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Problem Set #1
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Activity 17.4 Find, via a library or internet search, an object (building, tiling, painting, sculpture, mosaic, fractal, rug, etc.) that has significant symmetry. Then complete the following.

(a) Identify the object and the source through which you found it. Choose something other than a simple polygon; that is, find an object that is interesting to you and that possesses at least 6 symmetries, including both rotational and reflective symmetry.

The K_5 complete graph contains all of the same symmetries as a regular pentagon. It has a total of ten symmetries. The following figure is a drawing of a K_5 graph from wikipedia.
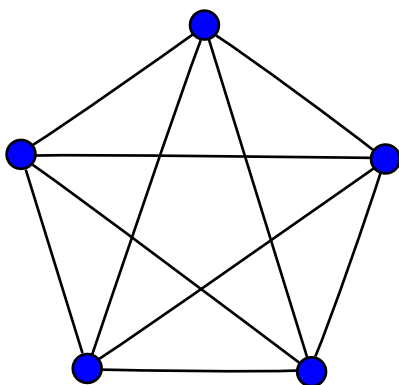
source: en.wikipedia.org/wiki/File:4-simplex_graph.svg

b) Describe all of the symmetries possessed by your object. Choose 6 symmetries (including at least one non-trivial rotation and one non-trivial reflection), and make a copy of the picture of your object for each symmetry. Find a convenient way to label your object so that you can use permutation notation to represent each symmetry. Then illustrate each symmetry on one of the copies of your picture.

The K_5 graph when drawn as it is from my source has reflective symmetries r_1, r_2, r_3, r_4, and r_5 where r_n's line of symmetry is through vertex n and the center of K_5. It also has the rotational symmetries R_{72}, R_{144}, R_{216}, and R_{288} where R_n is the rigid motion of rotation about the center of K_5 n degrees counter-clockwise. There is also I which is the rigid motion $f : O \to O$

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where $f(x) = x$ for each point of the object O. The following is a table of symmetries for this object.

c) Choose 3 of the symmetries, and find all of the compositions of these three symmetries. Is each composition a symmetry of your object? Explain.

The following is a list of all compositions of the symmetries r_1, r_2, and R_{72}.

$$r_1 \circ (r_2 \circ R_{72}) = (r_1 \circ r_2) \circ R_{72} = R_{216}$$
$$r_1 \circ (R_{72} \circ r_2) = (r_1 \circ R_{72}) \circ r_2 = R_{288}$$
$$r_2 \circ (r_1 \circ R_{72}) = (r_2 \circ r_1) \circ R_{72} = R_{288}$$
$$r_2 \circ (R_{72} \circ r_1) = (r_2 \circ R_{72}) \circ r_1 = R_{144}$$
$$R_{72} \circ (r_1 \circ r_2) = (R_{72} \circ r_1) \circ r_2 = R_{216}$$
$$R_{72} \circ (r_2 \circ r_1) = (R_{72} \circ r_2) \circ r_1 = R_{288}$$

Each composition is a symmetry. This is due to the fact that a set of all symmetries of an object is a group under operation composition.

(3)

(a) Find all of the symmetries of the letter B, and create the operation table for the set of symmetries of B.

There are two symmetries for the letter B, the horizontal reflection r_H and I. The following is the operation table of these two symmetries.

$\circ$	I	r_H
I	I	r_H
r_H	r_H	I

Symmetries of B

(b) Find all of the symmetries of the letter T, and create the operation table for the set of symmetries of T.

There are two symmetries for the letter T, the vertical reflection, r_v and I. The following is the operation table of these two symmetries.

$\circ$	I	r_V
I	I	r_V
r_V	r_V	I

Symmetries of T

(c) Find all of the symmetries of the letter Z, and create the operation table

for the set of symmetries of Z.

There are two symmetries for the letter Z, a $180°$ rotation, R_{180} and I. The following is the operation table of these two symmetries.

$\circ$	I	R_{180}
I	I	R_{180}
R_{180}	R_{180}	I

Symmetries of Z

(d) Compare the operation tables for B, T, and Z. Describe all of the similarities and differences you observe.

All of the operation tables are closed. The operation tables all have two symmetries, but Z is the only letter with rotational symmetry. All three operation tables have the identity I and are abelian.

(4) Is composition of symmetries a commutative operation? Prove your answer.

The composition of symmetries is not a commutative operation. A simple counter-example is that the symmetries r_1 and R_{120} of an equilateral triangle are not commutative.

$$r_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$R_{120} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$r_1 \circ R_{120} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$R_{120} \circ r_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

From this we see that $R_{120} \circ r_1 \neq r_1 \circ R_{120}$ and so composition of symmetries is not a commutative operation for all symmetries.

(5) Let A, B, and C be the objects shown in Figure 17.5.

(a) For each object, find all of the symmetries. Describe the symmetries in words and using the permutation notation introduced in this investigation.

The rectangle, A, has horizontal and vertical symmetry, a 180° rotational symmetry, and a symmetry where we do nothing. Notationally, the symmetries are $\bar{r}_1$, $\bar{r}_2$, R_{180}, and I.

The pinwheel, B, has the rotational symmetries of 90°, 180°, and 270°. We can also leave it alone for a symmetry. Notationally, the symmetries are R_{90}, R_{180}, R_{270}, and I.

The figure-eight, C, has a 180° rotational symmetry, a vertical, and horizontal symmetry, and itself as a symmetry. Notationally, the symmetries are r_1, $\bar{r}_2$, R_{180}, and I.

(b) Create the operation table for the set of symmetries of each object.

The following are the operation tables for each of the objects' symmetries.

$\circ$	I	$\bar{r}_1$	$\bar{r}_2$	R_{180}
I	I	$\bar{r}_1$	$\bar{r}_2$	R_{180}
$\bar{r}_1$	$\bar{r}_1$	I	R_{180}	$\bar{r}_2$
$\bar{r}_2$	$\bar{r}_2$	R_{180}	I	$\bar{r}_1$
R_{180}	R_{180}	$\bar{r}_2$	$\bar{r}_1$	I

Symmetries of A Operation Table

$\circ$	I	R_{90}	R_{180}	R_{270}
I	I	R_{90}	R_{180}	R_{270}
R_{90}	R_{90}	R_{180}	R_{270}	I
R_{180}	R_{180}	R_{270}	I	R_{90}
R_{270}	R_{270}	I	R_{90}	R_{180}

Symmetries of B Operation Table

$\circ$	I	r_1	$\bar{r}_2$	R_{180}
I	I	r_1	$\bar{r}_2$	R_{180}
r_1	r_1	I	R_{180}	$\bar{r}_2$
$\bar{r}_2$	$\bar{r}_2$	R_{180}	I	r_1
R_{180}	R_{180}	$\bar{r}_2$	r_1	I

(c) Describe the similarities and differences in the operation tables you made in part (b). Your description should include not only obvious attributes like the number of elements, but also how the elements interact within a given

set of symmetries.

Each of the operation tables are made up of four reflections and are commutative. The reason for B being commutative is that it contains only rotational symmetries. For A and C there are only two reflective symmetries.

Activity 18.12. Recall that a symmetry of an object O is a bijective, distance preserving function f such that $f(O) = O$. In this activity, we will verify that the set S of symmetries of an object O forms a group under the operation of composition, called the group of symmetries of O.

(a) Let f and g be bijective, distance-preserving functions with f (O) = O and $g(O) = O$. To show that S is closed, we need to verify that $f \circ g$ is a bijective, distance preserving function with $(f \circ g)(O) = O$.

(i) Prove that $|(f \circ g)(x)(f \circ g)(y)| = |xy|$ for all x, y in the domain of $f \circ g$.

Let S be the set of all symmetries of an object O. Let f and g be bijective, distance-preserving functions with $f(O) = O$ and $g(O) = O$.

Conjecture. The function $f \circ g$ is a distance-preserving function.

Proof. We know that for all $x, y \in \mathbb{R}^2$

$$g(x), g(y) \in \mathbb{R}^2 \tag{1}$$

from this fact that $g : \mathbb{R}^2 \to \mathbb{R}^2$. From (1) and the fact that f is a distance-preserving function we also know that

$$|f(g(x)) - f(g(y))| = |g(x) - g(y)|. \tag{2}$$

We also know that

$$|g(x) - g(y)| = |x - y| \tag{3}$$

from the fact that g is a distance-preserving function. Using the transitive property of equality with (**??**) and (**??**) we can conclude that

$$|(f \circ g)(x) - (f \circ g)(y)| = |x - y|.$$

In conclusion, we have proven that $|(f \circ g)(x) - (f \circ g)(y)| = |x - y|$ for alll $x, y \in \mathbb{R}^2$ and therefore $f \circ g$ is a distance-preserving function. $\square$

(ii) Use part (a) to deduce that $f \circ g$ is an injection.

Proof. Let $x, y \in \mathbb{R}^2$ and

$$(f \circ g)(x) = (f \circ g)(y) \tag{4}$$

From (??) we can conclude that

$$|(f \circ g)(x) - (f \circ g)(y)| = 0. \tag{5}$$

We have already shown that $f \circ g$ is a distance-preserving function and so

$$|(f \circ g)(x) - (f \circ g)(y)| = |x - y|. \tag{6}$$

From (??), (??), and the transitive property of equality

$$|x - y| = 0. \tag{7}$$

From (??) and the definition of equality we can conclude

$$x = y.$$

In conclusion, we have shown that $(f \circ g)(x) = (f \circ g)(y)$ implies $x = y$ and so $f \circ g$ is an injection. $\square$

(iii) Show that $(f \circ g)(O) = O$. Deduce that $f \circ g$ is a surjection.

Proof. We know that

$$f(O) - O \tag{8}$$

and

$$g(O) = O. \tag{9}$$

From (??) and (??) we can conclude that

$$f(g(O)) = O.$$

In conclusion, we have shown that $(f \circ g)(O) = O$ and therefore $f \circ g$ is surjective.

$\square$

(iv) Explain how parts (a)(c) establish that S is closed under composition.

From the fact that $f \circ g$ is a bijective function such that $(f \circ g)(O) = O$ we can conclude that $f \circ g \in S$. If there were some $x \in \mathbb{R}^2$ such that $(f \circ g)(x) \notin S$
(b) Prove that composition of functions is an associative operation.
(c) What is the identity element of S? Verify your answer.
(d) If f S, what is the inverse of f in S? Verify your answer. (Please note that there is quite a bit to do to complete this problem.)
(e) Explain why S is a group. Is S an Abelian group? Explain.