MATH3210

Exam 2

The following rules apply:

- **Exam must be typed**. Please organize your proofs in a reasonably neat and coherent way. Write in complete sentences.
- **Mysterious or unsupported claims will not receive full credit**. Unreasonably large gaps in logic or an argument will receive little credit. You may quote theorems from class or the book.
- **Your solutions must be your own.** You may use outside sources but your submitted solution must be in your own words.

1. Show that the determinant of a matrix is the product of its eigenvalues.

 Hint: You can use Corollary 6 in the determinant notes.

 Proof. Let A be a a square $n \times n$ matrix. Let $L_A : \mathbb{C}^n \to \mathbb{C}^n$ denote the operator given by multiplication by A. We note under the standard basis $\mathcal{E} = \{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ that

 $$[L_A]_{\mathcal{E}} = A.$$

 We note that $L_A \in \mathcal{L}(\mathbb{C}^n)$ has an upper triangular matrix with respect to some basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of $\mathbb{C}^n$. We note that the eigenvalues of L_A (and hence A) are the diagonal entries of $[L_A]_{\mathcal{B}}$. For convenience let $B = [L_A]_{\mathcal{B}}$. By corollary 6, we have that

 $$A = P^{-1}BP$$

 for some invertible matrix P. Moreover,

 $$\det(A) = \det(P^{-1}BP) = \det(P^{-1})\det(B)\det(P) = \det(B).$$

 Since the eigenvalues of L_A (and thus A) are the diagonal entries of B we note that

 $$\det(A) = \det(B) = \prod_{i=1}^{n} \lambda_n$$

 since B is upper triangular and where λ_i, $i \in \{1, \ldots, n\}$, are the eigenvalues of A. $\square$

2. For $u \in V$, let φ_u denote the linear functional on the inner product space V defined by

 $$\varphi_u(v) = \langle v, u \rangle$$

 for $v \in V$.

 (a) Show that if $\mathbb{F} = \mathbb{R}$ then the map $\Phi : V(\mathbb{R}) \to V(\mathbb{R})'$ defined by

 $$\Phi(u) = \varphi_u$$

 is a linear map.

 (b) Show that if $\mathbb{F} = \mathbb{C}$ and $V(\mathbb{C}) \neq \{0\}$, then Φ is not linear.

 (c) Suppose that $\mathbb{F} = \mathbb{R}$ and $V(\mathbb{R})$ is finite dimensional. Show Φ is an isomorphism.

 Proof.

 (a) Let $v \in V$ and consider $\Phi(\lambda u + w)$ where $u, w \in V$ and $\lambda \in \mathbb{R}$.

 $$\begin{aligned}(\Phi(\lambda u + w))(v) &= \varphi_{\lambda u + w}(v) \\ &= \langle v, \lambda u + w \rangle \\ &= \lambda \langle u, v \rangle + \langle w, v \rangle \\ &= \lambda \varphi_u(v) + \varphi_w(v) \\ &= (\lambda \Phi(u) + \Phi(w))(v)\end{aligned}$$

(b) Suppose $\mathbb{F} = \mathbb{C}$ and $V(\mathbb{C}) \neq \{0\}$. Since $V \neq \{0\}$, let $u, v \in V$ and consider $(\Phi(iu))(v)$. We have that

$$(\Phi(iu))(v) = \varphi_{iu}(v) = \langle v, iu \rangle = -\imath\langle v, u \rangle = -i(\varphi_u)(v) = -i(\Phi(u))(v).$$

Hence Φ is not linear.

(c) We need only show that the map is injective since $\dim(V) = \dim(V')$ by application of the fundamental theorem of linear maps the map is automatically surjective. Suppose $u, w \in V$ and $\Phi(u) = \Phi(w)$. We have that $(\Phi(u))(v) = (\Phi(w))(v)$ for all $v \in V$. Hence,

$$\langle v, u - w \rangle = 0$$

for all $v \in V$ and in particular for $v = u - w$. Therefore $\|u - w\| = 0$ and we have that $u = w$.

□

3. Let $P(\mathbb{R})$ be the vector space of polynomials and $\mathbb{R}^\infty$ be the vector space of sequences of real numbers (page 13). Show that $P(\mathbb{R})'$ and $\mathbb{R}^\infty$ are isomorphic.

Hint: Construct an explicit isomorphism between $P_n(\mathbb{R})'$ and $\mathbb{R}^{n+1}$ and extend this in the natural way to $P(\mathbb{R})'$ and $\mathbb{R}^\infty$.

Proof. We construct the following isomorphism. Let $\Gamma : P(\mathbb{R})' \to \mathbb{R}^\infty$ be defined by

$$\varphi \mapsto (\varphi(1), \varphi(x), \varphi(x^2), \ldots).$$

We will show that Γ is linear and a bijection. Let $\varphi, \psi \in P(\mathbb{R})'$ and $\lambda \in \mathbb{R}$. Consider

$$\begin{aligned}
\Gamma(\lambda\varphi + \psi) &= ((\lambda\varphi + \psi)(1), (\lambda\varphi + \psi)(x), (\lambda\varphi + \psi)(x^2), \ldots) \\
&= ((\lambda\varphi)(1), (\lambda\varphi)(x), (\lambda\varphi)(x^2), \ldots) + (\psi(1), \psi(x), \psi(x^2), \ldots) \\
&= \lambda(\varphi(1), \varphi(x), \varphi(x^2), \ldots) + (\psi(1), \psi(x), \psi(x^2), \ldots) \\
&= \lambda\Gamma(\varphi) + \Gamma(\psi)
\end{aligned}$$

This shows that Γ is linear. We now show surjectivity. Suppose $(a_0, a_1, a_2, \ldots) \in \mathbb{R}^\infty$ and define $\hat{\varphi} \in P(\mathbb{R})'$ by

$$\hat{\varphi}(c_0 + c_1 x + \ldots + c_n x^n) = c_0 a_0 + c_1 a_1 + \ldots + c_n a_n.$$

Hence we have that,

$$\hat{\varphi}(x^m) = a_m$$

by definition and $\hat{\varphi} \in P(\mathbb{R})'$. Moreover,

$$\Gamma(\hat{\varphi}) = (a_0, a_1, a_2, \ldots).$$

We now show that Γ is injective. Suppose that $\Gamma(\varphi) = (0, 0, \ldots)$ for some $\varphi \in P(\mathbb{R})'$. Hence $\varphi(x^m) = 0$ for all $m \in \mathbb{N}$. By linearity of φ we have that $\varphi(p) = 0$ for all $p \in P(\mathbb{R})$. Thus φ is the zero functional on $P(\mathbb{R})$ and Γ is injective. □

4. Consider the space $\mathbb{C}^\infty$. Let $B : \mathbb{C}^\infty \to \mathbb{C}^\infty$ be defined by the following:

$$B(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots).$$

Does B have eigenvalues? Rectify your answer with Theorem 5.21 (page 145).

Proof. The eigenvalue condition implies the following:

$$(\lambda x_1, \lambda x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Which gives us the following set of equations.

$$\lambda x_1 = 0$$

$$\lambda x_2 = x_1$$

$$\vdots$$

$$\lambda x_n = x_{n-1}$$

$$\vdots$$

If $\lambda = 0$ then $0 = x_1 = x_2 = \ldots$ and $(x_1, x_2, \ldots) = (0, 0, \ldots)$ which is not possible since the zero vector cannot be an eigenvector. If $\lambda \neq 0$ then $x_1 = \frac{0}{\lambda}$, $x_n = \frac{x_{n-1}}{\lambda}$ and $(x_1, x_2, \ldots) = (0, 0, \ldots)$. To rectify this with our prior knowledge we need only to note that these are infinite dimensional vector spaces and the theorem does not apply. □

5. Let $M_{2\times 2}(\mathbb{R})$ denote the vector space of 2×2 matrices with real entries. Define the trace of a matrix A as the sum of the diagonal entries, i.e.

$$\operatorname{tr}(A) = a_{1,1} + a_{2,2}$$

Show that

$$\langle A, B\rangle = \operatorname{tr}(B^\top A)$$

is an inner product on this space.

Proof. We must show the following:

(a) $\langle A, A\rangle \geq 0$ for all $A \in M_{2\times 2}(\mathbb{R})$
(b) $\langle A, A\rangle = 0$ if and only if $A = 0$
(c) $\langle A + B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$ for all $A, B, C \in M_{2\times 2}(\mathbb{R})$
(d) $\langle \lambda A, B\rangle = \lambda\langle A, B\rangle$ for all $\lambda \in \mathbb{R}$ and $A, B \in M_{2\times 2}(\mathbb{R})$
(e) $\langle A, B\rangle = \langle B, A\rangle$ for all $A, B \in M_{2\times 2}(\mathbb{R})$.

Let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}.$$

We have that

$$B^\top A = \begin{pmatrix} b_{1,1}a_{1,1} + b_{2,1}a_{2,1} & b_{1,1}a_{1,2} + b_{2,1}a_{2,2} \\ b_{12}a_{1,1} + b_{2,2}a_{2,1} & b_{1,2}a_{1,2} + b_{2,2}a_{2,2} \end{pmatrix}$$

and

$$\mathrm{tr}(B^\top A) = b_{1,1}a_{1,1} + b_{2,1}a_{2,1} + b_{1,2}a_{1,2} + b_{2,2}a_{2,2}.$$

To prove (a) and (b) note that $\mathrm{tr}(A^\top A) = a_{1,1}^2 + a_{2,1}^2 + a_{1,2}^2 + a_{2,2}^2 \geq 0$ and that $\mathrm{tr}(A^\top A) = 0$ if and only if $a_{i,j} = 0$ for $i, j \in \{1, 2\}$. To prove (c) note

$$\begin{aligned}
\mathrm{tr}(C^\top(A+B)) &= c_{1,1}(a_{1,1}+b_{1,1}) + c_{2,1}(a_{2,1}+b_{2,1}) + c_{1,2}(a_{1,2}+b_{1,2}) + c_{2,2}(a_{2,2}+b_{2,2}) \\
&= c_{1,1}a_{1,1} + c_{1,1}b_{1,1} + c_{2,1}a_{2,1} + c_{2,1}b_{2,1} + c_{1,2}a_{1,2} + c_{1,2}b_{1,2} + c_{2,2}a_{2,2} + c_{2,2}b_{2,2} \\
&= c_{1,1}a_{1,1} + c_{2,1}a_{2,1} + c_{1,2}a_{1,2} + c_{2,2}a_{2,2} + c_{1,1}b_{1,1} + c_{2,1}b_{2,1} + c_{1,2}b_{1,2} + c_{2,2}b_{2,2} \\
&= \mathrm{tr}(C^\top A) + \mathrm{tr}(C^\top B).
\end{aligned}$$

To prove (d) note

$$\begin{aligned}
\mathrm{tr}(B^\top(\lambda A)) &= b_{1,1}\lambda a_{1,1} + b_{2,1}\lambda a_{2,1} + b_{1,2}\lambda a_{1,2} + b_{2,2}\lambda a_{2,2} \\
&= \lambda(b_{1,1}a_{1,1} + b_{2,1}a_{2,1} + b_{1,2}a_{1,2} + b_{2,2}a_{2,2}) \\
&= \lambda\mathrm{tr}(B^\top A).
\end{aligned}$$

Finally to prove (e) note

$$\mathrm{tr}(B^\top A) = b_{1,1}a_{1,1} + b_{2,1}a_{2,1} + b_{1,2}a_{1,2} + b_{2,2}a_{2,2} = \mathrm{tr}(A^\top B).$$

□

6. Show via Cauchy-Schwarz that

$$\left(\frac{a_1 + \ldots + a_n}{n}\right)^2 \leq \frac{a_1^2 + \ldots + a_n^2}{n}$$

i.e. the square of an average is less than or equal to the average of the squares.

Proof. Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{n} = (\frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n})$. We note

$$|\langle \mathbf{a}, \mathbf{n} \rangle|^2 = \left(\frac{a_1 + \ldots + a_n}{n}\right)^2$$

$$\|\mathbf{a}\|^2 = a_1^2 + \ldots + a_n^2,$$

and

$$\|\mathbf{n}\|^2 = \left(\frac{1}{n}\right)^2 + \ldots + \left(\frac{1}{n}\right)^2 = \frac{n}{n^2} = \frac{1}{n}.$$

By Cauchy-Schwarz

$$|\langle \mathbf{a}, \mathbf{n} \rangle|^2 \leq \|\mathbf{a}\|^2 \|\mathbf{n}\|^2$$

and

$$\left(\frac{a_1 + \ldots + a_n}{n}\right)^2 \leq \frac{a_1^2 + \ldots + a_n^2}{n}.$$

□

7. Suppose that V is finite dimensional and U is a subspace of V. Show that

$$P_{U^\perp} = I - P_U,$$

where I is the identity operator on V.

Proof. Suppose that $v \in V$. We note that $v = u + w$ where $u \in U$ and $w \in U^\perp$. By definition of P_U we have that

$$P_U(v) = u.$$

Since $U^{\perp\perp} = U$ we have that

$$P_{U^\perp}(v) = w = v - u = v - P_u(v) = (I - P_U)v$$

as desired. □