
Note: Let Γ be an arbitrary indexing set (possibly infinite and possibly uncountable). A collection of subspaces indexed by Γ is $\{U_\gamma \mid \gamma \in \Gamma, U_\gamma \text{ is a subspace of } V\}$.

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1. (§1.C #11) Prove that the intersection of every collection of subspaces of V is a subspace of V.

   Proof. A vector $u \in V$ is in $\bigcap_{\gamma\in\Gamma} U_\gamma$ if and only if $u \in U_\gamma$ for every $\gamma \in \Gamma$. To prove that $\bigcap_{\gamma\in\Gamma} U_\gamma$ is a subspace we will show that $0 \in \bigcap_{\gamma\in\Gamma} U_\gamma$ and that $\bigcap_{\gamma\in\Gamma} U_\gamma$ is closed under addition and scalar multiplication. Since each U_γ is a subspace then $0 \in U_\gamma$ for all $\gamma \in \Gamma$. Hence $0 \in \bigcap_{\gamma\in\Gamma} U_\gamma$. Likewise, let x and y be arbitrary vectors in $\bigcap_{\gamma\in\Gamma} U_\gamma$. Then $x \in U_\gamma$ and $y \in U_\gamma$ for all $\gamma \in \Gamma$. Since each U_γ is a subspace we have $x + y \in U_\gamma$ for all $\gamma \in \Gamma$. Hence $x + y \in \bigcap_{\gamma\in\Gamma} U_\gamma$. Similarly, since each U_γ is a subspace we have that $\lambda x \in U_\gamma$ for each $\lambda \in \mathbb{F}$, $x \in U_\gamma$ and each $\gamma \in \Gamma$. Thus $\lambda x \in \bigcap_{\gamma\in\Gamma} U_\gamma$.

   $\square$

Definition:

We say that a vector space V is the direct sum of subspaces $U_1, \ldots, U_n$ if the following hold true:

(a) $U_i \neq \{0\}$ for each $i = 1, \ldots n$.

(b) $U_i \cap (U_1 + \ldots U_{i-1} + U_{i+1} + \ldots U_n) = \{0\}$ for $i = 1, \ldots n$.

(c) $V = U_1 + \ldots + U_n$.

Denote this by $V = U_1 \oplus \ldots \oplus U_n$.

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2. Prove the following theorem.

   Theorem 0.1. *If $U_1, \ldots U_n$ are non-trivial subspaces of V, then*

   $$V = U_1 \oplus \ldots \oplus U_n$$

   if and only if every $v \in V$ has a unique representation of the form

   $$v = u_1 + \ldots + u_n$$

   where $u_i \in U_i$ for each $i = 1, \ldots, n$.

Proof. First assume that $V = U_1 \oplus \ldots \oplus U_n$ for some non-trivial subspaces $U_1, \ldots U_n$ as defined above. Let $v \in V$ and suppose that

$$v = v_1 + \ldots + v_n$$

and

$$v = u_1 + \ldots + u_n$$

where $v_i, u_i \in U_i$ for $1 \leq i \leq n$. Let $j \in \{1, \ldots, n\}$ and note that

$$-(v_j - u_j) = (v_1 - u_1) + \ldots (v_{j-1} - u_{j-1}) + (v_{j+1} - u_{j+1}) + \ldots + (v_n - u_n).$$

Hence,

$$v_j - u_j \in U_j$$

and

$$v_j - u_j \in (U_1 + \ldots + U_{j-1} + U_{j+1} + \ldots U_n)$$

for each $j \in \{1, \ldots, n\}$. Hence $u_j = v_j$ for each $j \in \{1, \ldots, n\}$.

Conversely, suppose each $v \in V$ has a unique representation in the form

$$v = u_1 + \ldots + u_n \quad \text{where each} \quad u_i \in U_i \neq \{0\}.$$

Parts (a) and (c) are automatically satisfied. We need to show part (b) of the the definition holds. If

$$w \in U_i$$

and

$$w \in U_1 + \ldots + U_{i-1} + U_{i+1} + \ldots U_n$$

for some $i \in \{1, \ldots, n\}$ then,

$$0 = w_1 + \ldots w_{i-1} + w + w_{i+1} + \ldots + w_n$$

where each $w_j \in U_j$ for $j \in \{1, \ldots i-1\} \cup \{i+1, \ldots n\}$. By the unique representation of 0 we have that

$$w_1 = \ldots w_{i-1} = w = w_{i+1} = \ldots = w_n = 0.$$

Hence (b) is satisfied. □

3. (§2.A # 14) Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Proof. Suppose that V is infinite dimensional. Let $v_1 \neq 0$ and choose $v_2, v_3, \ldots$ by the following procedure: Suppose $v_1, \ldots v_{m-1}$ is chosen, and choose $v_m \in V$ such that $v_m \notin \text{span}\{v_1, \ldots v_{m-1}\}$. Since V is infinite dimensional this is always possible and $\{v_1, \ldots, v_m\}$ is linearly independent for each $m \in \mathbb{N}$. Conversely, suppose V is finite dimensional. Thus V has a finite spanning list. Since the length of every linearly independent list must be less than or equal to the length of any spanning list there does not exist a sequence of vectors such that $v_1, \ldots v_m$ is linearly independent for all $m \in \mathbb{N}$. □

4. (§2.A # 16) Prove that the real vector space of all continuous real-valued functions on $[0, 1]$ is infinite dimensional.

   Proof. For each $m \in \mathbb{N}$ we have that $\{1, x, \ldots, x^m\}$ is a linearly independent list of vectors in $C[0, 1]$ (continuous functions over [0,1]) since if

   $$a_0 \cdot 1 + \ldots + a_m \cdot x^m = 0 \quad \text{for all} \quad x \in [0, 1]$$

   then $a_0 = \ldots = a_m = 0$ (the only polynomial with an infinite number of zeros in $[0, 1]$ is the zero polynomial). Hence, by the above problem, polynomials over $[0, 1]$ form a infinite dimensional subspace and hence $C[0, 1]$ is infinite dimensional. □

5. (§2.B # 8) Suppose that U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

   $$u_1, \ldots u_m, w_1, \ldots, w_n$$

   is a basis of V.

   Proof. Clearly, $V \subseteq \text{span}\{u_1, \ldots u_m, w_1, \ldots w_n\}$ since $V = U \oplus W$. We need to show that $\{u_1, \ldots u_m, w_1, \ldots w_n\}$ is linearly independent. Suppose that

   $$a_1u_1 + \ldots a_mu_m + b_1w_1 + \ldots b_nw_n = 0.$$

   Note that

   $$a_1u_1 + \ldots a_mu_m = -(b_1w_1 + \ldots + b_nw_n).$$

   Hence,

   $$a_1u_1 + \ldots a_mu_m \in U \cap W$$

   and

   $$b_1w_1 + \ldots b_nw_n \in U \cap W.$$

   Thus,

   $$a_1u_1 + \ldots a_mu_m = b_1w_1 + \ldots b_nw_n = 0$$

   and by linear independence of $\{u_1, \ldots, u_m\}$ and $\{w_1, \ldots, w_n\}$ we have

   $$a_1 = \ldots a_m = b_1 = \ldots = b_n = 0.$$

   □