1. A sequence of integers $x_1, x_2, x_3, \ldots$ is defined recursively by $x_1 = 3$, $x_2 = 7$ and

$$x_k = 5x_{k-1} - 6x_{k-2} \quad \text{for all } k \geq 3$$

Prove by induction that $x_n = 2^n + 3^{n-1}$ for all positive integers n.

Solution:

Proof. **Base Case:** n=1

We note the following:

$$x_1 = 2^1 + 3^0 = 3$$

Induction Hypothesis: Suppose for $n \leq k$ that

$$x_k = 2^k + 3^{k-1}$$

We show for $n = k + 1$ that

$$x_{k+1} = 2^{k+1} + 3^k$$

By definition,

$$x_{k+1} = 5x_k - 6x_{k-1}$$

Hence by the induction hypothesis,

$$\begin{aligned}
x_{k+1} &= 5x_k - 6x_{k-1} \\
&= 5(2^k + 3^{k-1}) - 6(2^{k-1} + 3^{k-2}) \\
&= 5(2^k) + 5(3^{k-1}) - 6(2^{k-1}) - 6(3^{k-2}) \\
&= 5(2^k) + 5(3^{k-1}) - 3(2^k) - 2(3^{k-1}) \\
&= 2^{k+1} + 3^k
\end{aligned}$$

□

2. Prove by induction that a set of n elements contains 2^n subsets (including the set itself and $\emptyset$).

Solution: We prove this by induction on the number of elements.

Proof. **Base Case:** n=1

Let A be a set with $n = 1$ elements, i.e. $A = \{a\}$. Note that A has two subsets: $\{\ \}$ and $\{a\}$. Hence, our proposition is true for the base case.

Induction Hypothesis: Suppose for $n \leq k$ that a set with k elements has 2^k subsets.

We show for $n = k + 1$ that a set with $k + 1$ elements has 2^{k+1} subsets. Let A be a set with $k+1$ elements. For ease, label the elements with subscripts 1 through $k+1$, i.e. $A = \{a_1, \ldots a_{k+1}\}$. Now, if B is an arbitrary subset of A, then

$$a_{k+1} \in B$$

or

$$a_{k+1} \notin B$$

that is,

$$B = C \cup \{\}$$

or

$$B = C \cup \{a_{k+1}\}$$

where $C \subseteq \{a_1, \ldots a_k\}$. By our induction hypothesis, $\{a_1, \ldots a_k\}$ has 2^k subsets. Hence, there are 2^k subsets of A of the form

$$C \cup \{\ \} \qquad \text{where } C \subseteq \{a_1, \ldots a_k\}$$

and 2^k subsets of A of the form

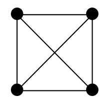
$$C \cup \{a_{k+1}\} \qquad \text{where } C \subseteq \{a_1, \ldots a_k\}.$$

Since every subset of A is exactly one of these forms we have that A has $2 \cdot 2^k = 2^{k+1}$ subsets.

$\square$

3. Prove by induction that if n points lie in a plane and no three are colinear, prove that there are $\frac{1}{2}n(n-1)$ lines joining these points.

Example:



Solution: We prove this by induction on the number of points in the plane.

Proof. **Base Case:** n=1

Suppose that there is one point p in the plane. Obviously, there are 0 lines connecting p to other points. Hence our proposition is true for the base case.

Induction Hypothesis: Suppose for $n = k$ points in the plane where no three are colinear that there is $\frac{1}{2}k(k-1)$ lines connecting them.

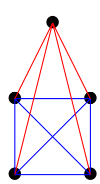
We show for $n = k+1$ points in the plane where no three are colinear that there are $\frac{1}{2}(k+1)k$ lines connecting them. Suppose there are $k+1$ points in the plane where no three are colinear. Label them $p_1, \ldots, p_{k+1}$. Consider the points, $p_1, \ldots p_k$. These are k points, in which no three are colinear. By our induction hypothesis, there are $\frac{1}{2}k(k-1)$ lines connecting the points $p_1, \ldots p_k$. There are k lines connecting p_{k+1} to $p_1, \ldots p_k$ (since there are k points). The total number of lines connecting the points $p_1, \ldots p_{k+1}$ is thus

$$\frac{1}{2}k(k-1) + k = \frac{k(k-1)+2k}{2} = \frac{k^2+k}{2} = \frac{1}{2}k(k+1)$$

lines connecting the $k+1$ points $p_1, \ldots p_{k+1}$.

$\square$

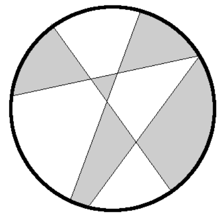
Example:



$k + \frac{1}{2}k(k-1) = \frac{1}{2}k(k+1)$

4. Suppose that n chords are drawn in a circle, dividing the circle into different regions. Prove that every region can be colored one of two colors such that adjacent regions are different colors.

Example:



Solution: We prove this by induction on the number of chords on the circle.

Proof. For convenience, we call the property that every region can be colored with one of two colors so that adjacent regions are different colors "2-colorable". If we have colored the regions with 2 colors so that the adjacent regions are different colors we call that a "2-coloring".

Base Case: n=1

One chord cuts a circle into two distinct regions. Obviously, the regions are 2-colorable.

Induction Hypothesis: Suppose for $n = k$ chords on the circle that the regions defined by the chords are 2-colorable.

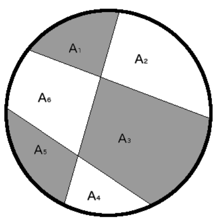
We show for $n = k + 1$ chords that the regions defined by the chords are 2-colorable. Label the chords $\{c_1, \ldots c_{k+1}\}$. The first k chords divide the circle into n regions, label them $A_1, \ldots A_n$. By our induction hypothesis, $A_1, \ldots A_n$ are 2-colorable, i.e. if A_i and A_j (where $i, j \in \{1, \ldots n\}$) are adjacent then A_i and A_j are two different colors. The chords $\{c_1, \ldots c_{k+1}\}$ divide the circle into m-regions $B_1, B_2, \ldots, B_m$. Notice, each B_i is contained in some A_j. The $k+1$-th chord c_{k+1} divides the circle into two regions C_1 and C_2. Again, each B_i is contained in either C_1 or C_2. We now produce a 2-coloring of $B_1, \ldots, B_m$ in a two step process.

Step 1. Given the 2-coloring of $A_1, \ldots A_n$, if B_i is contained in A_j then give it the same color.

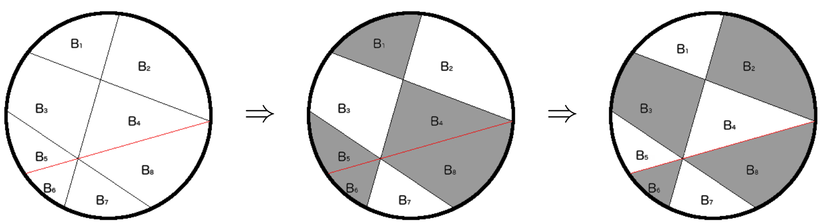
Step 2. Given this coloring of $B_1, \ldots B_m$ to produce a 2-coloring, if B_i is contained in C_1 then change the color to the opposite color of B_i.

After Step 1. the regions $B_1, \dots B_m$ are colored with 2 colors, however two adjacent regions B_i and B_j can be the same color (obviously, this is not a 2-coloring of $B_1, \dots, B_m$). After Step 2. we have a 2-coloring of $B_1, \dots B_m$. Since, if two adjacent regions B_i and B_j do not share c_{k+1} as a side, then since they were contained in some A_ℓ and A_d, they did not share a color after Step 1., changing to the opposite color in Step 2. does not change this. If B_i and B_j shared c_{k+1} as a side, then they were the same color in Step 1. and opposite colors in Step. 2. $\square$

Example of Process:



2-coloring of k chords



k+1 chords Step 1. Step 2.

5. Prove that multiplication is a well defined operation on $\mathbb{Q}$.

Solution:

Proof. Suppose that $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and $\frac{c_1}{d_1} = \frac{c_2}{d_2}$, we need to show that

$$\frac{a_1 c_1}{b_1 d_1} = \frac{a_2 c_2}{b_2 d_2}.$$

In the language of equivalence classes we need to show that if

$$(a_1, b_1) \sim (a_2, b_2) \qquad \text{and} \qquad (c_1, d_1) \sim (c_2, d_2)$$

then

$$(a_1 c_1, b_1 d_1) \sim (a_2 c_2, b_2 d_2).$$

This is equivalent to showing that

$$a_1 c_1 b_2 d_2 = a_2 c_2 b_1 d_1.$$

Since $(a_1, b_1) \sim (a_2, b_2)$ we have

$$a_1 b_2 = a_2 b_1.$$

Likewise, since $(c_1, d_1) \sim (c_2, d_2)$ we have

$$c_1 d_2 = c_2 d_1.$$

Hence,

$$a_1 c_1 b_2 d_2 = a_2 b_1 c_1 d_2 = a_2 b_1 c_2 d_1.$$

□

6. Prove that $\sqrt{3}$ is irrational.

Solution:

Proof. Suppose there is a rational number x such that $x^2 = 3$. Let $x = \frac{a}{b}$ where x is in lowest terms, i.e. $\gcd(a, b) = 1$. Now $\left(\frac{a}{b}\right)^2 = 3$ so we have that $a^2 = 3b^2$. Therefore $3 \mid a^2$ and since 3 is prime we have that $3 \mid a$. Hence $a = 3c$ for some $c \in \mathbb{Z}$. Therefore $9c^2 = 3b^2$ and thus $3c^2 = b^2$. It now follows $3 \mid b^2$ and hence $3 \mid b$. So 3 is a common divisor of a and b, but $\gcd(a, b) = 1$. We have a contradiction. □