Credit Markets

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Agenda: Multi-Name Credit

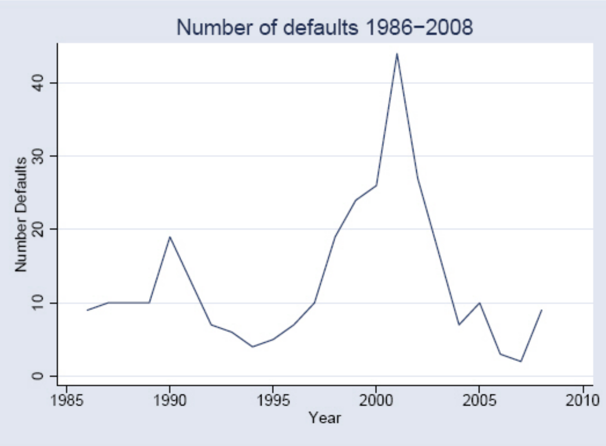
Default Dependence

- The underlying of a **single-name** credit derivative is one entity (e.g. single firm for a CDS contract).
- Therefore, default dependence is usual of second-order importance.
- The underlying of **multi-name** credit derivatives are portfolios of loans, bonds etc.
- Therefore, default dependence is crucial.
- In particular, so-called **contagion** effects are pricing relevant.

Example for Contagion Effects

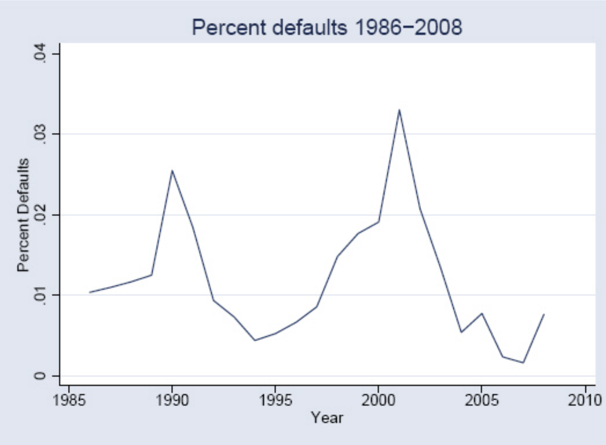
- On October 8, 2005 the auto parts maker Delphi Corp. filed for Chapter 11 bankruptcy protection.
- On December 12, 2005 the rating agency S&P cut GM's corporate credit rating to B, five steps below investment grade.
- On March 3, 2006 the auto parts maker Dana Corp. filed for bankruptcy protection defaulting on $2.5 billion of debt.

Default Clustering



Defaults of US Firms that are rated by Moody's
(Source: Moody's).

Default Clustering



Defaults of US Firms that are rated by Moody's (Source: Moody's).

A Note on Correlation vs. Dependence

Correlation

$$\rho = \frac{\text{Cov}[X, Y]}{\text{SD}[X]\text{SD}[Y]}$$

Uncorrelated vs. Independent

X and Y are **uncorrleted** if $\rho = 0$ ($\Longleftrightarrow$ $\text{Cov}[X, Y] = 0$).
X and Y are **independent** if $P(Y \leq y | X \leq x) = P(Y \leq y)$

- **1st Warning:** Correlation captures **linear** dependencies only!
- Nevertheless, correlation is often used to model dependencies.
- **2nd Warning:** People often use the word “correlation”, but mean “dependence”.

Modeling Default Dependencies

- The challenge is to set up a **realistic**, but still **tractable** model.
- Roughly speaking we need a model where the default probabilities of firms changes if other relevant firms default.
- Formally, one has to describe the **dependencies between the default times** τ_i, $i = 1, \dots, I$, of the firms in a portfolio.
- There are **several models** available:
  - Cox processes: Dependence via common factors driving all intensities.
  - Copula approach: Dependence via choice of copula.
  - Joint defaults: Dependence via simultaneous defaults of firm (joint point processes).
  - Markov chains: Dependence via jumps in default intensities upon transition of a Markov Chain.
  - Self-exciting processes: Dependence via intensities being exposed to jumps.

Agenda: Multi-Name Credit

Copula: What is it?

Copulas are a tool to **"glue together" marginal distributions**.

Example. **Default times** of firms

Q: Which joint distribution is easy to handle?

A: Multivariate normal distribution

Problem. Default times are **positive.**

With a **Gaussian Copula model** we can solve this problem.

Random Variables and Their Distributions

Recall the following helpful properties:

Construction of Distributions/Random Variables

Consider a random variable X with cdf F.

1. The random variable $F(X)$ is uniform on $[0, 1]$.
2. If U is uniform on $[0, 1]$, then the random variable $F^{-1}(U)$ has the cdf F.

Notice: (2) can be used to simulate random variables with a given distribution.

Factor Model for Normally Distributed Random Variables

Consider n standard normally distributed random variables X_j, $j = 1, \ldots, n$.

Q: Is there a simple way such that $(X_1, \ldots, X_n)$ is **multivariate normal**?

A: One-factor model

$$X_j = a_j W + \sqrt{1 - a_j^2} Z_j,$$

where W and Z_j are standard normal and uncorrelated.

Result: $(X_1, \ldots, X_n)$ are multivariate normal with

$$\mathrm{Cov}[X_j, X_k] = a_j a_k$$

Application: Factor-Copula Model for Credit Portfolios

Consider a credit portfolio consisting of I loans.
Some notation

τ_i	:	default time of firm i
F_i	:	cdf of τ_i

We already know that $F_i(\tau_i)$ is uniform on $[0,1]$ and thus (N: cdf of standard normal distr.)

$$U_i \equiv N^{-1}(F_i(\tau_i)) \sim \mathcal{N}(0,1)$$

We model the **dependence between the default times** via a **one-factor model**:

$$U_i = a_i W + \sqrt{1-a_i^2} Z_i \quad \Longrightarrow \quad P(U_i < u | W) = N\left(\frac{u - a_i W}{\sqrt{1-a_i^2}}\right)$$

Application: Factor-Copula Model for Credit Portfolios

Recall that $U_i \equiv N^{-1}(F_i(\tau_i))$ and $P(U_i < u|W) = N\left(\frac{u - a_i W}{\sqrt{1-a_i^2}}\right)$

Besides,

$$\begin{aligned} P(\tau_i < t|W) &= P(F_i(\tau_i) < F_i(t)|W) \\ &= P\left(N^{-1}(F_i(\tau_i)) < N^{-1}(F_i(t))|W\right) \\ &= P(U_i < N^{-1}(F_i(t))|W) \end{aligned}$$

Therefore,

$$P(\tau_i < t|W) = N\left(\frac{N^{-1}(F_i(t)) - a_i W}{\sqrt{1-a_i^2}}\right)$$

Application: Factor-Copula Model for Credit Portfolios

Recall that

$$P(\tau_i < t|W) = N\left(\frac{N^{-1}(F_i(t)) - a_i W}{\sqrt{1 - a_i^2}}\right)$$

Assumption. Copula correlations are identical, i.e. $a_i \cdot a_j = \rho$ and thus $a_i = \sqrt{\rho}$.

Therefore, we obtain

Default Rate

$$P(\tau_i < t|W) = N\left(\frac{N^{-1}(F_i(t)) - \sqrt{\rho}\, W}{\sqrt{1 - \rho}}\right)$$

Interpretation. The default rate equals the default probability of the i-th borrower until time t given the common factor W.

Loss of a Portfolio

We consider a credit portfolio with **identical notionals** and define

Gross Loss

The portfolio percentage gross loss is given by (before recoveries)

$$L_t = \frac{1}{I} \sum_{i=1}^{I} \mathbf{1}_{\{\tau_i < t\}}.$$

To understand the **risk characteristics** of a portfolio (e.g. expected loss, VaR), we need to know the **loss distribution**.

Finite Homogeneous Pool Model and Factor-Copula Model

- We study a **finite** pool of I firms and apply the **factor-copula model**.
- We already know that the cumulative probability of default for some firm **conditional** on a given realization of the common factor W is

$$P(\tau_i < t|W) = N\left(\frac{N^{-1}(F_i(t)) - \sqrt{\rho}W}{\sqrt{1-\rho}}\right).$$

- Conditioned on the common factor W, these probabilities are **independent**, and as we assume a homogeneous pool, they are also **identical** for all firms.

Default Probabilities

- Due to the homogeneity assumption, we can apply the **binomial distribution** to get the probability for exactly k defaults

$$P(\#defaults_t = k|W) = \binom{I}{k} P(\tau_i < t|W)^k (1 - P(\tau_i < t|W))^{I-k}.$$

- Recall that W was chosen to be **normally distributed**.
- We integrate over the common factor W to get **unconditional** probabilities

$$P(\#defaults_t = k) = \binom{I}{k} \int_{-\infty}^{\infty} F_i(t|w)^k (1 - F_i(t|w))^{I-k} n(w) dw,$$

where $F_i(t|w) \equiv P(\tau_i < t|w)$ and $n(w)$ represents the density of a standard normal random variable.

Portfolio Loss: Finite Pool

First Assumption. Finite pool of homogenous loans

Then the loss distribution is given by

$$F_L(\ell) = P(L_t \leq \ell) = P\Big(\sum_{i=1}^{I} \mathbf{1}_{\{\tau_i < t\}} \leq \ell I\Big) = \sum_{k=0}^{\lfloor \ell I \rfloor} P(\#defaults_t = k)$$

Second Assumption. Factor-copula model

Loss Distribution of Finite Pool

In the factor-copula model, the loss distribution of a finite pool reads

$$F_L(\ell) = \sum_{k=0}^{\lfloor \ell I \rfloor} \binom{I}{k} \int_{-\infty}^{\infty} F_i(t|w)^k (1 - F_i(t|w))^{I-k} n(w) dw.$$

This formula is explicit up to a numerical integration.

Portfolio Loss: Very Large Pool

If we additional assume that the number of loans is very large (in fact, infinity), then

Loss Distribution of Large Homogeneous Pool (LHP)

Under our previous assumptions, the loss distribution for a large pool reads

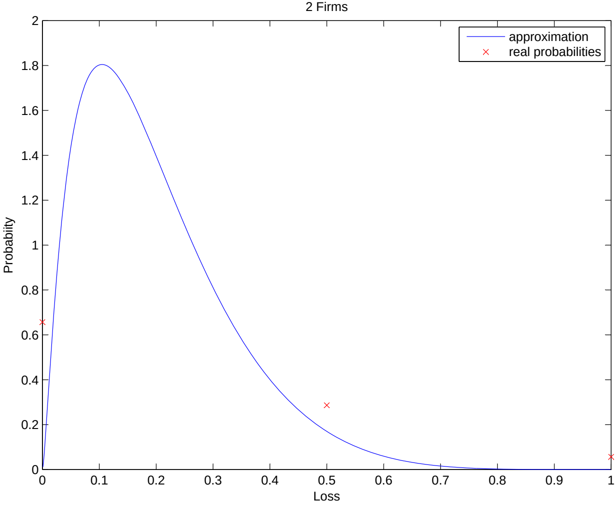
$$F_L^\infty(\ell) = F_L^\infty(\ell; p, \rho) = N\left(\frac{\sqrt{1-\rho}N^{-1}(\ell) - N^{-1}(p)}{\sqrt{\rho}}\right),$$

where $p \equiv F_i(t)$.

There are two ways to understand this result:

- For a large portfolio the default rate of an individual borrower approximates the loss of the portfolio well.
- We can take the limit in the representation for a finite pool.

Loss Distributions: Finite Pool vs. LHP Approximation



Horizon 5y, index spread 268bp, $\rho = 0.2$

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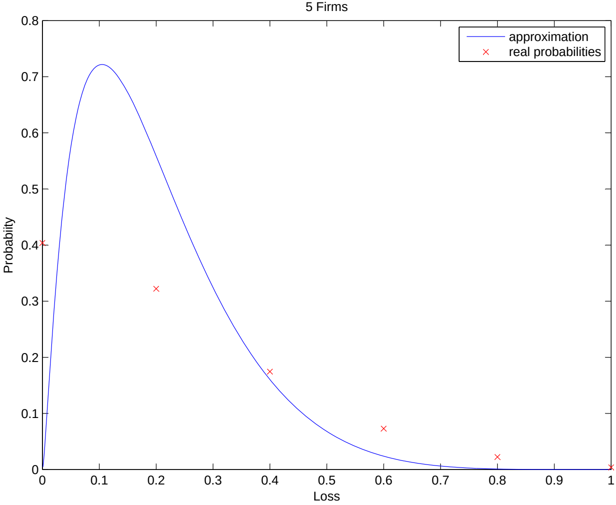

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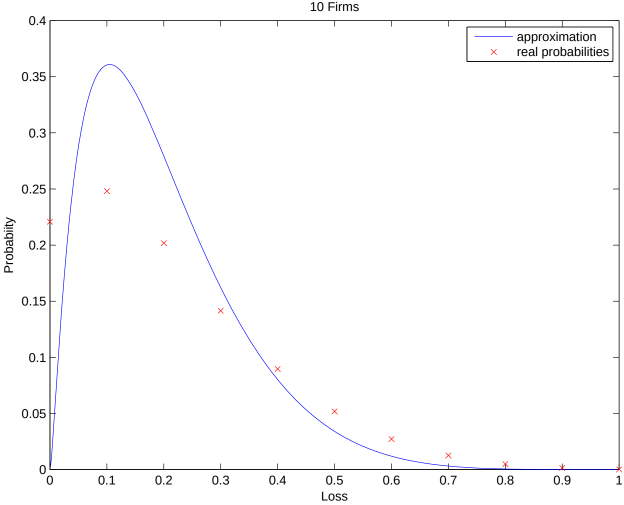

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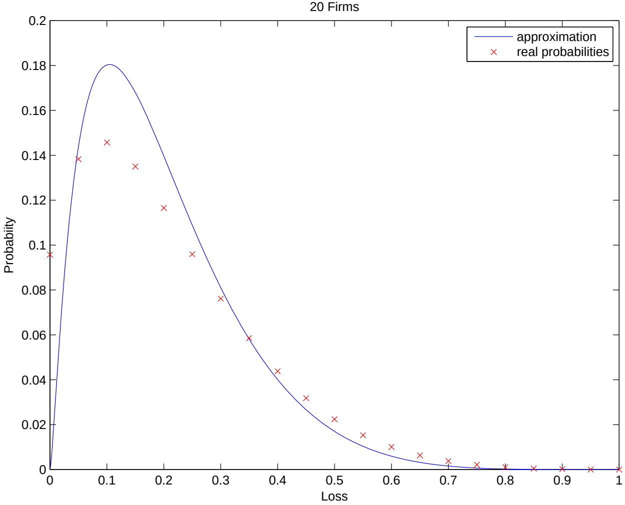

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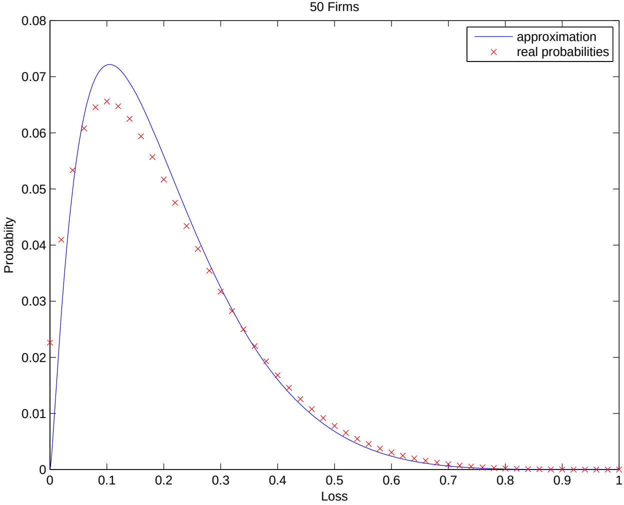

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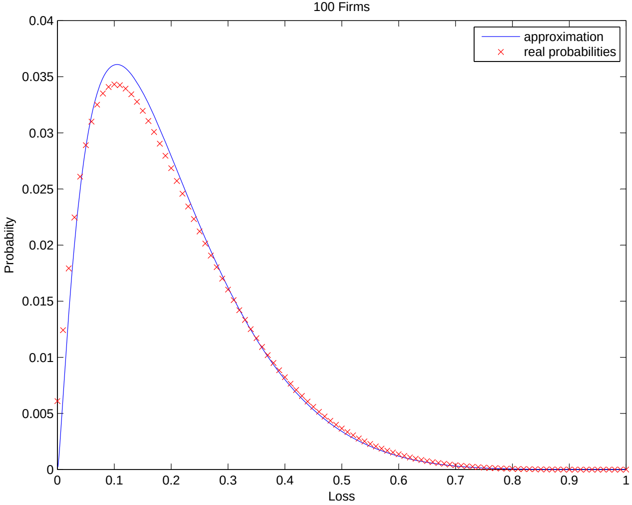

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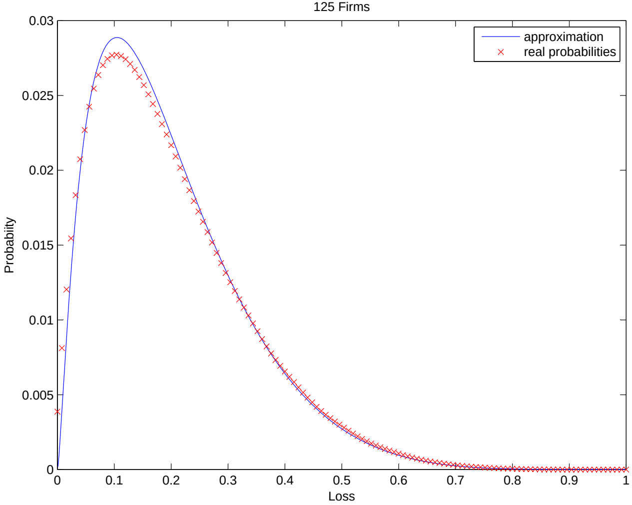
Loss Distributions: Finite Pool vs. LHP Approximation



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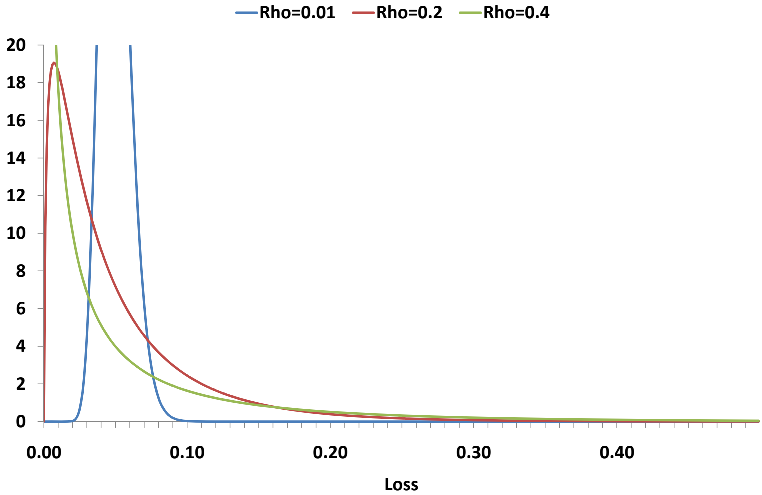
Loss Distribution: Density under LHP



Densities for different correlations ρ and $p = 0.05$.

Understanding the Loss Distribution under LHP

Recall that the default rate reads

$$DR \equiv P(\tau_i < t|W) = N\left(\frac{N^{-1}(F_i(t)) - \sqrt{\rho}\,W}{\sqrt{1-\rho}}\right)$$

Besides, $W \sim \mathcal{N}(0,1)$

Q: Is there an estimate for the default rate?

A: Since $P(W \leq N^{-1}(y)) = y$, we get

$$P\left(DR > N\left(\frac{N^{-1}(F_i(t)) - \sqrt{\rho}\,N^{-1}(y)}{\sqrt{1-\rho}}\right)\right) = y$$

$$\Longrightarrow P\left(DR \leq N\left(\frac{N^{-1}(F_i(t)) - \sqrt{\rho}\,N^{-1}(y)}{\sqrt{1-\rho}}\right)\right) = 1-y$$

Understanding the Loss Distribution under LHP

Recall that

$$P\left(DR \le N\left(\frac{N^{-1}(F_i(t)) - \sqrt{\rho}\,N^{-1}(y)}{\sqrt{1-\rho}}\right)\right) = 1 - y$$

Set $x = 1 - y$. Due to $-N^{-1}(1-x) = N^{-1}(x)$, we obtain

$$P\left(DR \le \underbrace{N\left(\frac{N^{-1}(F_i(t)) + \sqrt{\rho}\,N^{-1}(x)}{\sqrt{1-\rho}}\right)}_{\equiv dr(t,x)}\right) = x.$$

Interpretation: $dr(t,x)$ is the default rate that will not be exceeded with probability x.

Understanding the Loss Distribution under LHP

- Bank with retail portfolio of 100m dollars and copula correlation of 0.1 between the loans.
- Annual default probability for a single loan: 2%

$\Longrightarrow F_i(1) = P(\tau_i \leq 1) = 0.02$

This yields

$$dr(1, 0.999) = N\left(\frac{N^{-1}(0.02) + \sqrt{0.1}\,N^{-1}(0.999)}{\sqrt{1-0.1}}\right) = 0.128,$$

i.e. with probability 99.9% the DR will not exceed 12.8%.

Assuming a recovery rate of 60% this would result in losses of

$$100\text{m} \cdot 12.8\% \cdot (1 - 60\%) = 5.13\text{m}$$

Understanding the Loss Distribution under LHP

Recall ($p = F_i(t)$)

$$P\left(DR \leq \underbrace{N\left(\frac{N^{-1}(p) + \sqrt{\rho}\, N^{-1}(x)}{\sqrt{1-\rho}}\right)}_{\equiv dr(x)}\right) = x.$$

Therefore, the cdf of DR is

$$F_{DR}(z) = P\left(DR \leq z\right) = (dr)^{-1}(z) = N\left(\frac{\sqrt{1-\rho}N^{-1}(z) - N^{-1}(p)}{\sqrt{\rho}}\right),$$

which shows the result.