1. Assuming the elementary properties of the trigonometric functions show on the interval $(0, \pi/2)$ that the function $\tan(x) - x$ is strictly increasing and $\frac{\sin(x)}{x}$ is strictly decreasing.

   Solution: Consider the function $f(x) : (0, \pi/2) \to \mathbb{R}$ given by $f(x) = \tan(x) - x$. Note that $f'(x) = \tan^2(x)$. We note that the derivative is positive on the domain $(0, \pi/2)$. Hence, the function is strictly increasing. Let $g(x) = \frac{\sin(x)}{x}$ where $g : (0, \pi/2) \to \mathbb{R}$. We have $g'(x) = \frac{x\cos(x) - \sin(x)}{x^2}$. We must show that $x\cos(x) - \sin(x)$ is negative for all $x \in (0, \pi/2)$ to show that $g(x)$ is strictly decreasing. Let $h(x) = x\cos(x) - \sin(x)$ on $(0, \pi/2)$. Note that $h'(0) = 0$ and that $h'(x) = -x\sin(x)$. Hence, $h'(x) < 0$ on the interval $(0, \pi/2)$ so $h(x)$ is strictly decreasing. Therefore, the maximum for h occurs at $x = 0$ and $h(x)$ is negative on the interval.

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2. We first define limits at infinity.

   Definition 0.1. Given a metric space Y, a point $L \in Y$ and $f : [0, \infty) \to Y$ *has limit* $L \in Y$ *at infinity*, written
   $$\lim_{x \to \infty} f(x) = L,$$
   if for every $\varepsilon > 0$ there is a $C > 0$ such that if $x > C$ then $d_Y(f(x), L) < \varepsilon$.

   Warning: This is now a definition you will be expected to know

   Show that if $f : [0, \infty) \to Y$ is continuous and has a limit at infinity then f is uniformly continuous.

   Solution: We will prove the result directly. This argument is made slightly more complicated by a small technical detail. Compare and contrast this with the hints I gave in class.

   Proof. Let $f : [0, \infty) \to Y$ be a continuous function with a limit L at infinity. Let $\varepsilon > 0$ be given. There exists a $C > 0$ such that if $x > C$ then $d_Y(f(x), L) < \frac{\varepsilon}{2}$. Thus for $x, y \in (C, \infty)$ we have that
   $$d_Y(f(x), f(y)) \leq d_Y(f(x), L) + d_Y(f(y), L) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
   Let $\hat{C} = C + \hat{\varepsilon} > C$ where $\hat{\varepsilon} > 0$. We have that $[0, \infty) = [0, \hat{C}] \cup [\hat{C}, \infty)$. On the interval $[0, \hat{C}]$ the function $f|_{[0,\hat{C}]}$ is uniformly continuous since $[0, \hat{C}]$ is a compact set. Thus, there exists a $\delta > 0$ such that if $x, y \in [0, \hat{C}]$ and $|x - y| < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$. We can now claim that $f : [0, \infty) \to Y$ is uniformly continuous. Suppose $|x - y| < \delta/2$. If $x, y \in [0, \hat{C}]$ then clearly $d_Y(f(x), f(y)) < \varepsilon$ by uniform continuity. If $x, y \in [\hat{C}, \infty)$ then $d_y(f(x), f(y)) < \varepsilon$ by the existence of the limit at infinity. Now suppose without loss of generality that $x \in [0, \hat{C}]$ and $y \in [\hat{C}, \infty)$. We have that $x \leq \hat{C} \leq y$. Note since $|x - y| \leq \delta/2$ then $|x - \hat{C}| < \delta/2$ and $|y - \hat{C}| \leq \delta/2$. Thus
   $$d_Y(f(x), f(y)) \leq d_y(f(x), f(\hat{C})) + d_Y(f(\hat{C}), f(y)) < \varepsilon + \varepsilon = 2\varepsilon$$
   by combining the above two arguments. □

3. Let $f : [0, 1] \to [0, 1]$ be a continuous function. Show that f has a fixed point, i.e. there is a point $x \in [0, 1]$ such that $f(x) = x$.

Solution: We will prove the result by using the intermediate value theorem.

Proof. Let $g(x) = f(x) - x$ on $[0, 1]$. Since $f(x)$ is a continuous function so is $g(x)$. Note that f has a fixed point x_0 if and only if $g(x_0) = f(x_0) - x_0 = x_0 - x_0 = 0$. If $x = 0$ then $g(0) = f(0) - 0 = f(0) \geq 0$ since $f : [0, 1] \to [0, 1]$. Likewise, $g(1) = f(1) - 1 \leq 0$. By intermediate value theorem, since $g(1) \leq 0 \leq g(0)$ there exists a $x_0 \in (0, 1)$ such that $g(x_0) = 0$. Thus, $f(x)$ has a fixed point. □

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4. Formulate and prove a squeeze theorem for functions.

Solution: Here is one version of the squeeze theorem for functions.

Theorem 0.2. *Suppose $f, g, h : X \to \mathbb{R}$ and suppose that for all $x \in X$ we have that*

$$f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x),$$

then

$$\lim_{x \to a} g(x) = L.$$

Proof. We note that $\lim_{x \to a} f(x) = L$ if and only if for all sequences $x_n \to a$ we have that $f(x_n) \to L$. We note that if $x_n \to a$ then

$$\lim_{n \to \infty} f(x_n) = L = \lim_{n \to \infty} h(x_n).$$

Moreover, we have that

$$f(x_n) \leq g(x_n) \leq h(x_n).$$

Therefore, given any arbitrary $x_n \to a$ we have by the squeeze theorem for sequences that $g(x_n) \to L$. Therefore, $\lim_{x \to a} g(x) = L$. □

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5. We start with the following definition

Definition 0.3. Let X and Y be metric spaces. We call a function $f : X \to Y$ *Lipschitz continuous* if there exists a $K > 0$ such that

$$d_Y(f(p), f(q)) \leq K d_X(p, q)$$

for all $p, q \in X$.

Let U be an open interval of $\mathbb{R}$. Prove that if f is differentiable and $f' : U \to \mathbb{R}$ is bounded, then f is Lipschitz continuous.

Solution: We prove the result via Mean Value Theorem.

Proof. Let p and q be arbitrary points in U where $p < q$. Since f is differentiable on U it is differentiable on (p, q) and continuous on $[p, q]$. We have via the mean value theorem there exists a $r \in (p, q)$ such that

$$f(q) - f(p) = f'(r)(q - p)$$

Since $f' : U \to \mathbb{R}$ is bounded there exists a K such that $f'(x) \leq K$ for all $x \in U$. Thus,

$$f(q) - f(p) = K(q - p).$$

More generally,

$$|f(p) - f(q)| \leq K|p - q|.$$

Hence, f is Lipschitz continuous. $\square$