1. Let $p_n = a_m n^m + a_{m-1}n^{m-1} + \ldots + a_1 n + a_0$ and $q_n = b_m n^m + b_{m-1}n^{m-1} + \ldots + b_1 n + b_0$ for $n > 0$. Prove that
$$\lim_{n\to\infty} \frac{p_n}{q_n} = \frac{a_m}{b_m}.$$

Solution: Here we state two lemmas without proof. Both are left to the reader for verification.

Lemma 0.1. *Let $s \geq 1$ and $a_n = \frac{1}{n^s}$ for $n \geq 0$. The $\lim_{n\to\infty}(a_n)$ exists and $\lim_{n\to} \frac{1}{n^s} = 0$.*

Lemma 0.2. *Let $(a_n)_{n=k}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be sequences from a metric space (X, d). If there is an N and ℓ such that for $n \geq N$, $b_n = a_{n+\ell}$ then (a_n) converges if and only (b_n) converges and moreover in this case the sequences have the same limit.*

Proof. Let $c_n = \frac{p_n}{q_n}$. Let $x \in \mathbb{R}$ be the largest real number such that $q(x) = b_m x^m + \ldots + b_0$ is zero. For $n \geq x$ the sequence c_n is defined. By Lemma 0.2 we can assume without loss of generality that the sequence c_n is defined for all $n \geq 1$ i.e the numerator is non-zero. Define
$$d_n = \frac{\frac{1}{n^m}p_n}{\frac{1}{n^m}q_n} = \frac{a_m + a_{m-1}\frac{1}{n} + \ldots + a_0\frac{1}{n^m}}{b_m + b_{m-1}\frac{1}{n} + \ldots + b_0\frac{1}{n^m}}$$
We note that $d_n = c_n$ for all n. By Lemma 0.2 we have that $\lim_{n\to\infty}(c_n) = \lim_{n\to\infty}(d_n)$. By a repeated application of Lemma 0.1 and noting that the limit of a sum is the sum of the limits if both limits exist we have that
$$\lim_{n\to\infty} a_m + a_{m-1}\frac{1}{n} + \ldots + a_0\frac{1}{n^m} = a_m.$$
Likewise
$$\lim_{n\to\infty} b_m + b_{m-1}\frac{1}{n} + \ldots + b_0\frac{1}{n^m} = b_m$$
Applying another limit law we note that
$$\lim_{n\to\infty} d_n = \frac{a_m}{b_m}$$
and thus
$$\lim_{n\to\infty} c_n = \frac{a_m}{b_m}$$
by Lemma 0.2. □

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2. Suppose that y is a limit point of a metric space X. Show that $Y = X\setminus\{y\}$ is not complete.

Solution: We will remove a limit point from X and construct a Cauchy sequence that does not converge.

Proof. Suppose y is a limit point of the set X. Let $\varepsilon_n = \frac{1}{n}$ for $n \geq 1$. For each $\varepsilon_n > 0$ there is a point $s_n \neq y$ such that $s \in N_{\varepsilon_n}(y)$. The sequence (s_n) converges to y. Moreover, since (s_n) is convergent we have that (s_n) is a Cauchy sequence in X. We note in a metric space limits are unique. Thus $X \setminus \{y\}$ is not complete since (s_n) is a Cauchy sequence which does not converge. □

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3. Let (X, d_X) and (Y, d_Y) be complete metric spaces. Let $(X \times Y, d)$ be the metric space defined by the metric

$$d : (X \times Y) \times (X \times Y) \to \mathbb{R}; \qquad d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b)$$

is a complete metric space.

Solution: We will prove directly that if (a_n) is a Cauchy sequence on $(X \times Y, d)$ then it converges.

Proof. Suppose that $(a_n)_{n=0}^{\infty} = ((x_n, y_n))_{n=0}^{\infty}$ is a Cauchy sequence in $(X \times Y, d)$. We will first show that (x_n) and (y_n) are Cauchy sequences in (X, d_X) and (Y, d_Y) respectively. Let $\varepsilon > 0$ be given. Since (a_n) is a Cauchy sequence there exists an $N \geq 0$ such that if $n, m \geq N$ then

$$d(a_n, a_m) = d((x_n, y_n), (x_m, y_m)) = d_X(x_n, x_m) + d_Y(y_n, y_m) < \varepsilon$$

Hence, if $n, m > N$ we have that $d_X(x_n, x_m) < \varepsilon$ and $d_Y(y_n, y_m) < \varepsilon$. Thus (x_n) and (y_n) are Cauchy sequences in (X, d_X) and (Y, d_Y) respectively. Now we will show the given Cauchy sequence converges in $(X \times Y, d)$. Since (X, d_X) and (Y, d_Y) are complete metric spaces we have that (x_n) converges to a point $x \in X$ and (y_n) converges to a point $y \in Y$. By definition of the metric d it is clear that $((x_n, y_n))_{n=0}^{\infty}$ converges to (x, y). □

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4. Let S be a bounded subset of $\mathbb{R}$. Show that

$$\inf(S) = -\sup(-S)$$

where $-S = \{-s \ : \ s \in S\}$.

Solution: We will show that $-\inf(S) \leq \sup(-S)$ and $\inf(S) \geq -\sup(-S)$.

Proof. We note that since S is a bounded subset of $\mathbb{R}$ it is bounded above and below and thus has both a supremum and infimum. Moreover the same is true for $-S$. Note that for all $s \in S$ we have that $\inf(S) \leq s$ since $\inf(S)$ is a lower bound. Hence we have that

$$-\inf(S) \geq -s \quad \text{for all } -s \in -S.$$

Thus $-\inf(S)$ is an upper bound for $-S$ and we have that $\sup(-S) \geq -\inf(S)$. Likewise for all $-s \in -S$ we have $-s \leq \sup(-S)$. Thus,

$$-\sup(-S) \leq s \quad \text{for all } s \in S,$$

i.e. $-\sup(-S)$ is a lower bound. Hence $\inf(S) \geq -\sup(-S)$.

□

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5. A metric space (X, d) is called sequentially compact if every sequence has a convergent subsequence. Show that X is sequentially compact if and only if every infinite subset has a limit point in X.

Solution: We prove the result directly.

Proof. Suppose (X, d) be a sequentially compact metric space, we will show that every infinite subset have a limit point in X. Suppose that $Y \subseteq X$ is an infinite subset of X. We construct a sequence (s_n) in the following fashion. Pick an arbitrary point in Y and label it s_1. Suppose that s_n has been picked and we pick s_{n+1} such that $s_{n+1} \in Y \setminus \{s_1, \ldots, s_n\}$. Since (X, d) is sequentially compact we have that (s_n) has a convergent subsequence. Suppose that (s_{n_k}) converges to s, it is clear that s is a limit point of Y.

Note:
The first part of the proof was done with a repeated application of the Axiom of Choice, this may ruffle a couple of mathematician's feathers. See the following link:
`https://en.wikipedia.org/wiki/Axiom_of_choice`

We now suppose every infinite subset has a limit point in X. Suppose (s_n) is a sequence in X. If there exist an n_0 such that $s_m = s_{n_0}$ for infinitely many m_i then clearly (s_{m_i}) is a subsequence converging to s_{n_0}. Otherwise $\{s_n\}$ is an infinite subset of X and thus has a limit point. We construct a convergent subsequence from (s_n) as we did in question 2.

$\square$