MATH2210Q Name: Solution

Exam 1 Date:

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any unapproved calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, ask for an extra sheet of paper to continue the problem on; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	4	
2	5	
3	4	
4	2	
5	4	
6	6	
Total:	25	

1. Determine which of the following sets of vectors are linearly independent.

(a) (1 point) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix} \right\}$

linearly dependent, more vectors than entries.

(b) (1 point) $\left\{ \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 15 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 12 \\ 30 \\ 6 \end{bmatrix} \right\}$

linearly dependent, $2\begin{bmatrix} 3 \\ 6 \\ 15 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 30 \\ 6 \end{bmatrix}$.

(c) (2 points) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x_1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has only the trivial solution.

The vectors are linearly independent

2. Determine if b lies in the span of the given vectors.

(a) (1 point) $b = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$; $\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

yes, $\begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(b) (2 points) $b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$; $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} \right\}$

Solve : $x_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$

$$\Leftrightarrow \left[\begin{array}{cc|c} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -9 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 11 \end{array}\right]$$

inconsistent system; $\vec{b} \notin \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} \right\}$

(c) (2 points) Does every vector b in $\mathbb{R}^3$ lie in the span of the set below?

$$\left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix} \right\}$$

No ; $\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 14 & 10 \\ 0 & -7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$ zero row would give an inconsistent system, i.e. the matrix only has 2 pivots.

3. Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(a) (2 points) Compute $\det(2A \cdot B^2)$

$\det A = 1 \cdot 2 \cdot 3 \cdot 1 = 6$

$\det B = 1 \cdot 2 \cdot 1 \cdot 1 = 2$

$\det(2A) = 2^4 \cdot 6 = 16 \cdot 6 = 96$

$\det(B^2) = 4$

$\det(2A \cdot B^2) = 96 \cdot 4 = 384$

(b) (2 points) Which of the above matrices are invertible? (State your reasoning.)

$$\det(C) = 2 \det \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} = 2\left(-2 \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\right)$$

$$= -4 \cdot \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= 4$$

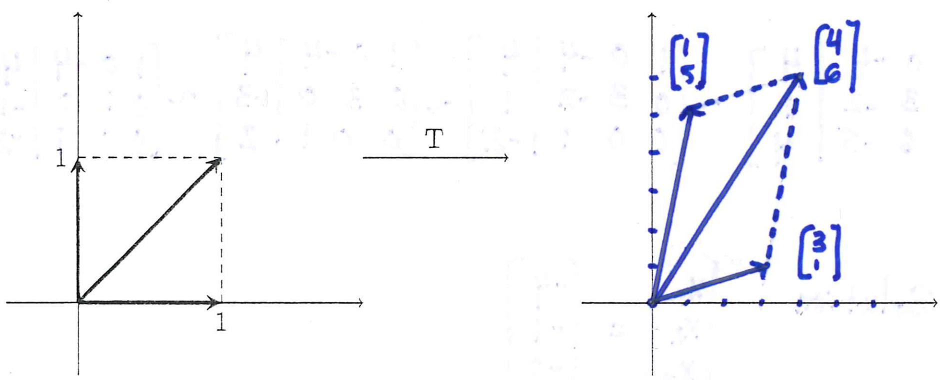
$\det(A) \neq 0$, $\det(B) \neq 0$, $\det(C) \neq 0$

All three are invertible.

4. Define the following two transformations

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix} \mathbf{x}; \quad S(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \mathbf{x}$$

(a) (1 point) Draw the image of the unit square under T



Under T : $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xmapsto{T} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \xmapsto{T} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Moreover, $\begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \xmapsto{T} \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

(b) (1 point) Find the standard matrix for $S(T(\mathbf{x}))$ and $T(S(\mathbf{x}))$.

$$[ST] = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix}$$

$$[TS] = \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix}$$

5. Consider the following matrix equation.

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$$

(a) (2 points) Find the solution set.

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Solution : $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ -2 \end{bmatrix}$

(b) (2 points) Write the equivalent linear system and vector equation.

$X_1 - 4X_3 = 4$

$3X_2 - 2X_3 = 1$

$-2X_1 + 6X_2 + 3X_3 = -4$

linear system

$X_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + X_2 \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} + X_3 \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$

Vector Equation

6. Consider the following matrix and one of its echelon forms.

$$A = \begin{bmatrix} 1 & -5 & 4 & 1 \\ 2 & -10 & 12 & 4 \\ -3 & 15 & -12 & 4 \end{bmatrix} \sim E = \begin{bmatrix} 1 & -5 & 4 & 1 \\ 2 & -10 & 12 & 4 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

(a) (2 points) Find a basis for Null(A).

$$[A|0] \sim [E|0] \sim \left[\begin{array}{cccc|c} 1 & -5 & 4 & 0 & 0 \\ 2 & -10 & 12 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -5 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ $x_1 = 5x_2$
$x_2 = x_2$
$x_3 = 0$
$x_4 = 0$

Basis for null(A): $\left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(b) (2 points) Find a basis for Col(A).

pivot columns : 1, 3, 4

Basis for col(A) : $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 12 \\ -12 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} \right\}$

(c) (1 point) If $T(\mathrm{x}) = A\mathrm{x}$ is T a one to one transformation?

No ; $A\vec{x} = \vec{0}$ has more than just the trivial solution

i.e. solutions : $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(d) (1 point) If $T(\mathrm{x}) = A\mathrm{x}$ is T onto?

Yes, $T : \mathbb{R}^4 \to \mathbb{R}^3$ and A has 3 pivots.