1. Let $f : X \to Y$ and $g : Y \to X$ be functions so that $g \circ f = 1_X$. Prove that f is injective and that g is surjective.

Solution:

Proof. Suppose that $f(x_1) = f(x_2)$ and apply g to both sides of the equality. Hence,

$$g \circ f(x_1) = g \circ f(x_2)$$

since g is a function. Since, $g \circ f = 1_X$ we have that

$$x_1 = x_2$$

and thus f is injective. Now let $x \in X$, to show g is surjective we must show there exists a $y \in Y$ such that $g(y) = x$. Note,

$$x = 1_x(x) = g \circ f(x) = g(f(x)).$$

Hence, if $y = f(x)$ then $g(y) = x$. $\square$

2. Prove that if $|A| = |B|$ then $|A \times A| = |B \times B|$.

Solution:

Proof. Since $|A| = |B|$ there exists a bijection $f : A \to B$. Define the function

$$F : A \times A \to B \times B$$

by

$$F(a_1, a_2) = (f(a_1), f(a_2)).$$

We will show that F is a bijection. To show that F is surjective, let $(b_1, b_2) \in B$. Since f is a bijection, f is surjective. Hence, there exists $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Note,

$$F(a_1, a_2) = (f(a_1), f(a_2)) = (b_1, b_2)$$

by definition. Thus, F is surjective. To show that F is injective suppose that

$$F(a_1, a_2) = F(a_1^*, a_2^*)$$

for some $(a_1, a_2), (a_1^*, a_2^*) \in A \times A$. We have that

$$(f(a_1), f(a_2)) = (f(a_1^*), f(a_2^*))$$

by definition of F. Since f is an injection, $f(a_1) = f(a_1^*)$ and $f(a_2) = f(a_2^*)$ implies that $a_1 = a_1^*$ and $a_2 = a_2^*$. Hence, $(a_1, a_2) = (a_1^*, a_2^*)$ and F is an injection. $\square$

3. If $f(x+y) = f(x)f(y)$ and f is a bijection, show that the inverse satisfies

$$f^{-1}(xy) = f^{-1}(x) + f^{-1}(y)$$

Solution: In truth, without some sort of information about the domain, this is not generally true. Consider the following counter example. Let $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$.

$$e^{(0+1)} = e^0 e^1$$

but

$$\ln(0 \cdot 1) = \ln(0) + \ln(1)$$

is not defined.

For the following proof, we will assume that f and f^{-1} are defined on the same domain.

Proof. Since $f(s+t) = f(s)f(t)$ for all s and t in the domain of f

$$xy = f(f^{-1}(x))f(f^{-1}(y)) = f(f^{-1}(x) + f^{-1}(y))$$

If we apply f^{-1} to both sides we get

$$f^{-1}(xy) = f^{-1}(f(f^{-1}(x) + f^{-1}(y))) = f^{-1}(x) + f^{-1}(y)$$

□

4. A card shuffling machine always rearranges cards in the same way relative to the order in which they were given to it. All of the hearts arranged in order from ace to king were put into the machine, and then the shuffled cards were put into the machine again to be shuffled. If the cards emerged in the order 10, 9, Q, 8, K, 3, 4, A, 5, J, 6, 2, 7, in what order were the cards after the first shuffle?

Solution: We know that after two shuffles,

$$10 \mapsto \text{position } 1$$

$$9 \mapsto \text{position } 2$$

$$Q \mapsto \text{position } 3$$

$$\vdots$$

$$A \mapsto \text{position } 8$$

$$\vdots$$

$$7 \mapsto \text{position } 13$$

If we start to write this in cycle notation we get

$$(A, \_\_, 8, \_\_, 4, \_\_, 7, \ldots, \_\_, 9).$$

Completed, the permutation written in cycle notation is written as

$$(A, 2, 8, Q, 4, 3, 7, 6, K, J, 5, 10, 9)$$

Which means after one shuffle the order of the cards is

$$9, A, 4, Q, J, 7, 3, 2, 10, 5, K, 8, 6.$$

5. Let $f : X \to Y$ and $g : Y \to Z$ and suppose that $g \circ f$ is onto. Prove that g is onto then prove or disprove that f is onto.

Solution:

Proof. Let $z_0 \in Z$, we will show that there exists a $y_0 \in Y$ such that $g(y_0) = z_0$. Since $g \circ f$ is onto, there exists an $x_0 \in X$ such that $g \circ f(x_0) = z_0$. If we let $y_0 = f(x_0)$ then

$$g(y_0) = g(f(x_0)) = g \circ f(x_0) = z_0.$$

□