MTH354
Homework 2

1. Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Define $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by
$$d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b).$$
Prove $(X \times Y, d)$ is a metric space.

2. Let X be a set with the following metric:
$$\rho(x, x) = 0$$
$$\rho(x, y) = 1, \quad x \neq y$$
Show that in (X, ρ) every subset is open.

3. Show that the function $f : \mathbb{R} \to \mathbb{R}$; $f(x) = |x|$ is continuous for all $x \in \mathbb{R}$.
Hint: Use the reverse triangle inequality.

4. Show that if $d : X \times X \to \mathbb{R}$ is a metric then d is a continuous function.

5. Find the limits and show by arguing directly from the definitions that the following sequences converge.

   a) $a_n = \dfrac{2n - 3}{n + 5}$, $n \geq 0$.

   b) $b_n = \dfrac{n + 5}{n^2 - n - 1}$, $n \geq 2$.

6. Suppose (a_n), (b_n) and (c_n) are sequences of real numbers. Show if $a_n \leq b_n \leq c_n$ for all n and both (a_n) and (c_n) converge to L then (b_n) converges to L.