Computational Skills

- Row reduction
- Change between matrix Eqns. / Vect Eqns. / Linear systems
- Solving Linear Systems
- Writing solutions in parametric form $\vec{x} = \vec{p} + \vec{h}$
- Determine linear independence
- calculate dimension
- recognize pivots & free var.
- Matrix algebra addition / scalar mult / matrix mult
- Finding inverses for matrices
- Taking determinants
- Finding Change of basis matrix / switching coordinate systems
- Find a basis for null(A), col(A), row(A)
- Find Eigen values / Eigen vect.
- Find P, D for diagonalization
- calculate dot prod.
- Calculate projections
- Find vector decomposition
- Calculate distance between vectors.
- Gram-Schmidt

Theory

Linear Systems

linear systems $\Leftrightarrow$ Vect Eqns $\Leftrightarrow$ matrix Eqns

Solns = particular soln + homogeneous soln.

Unique soln $\Leftrightarrow$ A is invertible

Vectorspaces

basis = linearly independent spanning set

$\beta = \{\vec{b}_1, \vec{b}_2, \dots \vec{b}_n\}$ is a basis for an n-dimensional space if

$B = [\vec{b}_1, \dots \vec{b}_n]$

- has n-pivots = spans
- has no free var. = lin indep

dim(V) = # basis elts

- if $U \leq V$ is a subspace $\dim(U) \leq \dim(V)$
- $\dim(U) = \dim(V) \Leftrightarrow V = U$

Every vector is made of basis elements

$V = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$

- $[V]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ = Coordinate Vector w.r.t β

- if B, C are bases

$$[V]_C = \underset{C \leftarrow B}{P} [V]_B$$

Change of basis matrix

Basis for null(A) = homogeneous solns

Basis for Col(A) = pivot columns

Basis for row(A) = nonzero rows of RREF

T is onto & 1 to 1

$\Updownarrow$

T is invertible

$\Updownarrow$

A is invertible

Linear Transformations

$T: \mathbb{R}^n \to \mathbb{R}^m$

- represented by matrix $T(\vec{x}) = A\vec{x}$ (mxn)

$A = [T(\vec{e}_1), \dots, T(\vec{e}_n)]$

$\{\vec{e}_i\}$ - standard basis

if $A = [\vec{a}_1 \dots \vec{a}_n]$ (mxn)

- $\mathrm{Col}(A) = \{x_1\vec{a}_1 + \dots + x_n\vec{a}_n \mid x_i \in \mathbb{R}\}$
  = output of T
  = $\mathrm{range}(T) \subseteq \mathbb{R}^m$
- $\mathrm{Null}(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$
  = homogeneous solns of $A\vec{x} = \vec{0}$
  = $\mathrm{null}(T) \subseteq \mathbb{R}^n$
- rank(A) = dim(Col(A))
  = # pivots
- nullity(A) = dim(Null(A))
  = # free var.
- rank + nullity = # cols
- dim(row(A)) = dim(col(A))
- T onto $\Leftrightarrow$ Pivot in Every row
- T 1 to 1 $\Leftrightarrow$ No free var.

More Theory

Determinants

- Only square matrices
- $\det(A) \neq 0$

  $\Updownarrow$

  A is invertible
- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = 1/\det(A)$
- $\det(A^T) = \det(A)$

row op	Effect
scale by K	$K\det(A)$
switch rows	$-\det(A)$
add multiples of rows	No change

Eigenvectors/Eigenvalues

- A $(n \times n)$ has n complex Eigenvalues (counting multiplicity)
- Eigenvalues = roots of $\det(A - \lambda I)$
- Eigenvectors = $\text{null}(A - \lambda I)$ w/o $\vec{v} = \vec{0}$
- $A\vec{x} = \underbrace{\lambda}_{\text{Eigenvalue}} \underbrace{\vec{x}}_{\text{Eigenvector}}$
- A is diagonalizable

  $\Updownarrow$

  $\dim(\text{null}(A - \lambda_i I)) = \text{mult}(\lambda_i)$
- $A = PDP^{-1}$ where

  $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\quad P = [\text{Eigenvects}]$

  change of basis matrix

Orthogonality

- $\vec{w} \perp \vec{v} \iff \vec{w} \cdot \vec{v} = 0$
- $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$ = length of $\vec{v}$
- $\vec{v}/\|\vec{v}\|$ is a unit vector
- If W = subspace then $W^\perp$ = vectors $\perp$ to all vectors in W

  Given a vector $\vec{y}$

  $\vec{y} = \vec{w} + \vec{w}_\perp$ where

  $\vec{w} = \text{proj}_W(\vec{y}) \in W$

  $\vec{w}_\perp = \vec{y} - \text{proj}_W(\vec{y}) \in W^\perp$
- Given an orthogonal basis for W; $\{\vec{w}_1 \ldots \vec{w}_n\}$

  $\text{proj}_W(\vec{y}) = c_1\vec{w}_1 + \ldots + c_n\vec{w}_n$

  $c_i = \dfrac{\vec{y} \cdot \vec{w}_i}{\vec{w}_i \cdot \vec{w}_i}$
- if $\vec{y} \in W$ then $\vec{y} = \text{proj}_W(\vec{y})$
- $\text{dist}(\vec{y}, W) = \|\vec{y} - \text{proj}_W(\vec{y})\|$