In class we have gone through the following theorem.

Theorem 1. *Let $T \in \mathcal{L}(V,W)$ and let $\beta = \{v_1, \dots v_n\}$ and $\gamma = \{w_1, \dots w_m\}$ be ordered bases for V and W respectively. There exists a matrix representation with respect to β and γ*

$$A = [T]_\beta^\gamma$$

defined by

$$Tv_k = A_{1,k}w_1 + \dots + A_{m,k}w_m \quad \text{for } k \in \{1, \dots n\}$$

such that the mapping $M : \mathcal{L}(V,W) \to \mathbb{F}^{m\times n}$ defined by

$$M(T) = [T]_\beta^\gamma$$

is a vector space isomorphism.

This matrix is unique for a given choice of β and γ. Moreover, we have the following multiplicative property.

Theorem 2. *Let $T \in \mathcal{L}(V,W)$ and $S \in \mathcal{L}(W,X)$. Let $\beta = \{v_1, \dots v_n\}$, $\gamma = \{w_1, \dots w_m\}$ and $\delta = \{x_1, \dots, x_p\}$ be ordered bases for V, W and X respectively. We have*

$$[ST]_\beta^\delta = [S]_\gamma^\delta [T]_\beta^\gamma$$

The multiplicative property above leads to the following theorem.

Definition 1. A matrix $A \in \mathbb{F}^{n\times n}$ is said to be invertible if there exists a matrix $B \in \mathbb{F}^{n\times n}$ such that

$$AB = I$$

$$BA = I$$

where I is the $n \times n$ identity matrix.

Theorem 3. *Let $T \in \mathcal{L}(V,W)$ and let $\beta = \{v_1, \dots v_n\}$, $\gamma = \{w_1, \dots w_m\}$ be bases for V and W respectively. The transformation T is invertible if and only if $[T]_\beta^\gamma$ is invertible.*

Proof. Assume T in invertible, then

$$I = [\mathrm{Id}_V]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$$

and

$$I = [\mathrm{Id}_W]_\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta.$$

Conversely, assume $[T]_\beta^\gamma$ is invertible. Then there is a matrix B such that

$$I = B[T]_\beta^\gamma = [T]_\beta^\gamma B$$

Since M is an isomorphism, there exists a linear transformation S such that $[S]_\gamma^\beta = B$. We must have that

$$S = T^{-1}$$

from the above equations. $\square$

We have that linear transformations on V to W are represented by matrix multiplication on coordinate vectors in the following way

Theorem 4. *Let $\phi_\beta : V \to \mathbb{F}^n$ be the coordinate transform with respect to $\beta = \{v_1, \ldots, v_n\}$ given by*

$$v = c_1 v_1 + \ldots c_n v_n; \quad \phi_\beta(v) = \begin{bmatrix} c_1, \\ \vdots \\ c_n \end{bmatrix}.$$

Likewise, let $\phi_\gamma : W \to \mathbb{F}^m$ be the coordinate transform with respect to γ. Define $\hat{T}_\beta^\gamma : \mathbb{F}^n \to \mathbb{F}^m$ by

$$\hat{T}_\beta^\gamma(x) = [T]_\beta^\gamma x \quad \text{for} \quad x \in \mathbb{F}^n$$

i.e. $\hat{T}_\beta^\gamma$ is the linear map of multiplication by $[T]_\beta^\gamma$. We have that

$$\phi_\gamma T = \hat{T}_\beta^\gamma \phi_\beta.$$

The above constructions rely on a choice of basis for both V and W. However, the following theorem shows that (for now) one choice of basis is as good as another.

Theorem 5. *Let $\alpha = \{v_1 \ldots v_n\}$ and $\beta = \{u_1, \ldots, u_n\}$ be two bases for V and let $\gamma = \{w_1, \ldots w_m\}$ and $\delta = \{z_1, \ldots, z_m\}$ be two bases for W. If $T \in \mathcal{L}(V, W)$ then there exists an invertible matrice P and Q such that*

$$[T]_\alpha^\gamma = Q[T]_\beta^\delta P.$$

Proof. Note that

$$T = \phi_\gamma^{-1} \hat{T}_\alpha^\gamma \phi_\alpha$$

and

$$T = \phi_\delta^{-1} \hat{T}_\beta^\delta \phi_\beta.$$

by construction. We have

$$\phi_\delta \phi_\gamma^{-1} \hat{T}_\alpha^\gamma = \hat{T}_\beta^\delta \phi_\beta \phi_\alpha^{-1}. \tag{1}$$

Consider the linear isomorphisms $\Phi : \mathbb{F}^n \to \mathbb{F}^n$ and $\Psi : \mathbb{F}^m \to \mathbb{F}^m$ defined by

$$\Phi = \phi_\beta \phi_\alpha^{-1}$$

and

$$\Psi = \phi_\delta \phi_\gamma^{-1}$$

Thus Equation (1) reads as

$$\Psi \hat{T}_\alpha^\gamma = \hat{T}_\beta^\delta \Phi.$$

Let P be a matrix representation for Φ and Q a matrix representation for Ψ (with respect to the standard basis for $\mathbb{F}^n$ and $\mathbb{F}^m$). By construction

$$[T]_\alpha^\gamma = Q^{-1}[T]_\beta^\delta P.$$

$\square$

Corollary 6.

$$[T]_\alpha = P^{-1}[T]_\beta P$$

for some invertible matrix P.

Example: Consider the following linear transformation. Let,

$$D : P_2(\mathbb{R}) \to P_1(\mathbb{R}); \qquad Dp = p'$$

i.e. D is differentiation on polynomials of degree ≤ 2 into polynomials of degree ≤ 1. Let

$$\alpha = \{1,\ t,\ t^2\}; \qquad \beta = \{2,\ t+1,\ t^2 - t\}$$

$$\gamma = \{1,\ t\}; \qquad \delta = \{2,\ t+1\}$$

be bases for $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ respectively. Note that

$$\phi_\alpha(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \phi_\alpha(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \phi_\alpha(t^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\phi_\beta(2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \phi_\beta(t+1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \phi_\beta(t^2 - t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\phi_\gamma(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \phi_\gamma(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\phi_\delta(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \phi_\delta(t+1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\phi_\alpha^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = 1 = \frac{1}{2}(2); \qquad \phi_\beta\phi_\alpha^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$$

$$\phi_\alpha^{-1}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = t = -\frac{1}{2}(2) + 1(t+1); \qquad \phi_\beta\phi_\alpha^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$$

$$\phi_\alpha^{-1}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = t^2 = -\frac{1}{2}(2) + 1(t+1) + 1(t^2 - t); \qquad \phi_\beta\phi_\alpha^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix},$$

and

$$\phi_\gamma^{-1}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1 = \frac{1}{2}(2); \qquad \phi_\delta\phi_\gamma^{-1}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$\phi_\gamma^{-1}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 1 = \frac{1}{2}(2); \qquad \phi_\delta\phi_\gamma^{-1}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}.$$

Thus,

$$[D]^\gamma_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \qquad [D]^\delta_\beta = \begin{bmatrix} 0 & 1/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\phi_\delta \phi_\gamma^{-1}] = \begin{bmatrix} 1/2 & -1/2 \\ 0 & 1 \end{bmatrix}; \qquad [\phi_\beta \phi_\alpha^{-1}] = \begin{bmatrix} 1/2 & -1/2 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$[\phi_\delta \phi_\gamma^{-1}]^{-1}[D]^\delta_\beta[\phi_\beta \phi^{-1}] = [D]^\gamma_\alpha$$

We know from Theorem (3) that a transformation is invertible if and only if its (necessarily square) matrix is invertible. In order to classify the invertible transformations we can consider classifying the invertible matrices. Suppose we wanted a function to differentiate between the invertible and non-invertible matrices. The function would have to be able to deal with a change of basis since the choice of representative matrix depends on the choice in basis. One way to achieve this is by using a monoid homomorphism.

Definition 2. Suppose S is a set with a binary operation $\cdot : S \times S \to S$, then S with $\cdot$ is a monoid if

(a) For all a, b and c in S

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

(b) There exists an identity element 1 in S such that for all $a \in S$

$$a \cdot 1 = 1 \cdot a = a.$$

Definition 3. Let A be a monoid and 1 its identity element. We say $a \in A$ is invertible if and only if there exists a $b \in A$ such that

$$a \cdot b = b \cdot a = 1$$

Definition 4. A monoid homomorphism between monoids A and B is a function $\phi : A \to B$ such that

$$\phi(a_1) \cdot \phi(a_2) = \phi(a_1 \cdot a_2)$$

for all $a_1, a_2 \in A$ and

$$\phi(1_A) = 1_B$$

for the identity elements $1_A \in A$ and $1_B \in B$.

It is clear from the definitions that both $n \times n$ matrices over a field $\mathbb{F}$ and the field $\mathbb{F}$ itself are both monoids (with their standard multiplications). Moreover, it is clear that a monoid homomorphism $\phi : \mathbb{F}^{n\times n} \to \mathbb{F}$ deals appropriately with a change of basis since

$$\phi(A^{-1}BA) = \phi(A^{-1})\phi(B)\phi(A) = \phi(B)$$

for matrices A and B. It is also possible to see that monoid homomorphism from matrices to the underlying field will take invertible matrices to invertible elements (i.e. non-zero) elements of the field.

We will now begin to talk about the determinant of a matrix. In particular we will show that the determinant is a monoid homomorphism from $\mathbb{F}^{n\times n}$ into $\mathbb{F}$.

Definition 5. Let $V_1, \dots V_n$ be vector spaces over a field $\mathbb{F}$. The product $V_1 \times \dots \times V_n$ is defined by

$$V_1 \times \dots \times V_n = \{(v_1, \dots v_n) \mid v_1 \in V_1, \dots v_n \in V_n\}$$

Proposition 7. *$V_1 \times \dots \times V_n$ is a vector space over $\mathbb{F}$.*

Proof. The proof is standard. □

Definition 6. Let $V_1, \dots, V_n, W$ be vector spaces over a field $\mathbb{F}$. A map $\varphi : V_1 \times \dots \times V_n \to W$ is called multilinear if for each fixed i and fixed elements $v_j \in V_j$, $j \neq i$, the map

$$V_i \to W \quad \text{defined by} \quad x \mapsto \varphi(v_1, \dots, v_{i-1}, x, v_{i+1}, \dots, v_n)$$

is linear. If each $V_i = V$, $i = 1, 2, \dots n$ then φ is called a n-multilinear function on V. If W is a field, then φ is called a multilinear form on V.

Definition 7. An n-multilinear function φ on V is called alternating if φ is zero whenever two consecutive arguments are equal, i.e. if $v_i = v_{i+1}$ for some $i \in \{1, \dots, n-1\}$, then $\varphi(v_1, \dots, v_n) = 0$.

Definition 8. Let $\hat{A}_{i,j}$ be the $(n-1)\times(n-1)$ matrix that results from A by removing the ith row and jth column. Consider the set of $n \times n$ matrices over $\mathbb{F}$. Define

$$\det([a]) = a$$

and

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} A_{i,1} \det(\hat{A}_{i,1})$$

We say that the determinant is calculated by cofactor expansion along column 1.

We will show that the function $\det : \mathbb{F}^{n\times n} \to \mathbb{F}$ defined above is both multilinear and alternating.

Proposition 8. *$\det : \mathbb{F}^{n\times n} \to \mathbb{F}$ is a multilinear function (viewing each matrix as a tuple of column vectors in $\mathbb{F}^n \times \dots \times \mathbb{F}^n$).*

Proof. We proceed by induction on n. The result is clear for $n = 1$, so suppose that for an integer $n \geq 2$ the determinant of any $(n-1)\times(n-1)$ matrix is a linear function of each column when the remaining columns are fixed. Suppose that $a_r = u + kv$ for some r, where $u, v \in \mathbb{F}^n$. Let A be a matrix and $a_1, \dots, a_n$ be the columns of A. We want to show that

$$\det(a_1 \dots a_{r-1}, u+kv, a_{r+1}, \dots a_n) = \det(a_1 \dots a_{r-1}, u, a_{r+1}, \dots a_n) + k\det(a_1 \dots a_{r-1}, v, a_{r+1}, \dots a_n).$$

Let

$$u = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Let B and C be the the matrices obtained by replacing column a_r with u and v respectively. We must then prove

$$\det(A) = \det(B) + k\det(C).$$

We leave the proof to the reader for the case that $a_r = a_1$. For $r > 1$ and $1 \leq i \leq n$ the columns of $\hat{A}_{i,1}$, $\hat{B}_{i,1}$, and $\hat{C}_{i,1}$ are the same except for column (now labeled) $r-1$. Moreover, column $r-1$ of $\hat{A}_{i,1}$ is

$$\begin{bmatrix} b_1 + kc_1 \\ \vdots \\ b_{i-1} + kc_{i-1} \\ b_{i+1} + kc_{i+1} \\ \vdots \\ b_n + kc_n \end{bmatrix}$$

which is the sum of column $r-1$ of $\hat{B}_{i,1}$ and k times column $r-1$ of $\hat{C}_{i,1}$. Since $\hat{B}_{i,1}$ and $\hat{C}_{i,1}$ are $(n-1)\times(n-1)$ by the induction hypothesis

$$\det(\hat{A}_{i,1}) = \det(\hat{B}_{i,1}) + k\det(\hat{C}_{i,1}).$$

Then

$$\begin{aligned}
\det(A) &= \sum_{i=1}^{n}(-1)^{i+1}A_{i,1}\det(\hat{A}_{i,1}) \\
&= \det(A) = \sum_{i=1}^{n}(-1)^{i+1}A_{i,1}(\det(\hat{B}_{i,1}) + k\det(\hat{C}_{i,1})) \\
&= \sum_{i=1}^{n}(-1)^{i+1}A_{i,1}\det(\hat{B}_{i,1}) + k\sum_{i=1}^{n}(-1)^{i+1}A_{i,1}\det(\hat{C}_{i,1}) \\
&= \det(B) + k\det(C)
\end{aligned}$$

□

Corollary 9. *If $A \in \mathbb{F}^{n\times n}$ has a column consisting entirely of zeros then* $\det(A) = 0$.

Proof. We see that,

$$\det(a_1, \ldots 0 \ldots a_n) = \det(a_1 \ldots 0 \ldots a_n) + \det(a_1, \ldots 0 \ldots a_n).$$

So $\det(A) = 2\det(A)$ and moreover $\det(A) = k\det(A)$ for all k. Hence we must have that $\det(A) = 0$. □

The following theorem says we can actually expand along any column. We begin by stating a technical lemma without proof.

Lemma 10. *Let $B \in \mathbb{F}^{n\times n}$, where $n \geq 2$. If the column j of B equals*

$$e_k = \begin{bmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{bmatrix} - \text{ } kth\ spot$$

for some k ($1 \leq k \leq n$), then

$$\det(B) = (-1)^{j+k} \det(\hat{B}_{k,j})$$

Theorem 11. *The determinant of a square matrix can be evaluated by cofactor expansion along any column, i.e.*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{i,j} \det(\hat{A}_{i,j})$$

Proof. Let column j of the matrix A be written as $\sum_{i=1}^{n} A_{i,j} e_i$. Let B_i denote the matrix obtained from A by replacing column j by e_i. Thus

$$\det(A) = \sum_{i=1^n} A_{i,j} \det(B_i) = \sum_{i=1}^{n} (-1)^{i+j} A_{i,j} \det(\hat{A}_{i,j}).$$

□

Proposition 12. *The determinant function* det : $\mathbb{F}^{n\times n} \to \mathbb{F}$ *is an alternating function.*

Proof. We proceed by induction. We leave the reader to prove the base case of $n \leq 2$. Let $n \geq 3$, and assume that for all matrices of size $(n-1) \times (n-1)$, that if the matrix has two identical columns the determinant is zero. Suppose column r and s of A are identical and that $a_l \neq a_m$ for $r \neq s$. Since,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{i,j} \det(\hat{A}_{i,j})$$

we can expand on any column that is not r or s. Then $\det(\hat{A}_{i,j}) = 0$ for each i and j since $\hat{A}_{i,j}$ is of size $(n-1) \times (n-1)$ and contains two identical columns. □

The determinant takes the identity element of one monoid to the identity element of the other.

Proposition 13. *We have that*

$$\det(I) = 1$$

Proof. Proceed by cofactor expansion along the first column. □

We have seen that the determinant is a multilinear alternating function which takes the identity matrix to $1 \in \mathbb{F}$. In fact this completely the determinant.

Theorem 14. *The determinant* $\det : \mathbb{F}^{n\times n} \to \mathbb{F}$ *is the* unique *multilinear alternating map taking the identity matrix to the mulitiplicative identity element in* $\mathbb{F}$

The proof is too far outside the scope of the course, but using the uniqueness property above we can show that the determinant is equal to cofactor expansion along any row (by a modification of the proofs above). We will now prove the multiplicative identity.

Proposition 15.

$$\det(AB) = \det(A)\det(B)$$

We begin by listing a series of definitions and lemmas. While they are not difficult to prove, the proofs will be omitted.

Definition 9 (Elementary Row Operations)**.** An elementary row operation is any one of the following operations performed on a matrix.

- Switching the position of two rows.
- Multiplying the entries of a row by a scalar.
- Replacing a row with its addition of a scalar multiple of another row.

Lemma 16. *Let* A *and* B *be matrices such that* $C = AB$ *is defined. Suppose* $e_1, \dots e_n$ *be a sequence of elementary row operations. Let* A' *be the matrix resulting from performing* $e_1, \dots, e_n$ *on* A *and* C' *be the matrix resulting from performing* $e_1, \dots, e_n$ *on* C*. Then*

$$C' = A'B$$

Lemma 17. *Suppose* $e_1, \dots e_n$ *be a sequence of elementary row operations. Let* A' *be the matrix resulting from performing* $e_1, \dots, e_n$ *on* A*. We have that*

$$\alpha \det(A') = \det A$$

for some $\alpha \in \mathbb{F}$ *depending only on* $e_1, \dots, e_n$*.*

Definition 10. Let A be a matrix. Define its transpose, denoted $A^\top$, by

$$(A^\top)_{i,j} = A_{j,i}$$

Lemma 18. *Let* A *and* B *be matrices such that* AB *is defined.*

$$(AB)^\top = B^\top A^\top$$

Lemma 19. *For any square matrix* A

$$\det(A) = \det(A^\top)$$

Definition 11. A lower triangular matrix is any matrix L such that

$$L_{i,j} = 0 \quad \text{for} \quad j > i$$

Definition 12. An upper triangular matrix is any matrix U such that

$$U_{i,j} = 0 \quad \text{for} \quad j < i$$

Lemma 20. *The product of upper (lower) triangular matrices is an upper (lower) triangular matrix.*

Lemma 21. *Let A be either an upper triangular matrix or lower triangular matrix. We have that*

$$\det(A) = \prod a_{i,i}.$$

We are now in a position to prove Proposition 15.

Proof. Let A and B be $n \times n$ matrices and $C = AB$. The matrix A can be turned into an upper triangular matrix by a finite sequence of row operations $e_1, \ldots e_n$. Let A' be the upper triangular matrix resulting from performing $e_1, \ldots e_n$ on A. Let C' be the the matrix resulting from performing $e_1, \ldots, e_n$ on C. Note,

$$C' = A'B$$

by Lemma 16. By Lemma 17 there exists an $\alpha \in \mathbb{F}$ such that

$$\alpha \det(A') = \det(A)$$

and

$$\alpha \det(C') = \det(C).$$

In addition,

$$(C')^\top = B^\top (A')^\top.$$

The matrix $B^\top$ by a sequence of elementary row operations $f_1, \ldots, f_m$. Let $(B^\top)'$ be the upper triangular matrix resulting from performing $f_1, \ldots f_m$ on $B^\top$. Let C'' be the the matrix resulting from performing $f_1, \ldots, f_m$ on $(C')^\top$. So

$$C'' = (B^\top)'(A')^\top$$

and there exists a $\beta \in \mathbb{F}$ such that

$$\beta \det((B^\top)') = \det(B^\top)$$

and

$$\beta \det(C'') = \det((C')^\top).$$

The product of lower triangular matrices is a lower triangular matrix. Hence, $(B^\top)'(A')^\top$ is a lower triangular matrix. By an application of Lemma 21

$$\det((B^\top)'(A')^\top) = \det((B^\top)') \det((A')^\top).$$

Using an application of Lemma 19

$$\begin{aligned}
\det(C) &= \alpha \det(C') \\
&= \alpha \det((C')^\top) \\
&= \alpha\beta \det(C'') \\
&= \alpha\beta \det((B^\top)'(A')^\top) \\
&= \alpha\beta \det((B^\top)') \det((A')^\top) \\
&= \beta \det((B^\top)')\alpha \det((A')^\top) \\
&= \det(A) \det(B)
\end{aligned}$$

□

With the previous theorem proven, we have seen that the determinant is a monoid homomorphism.

We conclude with an example. Shown below is the "sign" matrix for calculating determinants. It's a useful tool for calculating determinants.

Sign matrix for determinants:

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 1. Compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

Solution: We will do row expansion along the 2nd row.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{\text{sign matrix}} \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\det(A) = -2\det\left(\begin{bmatrix} \cancel{1} & 5 & 0 \\ \cancel{2} & \cancel{4} & \cancel{-1} \\ \cancel{0} & -2 & 0 \end{bmatrix}\right) + 4\det\left(\begin{bmatrix} 1 & \cancel{5} & 0 \\ \cancel{2} & \cancel{4} & \cancel{-1} \\ 0 & \cancel{-2} & 0 \end{bmatrix}\right) - (-1)\det\left(\begin{bmatrix} 1 & 5 & \cancel{0} \\ \cancel{2} & \cancel{4} & \cancel{-1} \\ 0 & -2 & \cancel{0} \end{bmatrix}\right)$$

$$\begin{aligned} &= -2\det\left(\begin{bmatrix} 5 & 0 \\ -2 & 0 \end{bmatrix}\right) + 4\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) - (-1)\det\left(\begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix}\right) \\ &= -2 \end{aligned}$$

Properties and Facts:

- If A is a triangular matrix then $\det(A)$ is the product of the entries on the main diagonal.
- If a multiple of one row of A is added to another row to produce B then $\det(A) = \det(B)$.
- If two rows of A are interchanged to produce B then $\det(A) = -\det(B)$.
- If one row of A is multiplied by k to produce B then $\det(B) = k \cdot \det(A)$.
- A square matrix is invertible if and only if $\det(A) \neq 0$.
- $\det(A^\top) = \det(A)$.
- $\det(AB) = \det(A)\det(B)$.