
Note: Let Γ be an arbitrary indexing set (possibly infinite and possibly uncountable). A collection of subspaces indexed by Γ is $\{U_\gamma \mid \gamma \in \Gamma, U_\gamma \text{ is a subspace of } V\}$.

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1. (§1.C #11) Prove that the intersection of every collection of subspaces of V is a subspace of V.

Definition:

We say that a vector space V is the direct sum of subspaces $U_1, \ldots, U_n$ if the following hold true:

(a) $U_i \neq \{0\}$ for each $i = 1, \ldots n$.

(b) $U_i \cap (U_1 + \ldots U_{i-1} + U_{i+1} + \ldots U_n) = \{0\}$ for $i = 1, \ldots n$.

(c) $V = U_1 + \ldots + U_n$.

Denote this by $V = U_1 \oplus \ldots \oplus U_n$.

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2. Prove the following theorem.

Theorem 0.1. *If $U_1, \ldots U_n$ are non-trivial subspaces of V, then*

$$V = U_1 \oplus \ldots \oplus U_n$$

if and only if every $v \in V$ has a unique representation of the form

$$v = u_1 + \ldots + u_n$$

where $u_i \in U_i$ for each $i = 1, \ldots, n$.

3. (§2.A # 14) Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

4. (§2.A # 16) Prove that the real vector space of all continuous real-valued functions on $[0, 1]$ is infinite dimensional.

5. (§2.B # 8) Suppose that U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots u_m, w_1, \ldots, w_n$$

is a basis of V.