Worksheet 3

1. Complete the following definitions.

   (a) We say for two sets A and B that $|A| = |B| \dots$

   (b) We say a sequence $x_n \to x \dots$

   (c) We say a sequence $x_n \to \infty \dots$

   (a) We say for two sets A and B that $|A| = |B|$ if there is a bijection $f : A \to B$.

   (b) We say a sequence $x_n \to x$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have that $|x_n - x| < \varepsilon$.

   (c) We say a sequence $x_n \to \infty$ if for all $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that for all $n > N$ we have that $x_n \geq M$.

2. True or False.

   (a) $|\mathbb{Q}| = |\mathbb{R}|$

   (b) $|\mathbb{N}| = |\mathbb{Q}|$

   (c) For any set A we have $|A| \neq |P(A)|$ where $P(A)$ is the power set of A.

   (d) $|x - y| \geq ||x| - |y||$

   (e) $|x - y| \geq |x| + |y|$

   (a) False

   (b) True

   (c) True

   (d) True

   (e) False

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3. Show that the even numbers have the same cardinality as the odd numbers.

Proof. Let E denote the even numbers and let O denote the odd numbers, i.e.

$$E = \{2n \mid n \in \mathbb{N}\}$$

$$O = \{2n + 1 \mid n \in \mathbb{N}\}.$$

Define the following bijection $f : E \to O$

$$f(2n) = 2n + 1$$

where $f^{-1} : O \to E$ is

$$f^{-1}(2n + 1) = 2n$$

$\square$

4. Show that $\{\frac{1}{n^2+1}\}_{n=1}^{\infty}$ converges by monotone sequence theorem.

Proof. This sequence is bounded below by 0 since

$$0 < \frac{1}{n^2+1} \text{ for all } n \in \mathbb{N}$$

We also note that

$$\frac{1}{n^2+1} \geq \frac{1}{(n+1)^2+1}$$

since

$$(n+1)^2 + 1 \geq n^2 + 1.$$

So the sequence is monotone decreasing. Hence this sequence converges. $\square$

Notes: There are some different versions of the monotone sequence theorem.

(a) A monotone increasing sequence that is bounded above converges and a monotone decreasing sequence that is bounded below converges.

(b) A bounded monotone sequence converges.

We could have alternatively said that the sequence $\{\frac{1}{n^2+1}\}_{n=1}^{\infty}$ is bounded by noting

$$\left|\frac{1}{n^2+1}\right| \leq 1$$

for all $n \in \mathbb{N}$.

5. Show that the sequence $\{\frac{n^2+1}{n}\}_{n=1}^{\infty}$ diverges to infinity.

Proof. Let $M \in \mathbb{R}$ be given. Choose an N such that $N > M$. For all $n > N$ we have that

$$\frac{n^2+1}{n} = \frac{n^2}{n} + \frac{1}{n} \geq \frac{N^2}{N} \geq M.$$

$\square$