

## MATH3210

### Exam 2

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The following rules apply:

- **Exam must be typed.** Please organize your proofs in a reasonably neat and coherent way. Write in complete sentences.
- **Mysterious or unsupported claims will not receive full credit.** Unreasonably large gaps in logic or an argument will receive little credit. You may quote theorems from class or the book.
- **Your solutions must be your own.** You may use outside sources but your submitted solution must be in your own words.

1. Show that the determinant of a matrix is the product of its eigenvalues.

**Hint:** You can use Corollary 6 in the determinant notes.

2. For  $u \in V$ , let  $\varphi_u$  denote the linear functional on the inner product space  $V$  defined by

$$\varphi_u(v) = \langle v, u \rangle$$

for  $v \in V$ .

- (a) Show that if  $\mathbb{F} = \mathbb{R}$  then the map  $\Phi : V(\mathbb{R}) \rightarrow V(\mathbb{R})'$  defined by

$$\Phi(u) = \varphi_u$$

is a linear map.

- (b) Show that if  $\mathbb{F} = \mathbb{C}$  and  $V(\mathbb{C}) \neq \{0\}$ , then  $\Phi$  is not linear.
  - (c) Suppose that  $\mathbb{F} = \mathbb{R}$  and  $V(\mathbb{R})$  is finite dimensional. Show  $\Phi$  is an isomorphism.
3. Let  $P(\mathbb{R})$  be the vector space of polynomials and  $\mathbb{R}^\infty$  be the vector space of sequences of real numbers (page 13). Show that  $P(\mathbb{R})'$  and  $\mathbb{R}^\infty$  are isomorphic.

**Hint:** Construct an explicit isomorphism between  $P_n(\mathbb{R})'$  and  $\mathbb{R}^{n+1}$  and extend this in the natural way to  $P(\mathbb{R})$  and  $\mathbb{R}^\infty$ .

4. Consider the space  $\mathbb{C}^\infty$ . Let  $B : \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$  be defined by the following:

$$B(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots).$$

Does  $B$  have eigenvalues? Rectify your answer with Theorem 5.21 (page 145).

5. Let  $M_{2 \times 2}(\mathbb{R})$  denote the vector space of  $2 \times 2$  matrices with real entries. Define the trace of a matrix  $A$  as the sum of the diagonal entries, i.e.

$$\text{tr}(A) = a_{1,1} + a_{2,2}$$

Show that

$$\langle A, B \rangle = \text{tr}(B^\top A)$$

is an inner product on this space.

6. Show via Cauchy-Schwarz that

$$\left( \frac{a_1 + \dots + a_n}{n} \right)^2 \leq \frac{a_1^2 + \dots + a_n^2}{n}$$

i.e. the square of an average is less than or equal to the average of the squares.

7. Suppose that  $V$  is finite dimensional and  $U$  is a subspace of  $V$ . Show that

$$P_{U^\perp} = I - P_U,$$

where  $I$  is the identity operator on  $V$ .