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Midterm test - Magomedov Rustam
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Q1: Can it be that $|A| = 5$, $|B| = 3$ and $|A \cup B| = 6$? If it is possible, provide an example, otherwise provide a proof that this is impossible.

We can prove that it is possible using a mathematical notation of a set union $A \cup B = \{x \mid x \in A$ or $x \in B\}$. Since element x must belong to at least one of the two sets A and B, such case as in question is possible. We just need to have one such element x that it would belong to both sets A and B simultaneously.

Consider the following example:
A = {1,2,3,4,5}, B = {4,5,6}
$\therefore |A| = 5, |B| = 3$
$\therefore$ to find the set union, we apply the rule of sum and obtain $A \cup B$ = {1, 2, 3, 4, 4, 5, 5, 6}
$\therefore$ since element repetitions are not important when determining the cardinality, we obtain that $|A \cup B| = 6$
$\because$ element 5 is in contained in both sets

Answer: possible

Q2: Can it be that $|A| = 5$, $|B| = 3$, $|A \cup B| = 6$, and $|A \cap B| = 1$? If it is possible, provide an example, otherwise provide a proof that this is impossible.

Solution: We can prove that such case is impossible by referring to the matematical notation of the cardinality of a set union $|A \cup B| = |A| + |B| - |A \cap B|$

- To count the number of unique elements in the union of two sets, we must count the sum of all unique numbers from two sets. Then, we need to remove the distinct elements that belong to both sets.
- Given the specified conditions of $|A| = 5, |B| = 3, |A \cup B| = 6, |A \cap B| = 1,$ we obtain that the equality $6 = 5 + 3 - 1$ **does not hold**
- **Hence, such case is impossible**. Such case would be possible, however, if it would hold true that $|A \cap B| = 2$

Answer: impossible

Q3: Check whether the following equivalence on sets holds by considering elements of each side and checking whether they are contained in the other side:
$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

Solution: Let us check the elements of the equivalence step by step.

Left side:

1. $x \in (A \cup B) \setminus C \iff x \in A$ or $x \in B$ or $x \in A \cap B$, and $x \notin C$
   $\implies$ element x is in the union of sets A and B, but not in the set C
   $\implies$ element x is in at least one of the two sets A or B, but not in the set C

Right side:

1. Let us denote $F = (A \setminus C) \cup (B \setminus C)$
2. $x \in F \iff (x \in A$ and $x \notin C)$ or $(x \in B$ and $x \notin C)$
   $\implies$ element x is in the set A but not in the set C, or element x is in the B but not in the set C
   $\implies$ element x is in at least one of the two sets A or B, but not in the set C

After the decomposition, we obtain that the conditions in the left & right sides are stating the same
$\therefore$ **the equivalence holds**.

Answer: the equivalence on sets holds

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*Q4: The mobile company has **6 mobile plans**. They make a survey among their clients asking for the client's **favorite** mobile plan, **second favorite** plan (that should be different from the first one) and the **most overpriced** mobile plan in client's opinion (that can be the same as his two favorite plans, or can be some other plan). What is the number of possible different outcomes of the survey? Provide a detailed explanation of your calculation. If you are using some of the rules (including the rule of sum and the rule of product), please explain why are you using these rules.*

Solution:
Let us decompose the problem first.

- When the person makes the decision of the favorite and 2-nd favorite mobile plan, this essentially becomes a problem of k**-permutations**, because
  - a person chooses $k = 2$ mobile plans out the available $n = 6$ plans
  - the order of choice matters, as (1,2) and (2,1) are *different* pairings of the mobile plans
  - the plans cannot be the same as stated by the restriction

We've decomposed & classified the part concerning **favourite** and **2-nd favourite** plans. Now, let us decompose the last part.

- When the person makes decision about the most overpriced mobile plan, it is, however, a problem of **combinations**, because
  - the order in this case does not matter because there is only one plan to be chosen
  - the repetitions cannot occur because a person makes only one decision

After the decomposition and classification of each part, we obtain that we need to find the

1. number of k-permutations such that $k = 2, n = 6$, meaning $\frac{n!}{(n-k)!}$, hence, we obtain $\frac{6!}{(6-2)!} = 30$
2. number of k-combinations such that $k = 1, n = 6$, meaning $\binom{n}{k}$, hence, we obtain $\binom{6}{1} = 6$

By the rule of product, if there exist j objects of the 1-st type, i objects of the 2-nd type, then there exist $i \times j$ pairs of objects of the given types. So, we need to multiply the number of all possible sequences for each section of choice (described above) a person makes. Simply put, for permutations part, we have sequences of length 2 (*two plans*). For combinations part, we have sequences of length 1 (*one plan*).

Applying the rule of product, we obtain $\frac{6!}{(6-2)!} \times \binom{6}{1} = 180$ possible outcomes of the survey

Answer: 180

Q5: Suppose mortgage company has 10 houses. They want to calculate distance between each two (distinct) houses among them (just usual distance on the map). We want to count how many distances we will calculate. What is wrong with the following solution attempt? Explain the mistake and provide a correct answer to the problem.

> Solution attempt: We need to count the number of pairs of houses. The first house can be picked in 10 ways. The second house can be picked in 9 possible ways, since one of the houses was already picked. By the rule of product the pairs of houses can be picked in 10 $\times$ 9 = 90 possible ways. So we will need to make 90 calculations.

Solution:
The solution attempt is wrong, as it misinterprets the problem and attempts to solve it using k**-permutations**. However, this problem belongs to **combinations** setting.

To prove it, let us decompose the problem and find its properties.

- **Repetitions are not allowed**, because we are interested in finding unique pairs of houses.
- **Order is not important**. When measuring the *absolute* distance between two houses - it really does not matter whether we measure the distance from A to B or from B to A.
- Hence, by looking at the properties, we can say that this is not a problem of k-permutations, where order of elements matters and $(A, B), (B, A)$ considered different sequences. Instead, ordering is irrelevant, thus it is a combinations problem.

To solve the problem, we simply apply the combinations formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ Applying the conditions, we get $\binom{10}{3} = 45.$

Equivalently, we can obtain the right answer by modifying to the given solution attempt: since we already know that we're interested in sequences of 2, we understand that in the solution attempt each calculation is counted twice. So, we simply need to divide the given answer by $k = 2$. Therefore, we again obtain $\frac{90}{2} = 45$

Answer: 45

Q6: How many ways are there to write down numbers from 0 to 9 in a sequence in such a way that even numbers are positioned in the sequence in the increasing order and odd numbers are positioned in the decreasing order? Provide a detailed explanation for your solution.

Solution:

- We are interested in sequences of length 10
- There exists only one right ordering of ascending even ditis, namely {0,2,4,6,8}
- There exists only one right ordering of descending odd ditis, namely {9,7,5,3,1}
- Since there exists only 1 ordering such that satisfies the conditions, we can infer that order is not important in this task. Repetitions are not allowed, so we can solve the problem applying the **combinations** notation.

When it comes to counting number of sequences, let us model the situation as follows:

- We can start with odd or even number
- Let's assume we start with an odd number. After we've placed it, we have 4 other odd numbers to distribute on the remaining 9 places. When we're finished with this task, the remaining 5 places will be occupied by even numbers in the specified ordering {0,2,4,6,8}.
- Likewise, assume that we start with an even number. After we're finished with distributing all 5 even numbers in the specified ordering, the 5 odd numbers will be distributed on 5 remining places in the only one possible order {9,7,5,3,1}.

Hence, we see that essentially problem can be narrowed down to determining the 5 places for either odd/even numbers, and the remaining 5 places are simply filled by remaining numbers so they satisfy the ordering.
Therefore, we obtain that we have $\binom{5}{10} = 252$ combinations of sequences that satisfy the conditions.

Answer: 252

Q7: Let A and B be events associated with some random experiment. Is it possible that these events never occur simultaneously and $P(A)=1/2$, $P(B)=1/3$? If yes, give an example of random experiment and events that satisfy these conditions. If no, give a proof.

Solution:
Consider a football penalty kick. For simplicity, let us model it as follows:

A striker has 2 options where to shoot

1. shoot at the corner
2. shoot at the center

Now, consider two following events:
A = "the striker shoots at the corner and the goal is scored"
B = "the striker shoots at the center and the goal is scored"

Fact: statistically, penalties taken at the corner have higher goal probability.

Applying the conditions above, we obtain that the case as follows is possible:

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$

These events by nature cannot occur simultaniously, hence such an experiment with listed probabilities is possible.

Answer: possible

Q8: There is a small village with only ten adult people living in it. Instead of elections, they form their "government" using random choice. Every adult villager has equal probability to be chosen. Consider two random experiments: The villagers want to select a Village Council that consists of three persons. The researcher is interested in the list of members of one Council. Every four months villagers select a President. The same person can be President arbitrary number of times. The same person can be a President and a member of Village Council at the same time. The researcher is interested in the ordered list of Presidents during one year.Describe the sample spaces (sets of all elementary outcomes) for both experiments. Use the corresponding notions from combinatorics. Find the number of elements in both sample spaces. Provide all calculations and detailed explanations. What is the difference between these experiments?

Solution:

- The first experiment has two distinctive properties:
  - since roles cannot be differentiated for the council members, order is not important
  - the same person cannot be picked twice for one council, so repetitions are not possible
- Considering the abovementioned properties, we obtain that the number of elements in the sample space can be obtained using **combinations** notation. Hence, the sample space can be described as $\binom{n}{k}$. Given that $n = 10, k = 3$, we obtain $\binom{10}{3} = 120$
- The second experiment has some important properties as well:
  - the president can be chosen again, so repetitions are allowed
  - the order matters for the researcher
  - the president can be a part of the council, so it remanins that $n = 10$
  - the president is chosen 12/4 = 3 times per year
- Considering the properties, we obtain that the number of elements in the sample space can be obtained using **tuples** notation, Hence, the number of elements in the sample space for this experiment can be described as n^k . Given that $n = 10$, $k = 3$, we obtain $10^3 = 1000$.

The difference between the experiments can be described as follows:

- Essentially, each person makes 3 choices in both experiments. However, due to different conditions of the experiments, the size of sample differs drastically. In the first experiment, we have a number 120 sequences of size 3. In the second experiment, we have 1000 sequences of size 3. Allowing for repetitions and making order important, thus, increases the number of outcomes substantially.

Q9: Consider the following random experiment. We roll two identical indistinguishable symmetric dices once and record the result of this tossing (i.e. "on one dice we obtained 1 and on another we obtained 2"). Describe sample space (set of all outcomes) of this experiment. What is the probability of event "on one dice we obtained 1 and on another we obtained 2"? What is the probability of event "we obtained two 1's"? What kind of sample space you consider in this problem: sample space with equal probabilities of outcomes or sample space with non-equal probabilities of outcomes? Explain your answer.

Solution:

1. Sample space can be represented as two ordered sets with repetitions, hence tuples, each of size 6. Hence, we obtain $|\Omega|$ = $6^2 = 36$

The list of possible outcomes is as follows:
Ω = { 1 1, 1 2, 1 3, 1 4, 1 5, 1 6
2 1, 2 2, 2 3, 2 4, 2 5, 2 6
3 1, 3 2, 3 3, 3 4, 3 5, 3 6
4 1, 4 2, 4 3, 4 4, 4 5, 4 6
5 1, 5 2, 5 3, 5 4, 5 5, 5 6
6 1, 6 2, 6 3, 6 4, 6 5, 6 6}

2. The total number of possible outcomes is $6^2 = 36$
   $A =$ "*on one dice we obtained 1 and on another we obtained 2*"
   $\implies$ only {1,2} and {2,1} satisfy the condition
   $\therefore P(A) = \frac{2}{36}$ = $\frac{1}{18}$
3. Given that dices are indistinguishable and symmetric, we assume that outcome {1,1} is present in our sample space only once. Hence, the probability of this event = $\frac{1}{36}$
4. Considering that 1) two dices are symmetric, 2) it is not specified that dices are unfair, 3) the result of one dice is not influenced by the result of another, one can assume that the probability of each outcome is the same.

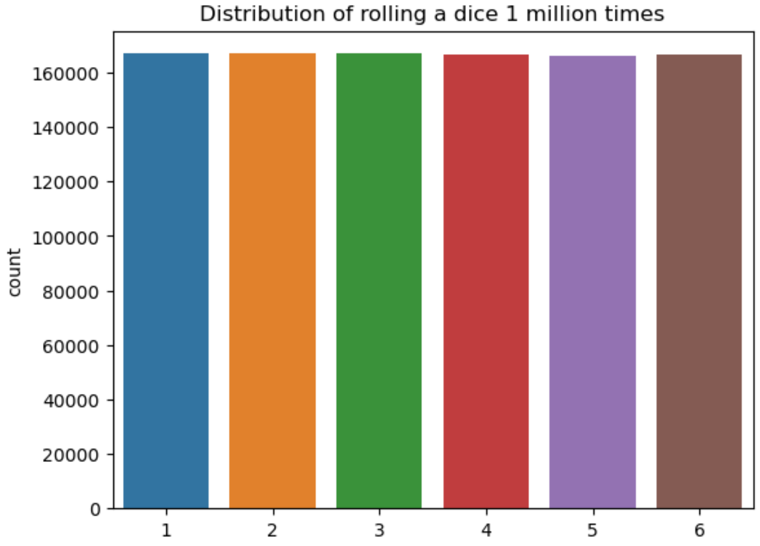
The code below shows the distribution of the 1 million rollings of a dice and of a pair of dices. Given that the first distribution is symmetric and the second is normal, we can conclude that the prbability of outcomes is approximately equal.

In [1]:
```python
from random import randint
import seaborn as sns
from warnings import filterwarnings
filterwarnings('ignore')

def dice_distribution():
    "Rolls the dice million times"
    return [randint(1, 6) for _ in range(1_000_000)]

sns.countplot(dice_distribution()).set(title = 'Distribution of rol
```

Out[1]:
```
[Text(0.5, 1.0, 'Distribution of rolling a dice 1 million times')]
```



In [2]:
```python
def roll_two_dices():
    "Rolls two dices"
    return randint(1, 6) + randint(1, 6)

def roll_million_times():
    "Rolls a pair of dices million times"
    return [roll_two_dices() for _ in range(1_000_000)]

sns.countplot(roll_million_times()).set(title = 'Distribution of ro
```

Out[2]:
```
[Text(0.5, 1.0, 'Distribution of rolling a pair of dices 1 million
times')]
```