1. Prove or give a counter example.

$$(S \cap T) \cup U = S \cap (T \cup U) \text{ for any sets } S,\ T,\ \text{and } U.$$

Solution: We will provide a counter example. Let

$$S = \{1, 2, 3\},\ T = \{2, 4, 5, 6\},\ \text{and } U = \{8, 9, 10\}.$$

We have that

$$S \cap T = \{2\}$$

and so

$$(S \cap T) \cup U = \{2, 8, 9, 10\}.$$

Moreover, we have

$$T \cup U = \{2, 4, 5, 6, 8, 9, 10\}$$

and so

$$S \cap (T \cup U) = \{2\}.$$

2. Prove or give a counter example.

$$S \cup T = T \iff S \subseteq T.$$

Solution: We will show that the statement is true.

Proof. We will first show that if $S \cup T = T$ then $S \subseteq T$. By definition of union we have that $S \subseteq S \cup T$. Since $S \cup T = T$ we have that $S \subseteq T$. Now suppose that $S \subseteq T$, we will show that this implies that $S \cup T = T$. To do this we show that $S \cup T \subseteq T$ and that $T \subseteq S \cup T$. However, we automatically will have that $T \subseteq S \cup T$. So let $x \in S \cup T$ be arbitrary. If $x \in S$ then since $S \subseteq T$ we have that $x \in T$. If $x \in T$ then obviously $x \in T$. In both cases we have that $x \in T$ and thus since $x \in S \cup T$ was arbitrary we have that $S \cup T \subseteq T$ if $S \subseteq T$. □

3. Prove the distributive law.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof. We will prove the proposition by show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Let $x \in A \cap (B \cup C)$ be arbitrary. We have that $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$ then $x \in B$ or $x \in C$ (or both). If $x \in B$ then $x \in A \cap B$ and hence $x \in (A \cap B) \cup (A \cap C)$. Similarly if $x \in C$ we have that $x \in (A \cap B) \cup (A \cap C)$. Hence, since $x \in A \cap (B \cup C)$ was arbitrary we have that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

Now let $x \in (A \cap B) \cup (A \cap C)$ be arbitrary. If $x \in (A \cap B) \cup (A \cap C)$ then $x \in (A \cap B)$ or $x \in (A \cap C)$ (or both). If $x \in A \cap B$ then $x \in A$ and $x \in B$ and therefore $x \in B \cup C$ as well. So if $x \in A \cap B$ then $x \in A \cap (B \cup C)$. Similarly, if $x \in A \cap C$ then $x \in A \cap (B \cup C)$. Hence since $x \in (A \cap B) \cup (A \cap C)$ was arbitrary we have that

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C).$$

□