Let's denote the nodes of the system by corresponding variables. We obtain the following system of equations:

$$\begin{cases} x_1 = \frac{1}{2} \cdot x_3 + \frac{1}{3} \cdot x_4 \\ x_2 = 1 \cdot x_2 + \frac{1}{3} \cdot x_4 \\ x_3 = \frac{1}{2} \cdot x_1 \\ x_4 = \frac{1}{3} \cdot x_4 + \frac{1}{2} \cdot x_1 + \frac{1}{2} \cdot x_3 \end{cases}$$

We can rewrite the system in the matrix form: $M = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$

As we know, each page gets the equal treatment, so the rank vector $\vec{v} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$

To obtain the most influenced vertex, we can iteratively scale the matrix by the rank vector for an arbitrary number of iterations i_n and update our rank vector after each iteration. Hence, the solution will have a following form:

$$\vec{v_1} = M \cdot \vec{v}$$
$$\vec{v_2} = M \cdot 2\vec{v} = M \cdot (M \cdot \vec{v})$$
$$\vec{v_3} = M \cdot 3\vec{v} = M \cdot (M \cdot M \cdot \vec{v})$$
$$\vec{v_i} = M \cdot i\vec{v} = M \cdot (M^i \cdot \vec{v})$$

Even after a few iterations, we obtain

$$\vec{v_1} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{5}{24} \\ \frac{1}{3} \\ \frac{1}{8} \\ \frac{1}{3} \end{pmatrix}$$

$$\vec{v_2} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{5}{24} \\ \frac{1}{3} \\ \frac{1}{8} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{25}{144} \\ \frac{4}{9} \\ \frac{5}{48} \\ \frac{5}{18} \end{pmatrix}$$, with the updated probabilities we obtain $p = \begin{pmatrix} 0.17 \\ 0.44 \\ 0.10 \\ 0.28 \end{pmatrix}$

$$\begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} {}^{25} \\ \\ \\ \end{pmatrix} \begin{pmatrix} {}^{125} \\ \\ \\ \end{pmatrix}$$

$$\vec{v_3} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{25}{144} \\ \frac{4}{9} \\ \frac{5}{48} \\ \frac{5}{18} \end{pmatrix} = \begin{pmatrix} \frac{125}{864} \\ \frac{29}{54} \\ \frac{25}{288} \\ \frac{25}{108} \end{pmatrix}, \text{ with the updated probabilities we obtain } p = \begin{pmatrix} 0.14 \\ 0.54 \\ 0.09 \\ 0.23 \end{pmatrix}$$

$$\vec{v_4} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} \frac{125}{864} \\ \frac{29}{54} \\ \frac{25}{288} \\ \frac{25}{108} \end{pmatrix} = \begin{pmatrix} \frac{625}{5184} \\ \frac{197}{324} \\ \frac{125}{1728} \\ \frac{125}{648} \end{pmatrix}, \text{ with the updated probabilities we obtain } p = \begin{pmatrix} 0.12 \\ 0.61 \\ 0.07 \\ 0.20 \end{pmatrix}$$

$\therefore$ the vertex 2 is growing at higher rates, both in terms of the probability increase and relative delta. I attach the *numpy* code below for consistency & sanity check.

In [50]:
```python
import numpy as np

# matrix & rank vector
m = np.array([[0,0,1/2,1/3], [0,1,0,1/3], [1/2,0,0,0], [1/2, 0, 1/2, 1/3]])
page_rank_vector = np.array([[0.25], [0.25], [0.25], [0.25]])

# 10 iterations
for i in range(1, 7):
    page_rank_vector = np.dot(m, page_rank_vector)
    print(f"Iteration: {i}\nRanks:\n{[float(i) for i in page_rank_vector]}\n")
print(f"most important page index: {np.argmax(page_rank_vector)}\nmost important page number: {np.argmax(page_rank_vector)+1}")
```

```
Iteration: 1
Ranks:
[0.20833333333333331, 0.3333333333333333, 0.125, 0.3333333333333333]

Iteration: 2
Ranks:
[0.1736111111111111, 0.4444444444444444, 0.10416666666666666, 0.2777777777777778]

Iteration: 3
Ranks:
[0.14467592592592593, 0.537037037037037, 0.08680555555555555, 0.23148148148148148]

Iteration: 4
Ranks:
[0.12056327160493827, 0.6141975308641975, 0.07233796296296297, 0.19290123456790123]

Iteration: 5
Ranks:
[0.10046939300411523, 0.6784979423868313, 0.06028163580246913, 0.16075102880658437]

Iteration: 6
Ranks:
[0.08372449417009603, 0.7320816186556927, 0.05023469650205761, 0.13395919067215364]

most important page index: 1
most important page number: 2
```