MTH 354
Homework 3

Definition 0.1. A point $p \in X$ is a limit point of $S \subseteq X$ if there exists a sequence (s_n) from $S \setminus \{p\}$ which converges to p.

1. Show if (a_n) is a bounded sequence of real numbers then
$$\limsup a_n \geq \liminf a_n.$$
2. Suppose that y is a limit point of a metric space X. Show that $Y = X \setminus \{y\}$ is not complete.
3. A metric space (X, d) is called sequentially compact if every sequence has a convergent subsequence. Show that X is sequentially compact if and only if every infinite subset has a limit point in X.
4. Let (X, d_X) and (Y, d_Y) be complete metric spaces. Let $(X \times Y, d)$ be the metric space defined by the metric
$$d : (X \times Y) \times (X \times Y) \to \mathbb{R}; \qquad d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b)$$
is a complete metric space.
5. Prove, if X is a metric space and (a_n) is a sequence in X which converge to $A \in X$ then $\{A, a_1, a_2, \ldots, \}$ is a compact set.
6. Show any finite set in a metric space is compact.