MATH3210 **Name:**________________________

Exam 1 **Date:**________________________

This exam contains 6 pages (including this cover page). Check to see if any pages are missing. Enter all requested information on the top of this page, and put your name on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any unapproved calculator on this exam.

The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Write in complete sentences.
- **Mysterious or unsupported claims will not receive full credit**. Unreasonably large gaps in logic or an argument will receive little credit. You may quote theorems from class or the book.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Definitions	
Question 1	
Question 2	
Question 3	
Question 4	
Question 5	

1. State the definitions of the following concepts:

(a) (1 point) Basis

Definition. A set of vectors $\{v_1, \ldots, v_n\} \subset V$ is a a basis for a (finite dimensional) vector space V if it is a linearly independent set that spans V

(b) (1 point) Linear independence of a set of vectors

Definition. A set of vectors $\{v_1, \ldots, v_n\}$ are linearly independent if

$$a_1 v_1 + \ldots a_n v_n = 0$$

implies that $a_1 = a_2 = \ldots = a_n = 0$.

(c) (1 point) Linear transformation

Definition. A linear transformation is a function from one vector space to another, $T: \ V(\mathbb{F}) \to W(\mathbb{F})$, such that

$$T(\lambda v_1 + v_2) = \lambda T(v_1) + T(v_2)$$

for all $\lambda \in \mathbb{F}$ and $v_1, v_2 \in V$.

(d) (1 point) Null space of a transformation

Definition. Let $T : V \to W$ be a linear transformation from V to W. Define the null space of T as

$$\mathrm{null}(T) = \{v \in V \mid T(v) = 0\}$$

(e) (1 point) Range of a transformation

Definition. Let $T : V \to W$ be a linear transformation from V to W. Define the range of T as

$$\mathrm{ran}(T) = \{w \in W \mid T(v) = w \text{ for some } v \in V\}$$

Choose 5 of the 6 questions below to turn in for your exam. Please label the questions and solutions clearly. They are worth 4 points each.

- Suppose that V is a finite dimensional vector space and U is a subspace of V. Prove that $\dim(U) = \dim(V)$ if and only if $U = V$.

- Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\operatorname{ran}(T) = \operatorname{null}(T).$$

- Consider $C[0, 1]$; the space of continuous real valued functions on $[0, 1]$. Show that

$$U = \{f \in C[0, 1] \mid f(0) = f(1)\}$$

  is a subspace of $C[0, 1]$.

- Suppose that V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists an $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

- Let V be a finite dimensional vector space and U be a subspace of V. Suppose T is a linear transformation defined on U. Show that T can be extended to a linear transformation $\hat{T}$ on V, i.e.

$$\hat{T}(u) = T(u) \quad \text{for } u \in U.$$

- Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1+w, \ldots, v_m+w$ is linearly dependent, then $w \in \operatorname{span}\{v_1, \ldots, v_m\}$.

Suppose that V is a finite dimensional vector space and U is a subspace of V. Prove that $\dim(U) = \dim(V)$ if and only if $U = V$.

Proof. Suppose that V is a vector space and U a subspace such that $\dim(U) = \dim(V)$. Let $\{u_1, \ldots, u_n\}$ be a basis for U. Note $\{u_1, \ldots, u_n\}$ is also a basis of V since it is a linearly independent set of vectors of length equal to the dimension of V. Hence,

$$V = \operatorname{span}\{u_1, \ldots, u_n\} = U.$$

Now suppose that V is a vector space and U is a subspace of V such that $U = V$. Clearly, $\dim(U) = \dim(V)$. $\square$

Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\operatorname{ran}(T) = \operatorname{null}(T).$$

Proof. Suppose for the sake of contradiction that there exists a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\operatorname{ran}(T) = \operatorname{null}(T).$$

By the fundamental theorem of linear maps we have that

$$\dim(\mathbb{R}^5) = 5 = \dim(\operatorname{ran}(T)) + \dim(\operatorname{null}(T)) = 2n$$

for some $n \in \mathbb{N}$. This is a contradiction. $\square$

Consider $C[0, 1]$; the space of continuous real valued functions on $[0, 1]$. Show that

$$U = \{f \in C[0, 1] \mid f(0) = f(1)\}$$

is a subspace of $C[0, 1]$.

Proof. We must show that $0 \in U$ and that U is closed under addition and scalar multiplication. Let $z(t) = 0$ for all $t \in [0, 1]$. Note that $z(t)$ is the zero function and clearly,

$$z(0) = 0 = z(1).$$

Now suppose that $f, g \in U$ and that $\lambda \in \mathbb{R}$. Note that

$$(f + g)(0) = f(0) + g(0) = f(1) + g(1) = (f + g)(1)$$

and

$$(\lambda f)(0) = \lambda \cdot f(0) = \lambda \cdot f(1) = (\lambda f)(1)$$

Hence, $(f + g) \in U$ and $(\lambda f) \in U$. Therefore U is a subspace of $C[0, 1]$. $\square$

Suppose that V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists an $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

Proof. Suppose that there exists a map $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W. To prove T is surjective, let $w \in W$ be arbitrary and note

$$w = (TS)(w) = T(S(w))$$

Hence T sends $S(w) \in V$ to $w \in W$. So T is surjective. Now suppose that $T \in \mathcal{L}(V, W)$ and suppose that T is surjective. For every $w \in W$ there exists a $v \in V$ such that $T(v) = w$. Since T is surjective by the fundamental theorem of linear maps W is finite dimensional. Let $\{w_1, \ldots w_m\}$ be a basis for W, there exists some $v_1, \ldots, v_m$ such that

$$T(v_i) = w_i \quad \text{for} \quad i \in \{1, \ldots, m\}.$$

Define a linear map $S : W \to V$ by $S(w_i) = v_i$ for all $i \in \{1, \ldots, m\}$, i.e.

$$S(\lambda_1 w_1 + \ldots + \lambda_m w_m) = \lambda_1 v_1 + \ldots + \lambda_m v_m.$$

Note that $(TS)(w_i) = w_i$ for every $i \in \{1, \ldots, m\}$. Hence (TS) is the identity map on W. □

Let V be a finite dimensional vector space and U be a subspace of V. Suppose T is a linear transformation defined on U. Show that T can be extended to a linear transformation $\hat{T}$ on V, i.e.

$$\hat{T}(u) = T(u) \quad \text{for } u \in U.$$

Proof. Suppose $T : U \to W$ is a linear transformation defined on U. Since V is finite dimension then U, a subspace of V, is as well. Let $\{u_1, \ldots, u_m\}$ be a basis for U and extend it to a basis $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ of V. Let $v \in V$ where $v = a_1 u_1 + \ldots a_m u_m + b_1 v_1 + \ldots b_n v_n$ and define a linear map $\hat{T} : V \to W$ by

$$\hat{T}(v) = \begin{cases} T(u_i) & v = u_i \quad i \in \{1, \ldots, m\} \\ \\ 0 & v = v_j \quad j \in \{1, \ldots, n\} \end{cases}$$

i.e.

$$\hat{T}(a_1 u_1 + \ldots a_m u_m + b_1 v_1 + \ldots b_n v_n) = T(a_1 u_1 + \ldots a_m u_m).$$

Note,

$$\begin{aligned} \hat{T}(a_1 u_1 + \ldots + a_m u_m + b_1 v_1 + \ldots + b_n v_n) &= T(a_1 u_1 + \ldots a_m u_m) \\ &= a_1 T(u_1) + \ldots + a_m T(u_m) + 0 + \ldots + 0 \\ &= a_1 \hat{T}(u_1) + \ldots + a_m \hat{T}(u_m) + b_1 \hat{T}(v_1) + \ldots + b_n \hat{T}(v_n) \end{aligned}$$

i.e. $\hat{T}$ is linear and clearly extends T. □

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}\{v_1, \ldots, v_m\}$.

Proof. Suppose that $v_1, \ldots, v_m$ is a linearly independent set of vectors in V and $w \in V$ is a vector such that $v_1 + w, \ldots, v_m + w$ are linearly dependent. There exist $a_1, \ldots a_m$ not all zero such that

$$a_1(v_1 + w) + \ldots a_m(v_m + w) = 0.$$

Note,

$$w = -\frac{1}{a_1 + \ldots + a_m}(a_1 v_1 + \ldots + a_m v_m)$$

and that $(a_1 + \ldots + a_m) = 0$ would contradict the linear independence of $v_1, \ldots, v_m$. Hence, $w \in \operatorname{span}\{v_1, \ldots, v_m\}$. $\square$