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Orbital Mechanics: The Three-Body Problem

Learn how to derive the most studied problem in orbital mechanics (used to design the James Webb space telescope orbit)

Zack Fizell · Following

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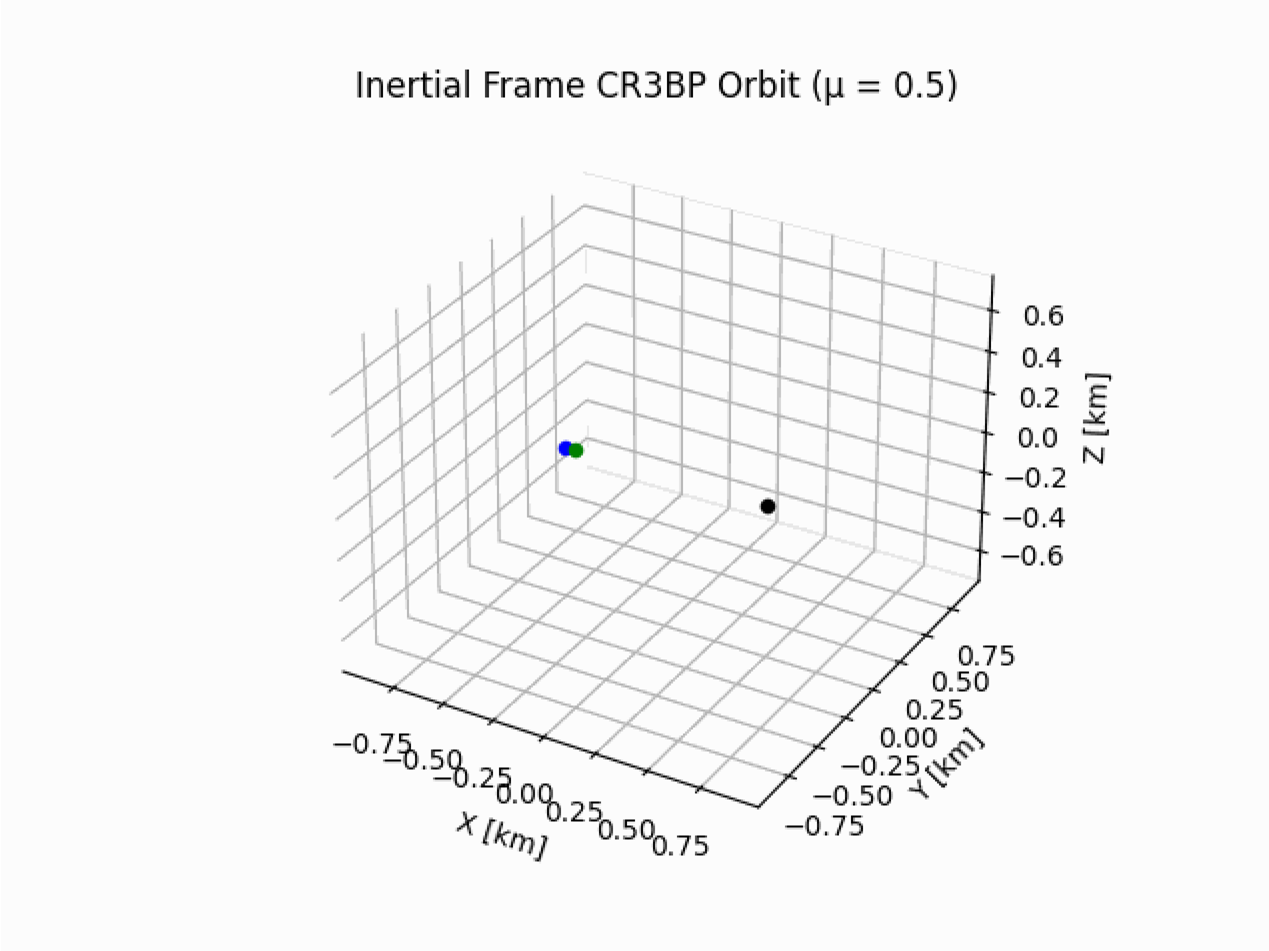

CR3BP Orbit [Created by Author]

Let's start by understanding what the three-body problem is. The three-body problem (or 3BP) is a special case of the broader *n*-body problem, which involves predicting the motion of celestial bodies under each other's gravitational influence. Unlike the simpler two-body problem (2BP), the three-body problem does not have a closed-form solution. This means the motion of the bodies must be estimated using initial conditions (position and velocity) and numerical methods. For practical applications, the 3BP can be focused on the motion of a satellite orbiting two larger masses (also called primaries); these could be moons, planets, or stars.

The motion of a satellite under the influence of two larger primaries is typically chaotic, meaning the motion is difficult to predict. This is why we use modern numerical methods to estimate/predict this motion as accurately as possible. In order to estimate 3BP motion, a model needs to be created, which involves using Newton's laws of motion and Newton's law of universal gravitation. The derivation can be challenging to understand, so I encourage you to check out the derivation of the 2BP here first. It is a simpler derivation and may help you understand the subsequent derivation better. Now, the best place to start to understand any physics problem is a well-drawn diagram.

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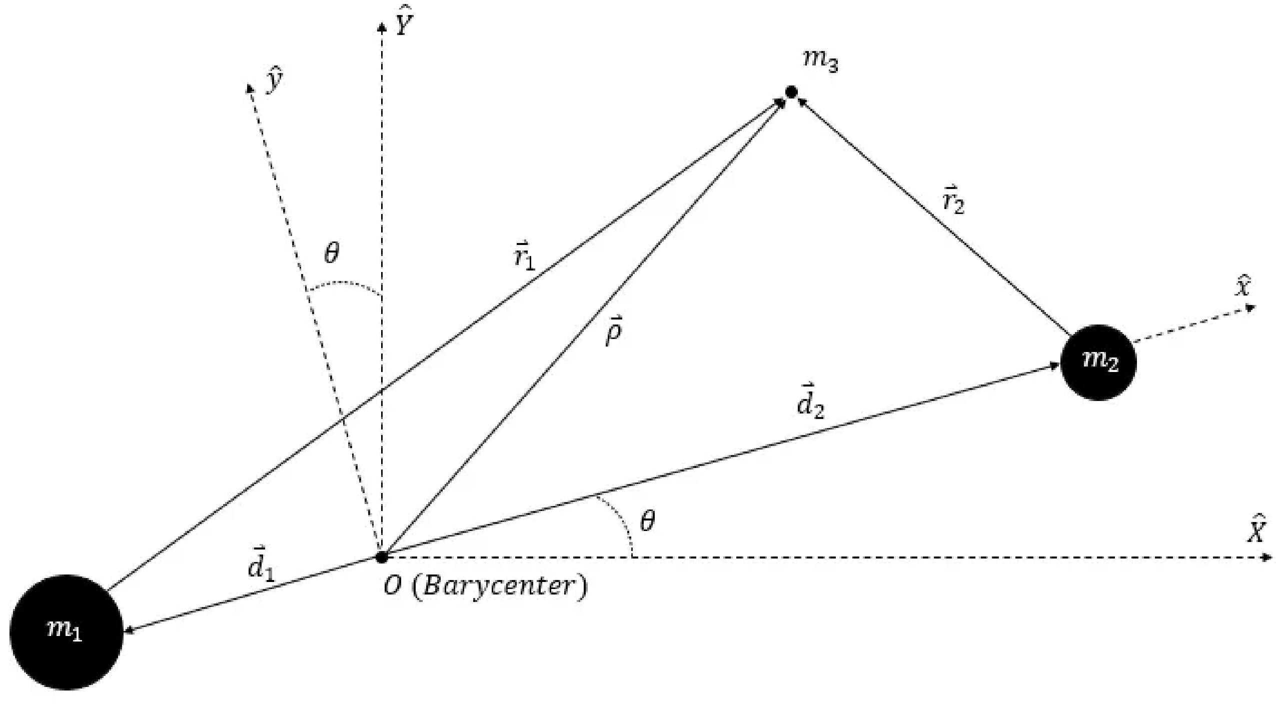
The 3BP Diagram



Three-Body Problem [Created by Author]

The diagram above shows the standard setup for the 3BP. The primaries are denoted as m_1 and m_2, where m_1 is typically the larger of the two masses. The satellite (or object whose motion is of interest) is labeled m_3. Even though we label the satellite as mass 3, this mass is considered negligible compared to the primaries for practical purposes. Since the third mass is considered negligible, the orbit of the larger two masses can be considered a conic (2BP) orbit. This simplifies the derivation substantially. Further, typically, the special cases of elliptic and circular primary orbits are studied, known as the elliptic-restricted 3BP (ER3BP) or circular-restricted 3BP (CR3BP).

With this in mind, the barycenter, or center of mass, of the two primaries can be considered an inertial point, labeled ***O***. There are two coordinate frames in this system fixed in the barycenter: a rotating frame that rotates with the primaries (***x***- and ***y***-hat) and an inertial frame that does not rotate (***X***- and ***Y***-hat). These two frames are separated by an angle, ***θ***, at any given time. There are also a few position vectors ($\boldsymbol{d_1}$, $\boldsymbol{d_2}$, $\boldsymbol{r_1}$, $\boldsymbol{r_2}$, and ***ρ***) that determine the position of the masses with respect to the inertial barycenter (important for using Newton's laws of motion) and the position of $\boldsymbol{m_3}$ in relation to the primaries. The vector of interest for this derivation is ***ρ***, as it will determine the satellite's inertial motion.

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Nondimensionalizing

Scary word, I know, but it's not as complicated as it looks. This is not a necessary step, but it does make deriving the equations of motion for the 3BP easier. Nondimensionalizing is a method to take physical dimensions out of a problem and is useful for simplifying mathematical expressions. Let's take the 3BP as an example of this. We can define nondimensionalizing parameters for mass, length, and time as follows (by convention):

$$M^* \triangleq m_1 + m_2$$
$$L^* \triangleq a$$
$$G^* \triangleq 1 = G\,\frac{M^*(T^*)^2}{(L^*)^3}$$
$$T^* \triangleq \left[\frac{(L^*)^3}{GM^*}\right]^{1/2} \qquad \text{(chosen so } G^* = 1\text{)}$$

Here, ***a*** is the semi-major axis of the motion of the two primaries and ***G*** is the universal gravitational constant. This might not make sense just yet, so I'll demonstrate how you could nondimensionalize a set of initial conditions in the Earth-Moon system (our two primaries of this example). The nondimensionalizing parameters for this particular system would be:

$$M^* \triangleq m_E + m_M = 5.97219 \times 10^{24}\ kg + 7.34767 \times 10^{22}\ kg = 6.045667 \times 10^{24}\ kg$$
$$L^* \triangleq a = 3.844 \times 10^5\ km$$
$$T^* \triangleq \left[\frac{(L^*)^3}{GM^*}\right]^{1/2} = 3.75195 \times 10^5\ s$$

Now, if we have a state vector (position and velocity vectors combined), then we could nondimensionalize it as follows:

$$\text{Dimensional State Vector: } \vec{x}_{dim} = \begin{Bmatrix} 10,000\ km \\ -6,500\ km \\ 0\ km \\ -2.3\ km/s \\ 0.5\ km/s \\ 0.05\ km/s \end{Bmatrix}$$

$$\text{Nondimensional State Vector: } \vec{x}_{ND} = \begin{Bmatrix} 10,000\ km \times \frac{1}{L^*} \approx 0.0260 \\ -6,500\ km \times \frac{1}{L^*} \approx -0.0169 \\ 0\ km \times \frac{1}{L^*} \approx 0.0000 \\ -2.3\ km/s \times \frac{T^*}{L^*} \approx -2.2449 \\ 0.5\ km/s \times \frac{T^*}{L^*} \approx 0.4880 \\ 0.05\ km/s \times \frac{T^*}{L^*} \approx 0.0488 \end{Bmatrix}$$

Notice the nondimensional vector does not have units and that we used our dimensional parameters to remove the *km* and *s* units. This process works in reverse, so if you wanted to add dimensions back in, you would simply multiply or divide by L^*, M^*, or T^*.

. . .

Deriving ER3BP Equations of Motion

The final and longest step to formulating the three-body problem is deriving the equations of motion for the negligible mass, m_3. First, we need to make a few assumptions, some of which have already been mentioned. We will assume that $m_3 << m_1$ and m_2; this means m_1 and m_2 move with unperturbed two-body motion (Keplerian motion). Also, m_1 and m_2 are considered point masses (this simplifies the derivation). We will start the derivation with the case of m_1 and m_2 moving in elliptic orbits about the barycenter (ER3BP) and then simplify this result to obtain the CR3BP. We start by defining a mass ratio for convenience:

$$\mu \triangleq \frac{m_2}{m_1 + m_2} = \frac{m_2}{M^*}$$

Therefore ND mass:

$$m_2 = \mu$$
$$m_1 = 1 - \mu$$

The next step is applying Newton's second law of motion and Newton's gravitational law to m_3.

$$m_3\vec{\rho}\,'' = -\frac{Gm_1m_3}{r_1^3}\vec{r_1} - \frac{Gm_2m_3}{r_2^3}\vec{r_2}$$
$$\vec{\rho}\,'' = -\frac{Gm_1}{r_1^3}\vec{r_1} - \frac{Gm_2}{r_2^3}\vec{r_2}$$

The equation above is the dimensional acceleration of ***m₃*** (the ticks stand for a 2nd time derivative). Now, the nondimensionalizing parameters can be used to remove the physical dimensions from the system by multiplying the acceleration by $(T^*)^2$ and dividing by L^* (since acceleration has units of length over time squared).

$$\begin{aligned}\ddot{\vec{\rho}} = \vec{\rho}\,''\frac{(T^*)^2}{L^*} &= \left(-\frac{Gm_1}{r_1^3}\vec{r}_1 - \frac{Gm_2}{r_2^3}\vec{r}_2\right)\frac{(T^*)^2}{L^*} \\ &= \left(-\frac{Gm_1}{r_1^3}\vec{r}_1 - \frac{Gm_2}{r_2^3}\vec{r}_2\right)\left[\frac{(L^*)^3}{GM^*}\right]\frac{1}{L^*} \\ &= -\frac{G}{G}\frac{m_1}{M^*}\frac{(L^*)^2}{r_1^3}\vec{r}_1 - \frac{G}{G}\frac{m_2}{M^*}\frac{(L^*)^2}{r_2^3}\vec{r}_2\end{aligned}$$

$$\ddot{\vec{\rho}} = -\frac{(1-\mu)}{r_1^3}\vec{r_1} - \frac{\mu}{r_2^3}\vec{r_2}$$

In the equation above, the dots above the $\boldsymbol{\rho}$ vector stand for the ND 2nd time derivative, and $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are ND vectors. Now we can use kinematics to determine the components of the velocity and acceleration of $\boldsymbol{m_3}$. This will be important to create a scalar form of the equations of motion. The basic kinematic equation, or BKE, can be used to do this:

BKE:
$$\frac{d}{dt}\vec{f} = \left(\frac{d}{dt}\vec{f}\right)_r + \vec{\Omega}\times\vec{f}$$

$$\vec{\rho} = x\hat{x} + y\hat{y} + z\hat{z}$$

Applying the BKE to $\boldsymbol{\rho}$ to get the first time derivative, or velocity:

$$\begin{aligned}\dot{\vec{\rho}} &= \left(\frac{d}{dt}\vec{\rho}\right)_r + \vec{\Omega} \times \vec{\rho} \\ &= \frac{d}{dt}(x\hat{x} + y\hat{y} + z\hat{z}) + \dot{\theta}\hat{z} \times (x\hat{x} + y\hat{y} + z\hat{z}) \\ &= \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} + (-\dot{\theta}y\hat{x} + \dot{\theta}x\hat{y})\end{aligned}$$

$$\dot{\vec{\rho}} = (\dot{x} - \dot{\theta}y)\hat{x} + (\dot{y} + \dot{\theta}x)\hat{y} + \dot{z}\hat{z}$$

where,

$$\dot{\theta} = \frac{\sqrt{1-e^2}}{(1-e\ cosE)^2} \qquad \text{(ND form, derived from 2BP)}$$

Using 2BP geometry for elliptic orbits, you can derive the rate of change of $\boldsymbol{\theta}$ as:

$$\begin{aligned}\dot{\theta} &= \frac{h}{R^2} \\ R &= a(1 - e\ cosE) && \text{or ND:} \quad R = (1 - e\ cosE) \\ h &= \sqrt{G(m_1 + m_2)\ a\ (1-e^2)} && \text{or ND:} \quad h = \sqrt{1-e^2} \\ \dot{\theta} &= \frac{\sqrt{1-e^2}}{(1-e\ cosE)^2}\end{aligned}$$

Here, $\boldsymbol{h}$ is the specific angular momentum of the $\boldsymbol{m_1}$-$\boldsymbol{m_2}$ system, $\boldsymbol{R}$ is the instantaneous distance between the two primary masses (it changes throughout time in the ER3BP), $\boldsymbol{e}$ is the eccentricity of the elliptic orbit, and $\boldsymbol{E}$ is the eccentric anomaly. Now, applying the BKE again to get acceleration:

$$
\begin{aligned}
\ddot{\vec{\rho}} &= \left(\frac{d}{dt}\dot{\vec{\rho}}\right)_r + \vec{\Omega} \times \dot{\vec{\rho}} \\
&= \frac{d}{dt}((\dot{x} - \dot{\theta}y)\hat{x} + (\dot{y} + \dot{\theta}x)\hat{y} + \dot{z}\hat{z}) + \dot{\theta}\hat{z} \times ((\dot{x} - \dot{\theta}y)\hat{x} + (\dot{y} + \dot{\theta}x)\hat{y} + \dot{z}\hat{z}) \\
&= (\ddot{x} - \ddot{\theta}y - \dot{\theta}\dot{y})\hat{x} + (\ddot{y} + \ddot{\theta}x + \dot{\theta}\dot{x})\hat{y} + \ddot{z}\hat{z} + ((\dot{\theta}\dot{x} - \dot{\theta}^2 y)\hat{y} - (\dot{\theta}\dot{y} + \dot{\theta}^2 x)\hat{x})
\end{aligned}
$$

$$
\ddot{\vec{\rho}} = (\ddot{x} - \ddot{\theta}y - 2\dot{\theta}\dot{y} - \dot{\theta}^2 x)\hat{x} + (\ddot{y} + \ddot{\theta}x + 2\dot{\theta}\dot{x} - \dot{\theta}^2 y)\hat{y} + \ddot{z}\hat{z}
$$

where,

$$
\ddot{\theta} = -2e\ sinE\ \frac{\sqrt{1-e^2}}{(1 - e\ cosE)^4} \qquad \text{(Time derivative of } \dot{\theta}\text{)}
$$

Next, we can combine the two equations for the acceleration vector, but first we should define the position vectors from the diagram. ***d_1*** and ***d_2*** are defined using the equation for the center of mass of a two-particle system (since ***m_3*** is negligible). ***r_1**, **r_2**,* and ***ρ*** can be defined using the diagram and vector subtraction.

$$
\begin{aligned}
\vec{d_1} &= -\mu R\ \hat{x} \\
\vec{d_2} &= (1-\mu)R\ \hat{x} \\
\vec{\rho} &= x\hat{x} + y\hat{y} + z\hat{z} \\
\vec{r_1} &= \vec{\rho} - \vec{d_1} = (x + \mu R)\hat{x} + y\hat{y} + z\hat{z} \\
\vec{r_2} &= \vec{\rho} - \vec{d_2} = [x - (1-\mu)R]\hat{x} + y\hat{y} + z\hat{z}
\end{aligned}
$$

where,

$$
R = 1 - e\ cosE \qquad \text{(ND form, derived from 2BP)}
$$

Now, we can use the definitions of ***r_1*** and ***r_2*** in the first acceleration equation and then combine the ***ρ*** acceleration equations as follows:

$$
\begin{aligned}
\ddot{\vec{\rho}} &= -\frac{(1-\mu)}{r_1^3}\vec{r_1} - \frac{\mu}{r_2^3}\vec{r_2} \\
&= -\frac{(1-\mu)}{r_1^3}\left[(x+\mu R)\hat{x} + y\hat{y} + z\hat{z}\right] - \frac{\mu}{r_2^3}\left[(x-(1-\mu)R)\hat{x} + y\hat{y} + z\hat{z}\right] \qquad (1) \\
\ddot{\vec{\rho}} &= (\ddot{x} - \ddot{\theta}y - 2\dot{\theta}\dot{y} - \dot{\theta}^2 x)\hat{x} + (\ddot{y} + \ddot{\theta}x + 2\dot{\theta}\dot{x} - \dot{\theta}^2 y)\hat{y} + \ddot{z}\hat{z} \qquad (2)
\end{aligned}
$$

Combining like terms (*x*-hat, *y*-hat, and *z*-hat) of equations (1) and (2):

$$
\begin{aligned}
\ddot{x} - \ddot{\theta}y - 2\dot{\theta}\dot{y} - \dot{\theta}^2 x &= -\frac{(1-\mu)(x+\mu R)}{r_1^3} - \frac{\mu[x-(1-\mu)R]}{r_2^3} \\
\ddot{y} + \ddot{\theta}x + 2\dot{\theta}\dot{x} - \dot{\theta}^2 y &= -\frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} \\
\ddot{z} &= -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}
\end{aligned}
$$

The above equations represent the ER3BP equations of motion for m_3 in the rotating frame. The ER3BP is challenging to numerically integrate with the θ terms; though, assumptions can be made to simplify the equations to obtain the CR3BP, an easier problem to integrate. In the CR3BP, the primaries move in circular orbits about the barycenter. This means:

$$
\begin{aligned}
&e = 0 \\
&R = a = 1 \text{ (ND)} = const \\
&\dot{\theta} = \sqrt{\frac{G(m_1+m_2)}{a^3}} = 1 \text{ (ND)} = const \\
&\ddot{\theta} = 0
\end{aligned}
$$

Then, simplifying the ER3BP equations using these new assumptions:

$$\ddot{x} = x + 2\dot{y} - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu[x-(1-\mu)]}{r_2^3}$$
$$\ddot{y} = y - 2\dot{x} - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}$$
$$\ddot{z} = -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}$$

These equations represent the CR3BP equations of motion for m_3. They can be numerically integrated to obtain a time history of the position and velocity of m_3 in the rotating frame. Note that for the CR3BP, the primaries will remain stationary in the rotating frame. In order to get an inertial vector from a rotating vector in the CR3BP, we can use the following equation:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} x\cos(t) - y\sin(t) \\ x\sin(t) + y\cos(t) \\ z \end{bmatrix}$$

Here, x, y, and z represent the rotating frame vector components and X, Y, and Z represent the inertial frame vector components. If you want to learn how to numerically integrate these equations of motion, check out my other article for a step-by-step guide:

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Thank you for reading the article! I really enjoy writing articles where I can show derivations and put solid information out to you as the reader. Orbital mechanics is not an easy topic to learn, so if you have any questions, make sure to comment, and I'll do my best to answer them! Thanks again and stay tuned for more orbital mechanics articles!

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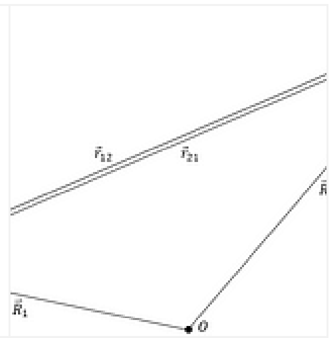

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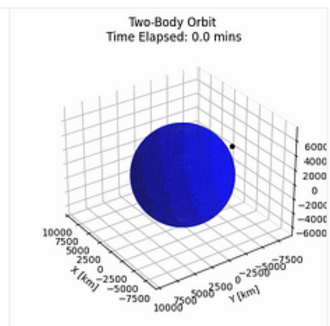

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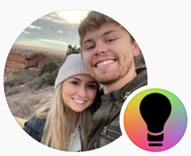

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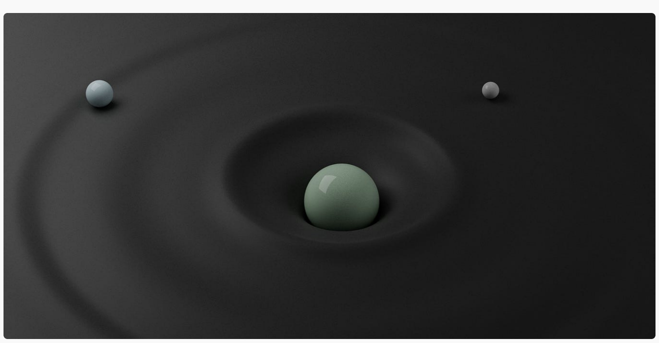

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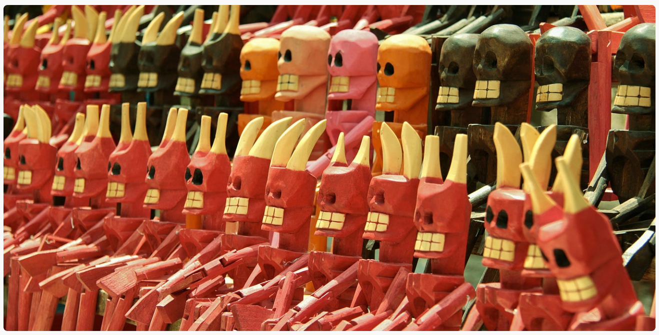

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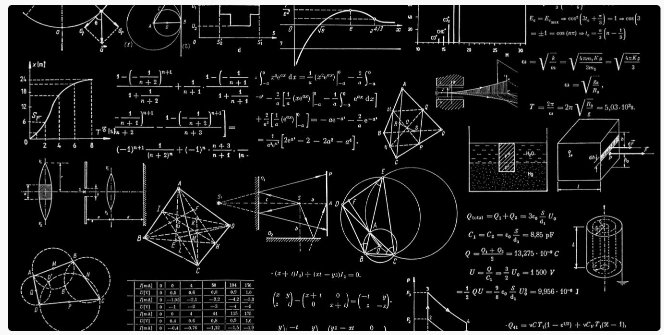

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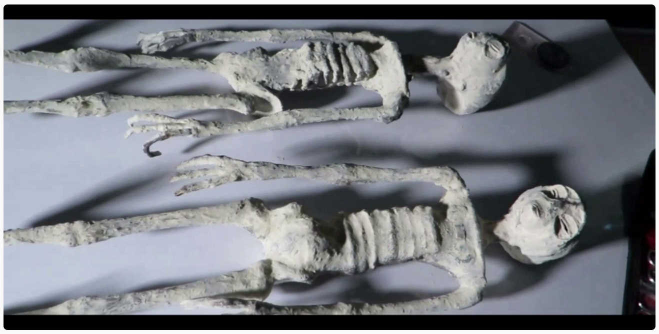

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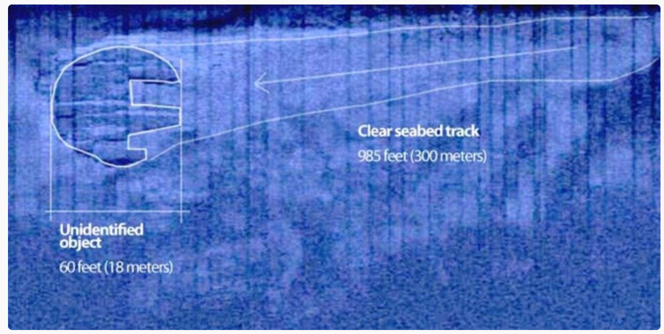

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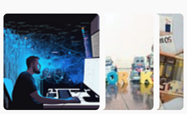

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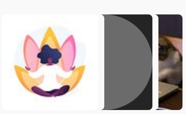

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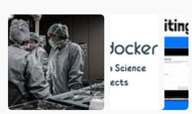

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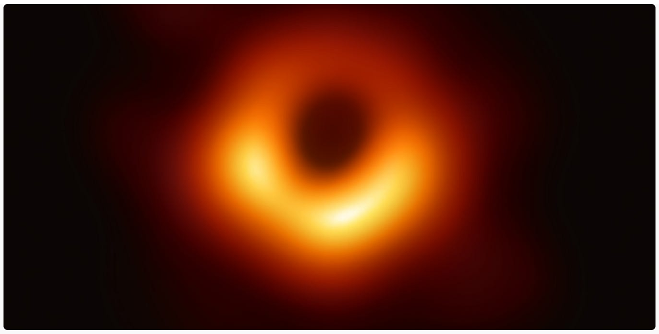

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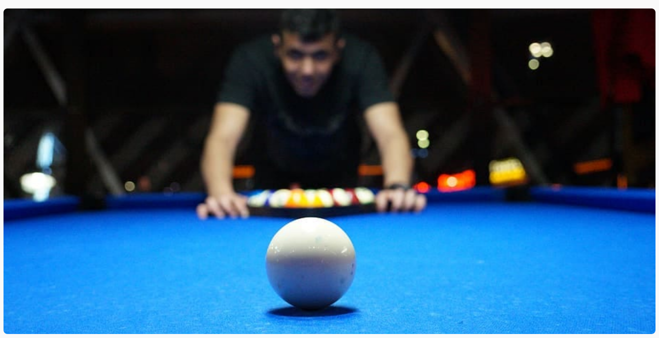

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