1 Exercise 1.13

まず

$$\mathrm{Fib(n)} = \frac{\phi^{\mathrm{n}} - \psi^{\mathrm{n}}}{\sqrt{5}} \tag{1}$$

を数学的帰納法で証明する

$n = 1$ のとき

$$\mathrm{Fib(1)} = \frac{\phi - \psi}{\sqrt{5}} \tag{2}$$

$$= \frac{\dfrac{1 + \sqrt{5}}{2} - \dfrac{1 - \sqrt{5}}{2}}{\sqrt{5}} \tag{3}$$

$$= 1 \tag{4}$$

となり成り立つ.

$n = 2$ のとき

$$\mathrm{Fib(2)} = \frac{\phi^2 - \psi^2}{\sqrt{5}} \tag{5}$$

$$= \frac{1 + 2\sqrt{5} + 5 - (1 - 2\sqrt{5} + 5))}{4\sqrt{5}} \tag{6}$$

$$= \frac{4\sqrt{5}}{4\sqrt{5}} \tag{7}$$

$$= 1 \tag{8}$$

となり成り立つ.

$n = 3$ のとき

$$\mathrm{Fib(3)} = \frac{\phi^3 - \psi^3}{\sqrt{5}} \tag{9}$$

$$= \frac{1 + 3\sqrt{5} + 15 + 5\sqrt{5} - (1 - 3\sqrt{5} + 15 - 5\sqrt{5})}{8\sqrt{5}} \tag{10}$$

$$= \frac{6\sqrt{5} + 10\sqrt{5}}{8\sqrt{5}} \tag{11}$$

$$= \frac{16\sqrt{5}}{8\sqrt{5}} \tag{12}$$

$$= 2 = 1 + 1 = \mathrm{Fib(1)} + \mathrm{Fib(2)} \tag{13}$$

となり成り立つ.

$n = 4$ のとき

$$\text{Fib}(4) = \frac{\phi^4 - \psi^4}{\sqrt{5}} \tag{14}$$

$$= \frac{1 + 4\sqrt{5} + 30 + 20\sqrt{5} + 25 - (1 - 4\sqrt{5} + 30 - 20\sqrt{5} + 25)}{16\sqrt{5}} \tag{15}$$

$$= \frac{8\sqrt{5} + 40\sqrt{5}}{16\sqrt{5}} \tag{16}$$

$$= 3 = 1 + 2 = \text{Fib}(2) + \text{Fib}(3) \tag{17}$$

となり成り立つ.

次に $n = k, k+1$ のとき成り立つとする

$$\text{Fib(k)} = \frac{\phi^{\text{k}} - \psi^{\text{k}}}{\sqrt{5}} \tag{18}$$

$$\text{Fib(k+1)} = \frac{\phi^{\text{k+1}} - \psi^{\text{k+1}}}{\sqrt{5}} \tag{19}$$

$n = k+2$ のときは

$$\text{Fib(k+2)} = \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}} + \frac{\phi^k - \psi^k}{\sqrt{5}} \tag{20}$$

$$= \frac{\phi^{k+1} - \psi^{k+1} + \phi^k - \psi^k}{\sqrt{5}} \tag{21}$$

$$= \frac{\phi^k(\phi + 1) - \psi^k(\psi + 1)}{\sqrt{5}} \tag{22}$$

ここで

$$\phi^2 = \phi + 1, \qquad \psi^2 = \psi + 1 \tag{23}$$

より

$$\text{Fib(k+2)} = \frac{\phi^k(\phi + 1) - \psi^k(\psi + 1)}{\sqrt{5}} \tag{24}$$

$$= \frac{\phi^k\phi^2 - \psi^k\psi^2}{\sqrt{5}} \tag{25}$$

$$= \frac{\phi^{k+2} - \psi^{k+2}}{\sqrt{5}} \tag{26}$$

となり成り立つ.

以上より,数学的帰納法から

$$\text{Fib(n)} = \frac{\phi^{\text{n}} - \psi^{\text{n}}}{\sqrt{5}} \tag{27}$$

が成り立つことを証明した.

次に

$$\psi = \frac{1 - \sqrt{5}}{2} \sim \frac{1 - 2.23606797749979}{2} \sim -0.618033988749895 \tag{28}$$

より

$$\lim_{n \to \infty} \psi^n = 0 \tag{29}$$

となるので,

$$\mathrm{Fib(n)} = \frac{\phi^n - \psi^n}{\sqrt{5}} \sim \frac{\phi^n - 0}{\sqrt{5}} = \phi^n/\sqrt{5} \tag{30}$$

となる