Worksheet 1

1. State the definition of the following.
   (a) relation
   (b) equivalence relation
   (c) function
   (d) injective function
   (e) surjective function
   (f) bijective function

(a) A relation R on a set $S$ is a subset of $S \times S$. If $(s_1, s_2) \in R$ we write $s_1 R\, s_2$ or $s_1 \sim s_2$.

(b) An equivalence relation is a relation on a set $S$ with the following properties:
   (i) $a \sim a$ for all $a \in S$.
   (ii) If $a \sim b$ then $b \sim a$ for all $a, b \in S$.
   (iii) If $a \sim b$ and $b \sim c$ then $a \sim c$ for all $a, b, c \in S$.

(c) A function $f : X \to Y$ is a subset of $X \times Y$ such that for every element $x \in X$ there exists at most one $y \in Y$ such that $(x, y) \in f$. If $(x, y) \in f$ we write $f(x) = y$.

(d) An injective function is a function $f : X \to Y$ such that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

(e) A surjective function is a function $f : X \to Y$ such that for all $y \in Y$ there is an $x \in X$ such that $f(x) = y$.

(f) A bijective function is a function $f : X \to Y$ such that $f$ is a surjection and injection.

2. Show that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$ is a bijection.

   *Proof.* Suppose that $f(x_1) = f(x_2)$, then

   $$2x_1 = 2x_2.$$

   If we divide by 2, we get that

   $$x_1 = x_2.$$

   Hence, $f$ is an injection. Now let $y \in \mathbb{R}$ and let $x = \frac{1}{2}y$. We have

   $$f(x) = 2 \cdot \frac{1}{2}y = y.$$

   Hence $f$ is a surjection. □

3. Is the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ a bijection ? Why or why not?

   This is not a bijection. There is no $x \in \mathbb{R}$ such that $f(x) = x^2 = -1$.

Worksheet 1

Define the following equivalence relation on $\mathbb{R}$: $x \sim y$ if and only if $x \cdot y \geq 0$. Is this an equivalence relation?

This is not an equivalence relation. It fails transitivity. Note that $5 \sim 0$ and $0 \sim (-5)$ but $5 \not\sim (-5)$.