1. Let X and Y be sets of positive real numbers which are bounded above. Define

$$XY = \{xy \mid x \in X,\, y \in Y\}.$$

Show that $\text{lub}(XY) = \text{lub}(X) \cdot \text{lub}(Y)$.

Hint: Do the following: Let $x = \text{lub}(X)$ and $y = \text{lub}(Y)$ and $\varepsilon > 0$.

(i) Show that XY is bounded above.
(ii) Show that there exists an $\hat{x} \in X$ such that $\hat{x} \geq x - \frac{\varepsilon}{x+y}$
(iii) Show that there exists an $\hat{y} \in Y$ such that $\hat{y} \geq y - \frac{\varepsilon}{x+y}$
(iv) Show $\hat{x}\hat{y} \geq xy - \varepsilon$
(v) Use the above to conclude $xy = \text{lub}(XY)$.

Solution:

Proof. Given that X and Y are sets of positive real numbers that are bounded above, by the least upper bound property we have that X and Y both have a least upper bound. Denote the least upper bounds of X and Y as x and y respectively. Note that $x \geq \tilde{x} > 0$ for all $\tilde{x} \in X$, and thus $x > 0$. Similarly, we have that $y > 0$.

Since X is bounded above by x and Y is bounded above by y. The set XY is bounded above by xy since $\tilde{x} < x$ for all $\tilde{x} \in X$ and $\tilde{y} < y$ for all $\tilde{y} \in Y$. Hence $\tilde{x}\tilde{y} < xy$ for all $\tilde{x}\tilde{y} \in XY$. Thus the set XY is bounded above and has a least upper bound. Denote the least upper bound of the set XY as α.

We claim that $\alpha = xy$. To see this we will show that given an $\varepsilon > 0$ the number $xy - \varepsilon$ is not an upper bound. Since x is the least upper bound of X we have that $x - \frac{\varepsilon}{x+y}$ is not an upper bound and thus there exists an $\hat{x} \in X$ such that

$$\hat{x} \geq x - \frac{\varepsilon}{x+y}.$$

Similarly, there exists a $\hat{y} \in Y$ such that

$$\hat{y} \geq y - \frac{\varepsilon}{x+y}.$$

Note that

$$\hat{x}\hat{y} \geq \left(x - \frac{\varepsilon}{x+y}\right)\left(y - \frac{\varepsilon}{x+y}\right) = xy - \varepsilon + \frac{\varepsilon^2}{(x+y)^2} > xy - \varepsilon$$

since $\frac{\varepsilon^2}{(x+y)^2} > 0$. Hence there exists an element $\hat{x}\hat{y} \in XY$ such that

$$\hat{x}\hat{y} > xy - \varepsilon$$

for any given $\varepsilon > 0$. Thus we must have that $xy = \alpha$, i.e. xy is the least upper bound of XY. $\square$

2. Show, that the sequence

$$a_n = \frac{2n-3}{n+5} \qquad n \geq 1$$

converges.

Solution:

Proof. We will show the sequence converges to 2. Let $\varepsilon > 0$ be given. Choose an $N \in \mathbb{Z}^+$ such that $\frac{13}{N} < \varepsilon$. If $n > N$, then

$$\left| \frac{2n-3}{n+5} - 2 \right| = \left| \frac{-13}{n+5} \right| = \frac{13}{n+5} \leq \frac{13}{n} \leq \frac{13}{N} < \varepsilon.$$

$\square$

3. Prove that $\{n^2+2\}_{n=1}^{\infty}$ diverges to infinity.

Solution:

Proof. Let $M > 0$ be given. Choose an $N \in \mathbb{Z}^+$ such that $N > M^{1/2}$. If $n \geq N$, then

$$n^2 + 2 \geq N^2 + 2 \geq N^2 > M.$$

$\square$

4. Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences with limits x and y respectively. Prove
   (a) $\{cx_n\}$ converges to cx where $c \in \mathbb{R}$.
   (b) $\{x_n + y_n\}$ converges to $x + y$.

Solution:

(a) *Proof.* Suppose $c \neq 0$, otherwise the statement is trivial. Let $\varepsilon > 0$ be given. Since x_n converges to x there exists an $N \in \mathbb{Z}^+$ such that if $n > N$, then

$$|x_n - x| < \frac{\varepsilon}{|c|}.$$

Hence, if $n > N$ then

$$|cx_n - cx| = |c||x_n - x| < |c|\frac{\varepsilon}{|c|} = \varepsilon.$$

$\square$

(b) *Proof.* Let $\varepsilon > 0$ be given. Since x_n converges to x there exists an $N_1 \in \mathbb{Z}^+$ such that if $n > N_1$, then
$$|x_n - x| < \frac{\varepsilon}{2}.$$
Simlarly, there exists an $N_2 \in \mathbb{Z}^+$ such that if $n > N_2$, then
$$|y_n - y| < \frac{\varepsilon}{2}.$$
Let $N = \max\{N_1, N_2\}$. If $n > N$, then
$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
by the triangle inequality. □

5. Use the monotone convergence theorem to show the sequence $\{x_n\}$ defined by
$$x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{2 + x_n} \quad \text{for } n > 1$$
converges.

Hint: Show by induction that the sequence is increasing and bounded above by 2.

Solution:

We first show by induction that the sequence is bounded above by 2.

Proof.
Base case: Clearly, $\sqrt{2} \leq 2$.

Induction Hypothesis: For $n = k$, $x_k \leq 2$.

We now show that $x_{k+1} \leq 2$. By definition
$$x_{k+1} = \sqrt{2 + x_k} \leq \sqrt{2 + 2} = 2.$$
□

Now, we show by induction that the sequence is increasing.

Proof.
Base case: Clearly, $\sqrt{2} \leq \sqrt{2 + \sqrt{2}}$.

Induction Hypothesis: For $n = k$, $x_k \leq x_{k+1}$.

We now show that $x_{k+1} \leq x_{k+2}$. By definition
$$x_{k+1} = \sqrt{2 + x_k} \leq \sqrt{2 + x_{k+1}} = x_{k+2}.$$
□

Since, the sequence is monotone increasing and bounded above it must converge.