MATH2210Q

Exam 2

Name:________________________

Date:________________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any unapproved calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	4	
2	3	
3	3	
4	6	
5	5	
6	4	
Total:	25	

1. Define $T : P_2(t) \to \mathbb{R}^3$ by $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$.

(a) (2 points) Let $\mathcal{B} = \{1, t, t^2\}$ be a basis for $P_2(t)$ and let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Find the matrix for T relative to $\mathcal{B}$ and $\mathcal{E}$.

$$T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T(t^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[T]_B^{\mathcal{E}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) (2 points) Let $p(t) = 1 + 2t + 3t^2$ and use the above matrix and the coordinate transformation to compute $[T(p)]_{\mathcal{E}}$.

$$[P]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

If $T : V \to W$

and B is a basis for V

and C is a basis for W

then

$$[T(\vec{x})]_C = [T]_B^C [\vec{x}]_B$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} = [T(p)]_{\mathcal{E}}$$

2. (3 points) Determine if the following are true or false. If a statement is false, explain why.

a) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

b) $\mathbb{P}_3(t)$ is a subspace of $C[0,1]$.

c) The line $y = 2x + 1$ is a subspace of $\mathbb{R}^2$.

a) False, $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$

b) True

c) False, $y = 2x + 1$ does not contain the origin

3. (3 points) Show that the following polynomials are a basis for $\mathbb{P}_3$

$$p_0(t) = 1, \quad p_1(t) = 1 - t, \quad p_2(t) = -2 + 4t^2, \quad p_3(t) = -12t + 8t^3.$$

Using the coordinate mapping

$$P_0(t) \longmapsto \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$P_1(t) \longmapsto \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$P_2(t) \longmapsto \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

$$P_3(t) \longmapsto \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$$

Since $\dim(\mathbb{P}_3(t)) = 4$ we only need to check these are linearly indep

$$\begin{bmatrix} 1 & 1 & -2 & 0 & 0 \\ 0 & -1 & 0 & -12 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -12 & 0 \\ 0 & 1 & 0 & -12 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\{P_0(t), P_1(t), P_2(t), P_3(t)\}$ are linearly indep.

and thus a basis.

4. Let

$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

and note that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are bases for $\mathbb{R}^2$.

(a) (2 points) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. $[c_1 \; c_2 \; \vdots \; b_1 \; b_2] \sim [I \; \vdots \; \underset{C\leftarrow B}{P}]$

$$\left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ 0 & -8 & 40 & -16 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right] \Rightarrow \boxed{\underset{C\leftarrow B}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}}$$

(b) (2 points) Find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

$$\underset{B\leftarrow C}{P} = \underset{C\leftarrow B}{P}^{-1} = \frac{1}{\det(\underset{C\leftarrow B}{P})} \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \frac{1}{(-6+5)} \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

$$\Rightarrow \boxed{\underset{B\leftarrow C}{P} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}}$$

(c) (2 points) Let $[x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find $[x]_{\mathcal{C}}$.

$$\underset{C\leftarrow B}{P} [X]_B = [X]_C$$

So $\begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$

$$\boxed{\text{Hence } [X]_C = \begin{bmatrix} -2 \\ -3 \end{bmatrix}}$$

5. (a) (2 points) If A is a 3×7 matrix, what is the smallest possible dimension of Null(A)?

$\text{rank}(A) = \dim(\text{row}(A)) \leq 3$

Since $\text{rank}(A) + \text{nullity}(A) = 7$

We need $\text{nullity}(A) \geq 4$

So the smallest possible dimension of Null(A) is 4.

(b) (1 point) Let B be a 5×5 matrix with rank$(B) = 4$. Is B invertible?

No, B is invertible if and only if

$\text{rank}(B) = 5$.

(c) (2 points) If C is a 4×7 matrix with rank$(C) = 4$ is the linear transformation $T(\mathbf{x}) = C\mathbf{x}$ onto ?

Yes, T is onto if and only if $\text{rank}(C) = 4$

6. (4 points) Let

$$A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix} \qquad \lambda = 3$$

where $\lambda = 3$ is an eigenvalue. Find a basis for the eigenspace corresponding to λ.

$$[A - 3I \mid 0] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 3 & -3 & 3 & 0 \\ 2 & -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Longleftrightarrow \quad \begin{aligned} x_1 &= x_3 \\ x_2 &= 2x_3 \\ x_3 &= x_3 \end{aligned}$$

Basis for Eigenspace of $\lambda = 3$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$