MATH 2710 Your name
Homework 3 October 24, 2016

1. Let $A$ be a set and define $P(A)$ to be the set of all subsets of $A$. Let $C$ be a fixed subset of the set $A$ and define relation $R$ on the set $P(A)$ by $XRY$ if and only if $X \cap C = Y \cap C$. Prove that this is an equivalence relation.

   **Solution:** We will show that this is an equivalence relation.

   *Proof.* We must show that for all $X, Y, Z \in P(A)$ that

   (a) $XRX$

   (b) If $XRY$ then $YRX$

   (c) If $XRY$ and $YRZ$ then $XRZ$.

   To show (a), notice that $X \cap C = X \cap C$ and thus $XRX$. To show (b) suppose that $XRY$, then $X \cap C = Y \cap C$. Obviously, $Y \cap C = X \cap C$ so $YRX$. To show (c), suppose that $XRY$ and $YRZ$. Thus $X \cap C = Y \cap C$ and $Y \cap C = Z \cap C$. By transitivity of set equality we have that $X \cap C = Z \cap C$. Hence, $XRZ$. □

2. Let $A$ be a set and let $P$ be a partition of the set $A$ i.e. $P = \{A_1, A_2, \ldots A_n\}$ where

   i) $A_i \subset A$,

   ii) $\emptyset \notin P$

   iii) $A_1 \cup A_2 \cup \ldots \cup A_n = A$

   iv) $A_i \cap A_j = \emptyset$ when $i \neq j$.

   For $x, y \in A$ we say that $xRy$ if and only if $x \in A_i$ and $y \in A_i$ for the same $i$. Prove this is an equivalence relation.

   **Solution:** We will show that this is an equivalence relation.

   *Proof.* We must show that for all $x, y, z \in A$ that

   (a) $xRx$

   (b) If $xRy$ then $yRx$

   (c) If $xRy$ and $yRz$ then $xRz$.

   To show (a) notice that if $x \in A_i$ for some $i$ that $x \in A_i$, i.e. $xRx$. To show (b) suppose that $xRy$, then $x \in A_i$ and $y \in A_i$ for the same $i$. Hence $yRx$. To show (c), suppose that $xRy$ and $yRz$. Then $x \in A_i$ and $y \in A_i$ for some $i$ and $y \in A_j$ and $z \in A_j$ for some $j$. We must have that $A_i = A_j$ since $y \in A_i$, $y \in A_j$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. Hence $x$ and $z$ lie in the same $A_i$ and thus $xRz$. □

3. Prove or disprove: The relation $R$ defined on the set $\mathbb{Z}$ by $xRy$ if and only if $xy > 0$ is an equivalence relation.

**Solution:** We will show that $R$ is not an equivalence relation by providing a counterexample. For $R$ to be an equivalence relation on $\mathbb{Z}$ we need that $xRx$ for all $x \in \mathbb{Z}$, i.e.

$$x^2 > 0 \text{ for all } x \in \mathbb{Z}$$

However, if $x = 0$ then $0^2 \not> 0$ and we have a counterexample.

4. Find all the $x$ that satisfy the following equation. (Hint: Use Fermat's Little theorem and notice that if $x_0$ is a solution then it's entire residue class is a solution.)

$$x^{86} \equiv 2 \pmod 7$$

**Solution:** We note that $7 \nmid x$ since if $x = 7 \cdot n$ for some $n \in \mathbb{Z}$ then

$$x^{86} \equiv 0 \mod 7.$$

Hence, by Fermat's Little theorem

$$x^6 \equiv 1 \mod 7$$

and

$$x^{86} = x^{6(14)+2} \equiv x^2.$$

Thus, we can reduce the problem to solving

$$x^2 \equiv 2 \mod 7.$$

Upon inspection we see that the solution sets are [3] and [4].

5. Prove that every integer of the form $5n + 3$ for $n \in \mathbb{Z}$, $n \geq 1$, cannot be a perfect square.

**Solution:**

*Proof.* Suppose for the sake of contradiction that there exists an integer $q$ such that

$$q^2 = 5n + 3 \text{ for some } n \in \mathbb{Z}$$

Thus,

$$[q^2] = [q]^2 = 3 \mod 5$$

However,

$$[0]^2 = 0 \mod 5$$
$$[1]^2 = 1 \mod 5$$
$$[2]^2 = 4 \mod 5$$
$$[3]^2 = 4 \mod 5$$
$$[4]^2 = 1 \mod 5$$

Thus, we arrive at a contradiction. $\square$

**Bonus Question: (+3 Points added to exam)**

Jim is looking to have a easy life and make a lot of money. Jim goes looking for employment and finds a mysterious man. The man points to a bridge and says the following to Jim: "The work I have for you is light and you will get rich. Do you see the bridge? Each time you cross it I will double the money in your pocket. But since I am so generous you must give me back \$ 24 after each crossing." Jim accepts and walks across the bridge. Miraculously the money in his pocket doubled! He threw \$ 24 dollars to the mystery man for the first crossing and crossed again. Amazingly his money doubled! He paid the mystery man \$ 24 again for the second crossing. He crossed a third time, again his money doubles. He goes to pay the mystery man, but the mystery man laughs because Jim only had \$ 24 dollars in his pocket and had to give it all away. How much money did Jim start with ?

**Solution:** \$21