MATH2210Q

Name: Solution

Exam 2

Date:

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any unapproved calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	4	
2	4	
3	4	
4	4	
5	5	
6	4	
Total:	25	

1. Let $T : P_2(t) \to P_1(t)$ be the following linear transformation

$$T(p(t)) = T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$$

(i.e. T is differentiation of the polynomial).

Let $\mathcal{B} = \{1, t, t^2\}$ be a basis for $P_2(t)$ and $\mathcal{C} = \{1, t\}$ be a basis for $P_1(t)$.

(a) (2 points) Find a matrix for T relative to $\mathcal{B}$ and $\mathcal{C}$.

$T(1) = 0 + 0t$

$T(t) = 1 + 0t$

$T(t^2) = 0 + 2t$

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) (2 points) Given your answer in part (a), is the transformation T onto or one to one?

rank $= 2$ so T is onto.

nullity $= 1$ so T is not one to one.

2. Find the dimension of the following vector spaces.

(a) (1 point) Null(A) where $A = \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

nullity = dim(Null(A)) = 2

(b) (1 point) Col(A) where A is the matrix above.

rank = dim(Col(A)) = 3

(c) (2 points) span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$

$\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\} = \text{Col}(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so $\dim\left(\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}\right) = 2$

3. (4 points) Use the coordinate transformation to determine the dimension of $H = \text{span}\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$.

$$\mathbf{p}_0(t) = 5t + t^2, \quad \mathbf{p}_1(t) = 1 - 8t - 2t^2, \quad \mathbf{p}_2(t) = -3 + 4t + 2t^2.$$

Coordinate Transform:

$$P_0(t) \longmapsto \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix}$$

$$P_1(t) \longmapsto \begin{bmatrix} 1 \\ -8 \\ -2 \end{bmatrix}$$

$$P_2(t) \longmapsto \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}.$$

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 5 & -8 & 4 \\ 1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 5 & -8 & 4 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 2 & -6 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2$

So $\dim(H) = 2$

4. Let
$$\mathbf{b}_1 = \begin{bmatrix}2\\0\\1\end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix}1\\2\\0\end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix}6\\3\\3\end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix}4\\-1\\3\end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix}5\\5\\2\end{bmatrix}$$
and note that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ are bases for $\mathbb{R}^3$.

(a) (2 points) Find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

$$\left[\begin{array}{ccc|ccc} 2&1&1&6&4&5\\ 0&2&1&3&-1&5\\ 1&0&1&3&3&2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1&0&1&3&3&2\\ 0&2&1&3&-1&5\\ 2&1&1&6&4&5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1&0&1&3&3&2\\ 0&2&1&3&-1&5\\ 0&1&-1&0&-2&1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1&0&1&3&3&2\\ 0&1&-1&0&-2&1\\ 0&2&1&3&-1&5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1&0&1&3&3&2\\ 0&1&-1&0&-2&1\\ 0&0&3&3&3&3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1&0&1&3&3&2\\ 0&1&-1&0&-2&1\\ 0&0&1&1&1&1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1&0&0&2&2&1\\ 0&1&0&1&-1&2\\ 0&0&1&1&1&1 \end{array}\right]$$

So, $\underset{B\leftarrow C}{P} = \begin{bmatrix}2&2&1\\1&-1&2\\1&1&1\end{bmatrix}$

(b) (2 points) Let $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, find $[\mathbf{x}]_{\cancel{\mathcal{C}}B}$

$$\begin{bmatrix}2&2&1\\1&-1&2\\1&1&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\2\\3\end{bmatrix} = [x]_B$$

5. (a) (2 points) Let A be a 3×6 matrix. Is it possible for the transformation $T(\vec{x}) = A \cdot \vec{x}$ to be one to one?

rank(A) ≤ 3

So nullity(A) ≥ 3

T cannot be 1 to 1

(b) (1 point) Let B be a 5×6 matrix. What is the maximal rank of B?

rank(B) = dim (row(B)) ≤ 5

Max possible rank is 5.

(c) (2 points) Suppose A has the following eigenvalues:

Eigenvalue	1	2	-5
multiplicity	1	2	1

What would the dimensions of the corresponding eigen-spaces need to be to make A diagonalizable?

Need

dim (Eigenspace(1)) = 1

dim (Eigenspace (2)) = 2

dim (Eigenspace (-5)) = 1

6. (4 points) Diagonalize the following matrix, i.e. find matrices P and D such that $A = PDP^{-1}$ where D is a diagonal matrix.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}.$$

$\lambda = 1, -1$ are the Eigenvalues.

$\lambda = 1$: $\left[A - I \mid 0\right] = \left[\begin{array}{cc|c} 0 & 4 & 0 \\ 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = x_1 \\ x_2 = 0 \end{array}$

$\text{Eigenspace}(1) = \left\{ t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

$\lambda = -1$: $\left[A + I \mid 0\right] = \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -2x_2 \\ x_2 = x_2 \end{array}$

$\text{Eigenspace}(-1) = \left\{ t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

$$\boxed{P = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$