MATH 2710
Exam 2 questions

1. Prove that $a \equiv b \mod m$ is an equivalence relation.

2. Prove the following theorem: If $[a]$ is any non-zero element in $\mathbb{Z}_p$, where p is prime, then there exists an element $[b] \in \mathbb{Z}_p$ such that
$$[a] \cdot [b] = [1].$$

3. Let A be a set and define $P(A)$ to be the set of all subsets of A. Let C be a fixed subset of the set A and define relation R on the set $P(A)$ by XRY if and only if $X \cap C = Y \cap C$. Prove that this is an equivalence relation.

4. Let A be a set and let P be a partition of the set A i.e. $P = \{A_1, A_2, \ldots A_n\}$ where
   i) $A_i \subset A$,
   ii) $\emptyset \notin P$
   iii) $A_1 \cup A_2 \cup \ldots \cup A_n = A$
   iv) $A_i \cap A_j = \emptyset$ when $i \neq j$.

   For $x, y \in A$ we say that xRy if and only if $x \in A_i$ and $y \in A_i$ for the same i. Prove this is an equivalence relation.

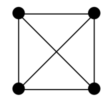
5. Prove or disprove: The relation R defined on the set $\mathbb{Z}$ by xRy if and only if $xy > 0$ is an equivalence relation.

6. A sequence of integers $x_1, x_2, x_3, \ldots$ is defined recursively by $x_1 = 3$, $x_2 = 7$ and
$$x_k = 5x_{k-1} - 6x_{k-2} \quad \text{for all } k \geq 3$$
   Prove by induction that $x_n = 2^n + 3^{n-1}$ for all positive integers n.

7. Prove by induction that a set of n elements contains 2^n subsets (including the set itself and $\emptyset$).

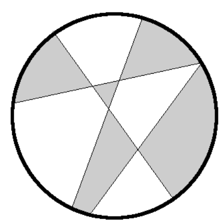
8. Prove by induction that if n points lie in a plane and no three are colinear, prove that there are $\frac{1}{2}n(n-1)$ lines joining these points.

   Example:



9. Suppose that n chords are drawn in a circle, dividing the circle into different regions. Prove that every region can be colored one of two colors such that adjacent regions are different colors.

   Example:



10. Let $\phi(m) : \mathbb{Z}^+ \to \mathbb{Z}^+$ denote the Euler ϕ-function.

$$\phi(m) = \# \text{ of positive integers less or equal to } m \text{ that are relatively prime to } m$$

   Prove $\phi(m) = m - 1$ if and only if m is prime.

11. Prove by induction the Leibniz rule for calculus

$$\frac{d^n}{dx^n}(f \cdot g) = \sum_{r=0}^{n} \binom{n}{r} \frac{d^{n-r}}{dx^{n-r}} f \frac{d^r}{dx^r} g$$

12. Prove that if $x \equiv 1 \pmod 2$ that

$$x^{2^n} \equiv 1 \mod 2^{n+2} \text{ for all } n \in \mathbb{P}$$

13. If p is prime prove that

$$(a+b)^p \equiv a^p + b^p \mod p$$

   for all $a, b \in \mathbb{Z}$.

14. Prove that multiplication is a well defined operation on $\mathbb{Q}$.

15. Prove that $\sqrt{3}$ is irrational.

16. If $f : X \to Y$ is a bijective function, prove that the inverse is unique.

17. If $f : X \to Y$ and $g : Y \to Z$ prove that

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

18. Prove that if A and B are disjoint finite sets that $|A \cap B| = |A| + |B|$.

19. Prove that $|\mathbb{Z}^+| = |\mathbb{Z}|$.

20. Prove that $|\mathbb{Z}^+| \neq |\mathbb{R}|$.

21. Let $S = P(A)$ be the power set of A. Define the following relation on S: Say that $A \sim B$ if and only if $|A| = |B|$. Show that this is an equivalence relation.

22. Let $f : X \to X$ be a function on a finite set. Show that f is injective if and only if it is surjective.

23. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are injective that $g \circ f$ is injective.

24. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are surjective that $g \circ f$ is surjective.

25. Show that if $f : A \to P(A)$ is a function then it cannot be surjective.
Hint: Let $D = \{a \in A \mid a \notin f(a)\}$ and show that $f(a) \neq D$ for all $a \in A$.