1. Let X and Y be sets of positive real numbers which are bounded above. Define

$$XY = \{xy \mid x \in X,\, y \in Y\}.$$

Show that $\mathrm{lub}(XY) = \mathrm{lub}(X) \cdot \mathrm{lub}(Y)$.

Hint: Do the following: Let $x = \mathrm{lub}(X)$ and $y = \mathrm{lub}(Y)$ and $\varepsilon > 0$.

(i) Show that XY is bounded above.
(ii) Show that there exists an $\hat{x} \in X$ such that $\hat{x} \geq x - \frac{\varepsilon}{x+y}$
(iii) Show that there exists an $\hat{y} \in Y$ such that $\hat{y} \geq y - \frac{\varepsilon}{x+y}$
(iv) Show $\hat{x}\hat{y} \geq xy - \varepsilon$
(v) Use the above to conclude $xy = \mathrm{lub}(XY)$.

2. Show, that the sequence

$$a_n = \frac{2n-3}{n+5} \qquad n \geq 1$$

converges.

3. Prove that $\{n^2+2\}_{n=1}^{\infty}$ diverges to infinity.

4. Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences with limits x and y respectively. Prove

(a) $\{cx_n\}$ converges to cx where $c \in \mathbb{R}$.
(b) $\{x_n + y_n\}$ converges to $x + y$.

5. Use the monotone convergence theorem to show the sequence $\{x_n\}$ defined by

$$x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{2 + x_n} \quad \text{for } n > 1$$

converges.

Hint: Show by induction that the sequence is increasing and bounded above by 2.