C^*-algebras and the Category of Stochastic Maps

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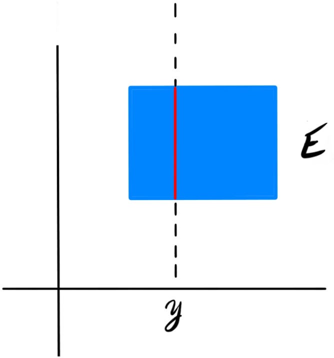
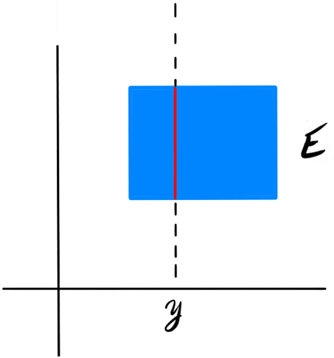
Measure Theory



Suppose one wanted to decompose the product measure of this set in terms of the measures of the slices.

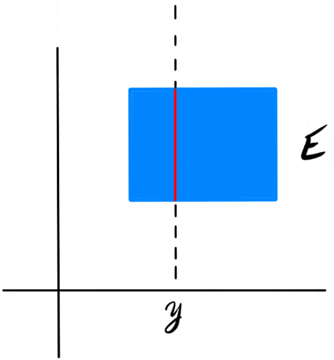
Measure Theory



Suppose one wanted to decompose the product measure of this set in terms of the measures of the slices.

Obviously, the slice has zero area measure.

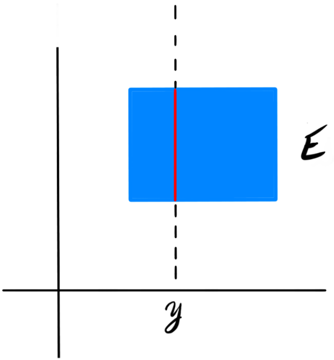
Measure Theory



A <u>*transition kernel*</u> r from (Y, Ω) to (X, Σ), written $r : Y \rightsquigarrow X$, is a function $r : Y \times \Sigma \to [0, \infty]$ such that

i. $r(y, \cdot\,) : \Sigma \to [0, \infty]$ is a measure for all $y \in Y$ and
ii. $r(\,\cdot\,, E) : Y \to [0, \infty]$ is measurable for all $E \in \Sigma$.

Measure Theory



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With, $r_y(E) := r(y, E)$

$$\mu(E) = \int_Y r_y(E)\, d\nu(y)$$

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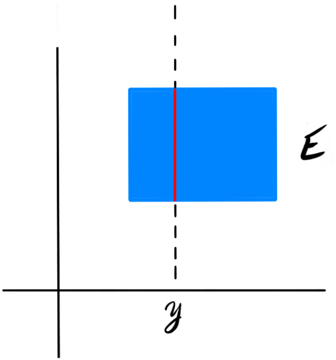

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With, $r_y(E) := r(y, E)$

$$\mu(E) = \int_Y r_y(E)\, d\nu(y)$$

We will call the transition kernel $r_y(E)$ a *disintegration* of μ over ν.

A Probabilistic Concept

Now let's have X and Y be finite sets. In this instance, transition kernels become *stochastic maps*

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Definition

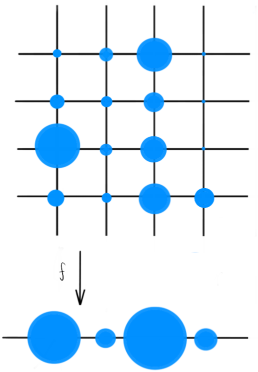
Let X and Y be two finite sets (equipped with the discrete σ-algebra). Let $\mathrm{PM}(Y)$ denote the set of probability measures on Y. A <u>stochastic map</u> from X to Y is a function

$$\begin{aligned} X &\xrightarrow{f} \mathrm{PM}(Y) \\ x &\mapsto f(x) \end{aligned}$$

whose evaluation on subsets of Y is written as

$$Y \supseteq E \xmapsto{f(x)} f_x(E) = \sum_{y \in E} f_x(\{y\}) \in \mathbb{R}_{\geq 0}.$$

Occasionally, we have the additional datum of having a measure preserving function $f : X \to Y$.



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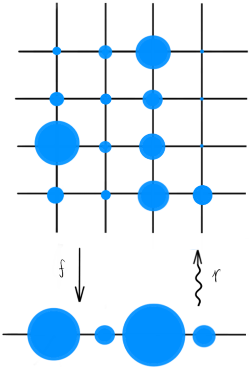

Given a measure preserving function f we say the disintegration of μ over ν is consistent with f if for each $F \in \Omega$ there exists a ν-null set $N_F \in \Omega$ such that $r_y(f^{-1}(F)) = 1$ for all $y \in (Y \setminus N_F) \cap F$

Category of Stochastic Maps

There is a one to one correspondence between stochastic maps $f : X \rightsquigarrow Y$ and stochastic matrices $F = (f_{xy})$ given by

$$f_{yx} := f_x(\{y\})$$

FinStoch :

- Objects: Finite sets
- Morphisms: Stochastic maps
- Identity Morphism: Stochastic map associated to $\mathrm{Id}_{|X|}$
- Composition: $(f \circ g)$ is the stochastic map associated to $F \cdot G$

Diagrammatic Disintegrations

Diagrammatic Disintegrations

Let

$$\begin{tikzcd} & \{\bullet\} \arrow[dl, rightsquigarrow, "p"'] \arrow[dr, rightsquigarrow, "q"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

be a commutative diagram.

Diagrammatic Disintegrations

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be a commutative diagram.

A disintegration of p over q is a stochastic map $r : Y \rightsquigarrow X$ such that

$$\begin{tikzcd} & \{\bullet\} \arrow[dl, squiggly, "p"'] \arrow[dr, squiggly, "q"] & \\ X & & Y \arrow[ll, squiggly, "r"'] \end{tikzcd}$$

commutes.

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A disintegration of p over q is consistent with f if in addition

$$\begin{tikzcd} & X \arrow[dl, "f"'] & \\ Y & & Y \arrow[ll, "\mathrm{id}_Y"] \arrow[ul, squiggly, "r"'] \end{tikzcd}$$

commutes.

Category of C^*-algebras

We'll also define the category **C*-AlgCPU**.

C*-AlgCPU :

- Objects: C^*-algebras
- Morphisms: Unital completely positive maps
- Identity Morphism: Identity map
- Composition: composition of maps

A Contravariant Functor

Theorem

There is a fully faithful contravariant functor F going from **FinStoch** *into* **C*-AlgCPU** *obtained by*

$$F(X) = \mathbb{C}^X$$

$$F(f) : \mathbb{C}^Y \to \mathbb{C}^X$$

Where $F(f)$ is the map given by sending the basis vector

$$e_y \mapsto \sum_{x \in X} f_{yx} e_x$$

and extended linearly to all of $\mathbb{C}^Y$.

Main Idea

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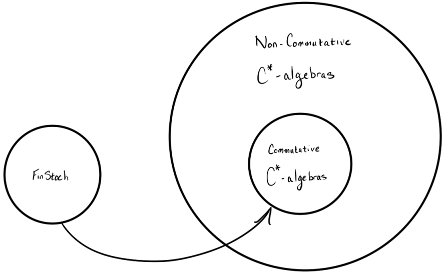
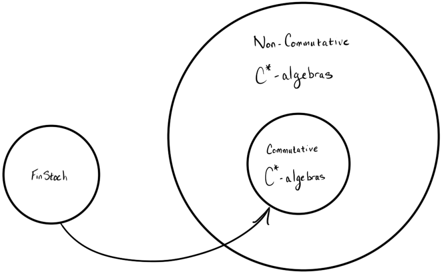


Non-Commutative
C*-algebras
Commutative
C*-algebras
FinStoch

Main Idea



Functors preserve commutative diagrams between categories.

$$
\begin{tikzcd}
 & Z \arrow[dl, "g"'] \arrow[dr, "h"] & & & & F(Z) \arrow[dl, "F(g)"'] \arrow[dr, "F(h)"] & \\
X \arrow[rr, "f"'] & & Y & \arrow[r, "\mathsf{F}"'] & F(X) \arrow[rr, "F(f)"'] & & F(Y)
\end{tikzcd}
$$

A Dictionary

Functions	$\Rightarrow$	*-homomorphims
Prob. measures	$\Rightarrow$	states
Stochastic maps	$\Rightarrow$	CPU maps

Diagrammatic Disintegrations

Diagrammatic Disintegrations

Let

$$\begin{tikzcd} \mathcal{A} & & \mathcal{B} \arrow[ll, "F"'] \arrow[dl, squiggly, "\xi"] \\ & \mathbb{C} & \arrow[from=1-1, to=2-2, squiggly, "\omega"'] \end{tikzcd}$$

be a commutative diagram.

Diagrammatic Disintegrations

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$$\begin{tikzcd} \mathcal{A} \arrow[dr, rightsquigarrow, "\omega"'] & & \mathcal{B} \arrow[ll, "F"'] \arrow[dl, rightsquigarrow, "\xi"] \\ & \mathbb{C} & \end{tikzcd}$$

be a commutative diagram.

A disintegration of ω over ξ is a unital completely positive map $R : \mathcal{A} \rightsquigarrow \mathcal{B}$ such that

$$\begin{tikzcd} \mathcal{A} \arrow[rr, rightsquigarrow, "R"] \arrow[dr, rightsquigarrow, "\omega"'] & & \mathcal{B} \arrow[dl, rightsquigarrow, "\xi"] \\ & \mathbb{C} & \end{tikzcd}$$

commutes.

Diagrammatic Disintegrations

Let

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\mathcal{A} \arrow[dr, squiggly, "\omega"'] & & \mathcal{B} \arrow[ll, "F"'] \arrow[dl, squiggly, "\xi"] \\
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& \mathbb{C} &
\end{tikzcd}$$

commutes.

A disintegration of ω over ξ is consistent with F if in addition the diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "\mathrm{id}_{\mathcal{B}}"] \arrow[dr, "F"'] & & \mathcal{B} \\
& \mathcal{A} \arrow[ur, squiggly, "R"'] &
\end{tikzcd}$$

commutes.

Note:

For finite sets disintegrations always exists and thus always exist in the commutative $C*$-algebra category as well given by the contravariant functor.

However, the existence in the non-commutative case is a bit more complicated.

A Counter Example

Let

$$\rho := \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \sigma := \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ be a $*$-homomorphism defined by

$$\mathcal{M}_2(\mathbb{C}) \ni A \mapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

and $\omega := \mathrm{tr}(\rho\,\cdot\,)$ and $\xi := \mathrm{tr}(\sigma\,\cdot\,)$ be the corresponding states.

A Counter Example

Theorem (Parzygnat-R)

The diagram

$$\begin{tikzcd} \mathcal{M}_4(\mathbb{C}) \arrow[dr, squiggly, "\omega=\mathrm{tr}(\rho\,\cdot\,)"'] & & \mathcal{M}_2(\mathbb{C}) \arrow[ll, "F"'] \arrow[dl, squiggly, "\xi=\mathrm{tr}(\sigma\,\cdot\,)"] \\ & \mathbb{C} & \end{tikzcd}$$

commutes, but there does not exist a CPU disintegration of ω *over* ξ *consistent with* F, *i.e. a CPU map* $R : \mathcal{M}_4(\mathbb{C}) \rightsquigarrow \mathcal{M}_2(\mathbb{C})$ *such that*

$$R \circ F = \mathrm{id}_{\mathcal{M}_2(\mathbb{C})} \quad \textit{and} \quad \xi \circ R = \omega.$$

Theorem (Parzygnat-R)

Fix $n, p \in \mathbb{N}$. Let F be the $*$-homomorphism given by the block diagonal inclusion

$$\mathcal{M}_n(\mathbb{C}) \ni A \overset{F}{\mapsto} \begin{bmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{bmatrix} \in \mathcal{M}_{np}(\mathbb{C})$$

and let

$$\begin{tikzcd} \mathcal{M}_{np}(\mathbb{C}) \arrow[dr, squiggly, "\mathrm{tr}(\rho\,\cdot\,)\equiv\omega"'] & & \mathcal{M}_n(\mathbb{C}) \arrow[ll, "F"'] \arrow[dl, squiggly, "\xi\equiv\mathrm{tr}(\sigma\,\cdot\,)"] \\ & \mathbb{C} & \end{tikzcd}$$

be a state-preserving $*$-homomorphism with σ a density matrix that has strictly positive eigenvalues. A CPU disintegration of ω over ξ consistent with F exists if and only if there exists a density matrix $\tau \in \mathcal{M}_p(\mathbb{C})$ such that $\rho = \tau \otimes \sigma$.

Example

Fix $p_1, p_2, p_3, p_4 \geq 0$ with $p_1 + p_2 + p_3 + p_4 = 1$, $p_1 + p_3 > 0$, and $p_2 + p_4 > 0$. Let

$$\rho = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \qquad \& \qquad \sigma = \begin{bmatrix} p_1 + p_3 & 0 \\ 0 & p_2 + p_4 \end{bmatrix}$$

there exists a CPU disintegration $R : \mathcal{M}_4(\mathbb{C}) \rightsquigarrow \mathcal{M}_2(\mathbb{C})$ of ω over ξ consistent with

$$\mathcal{M}_2(\mathbb{C}) \ni A \mapsto \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

if and only if

$$p_1 p_4 = p_2 p_3.$$

Pure States

Corollary (Parzygnat-R)

Given a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{np}(\mathbb{C}) \arrow[dr, squiggly, "\operatorname{tr}(\rho\,\cdot\,)"'] & & \mathcal{M}_n(\mathbb{C}) \arrow[ll, "F"'] \arrow[dl, squiggly, "\operatorname{tr}(\sigma\,\cdot\,)"] \\
& \mathbb{C} &
\end{tikzcd}$$

of CPU maps with ρ pure, if a disintegration exists, then σ must necessarily be pure as well.

Direct Sums

Theorem (Parzygnat-R)

Let,

$$\begin{tikzcd}
\mathcal{M}_{m_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{m_s}(\mathbb{C}) \arrow[dr, squiggly, "\omega"'] & & \mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_t}(\mathbb{C}) \arrow[ll, "F"'] \arrow[dl, squiggly, "\xi"] \\
& \mathbb{C} &
\end{tikzcd}$$

A disintegration R of ω over ξ consistent with F exists if and only if for each $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$ there exist non-negative matrices $\tau_{ji} \in \mathcal{M}_{c_{ij}}(\mathbb{C})$ such that

$$\operatorname{tr}\left(\sum_{i=1}^{s} \tau_{ji}\right) = 1 \qquad \forall\, j \in \{1, \ldots, t\}$$

and $\quad p_i \rho_i = \operatorname{diag}(q_1 \tau_{1i} \otimes \sigma_1, \ldots, q_t \tau_{ti} \otimes \sigma_t) \qquad \forall\, i \in \{1, \ldots, s\}.$

Thanks!

Non-commutative disintegrations:
existence and uniqueness in finite dimensions

https://arxiv.org/abs/1907.09689