MAP2302 **Name:**______________________

Final Exam **Section:**______________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
Total:	35	

1. (5 points) Solve the following equation.

$$\underbrace{(2xy+3)}_{M}\,dx + \underbrace{(x^2-1)}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x \qquad \text{Exact}$$

$$\left\{\begin{aligned} \frac{\partial F}{\partial x} &= 2xy+3 \\ \frac{\partial F}{\partial y} &= x^2-1 \end{aligned}\right\} \Rightarrow F(x,y) = \int 2xy+3\,dx + h(y)$$

$$F(x,y) = x^2y + 3x\ h(y)$$

$$\frac{\partial F}{\partial y} = x^2 + h'(y) = x^2 - 1$$

$$\therefore\ h'(y) = -1$$

$$h(y) = -y$$

$$\boxed{F(x,y) = x^2y + 3x - y = C}$$

2. (5 points) Find the general solution to the Cauchy Euler equation.

$$t^2y'' + 5ty' + 3y = t^2 + 1 \quad t > 0$$

$r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$\therefore \quad r = -1, -3$

$y_1 = t^{-1} \quad y_2 = t^{-3} \quad W[y_1, y_2] = \begin{vmatrix} t^{-1} & t^{-3} \\ -t^{-2} & -3t^{-4} \end{vmatrix} = -3t^{-5} + t^{-5} = -2t^{-5}$

$g(t) = \frac{t^2+1}{t} = 1 + t^{-2}$

$V_1 = -\int \frac{(1+t^{-2})t^{-3}}{-2t^{-5}} dt = \frac{1}{2}\int t^2+1 \, dt = \frac{1}{2}\left(\frac{t^3}{3} + t\right) = \frac{t^3}{6} + \frac{t}{2}$

$V_2 = -\int \frac{(1+t^{-2})t^{-3}}{2t^{-5}} dt = -\int \frac{t^4+t^2}{2} dt = -\frac{t^5}{10} - \frac{t^3}{6}$

$y_p = \left(\frac{t^3}{6} + \frac{t}{2}\right)t^{-1} + \left(-\frac{t^5}{10} - \frac{t^3}{6}\right)t^{-3}$

$y_p = \left(\frac{t^2}{6} + \frac{1}{2}\right) - \frac{t^2}{10} - \frac{1}{6} = \frac{t^2}{15} + \frac{1}{3}$

$\boxed{y_g = \left(\frac{t^2}{15} + \frac{1}{3}\right) + c_1 t^{-1} + c_2 t^{-3}}$

3. (5 points) True or false:

a) Two functions are linearly independent if their Wronskian is 0.

b) The fundamental solution set for a fourth order linear equation has four functions.

c) The functions $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are linearly independent.

d) The functions $y_1 = e^{2t}$ and $y_2 = e^{3t}$ are linearly independent.

e) The functions $y_1 = \cos^2(x)$, $y_2 = \sin^2(x)$, and $y_3 = 2$ are linearly independent.

(a) F

(b) T

(c) T

(d) T

(e) F $2\cos^2(x) + 2\sin^2 x = 2$

5. (5 points) Let $g(t)$ be a piece-wise continuous function on $[0, \infty)$ of exponential order. Use convolution to obtain a formula for the solution of the following differential equation.

$$y'' - 2y' + y = g(t); \qquad y(0) = 0, \quad y'(0) = 1$$

$\mathcal{L}\{y\} = Y$

$\mathcal{L}\{g\} = G$

$$s^2 Y - 1 - 2(sY) + Y = G(s)$$

$$(s^2 - 2s + 1) Y = G(s) + 1$$

$$Y = G\left(\frac{1}{(s-1)^2}\right) + \left(\frac{1}{(s-1)^2}\right)$$

$$\boxed{y = g * e^t t + e^t t}$$

4. (a) (3 points) Solve the following initial value problem.

$$y'' + y = \delta(t - 2\pi); \qquad y(0) = 0, \ \ y'(0) = 1$$

$$s^2 Y - 1 + Y = \mathcal{L}\{\delta(t-2\pi)\} = e^{-2\pi s}$$

$$(s^2+1) Y = e^{-2\pi s} + 1$$

$$Y = \frac{e^{-2\pi s}}{s^2+1} + \frac{1}{s^2+1}$$

Note $\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a) u(t-a)$

$$\boxed{y = u(t-2\pi)\sin(t-2\pi) + \sin t}$$

(b) (2 points) Let $f(t)$ be the square wave function of period 2, i.e.

$$f(t) = \begin{cases} 1 : 0 < t < 1 \\ 0 : 1 < t < 2 \end{cases}.$$

Find its Laplace transform.

$$f_T(t) = u(t) - u(t-1)$$

$$F_T(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s}$$

$$\boxed{F(s) = \frac{F_T(s)}{1-e^{-sT}} = \frac{1-e^{-s}}{(1-e^{-2s})s}}$$

6. (5 points) Find formula(s) for the coefficients of the general power series solution to the following differential equation.

$$y'' + y = 0$$

Expand about $x=0$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

For $k \geq 0$ $\quad a_{k+2} = \dfrac{-a_k}{(k+2)(k+1)}$

a_0, a_1 - free

$$a_2 = \frac{-a_0}{(2)(1)} \qquad a_4 = \frac{-a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)}$$

$$a_3 = \frac{-a_1}{(3)(2)} \qquad a_5 = \frac{-a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)(1)}$$

For $n = 0, 1, 2 \ldots$

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!} \qquad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}$$

7. Consider the following differential equation.

$$(x^2-1)y'' + (x+1)y' + 3y = 0$$

(a) (3 points) Classify the singular points of the equation.

$p(x) = \frac{(x+1)}{(x-1)(x+1)} = \frac{1}{(x-1)}$ $\quad q(x) = \frac{3}{(x-1)(x+1)}$

$x=1$ $\quad (x-1)p(x) = 1$ analytic & $(x-1)^2 q(x) = \frac{3(x-1)}{(x+1)}$ analytic $\quad x=1$ is regular singular

$x=-1$ $\quad (x+1)p(x) = \frac{(x+1)}{(x-1)}$ analytic $\quad (x+1)^2 q(x) = \frac{3(x+1)}{(x-1)}$ analytic $\quad x=-1$ is regular singular

(b) (2 points) For the regular singular points find a value of r such that

$$w(r,x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

is the form for an analytic solution.

$x=1$ $\quad \lim_{x\to 1} (x-1)p(x) = 1$ $\quad \lim_{x\to 1} (x-1)^2 q(x) = 0$

ind Eqn: $r(r-1) + r = 0$

$r^2 = 0$

$\boxed{r=0}$

$x=-1$ $\quad \lim_{x\to -1} (x+1)p(x) = 0$ $\quad \lim_{x\to -1} (x+1)^2 q(x) = 0$

ind Eqn: $r(r-1) = 0$

$r=0$ $\quad \boxed{r=1}$