MATH3210

Final Exam

The following rules apply:

- **Exam must be typed**. Please organize your proofs in a reasonably neat and coherent way. Write in complete sentences.
- **Mysterious or unsupported claims will not receive full credit**. Unreasonably large gaps in logic or an argument will receive little credit. You may quote theorems from class or the book.
- **Your solutions must be your own.** You may use outside sources but your submitted solution must be in your own words.

1. Let $\mathcal{A}(\mathbb{R})$ be the set of real analytic functions over $\mathbb{R}$, this is a real vector space with the usual operations of function addition and scalar multiplication.

$$\mathcal{A}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; f(x) = \sum_{n=0}^{\infty} a_n x^n \right\}$$

Consider the following operator on $\mathcal{A}(\mathbb{R})$;

$$B : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$$

$$Bf(x) = xf(x).$$

Show that this operator has no eigenvalues. Note that two analytic functions $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are equal if and only if $a_n = b_n$ for all n.

2. Let V be a finite dimensional complex vector space. A linear map $\Gamma : \mathcal{L}(V) \to \mathcal{L}(V)$ is called positive if it takes positive operators to positive operators i.e. if A is positive then $\Gamma(A)$ is positive. Let, $\Psi : \mathcal{L}(V) \to \mathcal{L}(V)$ denote the following map,

$$\Psi(A) = \sum_{i=0}^{n} B_i^* A B_i$$

where each $B_i \in \mathcal{L}(V)$. Show Ψ is a positive map.

3. Suppose T is a positive operator on V. Prove that T is invertible if and only if

$$\langle Tv, v\rangle > 0$$

for every $v \in V$ with $v \neq 0$.

4. Suppose V and W are finite dimensional and U is a subspace of V. Prove there exists a $T \in \mathcal{L}(V, W)$ such that $\text{null}(T) = U$ if and only if $\dim(U) \geq \dim(V) - \dim(W)$.

5. Suppose V is a complex finite dimensional vector space and $T \in \mathcal{L}(V)$. Let $p \in \mathcal{P}(\mathbb{C})$ be a polynomial and $\alpha \in \mathbb{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of T.

6. Show that every self adjoint operator on a complex vector space has a cube root, i.e. if T is self adjoint then there exists an operator S such that $S^3 = T$.

7. Let V and W be vector spaces over some field $\mathbb{F}$ and let

$$V \times W = \{(v, w) \mid v \in V, w \in W\}.$$

$V \times W$ is a vector space with the following operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$

and

$$c(v, w) = (cv, cw)$$

where $c \in \mathbb{F}$, $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$. Let $T : V \to W$ be a map. The graph of T is the subset of $V \times W$ defined by

$$\text{graph}(T) = \{(v, Tv) \in V \times W : v \in V\}.$$

Show that T is a linear map if and only if graph(T) is a subspace of $V \times W$.