1. Basic Notions

Definition 1.1. A binary operation on a set S is a function $f : S \times S \to S$.

Definition 1.2. A *field* is a set $\mathbb{F}$ together with two binary operations $+$, and $\cdot$ called addition and multiplication (respectively) such that

1. For all $a, b, c \in \mathbb{F}$ we have
$$a + (b + c) = (a + b) + c$$
and
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$
2. For all $a, b \in \mathbb{F}$ we have
$$a + b = b + a$$
and
$$a \cdot b = b \cdot a.$$
3. There exists an element $0 \in \mathbb{F}$, called an additive identity, such that for all $a \in \mathbb{F}$ we have $a + 0 = a$.
4. There exists an element $1 \in \mathbb{F}$, called a multiplicative identity, such that for all $a \in \mathbb{F}$ we have $a \cdot 1 = a$.
5. For all $a \in \mathbb{F}$ there exists an element $b \in \mathbb{F}$, called an additive inverse, such that $a + b = 0$.
6. For all $a \in \mathbb{F}$ such that $a \neq 0$ there exists an element $c \in \mathbb{F}$, called a multiplicative inverse, such that $a \cdot c = 1$.
7. For all $a, b, c \in \mathbb{F}$
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

Note: Fields have a unique additive and multiplicative identity denoted 0 and 1 respectively. Moreover, when the additive and multiplicative inverses exist they are unique.

Some examples: All of the following examples are with their standard operations.
1. $\mathbb{Q}$ (rational numbers)
2. $\mathbb{R}$ (real numbers)
3. $\mathbb{C}$ (complex numbers)
4. $\mathbb{Z}/p\mathbb{Z}$ for p prime (Integers modulo p)

Non example: $\mathbb{Z}$ is not a field, it lacks multiplicative inverses.

Definition 1.3. A *vector space* V over a field $\mathbb{F}$ is a set V with two operations called *vector addition* and *scalar multiplication* where vector addition is a function $+ : V \times V \to V$ and scalar multiplication is a function $\cdot : \mathbb{F} \times V \to V$ such that

1. For all $u, v \in V$ we have
$$u + v = v + u$$

2. For all $u, v, w \in V$ and for all $a, b \in \mathbb{F}$ we have
$$(u + v) + w = u + (v + w)$$
and
$$(ab) \cdot v = a \cdot (b \cdot v)$$
3. There exists a vector $0 \in V$, called an additive identity, such that for all $v \in V$ we have
$$v + 0 = v$$
4. For all $v \in V$ we have a vector $w \in V$, called an additive inverse, such that
$$v + w = 0$$
5. For all $v \in V$ we have
$$1 \cdot v = v$$
6. For all $a, b \in \mathbb{F}$ and for all $u, v \in V$ we have
$$a \cdot (u + v) = a \cdot u + a \cdot v$$

Some examples: All of the following examples are with their standard operations.

1. $\mathbb{F}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in \mathbb{F} \right\}$ where $\mathbb{F}$ is a field.
2. Polynomials with coefficients in a field $\mathbb{F}$.
3. Polynomials (with coefficients in a field $\mathbb{F}$) of degree $\leq n$
4. Continuous functions $f : X \to Y$, $C(X, Y)$, where X and Y are fields.
5. Functions from a field X into a field Y.
6. $\mathbb{F}^\infty = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{F}\}$.

Proposition 1.1. *Every vector space V has a unique additive identity. The unique additive identity is denoted 0.*

Proposition 1.2. *Every element $v \in V$ has a unique additive inverse. For all $v \in V$ its unique additive inverse is denoted $-v$.*

Proposition 1.3. *For all $v \in V$ we have $0 \cdot v = 0$.*

Proposition 1.4. *For all $a \in \mathbb{F}$ and $0 \in V$ we have $a \cdot 0 = 0$.*

Proposition 1.5. *For every $v \in V$ we have $(-1) \cdot v = -v$*

2. Basis for a Vector Space

Definition 2.1. A *linear combination* of a list of vectors $v_1, \ldots, v_m$ in V is a vector of the form

$$a_1 v_1 + \ldots + a_m v_m$$

where $a_1, \ldots, a_m \in \mathbb{F}$.

Definition 2.2. The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the *span* of $v_1, \ldots, v_m$ denoted by $\operatorname{span}\{v_1, \ldots, v_m\}$.

$$\operatorname{span}\{v_1, \ldots, v_m\} = \{a_1 v_1 + \ldots + a_m v_m \mid a_i \in \mathbb{F}\}$$

Definition 2.3. If V is a vector space and $V = \operatorname{span}\{v_1, \ldots, v_m\}$ then we say that $v_1, \ldots, v_m$ span V.

Definition 2.4. We say that a vectors space is *finite dimensional* if there exists a finite list of vectors $v_1, \ldots, v_m$ such that

$$\operatorname{span}\{v_1, \ldots v_m\} = V$$

Otherwise we say that V is *infinite dimensional.*

Definition 2.5. A list of vectors $v_1, \ldots, v_m$ in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$a_1 v_1 + \ldots + a_m v_m = 0$$

is $a_1 = a_2 = \ldots = a_m$. A list is called *linearly dependent* if it is not linearly independent.

Lemma 2.6. *Suppose that $v_1, \ldots, v_m$ is a linearly dependent list in V. There exists a $j \in \{1, \ldots, m\}$ such that*

1) $v_j \in span\{v_1, \ldots v_{j-1}\}$
2) $span\{v_1, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots v_m\} = span\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots v_m\}$

Proposition 2.1. *In a finite dimensional vector space the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.*

Definition 2.7. A basis for a vector space V is a list of vectors $\{v_1, \ldots, v_n\}$ such that
1. $\{v_1, \ldots, v_n\}$ is linearly independent
2. $\operatorname{span}\{v_1, \ldots, v_n\} = V$.

Proposition 2.2. *A list of vectors $\{v_1, \ldots, v_n\}$ in V is a basis for V if and only if every vector $v \in V$ can be written* uniquely *in the form*

$$v = a_1 v_1 + \ldots + a_n v_n$$

for some $a_1, \ldots, a_n \in \mathbb{F}$.

Proposition 2.3. *Every spanning list of vectors in V can be reduced down to a basis.*

Proposition 2.4. *Every linearly independent list of vectors in V can be extended to a basis.*

Proposition 2.5. *Any two basis of a finite dimensional vector space V have the same length.*

Definition 2.8. The dimension of a finite dimensional vector space is the length of any basis of the vector space. Denoted $\dim(V)$.

Proposition 2.6. *Suppose V is finite dimensional. Every list of linearly independent vectors whose length is equal to the dimension of V is a basis.*

Proposition 2.7. *Suppose V is finite dimensional. Every spanning list vectors whose length is equal to the dimension of V is a basis.*

3. Subspaces

Definition 3.1. A subspace of a vector space V is a subset H such that H is a vector space under the same binary relations and field as V.

Proposition 3.1 (Subspace Test)**.** *A subset H is a subspace of V if and only if*
1. $0 \in H$.
2. For all $u, v \in H$ we have $u + v \in H$
3. For all $u \in H$ and $a \in \mathbb{F}$ we have $au \in H$.

Proposition 3.2. *If U is a subspace of a finite dimensional vector space V then $dim(U) \leq dim(V)$. Moreover, $dim(U) = dim(V)$ if and only if $V = U$.*

Definition 3.2. Suppose $U_1, \ldots, U_m$ are subsets of V. The *sum* of $U_1, \ldots, U_m$ denoted $U_1 + \ldots + U_m$ is the set of all possible sums i.e.,

$$U_1 + \ldots + U_m = \{u_1 + \ldots + u_m \mid u_i \in U_i, i = 1, \ldots, m\}$$

Proposition 3.3. *If $U_1, \ldots, U_m$ are subspaces then so is $U_1 + \ldots + U_m$.*

Definition 3.3. Suppose $U_1, \ldots, U_m$ are subsspaces of V. The sum $U_1 + \ldots + U_m$ is a *direct sum* if each element of $U_1 + \ldots + U_m$ can be written in only one way as a sum $u_1 + \ldots + u_m$ where $u_i \in U_i$, $i = 1, \ldots, m$. The direct sum is denoted $U_1 \oplus \ldots \oplus U_m$.

Proposition 3.4. *$U_1 + \ldots + U_m$ is a direct sum if and only if the only way to write 0 as a sum is by taking each u_i where $i = 1, \ldots, m$ to be 0.*

Proposition 3.5. *The sum of two subspaces U and W is a direct sum if and only if $U \cap W = \emptyset$.*

Proposition 3.6. *If V is a finite dimensional vector space and U is a subspace of V then there exists a W which is a subspace of V such that $V = U \oplus W$.*

3.1. **Quotient Spaces.**

Definition 3.4. Let V be a vectors space and U a subspace. For every $v \in V$ define

$$v + U = \{v + u \mid u \in U\}$$

and

$$V/U = \{v + U \mid v \in V\}$$

Proposition 3.7. *Let V be a vectorspace, U a subspace and $v, w \in V$. The following are equivalent.*
(a) $v - w \in U$
(b) $v + U = w + U$
(c) $(v + U) \cap (w + U) \neq \emptyset$

Proposition 3.8. *Let V be a vectorspace, U a subspace, $\lambda \in \mathbb{F}$, and $v, w \in V$ The set V/U is a vector space with the following operations:*

$$(v + U) + (w + U) = (v + w) + U$$
$$\lambda(v + U) = (\lambda v) + U$$

Proposition 3.9. *Let V be a finite dimensional vector space and U be a subspace.*

$$dim(V/U) = dim(V) - dim(U)$$

4. Linear Maps

Definition 4.1. A linear map from $V(\mathbb{F})$ to $W(\mathbb{F})$ is a function $T : V \to W$ such that

$$T(\lambda x + y) = \lambda T(x) + T(y)$$

for every $x, y \in V$ and $\lambda \in \mathbb{F}$. Denote the set of all linear maps from V to W as $\mathcal{L}(V, W)$.