How to Solve the Double Pendulum Problem

Learn to solve for the equations of motion in the double pendulum problem using Lagrangian Mechanics

Zack Fizell · Following

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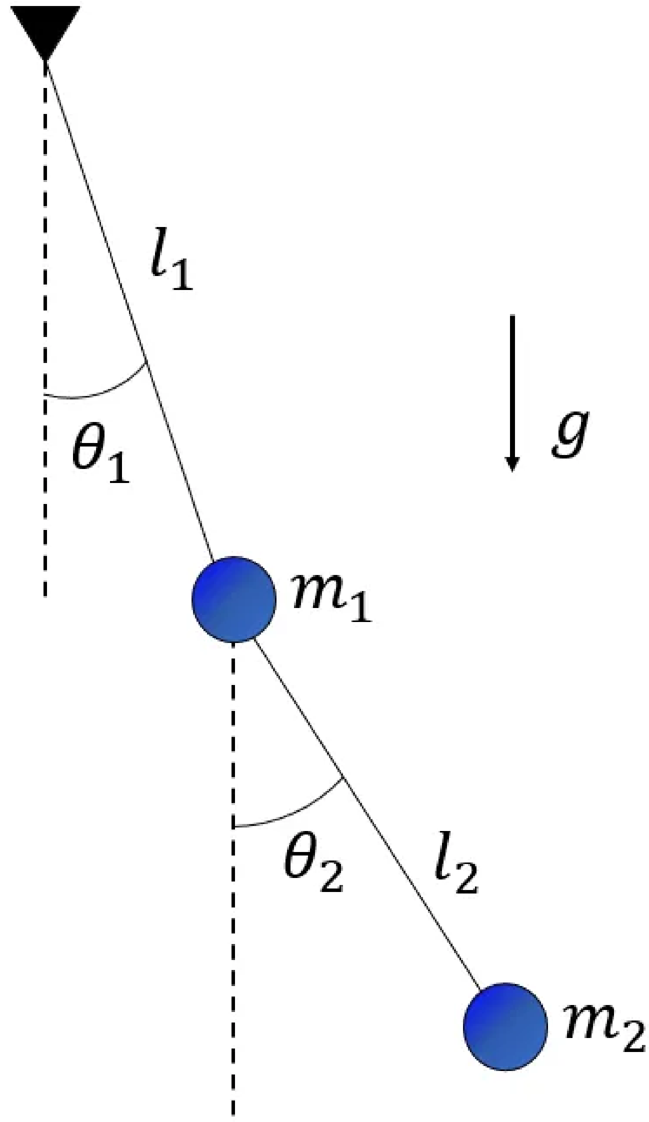

Double Pendulum Problem [Created by Author]

Dynamics can be a pesty topic for even the sharpest minds. Most people do not know what dynamics means, but for engineers and physicists (and students of course), it can be a haunting memory. We all remember textbooks forcing us to derive equations that determine how a particle moves under the influence of very weird forces. Fear not, I will help you conquer dynamics and teach you how to use Lagrangian mechanics for complicated dynamic systems, such as the double pendulum.

While this article focuses on the double pendulum problem, the concepts can be expanded to any number of problems. Using the diagram above as reference, the double pendulum problem is a system of two mass particles, labeled m_1 and m_2. They are joined by two massless, rigid rods that allow the masses to rotate freely without friction. The end goal will be to solve for the equations of motion for each of the masses using Lagrangian mechanics.

To get a refresher or to learn about Lagrangian mechanics in an easier example, check out this other article. It's a simple pendulum with a single mass, instead of two masses; this makes the equations of motion derivation much simpler. With the basics of the problem described, let's start solving.

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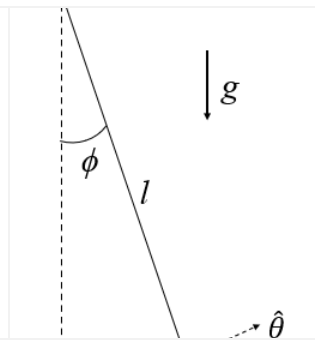

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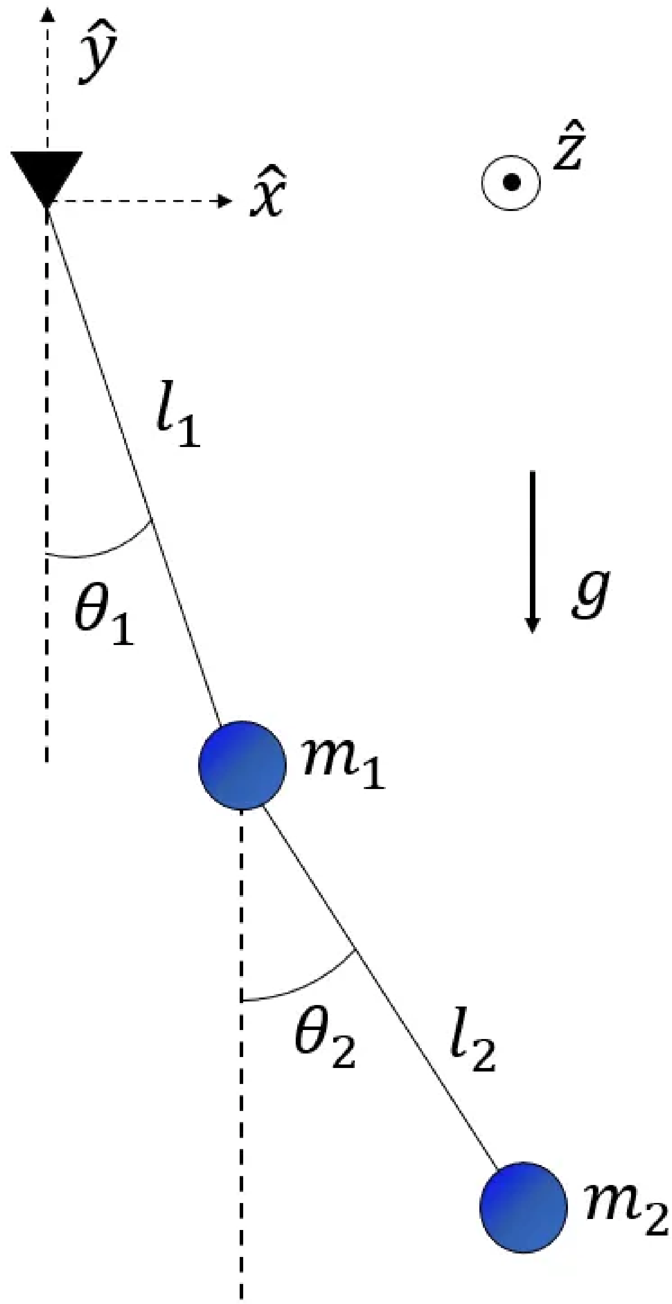
1. Knowns and Unknowns

The best practice for starting any physics, engineering, or dynamics problem is writing out what you know and do not know. For this problem, we will say that we know the lengths of the massless rods and the masses of the particles. Further simplifying, we can say that $m_1=m_2=m$ and $l_1=l_2=l$ (this is not a necessary assumption, but it simplifies the final results). Additionally, we will say that we are at the surface of the Earth, where the gravitational acceleration is a constant 9.81 m/s^2. We will make one more assumption and say that the pivot points are frictionless (an okay assumption for our purposes).

With our knowns defined, we need to determine our unknowns, which are what we want to gain by looking at a problem. For the double pendulum problem, we want to know how each of the masses move throughout time. We can get valuable information from this, such as how they move in relation to one another, what speeds are they seeing, etc. This could be defined using the two angles evolution over time, $\theta_1(t)$ and $\theta_2(t)$.

2. Adding a Coordinate Frame

The next step for solving this problem is setting up an inertial coordinate system (x-y axes). The utility of the coordinate system will be seen as we begin deriving the equations of motion for each of the masses. To complete the coordinate system, we can use the right hand-rule to define a z-axis coming out of the page. The following diagram shows our system with the added coordinate system.



Coordinate Systems Defined [Created by Author]

3. Defining the Lagrangian for the System

The Lagrangian, L (not to be confused with the massless rod lengths, l), may be a foreign term, but I assure you it is not as complicated as it sounds. The Lagrangian is simply the the difference between kinetic and potential energy of a system. It can be used to derive the equations of motion as we will see soon.

$$L = T - V$$

In this equation, *T* is the kinetic energy and *V* is the potential energy of the system of interest. Our system is the masses, m_1 and m_2. Though, the concept can be expanded to any number of masses. To obtain an expression for the Lagrangian, we need to determine the kinetic and potential energies of each of the particles. Kinetic energy can be defined as the sum of the individual kinetic energies of each of the particles.

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

We can use our coordinate system to first define the position of each of the particles. Then, we can take a time derivative to arrive at the velocity for each particles. First, the positions of each of the particles can be defined using geometry.

$$\begin{aligned}\vec{r}_1 &= l\sin\theta_1\hat{x} - l\cos\theta_1\hat{y}\\ \vec{r}_2 &= (l\sin\theta_1 + l\sin\theta_2)\hat{x} - (l\cos\theta_1 + l\cos\theta_2)\hat{y}\\ &= l(\sin\theta_1 + \sin\theta_2)\hat{x} - l(\cos\theta_1 + \cos\theta_2)\hat{y}\end{aligned}$$

Then, taking the time derivative of each with respect to the inertial frame of reference (*x*-*y* axes), we arrive at the following velocities of each of the particles:

$$\begin{aligned} \vec{v}_1 = \frac{d}{dt}(\vec{r}_1) &= \frac{d}{dt}(l\sin\theta_1\hat{x} - l\cos\theta_1\hat{y}) \\ &= l\dot{\theta}_1\cos\theta_1\hat{x} + l\dot{\theta}_1\sin\theta_1\hat{y} \\ v_1 &= l\dot{\theta}_1 \end{aligned}$$

$$\begin{aligned} \vec{v}_2 = \frac{d}{dt}(\vec{r}_2) &= \frac{d}{dt}((l\sin\theta_1 + l\sin\theta_2)\hat{x} - (l\cos\theta_1 + l\cos\theta_2)\hat{y}) \\ &= l(\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2)\hat{x} + l(\dot{\theta}_1\sin\theta_1 + \dot{\theta}_2\sin\theta_2)\hat{y} \\ v_2 &= l\sqrt{\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)} \end{aligned}$$

Notice that the magnitudes of the velocity vectors were used to calculate v_1 and v_2. Now, we can plug these into the kinetic energy equation to get the total kinetic energy of the system:

$$T = \frac{1}{2}ml^2\dot{\theta}_1^{\,2} + \frac{1}{2}ml^2\left(\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)\right)$$

Next, we need to derive the potential energy of the system. Since gravity is the only conservative force acting on the particles, we only need to consider gravitational potential energy. We can use the angles, θ_1 and θ_2, to define the potential heights, h_1 and h_2, from the pivot point (note they are negative since we are below the *x*-axis).

$$\begin{aligned} V &= mgh_1 + mgh_2 \\ &= mg(-l\cos\theta_1) + mg(-l\cos\theta_1 - l\cos\theta_2) \\ &= -mgl\cos\theta_1 - mgl(\cos\theta_1 + \cos\theta_2) \end{aligned}$$

Now, subtracting the potential energy from the kinetic energy, we arrive at the Lagrangian for the double pendulum system.

$$L = \frac{1}{2}ml^2\dot{\theta}_1^{\,2} + \frac{1}{2}ml^2\left(\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)\right) + mgl\cos\theta_1 + mgl(\cos\theta_1 + \cos\theta_2)$$

4. Deriving the Equations of Motion Using the Euler-Lagrange Equation

Now that we have the Lagrangian in terms of θ_1 and θ_2, we can use it along with the Euler-Lagrange equation of the following form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \frac{\partial L}{\partial \theta_1}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{\partial L}{\partial \theta_2}$$

Notice we apply the Euler-Lagrange equation to each of our angles. Also, make note we can apply these equations since we have a conservative system (system without dissipative forces such as friction). Let's start with the derivatives in the θ_1 equation:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= \frac{d}{dt}\left(ml^2\dot{\theta}_1 + ml^2\dot{\theta}_1 + ml^2\dot{\theta}_2\cos(\theta_2-\theta_1)\right) \\ &= 2ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2\cos(\theta_2-\theta_1) - ml^2\dot{\theta}_2^2\sin(\theta_2-\theta_1) + ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial \theta_1} &= ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) - mgl\sin\theta_1 - mgl\sin\theta_1 \\ &= ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) - 2mgl\sin\theta_1\end{aligned}$$

Combining and simplifying:

$$2l\ddot{\theta}_1 + l\ddot{\theta}_2\cos(\theta_2-\theta_1) - l\dot{\theta}_2^2\sin(\theta_2-\theta_1) = -2g\sin\theta_1$$

Similarly for the second Euler-Lagrange equation for θ_2:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{d}{dt}\left(ml^2\dot{\theta}_2 + ml^2\dot{\theta}_1\cos(\theta_2-\theta_1)\right)$$
$$= ml^2\ddot{\theta}_2 + ml^2\ddot{\theta}_1\cos(\theta_2-\theta_1) - ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) + ml^2\dot{\theta}_1^2\sin(\theta_2-\theta_1)$$

$$\frac{\partial L}{\partial \theta_2} = -ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) - mgl\sin\theta_2$$

Combining and simplifying both Euler-Lagrange equations, you will arrive at the following equations of motion for the masses:

$$\ddot{\theta}_1 = \frac{l\dot{\theta}_1^2\cos(\theta_1-\theta_2)\sin(\theta_1-\theta_2) + l\dot{\theta}_2^2\sin(\theta_1-\theta_2) + 2g\sin\theta_1 - g\cos(\theta_1-\theta_2)\sin\theta_2}{l(\cos^2(\theta_1-\theta_2)-2)}$$

$$\ddot{\theta}_2 = \frac{2l\dot{\theta}_1^2\sin(\theta_1-\theta_2) + l\dot{\theta}_2^2\cos(\theta_1-\theta_2)\sin(\theta_1-\theta_2) - 2g\sin\theta_2 + 2g\cos(\theta_1-\theta_2)\sin\theta_1}{l(2-\cos^2(\theta_1-\theta_2))}$$

These equations can be solved using numerical methods to get a time history of how θ_1 and θ_2 evolve over time, which is what we've sought out to do. The double pendulum problem can be used as an example of chaotic motion and can provide many interesting results once numerically solved. I encourage you to attempt to check out these other articles if you want to extend this problem further:

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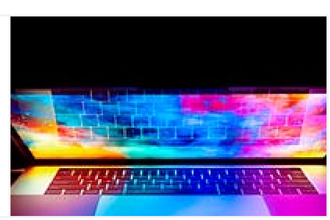

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That's all for this derivation. If you have any questions, feel free to comment and I'll be happy to answer! Give me a follow and check out my other articles on orbits, physics, and coding! Thank you!

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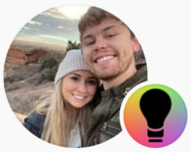

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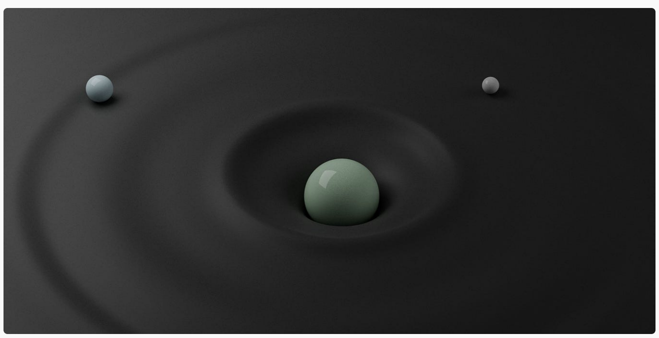

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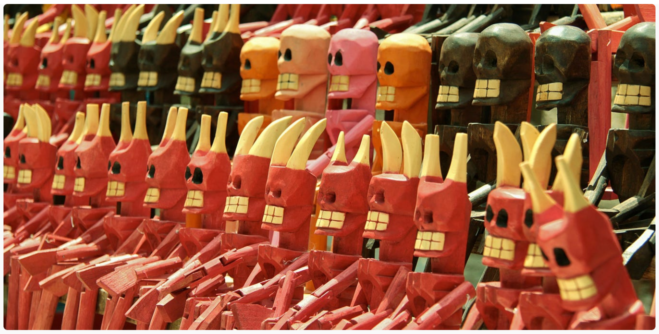

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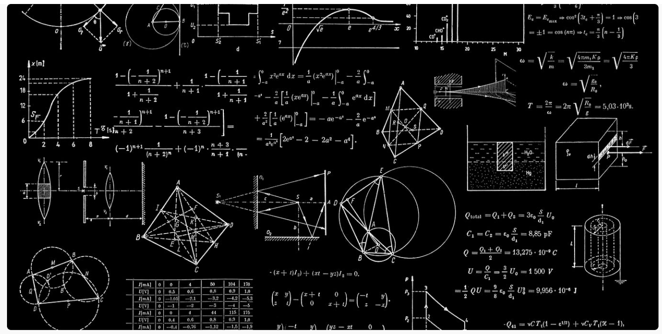

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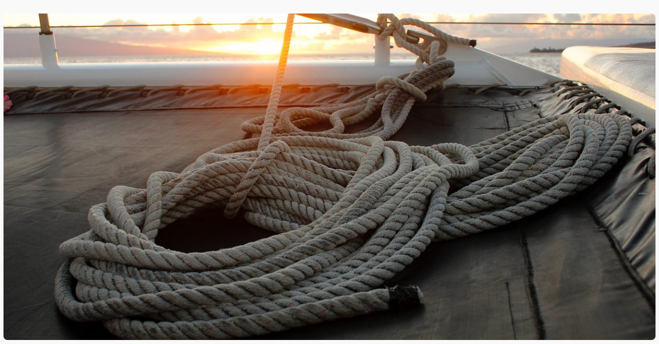

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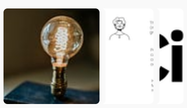

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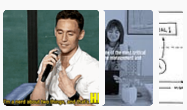

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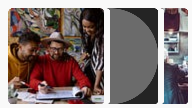

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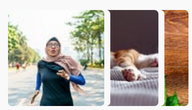

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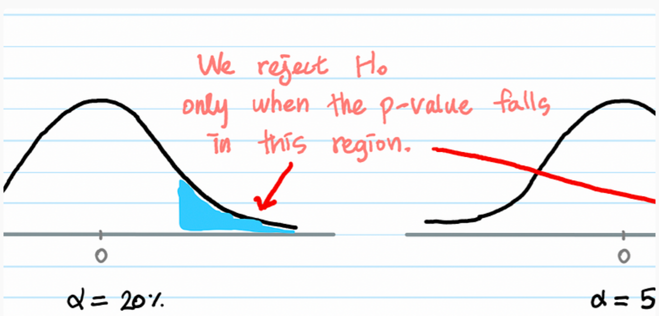

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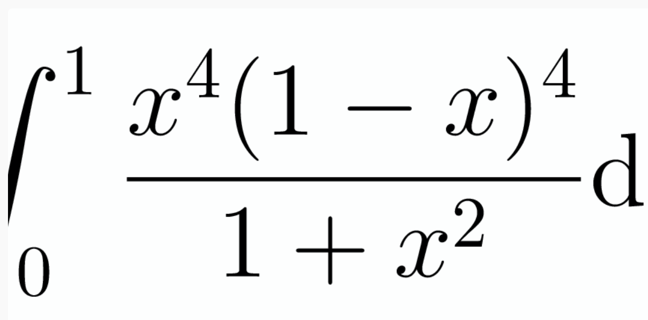

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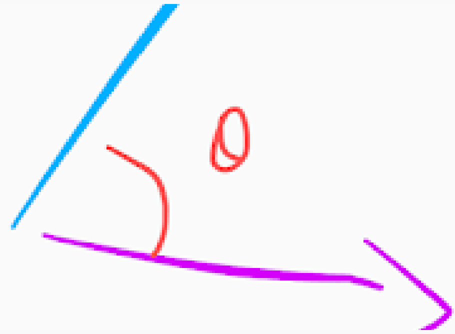

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