1. Let $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Note that $\mathbb{Q}(\sqrt{2})$ is field and more specifically it is known as an algebraic number field. The binary operations on $\mathbb{Q}(\sqrt{2})$ are the standard addition and multiplication of numbers. Verify for all $\alpha \neq 0$ in $\mathbb{Q}(\sqrt{2})$ that there exists a $\beta \in \mathbb{Q}(\sqrt{2})$ such that $\alpha \cdot \beta = 1$.

   Solution: Consider $\alpha = a + b\sqrt{2} \neq 0$, where $a, b \in \mathbb{Q}$. Let

$$\beta = \frac{1}{\alpha} = \frac{1}{a + b\sqrt{2}} \cdot \left(\frac{a - b\sqrt{2}}{a - b\sqrt{2}}\right) = \left(\frac{a}{a^2 - 2b^2}\right) - \left(\frac{b}{a^2 - 2b^2}\right)\sqrt{2} \in \mathbb{Q}(\sqrt{2}).$$

   Clearly,

$$\alpha \cdot \beta = \frac{a^2 + ab\sqrt{2} - ab\sqrt{2} - 2b^2}{a^2 - 2b^2} = 1.$$

   Note that if $a + b\sqrt{2} \neq 0$ where $b \neq 0$ then $a - b\sqrt{2} \neq 0$ (otherwise this implies $\sqrt{2} = \frac{a}{b}$) and

$$a^2 - 2b^2 = \left(a + b\sqrt{2}\right) \cdot \left(a - b\sqrt{2} \neq 0\right) \neq 0.$$

For the next two problems let $\mathbb{F}$ be an arbitrary field. We define the following vector space over $\mathbb{F}$. Let

$$\mathbb{F}^n = \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{F},\ j = 1, \ldots n\}$$

where scalar multiplication and vector addition is defined thusly,

$$\lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots x_n + y_n).$$

---

2. (#13 §1.A) Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and all $a, b \in \mathbb{F}$.

   Proof. Note,

$$\begin{aligned}
(ab)x &= (ab) \cdot (x_1, x_2, \ldots, x_n) \\
&= ((ab)x_1, (ab)x_2, \ldots, (ab)x_n) \\
&= (a(bx_1), a(bx_2), \ldots, a(bx_n)) \\
&= a(bx_1, bx_2, \ldots, bx_n) \\
&= a(bx).
\end{aligned}$$

   The above calculation relies on the definition of the scalar multiplication and from the associativity of the field multiplication. □

3. (# 15§1.A) Show that $\lambda \cdot (x+y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and all $x, y \in \mathbb{F}^n$.

Proof. Note,

$$\begin{aligned}
\lambda \cdot (x+y) &= \lambda \cdot ((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) \\
&= \lambda \cdot (x_1 + y_1, \ldots x_n + y_n) \\
&= (\lambda(x_1 + y_1), \ldots \lambda(x_n + y_n)) \\
&= (\lambda x_1 + \lambda y_1, \ldots \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, \ldots \lambda x_n) + (\lambda y_1, \ldots \lambda y_n) \\
&= \lambda x + \lambda y.
\end{aligned}$$

The above calculation relies on the definition of the binary operations and from the distribution property of the field multiplication. □

For the next two problems let $\mathbb{F}$ be an arbitrary field and V a vector space over $\mathbb{F}$.

---

4. (#1 §1.B) Prove that $-(-v) = v$ for every $v \in V$.

Proof. Note,

$$\begin{aligned}
-(-v) = -((-1) \cdot v) &= (-1) \cdot ((-1) \cdot v) \\
&= (-1)(-1) \cdot (v) \\
&= 1 \cdot v \\
&= v
\end{aligned}$$

The above calculation is done by two applications on Proposition 1.31 (pg 17). As an aside, if 1 is the multiplicative identity in the field and -1 is the additive inverse of 1, then

$$(-1)(-1+1) = (-1)(0) = 0.$$

So

$$((-1)(-1) + (-1)) = 0.$$

This would show that $(-1)(-1) = 1$ if we also showed that the additive inverses in a field are unique and that $a0 = 0$ for all $a \in \mathbb{F}$. However, this was not necessary for the problem. □

5. (#2 §1.B) Suppose $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Prove $a = 0$ or $v = 0$.

Proof. Suppose $a \neq 0$ and show that $v = 0$. If $a \neq 0$ then there exists a unique multiplicative inverse element in the field, call it a^{-1}. If

$$av = 0$$

then

$$a^{-1}(av) = a^{-1}0 = 0,$$

and thus

$$v = 0.$$

$\square$