MTH150 **Name:**________________________

Practice Final **Section:**________________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	3	
2	4	
3	3	
4	3	
5	5	
6	5	
7	2	
Total:	25	

1. (3 points) Differentiate the following functions

a) $f(x) = \dfrac{\sin(x^2)}{x^3+1}$

$$f'(x) = \frac{(x^3+1)\,2x\cos(x^2) - \sin(x^2)\cdot(3x^2)}{(x^3+1)^2}$$

$$\Rightarrow f'(x) = \frac{(2x^4+2x)\cos(x^2) - 3x^2\sin(x^2)}{(x^3+1)^2}$$

b) $f(x) = \sin\left(\tan(\sqrt{1+x^2})\right)$

$$f'(x) = \cos\left(\tan(\sqrt{1+x^2})\right)\cdot \tfrac{d}{dx}\left(\tan(\sqrt{1+x^2})\right)$$

$$= \cos\left(\tan(\sqrt{1+x^2})\right)\cdot \sec^2(\sqrt{1+x^2})\cdot \tfrac{d}{dx}\left(\sqrt{1+x^2}\right)$$

$$= \cos\left(\tan(\sqrt{1+x^2})\right)\cdot \sec^2(\sqrt{1+x^2})\cdot\left(\tfrac{1}{2}(x^2+1)^{-1/2}\cdot 2x\right)$$

$$= \frac{x}{\sqrt{1+x^2}}\cdot \cos\left(\tan(\sqrt{1+x^2})\right)\cdot \sec^2(\sqrt{1+x^2})$$

c) $f(x) = x^{2x}$

$$\ln(f(x)) = \ln(x^{2x}) = 2x\ln x$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{2x}{x} + 2\ln x$$

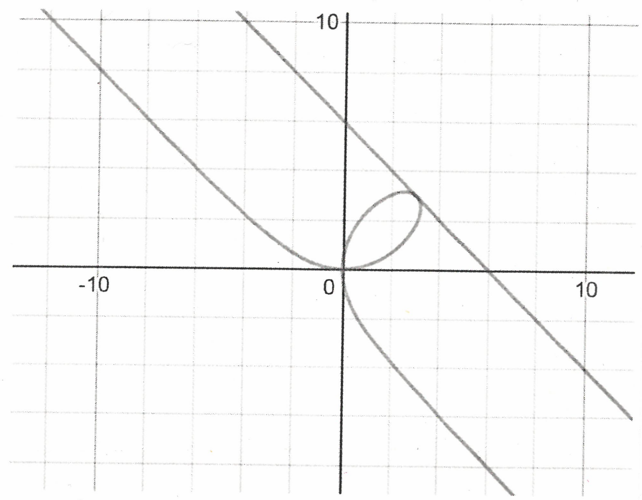
$$\Rightarrow f'(x) = (2+2\ln x)\,x^{2x}$$

2. (4 points) Use implicit differentiation to find the tangent line to the curve

$$x^3 + y^3 = 6xy$$

at the point $(3, 3)$.

$$x^3 + y^3 = 6xy$$



$$3x^2 + 3y^2 \frac{dy}{dx} = 6x \frac{dy}{dx} + 6y$$

$$3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$

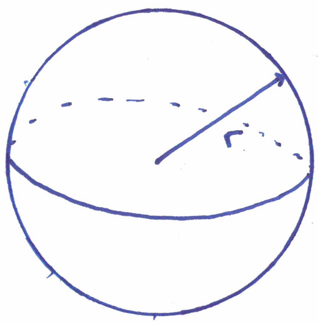
$$\Rightarrow \; dy/dx = \frac{6y - 3x^2}{3y^2 - 6x}$$

@ $(3,3)$ $\quad dy/dx = \dfrac{6(3) - 3(3)^2}{3(3)^2 - 6(3)} = -1$

Tangent line: $y - 3 = -1(x - 3)$

or $y = -x + 6$

3. (3 points) Air is being pumped into a spherical ballon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the ballon increasing when the diameter is 50 cm?



$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

$$\Rightarrow \frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{1}{4\pi r^2}$$

$$\Rightarrow \frac{dr}{dt} = 100 \cdot \frac{1}{4\pi(50)^2} = \frac{1}{100\pi}$$

4. (3 points) Find the linearization $L(x)$ of the function below at $a = 10$ then use the linearization found to estimate the value of $\frac{1}{9.98}$.

$$f(x) = \frac{1}{x}$$

$f'(x) = -1/x^2$; @ $x=10$ $\Rightarrow f'(10) = -1/100$

$L(x) = f(a) + f'(a)(x-a)$

$\Rightarrow \boxed{L(x) = 1/10 - \frac{1}{100}(x-10)}$

$f(9.98) = \frac{1}{9.98}$

$f(9.98) \approx L(9.98) = 1/10 - 1/100\,(9.98-10) = \frac{1}{10} + \frac{.02}{100} = \frac{1}{10} + \frac{2}{10000} = 0.1002$

Note:

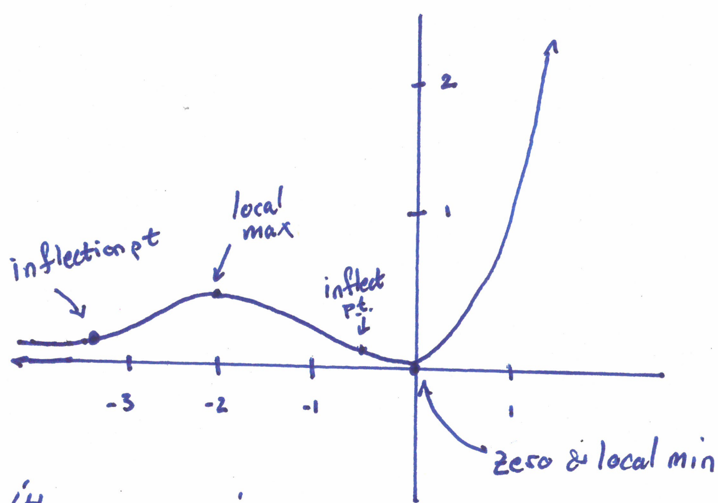
"Exact" Value: 0.1002004008...

Estimated value: 0.1002

5. (5 points) Sketch the function

$$f(x) = x^2e^x$$

Sketch:



$f(x) = x^2e^x$

domain : $(-\infty,\infty)$

zeros : $x=0$

V.A. : None

H.A. : $\lim_{x\to\infty} x^2e^x = \infty$

$\lim_{x\to-\infty} x^2e^x = \lim_{x\to-\infty} \frac{x^2}{e^{-x}} \overset{LH}{=} \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \overset{LH}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0$

$y=0$ is a Horizontal asymptote

$f'(x) = x^2e^x + 2xe^x = xe^x(x+2)$

Critical #'s : $x=0$, $x=-2$

f': $+$ on $(-\infty,-2)$ $\nearrow$; $-$ on $(-2,0)$ $\searrow$; $+$ on $(0,\infty)$ $\nearrow$

local max : $x=-2$ $f(-2) = 4e^{-2} \approx 0.54$

local min : $x=0$ $f(0)=0$

$f''(x) = x^2e^x + 2xe^x + 2xe^x + 2e^x = e^x(x^2+4x+2)$

possible inflection points : $x^2+4x+2=0$

$\Rightarrow x = \frac{-4\pm\sqrt{8}}{2} = -2\pm\sqrt{2}$

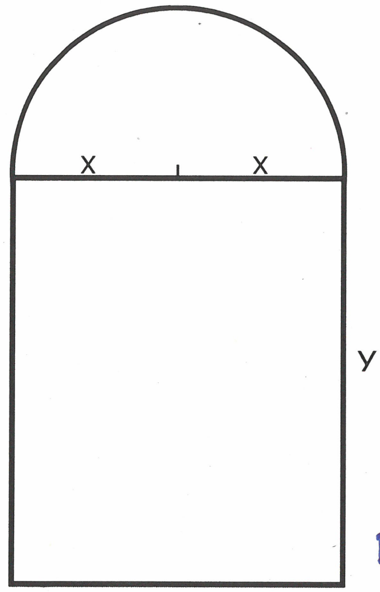
f'': $+$ $\cup$ on $(-\infty,-2-\sqrt{2})$; $-$ $\cap$ on $(-2-\sqrt{2},-2+\sqrt{2})$; $+$ $\cup$ on $(-2+\sqrt{2},\infty)$

$\Rightarrow x = -2-\sqrt{2} \approx -3.41$

$\Rightarrow x = -2+\sqrt{2} \approx -0.58$

6. (5 points) A Norman window, diagrammed below, has the shape of a semi-circle placed on top of a rectangle. If the perimeter of the window is to be fixed at 30 ft, find the dimensions of the window with the greatest area.

Goal: Maximize area w/ perimeter = 30



Area: $A(x,y) = 2xy + \frac{\pi x^2}{2}$

Perimeter: $P(x,y) = \pi x + 2y + 2x$

$\Rightarrow 30 = 2y + (2+\pi)x$

$\Rightarrow y = \frac{30-(2+\pi)x}{2}$

Domain for application: We need $x, y \geq 0$ and $A \geq 0$

For $y \geq 0$ we need $30-(2+\pi)x \geq 0$ i.e. $x \leq \frac{30}{2+\pi} \approx 5.83$

$A(x) = 30x - (2+\pi)x^2 + \frac{\pi x^2}{2} = 30x - (2+\frac{\pi}{2})x^2$

For $A \geq 0$ we need $30-(2+\frac{\pi}{2})x \geq 0$ i.e. $x \leq \frac{30}{(2+\frac{\pi}{2})} \approx 8.40$

Hence we want: $0 \leq x \leq \frac{30}{2+\pi}$

$A(x) = 30x - (2+\frac{\pi}{2})x^2$

$\Rightarrow A'(x) = 30 - 2(2+\frac{\pi}{2})x$

Critical #: $x = \frac{30}{4+\pi} \approx 4.2$

$+ \quad | \quad - \quad \to A'$

$30/4+\pi$

local max @ $x = 30/4+\pi$

$A(30/4+\pi) \approx 63.0$

Test End points of domain

$A(0) = 0$

$A(\frac{30}{2+\pi}) \approx 53.5$

Dimensions: $x = 30/4+\pi$

$y = \frac{30-(2+\pi)(30/4+\pi)}{2} = \frac{30}{2}\left(1 - \frac{2+\pi}{4+\pi}\right) = \frac{30}{4+\pi}$

7. (2 points) Compute the net area under the given curve on the indicated interval.

a) $f(x) = 3x^2 + 2x + 5$ on $[0, 2]$.

$$\int_0^2 3x^2 + 2x + 5 \, dx = x^3 + x^2 + 5x \Big|_0^2 = (8 + 4 + 10) - (0 + 0 + 0) = 22$$

b) $f(x) = 3\sin(x)$ on $[0, \pi]$

$$\int_0^\pi 3\sin(x) \, dx = -3\cos(x) \Big|_0^\pi = -3\cos(\pi) - (-3\cos(0)) = 3 + 3 = 6$$