Are you Smart Enough to Solve this 300-Year-Old Problem?

A look into the problem that gave birth to modern optimal control theory

Zack Fizell · Following

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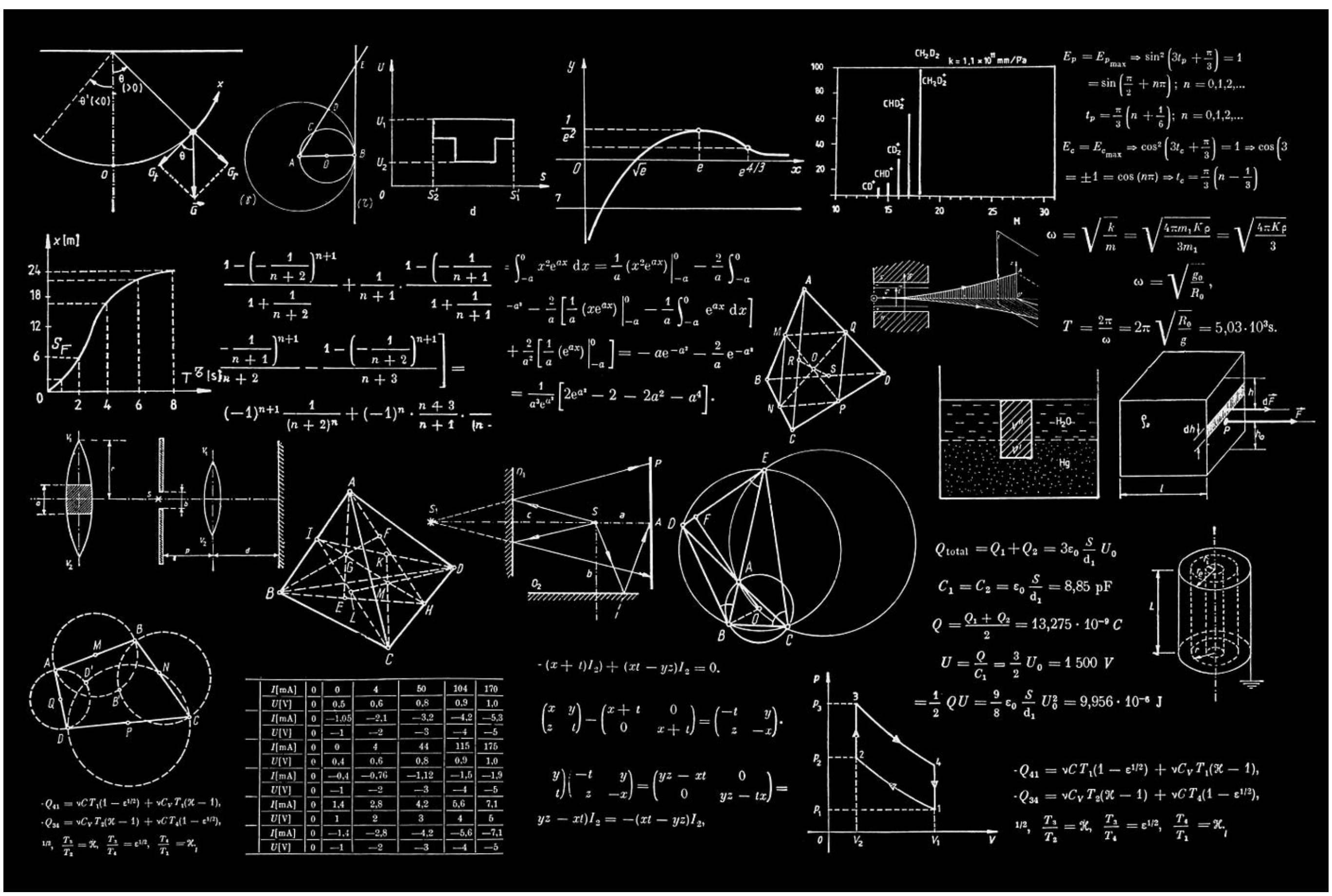

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Nature always tends to act in the simplest way — Daniel Bernoulli

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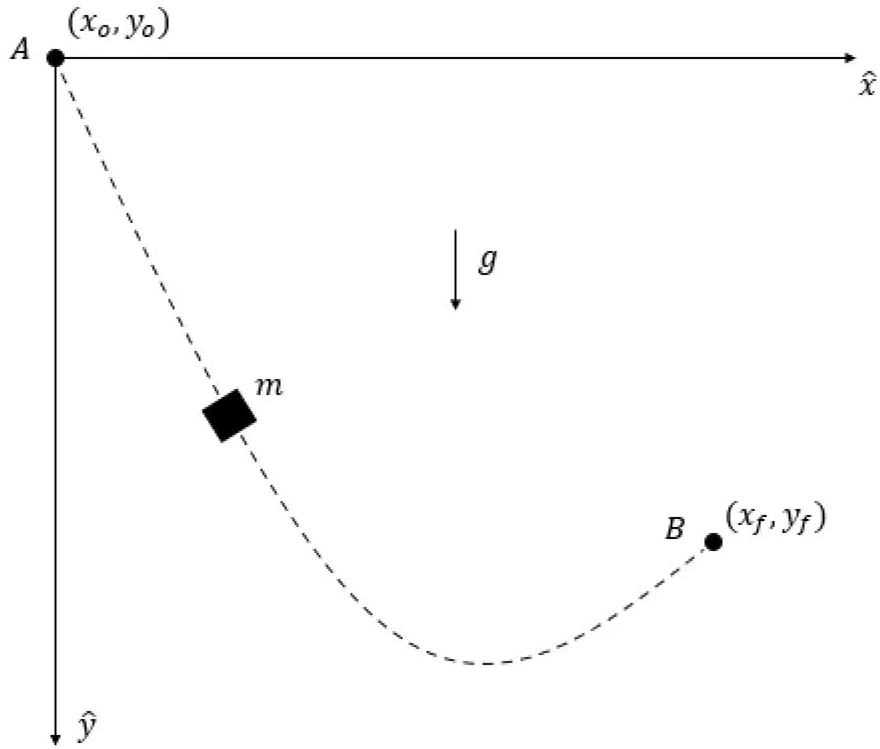
Johann Bernoulli's son, Daniel, wrote the elegant quote above. It implies nature follows an optimized path, or the simplest path. In 1697, Johann Bernoulli, a famous Swiss mathematician, posed the Brachistochrone ("the shortest time") problem to any mathematicians of the time. The problem stated, "If in a vertical plane two points *A* and *B* are given, then it is required to specify the orbit *AMB* of the movable point *M*, along which it, starting from *A*, and under the influence of its own weight, arrives at *B* in the shortest time possible."

If that sounds overwhelming, I'm right there with you. The problem statement is actually a lot simpler than it sounds (the solution, not so much). What is being asked is what path, or trajectory, must the mass take in order to get from point A to B is the shortest time while being only affected by a constant gravity (see diagram below).



Brachistochrone Problem Diagram [Created by Author]

Some people think the optimal path would be a straight line from *A* to *B*. Some might think is a quarter circle. Both are good guesses, but you may be surprised to know that both are wrong. While a straight line is the **shortest** distance between *A* and *B*, it is not the **fastest**. The quarter circle is a "more correct" answer, but still incorrect. The consideration of gravitational acceleration is what makes the optimized trajectory challenging to solve for.

Enough about the problem, let's cover the solution now and learn about the optimal trajectory that minimizes the time for the mass to reach the desired point.

Solution

To begin the solution, we consider the total energy, *E,* of the mass, *m*. Since the trajectory has no forces other than the conservative gravitational force, the energy of the system is conserved. Also, since the mass is at rest at point

A, the total energy of the system can be defined as the potential energy between *A* and the final point, *B*:

$$E = mgy(x)$$

Here, *g* is a constant gravitational acceleration, *m* is the mass, and *y(x)* describes the vertical position of the mass relative to point *B*. With the potential energy being zero at the final point, we obtain the following relationship:

$$E = \frac{1}{2}mv^2 = mgy(x)$$

Using algebra to solve for the velocity of the mass at a location *y(x)*, we get the following expression:

$$v(x) = \sqrt{2gy(x)}$$

Let's also consider the differential arc length of the optimal trajectory in terms of differential *x* and *y*:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)} = \sqrt{1 + y'^2}\,dx$$

If we take the time derivative of the differential arc length, this can be equated to the velocity equation derived earlier.

$$\frac{ds}{dt} = v(x)$$
or
$$dt = \frac{ds}{v} = \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}} = \sqrt{\frac{1+y'^2}{2gy}}\,dx$$

If we integrate this to get the time to travel from the initial point to the final point, we get the following expression:

$$t_{12} = \int_{x_0}^{x_f} \sqrt{\frac{1+y'^2}{2gy}}\,dx$$

Here, t_{12} is the time to go from point A to point B. Our goal is to minimize this time, so we can say the integral is the function we want to minimize. From calculus of variations, the integral has a stationary value (minimum or maximum for physical problems) if the Euler-Lagrange differential equation is satisfied:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Here, F is the integrand of the integral of interest. We are seeking to find the path that minimizes the time for the mass to travel from point A to point B, so we must satisfy the Euler-Lagrange equation to find such path. We will use the integrand of the time equation defined above. So, F in our case is as follows:

$$F = \sqrt{\frac{1+y'^2}{2gy}}$$

Thankfully, a smart mathematician named Beltrami derived the following identity that simplifies the Euler-Lagrange equation given *x* does not explicitly appear in the function, *F*, as is the case for us.

$$F - y'\left(\frac{\partial F}{\partial y'}\right) = C$$

Here, *C* is some unknown constant. Now, we apply Beltrami's identity to our function. You will arrive at the following result:

$$y(1 + y'^2) = \frac{1}{2gC^2}$$

While not entirely obvious, the solutions for *x* and *y* in this equation are parametric equations. Upon substituting and integrating, you arrive at the following expressions for *x* and *y*:

$$x = a(\theta - \cos\theta)$$
$$y = a(1 - \sin\theta)$$
where,
$$a = \frac{1}{4gC^2}$$

These are the equations for a cycloid. This result means the fastest, or optimized, route for the mass to take from point *A* to point *B* is a cycloid-like path! Optimization is a powerful tool in not only engineering, but business, finance, etc. While Bernoulli set the ball rolling for the field of optimization, modern mathematics have really taken hold of this concept and made it practical. Optimal control theory has been devised as a result. Using this theory, one can determine, for example, the optimal thrust steering angle for a launch trajectory to minimize fuel consumption. The field is math intensive, but once you understand the principles, you will be on your way to optimizing any problem!

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Thank you for taking the time to learn a little about the Brachistochrone problem and how it has led to modern optimal control theory! Check out my other articles on astrodynamics, mathematics, programming, etc. Leave a comment and a clap if you found the article interesting!

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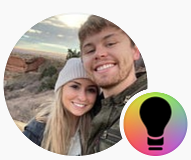

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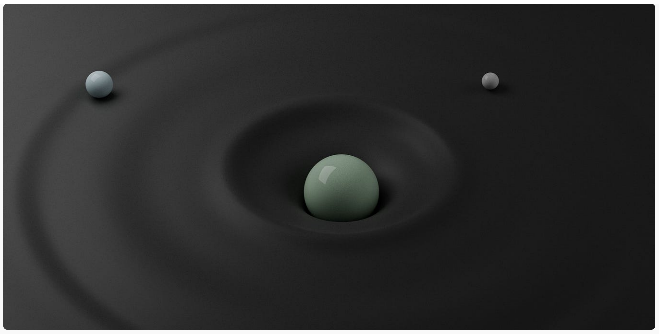

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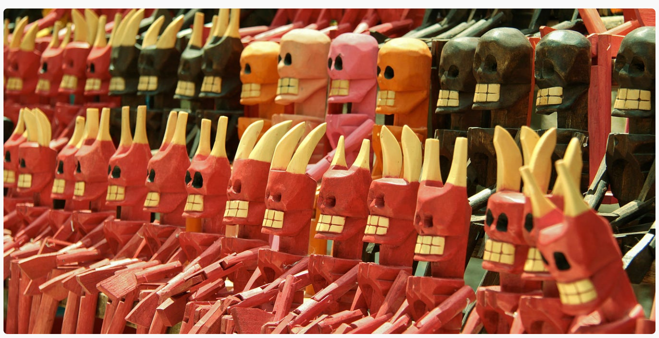

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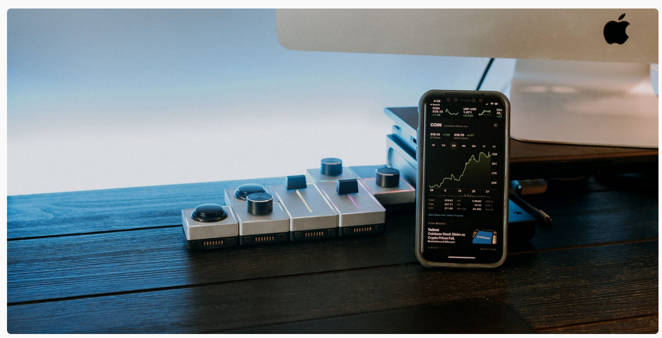

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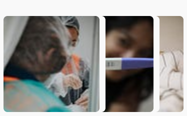

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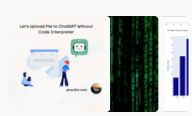

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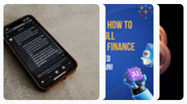

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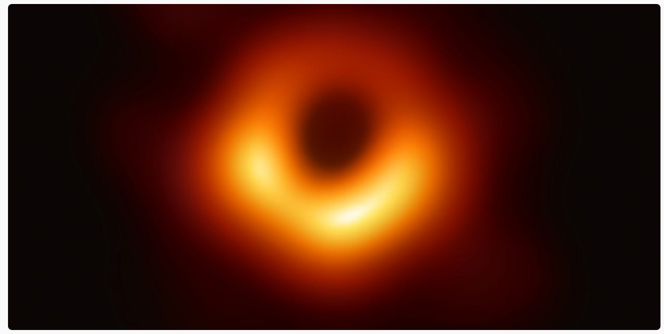

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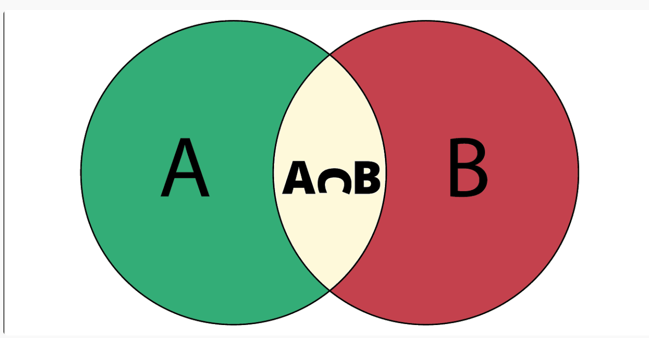

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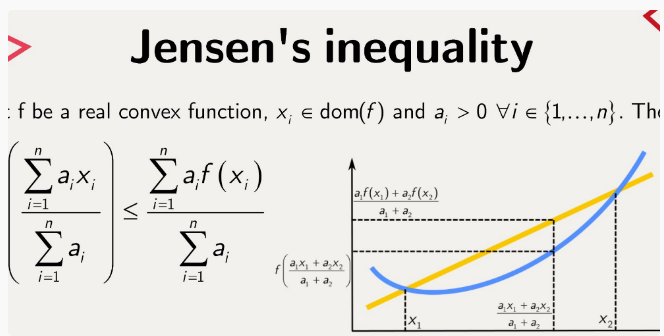

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