# Credit Markets

Holger Kraft

UCLA Anderson

Fall 2019

# Agenda: Multi-Name Credit

# Longstaff-Rajan Model

- The model of Longstaff and Rajan is a **top-down model**.
- This means that the **loss process** itself is modeled.
- Losses are not directly attributed to particular firms.
- Contagion effects are modeled via **joint defaults**.
- More precisely, the model allows for
  - **small** losses (single firms)
  - **medium** losses (sector)
  - **large** losses (economy)

- Let $L_t$ denote the **total portfolio loss**. By definition, $L_0 = 0$.
- To model the dynamic evolution of $L_t$ in a top down framework, we assume

$$dL_t = (1 - L_{t-}) \left\{ \bar{\gamma}_1 dN_{1t} + \bar{\gamma}_2 dN_{2t} + \bar{\gamma}_3 dN_{3t} \right\},$$

  with $\bar{\gamma}_i = 1 - e^{-\gamma_i}$, $i = 1, 2, 3$.
- The parameters $\gamma_i$ are nonnegative constants that define the **jump sizes** and $N_{it}$ are **independent Cox processes**.
- Note that for small values of $\gamma_i$, the jump size $\bar{\gamma}_i$ is essentially just $\gamma_i$.

# Explicit Solution

Applying Ito's lemma to $H(L_t) = \ln(1 - L_t)$ yields

**Loss Process**

$$L_t = 1 - e^{-\gamma_1 N_{1t}} e^{-\gamma_2 N_{2t}} e^{-\gamma_3 N_{3t}}.$$

- The dynamics show that **three factors** generate portfolio losses.
- Obviously, the consistency requirement $0 \leq L_t \leq 1$ is satisfied.
- Furthermore, we see that the total loss process is increasing in time which is also reasonable.

# Example: Three Factors

- To illustrate the model, consider an example with three jump sizes 0.01, 0.1, and 0.5.
- For simplicity, recovery is zero.
- A realization of the first Cox process will result in a 1% portfolio loss $\rightarrow$ **isolated default** that affects only one firm.
- A realization of the second Cox process will result in a 10% portfolio loss $\rightarrow$ **impact of a major event** that decimates the ranks of firms in a specific sector or industry.
- After a jump in the third Cox process 50% of the remaining firms default $\rightarrow$ **catastrophic event** affecting the entire economy.
- Thus, the model **captures** both **idiosyncratic and systematic risk**.

# Calculating Default Probabilities

- To specify the model, we choose a CIR process to model the **intensities** of the three Cox processes

$$
\begin{aligned}
d\lambda_{1t} &= \kappa_1(\theta_1 - \lambda_{1t})dt + \sigma_1\sqrt{\lambda_{1t}}dW_{1t}, \\
d\lambda_{2t} &= \kappa_2(\theta_2 - \lambda_{2t})dt + \sigma_2\sqrt{\lambda_{2t}}dW_{2t}, \\
d\lambda_{3t} &= \kappa_3(\theta_3 - \lambda_{3t})dt + \sigma_3\sqrt{\lambda_{3t}}dW_{3t},
\end{aligned}
$$

  where $W_{it}$ are independent Brownian motions.
- As seen before, $L_t$ is a simple function of the values of the three Cox processes. Therefore, it is sufficient to compute the **probability distribution** for the single Cox processes.
- From previous results, we know that for $i = 1, 2, 3$

$$
P(N_{iT} = k|\mathcal{G}_T) = \frac{1}{k!}e^{-\int_0^T \lambda_{is}ds}\left(\int_0^T \lambda_{is}ds\right)^k,
$$

  where $\mathcal{G}_T$ is generated by $\lambda_i$.

# Calculating Default Probabilities

- Let $P_k^i(\lambda_i, T)$ denote $k!$ times the probability that $N_{iT} = k$ conditioned on $\lambda_{i0} = \lambda_i$. Then

$$P_k^i(\lambda_i, T) = \mathrm{E}\left[ e^{-\int_0^T \lambda_{is} ds} \left( \int_0^T \lambda_{is} ds \right)^k \right].$$

- For $k = 0$ we know the explicit solution since the model is **affine**:

$$P_0^i(\lambda_i, T) = e^{A^i(T) - B^i(T)\lambda_i},$$

$$e^{A^i(T)} = \left( \frac{2\gamma_i e^{(\kappa_i + \gamma_i)\frac{T}{2}}}{2\gamma_i + (\kappa_i + \gamma_i)(e^{\gamma_i T} - 1)} \right)^{\frac{2\kappa_i \theta_i}{\sigma_i^2}}, \; B^i(T) = \frac{2(e^{\gamma_i T} - 1)}{2\gamma_i + (\kappa_i + \gamma_i)(e^{\gamma_i T} - T)},$$

with $\gamma_i = \sqrt{\kappa_i^2 + 2\sigma_i^2}$.

# Calculating Default Probabilities

Since $P_0^i(\lambda_i, T) = e^{A^i(T) - B^i(T)\lambda_i}$, we conjecture

**Closed-form Solution**

$$P_k^i(\lambda_i, T) = e^{A^i(T) - B^i(T)} \sum_{j=1}^{k} C_{k,j}^i(T)\lambda_i^k.$$

Francis shows that $A^i$, $B^i$, and $C_{k,j}$ satisfy a system of **first-order ODE's**

$$\partial_t C_{k,k}^i = kC_{k-1,k-1}^i - (\sigma_i^2 B^i(t) + \kappa_i)kC_{k,k}^i$$
$$\partial_t C_{k,j}^i = kC_{k-1,j-1}^i - (\sigma_i^2 B^i(t) + \kappa_i)jC_{k,j}^i + (j+1)(\kappa_i\theta_i + 0.5j\sigma_i^2)C_{k,j+1}^i$$
$$\partial_t C_{k,0}^i = \kappa_i\theta_i C_{k,1}^i$$

where $1 \leq j \leq k-1$ and $C_{k,j}^i(0) = 0$ for all $k > 0$.

# Expected Portfolio Loss

- Following a **recursive** algorithm, this system of ODEs can be solved **numerically**: For each $k$, one has to solve for $C^i_{k,j}$ where $j$ runs backwards from $k$ to 0.
- After computing the solutions, the expectation of every function $G(L_t)$ of the **portfolio loss** at time $t$ can be calculated.

**Expected portfolio loss**

$$\mathrm{E}[G(L_t)] = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{P^1_j(\lambda_{10}, t)}{j!}\frac{P^2_k(\lambda_{20}, t)}{k!}\frac{P^3_l(\lambda_{30}, t)}{l!} G(j,k,l),$$

with $G(j,k,l) \equiv G\left(1 - e^{-\gamma_1 j} e^{-\gamma_2 k} e^{-\gamma_3 l}\right)$.

- One only has to **compute** the **first few arguments** of the sum since higher-order terms are negligible.

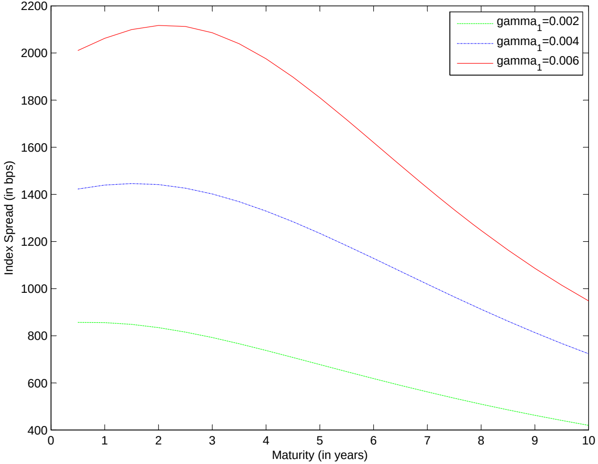
# Example: Pricing CDO Tranches

- As a numerical illustration, let $r = 0.05$, $\delta = 0.25$, $T = 5$.
- Furthermore, the tranches are characterized by the following attachment and detachment points $K_0 = 0$, $K_1 = 0.03$, $K_2 = 0.07$, $K_3 = 0.1$, $K_4 = 0.15$, $K_5 = 0.3$, and $K_6 = 1$.
- The jump sizes and volatilities of the three intensities read

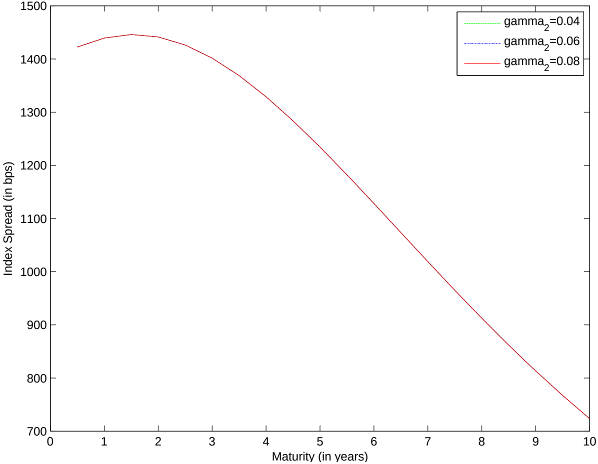
$$\gamma_1 = 0.004,\ \gamma_2 = 0.06,\ \gamma_3 = 0.3,\ \sigma_1 = 0.17,\ \sigma_2 = 0.25,\ \sigma_3 = 0.3.$$

- The mean reversion speed is $\kappa_i = 0.5$ for all processes.
- We set the start intensities as $\lambda_1 = 0.8$, $\lambda_2 = 0.03$, $\lambda_3 = 0.001$, which are assumed to be equal to the mean reversion levels.
- Since each firm has a weight of $1/125 = 0.008$ in the portfolio, a jump size in the first Poisson process of 0.004 represents the idiosyncratic default of an individual firm, where the implicit recovery rate for the firm's debt is 50 percent.

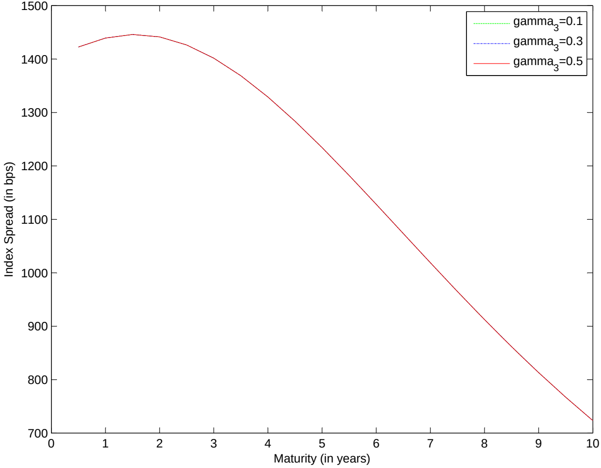
# 0-3-Tranche Spread: Jump Size of First Poisson Proc.



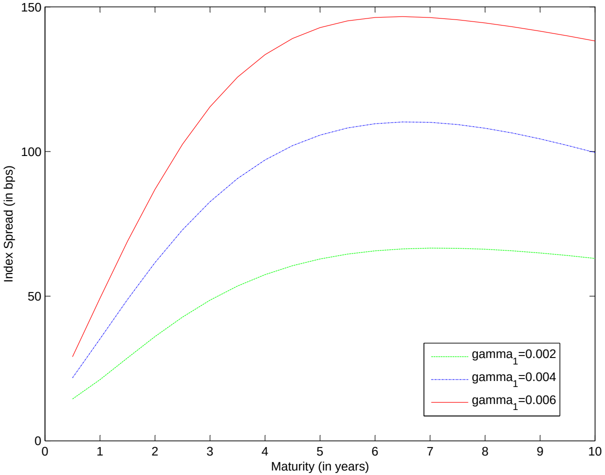
# 0-3-Tranche Spread: Jump Size of Second Poisson Proc.



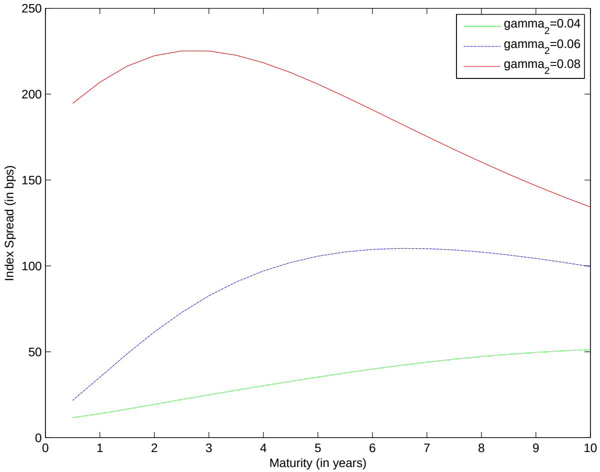
# 0-3-Tranche Spread: Jump Size of Third Poisson Proc.



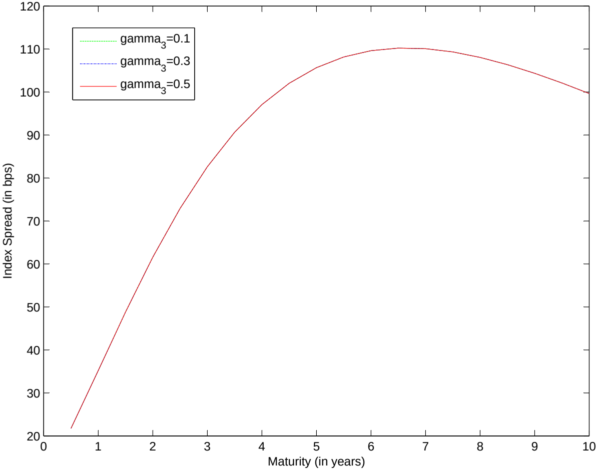
# 6-9-Tranche Spread: Jump Size of First Poisson Proc.



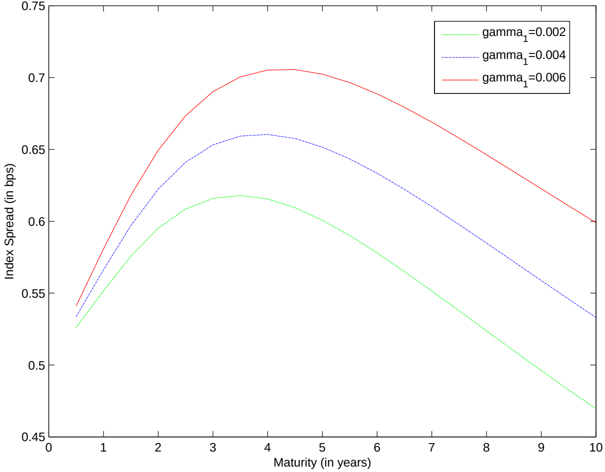
# 6-9-Tranche Spread: Jump Size of Second Poisson Proc.



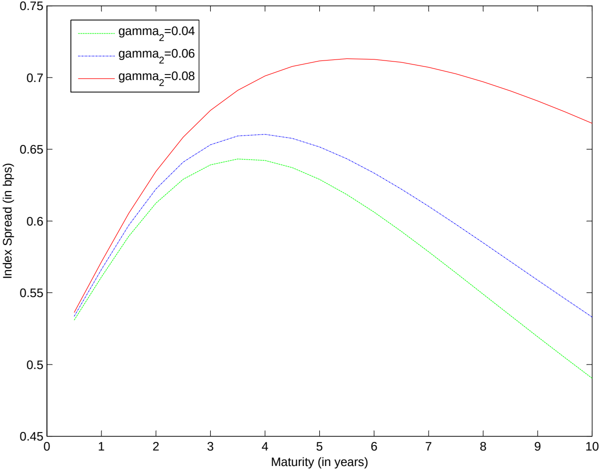
# 6-9-Tranche Spread: Jump Size of Third Poisson Proc.



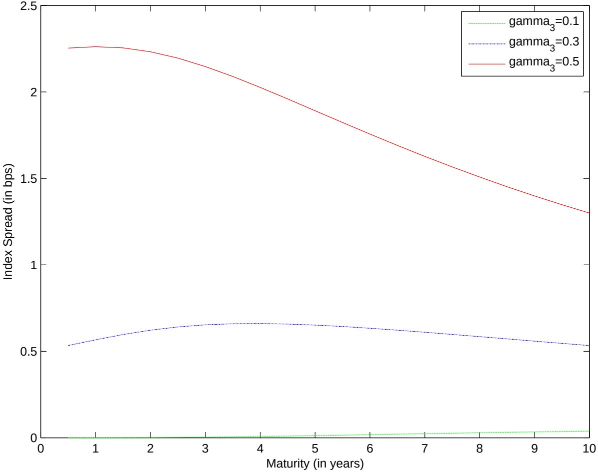
# 22-100-Tranche Spread: Jump Size of First Poisson Proc.



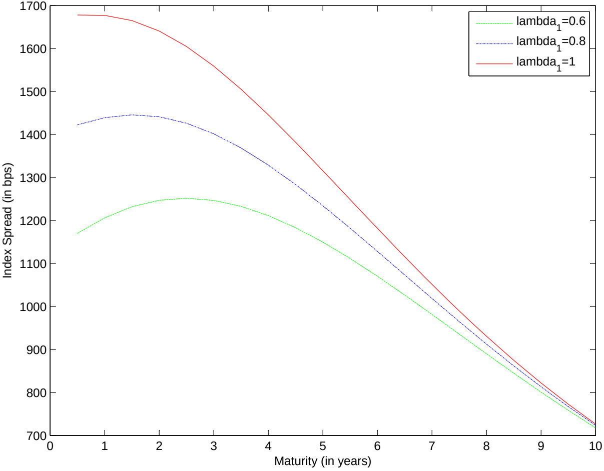
# 22-100-Tranche Spread: Jump Size of Second Poisson Proc.



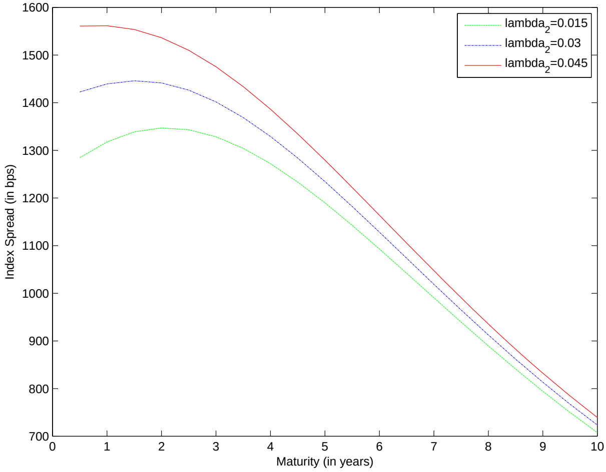
# 22-100-Tranche Spread: Jump Size of Third Poisson Proc.



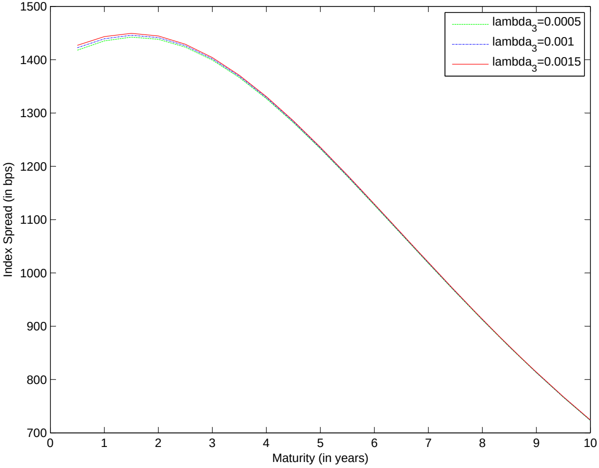
# 0-3-Tranche Spread: Intensity of First Poisson Proc.



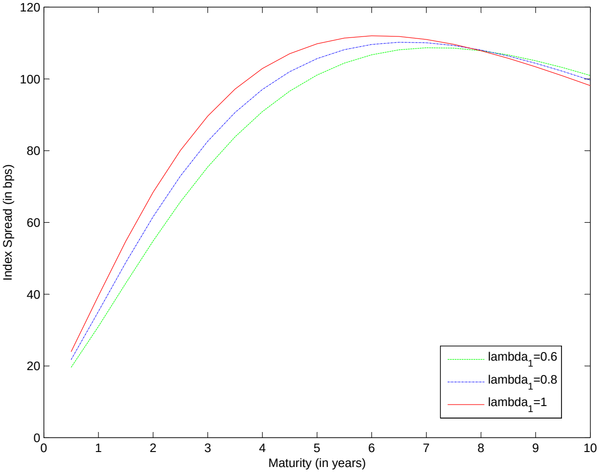
# 0-3-Tranche Spread: Intensity of Second Poisson Proc.



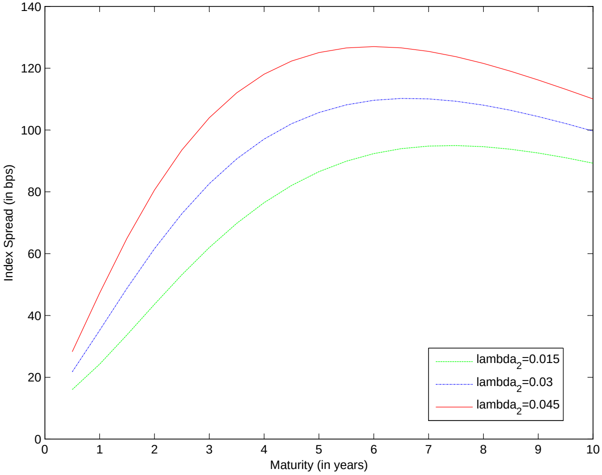
# 0-3-Tranche Spread: Intensity of Third Poisson Proc.



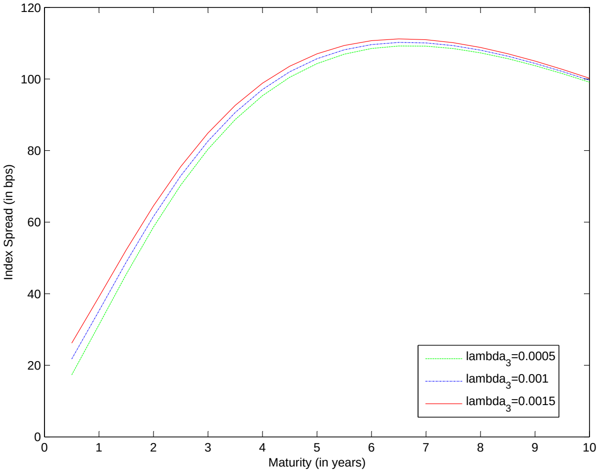
# 6-9-Tranche Spread: Intensity of First Poisson Proc.



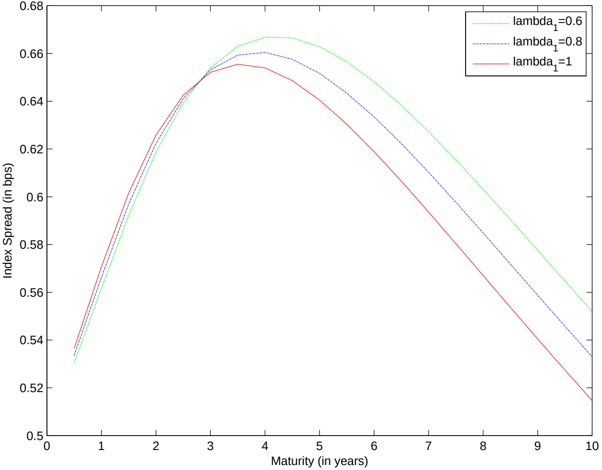
# 6-9-Tranche Spread: Intensity of Second Poisson Proc.



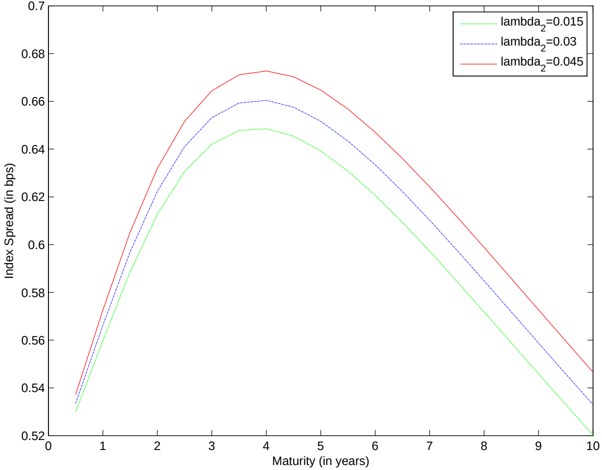
# 6-9-Tranche Spread: Intensity of Third Poisson Proc.



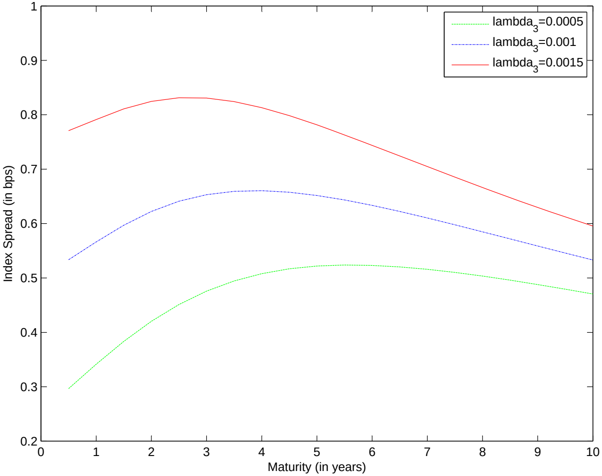
# 22-100-Tranche Spread: Intensity of First Poisson Proc.



# 22-100-Tranche Spread: Intensity of Second Poisson Proc.



# 22-100-Tranche Spread: Intensity of Third Poisson Proc.



# Agenda: Multi-Name Credit

# Self-Exciting Framework

- Empirical evidence suggests that **defaults cluster**.
- A single default sometimes triggers a widening of credit spreads across the board.
- In this section, we use a **top down approach** where credit derivatives are path-dependent claims on the total portfolio loss $L$.
- In order to evaluate these derivatives, we need a **model for $N$ and $L$**.
- We specify these processes in terms of a risk-neutral **intensity process** $\lambda$ and a distribution $\nu$ for the random loss at default.
- The intensity process is modeled as an **affine jump diffusion** model where the loss process itself is a risk factor.
- This so-called **self-exciting** property captures default clustering.

# Specification of Intensity Process

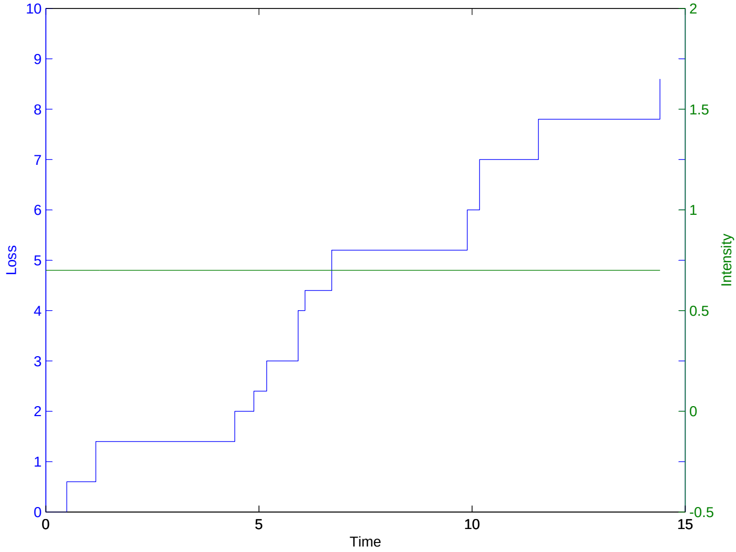
**Self-Exciting Intensity Model**

The intensity process has the dynamics

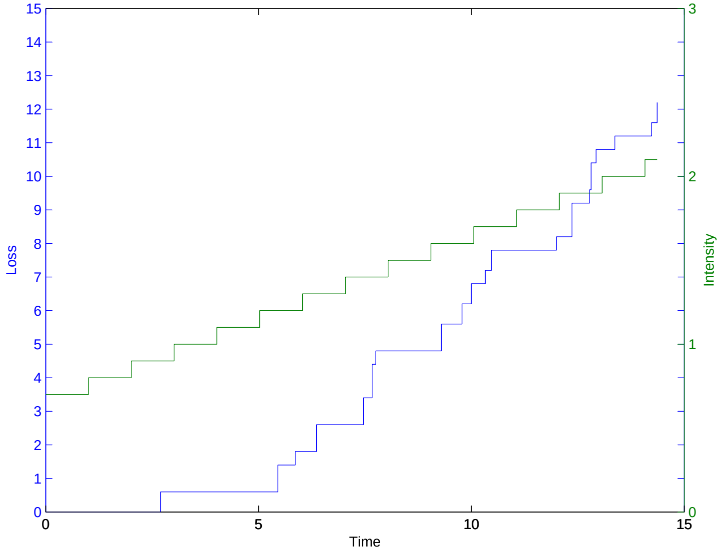
$$d\lambda_t = \kappa(\theta - \lambda_t)\, dt + \delta\, dL_t.$$

- Applying Ito's lemma yields
$$\lambda_t = \theta + (\lambda_0 - \theta)e^{-\kappa t} + \delta \int_0^t e^{-\kappa(t-s)}\, dL_s.$$
- Upon default the intensity **increases by the realized loss** scaled by the sensitivity parameter $\delta$.
- The impact of an default event **exponentially decays** over time with rate $\kappa$.
- The processes are said to be self-exciting since **defaults trigger jumps of** the **default intensity**.
- This makes future defaults more likely, i.e. **defaults** are **positively correlated**.

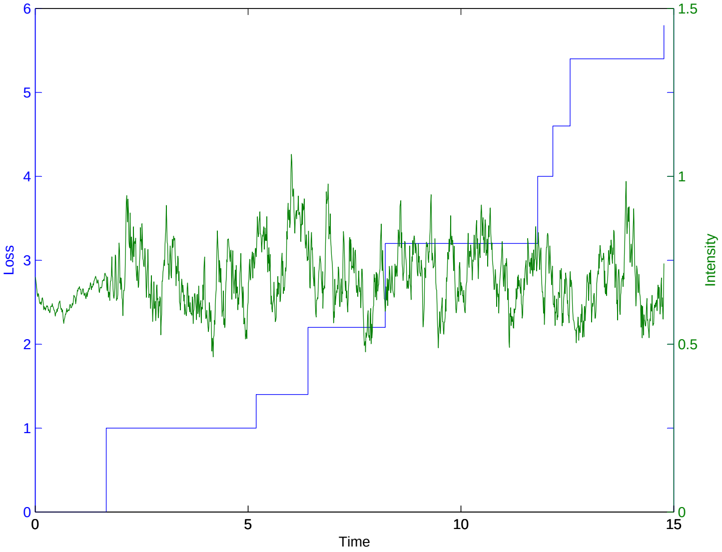
# Loss Process: Poisson



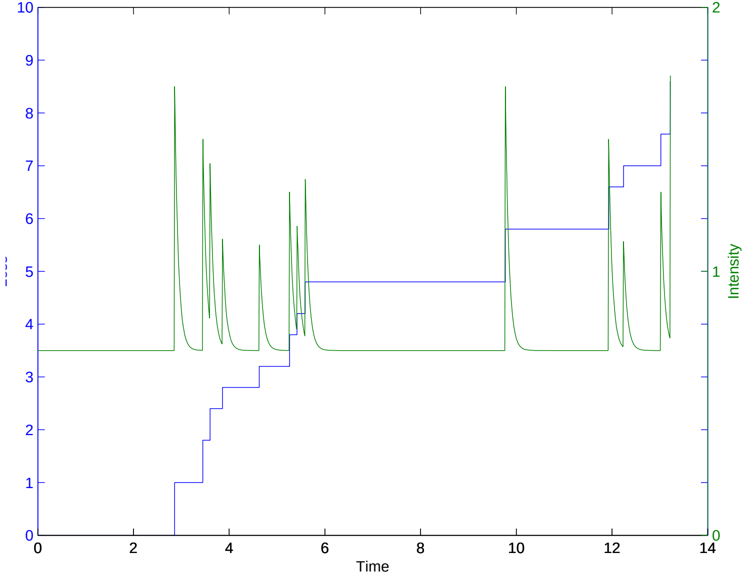
# Loss Process: Inhomogeneous Poisson



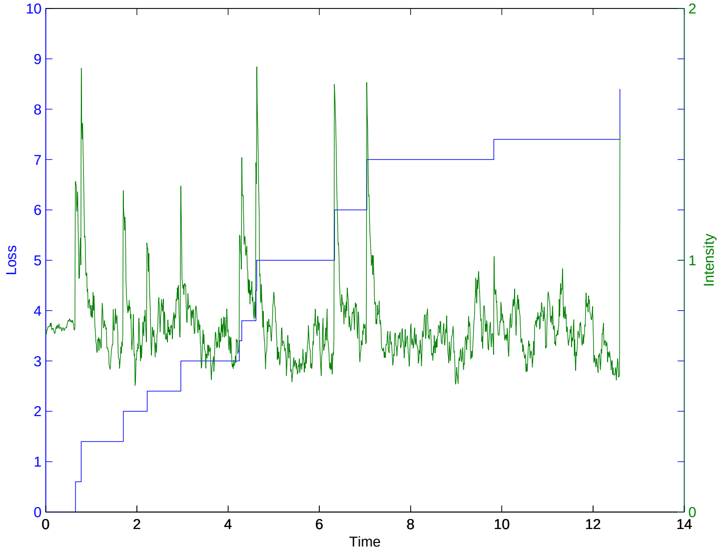
# Loss Process: CIR



# Loss Process: Self-Exciting without Diffusion



# Loss Process: Self-Exciting with Diffusion



# Some Properties

- The process $\lambda_t$ is the **intensity** of $N_t$ and $L_t$.
- $N_t$ and $L_t$ have **common event times**.
- Whereas the jumps of the default process are unit-sized, the jump sizes of the loss process are drawn from the independent jump size process $\nu$.
- The two dimensional process $J_t \equiv (L_t, N_t)^\top$ is called a **Hawkes Process**.
- There are two **special cases**:
  1. $\kappa = 0$: Birth process
  2. $\kappa = 0$, $\delta = 0$: Poisson process

# Expected Portfolio Losses and Defaults

Since we are an an **affine** framework, we can compute the expected losses explicitly:

## Expected Loss

With $\ell$ being loss given default we get that

$$\mathrm{E}_t[L_T] = \mathcal{A}(t,T) + \mathcal{B}(t,T)\lambda_t + L_t,$$

where

$$\mathcal{B}(t,T) = \frac{\ell}{\kappa - \delta\ell}\left(1 - e^{-(\kappa-\delta\ell)(T-t)}\right),$$

$$\mathcal{A}(t,T) = \frac{\ell\kappa\theta}{\kappa - \delta\ell}\left(\frac{e^{-(\kappa-\delta\ell)(T-t)} - 1}{\kappa - \delta\ell} + T - t\right).$$

The expected number of jumps $\mathrm{E}_t[N_T]$ can be computed in the same way by setting $\ell = 1$ and replacing $L_t$ by $N_t$.

# Pricing Multi-name Products

- Recall that the fair spread of an **index CDS** reads

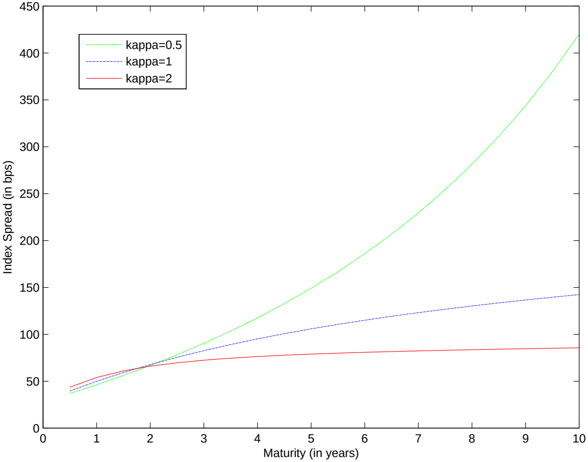
$$S_t = \frac{p(t,T)\mathrm{E}_t[L_T] - L_t - \int_t^T \mathrm{E}_t[L_s]\partial_s p(t,s)\,ds}{\int_t^T p(t,s)\left(1 - \mathrm{E}_t[N_s]/I\right)\,ds}.$$

- With the previous results, we are able to price this contract.
- We have already seen that **CDO tranches** consist of **options on the loss process**

$$\mathrm{E}[(L_T - K)^+] = \int_K^\infty (x-K)f(x)\,dx.$$

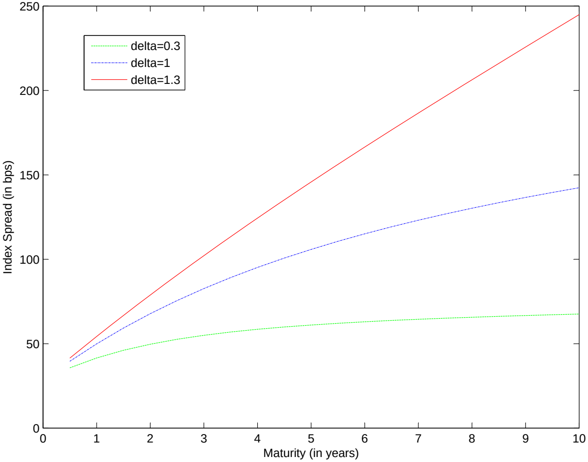
  The density $f$ can be obtained by **inverting the conditional transform** of the loss process.
- Hence, we can compute both index spreads and CDO spreads in this model.

# Index Spread and Mean Reversion Speed



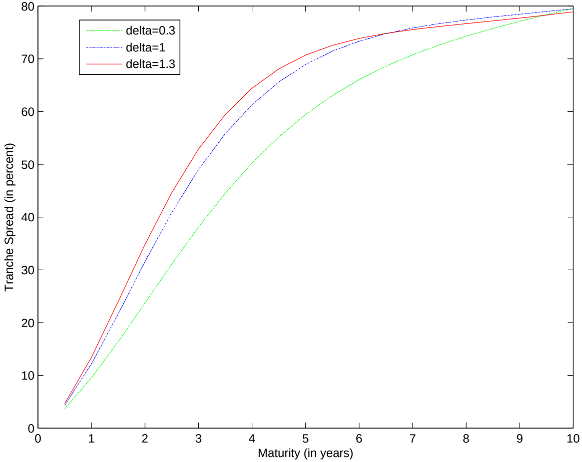

125 Firms, $r = 0.05$, $\theta = 1$, $\lambda_0 = 0.5$, $\delta = 1$, $\ell \in \{0.4, 0.6, 0.8, 1\}$ uniform, quarterly payments

# Index Spread and Jump Size



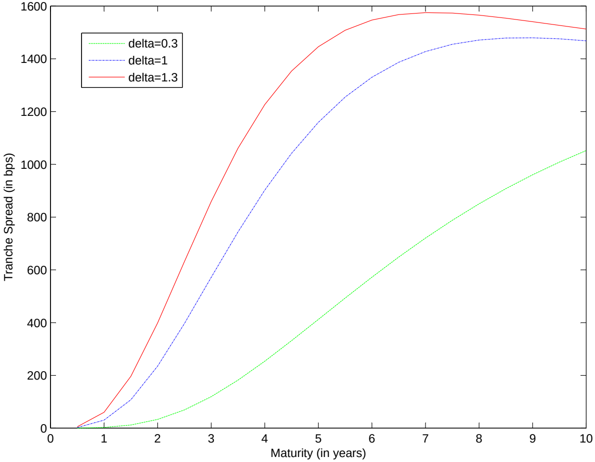

125 Firms, $r = 0.05$, $\kappa = 1$ $\theta = 1$, $\lambda_0 = 0.5$, $\ell \in \{0.4, 0.6, 0.8, 1\}$ uniform, quarterly payments

# 0-3-Tranche Spread and Jump Size



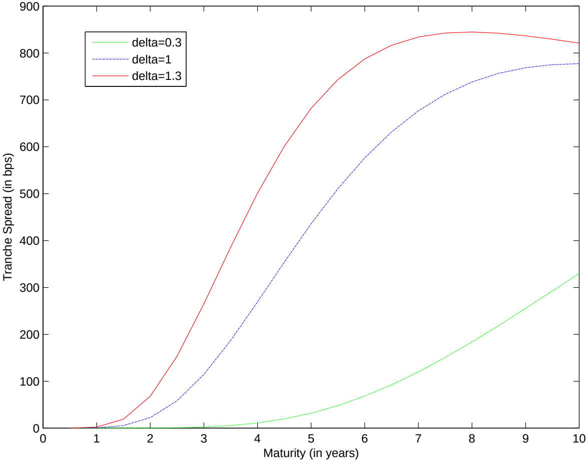

125 Firms, $r = 0.05$, $\kappa = 1$ $\theta = 1$, $\lambda_0 = 0.5$, $\ell \in \{0.4, 0.6, 0.8, 1\}$ uniform, quarterly payments

# 3-6-Tranche Spread and Jump Size



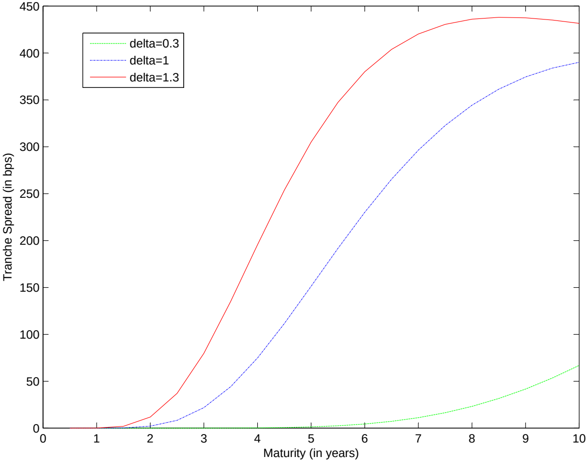

125 Firms, $r = 0.05$, $\kappa = 1$ $\theta = 1$, $\lambda_0 = 0.5$, $\ell \in \{0.4, 0.6, 0.8, 1\}$ uniform, quarterly payments

# 6-9-Tranche Spread and Jump Size



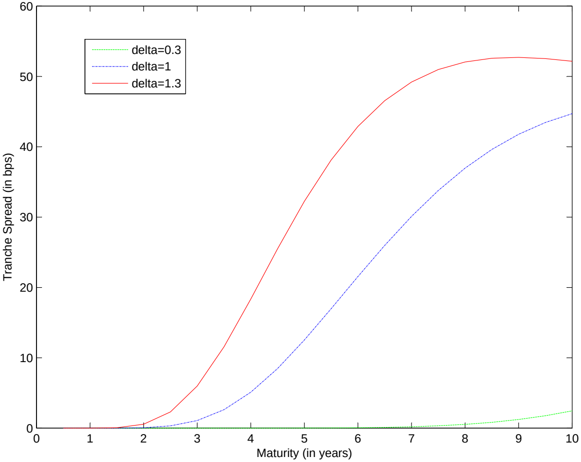

125 Firms, $r = 0.05$, $\kappa = 1$ $\theta = 1$, $\lambda_0 = 0.5$, $\ell \in \{0.4, 0.6, 0.8, 1\}$ uniform, quarterly payments

# 9-12-Tranche Spread and Jump Size



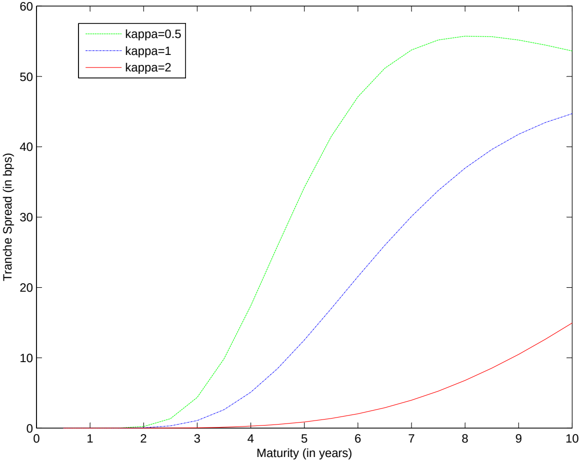

125 Firms, $r = 0.05$, $\kappa = 1$ $\theta = 1$, $\lambda_0 = 0.5$, $\ell \in \{0.4, 0.6, 0.8, 1\}$ uniform, quarterly payments

# 12-22-Tranche Spread and Jump Size




125 Firms, $r = 0.05$, $\kappa = 1$ $\theta = 1$, $\lambda_0 = 0.5$, $\ell \in \{0.4, 0.6, 0.8, 1\}$ uniform, quarterly payments

# 12-22-Tranche Spread and Mean Reversion Speed




125 Firms, $r = 0.05$, $\theta = 1$, $\lambda_0 = 0.5$, $\delta = 1$, $\ell \in \{0.4, 0.6, 0.8, 1\}$ uniform, quarterly payments