1. Suppose that A and B are monoids and that $\phi : A \to B$ is a monoid homomorphism. Show that ϕ sends the invertible elements of A to the invertible elements of B. Use this to show that the determinant of an invertible matrix is non-zero.

2. Show by induction that the determinant of an upper triangular matrix is the product of the diagonal entries.

3. Call a matrix A nilpotent if $A^k = 0$ for some positive integer k. Show that every square nilpotent matrix has determinant zero.

4. Suppose A is square non-invertible. Note that there exists a sequence of elementry row operations $e_1, \ldots, e_n$ such that B the matrix resulting from applying $e_1, \ldots, e_n$ to A is upper triangular and contains a 0 along the diagonal. Use this to prove that the determinant of a square non-invertible matrix is zero.

5. Use concepts in Example 3.104 on page 105 of your text to prove Theorem 3.106.

6. (§3.F # 12) Show that the dual map of the identity map on V is the identity map on V'.

7. (§3.F # 34) The *double dual* of V denoted V'', is defined to be the dual space of V'. In other words $V' = (V')'$. Define $\Lambda : V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for $v \in V$ and $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(b) Show that if $T \in \mathcal{L}(V)$ then $T'' \circ \Lambda = \Lambda \circ T$ where $T'' = (T')'$.

(c) Show that if V is finite dimensional, then Λ is an isomorphism from V onto V''.