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Problem Set #5
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Activity 24.12 Write a formal proof of Lagrange's Theorem (Theorem 24.4).

Proof. Let G be a finite group and let H be a subgroup of G. From Theorem 24.3 we know that the left cosets of H form a partition of G. Let n be the number of distinct cosets of H in G. Each left coset of H contains $|H|$ elements. (Activity 24.7). And so, $|G| = n|H|$. In conclusion, $|H|$ divides $|G|$. $\square$

Activity 24.13. Let G be a group with identity e and assume that $\sim$ is a congruence relation on G.
(a) Let $H = \{x \in G : x \sim e\}$. Show that H is a subgroup of G.

First of all, we know that $e \sim e$ from the reflexive property of the congruence relation $\sim$. And so, the identity element of G is also in H.

Next, let $a \in H$. From the definition of the set H, we know that $a \sim e$. Because $\sim$ is a congruence relation, $a^{-1} \sim e^{-1}$. And so, $a^{-1} \sim e$ which means that $a^{-1} \in H$. In conclusion, every element in H has an inverse that is also in H. (Which is the same as the inverse of this element in G).

Finally, let $a, b \in H$. From the definition of the set H, we know that $a \sim e$ and $a \sim e$. Because $\sim$ is a congruence relation, $ab \sim e^2$. And so $ab \sim e$ which means that $ab \in H$. In conclusion, the operation in G is closed in H. The identity of G is in H, every element in H has an inverse, and the operator of G is closed in the set H. And so, the set H is a subgroup of G.

(b) Let $a, b \in G$. Prove that $a \sim b$ if and only if $a^{-1}b \in H$.

Let $a, b \in G$ such that $a \sim b$. From the reflexive property of the congruence relation $\sim$ we know that $a^{-1} \sim a^{-1}$. Applying (c) Explain why $\sim_H$ is the only possible congruence relation on a group G.

Activity 24.14.
(a) State the converse of Lagrange's Theorem. What do we need to do to show that the converse of Lagrange's Theorem is not true?
(b) Consider the group $G = A_4$. List the elements of A_4 in cycle notation and determine the order of A_4.
(c) Assume that H is a subgroup of A_4 of order 6.
(i) Explain why the nonidentity elements of H must have order 2 or 3.
(ii) Explain why there must be an element α of A_4 of order 3 that is not in

H.
(iii) Explain why the left cosets H, αH and $\alpha^2 H$ cannot all be distinct.
(iv) Show that it is not possible for any two of H, αH and $\alpha^2 H$ to be equal.
(d) Explain why the converse of Lagrange's Theorem is not true.

(4) A group G contains elements of every order from 1 to 10. What is the smallest order G could have? Find a group G of that order that contains elements of every order from 1 through 10.

(6) Let $H = \{I, r\}$ in D_4.
(a) Determine all of the distinct left cosets of H in D_4.
First of all, $D_4 = \{I, R, R^2, R^3, r, Rr, R^2r, R^3r\}$ and so $|D_4| = 8$. The number of distinct left cosets of H in D_4 is referred to as the index and is $\dfrac{|D_4|}{|H|} = 4$. These four distinct left cosets are

$$IH = rH = \{I, r\},$$

$$RH = RrH = \{R, Rr\},$$

$$R^2H = R^2rH = \{R^2, R^2r\},$$

and

$$R^3H = R^3rH = \{R^3, R^3r\}.$$

(b) Determine all of the distinct right cosets of H in D_4.
Like in part (a), there are four distinct right cosets. These right cosets are

$$HI = Hr = \{I, r\},$$

$$HR = HR^3r = \{R, R^3\},$$

$$HR^2 = HR^2r = \{R^2, R^2r\},$$

and

$$HR^3 = HRr = \{R^3, Rr\}.$$