Credit Markets

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Agenda: Single-Name Credit

Agenda: Multi-Name Credit

Agenda: Single-Name Credit

1. Introduction
2. Model-free Results for Corporate Bonds
3. Toolbox for Default Risk
4. Pricing of Corporate Bonds and CDS in a Simple Model
5. Pricing Defaultable Bonds with Stochastic Intensity
6. Pricing CDS
7. CDS Derivatives
8. Firm Value Models

Types of Risks

A firm is exposed to several types of risks. The most important ones are

Market Risk: Risk related to movements in market variables such as stock indices, interest rates, exchange rates etc.

Credit Risk: Risk that a loss occurs because of a default by a counterparty.

Both are very **different** since

- default is a 0-1-event
- default risk is harder to measure
- default risk cannot be hedged away by a market index

Credit Risk – What is it?

Credit Risk

Credit risk is the risk that a loss will be experienced because of

- a debtor's non-payment of a loan or other line of credit (either the principal or interest (coupon) or both)
- a default by the counterparty in a derivatives transaction.

Examples:

- The issuer of a bond does not meet his obligations (e.g. misses a coupon payment).
- Seller of a call option is not able to deliver the underlying.

Default Events

Default does not mean bankruptcy

- On March 1, 1985 Dole Food Company defaulted on missed interest payments
- Dole made interest payment on March 29, 1985

Default does not mean that the firm is liquidated

- An explosive expansion and conversion of gas stations to convenience stores between 1983 and 1989 had left company overburdened by debt. After banks refused to extend standstill agreement on April 4, 1990 Circle K Corp. defaulted on missed interest payment.
- On May 15, 1990 Circle K filed for bankruptcy under Chapter 11
- On July 26, 1993 Circle K emerged from Chapter 11

Default Events

But a default event **can lead to liquidation**:

- On March 6th 2002, National Steel Corporation filed for Chapter 11.
- The company's operating performance and cash flow have been severely impacted by very difficult market conditions. In particular, prices for its steel products have been depressed for two years, which is primarily driven by weakness in the U.S. economy and large volumes of imported steel products. National Steel, based in Mishawaka, Indiana, was one of the largest domestic producers of carbon flat-rolled steel.
- The firm was liquidated in October 2003

How to Quantify Credit Risk?

There are **two** important **dimensions** of credit risk:

1. How likely is a default?
2. How big is the loss if a default occurs?

These dimensions are captured by the

1. default probability (PD),
2. loss given default (LGD).

Note that LGD $=$ 1 - recovery rate.

Refresher: Pricing of Treasury Bonds

Recall that at maturity a Treasury zero-coupon bond pays its notational ($N = 1$) with certainty:

Time	0	1	2	$\cdots$	T
Zero-coupon Bond					1

We know that its time-0 price is given by

$$p(0, T) = e^{-y(0,T)\cdot T} \cdot 1,$$

where $y(0, T)$ denotes the default-free zero yield.

Q: What happens if the issuer could default?

Pricing of Defaultable Bonds

Consider a zero with maturity T issued by a firm.

Q: What is its price $p^d(0, T)$?

Payoff of a Defaultable Zero

$$p^d(T,T) = \begin{cases} 1 & \text{if} \quad \text{firm is solvent at } T \\ R = 1 - L & \text{if} \quad \text{firm is insolvent at } T \end{cases},$$

where R is the recovery rate and L the loss rate.

Alternatively, we can write the payoff as

$$\begin{aligned} p^d(T,T) &= \mathbf{1}_{\{\tau > T\}} + R \cdot \mathbf{1}_{\{\tau \leq T\}} \\ &= 1 - \mathbf{1}_{\{\tau \leq T\}} + R \cdot \mathbf{1}_{\{\tau \leq T\}} \\ &= 1 - (1 - R) \cdot \mathbf{1}_{\{\tau \leq T\}} \\ &= 1 - L \cdot \mathbf{1}_{\{\tau \leq T\}} \end{aligned}$$

where τ denotes the default time of the firm.

Pricing of Defaultable Bonds

For simplicity, assume for the moment that the default-free interest rates are constant, i.e. $p(0, T) = e^{-r \cdot T}$.

Two situations can occur:

1. "good state" (="boom state"): Firm pays one dollar.
2. "bad state" (="depression state"): Firm pays only R cent.

Key Insight

Prices are expected discounted payoffs under the risk-neutral measure.

So, let's use this insight and calculate the price of a defaultable zero:

$$\begin{aligned} p^d(0, T) &= (q_u \cdot 1 + q_d \cdot R) \cdot e^{-rT} \\ &= (q_u \cdot 1 + q_d \cdot R) \cdot p(0, T) \end{aligned}$$

Pricing of Defaultable Bonds

Since $R = 1 - L$, we get

$$p^d(0,T) = (q_u \cdot 1 + q_d \cdot (1-L)) \cdot p(0,T)$$
$$= (1 - L \cdot q_d) \cdot p(0,T).$$

Q: What is q_d?

A: This is the (risk-neutral) probability that the firm defaults before time T!

This leads to

Rule: Price of a Corporate Zero

$$p^d(0,T) = (1 - L \cdot \mathrm{PD}^Q) \cdot p(0,T).$$

If $L = 1$, then $p^d(0,T) = \mathrm{SP}^Q \cdot p(0,T)$, where SP^Q is the risk-neutral survival probability.

Risk-neutral Default Probabilities

Q: How can we use this insight in practice?

A: We could use **bond data** to back out default probabilities:

$$\mathrm{PD}^Q = \frac{p(0,T) - p^d(0,T)}{L \cdot p(0,T)},$$

which is the (risk-neutral) default probability that the firm defaults before time T.

Example: Calculating Risk-neutral Default Probabilities

Goal: Calculate the 1-year default probability of firm XY

Data: Prices of zeros with maturity one year

$$p(0,1) = 0.97, \quad p^d(0,1) = 0.955$$

Rule of Thumb: Loss rate is 60%.

$$\Longrightarrow \text{PD}^Q = \frac{p(t,T) - p^d(t,T)}{L \cdot p(t,T)} = \frac{0.97 - 0.955}{0.6 \cdot 0.97} = 0.0258,$$

i.e. the 1-year default probability of firm XY is 2.58% and thus the 1-year survival probability is 97.42%.

Calculating the Spread of a Corporate Bond

Spread

The difference between a default-free and a corresponding defaultable interest rate is said to be the spread.

In our example,

$$p(0,1) = 0.97, \quad p^d(0,1) = 0.955,$$

we get

$$r(1) = \ln(1/0.97) = 3.05\%, \quad r^d(1) = \ln(1/0.955) = 4.60\%$$

and thus the spread for a 1-year investment in XY bonds is

$$4.60 - 3.05 = 1.55\% = 155\text{bp}$$

What is the Interpretation of the Spread?

For simplicity, $r \equiv r(1)$, $r^d \equiv r^d(1)$, $p \equiv p(0,1)$, $p^d \equiv p^d(0,1)$. Then

$$r = \ln(1/p) \qquad r^d = \ln(1/p^d) \qquad p^d = (1 - L \cdot \mathrm{PD}^Q)p$$

Therefore, the spread is given by

$$\begin{aligned} r^d - r = \ln(1/p^d) - \ln(1/p) &= \ln(p/p^d) = -\ln(1 - L \cdot \mathrm{PD}^Q) \\ &\approx L \cdot \mathrm{PD}^Q, \end{aligned}$$

which is the risk-neutral expected loss EL^Q of the corporate zero since

$$\mathrm{EL}^Q = 0 \cdot (1 - \mathrm{PD}^Q) + L \cdot \mathrm{PD}^Q = L \cdot \mathrm{PD}^Q$$

Interpretation of Spread

Spread of corporate zero is approx. the risk-neutral expected loss.

Reduced-form vs. Firm Value Models

- The model considered so far is a so-called **reduced-form model**.
- **Default** happens as a **sudden surprise**.
- Such a model is well-suited to calibrate CDS prices etc.
- Certain **information** is however **disregarded** (stock and option prices).
- For instance, a default is sometimes preceded by a period of a declining stock price.
- **Firm value models** (syn. structural models) explicitly model the **link between equity and debt**.

Firm Value Model

Q: What is the **economic intuition** behind a default?

Firm value models shed some light on the mechanism driving a default.

We consider a firm with a simple **capital structure**:

- Debt: Zero with face value F and maturity T
- Equity: Stock
- Firm Value $V =$ Value Equity + Value Debt

Firm Value Model

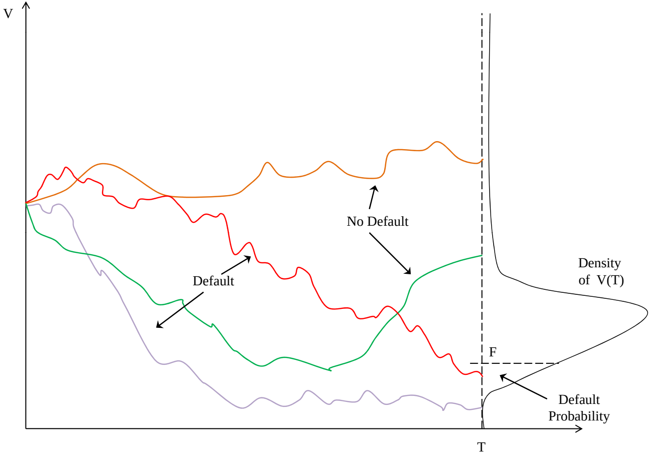
Upon maturity T of the zero, two scenarios can occur:

	Stock Holders	Bond Holders	
$V_T \geq F$	$V_T - F$	F	Zero redeemed
$V_T < F$	0	V_T	Zero defaults

Equity and Debt in a Firm Value Model

$$\textbf{Equity} \quad : \quad \underbrace{\max\{V_T - F; 0\}}_{\text{“Call on firm value”}}$$

$$\textbf{Debt} \quad : \quad F - \underbrace{\max\{F - V_T; 0\}}_{\text{“Put on firm value”}}$$


V
No Default
Default
Density
of V(T)
F
Default
Probability
T

Firm Value Model: Lognormal Firm Value

In a Black-Scholes world, one can easily calculate the value of equity and debt since the risk-neutral firm value dynamics read

$$dV_t = V_t[rdt + \sigma dW_t].$$

Merton (1974)

$$\begin{aligned} E_t &= V_t N(d_1) - Fe^{-r(T-t)} N(d_2) \\ D_t &= V_t N(-d_1) + Fe^{-r(T-t)} N(d_2) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{V_t}{F}\right) + (r + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Default Probability in Merton's Model

Q: When do we have a default in this model?
A:

- Default can only happen at maturity (no premature default).
- Default if firm value smaller than bond's face value.

Q: What is the probability that $V_T < F$?

Default Probability

$$\mathrm{PD}^Q = Q(V_T < F) = 1 - N(d_2) = N(-d_2).$$

KMV calls d_2 the distance to default.

Risk-neutral vs. Historical Default Probabilities

In practice, we often need real (=historical) probabilities

$$\text{PD} = \text{Prob}(\text{“default”}),$$

instead of the risk-neutral probability

$$\text{PD}^Q = \text{Prob}^Q(\text{“default”}).$$

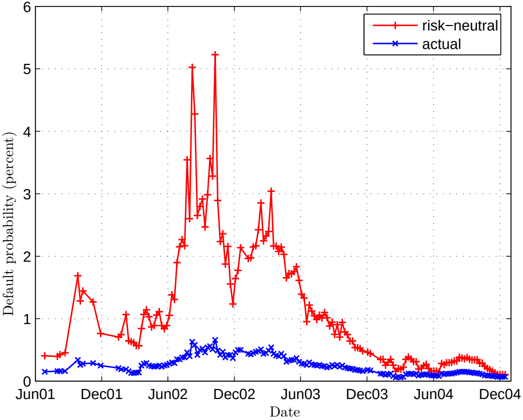
Rule

For pricing purposes, we use risk-neutral probabilities and, for risk management purposes, we use historical probabilities.

Crucial Q: How are these two probabilities related?

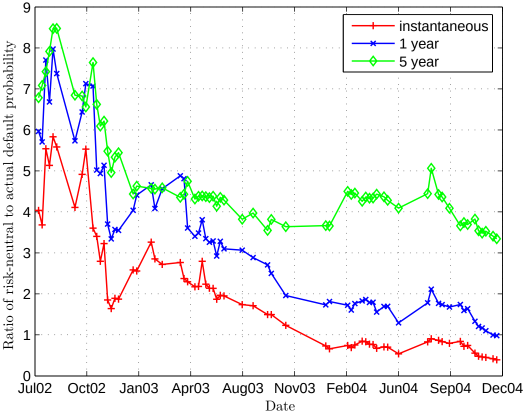
Empirical evidence shows that $\text{PD}^Q > \text{PD}$.

Risk-neutral vs. Historical Default Probabilities



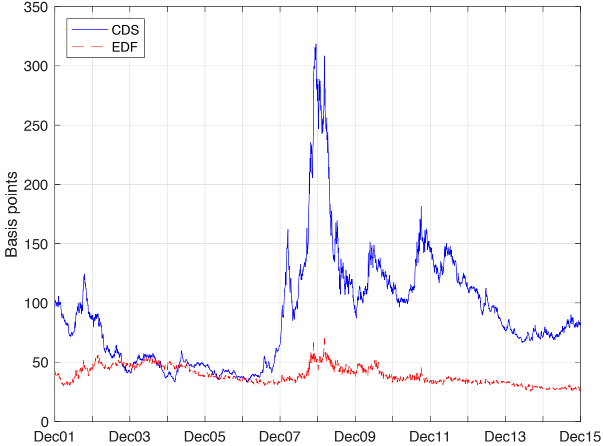
Estimated actual and risk-neutral 1-year default probabilities for Disney (Source: Berndt et al. (2005)).

Risk-neutral vs. Historical Default Probabilities



Within-sector medians of the ratios of risk-neutral to actual default probabilities, for the broadcasting-and-entertainment sector, at various maturities (Source: Berndt et al. (2005)).

Risk-neutral vs. Historical Default Probabilities



The figure shows the daily times series of median five-year CDS rates and median five-year EDF rates. The data cover 505 public U.S. firms, over 2002-2015. (Source: Berndt et al. (2017))

Risk-neutral vs. Historical Default Probabilities

Year	CDS	EDF	CDS/EDF
2002	88	39	2.26
2003	60	48	1.25
2004	49	49	1.00
2005	46	44	1.05
2006	41	37	1.11
2007	44	33	1.33
2008	134	40	3.35
2009	156	47	3.32
2010	120	43	2.79
2011	121	37	3.27
2012	129	35	3.69
2013	98	33	2.97
2014	73	30	2.43
2015	77	28	2.75
Mean	88.29	38.79	2.33

The table reports median five-year CDS rates and EDFs. CDS rates and EDFs are reported as annualized rates, in basis points. The data cover 505 public U.S. firms, over 2002-2015. (Source: Berndt et al. (2017))

Historical Default Probabilities

As we have seen, from bond prices we can back out risk-neutral default probabilities.

Q: How can we estimate the default probability of a firm?

Example "Unbalanced Coin": To estimate the probability of head, one can toss a coin several times (e.g. 100) and count the numbers of heads (e.g. 40).

$$\Longrightarrow \text{Prob}(\text{“head”}) = 40\%$$

Q: Can we apply this idea to estimate a firms's default probability?

Example "General Electric": For more than 100 years, we do not observe any default of GE.

$$\Longrightarrow \text{Prob}(\text{“default GE”}) = 0\% \qquad ???$$

This is obviously not true.

Historical Default Probabilities

Idea: Group firms into classes of firms with the same credit qualities and count the number of defaults in every class.

This leads to the concept of ratings.

Credit Ratings

Rating

A rating is a measure of the creditworthiness of a firm.

There are two types of ratings

- **External ratings**: published by a rating agency
- **Internal ratings**: provided by banks for internal use

The most important external ones are the following:

S&P	AAA	AA	A	BBB	BB	B	CCC
Moody's	Aaa	Aa	A	Baa	Ba	B	Caa

- Both agency do fine tuning like A+, A, A-
- **Investment grade** (IG): at least BBB or Baa
- Otherwise: speculative grade

Credit Ratings

- Formally, a rating is nothing else than an **estimate of a firm's default probability** disguised in shortcuts like AA or B.
- Rating agencies also publish the average default probabilities of the rating classes.

Problem: This could lead to volatile ratings.

- **However**, the major users of ratings (e.g. funds, bond traders) are often subject to rules governing what the credit ratings of the bonds they hold must be.
- Consequently, **frequent rating changes** would induce a **large amount of trading** (and incur high transaction costs).
- For this reason, rating agencies try to provide stable ratings.

Rule of Thumb

When rating agencies assign ratings, one of their objectives is ratings stability.

Example: Ratings and Historical Default Probabilities

Average cumulative default rates (%), 1970-2003 (Source: Moody's)

term (years)	1	2	3	4	5	...
A	0.02	0.09	0.23	0.38	0.54	...
Caa	23.65	37.20	48.02	55.56	60.83	...

Rule of Thumb

Annual default probabilities tend to be increasing (decreasing) for investment-grade (speculative-grade) bonds over time.

Annual default probabilities (%)

year	1	2	3	4	5	...
A	0.02	0.07	0.14	0.15	0.16	...
Caa	23.65	13.55	10.82	7.54	5.24	...

Internal vs. External Ratings

Rating Agencies only provide ratings for bigger firms.

Problem: To assess the creditworthiness, banks need ratings for all firms.

- This is also one of the requirements of Basel 2 (e.g. Standardized Approach).
- For this reason, most banks have procedures for rating the creditworthiness of their corporate clients.
- These procedures usually involve accounting ratios like the firm's debt-to-equity ratio.
- Example: Altman's Z-Score

Why Credit Markets?

- You might want to **change exposure** to credits that you cannot get rid of.
- You may want to **take exposures** to credits even if that leaves you less diversified.
- **Diversification** reduces credit risk.
- In all these cases, credit derivatives can help.

The Market for Credit Derivatives

OTC, credit default swaps, by type of position

In billions of US dollars — Table D10.1

	Total	Reporting dealers	Other financial institutions: Total	CCPs	Banks and securities firms	Insurance and financial guaranty firms	SPVs, SPCs and SPEs	Hedge funds	Other	Non-financial institutions
	H2 18	H2 18	H2 18	H2 18	H2 18	H2 18	H2 18	H2 18	H2 18	H2 18
Total CDS contracts										
Notional amounts outstanding	8,143	1,809	6,063	4,445	402	127	48	383	659	270
Bought (gross basis)	5,101	1,821	3,133	2,183	238	72	37	206	396	148
Sold (gross basis)	4,851	1,798	2,930	2,262	164	55	11	177	263	122
Gross market values	187	52	129	75	8	5	2	18	20	6
Positive (gross basis)	118	52	64	38	4	2	1	8	10	2
Negative (gross basis)	119	50	65	38	4	2	1	10	10	4
Net market values	55	14	37	5	4	3	2	11	12	4
Positive (gross basis)	34	14	18	3	2	2	1	5	6	1
Negative (gross basis)	33	12	18	2	2	1	1	7	6	3
Single-name instruments										
Notional amounts outstanding	3,954	1,268	2,633	1,920	222	55	29	150	257	53
Bought (gross basis)	2,644	1,281	1,329	933	142	33	26	55	140	35
Sold (gross basis)	2,578	1,256	1,304	987	80	23	3	95	117	18
Gross market values	105	32	70	45	5	3	2	7	9	3
Positive (gross basis)	68	32	36	22	2	1	1	4	5	1
Negative (gross basis)	69	32	35	23	3	1	1	4	4	2
Multi-name instruments										
Notional amounts outstanding	4,189	541	3,431	2,525	180	72	19	233	402	217
Bought (gross basis)	2,457	540	1,804	1,250	96	40	11	152	256	113
Sold (gross basis)	2,273	542	1,627	1,275	84	32	8	82	146	104
Gross market values	82	20	59	31	3	2	1	11	11	3
Positive (gross basis)	50	20	29	15	2	1	0	4	6	1
Negative (gross basis)	50	18	30	15	2	1	0	7	6	2
Of which: index products										
Notional amounts outstanding	3,899	468	3,226	2,456	134	63	13	185	374	206
Bought (gross basis)	2,229	458	1,667	1,220	67	33	6	108	234	104
Sold (gross basis)	2,138	477	1,559	1,237	66	30	8	78	140	101

Further information is available at www.bis.org/statistics/index.htm

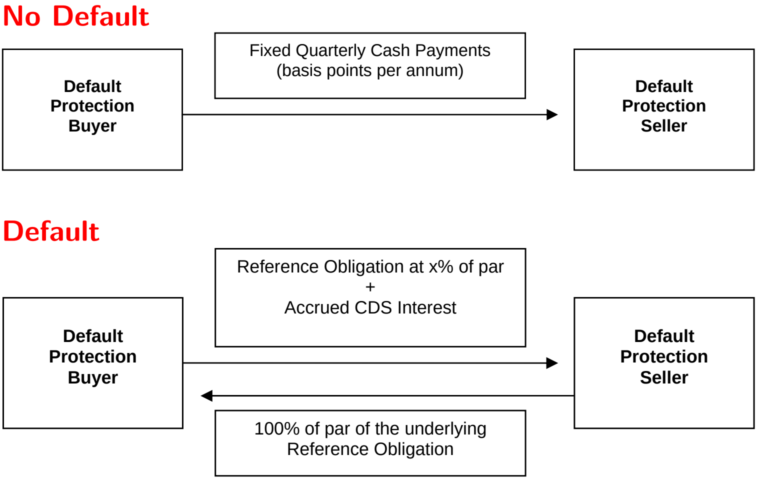
What is a CDS Contract?

Credit Default Swap

The buyer of a CDS acquires protection from the seller against a default by a particular company or country (reference entity)

- A CDS contract can be interpreted as an **insurance contract against the default of a third party**.
- A credit default swap is an agreement between two counterparties that allows one counterparty to "long" a third-party credit risk, and the other counterparty to "short" the credit risk.
- The most controversial area of the specification of the contract is **definition of the credit event** (filing for bankruptcy, failure to pay, etc.)

CDS Mechanics



Example: AT&T

- On 09/16/19 two counterparties A and B enter into a 5 year CDS on a notional value of USD 100m in which A pays B 76.67bps (=0.7667 %) per year, i.e. 766,700 dollars.
- The reference entity is AT&T Corp.
- If AT&T defaults at any time within that five years, then A will get an insurance payment from B.

Q: Why could such a contract cause problems?

Some CDS Quotes from 09/16/19

- USA: 14.28bps
- Argentina: 5138.52bps
- Citigroup: 45.37bps
- IBM: 34.60bps
- Home Depot: 16.78bps
- Kraft Heinz Foods Co: 87.60bps
- Whirlpool: 81.86bps
- ...

Notice that these are 'conventional spreads'. Since the Big Bang CDS are traded 'upfront + coupon'.

Example. A spread of 123.74bps translates into an upfront payment of 1.1105 and a running spread of 100bps.

What if a Credit Event Occurs?

Two possible **settlement** procedures:

1. **Physical** settlement: Protection buyer delivers bond, seller pays notional
2. **Cash** settlement: Protection buyer receives loss in cash from seller

Some **recent credit events** and their **recovery rates**

- Lehman Brothers (Oct 08): 8.625%
- Washington Mutual (Oct 08): 57%
- General Motors (Jun 09): 12.5%
- FGIC (Jan 7, 10): 26%
- AMR Corp (Dec 15, 11): 23.5%
- Eastman Kodak (Feb 22, 12): 23.875%
- Argentine Republic (Sep 3, 14): 39.5%
- RadioShack (Mar 5, 15): 11.5%
- Peabody Energy Corporation (April 13, 16): 11%

Doc Clause: What Credit Event Triggers Settlement?

Restructuring Clause - Defines the credit events that trigger settlement. This is a key element as CDS spreads are higher for contracts with a broader range of credit events (i.e. more events can trigger the payment to the Protection Buyer, therefore the CDS protection is more valuable), and/or fewer restrictions on the Protection Buyer's settlement obligations (i.e. the more flexibility a Protection Buyer has to deliver a bond, the more valuable the CDS contract). Variations include:

CR – Complete Restructuring (a.k.a. full restructuring, **FR**): Any restructuring event qualifies as a credit event and any bond of maturity up to 30 years is deliverable. This is standard for EM and MCDX trades. It was the standard for IG and HY trades but was replaced by MR in 2001.

MR – Modified Restructuring: Restructuring agreements count as a credit event, but the deliverable obligation against the contract has to be limited to those with a maturity of 30 months or less after the termination date of the CDS contract or the reference obligation that is restructured (regardless of maturity). Generally used for IG trades in the US. This doc-clause started in 2001.

MM – "Modified-Modified" restructuring: In 2003, market participants in Europe found the 30 months limit on deliverable bonds to be too restrictive, so MM was introduced with a maturity limit of 60 months for restructured obligations and 30 months for all other obligations. This is used mostly in Europe.

XR – No Restructuring (a.k.a. **NR**): All restructuring events are excluded as trigger events. This is prevalent in the high yield market.

Source: Markit

In 2014 these definitions were updated, but the main ideas are similar.

Credit Indices: CDX and iTraxx

- The **CDX-NAIG** index is an equally weighted portfolio of 125 investment grade North American companies.
- The **CDX-NAHY** index is an equally weighted portfolio of 100 below investment grade North American companies.
- The **iTraxx Europe** is an equally weighted portfolio of 125 investment grade European companies.
- On 09/16/19, the spread of the 5y CDX-NAIG, Series 32, was 50.13bps (price quote: 102.2707, running spread 100bps).
- This means that a portfolio of 125 CDS contracts on the CDX companies can be bought for 50.13bps, e.g., USD 800,000 of 5-year protection on each name (total USD 100m) could be purchased for USD 501,300 per year. When a company defaults, the annual payment is reduced by 1/125.
- On 09/16/19 the conventional spread on the 5y CDX-NAHY, Series 32, was about 316.65bps (price quote: 107.6604, running spread 500bps).

CDX-NAIG: Some Constituents over Time

Source: Markit

Clearable Constituents	IG.9	IG.10	IG.11	IG.12	IG.13	IG.14	IG.15	IG.16	IG.17	IG.18	IG.19	IG.20	IG.21	IG.22	IG.23	IG.24	IG.25	IG.26	IG.27	IG.28
21st Century Fox America, Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
ACE INA Holdings Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X			
ADVANCED MICRO DEVICES, INC.																				
Aetna Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
AK Steel Corporation																				
Alcatel-Lucent USA Inc.																				
Arconic Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X						
Ally Financial Inc.																				
Altria Group, Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
American Axle & Manufacturing, Inc.																				
AMERICAN ELECTRIC POWER COMPA	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
American Express Company	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
American International Group, Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Amgen Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Amkor Technology, Inc.																				
Anadarko Petroleum Corporation	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
APACHE CORPORATION																X	X	X	X	X
ARROW ELECTRONICS, INC.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
ASSURED GUARANTY MUNICIPAL CORP.													X	X	X	X	X	X	X	X
AT&T Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
AutoZone, Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Avis Budget Group, Inc.																				
Avnet, Inc.				X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Avon Products, Inc.													X	X	X					
Bank of America Corporation	X																			
BARRICK GOLD CORPORATION			X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Baxter International Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Beam Suntory Inc.	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X				
BEAZER HOMES USA, INC.																				
Berkshire Hathaway Inc.								X	X	X	X	X	X	X	X	X	X	X	X	X
Best Buy Co., Inc.																	X	X	X	X
Block Financial LLC												X	X	X	X	X	X	X	X	X
Boeing Capital Corporation	X	X	X	X	X	X	X	X	X	X	X	X	X	X						
BOMBARDIER INC.																				
Boston Properties Limited Partnership				X	X	X	X													
Boston Scientific Corporation									X	X	X	X	X	X	X	X	X	X	X	X
Bristol-Myers Squibb Company	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
BRUNSWICK CORPORATION		X																		
Burlington Northern Santa Fe, LLC	X	X	X	X	X	X	X													
CA, Inc.							X	X	X	X										
CalAtlantic Group, Inc.																				
Campbell Soup Company	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X

Portfolio of Corporate Bond and CDS

Assume that you hold a corporate bond and a corresponding CDS.
Two cases:

1. **No default** until maturity: You receive "coupon minus CDS premium" and the notional at maturity.
2. Premature **default**: You receive "coupon minus CDS premium" until default. At default you are suffering a loss, but this loss is covered by the seller of the CDS contract.

Therefore, your portfolio is similar to a **default-free bond**.

Q: Can we quantify the difference between the coupon and the CDS premium?

What is the Interpretation of the CDS Spread?

We consider a one-period model. In this model

Payoff of Defaultable Zero and CDS

State at $t = 1$	Zero	CDS
firm solvent	1	0
firm insolvent	R	L

PV of protection payment is

$$V^{prot} = p(0,1) \cdot \Big(0 \cdot (1 - \mathrm{PD}^Q) + L \cdot \mathrm{PD}^Q\Big) = p(0,1) \cdot \mathrm{EL}^Q$$

PV of fee payment is $\qquad \widehat{V}^{fee} = p(0,1)S$

where S is CDS spread (paid in arrear). S is fixed such that

$$\widehat{V}^{fee} = V^{prot} \quad \Longrightarrow$$

Rule: CDS Spread $\approx$ Bond Spread

$$S = \mathrm{EL}^Q$$

Collateralized Debt Obligations: Example

Consider four investors, A, B, C, and D, investing USD 100m each in corporate bonds issued by JC Penney, Sprint Nextel, Macy's, and US Steel.

1st Setting.

- A holds the bond issued by JC Penney.
- B holds the bond issued by Sprint.
- C holds the bond issued by Macy's.
- D holds the bond issued by US Steel.

Q: What happens if US Steel defaults and the recovery rate is 50%?

$\longrightarrow$ Investor D loses USD 50m, the other investors are not affected.

Example: Diversification

Consider four investors, A, B, C, and D, investing USD 100m each in corporate bonds issued by JC Penney, Sprint, Macy's, and US Steel.

2nd Setting: The investors invest in a fund consisting of these bonds:

- A holds $1/4$ of each bond.
- B holds $1/4$ of each bond.
- C holds $1/4$ of each bond.
- D holds $1/4$ of each bond.

Q: What happens if US Steel defaults and the recovery rate is 50%?

$\longrightarrow$ All investors lose USE 12.5m (=50/4)

Example: Tranching

Possible problem: Some investors are legally constrained to hold below investment grade bonds (high yield bond, junk bond).

In our example: All bonds are below investment grade (S&P rating: BB).

Q: How can we create a security that meets these requirements?
3rd Setting: We create tranches of the portfolio.

For instance, three tranches with notionals of USD 50m and one tranche with notional USD 250m

- 1st tranche absorbs losses up to USD 50m.
- 2nd tranche absorbs losses from USD 50m to 100m.
- 3rd tranche absorbs losses from USD 100m to 150m.
- 4th tranche absorbs losses from USD 150m to 400m.

Example: Tranching

Q: What happens if US Steel defaults and the recovery rate is 50%?
$\longrightarrow$ Holder of 1st tranche loses all his money, the other investors are not effected.

This is true no matter which firm defaults!

Important: In practice, portfolios of 100 loans or more are tranched.

Result

By tranching a portfolio of high-yield bonds in a way that transfers virtually all default risk to junior tranches, it would still be possible for a senior tranche to carry a AAA rating.

Tranching

Question

When is tranching most efficient?

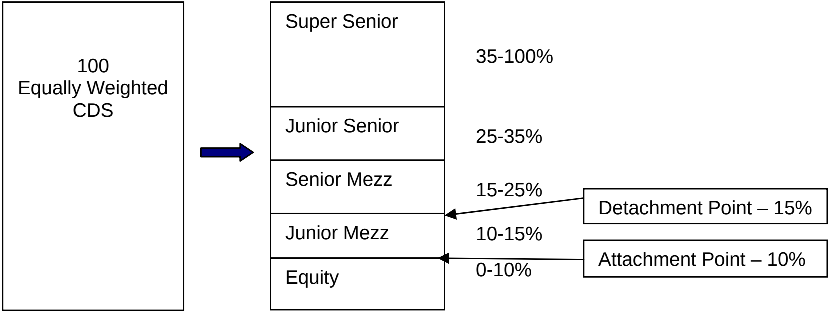
For instance, compare the following situations:

- Tranching a portfolio consisting of bonds issued by one particular firm.
- Tranching a portfolio consisting of bonds issued by firms of the same industry (e.g. department stores).
- Tranching a portfolio consisting of bonds issued by firms of different industries.

CDO Structuring

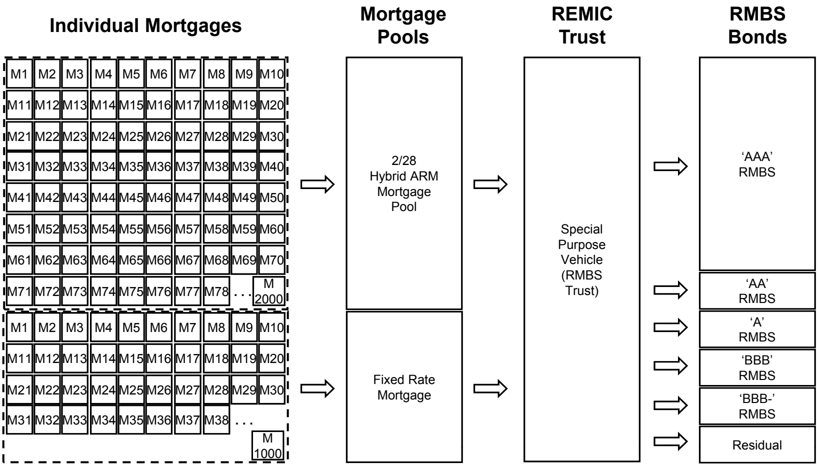
- A pool of debt issues are put into a special purpose trust
- Trust issues claims against the debt in a number of tranches
- First tranche covers x% of notional and absorbs first x% of default losses
- Second tranche covers y% of notional and absorbs next y% of default losses, and so on...
- A tranche earns a promised yield on remaining principal in the tranche

CDO Structuring



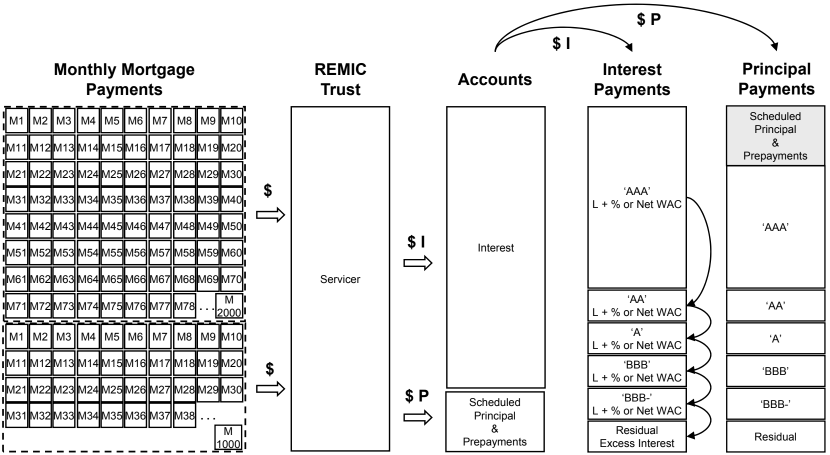
Source: Markit

CDO Structuring: Mortgage-Backed Securities



Source: Fitch, RMBS: Residential Mortgage-Backed Securities,
REMIC: Real Estate Mortgage Investment Conduit

CDO: Waterfall



Source: Fitch

Different Kinds of CDOs

In general, CDOs are financial claims to the cash flows generated by a portfolio of debt securities.

Therefore, there exist a lot of different versions.

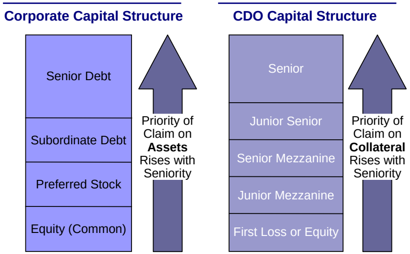
- Corporate vs. asset backed CDOs
- Different underlyings: Home-equity loans, credit-card receivables, commercial mortgages, ...
- Synthetic CDOs: Underlying are CDS contracts.
- For instance, indices: CDX, iTraxx

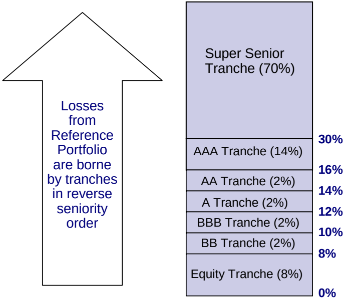
Example: ABS CDO

Tranche	Notional Amount	Price to Public	Under-Writer Fee	Initial Pass-thru Rate	Maturity	Initial Moody's Rating	Initial S&P Rating	Seniority Ranking
AF-1	147,232,000	100.0000	0.0521	L+ 0.130%	Nov 2025	Aaa	AAA	1/7
AF-2	22,857,000	99.9995	0.1042	5.281%	May 2027	Aaa	AAA	1/7
AF-3	90,995,000	99.9998	0.1563	5.384%	Jul 2033	Aaa	AAA	1/7
AF-4	21,633,000	99.9985	0.2500	5.714%	Sep 2034	Aaa	AAA	1/7
AF-5	38,617,000	99.9987	0.3333	5.884%	Jul 2036	Aaa	AAA	1/7
AF-6	44,200,000	99.9980	0.4167	5.526%	May 2036	Aaa	AAA	1/7
MF-1	13,260,000	99.9981	0.4167	5.917%	May 2036	Aa1	AA+	2/7
MF-2	12,155,000	99.9972	0.5000	6.016%	May 2036	Aa2	AA+	3/7
MF-3	7,293,000	99.9965	0.5833	6.115%	Apr 2036	Aa3	AA	4/7
MF-4	6,409,000	99.4627	0.8333	6.200%	Apr 2036	A1	AA-	5/7
MF-5	6,188,000	98.9985	1.0000	6.200%	Mar 2036	A2	A+	6/7
MF-6	5,525,000	98.5371	1.2500	6.200%	Feb 2036	A3	A	7/7
AV-1	139,560,000	100.0000	0.0522	L+0.080%	Jul 2028	Aaa	AAA	1/8
AV-2	115,712,000	100.0000	0.1033	L+0.190%	May 2035	Aaa	AAA	1/8
AV-3	25,042,000	100.0000	0.1033	L+0.300%	Jun 2036	Aaa	AAA	1/8
MV-1	14,320,000	100.0000	0.4167	L+0.390%	May 2036	Aa1	AA+	2/8
MV-2	13,067,000	100.0000	0.5000	L+0.410%	May 2036	Aa2	AA+	3/8
MV-3	7,518,000	100.0000	0.8333	L+0.440%	May 2036	Aa3	AA	4/8
MV-4	6,802,000	100.0000	0.9167	L+0.560%	Apr 2036	A1	AA-	5/8
MV-5	6,802,000	100.0000	0.9667	L+0.600%	Apr 2036	A2	A+	6/8
MV-6	5,907,000	100.0000	1.0000	L+0.660%	Mar 2036	A3	A	7/8
MV-7	5,549,000	100.0000	1.0833	L+1.300%	Mar 2036	Baa1	A	8/8

Longstaff (2008), Subprime Credit Crisis and Contagion in Financial Markets: Countrywide Subprime ABS CDO Structure CWABS 2006-1.

CDO vs. Bank





Source: Markit

Agenda: Single-Name Credit

Independence Assumption

- In many credit risk models, it is assumed that
  - **interest** rate risk, i.e. the short rate r
  - **default** risk, i.e. the default time τ of an entity

  are independent
- Under this assumption pricing of zeros **with zero recovery** is simplified considerably:

$$\begin{aligned} p^d(0,T) &= \mathrm{E}[e^{-\int_0^T r_s\,ds}\mathbf{1}_{\{\tau>T\}}] \\ &= \mathrm{E}[e^{-\int_0^T r_s\,ds}]\mathrm{E}[\mathbf{1}_{\{\tau>T\}}] = p(0,T)Q(\tau>T) \end{aligned}$$

- Empirically, this assumption is not correct
- However, imposing this assumption makes models much more **tractable**
- Besides, calibration takes care of most of the problem.

Pricing a Defaultable Zero-Coupon Bond with Recovery

- Now, consider a zero-coupon bond and assume that **recovery** occurs **at maturity**
- For a zero bond this is a reasonable assumption since no payment is due before maturity
- Then the payoff at maturity is given by

$$p^d(T,T) = 1_{(\tau > T)} + (1-\ell)\,1_{(\tau \leq T)} = 1 - \ell 1_{(\tau \leq T)}$$

- Furthermore, **assume** that
  - ℓ is constant,
  - the short rate r is independent of τ.
- Then we obtain

$$\begin{aligned} p^d(0,T) &= \mathrm{E}\left[\frac{p^d(T,T)}{B(T)}\right] = p(0,T) - \ell \mathrm{E}\left[\frac{1_{(\tau \leq T)}}{B(T)}\right] \\ &= p(0,T) - \ell p(0,T)\, Q(\tau \leq T) \end{aligned}$$

Defaultable Coupon Bonds

- A coupon bond consists of
  - coupon payments c_k at times t_k, $k = 1, \dots, K$,
  - a notional payment at maturity T.
- If a default occurs, then a recovery payment is received.
- We assume that the bond has a unit face value.
- Notice that $t_K = T$.
- If Δ_k denotes the time span between coupon payments c_{k-1} and c_k, then $c_k = \Delta_k C$ where C denotes the annualized coupon rate (e.g. 5%).
- We disregard accrued interest payments.

Pricing of Defaultable Coupon Bonds

Intuitively, the price of a defaultable coupon bond is given by:

Price of Defaultable Coupon Bond

$$
\begin{aligned}
\text{Price} &= \text{PV of coupons with zero recovery} \\
&+ \text{PV of notional with zero recovery} \\
&+ \text{PV of recovery payment}
\end{aligned}
$$

- Modeling recovery is more tricky for coupon bonds
- This is because it does not make sense to assume that recovery happens at maturity
- We thus postpone a more detailed discussion to a later section
- It is however easier to formalize the pricing with zero recovery

Pricing of Defaultable Coupon Bonds with Zero Recovery

The price of coupon bond with **zero recovery** is given by

$$p_c^d(0,T) = \sum_{k=1}^{K} \mathrm{E}\left[e^{-\int_0^{t_k} r_u\,du} c_k \mathbf{1}_{\{\tau > t_k\}}\right] + \mathrm{E}\left[e^{-\int_0^{T} r_u\,du} \mathbf{1}_{\{\tau > T\}}\right]$$

Under **independence** this becomes

$$p_c^d(0,T) = \sum_{k=1}^{K} c_k p(0,t_k) Q(\tau > t_k) + p(0,T) Q(\tau > T)$$

Outlook: Why Digging Deeper?

- We have already obtained pretty explicit results
- However, we cannot model recovery in sophisticated way
- Besides, the survival probability $Q(\tau > t)$ is still a black box
- To get **more explicit results** we now make **additional assumptions** about the survival probability
- This is also important if we want to price other contracts like CDS

Agenda: Single-Name Credit

Disclaimer

Unless otherwise stated

- all expectations are taken with respect to the risk-neutral measure Q,
- all intensities are risk-neutral intensities.

Modeling Default Risk

- Consider a firm that has issued a defaultable bond
- The default time of the firm is denoted by τ
- Disregarding recovery the payoff of the bond at maturity T is

$$p^d(T,T) = \mathbf{1}_{\{\tau > T\}}$$

- Assume that default-free interest rates are constant.
- Then the time-0 price of a Treasury bond is

$$p(0,T) = e^{-rT}$$

- The time-0 price of the corporate bond is thus

$$p^d(0,T) = e^{-rT}\mathrm{E}^Q[p^d(T,T)] = e^{-rT}\mathrm{E}^Q[\mathbf{1}_{\{\tau > T\}}] = p(0,T)Q(\tau > T)$$

Moral

We need to model the (risk-neutral) survival probability $Q(\tau > T)$

Modeling Default Risk

Q: What is a reasonable model for $Q(\tau > t)$?

Important Properties of $Q(\tau > t)$

- $Q(\tau > 0) = 1$
- $Q(\tau > t) > 0$
- $Q(\tau > t)$ monotonously decreasing in t

Q: Have we encountered something similar in Finance?

A: Yes, zero-bond prices (= discount factors) $p(0, t) = e^{-rt}$

Idea: Let's try a similar model for survival probabilities

$$Q(\tau > t) = e^{-\lambda t}$$

with a constant λ

Q: What is the meaning of the so-called intensity λ?

What Is the Default Intensity?

- The default probability (until t) is given by

$$F(t) \equiv Q(\tau \leq t) = 1 - e^{-\lambda t}$$

- Therefore,

$$F'(t) = \lambda e^{-\lambda t}$$

- and we can approximate the default probability

$$F(t) \approx \underbrace{F(0)}_{=0} + \underbrace{F'(0)}_{=\lambda}(t - 0) = \lambda t$$

- Consequently, we arrive at

$$F(1) = Q(\tau \leq 1) = \lambda$$

First Interpretation of the Intensity λ

λ is approximately the one-year default probability

What Is the Default Intensity?

- Now, back to pricing
- The price of a defaultable zero with zero recovery is

$$p^d(0,T) = e^{-rT} Q(\tau > T) = e^{-(r+\lambda)T}$$

- Compare this with the price of a Treasury zero

$$p(0,T) = e^{-rT}$$

- The yields are defined as

$$y^d(T) \equiv \frac{1}{T} \ln\left(\frac{1}{p^d(0,T)}\right) = r + \lambda \quad y(T) \equiv \frac{1}{T} \ln\left(\frac{1}{p(0,T)}\right) = r$$

- Therefore, the spread is given by

$$y^d(T) - y(T) = \lambda$$

Second Interpretation of the Intensity λ

With zero recovery λ is the spread of the defaultable bond

1. Time-t price of default-free zero with maturity $t+\Delta$:

$$p(t,t+\Delta) = e^{-r\Delta}$$

2. Time-t survival probability until $t+\Delta$:

$$q(t,t+\Delta) \equiv Q_t(\tau > t+\Delta) = e^{-\lambda\Delta}$$

Therefore, we obtain

1. Interest earned from t to $t+\Delta$:

$$1 - \underbrace{e^{-r\Delta}}_{\approx 1 - r\Delta} \approx r\Delta$$

2. Default probability from t to $t+\Delta$:

$$1 - \underbrace{e^{-\lambda\Delta}}_{\approx 1 - \lambda\Delta} \approx \lambda\Delta$$

Keeping Track of Defaults

- In frameworks modeling default risk, we must be able to keep track of the number of defaults:
  - For a single bond, the model should tell us whether at a certain point of time t the firm has defaulted
  - For a portfolio of bonds, we must know how many firms have already defaulted
- Therefore, we need a default counter:

$$N_t \equiv \# \text{ of defaults until } t$$

Properties of the Default Counter

1. N starts at zero, i.e. $N_0 = 0$
2. At every point in time, either a default occurs or not, i.e. $\Delta N_t = 0$ or 1.
3. Every path is a step function

Formally, this counter is said to be a counting process

Example for a Default Counter: Single Bond

- Consider a firm that has issued a defaultable zero bond
- At a particular point in time $t \geq 0$ the firm is either
  - solvent, i.e. $\tau > t$ and $N_t = 0$ or
  - in default, i.e. $\tau \leq t$ and $N_t = 1$
- So N just tells us whether the firm is in default or not
- We can express default and survival probabilities in terms of N
- The survival probability is given by

$$Q(\tau > t) = Q(N_t = 0)$$

- The default probability is given by

$$Q(\tau \leq t) = Q(N_t = 1)$$

- In our simple model with constant intensity λ we have

$$Q(\tau > t) = Q(N_t = 0) = e^{-\lambda t} \quad Q(\tau \leq t) = Q(N_t = 1) = 1 - e^{-\lambda t}$$

Formalizing the Previous Ideas

Poisson Distribution

A random variable X is called Poisson distributed with parameter α if

$$P(X = k) = e^{-\alpha}\frac{\alpha^k}{k!}, \qquad k \in \mathbb{N}_0.$$

Formalizing the Previous Ideas

Let $\mathcal{F}_t$ denote the information until t.

Poisson Process

Fix $\lambda > 0$. A stochastic process $N = \{N_t\}_{t \geq 0}$ is said to be a Poisson process with intensity λ if

1. N has right-continuous paths and is piecewise constant.
2. $N_0 = 0$, and $\Delta N_t \equiv N_t - N_{t-} = 0$ or 1.
3. For all $s \leq t$, $N_t - N_s$ is independent of $\mathcal{F}_s$. Furthermore, $N_t - N_s$ is Poisson distributed with parameter $\lambda(t-s)$, i.e. for $k \in \mathbb{N}_0$

$$P(N_t - N_s = k \mid \mathcal{F}_s) = P(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{\lambda^k (t-s)^k}{k!}.$$

A stochastic process that satisfies conditions (i)-(ii) is called a (univariate) point process (syn. counting process). A process that only fulfils (i) is called a jump process.

Typical Application of a Counting Process

- Situation where one is interested in certain events that happen at stochastic points in time.
- Standard application: Arrivals of customers at a queue.
- Finance: Number of defaults in a portfolio.
- In this case, N_t counts the number of defaults that have occurred in the interval $[0, t]$.

What Is an Intensity?

- We can loosely interpret dN_t as $N_{t+dt} - N_t$.
- If N is a Poisson process, then the conditional jump probability over the interval $[t, t+h]$ is given by

$$P(N_{t+h} - N_t = 1 \mid \mathcal{F}_t) = e^{-\lambda h}\lambda h = \lambda h \sum_{n \geq 0} \frac{(-\lambda h)^n}{n!}.$$

- If h becomes small, then terms of higher order can be disregarded:

$$P(dN_t = 1 \mid \mathcal{F}_t) = \lambda dt.$$

- Therefore, λ is the conditional jump probability per time unit.
- Alternatively, one can interpret λ as the expected number of jumps per time unit.

Is a Poisson Process a Martingale?

A: NO!!!

- A Poisson process is increasing (paths are step functions)
- This violates the martingale property.
- But one can turn a Poisson process into a martingale by subtracting the expected number of jumps.
- The fancy way of saying this is that the Poisson process is compensated.

Compensated Poisson Process

Let N be an Poisson process with intensity λ. Then

$$M_t = N_t - \lambda t.$$

is a martingale.

Integration and Poisson Processes

- A stochastic integral is given by $\int_0^t X_s dW_s$, where the Brownian motion W is a martingale.
- Therefore, it makes sense to study $\int_0^t X_s dM_s$, where the compensated Poisson process M is a martingale.
- Generic example: Asset dynamics of the form

$$dS_t = S_t[\mu dt + \sigma dW_t - L dM_t]$$

Martingale Property

Let M be a compensated Poisson process with intensity λ, i.e. $M_t = N_t - \lambda t$. Then

$$\int_0^t X_s dM_s \equiv \int_0^t X_s dN_s - \int_0^t X_s \lambda ds$$

is a martingale if $\mathrm{E}[\int_0^t X_s \lambda \, ds] < \infty$ and X is predictable.

Ito's Formula with Jumps

Consider asset dynamics of the form

$$dS_t = S_t[\mu dt + \sigma dW_t - L dM_t]$$

Q: How do the price dynamics of a derivative on this asset look like?

A: We need an Ito formula with jumps!

Goal

Derive the dynamics $dZ_t = ...$ if $Z_t = F(t, X_t)$ and

$$dX_t = f_t dt + g_t dW_t + h_t dN_t.$$

Ito's Formula with Jumps

For the moment, disregard the Brownian motion, i.e.

$$dX_t = f_t dt + h_t dN_t,$$

1. **Between two jumps** of N we have

$$dX_t = f_t dt.$$

2. If N **jumps** at time t, then the path of X has discontinuity with jump size

$$\Delta X_t \equiv X_t - X_{t-} = h_t \Delta N_t = h_t.$$

   Note that the jump size of a counting process is always $+1$.

Ito's Formula with Jumps

Now, consider $Z_t = F(t, X_t)$.

1. **Between two jumps** of N we have (ordinary Ito formula!)

$$dZ_t = \frac{\partial F}{\partial t}(t, X_t)dt + \frac{\partial F}{\partial x}(t, X_t)dX_t = \left\{ \frac{\partial F}{\partial t}(t, X_t) + \frac{\partial F}{\partial x}(t, X_t) f_t \right\} dt.$$

2. If N **jumps** at time t, then Z jumps by

$$\Delta Z_t \equiv Z_t - Z_{t-} = F(t, X_t) - F(t-, X_{t-}).$$

Since $X_t = X_{t-} + h_t$, we get

$$\Delta Z_t = F(t, X_{t-} + h_t) - F(t, X_{t-}).$$

Ito Formula

$$dF(t, X_t) = \left\{ \frac{\partial F}{\partial t}(t, X_t) + \frac{\partial F}{\partial x}(t, X_t) f_t \right\} dt + \left\{ F(t, X_{t-} + h_t) - F(t, X_{t-}) \right\} dN_t.$$

Notice that $dN_t = 1$ if N jumps (and zero else).

Ito's Formula vs. Taylor

A Taylor expansion yields

$$\begin{aligned}\frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}\{f_t dt + h_t dN\} \\ &= \left\{\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}f_t\right\}dt + \frac{\partial F}{\partial x}h_t dN.\end{aligned}$$

Compare with Ito formula

$$dF = \left\{\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}f_t\right\}dt + \Delta F,$$

where $\Delta F(t, X_t) \equiv \{F(t, X_{t-} + h_t) - F(t, X_{t-})\}dN_t$.

Ito Formula Again

$$\begin{aligned}dF(t, X_t) &= \frac{\partial F}{\partial t}(t, X_t)dt + \frac{\partial F}{\partial x}(t, X_{t-})dX_t \\ &\quad + \left\{\Delta F(t, X_t) - \frac{\partial F}{\partial x}(t, X_{t-})\Delta X_t\right\}.\end{aligned}$$

Ito's Formula with Brownian Motion and Jumps

Ito Formula II

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \left\{ \frac{\partial F}{\partial t}(s, X_s) + \frac{1}{2}{\beta_s}^2 \frac{\partial^2 F}{\partial x^2}(s, X_s) \right\} ds \\ &\quad + \int_0^t \frac{\partial F}{\partial x}(s, X_{s-}) dX_s \\ &\quad + \sum_{s \le t} \left\{ \Delta F(s, X_s) - \frac{\partial F}{\partial x}(s, X_{s-}) \Delta X_s \right\}, \end{aligned}$$

where

$$dX_t = \alpha_t dt + \beta_t dW_t + \gamma_t dN_t.$$

Ito's Formula with Brownian Motion and Jumps

1. The differential version

$$dF(t,X_t) = [F_t(t,X_t) + 0.5{\beta_t}^2 F_{xx}(t,X_t)]dt + F_x(t,X_{t-})dX_t + \Delta F(t,X_t) - F_x(t,X_{t-})\Delta X_t$$

2. Set $dX_t^c = \alpha_t dt + \beta_t dW_t$. Then $d < X^c >_t = \beta_t^2 dt$ and

$$\begin{aligned} dF(t,X_t) &= F_t(t,X_t)dt + F_x(t,X_t)dX_t^c + 0.5F_{xx}(t,X_t)d < X^c >_t \\ &\quad + \Delta F(t,X_t) \\ &= F_t(t,X_t)dt + F_x(t,X_t)dX_t^c + 0.5F_{xx}(t,X_t)d < X^c >_t \\ &\quad + [F(t,X_t) - F(t,X_{t-})]dN_t \end{aligned}$$

Short Hand Version

$$\begin{aligned} dF &= F_t dt + F_x dX^c + 0.5F_{xx} d < X^c > + \Delta F \\ &= F_t dt + F_x dX^c + 0.5F_{xx} d < X^c > + [F - F^-]dN \end{aligned}$$

Ito Formula: Product Rule

For two processes

$$\begin{aligned} dX_t &= \alpha_t dt + \beta_t dW_t + \gamma_t dN_t, \quad X_0 = x_0, \\ dY_t &= \mu_t dt + \sigma_t dW_t + \chi_t dN_t, \quad Y_0 = y_0, \end{aligned}$$

we obtain

$$d(X_t Y_t) = X_{t-} dY_t + Y_{t-} dX_t + \Delta X_t \Delta Y_t + d < X^c, Y^c >_t,$$

$X_0 Y_0 = x_0 y_0$, where

$$\begin{aligned} d < X^c, Y^c >_t &= \beta_t \sigma_t dt, \\ \Delta X_t \Delta Y_t &= \gamma_t \chi_t dN_t. \end{aligned}$$

Linear Stochastic Differential Equations with Jumps

Variation of Constants

The solution to the linear inhomogenous SDE

$$dX_t = [A_t X_t + a_t]dt + [B_t X_t + b_t]dW_t + [C_t X_{t-} + c_t]dN_t, \quad X_0 = \xi,$$

where A, a, B, b, $C > -1$ and c are stochastic processes, is given by

$$X_t = Z_t \left(\xi + \int_0^t \frac{1}{Z_u}[a_u - B_u b_u]du + \int_0^t \frac{1}{Z_u} b_u \, dW_u + \int_0^t \frac{1}{Z_{u-}} \frac{c_u}{1 - C_u} dN_u \right),$$

where

$$Z_t = \exp\left(\int_0^t [A_u - \frac{1}{2} B_u^2] \, du + \int_0^t B_u \, dW_u \right) \prod_{\tau_n \leq t} (1 + C_{\tau_n})$$

is the solution to $dZ_t = A_t Z_t dt + B_t Z_t dW_t + C_t Z_{t-} dN_t$, $Z_0 = 1$, (linear SDE) and τ_n is the n-th jump time of N.

Application: Jump-Diffusion Model

The asset dynamics (constant μ, σ, L)

$$dS_t = S_t[\mu dt + \sigma dW_t - L dN_t]$$

have the solution ($L > -1$)

$$\begin{aligned} S_t &= s_0 \prod_{\tau_n \leq t} (1 - L) e^{(\mu - 0.5\sigma^2)t + \sigma W_t} \\ &= s_0 (1 - L)^{N_t} e^{(\mu - 0.5\sigma^2)t + \sigma W_t}. \end{aligned}$$

Martingale Representation with Poisson Processes

We already know that a compensated Poisson process is a martingale. Applying Ito's formula, we can also show the converse.

Assume that **all randomness** (more precisely, the filtration) is **generated by a Poisson process**.

Martingales

Let N be an (adapted) counting process. Assume that there exists a real valued $\lambda > 0$ such that the process M given by

$$M_t = N_t - \lambda t$$

is a martingale. Then, N is an Poisson process with intensity λ.

Point Process

Q: Why point processes?
A:

- In applications, the assumption of a constant intensity is sometimes too restrictive.
- For instance, we will see later on that spreads are basically default intensities.
- However, it is unrealistic to assume that spreads are constant over time or maturities.
- Therefore, we need to generalize our setting to stochastic intensities.
- Important: In the derivation of the Ito formula and the Variation of Constants Theorem, we have not used that intensities are constant. Consequently, the Ito formula also holds in this more general setting.

Point Process

Definition: Intensity

Let λ be a nonnegative process such that $\int_0^t \lambda_s ds < \infty$, for every $t > 0$. If for every nonnegative process h

$$\mathrm{E}\left[\int_0^\infty h_t \lambda_t dt\right] = \mathrm{E}\left[\int_0^\infty h_t dN_t\right]$$

holds, then N has the intensity λ.

One can show that the following is true.

Martingale Property

Assume that N has an intensity λ and that $\mathrm{E}[N_t] < \infty$, $t \geq 0$. Then the following process is a martingale

$$M_t = N_t - \int_0^t \lambda_s ds.$$

Point Process

- The previous proposition provides a nice interpretation of an intensity.
- In differential form, we have

$$dM_t = dN_t - \lambda_t dt.$$

- Notice that $\mathsf{E}[dM_t|\mathcal{F}_t] = 0$ since M is a martingale.
- Taking expectations thus yields

$$\mathsf{E}[dN_t|\mathcal{F}_t] = \lambda_t dt.$$

- Hence, λ can be interpreted as the expected number of jumps over the interval $[t, t + dt]$ conditioned on the information $\mathcal{F}_t$.

Martingale Representation

1. Define the filtration $\underline{\mathcal{F}}$ via $\mathcal{F}_t = \sigma\{N_s; 0 \leq s \leq t\}$, and assume that N has the $\underline{\mathcal{F}}$-intensity λ. Furthermore, let X be a right-continuous $\underline{\mathcal{F}}$-martingale. Then there exists a predictable process h such that

$$X_t = X_0 + \int_0^t h_s dM_s,$$

   where $M_t \equiv N_t - \int_0^t \lambda_s ds$.

2. Besides, h is uniquely determined and satisfies $\int_0^t |h_s| \cdot \lambda_s ds < \infty$, for all t. If M is square-integrable, i.e. $\sup_{t \geq 0} \mathrm{E}[M_t^2] < \infty$, then $\mathrm{E}\Big[\int_0^t h_s^2 \cdot \lambda_s ds\Big] < \infty$, for all t.

Change of Measure: Point Processes vs. Brownian Motion

1. Pick a risk premium θ or h
2. Define a density

$$\begin{aligned} dZ_t &= -Z_t\theta_t dW_t \quad \text{or} \\ dZ_t &= Z_{t-}h_t dM_t. \end{aligned}$$

3. Define a measure by

$$Q(A) \equiv \mathrm{E}[\mathbf{1}_A Z_T], \quad A \in \mathcal{F}_T,$$

4. Adjust the Brownian motion

$$W_t^Q = W_t + \int_0^t \theta_s \, ds.$$

or the intensity

$$\lambda_t^Q = \lambda_t(1 + h_t).$$

Change of Measure for Point Processes

Girsanov Theorem for Point Processes

Let N be a point process on $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ with $\underline{\mathcal{F}}$-intensity λ under P. Fix a predictable process h such that $h > -1$ and define L via

$$dL_t = h_t L_{t-}[dN_t - \lambda_t dt], \quad L_0 = 1.$$

Assume that $E_P[L_T] = 1$. We can then define a new probability measure $Q \prec P$ on $\mathcal{F}_T$ via $Q(A) \equiv \mathrm{E}[\mathbf{1}_A L_T]$ and N has the following $\underline{\mathcal{F}}$-intensity under Q:

$$\lambda_t^Q = \lambda_t(1 + h_t).$$

Stochastic Representation for Point Processes

Consider

$$\begin{aligned} 0 \;&=\; F_t(t,x) + \alpha(t,x)F_x(t,x) + 0.5\beta^2(t,x)F_{xx}(t,x) - r(t,x)F(t,x) \\ &\quad + \lambda(t,x)[F(t,x+\gamma(t,x)) - F(t,x)], \qquad F(T,x) = \Phi(x). \end{aligned}$$

We define

$$dX_t = \alpha(t,X_t)dt + \beta(t,X_t)dW_t + \gamma(t,X_{t-})dN_t,$$

where N is a point process with intensity $\lambda_t = \lambda(t,X_t)$.

Feynman and Kac

Assume that $F \in C^{1,2}$ solves the above PDE. Under suitable integrability conditions,

$$F(t,x) = \mathrm{E}_{t,x}\left[e^{-\int_t^T r(s,X_s)\,ds}\Phi(X_T)\right].$$

The proof works similar to the pure Brownian case and relies on the fact that $dM_t = dN_t - \lambda_t$ is a martingale increment.