Worksheet 2

1. Consider the sequence defined recursively as

$$a_0 = 1$$

$$a_n = a_{n-1} + a_{n-2} + \ldots + a_0 + 1$$

Show $a_n = 2^n$ for all positive integers $n$.

*Proof.*

**Base case:** Note that,

$$a_1 = a_0 + 1 = 1 + 1 = 2^1$$

**Induction Hypothesis:** Suppose that for $n = k$ that $a_k = 2^k$.

We show that for $n = k + 1$ that

$$a_{k+1} = 2^{k+1}.$$

By definition

$$a_{k+1} = a_k + a_{k-1} + \ldots + a_0 + 1$$

Since $a_k = a_{k-1} + \ldots + a_0 + 1$ we have,

$$a_{k+1} = a_k + a_k = 2a_k = 2(2^k) = 2^{k+1}$$

by our induction hypothesis.

$\square$

2. Prove that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all positive integers $n$.

*Proof.*

**Base case:** Note that,

$$1^2 = \frac{1(1+1)(2(1)+1)}{6}$$

**Induction Hypothesis:** Suppose that for $n = k$ that $1^1 + 2^2 + \ldots k^2 = \dfrac{k(k+1)(2k+1)}{6}$.

We show that for $n = k+1$ that

$$1^1 + 2^2 + \ldots k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

By our induction hypothesis,

$$1^1 + 2^2 + \ldots k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2.$$

Hence,

$$\begin{aligned}
1^1 + 2^2 + \ldots k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2. \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

$\square$

3. Prove that for any positive integer $n$ that $n^3 + 2n$ is divisible by 3.

*Proof.*

**Base case:** Note that,

$$1^3 + 2(1) = 3$$

which is divisible by 3.

**Induction Hypothesis:** Suppose that for $n = k$ that $k^3 + 2k$ is divisible by 3.

We show that for $n = k + 1$ that

$$(k + 1)^3 + 2(k + 1)$$

is divisible by 3.

Notice that,

$$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3 = (k^3 + 2k) + (3k^2 + 3k + 3).$$

By our induction hypothesis,

$$k^3 + 2k = 3m \qquad \text{for some } m \in \mathbb{Z}.$$

Hence,

$$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3 = 3m + 3(k^2 + k + 1).$$

Since both terms on the right hand side are divisible by 3 we can conclude that $(k+1)^3+2(k+1)$ is divisible by 3.

$\square$