MAP2302 Name: Key

Final Exam Section:

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
Total:	35	

1. (a) (1 point) Find the general solution the differential equation

$$\frac{dy}{dx} = y/x^2$$

$$\frac{dy}{y} = x^{-2}dx$$

$$\ln y = -x^{-1} + C$$

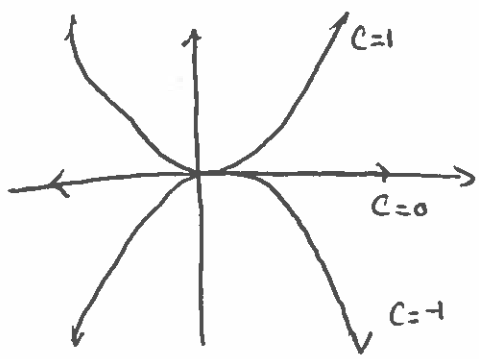
$$\Rightarrow y = Ce^{-1/x}$$

(b) (1 point) Draw the set of isoclines for $c = -1, 0, 1$.

$$-1 = \frac{y}{x^2} \Rightarrow y = -x^2$$

$$0 = \frac{y}{x^2} \Rightarrow y = 0$$

$$1 = \frac{y}{x^2} \Rightarrow y = x^2$$



(c) (3 points) Solve the differential equation.

$$\underbrace{(ye^{xy} - 1/y)}_{M}\,dx + \underbrace{(xe^{xy} + x/y^2)}_{N}\,dy = 0.$$

Exact Eqn $\quad \frac{\partial M}{\partial y} = xye^{xy} + \frac{1}{y^2} + e^{xy} \qquad \frac{\partial N}{\partial x} = yxe^{xy} + e^{xy} + \frac{1}{y^2}$

There exists an $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = ye^{xy} - 1/y$$

$$\frac{\partial F}{\partial y} = xe^{xy} + \frac{x}{y^2}$$

$$F(x,y) = \int ye^{xy} - \frac{1}{y}\,dx$$

$$F(x,y) = e^{xy} - \frac{x}{y} + h(y)$$

$$\frac{\partial F}{\partial y} = xe^{xy} + \frac{x}{y^2} + h'(y) = xe^{xy} + \frac{x}{y^2}$$

$\Rightarrow h'(y) = 0$ so $h(y) = C$

$$\boxed{F(x,y) = e^{xy} - \frac{x}{y} = C}$$

2. (5 points) Show the homogeneous solutions

$$y_1(t) = 2t - 1; \qquad y_2(t) = e^{-2t}$$

to

$$ty'' + (2t-1)y' - 2y = t^2 e^{-2t}, \qquad t > 0$$

are linearly independent then use variation of parameters to find a general solution.

$$y_1 = 2t - 1 \quad y_2 = e^{-2t} \quad W[y_1, y_2] = \begin{vmatrix} 2t-1 & e^{-2t} \\ 2 & -2e^{-2t} \end{vmatrix} = -4te^{-2t} + 2e^{-2t} - 2e^{-2t}$$

$$g(t) = te^{-2t} \qquad \therefore\ W[y_1, y_2] = -4te^{-2t}$$

$$V_1 = -\int \frac{te^{-2t}(e^{-2t})}{-4te^{-2t}}\,dt = \frac{1}{4}\int e^{-2t}\,dt = \frac{-1}{8}e^{-2t}$$

$$V_2 = \int \frac{te^{-2t}(2t-1)}{-4te^{-2t}}\,dt = -\frac{1}{4}\int 2t-1\,dt = \frac{-t^2}{4} + \frac{t}{4}$$

$$y_p = V_1 y_1 + V_2 y_2 = \left(\frac{-1}{8}e^{-2t}\right)(2t-1) + \left(-\frac{t^2}{4} + \frac{t}{4}\right)e^{-2t} = \frac{1}{8}e^{-2t} - \frac{t^2 e^{-2t}}{4} = e^{-2t}\left(\frac{-t^2}{4} + \frac{1}{8}\right)$$

$$\boxed{y_g = e^{-2t}\left(\frac{-t^2}{4} + \frac{1}{8}\right) + C_1(2t-1) + C_2(e^{-2t})}$$

3. (5 points) True or false:

T a) Approximations via Euler's method typically get worse the further away from the starting point.

F b) The integrating factor for the first order differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ is $\mu(x) = \exp\left(\int Q(x)dx\right)$

T c) The differential equation $(2x + y)\,dx + (x - 2y)\,dy = 0$ is exact. $\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$

F d) The auxiliary equation for $y'' + 5y' - 6y = 0$ is $r^2 + 4r - 6 = 0$.

F e) The minimum radius of convergence for a power series solution about $x_0 = 0$ for $y''+y = 0$ is 1.

4. Find the following Laplace transforms.

(a) (1 point) $\mathscr{L}\{\sin(t) * \cos(t)\}$

$$\mathcal{L}\{\sin(t) * \cos(t)\} = \left(\frac{1}{s^2+1}\right)\left(\frac{s}{s^2+1}\right) = \frac{s}{(s^2+1)^2}$$

(b) (1 point) $\mathscr{L}\{t^2\delta(t-2)\}$

$$\mathcal{L}\{t^2\delta(t-2)\} = \int_0^\infty e^{-st} t^2 \delta(t-2)\,dt = \int_{-\infty}^{\infty} e^{-st} t^2 \delta(t-2)\,dt = 4e^{-2s}$$

(c) (3 points) $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 2t, & 0 < t < 1, \\ 1, & 1 < t < 2, \end{cases} \qquad \text{where } f(t) \text{ has period 2.}$$

$$f_T(t) = 2t\,(u(t) - u(t-1)) + (u(t-1) - u(t-2))$$

$$\mathcal{L}\{f_T(t)\} = 2\mathcal{L}\{t\} - 2e^{-s}\mathcal{L}\{t+1\} + \mathcal{L}\{u(t-1)\} - \mathcal{L}\{u(t-2)\}$$

$$F_T(s) = \frac{2}{s^2} - \frac{2e^{-s}}{s^2} - \frac{2e^{-s}}{s} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$F(s) = \frac{F_T(s)}{1-e^{-2s}} = \frac{2}{s^2(1-e^{-2s})} - \frac{2e^{-s}}{s^2(1-e^{-2s})} - \frac{2e^{-s}}{s(1-e^{-2s})} + \frac{e^{-s}}{s(1-e^{-2s})} - \frac{e^{-2s}}{s(1-e^{-2s})}$$

5. (5 points) Use convolution to find a solution to the integral equation.

$$y(t) + \int_0^t (t-\nu)y(\nu)d\nu = t, \qquad y(0) = 0$$

$$= y + t * y = t$$

$$\Rightarrow Y + \frac{1}{s^2}Y = \frac{1}{s^2}$$

so $$\frac{s^2 Y + Y}{s^2} = \frac{1}{s^2}$$

$$(s^2+1)Y = 1$$

$$Y = \frac{1}{s^2+1}$$

$$\boxed{y(t) = \sin(t)}$$

6. (5 points) Find the general power series solution about $x_0 = 0$ to the following differential equation.

$$y'' - 2y = 0$$

$$\sum_{n=2}^{\infty} (n-1)(n) a_n x^{n-2} - \sum_{n=0}^{\infty} 2a_n x^n = 0$$

let $k = n-2$, $k+2 = n$ let $k = n$

$$\Rightarrow \sum_{k=0}^{\infty} (k+1)(k+2) a_{k+2} x^k - \sum_{k=0}^{\infty} 2a_k x^k = 0$$

$$\sum_{k=0}^{\infty} [(k+1)(k+2) a_{k+2} - 2a_k] x^k = 0$$

For $k \geq 0$ $\quad a_{k+2} = \dfrac{2a_k}{(k+1)(k+2)}$

$k=0$ $\quad a_2 = \dfrac{2a_0}{(1)(2)}$

$k=1$ $\quad a_3 = \dfrac{2a_1}{(2)(3)}$

$k=2$ $\quad a_4 = \dfrac{2a_2}{(4)(3)} = \dfrac{2^2 a_0}{(4)(3)(2)(1)}$

$k=3$ $\quad a_5 = \dfrac{2a_3}{(4)(5)} = \dfrac{2^2 a_1}{(5)(4)(3)(2)}$

$n = 0, 1, 2 \ldots$ $\quad a_{2n} = \dfrac{2^n a_0}{(2n)!}$ $\quad a_{2n+1} = \dfrac{2^n a_1}{(2n+1)!}$

$$y = a_0 \sum_{n=0}^{\infty} \frac{2^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{(2n+1)!}$$

7. Consider the following differential equation.

$$x^2(x+4)y'' + x(1+x)y' - (1+x)y = 0$$

(a) (2 points) Find and classify the singular points of the equation.

$p(x) = \dfrac{(1+x)}{x(x+4)}$ $q(x) = \dfrac{-(1+x)}{x^2(x+4)}$ $x=0,\ x=-4$ singular points

$x\,p(x) = \dfrac{(1+x)}{(x+4)}$ $x^2 q(x) = \dfrac{-(1+x)}{(x+4)}$ both analytic at $x=0$
$x=0$ is regular

$(x+4)\,p(x) = \dfrac{1+x}{x}$ $(x+4)^2 q(x) = \dfrac{-(1+x)(x+4)}{x^2}$ both analytic at $x=-4$
$x=-4$ is regular.

(b) (3 points) For each regular singular point find the associated indicial equation.

$\lim_{x\to 0} x\,p(x) = \dfrac{1}{4}$ $\lim_{x\to 0} x^2 q(x) = \dfrac{-1}{4}$ $\Rightarrow$ $r(r-1) + \frac{1}{4}r - \frac{1}{4} = 0$

$\lim_{x\to -4} (x+4)p(x) = \dfrac{3}{4}$ $\lim_{x\to -4} (x+4)^2 q(x) = 0$ $\Rightarrow$ $r(r-1) + \frac{3}{4}r = 0$