Graph test – Magomedov Rustam

Q1. In a country there are several airports. Airport A is directly connected to 23 other airports. Airport B has a direct connection to 3 other airports. Each airport, except A and B, is directly connected to 10 other airports. Prove that there is an airline route (maybe with flight changes) between A and B.

Proof: let us consider that airports A and B are disconnected. Since airport A is connected to 23 other airports, in a simple graph we obtain that the sum of its vertex degree = 23. This means that it contains exactly 1 vertex of the **odd** degree. However, the property of the handshaking lemma states that the graph must have an **even number of odd vertices**. Otherwise, the number of edges in the graph will not be a whole number. This can be proved using the formula to find the number of edges in a graph. We obtain that number of edges in a graph is the sum of vertex degree divide by two, and in this case we obtain that if A is disconnected from B, then $m(A) = 23/2 = 11.5$. Likewise, $m(B) = 3/2 = 1.5$. Both of these results violate the handshaken lemma, as can be proved by the fractional number of edges.

Considering that other n airports are connected to 10 other airports, it means that each of that airport has a vertex degree of 10. We can represent the set of airports without A and B as $Airports$ = {A, C, D, ..., k}, | A,B $\notin$ N}. From that, we derive that the number of edges of airports in the set $Airports$ can be represented by the formula $m = \frac{10n}{2}$, wher n is the number of elements in the set $Airports$. Thus, the number of edges in such a subset is always **even**.

From that, we obtain that only airports A and B each have an odd number of odd degrees. Once again, by the property of the handshaking lemma the graph must have an even number of odd vertices. Thus, A and B must be connected to satisfy the condition. We prove it by calculating the number of edges in such a graph, $m = \frac{10n+23+3}{2} = \frac{10n+26}{2}$. The obtained formula will always produce the positive whole number of edges in a graph.
$\therefore$ airports A and B are connected

The below code visually describes the problem.

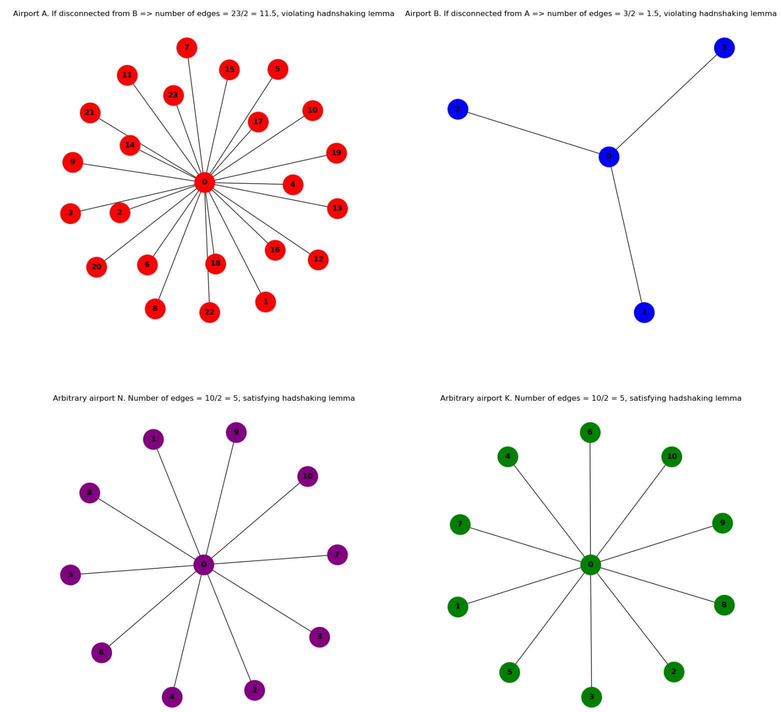
In [8]:
```python
# Demonstration of proof for why airports A and B are connected
# plotting graphs
plt.figure(figsize=(20, 20))

    # Airport A
plt.subplot(221, title = 'Airport A. \
If disconnected from B => number of edges = 23/2 = 11.5, \
violating hadnshaking lemma' )
nx.draw(AG, with_labels=True,
        font_weight='bold', node_size=1000, node_color='red')

    # Airport B
plt.subplot(222, title = 'Airport B. \
If disconnected from A => number of edges = 3/2 = 1.5, \
violating hadnshaking lemma' )
nx.draw(BG, with_labels=True,
        font_weight='bold', node_size=1000, node_color='blue')

    # Arbitrary airport N
plt.subplot(223, title = 'Arbitrary airport N. \
Number of edges = 10/2 = 5, satisfying hadshaking lemma')
nx.draw(NG, with_labels=True, font_weight='bold',
        node_size=1000, node_color='purple')

    # Arbitrary airport K - with the same properties as N
KG = NG
plt.subplot(224, title = 'Arbitrary airport K. \
Number of edges = 10/2 = 5, satisfying hadshaking lemma')
nx.draw(KG, with_labels=True, font_weight='bold',
        node_size=1000, node_color='green')
```



Q2. Consider an Erdős–Rényi random graph on 4 vertices with p=1/2. Calculate the probability that this graph is connected.

Let us firstly denote the sample space of the problem. To do it, let us consider all possible combinations of a connected graph with 4 vertices. Such a graph at max consists of 6 edges. Thus, there exists $\Omega = 2^6 = 64$ total graphs (connected and disconnected) with 4 vertices. Hence, we have 64 in our denominator.

For nominator, we need to consider all cases of the connected graph with 4 vertices. Firstly, there exists a one complete graph, such that it has 4 vertices and 6 edges. By the same token, there exists only $\binom{6}{1} = 6$ connected graphs with 4 vertices and 5 edges. By applying the same logic, we obtain $\binom{6}{2} = 15$ connected graphs with 4 edges. We have $\binom{6}{3} - \binom{4}{1} = 16$ connected graphs with 3 edges. From $m \geq \frac{(n-1)(n-2)}{2} \implies m \geq 3$ it follows that any graph with less than 3 edges will be disconnected.

Thus, the probability of an Erdős–Rényi random graph on 4 vertices to be connected is obtained by $\frac{1+6+15+16}{64} = \frac{38}{64} = 0.59375$

Q3. Which of the following pictures can be drawn in one stroke of pen, without traversing a line twice (like a Euler path in a graph)?

Let us check both graphs against the conditions for a Euler Path to exist in an undirected graph.

Such a path exists $\iff$

1. All vertices with non-zero degree in a graph are connected and
2. Zero or two vertices in a graph have an odd degree, while all other vertices have an even degree.

Consider the graph on the left

- All vertices of non-zero degree are connected.
- The graph has 9 vertices of degree 2 and 6 vertices of degree 4.
  $\therefore$ there exists a Euler path in this graph

Consider the graph on the right

- All vertices of non-zero degree are connected.
- The graph has 1 vertex of degree 8, 4 veritces of degree 5, 4 vertices of degree 4, 4 vertices of degree 3. Therefore, we obtain that there are 8 vertices with odd degree, violating the second condition for the existence of the Euler path.
  $\therefore$ there is no Euler path in this graph

Q4. Does there exist a graph with 5 vertices which have the following degrees: 2, 4, 4, 4, 4?

No, a simple connected graph with such number of vertices and vertex degrees cannot exist.
Proof: if 4 vertices in a 5-vertex graph have a vertex degree of 4, it means that each of this vertex must be adjacent to every other vertex in a graph. However, the presence of a vertex with a degree 2 violates that condition, since you 4 verteces cannot be at the same time be adjacent to the vertex with a degree 2. This would force a vertex to be of degree 4, not 2.

We can also prove that such sequence of degrees is not graphical by applying the *Havel-Hakimi* algorithm. We have the following initial ordered sequence: (4 4 4 4 2). Applying the algorithm, we obtain

- 4 4 4 4 2
- 3 3 3 1
- 2 2 0
- 1 -1

We obtain a negative number in the sequence, and by the rule of the Havel-Hakimi theorem, such a sequence is not graphical
$\therefore$ such a graph does not exist

Q5. A connected graph on 10 vertices has 15 edges. What is the maximal number of edges one can remove so that the graph remains connected? Note that if your answer is N, then you need to explain that:
a) after removing some N edges from any connected graph on 10 vertices with 15 edges the resulting graph remains connected;
b) there exists a connected graph on 10 vertices with 15 edges such that after removing any (N + 1) edges it becomes disconnected.

a) At max 6 non-random edges (edges of our choice) can be disconnected from the graph to preserve a connected graph.
Proof: for a $G = (V, E)$ such that G is a tree, we can disconect $(n - m \mid m = n - 1)$ edges. Thus, we obtain $m = 10 - 1 = 9 \implies 15 - 9 = 6$ edges. Thus, disconnecting 6 edges from such a graph when the graph is a tree still preserve a connectedness condition.

b) A minimal number of edges for a tree graph to be connected is derived from the formula $m = n - 1$, where m is the number of edges, n is the number of vertices. Following the argument made in the above subtask, we obtain that removing $6 + 1 = 7$ edges always results in a disconnected graph.

Q6. A graph on 6 vertices has 11 edges. Prove that this graph is connected.

For a graph $G = (V, E)$ to be connected, the following inequality should hold: $m \geq \frac{(n-1)\times(n-2)}{2}$, where m is the number of edges, n is the number of vertices.
Applying the condition, we obtain $11 \geq \frac{(6-1)\times(6-2)}{2} \implies$ the inequality holds as $11 \geq 10$
$\therefore$ a graph is connected

Q7. A graph on 10 vertices has 3 isolated vertices (degree 0) and 7 vertices of degree 2. Could such a graph be bipartite? How many vertices are there in an optimal vertex cover for this graph? (Consider all possible cases.)

Such a graph cannot be bipartite. We can infer it by checking the properties of a bipartite graph. Firstly, a graph cannot be bipartite if it contains an odd cycle. Since we have 7 vertices of degree 2, the exists only two ways to construct such a graph:

- a connected heptagon
- a disconnected graph consisting of 1) a triangle and 2) a square.
  $\therefore$ in both of these cases connected components will contain an odd cycle, a graph with 7 vertices of degree 2 **cannot** be bipartite

C_{opt} for such graph is 4. It can be inferred from the following conditions:

- In a heptagon a vertex cover of $\frac{7}{2} \leq 4$ would include an endpoint for an each edge for a graph
- For a triangle and a square, you'll need a closest even number such that $\frac{3}{2} \leq 2$ and the closest even number such that $\frac{4}{2} \leq 2$, so $2 + 2 = 4$ vertex cover would include an endpoint for an each edge in two components of the graph.
- 3 isolated vertices with degree 0 do not affect the C_{opt} since they do not have any edges.
  $\therefore C_{opt} = 4$