

Final Sheet

December 2021

1 Data

1.a Types of Variables

Qualitative/Categorical

- Outcomes fall into different categories
- Categories can be ordered

Quantitative

- Measured on a numeric scale

1.b Summarizing Data Visually

Qualitative/Categorical Data

- Frequency tables - displays all categories of a single categorical variable with associated frequencies
- Contingency tables - display two categorical variables simultaneously
- Marginal distributions - display distribution of one of the two variables only
- Conditional distributions - display distribution of one variable, satisfying a condition of the other variable
- Bar charts
- Pie charts

Quantitative Data

• Graphically

- Histogram
- Stem-and-leaf displays
- Boxplots

• Shape of the Distribution

- Modality (number of peaks):
 - * unimodal
 - * bimodal
 - * multimodal
- Symmetry of distribution:
 - * unimodal

- * skewed to right (long right tail)
- * skewed to left (long left tail)
- Presence of outliers

- **Numerically**

- Measures of center:
 - * mean
 - * median
- Measures of spread:
 - * variance: $s^2 = \frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n-1}$
 - * standard deviation: $s = \sqrt{\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{n-1}}$
 - * interquartile range $IQR = Q3 - Q1$
- Percentiles (also called quantiles)
- 5-number summary:
 - * minimum
 - * first quartile (Q1)
 - * second quartile (Q2)
 - * third quartile (Q3)
 - * maximum

- **Sensitivity to Outliers**

- Sensitive to outliers:
 - * mean
 - * range, variance, standard deviation
- Not sensitive to outliers
 - * median
 - * IQR

2 Normal Distribution

Characteristics of the Normal Model

- bell-shaped; unimodal
- perfectly symmetric about the mean
- spread of distribution determined by value of standard deviation
- mean μ and the standard deviation σ are parameters (numerical characteristics of a model)
- mean \bar{y} and standard deviation s are statistics (numerical characteristics of a sample)

The 68-95-99.7 Rule

- 68% of data falls within 1σ of μ
- 95% of data falls within 2σ of μ
- 99.7% of data falls within 3σ of μ

Finding Areas Under the Normal Model

Algorithm

- Identify the:
 - μ - mean of the model
 - σ - standard deviation of the model
 - y - observed value
- Construct the normal model: $N(\mu, \sigma)$
- Calculate the z-score (z): $z = \frac{y-\mu}{\sigma}$
- Using R compute the p-value:
 - Area below y : `pnorm(z)`
 - Area above y : `pnorm(z, lower.tail = F)`
 - Area in between y_1 and y_2 (where $y_1 > y_2$): `pnorm(z_1) - pnorm(z_2)`
- **Finding Z-Score from the Area Under the Normal Model**
 - Area above unknown y : `qnorm(p, lower.tail = F)`
 - Area below unknown y : `qnorm(p)`

3 Probability and Random Variables

Determining Independence of Events

The Binomial Model

- Used for discrete random variables
- Binomial Model:
 - Experiment must consist of n identical trials (number of trials is fixed in advance)
 - Outcomes of each trial are either success or failure
 - Probability of success p is constant
 - Probability of failure is $q = 1 - p$
 - The trials are independent
 - The random variable X represents the number of successes out of n trials

Algorithm for the Probability of Binomials

- Identify the parameters:
 - n - number of trials
 - p - probability of success
- Construct the binomial model: $X \sim Bin(n, p)$
- Calculate the probability:
Where the probability that X will take on value x is given by:
$$P(X = x) = {}_n C_x p^x * (1 - p)^{n-x}, x = 0, 1, 2, \dots, n$$

Where:
$${}_n C_x = \frac{n!}{x!(n-x)!}$$

Mean, Variance and Standard Deviation for a Random Binomial Variable

- Mean: np
Interp. average number of successes if you were to repeat experiment many times

- Variance: $np(1 - p)$
Interp. measure of variability of numbers of successes you were to repeat experiment many times
- Standard deviation: $\sqrt{np(1 - p)}$

Uniform Random Variable

4 Correlation and Association

Scatterplots

- Direction:
 - Positive (x and y) values tend to go in the same direction)
 - Negative (x and y values tend to go in the opposite direction)
- Form:
 - Linear
 - Non-linear
- Point relationship:
 - Strong relationship between points
 - weak or no relationship between points (randomly scattered)
- Outliers

Correlation (r)

- Positive correlation: large x values are linearly associated with large y values (r is positive)
- Negative correlation: large x values are linearly associated with small y values (r is negative)
- r has a value between 1 and -1, and has no units
- $r = \frac{\sum z_x * z_y}{n-1}$

Association vs Causality

- Association does not imply causation. There may be a lurking variable

5 Regression Analysis

The Regression Line

- Equation for regression line: $\hat{y} = \text{intercept} + (\text{slope} * x)$
- Equation for slope: $\text{slope} = r * \frac{s_y}{s_x}$
(where s_y and s_x are the standard deviations of y and x respectively)
- Equation for intercept: $\text{intercept} = \bar{y} - (\text{slope} * \bar{x})$
(where \bar{y} and \bar{x} are the mean y and x values respectively)

“The Residuals

- The residual (e) is the difference between observed value y and the predicted value \hat{y} . Therefore:
 $e = y - \hat{y}$ (from data) - \hat{y} (from model)
- The sum of residuals is equal to zero
- Linear model is obtained by minimizing the sum of the squared residuals. Therefore, also referred to as the least squares regression line
- To assess appropriateness of regression model, we use the residual plot (plots residuals against explanatory variable data). If plot shows no pattern, model is appropriate.

6 Experiments and Observational Studies

Types of Studies

- Observational Studies
 - Investigators have no control over either variable
 - No deliberate human intervention
 - Retrospective study: based on information from events that have taken place in the past
 - Prospective study: data and information is gathered in real time
- Experiments
 - Involves planned intervention on the exposure to a condition suspected of altering the response outcome
 - Most often control group(s) will be used

Randomized, Comparative Experiments

- Involves assessing the effect of an explanatory variable, called a factor, on a response variable
- Compares the response variable between different levels of the factor
- Experimenters control what type of treatment individuals receive, the treatment assignment is random
- Participants referred to as subjects or experimental units
- The treatment a subject receives will be a combination of the levels from different factors

Principles of Experimental Design

- Randomize
 - Treatments are randomly assigned to subjects
- Replicate
 - Comparison between different treatment groups will not be reliable unless more individuals receive each treatment
- Blocking
 - May be beneficial to control for variables that are not factors but are believed to have some influence on the response variable
 - Subjects are divided into blocks (ex. male and female groups). Treatment assignment and comparisons are done within each block separately

Blinding and Placebo

- Single Blind: either the subjects or the evaluators are blinded as to treatment assignment
- Double Blind: neither the subjects nor the evaluators knows the treatment assignments
- Blinding is usually done using a placebo which is designed to look like the treatment but has no real treatment value

7 Types of Sampling

Sampling Methods

- Simple Random Sampling
 - Consists of n individuals sampled at random from the population
 - Each individual has an equal chance of being selected
 - Each possible sample size n is equally likely
- Stratified Sampling
 - Population is divided into strata (a stratum is a subset of the population that shares a particular characteristic)
 - Simple random sample is drawn from each stratum
 - Stratified sample has smaller variability across samples and hence give more reliable results
- Cluster Sampling
 - Can be used when natural groups in a population exist
 - Population is divided into those groups/clusters
 - Simple random sample from all clusters is obtained
 - If all individuals in a selected cluster are included, final sample is a one-stage cluster sample
 - If additional simple random sample is drawn from selected clusters, final sample is a two-stage cluster sample
 - This method is used for the sake of convenience, practicality, and cost-efficiency
- Multistage Sampling
 - Involves more than one stage or more than one sampling procedure in obtaining a sample
- Systematic Sampling
 - Obtained by selecting every k th individual from the sampling frame
 - Method can be used as long as list being sampled from does not contain a hidden order

Bad Sampling Procedures and Biases

- Undercoverage
 - When sampling frame or sampling procedure excludes or under-represents certain types of individuals from the population
- Convenience Sampling
 - Selecting individuals from a population based on availability and access
- Voluntary Response Bias
 - If responses are voluntary, those with strong opinions tend to be over-represented
- Non-response Bias
 - Individuals who do not respond in a survey might differ from the respondents in certain aspects
 - Including only the respondents in a sample will result in non-response bias
- Response Bias
 - Subject's response is influenced by how the question was phrased or asked, or due to misunderstanding of a question, or unwillingness to disclose the truth

8 Sampling Distribution Models

Basic Information

- Population: all individuals who want to be studied
- Sample: a subset of individuals selected from a population
- Parameter: a numerical summary of a population
- Statistic: a numerical summary of the sample

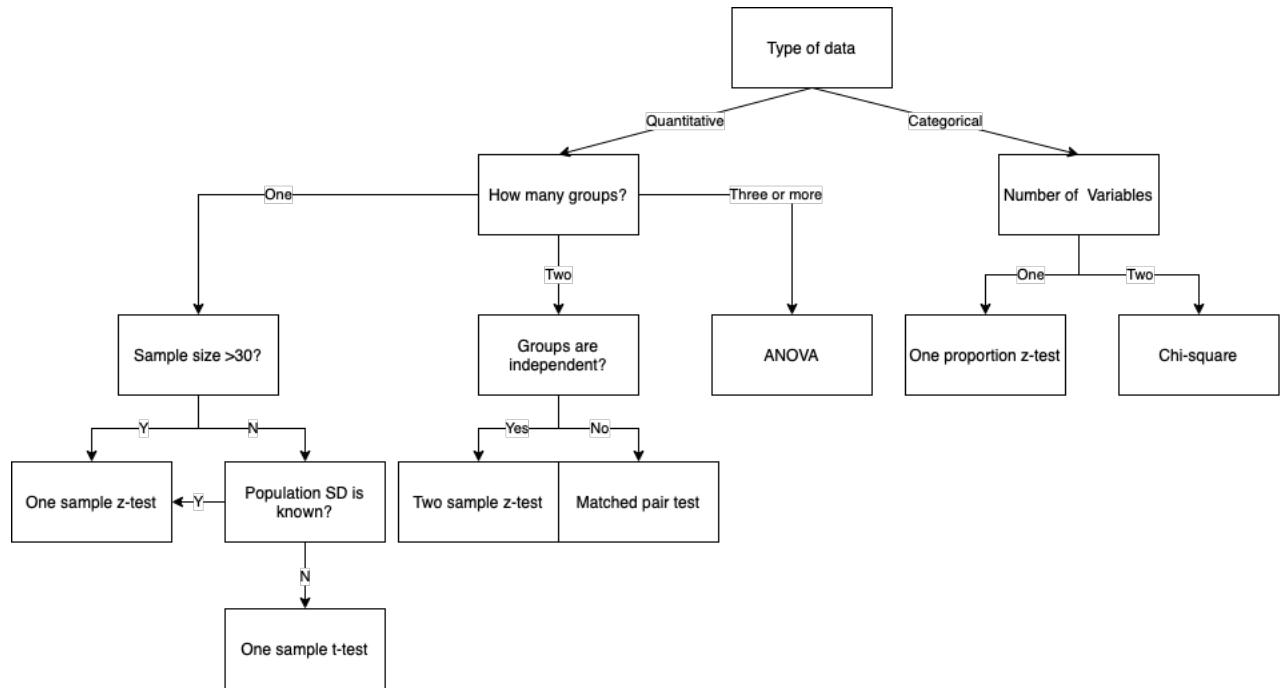
Sampling Distribution of Proportions

- The sample proportion (a) statistic is given by:
$$\hat{p} = \frac{\text{number of individuals sampled who have the characteristic}}{\text{sample size } n}$$
- Value of population proportion p is fixed, usually unknown. Therefore, sample proportion \hat{p} used to estimate
- Sampling distribution of \hat{p} :
 - mean $\mu(\hat{p})$: mean of \hat{p} = mean of p
 - standard deviation $\sigma(\hat{p})$: $\sqrt{\frac{p(1-p)}{n}}$
 - Sampling distribution of \hat{p} approximately normal when:
 - * Sample is random
 - * Individual values are independent (sample size $\leq 10\%$ of population)
 - * Sample size is large ($np \geq 10$ and $n(1-p) \geq 10$)

Sampling Distribution of Means

- The sample mean (a statistic) is given by:
$$\bar{y} = \frac{y_1+y_2+\dots+y_n}{n}$$
- Population mean μ is a parameter, fixed and usually unknown
- Sampling distribution of means:
 - mean $\mu(\bar{y}) = \mu$
 - standard deviation $\sigma(\bar{y}) = \frac{\sigma}{\sqrt{n}}$
- Central limit theorem (CLT)
 - For sufficiently large samples, sample mean approximately follows the normal model
 - Assumption for CLT are:
 - * Sample is random
 - * Individual values are independent (sample size $\leq 10\%$ of population)
 - * Sample size is sufficiently large (generally $n \geq 30$)

9 Hypothesis Testing



9.a One sample z-test

Algorithm

- Identify parameter of interest. Find the null and alternative hypotheses.
 s - The standard deviation of the sample.
 n - The sample size.
 μ - Hypothesized population mean.
 $\text{SE}(\bar{y}) = \frac{s}{\sqrt{n}}$ - Standard error of the statistic.
- Construct the null-model: $\mathbf{N}(\mu, \frac{s}{\sqrt{n}})$
- Find the test-statistic(t): $Z = \frac{\bar{x} - \mu}{\text{SE}(\bar{y})}$
- Using R compute the p-value:
 - One-sided hypothesis : `pnorm(t)`
 - Two-sided hypothesis : $2 \cdot \text{pnorm}(t)$
- If the p-value is less than α - reject the null-hypothesis. Otherwise, you fail to reject the null-hypothesis.

9.b One proportion z-test

Algorithm

- Identify parameter of interest. Find the null and alternative hypotheses.
 n - The sample size.
 p_0 - Hypothetised proportion.
 $\text{SD} = \sqrt{\frac{p_0(1-p_0)}{n}}$ - Standard error of the statistic.
- Construct the null-model: $\mathbf{N}(p_0, \sqrt{\frac{p_0(1-p_0)}{n}})$

- Find the test-statistic(t): $Z = \frac{x - p_0}{SD}$
- Using R compute the p-value:
 - One-sided hypothesis : `pnorm(t)`
 - Two-sided hypothesis : $2 \cdot \text{pnorm}(t)$
- If the p-value is less than α - reject the null-hypothesis. Otherwise, you fail to reject the null-hypothesis.

9.c Two sample z-test

Algorithm

- Identify parameter of interest. Find the null and alternative hypotheseses.
 - s - The standard deviation of the sample.
 - n_1 - The sample size of the first sample.
 - n_2 - The sample size of the second sample.
 - Δ_0 - Hypoththesised mean of difference between two populations.
 - $SD(\bar{y}_1 - \bar{y}_2) = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ - Standard error of the statistic.
 - $df = \min(n_1 - 1, n_2 - 1)$
- Construct the null-model: $N(\mu, \frac{s}{\sqrt{n}})$
- Using R compute the p-value:
 - One-sided hypothesis : `pnorm(t)`
 - Two-sided hypothesis : $2 \cdot \text{pnorm}(t)$
- If the p-value is less than α - reject the null-hypothesis. Otherwise, you fail to reject the null-hypothesis.

9.d Matched pair

Algorithm

- Identify parameter of interest. Find the null and alternative hypotheseses.
 - Δ_0 - Hypoththesised population mean difference (usually 0)
 - \bar{d} - The mean of the differences.
 - $s_d = \sqrt{\frac{\sum_{i=1}^n (d_i - \bar{d})^2}{n-1}}$ = The standard deviation of the sample differences.
 - n - The sample size.
- Find the test-statistic(t): $t = \frac{\bar{d} - \Delta_0}{\frac{s_d}{\sqrt{n}}}$
- Using R compute find the p-value using `pt` function:
 - For one sided tests use: `pt(t, n-1)`
 - For two sided tests use: $2 \cdot \text{pt}(t, n-1)$
- If the p-value is less than α - reject the null-hypothesis. Otherwise, you fail to reject the null-hypothesis.

9.e One sample t-test

Algorithm

- Identify parameter of interest. Find the null and alternative hypotheses.
 s - The standard deviation of the sample.
 n - The sample size.
 μ - Hypothesized population mean.
 $\text{SE}(\bar{y}) = \frac{s}{\sqrt{n}}$ - Standard error of the statistic.
- Construct the null-model: $\mathbf{N}(\mu, \frac{s}{\sqrt{n}})$
- Find the test-statistic(t): $Z = \frac{\bar{x} - \mu}{\text{SE}(\bar{y})}$
- Using R compute find the p-value using pt function:
 - For one sided tests use: $\text{pt}(t, n-1)$
 - For two sided tests use: $2 \cdot \text{pt}(t, n-1)$

9.f ANOVA

- k = number of groups.
- N = number of subjects in total.
- $SSTo$ = Sum of Squares Total.
- $SSTo = SSTR + SSEr$.

Source of Variation	df	Sum of Squares (SS)	Mean Sum of Squares (MSS)	F-test	p-value
Treatment	$k-1$	$SSTR$	$MSTR=SSTR/(k-1)$	$F=MSTR/MSE$	
Error	$N-k$	SSE	$MSE=SSE/(N-k)$		
Total	$N-1$	$SSTo$			

- After filling the table with the values and finding F statistic, you can decide to reject the null-hypothesis or not based on two methods:
 - Method 1:
 - Compute critical value using R : $qf(1 - \alpha, k - 1, N - k, \text{lower.tail=TRUE})$
 - If F-statistic is greater than critical value - reject the null-hypothesis. Otherwise, you fail to reject the null-hypothesis.
 - Method 2:
 - Compute the pvalue using R: $pf(F, k - 1, N - k, \text{lower.tail=FALSE})$
 - If pvalue is less than α - reject the null-hypothesis. Otherwise, you fail to reject the null-hypothesis.

9.g Chi-Square Test

- n = number of groups.
- $c(kj\dots)$ = all observations from the table given where you enter the data for each row from left to right.
- Enter your data: `matrix(c($k_{ij}\dots$), ncol=n, byrow=TRUE)`
- Using R compute the p-value and test-statistic: `chisq.test(sample.data, correct=FALSE)`
- If the p-value is less than α - reject the null-hypothesis. Otherwise, you fail to reject the null-hypothesis.
- Example:

	Heart disease	No heart disease	Total
High cholesterol diet	(i) 11	(iii) 4	15
Low cholesterol diet	(ii) 2	(iv) 6	8
	13	10	23

- Enter data into R: `matrix(c(11,4,2,6), ncol=2, byrow=TRUE)`
- Using chisq function compute test-statistic: `chisq.test(sample.data, correct=FALSE)`

Pearson's Chi-squared test

```
data: sample_data  
X-squared = 4.9597, df = 1, p-value = 0.02594
```

- At 5% confidence level we would reject the null-hypothesis, because the p-value is less than 5%.