Calculus, SGA – [Extra] Open Question: Multivariate limit, by Magomedov Rustam

Consider the following function from the Final Quiz of the week 2: $f(x, y) = \frac{x^2 y}{x^3+y^3}$

Let us find sequential (iterated) limits given $x \to 0$ and $y \to 0$

Solving for the sequential (itetated) limits, in both instances in the numerator we eventually multiply by 0, hence we obtain

$\lim_{x\to 0}(\lim_{y\to 0} \frac{x^2 y}{x^3+y^3}) = 0$

$\lim_{y\to 0}(\lim_{x\to 0} \frac{x^2 y}{x^3+y^3}) = 0$

Although we obtained the limits when solving for iterated forms, when solving for the ordinary multivariate limit, such limit does not exist. Let us prove it.

We need to solve $\lim_{x,y\to(0,0)} \frac{x^2 y}{x^3+y^3}$ = $\lim_{x,y\to(0,0)} \frac{0}{0}$ $\implies$ indeterminate. To solve, let us substitute $y = kx$

$\lim_{x\to 0, y=kx} \frac{x^2 y}{x^3+y^3}$ =

= $\lim_{x\to 0, y=kx} \frac{x^2 kx}{x^3+k^3 x^3}$ =

= $\lim_{x\to 0, y=kx} \frac{kx^3}{x^3+k^3 x^3}$ =

= $\lim_{x\to 0, y=kx} \frac{x^3(k)}{x^3(1+k^3)}$ =

= $\lim_{x\to 0, y=kx} \frac{k}{1+k^3}$ $\implies$ the result depends on k

$\therefore$ $\lim_{x\to 0, y=kx} \frac{x^2 y}{x^3+y^3}$ = DNE

$\therefore$ sequential (iterated) limits exist, the ordinary limit DNE