Polar Area Example

Here we are going to calculate the area that lies within the graph of one polar curve and outside another polar curve. The curves in question are,

$$r^2 = \sin(2\theta) \tag{1}$$

$$r^2 = \cos(2\theta) \tag{2}$$

For convenience we will label the top equation "curve 1" and the bottom equation "curve 2". If we isolate r on either equation we get the following,

$$r = \pm\sqrt{\sin(2\theta)}; \quad r = \pm\sqrt{\cos(2\theta)}$$

We will choose

$$r = \sqrt{\sin(2\theta)} \quad \text{for curve 1}$$

and

$$r = \sqrt{\cos(2\theta)} \quad \text{for curve 2.}$$

Note we could have chosen the negative versions, and this would result in an equivalent graph.

If the equations were

$$r = \sin(2\theta) \;\text{ and }\; r = \cos(2\theta)$$

these would be the standard roses with four petals each. However, notice that if either $\sin(2\theta) < 0$ or $\cos(2\theta) < 0$, then

$$r = \sqrt{\sin(2\theta)} \;\text{ and }\; r = \sqrt{\cos(2\theta)}$$

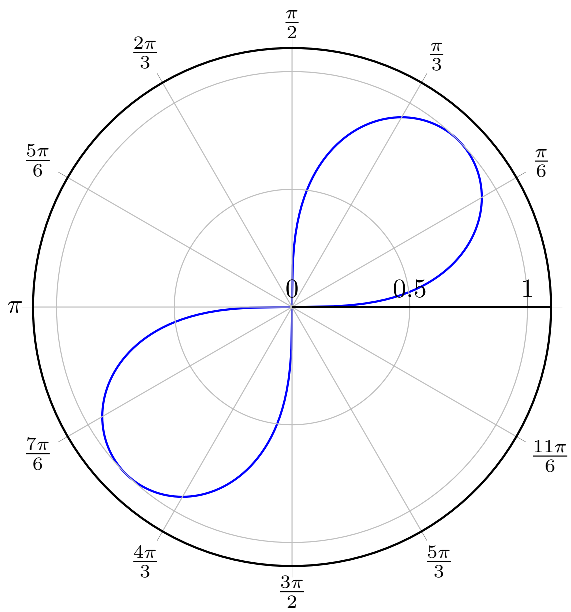
are not defined in the real numbers. In fact, they take on imaginary values. For example here is a chart of values for curve 1.

$$r^2 = \sin(2\theta)$$

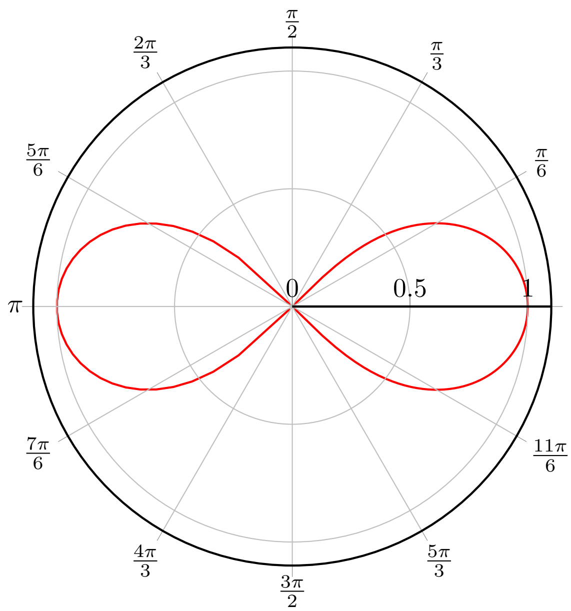
θ	r^2	r	θ	r^2	r
0	0	0	$\frac{5\pi}{4}$	1	1
$\frac{\pi}{4}$	1	1	$\frac{3\pi}{2}$	0	0
$\frac{\pi}{2}$	0	0	$\frac{3\pi}{2}$	0	0
$\frac{3\pi}{4}$	-1	i	$\frac{7\pi}{4}$	-1	i
π	0	0	2π	0	0

In the above it is standard notation to let, $i = \sqrt{-1}$.

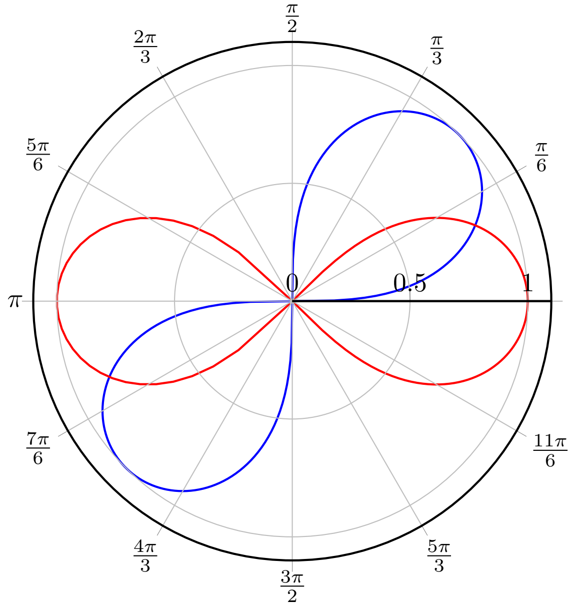
If we graph curve 1 we get the following.



If we graph curve 2 we get the following.



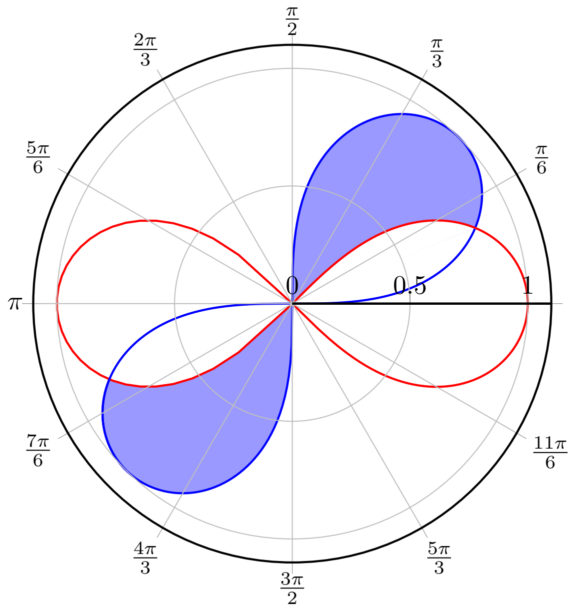
When graphed together we have the following.



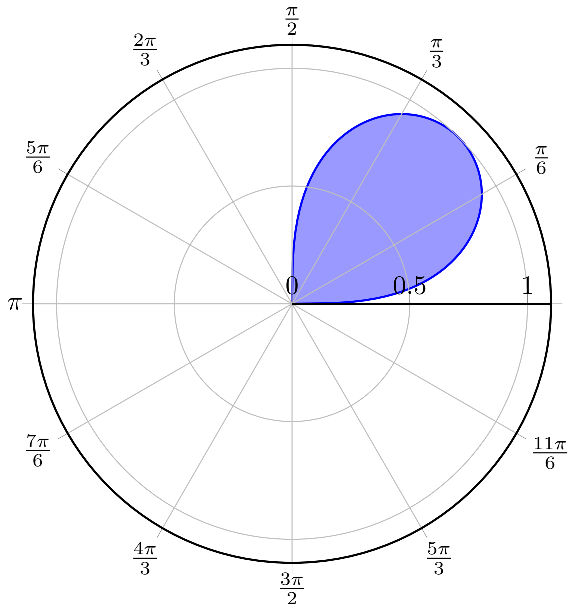
The intersection is found by setting the equations equal to each other.

$$\begin{aligned}\sin(2\theta) &= \cos(2\theta)\\ \tan(2\theta) &= 1\\ 2\theta &= \frac{\pi}{4}, \frac{5\pi}{4}, \ldots\\ \theta &= \frac{\pi}{8}, \frac{5\pi}{8}, \ldots\end{aligned}$$

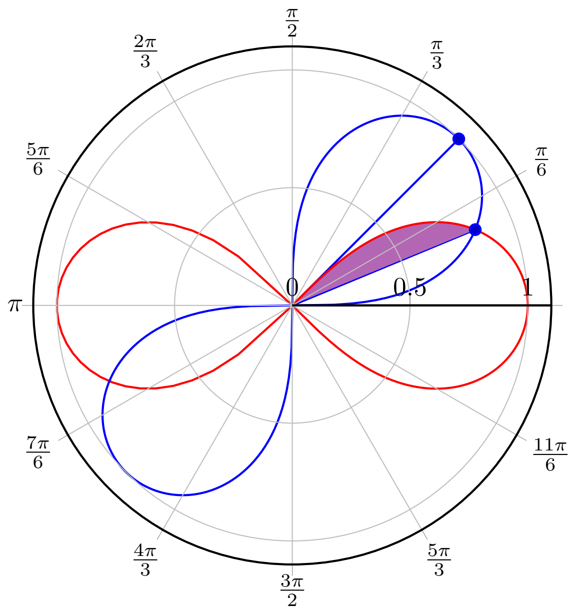
The area we are looking for is the shaded area below.



To find it we are going to calculate this area,



and this area.



So we have the following,

$$\begin{aligned}
\text{Area we want } &= 2\cdot \text{ Blue area} - 4\cdot \text{ Purple area}\\
A &= 2\cdot \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin(2\theta)\ d\theta - 4\cdot \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{2}\cos(2\theta)\ d\theta\\
A &= \int_0^{\frac{\pi}{2}} \sin(2\theta)\ d\theta - 2\cdot \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cos(2\theta)\ d\theta\\
A &= -\frac{1}{2}\cos(2\theta)\Big|_0^{\frac{\pi}{2}} - 2\cdot\frac{1}{2}\sin(2\theta)\Big|_{\frac{\pi}{8}}^{\frac{\pi}{4}}\\
A &= -\frac{1}{2}\cos(2\theta)\Big|_0^{\frac{\pi}{2}} - \sin(2\theta)\Big|_{\frac{\pi}{8}}^{\frac{\pi}{4}}\\
A &= \left[-\frac{1}{2}\cos(\pi) + \frac{1}{2}\cos(0)\right] - \left[\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right)\right]\\
A &= \left[\frac{1}{2} + \frac{1}{2}\right] - \left[1 - \frac{\sqrt{2}}{2}\right]\\
A &= \frac{\sqrt{2}}{2}
\end{aligned}$$

Here we note the bounds for the first area are from 0 to $\frac{\pi}{2}$, and the bounds of the second area are from $\frac{\pi}{8}$ to $\frac{\pi}{4}$. We get the bounds on the second area from the angle of the intersection point and the angle of the zero for $r^2 = \cos(2\theta)$.