MTH322 **Name:** Soln

Exam 2 **Section:**

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books or notes on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	4	
2	4	
3	5	
4	5	
5	4	
6	3	
Total:	25	

1. (4 points) The following questions consider homogeneous differential equations.

   a) Find the general solution to the equation below.

$$y'' + 16y = 0$$

Aux Eqn : $r^2 + 16 = 0$

$\Rightarrow r = \pm 4i$

$y(t) = C_1 \cos(4t) + C_2 \sin(4t)$

   b) Suppose the auxilliary equation for a differential equation is given below:

$$(r+1)(r-2)(r-5)(r-3i)(r+3i) = 0$$

   Determine the order of the differential equation and write the general solution.

5th Order

$y(t) = C_1 e^{-t} + C_2 e^{2t} + C_3 e^{5t} + C_4 \cos(3t) + C_5 \sin(3t)$

2. (4 points) Determine the form of the particular solution for the differential equations below. DO NOT FIND THE COEFFICIENTS.

a) $y'' + 16y = 6t^2 \sin(4t) + 4t^3 \cos(4t)$ $r^2 + 16 = 0$ $r = \pm 4i$; $s = 1$

$y_p(t) = t\,(A_3t^3 + A_2t^2 + A_1t + A_0)\sin(4t) + t\,(B_3t^3 + B_2t^2 + B_1t + B_0)\cos(4t)$

b) $y'' + y' - 6y = (t^5 + 4t^3 + 2t + 1)e^{-3t}$ $r^2 + r - 6 = 0$ $s = 1$

$\Rightarrow (r-2)(r+3) = 0$

$\Rightarrow r = 2 \quad r = -3$

$y_p(t) = t\,(A_5t^5 + A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0)e^{-3t}$

c) $y'' - 4y' + 5y = (5t + 2)e^{5t} + (2t^2 + 2 + 1)e^{2t} \sin(t)$ $r^2 - 4r + 5 = 0$

$r = \dfrac{4 \pm \sqrt{16 - 4(5)}}{2}$

$r = 2 \pm i$

$y_p(t) = (A_1t + A_0)e^{5t} + t\,(B_2t^2 + B_1t + B_0)e^{2t}\sin(t) + t\,(C_2t^2 + C_1t + C_0)e^{2t}\cos(t)$

3. (5 points) Use the method of undetermined coefficients to find the solution to the I.V.P.

$$y'' - 7y' + 12y = 4e^{2x}; \qquad y(0) = 1,\ y'(0) = 0$$

Aux Eqn: $r^2 - 7r + 12 = 0$
$(r-3)(r-4) = 0$
$r = 3 \quad r = 4$

Homogeneous Soln:
$\boxed{y_H(x) = C_1 e^{3x} + C_2 e^{4x}}$

particular Soln: Form

$y_p(x) = Ae^{2x}$
$y_p'(x) = 2Ae^{2x}$
$y_p''(x) = 4Ae^{2x}$

$4Ae^{2x} - 7(2Ae^{2x}) + 12Ae^{2x} = 4e^{2x}$

$\Rightarrow (4A - 14A + 12A)e^{2x} = 4e^{2x}$

$\Rightarrow 2Ae^{2x} = 4e^{2x}$

$\Rightarrow \boxed{A = 2}$

General Soln: $\boxed{y(x) = 2e^{2x} + C_1 e^{3x} + C_2 e^{4x}}$

$y'(x) = 4e^{2x} + 3C_1 e^{3x} + 4C_2 e^{4x}$

Fit: $1 = y(0) = 2 + C_1 + C_2$
$0 = y'(0) = 4 + 3C_1 + 4C_2$

$\Rightarrow 3C_1 + 4C_2 = -4$
$C_1 + C_2 = -1$

$\Rightarrow 3C_1 + 4C_2 = -4$
$-3C_1 - 3C_2 = 3$
$C_2 = -1$
& $C_1 = 0$

Soln: $\boxed{y(x) = 2e^{2x} - e^{4x}}$

4. (5 points) Find the Laplace transform of the following functions.

a) $\mathscr{L}\left\{t^3 + e^{-t}\sin(4t)\right\}(s)$ $= \dfrac{3!}{s^4} + \dfrac{4}{(s+1)^2+16}$

b) $\mathscr{L}\left\{2t^3e^t - e^{2t}\cos(2t)\right\}(s)$ $= 2\left(\dfrac{3!}{(s-1)^4}\right) - \dfrac{(s-2)}{(s-2)^2+4}$

c) $\mathscr{L}\left\{3\sin(t) - 2t^2 + 1\right\}(s)$ $= 3\left(\dfrac{1}{s^2+1}\right) - 2\left(\dfrac{2}{s^3}\right) + \dfrac{1}{s}$

d) $\mathscr{L}\left\{t^3e^{-2t} - 5t + \cos(2t)\right\}(s)$ $= \dfrac{3!}{(s+2)^4} - \dfrac{5}{s^2} + \dfrac{s}{s^2+4}$

5. (4 points) Write down the form of the partial fraction decomposition. DO NOT FIND THE COEFFICIENTS.

a) $\dfrac{s^3 - 2s^2 - 7s + 1}{(s+1)(s+2)(s-4)(s-2)} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+2)} + \dfrac{C}{(s-4)} + \dfrac{D}{(s-2)}$

b) $\dfrac{5s^2 - 2s - 2}{(s-1)^2(s+2)^3} = \dfrac{A}{(s-1)} + \dfrac{B}{(s-1)^2} + \dfrac{C}{(s+2)} + \dfrac{D}{(s+2)^2} + \dfrac{E}{(s+2)^3}$

c) $\dfrac{4s^2 + 2s + 8}{(s+1)(s^2 - 2s + 10)} = \dfrac{A}{(s+1)} + \dfrac{B(s-1) + 3C}{((s-1)^2 + 9)}$

Complete the square

$s^2 - 2s + 1 + 10 - 1$

$= (s-1)^2 + 9$

d) $\dfrac{s^2 + 5s + 6}{(s-2)^3((s-1)^2 + 4)^2} = \dfrac{A}{(s-2)} + \dfrac{B}{(s-2)^2} + \dfrac{C}{(s-2)^3} + \dfrac{D(s-1) + 2E}{(s-1)^2 + 4} + \dfrac{F(s-1) + 2G}{((s-1)^2 + 4)^2}$

6. (3 points) Find the inverse Laplace transform.

a) $\dfrac{4s^2+3s-7}{(s-2)(s+1)(s+3)} = \dfrac{A}{(s-2)} + \dfrac{B}{(s+1)} + \dfrac{C}{(s+3)}$

$4s^2+3s-7 = A(s+1)(s+3) + B(s-2)(s+3) + C(s-2)(s+1)$

$s=-1$: $4-3-7 = B(-3)(2) \Rightarrow B=1$

$s=2$: $16+6-7 = A(3)(5) \Rightarrow A=1$

$s=-3$: $4(9)-9-7 = C(-5)(-2) \Rightarrow C=2$

$$\mathcal{L}^{-1}\left\{\frac{4s^2+3s-7}{(s-2)(s+1)(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+3)}\right\}$$

$$= e^{2t} + e^{-t} + 2e^{-3t}$$

b) $\dfrac{2s^2+2s+9}{(s-1)[(s+1)^2+9]} = \dfrac{A}{(s-1)} + \dfrac{B(s+1)+3C}{(s+1)^2+9}$

$2s^2+2s+9 = A[(s+1)^2+9] + B(s+1)(s-1) + 3C(s-1)$

$$\left.\begin{aligned} s=-1 &: 2-2+9 = 9A-6C \\ s=1 &: 2+2+9 = 13A \\ s=0 &: 9 = 10A - B - 3C \end{aligned}\right\} \quad \begin{aligned} A&=1 \\ C&=0 \\ B&=1 \end{aligned}$$

$$\mathcal{L}^{-1}\left\{\frac{2s^2+2s+9}{(s-1)((s+1)^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+9}\right\}$$

$$= e^{t} + e^{-t}\cos(3t)$$