Q1

Converting the SLAE to the augmented matrix form, we obtain $\begin{bmatrix} 5 & 1 & -2 & 6 & 0 \\ 1 & 3 & -2 & 4 & 0 \\ 3 & 2 & -2 & 5 & 0 \\ 4 & 5 & -4 & 9 & 0 \end{bmatrix}$.

By definition, this system is homogenous given that for each linear equation we have 0 on the right side.

To obtain a system of solutions, as a first step let us transofrm the coefficient matrix to the row reduced echelon form. We obtain

$$\begin{pmatrix} 5 & 1 & -2 & 6 \\ 1 & 3 & -2 & 4 \\ 3 & 2 & -2 & 5 \\ 4 & 5 & -4 & 9 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{5} & \frac{-2}{5} & \frac{6}{5} \\ 1 & 3 & -2 & 4 \\ 3 & 2 & -2 & 5 \\ 4 & 5 & -4 & 9 \end{pmatrix}^{I=\frac{1}{5}\cdot I} = \begin{pmatrix} 1 & \frac{1}{5} & \frac{-2}{5} & \frac{6}{5} \\ 0 & \frac{14}{5} & \frac{-8}{5} & \frac{14}{5} \\ 3 & 2 & -2 & 5 \\ 4 & 5 & -4 & 9 \end{pmatrix}^{II=II-I} =$$

$$\begin{pmatrix} 1 & \frac{1}{5} & \frac{-2}{5} & \frac{6}{5} \\ 0 & \frac{14}{5} & \frac{-8}{5} & \frac{14}{5} \\ 0 & \frac{7}{5} & \frac{-4}{5} & \frac{7}{5} \\ 4 & 5 & -4 & 9 \end{pmatrix}^{III=III-3\cdot I} = \begin{pmatrix} 1 & \frac{1}{5} & \frac{-2}{5} & \frac{6}{5} \\ 0 & \frac{14}{5} & \frac{-8}{5} & \frac{14}{5} \\ 0 & \frac{7}{5} & \frac{-4}{5} & \frac{7}{5} \\ 0 & \frac{21}{5} & \frac{-12}{5} & \frac{21}{5} \end{pmatrix}^{IV=IV-4\cdot I} = \dots =$$

$$\begin{pmatrix} 1 & 0 & \frac{-2}{7} & 1 \\ 0 & 1 & \frac{-4}{7} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let us also make a sanity check for row reduced echelon form using *sympy*.

In [1]:

```python
# sanity check
import sympy as sp
import numpy as np

SLAE = np.array([[5,1,-2,6], [1,3,-2,4], [3,2,-2,5], [4,5,-4,9]])

sp.Matrix(SLAE).rref()[0]
```

Out[1]:

$$\begin{bmatrix} 1 & 0 & -\frac{2}{7} & 1 \\ 0 & 1 & -\frac{4}{7} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the obtained RREF matrix is correct. As we can see from the obtained matrix, we have a matrix rank of 2, and consequently 2 free variables. Let us next present this in a variable form:

$$\begin{cases} x_1 - \frac{2}{7}x_3 + x_4 = 0 \\ x_2 - \frac{4}{7}x_3 + x_4 = 0 \end{cases}$$

When transforming this to vectors, we get a general solution:

$$\begin{cases} x_1 = \frac{2}{7}x_3 - x_4 \\ x_2 = \frac{4}{7}x_3 - x_4 \end{cases}$$

$\implies$ fundamental system of solutions of a form $v1, v2$: $\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{2}{7} \\ \frac{4}{7} \\ 1 \\ 0 \end{pmatrix}$

Let us check whether matrices A and/or B form a fundamental system of solutions when checked against obtained vectors.

Matrix A we get the sum of vectors $\begin{pmatrix} 4 \\ 8 \\ 14 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 7 \\ 0 \end{pmatrix} \implies$ linear dependence + violation of the presence of 1 in 4-th row of the 1-st column of $v1$ in the system of solutions, which cannot be obtained by either scaling or summing $\therefore$ not a basis

For matrix B, we get $\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \\ 14 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + 14 \cdot \begin{pmatrix} \frac{2}{7} \\ \frac{4}{7} \\ 0 \\ 1 \end{pmatrix} \implies$ matrix B appears to be a part of the fundamental system of solutions given the value of $v1 = 1$ and $v2 = 14$; it also follows that $\begin{cases} 1 \cdot -1 + 1 \cdot 1 = 0 \\ 1 \cdot 8 - 14 \cdot \frac{4}{7} = 0 \end{cases}$

`∴ matrix B forms a fundamental system of solutions for a given SLAE.`