1 Exercise 1.13

$$\mathrm{Fib(n)} = \frac{\phi^{\mathrm{n}} - \psi^{\mathrm{n}}}{\sqrt{5}} \tag{1}$$

数学的帰納法で証明する
　$n = 1$ のとき

$$\begin{align}
\mathrm{Fib}(1) &= \frac{\phi - \psi}{\sqrt{5}} \tag{2}\\
&= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \tag{3}\\
&= 1 \tag{4}
\end{align}$$

となり成り立つ.
　$n = 2$ のとき

$$\begin{align}
\mathrm{Fib}(2) &= \frac{\phi^2 - \psi^2}{\sqrt{5}} \tag{5}\\
&= \frac{1 + 2\sqrt{5} + 5 - (1 - 2\sqrt{5} + 5))}{4\sqrt{5}} \tag{6}\\
&= \frac{4\sqrt{5}}{4\sqrt{5}} \tag{7}\\
&= 1 \tag{8}
\end{align}$$

となり成り立つ.
　$n = 3$ のとき

$$\begin{align}
\mathrm{Fib}(3) &= \frac{\phi^3 - \psi^3}{\sqrt{5}} \tag{9}\\
&= \frac{1 + 3\sqrt{5} + 15 + 5\sqrt{5} - (1 - 3\sqrt{5} + 15 - 5\sqrt{5})}{8\sqrt{5}} \tag{10}\\
&= \frac{6\sqrt{5} + 10\sqrt{5}}{8\sqrt{5}} \tag{11}\\
&= \frac{16\sqrt{5}}{8\sqrt{5}} \tag{12}\\
&= 2 = 1 + 1 = \mathrm{Fib}(1) + \mathrm{Fib}(2) \tag{13}
\end{align}$$

となり成り立つ.

$n=4$ のとき

$$
\begin{aligned}
\mathrm{Fib}(4) &= \frac{\phi^4-\psi^4}{\sqrt{5}} && (14)\\
&= \frac{1+4\sqrt{5}+30+20\sqrt{5}+25-(1-4\sqrt{5}+30-20\sqrt{5}+25)}{16\sqrt{5}} && (15)\\
&= \frac{8\sqrt{5}+40\sqrt{5}}{16\sqrt{5}} && (16)\\
&= 3 = 1+2 = \mathrm{Fib}(2)+\mathrm{Fib}(3) && (17)
\end{aligned}
$$

となり成り立つ.

次に $n=k,k+1$ のとき成り立つとする

$$\mathrm{Fib(k)} = \frac{\phi^{\mathrm{k}}-\psi^{\mathrm{k}}}{\sqrt{5}} \tag{18}$$

$$\mathrm{Fib(k+1)} = \frac{\phi^{\mathrm{k+1}}-\psi^{\mathrm{k+1}}}{\sqrt{5}} \tag{19}$$

$n=k+2$ のときは

$$
\begin{aligned}
\mathrm{Fib(k+2)} &= \frac{\phi^{k+1}-\psi^{k+1}}{\sqrt{5}} + \frac{\phi^k-\psi^k}{\sqrt{5}} && (20)\\
&= \frac{\phi^{k+1}-\psi^{k+1}+\phi^k-\psi^k}{\sqrt{5}} && (21)\\
&= \frac{\phi^k(\phi+1)-\psi^k(\psi+1)}{\sqrt{5}} && (22)
\end{aligned}
$$

ここで

$$\phi^2=\phi+1, \qquad \psi^2=\psi+1 \tag{23}$$

より

$$
\begin{aligned}
\mathrm{Fib(k+2)} &= \frac{\phi^k(\phi+1)-\psi^k(\psi+1)}{\sqrt{5}} && (24)\\
&= \frac{\phi^k\phi^2-\psi^k\psi^2}{\sqrt{5}} && (25)\\
&= \frac{\phi^{k+2}-\psi^{k+2}}{\sqrt{5}} && (26)
\end{aligned}
$$

となり成り立つ.

以上より，数学的帰納法から

$$\mathrm{Fib(n)} = \frac{\phi^{\mathrm{k}}-\psi^{\mathrm{k}}}{\sqrt{5}} \tag{27}$$

が成り立つことを証明した.