1. (§5.A #3) Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{ran}(S)$ is invariant under T.

Proof. If $v \in \operatorname{ran}(S)$ then $v = S(u)$ for some $u \in V$. We apply T to v. Hence we have that
$$T(v) = TS(u) = ST(u).$$
Thus $T(v) = S(T(u))$ and is in the range of S. □

2. (§5.B #1) Suppose that $T \in \mathcal{L}(V)$ and there exists a positive integer n such that $T^n = 0$. Prove that $(I - T)$ is invertible and that
$$(I - T)^{-1} = I + T + \cdots + T^{n-1}$$

Proof. We proceed by computation.
$$(I - T)(I + T + \cdots + T^{n-1}) = I + T + \cdots + T^{n-1} - (T + T^2 + \cdots + T^n) = I + T^n = I.$$
Since I commutes with T and T commutes with itself we have that
$$(I + T + \cdots + T^{n-1})(I - T) = I.$$
□

3. Suppose that $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose that $p \in \mathcal{P}(\mathbb{F})$ is a polynomial. Prove that
$$p(STS^{-1}) = Sp(T)S^{-1}.$$

Lemma 0.1.
$$(STS^{-1})^n = ST^nS^{-1}$$
for all $n \in \mathbb{N}$.

Proof. We proceed by induction on n

Base Case: $n = 1$

This case is clear.

Induction Hypothesis: For $n = k - 1$ we have $(STS^{-1})^{k-1} = ST^{k-1}S^{-1}$.

Now for $n = k$ we have that
$$(STS^{-1})^k = (STS^{-1})^{k-1}(STS^{-1}) = ST^{k-1}S^{-1}STS^{-1} = ST^kS^{-1}$$
by our induction hypothesis. □

We now prove our main result.

Proof. Suppose $p(z) = a_0 + a_1 z + \ldots + a_n z^n$. We then have that

$$\begin{aligned} p(STS^{-1}) &= a_0 I + a_1(STS^{-1}) + a_2(STS^{-1})^2 + \ldots + a_n(STS^{-1})^n \\ &= a_0 SS^{-1} + a_1 STS^{-1} + a_2 ST^2 S^{-1} + \ldots + a_n ST^n S^{-1} \\ &= Sp(T)S^{-1} \end{aligned}$$

by a repeated application of our lemma. □

4. (§5.C # 16) The Fibonacci sequence $F_1, F_2, \ldots$ is defined by

$$F_1 = 1, F_2 = 1, \qquad \text{and} \qquad F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 3$$

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x+y \end{bmatrix}.$$

(a) Show that $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$

(b) Find the eigenvalues of T.

(c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

(d) Use the solution to part (c) to compute $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each positive integer n.

Proof.

(a) We proceed by induction on n.

Base case: n=1

Note that $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$.

Induction Hypothesis: Suppose for $n = k-1$ we have that

$$T^{k-1}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_{k-1} \\ F_k \end{bmatrix}$$

Now we apply T to T^{k-1} and we have that

$$T^k\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} F_{k-1} \\ F_k \end{bmatrix}\right) = \begin{bmatrix} F_k \\ F_{k-1} + F_k \end{bmatrix} = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}$$

by application of the recurrence relation and induction hypothesis.

(b) The eigenvector equation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

is equivalent to the system

$$y = \lambda x \quad \text{and} \quad x + y = \lambda y.$$

By substitution we have that

$$x + \lambda x = \lambda^2 x.$$

We note that $x \neq 0$ since this would imply $y = 0$ by the system of equations and the zero vector is not a candidate for an eigenvector. Hence we can dived both sides by x and get that

$$\lambda^2 - \lambda - 1 = 0.$$

The only solutions to this equation are

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

(c) We find the eigenvectors corresponding to the above eigenvectors. Substituting $\lambda = \frac{1\pm\sqrt{5}}{2}$ into the above system and solving for x and y shows that the eigenvectors are

$$\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}. \quad \text{and} \quad \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

These vectors are clearly linearly independent and thus form a basis.

(d) Note that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} - \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

Hence,

$$T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{5}} T^n\left(\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}\right) - \frac{1}{\sqrt{5}} T^n\left(\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}\right).$$

By our eigenvalue relation ship we have that

$$T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

By part (a) we have

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each positive integer n.

$\square$