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Problem Set #3
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(15) Let $\mathcal{F}(\mathbb{R})$ denote the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Define addition and multiplication on $\mathcal{F}(\mathbb{R})$ as follows:

- For all $f, g \in \mathcal{F}(\mathbb{R}), (f+g) : \mathbb{R} \to \mathbb{R}$ is the function defined by
$$(f+g)(x) = f(x) + g(x)$$
for all $x \in \mathbb{R}$.

- $f, g \in \mathcal{F}(\mathbb{R}), (fg) : \mathbb{R} \to \mathbb{R}$ is the function defined by
$$(fg)(x) = f(x)g(x)$$
for all $x \in \mathbb{R}$.

(a) Prove that $\mathcal{F}(\mathbb{R})$ is an Abelian group under addition.

For this proof we will first show that $\mathcal{F}(\mathbb{R})$ is closed under addition, that addition is associative in $\mathcal{F}(\mathbb{R})$, that $\mathcal{F}(\mathbb{R})$ contains an identity element, and that each element in $\mathcal{F}(\mathbb{R})$ has an inverse. Finally, we will show that addition is commutative in $\mathcal{F}(\mathbb{R})$.

First of all, from the definition of addition in $\mathcal{F}(\mathbb{R})$ we see that this operation will always give us another function from the reals to the reals which is also an element in $\mathcal{F}(\mathbb{R})$. Therefore, $\mathcal{F}(\mathbb{R})$ is closed under addition. Next, we will prove that addition is associative in $\mathcal{F}(\mathbb{R})$.

Proof. Let $f, g, h \in \mathcal{F}(\mathbb{R})$ and let $x \in \mathbb{R}$. From the definition of addition in $\mathcal{F}(\mathbb{R})$ we know that
$$((f+g)+h)(x) = (f(x)+g(x)) + h(x) \tag{1}$$
and
$$(f+(g+h))(x) = f(x) + (g(x)+h(x)) \tag{2}$$
We also know that
$$f(x) \in \mathbb{R}, \tag{3}$$
$$g(x) \in \mathbb{R}, \tag{4}$$
and
$$h(x) \in \mathbb{R} \tag{5}$$

from the fact that the codomain of f, g, and h is $\mathbb{R}$. We already know that addition is associative in $\mathbb{R}$. From this fact and the equations (3), (4), and (5) we also know that

$$(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x)). \tag{6}$$

Applying the transitive property of equality to (6) and (1), we obtain

$$((f + g) + h)(x) = f(x) + (g(x) + h(x)). \tag{7}$$

Applying this same property again to (7) and (2), we obtain

$$((f + g) + h)(x) = (f + (g + h))(x). \tag{8}$$

Finally, from (8) we see that addition is associative in $\mathcal{F}(\mathbb{R})$. □

Next we will prove that $\mathcal{F}(\mathbb{R})$ contains an identity element.

Proof. Let $e \in \mathcal{F}(\mathbb{R})$ such that

$$e(x) = 0 \tag{9}$$

for all $x \in \mathbb{R}$. Let $f \in \mathcal{F}(\mathbb{R})$ and let $a \in \mathbb{R}$. We see that

$$(f + e)(a) = f(a) + e(a) = f(a) + 0 = f(a)$$

and

$$(e + f)(a) = e(a) + f(a) = 0 + f(a) = f(a).$$

In conclusion, we have shown that $(f + e)(a) = (e + f)(a) = f(a)$ for some $a \in \mathbb{R}$ and therefore the function e defined $e(x) = 0$ for all $x \in \mathbb{R}$ is the identity element in $\mathcal{F}(\mathbb{R})$. □

Next will show that each function in $\mathcal{F}(\mathbb{R})$ has an inverse under addition.

Proof. Let $f \in \mathcal{F}(\mathbb{R})$ and let $x \in \mathbb{R}$. First we note that $f(x) \in \mathbb{R}$. Because the reals are a group under addition we know that there exits some $g \in \mathcal{F}(\mathbb{R})$ such that $f(x) + g(x) = 0$ and $g(x) + f(x) = 0$. Therefore, the arbitrary element $f \in \mathcal{F}(\mathbb{R})$ has an inverse.

In conclusion we have shown that every element in $\mathcal{F}(\mathbb{R})$ has an inverse. □

Finally, we will show that every element in $\mathcal{F}(\mathbb{R})$ commutes under addition.

Proof. Let $f, g \in \mathcal{F}(\mathbb{R})$ and let $x \in \mathbb{R}$. We know that

$$(f + g)(x) = f(x) + g(x) \tag{10}$$

and

$$(g + f)(x) = g(x) + f(x) \tag{11}$$

from the definition of a function in $\mathcal{F}(\mathbb{R})$. We also know that

$$f(x) \in \mathbb{R} \tag{12}$$

and

$$g(x) \in \mathbb{R} \tag{13}$$

from the fact that a function in $\mathcal{F}(\mathbb{R})$ has the codomain $\mathbb{R}$. Because addition is commutative in $\mathbb{R}$, from (12) and (13) we can conclude that

$$f(x) + g(x) = g(x) + f(x). \tag{14}$$

Applying the transitive property of equality to (10) and (14) we obtain

$$(f + g)(x) = g(x) + f(x). \tag{15}$$

Applying this same property to (15) and (11) we obtain

$$(g + f)(x) = (f + g)(x).$$

In conclusion, we have shown that $(g + f)(x) = (f + g)(x)$ for all $x \in \mathbb{R}$ and therefore addition is commutative in $\mathcal{F}(\mathbb{R})$. $\square$

In conclusion, we have shown that $\mathcal{F}(\mathbb{R})$ is closed under addition, addition is associative in $\mathcal{F}(\mathbb{R})$, $\mathcal{F}(\mathbb{R})$ contains an identity element, each element in $\mathcal{F}(\mathbb{R})$ has an inverse, and addition is commutative in $\mathcal{F}(\mathbb{R})$. From this we know that $\mathcal{F}(\mathbb{R})$ is an Abelian group under addition.

(b) Does $\mathcal{F}(\mathbb{R})$ have an identity element for multiplication?

Yes, let $e \in \mathcal{F}(\mathbb{R})$ such that

$$e(x) = 1$$

for all $x \in \mathbb{R}$. Let $f \in \mathcal{F}(\mathbb{R})$ and let $a \in \mathbb{R}$. We see that

$$(fe)(a) = f(a)e(a) = f(a) \cdot 1 = f(a)$$

and

$$(ef)(a) = e(a)f(a) = 1 \cdot f(a) = f(a).$$

Therefore, e is the identity in $\mathcal{F}(\mathbb{R})$ under multiplication.

(c) Find an element in $\mathcal{F}(\mathbb{R})$ that does not have a multiplicative inverse in $\mathcal{F}(\mathbb{R})$. Explain how this shows $\mathcal{F}(\mathbb{R})$ is not a group under multiplication.

Let $f \in \mathcal{F}(\mathbb{R})$ such that

$$f(x) = 0$$

for all $x \in \mathbb{R}$. Let $g \in \mathcal{F}(\mathbb{R})$ and let $a \in \mathbb{R}$. We see that

$$(fg)(a) = 0 \cdot g(a) = 0 \neq 1.$$

And from this we can conclude that the function f defined as $f(x) = 0$ for all $x \in \mathbb{R}$ has no inverse. Because a group must have an inverse for every element and f is an element in $\mathcal{F}(\mathbb{R})$, $\mathcal{F}(\mathbb{R})$ is not a group under multiplication.

(d) Find necessary and sufficient conditions for an element in $\mathcal{F}(\mathbb{R})$ to be a unit in $\mathcal{F}(\mathbb{R})$. State your result in a lemma of the form "The function $f \in \mathcal{F}(\mathbb{R})$ is a unit in $\mathcal{F}(\mathbb{R})$ if and only if ...". Your lemma must say something more than just a rehash of the definition of a unit; rather, it must actually characterize the functions that are invertible under multiplication in $\mathcal{F}(\mathbb{R})$.

Conjecture. An element in $\mathcal{F}(\mathbb{R})$ is a unit if and only if $f(x) \neq 0$ for all $x \in \mathbb{R}$.

First we will show that if f is a function in $\mathcal{F}(\mathbb{R})$ such that $f(x) \neq 0$ for all $x \in \mathbb{R}$, then f is a unit.

Proof. Let f be a function in $\mathcal{F}(\mathbb{R})$ such that $f(x) \neq 0$ for all $x \in \mathbb{R}$ and let $a \in \mathbb{R}$. Because the set $\mathbb{R} - \{0\}$ is a group under multiplication and $f(x)$ is in this group we know that there exists some g in $\mathcal{F}(\mathbb{R})$ such that $f(x)g(x) = 1$ and $g(x)f(x) = 1$. From this we see that f has an inverse in $\mathcal{F}(\mathbb{R})$. $\square$

Next we will show that if f is a unit, then it must be true that $f(x) \neq 0$ for all $x \in \mathbb{R}$.

Proof. We assume to the contrary that f is a unit in $\mathcal{F}(\mathbb{R})$ such that there exists some $a \in \mathbb{R}$ such that $f(a) = 0$. Because f is a unit, there must exist some g in $\mathcal{F}(\mathbb{R})$ such that $g(x)f(x) = 1$ for all $x \in \mathbb{R}$. This implies that there exists a real number $g(a)$ such that $g(a) \cdot 0 = 1$. This is a contradiction and so it must be necessary for $f(x) \neq 0$ for all $x \in \mathbb{R}$ in order for f to be a unit. $\square$

Activity 20.12. In this activity, we will explore a simple relationship between the order of a group element and the order of its inverse.

(a) Determine the order of $[2]$ in $\mathbb{Z}_6$. What is the inverse of $[2]$ in $\mathbb{Z}_6$? Directly compute the order of the inverse of $[2]$ in $\mathbb{Z}_6$. What do you notice?

First of all, we note that $\langle [2] \rangle = \{[0], [2], [4]\}$. The magnitude of this set is 3 and therefore the order of $[2]$ in $\mathbb{Z}_6$ is 3. The inverse of $[2]$ is $[4]$ ($[2]+[4] = [0]$).

The order of $[4]$ in $\mathbb{Z}_6$ is equal to the magnitude of $\langle[4]\rangle = \{[0],[2],[4]\}$, and so the order of $[4]$ is 3. The sets $\langle[2]\rangle$ and $\langle[4]\rangle$ are equal and therefore the orders $[2]$ and $[4]$ must be equal as well.

(b) Determine the order of $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ in the group D_4 of symmetries of a square. What is the inverse of α in D_4 ? Directly compute the order of the inverse of α in D_4 . What do you notice?

First of all, we note that

$$\langle\alpha\rangle = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \right\}.$$

From this we see that the magnitude of α is 4. The inverse of α is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$. The cyclic group generated by α^{-1} is

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \right\}.$$

Therefore the order of α^{-1} is 4. Again, this is equal to α.

(c) Based on your observations in parts (a) and (b), what relationship do you think exists between $|a|$ and $|a^{-1}|$ in a group G?

The order of a is equal to the order of a^{-1}.

(d) Let G be a group with identity e, and let $a \in G$. Show that if $a^n = e$ for some positive integer n, then $(a^{-1})^n = e$.

Proof. Let G be a group with identity e and let $a \in G$ such that

$$a^n = e \tag{16}$$

for some positive integer n. Applying the definitions of an integer power of an element in a group we see that

$$(a^{-1})^n = a^{-1\cdot n} = a^{-n} = (a^n)^{-1} \tag{17}$$

Now applying the transitive property of equality to (16) and (17) we obtain

$$(a^{-1})^n = e^{-1} = e.$$

In conclusion, we have shown that if G be a group with identity e and $a \in G$ such that $a^n = e$ for some positive integer n, then $(a^{-1})^n = e$. $\square$

(e) Let G be a group with identity e, and let a be an element of G with finite order. For this case, prove the relationship you conjectured between $|a|$ and $|a^{-1}|$ in part (c).

Conjecture. Let G be a group with identity e with element a of finite order. The order of a is equal to the order of a^{-1}.

Proof. We know that $\langle a \rangle$ contains e and so there must exist some positive integer n such that

$$a^n = e. \tag{18}$$

Let $S = \{x \in \mathbb{Z}^+ : a^x = e\}$. From (18) we know that S is not empty. Therefore, from the Axiom of Choice we are able to choose the a smallest element $k \in S$. The set $\{a, a^2, \ldots, a^k\}$ is equal to the set $\langle a \rangle$ (it contains a and all of the elements up to e) and so the order of a is k. From part (d) we know that $(a^{-1})^k = e$ and so $|a^{-1}| \leq |a|$.

We also know that $\langle a^{-1} \rangle$ contains e and so there must exist some positive integer m such that

$$(a^{-1})^m = e. \tag{19}$$

Let $T = \{x \in \mathbb{Z}^+ : ((a)^{-1})^x = e\}$. From (19) we know that T is not empty. Therefore, from the Axiom of Choice we are able to choose the a smallest element $p \in T$. The set $\{a^{-1}, (a^{-1})^2, \ldots, (a^{-1})^p\}$ is equal to the set $\langle a^{-1} \rangle$ (it contains a^{-1} and all of the elements up to e) and so the order of a^{-1} is p. From part (d) we know that $a^p = e$ and so $|a| \leq |a^{-1}|$.

In conclusion we have shown that the order of a is less than or equal to the order of a^{-1} and the order of a^{-1} is less than or equal to the order of a. Therefore, the order of a and the order of a^{-1} is equal. $\square$

(f) Let G be a group with identity e, and let $a \in G$. Prove that if a has infinite order, then a^{-1} has infinite order.

Proof. Assume to the contrary that a has infinite order, but a^{-1} does not. Because the order of a^{-1} is finite, there must exist some $n \in \mathbb{Z}^+$ such that

$$|a^{-1}| < n \tag{20}$$

The inverse of a^n is $a^{-1})^n$, but because the order of $(a^{-1}$ is less than n there must exist some $k < n$ such that $(a^{-1})^n = (a^{-1})^k$. But this would mean that $(a^{-1})^k$ has two unique inverses, a^k and a^n. This is a contradiction and so a^{-1} has infinite order. $\square$

(3) Let H denote the set of all 2×2 matrices of the form

$$\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}$$

where $x, y \in \mathbb{R}$. Is H a subgroup of $\mathcal{M}_{2\times 2}(\mathbb{R})$?

Conjecture. The set H is a subgroup of $\mathcal{M}_{2\times 2}(\mathbb{R})$.

Proof. First of all, the identity of $\mathcal{M}_{2\times 2}(\mathbb{R})$ under addition is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and this is in H ($x = 0, y = 0$). Next, let $a, b, c, d \in \mathbb{R}$. Then $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ and $\begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}$ are both in H. When we add these two matrices together we get

$$\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} a+c & 0 \\ b+d & 0 \end{bmatrix}.$$

Because the reals are closed under addition, we know that both $a + c$ and $b + d$ are in $\mathbb{R}$. Therefore, $\begin{bmatrix} a+c & 0 \\ b+d & 0 \end{bmatrix}$ is in H and from this we can conclude that H is closed under addition. Finally let $x, y \in \mathbb{R}$. The inverses of x and y are also in $\mathbb{R}$ and so both $\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}$ and $\begin{bmatrix} -x & 0 \\ -y & 0 \end{bmatrix}$ are in H. We also see that

$$\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} + \begin{bmatrix} -x & 0 \\ -y & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From this, we can conclude that every element in H has an inverse. In conclusion, we have shown that the the subset H of $\mathcal{M}_{2\times 2}(\mathbb{R})$ is closed under addition, the identity of $\mathcal{M}_{2\times 2}(\mathbb{R})$ under addition is in H, and every element of H has an inverse. From this we have shown that H is a subgroup of $\mathcal{M}_{2\times 2}(\mathbb{R})$ under addition. $\square$

(4) Let G be a group and H a subgroup of G. Which of the following conjectures do you think are true, and which do you think are false? Provide brief arguments or examples to justify your answers.
(a) If G is finite, then H is finite.

This statement is true due to the fact that H is a subset of G and therefore cannot have a magnitude greater than its superset G.

(b) If H is finite, then G is finite.

This is not necessarily true. For example, $\{0\}$ is a subgroup of $\mathbb{Z}$ under addition. In this case H is finite, but G is infinite.

(c) If G is Abelian, then H is Abelian.

This is a property of the operator of G and therefore it will also be true for H whose elements are all in G.

(d) If H is Abelian, then G is Abelian.

This is not always true. For example, we have the group G containing the symmetries of a square with the subgroup H containing only its identity. In this case H is Abelian, but G is not.

(8) **Intersections of subgroups.** Let G be a group with subgroups H and K.
(a) Is $H \cap K$ a subgroup of G? Prove your answer.

Conjecture. Let G be a group with subgroups H and K. Then $H \cap K$ is a subgroup of G.

Proof. Let G be a group with identity e and let H and K be subgroups of G. We know from the fact that H and K are subgroups of G that $e \in H$ and $e \in K$, and so $e \in H \cap K$. Next, let $a, b \in H \cap K$. From the definition of intersection of sets we know that $a, b \in H$ and $a, b \in K$. Because H and K are groups, $ab \in H$ and $ab \in K$. Therefore $ab \in H \cap K$, coming from the definition of intersection. From this we have shown that $H \cap K$ are closed under the operation of G. Finally, let $x \in H \cap K$. From the definition of intersection of sets we know that $x \in H$ and $x \in K$. Because H and K are groups, $x^{-1} \in H$ and $x^{-1} \in K$. Therefore $x^{-1} \in H \cap K$, coming from the definition of intersection of sets.

In conclusion, we have shown that the identity of G is in $H \cap K$, the set $H \cap K$ is closed under the operation of G, and each element in $H \cap K$ has an inverse in $H \cap K$. Therefore, $H \cap K$ is a subgroup of G. $\square$

(b) Can we generalize? That is, if H_α is a collection of subgroups of G indexed by α in an indexing set I, is it the case that $\cap_{\alpha \in I} H_\alpha$ is a subgroup of G? Prove your answer.

Conjecture. Let H be a subgroup of G in the collection H_α. We know that $e \in H$ from the fact that H is a subgroup of G. Because H is an arbitrary element in H_α we have shown that e is in every element of H_α.

Therefore, from the definition of intersection of sets $e \in \cap_{\alpha \in I} H_\alpha$. Next let $a, b \in \cap_{\alpha \in I} H_\alpha$. From the definition of intersection we know that $a, b \in H$. Because H is a group, $ab \in H$. Again, H is an arbitrary element in H_α and so $ab \in \cap_{\alpha \in I} H_\alpha$. From this, we see that $\cap_{\alpha \in I} H_\alpha$ is closed under the operation of the group G. Finally, let $x \in \cap_{\alpha \in I} H_\alpha$. From the definition of intersection we know that $x \in H$. Because H is a group, $x^{-1} \in H$. And because H is an arbitrary element in H_α, $x^{-1} \in \cap_{\alpha \in I} H_\alpha$.

In conclusion, we have shown that $\cap_{\alpha \in I} H_\alpha$ contains the identity $e \in G$, it is closed under the operation of G, and each element in $\cap_{\alpha \in I} H_\alpha$ has an inverse. Therefore, $\cap_{\alpha \in I} H_\alpha$ is a subgroup of G.

Proof. We will prove this by induction. Let H_α be a collection of subgroups of G indexed by α in an indexing set I with $|I| = 1$. In this case, $\cap_{\alpha \in I} H_\alpha = H_1$ where H_1 is the one and only element in the collection H_α. We already know that H_1 is a subgroup of G and so $\cap_{\alpha \in I} H_\alpha$ is a subgroup of G. For the next step in the induction proof we must show that if $\cap_{\alpha \in I} H_\alpha$ is a subgroup of □

(12) Determine whether H is a subgroup of G.
(a) $G = \mathbb{Z}_{20}$ under addition, $H = \{[0], [3], [6], [9], [12], [15], [18]\}$.

The inverse of [3] is [17] which is not in H. Therefore H is not a subgroup of G.

(b) $G = U_7$ under multiplication, $H = \{[1], [2], [4]\}$.

First of all, the identity [1] is in H. Now we will construct an operation table to see if H is closed under multiplication and each element of H has an inverse.

$\cdot$	[1]	[2]	[4]
[1]	[1]	[2]	[4]
[2]	[2]	[4]	[1]
[4]	[4]	[1]	[2]

From this operation table we see that H is closed under multiplication and each element of H has an inverse. And so, we can conclude that H is a subgroup of G.

(c) $G = U_{16}$ and $H = \{[1], [7], [9], [15]\}$.

First of all, the identity [1] is in H. Now we will construct an operation table to see if H is closed under multiplication and each element of H has an inverse.

$\cdot$	[1]	[7]	[9]	[15]
[1]	[1]	[7]	[9]	[15]
[7]	[7]	[1]	[15]	[19]
[9]	[9]	[15]	[1]	[7]
[15]	[15]	[9]	[7]	[1]

From this operation table we see that H is closed under multiplication and each element of H has an inverse. And so, we can conclude that H is a subgroup of G.