Scalar Line integrals

2D:

$$\int_C \underbrace{f(x,y)}_{\text{scalar function}} \underbrace{ds}_{\text{differential arc length}} = \int_a^b f(x(t),y(t)) \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}_{ds}$$

Given C is parameterized by $\vec{r}(t) = \langle x(t), y(t) \rangle \quad t: a \to b$

Note: $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\Rightarrow ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Compact form: $\int_C f(x,y)\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$

3D:

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

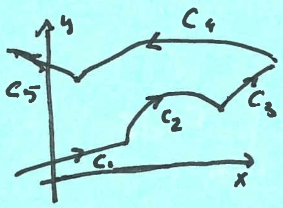
$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

Compact form: $\int_C f(x,y,z)\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$

Note: For piecewise smooth curves

Entire path – $\int_C f(x,y,z)\, ds = \underbrace{\int_{C_1} f(x,y,z)\, ds + \cdots + \int_{C_n} f(x,y,z)\, ds}_{\text{sum of pieces}}$

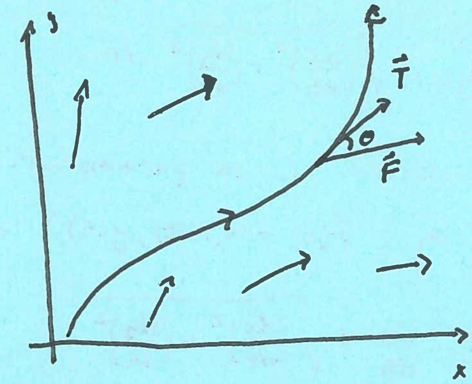
Ex//



Common parameterizations: $\vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) \quad t: 0 \to 1$ line segment from $\vec{r}_0$ to $\vec{r}_1$

$\vec{r}(t) = \langle a\cos t, b\sin t \rangle \quad t: 0 \to 2\pi$ planar ellipses (circles) (counter-clockwise) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

Line integrals of Vector fields



Defined to be

$$\int_C \underbrace{\vec F \cdot \vec T}_{\text{scaler valued}}\, \overbrace{ds}^{\text{arclength}} \qquad \vec F \cdot \vec T = |F|\cos\theta$$

Intuitively can be thought of as the "summation" of the vector field which lies along the path or the work done by the force field $\vec F$ as you move an object along the path.

Given a parameterization $\vec r(t)$; $t: a \to b$

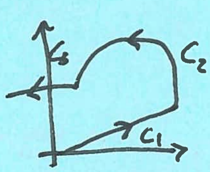
$$\int_C \vec F \cdot \vec T\, ds = \int_a^b \underbrace{\vec F(\vec r(t))}_{\vec F} \cdot \underbrace{\frac{\vec r\,'(t)}{|\vec r\,'(t)|}}_{\vec T}\, \underbrace{|\vec r\,'(t)|\, dt}_{ds} = \int_a^b \vec F(\vec r(t)) \cdot \vec r\,'(t)\, dt$$

Alternate forms

$$\int_C \vec F \cdot \vec T\, ds = \int_a^b \vec F(\vec r(t))\, \vec r\,'(t)\, dt = \int_C \vec F \cdot \underbrace{d\vec r}_{d\vec r = \vec r\,'(t)dt} = \int_C P\,dx + Q\,dy + R\,dz$$

$$\vec F = P\hat i + Q\hat j + R\hat k$$

Properties: Piecewise curves



$$\underbrace{\int_C \vec F \cdot d\vec r}_{\text{entire path}} = \underbrace{\int_{C_1} \vec F \cdot d\vec r + \dots + \int_{C_n} \vec F \cdot d\vec r}_{\text{sum of individual pieces}}$$

Orientation Reversing

$$\underbrace{\int_{-C} \vec F \cdot d\vec r}_{\text{backwords}} = \underbrace{-\int_C \vec F \cdot d\vec r}_{\text{negative of forward direction}}$$

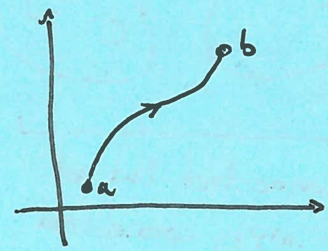
Conservative Vector fields

A vector field $\vec{F}$ is a <u>conservative</u> or <u>gradient</u> field if there is a scalar function ϕ such that

$$\underbrace{\vec{F}}_{\text{vectfield}} = \nabla \overbrace{\underbrace{\phi}_{\text{scalarfunction}}}^{\text{vect field}}$$

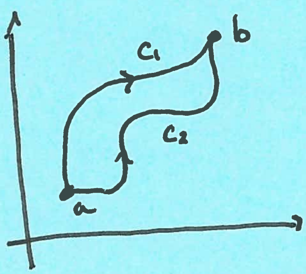
Fundamental Theorem of Conservative Vector fields

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\phi\cdot d\vec{r} = \phi(\underbrace{\vec{r}(b)}_{\text{End}}) - \phi(\underbrace{\vec{r}(a)}_{\text{start}})$$



C is parameterized by $\vec{r}(t) : t : a \longrightarrow b$

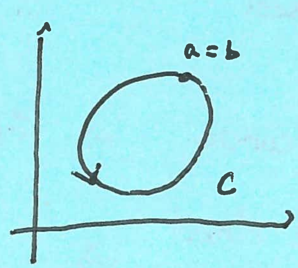
Path independence: $\int_C \vec{F}\cdot d\vec{r}$ is independent of the path C if $\vec{F}$ is a conservative vector field (all that matters is the end points)



$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = \phi(\vec{r}(b)) - \phi(\vec{r}(a))$$

for $\vec{F} = \nabla\phi$

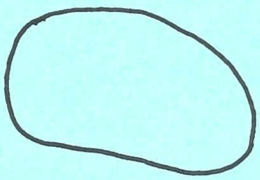
A loop is <u>closed</u> if $\vec{r}(b) = \vec{r}(a)$



closed loop $\rightarrow \oint_C \vec{F}\cdot d\vec{r} = 0$ if $\vec{F} = \nabla\phi$

Regions, Loops, and paths.

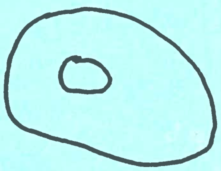
We will assume regions are open (no boundary points)

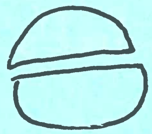


simply connected

simply connected $\Rightarrow$ connected

connected $\not\Rightarrow$ simply connected

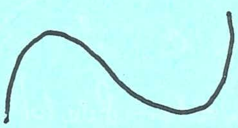




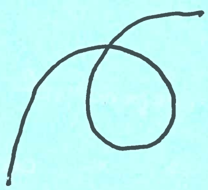
Not connected

not simply connected.

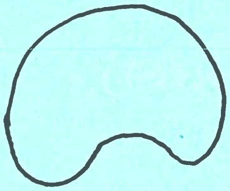
Connected but not simply connected



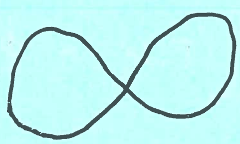
simple not closed



Not simple not closed



Simple closed



Not simple closed

Simple curve $\Rightarrow$ $\vec{r}(t_1) \neq \vec{r}(t_2)$ when $a < t_1 < t_2 < b$

simple closed curve $\Rightarrow$ $\vec{r}(t_1) \neq \vec{r}(t_2)$ when $a < t_1 < t_2 < b$

but $\vec{r}(a) = \vec{r}(b)$

Theorems

- Suppose $\vec{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D then $\vec{F}$ is conservative.

- If $\vec{F}$ is a conservative vector field with components having continuous first order partials on a domain D (open connected) then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \; ; \; \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \; ; \; \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

- If $\vec{F}$ is a vector field with components having continuous first order partials on an open <u>simply</u> connected domain D and

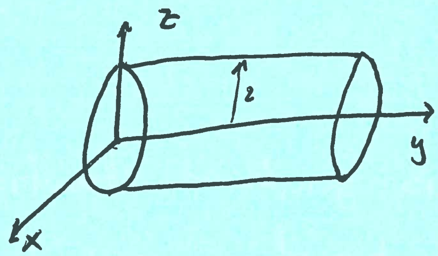
$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \; ; \; \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \; ; \; \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

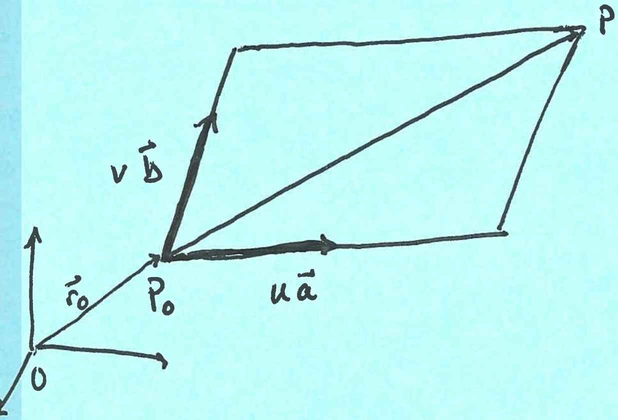
then F is conservative

Parametric Surfaces.

Common Examples

$\vec{r}(u,v) = \langle 2\cos u,\ v,\ 2\sin u \rangle$



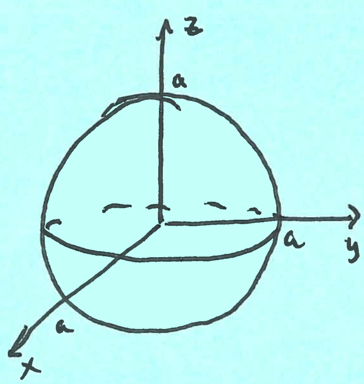


Plane that passes through P_0 with position vector $\vec{r}_0$ and contains two nonparallel vectors $\vec{a}$ and $\vec{b}$

$$\vec{r} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$$

$$\vec{r} = \vec{r}_0 + u\vec{a} + v\vec{b}$$

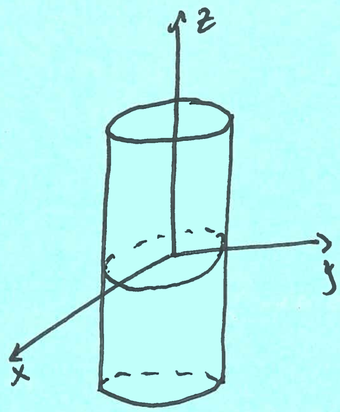
$$\Rightarrow \vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$$



Sphere of radius a

$$\vec{r}(\phi,\theta) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle$$

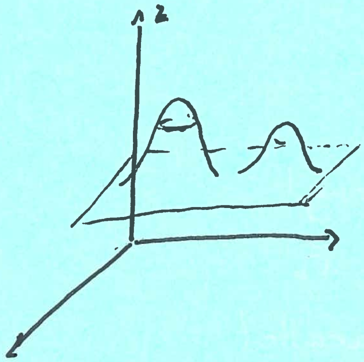
$0 \le \phi \le \pi \quad 0 \le \theta \le 2\pi$



Cylinder

$x^2 + y^2 = a^2$

$\vec{r}(z, \theta) = \langle a\cos\theta, a\sin\theta, z \rangle$

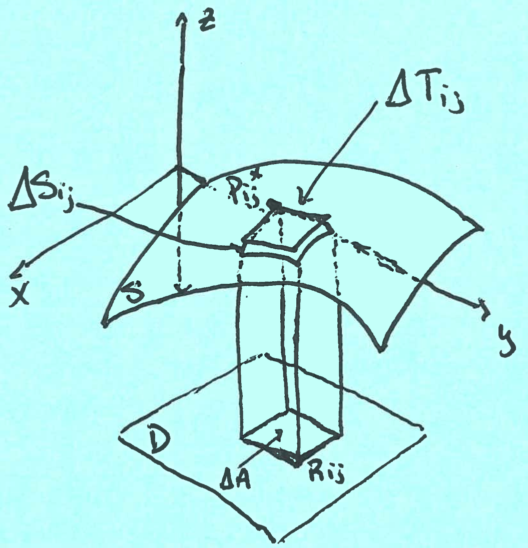


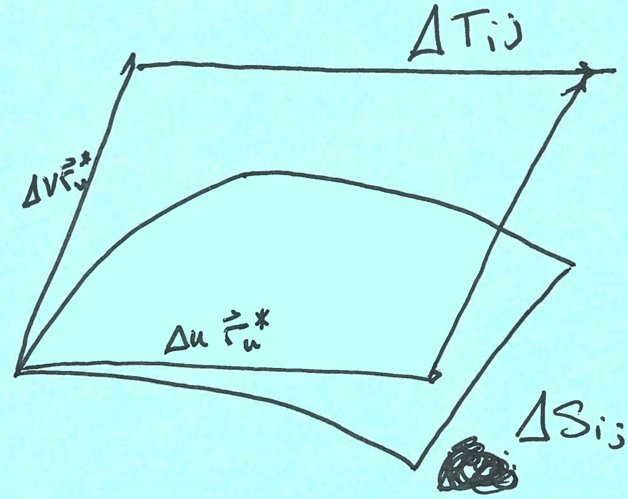
surface is graph of function.

$z = K(x,y)$ $\quad$ Z is a function of 2-variables

$\vec{r}(x, y) = \langle x, y, K(x,y) \rangle$

Scalar Surface integrals





Given a parameterization $\vec{r}(u,v)$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,\underbrace{|\vec{r}_u \times \vec{r}_v|\,dA}_{dS}$$

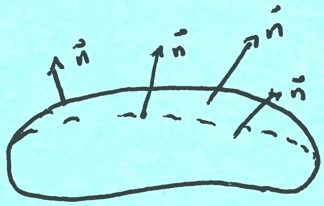
If $\quad z = K(x,y) \qquad \vec{r}(x,y) = \langle x,\ y,\ K(x,y) \rangle$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

Surface integrals of Vector fields

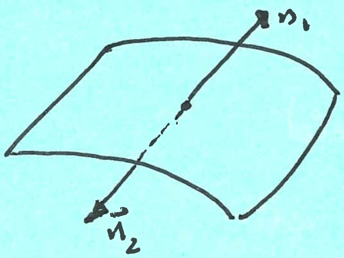
To do surface integrals for vector fields

we need orientable surfaces



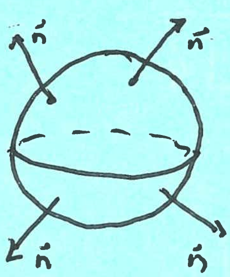
$\vec{n}$ - normal vector

Surfaces can have two possible orientations



For problems we have to tell you which orientation to use

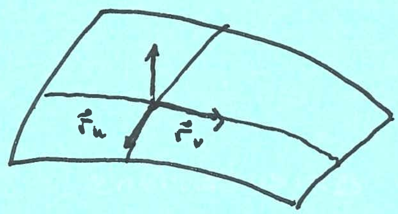
Exception! If the surface is closed we pick the "outward" pointing orientation.



For a parameterized surface

$\vec{r}(u,v)$ the normal (unit) vector $\hat{n}$

is given by $\dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ or $\dfrac{-(\vec{r}_u \times \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|}$



If $\vec{F}$ is a continuous vector field

defined on an oriented surface S with

unit normal $\hat{n}$ the surface integral of $\vec{F}$ over S

is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\hat{n}\, dS \qquad \text{(called Flax)}$$

Given parameterization

$$= \iint_D \vec{F}\cdot \overbrace{\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}}^{\hat{n}}\ \overbrace{|\vec{r}_u\times\vec{r}_v|\, dA}^{dS}$$

u,v region.

$$= \iint_D \vec{F}(\vec{r}(u,v))\cdot \underbrace{(\vec{r}_u\times\vec{r}_v)}\, dA$$

or $-(\vec{r}_u\times\vec{r}_v)$

depending on orientation

For a surface $z = K(x, y)$

$\vec{r}(x,y) = \langle x, y, K(x,y) \rangle$ is the parameterization

$$\hat{n}\, dS = \frac{\vec{r}_x \times \vec{r}_y}{\cancel{|\vec{r}_x \times \vec{r}_y|}} \cdot \cancel{|\vec{r}_x \times \vec{r}_y|}\, dA = \langle -K_x, -K_y, 1 \rangle\, dA$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, dS = \iint_S \vec{F} \cdot \langle -K_x, -K_y, 1 \rangle\, dA$$

Curl and divergence

Let $\vec{F} = \langle f, g, h \rangle$

$$\text{curl } \vec{F} = \underbrace{\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}}$$

↳ result is a vector

define $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$

Then $\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f & g & h \end{vmatrix}$

Theorem: If f is a function of three variables that has continuous 2nd order partials, then

$$\text{curl}(\nabla f) = \nabla \times (\nabla f) = \vec{0}$$ — zero vector

ie. If $\vec{F}$ is conservative then $\text{curl } \vec{F} = \vec{0}$

Theorem: If $\vec{F}$ is a vector field over a simply connected domain whose components have continuous partials and $\text{curl } \vec{F} = \vec{0}$ then $\vec{F}$ is conservative.

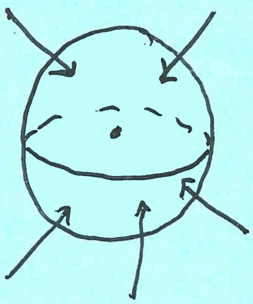
If $\text{curl } \vec{F} = \vec{0}$ then $\vec{F}$ is called <u>irrotational</u>

Define, for $\vec{F} = \langle f, g, h \rangle$

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \underbrace{\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}}_{\text{result is a scalar.}}$$

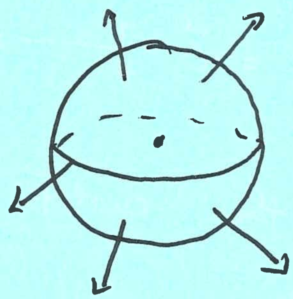
Theorem: If $\vec{F}$ has continuous 2nd order partials then

$$\operatorname{div}(\operatorname{curl} \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = \underbrace{0}_{\text{scalar}}$$



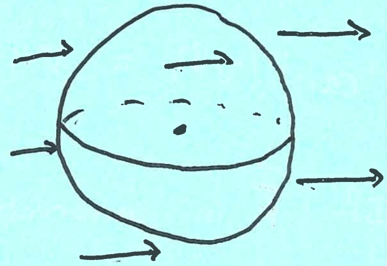
$\operatorname{div} \vec{F} < 0$

sink



$\operatorname{div} \vec{F} > 0$

source



$\operatorname{div} \vec{F} = 0$

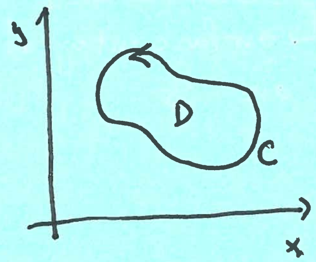
source free.

(incompressible)

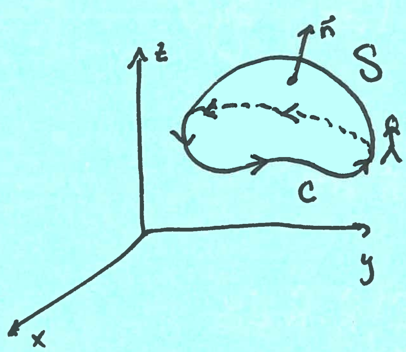
Greens, Stokes, and Divergence Theorems

Green's Theorem: Let C be a positively (counterclockwise) piecewise smooth, simple closed curve in $\mathbb{R}^2$ and let D be the region bounded by C. Suppose $\vec{F} = \langle P, Q \rangle$ and that P and Q have continuous partials on an open region containing D.

$$\underbrace{\oint_C P\,dx + Q\,dy}_{= \oint_C \vec{F} \cdot d\vec{r}} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

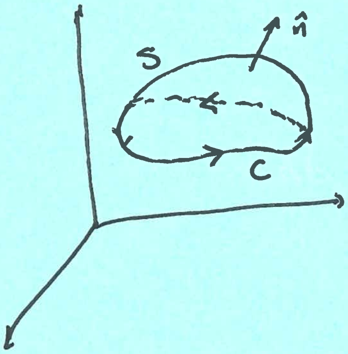


Note Green's Thm is a specific case of Stoke's Theorem.



Suppose you have an oriented surface S. The orientation of S induces a positive ("counter-clockwise") orientation of the boundary curve C. If you walk in the positive direction around C, your head will point in the direction of $\hat{n}$ and the surface will be on your left.

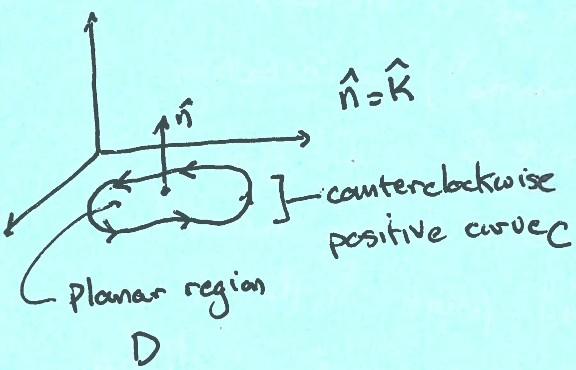
Stoke's Thm : Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation. Let $\vec{F}$ be a vector field w/ components having continuous partials on a region of $\mathbb{R}^3$ containing S. Then



$$\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\text{Line integral around } C} = \underbrace{\iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, dS}_{\text{surface integral over } S}$$

Green's Thm via Stoke's Thm

$$\vec{F} = \langle P, Q \rangle \qquad \operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \hat{k}$$



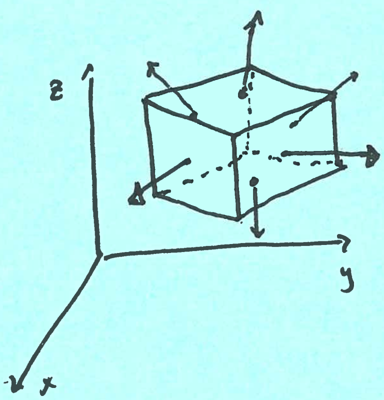
$\hat{n} = \hat{k}$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\operatorname{curl} \vec{F}) \cdot \hat{n}\, dS$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \hat{k} \cdot \hat{k}\, dS$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad (dA = dS)$$

Divergence Thm: Let D be a $\overbrace{\text{simple solid}}^{\text{no holes}}$ region and let S be the boundary surface of D with positive (outward) orientation.



Let $\vec{F}$ be a vector field whose component functions have continuous partials on an open region containing D.

$$\underbrace{\iint_S \vec{F}\cdot\hat{n}\,dS}_{\substack{\text{Flux} \\ \text{across } S}} = \underbrace{\iiint_D \operatorname{div}\vec{F}\,dV}_{\substack{\text{Scalar} \\ \text{triple integral}}}$$