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Final Exam
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2.
(a) The operation table for U_{44} is given in Table 1. Explain why $U_{44} = \langle[3]\rangle \times \langle[21]\rangle$, the internal direct product of the subgroups $\langle[3]\rangle$ and $\langle[21]\rangle$.

First of all, $\langle[3]\rangle = \{[1],[3],[9],[15],[23],[25],[27],[31],[37]\}$ and $\langle[21]\rangle = \{[1],[21]\}$. From this we see that the intersection of $\langle[3]\rangle$ and $\langle[21]\rangle$ contains only [1]. From Theorem 26.6 (2) we can conclude that each element in $\langle[3]\rangle \times \langle[21]\rangle$ has a unique representation kn where $k \in \langle[3]\rangle$ and $n \in \langle[21]\rangle$. And so, $|\langle[3]\rangle \times \langle[21]\rangle| = |\langle[3]\rangle| \cdot |\langle[21]\rangle| = 10 \cdot 2 = 20$. From the closure property of the group G and the fact that $\langle[3]\rangle$ and $\langle[21]\rangle$ are subgroups of G we know that each element in $\langle[3]\rangle \times \langle[21]\rangle$ is also in G. From this fact and the fact that $\langle[3]\rangle \times \langle[21]\rangle$ and G have the same order, it must be the case that $G = \langle[3]\rangle \times \langle[21]\rangle$

(b) When we decompose a group as an internal direct product, it is convenient for classification purposes to identify that internal direct product with an external direct product. Let G be an arbitrary group with identity element e and let K and N be normal subgroups of G with $K \cap N = \{e\}$. Prove that

$$K \times N \cong (K \oplus N).$$

Proof. We will prove that $\phi : K \oplus N \to K \times N$ such that $\phi((k,n)) = kn$ for all $k \in K$ and for all $n \in N$ is an isomorphism. In doing so we will have proven that $K \oplus N$ and $K \times N$ are isomorphic and therefore $(K \oplus N) \cong (K \times N)$.

First we will show that the ϕ is in fact a function. This means that it is well-defined. Let $(k,n) = (k',n') \in K \oplus N$. We will prove that $\phi((k,n)) = \phi((k',n'))$. We see that

$$\phi((k,n)) = kn = k'n' = \phi((k',n'))$$

and therefore ϕ is a function.

Next we will show that ϕ preserves structure and therefore is a homomorphism. Let $(k,n),(k',n') \in K \oplus N$. First we see that

$$\phi((k,n)(k',n')) = \phi((kk',nn')) = (kk')(nn').$$

From Theorem 26.6 (1) we know that $k'n = nk'$. Therefore,

$$\phi((k,n)(k',n')) = (kk')(nn') = k(k'n)n' = k(nk')n' = (kn)(k'n') = \phi((k,n))\phi((k'n'))$$

and ϕ preserves the operation in G.

Finally, we will show that ϕ is both injective and surjective. Let $\phi((k,n)) = \phi((k',n'))$. First we note that

$$kn = \phi((k,n)) = \phi((k',n')) = k'n'$$

and so $kn = k'n'$. From Theorem 26.6 (2) we know that kn is a unique representation of an element in $K \times N$, which means that k must be equal to k' and n must be equal to n'. Therefore, $(k,n) = (k',n')$ and ϕ is injective.

Let $kn \in K \times N$. The element $(k,n) \in K \oplus N$ is such that $\phi((k,n)) = kn$ and so ϕ is surjective. $\square$

Explain why $U_{44} \cong (\mathbb{Z}_{10} \oplus \mathbb{Z}_2)$.

Because $\mathbb{Z}_{10} \cap \mathbb{Z}_2 = [1]$, it follows that $\mathbb{Z}_{10} \times \mathbb{Z}_2 = U_{44}$ in a similar manner to part (a). Also from part (b) we know that $U_{44} = \mathbb{Z}_{10} \times \mathbb{Z}_2 \cong \mathbb{Z}_{10} \oplus \mathbb{Z}_2$. Therefore, $U_{44} \cong (\mathbb{Z}_{10} \oplus \mathbb{Z}_2)$.

3.
(a) Let $G = U_{44}$. Let $N = \langle [3] \rangle$ and let $K = \langle [9] \rangle$. The operation table for U_{44} is shown in Table 1. Find G/K, G/N, N/K, and $(G/K)/(N/K)$. Explain why $(G/K)/(N/K) \cong G/N$. First of all,

$$G/K = \{K, [3]K, [7]K, [13]K\},$$

$$G/N = \{N, [7]N\},$$
$$N/K = \{K, [3]K\},$$

and

$$(G/K)/(N/K) = \{N/K, ([7]K)(N/K)\}.$$

The groups $(G/K)/(N/K)$ and G/N both have only two elements and therefore are both isomorphic with $\mathbb{Z}_2$. And so, $(G/K)/(N/K) \cong G/N$.

(b) Prove the Third isomorphism theorem as stated below.

Theorem (The Third Isomorphism Theorem). *Let G be a group, K and N normal subgroups of G with $K \subseteq N$. Then $(G/K)/(N/K) \cong G/N$.*

Proof. Let G be a group, K and N normal subgroups of G with $K \subseteq N$. Let $\phi : G/K \to G/N$ such that $\phi(gK) = gN$. First we need to make sure that ϕ is well-defined. Let $xK = yK$. In other words, $y^{-1}x \in K$. Because K is a subset of N we know that $y^{-1}x \in N$. And so, $xN = yN$. Therefore, $\phi(xK) = \phi(yK)$ and ϕ is well-defined.

Next let $xK, yK \in G/K$. We see that

$$\begin{aligned} \phi(xK)\phi(yK) &= (xN)(yN) \\ &= (xy)N \\ &= \phi((xy)K). \end{aligned}$$

Therefore, ϕ preserves structure and is a homomorphism.

Next we note that,

$$\begin{aligned} Ker(\phi) &= \{gK \in G/K : \phi(gK) = e_{G/N}\} \\ &= \{gK \in G/K : gN = N\} \\ &= \{gK \in G/K : g \in N\} \\ &= N/K. \end{aligned}$$

Finally, we will show that $Im(\phi) = G/N$ and apply The First Isomorphism Theorem to obtain $(G/K)/(N/K) \cong G/N$. Let $aN \in G/N$. The element $aK \in G/K$ is such that $\phi(aK) = aN$ and so ϕ is surjective. Because ϕ is surjective, the image of ϕ contains the entire codomain. In other words $Im(\phi) = G/N$. Applying The First Isomorphism Theorem we obtain

$$(G/K)/(N/K) = (G/K)/Ker(\phi) \cong Im(\phi) = G/N.$$

In other words, $(G/K)/(N/K) \cong G/N$.

$\square$

EXTRA CREDIT We showed that if N is a normal subgroup of a group G, then the operation

$$(aN)(bN) = (ab)N \tag{1}$$

on G/N (the collection of left cosets of N in G) is well-defined. Is the converse true? That is, if G is a group and N is a subgroup of G so that (1) is well-defined, must N be a normal subgroup of G? Prove your answer.

Conjecture. If G is a group and N is a subgroup of G such that the operation $(aN)(bN) = (ab)N$ is well-defined, then N

Proof. Let G be group with identity e and let N be a subgroup of G such that the operation $(aN)(bN) = (ab)N$ is well-defined. Let $n \in N$. We know that $nN = eN$ from the definition of a coset. Let $g \in G$. We know that $(eN)(gN) = (eg)N = gN$, because the operation $(aN)(bN) = (ab)N$ is well-defined. Also, because $nN = eN$ we see that

$$gN = (eN)(gN) = (nN)(gN) = (ng)N.$$

And so, $gN = (ng)N$. Multiplying the inverse of g on the left we obtain $N = (g^{-1}ng)N$. From the definition of a coset we now know that $g^{-1}ng \in N$. Finally, from this we can conclude that $g^{-1}Ng \subseteq N$, which makes N a normal subgroup of G. $\square$