1. We define the following:

   Definition 0.1. Let $I = (c, d)$ be an interval and suppose $a \in I$. Let E denote either I or $I \setminus \{a\}$ and suppose $f : E \to \mathbb{R}$. We say f has a limit as x approaches a from the right if the function $f|_{(a,d)} : (a, d) \to \mathbb{R}$ has a limit at a. The limit if it exists is denoted

   $$\lim_{x \to a^+} f(x) = \lim_{a < x \to a} f(x).$$

   The limit from the left is defined similarly.

   Show f has a limit at a if and only if both the limit from the right and the limit from the left exist and are equal.

2. We will start with a definition.

   Definition 0.2. Let (f_n) be a sequence of functions in $C(X)$ where X is a compact set. We say the sequence (f_n) is *equicontinuous* if given an $\varepsilon > 0$ there is a $\delta > 0$ such that, for every n, if $d(x, y) < \delta$ we have that $d(f_n(x), f_n(y)) < \varepsilon$.

   Show that if (f_n) converges to f (in the sense of the uniform metric) then (f_n) is equicontinuous.

3. A simple pendulum consists of a mass m hanging from a string of length L. When displaced to an initial angle and released the pendulum will have the following equation of motion:

   $$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin\theta = 0,$$

   where θ is the displacement angle and g is the acceleration due to gravity. To solve this equation a physicist will employ the small angle approximation

   $$\sin\theta \approx \theta$$

   for small θ. The equation of motion then becomes simplified:

   $$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0.$$

   Show this step is justified by showing that

   $$|\sin\theta - \theta| \leq \frac{1}{6}|\theta|^3$$

   via Taylor's theorem.

4. Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable then between any two zeros of f there is a zero of f'. Use this result to show via induction that any polynomial of degree n can have *at most* n distinct real roots. (You may not use the Fundamental Theorem of Algebra to do this problem.)

5. Let $f(x) = \log(x)$. Given $a > 0$, let $g(x) = f(ax)$. Prove $g'(x) = f'(x)$ and thus there exists a c such that $g(x) = f(x) + c$. Prove, $c = \log(a)$ and thus $\log(ax) = \log(x) + \log(a)$. (Use the integral definition of logarithms.)