Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2010)

Problem Set 1
Due: September 15, 2010

1. Express each of the following events in terms of the events A, B and C as well as the operations of complementation, union and intersection:

   (a) at least one of the events A, B, C occurs;
   (b) at most one of the events A, B, C occurs;
   (c) none of the events A, B, C occurs;
   (d) all three events A, B, C occur;
   (e) exactly one of the events A, B, C occurs;
   (f) events A and B occur, but not C;
   (g) either event A occurs or, if not, then B also does not occur.

   In each case draw the corresponding Venn diagrams.

2. You flip a fair coin 3 times, determine the probability of the below events. Assume all sequences are equally likely.

   (a) Three heads: HHH
   (b) The sequence head, tail, head: HTH
   (c) Any sequence with 2 heads and 1 tail
   (d) Any sequence where the number of heads is greater than or equal to the number of tails

3. Bob has a peculiar pair of four-sided dice. When he rolls the dice, the probability of any particular outcome is proportional to the sum of the results of each die. All outcomes that result in a particular sum are equally likely.

   (a) What is the probability of the sum being even?
   (b) What is the probability of Bob rolling a 2 and a 3, in any order?

4. Alice and Bob each choose at random a number in the interval $[0, 2]$. We assume a uniform probability law under which the probability of an event is proportional to its area. Consider the following events:

   A : The magnitude of the difference of the two numbers is greater than 1/3.
   B : At least one of the numbers is greater than 1/3.
   C : The two numbers are equal.
   D : Alice's number is greater than 1/3.

   Find the probabilities $\mathbf{P}(B)$, $\mathbf{P}(C)$, and $\mathbf{P}(A \cap D)$.

5. Mike and John are playing a friendly game of darts where the dart board is a disk with radius of 10in.

   Whenever a dart falls within 1in of the center, 50 points are scored. If the point of impact is between 1 and 3in from the center, 30 points are scored, if it is at a distance of 3 to 5in 20 points are scored and if it is further that 5in, 10 points are scored.

   Assume that both players are skilled enough to be able to throw the dart within the boundaries of the board.

   Mike can place the dart uniformly on the board (i.e., the probability of the dart falling in a given region is proportional to its area).

   (a) What is the probability that Mike scores 50 points on one throw?
   (b) What is the probability of him scoring 30 points on one throw?
   (c) John is right handed and is twice more likely to throw in the right half of the board than in the left half. Across each half, the dart falls uniformly in that region. Answer the previous questions for John's throw.

6. Prove that for any three events A, B and C, we have

$$\mathbf{P}(A \cap B \cap C) \geq \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C) - 2.$$

G1†. Consider an experiment whose sample space is the real line.

   (a) Let $\{a_n\}$ be an increasing sequence of numbers that converges to a and $\{b_n\}$ a decreasing sequence that converges to b. Show that

$$\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = \mathbf{P}([a, b]).$$

   Here, the notation $[a, b]$ stands for the closed interval $\{x \mid a \leq x \leq b\}$. *Note:* This result seems intuitively obvious. The issue is to derive it using the axioms of probability theory.

   (b) Let $\{a_n\}$ be a decreasing sequence that converges to a and $\{b_n\}$ an increasing sequence that converges to b. Is it true that

$$\lim_{n\to\infty} \mathbf{P}([a_n, b_n]) = \mathbf{P}([a, b])?$$

Note: You may use freely the results from the problems in the text in your proofs.

Problem Set 2
Due September 22, 2010

1. Most mornings, Victor checks the weather report before deciding whether to carry an umbrella. If the forecast is "rain," the probability of actually having rain that day is 80%. On the other hand, if the forecast is "no rain," the probability of it actually raining is equal to 10%. During fall and winter the forecast is "rain" 70% of the time and during summer and spring it is 20%.

   (a) One day, Victor missed the forecast and it rained. What is the probability that the forecast was "rain" if it was during the winter? What is the probability that the forecast was "rain" if it was during the summer?

   (b) The probability of Victor missing the morning forecast is equal to 0.2 on any day in the year. If he misses the forecast, Victor will flip a fair coin to decide whether to carry an umbrella. On any day of a given season he sees the forecast, if it says "rain" he will always carry an umbrella, and if it says "no rain," he will not carry an umbrella. Are the events "Victor is carrying an umbrella," and "The forecast is no rain" independent? Does your answer depend on the season?

   (c) Victor is carrying an umbrella and it is not raining. What is the probability that he saw the forecast? Does it depend on the season?

2. You have a fair five-sided die. The sides of the die are numbered from 1 to 5. Each die roll is independent of all others, and all faces are equally likely to come out on top when the die is rolled. Suppose you roll the die twice.

   (a) Let event A to be "the total of two rolls is 10", event B be "at least one roll resulted in 5", and event C be "at least one roll resulted in 1".

      i. Is event A independent of event B?

      ii. Is event A independent of event C?

   (b) Let event D be "the total of two rolls is 7", event E be "the difference between the two roll outcomes is exactly 1", and event F be "the second roll resulted in a higher number than the first roll".

      i. Are events E and F independent?

      ii. Are events E and F independent given event D?

3. The local widget factory is having a blowout widget sale. Everything must go, old and new. The factory has 500 old widgets, and 1500 new widgets in stock. The problem is that 15% of the old widgets are defective, and 5% of the new ones are defective as well. You can assume that widgets are selected at random when an order comes in. You are the first customer since the sale was announced.

   (a) You flip a fair coin once to decide whether to buy old or new widgets. You order two widgets of the same type, chosen based on the outcome of the coin toss. What is the probability that they will both be defective?

   (b) Given that both widgets turn out to be defective, what is the probability that they were old widgets?

4. Oscar has lost his dog in either forest A (with a priori probability 0.4) or in forest B (with a priori probability 0.6).

   On any given day, if the dog is in A and Oscar spends a day searching for it in A, the conditional probability that he will find the dog that day is 0.25. Similarly, if the dog is in B and Oscar spends a day looking for it there, the conditional probability that he will find the dog that day is 0.15.

   The dog cannot go from one forest to the other. Oscar can search only in the daytime, and he can travel from one forest to the other only at night.

   (a) In which forest should Oscar look to maximize the probability he finds his dog on the first day of the search?

   (b) Given that Oscar looked in A on the first day but didn't find his dog, what is the probability that the dog is in A?

   (c) If Oscar flips a fair coin to determine where to look on the first day and finds the dog on the first day, what is the probability that he looked in A?

   (d) If the dog is alive and not found by the Nth day of the search, it will die that evening with probability $\frac{N}{N+2}$. Oscar has decided to look in A for the first two days. What is the probability that he will find a live dog for the first time on the second day?

5. In solving this problem, feel free to browse problems 43-45 in Chapter 1 of the text for ideas. If you need to, you may quote the results of these problems.

   (a) Suppose that A, B, and C are independent. Use the definition of independence to show that A and $B \cup C$ are independent.

   (b) Prove that if $A_1, \ldots, A_n$ are independent events, then

$$\mathbf{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - \prod_{i=1}^{n}(1 - \mathbf{P}(A_i)).$$

G1[†]. Alice, Bob, and Caroll play a chess tournament. The first game is played between Alice and Bob. The player who sits out a given game plays next the winner of that game. The tournament ends when some player wins two successive games. Let a tournament history be the list of game winners, so for example $ACBAA$ corresponds to the tournament where Alice won games 1, 4, and 5, Caroll won game 2, and Bob won game 3.

   (a) Provide a tree-based sequential description of a sample space where the outcomes are the possible tournament histories.

   (b) We are told that every possible tournament history that consists of k games has probability $1/2^k$, and that a tournament history consisting of an infinite number of games has zero probability. Demonstrate that this assignment of probabilities defines a legitimate probability law.

   (c) Assuming the probability law from part (b) to be correct, find the probability that the tournament lasts no more than 5 games, and the probability for each of Alice, Bob, and Caroll winning the tournament.

[†]Required for 6.431; optional for 6.041

Problem Set 3
Due September 29, 2010

1. The hats of n persons are thrown into a box. The persons then pick up their hats at random (i.e., so that every assignment of the hats to the persons is equally likely). What is the probability that

   (a) every person gets his or her hat back?

   (b) the first m persons who picked hats get their own hats back?

   (c) everyone among the first m persons to pick up the hats gets back a hat belonging to one of the last m persons to pick up the hats?

   Now assume, in addition, that every hat thrown into the box has probability p of getting dirty (independently of what happens to the other hats or who has dropped or picked it up). What is the probability that

   (d) the first m persons will pick up clean hats?

   (e) exactly m persons will pick up clean hats?

2. Alice plays with Bob the following game. First Alice randomly chooses 4 cards out of a 52-card deck, memorizes them, and places them back into the deck. Then Bob randomly chooses 8 cards out of the same deck. Alice wins if Bob's cards include all cards selected by her. What is the probability of this happening?

3. (a) Let X be a random variable that takes nonnegative integer values. Show that

   $$\mathbf{E}[X] = \sum_{k=1}^{\infty} \mathbf{P}(X \geq k).$$

   Hint: Express the right-hand side of the above formula as a double summation then interchange the order of the summations.

   (b) Use the formula in the previous part to find the expectation of a random variable Y whose PMF is defined as follows:

   $$p_Y(y) = \frac{1}{b-a+1}, \qquad y = a, a+1, \ldots, b$$

   where a and b are nonnegative integers with $b > a$. Note that for $y = a, a+1, \ldots, b$, $p_Y(y)$ does not depend explicitly on y since it is a uniform PMF.

4. Two fair three-sided dice are rolled simultaneously. Let X be the difference of the two rolls.

   (a) Calculate the PMF, the expected value, and the variance of X.

   (b) Calculate and plot the PMF of X^2.

5. Let $n \geq 2$ be an integer. Show that

   $$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}.$$

   Hint: As one way of solving the problem, following from Example 1.31 in the text, think of a committee that includes a chair and a vice-chair.

G1†. A candy factory has an endless supply of red, orange, yellow, green, blue, black, white, and violet jelly beans. The factory packages the jelly beans into jars in such a way that each jar has 200 beans, equal number of red and orange beans, equal number of yellow and green beans, one more black bean than the number blue beans, and three more violet beans than the number of white beans. One possible color distribution, for example, is a jar of 50 yellow, 50 green, one black, 48 white, and 51 violet jelly beans. As a marketing gimmick, the factory guarantees that no two jars have the same color distribution. What is the maximum number of jars the factory can produce?

†Required for 6.431; optional for 6.041

Problem Set 4
Due October 6, 2010

1. Random variables X and Y have the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} c(x^2+y^2), & \text{if } x \in \{1,2,4\} \text{ and } y \in \{1,3\}, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the value of the constant c?
   (b) What is $\mathbf{P}(Y < X)$?
   (c) What is $\mathbf{P}(Y > X)$?
   (d) What is $\mathbf{P}(Y = X)$?
   (e) What is $\mathbf{P}(Y = 3)$?
   (f) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.
   (g) Find the expectations $\mathbf{E}[X]$, $\mathbf{E}[Y]$ and $\mathbf{E}[XY]$.
   (h) Find the variances $\text{var}(X)$, $\text{var}(Y)$ and $\text{var}(X+Y)$.
   (i) Let A denote the event $X \geq Y$. Find $\mathbf{E}[X \mid A]$ and $\text{var}(X \mid A)$.

2. The newest invention of the 6.041/6.431 staff is a three-sided die with faces numbered 1, 2, and 3. The PMF for the result of any one roll of this die is

$$p_X(x) = \begin{cases} 1/2, & \text{if } x = 1, \\ 1/4, & \text{if } x = 2, \\ 1/4, & \text{if } x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

   Consider a sequence of six independent rolls of this die, and let X_i be the random variable corresponding to the ith roll.

   (a) What is the probability that exactly three of the rolls have result equal to 3?
   (b) What is the probability that the first roll is 1, given that exactly two of the six rolls have result of 1?
   (c) We are told that exactly three of the rolls resulted in 1 and exactly three resulted in 2. Given this information, what is the probability that the sequence of rolls is 121212?
   (d) Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3's.

3. Suppose that X and Y are independent, identically distributed, geometric random variables with parameter p. Show that

$$\mathbf{P}(X = i \mid X + Y = n) = \frac{1}{n-1}, \qquad \text{for } i = 1, 2, \ldots, n-1.$$

4. Consider 10 independent tosses of a biased coin with a probability of heads of p.

   (a) Let A be the event that there are 6 heads in the first 8 tosses. Let B be the event that the 9th toss results in heads. Show that events A and B are independent.

   (b) Find the probability that there are 3 heads in the first 4 tosses and 2 heads in the last 3 tosses.

   (c) Given that there were 4 heads in the first 7 tosses, find the probability that the 2nd head occurred during the 4th trial.

   (d) Find the probability that there are 5 heads in the first 8 tosses and 3 heads in the last 5 tosses.

5. Consider a sequence of independent tosses of a biased coin at times $t = 0, 1, 2, \ldots$. On each toss, the probability of a 'head' is p, and the probability of a 'tail' is $1 - p$. A reward of one unit is given each time that a 'tail' follows immediately after a 'head.' Let R be the total reward paid in times 1, 2, ..., n. Find $\mathbf{E}[R]$ and $\text{var}(R)$.

G1†. A simple example of a random variable is the *indicator* of an event A, which is denoted by I_A:

$$I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Prove that two events A and B are independent if and only if the associated indicator random variables, I_A and I_B are independent.

   (b) Show that if $X = I_A$, then $\mathbf{E}[X] = \mathbf{P}(A)$.

†Required for 6.431; optional for 6.041

Problem Set 5
Due October 18, 2010

1. Random variables X and Y are distributed according to the joint PDF

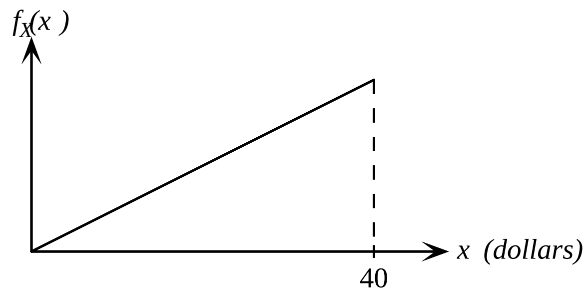
$$f_{X,Y}(x,y) = \begin{cases} ax, & \text{if } 1 \le x \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Evaluate the constant a.

   (b) Determine the marginal PDF $f_Y(y)$.

   (c) Determine the expected value of $\frac{1}{X}$, given that $Y = \frac{3}{2}$.

2. Paul is vacationing in Monte Carlo. The amount X (in dollars) he takes to the casino each evening is a random variable with the PDF shown in the figure. At the end of each night, the amount Y that he has on leaving the casino is uniformly distributed between zero and twice the amount he took in.

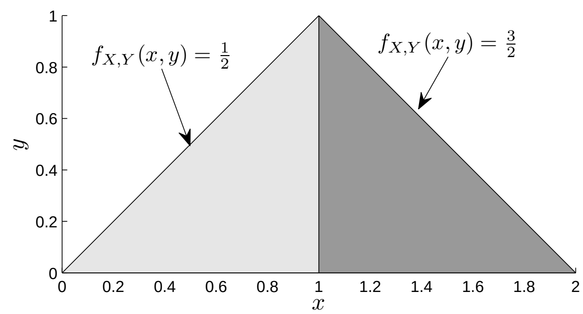


   (a) Determine the joint PDF $f_{X,Y}(x,y)$. Be sure to indicate what the sample space is.

   (b) What is the probability that on any given night Paul makes a positive profit at the casino? Justify your reasoning.

   (c) Find and sketch the probability density function of Paul's profit on any particular night, $Z = Y - X$. What is $\mathbf{E}[Z]$? Please label all axes on your sketch.

3. X and Y are continuous random variables. X takes on values between 0 and 2 while Y takes on values between 0 and 1. Their joint pdf is indicated below.



   (a) Are X and Y independent? Present a convincing argument for your answer.
   (b) Prepare neat, fully labelled plots for $f_X(x)$, $f_{Y|X}(y \mid 0.5)$, and $f_{X|Y}(x \mid 0.5)$.
   (c) Let $R = XY$ and let A be the event $X < 0.5$. Evaluate $\mathbf{E}[R \mid A]$.
   (d) Let $W = Y - X$ and determine the cumulative distribution function (CDF) of W.

4. **Signal Classification:** Consider the communication of binary-valued messages over some transmission medium. Specifically, any message transmitted between locations is one of two possible symbols, 0 or 1. Each symbol occurs with equal probability. It is also known that any numerical value sent over this wire is subject to distortion; namely, if the value X is transmitted, the value Y received at the other end is described by $Y = X + N$ where the random variable N represents additive noise that is independent of X. The noise N is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$.

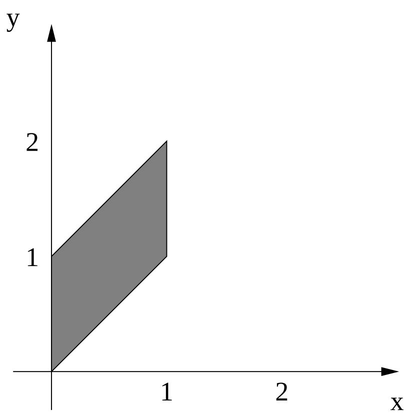
   (a) Suppose the transmitter encodes the symbol 0 with the value $X = -2$ and the symbol 1 with the value $X = 2$. At the other end, the received message is decoded according to the following rules:
   - If $Y \geq 0$, then conclude the symbol 1 was sent.
   - If $Y < 0$. then conclude the symbol 0 was sent.

   Determine the probability of error for this encoding/decoding scheme. Reduce your calculations to a single numerical value.

   (b) In an effort to reduce the probability of error, the following modifications are made. The transmitter encodes the symbols with a repeated scheme. The symbol 0 is encoded with the vector $\overline{X} = [-2, -2, -2]^\mathsf{T}$ and the symbol 1 is encoded with the vector $\overline{X} = [2, 2, 2]^\mathsf{T}$. The vector $\overline{Y} = [Y_1, Y_2, Y_3]^\mathsf{T}$ received at the other end is described by $\overline{Y} = \overline{X} + \overline{N}$. The vector $\overline{N} = [N_1, N_2, N_3]^\mathsf{T}$ represents the noise vector where each N_i is a random variable assumed to be normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$. Assume each N_i is independent of each other and independent of the X_i's. Each component value of $\overline{Y}$ is decoded with the same rule as in part (a). The receiver then uses a majority rule to determine which symbol was sent. The receiver's decoding rules are:
   - If 2 or more components of $\overline{Y}$ are greater than 0, then conclude the symbol 1 was sent.
   - If 2 or more components of $\overline{Y}$ are less than 0, then conclude the symbol 0 was sent.

   Determine the probability of error for this modified encoding/decoding scheme. Reduce your calculations to a single numerical value.

5. The random variables X and Y are described by a joint PDF which is constant within the unit area quadrilateral with vertices $(0,0)$, $(0,1)$, $(1,2)$, and $(1,1)$.



   (a) Are X and Y independent?
   (b) Find the marginal PDFs of X and Y.
   (c) Find the expected value of $X+Y$.
   (d) Find the variance of $X+Y$.

6. A defective coin minting machine produces coins whose probability of heads is a random variable P with PDF

$$f_P(p) = \begin{cases} 1+\sin(2\pi p), & \text{if } p \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

   In essence, a specific coin produced by this machine will have a fixed probability $P = p$ of giving heads, but you do not know initially what that probability is. A coin produced by this machine is selected and tossed repeatedly, with successive tosses assumed independent.

   (a) Find the probability that the first coin toss results in heads.
   (b) Given that the first coin toss resulted in heads, find the conditional PDF of P.
   (c) Given that the first coin toss resulted in heads, find the conditional probability of heads on the second toss.

G1†. Let C be the circle $\{(x,y) \mid x^2+y^2 \leq 1\}$. A point a is chosen randomly on the boundary of C and another point b is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x- and y-axes with diagonal ab. What is the probability that no point of R lies outside of C?

†Required for 6.431; optional for 6.041

Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

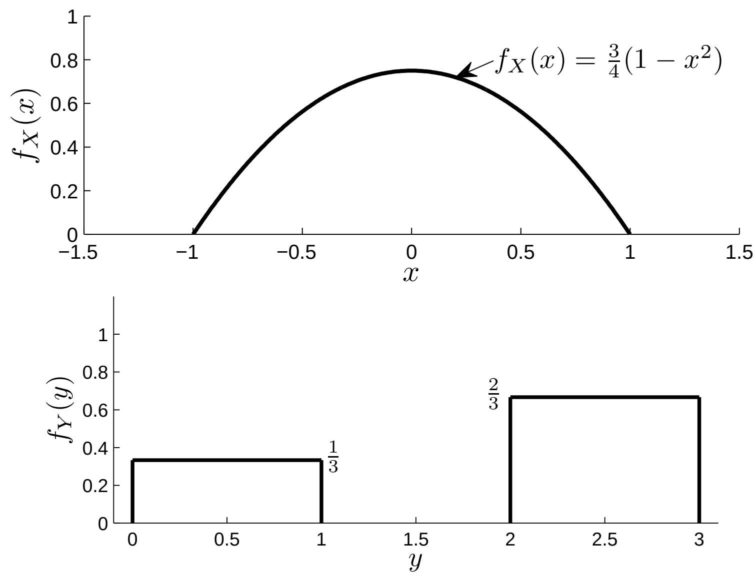
Problem Set 6
Due October 27, 2010

1. Random variables X and Y are distributed according to the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} ax, & \text{if } 1 \le x \le 2 \text{ and } 0 \le y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Evaluate the constant a.
   (b) Determine the marginal PDF $f_Y(y)$.
   (c) Determine the conditional expectation of $1/X$ given that $Y = 3/2$.
   (d) Random variable Z is defined by $Z = Y - X$. Determine the PDF $f_Z(z)$.

2. Let X and Y be two independent random variables. Their probability densities functions are shown below.

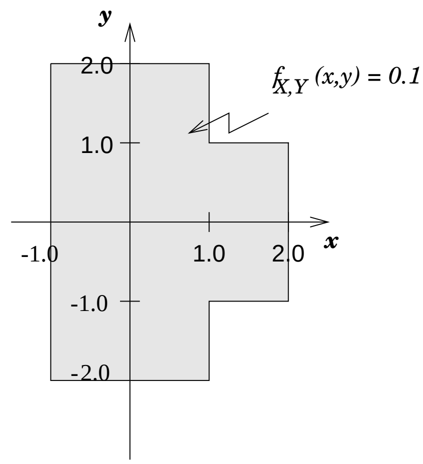


   Let $Z = X + Y$. Determine $f_Z(z)$.

3. Consider n independent tosses of a k-sided fair die. Let X_i be the number of tosses that result in i.

   (a) Are X_1 and X_2 uncorrelated, positively correlated, or negatively correlated? Give a one-line justification.
   (b) Compute the covariance $\text{cov}(X_1, X_2)$ of X_1 and X_2.

4. Random variables X and Y have the joint PDF shown below:



   (a) Find the conditional PDFs $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$, for various values of x and y, respectively.
   (b) Find $\mathbf{E}[X \mid Y = y]$, $\mathbf{E}[X]$, and $\text{var}(X \mid Y = y)$. Use these to calculate $\text{var}(X)$.
   (c) Find $\mathbf{E}[Y \mid X = x]$, $\mathbf{E}[Y]$, and $\text{var}(Y \mid X = x)$. Use these to calculate $\text{var}(Y)$.

5. The wombat club has N members, where N is a random variable with PMF

$$p_N(n) = p^{n-1}(1-p) \quad \text{for } n = 1, 2, 3, \ldots.$$

On the second Tuesday night of every month, the club holds a meeting. Each wombat member attends the meeting with probability q, independently of all the other members. If a wombat attends the meeting, then it brings an amount of money, M, which is a continuous random variable with PDF

$$f_M(m) = \lambda e^{-\lambda m} \quad \text{for } m \geq 0.$$

N, M, and whether each wombat member attends are all independent. Determine:

   (a) The expectation and variance of the number of wombats showing up to the meeting.
   (b) The expectation and variance for the total amount of money brought to the meeting.

G1†. (a) Let $X_1, X_2, \ldots, X_n, X_{n+1}, \ldots, X_{2n}$ be independent and identically distributed random variables.
Find

$$\mathbf{E}[X_1 \mid X_1 + X_2 + \ldots + X_n = x_0],$$

where x_0 is a constant.

(b) Define

$$S_k = X_1 + X_2 + \ldots + X_k, 1 \leq k \leq 2n.$$

Find

$$\mathbf{E}[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots, S_{2n} = s_{2n}],$$

where $s_n, s_{n+1}, \ldots, s_{2n}$ are constants.

†Required for 6.431; optional for 6.041

Problem Set 7
Due November 8, 2010

1. Consider a sequence of mutually independent, identically distributed, probabilistic trials. Any particular trial results in either a success (with probability p) or a failure.

   (a) Obtain a simple expression for the probability that the ith success occurs before the jth failure. You may leave your answer in the form of a summation.

   (b) Determine the expected value and variance of the number of successes which occur before the jth failure.

   (c) Let L_{17} be described by a Pascal PMF of order 17. Find the numerical values of a and b in the following equation. Explain your work.

$$\sum_{l=42}^{\infty} p_{L_{17}}(l) = \sum_{x=0}^{a} \binom{b}{x} p^x (1-p)^{(b-x)}$$

2. Fred is giving out samples of dog food. He makes calls door to door, but he leaves a sample (one can) only on those calls for which the door is answered *and* a dog is in residence. On any call the probability of the door being answered is 3/4, and the probability that any household has a dog is 2/3. Assume that the events "Door answered" and "A dog lives here" are independent and also that the outcomes of all calls are independent.

   (a) Determine the probability that Fred gives away his first sample on his third call.

   (b) Given that he has given away exactly four samples on his first eight calls, determine the conditional probability that Fred will give away his fifth sample on his eleventh call.

   (c) Determine the probability that he gives away his second sample on his fifth call.

   (d) Given that he did not give away his second sample on his second call, determine the conditional probability that he will leave his second sample on his fifth call.

   (e) We will say that Fred "needs a new supply" immediately *after* the call on which he gives away his last can. If he starts out with two cans, determine the probability that he completes at least five calls before he needs a new supply.

   (f) If he starts out with exactly m cans, determine the expected value and variance of D_m, the number of homes with dogs which he passes up (because of no answer) before he needs a new supply.

3. Let T_1 and T_2 be exponential random variables with parameter λ, and let S be an exponential random variable with parameter μ. We assume that all three of these random variables are independent. Derive an expression for the expected value of $\min\{T_1+T_2, S\}$. *Hint:* See Problem 6.19 in the text.

4. A single dot is placed on a very long length of yarn at the textile mill. The yarn is then cut into lengths requested by different customers. The lengths are independent of each other, but all distributed according to the PDF $f_L(\ell)$. Let R be be the length of yarn purchased by that customer whose purchase included the dot. Determine the expected value of R in the following cases:

(a) $f_L(\ell) = \lambda e^{-\lambda \ell}, \quad \ell \geq 0$

(b) $f_L(\ell) = \frac{\lambda^3 \ell^2 e^{-\lambda \ell}}{2}, \quad \ell \geq 0$

(c) $f_L(\ell) = \ell e^{\ell}, \quad 0 \leq \ell \leq 1$

5. Consider a Poisson process of rate λ. Let random variable N be the number of arrivals in $(0, t]$ and M be the number of arrivals in $(0, t+s]$, where $t, s \geq 0$.
   (a) Find the conditional PMF of M given N, $p_{M|N}(m|n)$, for $m \geq n$.
   (b) Find the joint PMF of N and M, $p_{N,M}(n, m)$.
   (c) Find the conditional PMF of N given M, $p_{N|M}(n|m)$, for $n \leq m$, using your answer to part (b).
   (d) Rederive your answer to part (c) without using part (b). As a hint, consider what kind of distribution the k^{th} arrival time has if we are given the event $\{M = m\}$, where $k \leq m$.
   (e) Find $E[NM]$.

6. The interarrival times for cars passing a checkpoint are independent random variables with PDF

$$f_T(t) = \begin{cases} 2e^{-2t}, & \text{for } t > 0 \\ 0, & \text{otherwise.} \end{cases}$$

   where the interarrival times are measured in minutes. The successive experimental values of the durations of these interarrival times are recorded on small computer cards. The recording operation occupies a negligible time period following each arrival. Each card has space for three entries. As soon as a card is filled, it is replaced by the next card.
   (a) Determine the mean and the third moment of the interarrival times.
   (b) Given that no car has arrived in the last four minutes, determine the PMF for random variable K, the number of cars to arrive in the next six minutes.
   (c) Determine the PDF and the expected value for the total time required to use up the first dozen computer cards.
   (d) Consider the following two experiments:
      i. Pick a card at random from a group of completed cards and note the total time, Y, the card was in service. Find $\mathbf{E}[Y]$ and $\text{var}(Y)$.
      ii. Come to the corner at a random time. When the card in use at the time of your arrival is completed, note the total time it was in service (the time from the start of its service to its completion). Call this time W. Determine $\mathbf{E}[W]$ and $\text{var}(W)$.

G1†. Consider a Poisson process with rate λ, and let $N(G_i)$ denote the number of arrivals of the process during an interval $G_i = (t_i, t_i + c_i]$. Suppose we have n such intervals, $i = 1, 2, \cdots, n$, mutually disjoint. Denote the union of these intervals by G, and their total length by $c = c_1 + c_2 + \cdots + c_n$. Given $k_i \geq 0$ and with $k = k_1 + k_2 + \cdots + k_n$, determine

$$\mathbf{P}\Big(N(G_1) = k_1, N(G_2) = k_2, \ldots, N(G_n) = k_n \,|\, N(G) = k\Big).$$

†Required for 6.431; optional for 6.041

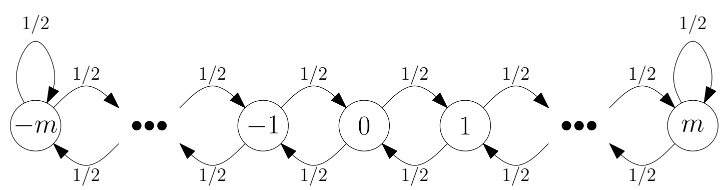
Problem Set 8
Due November 15, 2010

1. Oscar goes for a run each morning. When he leaves his house for his run, he is equally likely to go out either the front or back door; and similarly, when he returns, he is equally likely to go to either the front or back door. Oscar owns only five pairs of running shoes which he takes off immediately after the run at whichever door he happens to be. If there are no shoes at the door from which he leaves to go running, he runs barefooted. We are interested in determining the long-term proportion of time that he runs barefooted.

   (a) Set the scenario up as a Markov chain, specifying the states and transition probabilities.

   (b) Determine the long-run proportion of time Oscar runs barefooted.

2. Consider a Markov chain $X_1, X_2, \ldots$ modeling a *symmetric simple random walk with barriers*, as shown below:



   (a) Explain why $|X_1|, |X_2|, |X_3|, \ldots$ also satisfies the Markov property and draw the associated chain.

   (b) Suppose that we also wish to keep track of the largest deviation from the origin, i.e., define the largest deviation at time t as $Y_t = \max\{|X_1|, |X_2|, \ldots, |X_t|\}$. Draw a Markov chain that keeps track of the largest deviation and explain why it satisfies the Markov property.

3. As flu season is upon us, we wish to have a Markov chain that models the spread of a flu virus. Assume a population of n individuals. At the beginning of each day, each individual is either infected or susceptible (capable of contracting the flu). Suppose that each pair (i, j), $i \neq j$, independently comes into contact with one another during the daytime with probability p. Whenever an infected individual comes into contact with a susceptible individual, he/she infects him/her. In addition, assume that overnight, any individual who has been infected for at least 24 hours will recover with probability $0 < q < 1$ and return to being susceptible, independently of everything else (i.e., assume that a newly infected individual will spend at least one restless night battling the flu).

   (a) Suppose that there are m infected individuals at daybreak. What is the distribution of the number of new infections by day end?

   (b) Draw a Markov chain with as few states as possible to model the spread of the flu for $n = 2$. In epidemiology, this is called an SIS (Susceptible-Infected-Susceptible) model.

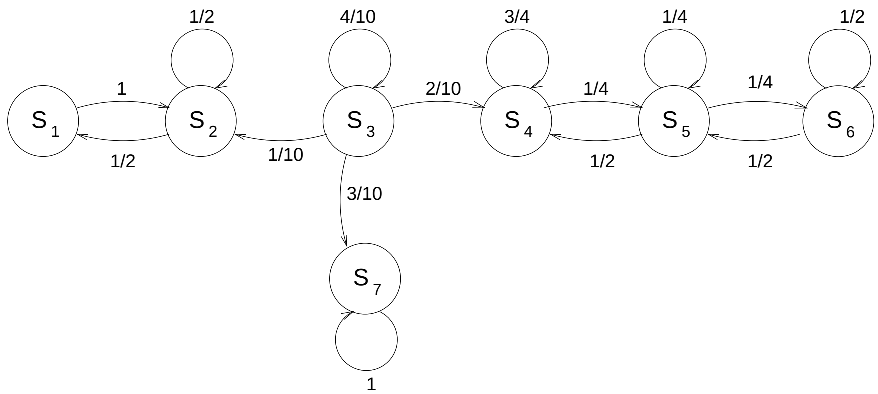
   (c) Identify all recurrent states.

   Due to the nature of the flu virus, individuals almost always develop immunity after contracting the virus. Consequently, we improve our model and assume that individuals become infected at most one time. Thus, we consider individuals as either infected, susceptible, or recovered.

(d) Draw a Markov chain to model the spread of the flu for $n = 2$. In epidemiology, this is called an SIR (Susceptible-Infected-Recovered) model.

(e) Identify all recurrent states.

4. Consider the Markov chain below. For all parts of this problem, the process is in state 3 immediately before the first transition. Be sure to comment on any unusual results.



(a) Find the variance for J, the number of transitions up to and including the transition on which the process leaves state 3 for the last time.

(b) Find the expectation for K, the number of transitions up to and including the transition on which the process enters state 4 for the first time.

(c) Find π_i for $i = 1, 2, \ldots, 7$, the probability that the process is in state i after 10^{10} transitions.

(d) Given that the process never enters state 4, find the π_i's as defined in part (c).

G1†. Consider a Markov chain $\{X_k\}$ on the state space $\{1, \ldots, n\}$, and suppose that whenever the state is i, a reward $g(i)$ is obtained. Let R_k be the total reward obtained over the time interval $\{0, 1, \ldots, k\}$, that is, $R_k = g(X_0) + g(X_1) + \cdots + g(X_k)$. For every state i, let

$$m_k(i) = E[R_k \mid X_0 = i],$$

and

$$v_k(i) = \mathrm{var}(R_k \mid X_0 = i)$$

respectively be the conditional mean and conditional variance of R_k, conditioned on the initial state being i.

(a) Find a recursion that, given the values of $m_k(1), \ldots, m_k(n)$, allows the computation of $m_{k+1}(1), \ldots, m_{k+1}(n)$.

(b) Find a recursion that, given the values of $m_k(1), \ldots, m_k(n)$ and $v_k(1), \ldots, v_k(n)$, allows the computation of $v_{k+1}(1), \ldots, v_{k+1}(n)$. *Hint*: Use the law of total variance.

†Required for 6.431; optional for 6.041

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Problem Set 9

Due November 22, 2010

1. Random variable X is uniformly distributed between -1.0 and 1.0. Let $X_1, X_2, \ldots$, be independent identically distributed random variables with the same distribution as X. Determine which, if any, of the following sequences (all with $i = 1, 2, \ldots$) are convergent in probability. Fully justify your answers. Include the limits if they exist.

   (a) $U_i = \dfrac{X_1 + X_2 + \ldots + X_i}{i}$

   (b) $W_i = \max(X_1, \ldots, X_i)$

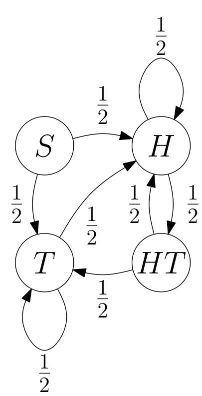
   (c) $V_i = X_1 \cdot X_2 \cdot \ldots \cdot X_i$

2. Demonstrate that the Chebyshev inequality is tight, that is, for every μ, $\sigma > 0$, and $c \geq \sigma$, construct a random variable X with mean μ and standard deviation σ such that

$$\mathbf{P}(|X - \mu| \geq c) = \frac{\sigma^2}{c^2}$$

   Hint: You should be able to do this with a discrete random variable that takes on only 3 distinct values with nonzero probability.

3. Assume that a fair coin is tossed repeatedly, with the tosses being independent. We want to determine the expected number of tosses necessary to first observe a head directly followed by a tail. To do so, we define a Markov chain with states S, H, T, HT, where S is a starting state, H indicates a head on the current toss, T indicates a tail on the current toss (without heads on the previous toss), and HT indicates heads followed by tails over the last two tosses. This Markov chain is illustrated below:



   We can find the expected number of tosses necessary to first observe a heads directly followed by tails by solving a mean first passage time problem for this Markov chain.

   (a) What is the expected number of tosses necessary to first observe a head directly followed by tails?

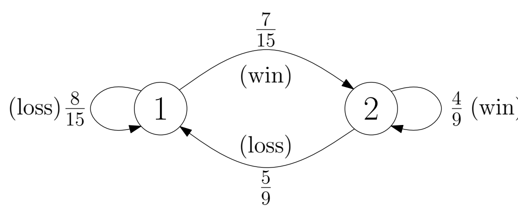
   (b) Assuming we have just observed a head followed by a tail, what is the expected number of additional tosses until we again observe a head followed directly by a tail?

Next, we want to answer the same questions for the event tails directly followed by tails. Set up a different Markov chain from which we could calculate the expected number of tosses necessary to first observe tails directly followed by tails.

(c) What is the expected number of tosses necessary to first observe a tail directly followed by a tail?

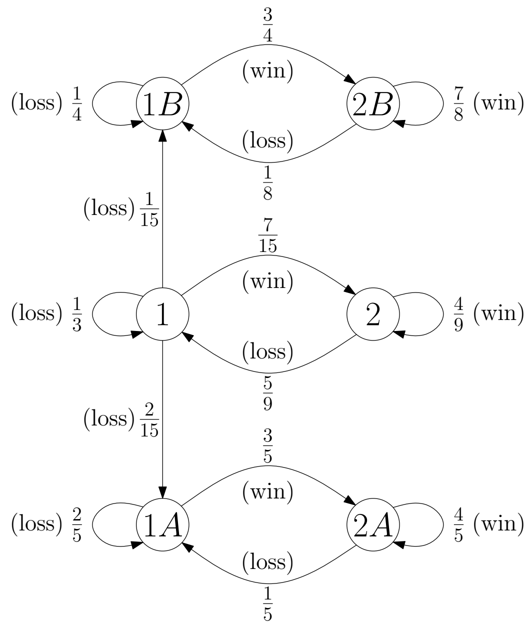
(d) Assuming we have just observed a tail followed by a tail, what is the expected number of additional tosses until we again observe a tail followed directly by a tail? Note that the number of additional tosses could be as little as one, if tails were to come up again.

4. Jack is a gambler who pays for his MIT tuition by spending weekends in Las Vegas. Lately he's been playing 21 at a table that returns cards to the deck and reshuffles them all before each hand. As he has a fixed policy in how he plays, his probability of winning a particular hand remains constant, and is independent of all other hands. There is a wrinkle, however; the dealer switches between two decks (deck #2 is more unfair to Jack than deck #1), depending on whether or not Jack wins. Jack's wins and losses can be modeled via the transitions of the following Markov chain, whose states correspond to the particular deck being used.



(a) What is Jack's long term probability of winning?

Given that Jack loses and the dealer is not occupied with switching decks, with probability $\frac{2}{8}$ the dealer looks away for one second and with probability $\frac{1}{8}$ the dealer looks away for two seconds, independently of everything else. When this happens, Jack secretly inserts additional cards into both of the dealer's decks, transforming the decks into types $1A$ & $2A$ (when he has 1 second) or $1B$ & $2B$ (when he has 2 seconds). Jack slips cards into the decks at most once. The process can be described by the modified Markov chain in the picture. Assume in all future problems that play begins with the dealer using deck #1.



(b) What is the probability of Jack eventually playing with decks $1A$ and $2A$?

(c) What is Jack's long-term probability of winning?

(d) What is the expected time (as in number of hands) until Jack slips additional cards into the deck?

(e) What is the distribution of the number of times that the dealer switches from deck 2 to deck 1?

(f) What is the distribution of the number of wins that Jack has before he slips extra cards into the deck? *Hint*: Note that after some conditioning, we have a geometric number of geometric random variables, all of which are independent.

(g) What is the average net losses (number of losses minus the number of wins, sometimes negative) prior to Jack slipping additional cards into the deck?

(h) Given that after a very long period of time Jack is playing a hand with deck $1A$, what is the approximate probability that his previous hand was played with deck $2A$?

G1†. Show the following one-sided version of Chebyshev's inequality:

$$\mathbf{P}(X - \mu \geq a) \leq \frac{\sigma^2}{(\sigma^2 + a^2)}$$

where μ and σ^2 are the mean and variance of X respectively, and $a > 0$. Hint: Start by finding a bound on $\mathbf{P}(X - \mu + c \geq a + c)$ with $c \geq 0$. Then find the c that 'tightens' your bound.

†Required for 6.431; optional challenge problem for 6.041

Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Problem Set 10

Due December 2, 2010 (in recitation)

1. **A financial parable.** An investment bank is managing \$1 billion, which it invests in various financial instruments ("assets") related to the housing market (e.g., the infamous "mortgage backed securities"). Because the bank is investing with borrowed money, its actual assets are only \$50 million (5%). Accordingly, if the bank loses more than 5%, it becomes insolvent. (Which means that it will have to be bailed out, and the bankers may need to forgo any huge bonuses for a few months.)

   (a) The bank considers investing in a single asset, whose gain (over a 1-year period, and measured in percentage points) is modeled as a normal random variable R, with mean 7 and standard deviation 10. (That is, the asset is expected to yield a 7% profit.) What is the probability that the bank will become insolvent? Would you accept this level of risk?

   (b) In order to safeguard its position, the bank decides to diversify its investments. It considers investing \$50 million in each of twenty different assets, with the ith one having a gain R_i, which is again normal with mean 7 and standard deviation 10; the bank's gain will be $(R_1 + \cdots + R_{20})/20$. These twenty assets are chosen to reflect the housing sectors at different states or even countries, and the bank's rocket scientists choose to model the R_i as independent random variables. According to this model, what is the probability that the bank becomes insolvent?

   (c) Based on the calculations in part (b), the bank goes ahead with the diversified investment strategy. It turns out that a global economic phenomenon can affect the housing sectors in different states and countries simultaneously, and therefore the gains R_i are in fact positively correlated. Suppose that for every i and j where $i \neq j$, the correlation coefficient $\rho(R_i, R_j)$ is equal to 1/2. What is the probability that the bank becomes insolvent? You can assume that $(R_1 + \cdots + R_{20})/20$ is normal.

2. The adult population of Nowhereville consists of 300 males and 196 females. Each male (respectively, female) has a probability of 0.4 (respectively, 0.5) of casting a vote in the local elections, independently of everyone else. Find a good numerical approximation for the probability that more males than females cast a vote.

3. Let S_n be the number of successes in n independent Bernoulli trials, where the probability of success in each trial is $p = \frac{1}{2}$. Provide a numerical value for the limit as n tends to infinity for each of the following three expressions:

   (a) $\mathbf{P}(\frac{n}{2} - 10 \leq S_n \leq \frac{n}{2} + 10)$

   (b) $\mathbf{P}(\frac{n}{2} - \frac{n}{10} \leq S_n \leq \frac{n}{2} + \frac{n}{10})$

   (c) $\mathbf{P}(\frac{n}{2} - \frac{\sqrt{n}}{2} \leq S_n \leq \frac{n}{2} + \frac{\sqrt{n}}{2})$

4. Alice has two coins. The probability of heads for the first coin is 1/3; the probability of heads for the second coin is 2/3. Other than this difference in their bias, the coins are indistinguishable through any measurement known to man. Alice chooses one of the coins randomly and sends it to Bob. Let p be the probability that Alice chose the first coin. Bob tries to guess which of the two coins he received by flipping it 3 times in a row and observing the outcome. Assume that all coin flips are independent. Let Y be the number of heads Bob observed.

(a) Given that Bob observed k heads, what is the probability that he received the first coin?

(b) Find values of k for which the probability that Alice sent the first coin increases after Bob observes k heads out of 3 tosses. In other words, for what values of k is the probability that Alice sent the first coin given that Bob observed k heads greater than p? If we increase p, how does your answer change (goes up, goes down, or stays unchanged)?

(c) Help Bob develop the rule for deciding which coin he received based on the number of heads k he observed in 3 tosses if his goal is to minimize the probability of error.

(d) For this part, assume $p = 2/3$.

i. Find the probability that Bob will guess the coin correctly using the rule above.

ii. How does this compare to the probability of guessing correctly if Bob tried to guess which coin he received before flipping it?

(e) If we increase p, how does that affect the decision rule?

(f) Find the values of p for which Bob will never guess he received the first coin, regardless of the outcome of the tosses.

(g) Find the values of p for which Bob will always guess he received the first coin, regardless of the outcome of the tosses.

5. Consider a Bernoulli process $X_1, X_2, X_3, \ldots$ with unknown probability of success q. As usual, define the kth inter-arrival time T_k as

$$T_1 = Y_1, \qquad T_k = Y_k - Y_{k-1}, \quad k = 2, 3, \ldots$$

where Y_k is the time of the kth success. This problem explores estimation of q from observed inter-arrival times $\{t_1, t_2, t_3, \ldots\}$.

You may find the following integral useful: For any non-negative integers k and m,

$$\int_0^1 q^k (1-q)^m dq = \frac{k!\,m!}{(k+m+1)!}$$

Assume q is sampled from the random variable Q which is uniformly distributed over $[0, 1]$.

(a) Compute the PMF of T_1, $p_{T_1}(t_1)$

(b) Compute the least squares estimate (LSE) of Q from the first recording $T_1 = t_1$.

(c) Compute the maximum a posteriori (MAP) estimate of Q given the k recordings, $T_1 = t_1, \ldots, T_k = t_k$.

For this part only assume q is sampled from the random variable Q which is now uniformly distributed over $[0.5, 1]$

(d) Find the linear least squares estimate (LLSE) of the second inter-arrival time (T_2), from the observed first arrival time ($T_1 = t_1$).

6. The joint PDF of X and Y is defined as follows:

$$f_{X,Y}(x, y) = \begin{cases} cxy & \text{if } 0 < x \leq 1,\ 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the normalization constant c.

(b) Compute the conditional expectation estimator of X based on the observed value $Y = y$.

(c) Is this estimate different from what you would have guessed before you saw the value $Y = y$? Explain.

(d) Repeat (b) and (c) for the MAP estimator.

Problem Set 11
Never Due
Covered on Final Exam

1. **Problem 7, page 509 in textbook**
   Derive the ML estimator of the parameter of a Poisson random variable based of i.i.d. observations $X_1, \ldots, X_n$. Is the estimator unbiased and consistent?

2. Caleb builds a particle detector and uses it to measure radiation from far stars. On any given day, the number of particles Y that hit the detector is conditionally distributed according to a Poisson distribution conditioned on parameter x. The parameter x is unknown and is modeled as the value of a random variable X, exponentially distributed with parameter μ as follows.

$$f_X(x) = \begin{cases} \mu e^{-\mu x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Then, the conditional PDF of the number of particles hitting the detector is,

$$p_{Y|X}(y \mid x) = \begin{cases} \frac{e^{-x}x^y}{y!} & \text{y = 0, 1, 2, ...} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the MAP estimate of X from the observed particle count y.

   (b) Our goal is to find the conditional expectation estimator for X from the observed particle count y.

      i. Show that the posterior probability distribution for X given Y is of the form

$$f_{X|Y}(x \mid y) = \frac{\lambda^{y+1}}{y!} x^y e^{-\lambda x}, \quad x > 0$$

      and find the parameter λ. You may find the following equality useful (it is obviously true if the equation above describes a true PDF):

$$\int_0^\infty a^{y+1} x^y e^{-ax}\, dx = y! \quad \text{for any } a > 0$$

      ii. Find the conditional expectation estimate of X from the observed particle count y. *Hint:* you might want to express $x f_{X|Y}(x \mid y)$ in terms of $f_{X|Y}(x \mid y+1)$.

   (c) Compare the two estimators you constructed in part (a) and part (b).

3. Consider a Bernoulli process $X_1, X_2, X_3, \ldots$ with unknown probability of success q. Define the kth inter-arrival time T_k as

$$T_1 = Y_1, \qquad T_k = Y_k - Y_{k-1}, \quad k = 2, 3, \ldots$$

   where Y_k is the time of the kth success. This problem explores estimation of q from observed inter-arrival times $\{t_1, t_2, t_3, \ldots\}$. In problem set 10, we solved the problem using Bayesian inference. Our focus here will be on classical estimation.

We assume that q is an unknown parameter in the interval $(0, 1]$. Denote the true parameter by q^*. Denote by $\widehat{Q}_k$ the maximum likelihood estimate (MLE) of q given k recordings, $T_1 = t_1, \ldots, T_k = t_k$.

(a) Compute $\widehat{Q}_k$. Is this different from the MAP estimate you found in problem set 10?

(b) Show that for all $\epsilon > 0$

$$\lim_{k \to \infty} \mathbf{P}\left(\left|\frac{1}{\widehat{Q}_k} - \frac{1}{q^*}\right| > \epsilon\right) = 0$$

(c) Assume $q^* \geq 0.5$. Give a lower bound on k such that

$$\mathbf{P}\left(\left|\frac{1}{\widehat{Q}_k} - \frac{1}{q^*}\right| \leq 0.1\right) \geq 0.95$$

4. A body at temperature θ radiates photons at a given wavelength. This problem will have you estimate θ, which is fixed but unknown. The PMF for the number of photons K in a given wavelength range and a fixed time interval of one second is given by,

$$p_K(k; \theta) = \frac{1}{Z(\theta)} e^{-\frac{k}{\theta}}, k = 0, 1, 2, \ldots$$

$Z(\theta)$ is a normalization factor for the probability distribution (the physicists call it the partition function). You are given the task of determining the temperature of the body to two significant digits by photon counting in non-overlapping time intervals of duration one second. The photon emissions in non-overlapping time intervals are statistically independent from each other.

(a) Determine the normalization factor $Z(\theta)$.

(b) Compute the expected value of the photon number measured in any 1 second time interval, $\mu_K = \mathbf{E}_\theta[K]$, and its variance, $\mathrm{var}_\theta(K) = \sigma_K^2$.

(c) You count the number k_i of photons detected in n non-overlapping 1 second time intervals. Find the maximum likelihood estimator, $\hat{\Theta}_n$, for temperature Θ. Note, it might be useful to introduce the average photon number $s_n = \frac{1}{n}\sum_{i=1}^{n} k_i$. In order to keep the analysis simple we assume that the body is hot, i.e. $\theta \gg 1$.
You may use the approximation: $\frac{1}{e^{\frac{1}{\theta}} - 1} \approx \theta$ for $\theta \gg 1$.

In the following questions we wish to estimate the mean of the photon count in a one second time interval using the estimator $\hat{K}$, which is given by,

$$\hat{K} = \frac{1}{n}\sum_{i=1}^{n} K_i.$$

(d) Find the number of samples n for which the noise to signal ratio for $\hat{K}$, (i.e.,$\frac{\sigma_{\hat{K}}}{\mu_{\hat{K}}}$), is 0.01.

(e) Find the 95% confidence interval for the mean photon count estimate for the situation in part (d). (You may use the central limit theorem.)

5. The RandomView window factory produces window panes. After manufacturing, 1000 panes were loaded onto a truck. The weight W_i of the i-th pane (in pounds) on the truck is modeled as a random variable, with the assumption that the W_i's are independent and identically distributed.

(a) Assume that the measured weight of the load on the truck was 2340 pounds, and that $\text{var}(W_i) \leq 4$. Find an approximate 95 percent confidence interval for $\mu = \mathbf{E}[W_i]$, using the Central Limit Theorem.

(b) Now assume instead that the random variables W_i are i.i.d., with an exponential distribution with parameter $\theta > 0$, i.e., a distribution with PDF

$$f_W(w;\theta) = \theta e^{-\theta w}.$$

What is the maximum likelihood estimate of θ, given that the truckload has weight 2340 pounds?

6. Given the five data pairs (x_i,y_i) in the table below,

x	0.8	2.5	5	7.3	9.1
y	-2.3	20.9	103.5	215.8	334

we want to construct a model relating x and y. We consider a linear model

$$Y_i = \theta_0 + \theta_1 x_i + W_i, \qquad i = 1, \cdots, 5,$$

and a quadratic model

$$Y_i = \beta_0 + \beta_1 x_i^2 + V_i, \qquad i = 1, \cdots, 5.$$

where W_i and V_i represent additive noise terms, modeled by independent normal random variables with mean zero and variance σ_1^2 and σ_2^2, respectively.

(a) Find the ML estimates of the linear model parameters.

(b) Find the ML estimates of the quadratic model parameters.

Note: You may use the regression formulas and the connection with ML described in pages 478-479 of the text. However, the regression material is outside the scope of the final.

6.041/6.431 Spring 2009 Quiz 1
Wednesday, March 11, 7:30 - 9:30 PM.

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

Question	Part	Score	Out of
0			3
1	all		40
2	a		5
	b		5
	c		6
	d		6
3	a		5
	b		6
	c		6
	d		6
	e		6
	f		6
	g		10
6.041 Total			100
6.431 Total			110

- Write your solutions in this quiz packet, only solutions in the quiz packet will be graded.
- Question one, multiple choice questions, will receive no partial credit. Partial credit for question two and three will be awarded.
- You are allowed one two-sided 8.5 by 11 formula sheet plus a calculator.
- You have 120 minutes to complete the quiz.
- Be neat! You will not get credit if we can't read it.
- Graded quizzes will be returned in recitation on Tuesday 3/17.

Question 1

Multiple Choice Questions: **CLEARLY** circle the appropriate choice. Scratch paper is available if needed, though **NO** partial credit will be given for the Multiple Choice.

a. Which of the following statements is NOT true?

(i) If $A \subset B$, then $\mathbf{P}(A) \leq \mathbf{P}(B)$.

(ii) If $\mathbf{P}(B) > 0$, then $\mathbf{P}(A|B) \geq \mathbf{P}(A)$.

(iii) $\mathbf{P}(A \cap B) \geq \mathbf{P}(A) + \mathbf{P}(B) - 1$.

(iv) $\mathbf{P}(A \cap B^c) = \mathbf{P}(A \cup B) - \mathbf{P}(B)$.

b. We throw n identical balls into m urns at random, where each urn is equally likely and each throw is independent of any other throw. What is the probability that the i-th urn is empty?

(i) $\left(1 - \frac{1}{m}\right)^n$

(ii) $\left(1 - \frac{1}{n}\right)^m$

(iii) $\left(\begin{array}{c} m \\ n \end{array}\right)\left(1 - \frac{1}{n}\right)^m$

(iv) $\left(\begin{array}{c} n \\ m \end{array}\right)\left(\frac{1}{m}\right)^n$

c. We toss two fair coins simultaneously and independently. If the outcomes of the two coins are the same, we win; otherwise, we lose. Let A be the event that the first coin comes up heads, B be the event that the second coin comes up heads, and C be the event that we win. Which of the following statements is false?

(i) Events A and B are independent.

(ii) Events A and C are *not* independent.

(iii) Events A and B are *not* conditionally independent given C.

(iv) The probability of winning is 1/2.

d. For a biased coin, the probability of "heads" is 1/3. Let h be the number of heads in five independent coin tosses. What is the probability $\mathbf{P}$(first toss is a head | $h = 1$ or $h = 5$)?

(i) $\frac{\frac{1}{3}(\frac{2}{3})^4}{5\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$

(ii) $\frac{\frac{1}{3}(\frac{2}{3})^4}{\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$

(iii) $\frac{\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}{5\frac{1}{3}(\frac{2}{3})^4+(\frac{1}{3})^5}$

(iv) $\frac{1}{5}$

e. A well-shuffled deck of 52 cards is dealt evenly to two players (26 cards each). What is the probability that player 1 gets all the aces?

(i) $\dfrac{\binom{48}{22}}{\binom{52}{26}}$

(ii) $\dfrac{4\binom{48}{22}}{\binom{52}{26}}$

(iii) $\frac{48!}{22!}\frac{52!}{26!}$

(iv) $\dfrac{4!\binom{48}{22}}{\binom{52}{26}}$

f. Suppose X, Y and Z are three independent discrete random variables. Then, X and $Y + Z$ are

(i) always
(ii) sometimes
(iii) never

independent.

g. To obtain a driving licence, Mina needs to pass her driving test. Every time Mina takes a driving test, with probability 1/2, she will clear the test independent of her past. Mina failed her first test. Given this, let Y be the additional number of tests Mina takes before obtaining a licence. Then,

(i) $E[Y] = 1$.
(ii) $E[Y] = 2$.
(iii) $E[Y] = 0$.

h. Consider two random variables X and Y, each taking values in $\{1, 2, 3\}$. Let their joint PMF be such that for any $1 \leq x, y \leq 3$,

$$P_{X,Y}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in \{(1, 3), (2, 1), (3, 2)\} \\ \textit{strictly positive} & \text{otherwise.} \end{cases}$$

Then,

(i) X and Y can be independent or dependent depending upon the *strictly positive* values.
(ii) X and Y are always independent.
(iii) X and Y can never be independent.

i. Suppose you play a *matching coins* game with your friend as follows. Both you and your friend have a coin. Each time, you two reveal a side (i.e. H or T) of your coin to each other simultaneously. If the sides match, you WIN a 1 from your friend and if sides do not match then you lose a 1 to your friend. Your friend has a complicated (unknown) strategy in selecting the sides over time. You decide to go with the following simple strategy. Every time, you will toss your unbiased coin independently of everything else, and you will reveal its outcome to your friend (of course, your friend does not know the outcome of your random toss until you reveal it). Then,

(i) On average, you will lose money to your smart friend.

(ii) On average, you will neither lose nor win. That is, your average gain/loss is 0.

(iii) On average, you will make money from your friend.

j. Let $X_i, 1 \leq i \leq 4$ be independent Bernoulli random variable each with mean $p = 0.1$. Let $X = \sum_{i=1}^{4} X_i$. That is, X is a Binomial random variable with parameters $n = 4$ and $p = 0.1$. Then,

(i) $E[X_1 | X = 2] = 0.1$.

(ii) $E[X_1 | X = 2] = 0.5$.

(iii) $E[X_1 | X = 2] = 0.25$.

Question 2:

Alice and Bob both need to buy a bicycle. The bike store has a stock of four green, three yellow, and two red bikes. Alice randomly picks one of the bikes and buys it. Immediately after, Bob does the same. The sale price of the green, yellow, and red bikes are \$300, \$200 and \$100, respectively.

Let A be the event that Alice bought a green bike, and B be the event that Bob bought a green bike.

a. What is $\mathbf{P}(A)$? What is $\mathbf{P}(A|B)$?

b. Are A and B independent events? Justify your answer.

c. What is the probability that at least one of them bought a green bike?

d. What is the probability that Alice and Bob bought bicycles of different colors?

e. Given that Bob bought a green bike, what is the expected value of the amount of money spent by Alice?

f. Let G be the number of green bikes that remain on the store after Alice and Bob's visit. Compute $\mathbf{P}(B|G = 3)$.

Question 3:

Magic Games Inc. is a store that sells all sorts of fun games. One of its popular products is its magic 4-sided dice. The dice come in pairs; each die can be fair or crooked, and the dice in any pair can function independently or, in some cases, can have magnets inside them that cause them to behave in unpredictable ways when rolled together.

Xavier and Yvonne together buy a pair of dice from this store. Each of them picks a die in the pair; one of them then rolls the two dice together. Let X be the outcome of Xavier's die and Y the outcome of Yvonne's die. The joint PMF of X and Y, $p_{X,Y}(x,y)$, is given by the following figure:

Y \ X	1	2	3	4
4	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$
3	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{20}$
2	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$
1	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$

(a) Find the PMF of the outcome of Xavier's die, $p_X(x)$.

(b) Find the PMF of the outcome of Yvonne's die, $p_Y(y)$.

(c) Are X and Y independent?

Zach and Wendy are intrigued by Xavier and Yvonne's dice; they visit the store and buy a pair of dice of their own. Again, each of them picks a die in the pair; one of them then rolls the two dice together. Let Z be the outcome of Zach's die and W the outcome of Wendy's die. The joint PMF of Z and W, $p_{Z,W}(z,w)$, is given by the following figure:

W \ Z	1	2	3	4
4	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$
3	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$
2	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$
1	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$

The store also sells a variety of magic coins, some fair and some crooked. Alice buys a coin that on each toss comes up heads with probability 3/4.

(d) Wondering whether to buy some dice as well, Alice decides to try out her friends' dice first. She does the following. First, she tosses her coin. If the coin comes up heads, she borrows Xavier and Yvonne's dice pair and rolls the two dice; if the coin comes up tails, she borrows Zach and Wendy's dice pair and rolls those instead. What is the probability that she rolls a double, i.e., that both dice in the pair she rolls show the same number?

(e) Alice has still not made up her mind about the dice. She tries another experiment. First, she tosses her coin. If the coin comes up heads, she takes Xavier and Yvonne's dice pair and rolls the dice repeatedly until she gets a double; if the coin comes up tails, she does the same with Zach and Wendy's dice. What is the expected number of times she will need to roll the dice pair she chooses? (Assume that if a given pair of dice is rolled repeatedly, the outcomes of the different rolls are independent.)

(f) Alice is bored with the dice and decides to experiment with her coin instead. She tosses the coin until she has seen a total of 11 heads. Let R be the number of tails she sees. Find $\mathbf{E}[R]$. (Assume independent tosses.)

(g) Alice is still playing with her coin. Let A be the event that the second head she sees occurs on the 7th coin toss, and let S be the position of the first head. Find the conditional PMF of S given the event A, $p_{S|A}(s)$.

(h) Alice's friend Bob buys a coin from the same store that turns out to be fair, i.e., that on any toss comes up heads with probability 1/2. He tosses the coin repeatedly until he has seen either a total of 11 heads or a total of 11 tails. Let U be the number of times he will need to toss the coin. Find the PMF of U, $p_U(u)$. (Assume independent tosses.)

6.041/6.431 Fall 2009 Quiz 1
Tuesday, October 13, 12:05 - 12:55 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ______________________

Recitation Instructor: ______________________

TA: ______________________

Question	Score	Out of
A		2
B.1		10
B.2 (a)		10
B.2 (b i)		12
B.2 (b ii)		12
B.2 (c)		10
B.3 (a)		10
B.3 (b)		12
B.3 (c)		12
B.3 (d i)		5
B.3 (d ii)		5
Your Grade		100

- This quiz has 2 problems, worth a total of 100 points.
- You may tear apart pages 3 and 4, as per your convenience.
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed one two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- Parts B.2 and B.3 can be done independently.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of <u>numbers</u> that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- You have 50 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 10/15.

Summary of Results for Special Random Variables

Discrete Uniform over $[a, b]$**:**

$$p_X(k) = \begin{cases} \dfrac{1}{b-a+1}, & \text{if } k = a, a+1, \ldots, b, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{E}[X] = \frac{a+b}{2}, \qquad \text{var}(X) = \frac{(b-a)(b-a+2)}{12}.$$

Bernoulli with Parameter p**:** (Describes the success or failure in a single trial.)

$$p_X(k) = \begin{cases} p, & \text{if } k = 1, \\ 1-p, & \text{if } k = 0, \end{cases}$$

$$\mathbf{E}[X] = p, \qquad \text{var}(X) = p(1-p).$$

Binomial with Parameters p **and** n**:** (Describes the number of successes in n independent Bernoulli trials.)

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n,$$

$$\mathbf{E}[X] = np, \qquad \text{var}(X) = np(1-p).$$

Geometric with Parameter p**:** (Describes the number of trials until the first success, in a sequence of independent Bernoulli trials.)

$$p_X(k) = (1-p)^{k-1} p, \qquad k = 1, 2, \ldots,$$

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \text{var}(X) = \frac{1-p}{p^2}.$$

Problem B: (98 points) As a way to practice his probability skills, Bob goes apple picking. The orchard he goes to grows two varieties of apples: gala and honey crisp.

The **proportion of the gala apples** in the orchard is p $(0 < p < 1)$, the **proportion of the honey crisp apples** is $1 - p$. The number of apples in the orchard is so large that you can assume that picking a few apples does not change the proportion of the two varieties.

Independent of all other apples, the probability that a randomly picked **gala apple is ripe** is g and the probability that a randomly picked **honey crisp apple is ripe** is h.

1. **(10 points)** Suppose that Bob picks an apple at random (uniformly) and eats it. Find the probability that it was a **ripe gala** apple.

   Note: Parts 2 and 3 below can be done independently.

2. Suppose that Bob picks n apples at random (independently and uniformly).

   (a) **(10 points)** Find the probability that exactly k of those are **gala apples.**

   (b) Suppose that there are **exactly** k **gala apples** among the n apples Bob picked. Caleb comes by and gives Bob a **ripe gala** apple to add to his bounty. Bob then picks an apple at random from the $n + 1$ apples and eats it.

      (i) **(12 points)** What is the probability that it was a **ripe apple?**

      (ii) **(12 points)** What is the probability that it was a **gala apple if it was ripe?**

   (c) **(10 points)** Let $n = 20$, and suppose that Bob picked exactly 10 gala apples. What is the probability that the first 10 apples that Bob picked were all gala?

3. Next, Bob tries a different strategy. He starts with a tree of the **gala** variety and picks apples at random from that tree. Once Bob picks an apple off the tree, he carefully examines it to make sure it is ripe. Once he comes across an apple that is not ripe, he moves to **another gala tree.** He does this until he encounters an unripe apple on that second tree. Assume that each tree has a very large, essentially infinite, number of apples.

   (a) **(10 points)** Let X_i be the **number of apples** Bob picks off the ith tree, $(i = 1, 2)$. Write down the PMF, expectation, and variance of X_i.

   (b) **(12 points)** For $i = 1, 2$, let Y_i be the **total** number of **ripe apples** Bob picked from the first i trees. Find the expectation and the variance of Y_2. (Note that $Y_1 = X_1 - 1$ and $Y_2 = (X_1 - 1) + (X_2 - 1)$.)

   (c) **(12 points)** Find the joint PMF of Y_1 and Y_2.

   (d) In the following, answer just "yes" or "no." (Explanations will not be taken into account in grading.)

      (i) **(5 points)** Are X_1 and Y_2 independent?

      (ii) **(5 points)** Are X_2 and Y_1 independent?

Each question is repeated in the following pages. Please write your answer on the appropriate page.

1. **(10 points)** Suppose that Bob picks an apple at random (uniformly) and eats it. Find the probability that it was a **ripe gala** apple.

Note: Parts 2 and 3 can be done independently.

2. Suppose that Bob picks n apples at random (independently and uniformly).

   (a) **(10 points)** Find the probability that exactly k of those are **gala apples.**

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(c) **(12 points)** Find the joint PMF of Y_1 and Y_2.

(d) In the following, answer just "yes" or "no." (Explanations will not be taken into account in grading.)

(i) **(5 points)** Are X_1 and Y_2 independent?

(ii) **(5 points)** Are X_2 and Y_1 independent?

6.041/6.431 Fall 2010 Quiz 1
Tuesday, October 12, 7:30 - 9:00 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

Question	Score	Out of
1.1		10
1.2		10
1.3		10
1.4		10
1.5		5
1.6		10
1.7		10
1.8		10
2.1		10
2.2		10
2.3		10
Your Grade		105

- This quiz has 2 problems, worth a total of 105 points.
- You may tear apart pages 3, 4 and 5, as per your convenience, **but you must turn them in together with the rest of the booklet.**
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed one two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- **You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of** <u>numbers</u> **that could be evaluated using a calculator.** Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- You have 90 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 10/14.

Summary of Results for Special Random Variables

Discrete Uniform over $[a, b]$:

$$p_X(k) = \begin{cases} \dfrac{1}{b-a+1}, & \text{if } k = a, a+1, \ldots, b, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{E}[X] = \frac{a+b}{2}, \qquad \operatorname{var}(X) = \frac{(b-a)(b-a+2)}{12}.$$

Bernoulli with Parameter p: (Describes the success or failure in a single trial.)

$$p_X(k) = \begin{cases} p, & \text{if } k = 1, \\ 1-p, & \text{if } k = 0, \end{cases}$$

$$\mathbf{E}[X] = p, \qquad \operatorname{var}(X) = p(1-p).$$

Binomial with Parameters p and n: (Describes the number of successes in n independent Bernoulli trials.)

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n,$$

$$\mathbf{E}[X] = np, \qquad \operatorname{var}(X) = np(1-p).$$

Geometric with Parameter p: (Describes the number of trials until the first success, in a sequence of independent Bernoulli trials.)

$$p_X(k) = (1-p)^{k-1} p, \qquad k = 1, 2, \ldots,$$

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \operatorname{var}(X) = \frac{1-p}{p^2}.$$

Problem 1: (75 points)

Note: All parts can be done independently, with the exception of the last part. Just in case you made a mistake in the previous part, you can use a symbol for the expression you found there, and use that symbol in the formulas for the last part.

Note: **Algebraic or numerical expressions do not need to be simplified in your answers.**

Jon and Stephen cannot help but think about their commutes using probabilistic modeling. Both of the them start promptly at 8am.

Stephen drives and thus is at the mercy of traffic lights. When all traffic lights on his route are green, the entire trip takes 18 minutes. Stephen's route includes 5 traffic lights, each of which is red with probability 1/3, independent of every other light. Each red traffic light that he encounters adds 1 minute to his commute (for slowing, stopping, and returning to speed).

1. **(10 points)** Find the PMF, expectation, and variance of the length (in minutes) of Stephen's commute.
2. **(10 points)** Given that Stephen's commute took him at most 19 minutes, what is the expected number of red lights that he encountered?
3. **(10 points)** Given that the last red light encountered by Stephen was the fourth light, what is the conditional variance of the total number of red lights he encountered?
4. **(10 points)** Given that Stephen encountered a total of three red lights, what is the probability that exactly two out of the first three lights were red?

Jon's commuting behavior is rather simple to model. Jon walks a total of 20 minutes from his home to a station and from a station to his office. He also waits for X minutes for a subway train, where X has the discrete uniform distribution on $\{0, 1, 2, 3\}$. (All four values are equally likely, and independent of the traffic lights encountered by Stephen.)

5. **(5 points)** What is the PMF of the length of Jon's commute in minutes?
6. **(10 points)** Given that there was exactly one person arriving at **exactly** 8:20am, what is the probability that this person was Jon?
7. **(10 points)** What is the probability that Stephen's commute takes at most as long as Jon's commute?
8. **(10 points)** Given that Stephen's commute took at most as long as Jon's, what is the conditional probability that Jon waited 3 minutes for his train?

Problem 2. (30 points) For each one of the statements below, give either a proof or a counterexample showing that the statement is not always true.

1. **(10 points)** If events A and B are independent, then the events A and B^c are also independent.
2. **(10 points)** Let A, B, and C be events associated with a common probabilistic model, and assume that $0 < \mathbf{P}(C) < 1$. Suppose that A and B are conditionally independent given C. Then, A and B are conditionally independent given C^c.

3. **(10 points)** Let X and Y be independent random variables. Then, $\text{var}(X+Y) \geq \text{var}(X)$.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

Problem 1: (75 points)

Note: All parts can be done independently, with the exception of the last part. Just in case you made a mistake in the previous part, you can use a symbol for the expression you found there, and use that symbol in the formulas for the last part.

Note: **Algebraic or numerical expressions do not need to be simplified in your answers.**

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6. **(10 points)** Given that there was exactly one person arriving at **exactly** 8:20am, what is the probability that this person was Jon?

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1. **(10 points)** If events A and B are independent, then the events A and B^c are also independent.

2. **(10 points)** Let A, B, and C be events associated with a common probabilistic model, and assume that $0 < \mathbf{P}(C) < 1$. Suppose that A and B are conditionally independent given C. Then, A and B are conditionally independent given C^c.

(Additional space for Problem 2.2)

3. **(10 points)** Let X and Y be independent random variables. Then, $\text{var}(X+Y) \geq \text{var}(X)$.

6.041/6.431 Spring 2008 Quiz 2
Wednesday, April 16, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Question	Part	Score	Out of
0			3
1	all		36
2	a		4
	b		5
	c		5
	d		8
	e		5
	f		6
3	a		4
	b		6
	c		6
	d		6
	e		6
Total			100

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

6.041/6.431: ____________________

- Write your solutions in this quiz packet, only solutions in the quiz packet will be graded.
- Question one, multiple choice questions, will receive no partial credit. Partial credit for question two and three will be awarded.
- You are allowed 2 two-sided 8.5 by 11 formula sheet plus a calculator.
- You have 120 minutes to complete the quiz.
- Be neat! You will not get credit if we can't read it.
- We will send out an email with more information on how to obtain your quiz before drop date.
- **Good Luck!**

Question 1: Multiple choice questions. **CLEARLY** circle the best answer for each question below. Each question is worth 4 points each, with no partial credit given.

a. (4 pts) Let X_1, X_2, and X_3 be independent random variables with the continuous uniform distribution over $[0, 1]$. Then $\mathbf{P}(X_1 < X_2 < X_3) =$

  (i) 1/6
  (ii) 1/3
  (iii) 1/2
  (iv) 1/4

b. (4 pts) Let X and Y be two continuous random variables. Then

  (i) $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$
  (ii) $\mathbf{E}[X^2 + Y^2] = \mathbf{E}[X^2] + \mathbf{E}[Y^2]$
  (iii) $f_{X+Y}(x + y) = f_X(x)f_Y(y)$
  (iv) $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$

c. (4 pts) Suppose X is uniformly distributed over $[0, 4]$ and Y is uniformly distributed over $[0, 1]$. Assume X and Y are independent. Let $Z = X + Y$. Then

  (i) $f_Z(4.5) = 0$
  (ii) $f_Z(4.5) = 1/8$
  (iii) $f_Z(4.5) = 1/4$
  (iv) $f_Z(4.5) = 1/2$

d. (4 pts) For the random variables defined in part (c), $\mathbf{P}(\max(X, Y) > 3)$ is equal to

  (i) 0
  (ii) 9/4
  (iii) 3/4
  (iv) 1/4

e. (4 pts) Consider the following variant of the hat problem from lecture: N people put their hats in a closet at the start of a party, where each hat is uniquely identified. At the end of the party each person randomly selects a hat from the closet. Suppose N is a Poisson random variable with parameter λ. If X is the number of people who pick their own hats, then $\mathbf{E}$[X] is equal to

  (i) λ
  (ii) $1/\lambda^2$
  (iii) $1/\lambda$
  (iv) 1

f. (4 pts) Suppose X and Y are Poisson random variables with parameters λ_1 and λ_2 respectively, where X and Y are independent. Define $W = X + Y$, then

(i) W is Poisson with parameter $\min(\lambda_1, \lambda_2)$

(ii) W is Poisson with parameter $\lambda_1 + \lambda_2$

(iii) W may not be Poisson but has mean equal to $\min(\lambda_1, \lambda_2)$

(iv) W may not be Poisson but has mean equal to $\lambda_1 + \lambda_2$

g. (4 pts) Let X be a random variable whose transform is given by $M_X(s) = (0.4 + 0.6e^s)^{50}$. Then

(i) $\mathbf{P}(X = 0) = \mathbf{P}(X = 50)$

(ii) $\mathbf{P}(X = 51) > 0$

(iii) $\mathbf{P}(X = 0) = (0.4)^{50}$

(iv) $\mathbf{P}(X = 50) = 0.6$

h. (4 pts) Let X_i, $i = 1, 2, \ldots$ be independent random variables all distributed according to the pdf $f_X(x) = x/8$ for $0 \leq x \leq 4$. Let $S = \frac{1}{100}\sum_{i=1}^{100} X_i$. Then $\mathbf{P}(S > 3)$ is approximately equal to

(i) $1 - \Phi(5)$

(ii) $\Phi(5)$

(iii) $1 - \Phi\left(\frac{5}{\sqrt{2}}\right)$

(iv) $\Phi\left(\frac{5}{\sqrt{2}}\right)$

i. (4 pts) Let X_i, $i = 1, 2, \ldots$ be independent random variables all distributed according to the pdf $f_X(x) = 1$, $0 \leq x \leq 1$. Define $Y_n = X_1 X_2 X_3 \ldots X_n$, for some integer n. Then $\mathrm{var}(Y_n)$ is equal to

(i) $\frac{n}{12}$

(ii) $\frac{1}{3^n} - \frac{1}{4^n}$

(iii) $\frac{1}{12^n}$

(iv) $\frac{1}{12}$

Question 2: Each Mac book has a lifetime that is exponentially distributed with parameter λ. The lifetime of Mac books are independent of each other. Suppose you have two Mac books, which you begin using at the same time. Define T_1 as the time of the first ~~laptop~~ failure and T_2 as the time of the second ~~laptop~~ failure.

a. (4 pts) Compute $f_{T_1}(t_1)$

b. (5 pts) Let $X = T_2 - T_1$. Compute $f_{X|T_1}(x|t_1)$

c. (5 pts) Is X independent of T_1? Give a mathematical justification for your answer.

d. (8 pts) Compute $f_{T_2}(t_2)$ and $\mathbf{E}[T_2]$

e. (5 pts) Now suppose you have 100 Mac books, and let Y be the time of the first laptop failure. Find the best answer for $\mathbf{P}(Y < 0.01)$

Your friend, Charlie, loves Mac books so much he buys S new Mac books every day! On any given day S is equally likely to be 4 or 8, and all days are independent from each other. Let S_{100} be the number of Mac books Charlie buys over the next 100 days.

f. (6 pts) Find the best approximation for $\mathbf{P}(S_{100} \leq 608)$. Express your final answer in terms of $\Phi(\cdot)$, the CDF of the standard normal.

Question 3: Saif is a well intentioned though slightly indecisive fellow. Every morning he flips a coin to decide where to go. If the coin is heads he drives to the mall, if it comes up tails he volunteers at the local shelter. Saif's coin is not necessarily fair, rather it possesses a probability of heads equal to q. We do not know q, but we do know it is well-modeled by a random variable Q where the density of Q is

$$f_Q(q) = \begin{cases} 2q & \text{for } \ 0 \leq q \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Assume conditioned on Q each coin flip is independent. Note parts a, b, c, and $\{d, e\}$ may be answered independent of each other.

a. (4 pts) What's the probability that Saif goes to the local shelter if he flips the coin once?

In an attempt to promote virtuous behavior, Saif's father offers to pay him \$4 every day he volunteers at the local shelter. Define X as Saif's payout if he flips the coin every morning for the next 30 days.

b. (6 pts) Find $\text{var}(X)$

Let event B be that Saif goes to the local shelter at least once in k days.

c. (6 pts) Find the conditional density of Q given B, $f_{Q|B}(q)$

While shopping at the mall, Saif gets a call from his sister Mais. They agree to meet at the Coco Cabana Court yard at exactly 1:30PM. Unfortunately Mais arrives Z minutes late, where Z is a continuous uniform random variable from zero to 10 minutes. Saif is furious that Mais has kept him waiting, and demands Mais pay him R dollars where $R = \exp(Z + 2)$.

d. (6 pts) Find Saif's expected payout, $\mathbf{E}[R]$

e. (6 pts) Find the density of Saif's payout, $f_R(r)$

6.041 Fall 2009 Quiz 2
Tuesday, November 3, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ______________________

Recitation Instructor: ______________________

TA: ______________________

Question	Score	Out of
1		2
2 (a)		7
2 (b)		7
2 (c)		7
2 (d)		7
2 (e)		7
2 (f)		7
3		10
4 (a i)		5
4 (a ii)		5
4 (b)		7
4 (c)		8
5 (a)		7
5 (b)		7
5 (c)		7
Your Grade		100

- This quiz has 5 problems, worth a total of 100 points.
- **When giving a formula for a PDF, make sure to specify the range over which the formula holds.**
- Please make sure to return the entire exam booklet intact.
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed 2 two-sided, handwritten, formulae sheets. Calculators not allowed.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of <u>numbers</u> that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- You have 2 hrs. to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 11/5.

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \leq y)$, where Y is a standard normal random variable, for y between 0 and 1.99. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When y is negative, the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

Problem 2. (42 points)
The random variable X is exponential with parameter 1. Given the value x of X, the random variable Y is exponential with parameter equal to x (and mean $1/x$).

Note: Some useful integrals, for $\lambda > 0$:

$$\int_0^\infty x e^{-\lambda x}\, dx = \frac{1}{\lambda^2}, \qquad \int_0^\infty x^2 e^{-\lambda x}\, dx = \frac{2}{\lambda^3}.$$

(a) (7 points) Find the joint PDF of X and Y.

(b) (7 points) Find the marginal PDF of Y.

(c) (7 points) Find the conditional PDF of X, given that $Y = 2$.

(d) (7 points) Find the conditional expectation of X, given that $Y = 2$.

(e) (7 points) Find the conditional PDF of Y, given that $X = 2$ and $Y \geq 3$.

(f) (7 points) Find the PDF of e^{2X}.

Problem 3. (10 points)
For the following questions, mark the correct answer. If you get it right, you receive 5 points for that question. You receive no credit if you get it wrong. A justification is not required and will not be taken into account.

Let X and Y be continuous random variables. Let N be a discrete random variable.

(a) (5 points) The quantity $\mathbf{E}[X \mid Y]$ is always:

- (i) A number.
- (ii) A discrete random variable.
- (iii) A continuous random variable.
- (iv) Not enough information to choose between (i)-(iii).

(b) (5 points) The quantity $\mathbf{E}[\,\mathbf{E}[X \mid Y, N] \mid N]$ is always:

- (i) A number.
- (ii) A discrete random variable.
- (iii) A continuous random variable.
- (iv) Not enough information to choose between (i)-(iii).

Problem 4. (25 points)
The probability of obtaining heads in a single flip of a certain coin is itself a random variable, denoted by Q, which is uniformly distributed in $[0, 1]$. Let $X = 1$ if the coin flip results in heads, and $X = 0$ if the coin flip results in tails.

(a) (i) (5 points) Find the mean of X.
(ii) (5 points) Find the variance of X.

(b) (7 points) Find the covariance of X and Q.

(c) (8 points) Find the conditional PDF of Q given that $X = 1$.

Problem 5. (21 points)
Let X and Y be **independent continuous** random variables with marginal PDFs f_X and f_Y, and marginal CDFs F_X and F_Y, respectively. Let

$$S = \min\{X, Y\}, \qquad L = \max\{X, Y\}.$$

(a) (7 points) If X and Y are standard normal, find the probability that $S \geq 1$.

(b) (7 points) Fix some s and ℓ with $s \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \text{ and } L \leq \ell)$$

involving F_X and F_Y, and no integrals.

(c) (7 points) Assume that $s \leq s + \delta \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \leq s + \delta,\ \ell \leq L \leq \ell + \delta),$$

as an integral involving f_X and f_Y.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

Problem 2. (42 points)
The random variable X is exponential with parameter 1. Given the value x of X, the random variable Y is exponential with parameter equal to x (and mean $1/x$).

Note: Some useful integrals, for $\lambda > 0$:

$$\int_0^\infty xe^{-\lambda x}\,dx = \frac{1}{\lambda^2}, \qquad \int_0^\infty x^2 e^{-\lambda x}\,dx = \frac{2}{\lambda^3}.$$

(a) (7 points) Find the joint PDF of X and Y.

(b) (7 points) Find the marginal PDF of Y.

(c) (7 points) Find the conditional PDF of X, given that $Y = 2$.

(d) (7 points) Find the conditional expectation of X, given that $Y = 2$.

(e) (7 points) Find the conditional PDF of Y, given that $X = 2$ and $Y \geq 3$.

(f) (7 points) Find the PDF of e^{2X}.

Problem 3. (10 points)
For the following questions, mark the correct answer. If you get it right, you receive 5 points for that question. You receive no credit if you get it wrong. A justification is not required and will not be taken into account.

Let X and Y be continuous random variables. Let N be a discrete random variable.

(a) (5 points) The quantity $\mathbf{E}[X \mid Y]$ is always:

- (i) A number.
- (ii) A discrete random variable.
- (iii) A continuous random variable.
- (iv) Not enough information to choose between (i)-(iii).

(b) (5 points) The quantity $\mathbf{E}[\,\mathbf{E}[X \mid Y, N] \mid N]$ is always:

- (i) A number.
- (ii) A discrete random variable.
- (iii) A continuous random variable.
- (iv) Not enough information to choose between (i)-(iii).

Problem 4. (25 points)
The probability of obtaining heads in a single flip of a certain coin is itself a random variable, denoted by Q, which is uniformly distributed in $[0, 1]$. Let $X = 1$ if the coin flip results in heads, and $X = 0$ if the coin flip results in tails.

(a) (i) (5 points) Find the mean of X.

(ii) (5 points) Find the variance of X.

(b) (7 points) Find the covariance of X and Q.

(c) (8 points) Find the conditional PDF of Q given that $X = 1$.

Problem 5. (21 points)
Let X and Y be **independent continuous** random variables with marginal PDFs f_X and f_Y, and marginal CDFs F_X and F_Y, respectively. Let

$$S = \min\{X, Y\}, \qquad L = \max\{X, Y\}.$$

(a) (7 points) If X and Y are standard normal, find the probability that $S \geq 1$.

(b) (7 points) Fix some s and ℓ with $s \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \text{ and } L \leq \ell)$$

involving F_X and F_Y, and no integrals.

(c) (7 points) Assume that $s \leq s + \delta \leq \ell$. Give a formula for

$$\mathbf{P}(s \leq S \leq s + \delta,\ \ell \leq L \leq \ell + \delta),$$

as an integral involving f_X and f_Y.

6.041/6.431 Fall 2010 Quiz 2
Tuesday, November 2, 7:30 - 9:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

Question	Score	Out of
1.1		10
1.2		10
1.3		10
1.4		10
1.5		10
1.6		10
1.7		10
1.8		10
2.1		10
2.2		10
2.3		5
2.4		5
Your Grade		110

- **For full credit, answers should be algebraic expressions (no integrals), in simplified form. These expressions may involve constants such as π or e, and need not be evaluated numerically.**
- This quiz has 2 problems, worth a total of 110 points.
- You may tear apart page 3, as per your convenience, **but you must turn them in together with the rest of the booklet.**
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed two two-sided, handwritten, 8.5 by 11 formula sheets. Calculators are not allowed.
- You have 120 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 11/4.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Problem 1. (80 points) In this problem:
(i) X is a (continuous) uniform random variable on $[0,4]$.
(ii) Y is an exponential random variable, independent from X, with parameter $\lambda = 2$.

1. **(10 points)** Find the mean and variance of $X - 3Y$.
2. **(10 points)** Find the probability that $Y \geq X$.
   (Let c be the answer to this question.)
3. **(10 points)** Find the conditional joint PDF of X and Y, given that the event $Y \geq X$ has occurred.
   (You may express your answer in terms of the constant c from the previous part.)
4. **(10 points)** Find the PDF of $Z = X + Y$.
5. **(10 points)** Provide a fully labeled sketch of the conditional PDF of Z given that $Y = 3$.
6. **(10 points)** Find $\mathbf{E}[Z \mid Y = y]$ and $\mathbf{E}[Z \mid Y]$.
7. **(10 points)** Find the joint PDF $f_{Z,Y}$ of Z and Y.
8. **(10 points)** A random variable W is defined as follows. We toss a fair coin (independent of Y). If the result is "heads", we let $W = Y$; if it is tails, we let $W = 2 + Y$. Find the probability of "heads" given that $W = 3$.

Problem 2. (30 points) Let $X, X_1, X_2, \ldots$ be independent normal random variables with mean 0 and variance 9. Let N be a positive integer random variable with $\mathbf{E}[N] = 2$ and $\mathbf{E}[N^2] = 5$. We assume that the random variables $N, X, X_1, X_2, \ldots$ are independent. Let $S = \sum_{i=1}^{N} X_i$.

1. **(10 points)** If δ is a small positive number, we have $\mathbf{P}(1 \leq |X| \leq 1 + \delta) \approx \alpha\delta$, for some constant α. Find the value of α.
2. **(10 points)** Find the variance of S.
3. **(5 points)** Are N and S uncorrelated? Justify your answer.
4. **(5 points)** Are N and S independent? Justify your answer.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

Problem 1. (80 points) In this problem:
(i) X is a (continuous) uniform random variable on $[0, 4]$.
(ii) Y is an exponential random variable, independent from X, with parameter $\lambda = 2$.

1. **(10 points)** Find the mean and variance of $X - 3Y$.

2. **(10 points)** Find the probability that $Y \geq X$.
   (Let c be the answer to this question.)

3. **(10 points)** Find the conditional joint PDF of X and Y, given that the event $Y \geq X$ has occurred.
(You may express your answer in terms of the constant c from the previous part.)

4. **(10 points)** Find the PDF of $Z = X + Y$.

5. **(10 points)** Provide a fully labeled sketch of the conditional PDF of Z given that $Y = 3$.

6. **(10 points)** Find $\mathbf{E}[Z \mid Y = y]$ and $\mathbf{E}[Z \mid Y]$.

7. **(10 points)** Find the joint PDF $f_{Z,Y}$ of Z and Y.

8. **(10 points)** A random variable W is defined as follows. We toss a fair coin (independent of Y). If the result is "heads", we let $W = Y$; if it is tails, we let $W = 2 + Y$. Find the probability of "heads" given that $W = 3$.

Problem 2. (30 points) Let $X, X_1, X_2, \ldots$ be independent normal random variables with mean 0 and variance 9. Let N be a positive integer random variable with $\mathbf{E}[N] = 2$ and $\mathbf{E}[N^2] = 5$. We assume that the random variables $N, X, X_1, X_2, \ldots$ are independent. Let $S = \sum_{i=1}^{N} X_i$.

1. **(10 points)** If δ is a small positive number, we have $\mathbf{P}(1 \leq |X| \leq 1 + \delta) \approx \alpha\delta$, for some constant α. Find the value of α.

2. **(10 points)** Find the variance of S.

3. **(5 points)** Are N and S uncorrelated? Justify your answer.

4. **(5 points)** Are N and S independent? Justify your answer.

6.041/6.431 Spring 2009 Final Exam
Thursday, May 21, 1:30 - 4:30 PM.

Question	Part	Score	Out of
0			2
1	all		18
2	all		24
3	a		4
	b		4
	c		4
4	a		6
	b		6
	c		6
5	a		6
	b		6
6	a		4
	b		4
	c		4
	d		5
	e		5
7	a		6
	b		6
Total			120

Name: ______________________

Recitation Instructor: ______________________

- Write your solutions in this quiz packet, only solutions in the quiz packet will be graded.
- You are allowed three two-sided 8.5 by 11 formula sheet plus a calculator.
- You have 180 minutes to complete the quiz.
- Be neat! You will not get credit if we can't read it.

Problem 1: True or False (2pts. each, 18 pts. total)

No partial credit will be given for individual questions in this part of the quiz.

a. Let $\{X_n\}$ be a sequence of i.i.d random variables taking values in the interval $[0, 0.5]$. Consider the following statements:

(A) If $\mathbf{E}[X_n^2]$ converges to 0 as $n \to \infty$ then X_n converges to 0 in probability.

(B) If all X_n have $\mathbf{E}[X_n] = 0.2$ and $\text{var}(X_n)$ converges to 0 as $n \to \infty$ then X_n converges to 0.2 in probability.

(C) The sequence of random variables Z_n, defined by $Z_n = X_1 \cdot X_2 \cdots X_n$, converges to 0 in probability as $n \to \infty$.

Which of these statements are **always** true? Write **True** or **False** in each of the boxes below.

A:	**B**:	**C**:

b. Let X_i $(i = 1, 2, \ldots)$ be i.i.d. random variables with mean 0 and variance 2; Y_i $(i = 1, 2, \ldots)$ be i.i.d. random variables with mean 2. Assume that all variables X_i, Y_j are independent. Consider the following statements:

(A) $\frac{X_1+\cdots+X_n}{n}$ converges to 0 in probability as $n \to \infty$.

(B) $\frac{X_1^2+\cdots+X_n^2}{n}$ converges to 2 in probability as $n \to \infty$.

(C) $\frac{X_1Y_1+\cdots+X_nY_n}{n}$ converges to 0 in probability as $n \to \infty$.

Which of these statements are **always** true? Write **True** or **False** in each of the boxes below.

A:	**B**:	**C**:

c. We have i.i.d. random variables $X_1 \ldots X_n$ with an unknown distribution, and with $\mu = \mathbf{E}[X_i]$. We define $M_n = (X_1 + \ldots + X_n)/n$. Consider the following statements:

(A) M_n is a maximum-likelihood estimator for μ, irrespective of the distribution of the X_i's.

(B) M_n is a consistent estimator for μ, irrespective of the distribution of the X_i's.

(C) M_n is an asymptotically unbiased estimator for μ, irrespective of the distribution of the X_i's.

Which of these statements are **always** true? Write **True** or **False** in each of the boxes below.

A:	**B**:	**C**:

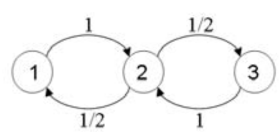
Problem 2: Multiple Choice (4 pts. each, 24 pts. total)

Clearly circle the appropriate choice. **No partial credit** will be given for individual questions in this part of the quiz.

a. Earthquakes in Sumatra occur according to a Poisson process of rate $\lambda = 2/\text{year}$. Conditioned on the event that exactly two earthquakes take place in a year, what is the probability that both earthquakes occur in the first three months of the year? (for simplicity, assume all months have 30 days, and each year has 12 months, i.e., 360 days).

(i) 1/12
(ii) 1/16
(iii) 64/225
(iv) $4e^{-4}$
(v) There is not enough information to determine the required probability.
(vi) None of the above.

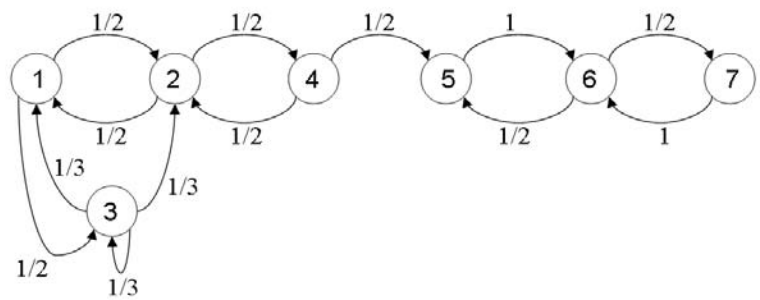
b. Consider a continuous-time Markov chain with three states $i \in \{1, 2, 3\}$, with dwelling time in each visit to state i being an exponential random variable with parameter $\nu_i = i$, and transition probabilities p_{ij} defined by the graph



What is the long-term expected fraction of time spent in state 2?

(i) 1/2
(ii) 1/4
(iii) 2/5
(iv) 3/7
(v) None of the above.

c. Consider the following Markov chain:



Starting in state 3, what is the steady-state probability of being in state 1?

(i) 1/3
(ii) 1/4
(iii) 1
(iv) 0
(v) None of the above.

d. Random variables X and Y are such that the pair (X, Y) is uniformly distributed over the trapezoid A with corners $(0, 0)$, $(1, 2)$, $(3, 2)$, and $(4, 0)$ shown in Fig. 1:

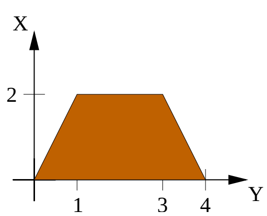


Figure 1: $f_{X,Y}(x, y)$ is constant over the shaded area, zero otherwise.

i.e.

$$f_{X,Y}(x,y) = \begin{cases} c\,, & (x,y) \in A \\ 0\,, & \text{else}\,. \end{cases}$$

We observe Y and use it to estimate X. Let $\hat{X}$ be the least mean squared error estimator of X given Y. What is the value of $\text{var}(\hat{X} - X|Y = 1)$?

(i) 1/6
(ii) 3/2
(iii) 1/3
(iv) The information is not sufficient to compute this value.
(v) None of the above.

e. $X_1 \ldots X_n$ are i.i.d. normal random variables with mean value μ and variance v. Both μ and v are unknown. We define $M_n = (X_1 + \ldots + X_n)/n$ and

$$V_n = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - M_n)^2$$

We also define $\Phi(x)$ to be the CDF for the standard normal distribution, and $\Psi_{n-1}(x)$ to be the CDF for the t-distribution with $n-1$ degrees of freedom. Which of the following choices gives an exact 99% confidence interval for μ for all $n > 1$?

(i) $[M_n - \delta\sqrt{\frac{V_n}{n}}, M_n + \delta\sqrt{\frac{V_n}{n}}]$ where δ is chosen to give $\Phi(\delta) = 0.99$.

(ii) $[M_n - \delta\sqrt{\frac{V_n}{n}}, M_n + \delta\sqrt{\frac{V_n}{n}}]$ where δ is chosen to give $\Phi(\delta) = 0.995$.

(iii) $[M_n - \delta\sqrt{\frac{V_n}{n}}, M_n + \delta\sqrt{\frac{V_n}{n}}]$ where δ is chosen to give $\Psi_{n-1}(\delta) = 0.99$.

(iv) $[M_n - \delta\sqrt{\frac{V_n}{n}}, M_n + \delta\sqrt{\frac{V_n}{n}}]$ where δ is chosen to give $\Psi_{n-1}(\delta) = 0.995$.

(v) None of the above.

f. We have i.i.d. random variables X_1, X_2 which have an exponential distribution with unknown parameter θ. Under hypothesis H_0, $\theta = 1$. Under hypothesis H_1, $\theta = 2$. Under a likelihood-ratio test, the rejection region takes which of the following forms?

(i) $R = \{(x_1, x_2) : x_1 + x_2 > \xi\}$ for some value ξ.

(ii) $R = \{(x_1, x_2) : x_1 + x_2 < \xi\}$ for some value ξ.

(iii) $R = \{(x_1, x_2) : e^{x_1} + e^{x_2} > \xi\}$ for some value ξ.

(iv) $R = \{(x_1, x_2) : e^{x_1} + e^{x_2} < \xi\}$ for some value ξ.

(v) None of the above.

Problem 3 (12 pts. total)

Aliens of two races (blue and green) are arriving on Earth independently according to Poisson process distributions with parameters λ_b and λ_a respectively. The Alien Arrival Registration Service Authority (AARSA) will begin registering alien arrivals soon.

Let T_1 denote the time AARSA will function until it registers its first alien. Let G be the event that the first alien to be registered is a green one. Let T_2 be the time AARSA will function until at least one alien of both races is registered.

(a) (4 points.) Express $\mu_1 = \mathbf{E}[T_1]$ in terms of λ_g and λ_b. **Show your work.**

(b) (4 points.) Express $p = \mathbf{P}(G)$ in terms of λ_g and λ_b. **Show your work.**

(c) (4 points.) Express $\mu_2 = \mathbf{E}[T_2]$ in terms of λ_g and λ_b. **Show your work.**

Problem 4 (18 pts. total)

Researcher Jill is interested in studying employment in technology firms in Dilicon Valley. She denotes by X_i the number of employees in technology firm i and assumes that X_i are independent and identically distributed with mean p. To estimate p, Jill randomly interviews n technology firms and observes the number of employees in these firms.

(a) (6 points.) Jill uses

$$M_n = \frac{X_1 + \cdots + X_n}{n}$$

as an estimator for p. Find the limit of $\mathbf{P}(M_n \leq x)$ as $n \to \infty$ for $x < p$. Find the limit of $\mathbf{P}(M_n \leq x)$ as $n \to \infty$ for $x > p$. **Show your work.**

(b) (6 points.) Find the smallest n, the number of technology firms Jill must sample, for which the Chebyshev inequality yields a guarantee

$$\mathbf{P}(|M_n - p| \geq 0.5) \leq 0.05.$$

Assume that $\mathrm{var}\,(X_i) = v$ for some constant v. State your solution as a function of v. **Show your work.**

(c) (6 points.) Assume now that the researcher samples $n = 5000$ firms. Find an approximate value for the probability

$$\mathbf{P}(|M_{5000} - p| \geq 0.5)$$

using the Central Limit Theorem. Assume again that $\text{var}(X_i) = v$ for some constant v. Give your answer in terms of v, and the standard normal CDF Φ. **Show your work.**

Problem 5 (12 pts. total)

The RandomView window factory produces window panes. After manufacturing, 1000 panes were loaded onto a truck. The weight W_i of the i-th pane (in pounds) on the truck is modeled as a random variable, with the assumption that the W_i's are independent and identically distributed.

(a) (6 points.) Assume that the measured weight of the load on the truck was 2340 pounds, and that $\text{var}(W_i) \leq 4$. Find an approximate 95 percent confidence interval for $\mu = \mathbf{E}[W_i]$, using the Central Limit Theorem (you may use the standard normal table which was handed out with this quiz). **Show your work.**

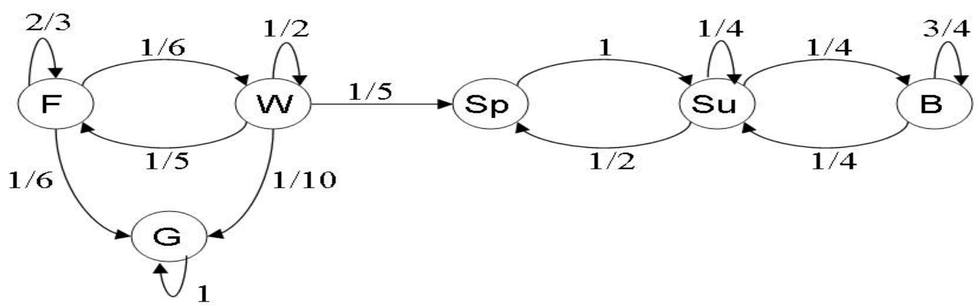
(b) (6 points.) Now assume instead that the random variables W_i are i.i.d, with an exponential distribution with parameter $\theta > 0$, i.e., a distribution with PDF

$$f_W(w; \theta) = \theta e^{-\theta w}$$

What is the maximum likelihood estimate of θ, given that the truckload has weight 2340 pounds? **Show your work.**

Problem 6 (21 pts. total)

In Alice's Wonderland, there are six different seasons: Fall (F), Winter (W), Spring (Sp), Summer (Su), Bitter Cold (B), and Golden Sunshine (G). The seasons do not follow any particular order, instead, at the beginning of each day the Head Wizard assigns the season for the day, according to the following Markov chain model:



Thus, for example, if it is Fall one day then there is 1/6 probability that it will be Winter the next day (note that it is possible to have the same season again the next day).

(a) (4 points.) For each state in the above chain, identify whether it is recurrent or transient. **Show your work.**

(b) (4 points.) If it is Fall on Monday, what is the probability that it will be Summer on Thursday of the same week? **Show your work.**

(c) (4 points.) If it is Spring today, will the chain converge to steady-state probabilities? If so, compute the steady-state probability for each state. If not, explain why these probabilities do not exist. **Show your work.**

(d) (5 points.) If it is Fall today, what is the probability that Bitter Cold will never arrive in the future? **Show your work.**

(e) (5 points.) If it is Fall today, what is the expected number of days till either Summer or Golden Sunshine arrives for the first time? **Show your work.**

Problem 7 (12 pts. total)

A newscast covering the final baseball game between Sed Rox and Y Nakee becomes noisy at the crucial moment when the viewers are informed whether Y Nakee won the game.

Let a be the parameter describing the actual outcome: $a = 1$ if Y Nakee won, $a = -1$ otherwise. There were n viewers listening to the telecast. Let Y_i be the information received by viewer i ($1 \leq i \leq n$). Under the noisy telecast, $Y_i = a$ with probability p, and $Y_i = -a$ with probability $1 - p$. Assume that the random variables Y_i are independent of each other.

The viewers as a group come up with a joint estimator

$$Z_n = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} Y_i \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

(a) (6 points.)

Find $\lim_{n\to\infty} \mathbf{P}(Z_n = a)$ assuming that $p > 0.5$ and $a = 1$. **Show your work.**

(b) (6 points.) Find $\lim_{n\to\infty} \mathbf{P}(Z_n = a)$, assuming that $p = 0.5$ and $a = 1$. **Show your work.**

6.041 Fall 2009 Final Exam
Tuesday, December 15, 1:30 - 4:30 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________

Recitation Instructor: ____________________

TA: ____________________

Question	Score	Out of
1		
2 (a)		5
2 (b)		5
2 (c)		5
2 (d)		5
3 (a)		5
3 (b)		5
3 (c)		5
3 (d)		5
3 (e)		5

Question	Score	Out of
4 (a)		5
4 (b)		5
4 (c)		5
4 (d)		5
4 (e)		5
4 (f)		5
5 (a)		5
5 (b)		5
5 (c)		5
5 (d)		5
5 (e)		5
Your Grade		100

- This exam has 5 problems, worth a total of 100 points.
- **When giving a formula for a PDF, make sure to specify the range over which the formula holds.**
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed three two-sided, handwritten, formula sheets plus a calculator.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of <u>numbers</u> that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- The last page of this final contains a standard normal table.

Problem 2. (20 points)
A pair of jointly continuous random variables, X and Y, have a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} c, & \text{in the shaded region of Fig. 1} \\ 0, & \text{elsewhere.} \end{cases}$$

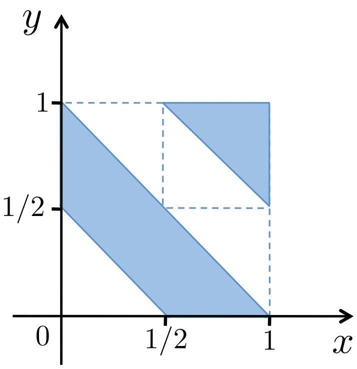


Figure 1: The shaded region is the domain in which $f_{X,Y}(x,y) = c$.

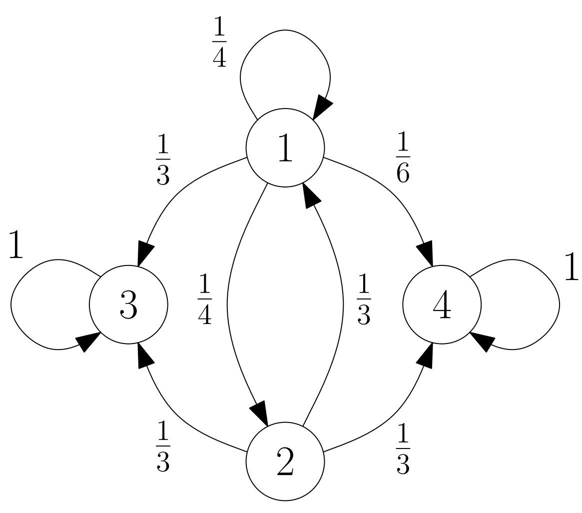
(a) (5 points) Find c.

(b) (5 points) Find the marginal PDFs of X and Y, i.e., $f_X(x)$ and $f_Y(y)$.

(c) (5 points) Find $\mathbf{E}[X \mid Y = 1/4]$ and $\mathrm{Var}[X \mid Y = 1/4]$, that is, the conditional mean and conditional variance of X given that $Y = 1/4$.

(d) (5 points) Find the conditional PDF for X given that $Y = 3/4$, i.e., $f_{X|Y}(x \mid 3/4)$.

Problem 3. (25 points)
Consider a Markov chain X_n whose one-step transition probabilities are shown in the figure.



(a) (5 points) What are the recurrent states?

(b) (5 points) Find $\mathbf{P}(X_2 = 4 \mid X_0 = 2)$.

(c) (5 points) Suppose that you are given the values of $r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i)$. Give a formula for $r_{11}(n+1)$ in terms of the $r_{ij}(n)$.

(d) (5 points) Find the steady-state probabilities $\pi_j = \lim_{n\to\infty} \mathbf{P}(X_n = j \mid X_0 = i)$, or explain why they do not exist.

(e) (5 points) What is the probability of eventually visiting state 4, given that the initial state is $X_0 = 1$?

Problem 4. (30 points)
Al, Bonnie, and Clyde run laps around a track, with the duration of each lap (in hours) being exponentially distributed with parameters $\lambda_A = 21$, $\lambda_B = 23$, and $\lambda_C = 24$, respectively. Assume that all lap durations are independent. At the completion of each lap, a runner drinks either one or two cups of water, with probabilities 1/3 and 2/3, respectively, independent of everything else, including how much water was consumed after previous laps. (The time spent drinking is negligible, assumed zero.)

(a) (5 points) Write down the PMF of the total number of completed laps over the first hour.

(b) (5 points) What is the expected number of cups of water to be consumed by the three runners, in total, over the first hour.

(c) (5 points) Al has amazing endurance and completed 72 laps. Find a good approximation for the probability that he drank at least 130 cups. (You do not have to use 1/2-corrections.)

(d) (5 points) What is the probability that Al finishes his first lap before any of the others?

(e) (5 points) Suppose that the runners have been running for a very long time when you arrive at the track. What is the distribution of the duration of Al's current lap? (This includes the duration of that lap both before and after the time of your arrival.)

(f) (5 points) Suppose that the runners have been running for 1/4 hours. What is the distribution of the time Al spends on his second lap, given that he is on his second lap?

Problem 5. (25 points)
A pulse of light has energy X that is a second-order Erlang random variable with parameter λ, i.e., its PDF is

$$f_X(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & \text{for } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

This pulse illuminates an ideal photon-counting detector whose output N is a Poisson-distributed random variable with mean x when $X = x$, i.e., its conditional PMF is

$$p_{N|X}(n \mid x) = \begin{cases} \dfrac{x^n e^{-x}}{n!}, & \text{for } n = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (5 points) Find $\mathbf{E}[N]$ and $\text{Var}[N]$, the unconditional mean and variance of N

(b) (5 points) Find $p_N(n)$, the unconditional PMF of N.

(c) (5 points) Find $\hat{X}_{\text{lin}}(N)$, the linear least-squares estimator of X based on an observation of N.

(d) (5 points) Find $\hat{X}_{\text{MAP}}(N)$, the MAP estimator of X based on an observation of N.

(e) (5 points) Instead of the prior distribution in Eq. (1), we are now told that

$$\mathbf{P}(X = 2) = 3^3/35, \qquad \mathbf{P}(X = 3) = 2^3/35.$$

Given the observation $N = 3$, and in order to minimize the probability of error, which one of the two hypotheses $X = 2$ and $X = 3$ should be chosen?

Useful integral and facts:

$$\int_0^\infty y^k e^{-\alpha y}\, dy = \frac{k!}{\alpha^{k+1}}, \quad \text{for } \alpha > 0 \text{ and } k = 0, 1, 2, \ldots \text{ (recall that 0!=1)}$$

The second-order Erlang random variable satisfies:

$$\mathbf{E}[X] = 2/\lambda, \qquad \mathrm{Var}(X) = 2/\lambda^2.$$

Each question is repeated in the following pages. Please write your answer on the appropriate page.

Problem 2. (20 points)
A pair of jointly continuous random variables, X and Y, have a joint probability density function given by

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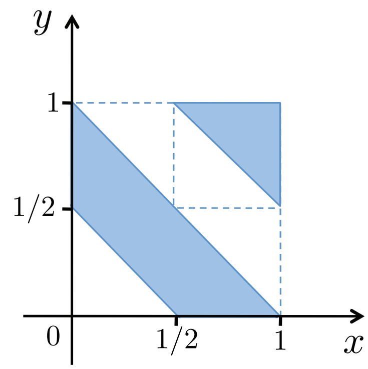

Figure 2: The shaded region is the domain in which $f_{X,Y}(x,y) = c$.

(a) (5 points) Find c.

(b) (5 points) Find the marginal PDFs of X and Y, i.e., $f_X(x)$ and $f_Y(y)$.

(c) (5 points) Find $\mathbf{E}[X \mid Y = 1/4]$ and $\text{Var}[X \mid Y = 1/4]$, that is, the conditional mean and conditional variance of X given that $Y = 1/4$.

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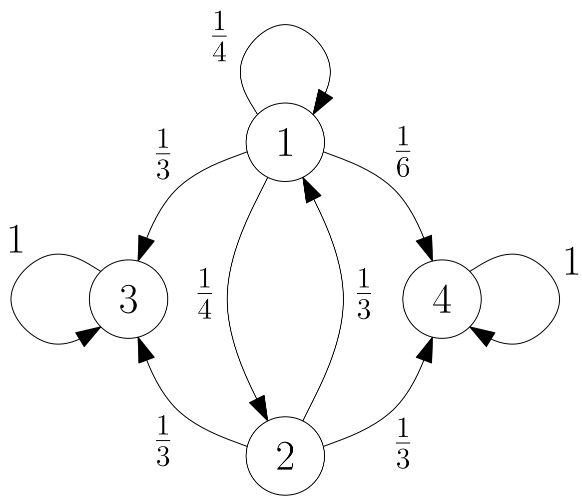

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(b) (5 points) What is the expected number of cups of water to be consumed by the three runners, in total, over the first hour.

(c) (5 points) Al has amazing endurance and completed 72 laps. Find a good approximation for the probability that he drank at least 130 cups. (You do not have to use 1/2-corrections.)

(d) (5 points) What is the probability that Al finishes his first lap before any of the others?

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6.041/6.431 Fall 2010 Final Exam
Wednesday, December 15, 9:00AM - 12:00noon.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Name: ____________________

Recitation Instructor: ____________________

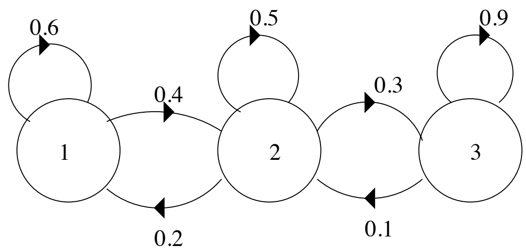
TA: ____________________

Question	Score	Out of
1.1		4
1.2		4
1.3		4
1.4		4
1.5		4
1.6		4
1.7		4
1.8		4
2.1		3
2.2 (a)		3
2.2 (b)		3
2.2 (c)		3

Question	Score	Out of
2.3 (a)		3
2.3 (b)		3
2.4 (a)		5
2.4 (b)		5
2.4 (c)		5
2.4 (d)		5
2.5		5
2.6		5
2.7		5
2.8		5
2.9 (a)		5
2.9 (b)		5
Your Grade		100

- **Show your work and provide brief justifications for your answers, except for parts where you are told that a justification is not needed.**
- **Unless instructed otherwise, for full credit answers should be algebraic expressions (no integrals), in simplified form. These expressions may involve constants such as π or e, and need not be evaluated numerically.**
- This quiz has 2 problems, worth a total of 100 points.
- You may tear apart page 3 and 4, as per your convenience, **but you must turn them in together with the rest of the booklet.**
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed three two-sided, handwritten, 8.5 by 11 formula sheets. Calculators are not allowed.

Problem 1. (32 points) Consider a Markov chain $\{X_n; n = 0, 1, \ldots\}$, specified by the following transition diagram.



1. **(4 points)** Given that the chain starts with $X_0 = 1$, find the probability that $X_2 = 2$.

2. **(4 points)** Find the steady-state probabilities π_1, π_2, π_3 of the different states.

   In case you did not do part (b) correctly, in **all** *subsequent parts of this problem you can just use the symbols π_i: you do not need to plug in actual numbers.*

3. **(4 points)** Let $Y_n = X_n - X_{n-1}$. Thus, $Y_n = 1$ indicates that the nth transition was to the right, $Y_n = 0$ indicates it was a self-transition, and $Y_n = -1$ indicates it was a transition to the left. Find $\lim_{n\to\infty} \mathbf{P}(Y_n = 1)$.

4. **(4 points)** Is the sequence Y_n a Markov chain? Justify your answer.

5. **(4 points)** Given that the nth transition was a transition to the right ($Y_n = 1$), find the probability that the previous state was state 1. (You can assume that n is large.)

6. **(4 points)** Suppose that $X_0 = 1$. Let T be defined as the first *positive time* at which the state is again equal to 1. Show how to find $\mathbf{E}[T]$. (It is enough to write down whatever equation(s) needs to be solved; you do not have to actually solve it/them or to produce a numerical answer.)

7. **(4 points)** Does the sequence $X_1, X_2, X_3, \ldots$ converge in probability? If yes, to what? If not, just say "no" without explanation.

8. **(4 points)** Let $Z_n = \max\{X_1, \ldots, X_n\}$. Does the sequence $Z_1, Z_2, Z_3, \ldots$ converge in probability? If yes, to what? If not, just say "no" without explanation.

Problem 2. (68 points) Alice shows up at an Athena* cluster at time zero and spends her time exclusively in typing emails. The times that her emails are sent are a Poisson process with rate λ_A per hour.

1. **(3 points)** What is the probability that Alice sent exactly three emails during the time interval $[1, 2]$?

2. Let Y_1 and Y_2 be the times at which Alice's first and second emails were sent.

   (a) **(3 points)** Find $\mathbf{E}[Y_2 \mid Y_1]$.
   (b) **(3 points)** Find the PDF of Y_1^2.
   (c) **(3 points)** Find the joint PDF of Y_1 and Y_2.

*Athena is MIT's UNIX-based computing environment. OCW does not provide access to it.

3. You show up at time 1 and you are told that Alice has sent exactly one email so far. (Only give answers here, no need to justify them.)
   (a) **(3 points)** What is the conditional expectation of Y_2 given this information?
   (b) **(3 points)** What is the conditional expectation of Y_1 given this information?

4. Bob just finished exercising (without email access) and sits next to Alice at time 1. He starts typing emails at time 1, and fires them according to an independent Poisson process with rate λ_B.
   (a) **(5 points)** What is the PMF of the total number of emails sent by the two of them together during the interval $[0, 2]$?
   (b) **(5 points)** What is the expected value of the total typing time associated with the email that Alice is typing at the time that Bob shows up? (Here, "total typing time" includes the time that Alice spent on that email both before and after Bob's arrival.)
   (c) **(5 points)** What is the expected value of the time until each one of them has sent at least one email? (Note that we count time starting from time 0, and we take into account any emails possibly sent out by Alice during the interval $[0, 1]$.)
   (d) **(5 points)** Given that a total of 10 emails were sent during the interval $[0, 2]$, what is the probability that exactly 4 of them were sent by Alice?

5. **(5 points)** Suppose that $\lambda_A = 4$. Use Chebyshev's inequality to find an upper bound on the probability that Alice sent at least 5 emails during the time interval $[0, 1]$. Does the Markov inequality provide a better bound?

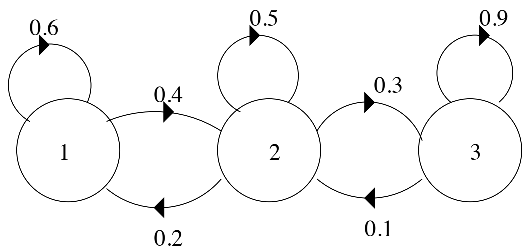
6. **(5 points)** You do not know λ_A but you watch Alice for an hour and see that she sent exactly 5 emails. Derive the maximum likelihood estimate of λ_A based on this information.

7. **(5 points)** We have reasons to believe that λ_A is a large number. Let N be the number of emails sent during the interval $[0, 1]$. Justify why the CLT can be applied to N, and give a precise statement of the CLT in this case.

8. **(5 points)** Under the same assumption as in last part, that λ_A is large, you can now pretend that N is a normal random variable. Suppose that you observe the value of N. Give an (approximately) 95% confidence interval for λ_A. State precisely what approximations you are making.
   Possibly useful facts: The cumulative normal distribution satisfies $\Phi(1.645) = 0.95$ and $\Phi(1.96) = 0.975$.

9. You are now told that λ_A is actually the realized value of an exponential random variable Λ, with parameter 2:
$$f_\Lambda(\lambda) = 2e^{-2\lambda}, \qquad \lambda \geq 0.$$
   (a) **(5 points)** Find $\mathbf{E}[N^2]$.
   (b) **(5 points)** Find the linear least squares estimator of Λ given N.

Problem 1. (32 points) Consider a Markov chain $\{X_n; n = 0, 1, \ldots\}$, specified by the following transition diagram.



1. **(4 points)** Given that the chain starts with $X_0 = 1$, find the probability that $X_2 = 2$.

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4. **(4 points)** Is the sequence Y_n a Markov chain? Justify your answer.

5. **(4 points)** Given that the nth transition was a transition to the right ($Y_n = 1$), find the probability that the previous state was state 1. (You can assume that n is large.)

6. **(4 points)** Suppose that $X_0 = 1$. Let T be defined as the first *positive time* at which the state is again equal to 1. Show how to find $\mathbf{E}[T]$. (It is enough to write down whatever equation(s) needs to be solved; you do not have to actually solve it/them or to produce a numerical answer.)

7. **(4 points)** Does the sequence $X_1, X_2, X_3, \ldots$ converge in probability? If yes, to what? If not, just say "no" without explanation.

8. **(4 points)** Let $Z_n = \max\{X_1, \ldots, X_n\}$. Does the sequence $Z_1, Z_2, Z_3, \ldots$ converge in probability? If yes, to what? If not, just say "no" without explanation.

Problem 2. (68 points) Alice shows up at an Athena* cluster at time zero and spends her time exclusively in typing emails. The times that her emails are sent are a Poisson process with rate λ_A per hour.

1. **(3 points)** What is the probability that Alice sent exactly three emails during the time interval $[1, 2]$?

2. Let Y_1 and Y_2 be the times at which Alice's first and second emails were sent.

   (a) **(3 points)** Find $\mathbf{E}[Y_2 \mid Y_1]$.

*Athena is MIT's UNIX-based computing environment. OCW does not provide access to it.

(b) **(3 points)** Find the PDF of Y_1^2.

(c) **(3 points)** Find the joint PDF of Y_1 and Y_2.

3. You show up at time 1 and you are told that Alice has sent exactly one email so far. (Only give answers here, no need to justify them.)

   (a) **(3 points)** What is the conditional expectation of Y_2 given this information?

   (b) **(3 points)** What is the conditional expectation of Y_1 given this information?

4. Bob just finished exercising (without email access) and sits next to Alice at time 1. He starts typing emails at time 1, and fires them according to an independent Poisson process with rate λ_B.

   (a) **(5 points)** What is the PMF of the total number of emails sent by the two of them together during the interval $[0, 2]$?

(b) **(5 points)** What is the expected value of the total typing time associated with the email that Alice is typing at the time that Bob shows up? (Here, "total typing time" includes the time that Alice spent on that email both before and after Bob's arrival.)

(c) **(5 points)** What is the expected value of the time until each one of them has sent at least one email? (Note that we count time starting from time 0, and we take into account any emails possibly sent out by Alice during the interval $[0, 1]$.)

(d) **(5 points)** Given that a total of 10 emails were sent during the interval $[0, 2]$, what is the probability that exactly 4 of them were sent by Alice?

5. **(5 points)** Suppose that $\lambda_A = 4$. Use Chebyshev's inequality to find an upper bound on the probability that Alice sent at least 5 emails during the time interval $[0, 1]$. Does the Markov inequality provide a better bound?

6. **(5 points)** You do not know λ_A but you watch Alice for an hour and see that she sent exactly 5 emails. Derive the maximum likelihood estimate of λ_A based on this information.

7. **(5 points)** We have reasons to believe that λ_A is a large number. Let N be the number of emails sent during the interval $[0, 1]$. Justify why the CLT can be applied to N, and give a precise statement of the CLT in this case.

8. **(5 points)** Under the same assumption as in last part, that λ_A is large, you can now pretend that N is a normal random variable. Suppose that you observe the value of N. Give an (approximately) 95% confidence interval for λ_A. State precisely what approximations you are making.
Possibly useful facts: The cumulative normal distribution satisfies $\Phi(1.645) = 0.95$ and $\Phi(1.96) = 0.975$.

9. You are now told that λ_A is actually the realized value of an exponential random variable Λ, with parameter 2:

$$f_\Lambda(\lambda) = 2e^{-2\lambda}, \qquad \lambda \geq 0.$$

(a) **(5 points)** Find $\mathbf{E}[N^2]$.

(b) **(5 points)** Find the linear least squares estimator of Λ given N.