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How to Solve the Two-Body Problem

Learn the fundamentals of orbital mechanics by deriving the equations of motion for a two-body system

Zack Fizell · Following

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Photo by SpaceX on Unsplash

Orbital mechanics can be an intimidating topic for any person. While it can get complicated and intensive, the fundamentals are quite intuitive. Let's start with what is orbital mechanics? Orbital mechanics, or astrodynamics, is the study of the motion spacecraft and natural bodies under the influence of gravity from one or more large natural bodies. The field is broad and encompasses many niche topics, some of which include determining where an object is in space, attitude dynamics, and maneuvers.

If you are just getting into astrodynamics or are curious to learn more, the best place to start is the two-body problem, or 2BP. The two-body problem is an astrodynamics model that considers only two masses. One is usually a celestial body, and the other is usually a spacecraft, whose motion is of interest. The 2BP is considered a low-fidelity model, meaning a model that does not have good accuracy compared to the real world, but it can be used as a building block to create higher fidelity models. Such models include other forces affecting a spacecraft such as other natural bodies (other planets, moons, etc.), solar radiation, oblateness and aerodynamic drag (if its orbit is touching an atmosphere).

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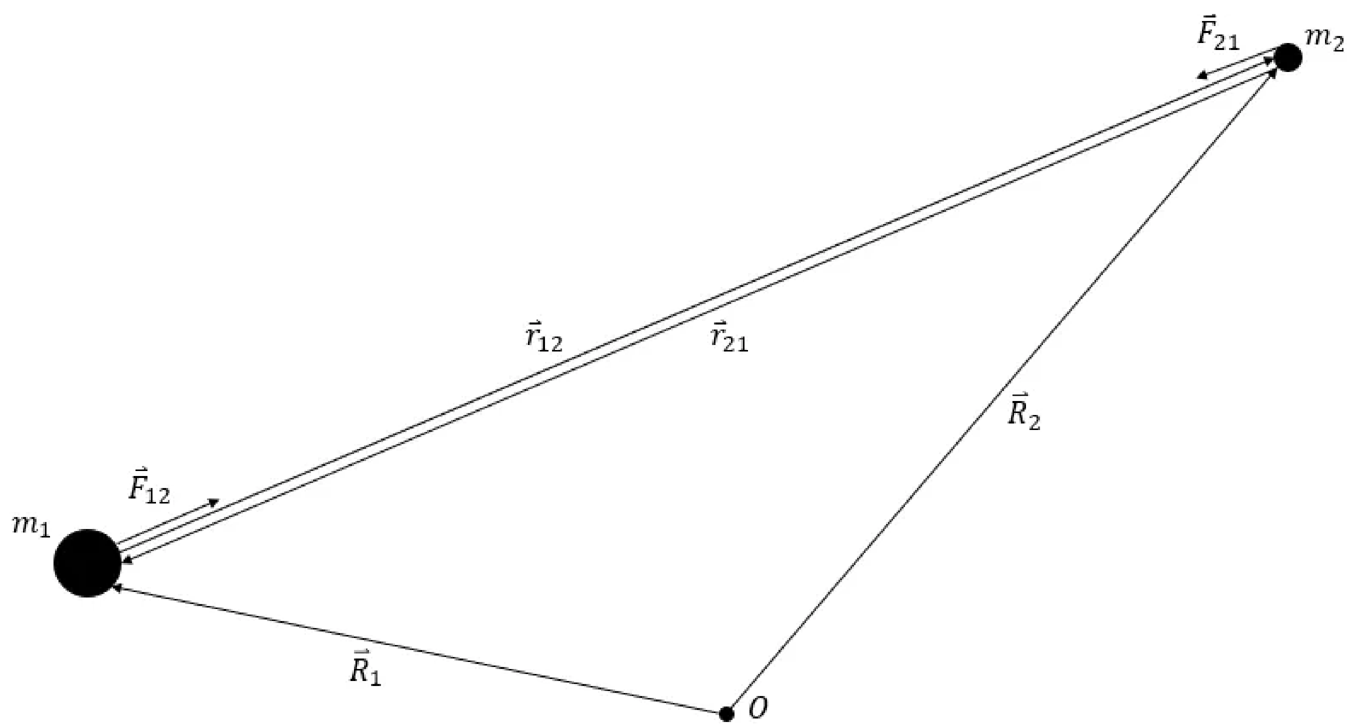
The Diagram

To begin to derive the equations of motion for a satellite in the two-body problem, we need a diagram to visualize the system. In the diagram below, you see a few vectors, our two masses, and a point labeled "*O*". The point "*O*" is an inertial origin. This means it is a point that is not accelerating (it can have a constant velocity) or rotating. This inertial point is important; we will be using it to apply Newton's laws of motion, which only apply in an inertial frame. For the two-body problem, we are typically interested in the motion of the smaller mass, m_2, with respect to the larger mass, m_1. This motion can be described by the vector r_{12} in the diagram below. Also in the diagram, we have the gravity force of m_2 acting on m_1 as F_{21}, and the other gravity force, respectively. We also have the position R vectors that locate m_1 and m_2 with respect to the inertial origin.



Two-Body Problem Diagram [Created by Author]

You may have noticed we do not have a coordinate frame attached to our inertial origin. We do not need that coordinate frame to define the equations of motion of m_2 as you will see.

Deriving Equations of Motion

The first thing we need to do is gather the equations we need for this derivation. Those equations are Newton's second law of motion and Newton's gravitational law:

$$\vec{F}_{net} = m\vec{a} = m\ddot{\vec{r}}$$
$$\vec{F}_G = \vec{F}_{12} = -\frac{GM_1M_2}{r_{21}^3}\,\vec{r}_{21}$$

where,

$$\vec{F}_{12} = \text{Gravitational force exerted on mass 1 by mass 2}$$
$$\vec{r}_{21} = \text{Vector from mass 2 to mass 1}$$

If we apply these equations to each of the particles, using the diagram to assist, we will get the following equations of motion for each of the masses:

$$m_1\ddot{\vec{R}}_1 = -\frac{Gm_1m_2}{r_{21}^3}\,\vec{r}_{21} = \frac{Gm_1m_2}{r_{12}^3}\,\vec{r}_{12}$$

$$m_2\ddot{\vec{R}}_2 = -\frac{Gm_1m_2}{r_{12}^3}\,\vec{r}_{12}$$

Note that,

$$\vec{r}_{21} = -\vec{r}_{12}$$

The *R* vectors give the inertial, second time derivative for each of the masses. If we cancel the masses on each side of the equations and change the r_{12} vector to just *r*, we get the following:

$$\ddot{\vec{R}}_1 = \frac{Gm_2}{r^3}\,\vec{r}$$

$$\ddot{\vec{R}}_2 = -\frac{Gm_1}{r^3}\,\vec{r}$$

Now let's look at the *r* vector (previously r_{12}). Using the diagram and vector subtraction, we arrive at the following relation for the *r* vector:

$$\vec{R}_2 = \vec{R}_1 + \vec{r} \qquad \text{or} \qquad \vec{r} = \vec{R}_2 - \vec{R}_1$$

Next, let's take a time derivative of the second equation twice to arrive at:

$$\ddot{\vec{r}} = \ddot{\vec{R}}_2 - \ddot{\vec{R}}_1$$

We've already solved for the second time derivative for the two *R* vectors, so we can directly plug those in to get a our result for the vector of interest, *r*.

$$\ddot{\vec{r}} = -\frac{Gm_1}{r^3}\,\vec{r} \; - \; \frac{Gm_2}{r^3}\,\vec{r}$$

$$= -\frac{G(m_1 + m_2)}{r^3}\,\vec{r}$$

A lot of the time you will see μ in this equation. This is known as the gravitational parameter, which is the product of the universal gravitational constant and our larger mass. We are typically trying to solve for the trajectory of a satellite around a much larger mass (which would be the time history of the vector *r*), so we can make the assumption that the mass of the satellite is negligible compared to the major body ($m_2 << m_1$). Let's apply this assumption and add the gravitational parameter to the equation we previously derived. Now we have:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\,\vec{r}$$

Another form,

$$\ddot{x} = -\frac{\mu}{r^3}\,x$$
$$\ddot{y} = -\frac{\mu}{r^3}\,y$$
$$\ddot{z} = -\frac{\mu}{r^3}\,z$$

where,

$$\mu = Gm_1 \qquad (m_2 \ll m_1)$$
$$r = \sqrt{x^2 + y^2 + z^2}$$

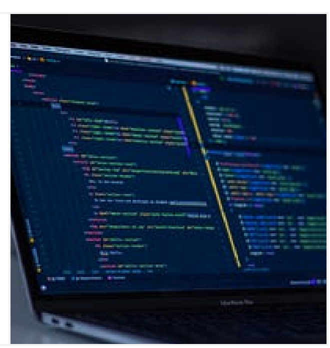
The equation above represents our final result for the equations of motion of the smaller mass, m_2, with respect to the larger mass, m_1. These equations can be used with a numerical integrator to solve for the motion of a spacecraft around a planet, moon, star, etc. Keep in mind that this is an approximation since there are other forces at play. We have simplified our system substantially, but this is a good pathway to learning more about astrodynamics.

If you want to numerically integrate this equation, check out my other articles for help:

Numerical Integration using Python

A simple method to numerically integrate equations and visualize results in Python

towardsdatascience.com



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Thank you for reading the article! I really enjoy writing these types of articles, where I can show derivations and put solid information out to you as the reader. If you have any questions or want to learn more, leave a comment and I'd be happy to answer. Stay tuned for more articles on astrodynamics!

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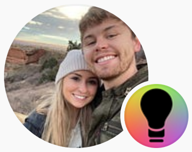

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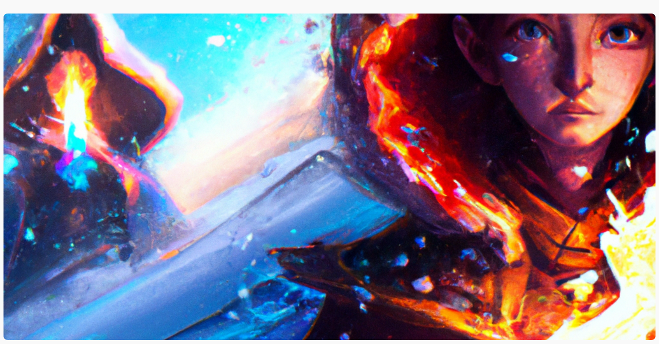

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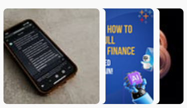

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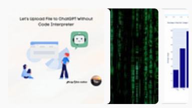

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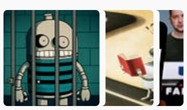

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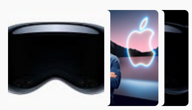

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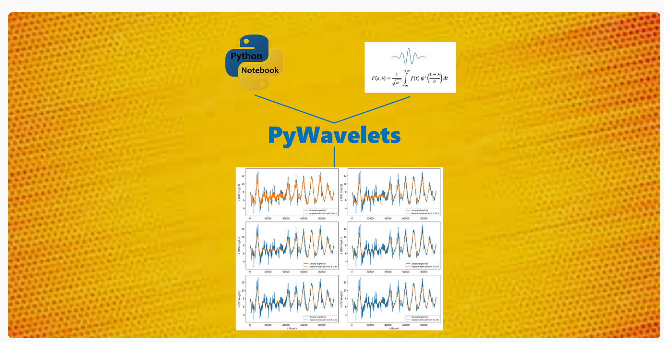

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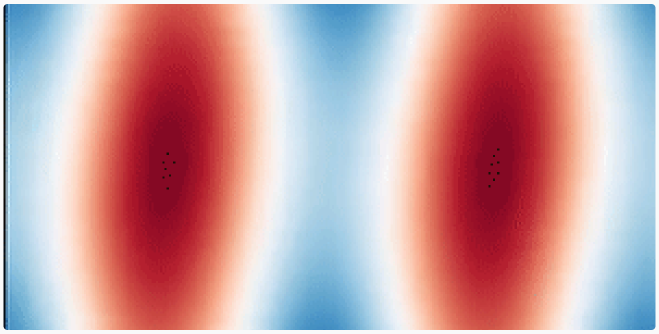

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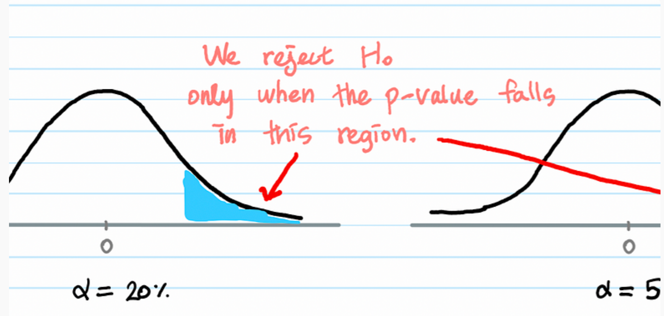

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