

1 Basic Notions

Definition 1. Let $\vec{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be two vectors in \mathbb{R}^n . We define the *dot product* to be

$$\vec{u} \cdot \vec{v} = a_1b_1 + a_2b_2 + \dots + a_nb_n$$

Example 1. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$$\vec{u} \cdot \vec{v} = 1(4) + 2(5) + 3(6)$$

Theorem 1. Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in \mathbb{R}^n and $c \in \mathbb{R}$.

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
- $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot c\vec{v}$
- $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = 0$.

Definition 2. We define the *norm* (magnitude, length) of a vector \vec{u} to be $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$. A vector with length 1 is called a *unit vector*. We define the *distance* between two vectors \vec{v} and \vec{u} is defined to be $\|\vec{v} - \vec{u}\|$.

Note that $\frac{\vec{v}}{\|\vec{v}\|}$ is always a unit vector.

Definition 3. We say that \vec{u} is *orthogonal* (perpendicular) to \vec{v} , written $\vec{u} \perp \vec{v}$, if and only if $\vec{u} \cdot \vec{v} = 0$.

Definition 4. If W is a subspace, we say \vec{v} is orthogonal to W , written $\vec{v} \perp W$ if and only if $\vec{v} \perp \vec{w}$ for all $\vec{w} \in W$. We call the set of all vectors $\vec{v} \perp W$ the *orthogonal complement* of W and denote this set W^\perp .

Proposition 2. W^\perp is a subspace. Moreover, $\vec{v} \perp W$ if and only if $\vec{v} \perp \vec{w}_i$ for $i = 1, \dots, p$ where $\{\vec{w}_1, \dots, \vec{w}_p\}$ is a basis for W .

Definition 5. A set of vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ is orthogonal if and only if

$$\vec{u}_i \cdot \vec{u}_j = 0 \text{ when } i \neq j.$$

An *orthogonal basis* is a basis which is also an orthogonal set. An *orthonormal basis* is an orthogonal basis consisting of unit vectors.

Theorem 3. An orthogonal set of vectors is linearly independent. Moreover, if an orthogonal set $\{\vec{u}_1, \dots, \vec{u}_p\}$ spans a subspace U then $\{\vec{u}_1, \dots, \vec{u}_p\}$ is a basis for U .

Note that if W is an n dimensional vector space, any set of n orthogonal vectors is automatically a basis for W .

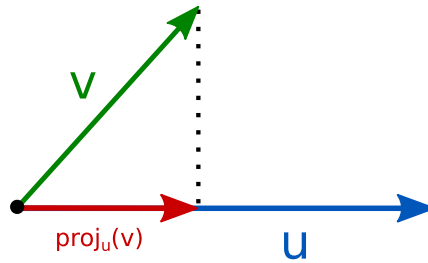
2 Orthogonal Projections

Definition 6. Given two vectors \vec{v} and \vec{u} we define

$$\hat{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \right) \vec{u}$$

to be the *orthogonal projection* of \vec{v} onto \vec{u} . This is sometimes denoted

$$\hat{v} = \text{proj}_{\vec{u}}(\vec{v}).$$



In the above picture the green vector \vec{v} is being projected onto the blue vector \vec{u} and the resulting red vector is the projection. Notice that the projection points in the same direction as \vec{u} .

Moreover,

$$\vec{z} = (\vec{v} - \hat{v}) \perp \vec{u}$$

and the length of $\vec{z} = \vec{v} - \hat{v}$ is represented by the dashed line.

Definition 7. Let \vec{v} and \vec{u} be two vectors let \hat{v} be the projection of \vec{v} onto \vec{u} . If we define $\vec{z} = \vec{v} - \hat{v}$ then

$$\vec{v} = \hat{v} + \vec{z}$$

where \hat{v} is parallel to \vec{u} and \vec{z} is perpendicular to \vec{u} . We call \hat{v} the component of \vec{v} parallel to \vec{u} and \vec{z} is the component of \vec{v} perpendicular to \vec{u} .

Example 2. Let $\vec{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

$$\vec{v} \cdot \vec{u} = 40, \quad \vec{u} \cdot \vec{u} = 20, \quad \hat{v} = \frac{40}{20} \vec{u} = 2\vec{u}$$

Hence we have that,

$$\vec{z} = \vec{v} - \hat{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Notice that,

$$\vec{z} \cdot \vec{u} = 0$$

and

$$\vec{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \hat{v} + \vec{z}$$

Orthogonal basis are convenient for the following reason.

Theorem 4. Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthogonal basis for W . For all $\vec{y} \in W$ we have

$$\vec{y} = c_1 \vec{u}_1 + \dots + c_p \vec{u}_p \quad (1)$$

and

$$c_j = \left(\frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j} \right) \text{ for } j = 1, \dots, p \quad (2)$$

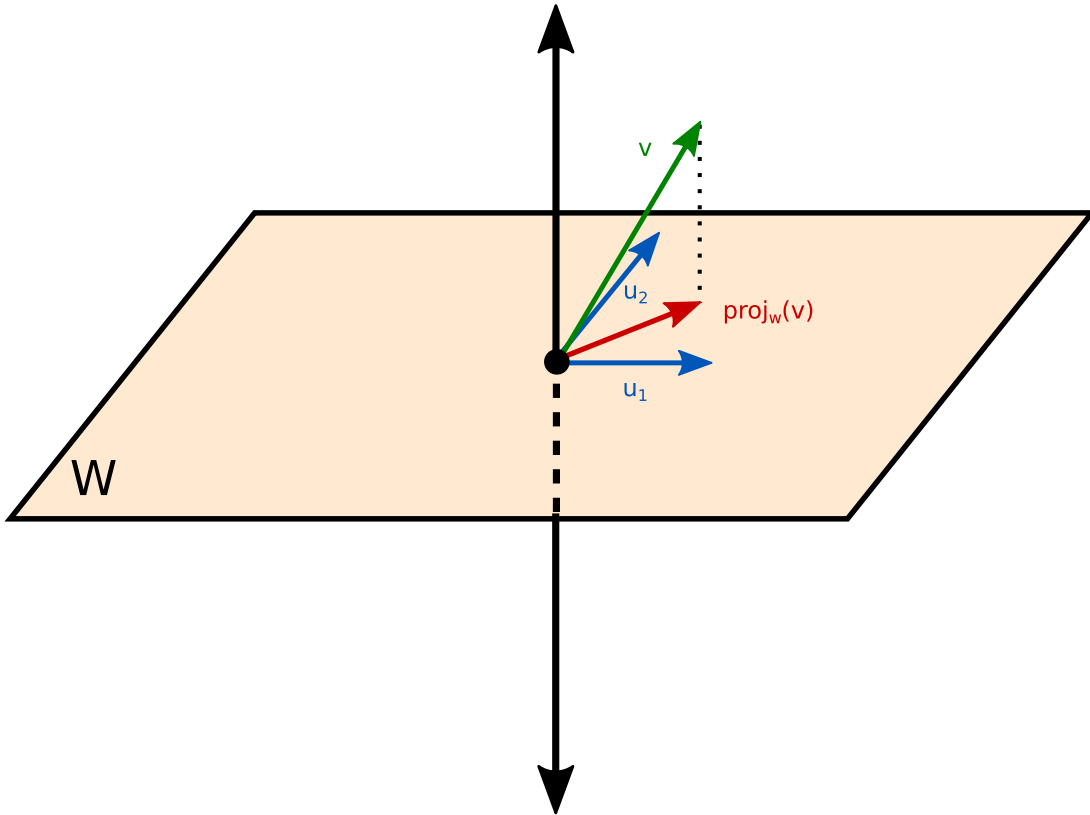
The above formulas (1) and (2) will make a bit more sense after the following definition.

Definition 8. Let W be a subspace of \mathbb{R}^n . Then each $\vec{v} \in W$ can be written uniquely in the form

$$\vec{v} = \hat{v} + \vec{z}$$

where $\hat{v} \in W$ and $\vec{z} \in W^\perp$. We call \hat{v} the orthogonal projection of \vec{v} onto W , sometimes denoted $\hat{v} = \text{proj}_W(\vec{v})$. Moreover, if $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthogonal basis for W then

$$\hat{v} = \left(\frac{\vec{v} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \right) \vec{u}_1 + \dots + \left(\frac{\vec{v} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p} \right) \vec{u}_p$$



In the above picture, W is a 2 dimensional subspace with basis vectors \vec{u}_1 and \vec{u}_2 (blue vectors). The red vector, \hat{v} , is the projection of the green vector, \vec{v} onto W . Moreover, the dashed line is the length of the vector $\vec{v} - \hat{v}$.

Definition 9. The distance between a vector \vec{v} and a subspace W is given by $\|\vec{z}\| = \|\vec{v} - \hat{v}\|$.

Notice, (1) and (2) makes some more sense in the context of the above definition. If \vec{y} is in the subspace W then it is *equal* to its projection onto W , i.e. $\vec{y} = \hat{y}$.

3 Gram Schmidt

The Gram Schmidt process allows us to get an orthogonalized version of any basis. Essentially, at each step of the process you subtract off the projection onto a subspace. Moreover, we can rescale at each step since if $\vec{v} \perp \vec{u}$ the $c\vec{v} \perp \vec{u}$.

Theorem 5. Given a basis $\{\vec{x}_1, \dots, \vec{x}_p\}$ for a subspace W of \mathbb{R}^n . Define,

$$\begin{aligned}\vec{v}_1 &= \vec{x}_1 \\ \vec{v}_2 &= \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 \\ \vec{v}_3 &= \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \right) \vec{v}_2 \\ &\vdots \\ \vec{v}_p &= \vec{x}_p - \left(\frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 - \left(\frac{\vec{x}_p \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \right) \vec{v}_2 - \dots - \left(\frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \right) \vec{v}_{p-1}\end{aligned}$$

Then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is an orthogonal basis for W and

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{span}\{\vec{x}_1, \dots, \vec{x}_k\} \quad \text{for } k = 1, \dots, p.$$

Example 3. Let

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Step 1:

Set $\vec{v}_1 = \vec{x}_1$.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Step 2:

Set $\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{3}{4} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

Chapter 6
Notes

Step 3:

Rescale

$$\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Step 4:

Set $\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right) \vec{v}_1 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right) \vec{v}_2$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{4}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{12}\right) \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

Step 5:

Rescale

$$\vec{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

Result:

We have that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$