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1 Basic Notions

Definition 1.1. A binary operation on a set S is a function $f : S \times S \to S$.

Definition 1.2. A *field* is a set $\mathbb{F}$ together with two binary operations $+$, and $\cdot$ called addition and multiplication (respectively) such that

1. For all $a, b, c \in \mathbb{F}$ we have
$$a + (b + c) = (a + b) + c$$
and
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

2. For all $a, b \in \mathbb{F}$ we have
$$a + b = b + a$$
and
$$a \cdot b = b \cdot a.$$

3. There exists an element $0 \in \mathbb{F}$, called an additive identity, such that for all $a \in \mathbb{F}$ we have $a + 0 = a$.

4. There exists an element $1 \in \mathbb{F}$, called a multiplicative identity, such that for all $a \in \mathbb{F}$ we have $a \cdot 1 = a$.

5. For all $a \in \mathbb{F}$ there exists an element $b \in \mathbb{F}$, called an additive inverse, such that $a+b = 0$.

6. For all $a \in \mathbb{F}$ such that $a \neq 0$ there exists an element $c \in \mathbb{F}$, called a multiplicative inverse, such that $a \cdot c = 1$.

7. For all $a, b, c \in \mathbb{F}$
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

Note: Fields have a unique additive and multiplicative identity denoted 0 and 1 respectively. Moreover, when the additive and multiplicative inverses exist they are unique.

Some examples: All of the following examples are with their standard operations.

1. $\mathbb{Q}$ (rational numbers)

2. $\mathbb{R}$ (real numbers)

3. $\mathbb{C}$ (complex numbers)

4. $\mathbb{Z}/p\mathbb{Z}$ for p prime (Integers modulo p)

Non example: $\mathbb{Z}$ is not a field, it lacks multiplicative inverses.

Definition 1.3. A *vector space* V over a field $\mathbb{F}$ is a set V with two operations called *vector addition* and *scalar multiplication* where vector addition is a function $+ : V \times V \to V$ and scalar multiplication is a function $\cdot : \mathbb{F} \times V \to V$ such that

1. For all $u, v \in V$ we have
$$u + v = v + u$$

2. For all $u, v, w \in V$ and for all $a, b \in \mathbb{F}$ we have

$$(u + v) + w = u + (v + w)$$

and

$$(ab) \cdot v = a \cdot (b \cdot v)$$

3. There exists a vector $0 \in V$, called an additive identity, such that for all $v \in V$ we have

$$v + 0 = v$$

4. For all $v \in V$ we have a vector $w \in V$, called an additive inverse, such that

$$v + w = 0$$

5. For all $v \in V$ we have

$$1 \cdot v = v$$

6. For all $a, b \in \mathbb{F}$ and for all $u, v \in V$ we have

$$a \cdot (u + v) = a \cdot u + a \cdot v$$

Some examples: All of the following examples are with their standard operations.

1. $\mathbb{F}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in \mathbb{F} \right\}$ where $\mathbb{F}$ is a field.

2. Polynomials with coefficients in a field $\mathbb{F}$.

3. Polynomials (with coefficients in a field $\mathbb{F}$) of degree $\leq n$

4. Continuous functions $f : X \to Y$, $C(X, Y)$, where X and Y are fields.

5. Functions from a field X into a field Y.

6. $\mathbb{F}^\infty = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{F}\}$.

Proposition 1.1. *Every vector space V has a unique additive identity. The unique additive identity is denoted* 0.

Proposition 1.2. *Every element $v \in V$ has a unique additive inverse. For all $v \in V$ its unique additive inverse is denoted $-v$.*

Proposition 1.3. *For all $v \in V$ we have $0 \cdot v = 0$.*

Proposition 1.4. *For all $a \in \mathbb{F}$ and $0 \in V$ we have $a \cdot 0 = 0$.*

Proposition 1.5. *For every $v \in V$ we have $(-1) \cdot v = -v$*

2 Basis for a Vector Space

Definition 2.1. A *linear combination* of a list of vectors $v_1, \ldots, v_m$ in V is a vector of the form

$$a_1 v_1 + \ldots + a_m v_m$$

where $a_1, \ldots, a_m \in \mathbb{F}$.

Definition 2.2. The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the *span* of $v_1, \ldots, v_m$ denoted by $\operatorname{span}\{v_1, \ldots, v_m\}$.

$$\operatorname{span}\{v_1, \ldots, v_m\} = \{a_1 v_1 + \ldots + a_m v_m \mid a_i \in \mathbb{F}\}$$

Definition 2.3. If V is a vector space and $V = \operatorname{span}\{v_1, \ldots, v_m\}$ then we say that $v_1, \ldots, v_m$ span V.

Definition 2.4. We say that a vectors space is *finite dimensional* if there exists a finite list of vectors $v_1, \ldots, v_m$ such that

$$\operatorname{span}\{v_1, \ldots v_m\} = V$$

Otherwise we say that V is *infinite dimensional.*

Definition 2.5. A list of vectors $v_1, \ldots, v_m$ in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$a_1 v_1 + \ldots + a_m v_m = 0$$

is $a_1 = a_2 = \ldots = a_m$. A list is called *linearly dependent* if it is not linearly independent.

Lemma 2.6. *Suppose that $v_1, \ldots, v_m$ is a linearly dependent list in V. There exists a $j \in \{1, \ldots, m\}$ such that*

1) $v_j \in span\{v_1, \dots v_{j-1}\}$

2) $span\{v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots v_m\} = span\{v_1, \dots, v_{j-1}, v_{j+1}, \dots v_m\}$

Proposition 2.1. *In a finite dimensional vector space the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.*

Definition 2.7. A basis for a vector space V is a list of vectors $\{v_1, \dots, v_n\}$ such that

1. $\{v_1, \dots, v_n\}$ is linearly independent

2. $\text{span}\{v_1, \dots, v_n\} = V$.

Proposition 2.2. *A list of vectors $\{v_1, \dots, v_n\}$ in V is a basis for V if and only if every vector $v \in V$ can be written* uniquely *in the form*

$$v = a_1 v_1 + \dots + a_n v_n$$

for some $a_1, \dots, a_n \in \mathbb{F}$.

Proposition 2.3. *Every spanning list of vectors in V can be reduced down to a basis.*

Proposition 2.4. *Every linearly independent list of vectors in V can be extended to a basis.*

Proposition 2.5. *Any two basis of a finite dimensional vector space V have the same length.*

Definition 2.8. The dimension of a finite dimensional vector space is the length of any basis of the vector space. Denoted $\dim(V)$.

Proposition 2.6. *Suppose V is finite dimensional. Every list of linearly independent vectors whose length is equal to the dimension of V is a basis.*

Proposition 2.7. *Suppose V is finite dimensional. Every spanning list vectors whose length is equal to the dimension of V is a basis.*

3 Subspaces

Definition 3.1. A subspace of a vector space V is a subset H such that H is a vector space under the same binary relations and field as V.

Proposition 3.1 (Subspace Test)**.** *A subset H is a subspace of V if and only if*

1. $0 \in H$.

2. For all $u, v \in H$ we have $u + v \in H$

3. For all $u \in H$ and $a \in \mathbb{F}$ we have $au \in H$.

Proposition 3.2. *If U is a subspace of a finite dimensional vector space V then $dim(U) \leq dim(V)$. Moreover, $dim(U) = dim(V)$ if and only if $V = U$.*

3.1 Direct Sums

Definition 3.2. Suppose $U_1, \ldots, U_m$ are subsets of V. The *sum* of $U_1, \ldots, U_m$ denoted $U_1 + \ldots + U_m$ is the set of all possible sums i.e.,

$$U_1 + \ldots + U_m = \{u_1 + \ldots + u_m \mid u_i \in U_i, i = 1, \ldots, m\}$$

Proposition 3.3. *If $U_1, \ldots, U_m$ are subspaces then so is $U_1 + \ldots + U_m$.*

Definition 3.3. Suppose $U_1, \ldots, U_m$ are subspaces of V. The sum $U_1 + \ldots + U_m$ is a *direct sum* if each element of $U_1 + \ldots + U_m$ can be written in only one way as a sum $u_1 + \ldots + u_m$ where $u_i \in U_i$, $i = 1, \ldots, m$. The direct sum is denoted $U_1 \oplus \ldots \oplus U_m$.

Proposition 3.4. *$U_1 + \ldots + U_m$ is a direct sum if and only if the only way to write 0 as a sum is by taking each u_i where $i = 1, \ldots, m$ to be 0.*

Proposition 3.5. *The sum of two subspaces U and W is a direct sum if and only if $U \cap W = \{0\}$.*

Proposition 3.6. *If V is a finite dimensional vector space and U is a subspace of V then there exists a W which is a subspace of V such that $V = U \oplus W$.*

3.2 Quotient Spaces

Definition 3.4. Let V be a vectors space and U a subspace. For every $v \in V$ define

$$v + U = \{v + u \mid u \in U\}$$

and

$$V/U = \{v + U \mid v \in V\}$$

Proposition 3.7. *Let V be a vectorspace, U a subspace and $v, w \in V$. The following are equivalent.*

(a) $v - w \in U$

(b) $v + U = w + U$

(c) $(v + U) \cap (w + U) \neq \emptyset$

Proposition 3.8. *Let V be a vectorspace, U a subspace, $\lambda \in \mathbb{F}$, and $v, w \in V$ The set V/U is a vector space with the following operations:*

$$(v + U) + (w + U) = (v + w) + U$$

$$\lambda(v + U) = (\lambda v) + U$$

Proposition 3.9. *Let V be a finite dimensional vector space and U be a subspace.*

$$dim(V/U) = dim(V) - dim(U)$$

4 Linear Maps

Definition 4.1. A linear map from $V(\mathbb{F})$ to $W(\mathbb{F})$ is a function $T : V \to W$ such that

$$T(\lambda x + y) = \lambda T(x) + T(y)$$

for every $x, y \in V$ and $\lambda \in \mathbb{F}$. Denote the set of all linear maps from V to W as $\mathcal{L}(V, W)$.

Proposition 4.1. *Suppose $\{v_1, \ldots, v_n\}$ is a basis of V and $w_1, \ldots, w_n \in W$. There is a unique linear map $T : V \to W$ such that*

$$T(v_j) = w_j \qquad j = 1, \ldots, n.$$

Proposition 4.2. *If $T : V \to W$ is linear then*

$$T(0_V) = 0_W.$$

Definition 4.2. Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$ define

$$(S + T)(v) = S(v) + T(v) \text{ for all } v \in V$$

and

$$(\lambda T)(v) = \lambda T(v) \text{ for all } v \in V.$$

Proposition 4.3. *$\mathcal{L}(V, W)$ is a vector space with the above operations.*

4.1 One to One, Onto, Invertibility, and Isomorphisms

Definition 4.3. For $T \in \mathcal{L}(V, W)$ define the *null space* of T (or kernel of T) to be

$$\text{null}(T) = \{v \in V \mid Tv = 0\} \subseteq V$$

and define the *range* of T to be

$$\text{ran}(T) = \{Tv \mid v \in V\} \subseteq W.$$

Theorem 4.4. *For $T \in \mathcal{L}(V, W)$ both null(T) and ran(T) are subspaces. A linear transformation is injective if and only if*

$$null(T) = \{0\}.$$

A linear transformation is surjective if and only if ran(T) $= W$.

Definition 4.5. The *rank* of a linear transformation is the dimension of its range. The *nullity* of a transformation is the dimension of its null space.

Definition 4.6. A vector space isomorphism from V onto W is a bijective linear map $T : V \to W$. If there is a vector space isomorphism from V onto W we say V is isomorphic to W and write $V \cong W$.

Theorem 4.7. *Let V and W be two finite dimensional vector spaces. V is isomorphic to W if and only if dim(V) $=$ dim(W).*

Definition 4.8. A linear map $T : V \to W$ is *invertible* if there exists an map $S : W \to V$ such that

$$S \circ T = \text{Id}_V$$

$$T \circ S = \text{Id}_W$$

Theorem 4.9. *A map $T : V \to W$ is invertible if and only if its bijective.*

4.2 Fundamental Theorem of Linear Maps

Proposition 4.4. *Suppose $T \in \mathcal{L}(V, W)$ and define*

$$\tilde{T} : V/null(T) \to W$$

by

$$\tilde{T}(v + null(T)) = Tv.$$

The following hold:

a) $\tilde{T}$ *is linear*

b) $\tilde{T}$ *is injective*

c) $ran(\tilde{T}) = ran(T)$

d) $V/null(T) \cong ran(T)$.

Theorem 4.10 (Fundamental Theorem of Linear Maps/ Rank-Nullity)**.** *Suppose V is a finite dimensional vector space and $T \in \mathcal{L}(V, W)$. We have $ran(T)$ is finite dimensional and*

$$dim(V) = dim(ran(T)) + dim(null(T)).$$

Proposition 4.5. *Suppose V and W are finite dimensional vector spaces. Let $T \in \mathcal{L}(V, W)$.*

a) If $dim(V) < dim(W)$ then T is not surjective.

b) If $dim(V) > dim(W)$ then T is not injective.

Definition 4.11. A linear map $T \in \mathcal{L}(V, V)$ is called an *operator*.

Theorem 4.12. *Suppose T is an operator over a vector space V. If V is finite dimension the following are equivalent:*

a) T *is injective*

b) T *is surjective*

c) T *is bijective*

4.3 The Matrix of a Linear Map and the Coordinate Transform

Definition 4.13. Let V be an n-dimensional vector space over the field $\mathbb{F}$. Let $\beta = \{v_1, \ldots, v_n\}$ be an *ordered* basis for V. The coordinate transform $\varphi_\beta : V \to \mathbb{F}^n$ is defined by

$$v = a_1 v_1 + \ldots + a_n v_n \overset{\varphi_\beta}{\longmapsto} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Denote the column vector $\varphi_\beta(v)$ by $[v]_\beta$.

Proposition 4.6. *The coordinate transform from V onto $\mathbb{F}^n$ is a vector space isomorphism.*

Definition 4.14. Suppose $T : V \to W$ is a linear maps with $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ as ordered basis for V and W respectively. Define the matrix of T, denoted by $[T]_\beta^\gamma$, by the following If

$$T(v_k) = a_{1,k}w_1 + \ldots + a_{m,k}w_k$$

then the k-th column of $[T]_\beta^\gamma$ is given by $\begin{bmatrix} a_{1,k} \\ \vdots \\ a_{m,k} \end{bmatrix}$.

Theorem 4.15. *The following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \\ \mathbb{F}^n \arrow[r, "{[T]_\beta^\gamma}"] & \mathbb{F}^m \arrow[u, "\phi_\gamma^{-1}"'] \end{tikzcd}$$

More specifically,

$$[T]_\beta^\gamma [v]_\beta = [T(v)]_\gamma.$$

5 Determinants

Definition 5.1. Let $\hat{A}_{i,j}$ be the $(n-1) \times (n-1)$ matrix that results from A by removing the ith row and jth column and let $a_{i,j}$ be the entry in the ith row and jth column. Consider the set of $n \times n$ matrices over $\mathbb{F}$. Define

$$\det([a]) = a$$

and

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i,1} \det(\hat{A}_{i,1})$$

5.1 Multilinear and Alternating

Definition 5.2. Let $V_1, \ldots V_n$ be vector spaces over a field $\mathbb{F}$. The product $V_1 \times \ldots \times V_n$ is defined by

$$V_1 \times \ldots \times V_n = \{(v_1, \ldots v_n) \mid v_1 \in V_1, \ldots v_n \in V_n\}$$

Of course, with the appropriate operations $V_1 \times \ldots \times V_n$ is a vector space.

Proposition 5.1. *$V_1 \times \ldots \times V_n$ is a vector space over $\mathbb{F}$ with the following operations:*

$$(v_1, \ldots, v_n) + (u_1, \ldots, u_n) = (v_1 + u_1, \ldots, v_n + u_n)$$

$$c(v_1, \ldots, v_n) = (cv_1, \ldots, cv_n).$$

The proof of the above proposition is standard, we will omit it.

Definition 5.3. Let $V_1, \ldots, V_n, W$ be vector spaces over a field $\mathbb{F}$. A map $\varphi : V_1 \times \ldots \times V_n \to W$ is called *multilinear* if for each fixed i and fixed elements $v_j \in V_j$, $j \neq i$, the map

$$V_i \to W \quad \text{ defined by } \quad x \mapsto \varphi(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n)$$

is linear. If each $V_i = V$, $i = 1, 2, \ldots n$ then φ is called a n-multilinear function on V. If W is a field, then φ is called a multilinear form on V.

The function $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ is viewed as a multilinear map by viewing the columns of a matrix as column vectors and making the following identification.

$$M_{n\times n}(\mathbb{F}) \ni A = [v_1, \ldots, v_n] \mapsto (v_1, \ldots, v_n) \in \mathbb{F}^n \times \ldots \times \mathbb{F}^n.$$

Proposition 5.2. $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ *is a multilinear function (viewing each matrix as a tuple of column vectors in $\mathbb{F}^n \times \ldots \times \mathbb{F}^n$).*

Definition 5.4. An n-multilinear function φ on V is called alternating if φ is zero whenever two consecutive arguments are equal, i.e. if $v_i = v_{i+1}$ for some $i \in \{1, \ldots, n-1\}$, then $\varphi(v_1, \ldots, v_n) = 0$.

Lemma 5.5. *Let $B \in M_{n\times n}(\mathbb{F})$, where $n \geq 2$. If the column j of B equals*

$$e_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} - \textit{kth spot}$$

for some k $(1 \leq k \leq n)$, then

$$\det(B) = (-1)^{j+k} \det(\hat{B}_{k,j})$$

Theorem 5.6. *The determinant of a square matrix can be evaluated by cofactor expansion* *along any column or row, i.e.*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{i,j} \det(\hat{A}_{i,j})$$

or

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(\hat{A}_{i,j})$$

Proposition 5.3. *The determinant function* $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ *is an alternating function.*

Theorem 5.7. *The determinant* $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ *is the* unique *multilinear alternating map taking the identity matrix to the mulitiplicative identity element in* $\mathbb{F}$

5.2 The Multiplicative Property

Definition 5.8 (Elementary Row Operations)**.** An elementary row operation is any one of the following operations performed on a matrix.

- Switching the position of two rows.
- Multiplying the entries of a row by a scalar.
- Replacing a row with its addition of a scalar multiple of another row.

Lemma 5.9. *Let A and B be matrices such that $C = AB$ is defined. Suppose $e_1, \dots e_n$ be a sequence of elementary row operations. Let A' be the matrix resulting from performing $e_1, \dots, e_n$ on A and C' be the matrix resulting from performing $e_1, \dots, e_n$ on C. Then*

$$C' = A'B$$

Lemma 5.10. *Suppose $e_1, \dots e_n$ be a sequence of elementary row operations. Let A' be the matrix resulting from performing $e_1, \dots, e_n$ on A. We have that*

$$\alpha \det(A') = \det A$$

for some $\alpha \in \mathbb{F}$ depending only on $e_1, \dots, e_n$.

Definition 5.11. Let A be a matrix. Define its transpose, denoted $A^\top$, by

$$(A^\top)_{i,j} = A_{j,i}$$

Lemma 5.12. *Let A and B be matrices such that AB is defined.*

$$(AB)^\top = B^\top A^\top$$

Lemma 5.13. *For any square matrix A*

$$\det(A) = \det(A^\top)$$

Definition 5.14. A lower triangular matrix is any matrix L such that

$$L_{i,j} = 0 \quad \text{for} \quad j > i$$

Definition 5.15. An upper triangular matrix is any matrix U such that

$$U_{i,j} = 0 \quad \text{for} \quad j < i$$

Lemma 5.16. *The product of upper (lower) triangular matrices is an upper (lower) triangular matrix.*

Lemma 5.17. *Let A be either an upper triangular matrix or lower triangular matrix. We have that*

$$\det(A) = \prod a_{i,i}.$$

Proposition 5.4.

$$\det(AB) = \det(A)\det(B)$$

5.3 Invertibility of a Matrix

Definition 5.18. A matrix $A \in M_{n\times n}(\mathbb{F})$ is said to be invertible if there exists a matrix $B \in M_{n\times n}(\mathbb{F})$ such that

$$AB = I$$

$$BA = I$$

where I is the $n \times n$ identity matrix.

Definition 5.19. Suppose S is a set with a binary operation $\cdot : S \times S \to S$, then S with $\cdot$ is a monoid if

(a) For all a, b and c in S

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

(b) There exists an identity element 1 in S such that for all $a \in S$

$$a \cdot 1 = 1 \cdot a = a.$$

Definition 5.20. Let A be a monoid and 1 its identity element. We say $a \in A$ is invertible if and only if there exists a $b \in A$ such that

$$a \cdot b = b \cdot a = 1$$

Definition 5.21. A monoid homomorphism between monoids A and B is a function $\phi : A \to B$ such that

$$\phi(a_1) \cdot \phi(a_2) = \phi(a_1 \cdot a_2)$$

for all $a_1, a_2 \in A$ and

$$\phi(1_A) = 1_B$$

for the identity elements $1_A \in A$ and $1_B \in B$.

Theorem 5.22. *Let A and B be two monoids and $\phi : A \to B$ a monoid homomorphism. If $a \in A$ is invertible then $\phi(a)$ is invertible.*

Theorem 5.23. *Let $T \in \mathcal{L}(V, W)$ and let $\beta = \{v_1, \dots v_n\}$, $\gamma = \{w_1, \dots w_m\}$ be bases for V and W respectively. The transformation T is invertible if and only if $[T]_\beta^\gamma$ is invertible.*

Definition 5.24. An elementary matrix is any matrix obtained from performing a single row operation on the identity matrix.

Proposition 5.5. *Let A be a matrix and let B be the matrix obtained from A by performing a row operation with corresponding elementary matrix E. We have*

$$EA = B$$

Proposition 5.6. *If A is a matrix and B is an upper triangular matrix obtained from A via a finite sequence of row operations, then A is invertible if and only if B is invertible. Moreover, B is invertible if and only if the entries on the diagonal are non-zero.*

Theorem 5.25. *A matrix A is invertible if and only if* $\det(A) \neq 0$.

5.4 Properties and Facts

- If A is a triangular matrix then $\det(A)$ is the product of the entries on the main diagonal.

- If a multiple of one row of A is added to another row to produce B then $\det(A) = \det(B)$.
- If two rows of A are interchanged to produce B then $\det(A) = -\det(B)$.
- If one row of A is multiplied by k to produce B then $\det(B) = k \cdot \det(A)$.
- A square matrix is invertible if and only if $\det(A) \neq 0$.
- $\det(A^\top) = \det(A)$.
- $\det(AB) = \det(A)\det(B)$.
- $\det(I) = 1$.

6 Eigenvalues and Eigenvectors

Definition 6.1. Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called *invariant under* T if $u \in U$ implies that $Tu \in U$.

Definition 6.2. Suppose $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is an *eigenvalue* of T if there exists a $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$. The vector v is called an *eigenvector*.

Proposition 6.1. *$T \in \mathcal{L}(V)$ has a one dimensional subspace if and only if T has an eigenvalue.*

Theorem 6.3. *Suppose V is a finite dimensional vector space and $T \in \mathcal{L}(V)$. The following are equivalent:*

(a) $\lambda \in \mathbb{F}$ is an eigenvalue.

(b) $T - \lambda I$ is not injective.

(c) $T - \lambda I$ is not surjective.

(d) $T - \lambda I$ is not invertible.

Proposition 6.2. *Suppose $T \in \mathcal{L}(V)$. If $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T and $v_1, \dots, v_m$ are corresponding eigenvectors then $\{v_1, \dots, v_m\}$ is a linearly independent set.*

Definition 6.4. Suppose $T \in \mathcal{L}(V)$. Define for positive m

$$T^m = \underbrace{T \circ \ldots \circ T}_{m \text{ times}}$$

$$T^0 = Id$$

and if T is invertible

$$T^{-m} = \underbrace{T^{-1} \circ \ldots \circ T^{-1}}_{m \text{ times}}$$

Definition 6.5. Suppose $T \in \mathcal{L}(V)$ and let $p(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial over $\mathbb{F}$. Define

$$p(T) = a_n T^n + \ldots + a_1 T + a_0 Id$$