

MTH354  
Homework 1

1. Let  $\mathbb{C}$  denote the complex numbers with the standard addition and multiplication. Show that there is no order relation  $>$  such that  $\mathbb{C}$  is an ordered field. As a reminder:

**Definition 0.1.** An *ordered field*  $\mathbb{F} = (\mathbb{F}, +, \cdot, <)$  consists of a field  $(\mathbb{F}, +, \cdot)$  together with a relation  $<$  on  $\mathbb{F}$ , called an *order*, satisfying

- (i) (*trichotomy*) for each  $x, y \in S$ , exactly one of the following hold,

$$x < y, \quad y < x, \quad x = y;$$

- (ii) (*transitivity*) for  $x, y, z \in S$ , if  $x < y$  and  $y < z$ , then  $x < z$ .  
(iii) if  $x, y, z \in \mathbb{F}$  and  $x < y$ , then  $x + z < y + z$ ;  
(iv) if  $x, y \in \mathbb{F}$  and  $x, y > 0$ , then  $xy > 0$ .

2. Find the supremum and infimum of the following set:  $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ . Prove your claim.

3. Show if  $A \subset B$  are subsets of  $\mathbb{R}$  where  $B$  is bounded above, then  $A$  and  $B$  have least upper bounds and

$$\sup(A) \leq \sup(B).$$

Find an example where  $\sup(A) = \sup(B)$ .

4. Let  $A$  and  $B$  be two non-empty subsets of  $\mathbb{R}$  which are bounded below. Show

$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}.$$

5. Suppose that  $A$  and  $B$  are non-empty subsets of  $\mathbb{R}$  that are bounded above. Let

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Show  $A + B$  has a supremum and that  $\sup(A + B) = \sup(A) + \sup(B)$ .