1. (§3.A #7) Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim(V) = 1$ and $T \in \mathcal{L}(V)$, then there exists a $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

   Proof. Let V be a one dimensional vector space, $\beta = \{v\}$ be its basis, and $T \in \mathcal{L}(V)$. If $u \in V$ then $u = \alpha v$ for some $\alpha \in \mathbb{F}$. Since $T : V \to V$, if $u \in V$ then

   $$Tv = \lambda v \quad \text{for some} \quad \lambda \in \mathbb{F}$$

   and

   $$Tu = T(\alpha v) = \alpha T(v) = \alpha(\lambda v).$$

   □

2. (§3.B # 9) Suppose that $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

   Proof. Suppose for the sake of contradiction that $Tv_1, \ldots, Tv_n$ is linearly dependent. Hence there exists $a_1, \ldots, a_n$ not all zero such that

   $$0 = a_1Tv_1 + \ldots + a_nTv_n = T(a_1v_1 + \ldots + a_nv_n).$$

   Since T is injective, we have that $\text{null}(T) = \{0\}$ and

   $$a_1v_1 + \ldots + a_nv_n = 0$$

   and we have a contradiction. □

3. (§3.B # 10) Suppose that $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Prove that $Tv_1, \ldots, Tv_n$ spans $\text{ran}(T)$.

   Proof. Since $v_1, \ldots, v_n$ spans V, if $v \in V$ then

   $$v = b_1v_1 + \ldots + b_nv_n \quad \text{for some} \quad b_1, \ldots, b_n \in \mathbb{F}.$$

   Hence, if $w \in \text{ran}(T)$ then $w = T\hat{v}$ for some $\hat{v} \in V$ and

   $$w = T\hat{v} = a_1Tv_1 + \ldots + a_nTv_n$$

   for some $a_1, \ldots, a_n \in \mathbb{F}$. Thus $\{Tv_1, \ldots, Tv_n\}$ spans $\text{ran}(T)$. □

4. (§3.B #12) Suppose that V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null}(T) = \{0\}$ and $\operatorname{ran}(T) = \{Tu : u \in U\}$.

   Proof. Since $\operatorname{null}(T)$ is a subspace of V then it has some basis $\beta = \{u_1, \ldots, u_p\}$. Expand the basis β to a basis $\gamma = \{v_1, \ldots, v_p, u_1, \ldots u_n\}$ of V. Let $U = \operatorname{span}\{u_1, \ldots, u_n\}$. Since γ is linearly independent we have that $U \cap \operatorname{null}(T) = \{0\}$. Suppose that $v \in V$, since γ spans V we have that

$$v = a_1 v_1 + \ldots + a_p v_p + b_1 u_1 + \ldots + b_n u_n.$$

   Thus

$$Tv = a_1 T v_1 + \ldots + a_p T v_p + b_1 T u_1 + \ldots + b_n T u_n = 0 + b_1 T u_1 + \ldots + b_n T u_n = T(b_1 u_1 + \ldots b_n u_n).$$

   Clearly, $\operatorname{ran}(T) = \{Tu \ : \ u \in U\}$. $\square$

5. (§3.D # 8) Suppose V is finite dimensional and $T : V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T \mid_U$ is an isomorphism of U onto W. (Here $T \mid_U$ means the function T restricted to U. In other words, $T \mid_U$ is the function whose domain is U, with $T \mid_U$ defined by $T \mid_U (u) = T(u)$ for every $u \in U$.)

   Proof. By the above problem there exists a subspace U such that $U \cap \operatorname{null}(T) = \{0\}$ and that $\operatorname{ran}(T) = \{Tu \ : u \in U\}$. By surjectivity of T we have that $\operatorname{ran}(T) = W$ and thus $W = \{Tu \ : \ u \in U\}$. Clearly $T \mid_U : U \to W$ is surjective. We must check injectivity of $T \mid_U$. Suppose that $T \mid_U$ is not injective, then there exists a $u \in U$ such that $T \mid_U (u) = 0$. But

$$T \mid_U (u) = T(u) = 0$$

   and therefore $u \in \operatorname{null}(T)$. This is a contradiction on the definition of U. Hence $T \mid_U$ is injective and surjective from U onto W and thus is a linear isomorphism from U onto W. $\square$

6. (§3.D # 18) Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

   Proof. We note that if V was finite dimensional we would have that

$$\dim(V) = \dim(V) \cdot 1 = \dim(V) \cdot \dim(\mathbb{F}) = \dim(\mathcal{L}(\mathbb{F}, V)).$$

   The result is then true since two finite dimensional vector spaces are isomorphic if and only if they are the same dimension. However, we will do a proof without using an argument which relies on finite dimensionality. Since $\{1\}$ is a basis for $\mathbb{F}$ over itself then T is completely determined by what happens to 1. That is,

$$T(\lambda \cdot 1) = \lambda T(1) = \lambda v.$$

For every $v \in V$ let $T_v \in \mathcal{L}(\mathbb{F}, V)$ be the linear transformation such that $T_v(1) = v$. We define the following linear isomorphism

$$\Phi : V \to \mathcal{L}(\mathbb{F}, V),$$

$$\Phi(v) = T_v \in V.$$

Note that Φ is linear since

$$\Phi(\lambda v + u) = T_{\lambda v + u} = \lambda T_v + T_u = \lambda \Phi(v) + \Phi(u).$$

Suppose that $v_1 \neq v_2$, then $T_{v_1} \neq T_{v_2}$ since

$$T_{v_1}(1) = v_1 \neq v_2 = T_{v_2}(1).$$

Thus Φ is injective. Let $T \in \mathcal{L}(\mathbb{F}, V)$. Let $\hat{v} = T(1)$. Then

$$\Phi(\hat{v}) = T$$

by definition of Φ. Thus, Φ is surjective. Therefore, Φ is a linear isomorphism. $\square$