# Credit Markets

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# Agenda: Single-Name Credit


1. Introduction
2. Model-free Results for Corporate Bonds
3. Toolbox for Default Risk
4. Pricing of Corporate Bonds and CDS in a Simple Model
5. Pricing Defaultable Bonds with Stochastic Intensity
6. Pricing CDS
7. CDS Derivatives
8. Firm Value Models

# Simple Model

## Assumptions in the Simple Model

We first assume that $r$ is constant and that $Q(\tau > t) = e^{-\lambda t}$ with constant intensity $\lambda$.

- Notice that the independence assumption holds if interest rates are constant.
- Whereas assuming constant interest rates is not such an issue, assuming a constant intensity is restrictive.

# Building Blocks

Most defaultable claims are a blend of the following ingredients:

1. **Payment at a fixed time point** if no default has occurred
   - Examples: Coupon or notional payment of defaultable bond, vulnerable option
2. **Continuous stream** of payments until default or maturity
   - Examples: Fee leg of CDS, coupon payments of defaultable bond
3. Recovery **payment at** the **default** time $\tau$.
   - Examples: Recovery payment of bond or CDS contract

In our simple model, one can calculate the values of these building blocks explicitly.

# Formalizing the Building Blocks

Formally, the payoffs of the building blocks are given by:

1. **Payment at a fixed time point** $t_k$ or $T$ if no default occurs until time $t_k$ or $T$.
   - Payoff: $c_k \mathbf{1}_{\{\tau > t_k\}}$ or $\mathbf{1}_{\{\tau > T\}}$
2. **Continuous stream** of payments until default or maturity
   - Payoff: $c \mathbf{1}_{\{\tau > t\}} dt$
3. Recovery **payment at** the **default** time $\tau$.
   - Payoff: $R_\tau \mathbf{1}_{\{\tau \leq T\}}$

# Pricing: Payment at a Fixed Time Point

Lump-sum coupon payment of $c_k$ at coupon date $t_k$

$$\mathrm{E}[e^{-\int_0^{t_k} r_s\,ds} c_k \mathbf{1}_{\{\tau > t_k\}}] = c_k e^{-r t_k} Q(\tau > t_k) = c_k e^{-(r+\lambda) t_k}$$

Notional payment at maturity $T$

$$\mathrm{E}[e^{-\int_0^{T} r_s\,ds} \mathbf{1}_{\{\tau > T\}}] = e^{-rT} Q(\tau > T) = e^{-(r+\lambda) T}$$

# Pricing: Continuous Stream of Payments

Continuous stream of payments until default or maturity whatever comes first

$$
\begin{aligned}
\mathrm{E}\Big[\int_0^T \mathbf{1}_{\{\tau>s\}} e^{-\int_0^s r_u\,du} ds\Big] &= \int_0^T \mathrm{E}[\mathbf{1}_{\{\tau>s\}} e^{-\int_0^s r_u\,du}] ds \\
&= \int_0^T Q(\tau > s) p(0,s) ds \\
&= \int_0^T e^{-\lambda s} e^{-rs} ds \\
&= \int_0^T e^{-(r+\lambda)s} ds \\
&= \frac{1}{r+\lambda}\Big(1 - e^{-(r+\lambda)T}\Big)
\end{aligned}
$$

Notice that the price of a perpetual defaultable payment is thus

$$
\frac{1}{r+\lambda}
$$

# Pricing: Constant Recovery Payment

Constant recovery payment at default $\tau$ if $\tau$ occurs before maturity

$$
\begin{aligned}
\mathrm{E}\left[e^{-\int_0^\tau r_s\,ds} R_\tau \mathbf{1}_{\{\tau\le T\}}\right] &= R\mathrm{E}\left[e^{-r\tau}\mathbf{1}_{\{\tau\le T\}}\right] \\
&= R\int_0^\infty e^{-rs} f(s)\mathbf{1}_{\{s\le T\}}\,ds \\
&= R\int_0^T e^{-rs} f(s)\,ds \\
&= R\int_0^T e^{-rs}\lambda e^{-\lambda s}\,ds \\
&= R\lambda\int_0^T e^{-(r+\lambda)s}\,ds \\
&= \frac{R\lambda}{r+\lambda}\left(1-e^{-(r+\lambda)T}\right)
\end{aligned}
$$

where $f$ is the density of $\tau$. Notice that the cdf of $\tau$ is

$$F(t) \equiv Q(\tau \le t) = 1 - e^{-\lambda t} \quad \Longrightarrow \quad f(t) = F'(t) = \lambda e^{-\lambda t}$$

# Application: Pricing of CDS

- Consider a CDS with maturity $T$
- The time-0 CDS spread is denoted by $S$
- During the lifetime of the CDS fee payments are made continuously if default has not occurred before $t$.
- The notional is normalized to one.
- The default time is modeled via a stopping time $\tau$.
- If a default occurs before maturity $T$, the protection buyer receives the loss $\ell$ from the protection seller
- We assume that $\ell$ is constant

**Definition: Fair Spread**

At time 0 the fair spread $S$ is fixed such that the PV of the fee payments (fee leg) is equal to the PV of the potential protection payment (protection leg)

# Application: Pricing of CDS

- The fee leg is an continuous payment stream until default or maturity whatever comes first
- Therefore, the **PV of the fee leg** is

$$\hat{V}^{fee} \equiv \mathrm{E}\Big[\int_0^T \mathbf{1}_{\{\tau > s\}} S e^{-rs} ds\Big] = \tfrac{S}{r+\lambda}\Big(1 - e^{-(r+\lambda)T}\Big)$$

- The protection leg is a recovery payment at default if default occurs before maturity
- Notice that the loss $\ell$ is recovered
- Therefore, the **PV of the protection leg**

$$V^{prot} \equiv \mathrm{E}\Big[e^{-\int_0^\tau r_s\, ds} \ell \mathbf{1}_{\{\tau \leq T\}}\Big] = \tfrac{\ell\lambda}{r+\lambda}\Big(1 - e^{-(r+\lambda)T}\Big)$$

- By definition of $S$, we must have $\hat{V}^{fee} = V^{prot}$ and thus

**Proposition: Fair CDS Spread in Simple Model**

The fair spread of a CDS contract is $S = \ell\lambda$

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# Deterministic Intensity and Spread

With zero recovery we get for all maturities $T$

$$\text{Spread}(t,T) \equiv y^d(t,T) - y(t,T) = \lambda = \mathrm{const}$$

where $p(t,T) = e^{-y(t,T)(T-t)}$ and $p^d(t,T) = e^{-y^d(t,T)(T-t)}$.

**1st problem:** Yield spreads are not constant over maturities.

**Quick fix:** Use a deterministic intensity $\lambda(t)$:

$$\text{Spread}(t,T) = \int_t^T \lambda(s)\,ds$$

**Additional problems:** Spreads are stochastic

**Moral**

We need a model with stochastic intensity

# Survey: Bond Prices vs. Survival Probabilities

| | Constant | Deterministic | Stochastic |
|---|---|---|---|
| Short rate | $r$ | $r(t)$ | $r_t$ |
| Bond price | $e^{-r(T-0)}$ | $e^{-\int_0^T r(s)\,ds}$ | $\mathrm{E}[e^{-\int_0^T r_s\,ds}]$ |
| Default intensity | $\lambda$ | $\lambda(t)$ | $\lambda_t$ |
| Survival probability | $e^{-\lambda(T-0)}$ | $e^{-\int_0^T \lambda(s)\,ds}$ | $\mathrm{E}[e^{-\int_0^T \lambda_s\,ds}]$ |

- The case with constant $r$ and $\lambda$ is our simple model
- The deterministic case is rarely used
- The stochastic case will be discussed in this section

# Building Blocks Again

Most defaultable claims are a blend of the following ingredients:

1. **Payment at a fixed time point** $t_k$ or $T$ if no default occurs until time $t_k$ or $T$.
   - Payoff: $c_k \mathbf{1}_{\{\tau > t_k\}}$ or $\mathbf{1}_{\{\tau > T\}}$
2. **Continuous stream** of payments until default or maturity
   - Payoff: $c \mathbf{1}_{\{\tau > t\}} dt$
3. Recovery **payment at** the **default** time $\tau$.
   - Payoff: $R_\tau \mathbf{1}_{\{\tau \leq T\}}$

# Independence vs. Dependence of Interest Rate and Default

We can distinguish two cases

1. $r$ and $\tau$ are stochastic, but independent
2. $r$ and $\tau$ are stochastic and dependent

The first case is very tractable and we start with

**Independence Assumption**

$r$ and $\tau$ can be stochastic, but are independent

- Later on we will discuss how to deal with the second case
- This leads to a Cox-process framework

# Pricing with Stochastic $\lambda$: Independence

**Payment at a fixed time point**

- **Lump-sum coupon payment** of $c_k$ at coupon date $t_k$

$$\begin{aligned} \mathrm{E}[e^{-\int_0^{t_k} r_s\,ds} c_k \mathbf{1}_{\{\tau > t_k\}}] &= c_k \mathrm{E}[e^{-\int_0^{t_k} r_s\,ds}]\mathrm{E}[\mathbf{1}_{\{\tau > t_k\}}] \\ &= c_k p(0, t_k) Q(\tau > t_k) \\ &= c_k p(0, t_k) \mathrm{E}[e^{-\int_0^{t_k} \lambda_s\,ds}] \end{aligned}$$

- **Notional payment** at maturity $T$

$$\begin{aligned} \mathrm{E}[e^{-\int_0^{T} r_s\,ds} \mathbf{1}_{\{\tau > T\}}] &= \mathrm{E}[e^{-\int_0^{T} r_s\,ds}]\mathrm{E}[\mathbf{1}_{\{\tau > T\}}] \\ &= p(0, T) Q(\tau > T) \\ &= p(0, T) \mathrm{E}[e^{-\int_0^{T} \lambda_s\,ds}] \end{aligned}$$

# Pricing with Stochastic $\lambda$: Independence

**Continuous stream of payments** until default or maturity whatever comes first

$$\mathrm{E}\Big[\int_0^T e^{-\int_0^s r_u\,du}\mathbf{1}_{\{\tau>s\}}ds\Big] = \int_0^T \mathrm{E}[e^{-\int_0^s r_u\,du}\mathbf{1}_{\{\tau>s\}}]ds$$

$$= \int_0^T \mathrm{E}[e^{-\int_0^s r_u\,du}]\mathrm{E}[\mathbf{1}_{\{\tau>s\}}]ds$$

$$= \int_0^T p(0,s)Q(\tau>s)ds$$

$$= \int_0^T p(0,s)\mathrm{E}[e^{-\int_0^s \lambda_s\,ds}]ds$$

# Pricing with Stochastic $\lambda$: Independence

**Const. recovery payment** at default $\tau$ if $\tau$ occurs before maturity

$$
\begin{aligned}
\mathrm{E}\Big[e^{-\int_0^\tau r_s\,ds} R_\tau \mathbf{1}_{\{\tau\leq T\}}\Big] &= R\mathrm{E}\Big[\mathrm{E}\big[e^{-\int_0^\tau r_s\,ds}\mathbf{1}_{\{\tau\leq T\}}|\lambda\big]\Big] \\
&= R\mathrm{E}\Big[\mathrm{E}\big[\int_0^\infty f(s|\lambda)e^{-\int_0^s r_u\,du}\mathbf{1}_{\{s\leq T\}}\,ds|\lambda\big]\Big] \\
&= R\mathrm{E}\Big[\int_0^T \lambda_s e^{-\int_0^s \lambda_u\,du} e^{-\int_0^s r_u\,du}\,ds\Big] \\
&= R\int_0^T E\Big[\lambda_s e^{-\int_0^s \lambda_u\,du} e^{-\int_0^s r_u\,du}\Big]\,ds \\
&= R\int_0^T E\Big[\lambda_s e^{-\int_0^s \lambda_u\,du}\Big] E\Big[e^{-\int_0^s r_u\,du}\Big]\,ds \\
&= R\int_0^T E\Big[\lambda_s e^{-\int_0^s \lambda_u\,du}\Big] p(0,s)\,ds
\end{aligned}
$$

where $f$ is conditional density of $\tau$. The conditional cdf of $\tau$ is

$$F(t|\lambda) \equiv Q(\tau\leq t|\lambda) = 1 - e^{-\int_0^t \lambda_s\,ds} \implies f(t|\lambda) = \lambda_t e^{-\int_0^t \lambda_s\,ds}$$

# Summary: Pricing with Stochastic $\lambda$ under Independence

## Proposition: Building Blocks

If $r$ and $\tau$ are independent, then we get:

- Notional payment

$$\mathrm{E}[e^{-\int_0^T r_s\,ds}\mathbf{1}_{\{\tau>T\}}] = p(0,T)\mathrm{E}[e^{-\int_0^T \lambda_s\,ds}]$$

- Continuous stream of defaultable payments

$$\mathrm{E}\Big[\int_0^T e^{-\int_0^s r_u\,du}\mathbf{1}_{\{\tau>s\}}ds\Big] = \int_0^T p(0,s)\mathrm{E}[e^{-\int_0^s \lambda_u\,du}]\,ds$$

- Constant recovery payment at default

$$\mathrm{E}\Big[e^{-\int_0^\tau r_s\,ds}R_\tau\mathbf{1}_{\{\tau\le T\}}\Big] = R\int_0^T p(0,s)\mathrm{E}\Big[\lambda_s e^{-\int_0^s \lambda_u\,du}\Big]\,ds$$

# Summary: Pricing with Stochastic $\lambda$ under Independence

We are thus interested in calculating

$$\mathrm{E}\left[e^{-\int_0^t \lambda_u \, du}\right] \quad \text{and} \quad \mathrm{E}\left[\lambda_t e^{-\int_0^t \lambda_u \, du}\right]$$

Notice that the second expression is related to the first:

$$\begin{aligned}\mathrm{E}\left[\lambda_t e^{-\int_0^t \lambda_u \, du}\right] &= \mathrm{E}\left[\partial_t \left(1 - e^{-\int_0^t \lambda_u \, du}\right)\right] = -\mathrm{E}\left[\partial_t e^{-\int_0^t \lambda_u \, du}\right] \\ &= -\partial_t \mathrm{E}\left[e^{-\int_0^t \lambda_u \, du}\right],\end{aligned}$$

i.e. we can just differentiate w.r.t. time $t$.

In **affine models** we obtain explicit representations!

# Affine Models

The canonical one-dimensional cases are

- Arithmetic Brownian motion (**Ho-Lee** model)

$$d\lambda_t = adt + bdW_t$$

  with $a$, $b$ constants.

- OU-process (**Vasicek** model)

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma dW_t$$

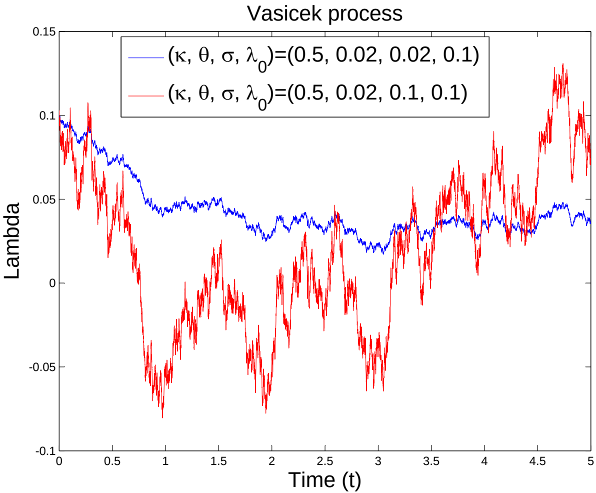
  with $\kappa$, $\theta$, and $\sigma$ constants.

- Square root process (**Cox-Ingersoll-Ross** model)

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}\, dW_t$$

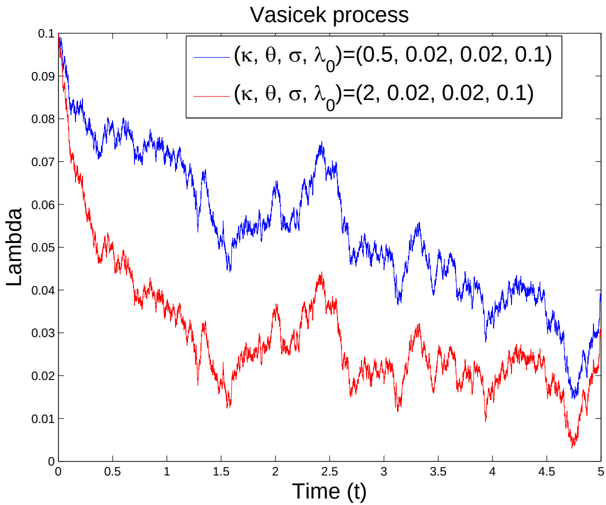
  with $\kappa$, $\theta$, and $\sigma$ constants.

# Vasicek Process



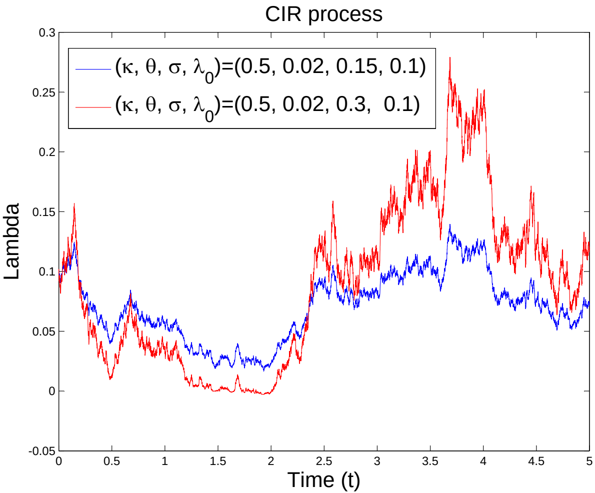

Paths of Vasicek processes with different volatilities.

# Vasicek Process



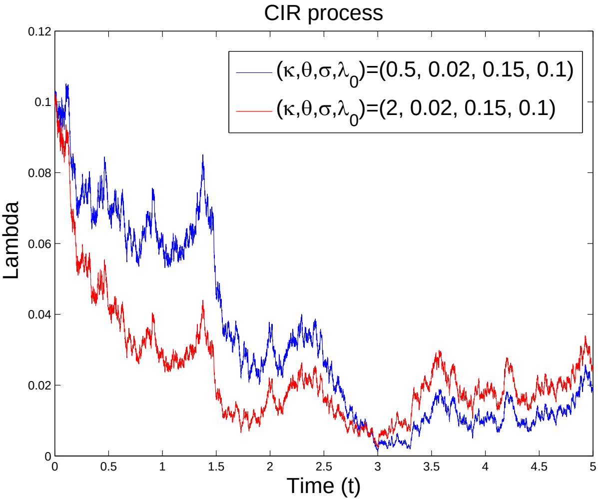

Paths of Vasicek processes with different mean-reversion speeds.

# CIR Process



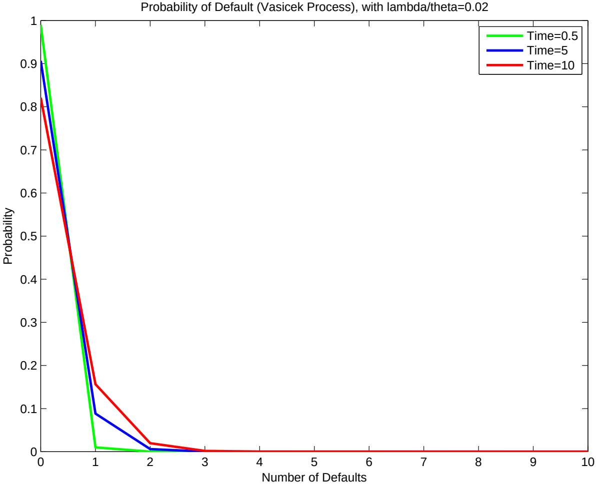

Paths of CIR processes with different volatilities.

# CIR Process



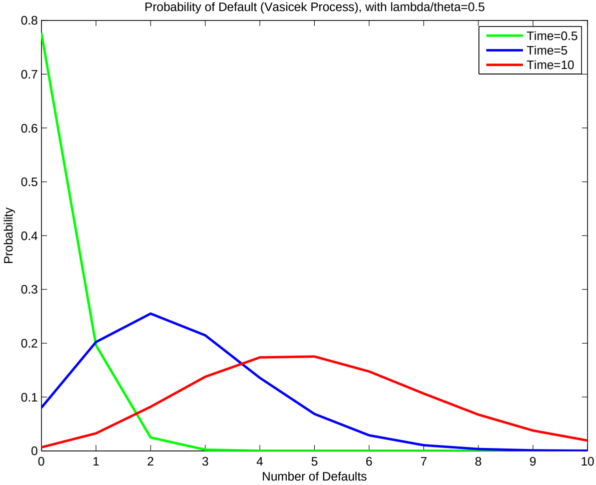

Paths of CIR processes with different mean-reversion speeds.

# Vasicek: Probability of $n$ Jumps



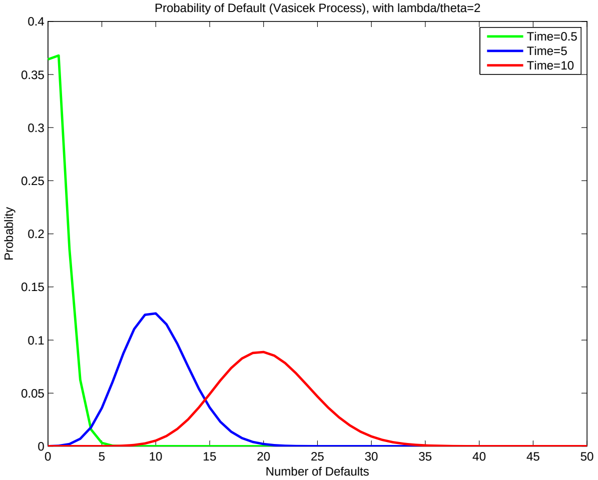

$\kappa = 0.5$, $\sigma = 0.02$

# Vasicek: Probability of $n$ Jumps



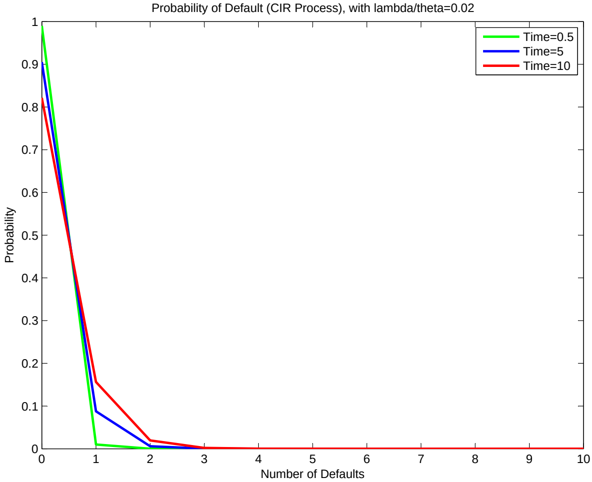

$\kappa = 0.5$, $\sigma = 0.02$

# Vasicek: Probability of $n$ Jumps



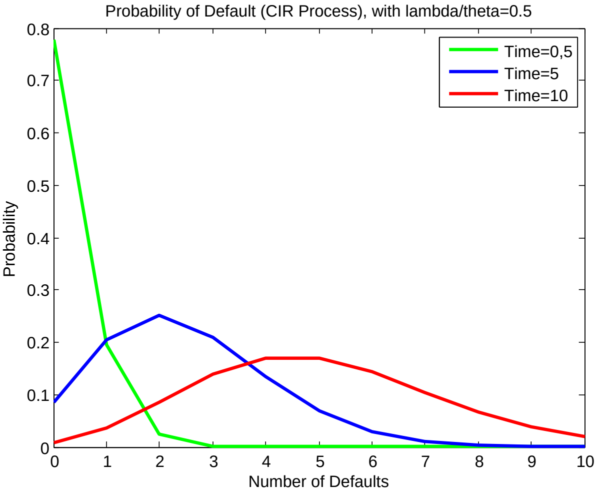

$\kappa = 0.5$, $\sigma = 0.02$

# CIR: Probability of $n$ Jumps



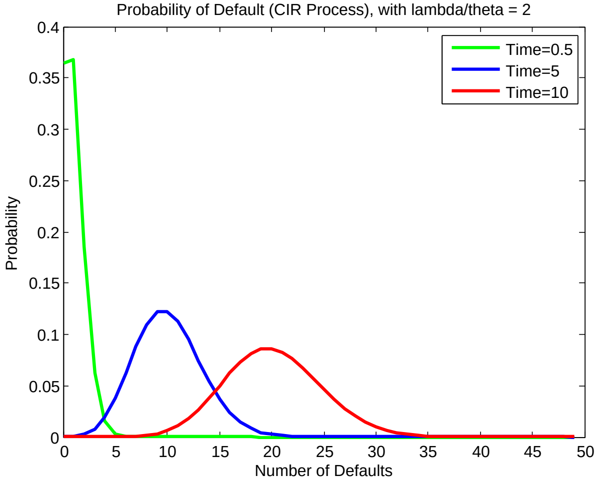

$\kappa = 0.5$, $\sigma = 0.15$

# CIR: Probability of $n$ Jumps




$\kappa = 0.5$, $\sigma = 0.15$

# CIR: Probability of $n$ Jumps




$\kappa = 0.5$, $\sigma = 0.15$

# Affine Models

For affine models, it is well-known from interest rate theory that

$$\mathrm{E}\left[e^{-\int_0^t \lambda_u \, du}\right] = e^{A(t)-B(t)\lambda_0}$$

**Ho-Lee model**

$$A(t) = -0.5at^2 + b^2 t^3/6, \quad B(t) = t.$$

**Vasicek model**

$$A(t) = \left(\theta - \frac{\sigma^2}{2\kappa^2}\right)(B(t) - t) - \frac{\sigma^2}{4\kappa}B^2(t), \;\; B(t) = \frac{1}{\kappa}(1 - e^{-\kappa t}).$$

**Cox-Ingersoll-Ross model.** Let $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$

$$e^{A(t)} = \left(\frac{2\gamma e^{(\kappa+\gamma)t/2}}{2\gamma + (\kappa+\gamma)(e^{\gamma t} - 1)}\right)^{\frac{2\kappa\theta}{\sigma^2}}, \; B(t) = \frac{2(e^{\gamma t} - 1)}{2\gamma + (\kappa+\gamma)(e^{\gamma t} - 1)}$$

# Affine Models

Now we obtain

$$
\begin{aligned}
E\left[\lambda_t e^{-\int_0^t \lambda_u\,du}\right] &= -\partial_t E\left[e^{-\int_0^t \lambda_u\,du}\right] = -\partial_t \left\{e^{A(t)-B(t)\lambda_0}\right\} \\
&= (-A'(t) + B'(t)\lambda_0)e^{A(t)-B(t)\lambda_0} \\
&= \left(C(t) + H(t)\lambda_0\right) e^{A(t)-B(t)\lambda_0},
\end{aligned}
$$

where we set $C = -A'$ and $H = B'$.

## Building Blocks

In an affine setting, we obtain

$$
\begin{aligned}
E\left[e^{-\int_0^t \lambda_u\,du}\right] &= e^{A(t)-B(t)\lambda_0}, \\
E\left[\lambda_t e^{-\int_0^t \lambda_u\,du}\right] &= \left(C(t) + H(t)\lambda_0\right) e^{A(t)-B(t)\lambda_0},
\end{aligned}
$$

where $A$ and $B$ can be calculated explicitly and $C = -A'$ and $H = B'$.

# Affine Models

**Ho-Lee model**

$$\begin{aligned} C(t) &= (at - 0.5b^2t^2), \\ H(t) &= 1. \end{aligned}$$

**Vasicek model**

$$\begin{aligned} C(t) &= -\kappa B(t)\left\{\frac{\sigma^2}{2\kappa}B(t) - \theta\right\} \\ H(t) &= e^{-\kappa t}. \end{aligned}$$

**Cox-Ingersoll-Ross model.** Let $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$.

$$\begin{aligned} C(t) &= \frac{-\frac{2\kappa\theta}{\sigma^2}(\kappa^2 - \gamma^2)(e^{\gamma t} - 1)}{4\gamma + 2(\kappa + \gamma)(e^{\gamma t} - 1)} \\ H(t) &= \frac{4\gamma^2 e^{\gamma t}}{[2\gamma + (\kappa + \gamma)(e^{\gamma t} - 1)]^2} \end{aligned}$$

# Log-normal Models

- Affine models are the workhorse of interest rate and default frameworks.
- The reason is that there are explicit solutions for several relevant building blocks.
- Nevertheless, some researchers also use log-normal models such as
$$d \ln(\lambda_t) = \kappa(\theta - \ln(\lambda_t))dt + \sigma dW_t.$$
- Since the logarithm of the intensity satisfies a Vasicek model, the intensity is log-normally distributed.

# Cox Process Setting

- Now, we consider a situation where $r$ and $\lambda$ can be correlated.
- For instance, $\lambda$ could be a function of $r$.
- In this case,

$$\mathrm{E}\left[e^{-\int_0^T r_t\,dt}\mathbf{1}_{\{\tau>T\}}\right] \neq p(0,t)Q(\tau > T)$$

- Formally, we assume that the default intensity is driven by process that characterizes the state of the economy.
- For instance, one component of this process can be $r$.

# Cox Process Setting

- $K$-dimensional state process

$$dY_t = \alpha(t, Y_t)\,dt + \sum_{k=1}^{K} \beta_k(t, Y(t))\,dW_t^k$$

where $W$ is a $K$-dimensional Brownian motion.

- Define $\mathcal{G}_t = \sigma\{Y_s, 0 \le s \le t\}$.
- Let $\lambda_t = \lambda(t, Y_t)$ be an intensity process adapted to $\mathcal{G}_t$.

**Cox Process**

A counting process $N$ with intensity $\lambda$ is a Cox process if, conditioned on $\mathcal{G}$, $N$ is a time-inhomogeneous Poisson process with intensity $\lambda$, i.e

$$P(N(T) = n \mid \mathcal{G}_T) = \frac{1}{n!}\left(\int_0^T \lambda(s)\,ds\right)^n e^{-\int_0^T \lambda(s)ds}$$

# Construction of Default Time

## Information Flow

- "Background information": $\mathcal{G}_t = \sigma\{Y_s, 0 \le s \le t\}$
- "Default information": $\mathcal{H}_t = \sigma\{N_s, 0 \le s \le t\}$
- "Full information": $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$

The first jump of $N$ can be simulated by

$$\tau \equiv \inf\left\{ t : \int_0^t \lambda_s \, ds \ge E_1 \right\},$$

where $E_1$ is a exponential random variable with mean 1 being independent of $\mathcal{G}_t$.

# Cox Process Setting: Defaultable Zero

Assume that the default-free short rate is a function of the state variables, i.e. $r_t = r(t, Y_t)$. Furthermore, set $\ell = 1$. Then

$$
\begin{aligned}
p^d(0,T) &= \mathrm{E}\left[\frac{\mathbf{1}_{\{\tau>T\}}}{B_T}\right] = \mathrm{E}\left[e^{-\int_0^T r(s,Y_s)ds}\mathbf{1}_{\{\tau>T\}}\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[e^{-\int_0^T r(s,Y_s)\,ds}\mathbf{1}_{\{\tau>T\}}\middle|\mathcal{G}_T\right]\right] \\
&= \mathrm{E}\left[e^{-\int_0^T r(s,Y_s)ds}Q(\tau>T\mid\mathcal{G}_T)\right] \\
&= \mathrm{E}\left[e^{-\int_0^T r(s,Y_s)+\lambda(s,Y_s)ds}\right]
\end{aligned}
$$

since $\exp(-\int_0^T r(s, Y_s)\,ds)$ is measurable with respect to $\mathcal{G}_t$ and $Q(\tau > T \mid \mathcal{G}_T) = \exp(-\int_0^T \lambda(s, Y_s)\,ds)$.

# Building Blocks Again

Most defaultable claims are a blend of the following ingredients:

1. **Payment at a fixed time point** $T$ if no default occurs until time $T$.
   - Examples: Notional payment of defaultable bond, vulnerable option
2. **Continuous stream** of payments until default occurs.
   - Examples: Fee leg of a CDS, coupon payments of a defaultable bond
3. Recovery **payment at** the **default** time $\tau$.
   - Examples: Recovery payment of a bond or a CDS contract

In Cox-process frameworks, one can calculate the values of these building blocks explicitly.

## Lando(1998)

Prior to default ($\tau > t$), we have for a $\mathcal{G}$-measurable process $X$

$$\mathrm{E}\left[e^{-\int_t^T r_s\,ds} X_T \mathbf{1}_{\{\tau > T\}} \middle| \mathcal{F}_t\right] = \mathrm{E}\left[e^{-\int_t^T r_s + \lambda_s\,ds} X_T \middle| \mathcal{G}_t\right]$$

$$\mathrm{E}\left[\int_t^T X_s \mathbf{1}_{\{\tau > s\}} e^{-\int_t^s r_u\,du}\,ds \middle| \mathcal{F}_t\right] = \mathrm{E}\left[\int_t^T X_s e^{-\int_t^s r_u + \lambda_u\,du}\,ds \middle| \mathcal{G}_t\right]$$

$$\mathrm{E}\left[e^{-\int_t^\tau r_s\,ds} X_\tau \mathbf{1}_{\{\tau \le T\}} \middle| \mathcal{F}_t\right] = \mathrm{E}\left[\int_t^T X_s \lambda_s e^{-\int_t^s r_u + \lambda_u\,du}\,ds \middle| \mathcal{G}_t\right]$$

# Understanding the Building Blocks

For simplicity, $t = 0$. Firstly, notice that

$$E[\mathbf{1}_{\{\tau \geq T\}} | \mathcal{G}_T] = e^{-\int_0^T \lambda_s \, ds}.$$

The **first** relation follows analogously as above.

The **second** one follows from

$$\begin{aligned}
E\Big[\int_0^T X_s \mathbf{1}_{\{\tau > s\}} e^{-\int_0^s r_u \, du} \, ds\Big] &= E\Big[E\Big[\int_0^T X_s \mathbf{1}_{\{\tau > s\}} e^{-\int_0^s r_u \, du} \, ds \Big| \mathcal{G}_T\Big]\Big] \\
&= E\Big[\int_0^T X_s E[\mathbf{1}_{\{\tau > s\}} | \mathcal{G}_T] e^{-\int_0^s r_u \, du} \, ds\Big] \\
&= E\Big[\int_0^T X_s e^{-\int_0^s r_u + \lambda_u \, du} \, ds\Big]
\end{aligned}$$

The **third** one follows from

$$
\begin{aligned}
\mathrm{E}\Big[e^{-\int_0^\tau r_s\,ds}X_\tau \mathbf{1}_{\{\tau\le T\}}\Big] &= \mathrm{E}\Big[\mathrm{E}\Big[e^{-\int_0^\tau r_s\,ds}X_\tau \mathbf{1}_{\{\tau\le T\}}\Big|\mathcal{G}_T\Big]\Big] \\
&= \mathrm{E}\Big[\mathrm{E}\Big[\int_0^\infty \lambda_s e^{-\int_0^s r_u+\lambda_u\,du}X_s \mathbf{1}_{\{s\le T\}}\,ds\Big|\mathcal{G}_T\Big]\Big] \\
&= \mathrm{E}\Big[\mathrm{E}\Big[\int_0^T \lambda_s e^{-\int_0^s r_u+\lambda_u\,du}X_s\,ds\Big|\mathcal{G}_T\Big]\Big] \\
&= \mathrm{E}\Big[\int_0^T X_s\lambda_s e^{-\int_0^s r_u+\lambda_u\,du}\,ds\Big],
\end{aligned}
$$

since the density of $\tau$ can be calculated as follows

$$
Q(\tau\le s|\mathcal{G}_T) = 1 - e^{-\int_0^s \lambda_u\,du} \quad\Longrightarrow\quad \frac{\partial Q(\tau\le s|\mathcal{G}_T)}{\partial s} = \lambda_s e^{-\int_0^s \lambda_u\,du}
$$

# Modeling Recovery

- If a default occurs, then the corresponding claims lose value.
- In **practice**, the **loss rate** is **random**.
- If the loss rate is assumed to be stochastic, then all recovery assumptions are equivalent.
- For **simplicity**, some models assume that the loss rate is **deterministic**.
- The loss rate is then usually fixed at **50-60%**.
- However, empirical evidence suggests that the loss rate depends on a lot of factors (rating, business cycle, CH 11 vs. CH 7)

# Modeling Recovery

- Modeling recovery means that we must simplify a situation that is actually pretty involved
- More precisely, one must specify
  - **when** recovery happens
  - **how much** money is recovered
- For pricing purposes, one typically assumes that recovery happens either at maturity or at default
- We first consider zero-coupon bonds

# Defaultable Zeros: Recovery of Treasury

- For zero-coupon assumptions a reasonable assumption is that one recovers a **fraction of par at maturity** if a default occurs
- Therefore, the payoff at maturity is $(R = 1 - \ell)$

$$p^d(T,T) = \mathbf{1}_{\{\tau > T\}} + \mathbf{1}_{\{\tau \le T\}} R = \mathbf{1}_{\{\tau > T\}} + (1 - \mathbf{1}_{\{\tau > T\}}) R = R + \ell \mathbf{1}_{\{\tau > T\}}$$

- This recovery assumption is also called recovery of Treasury.
- If $r$ and $\tau$ are independent, then

$$\begin{aligned} p^d(0,T) &= p(0,T)R + \mathrm{E}\left[e^{-\int_0^T r_s\,ds} \ell \mathbf{1}_{\{\tau > T\}}\right] \\ &= p(0,T)R + \ell p(0,T) Q(\tau > T) \end{aligned}$$

- Compared to Treasury bonds we now have **two terms** which is not very tractable!

# Defaultable Coupon Bonds

- A coupon bond consists of
  - coupon payments $c_k$ at times $t_k$, $k = 1, \ldots, K$,
  - a notional payment at maturity $T$.
- If a default occurs, then a recovery payment is received.
- We assume that the bond has a unit face value.
- Notice that $t_K = T$.
- If $\Delta_k$ denotes the time span between coupon payments $c_{k-1}$ and $c_k$, then $c_k = \Delta_k C$ where $C$ denotes the annualized coupon rate (e.g. 5%).
- We disregard accrued interest payments.

# Pricing of Defaultable Coupon Bonds

Recall that the value of defaultable coupon bond is the sum of the values of

1. the payment stream of defaultable coupon bond with zero recovery,
2. a recovery payment.

The price of coupon bond with **zero recovery** is given by

$$p_c^d(0,T) = \sum_{k=1}^{K} \mathrm{E}\left[e^{-\int_0^{t_k} r_u\,du} c_k \mathbf{1}_{\{\tau > t_k\}}\right] + \mathrm{E}\left[e^{-\int_0^{T} r_u\,du} \mathbf{1}_{\{\tau > T\}}\right]$$

Under independence this becomes

$$p_c^d(0,T) = \sum_{k=1}^{K} c_k p(0,t_k) Q(\tau > t_k) + p(0,T) Q(\tau > T)$$

**Q:** How can we account for recovery payments?

# Pricing of Defaultable Coupon Bonds and Recovery

- For defaultable coupon bonds it is **not reasonable** that **recovery** happens **at maturity**
- This is because there are coupon payments before maturity
- A **recovery** payment is thus a random payment that can occur at **any time** $\tau \leq T$.
- Its value is

$$\mathrm{E}\left[e^{-\int_0^\tau r_s\,ds} R_\tau \mathbf{1}_{\{\tau \leq T\}}\right]$$

- The price of coupon bond with **recovery** is thus given by

$$\begin{aligned} p_c^d(0,T) &= \sum_{k=1}^{K} \mathrm{E}\left[e^{-\int_0^{t_k} r_u\,du} c_k \mathbf{1}_{\{\tau > t_k\}}\right] + \mathrm{E}\left[e^{-\int_0^T r_u\,du} \mathbf{1}_{\{\tau > T\}}\right] \\ &\quad + \mathrm{E}\left[e^{-\int_0^\tau r_s\,ds} R_\tau \mathbf{1}_{\{\tau \leq T\}}\right] \end{aligned}$$

# Pricing of Defaultable Coupon Bonds and Recovery of Par

- In general, $R_\tau$ can be involved
- However, one reasonable simplifying assumption is that a **fraction of par** is **recovered at default**:

$$R_\tau = (1 - \ell)$$

where $\ell = const$

- **Warning:** In the literature, this is simply referred to as recovery of par

**Reminder: Wording**

- Recovery of Treasury: recovery of par at maturity
  $\longrightarrow$ used for zero-coupon bonds
- Recover of Par: recovery of par at default
  $\longrightarrow$ used for coupon bonds

# Pricing of Defaultable Coupon Bonds and Recovery of Par

Under recovery of par (at default) we obtain

$$\mathrm{E}\Big[e^{-\int_0^\tau r_s\,ds} R_\tau \mathbf{1}_{\{\tau \le T\}}\Big] = (1-\ell)\mathrm{E}\Big[\int_0^T \lambda_s e^{-\int_0^s r_u + \lambda_u\,du}\,ds\Big]$$

### Recovery of Par

If $r$ and $\lambda$ are independent, then

$$\begin{aligned}
&\mathrm{E}\Big[e^{-\int_0^\tau r_s\,ds} R_\tau \mathbf{1}_{\{\tau \le T\}}\Big] \\
&= (1-\ell)\bigg(p(0,T)Q(\tau \le T) - \int_0^T \partial_s\{p(0,s)\}Q(\tau \le s)\,ds\bigg).
\end{aligned}$$

- This representation is explicit if we can quantify the default probabilities $Q(\tau \le s)$.
- Notice that for constant interest rates we get $\partial_s\{p(0,s)\} = -re^{-rs}$.

# Understanding the Recovery Value

Applying integration by parts yields

$$\int_0^T e^{-\int_0^s r_u\,du}\lambda_s e^{-\int_0^s \lambda_u\,du}\,ds$$
$$= e^{-\int_0^T r_u\,du}\Big(1-e^{-\int_0^T \lambda_u\,du}\Big) + \int_0^T r_s e^{-\int_0^s r_u\,du}\Big(1-e^{-\int_0^s \lambda_u\,du}\Big)\,ds$$

Taking expectations leads to

$$p(0,T)Q(\tau\leq T) + \int_0^T \mathrm{E}\Big[r_s e^{-\int_0^s r_u\,du}\Big]Q(\tau\leq s)\,ds$$

The result follows since

$$\partial_s p(0,s) = \partial_s \mathrm{E}\Big[e^{-\int_0^s r_u\,du}\Big] = -\mathrm{E}\Big[r_s e^{-\int_0^s r_u\,du}\Big].$$

# Modeling Recovery

- We have considered recovery of Treasury and recovery of par
- Recovery of Treasury is used for zero-coupon bonds
- Recovery of par is used for coupon bonds
- Compared to default-free bonds, both recovery assumptions lead to one **extra term**
- This is not handy
- **Q**:Is there an alternative?
- **A**: Yes, recovery of market value (RMV), which is equivalent to a certain framework with multiple defaults

# Defaultable Zeros: Multiple Defaults and RMV

- RMV assumes that a **fraction** of the **pre-default value** is recovered **at default**
- For zeros, this means $p^d(\tau, T) = (1 - \ell)p^d(\tau-, T)$
- Assume now that multiple defaults are possible and that the notional is multiplied every time by the fraction $R = 1 - \ell$.
- For a defaultable zero this means that at maturity

$$p^d(T, T) = (1 - \ell)^{N(T)} \cdot 1,$$

where $N(T)$ equals the number of defaults until time $T$.

## Defaultable Zero and Multiple Defaults

If multiple defaults can occur and every time a constant fraction of $\ell$ is lost, then the value of a zero-coupon bond is given by

$$p^d(0, T) = \mathrm{E}^Q\left[e^{-\int_0^T r_s + \ell\lambda_s \, ds}\right],$$

when no default has occurred yet.

# Understanding the Pricing Formula

For simplicity, assume that $N$ is a Poisson process ($\lambda = const$). Thus

$$Q(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

The price of a zero can be written as ($t = 0$)

$$\begin{aligned}
p^d(0,T) &= \mathrm{E}\left[e^{-\int_0^T r_s\,ds}(1-\ell)^{N(T)}\right] = p(0,T)\mathrm{E}\left[(1-\ell)^{N_T}\right] \\
&= p(0,T)\sum_{n=0}^{\infty} Q(N_T = n)(1-\ell)^n = p(0,T)\sum_{n=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}(1-\ell)^n \\
&= p(0,T)e^{-\lambda T}\sum_{n=0}^{\infty}\frac{(\lambda T(1-\ell))^n}{n!} = p(0,T)e^{-\lambda T}e^{\lambda T(1-\ell)} \\
&= \mathrm{E}[\left[e^{-\int_0^T r_s\,ds}\right]e^{-\ell\lambda T} = \mathrm{E}[\left[e^{-\int_0^T r_s + \ell\lambda\,ds}\right]
\end{aligned}$$

One can show that a similar argument works for stochastic $\lambda$

# Multiple Defaults and Recovery of Market Value

To understand the similarity of MD and RMV, we consider situations where at time $t$

1. no default has occurred yet,
2. one default has already occurred.

**1st case.** The price of a zero reads

$$p^d(t,T;0) \equiv \mathrm{E}_t\left[e^{-\int_t^T r_s\,ds}(1-\ell)^{N_T}\middle| N_t = 0\right].$$

**2nd case.** The price of a zero reads

$$\begin{aligned} p^d(t,T;1) &= \mathrm{E}_t\left[e^{-\int_t^T r_s\,ds}(1-\ell)^{N_T}\middle| N_t = 1\right] \\ &= \mathrm{E}_t\left[e^{-\int_t^T r_s\,ds}(1-\ell)^{1+N_T}\middle| N_t = 0\right] = (1-\ell)p^d(t,T;0) \end{aligned}$$

In particular, at the first default we get $p^d(\tau,T) = (1-\ell)p^d(\tau-,T)$, which is RMV.

# Spreads

## Spread

The spread of a defaultable zero-coupon bond is defined as

$$-\frac{1}{T-t}\ln\left(\frac{p^d(t,T)}{p(t,T)}\right)$$

Under independence of $r$ and $\lambda$ and RMV the spread becomes

$$\begin{aligned}
-\frac{1}{T-t}\ln\left(\frac{p^d(t,T)}{p(t,T)}\right) &= -\frac{1}{T-t}\ln\left(\frac{p(t,T)\mathrm{E}_t\left[e^{-\int_t^T \ell\lambda_u\,du}\right]}{p(t,T)}\right)\\
&= -\frac{\ln\left(\mathrm{E}_t\left[e^{-\int_t^T \ell\lambda_u\,du}\right]\right)}{T-t}
\end{aligned}$$

# Spreads and Recovery of Market Value

L'Hospital yields

$$\lim_{T \searrow t} -\frac{\ln\left(\mathrm{E}_t\left[e^{-\int_t^T \ell\lambda_u\,du}\right]\right)}{T-t} = \lim_{T \searrow t} -\frac{\mathrm{E}_t\left[e^{-\int_t^T \ell\lambda_u\,du}(-\ell\lambda_T)\right]}{\mathrm{E}_t\left[e^{-\int_t^T \ell\lambda_u\,du}\right]}$$

$$= \ell\lambda_{t+} = \ell\lambda_t$$

- Therefore, the spread equals

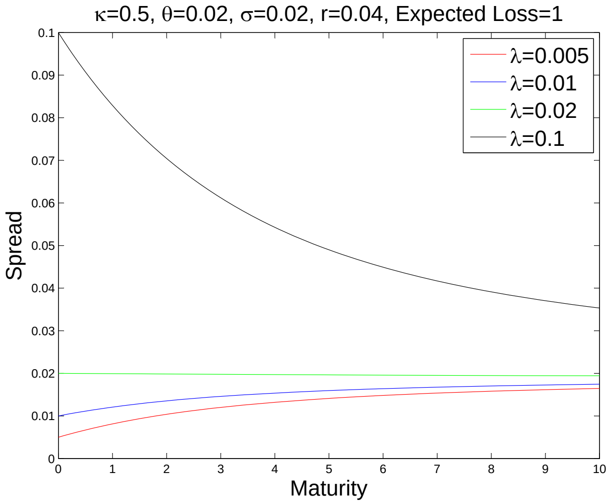
$$\text{loss rate} \times \text{intensity}$$

- This gives the discount factors a very natural interpretation:

$$e^{-\int_t^T r_s\,ds} \quad \text{vs.} \quad e^{-\int_t^T r_s^d\,ds},$$

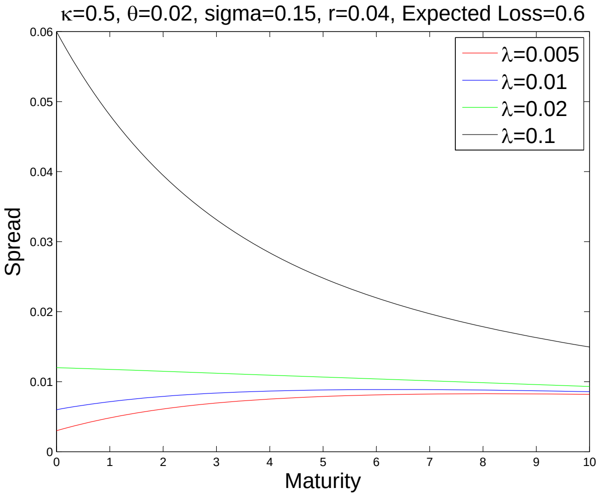
  where $r_t^d \equiv r_t + \ell\lambda_t$.
- If we use a model where $r$ and $\lambda$ are affine (e.g. Vasicek), then we can immediately write down bond prices etc.

# Spreads for Vasicek Intensity and Zero Recovery




Spreads of zero-coupon bonds for different maturities and initial intensity values.

# Spreads for CIR Intensity and Recovery of Market Value




Spreads of zero-coupon bonds for different maturities and initial intensity values.

# Backing Out Default Probabilities

- Under RMV and independence, bond prices are given by

$$\begin{aligned} p^d(0,T) &= p(0,T)\mathrm{E}\left[e^{-\int_0^T \ell\lambda_u\,du}\right] \\ &= p(0,T)\mathrm{E}\left[\left(e^{-\int_0^T \lambda_u\,du}\right)^{\ell}\right] \end{aligned}$$

- Therefore, we can only directly compute the $\ell$-th moment, but not the survival probability.
- Consequently, we have to impose additional assumptions on $\lambda$.
- Under RT and independence, bond prices read

$$p^d(0,T) = p(0,T)\left(R + Q(\tau > T)\ell\right) \Longrightarrow Q(\tau > T) = \frac{p^d(0,T) - p(0,T)R}{\ell p(0,T)}$$

- Therefore, we can directly back out survival probabilities without assuming a model for $\lambda$

# Pricing Coupon Bonds under RMV

Analogously to a zero-coupon bond, one obtains:

**Defaultable Coupon Bond**

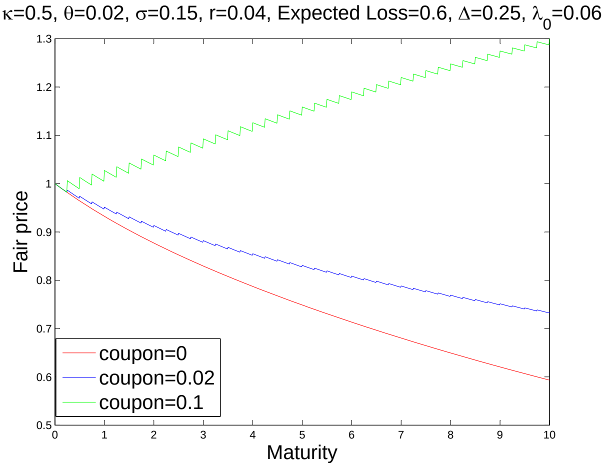
$$p_c^d(0,T) = \sum_{k=1}^{K} c_k \mathrm{E}\left[e^{-\int_0^{t_k} r_u + \ell\lambda_u \, du}\right] + \mathrm{E}\left[e^{-\int_0^{T} r_u + \ell\lambda_u \, du}\right]$$

- The relevant discount factor is again

$$e^{-\int_t^T r_s^d \, ds},$$

  where $r_t^d \equiv r_t + \ell\lambda_t$, i.e. the same intuition as above applies.
- Due to the adjustment of the interest rate, there is **no additional recovery term**.
- This is very convenient since the formula is additive.
- Notice that the case with zero recovery ($\ell = 1$) is included as a special case.

# Bond Prices for CIR Intensity and RMV




Coupon-bond prices for CIR intensity and different coupon sizes.

# Understanding the Pricing Formula: Multiple Defaults

- Again we draw the analogy to the multiple defaults framework.
- Under MD it is assumed that
  - multiple defaults are possible,
  - at default every promised payment is reduced by the fraction $1 - \ell$.
- Therefore, the price of a coupon bond can be written as

$$p_c^d(0, T) = \sum_{k=1}^{K} \mathrm{E}\left[e^{-\int_0^{t_k} r_u\, du} c_k (1-\ell)^{N_{t_k}}\right] + \mathrm{E}\left[e^{-\int_0^{T} r_u\, du} (1-\ell)^{N_T}\right]$$

- Recall that

$$\mathrm{E}\left[e^{-\int_0^{t_k} r_u\, du} (1-\ell)^{N_{t_k}}\right] = \mathrm{E}\left[e^{-\int_0^{t_k} r_u + \ell\lambda_u\, du}\right].$$

- Consequently, the above stated formula follows.

# Recovery Assumptions: Formal Summary

Let $\tau$ be the default time and $\ell$ the loss rate

## Common Recovery Models

1. **Recovery of Treasury**
   - Jarrow and Turnbull (1995)
   - $p^d(\tau, T) = (1-\ell)p(\tau, T)$
2. **Recovery of Par**
   - e.g. Lando (1998)
   - $p^d(\tau, T) = (1-\ell)$
3. **Recovery of Market Value**
   - Duffie and Singleton (1999)
   - $p^d(\tau, T) = (1-\ell)p^d(\tau-, T)$

- Recovery of par is used for coupon bonds
- Recovery of Treasury is used for zero-coupon bonds
- Recovery of market values is very tractable and is used for zero-coupon and coupon bonds

# A Simple Gaussian Model with Correlation

Consider a Gaussian two-factor model for the default-free short rate and the intensity:

$$\begin{aligned} dr_t &= (\theta_r - \kappa_r r_t)dt + \sigma_r dW_{rt}, \\ d\lambda_t &= (\theta_\lambda - \kappa_\lambda \lambda_t)dt + \sigma_\lambda dW_{\lambda t}, \end{aligned}$$

where $d < W_r, W_\lambda >_t = \rho dt$.

As above, default-free bond prices and survival probabilities are given by

$$p(t, T) = e^{A(T-t) - B(T-t)r_t}, \quad Q_t(\tau > T) = e^{A_\lambda(T-t) - B_\lambda(T-t)\lambda_t}.$$

Notice that the correlation is irrelevant for these results.

# A Simple Gaussian Model with Correlation

Consider the process

$$dx_t = (\theta(t) - \kappa x_t)dt + \sigma dW_t,$$

where $\theta$ is a time-dependent function.

## Hull-White Model

Under the above assumption, we obtain

$$\mathrm{E}_t\left[e^{-\int_t^T x_u\,ds}\right] = e^{\mathcal{A}(t,T;\theta,\sigma)-\mathcal{B}(t,T;\kappa)x_t},$$

where $\mathcal{B}(t, T; \kappa) = (1 - e^{-\kappa(T-t)})/\kappa$ and

$$\begin{aligned}\mathcal{A}(t, T; \theta, \sigma) &= \frac{\sigma^2}{2\kappa^2}\Big(T - t - \mathcal{B}(t, s; \kappa) - 0.5\kappa\mathcal{B}(t, T; \kappa)^2\Big) \\ &\quad - \int_t^T \mathcal{B}(t, s; \kappa)\theta(s)\,ds\end{aligned}$$

# A Simple Gaussian Model with Correlation

The correlation is however relevant for defaultable bond prices and continuous fee payments.

## Building Blocks

$$
\begin{aligned}
\mathrm{E}_t\left[e^{-\int_t^T r_u+\ell\lambda_u\,du}\right] &= p(t,T)e^{\mathcal{A}(t,T;\ell\tilde{\theta}_\lambda,\ell\sigma_\lambda)-\mathcal{B}(t,T;\kappa_\lambda)\ell\lambda_t} \\
\mathrm{E}_t\left[\lambda_T e^{-\int_t^T r_u+\lambda_u\,du}\right] &= p^d(t,T)\Big(\lambda_t e^{-\kappa_\lambda(T-t)}, \\
&\qquad + \int_t^T e^{-\kappa_\lambda(T-s)}\hat{\theta}_\lambda(s)\,ds\Big),
\end{aligned}
$$

where $\tilde{\theta}_\lambda(t) = \theta_\lambda - \rho\sigma_\lambda\sigma_r\mathcal{B}(t,T;\kappa_r)$ and $\hat{\theta}_\lambda(t) = \tilde{\theta}_\lambda(t) - \sigma_\lambda^2\mathcal{B}(t,T;\kappa_\lambda)$.

Notice that $p^d(t,T)$ denotes a defaultable zero-coupon bond with zero recovery.

We change to the $T$-forward measure:

$$\mathrm{E}_t\left[e^{-\int_t^T r_u+\ell\lambda_u\,ds}\right] = p(t,T)\mathrm{E}_t^T\left[e^{-\int_t^T \ell\lambda_u\,ds}\right]$$

Now define $\tilde{\lambda} \equiv \ell\lambda$ such that

$$d\tilde{\lambda}_t = \ell d\lambda_t = (\ell\theta_\lambda - \kappa_\lambda\tilde{\lambda}_t)dt + \ell\sigma_\lambda dW_{\lambda t}$$

We can rewrite this equation as

$$d\tilde{\lambda}_t = (\ell\theta_\lambda - \kappa_\lambda\tilde{\lambda}_t)dt + \ell\sigma_\lambda\left(\rho dW_{rt} + \sqrt{1-\rho^2}d\hat{W}_{\lambda t}\right),$$

where $\hat{W}_\lambda$ is a Brownian motion independent of $W_r$.

Under the $T$-forward measure

$$dW_{rt}^T = dW_{rt} - \sigma_B(t,T)dt$$

is a martingale increment, where $\sigma_B(t,T) = -\sigma_r \mathcal{B}(t,T;\kappa_r)$ is the volatility of the $T$-bond. Therefore,

$$d\tilde{\lambda}_t = (\ell\theta_\lambda + \ell\rho\sigma_\lambda\sigma_B(t,T) - \kappa_\lambda\tilde{\lambda}_t)dt + \ell\sigma_\lambda dW_{\lambda t}^T,$$

where $dW_{\lambda t}^T \equiv \rho dW_{rt}^T + \sqrt{1-\rho^2}d\hat{W}_{\lambda t}$ is a Brownian increment under the $T$-forward measure. Rewriting the equation yields

$$d\tilde{\lambda}_t = (\ell\tilde{\theta}_\lambda(t) - \kappa_\lambda\tilde{\lambda}_t)dt + \ell\sigma_\lambda dW_{\lambda t}^T.$$

Applying the Hull-White result to $\mathrm{E}_t^T[e^{-\int_t^T \tilde{\lambda}_u\, ds}]$ gives the desired representation.

# Understanding the Pricing Formulas

For the second relation, assume that we are in an artificial market without default risk where the short rate is $r^d = r + \lambda$. Bond prices are given by $p^d(t, T) = \mathrm{E}_t[e^{-\int_t^T r_u + \lambda_u \, du}]$. The money market account in this market is given by

$$dB_t^d = B_t^d (r_t + \lambda_t) dt.$$

We now change to the $T$-forward measure in this market:

$$\begin{aligned} \mathrm{E}_t\left[\lambda_T e^{-\int_t^T r_u + \lambda_u \, du}\right] &= B_t^d \mathrm{E}_t[\lambda_T / B_T^d] \\ &= p^d(t, T) \mathrm{E}_t^T[\lambda_T / p^d(T, T)] \\ &= p^d(t, T) \mathrm{E}_t^T[\lambda_T]. \end{aligned}$$

# Understanding the Pricing Formulas

Since bond prices are given by

$$p^d(t,T) = e^{A(t,T)-B(t,T)r_t} e^{\mathcal{A}(t,T;\tilde{\theta}_\lambda,\sigma_\lambda)-\mathcal{B}(t,T;\kappa_\lambda)\lambda_t},$$

the bond dynamics are given by

$$\begin{aligned} dp^d(t,T) &= p^d(t,T)[\dots dt - B(t,T)dr_t - \mathcal{B}(t,T;\kappa_\lambda)d\lambda_t] \\ &= p^d(t,T)[\dots dt - B(t,T)\sigma_r dW_{rt} - \mathcal{B}(t,T;\kappa_\lambda)dW_{\lambda t}] \\ &= p^d(t,T)[\dots dt - (B(t,T)\sigma_r + \mathcal{B}(t,T;\kappa_\lambda)\sigma_\lambda\rho)dW_{rt} \\ &\quad -\mathcal{B}(t,T;\kappa_\lambda)\sigma_\lambda\sqrt{1-\rho^2}d\hat{W}_{\lambda t}] \end{aligned}$$

The drift of $\lambda$ under the $T$-forward measure is shifted by

$$(\sigma_\lambda\rho, \sigma_\lambda\sqrt{1-\rho^2})\cdot(-B\sigma_r - \mathcal{B}\sigma_\lambda\rho, -\mathcal{B}\sigma_\lambda\sqrt{1-\rho^2}) = -B\sigma_r\sigma_\lambda\rho - \mathcal{B}\sigma_\lambda^2.$$

Using this one can calculate $\mathrm{E}_t^T[\lambda_T]$.

# Understanding the Pricing Formulas

Under the $T$-forward measure we get

$$\lambda_T = \lambda_t e^{-\kappa_\lambda (T-t)} + \int_t^T \hat{\theta}_\lambda(s) e^{-\kappa_\lambda (T-s)}\, ds + \sigma_\lambda \int_t^T e^{-\kappa_\lambda (T-s)}\, dW^T_{\lambda s}.$$

Therefore,

$$\mathrm{E}_t^T[\lambda_T] = \lambda_t e^{-\kappa_\lambda (T-t)} + \int_t^T \hat{\theta}_\lambda(s) e^{-\kappa_\lambda (T-s)}\, ds$$

and thus the claim follows.

# Other Models with Correlation: CIR

- In the above model, there is a positive probability that the intensity becomes negative.
- **Idea:** Use CIR-processes instead of Vasicek processes.
- There are however technical problems:
  - The processes might not be jointly affine.
  - The processes might not be well-defined at all since volatility does not vanish at zero.
- Alternatively, one could consider an intensity model of the form

  $$\lambda_t = \bar{\lambda} + a r_t + s_t,$$

  where $r$ and $s$ are independent CIR-processes.
- This is fine if $a > 0$.
- Otherwise, $\lambda$ can become negative and

  $$\mathrm{E}\left[e^{-\int_0^t a r_s \, ds}\right] = \infty$$

  if $|a|$ is large enough.

# Other Models with Correlation: Quadratic Gaussian

Set $r_t = (X_{1t})^2$ and $\lambda_t = (X_{2t})^2$, where

$$\begin{aligned} dX_{1t} &= a_1(b_1 - X_{1t})dt + \sigma_1 dW_{1t}, \\ dX_{2t} &= a_2(b_2 - X_{2t})dt + \sigma_2\Big(\rho dW_{1t} + \sqrt{1-\rho^2}dW_{2t}\Big). \end{aligned}$$

**Q:** Are $r$ and $\lambda$ negatively correlated if $\rho < 0$, i.e. is it true that for the instantaneous correlation

$$\text{corr}(dr_t, d\lambda_t) \equiv \frac{d < r, \lambda >_t}{\sqrt{d < r >_t}\sqrt{d < \lambda >_t}} < 0 \quad ?$$

Ito's lemma yields

$$\begin{aligned} dr_t &= \ldots dt + 2X_{1t}\sigma_1 dW_{1t}, \\ d\lambda_t &= \ldots dt + 2X_{2t}\sigma_2\Big(\rho dW_{1t} + \sqrt{1-\rho^2}dW_{2t}\Big). \end{aligned}$$

# Other Models with Correlation: Quadratic Gaussian

From the last slide

$$
\begin{aligned}
dr_t &= \dots dt + 2X_{1t}\sigma_1 dW_{1t}, \\
d\lambda_t &= \dots dt + 2X_{2t}\sigma_2\Big(\rho dW_{1t} + \sqrt{1-\rho^2}dW_{2t}\Big).
\end{aligned}
$$

$$
\Longrightarrow \qquad d<r>_t = 4(X_{1t})^2\sigma_1^2 dt, \quad d<\lambda>_t = 4(X_{2t})^2\sigma_2^2 dt
$$

$$
d<r,\lambda>_t = 4X_{1t}X_{2t}\rho\sigma_1\sigma_2 dt
$$

**Instantaneous Correlation**

$$
\mathrm{corr}(dr_t, d\lambda_t) \equiv \frac{d<r,\lambda>_t}{\sqrt{d<r>_t}\sqrt{d<\lambda>_t}} = \rho\frac{X_{1t}X_{2t}}{|X_{1t}||X_{2t}|}
$$

- Hence, the correlation keeps its sign if $X_1$ and $X_2$ are positive.
- Since both have a Gaussian distribution, this is not always the case.
- However, the probability can be maximized by choosing small volatilities $\sigma_i$ and positive mean reversion levels $b_i$, $i = 1, 2$.