Strassen, 1969)[2] $(2, 2, 2) \otimes (2, 2, 2)$	49	47	49
$(5, 5, 5)$	$(3, 5, 5) + (2, 5, 5)$	98	96	98
$(2, 2, 3)$	$(2, 2, 2) + (2, 2, 1)$	11	11	11
$(2, 2, 4)$	$(2, 2, 2) + (2, 2, 2)$	14	14	14
$(2, 2, 5)$	$(2, 2, 2) + (2, 2, 3)$	18	18	18
$(2, 3, 3)$	(Hopcroft and Kerr, 1971)[16]	15	15	15
$(2, 3, 4)$	(Hopcroft and Kerr, 1971)[16]	20	20	20
$(2, 3, 5)$	(Hopcroft and Kerr, 1971)[16]	25	25	25
$(2, 4, 4)$	(Hopcroft and Kerr, 1971)[16]	26	26	26
$(2, 4, 5)$	(Hopcroft and Kerr, 1971)[16]	33	33	33
$(2, 5, 5)$	(Hopcroft and Kerr, 1971)[16]	40	40	40
$(3, 3, 4)$	(Smirnov, 2013)[18]	29	29	29
$(3, 3, 5)$	(Smirnov, 2013)[18]	36	36	36
$(3, 4, 4)$	(Smirnov, 2013)[18]	38	38	38
$(3, 4, 5)$	(Smirnov, 2013)[18]	48	47	47
$(3, 5, 5)$	(Sedoglavic and Smirnov, 2021)[19]	58	58	58
$(4, 4, 5)$	$(4, 4, 2) + (4, 4, 3)$	64	63	63
$(4, 5, 5)$	$(2, 5, 5) \otimes (2, 1, 1)$	80	76	76

What made those systems Superhuman?

Scaled up trial and error

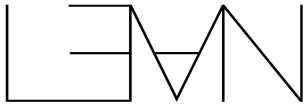
Grounded feedback signal

Search

Curriculum

What made those systems Superhuman?

✅ **Scaled up trial and error**



Grounded feedback signal

Search

Curriculum

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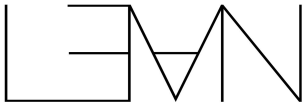
✅ **Grounded feedback signal** 

Search

Curriculum

What made those systems Superhuman?

✅ **Scaled up trial and error**

✅ **Grounded feedback signal**

✅ Search

Curriculum

What made those systems Superhuman?

✅ **Scaled up trial and error**

✅ **Grounded feedback signal**

✅ Search

⁉️ Curriculum

AlphaZero for Mathematics

When RL meets Formal Maths

AlphaZero is the agent.

Lean is the environment.

AlphaProof: Master Plan

1. Lean gives us a way to **scale up trial and error** with
   a. An **environment** to explore mathematics completely in silico
   b. A **perfect feedback** signal for proving

2. We can therefore **reach superhuman intelligence and discover new truths**, just like in Go/chess/Tensor decomposition… provided:
   - We can generate high enough quality + quantity problems
   - RL works

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   ✅ RL works

Where do the problems come from?

Humans define the problems

- We want to help human mathematics
- There are already a lot of problems!

Where do the problems come from?

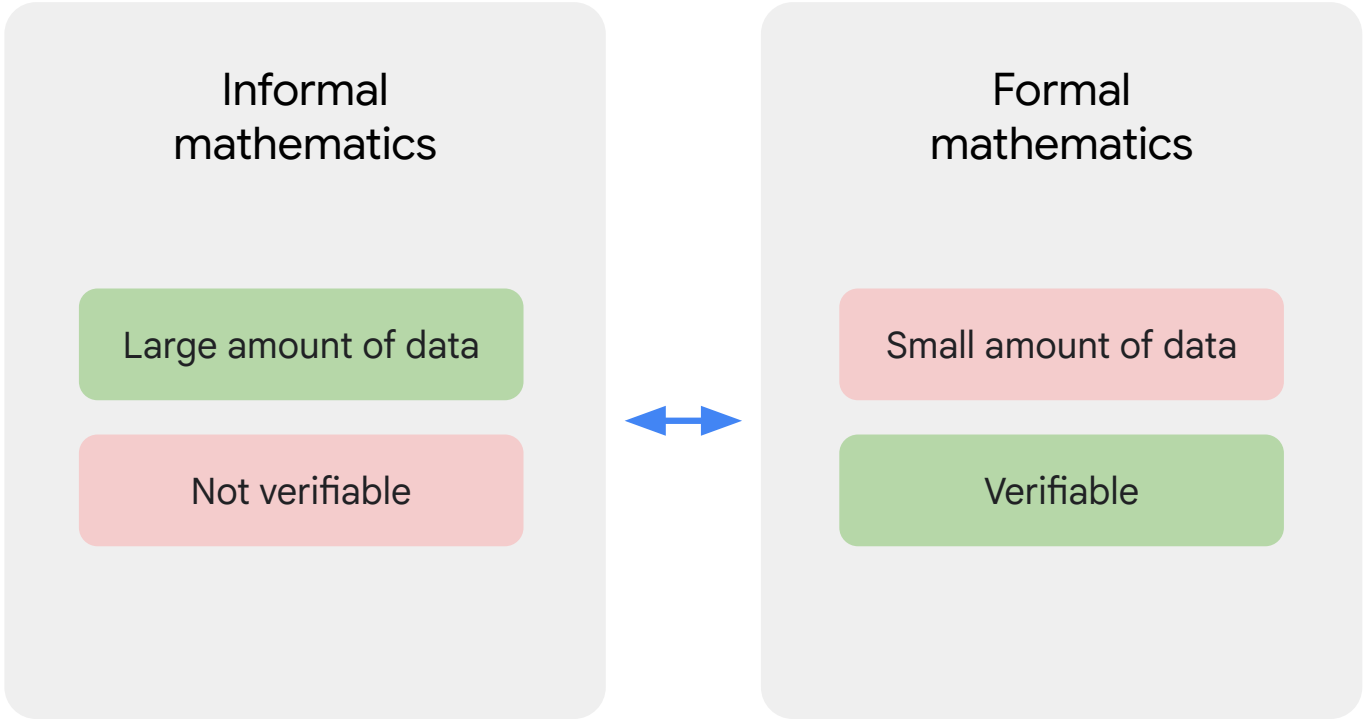
Humans define the problems

- We want to help human mathematics
- There are already a lot of problems!

AlphaProof defines the problems

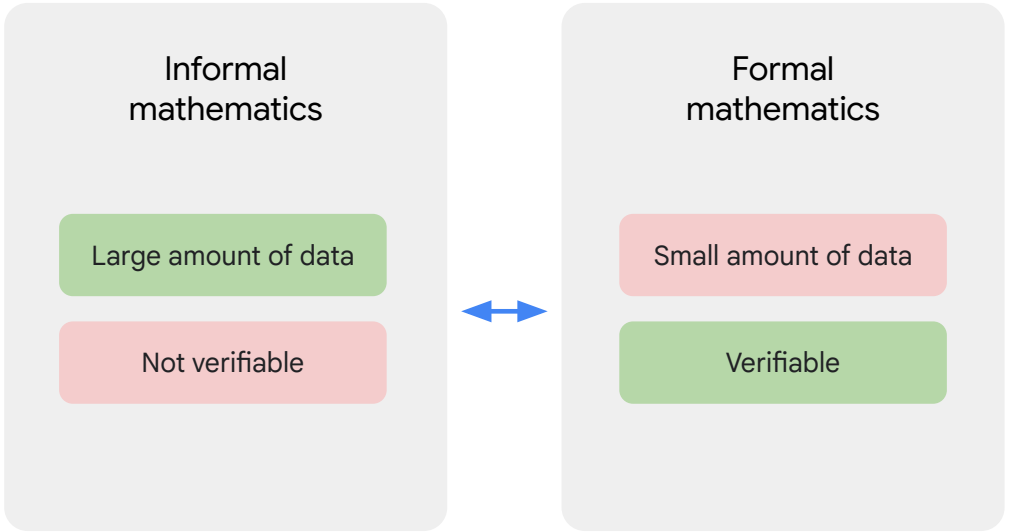
- Around human mathematics
  - Auto-formalisation
  - Test-time RL
- Augment human mathematics

Let's take a step back...



AlphaProof: Foundational Bet

Perfect verification will in the long run be the most important property for mathematics.



RL meets Formal Math

- We have a **proven recipe** to build agent that can discover knowledge by themselves and reach superhuman intelligence in certain domains
- Key ingredients are **scaled up trial and error** and **grounded feedback**
- **Formal Maths** provides us with those key ingredients

IMO 2024

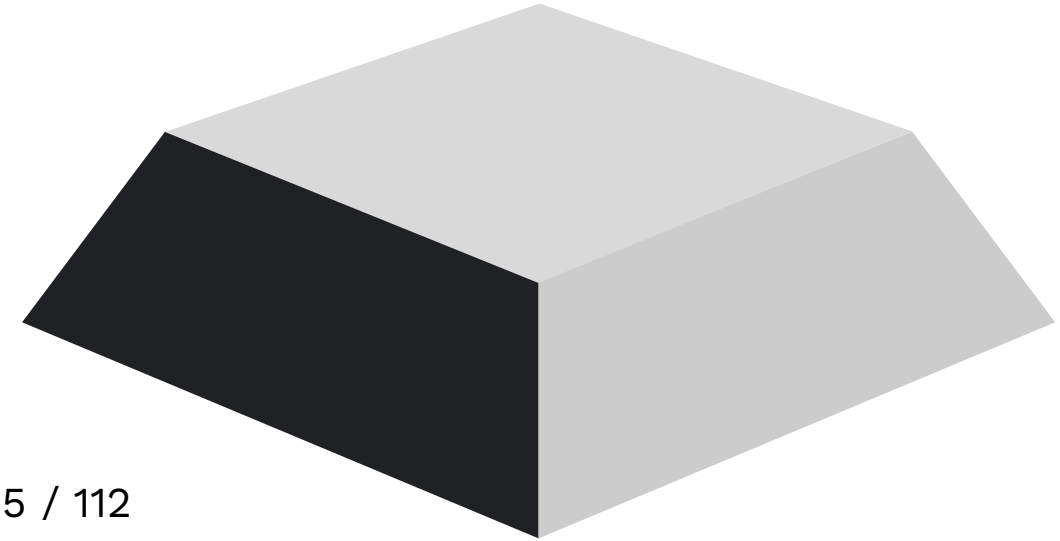
International Mathematics Olympiad

What's the IMO

The world-championship level event in mathematics that brings together the best young minds from around the world.

International Mathematics Olympiad

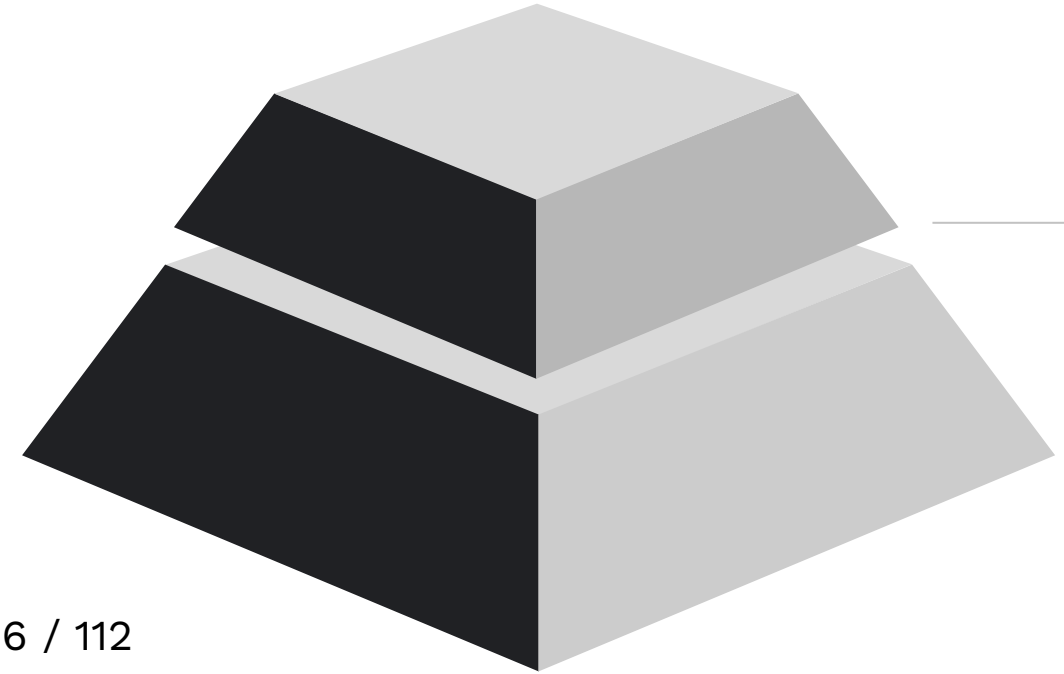
USA team selection



~17,000,000
High school students

45 / 112

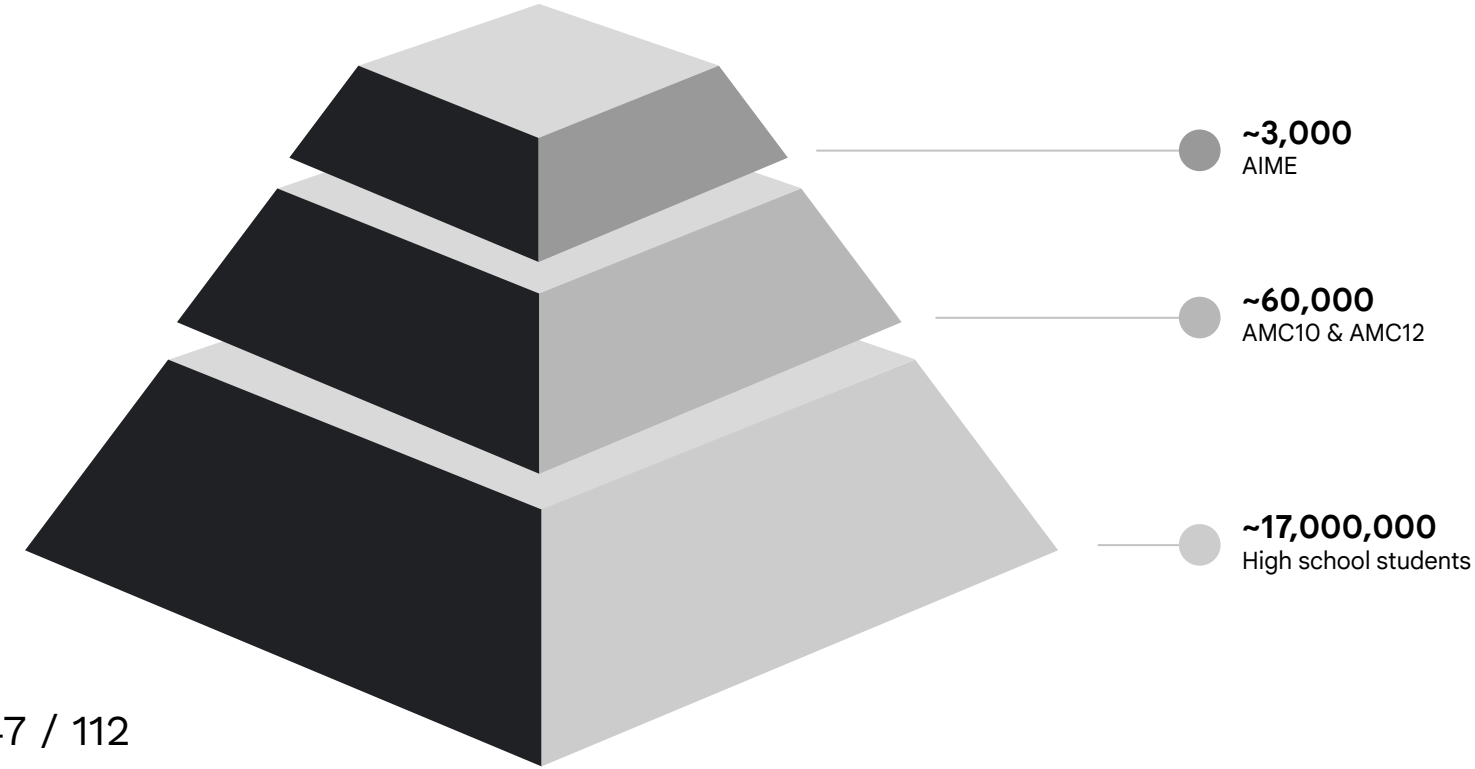
USA team selection



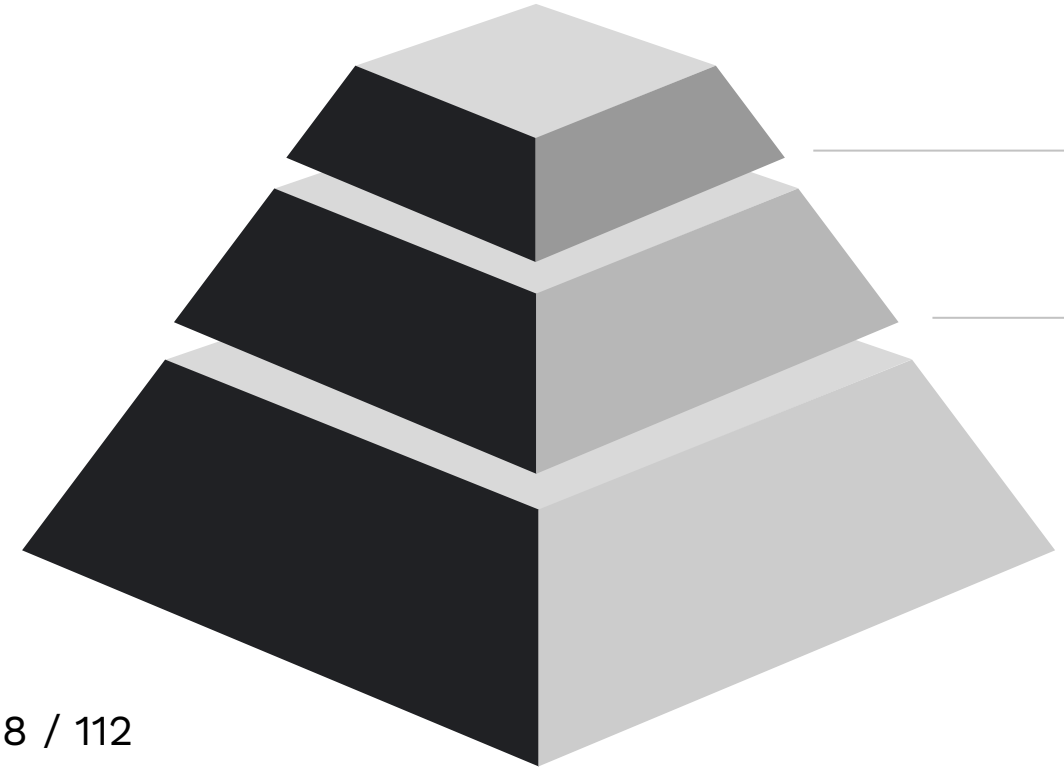
~60,000
AMC10 & AMC12

~17,000,000
High school students

USA team selection



USA team selection



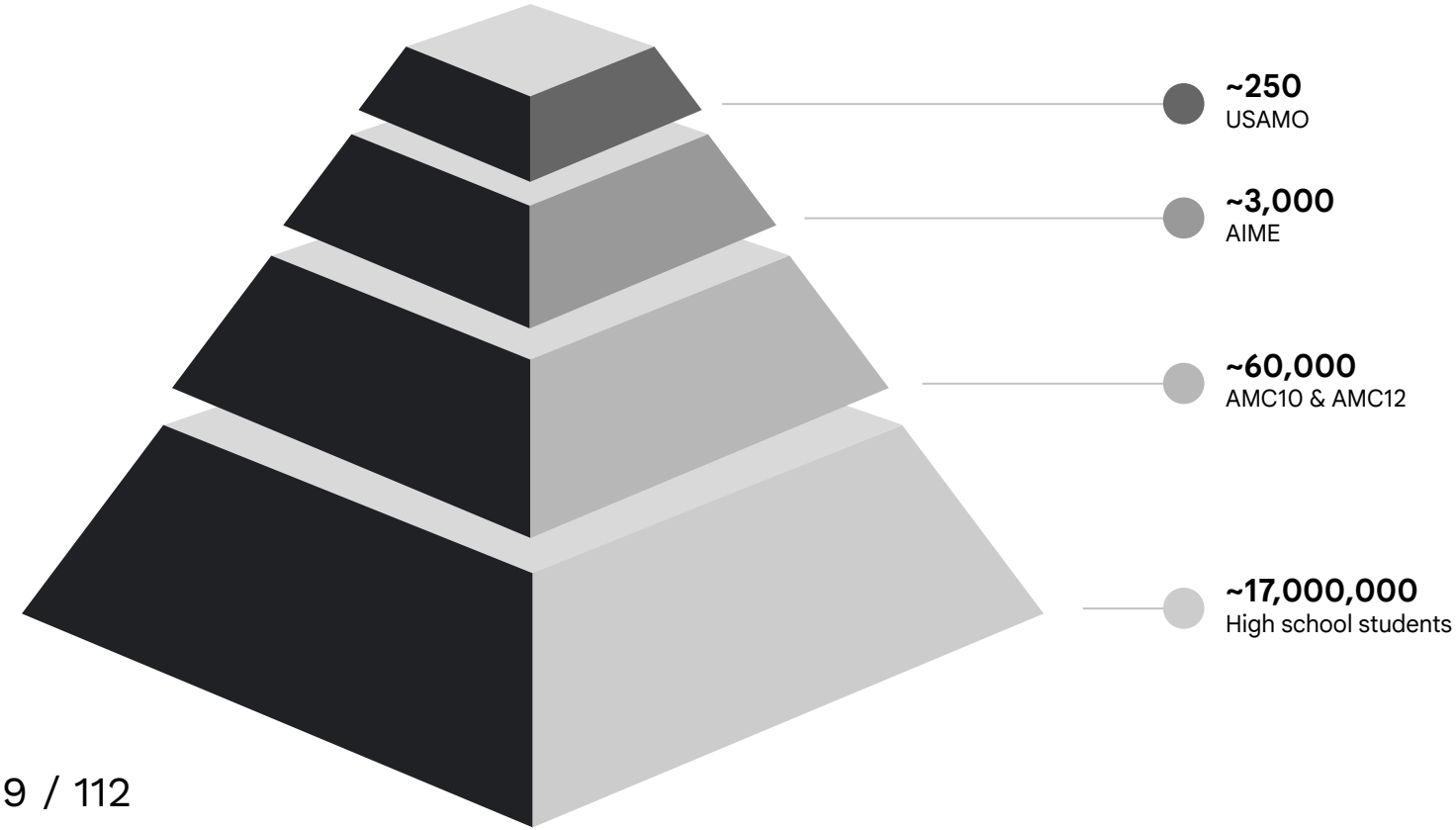
~3,000
AIME

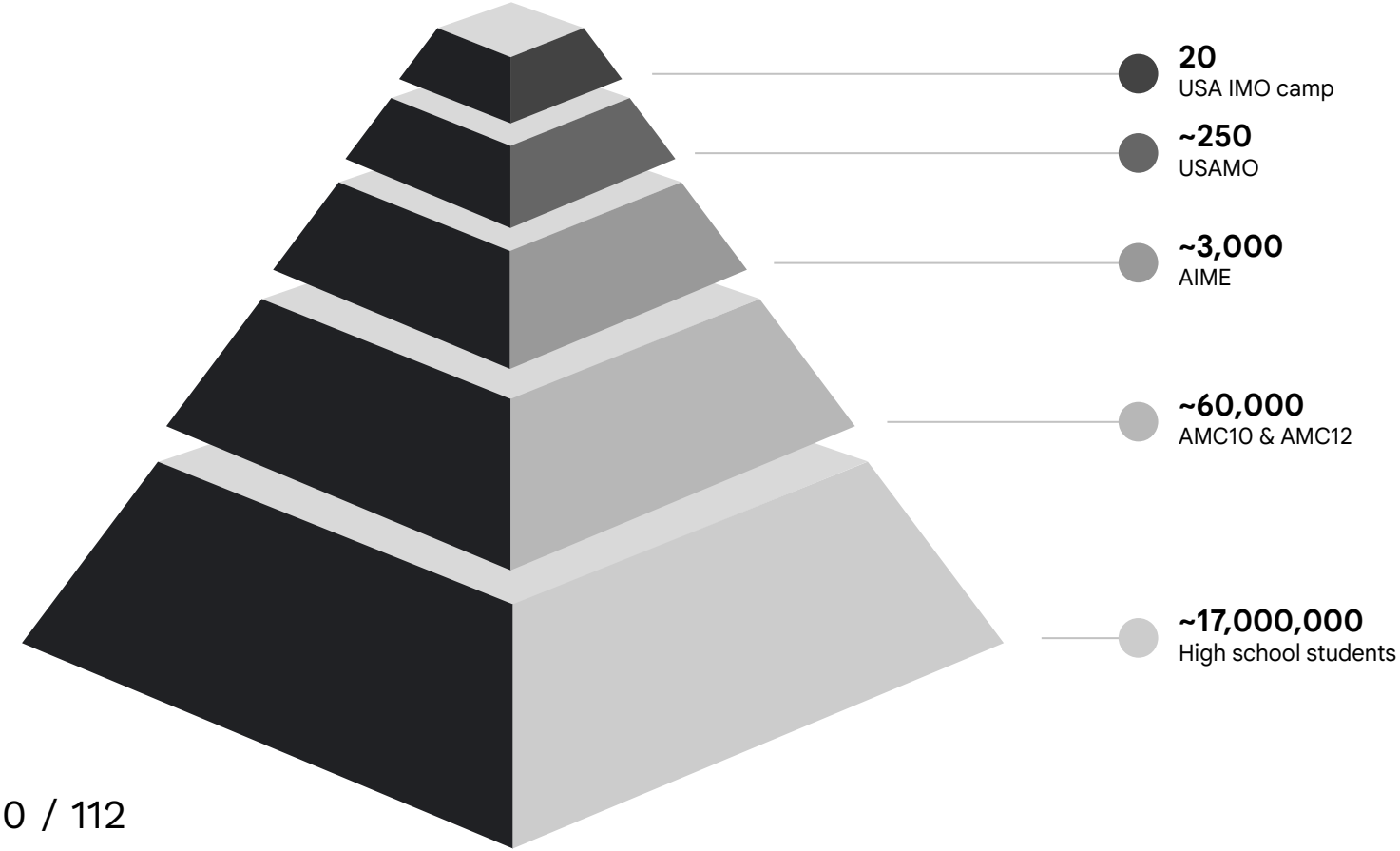
~60,000
AMC10 & AMC12

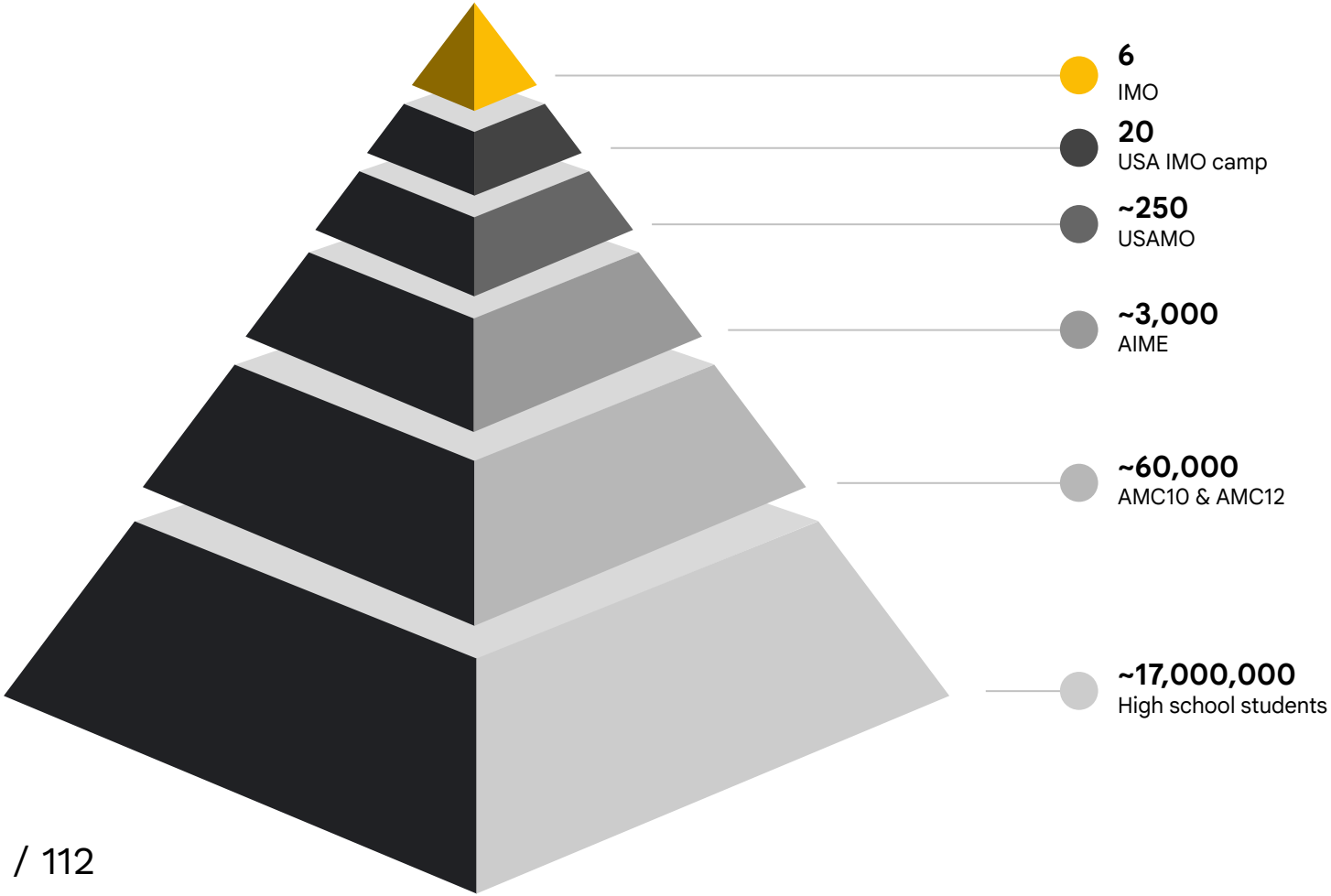
~17,000,000
High school students

MATH benchmark

USA team selection




20
USA IMO camp
~250
USAMO
~3,000
AIME
~60,000
AMC10 & AMC12
~17,000,000
High school students


6
IMO
20
USA IMO camp
~250
USAMO
~3,000
AIME
~60,000
AMC10 & AMC12
~17,000,000
High school students

The IMO Problems

These are hard problems!

- (Easy, Medium, Hard)
- Test of reasoning, not knowledge
- Take hours even for specialists
- Not leaked

Half of the participants, solve 2 or less problems.

Po-Shen Loh • 3rd+
Carnegie Mellon Math Professor • IMO Foundation VP for A...
1mo •

+ Follow

[...]

I tried this year's problems while at the International #Math Olympiad myself. Took me hours.

[...]

Context about IMO: problems are specifically selected to be non-standard. For the previous 10 years, I served as the national coach for USA (https://lnkd.in/ggXvNQHY). During IMO itself, national coaches meet to pick the problems to appear on the exam. One of most important tasks of that group is: avoid problems similar to problems that appeared anywhere before. National coaches would dig up an old obscure math contest with a similar problem, and show it to the group, after which the proposed problem would be struck down.

[...]

IMO2024

65TH INTERNATIONAL MATHEMATICAL OLYMPIAD

16-17TH JULY 2024

Our IMO Participation - Apollo program

Can we reach the moon?

Can our system solve the 2024 IMO problems at all, given

- the compute available to us.
- and enough time.

January 2024

We are 7 months away, we need to make a decision about Geometry:

- Mathlib is very sparse in 2D euclidean geometry
- Should we fill it ourselves?

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- Should we fill it ourselves?

Miraculously, another group within Google DeepMind has been working on IMO Geometry problem and the system is already very strong!

We decide to join forces for July:

- **AlphaGeometry** will tackle geometry
- **AlphaProof** will tackle algebra, number theory, combinatorics

March 2024

We have been training AlphaProof on proving theorems but not all questions are of the form *"Prove that"*. Some require an answer to be determined, e.g. *“Find all X such that ...”*

We debate what we should do:
Easy mode: Correct answer is given by an oracle; AI proves it is correct
Hard mode: AI determines the answer and proves it is correct

March 2024

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We debate what we should do:
Easy mode: Correct answer is given by an oracle; AI proves it is correct
Hard mode: AI determines the answer and proves it is correct

AlphaProof will operate in **hard mode**:

- Generates candidate answers using Gemini
- Attempts to prove/disprove all candidates

Formalised Problems in Input

We have a final debate around if the system should receive as input
- The natural language description of the problem or
- A manually formalised description of the problem

Formalised Problems in Input

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- The natural language description of the problem or
- A manually formalised description of the problem

After concertation with our judges, we decided to give the **formalised problems as input** to the system.
- The ability we cared about was mathematical reasoning and problem solving.
- We asked our lean experts to manually formalise the problems for the system.

Our Protocol

After officially receiving the problems at 1PM,

1. Lean experts manually formalize problems
2. Generate O(100) answer candidates with Gemini
3. Filter the easily disprovable ones
4. Run test-time RL

16th July, Day 1, Tuesday

1PM

P1*: Algebra
P2*: Number Theory
P3: Combinatorics

* require an answer to be determined

Problem 1. Determine all real numbers α such that, for every positive integer n, the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n. (Note that $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. For example, $\lfloor -\pi \rfloor = -4$ and $\lfloor 2 \rfloor = \lfloor 2.9 \rfloor = 2$.)

Problem 2. Determine all pairs (a, b) of positive integers for which there exist positive integers g and N such that

$$\gcd(a^n + b,\, b^n + a) = g$$

holds for all integers $n \geqslant N$. (Note that $\gcd(x, y)$ denotes the greatest common divisor of integers x and y.)

Problem 3. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers, and let N be a positive integer. Suppose that, for each $n > N$, a_n is equal to the number of times a_{n-1} appears in the list $a_1, a_2, \ldots, a_{n-1}$.

Prove that at least one of the sequences $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ is eventually periodic.

(An infinite sequence $b_1, b_2, b_3, \ldots$ is *eventually periodic* if there exist positive integers p and M such that $b_{m+p} = b_m$ for all $m \geqslant M$.)

16th July, Day 1, Tuesday

1PM

P1*: Algebra → Disproves 99% of guesses
P2*: Number Theory → Disproves 98% of guesses
P3: Combinatorics

* require an answer to be determined

17th July, Day 2, Wednesday

1PM

P4: Geometry
P5*: Combinatorics
P6*: Algebra

* require an answer to be determined

Problem 4. Let ABC be a triangle with $AB < AC < BC$. Let the incentre and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively.

Prove that $\angle KIL + \angle YPX = 180°$.

Problem 5. Turbo the snail plays a game on a board with 2024 rows and 2023 columns. There are hidden monsters in 2022 of the cells. Initially, Turbo does not know where any of the monsters are, but he knows that there is exactly one monster in each row except the first row and the last row, and that each column contains at most one monster.

Turbo makes a series of attempts to go from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then repeatedly moves to an adjacent cell sharing a common side. (He is allowed to return to a previously visited cell.) If he reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move, and Turbo remembers whether or not each cell he has visited contains a monster. If he reaches any cell in the last row, his attempt ends and the game is over.

Determine the minimum value of n for which Turbo has a strategy that guarantees reaching the last row on the n^{th} attempt or earlier, regardless of the locations of the monsters.

Problem 6. Let $\mathbb{Q}$ be the set of rational numbers. A function $f\colon \mathbb{Q} \to \mathbb{Q}$ is called *aquaesulian* if the following property holds: for every $x, y \in \mathbb{Q}$,

$$f(x + f(y)) = f(x) + y \qquad \text{or} \qquad f(f(x) + y) = x + f(y).$$

Show that there exists an integer c such that for any aquaesulian function f there are at most c different rational numbers of the form $f(r) + f(-r)$ for some rational number r, and find the smallest possible value of c.

17th July, Day 2, Wednesday

1PM

P4: Geometry → Solved in secs by AlphaGeometry

P5*: Combinatorics → Failed to formalise on the day

P6*: Algebra → Disproves 7% of guesses

* require an answer to be determined

18th July, Day 3, Thursday

We had a way to track progress and we observe:

- P1: **AlphaProof** solved half of the problem **3/7 points**
- P2: **AlphaProof** arguably solved the whole problem **6/7 points**
- P3: **AlphaProof** makes no progress

Day1: **9/21** points

18th July, Day 3, Thursday

We had a way to track progress and we observe:

- P1: **AlphaProof** solved half of the problem **3/7 points**
- P2: **AlphaProof** arguably solved the whole problem **6/7 points**
- P3: **AlphaProof** makes no progress

- P4: **AlphaGeometry** solved in seconds **7/7 points**
- P5: **AlphaProof** makes no progress
- P6: **AlphaProof** proves a case **2/7 points**

Day1: **9/21** points, Day2: **9/21** points, Total **18/42** points

19th July, Day 4, Friday

mehtarishi+alphaproof-agent@google.com

I have discovered a truly marvelous proof of imo_2024_p6.parts.ii.only_1588200855951509015 that this email is too short to contain.

19th July, Day 4, Friday

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Jk, here it is:

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theorem imo_2024_p6.parts.ii.only_1588200855951509015 (IsAquaesulian : (ℚ → ℚ) → Prop) (IsAquaesulian_def : ∀ f, IsAquaesulian f ↔ ∀ x y, f (x + f
y) = f x + y ∨ f (f x + y) = x + f y) : IsLeast {(c : ℤ) | ∀ f, IsAquaesulian f → {(f r + f (-r)) | (r : ℚ)}.Finite ∧ {(f r + f (-r)) | (r :
ℚ)}.ncard ≤ c} 2 := by
  exists@?_
  · useλu b=>if j:u 0=0then by_contra λc=>?_ else ?_
    · suffices:({J|∃k,u k+u (-k)= J}) ⊆{0}
      · simp_all[this.antisymm]
      rintro - ⟨a, rfl⟩
      contrapose! c
      simp_all
      suffices:{U|∃examples6, (u) ‹ℚ› +u ( -‹_›)= U} ⊆{0,(u (a : Rat)+ (u<|@@↑(( (-a ))))) } ..
      · use ( Set.toFinite ( _) ).subset ↑@@this , (Set.ncard_le_ncard$ (((this )) ) ).trans (Set.ncard_pair$ Ne.symm (↑ ( (c)) ) ).le
      rintro-⟨hz, rfl⟩
      induction b @hz a
      · have:=b (-a)$ hz+u a
        have:=b hz hz
        simp_all[add_comm]
        have:=b (-hz) (hz+u ↑(hz))
        simp_all[ add_assoc, C]
        induction this
        · simp_all
          have:=b hz (hz+(u a+u (-a)))
          have:=b (hz+(u a+u (-a)))$ hz+(u a+u (-a))
          use .inr$ by_contra$ by hint
        have:=b hz$ hz+(u hz+u (-hz))
        cases b (hz+(u hz+u (-hz)))$ hz+(u hz+u (-hz))with|_=>hint
      have:=b (-hz) (u hz+a)
      have:=b$ -a
      specialize this (u hz+a)
      simp_all[ ←add_assoc]
      have:=b 0
      have:=b
      specialize b a a
      simp_all[add_comm]
      have:=(this<| -a) (↑a + (((u a))): (↑_ :((( _) ) ) )) ..
      simp_all[add_assoc]
```

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T

Thomas Hubert <tkhubert@google.com>
to Demis, Sergey

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```
theorem imo_2024_p2_7949354807137552108 : {(a, b) | 0 < a ∧ 0 < b ∧ ∃ g N, 0 < g ∧ 0 < N ∧ ∀ n ≥ N, Nat.gcd (a ^ n + b) (b ^ n + a) = g} = {(a, b)
| Nat.gcd a b = 1 ∧ a = b} := by
  use subset antisymm (λt S=>S.2.2.rec λw o=>? ) ?   ..
```

19th July, Day 4, Friday

mehtarishi+alphaproof-agent@google.com

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```

mehtarishi+alphaproof-agent@google.com

I have discovered a truly marvelous proof of imo_2024_p1_1897952008761024785 that this email is too short to contain.
Jk, here it is:

```
theorem imo_2024_p1_1897952008761024785 : {(α : ℝ) | ∀ (n : ℕ), 0 < n → (n : ℤ) ∣ (∑ i in Finset.Icc 1 n, ⌊i * α⌋)} = {2 * k | k ∈ Set.range
(Int.cast : ℤ → ℝ)} := by
```

Final Results

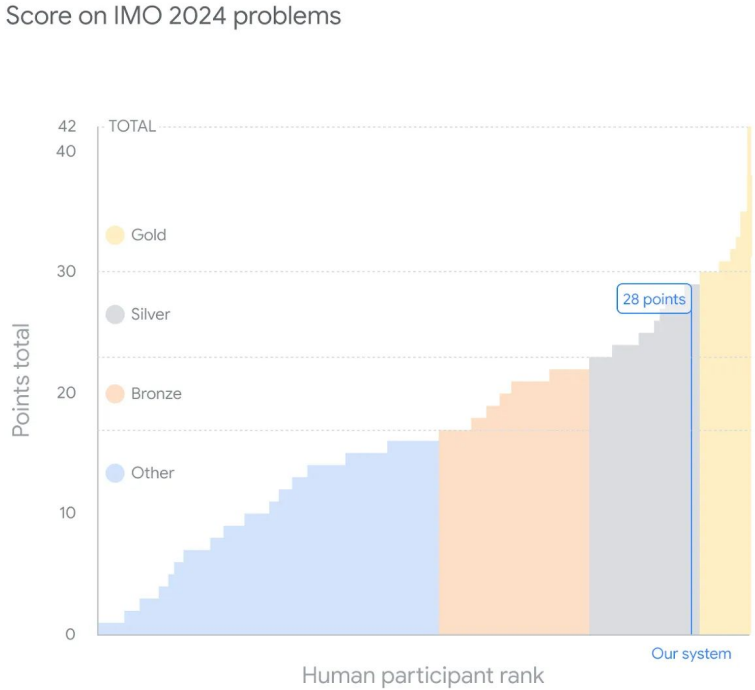
P1, P2, P6 fully solved by AlphaProof
P4 fully solved by AlphaGeometry

We keep running over the weekend in the hope of getting one point on P3. The agent made some progress but not enough for a partial point.

Final Results

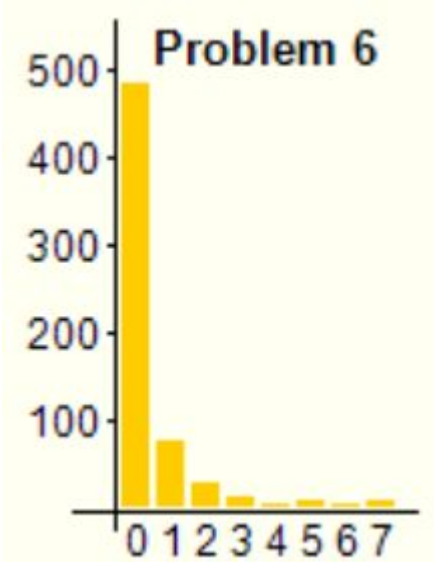
P1, P2, P6 fully solved by AlphaProof
P4 fully solved by AlphaGeometry

We reached the score of a Silver medallist and missed the Gold threshold by one point (with more time and more compute!).



P6

P6 was arguably one of the hardest problems in the last 10 years at the IMO. Only 5 / 609 solved the problem fully.



*"**I spent a couple of hours on this problem and did not solve it**. […] I find the fact that the program can come up with a somewhat complicated construction like this very impressive"*
Prof Sir Timothy Gowers, Fields medalist and IMO gold medalist

Challenges

Gaps in Mathlib:

- Geometry, even P1

Combinatorics problems were difficult:

- P5 was extremely hard to formalise
- Did not make progress on P3 & P5.

Used many orders of magnitude more compute than human contestants:

- Successfully landed on the moon
- And will want to be more efficient for the next trip!

AlphaProof

Methods

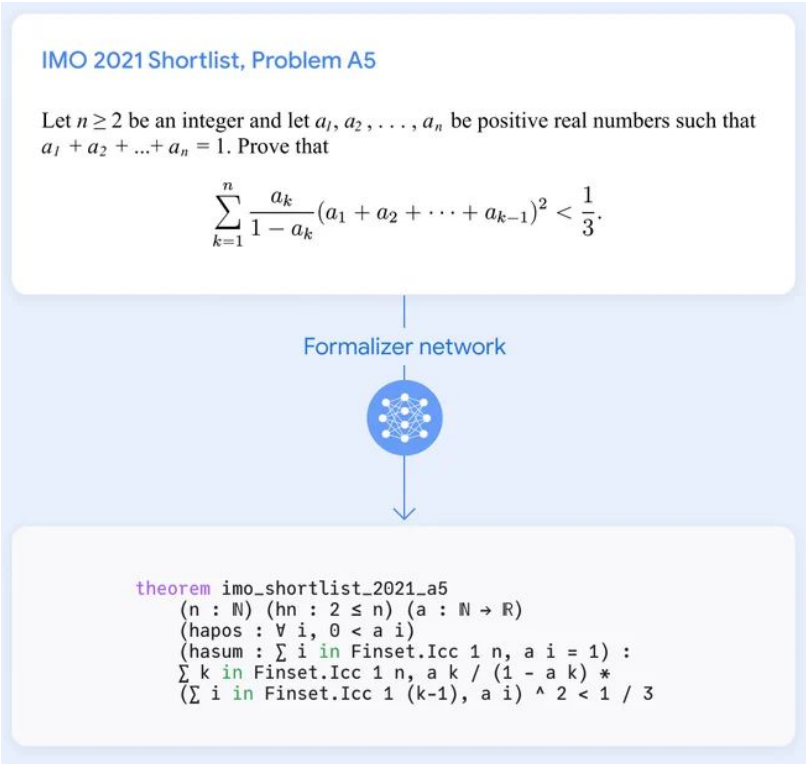
AlphaZero is the agent.

+

Lean is the environment.

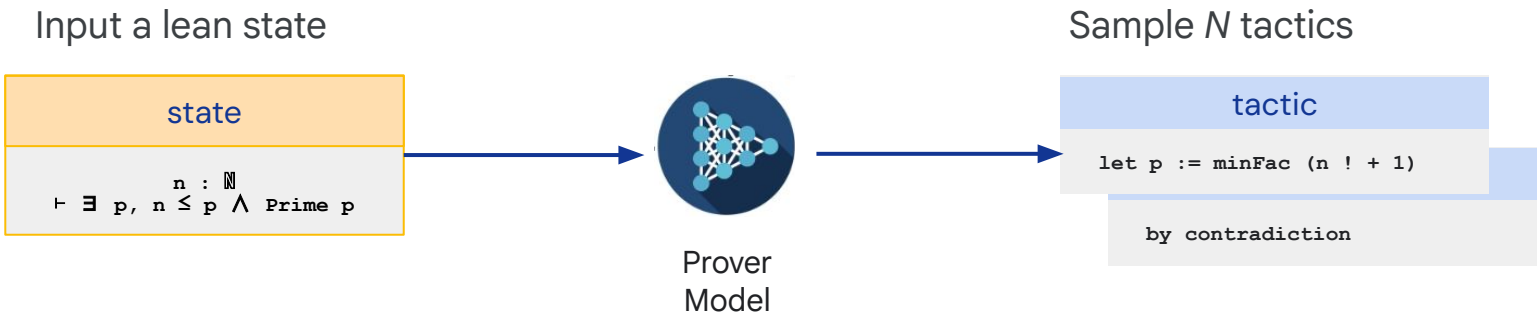
Formaliser Model

Input: a problem/theorem described in natural language



Output: a Lean formalisation

Prover Model

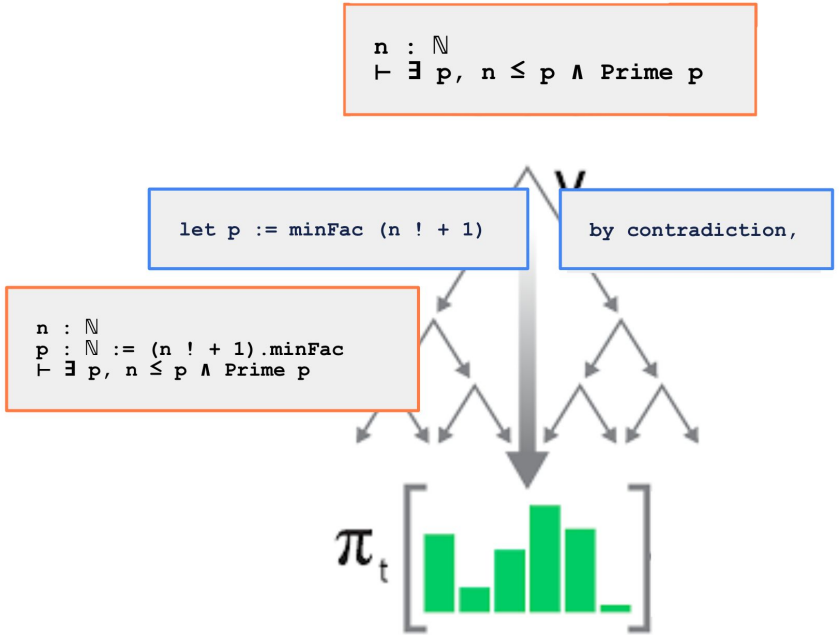


Prover Model + AlphaZero Search

Search over actions = Lean tactic

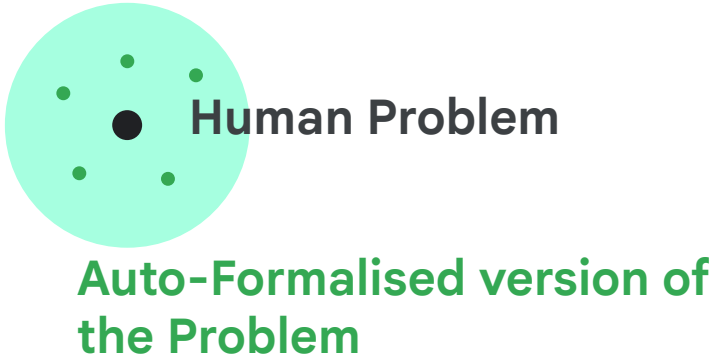
Compute new Lean state after every action / tactic application

Exploit high prior and high value paths
Explore low visited paths



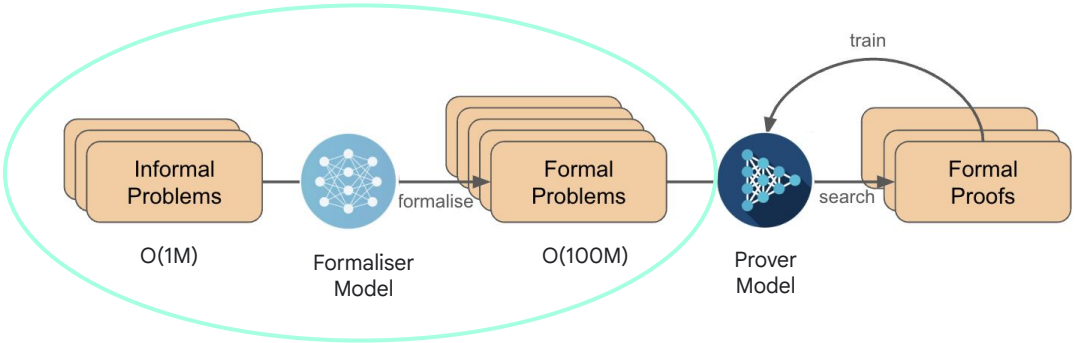
Step 1: Auto formalisation

1: Train a formalisation model and auto-formalise human created problems



Step 1: Auto formalisation

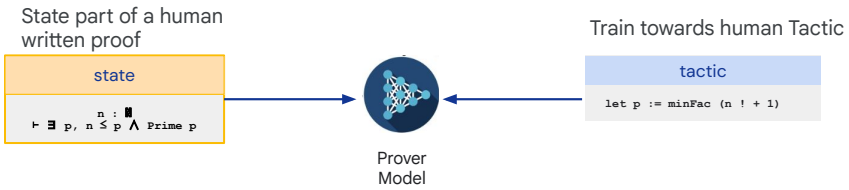
1: Train a statement formalisation model and auto-formalise human created problems



Step 2: Build on top of Mathlib

2: Train the prover model supervised on Mathlib

- 100k definitions
- 200k theorems
- 300k lines of proofs

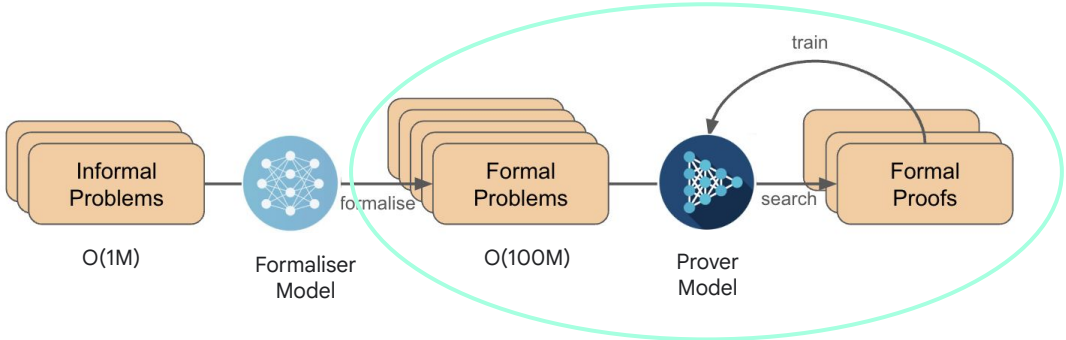


Learn a good prior of actions to take.

Step 3: AlphaZero Reinforcement Learning

3: Train the prover model by RL

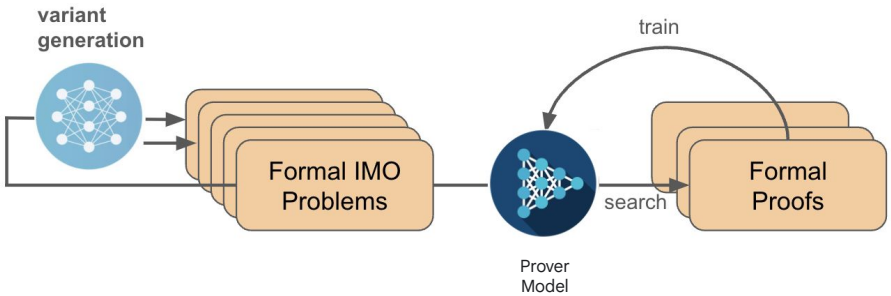
For each formal problem
- Generate experience of (dis)proving by searching over Lean steps.
- Use Lean to verify proofs
- Reinforce the prover network with each success



Final Step: Test-Time RL

4: Train the prover model on specific problems by RL

For each problem:
- Generate variants of the problem
- Run RL exactly as previous step



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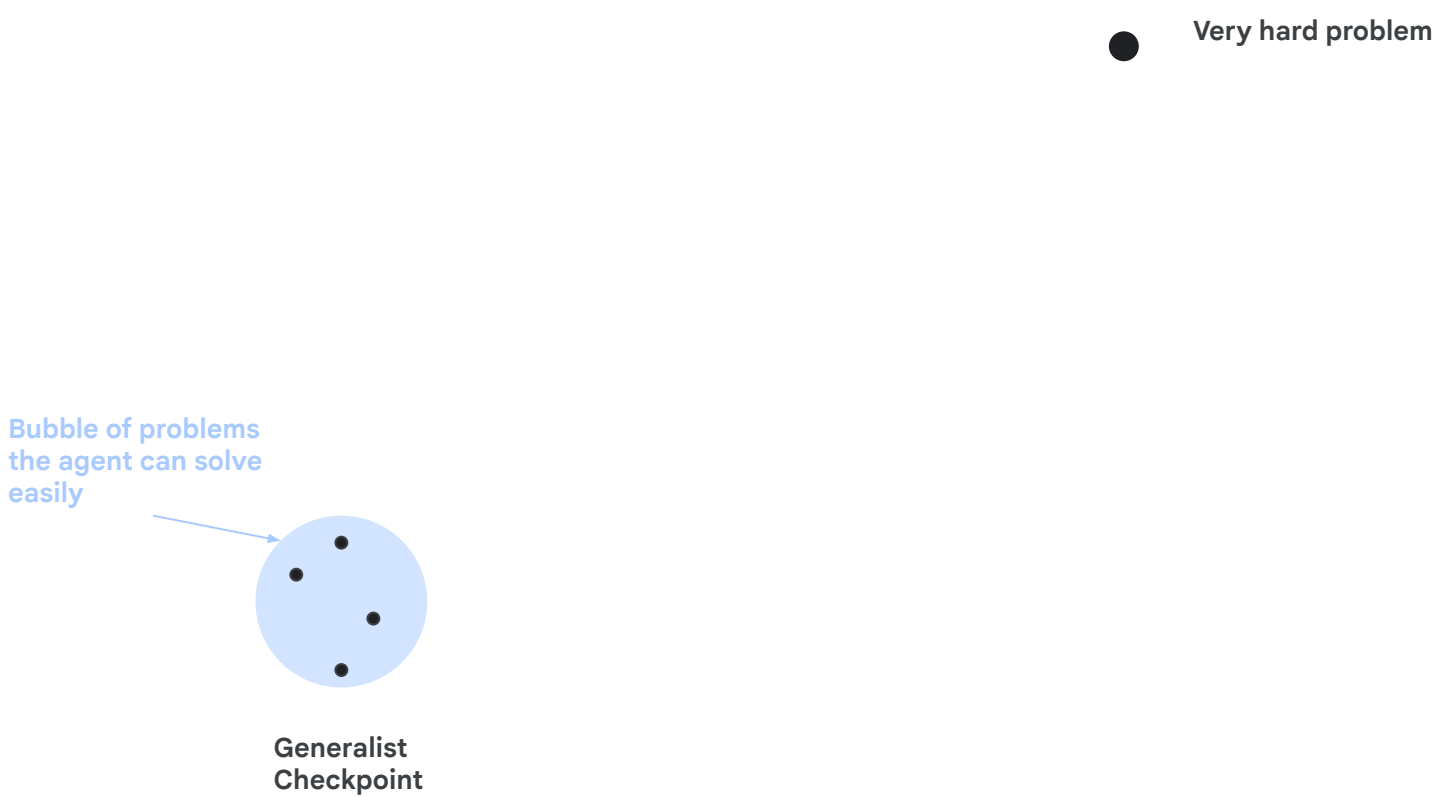
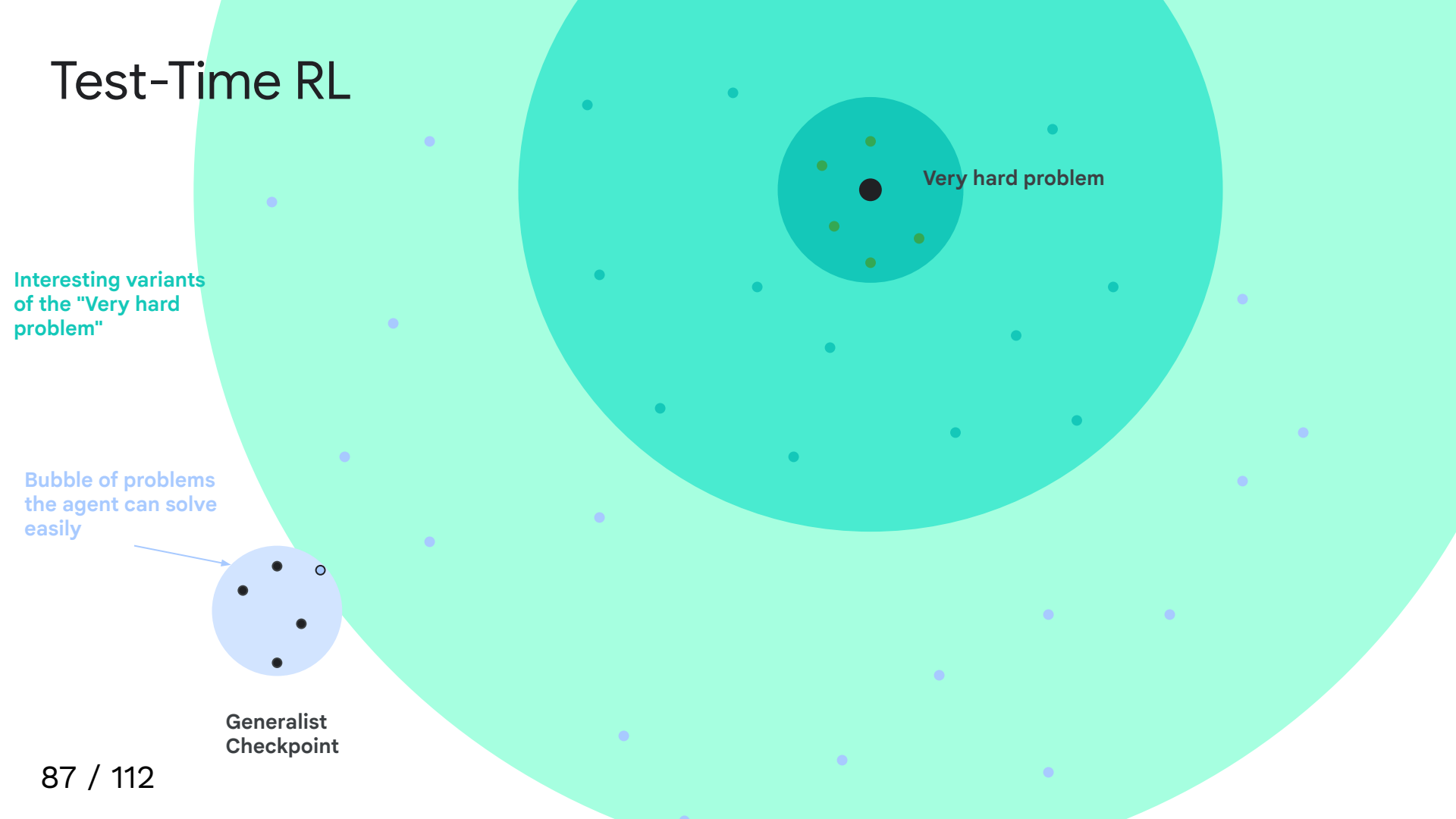
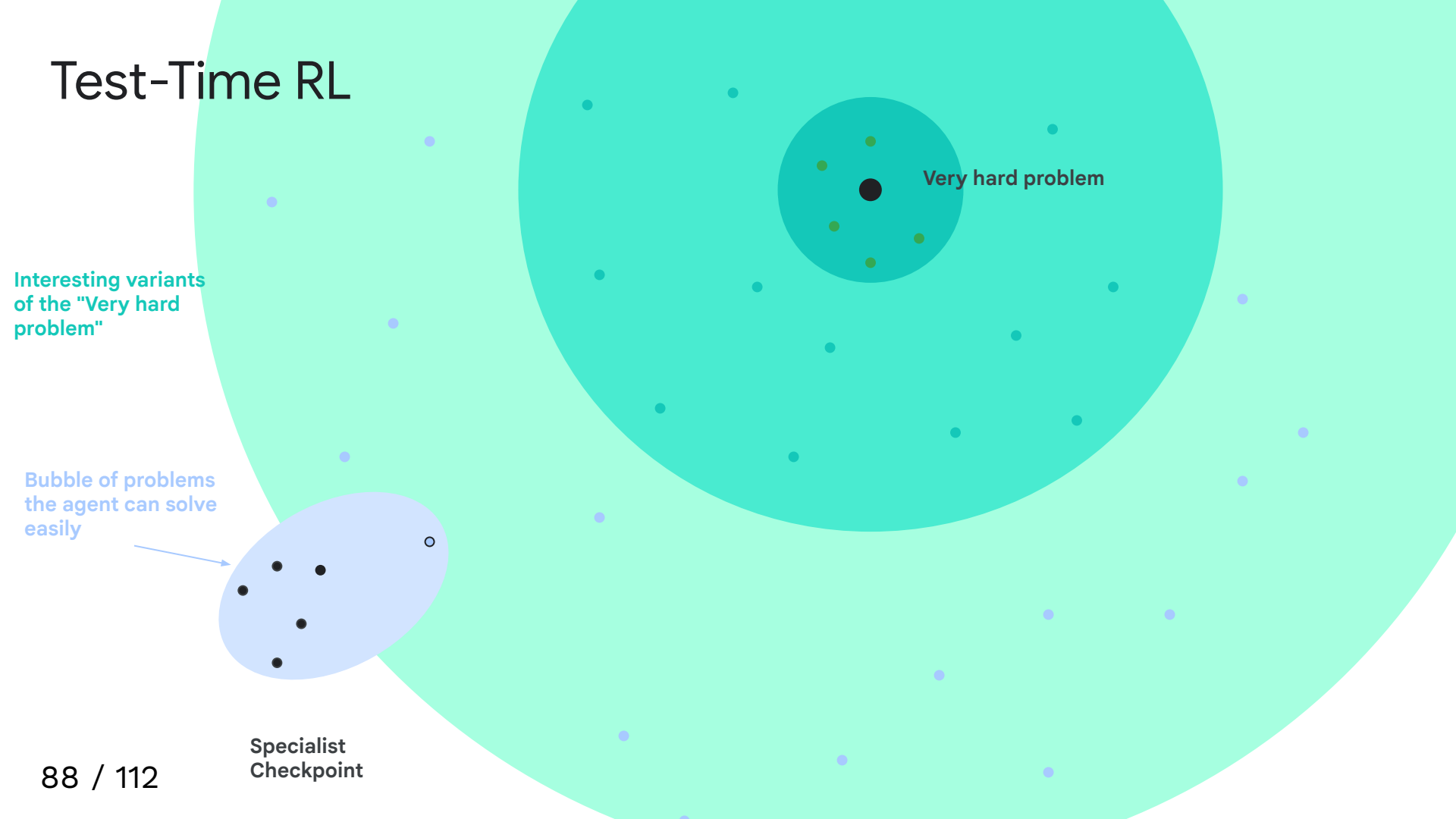


Test-Time RL


Very hard problem
Interesting variants of the "Very hard problem"
Bubble of problems the agent can solve easily
Generalist Checkpoint

Test-Time RL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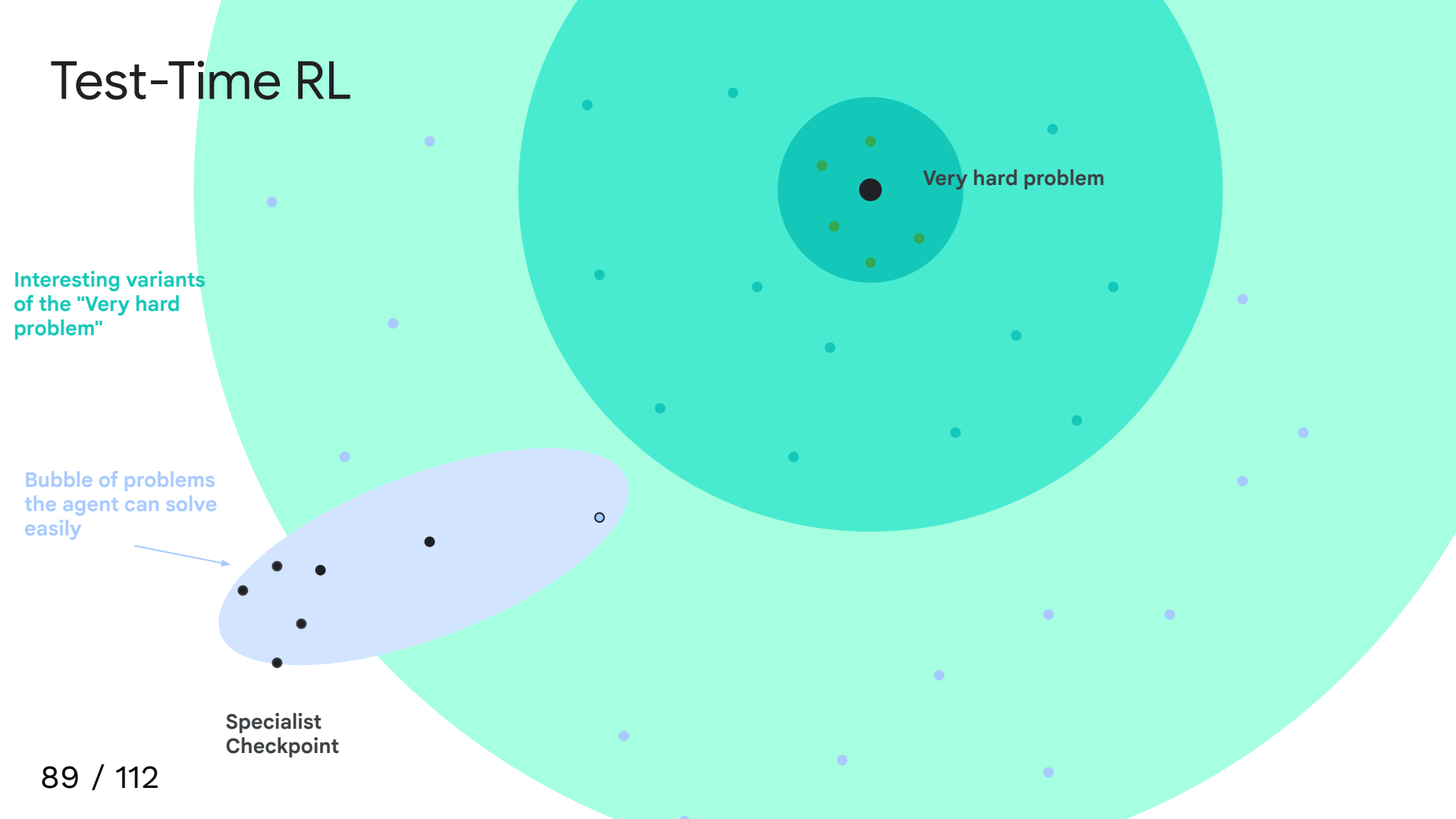
Specialist Checkpoint

Test-Time RL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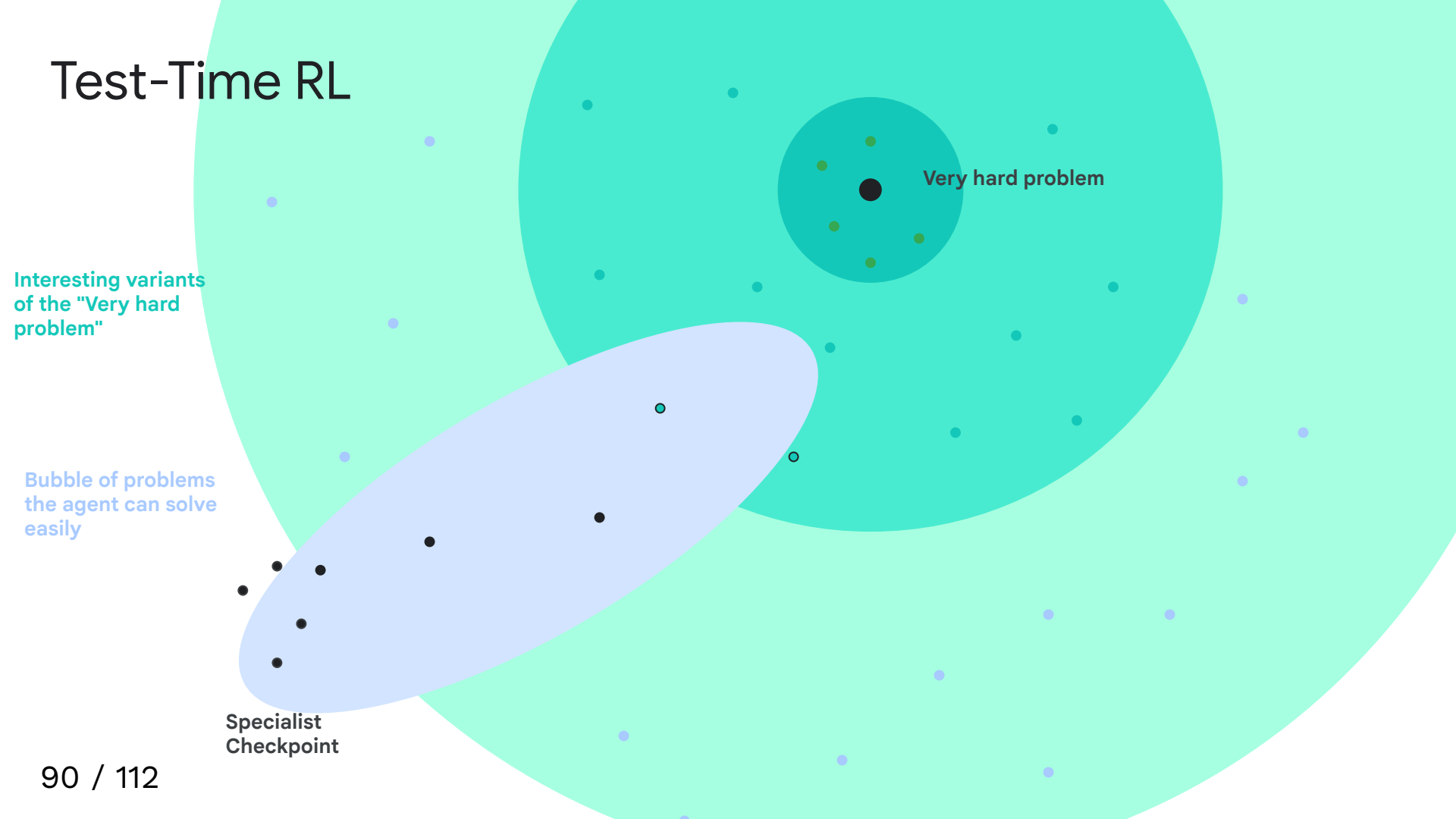
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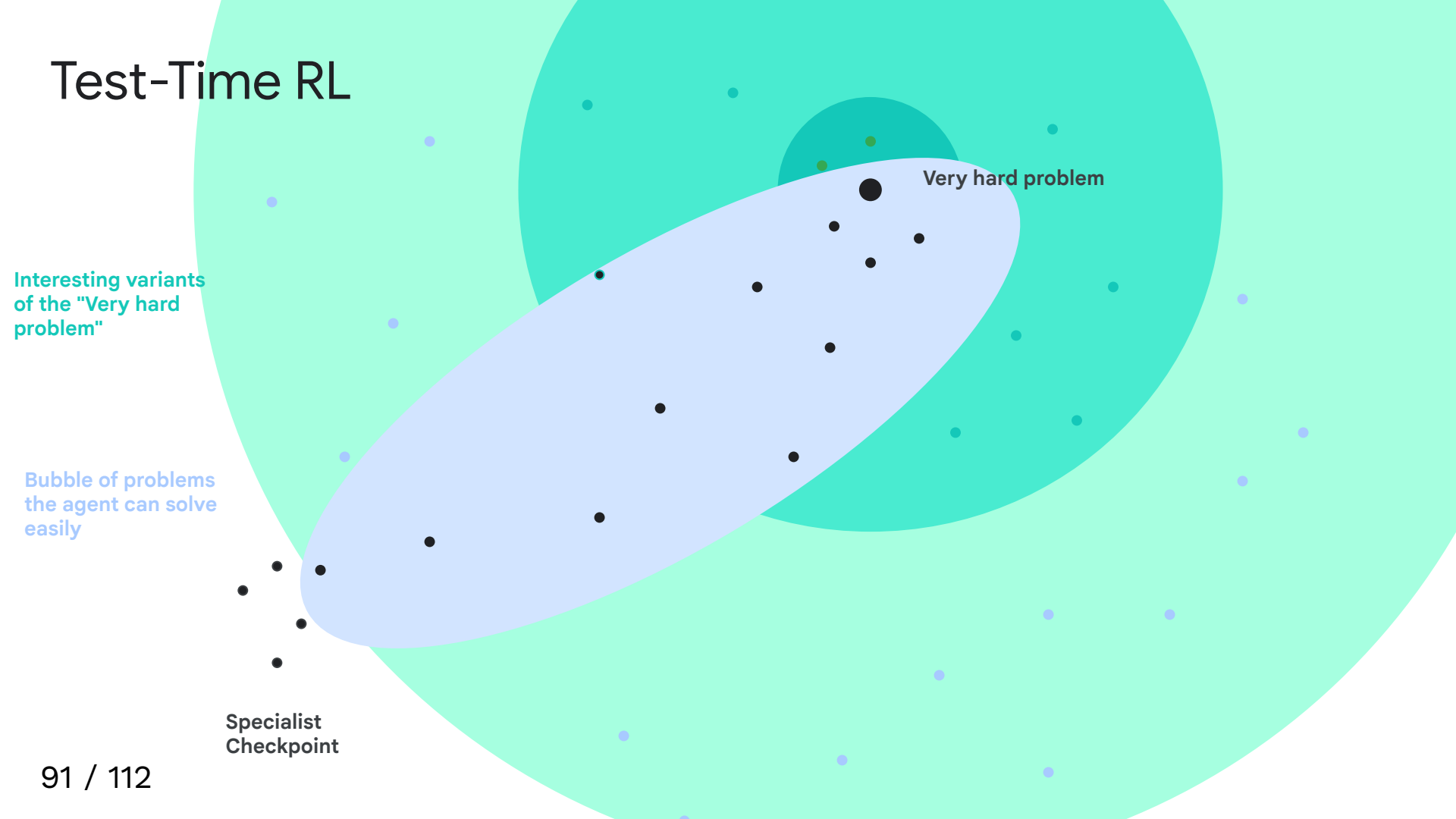
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Specialist Checkpoint

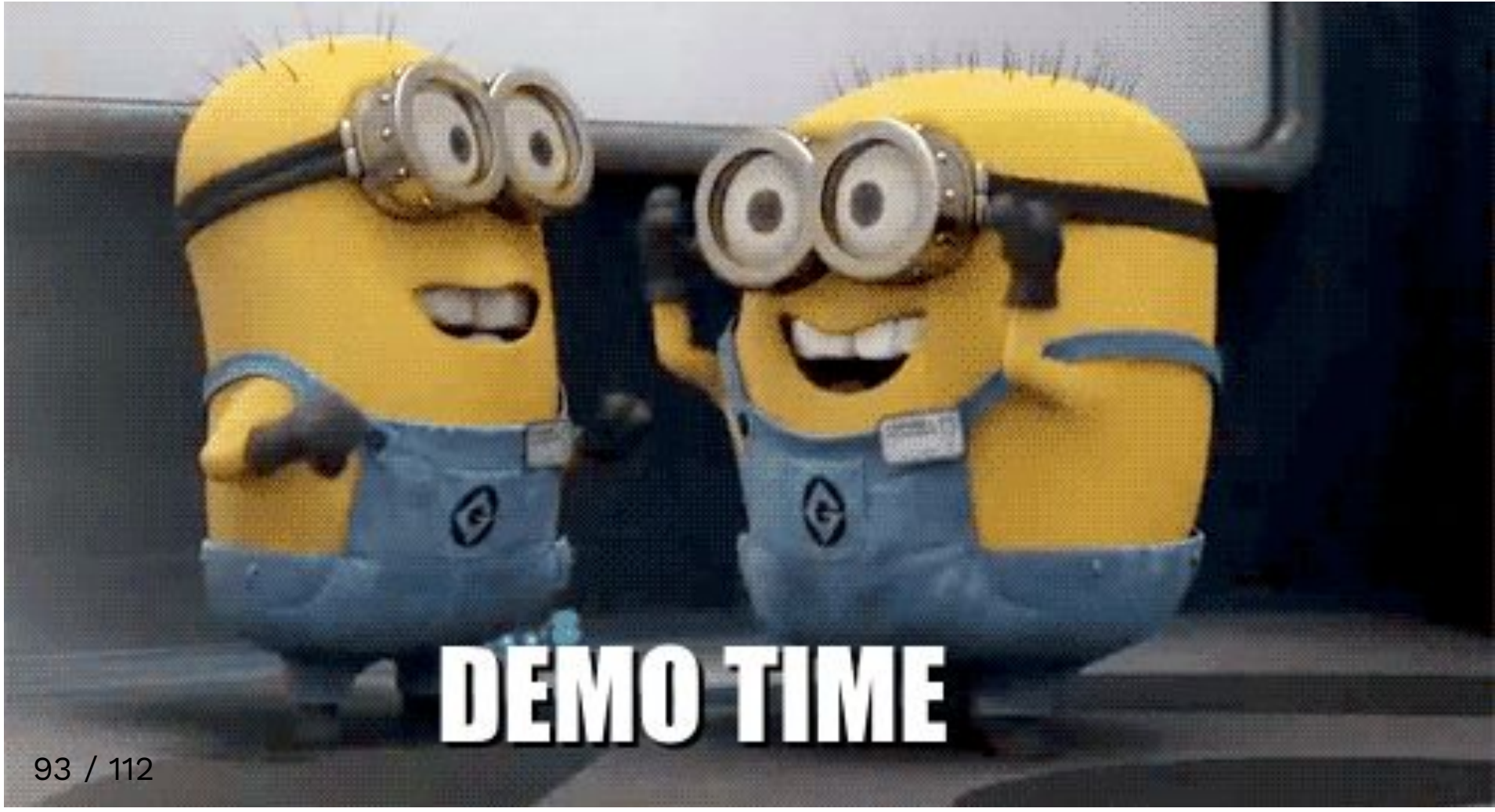
Challenges for AlphaProof

Inherited challenges from Formal Mathematics

- Most of human data in natural language
- Cannot easily learn and work on areas not supported by Mathlib.

Generally,

- Creative building of new objects and theories, interestingness and beauty in mathematics


DEMO TIME

Warm up: back to the infinitude of primes

AIME 2020 ii p10

Problem

Find the sum of all positive integers n such that when $1^3 + 2^3 + 3^3 + \cdots + n^3$ is divided by $n + 5$, the remainder is 17.

Solution 1

The formula for the sum of cubes, also known as Nicomachus's Theorem, is as follows:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2 = \left(\frac{k(k+1)}{2}\right)^2$$

for any positive integer k.

So let's apply this to this problem.

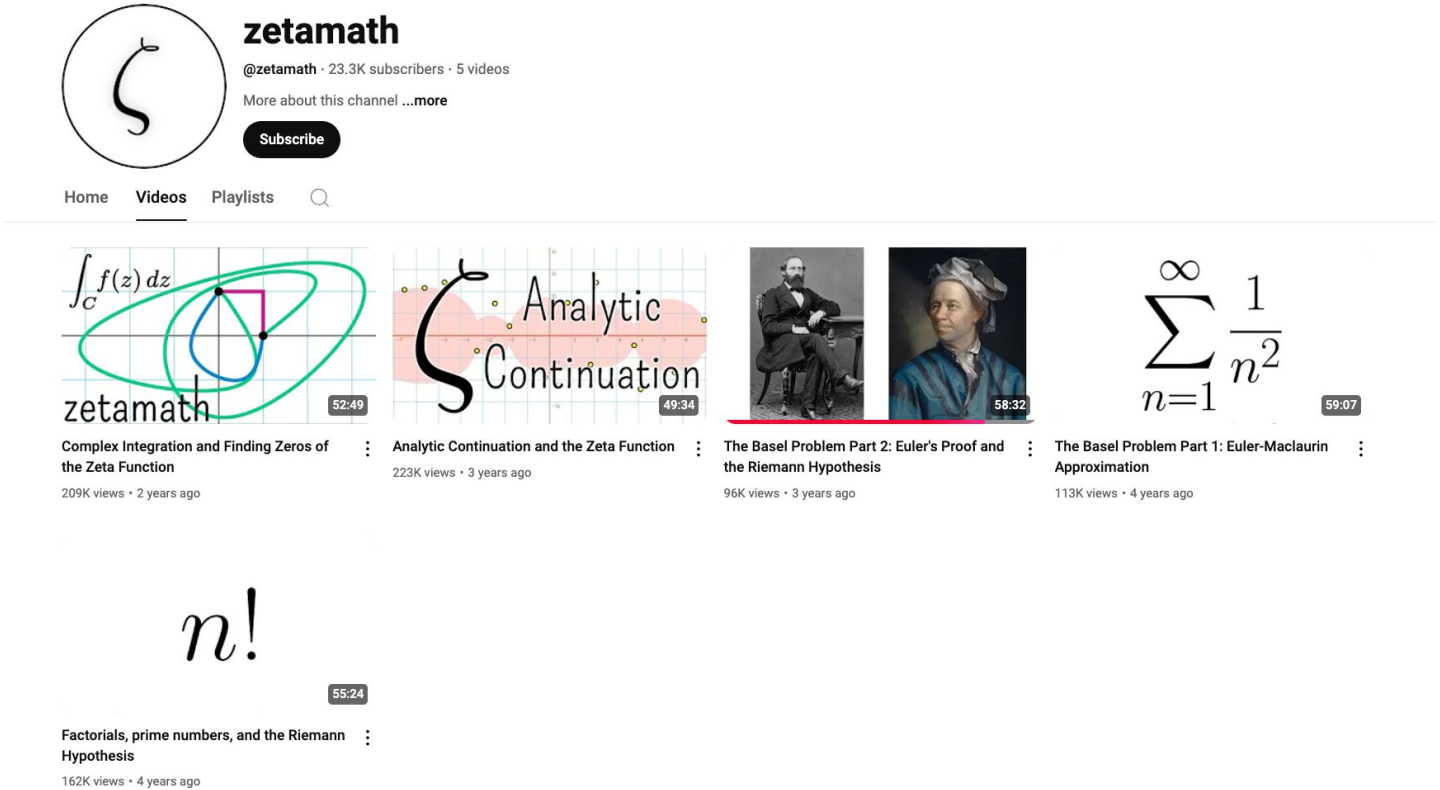
Let $m = n + 5$. Then we have

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + (m-5)^3 &\equiv 17 \mod m \\ \left(\frac{(m-5)(m-4)}{2}\right)^2 &\equiv 17 \mod m \\ \left(\frac{m(m-9)+20}{2}\right)^2 &\equiv 17 \mod m \\ (m(m-9)+20)^2 &\equiv 4 \cdot 17 \mod m \\ (20)^2 &\equiv 68 \mod m \\ 332 &\equiv 0 \mod m \end{aligned}$$

So, $m \in \{83, 166, 332\}$. Testing the cases, only 332 fails. This leaves $78 + 161 = \boxed{239}$.

Warm up: back to the infinitude of primes

Demo: Link between the Riemann Zeta and the primes



Link between the Riemann Zeta

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{5} + \frac{1}{2 \cdot 3} + \frac{1}{7} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{2 \cdot 5} + \frac{1}{11} + \frac{1}{2^2 \cdot 3} + \frac{1}{13} + \frac{1}{2 \cdot 7} + \frac{1}{3 \cdot 5} + \cdots$$

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots\right)$$
$$\cdot\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots\right)$$
$$\cdot\left(1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\dots\right)$$
$$\cdot\ \dots$$

$$\prod_{p} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \dots\right) = \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\prod_{p} \frac{1}{1 - \frac{1}{p}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right)$$

Let's prove the finite case

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots\right)$$
$$\cdot\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots\right)$$
$$\cdot\left(1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\dots\right)$$

Final Link with the root of the zeta function

$$\zeta(s) = \frac{e^{a+bs}}{s-1} \prod_{\zeta(\alpha)=0} \left(1 - \frac{s}{\alpha}\right) e^{s/\alpha}$$

Final Link with the root of the zeta function

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}}$$

$$\zeta(s) = \frac{e^{a+bs}}{s-1} \prod_{\zeta(\alpha)=0} \left(1 - \frac{s}{\alpha}\right) e^{s/\alpha}$$

Final Link with the root of the zeta function

$$\sum_{p^k \leq x} \log(p) = x - \log(2\pi) - \sum_{\zeta(\alpha)=0} \frac{x^\alpha}{\alpha}$$

What’s next

What's Next for AlphaProof?

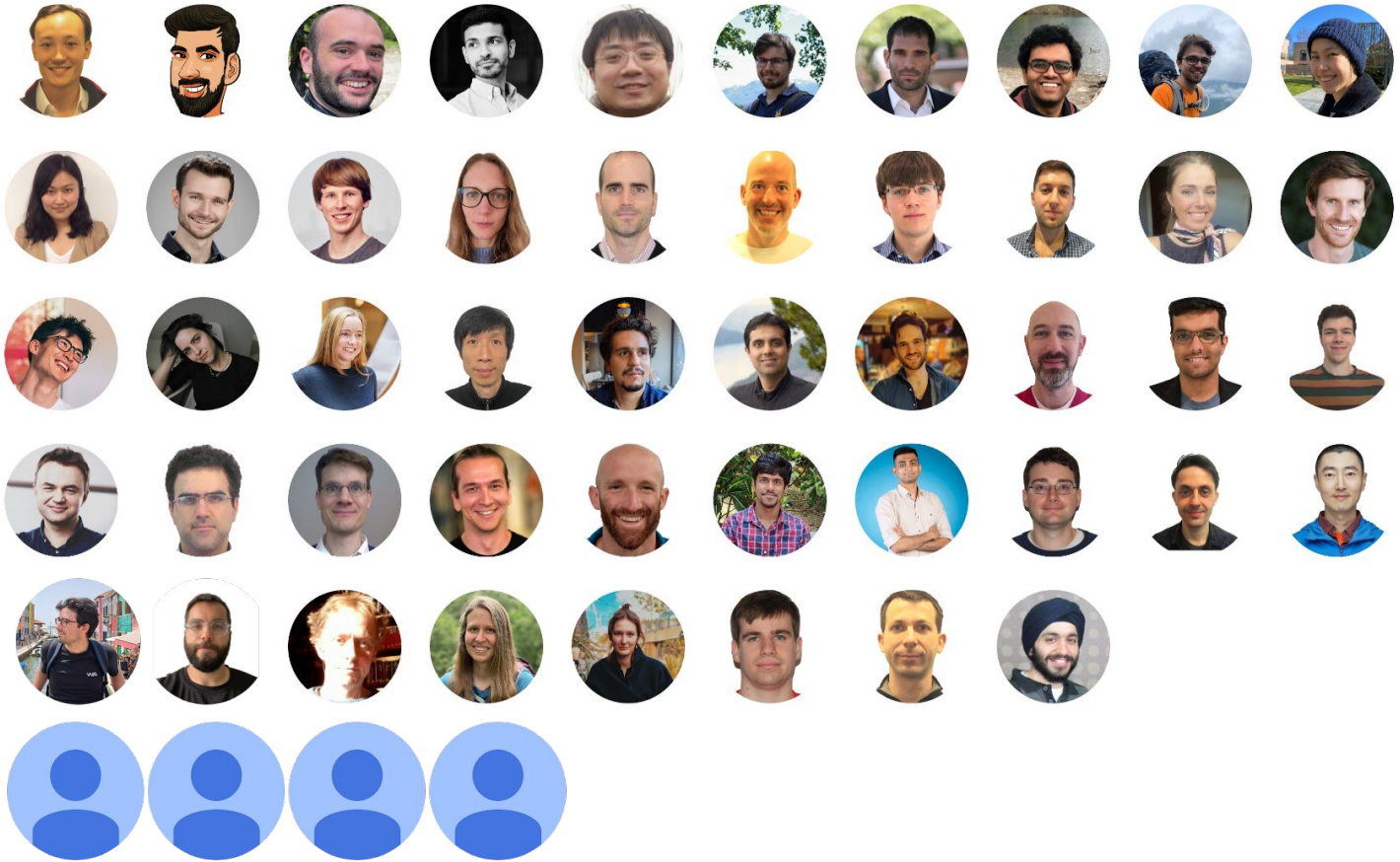
Broaden to the entire Mathematical landscape

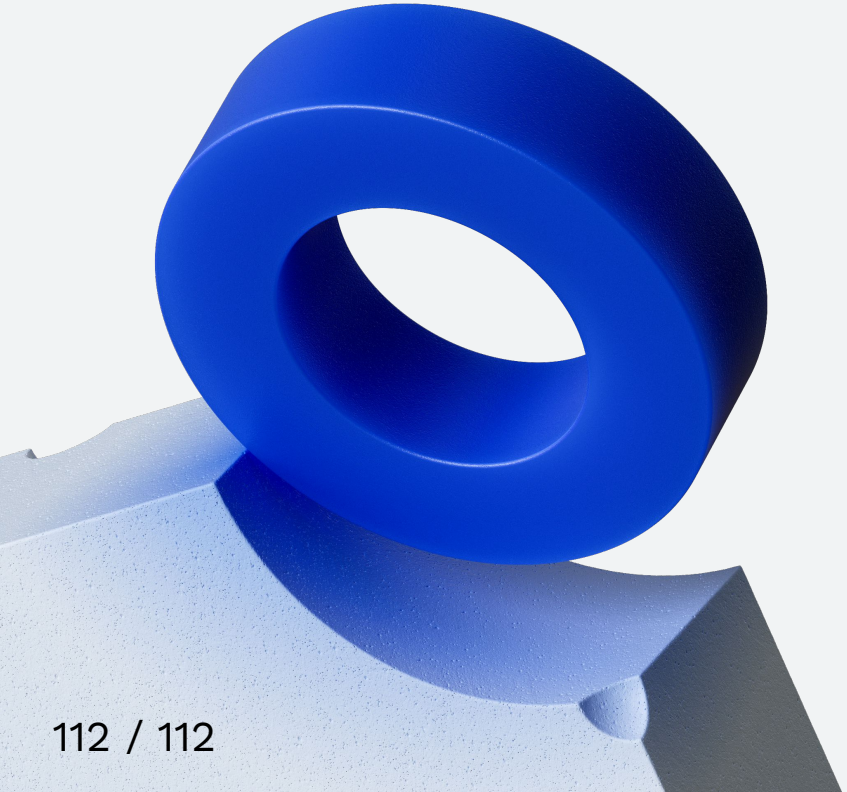
Contribute to the Frontiers of Research Math

AlphaProof as a useful tool for every thinker

Individually, this was almost impossible

Together, it felt impossible to fail





Thank you for Listening

Questions?