

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any unapproved calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	4	
2	4	
3	5	
4	2	
5	4	
6	6	
Total:	25	

1. (4 points) Find the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -3 & 1 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -3 & 1 \end{bmatrix}$$

2. (a) (2 points) Find the solution set of the following system.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 1 & -3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_3$$

$$x_2 = x_3$$

x_3 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

t is a real #.

- (b) (2 points) Can the following matrix be invertible? (your answer from part (a) is relevant)

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix}$$

No, $A\vec{x} = \vec{0}$ has more than just the trivial solution.

3. Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

(a) (1 point) Compute $2A + B$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 4 & 2 & 0 \\ 2 & 6 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 3 & 2 & 3 & 0 \\ 1 & 9 & 3 & 3 \end{bmatrix}$$

(b) (2 points) Compute $A \cdot B$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) (2 points) What is A^{-1}

$$A^{-1} = B$$

4. (2 points) Consider the following linear transformation.

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where A is the standard matrix for T i.e.

$$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3] = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ T(\mathbf{e}_3)]$$

In terms of the columns of A write $T\left(\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}\right)$.

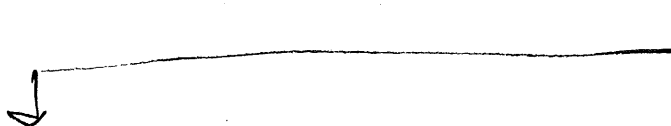
$$\begin{aligned} & T\left(2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ &= 2T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= 2\vec{a}_1 + \vec{a}_2 + 5\vec{a}_3 \end{aligned}$$

5. Consider the following matrix equation.

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$$

(a) (2 points) Find the solution set.

$$\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$



$$\sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -2 \end{bmatrix}$$

(b) (2 points) Write the equivalent linear system and vector equation.

$$\begin{aligned} x_1 - x_3 &= 4 \\ 3x_2 - 2x_3 &= 1 \\ -2x_1 + 6x_2 + 3x_3 &= -4 \end{aligned}$$

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$$

6. Consider the following matrix and one of its echelon forms.

$$A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \sim E = \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (2 points) Find a basis for $\text{Null}(A)$.

$$\begin{aligned} [A|0] &= [E|0] = \begin{bmatrix} 1 & -2 & 5 & 4 & 0 \\ 0 & 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & 5 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \text{Null}(A) \text{ basis} &= \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \\ &\sim \begin{bmatrix} 1 & -2 & 0 & -6 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(b) (2 points) Find a basis for $\text{Col}(A)$.

Pivot columns are 1 and 3

$$\text{Col}(A) \text{ basis} = \left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix} \right\}$$

(c) (1 point) If $T(\mathbf{x}) = A\mathbf{x}$ is T a one to one transformation?

No, $A\vec{x} = \vec{0}$ has more than just the trivial solution

(d) (1 point) If $T(\mathbf{x}) = A\mathbf{x}$ is T onto?

No, $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ $\text{col}(A) \neq \mathbb{R}^3$