MATH2210Q

Name: Solution

Exam 2

Date:

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any unapproved calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	4	
2	3	
3	3	
4	6	
5	5	
6	4	
Total:	25	

1. (a) (2 points) Calculate the determinant of the following matrix.

$$A = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & -7 & -5 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix} & \begin{array}{c} - \\ + \\ - \\ + \end{array} \end{array}$$

$$\det A = -2 \det \begin{bmatrix} 1 & -7 & -5 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix} + 4 \det \begin{bmatrix} 0 & 0 & 0 \\ 1 & -7 & 5 \\ 0 & 2 & 6 \end{bmatrix}$$

$$\det A = -2(2) + 4(0)$$

$$\det A = -4$$

(b) (2 points) Use your answer from part (a) to find $\det(A \cdot B^2)$, where

$$B = \begin{bmatrix} 1 & 4 & 4 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

$$\det B = 1 \cdot 1 \cdot 2 \cdot 2 = 4 \qquad \det B^2 = 4^2$$

$$\det(AB^2) = -4^3$$

2. (3 points) Consider the space of polynomials of degree less than or equal to 2, $\mathbb{P}_2(t)$. Let

$$V = \{p(t) = a_0 + a_1 t + a_2 t^2 \mid p(0) = 0\}$$

i.e. V is the subspace of $\mathbb{P}_2(t)$ of polynomials p such that $p(0) = 0$.

(a) Is it possible to have $\dim(V) = 3$? Why or why not.

No, since $\dim(\mathbb{P}_2(t)) = 3$, if $\dim(V) = 3$ then all polynomials would have $p(0) = 0$. Clearly false.

(b) Consider the set of vectors in V:

$$\mathbf{p}_1(t) = t; \qquad \mathbf{p}_2(t) = t(t+1)$$

Are $\{\mathbf{p}_1(t), \mathbf{p}_2(t)\}$ linearly independent?

Yes, since $p_1(t) \neq c\, p_2(t)$ for any c.

(c) Based on your answers from (a) and (b) what is the dimension of V?

By (a) $\dim(V) < 3$

By (b) $\dim(V) \geq 2$

So $\dim(V) = 2$

3. (3 points) Use the coordinate transformation to determine the dimension of $H = \text{span}\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$.

$$\mathbf{p}_0(t) = 5t + t^2, \quad \mathbf{p}_1(t) = 1 - 8t - 2t^2, \quad \mathbf{p}_2(t) = -3 + 4t + 2t^2.$$

$$P_0(t) \longmapsto \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

$$P_1(t) \longmapsto \begin{bmatrix} 1 \\ -8 \\ -2 \end{bmatrix}$$

$$P_2(t) \longmapsto \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 5 & -8 & 4 \\ 1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 5 & -8 & 4 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 2 & -6 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since A has 2 pivots and $\text{rank}(A) = \#$ pivots

$\dim(H) = 2$

4. Let

$$\mathbf{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ -7 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and note that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are bases for $\mathbb{R}^2$.

(a) (2 points) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

$$\left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 2 & 1 & 8 & -7 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & -1 & 10 & -9 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & 1 & -10 & 9 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 9 & -8 \\ 0 & 1 & -10 & 9 \end{array}\right]$$

$$\underset{C\leftarrow B}{P} = \begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix}$$

(b) (2 points) Find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

$$\det \underset{C\leftarrow B}{P} = 81 - 80 = 1$$

$$\underset{B\leftarrow C}{P} = \underset{C\leftarrow B}{P}^{-1} = \begin{bmatrix} 9 & 8 \\ 10 & 9 \end{bmatrix}$$

(c) (2 points) Let $[x]_\mathcal{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find $[x]_\mathcal{C}$.

$$\underset{C\leftarrow B}{P} [x]_B = [x]_C$$

$$\Rightarrow \begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{So } [x]_C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

5. (a) (2 points) If A is a 3×6 matrix, what is the smallest possible dimension of Null(A)?

$\dim(\text{row}(A)) = \text{rank}(A) \leq 3$

since $\text{rank}(A) + \text{nullity}(A) = 6$ we must have

$\text{nullity}(A) \geq 3$.

(b) (1 point) Let B be a 5×5 matrix with $\text{rank}(B) = 5$. Are the columns of B linearly independent?

Yes, since $\text{nullity}(B) = 0$ the columns are linearly independent.

(c) (2 points) If C is a 4×7 matrix with $\text{rank}(C) = 4$ is the linear transformation $T(\mathbf{x}) = C\mathbf{x}$ one to one?

No, $\text{nullity}(C) = 3$

T is one to one if and only if $\text{nullity}(C) = 0$.

6. (4 points) Diagonalize the following matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}.$$

Since A is lower triangular the Eigenvalues of A are $\lambda = 1$, $\lambda = -1$.

$\lambda = 1$: $[A - I \mid 0] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 6 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$$\Leftrightarrow \begin{aligned} 6x_1 &= 2x_2 \\ x_2 &= x_2 \end{aligned}$$

Eigenvectors for $\lambda = 1$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}$.

$\lambda = -1$: $[A + I \mid 0] = \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 6 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$$\Leftrightarrow \begin{aligned} x_1 &= 0 \\ x_2 &= x_2 \end{aligned}$$

Eigenvectors for $\lambda = -1$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}$.

The Eigenvectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly indep.

$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$