MATH 3210 Your name
Homework 5 date

1. (§5.A #3) Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{ran}(S)$ is invariant under T.

2. (§5.B #1) Suppose that $T \in \mathcal{L}(V)$ and there exists a positive integer n such that $T^n = 0$. Prove that $(I - T)$ is invertible and that
$$(I-T)^{-1} = I + T + \cdots + T^{n-1}$$

3. Suppose that $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose that $p \in \mathcal{P}(\mathbb{F})$ is a polynomial. Prove that
$$p(STS^{-1}) = Sp(T)S^{-1}.$$

4. (§5.C # 16) The Fibonacci sequence $F_1, F_2, \ldots$ is defined by
$$F_1 = 1, F_2 = 1, \quad \text{and} \quad F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 3$$
Define $T \in \mathcal{L}(\mathbb{R}^2)$ by
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x+y \end{bmatrix}.$$
   (a) Show that $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$
   (b) Find the eigenvalues of T.
   (c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.
   (d) Use the solution to part (c) to compute $T^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. Conclude that
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$
   for each positive integer n.