MATH2210Q - Quiz #3 Name: Soln

For full credit, you must show all work and circle your final answer.

1 Determine if the following set of vectors is orthogonal.

$$\left\{ \underset{\vec{u}_1}{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}, \underset{\vec{u}_3}{\begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}} \right\}$$

$\vec{u}_1 \cdot \vec{u}_2 = 0 + (-2) + 2 = 0$

$\vec{u}_1 \cdot \vec{u}_3 = -5 + 4 + 1 = 0$

$\vec{u}_2 \cdot \vec{u}_3 = 0 + (-2) + 2 = 0$

$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ are orthogonal

2 (a) Verify that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set then compute the orthogonal projection of $\mathbf{y}$ onto $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

$$\mathbf{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$$

$\vec{u}_1 \cdot \vec{u}_2 = -12 + 12 = 0$

$\vec{u}_1 \cdot \vec{u}_1 = 9 + 16 = 25$

$\vec{u}_2 \cdot \vec{u}_2 = 9 + 16 = 25$

$\vec{y} \cdot \vec{u}_1 = 18 + 12 = 30$

$\vec{y} \cdot \vec{u}_2 = -24 + 9 = -15$

$$\text{proj}_{\text{span}\{\vec{u}_1, \vec{u}_2\}} \vec{y} = \hat{y} = \left(\frac{30}{25}\right) \vec{u}_1 + \left(\frac{-15}{25}\right) \vec{u}_2$$

$$\hat{y} = \frac{6}{5} \vec{u}_1 - \frac{3}{5} \vec{u}_2$$

$$\hat{y} = \begin{bmatrix} 18/5 \\ 24/5 \\ 0 \end{bmatrix} - \begin{bmatrix} 12/5 \\ 9/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

(b) What is the distance between $\mathbf{y}$ and the plane formed from $\mathbf{u}_1$ and $\mathbf{u}_2$? (Do not simplify)

$$\text{dist} = \|y - \hat{y}\| = \left\| \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right\| = 2$$

$\boxed{3}$ (a) Use *matrix multiplication* to verify that

$$\left\{ \mathbf{u}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix} \right\}$$

is an orthonormal basis for $\mathbb{R}^3$.

$$U^T = \begin{bmatrix} 3/\sqrt{11} & 1/\sqrt{11} & 1/\sqrt{11} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{66} & -4/\sqrt{66} & 7/\sqrt{66} \end{bmatrix} \qquad U = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$$

Since $U^TU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we have $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal set.

Since $\dim(\mathbb{R}^3) = 3$, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an O.N. basis.

(b) Let $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Find c_1, c_2, and c_3 such that $y = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$.

$$c_1 = \vec{y} \cdot \vec{u}_1 = 3/\sqrt{11} + 1/\sqrt{11} = 4/\sqrt{11}$$

$$c_2 = \vec{y} \cdot \vec{u}_2 = 1/\sqrt{6} - 1/\sqrt{6} = 0$$

$$c_3 = \vec{y} \cdot \vec{u}_3 = -1/\sqrt{66} + 7/\sqrt{66} = 6/\sqrt{66}$$