Credit Markets

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Agenda: Multi-Name Credit

Multi-Name Credit Derivatives

- The **underlying** of multi-name credit derivatives is a **portfolio of loans**, bonds etc.
- In this section, we will briefly go through the **contractual specifications** of some multi-name credit derivatives.
- We will discuss the **following contracts**:
  - Index CDS
  - Basket CDS
  - CDOs

Modeling Default Counter and Default Stopping Times

- We consider a portfolio consisting of I entities.
- The **default stopping times** of the entities are denoted by

$$\tau_1, \tau_2, \ldots, \tau_I$$

- The **number of defaults** is counted by the default process (**bottom up**)

$$N_t \equiv \sum_{i=1}^{I} \mathbf{1}_{\{\tau_i \le t\}}$$

- The ***k*-th jump time** of N is denoted by T_k.
- Therefore, we can also write (**top down**)

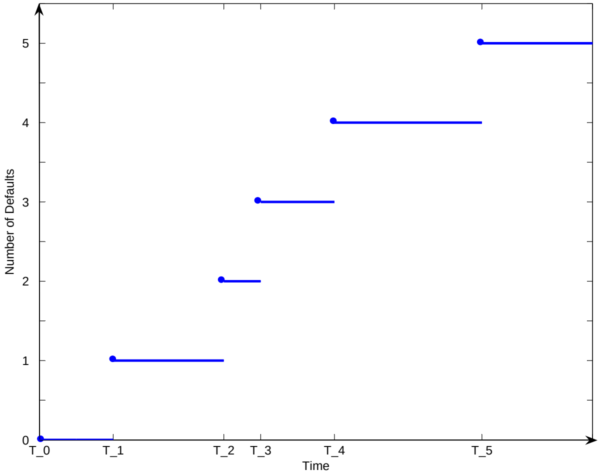
$$N_t = \sum_{k \ge 1} \mathbf{1}_{\{T_k \le t\}}$$

- The corresponding **loss process** reads

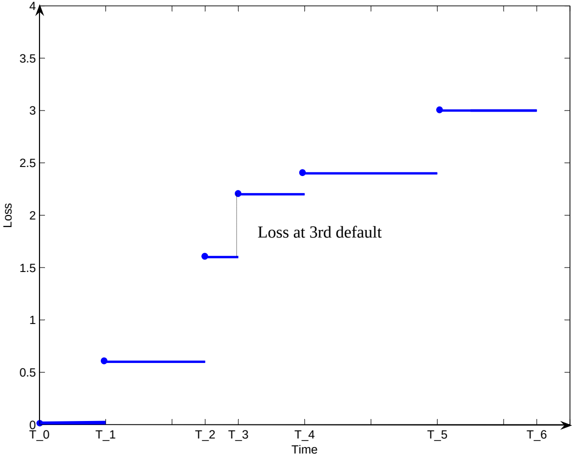
$$L_t = \sum_{k \ge 1} \mathbf{1}_{\{T_k \le t\}} \ell_k,$$

where ℓ_k is the loss associated with the k-th loss.

Default Process



Loss Process



Losses are assumed to be 0.2, 0.6 or 1.0

Index CDS

- The **underlying** of an index CDS is an **index** (e.g. CDX, iTraxx).
- The index portfolio **consists of single-name CDS** written on I firms.
- All CDS contracts have the **same maturity** T.
- Their **payment dates** are **identical**.
- All CDS contracts have the **same weights**, i.e. their notionals are $1/I$.
- Therefore, an index CDS corresponds to a portfolio of single-name CDS contracts.
- It offers full protection of this portfolio.

Index CDS: Fee Leg

- The **current notional** of the index CDS is given by

$$F_s \equiv 1 - N_s/I,$$

where N_s is the number of defaults until s and I denotes the number of constituents.

- The **fee payment** at time t_j is thus given by

$$S_t \, \delta \, F_{t_j},$$

where S_t denotes the index spread with maturity T which is contracted at initiation.

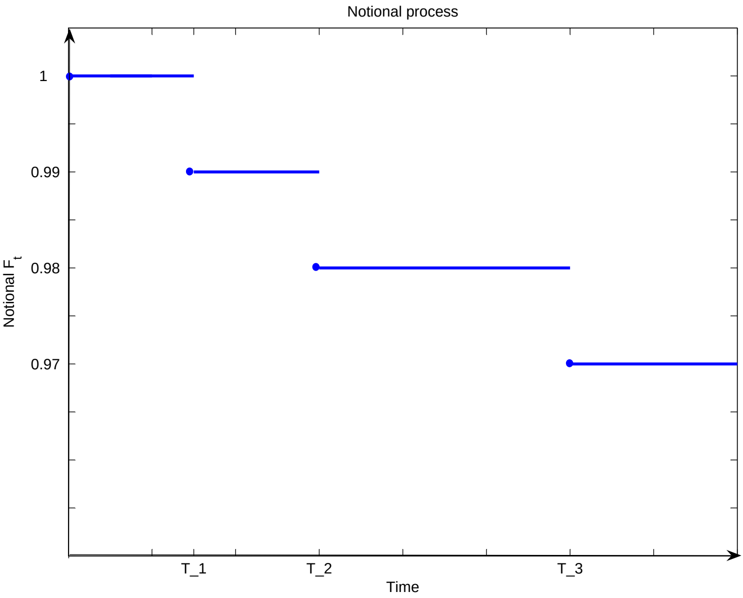
- Accrued payments are disregarded.

Value of the Fee Leg

The value of the fee leg per 1bp of fee payments reads

$$V_t^{fee} = \sum_{j=1}^{n} \delta \mathrm{E}_t \left[\frac{F_{t_j}}{B(t, t_j)} \right]$$

Index CDS: Current Notional



Index CDS: Fee Leg

If default-free interest rates and default risk are **independent**, then

$$V_t^{fee} = \sum_{j=1}^{n} \delta \mathrm{E}_t \left[\frac{F_{t_j}}{B(t, t_j)} \right] = \sum_{j=1}^{n} \delta \left(1 - \frac{\mathrm{E}_t \left[N_{t_j} \right]}{I} \right) p(t, t_j)$$

Sometimes it is assumed that the fee is paid continuously. In this case the value of the fee leg is given by

Fee Leg (Continuous Payment)

$$V_t^{fee} = \int_t^T \mathrm{E}_t \left[\frac{F_s}{B(t, s)} \right] ds.$$

If default-free interest rates and default risk are **independent**, then

$$V_t^{fee} = \int_t^T p(t, s) \mathrm{E}_t \left[F_s \right] ds = \int_t^T p(t, s) \left(1 - \frac{\mathrm{E}_t \left[N_s \right]}{I} \right) ds$$

Index CDS: Protection Leg

- The protection leg of a **single-name CDS** makes a payment if the reference entity defaults.
- Therefore, its value is

$$V_t^{prot} = \mathrm{E}_t\left[\frac{\ell\,\mathbf{1}_{\{t<\tau\leq T\}}}{B(t,\tau)}\right],$$

where τ is the default-time of the reference entity.

- The protection leg of an **index CDS** makes a payment whenever any entity of the reference pool defaults.
- Therefore, we obtain the following result.

Value of the Protection Leg (Top Down)

$$V_t^{prot} = \sum_{k\geq 1} \mathrm{E}_t\left[\frac{\ell_k\,\mathbf{1}_{\{t<T_k\leq T\}}}{B(t,\,T_k)}\right],$$

where T_k is the time of the k-th default.

Index CDS: Protection Leg

Notice that

$$\sum_{k\geq 1} \frac{\ell_k \, \mathbf{1}_{\{t<T_k\leq T\}}}{B(t,T_k)} = \sum_{k\geq 1} e^{-\int_t^{T_k} r_u\,du} \ell_k \, \mathbf{1}_{\{t<T_k\leq T\}} = \int_t^T e^{-\int_t^s r_u\,du} dL_s$$

Therefore, the protection leg has also the following representation.

Integral Representation

$$V_t^{prot} = \mathrm{E}_t\Big[\int_t^T e^{-\int_t^s r_u\,du} dL_s\Big]$$

where L is the loss process of the portfolio.

Applying Ito's product rule yields

$$e^{-\int_t^T r_u\,du} L_T = L_t + \int_t^T e^{-\int_t^s r_u\,du}\, dL_s + \int_t^T L_s \underbrace{de^{-\int_t^s r_u\,du}}_{=-r_s e^{-\int_t^s r_u\,du} ds}$$

Index CDS: Protection Leg

Therefore,

$$
\begin{aligned}
V_t^{prot} &= \mathrm{E}_t\Big[\int_t^T e^{-\int_t^s r_u\,du}dL_s\Big] \\
&= \mathrm{E}_t\Big[e^{-\int_t^T r_u\,du}L_T\Big] - L_t + \int_t^T \mathrm{E}_t\Big[L_s r_s e^{-\int_t^s r_u\,du}\Big]\,ds
\end{aligned}
$$

Under **independence** we get

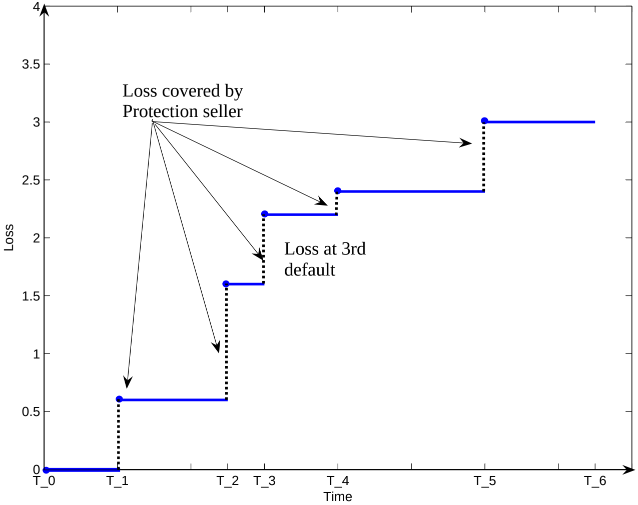
$$
V_t^{prot} = p(t,T)\mathrm{E}_t[L_T] - L_t - \int_t^T \mathrm{E}_t[L_s]\partial_s p(t,s)\,ds
$$

Recall that

$$
V_t^{fee} = \int_t^T p(t,s)\left(1 - \mathrm{E}_t\left[N_s\right]/I\right)\,ds.
$$

Therefore, we only need to calculate the **expected losses**, $\mathrm{E}_t[L_s]$, and the **expected number of defaults**, $\mathrm{E}_t[N_s]$.

Index CDS: Loss Process and Protection Leg



Pool size is 100, total notional is 100

Index CDS: Index Spread

Since the CDS spread S_t is computed such that the initial value of the CDS is zero, it is obtained by

Fair Spread

$$S_t = \frac{V_t^{prot}}{V_t^{fee}}.$$

Next Goal: Express the index spread in terms of the single-name CDS spreads.

Index CDS: Bottom Up

We can express the current notional in term of the individual default stopping times:

$$F_t = 1 - N_t/I = 1 - \sum_{i=1}^{I} \mathbf{1}_{\{\tau_i \le t\}}/I = 1 - \sum_{i=1}^{I} (1 - \mathbf{1}_{\{\tau_i > t\}})/I = \sum_{i=1}^{I} \mathbf{1}_{\{\tau_i > t\}}/I$$

Therefore,

$$V_t^{fee} = \sum_{j=1}^{n} \delta \mathsf{E}_t \left[\frac{F_{t_j}}{B(t, t_j)} \right] = \frac{1}{I} \sum_{i=1}^{I} \sum_{t_j > t} \delta \mathsf{E}_t \left[\frac{\mathbf{1}_{\{\tau_i > t_j\}}}{B(t, t_j)} \right] = \frac{1}{I} \sum_{i=1}^{I} V_t^{fee,i}$$

Value of the Fee Leg (Bottom Up)

The fee leg of an index CDS is the average of the fee legs of the single-name CDS contracts.

Index CDS: Bottom Up

The protection leg can also be expressed in terms of the individual default stopping times.

Value of the Protection Leg (Bottom Up)

$$V_t^{prot} = \sum_{i=1}^{I} \mathrm{E}_t \left[\frac{\ell_i \, \mathbf{1}_{\{t<\tau_i \le T\}}}{B(t,\tau_i)} \right],$$

where τ_i is the default stopping time of the i-th firm.

- **Warning.** This is not exactly the sum over the protection legs of the single-name CDS contracts since ℓ_i can be at most $1/I$ (not one).
- Therefore, multiplying by I yields the sum over the prot. legs

$$I \cdot V_t^{prot} = \sum_{i=1}^{I} \mathrm{E}_t \left[\frac{I \ell_i \, \mathbf{1}_{\{t<\tau_i \le T\}}}{B(t,\tau_i)} \right] = \sum_{i=1}^{I} V_t^{prot,i}$$

Index CDS: Bottom Up

Using these insights, we can rewrite the fair spread:

$$S_t = \frac{I \cdot V_t^{prot}}{I \cdot V_t^{fee}} = \frac{\sum_{i=1}^{I} V_t^{prot,i}}{\sum_{i=1}^{I} V_t^{fee,i}} = \frac{\sum_{i=1}^{I} S_t^i V_t^{fee,i}}{\sum_{i=1}^{I} V_t^{fee,i}}$$

Fair Spread (Bottom Up)

The fair index CDS spread is the weighted average over the single-name spreads

$$S_t = \sum_{i=1}^{I} w_i S_t^i,$$

where the weights are given by

$$w_i = \frac{V_t^{fee,i}}{\sum_{i=1}^{I} V_t^{fee,i}}.$$

Index CDS

- Theoretically, the index spread should be the weighted average of the single-name spreads as defined above.
- Practically, there might be a difference.
- This is said to be the **index basis**.
- There are several reasons why this can happen:
  - Different **liquidity** of the contracts
  - Different **contractual specifications** (settlement, credit events)

Index CDS: Mark-to-Market

- Assume that an investor bought protection on an index at time 0 for a spread S_0.
- At time t we usually have

$$V_t^{prot} - S_0 V_t^{fee} \neq 0.$$

- Since $V_t^{prot} = S_t V_t^{fee}$, we get the following result:

Mark-to-market of Index CDS

The time-t value of an index CDS long position initiated at time 0 reads

$$S_t V_t^{fee} - S_0 V_t^{fee} = V_t^{fee}\big(S_t - S_0\big).$$

Index CDS: Quotes

- In practice, index CDS contracts have a given **coupon** (like bonds) that is not continuously adjusted.
- At initiation, index CDS contracts are **launched at par**, i.e. the coupon is equal to the spread.
- Over time the spread and thus the price of a CDS contract changes.
- Therefore, an investor has to make an **upfront payment** to account for the movement in the spreads.
- Denoting the coupon by C the upfront payment is given by

$$V_t^{fee}(S_t - C)$$

Basket CDS

- **OTC**-traded credit derivatives
- **Underlying** of an n-to default basket: **pool** of defaultable bonds, loans etc
- **Protection payment** is triggered if the n-th entity in the pool defaults
- Default **correlations** play a **crucial** role for the pricing of a basket CDS
- Specifically, contagion effects influence its value significantly.

Basket CDS: Legs

- A basket CDS consists of **two legs** (fee and protection leg).
- During the lifetime of a CDS the buyer of a basket CDS pays a fee for a protection against a **default of the *n*-th entity** to the protection seller.
- This **fee** is **paid quarterly or semiannually in arrear** and is fixed at the time when the basket CDS is issued.
- It is chosen such that the **initial value** of the basket CDS is **zero**.
- This fee payment stops at the maturity of the CDS or at the default of the n-th entity, whichever occurs first.
- If ***n*-th default** occurs during the lifetime of the basket CDS, the **protection buyer** is **compensated for the loss** that is associated with this default.

Basket CDS: Notation

- We consider a basket CDS which **starts** at time t and has a **maturity** of T.
- The time-t **spread** of an n-to-default basket CDS is denoted by $S_t = S_t(T, n)$.
- During the lifetime of the basket CDS **fee payments** are made at times t_j, $j = 1, \ldots, n$ if default has not occurred before t_j.
- Note that $T = t_n$.
- Payments are made at equidistant points in time, i.e. $\delta = t_j - t_{j-1}$ for all $j = 1, \ldots, n$.
- Since we look at spot contracts, we have $t_0 = t$.
- The **notional** is normalized to **one**.
- The **time of the n-th default** is given by the stopping time T_n.

Basket CDS: Payment Streams

No Default until Maturity

Time	0	0.5	1	...	$T - 0.5$	T
Fee Leg	0	δS_0	δS_0	...	δS_0	δS_0
Protection Leg	0	0	0	...	0	0

Default before Maturity

Time	0	0.5	1	...	t_{j-1}	T_n	...	T
Fee Leg	0	δS_0	δS_0	...	δS_0	$(T_n - t_{j-1})S_0$	0	0
Protection Leg	0	0	0	...	0	ℓ_n	0	0

ℓ_n is the loss associated with the n-th default.

Basket CDS: Fee Leg

- The fee payment at time t_j is given by

$$S_t \, \delta \, \mathbf{1}_{\{T_n > t_j\}},$$

where S_t denotes the CDS spread with maturity T which is contracted today.

- We **disregard** accrued fees since contribution to PV negligible

Value of the Fee Leg

The value of the fee leg per 1bp of fee payments reads

$$V_t^{fee} = \sum_{j=1}^{n} \delta \mathrm{E}_t \left[\frac{\mathbf{1}_{\{T_n > t_j\}}}{B(t, t_j)} \right]$$

where $B(t,s) = e^{\int_t^s r(s)\, ds}$.

Basket CDS: Fee Leg

Sometimes it is assumed that the fee is paid continuously. In this case the value of the fee leg is given by

Fee Leg (Continuous Payment)

$$V_t^{fee} = \mathrm{E}_t\left[\int_t^T \frac{\mathbf{1}_{\{T_n > s\}}}{B(t,s)}\,ds\right] = \int_t^T \mathrm{E}_t\left[\frac{\mathbf{1}_{\{T_n > s\}}}{B(t,s)}\right] ds.$$

This simplification has the advantage that accrued fees are avoided.

Basket CDS: Protection Leg and Fair Spread

In any case, the value of the protection leg is given by

Value of the Protection Leg

$$V_t^{prot} = \ell \, \mathrm{E}_t \left[\frac{\mathbf{1}_{\{t_0 \le T_n \le T\}}}{B(t, T_n)} \right].$$

where ℓ denotes the loss which is assumed to be constant.

Since the CDS spread S_t is computed such that the initial value of the CDS is zero, it is obtained by

Basket CDS Spread

$$S_t = \frac{V_t^{prot}}{V_t^{fee}}.$$

Basket CDS

- At several places, we have expressions of the form

$$\mathrm{E}_t\left[\frac{\mathbf{1}_{\{T_n>s\}}}{B(t,s)}\right]$$

- If default-free interest rates and default risk are independent, we obtain

$$Q_t(T_n > s)p(t,s)$$

- Therefore, the following **probability** is **crucial**

$$Q_t(T_n > s) = Q_t(N_s < n)$$

- This shows that the value of a basket CDS significantly depends on the **default correlations** of the entities.

Basket CDS vs. Single-name CDS

- The sum over all n-to-default spreads has to be equal to the sum over all single-name CDS spreads.
- Therefore, we obtain the following result:

Portfolios of CDS Contracts

$$S(T,1)+S(T,2)+\cdots+S(T,I)=S^1(T)+S^2(T)+\cdots+S^I(T),$$

where $S^i(T)$ denotes the single-name CDS spread of the i-th firm and $S(T,n)$ denotes the spread of the n-to-default basket.

Nevertheless, it is not easy to hedge a particular basket CDS via single-name CDS.

CDO

- The **underlying** of a CDO is a pool of loans, CDS contracts etc.
- The pool is sliced into **tranches**.
- The **cash flows** generated by the pool are used to service the tranches according to the seniority (waterfall).
- Therefore, the holder of a tranche receives interest payments, but has to cover losses that are attributed to the tranche.
- Like Credit Defaut Swaps, a CDO tranche can thus be split up into **two legs**:
  - **Fee leg**: Interest rate payments
  - **Protection leg**: Payments upon default

CDO: Contractual Specifications

- The **underlying** is a pool of I corporate bonds
- The **tranches** of the CDO are modeled by the sequence $0 = K_0 < K_1 < \cdots < K_M = 1$ of **attachment and detachment points**.
- Thus the **equity tranche** is characterized by the interval $[K_0, K_1]$ and the **mezzanine tranche** by the interval $[K_1, K_2]$.
- CDX: $K_1 = 0.03$, $K_2 = 0.07$, $K_3 = 0.1$, $K_4 = 0.15$, $K_5 = 0.3$, $K_6 = 1$.
- Some notation

N_i	:	Notional of bond i
$\sum_{i=1}^{I} N_i$	:	Total notional of the pool
$F_m = (K_m - K_{m-1}) \sum_{i=1}^{I} N_i$	:	Face value of tranche m

- **Assumption.** **Notionals** are **identical**, i.e. $N_i = N_j = N$, and the **notional of** the **portfolio** is **one**, i.e. $\sum_{i=1}^{I} N_i = 1$

CDO: Contractual Specifications

- Initially, each tranche $m = 1, \ldots, M$ pays a fixed **coupon**.
- **Coupon dates**: $0 = t_0 < t_1 < \cdots < t_K = T$ with $\delta = t_k - t_{k-1} = const$
- Quarterly paid coupons ($\delta = 1/4$) and an interest per annum of $s^m = 5\%$ (for tranche m), for example, yield a coupon value of $0.05/4$.
- If the tranche has already been affected by subsequent defaults, then we adjust the coupon payments by a **thinning factor** (in %)

$$\theta_t^m \equiv \begin{cases} 1 & \text{if } L_t \leq K_{m-1}, \\ \frac{K_m - L_t}{K_m - K_{m-1}} & \text{if } K_{m-1} < L_t < K_m, \\ 0 & \text{otherwise} \end{cases}$$

where L_t denotes the **aggregated loss** at time t.

Example: Pool of 50 Loans with Notionals of 2m Dollars

- Three tranches (equity, mezzanine, senior): $K_1 = 0.03$, $K_2 = 0.06$, $K_3 = 1$.
- Face values of the tranches: $F_1 = 3$, $F_2 = 3$, $F_3 = 94$.
- Assume that all recovery rates are equal to $R = 0.4$.
- Therefore, the thinning factors θ_t^m, $m \in \{E, M, S\}$, are given by

# Defaults	0	1	2	3	4	5	6	7	...
θ_t^E	1	$\frac{1.8}{3}$	$\frac{0.6}{3}$	0	0	0	0	0	...
θ_t^M	1	1	1	$\frac{2.4}{3}$	$\frac{1.2}{3}$	0	0	0	...
θ_t^S	1	1	1	1	1	1	$\frac{92.8}{94}$	$\frac{91.6}{94}$	...

CDO: Loss Process

- At the **default time** τ_i of firm i, there is a **loss** of $\ell_i = (1 - R_i)N_i$ (expressed in dollars).
- The **total loss** until time t is given by

$$L_t \equiv \sum_{i=1}^{I} \ell_i \mathbf{1}_{\{\tau_i \le t\}}$$

- We are also interested in the **tranche loss** of tranche m at time t, which is "**one minus thinning factor**"

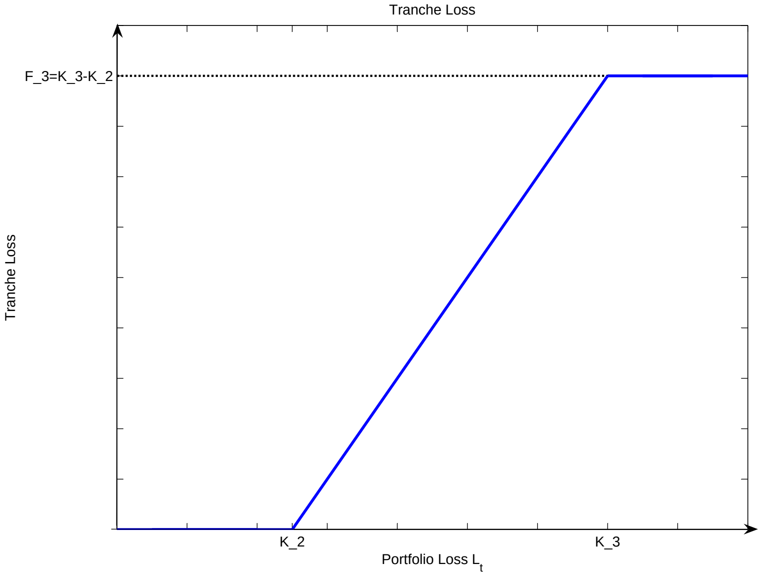
Relative Tranche Loss

The relative loss of the tranche $[K_{m-1}, K_m]$ can be written as

$$L_t^m \equiv \frac{\max\{L_t - K_{m-1}; 0\} - \max\{L_t - K_m; 0\}}{K_m - K_{m-1}}.$$

Notice that $L_t^m = \left(\min(L_t, K_m) - K_{m-1}\right)^+ / (K_m - K_{m-1})$.

Absolute Tranche Loss: Mezzanine



CDO: Fee Leg

Fee Leg

The value of the fee leg of tranche m per 1bp of fee payments is given by (disregarding accrued interest)

$$V_t^{f,m} = \mathrm{E}_t\Big[\sum_{t_k > t} \delta(1 - L_{t_k}^m) e^{-\int_t^{t_k} r_u\, du}\Big]$$

CDO: Protection Leg

Protection Leg

The value of the protection leg of tranche m is given by

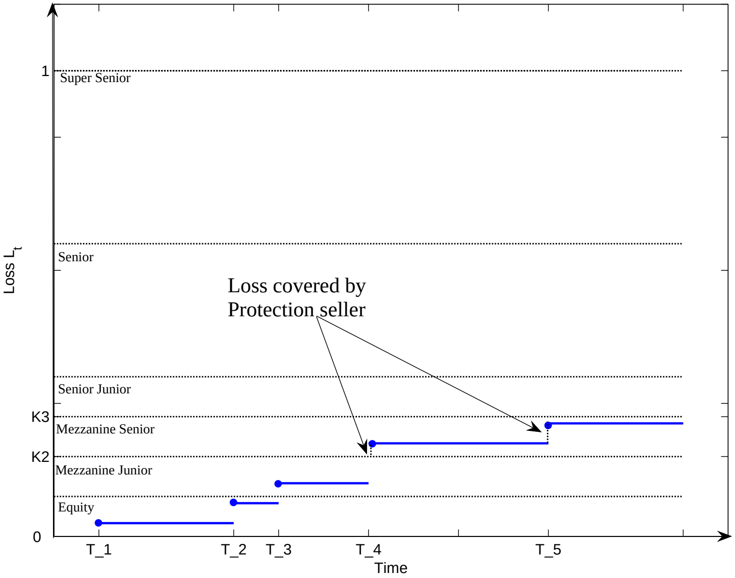
$$V_t^{p,m} = \mathrm{E}_t \left[\int_t^T e^{-\int_t^s r_u \, du} dL_s^m \right]$$

Fair Spread

The fair spread is given by

$$s_t^m = \frac{V_t^{p,m}}{V_t^{f,m}}$$

CDO: Protection Legs



CDO Legs

- The fee leg can be rewritten as

$$V_t^{f,m} = \delta \sum_{t_k > t} p(t, t_k)(1 - \mathrm{E}_t[L_{t_k}^m]).$$

- For simplicity, it is sometimes assumed that defaults occur in the **middle** of the period between two payments (see, e.g. Hull and White (2006)).
- In this case, the protection leg can be rewritten as

$$V_t^{p,m} = \sum_{t_k > t} p\left(t, \tfrac{t_{k-1}+t_k}{2}\right)\left(\mathrm{E}_t[L_{t_k}^m] - \mathrm{E}_t[L_{t_{k-1}}^m]\right).$$

- **Challenges.** Calculate the expected percentage tranche losses $\mathrm{E}_t[L_t^m]$.

- So far we have disregarded **accrual payments**. Assuming that default occurs in midway of two coupon dates, the **accrual leg** $V^{a,m}$ can be approximated by

$$V_t^{a,m} = 0.5\delta \sum_{t_k > t} p\left(t, \tfrac{t_{k-1}+t_k}{2}\right) \left(\mathrm{E}_t[L^m_{t_k}] - \mathrm{E}_t[L^m_{t_{k-1}}]\right).$$

- In this case, the fair spread becomes $s_t^m = \frac{V_t^{p,m}}{V_t^{a,m} + V_t^{f,m}}$.
- Some tranches (typically the equity tranche) are sometimes quoted in terms of an **upfront payment** u^m and a fixed **running spread** $s^{m,fix}$. Then, the fee leg can be written as

$$\tilde{V}_t^{f,m} = u^m + \delta s_t^{m,fix} \sum_{t_k > t} p(t, t_k)(1 - \mathrm{E}_t[L^m_{t_k}]),$$

and we have to compute u^m.

Agenda: Multi-Name Credit

Loss Distribution: Expected Losses

- In practice, **contingent claims** on loss distributions are traded (e.g. tranches of CDOs).
- For instance, the percentage loss on the $K_m - K_{m-1}$ **tranche** (e.g. $K_{m-1} = 3\%$ and $K_m = 7\%$) is given by

$$\frac{\max(0, L_t - K_{m-1}) - \max(0, L_t - K_m)}{K_m - K_{m-1}}$$

- Therefore, we are interested in calculating "truncated" means such as

$$\mathrm{E}[L_t \mathbf{1}_{\{L_t \leq K\}}],$$

which, for instance, are relevant for calculating the loss of an equity tranche (**first loss piece**).

Loss Distribution: Expected Losses

Expected Truncated Losses under LHP

We have

$$E[L_t \mathbf{1}_{\{L_t \leq K\}}] = N_2\left(N^{-1}(p), \psi(K), -\sqrt{\rho}\right),$$

where $\psi(K) \equiv N^{-1}(F_L^{\infty}(K; p, \rho))$ and $N_2(\cdot, \cdot, -\sqrt{\rho})$ is the cdf of the bivariate standard normal distribution with correlation $-\sqrt{\rho}$.

- This is a remarkable result since it allows us to calculate tranche losses of CDOs explicitly.
- As a special case, we get for $K = 1$

$$E[L_t] = E[L_t \mathbf{1}_{\{L_t \leq 1\}}] = N_2\left(N^{-1}(p), \infty, -\sqrt{\rho}\right) = p$$

Understanding the Form of the Expected Truncated Losses

Notice that $\psi(\ell) = \left(\sqrt{1-\rho}N^{-1}(\ell) - N^{-1}(p)\right) / \sqrt{\rho}$ and consider

$$
\begin{aligned}
\mathrm{E}[L_t \mathbf{1}_{\{L_t \leq K\}}] &= \int_0^K \ell f_L^\infty(\ell)\, d\ell = \int_0^K \ell n(\psi(\ell))\psi'(\ell)\, d\ell \\
&= \int_{\psi(0)}^{\psi(K)} \psi^{-1}(z) n(z) \underbrace{\psi'(\psi^{-1}(z))(\psi^{-1})'(z)}_{=1}\, dz \\
&= \int_{-\infty}^{\psi(K)} \psi^{-1}(z) n(z)\, dz \\
&= \int_{\infty}^{-\psi(K)} \psi^{-1}(-x) n(x)(-1)\, dx \\
&= \int_{-\psi(K)}^{\infty} N\left(\frac{-x\sqrt{\rho} + N^{-1}(p)}{\sqrt{1-\rho}}\right) n(x)\, dx
\end{aligned}
$$

Understanding the Form of the Expected Truncated Losses

From the last slide:

$$\mathrm{E}[L_t \mathbf{1}_{\{L_t \le K\}}] = \int_{-\psi(K)}^{\infty} N\,(a + bz)\, n(z)\, dz,$$

where $a \equiv N^{-1}(p)/\sqrt{1-\rho}$ and $b \equiv -\sqrt{\rho}/\sqrt{1-\rho}$. Now, define $y \equiv -bz + u$, where u is standard normal and independent from z. Conditional on z the variable y has mean $-bz$ and variance 1 implying $P(y \le a | z) = N(a + bz)$ and thus

$$\int_{-\psi(K)}^{\infty} N\,(a + bz)\, n(z)\, dz = P(y \le a, z \ge -\psi(K)).$$

Since $SD[y] = \sqrt{1+b}$ and $\mathrm{Corr}[y, -z] = b/\sqrt{1+b^2}$, we get

$$P(y \le a, z \ge -\psi(K)) = P(y \le a, -z \le \psi(K)) = N_2\left(\tfrac{a}{\sqrt{1+b^2}}, \psi(K), \tfrac{b}{\sqrt{1+b^2}}\right),$$

which gives the desired result.

Building Block: Loss Protection

- To compute the value of CDO tranches, we need to calculate **call-like payoffs**.
- This can be done using our above results:

$$
\begin{aligned}
\mathrm{E}[(L_t - K)^+] &= \mathrm{E}[L_t \mathbf{1}_{\{L_t \geq K\}}] - KP(L_t \geq K) \\
&= \mathrm{E}[L_t] - \mathrm{E}[L_t \mathbf{1}_{\{L_t \leq K\}}] - K(1 - F_L^\infty(K, p, \rho)).
\end{aligned}
$$

- Due to the symmetry $F_L^\infty(1 - K, 1 - p, \rho) = 1 - F_L^\infty(K, p, \rho)$, we arrive at

Loss Protection under LHP

$$
\mathrm{E}[(L_t - K)^+] = p - N_2\left(N^{-1}(p), \psi(K), -\sqrt{\rho}\right) - KF_L^\infty(1 - K, 1 - p, \rho)
$$

Building Block: Loss Protection

Using similar arguments as above one can derive **equivalent representations**

$$E[(L_t - K)^+] = N_2\left(-N^{-1}(K), N^{-1}(p), -\sqrt{1-\rho}\right)$$

or

$$E[(L_t - K)^+] = p - N_2\left(N^{-1}(p), N^{-1}(K), -\sqrt{1-\rho}\right).$$

The second representation can be found in Vasicek (2002).

Pricing CDOs: Standing Assumptions

- We assume the existence of a **risk-neutral pricing measure**. All expectations and probabilities are with respect to this measure.
- For $t \in [0, T]$, we take the **single-name default probabilities** $Q(\tau_i < t)$ as **given**, $i = 1, ..., I$. These probabilities can usually be bootstrapped from CDS quotes.
- Under the risk neutral probability measure, **credit** risk and **default-free** interest rates are **independent**.
- We have a **homogeneous pool** of firms.
- Firms' **default intensities** are **constant**.

Gaussian Copula Model

Main Task

Compute the expected loss of a specific tranche.

- Let λ be the constant default **intensity**, R the **recovery** rate, and ρ the **correlation** between two particular assets.
- Therefore, the **cumulative default probability** of firm i up to time t is given by $q(t) \equiv Q(\tau_i < t) = 1 - \exp(-\lambda t)$.
- Now, one can apply two models
  1. **Finite Homogeneous Pool Model**
  2. **Large Homogeneous Pool Model**

Explicit Representation

- By assumption, all firms have the **same recovery** R and thus the portfolio loss can be expressed as $L_t = \sum_{i=1}^{I}(1-R)\mathbf{1}_{\{\#defaults_t \geq i\}}$.
- Therefore, the **expected loss** at time t is

$$\mathrm{E}[L_t] = (1-R)\sum_{i=1}^{I} Q(\#defaults_t \geq i)$$

- The **loss of tranche m** is

$$\mathrm{E}[L_t^m] = \mathrm{E}\left[(\min(L_t, K_m) - K_{m-1})^+\right]/(K_m - K_{m-1})$$

- Using this representation of the expected losses one can **calculate** the **fair spreads** of the given tranches.

Example: CDO Pricing

- As a numerical illustration, let us consider the case

$$r = 0.05,\ \Delta t = 0.25,\ T = 5,\ I = 125,\ N = 8000,\ R = 40\%.$$

- Furthermore, the tranches are characterized by the following **attachment and detachment points** $K_0 = 0,\ K_1 = 0.03,\ K_2 = 0.06,\ K_3 = 0.09,\ K_4 = 0.12,\ K_5 = 0.22$, and $K_6 = 1$.
- This gives the following **tranche sizes**

tranche	attachment	detachment	tranche size F_m
Equity	0%	3%	30,000
Mezz Jun	3%	6%	30,000
Mezz Sen	6%	9%	30,000
Senior Jun	9%	12%	30,000
Senior	12%	22%	100,000
Super Senior	22%	100%	780,000

Tranche sizes for different attachment and detachment points.

Example continued

- To calculate the **single default probabilities**, we use CDS spreads.
- Due to the homogeneous-pool assumption, the **index spread** can be used.
- Consider $CDS_{index} = 100\text{bp}$. Then, we can use the formula $CDS_{index} = \lambda(1 - R)$ to get $\lambda = 166.67\text{bp}$. This gives us the single default probabilities $q(t)$ for all t.
- Applying the derived formulas, we get the following **fair spreads**

tranche	attachment	detachment	spread s^m
Equity	0%	3%	29.49%
Mezz Jun	3%	6%	963.56 bp
Mezz Sen	6%	9%	441.95 bp
Senior Jun	9%	12%	218.69 bp
Senior	12%	22%	59.98.bp
Super Senior	22%	100%	0.79 bp

Fair spreads of different tranches for a correlation of 20%.

Expected Tranche Losses

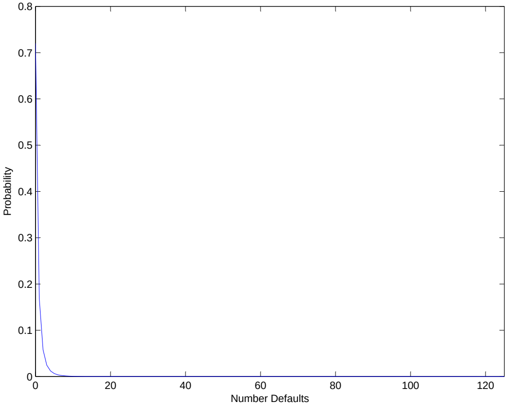
The following tabular displays the expected **percentage** tranche losses for coupon dates t_k.

date	Equity	Mezz Jun	Mezz Sen	Sen Jun	Sen	Sup Sen.
0.25	8.01%	0.26%	0.03%	0.01%	0%	0%
0.5	15.25%	1.10%	0.18%	0.04%	0%	0%
0.75	21.77%	2.41%	0.49%	0.12%	0.01%	0%
1	27.65%	4.06%	0.96%	0.27%	0.04%	0%
1.25	32.98%	5.96%	1.57%	0.48%	0.07%	0%
1.5	37.82%	8.06%	2.33%	0.76%	0.12%	0%
$\vdots$					$\vdots$	
4	68.70%	31.74%	14.90%	7.13%	1.81%	0.02%
4.25	70.63%	34.01%	16.44%	8.05%	2.10%	0.02%
4.5	72.43%	36.22%	18.00%	9.02%	2.43%	0.03%
4.75	74.10%	38.38%	19.57%	10.02%	2.77%	0.03%
5	75.66%	40.48%	21.16%	11.05%	3.15%	0.04%

Expected tranche losses for a correlation of 20%.

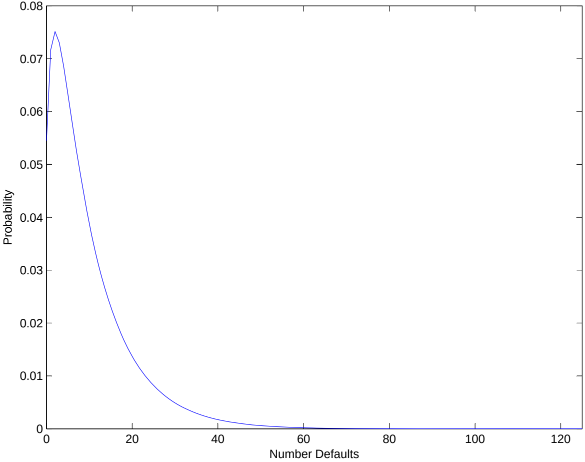
Probability Distributions

The **unconditional probabilities** that k firms default are depicted in the following graphs



Default probabilities at the first coupon date at $t = 0.25$.

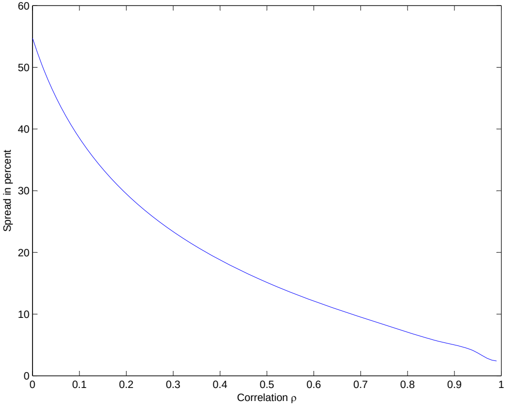
Probability Distributions



Default probabilities at the last coupon date at $t = 5$.

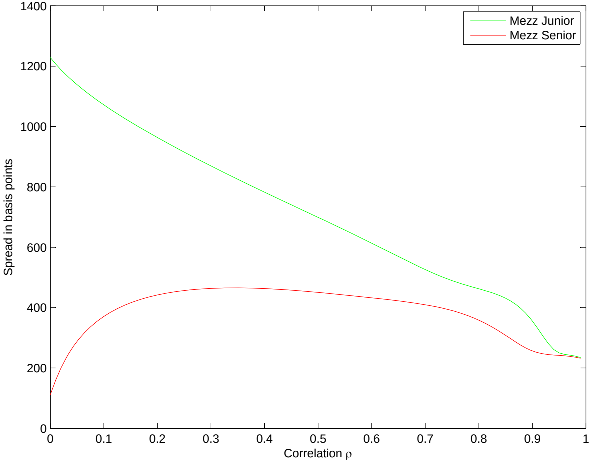
Contagion Effects: Equity

Via the correlation parameter ρ, we can include **contagion** effects in our model. The choice of ρ is crucial to the spread value.



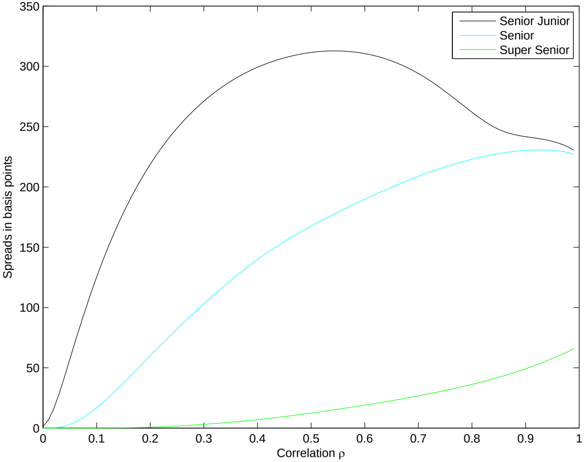
Fair spreads (in percent) for the equity tranche when varying the correlation parameter.

Contagion Effects: Mezzanine



Fair spreads (in basis points) for the mezzanine tranches when varying the correlation parameter.

Contagion Effects: Senior



Fair spreads (in basis points) for the senior tranches when varying the correlation parameter.

Base Correlation

- It is market practice to quote the prices of CDO (index) tranches in terms of implied copula correlations.
- If we fix a recovery rate, then one can try to back out the implied correlation of a one-factor Gaussian copula model.
- These are the so-called **tranche implied correlations**.
- **Problem:** Prices are linear in the correlation for equity tranches, but not for mezzanine tranches.
- **Idea:** Quote prices of artificial equity tranches $[0, K_n]$ in terms of their implied correlations.
- These are the so-called **base correlations**.
- The method is similar to calculating implied volatilities in the option market.
- We observe **correlation skews**, which suggests that the one-factor model is too simplistic.

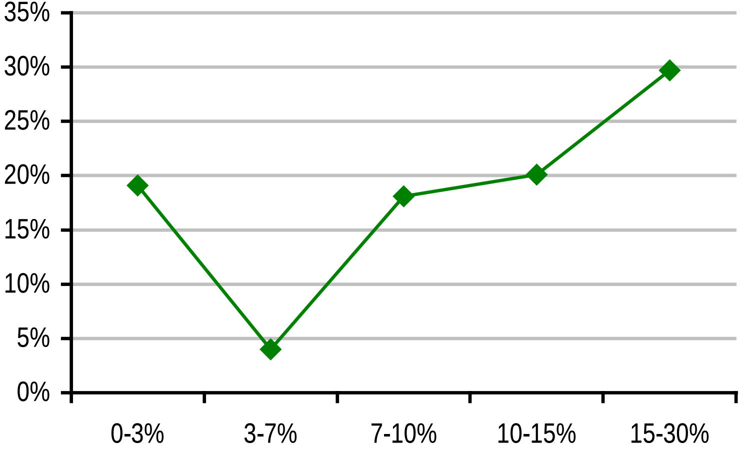
Base Correlation vs. Tranche Implied Correlation

North America (CDX.NA.IG3)

Std. Tranche	Bid	Ask	Mid	Tranche Correlation	Base Tranches	Base Correlation
0-3%	34.25%	35.25%	34.75	19.1%	0-3%	19.1%
3-7%	221	227	224	4.1%	0-7%	29.6%
7-10%	87	91	89	18.1%	0-10%	34.2%
10-15%	28	33	30.5	20.1%	0-15%	43.3%
15-30%	9.25	10.25	9.75	29.7%	0-30%	66.5%

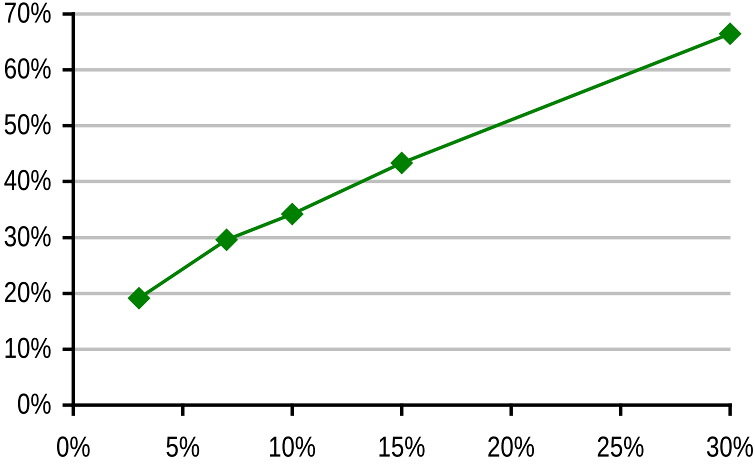
Base correlations and tranche implied correlations for standardized tranches. All 0-3% tranches quoted on upfront + 500bps running basis. (Source: Merrill Lynch)

Tranche Implied Correlation



Tranche implied correlations: CDX.NA.IG (Source: Merrill Lynch)

Base Correlation



Base correlations: CDX.NA.IG (Source: Merrill Lynch)