(§3.F # 34) The *double dual* of V denoted V'', is defined to be the dual space of V'. In other words $V' = (V')'$. Define $\Lambda : V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for $v \in V$ and $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(b) Show that if $T \in \mathcal{L}(V)$ then $T'' \circ \Lambda = \Lambda \circ T$ where $T'' = (T')'$.

(c) Show that if V is finite dimensional, then Λ is an isomorphism from V onto V''.

After talking to a couple of your classmates I decided that this question should get some hints.

(a) Part (a) is fairly straight forward

(b) Show that

$$((T'' \circ \Lambda)(v))(\varphi) = \varphi(Tv)$$

and

$$((\Lambda \circ T)(v))(\varphi) = \varphi(Tv)$$

for all $v \in V$ and $\varphi \in V'$.

For the first equality you will need to use the following: Let W be a vectorspace and W' its dual space, and let $T \in \mathcal{L}(W)$. We defined

$$(T'(\varphi))(w) = \varphi(T(w)) \quad \text{for } \varphi \in W' \text{ and } w \in W$$

(c) I would show that the map is injective.

Injective:

To do this let $v \in V$ and suppose that $\Lambda v = 0$. Show that this means that $\varphi(v) = 0$ for all $\varphi \in V'$. Now use the following Lemma.

Lemma 0.1. *Let V be a finite dimensional vectorspace. Suppose $v \neq 0$ then there exists a $\varphi \in V'$ such that $\varphi(v) = 1$.*

Proof. If $v \neq 0$, then extend $\{v\}$ into a basis $\beta = \{v, v_1, \dots v_n\}$ of V. Consider the dual basis to β. Then there exists a φ in the dual basis such that

$$\varphi(v) = 1$$

and

$$\varphi(v_i) = 0 \text{ for } i = 1, \dots n$$

Once you know the map is injective we can check surjectivity easily enough. Since

$$\dim(V) = \dim(\mathrm{null}(\Lambda)) + \dim(\mathrm{ran}(\Lambda))$$

then

$$\dim(V) = 0 + \dim(\mathrm{ran}(\Lambda)).$$

Since

$$\dim(V) = \dim(V') = \dim(V'')$$

we have that

$$\dim(V'') = \dim(\mathrm{ran}(\Lambda)).$$

□