Sequence Theorems

Theorem 1 (Squeeze Theorem)**.**

If $a_n \leq b_n \leq c_n$ (for $n \geq n_0$) and $\lim a_n = \lim c_n = L$ then $\lim b_n = L$.

Theorem 2 (Absolute Convergence Theorem)**.**

If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$.

Theorem 3 (Geometric Sequence Theorem)**.**

If r is a real number such that $|r| < 1$ then $\lim\limits_{n\to\infty} r^n = 0$.

Theorem 4 (Monotonic Sequence Theorem)**.**

If a sequence $\{a_n\}$ is monotonically increasing and bounded above then $\{a_n\}$ is convergent.
If a sequence $\{a_n\}$ is monotonically decreasing and bounded below then $\{a_n\}$ is convergent.

Series Tests

Theorem 5 (Geometric Series)**.**

The Geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots, \text{ where } a \neq 0$$

is absolutely convergent if $|r| < 1$ and its sum is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ where } a = \text{first term.}$$

If $|r| \geq 1$ then the geometric series is divergent.

Theorem 6 (Test for Divergence)**.**

If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then the series is divergent.

Theorem 7 (Integral Test)**.**

If f is a **continuous, positive,** and **decreasing** function on the interval $[0, \infty)$ and $a_n = f(n)$, then

1) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is finite then $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is not finite then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem 8 (P-series Test).

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent when } p > 1,$$
$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is divergent when } p \leq 1.$$

Theorem 9 (Direct Comparison Test).

Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms.

1) If $\sum b_n$ is convergent and $a_n \leq b_n$, for all $n \geq n_0$ where n_0 is some natural number, then $\sum a_n$ is absolutely convergent.

2) If $\sum b_n$ is divergent and $a_n \geq b_n$, for all $n \geq n_0$ where n_0 is some natural number, then $\sum a_n$ is divergent.

Theorem 10 (Limit Comparison Test).

Suppose that $\sum a_n$ and $\sum b_n$ are series with **positive** terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $0 < c < \infty$, then either both series converge absolutely or diverge.

Theorem 11 (Alternating Series Test).

If the alternating series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \ldots$$

where $b_n > 0$, satisfies

(i) $b_{n+1} \leq b_n$,

(ii) $\lim_{n\to\infty} b_n = 0$,

then the series is convergent.

Theorem 12 (Ratio Test).

Let $\sum a_n$ be a series and suppose that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

1) If $L < 1$ then the series is absolutely convergent.

2) If $L > 1$ then the series is divergent.

3) If $L = 1$ then the test is inconclusive.

Theorem 13 (Root Test)**.**

Let $\sum a_n$ be a series and suppose that
$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

1) If $L < 1$ then the series is absolutely convergent.

2) If $L > 1$ then the series is divergent.

3) If $L = 1$ then the test is inconclusive.

Theorem 14 (Absolute Convergence Implies Convergence)**.**

If the series $\sum a_n$ is absolutely convergent then the series is convergent.

Definitions and Remarks

Definition: The series $\sum_{n=0}^{\infty} a_n$ is **convergent** if the limit of the partial sums converges, i.e.

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} (a_1 + a_2 + \ldots + a_N) = S$$

where S is a finite number.

Remark: You can find the value of the sum when the series is geometric or telescopic.

Definition: A series $\sum a_n$ is **absolutely convergent** if the series $\sum |a_n|$ is convergent.

Definition: A series $\sum a_n$ is **conditionally convergent** if the sum $\sum a_n$ is convergent but $\sum |a_n|$ is **not** convergent.