Q2

$A = U\Sigma V^T$

First, let us obtain singular values. For that we need to find the square roots of eigenvalues for the AA^T.

Given $A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix}$, we get

$AA^T = \begin{pmatrix} 2 & 1 & 2 \\ 2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -2 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & -2 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 9 & -9 & 18 & 9 \\ -9 & 9 & -18 & -9 \\ 18 & -18 & 36 & 18 \\ 9 & -9 & 18 & 9 \end{pmatrix} \implies$ we can find eigenvectors and eigenvalues of AA^T

Given $\vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \vec{v_4} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\lambda_1 = \sqrt{63}, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$

$$\implies \Sigma = \begin{pmatrix} \sqrt{63} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To find U, we orthonormalize the obtained eigenvectors $\vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \vec{v_4} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ using the Gram-Schmidt process

$$\therefore \vec{u_1} = \vec{v_1}, \vec{e_1} = \frac{\vec{u_1}}{||\vec{u_1}||} = \begin{pmatrix} \frac{1}{\sqrt{7}} \\ -\frac{1}{\sqrt{7}} \\ \frac{2}{\sqrt{7}} \\ \frac{1}{\sqrt{7}} \end{pmatrix}$$

$$\overrightarrow{u_2} = \overrightarrow{v_2}, \overrightarrow{e_2} = \frac{\overrightarrow{u_2}}{||\overrightarrow{u_2}||} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\overrightarrow{u_3} = \overrightarrow{v_3}, \overrightarrow{e_3} = \frac{\overrightarrow{u_3}}{||\overrightarrow{u_3}||} = \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\overrightarrow{u_4} = \overrightarrow{v_4}, \overrightarrow{e_4} = \frac{\overrightarrow{u_4}}{||\overrightarrow{u_4}||} = \begin{pmatrix} \frac{1}{\sqrt{42}} \\ \frac{\sqrt{42}}{7} \\ \frac{\sqrt{42}}{21} \\ \frac{1}{\sqrt{42}} \end{pmatrix}$$

$$\implies U = \begin{pmatrix} \frac{1}{\sqrt{7}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{42}} \\ -\frac{1}{\sqrt{7}} & 0 & 0 & \frac{\sqrt{42}}{7} \\ \frac{2}{\sqrt{7}} & 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{42}}{21} \\ \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{sqrt42}{21} \end{pmatrix}$$

Given that one of our eigenvectors already satisfies the condition for upper corner of $V = \frac{1}{\sqrt{2}}$, we can get V through $A^T A$

$$A^T A = \begin{pmatrix} 2 & -2 & 4 & 2 \\ 1 & -1 & 2 & 1 \\ 2 & -2 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 2 \\ -2 & -1 & -2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 28 & 14 & 28 \\ 14 & 7 & 14 \\ 28 & 14 & 28 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}, \text{ and related eigenvectors } v_3 = \begin{pmatrix} -0.5 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \implies V = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{3} & \frac{2}{2\sqrt{2}} & 0 \\ \frac{2}{3} & -\frac{1}{3\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

The sanity check using *numpy* is below.

In [6]:
```
import numpy as np

AAT = np.array([[9, -9, 18, 9], [-9, 9, -18, -9], [18, -18, 36, 18], [9, -9, 18, 9]])
print(f"U:\n{np.round(np.linalg.svd(AAT, full_matrices=True)[0],4)}\n\nS:{np.round(np.linalg.svd(AAT, full_matrices=True)[1])}\n\n\nPls note that U and V sh

ATA = np.array([[28,14,28], [14,7,14], [28,14,28]])
print(f"V:\n{np.linalg.svd(ATA, full_matrices=True)[2] * -1}")
```

```
U:
[[-0.378  -0.9258  0.      0.    ]
 [ 0.378  -0.1543 -0.9129  0.    ]
 [-0.7559  0.3086 -0.3651 -0.4472]
 [-0.378   0.1543 -0.1826  0.8944]]

S:[63.  0.  0.  0.]


Pls note that U and V shall not necessarily align with the manually computed eigenvectors due to the numpy vector scaling or arbitrary rotation. In our ca
se U has flipped signs, meaning scaling by -1; also, there's a presence of arbitrary vector rotation

V:
[[ 0.66666667  0.33333333  0.66666667]
 [-0.74535599  0.2981424   0.59628479]
 [-0.          0.89442719 -0.4472136 ]]
```