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Midterm Exam
February 25, 2013

1.
Let H be a non-empty subset of a group G with identity e.
Conjecture. H is a subgroup of G if and only if whenever h and k are in H, then $h^{-1}k$ is in H.

We will first prove that H is a subgroup of G with identity e if whenever h and k are in H, then $h^{-1}k$ is in H.

Proof. Let $a \in H$. Because $a \in H$ we now that $a^{-1}a$ is also in H and coming from the fact that $a^{-1}a = e$, we know that $e \in H$.

Now that we now that H has an identity element, it becomes easy to prove that every element in H has an inverse. Again, let $a \in H$. We know that $e \in H$ and so $a^{-1}e = a^{-1} \in H$, and so every element in H has an inverse.

Finally, let $a, b \in H$. We also now that $a^{-1} \in H$, coming from the fact that every element in H has an inverse. Therefore, $(a^{-1})^{-1}b = ab \in H$ and from this we see that H is closed under the operation in group G. $\square$

Next we will prove that if H is a subgroup G, then whenever h and k are in H, $h^{-1}k$ is in H.

Proof. Let $h, k \in H$. We know that h^{-1} is in H, because H is a group and therefore every element in H has an inverse. We also know from the fact that H is a group it must be closed under its operation. Therefore, $h^{-1}k \in H$. $\square$

From these two proofs we can conclude that H is a subgroup of G if and only if whenever h and k are in H, then $h^{-1}k$ is in H.

2.
(a)
The set HK is equal to $\{[1], [2], [4], [8], [11], [16]\}$.

(b)
Let G be an Abelian group with identity e and subgroups H, and K. Let

$$HK = \{hk : h \in H \land k \in K\}.$$

Conjecture. The set HK is subgroup of G.

Proof. Let H and K be subgroups of G with identity e. We know that $e \in H$ and $e \in K$ from the fact that H and K are subgroups of G. We know from the definition of the set HK that $ee = e \in HK$. And so, HK contains the identity of G.

Let $a, b \in HK$. From the definition of HK we know that there exists some elements $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $ab = (h_1k_1)(h_2k_2)$. We know that the operation of G is commutative and so be applying this rule and the associative property of the group G we know that

$$ab = (h_1k_1)(h_2k_2) = h_1(k_1h_2)k_2 = h_1(h_2k_1)k_2 = (h_1h_2)(k_1k_2).$$

The groups H and K are closed under the operation in G and so $h_1h_2 \in H$ and $k_1k_2 \in K$. From the definition of HK we know that $ab = (h_1h_2)(k_1k_2) \in HK$. In conclusion, HK is closed under the operation of G.

Finally, let $a \in HK$. From the definition of HK we know that there exists some elements $h \in H$ and $k \in K$ such that $a = hk$. We know that $h^{-1} \in H$ and $k^{-1} \in H$ coming from the fact that H and K are subgroups of G. From this, the fact that the operation of G is commutative, and the associative property of the group G we know that

$$(k^{-1}h^{-1})a = a(k^{-1}h^{-1}) = (hk)(k^{-1}h^{-1}) = h(kk^{-1})h^{-1} = (he)h^{-1} = hh^{-1} = e.$$

And so, the inverse of a is $k^{-1}h^{-1}$ and from the definition of HK we know that $k^{-1}h^{-1} \in HK$. In conclusion, we have shown that every element in HK has an inverse. □

By proving that HK contains the identity element $e \in G$, the set HK is closed under the operation of G, and ever element in HK has an inverse, we have proven that HK is a subgroup of G.

(c)

The set HK is not necessarily a subgroup of G if G is non-Abelian. In our proof we relied on the fact that G is Abelian to show that HK is closed under the operation of G. We will provide a counter-example with the group G with identity e, operator $\cdot$, the operation table

$\cdot$	e	a	b	c
e	e	a	b	c
a	a	e	b	b
b	b	c	e	a
c	c	a	a	e

and the subgroups H and K with operation tables

$\cdot$	e	a
e	e	a
a	a	e

and

$\cdot$	e	b
e	e	b
b	b	e

respectively.
We see that $HK = \{e, a, b\}$ and $ba = c$. Therefore, HK is not closed under the operation $\cdot$ and is not a subgroup of G.

3.
The first symmetry is the identity,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

The next set of symmetries are rotations. The first of these three is a 180 degree rotation about the axis through the midpoints of edges 12 and 34. In permutation notation this is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34).$$

The next two are all 180 degree rotations about some axis. The first is through the midpoints of edges 13 and 24 and the second is through the midpoints of edges 14 and 23. In permutation notation these are

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24)$$

and

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23)$$

respectively.
The next set of symmetries are rotations. The first of these four is a 120 degree rotation counterclockwise about the axis through the vertex 1 and the center of $\triangle 234$. In permutation notation this is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = (234).$$

The next three are all 120 degree rotations about some axis. The first is through vertex 2 and the center of $\triangle 134$, the second is through vertex 3 and the center of $\triangle 124$, and the last is through vertex 4 and the center of $\triangle 123$. In permutation notation these are

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} = (143),$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (124),$$

and

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = (132)$$

respectively.

The last set of symmetries are rotations. The first of these four is a 240 degree rotation counterclockwise about the axis through the vertex 1 and the center of $\triangle 234$. In permutation notation this is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (243).$$

The next three are all 240 degree rotations about some axis. The first is through vertex 2 and the center of $\triangle 134$, the second is through vertex 3 and the center of $\triangle 124$, and the last is through vertex 4 and the center of $\triangle 123$. In permutation notation these are

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = (134),$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} = (142),$$

and

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = (123)$$

respectively.

All together we have the symmetries

$$I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

$$R_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34),$$

$$R_{13} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = (13)(24),$$

$$R_{14} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = (14)(23),$$

$$R_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = (234),$$

$$R_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} = (143),$$

$$R_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} = (124),$$

$$R_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} = (132),$$

$$R_1^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (243).$$

$$R_2^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = (134),$$

$$R_3^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} = (142),$$

and

$$R_4^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = (123).$$

The operation table for this group of symmetries is

$\circ$	I	R_{12}	R_{13}	R_{14}	R_1	R_2	R_3	R_4	R_1^2	R_2^2	R_3^2	R_4^2
I	I	R_{12}	R_{13}	R_{14}	R_1	R_2	R_3	R_4	R_1^2	R_2^2	R_3^2	R_4^2
R_{12}	R_{12}	I	R_{14}	R_{13}	R_3	R_4	R_1	R_2	R_4^2	R_3^2	R_2^2	R_1^2
R_{13}	R_{13}	R_{14}	I	R_{12}	R_4	R_3	R_2	R_1	R_2^2	R_1^2	R_4^2	R_3^2
R_{14}	R_{14}	R_{13}	R_{12}	I	R_2	R_1	R_4	R_3	R_3^2	R_4^2	R_1^2	R_2^2
R_1	R_1	R_4	R_2	R_3	R_1^2	R_4^2	R_2^2	R_3^2	I	R_{14}	R_{12}	R_{13}
R_2	R_2	R_3	R_1	R_4	R_3^2	R_2^2	R_4^2	R_1^2	R_{14}	I	R_{13}	R_{12}
R_3	R_3	R_2	R_4	R_1	R_4^2	R_1^2	R_3^2	R_2^2	R_{12}	R_{13}	I	R_{14}
R_4	R_4	R_1	R_3	R_2	R_2^2	R_3^2	R_1^2	R_4^2	R_{13}	R_{12}	R_{14}	I
R_1^2	R_1^2	R_3^2	R_4^2	R_2^2	I	R_{13}	R_{14}	R_{12}	R_1	R_3	R_4	R_2
R_2^2	R_2^2	R_4^2	R_3^2	R_1^2	R_{13}	I	R_{12}	R_{14}	R_4	R_2	R_1	R_3
R_3^2	R_3^2	R_1^2	R_2^2	R_4^2	R_{14}	R_{12}	I	R_{13}	R_2	R_4	R_3	R_1

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