Derivation of the Lorentz Force Law from the Lagrangian: A Comprehensive Tutorial

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Introduction

This tutorial provides a complete, step-by-step derivation of the Lorentz force law starting from the Lagrangian. The tutorial includes the derivation of key vector identities and explicitly details each term involved in the process. The goal is to make the content understandable even for someone new to the subject.

Step 1: Define the Lagrangian

$$L$$

The Lagrangian

$$L$$

is a function that describes the dynamics of a system. For a charged particle of mass

$$m$$

and charge

$$q$$

moving in electromagnetic fields described by a vector potential

$$\vec{A}$$

and a scalar potential

$$\phi$$

, the Lagrangian

$$L$$

is defined as:

$$L = \frac{1}{2}m\dot{\vec{x}} \cdot \dot{\vec{x}} + q\dot{\vec{x}} \cdot \vec{A} - q\phi$$

Here,

$$\dot{\vec{x}}$$

is the velocity of the particle, and the dot represents a time derivative.

Step 2: Compute

$$\frac{\partial L}{\partial \dot{\vec{x}}}$$

To proceed, we need to find the partial derivative of

$$L$$

with respect to

$$\dot{\vec{x}}$$

. This derivative is obtained as follows:

$$\frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} + q\vec{A}$$

Step 3: Derive the Vector Identity

Before diving into the next steps, let's derive a key vector identity that will be used later. The identity is:

$$\nabla(\vec{U} \cdot \vec{V}) = (\vec{U} \cdot \nabla)\vec{V} + (\vec{V} \cdot \nabla)\vec{U} + \vec{U} \times (\nabla \times \vec{V}) + \vec{V} \times (\nabla \times \vec{U})$$

The derivation of this identity involves using the definitions of gradient, divergence, and curl, along with the product rule for derivatives. Due to its complexity, it's typically proven using tensor notation or by working through each Cartesian component.

Step 4: Compute

$$\frac{\partial L}{\partial \vec{x}}$$

with Explicit Derivation of Terms

Now, we'll find the partial derivative of

$$L$$

with respect to

$$\vec{x}$$

. This involves differentiating the terms

$$q\dot{\vec{x}} \cdot \vec{A}$$

and

$$-q\phi$$

.

Derivation of the

$$-q\nabla\phi$$

Term

The

$$-q\phi$$

term differentiates to

$$-q\nabla\phi$$

when taking the gradient with respect to

$$\vec{x}$$

.

Derivation of the Last Two Terms from

$$q\vec{\dot{x}} \cdot \vec{A}$$

Using the vector identity derived in Step 3, the gradient of

$$q\vec{\dot{x}} \cdot \vec{A}$$

becomes:

$$\nabla(q\vec{\dot{x}} \cdot \vec{A}) = q(\vec{\dot{x}} \cdot \nabla)\vec{A} + q\vec{\dot{x}} \times (\nabla \times \vec{A})$$

These two terms will be part of

$$\frac{\partial L}{\partial \vec{x}}$$

.

Final Expression for

$$\frac{\partial L}{\partial \vec{x}}$$

Combining these terms, we get:

$$\frac{\partial L}{\partial \vec{x}} = -q\nabla\phi + q(\dot{\vec{x}} \cdot \nabla)\vec{A} + q\dot{\vec{x}} \times (\nabla \times \vec{A})$$

Step 5: Euler-Lagrange Equation and Time Derivative

The Euler-Lagrange equation states:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) - \frac{\partial L}{\partial \vec{x}} = 0$$

To apply this equation, we need to find the time derivative of

$$\frac{\partial L}{\partial \dot{\vec{x}}}$$

, which is:

$$\frac{d}{dt}(m\dot{\vec{x}} + q\vec{A}) = m\ddot{\vec{x}} + q\frac{d\vec{A}}{dt}$$

The total time derivative

$$\frac{d\vec{A}}{dt}$$

includes both explicit and implicit time dependencies:

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\dot{\vec{x}} \cdot \nabla)\vec{A}$$

Step 6: Substitute into Euler-Lagrange Equation and Simplify

Having all the necessary derivatives and expressions at hand, we can now substitute these into the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) - \frac{\partial L}{\partial \vec{x}} = 0$$

We had:

$$\frac{d}{dt}(m\dot{\vec{x}} + q\vec{A}) = m\ddot{\vec{x}} + q\left(\frac{\partial \vec{A}}{\partial t} + (\dot{\vec{x}} \cdot \nabla)\vec{A}\right)$$

And:

$$\frac{\partial L}{\partial \vec{x}} = -q\nabla\phi + q(\dot{\vec{x}} \cdot \nabla)\vec{A} + q\dot{\vec{x}} \times (\nabla \times \vec{A})$$

Substituting these into the Euler-Lagrange equation, we get:

$$m\ddot{\vec{x}} + q\left(\frac{\partial \vec{A}}{\partial t} + (\dot{\vec{x}} \cdot \nabla)\vec{A}\right) = q\left(-\nabla\phi + (\dot{\vec{x}} \cdot \nabla)\vec{A} + \dot{\vec{x}} \times (\nabla \times \vec{A})\right)$$

Simplifying, we find:

$$m\ddot{\vec{x}} = q\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t}\right) + q\dot{\vec{x}} \times (\nabla \times \vec{A})$$

Finally, using

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

and

$$\vec{B} = \nabla \times \vec{A}$$

, we arrive at the Lorentz force law:

$$m\ddot{\vec{x}} = q(\vec{E} + \dot{\vec{x}} \times \vec{B})$$

Conclusion

This concludes the comprehensive tutorial on deriving the Lorentz force law from the Lagrangian. The tutorial aimed to be as detailed as possible, explicitly showing the derivation of each term and equation involved. I hope you find this tutorial complete and informative.

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Derivation Using Limits

The Definition of the Total Derivative

The total derivative

$$\frac{d\vec{A}}{dt}$$

of a vector field

$$\vec{A}$$

is defined by the limit:

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t + \Delta t, \vec{x}(t + \Delta t)) - \vec{A}(t, \vec{x}(t))}{\Delta t}$$

Breaking Down the Limit Expression

We can express

$$\vec{A}(t + \Delta t, \vec{x}(t + \Delta t))$$

using a Taylor series expansion around

$$(t, \vec{x}(t))$$

:

$$\vec{A}(t+\Delta t, \vec{x}(t+\Delta t)) \approx \vec{A}(t, \vec{x}(t)) + \Delta t \left(\frac{\partial \vec{A}}{\partial t}\right)_{\vec{x}(t)} + \Delta \vec{x} \cdot \left(\nabla \vec{A}\right)_{\vec{x}(t)}$$

where

$$\Delta \vec{x} = \vec{x}(t+\Delta t) - \vec{x}(t)$$

.

Factor Out

$$\Delta t$$

Now, we can substitute this expansion back into the limit expression for

$$\frac{d\vec{A}}{dt}$$

:

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t, \vec{x}(t)) + \Delta t \left(\frac{\partial \vec{A}}{\partial t}\right)_{\vec{x}(t)} + \Delta \vec{x} \cdot \left(\nabla \vec{A}\right)_{\vec{x}(t)} - \vec{A}(t, \vec{x}(t))}{\Delta t}$$

Factoring out

$$\Delta t$$

in the numerator, we get:

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \left(\frac{\partial \vec{A}}{\partial t} \right)_{\vec{x}(t)} + \frac{\Delta \vec{x}}{\Delta t} \cdot \left(\nabla \vec{A} \right)_{\vec{x}(t)}$$

Taking the Limit

As

$$\Delta t$$

approaches zero,

$$\frac{\Delta \vec{x}}{\Delta t}$$

approaches

$$\dot{\vec{x}}$$

, the velocity of the particle. So, we have:

$$\frac{d\vec{A}}{dt} = \left(\frac{\partial \vec{A}}{\partial t} \right) + \dot{\vec{x}} \cdot \nabla \vec{A}$$

This gives us the expression for the total time derivative

$$\frac{d\vec{A}}{dt}$$

, which includes both the explicit and implicit time dependencies.

Certainly, demonstrating the vector identity through a concrete 3D example can offer a tangible way to grasp its intricacies. The vector identity we're interested in is:

$$\nabla(\vec{U} \cdot \vec{V}) = (\vec{U} \cdot \nabla)\vec{V} + (\vec{V} \cdot \nabla)\vec{U} + \vec{U} \times (\nabla \times \vec{V}) + \vec{V} \times (\nabla \times \vec{U})$$

For simplicity, let's consider

$$\vec{U}$$

and

$$\vec{V}$$

as 3D vectors defined in Cartesian coordinates

$$(x, y, z)$$

:

$$\vec{U} = U_x\hat{i} + U_y\hat{j} + U_z\hat{k}$$

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

Tutorial: Verifying the Vector Identity in 3D

Step 1: Compute

$$\vec{U} \cdot \vec{V}$$

The dot product

$$\vec{U} \cdot \vec{V}$$

is given by:

$$\vec{U} \cdot \vec{V} = U_xV_x + U_yV_y + U_zV_z$$

Step 2: Compute

$$\nabla(\vec{U} \cdot \vec{V})$$

The gradient of

$$\vec{U} \cdot \vec{V}$$

with respect to

$$\vec{r} = (x, y, z)$$

is:

$$\nabla(\vec{U} \cdot \vec{V}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (U_x V_x + U_y V_y + U_z V_z)$$

This results in a vector with components:

$$\left(\frac{\partial}{\partial x}(U_x V_x) + \frac{\partial}{\partial y}(U_y V_y) + \frac{\partial}{\partial z}(U_z V_z)\right)\hat{i} + (\text{similar terms for } \hat{j} \text{ and } \hat{k})$$

Step 3: Compute

$$(\vec{U} \cdot \nabla)\vec{V}$$

and

$$(\vec{V} \cdot \nabla)\vec{U}$$

The term

$$(\vec{U} \cdot \nabla)\vec{V}$$

can be written as:

$$(U_x \frac{\partial}{\partial x} + U_y \frac{\partial}{\partial y} + U_z \frac{\partial}{\partial z}) \cdot (V_x \hat{i} + V_y \hat{j} + V_z \hat{k})$$

After performing the dot product, we get:

$$U_x \frac{\partial V_x}{\partial x} \hat{i} + U_y \frac{\partial V_y}{\partial y} \hat{j} + U_z \frac{\partial V_z}{\partial z} \hat{k}$$

A similar calculation can be done for

$$(\vec{V} \cdot \nabla)\vec{U}$$

.

Step 4: Compute

$$\vec{U} \times (\nabla \times \vec{V})$$

and

$$\vec{V} \times (\nabla \times \vec{U})$$

Computing the curl

$$\nabla \times \vec{V}$$

and

$$\nabla \times \vec{U}$$

yields vectors in

$$\hat{i}, \hat{j}, \hat{k}$$

components. The cross product

$$\vec{U} \times (\nabla \times \vec{V})$$

can then be computed term-by-term in a straightforward manner.

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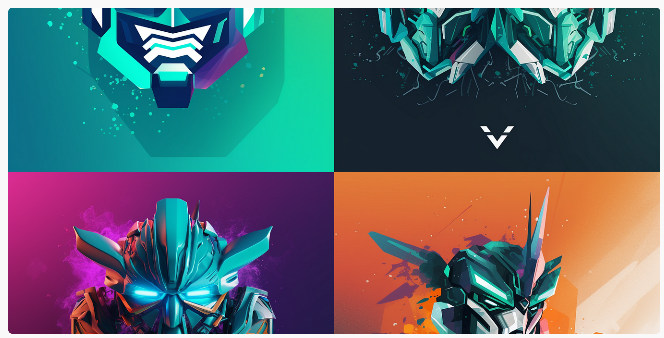

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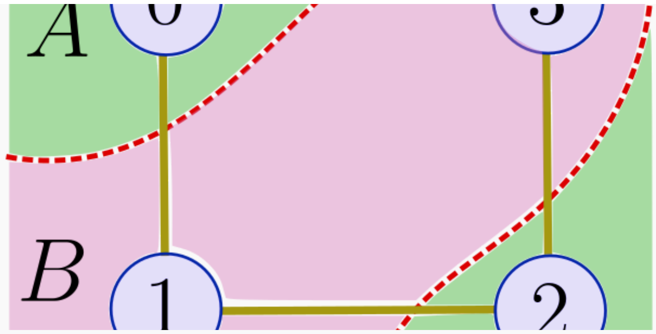

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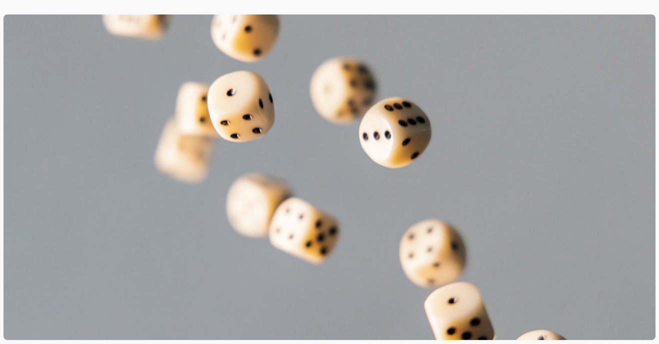

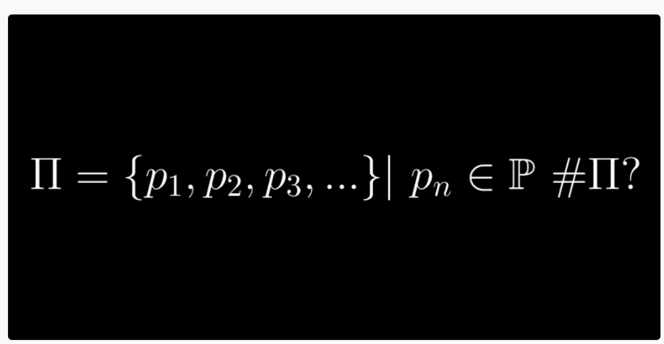

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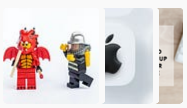

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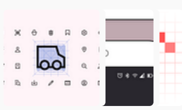

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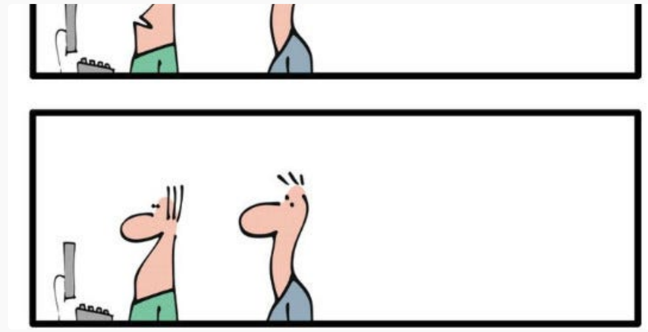

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