1. Basic Notions

Definition 1.1. A binary operation on a set S is a function $f : S \times S \to S$.

Definition 1.2. A *field* is a set $\mathbb{F}$ together with two binary operations $+$, and $\cdot$ called addition and multiplication (respectively) such that

1. For all $a, b, c \in \mathbb{F}$ we have
$$a + (b + c) = (a + b) + c$$
and
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$
2. For all $a, b \in \mathbb{F}$ we have
$$a + b = b + a$$
and
$$a \cdot b = b \cdot a.$$
3. There exists an element $0 \in \mathbb{F}$, called an additive identity, such that for all $a \in \mathbb{F}$ we have $a + 0 = a$.
4. There exists an element $1 \in \mathbb{F}$, called a multiplicative identity, such that for all $a \in \mathbb{F}$ we have $a \cdot 1 = a$.
5. For all $a \in \mathbb{F}$ there exists an element $b \in \mathbb{F}$, called an additive inverse, such that $a + b = 0$.
6. For all $a \in \mathbb{F}$ such that $a \neq 0$ there exists an element $c \in \mathbb{F}$, called a multiplicative inverse, such that $a \cdot c = 1$.
7. For all $a, b, c \in \mathbb{F}$
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

Note: Fields have a unique additive and multiplicative identity denoted 0 and 1 respectively. Moreover, when the additive and multiplicative inverses exist they are unique.

Some examples: All of the following examples are with their standard operations.

1. $\mathbb{Q}$ (rational numbers)
2. $\mathbb{R}$ (real numbers)
3. $\mathbb{C}$ (complex numbers)
4. $\mathbb{Z}/p\mathbb{Z}$ for p prime (Integers modulo p)

Non example: $\mathbb{Z}$ is not a field, it lacks multiplicative inverses.

Definition 1.3. A *vector space* V over a field $\mathbb{F}$ is a set V with two operations called *vector addition* and *scalar multiplication* where vector addition is a function $+ : V \times V \to V$ and scalar multiplication is a function $\cdot : \mathbb{F} \times V \to V$ such that

1. For all $u, v \in V$ we have
$$u + v = v + u$$
2. For all $u, v, w \in V$ and for all $a, b \in \mathbb{F}$ we have
$$(u + v) + w = u + (v + w)$$
and
$$(ab) \cdot v = a \cdot (b \cdot v)$$
3. There exists a vector $0 \in V$, called an additive identity, such that for all $v \in V$ we have
$$v + 0 = v$$
4. For all $v \in V$ we have a vector $w \in V$, called an additive inverse, such that
$$v + w = 0$$
5. For all $v \in V$ we have
$$1 \cdot v = v$$
6. For all $a, b \in \mathbb{F}$ and for all $u, v \in V$ we have
$$a \cdot (u + v) = a \cdot u + a \cdot v$$

Some examples: All of the following examples are with their standard operations.

1. $\mathbb{F}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in \mathbb{F} \right\}$ where $\mathbb{F}$ is a field.
2. Polynomials with coefficients in a field $\mathbb{F}$.
3. Polynomials (with coefficients in a field $\mathbb{F}$) of degree $\leq n$
4. Continuous functions $f : X \to Y$, $C(X, Y)$, where X and Y are fields.
5. Functions from a field X into a field Y.
6. $\mathbb{F}^\infty = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{F}\}$.

Proposition 1.1. *Every vector space V has a unique additive identity. The unique additive identity is denoted 0.*

Proposition 1.2. *Every element $v \in V$ has a unique additive inverse. For all $v \in V$ its unique additive inverse is denoted $-v$.*

Proposition 1.3. *For all $v \in V$ we have $0 \cdot v = 0$.*

Proposition 1.4. *For all $a \in \mathbb{F}$ and $0 \in V$ we have $a \cdot 0 = 0$.*

Proposition 1.5. *For every $v \in V$ we have $(-1) \cdot v = -v$*

2. Basis for a Vector Space

Definition 2.1. A *linear combination* of a list of vectors $v_1, \ldots, v_m$ in V is a vector of the form

$$a_1 v_1 + \ldots + a_m v_m$$

where $a_1, \ldots, a_m \in \mathbb{F}$.

Definition 2.2. The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the *span* of $v_1, \ldots, v_m$ denoted by $\text{span}\{v_1, \ldots, v_m\}$.

$$\text{span}\{v_1, \ldots, v_m\} = \{a_1 v_1 + \ldots + a_m v_m \mid a_i \in \mathbb{F}\}$$

Definition 2.3. If V is a vector space and $V = \text{span}\{v_1, \ldots, v_m\}$ then we say that $v_1, \ldots, v_m$ span V.

Definition 2.4. We say that a vectors space is *finite dimensional* if there exists a finite list of vectors $v_1, \ldots, v_m$ such that

$$\text{span}\{v_1, \ldots v_m\} = V$$

Otherwise we say that V is *infinite dimensional.*

Definition 2.5. A list of vectors $v_1, \ldots, v_m$ in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$a_1 v_1 + \ldots + a_m v_m = 0$$

is $a_1 = a_2 = \ldots = a_m$. A list is called *linearly dependent* if it is not linearly independent.

Lemma 2.6. *Suppose that $v_1, \dots, v_m$ is a linearly dependent list in V. There exists a $j \in \{1, \dots, m\}$ such that*

1) $v_j \in span\{v_1, \dots v_{j-1}\}$
2) $span\{v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots v_m\} = span\{v_1, \dots, v_{j-1}, v_{j+1}, \dots v_m\}$

Proposition 2.1. *In a finite dimensional vector space the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.*

Definition 2.7. A basis for a vector space V is a list of vectors $\{v_1, \dots, v_n\}$ such that
1. $\{v_1, \dots, v_n\}$ is linearly independent
2. $\text{span}\{v_1, \dots, v_n\} = V$.

Proposition 2.2. *A list of vectors $\{v_1, \dots, v_n\}$ in V is a basis for V if and only if every vector $v \in V$ can be written* uniquely *in the form*

$$v = a_1 v_1 + \dots + a_n v_n$$

for some $a_1, \dots, a_n \in \mathbb{F}$.

Proposition 2.3. *Every spanning list of vectors in V can be reduced down to a basis.*

Proposition 2.4. *Every linearly independent list of vectors in V can be extended to a basis.*

Proposition 2.5. *Any two basis of a finite dimensional vector space V have the same length.*

Definition 2.8. The dimension of a finite dimensional vector space is the length of any basis of the vector space. Denoted $\dim(V)$.

Proposition 2.6. *Suppose V is finite dimensional. Every list of linearly independent vectors whose length is equal to the dimension of V is a basis.*

Proposition 2.7. *Suppose V is finite dimensional. Every spanning list vectors whose length is equal to the dimension of V is a basis.*

3. Subspaces

Definition 3.1. A subspace of a vector space V is a subset H such that H is a vector space under the same binary relations and field as V.

Proposition 3.1 (Subspace Test)**.** *A subset H is a subspace of V if and only if*
1. $0 \in H$.
2. For all $u, v \in H$ we have $u + v \in H$
3. For all $u \in H$ and $a \in \mathbb{F}$ we have $au \in H$.

Proposition 3.2. *If U is a subspace of a finite dimensional vector space V then $dim(U) \leq dim(V)$. Moreover, $dim(U) = dim(V)$ if and only if $V = U$.*

Definition 3.2. Suppose $U_1, \ldots, U_m$ are subsets of V. The *sum* of $U_1, \ldots, U_m$ denoted $U_1 + \ldots + U_m$ is the set of all possible sums i.e.,

$$U_1 + \ldots + U_m = \{u_1 + \ldots + u_m \mid u_i \in U_i, i = 1, \ldots, m\}$$

Proposition 3.3. *If $U_1, \ldots, U_m$ are subspaces then so is $U_1 + \ldots + U_m$.*

Definition 3.3. Suppose $U_1, \ldots, U_m$ are subspaces of V. The sum $U_1 + \ldots + U_m$ is a *direct sum* if each element of $U_1 + \ldots + U_m$ can be written in only one way as a sum $u_1 + \ldots + u_m$ where $u_i \in U_i$, $i = 1, \ldots, m$. The direct sum is denoted $U_1 \oplus \ldots \oplus U_m$.

Proposition 3.4. *$U_1 + \ldots + U_m$ is a direct sum if and only if the only way to write 0 as a sum is by taking each u_i where $i = 1, \ldots, m$ to be 0.*

Proposition 3.5. *The sum of two subspaces U and W is a direct sum if and only if $U \cap W = \{0\}$.*

Proposition 3.6. *If V is a finite dimensional vector space and U is a subspace of V then there exists a W which is a subspace of V such that $V = U \oplus W$.*

3.1. **Quotient Spaces.**

Definition 3.4. Let V be a vectors space and U a subspace. For every $v \in V$ define

$$v + U = \{v + u \mid u \in U\}$$

and

$$V/U = \{v + U \mid v \in V\}$$

Proposition 3.7. *Let V be a vectorspace, U a subspace and $v, w \in V$. The following are equivalent.*
(a) $v - w \in U$
(b) $v + U = w + U$
(c) $(v + U) \cap (w + U) \neq \emptyset$

Proposition 3.8. *Let V be a vectorspace, U a subspace, $\lambda \in \mathbb{F}$, and $v, w \in V$ The set V/U is a vector space with the following operations:*

$$(v + U) + (w + U) = (v + w) + U$$

$$\lambda(v + U) = (\lambda v) + U$$

Proposition 3.9. *Let V be a finite dimensional vector space and U be a subspace.*

$$dim(V/U) = dim(V) - dim(U)$$

4. Linear Maps

Definition 4.1. A linear map from $V(\mathbb{F})$ to $W(\mathbb{F})$ is a function $T : V \to W$ such that

$$T(\lambda x + y) = \lambda T(x) + T(y)$$

for every $x, y \in V$ and $\lambda \in \mathbb{F}$. Denote the set of all linear maps from V to W as $\mathcal{L}(V, W)$.

Proposition 4.1. *Suppose $\{v_1, \ldots, v_n\}$ is a basis of V and $w_1, \ldots, w_n \in W$. There is a unique linear map $T : V \to W$ such that*

$$T(v_j) = w_j \qquad j = 1, \ldots, n.$$

Proposition 4.2. *If $T : V \to W$ is linear then*

$$T(0_V) = 0_W.$$

Definition 4.2. Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$ define

$$(S + T)(v) = S(v) + T(v) \text{ for all } v \in V$$

and

$$(\lambda T)(v) = \lambda T(v) \text{ for all } v \in V.$$

Proposition 4.3. *$\mathcal{L}(V, W)$ is a vector space with the above operations.*

Definition 4.3. For $T \in \mathcal{L}(V, W)$ define the *null space* of T (or kernel of T) to be

$$\text{null}(T) = \{v \in V \mid Tv = 0\} \subseteq V$$

and define the *range* of T to be

$$\text{ran}(T) = \{Tv \mid v \in V\} \subseteq W.$$

Theorem 4.4. *For $T \in \mathcal{L}(V, W)$ both null(T) and ran(T) are subspaces. A linear transformation is injective if and only if*

$$\textit{null}(T) = \{0\}.$$

A linear transformation is surjective if and only if ran(T) $= W$.

Definition 4.5. The *rank* of a linear transformation is the dimension of its range. The *nullity* of a transformation is the dimension of its null space.

Definition 4.6. A vector space isomorphism from V onto W is a bijective linear map $T : V \to W$. If there is a vector space isomorphism from V onto W we say V is isomorphic to W and write $V \cong W$.

Theorem 4.7. *Let V and W be two finite dimensional vector spaces. V is isomorphic to W if and only if dim(V) $=$ dim(W).*

Definition 4.8. A linear map $T : V \to W$ is *invertible* if there exists an map $S : W \to V$ such that

$$S \circ T = \mathrm{Id}_V$$

$$T \circ S = \mathrm{Id}_W$$

Theorem 4.9. *A map $T : V \to W$ is invertible if and only if its bijective.*

Proposition 4.4. *Suppose $T \in \mathcal{L}(V, W)$ and define*

$$\tilde{T} : V/null(T) \to W$$

by

$$\tilde{T}(v + null(T)) = Tv.$$

The following hold:

a) $\tilde{T}$ is linear

b) $\tilde{T}$ is injective

c) $ran(\tilde{T}) = ran(T)$

d) $V/null(T) \cong ran(T)$.

Theorem 4.10 (Fundamental Theorem of Linear Maps/ Rank-Nullity)**.** *Suppose V is a finite dimensional vector space and $T \in \mathcal{L}(V, W)$. We have $ran(T)$ is finite dimensional and*

$$dim(V) = dim(ran(T)) + dim(null(T)).$$

Proposition 4.5. *Suppose V and W are finite dimensional vector spaces. Let $T \in \mathcal{L}(V, W)$.*

a) If $dim(V) < dim(W)$ then T is not surjective.

b) If $dim(V) > dim(W)$ then T is not injective.

Definition 4.11. A linear map $T \in \mathcal{L}(V, V)$ is called an *operator*.

Theorem 4.12. *Suppose T is an operator over a vector space V. If V is finite dimension the following are equivalent:*

a) T is injective

b) T is surjective

c) T is bijective

4.1. **The Matrix of a Linear Map and the Coordinate Transform.**

Definition 4.13. Let V be an n-dimensional vector space over the field $\mathbb{F}$. Let $\beta = \{v_1, \dots, v_n\}$ be an *ordered* basis for V. The coordinate transform $\varphi_\beta : V \to \mathbb{F}^n$ is defined by

$$v = a_1 v_1 + \ldots + a_n v_n \xmapsto{\varphi_\beta} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Denote the column vector $\varphi_\beta(v)$ by $[v]_\beta$.

Proposition 4.6. *The coordinate transform from V onto $\mathbb{F}^n$ is a vector space isomorphism.*

Definition 4.14. Suppose $T : V \to W$ is a linear maps with $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ as ordered basis for V and W respectively. Define the matrix of T, denoted by $[T]_\beta^\gamma$, by the following If

$$T(v_k) = a_{1,k} w_1 + \ldots + a_{m,k} w_k$$

then the k-th column of $[T]_\beta^\gamma$ is given by $\begin{bmatrix} a_{1,k} \\ \vdots \\ a_{m,k} \end{bmatrix}$.

Theorem 4.15. *The following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varphi_\beta"'] & W \arrow[d, "\varphi_\gamma"] \\ \mathbb{F}^n \arrow[r, "{[T]_\beta^\gamma}"'] & \mathbb{F}^m \end{tikzcd}$$

More specifically,

$$[T]_\beta^\gamma [v]_\beta = [T(v)]_\gamma.$$