MATH2210Q   Name:___*Soln*_____

Exam 2   Date:_____

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any unapproved calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	3	
2	3	
3	4	
4	6	
5	5	
6	4	
Total:	25	

1. (a) (1 point) Calculate the determinant of the following matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

A contains a zero row so $\det(A) = 0$

(b) (2 points) Calculate $\det(A \cdot B^2)$, where

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 & 4 & 4 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} .$$

A and B are upper triangular

$\det(A) = 2 \cdot 1 \cdot 3 \cdot 2 = 12$

$\det(B) = 1 \cdot 1 \cdot 2 \cdot 2 = 4$

$\Rightarrow \det(AB^2) = 12 \cdot 4^2 = 192$

2. (3 points) Consider the space of polynomials of degree less than or equal to 2, $\mathbb{P}_2(t)$. Let

$$V = \{p(t) = a_0 + a_1 t + a_2 t^2 \mid p'(t) = c\}$$

i.e. V is the subspace of $\mathbb{P}_2(t)$ of polynomials p such that the derivative is constant.

(a) Does every vector in $\mathbb{P}_2(t)$ have constant derivative?

$$No \qquad t^2 \in \mathbb{P}_2(t) \qquad \frac{d}{dt}(t^2) = 2t$$

(b) Since $\dim(\mathbb{P}_2(t)) = 3$ what is an upper bound on $\dim(V)$?

$$\dim(V) \leq \underline{\quad 2 \quad}$$

if $\dim(V) = 3$ then

$V = \mathbb{P}_2(t)$ which is false by

part (a)

(c) Consider the set of vectors in V:

$$\mathbf{p}_1(t) = 5t + 2; \qquad \mathbf{p}_2(t) = 3$$

Is $\{\mathbf{p}_1(t), \mathbf{p}_2(t)\}$ linearly independent?

$$yes, \qquad P_1 \neq c\, P_2 \qquad \text{for any } c \in \mathbb{R}.$$

(d) Based on your answers from (a) and (b) write down a basis for V and calculate the dimension of the subspace.

$$\dim V = 2 \qquad by \ (b) \quad \dim V \leq 2$$
$$by \ (c) \quad \dim V \geq 2$$

$$Basis \ for \ V = \{5t+2, 3\} \quad \text{since they are}$$

linearly indep.

3. (a) (3 points) Find the dimension of the column space, row space and null space of A.

$$A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $\dim(\text{Col}(A)) = \underline{\quad 2 \quad}$ $\dim(Col(A)) = \#\ pivots$

- $\dim(\text{Row}(A)) = \underline{\quad 2 \quad}$ $\dim(Row(A)) = \dim(Col(A))$

- $\dim(\text{Null}(A)) = \underline{\quad 3 \quad}$ $rank(A) + nullity(A) = 5$

(b) (1 point) Find a basis for the column space of A.

Basis for $col(A)$ $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

4. Let

$$\mathbf{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ -7 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and note that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are bases for $\mathbb{R}^2$.

(a) (2 points) Find $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$.

$$\begin{bmatrix} 1 & 1 & | & -1 & 1 \\ 2 & 1 & | & 8 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & | & -1 & 1 \\ 0 & -1 & | & 10 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & | & -1 & 1 \\ 0 & 1 & | & -10 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 9 & -8 \\ 0 & 1 & | & -10 & 9 \end{bmatrix}$$

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix}$$

(b) (2 points) Find $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$.

$$\det \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = 81 - 80 = 1$$

$$\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P}^{-1} = \frac{1}{1} \begin{bmatrix} 9 & 8 \\ 10 & 9 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 10 & 9 \end{bmatrix}$$

(c) (2 points) Let $[x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find $[x]_{\mathcal{C}}$.

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [x]_{\mathcal{B}} = [x]_{\mathcal{C}}$$

$$\begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = [x]_{\mathcal{C}}$$

5. (a) (2 points) If A is a 3×6 matrix, what is the smallest possible dimension of Null(A)?

$$\dim (\text{row}(A)) = \text{rank}(A) \leq 3$$

$$\text{rank}(A) + \text{nullity}(A) = 6$$

$$\Rightarrow \text{nullity}(A) \geq 3$$

(b) (1 point) Let B be a 5×5 matrix with rank(B) = 5. Are the columns of B linearly independent?

Yes, since nullity(B) = 0 the columns are linearly indep

(c) (2 points) If C is a 4×7 matrix with rank(C) = 4 is the linear transformation $T(\mathbf{x}) = C\mathbf{x}$ one to one?

No since nullity(C) = 3

T is 1 to 1 if and only if nullity(C) = 0

6. (4 points) Find the eigenvalues and a basis for each eigenspace.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}.$$

A is upper triangular $\Rightarrow$ Eigenvalues: $\lambda = \pm 1$

$\boxed{\lambda = 1}$

$[A - I | 0] = \begin{bmatrix} 0 & 4 & | & 0 \\ 0 & -2 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \iff \begin{matrix} x_2 = 0 \\ x_1 = \text{free} \end{matrix} \iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$t \in \mathbb{R}$

Basis for $E(1) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\boxed{\lambda = -1}$

$[A + I | 0] = \begin{bmatrix} 2 & 4 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \iff \begin{matrix} x_1 = -2x_2 \\ x_2 = \text{free} \end{matrix} \iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$t \in \mathbb{R}$

Basis for $E(-1) = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$