MTH322 **Name:**________________________

Practice Final Exam **Section:**________________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
Total:	35	

1. (5 points) Compute the Laplace transform of the following functions.

a) $f(t) = \begin{cases} 2t+1 & : 0 < t < 1 \\ 3t^2 & : t > 1 \end{cases}$.

$$f(t) = (2t+1) - (2t+1)\,u(t-1) + 3t^2\,u(t-1)$$

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{2t+1\}(s) - e^{-s}\mathcal{L}\{2(t+1)+1\}(s) + e^{-s}\mathcal{L}\{3(t+1)^2\}$$

$$= \mathcal{L}\{2t+1\}(s) - e^{-s}\mathcal{L}\{2t+3\} + e^{-s}\mathcal{L}\{3t^2+6t+3\}$$

$$= \frac{2}{s^2} + \frac{1}{s} - e^{-s}\left(\frac{2}{s^2} + \frac{3}{s}\right) + e^{-s}\left(\frac{6}{s^3} + \frac{6}{s^2} + \frac{3}{s}\right)$$

$$= \frac{2}{s^2} + \frac{1}{s} + e^{-s}\left(\frac{4}{s^2} + \frac{6}{s^3}\right)$$

b) $f(t) = \begin{cases} t & : 0 < t < 1 \\ -t+2 & : 1 < t < 2 \end{cases}$.

$f(t) = f(t+2)$

$T = 2$

$$f_T(t) = t - t\,u(t-1) + (-t+2)\,u(t-1) - (-t+2)\,u(t-2)$$

$$\Rightarrow f_T(t) = t + u(t-1)(-2t+2) + u(t-2)\cdot(t-2)$$

$$\Rightarrow \mathcal{L}\{f_T\} = \frac{1}{s^2} + e^{-s}\mathcal{L}\{-2(t+1)+2\} + e^{-2s}\mathcal{L}\{(t+2)-2\}$$

$$= \frac{1}{s^2} + e^{-s}\mathcal{L}\{-2t\} + e^{-2s}\mathcal{L}\{t\}$$

$$= \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

So $\mathcal{L}\{f\}(s) = \dfrac{1}{s^2(1-e^{-2s})} - \dfrac{2e^{-s}}{s^2(1-e^{-2s})} + \dfrac{e^{-2s}}{s^2(1-e^{-2s})}$

2. (5 points) Solve the following symbolic problem.

$$x''(t) + 9x = 3\delta(t - \pi); \qquad x(0) = 1, \quad x'(0) = 1.$$

Since

$$\mathcal{L}\{x''\}(s) = s^2 X - s - 1$$

We get

$$s^2 X - s - 1 + 9X = 3e^{-\pi s}$$

$$\Rightarrow (s^2+9)X = 3e^{-\pi s} + s + 1$$

$$X = \frac{3e^{-\pi s}}{s^2+9} + \frac{s}{s^2+9} + \frac{1}{3}\frac{3}{s^2+9}$$

$$\Rightarrow \quad X(t) = \sin(3(t-\pi))\,u(t-\pi) + \cos(3t) + \frac{1}{3}\sin(3t)$$

3. (5 points) Consider the IVP below.

$$5w''(t) + 2w'(t) + 7w(t) = e^{2t-2\pi}; \qquad w(\pi) = 2, \quad w'(\pi) = 1.$$

a) Transform the equation into one solvable by Laplace transform techniques.

Let $y(t) = w(t+\pi)$

so $y'(t) = w'(t+\pi)$ & $y''(t) = w''(t+\pi)$

$y(0) = w(0+\pi) = 2$

$y'(0) = w(0+\pi) = 1$

The equation becomes

$$5y'' + 2y' + 7y = e^{2(t+\pi)-2\pi} \quad ; \; y(0)=2, \; y'(0)=1$$

$$\Rightarrow 5y'' + 2y' + 7y = e^{2t} \quad ; \; y(0)=2, \; y'(0)=1$$

b) Find the Laplace transform of the solution to the transformed equation.

Taking the Laplace transform,

$$5\left(s^2 Y - 2s - 1\right) + 2\left(sY - 2\right) + 7Y = \frac{1}{s-2}$$

$$\Rightarrow (5s^2 + 2s + 7)Y - 10s - 9 = \frac{1}{s-2}$$

$$\Rightarrow (5s^2 + 2s + 7)Y = \frac{1}{s-2} + 10s + 9$$

$$\Rightarrow Y(s) = \frac{1}{(s-2)(5s^2+2s+7)} + \frac{10s+9}{(5s^2+2s+7)}$$

4. (5 points) Use the Laplace transform to solve the IVP below.

$$y''(t) + 5y'(t) + 6y(t) = 6; \qquad y(0) = 0, \;\; y'(0) = 0.$$

Taking the $\mathcal{L}$aplace transform,

$$s^2 Y + 5sY + 6Y = \frac{6}{s}$$

$$Y(s^2+5s+6) = \frac{6}{s}$$

$$\Rightarrow Y = \frac{6}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+3)}$$

$$6 = A(s+2)(s+3) + B(s)(s+3) + C(s)(s+2)$$

$s=0$: $6 = 6A \Rightarrow A = 1$

$s=-2$: $6 = -2B \Rightarrow B = -3$

$s=-3$: $6 = 3C \Rightarrow C = 2$

So $y(t) = 1 - 3e^{-2t} + 2e^{-3t}$

5. (5 points) Use the method of undetermined coefficients to find the general solution to the equation below.

$$y'' + 5y' + 6y = e^{5t}$$

Homogeneous Eqn:

$y'' + 5y' + 6y = 0$

Aux eqn: $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2,\ r = -3$

Homogeneous soln: $y_H(t) = C_1 e^{-2t} + C_2 e^{-3t}$

Particular Soln: $y_p(t) = Ae^{5t}$

$\Rightarrow y_p'(t) = 5Ae^{5t}$

& $y_p''(t) = 25Ae^{5t}$

$y_p'' + 5y_p' + 6y_p = (25A + 25A + 6A)e^{5t} = 56Ae^{5t}$

if $y'' + 5y' + 6y = e^{5t}$ then $A = 1/56$

General soln: $y(t) = \frac{1}{56}e^{5t} + C_1 e^{-2t} + C_2 e^{-3t}$

6. (5 points) Find the general solution to the following first order differential equation.

$$\frac{dy}{dx} = \frac{3x^2 + 7x + \sin(x)}{2\cos(y) + y^2}.$$

$$2\cos(y) + y^2 \quad dy = 3x^2 + 7x + \sin(x) \quad dx$$

$$\Rightarrow \int 2\cos(y) + y^2 \, dy = \int 3x^2 + 7x + \sin(x) \, dx$$

$$2\sin(y) + \frac{y^3}{3} = x^3 + \frac{7}{2}x^2 - \cos(x) + C$$

is a general implicit soln

7. (5 points) Fill out the table using Euler's Method with step size $h = 1$ using the I.V.P below.

$$\frac{dy}{dx} = \frac{x}{y}, \qquad y(0) = 1$$

n	x_n	y_n
0	0	1
1	1	1
2	2	2
3	3	3
4	4	4

$x_{n+1} = x_n + h$

$y_{n+1} = y_n + h\, f(x_n, y_n)$

$x_0 = 0$

$y_0 = 1$

$x_1 = 1$

$y_1 = 1 + 1 \cdot \left(\frac{0}{1}\right) = 1$

$x_2 = 2$

$y_2 = 1 + 1 \cdot \left(\frac{1}{1}\right) = 2$

$x_3 = 3$

$y_3 = 2 + 1\left(\frac{2}{2}\right) = 3$

$x_4 = 4$

$y_4 = 3 + 1\left(\frac{3}{3}\right) = 4$

Table of Laplace Transformations

$f(t)$	$\mathscr{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n \;\; n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n \;\; n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}

Other Formulas:

- $\mathscr{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathscr{L}\{g(t+a)\}(s)$.
- $\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a)$.
- For $f(t)$ periodic with period T, $\mathscr{L}\{f\}(s) = \dfrac{F_T(s)}{1-e^{-sT}}$