MATH3210

Exam 2

The following rules apply:

- **Exam must be typed**. Please organize your proofs in a reasonably neat and coherent way. Write in complete sentences.
- **Mysterious or unsupported claims will not receive full credit**. Unreasonably large gaps in logic or an argument will receive little credit. You may quote theorems from class or the book.
- **Your solutions must be your own.** You may use outside sources but your submitted solution must be in your own words.

1. Show that the determinant of a matrix is the product of its eigenvalues.

   Hint: You can use Corollary 6 in the determinant notes.

2. For $u \in V$, let φ_u denote the linear functional on the inner product space V defined by

$$\varphi_u(v) = \langle v, u \rangle$$

   for $v \in V$.

   (a) Show that if $\mathbb{F} = \mathbb{R}$ then the map $\Phi : V(\mathbb{R}) \to V(\mathbb{R})'$ defined by

$$\Phi(u) = \varphi_u$$

   is a linear map.

   (b) Show that if $\mathbb{F} = \mathbb{C}$ and $V(\mathbb{C}) \neq \{0\}$, then Φ is not linear.

   (c) Suppose that $\mathbb{F} = \mathbb{R}$ and $V(\mathbb{R})$ is finite dimensional. Show Φ is an isomorphism.

3. Let $P(\mathbb{R})$ be the vector space of polynomials and $\mathbb{R}^\infty$ be the vector space of sequences of real numbers (page 13). Show that $P(\mathbb{R})'$ and $\mathbb{R}^\infty$ are isomorphic.

   Hint: Construct an explicit isomorphism between $P_n(\mathbb{R})'$ and $\mathbb{R}^{n+1}$ and extend this in the natural way to $P(\mathbb{R})$ and $\mathbb{R}^\infty$.

4. Consider the space $\mathbb{C}^\infty$. Let $B : \mathbb{C}^\infty \to \mathbb{C}^\infty$ be defined by the following:

$$B(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots).$$

   Does B have eigenvalues? Rectify your answer with Theorem 5.21 (page 145).

5. Let $M_{2\times 2}(\mathbb{R})$ denote the vector space of 2×2 matrices with real entries. Define the trace of a matrix A as the sum of the diagonal entries, i.e.

$$\operatorname{tr}(A) = a_{1,1} + a_{2,2}$$

   Show that

$$\langle A, B \rangle = \operatorname{tr}(B^\top A)$$

   is an inner product on this space.

6. Show via Cauchy-Schwarz that

$$\left(\frac{a_1 + \ldots + a_n}{n}\right)^2 \leq \frac{a_1^2 + \ldots + a_n^2}{n}$$

   i.e. the square of an average is less than or equal to the average of the squares.

7. Suppose that V is finite dimensional and U is a subspace of V. Show that

$$P_{U^\perp} = I - P_U,$$

   where I is the identity operator on V.